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On Realizing Modular Data

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Abstract. We use zesting and symmetry gauging of modular tensor categories to analyze some previously unrealized modular data obtained by Grossman and Izumi. In one case we find all realizations and in the other we determine the form of possible realizations; in both cases all realizations can be obtained from quantum groups at roots of unity.

1. Introduction

The modular data of a modular tensor category (MTC) is the most useful invariant of MTCs. Recently, several iterations of new constructions inspired by the doubled Haagerup MTC produced intriguing new potential modular data that have passed all known consistency conditions imposed on modular data from the full MTC structure [11], with the latest construction presented in [12]. These new potential modular data lead to an obvious question: can we construct MTCs that realize these modular data? The difficulty encountered in realizing MTCs with these potential modular data suggests that some major constructions for MTCs have yet to be discovered.

As suggested in [4, 6, 7], zesting and symmetry gauging can be considered as a new construction of new MTCs from old ones. In this short note, we point out that some of the potential modular data in [12] can be realized by zesting and gauging of known modular tensor categories.

1.1. Conventions

We will always assume the categories we study are pseudo-unitary, with spherical structure chosen to be the unique canonical choice with the prop-

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erty that dimensions of simple objects are equal to their Frobenius-Perron dimensions (see [10, Prop. 8.3]). As all modular data we are presented with has this property, we do not lose any generality.

From the quantum group $U_q\mathfrak{sl}_N$ with $q = e^{\pi i/\ell}$ one may construct a unitary modular category [1, Section 3.3] which we denote by $SU(N)_k$, setting $k = \ell - N$. The simple objects in $SU(N)_k$ are labeled by a certain finite subset of the highest weights of type A_{N-1} (which lie in $\frac{1}{N}\mathbb{Z}^N$) and the fusion rules are truncated versions of the tensor-product rules for $\mathfrak{sl}_N$. The subcategory of $SU(N)_k$ generated by simple objects labeled by integer weights is a ribbon category, which we denote by $PSU(N)_k$, (see [2] where the notation PGL_N is used). In fact, each $SU(N)_k$ is $\mathbb{Z}_N$-graded with trivial component $PSU(N)_k$. In [16] fusion categories with the same fusion rules are $SU(N)_k$ are classified: they call such a category an $SL_{N,\ell}$ category where $\ell = N + k$, and for historic reasons their q is our q^2. They show that any $SL_{N,\ell}$ category is equivalent to one obtained from $SU(N)_k$ by 1) changing q to another primitive root of unity of order ℓ and/or 2) twisting the associativity on each graded component by a 3-cocycle with values in the Nth roots of unity. We emphasize here that this associativity twisting restricts to the identity on the trivial component. While this category is *a priori* defined over $\mathbb{C}$, it is known [2, Corollaire 2.2.5] that $SU(N)_k$ and $PSU(N)_k$ may be defined over $\mathbb{Q}(q^{1/N})$. In particular, replacing q by a different primitive root of unity has the effect (after lifting) of a Galois conjugation on this cyclotomic field, and hence on the category itself. Of course Galois conjugation will generally not preserve (pseudo-)unitarity, but complex conjugation always does, and we denote the category obtained from $PSU(N)_k$ by replacing $q = e^{\pi i/\ell}$ with $q^{-1} = e^{-\pi i/\ell}$ by $\overline{PSU(N)}_k$.

We adopt standard physics notation for braided fusion categories with fusion rules like $\mathbb{Z}_2$: the two distinct unitary modular categories are denoted Sem and $\overline{\text{Sem}}$, the Tannakian symmetric category by $\text{Rep}(\mathbb{Z}_2)$ and the super-Tannakian symmetric category by sVec. The non-trivial simple object in such a category are called *semions, bosons*, and *fermions*, respectively.

2. Realizing Modular Data by Zesting and Gauging

2.1. Dualizing topological symmetry

Modular data of an MTC was once conjectured to determine uniquely an MTC. However, this conjecture was recently disproven in [17] by the construction of counterexamples. In retrospect, counterexamples to the conjecture were already suggested by the results of [8], which showed that there are some MTCs for which the map that sends an object X to its dual X^*, which we will refer to as "topological charge conjugation," cannot be extended to a braided tensor auto-equivalence. A braided tensor auto-equivalence that extends a topological charge conjugation will be called a dualizing topological symmetry. Since topological charge conjugation always preserves the modular data, it is probably true that those MTCs which are uniquely determined by their modular data would always have a dualizing topological symmetry. If true, then it would follow that the modular data cannot uniquely determine MTCs for which the topological charge conjugation cannot be extended to a dualizing topological symmetry. Even so, modular data is still a powerful and convenient invariant of MTCs.

2.2. Gauging and zesting

From a fixed modular category there are several ways to construct new modular categories. Two highly non-trivial constructions are *Gauging* (see [6]) and *zesting* (see [4]).

Given a categorical action of a finite group G on a modular category $\mathcal{C}$ one can try to gauge this symmetry to obtain a new category $\mathcal{C}_G^{\times,G}$ of dimension $|G|\dim(\mathcal{C})$. There are two cohomological obstructions and two cohomological choices in this process, so it is not always possible, and does not give a unique category when it is possible. However, there is a reverse process that is unique: *boson condensation.* Given a modular category containing a Tannakian category $\mathcal{D} \supset \mathcal{T} \cong \mathrm{Rep}(G)$, one first de-equivariantizes to obtain a faithfully G-graded category $\mathcal{D}_G = \bigoplus_g \mathcal{C}_g$. The trivially graded component $\mathcal{C}_e$ is modular, and $\mathcal{D}$ may be obtained as a G-gauging of $\mathcal{D}$.

A related construction known as zesting is as follows: if $\mathcal{C}$ is a G-graded modular category with group of $\otimes$-invertible objects in the trivial component $H = U(\mathcal{C}_e)$, then we may construct new fusion rules from those of $\mathcal{C}$ by a rule of the form $X \overset{\lambda}{\otimes} Y = X \otimes Y \otimes \lambda(a,b)$ where $X \in \mathcal{C}_a$, $Y \in \mathcal{C}_b$ and $\lambda(a,b) \in H$. Similarly as above, there are obstructions to extending this to a modular category, and choices to be made (see [9] for the full details) but this often leads to a new category with the same dimension and rank as $\mathcal{C}$, but with potentially new fusion rules. In the case where the category generated by H is sVec, the construction is found in [4].

2.3. Grossman-Izumi modular data

In [12], potential modular data is presented that generalizes the modular data of known Drinfeld centers of near-group fusion categories. In detail, the input for the data consists of: two involutive metric groups (G, q_1, θ_1) and (Γ, q_2, θ_2), i.e. (G, q_1) and (Γ, q_2) are metric groups, while θ_1 and θ_2 are, respectively, involutive automorphisms of G and Γ that preserve the quadratic forms q_1 and q_2. Since G and Γ are necessarily Abelian, we will write them as additive, but the co-domain of q_i is the multiplicative group $U(1)$. Additionally, these must satisfy:

(i) The pre-metric groups obtained by restriction of q_i to the θ_i fixed points of G and Γ coincide, i.e. $(G^{\theta_1}, q_1) \cong (\Gamma^{\theta_2}, q_2) =: (K, q_0)$.

(ii) $\mathcal{G}(q_1) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} q_1(g) = -\mathcal{G}(q_2) = \frac{1}{\sqrt{|\Gamma|}} \sum_{\gamma \in \Gamma} q_2(\gamma)$.

Given such $(G, q_1, \theta_1), (\Gamma, q_2, \theta_2)$, one furthermore chooses $G_* \subset G$ and $\Gamma_* \subset \Gamma$ so that $G = K \sqcup G_* \sqcup \theta_1(G_*)$ and $\Gamma = K \sqcup \Gamma_* \sqcup \theta_2(\Gamma_*)$, where $K = G^{\theta_1} \cong \Gamma^{\theta_2}$. The label set for the modular data is

$$J := K \sqcup (K \times \pi) \sqcup G_* \sqcup \Gamma_*.$$

In particular the rank of the modular data is $|K| + \frac{|G|+|\Gamma|}{2}$.

The non-degenerate bicharacter associated with (G, q_1) and (Γ, q_2) will be denoted $B_i(g, h) = \frac{q_i(g) q_i(h)}{q_i(g+h)}$ for $i = 1$ and 2, respectively, and we identify K and $K \times \{\pi\}$ with the appropriate subgroup of both G and Γ when computing these values. Define constants $a = 1/\sqrt{|G|}$ and $b = 1/\sqrt{|\Gamma|}$. The S matrix has the following block structure, where k, k' range over K, κ, κ' range over $K \times \{\pi\}$, g, g' range over G_*, and γ, γ' range over Γ_*:

$$S = \begin{bmatrix} \frac{a-b}{2} B_1(k,k') & \frac{a+b}{2} B_1(k,\kappa') & a B_1(k,g') & b B_2(k,\gamma') \\ \frac{a+b}{2} B_1(\kappa,k') & \frac{a-b}{2} B_1(\kappa,\kappa') & a B_1(\kappa,g') & -b B_2(\kappa,\gamma') \\ a B_1(g,k') & a B_1(g,\kappa') & a(B_1(g,g') + B_1(\theta_1(g),g')) & 0 \\ b B_2(\gamma,k') & -b B_2(\gamma,\kappa') & 0 & -b\big(B_2(\gamma,\gamma') + B_2(\theta_2(\gamma),\gamma')\big) \end{bmatrix} \tag{1}$$

The T matrix has the form:

$$T = \mathrm{Diag}[q_0(k), q_0(\kappa), q_1(g), g_2(\gamma)]. \tag{2}$$

In general, it is an open question whether this data is realized, i.e., if there is a modular category with this S and T matrices. Assuming a realization exists, it is clear that the objects labeled by K are invertible (have FP-dimension 1), the objects in $K \times \{\pi\}$ have dimension $\frac{a+b}{a-b}$, the objects labeled by G_* have dimension $\frac{2a}{a-b}$, and the objects labeled by Γ_* have dimension $\frac{2b}{a-b}$. Thus, there are 4 distinct dimensions, generically. Indeed, unless $9|G| = |\Gamma|$ the objects labeled by K are the only invertible objects, and the fusion rules of this pointed subcategory are the same as the group operation in K, i.e., it is the pointed ribbon fusion category associated with the pre-metric group (K, q_0).

2.4. Zesting to realize case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $G = \mathbb{Z}_4$ and $\Gamma = G \times \mathbb{Z}_3$

At the October 2018 BIRS workshop on fusion categories and subfactors, Izumi presented [15] the particular example $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ of their construction of potential modular data and asked if a categorification is known. He pointed out that any such categorification could not be a Drinfeld center as the (multiplicative) central charge is not 1. The second and third authors recognized and pointed out the similarities between the presented data and that of the rank=10 Drinfeld center of the $1/2E_6$ theory $\mathcal{Z}(\mathcal{E})$ in [13]. We checked later that all 16 zestings [4] of $\mathcal{Z}(\mathcal{E})$ and its complex conjugate appear as Grossman-Izumi modular data, which also appeared in [12]. We present the details of zesting here. The case of zesting spin modular categories by the sVec subcategory afforded by the distinguished fermion is worked out in [4], and the general case is found in [9].

When $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with $\theta_1(x, y) = (y, x)$ or $G = \mathbb{Z}_4$ with $\theta_1(x) = -x$ it is clear that $K \cong \mathbb{Z}_2$ in either case. The possible θ_1-invariant forms on $\mathbb{Z}_2 \times \mathbb{Z}_2$ are $q_1^f(x, y) = (-1)^{x^2+xy+y^2}$, $q_1^{tc}(x, y) = (-1)^{xy}$ and $q_1^s(x, y) = i^{\pm(x^2+y^2)}$, corresponding to the 3 fermion theory 3F, the toric code theory TC, and $(\mathrm{Sem}^{\boxtimes 2})^{\pm 1}$, respectively. The corresponding Gauss sums are $-1, 1$ and $\pm i$. For $G = \mathbb{Z}_4$ there are also 4 possible θ_1-invariant forms: $q_1^r(x) := \zeta_8^{rx^2}$ with r odd, corresponding to the 4 distinct $\mathbb{Z}_4$ theories, with Gauss sums ζ_8^r. In all cases the form on $K \cong \mathbb{Z}_2$ is $q_0(x) = (-1)^x$ corresponding to the pre-metric group associated with sVec.

As in [12], the smallest interesting case is $\Gamma = G \times \mathbb{Z}_3$ with G as above. In order to fulfil $G^{\theta_1} \cong \Gamma^{\theta_2}$, we clearly should take $\theta_2(A, z) = (\theta_1(A), -z)$ where $A \in G$. From this we can already see that $|G_*| = 1$ and $|\Gamma_*| = 5$. For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ we can choose $G_* = \{(1, 0)\}$ and $\Gamma_* = \{(1, 0, 1), (1, 0, 0), (0, 1, 1), (1, 1, 1), (0, 0, 1)\}$, and for $G = \mathbb{Z}_4$ we take $G_* = \{1\}$ and $\Gamma_* = \{(1, 0), (1, 1),$

$(0,1),(1,2),(2,1)\}$. In all cases, the object corresponding to the non-trivial element $\psi \in K$ is a fermion, as computed above. In particular, each category must be $\mathbb{Z}_2$-graded, with the objects X satisfying $B_i(X,\psi)=1$ forming the trivial component, i.e. the centralizer of the fermion ψ. This trivial component has the same fusion rules as $PSU(2)_{10}$: this can be checked directly from the S-matrix, or one simply observes that it is a super-modular category of rank 6 with an object of dimension $\frac{2b}{a-b}=1+\sqrt{3}$, which determines the fusion rules by [5]. It follows that trivial component is either equivalent to $PSU(2)_{10}$ or its complex conjugate, by [5, Theorem 3.1]. To give a little more detail, we note that braided fusion categories with fusion rules like $PSU(2)_{10}$ are shown in [18, Corollary 8.8] to be equivalent to a Galois conjugate of $PSU(2)_{10}$, i.e. by choosing a different q (note: they use the notation $SO(3)_q$). Since we require unitarity, checking the quantum dimension $q^2+q^{-2}=1+\sqrt{3}$ the only choices are $PSU(2)_{10}$ and its complex conjugate. We enumerate the cases, with the first three corresponding to $G=\mathbb{Z}_2\times\mathbb{Z}_2$ and the last to $G=\mathbb{Z}_4$:

(i) If $q_1(x,y)=(-1)^{xy}$ then $q_2(x,y,z)=i^{\pm(x^2+y^2)}\omega^{\pm z^2}$ where $\omega=e^{2\pi i/3}$. These cases are recognized as $\mathcal{Z}(\mathcal{E})$ and its complex conjugate: one sees that the (multiplicative) central charge is 1.

(ii) If $q_1(x,y)=(-1)^{x^2+xy+y^2}$ we similarly get two possible choices: $q_2(x,y,z)=i^{\pm(x^2+y^2)}\omega^{\mp z^2}$ where $\omega=e^{2\pi i/3}$.

(iii) If $q_1(x,y)=i^{\pm(x^2+y^2)}$ then either $q_2(x,y,z)=(-1)^{xy}\omega^{\mp z^2}$ or $q_2(x,y,z)=(-1)^{x^2+xy+y^2}\omega^{\pm z^2}$ which gives 4 more choices.

(iv) For $q_1^r(x)=\zeta_8^{rx^2}$ with r odd, $q_2(x,y)=\zeta_8^{sx^2}\omega^{\varepsilon z^2}$ with s odd, $\frac{r-s}{2}$ odd and $\varepsilon=\pm1$ which is determined by (s,r). This gives 8 possible distinct triples (r,s,ε).

The 16 Grossman-Izumi modular data constructed from the above can be compared with the 16 modular data constructed in [4, Section G] from zestings of $SU(2)_{10}$ and its complex conjugate, and one finds each appears. They each have distinct central charges, which makes the comparison practical. These are also exactly the 16 rank 10 minimal modular extensions of $PSU(2)_{10}$ and its complex conjugate (these also have rank 11 minimal modular extensions). Summarizing, we have:

THEOREM 2.1. *The* 16 *distinct Grossman-Izumi modular data associated with the involutive metric groups* (G, q_1, θ_1) *and* (Γ, q_2, θ_2) *with* $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $G = \mathbb{Z}_4$ *and* $\Gamma = G \times \mathbb{Z}_3$ *as described above are realized by the rank* 10 *minimal modular extensions of* $PSU(2)_{10}$ *and its complex conjugate.*

2.5. Gauging to realize case $G = \mathbb{Z}_4 \times \mathbb{Z}_4$, $\Gamma = \mathbb{Z}_{16} \times \mathbb{Z}_2$

2.5.1 Grossman-Izumi data for $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ *and* $\Gamma = \mathbb{Z}_{16} \times \mathbb{Z}_2$.

We follow the paper [12, Section 3.2.3]. In this case we start with

$$\theta_1(x, y) = (y, x), \quad \theta_2(x, y) = (3x + 8y, x + y). \tag{3}$$

It is clear that $G^{\theta_1} = \langle (1, 1) \rangle \cong \mathbb{Z}_4$ and $\Gamma^{\theta_2} = \langle (4, 1) \rangle \cong \mathbb{Z}_4$. We assume that $q_2(x, y) = \zeta_{32}^{rx^2} i^{-ry^2}$, with $r \in \{\pm 1, \pm 3\}$, as these preserve θ_2. Notice that the Gauss sum $\mathcal{G}(q_2)$ for $r = \pm 3$ is -1 whereas for $r = \pm 1$ we have $+1$. Since $q_2(4, 1) = q_2(12, 1) = i^r$, this gives us a further constraint on q_1. The possible q_1 and values of r are determined essentially as follows:

(i) First consider the form $q_1(x, y) = i^{sxy}$ on $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ for s odd. Since $q_1(1, 1) = q_1(3, 3) = i^s$, we see that $s \equiv r \pmod 4$ as $q_1(1, 1) = q_2(4, 1)$. The Gauss sums are always $\mathcal{G}(q_1) = 1$, so the $\mathcal{G}(q_1) = -\mathcal{G}(q_2)$ condition above shows we must take $r = \pm 3$.

(ii) Next we consider $q_1(x, y) = i^{s(x^2+xy+y^2)}$ for s odd. The considerations as above again give $q_1(1, 1) = i^{-s}$, so that $s \equiv -r \pmod 4$. The Gauss sums $\mathcal{G}(q_1) = 1$ for all s, so again we must choose $r = \pm 3$.

(iii) The case $q_1(x, y) = \zeta_8^{r(x^2+y^2)}$ gives $\mathcal{G}(q_1) = -i$ for all r, so there is no compatible choice of r.

Grossman and Izumi [12] have verified that that such categories exist, at least for the choices $q_1(x, y) = i^{\pm 3xy}$ with $r = \pm 3$ using [14]. For the sake of definiteness we will choose $r = 3$, so that $q_2(x, y) = \zeta_{32}^{3x^2} i^{y^2}$. There are two compatible choices of q_1, namely a hyperbolic form $q_1^h(x, y) = i^{-xy}$ and an elliptic form $q_1^e(x, y) = i^{(x^2+xy+y^2)}$. The other choices of r correspond to complex conjugation.

In all cases we obtain rank 28 modular data with labels

$$J := K \sqcup (K \times \pi) \sqcup G_* \sqcup \Gamma_*.$$

as above. Here we may take $K = \{(0,0),(1,1),(2,2),(3,3)\} = G^{\theta_1}$ or

$$K = \{(0,0),(4,1),(8,0),(12,1)\} = \Gamma^{\theta_2}.$$

We take transversals: $G_* = \{(1,0),(2,0),(2,1),(3,0),(3,1),(3,2)\}$ and

$$\Gamma_* = \{(1,0),(2,0),(3,0),(4,0),(5,0),(9,0),(10,0)$$
$$(0,1),(1,1),(2,1),(5,1),(6,1),(7,1),(13,1)\}.$$

When convenient we will decorate these labels with g or γ to indicate that they are in G or Γ respectively. The objects with labels in K are invertible, while those with labels in $K \times \{\pi\}$ are $3+2\sqrt{2}$, the objects labeled by G_* have dimension $4+2\sqrt{2}$ and those with labels in Γ_* have dimension $2+2\sqrt{2}$.

The pointed subcategory corresponding to labels in K is a $\mathbb{Z}_4$ pre-metric group $\mathcal{C}(\mathbb{Z}_4, q_0)$ with $q_0(x) = i^{-x^2}$. In particular there is a boson b given by the element $(2,2) \in G^{\theta_2}$ or $(8,0) \in \Gamma^{\theta_2}$, which can be condensed to give a modular category $[\mathcal{C}_{\mathbb{Z}_2}]_0$; that is, the trivial component of the $\mathbb{Z}_2$-de-equivariantization by $\mathrm{Rep}(\mathbb{Z}_2) \cong \langle b \rangle$. This is equivalent to the *modularization* [3] of the centralizer of the category generated by the boson, i.e., $[\langle b \rangle']_{\mathbb{Z}_2}$. The goal of this section is to prove the following:

THEOREM 2.2. *Let* $\mathbf{G} := (G, q_1^\epsilon, \theta_1)$ *and* $\Gamma := (\Gamma, q_2, \theta_2)$ *be involutive metric groups with* $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ *and* $\Gamma = \mathbb{Z}_{16} \times \mathbb{Z}_2$; $q_1^h(x,y) = i^{-xy}$, $q_1^e(x,y) = i^{(x^2+xy+y^2)}$ *and* $q_2(x,y) = \zeta_{32}^{3x^2} i^{y^2}$; *and* θ_1 *and* θ_2 *as in* (3). *If* $\mathcal{C}$ *is a modular category with* S *and* T *matrices as constructed above from* $(\mathbf{G}, \Gamma)$*, then* $[\mathcal{C}_{\mathbb{Z}_2}]_0 \cong \overline{\mathrm{Sem}} \boxtimes \mathcal{B}$ *where* $\mathcal{B}$ *is either* $\overline{PSU(3)_5}$ *or is obtained from* $\overline{PSU(3)_5}$ *by the Galois automorphism* $q \to -q$.

Before proceeeding to the proof we point out that since q is a primitive 16th root of unity that so is $-q$, hence $q \mapsto -q$ is indeed a Galois automorphism.

PROOF. The first step is to determine $\langle b \rangle'$, where b is the order 2 element of K, i.e. $(2,2) \in G$ or $(8,0) \in \Gamma$. By [3], the simple objects X that centralize b are precisely those with $\tilde{S}_{b,X} = d_X$, where $\tilde{S}$ is the S-matrix renormalized so that $\tilde{S}_{0,0} = 1$ and d_X is the dimension of the object labeled by X. From the form of the S matrix in Eqn. (1) we see that these correspond to those $X \in J$ such that $B_i^\epsilon(b, X) = 1$, where B_1^h, B_1^e is the form obtained from q_1^g/q_1^e and B_2 is obtained from q_2, depending on whether the label X is in G or Γ. Since $B_1^\epsilon((2,2),(x,y)) = (-1)^{x+y+2}$ for either choice $\epsilon \in \{h, e\}$ and $B_2((8,0),(x,y)) = (-1)^x$ We find that these are:

(i) K

(ii) $K \times \{\pi\}$

(iii) $\{(2,0)_g, (3,1)_g\} \subset G_*$ and

(iv) $\{(0,1)_\gamma, (2,1)_\gamma, (6,1)_\gamma, (4,0)_\gamma, (2,0)_\gamma, (10,0)_\gamma\} \subset \Gamma_*$.

In particular we find that the rank of $\mathcal{D} := \langle b \rangle'$ is $4+4+2+6 = 16$, and $\mathcal{D}$ is $\mathbb{Z}_2$-graded, inherited from the $\mathbb{Z}_4$ grading on $\mathcal{C}$. Since we have the S-matrix we can simply apply the Verlinde formula to determine the fusion rules and compute the fusion rules etc. for $\mathcal{D}$ directly. However, for future work we prefer to obtain the result more economically, so we take a more finessed approach.

Next we will show that $\mathcal{D}_{\mathbb{Z}_2}$ has a $\boxtimes$-factorization into two modular categories. Since the pointed subcategory with labels in K is $\mathcal{C}(\mathbb{Z}_4, q_0) \subset \langle b \rangle'$ and all other objects have non-integral dimension it is clear that the pointed subcategory of $\mathcal{D}_{\mathbb{Z}_2}$ is $\mathcal{C}(\mathbb{Z}_4, q_0)_{\mathbb{Z}_2}$ which has rank 2. In particular $\mathcal{D}_{\mathbb{Z}_2}$ is $\mathbb{Z}_2$-graded. Moreover, since modularization is a ribbon functor F_b [3] the non-trivial object $z := F_b(1,1)$ has $S_{z,z} = S_{(1,1),(1,1)} = -1$ and $\theta_z = \theta_{(1,1)} = q_1^\epsilon((1,1)) = -i$ for $\epsilon \in \{e, h\}$. We can then conclude that z is (conjugate to) a semion, i.e., $\overline{\text{Sem}} \cong \langle z \rangle \subset \mathcal{D}_{\mathbb{Z}_2}$ is the pointed subcategory. Thus $\mathcal{D} \cong \overline{\text{Sem}} \boxtimes \mathcal{B}$ for some modular category $\mathcal{B}$.

We now observe that $\mathcal{B}$ is the trivial component of the $\mathbb{Z}_2$-grading on $\mathcal{D}_{\mathbb{Z}_2}$, obtained as the $\mathbb{Z}_2$-de-equivariantization of $\mathcal{C}_{ad} \subset \mathcal{D}$. Now since $\mathcal{C}_{ad} = \mathcal{C}_{pt}'$ and $\mathcal{C}_{pt} \cong \mathcal{C}(\mathbb{Z}_4, q_0)$, we employ the same technique above showing that a simple object $X \in \mathcal{C}_{ad}$ if and only if $B_i^\epsilon(a, X) = 1$ where the tensor generator $a \in \mathcal{C}_{pt}$ is labeled by $(1,1) \in G$ or $(4,1) \in \Gamma$. These are:

$$J_{ad} = \{(0,0), (2,2), (0,0,\pi), (2,2,\pi), (3,1)_g, (2,1)_\gamma, (6,1)_\gamma, (4,0)_\gamma\}.$$

Let us denote the corresponding objects by $\mathbf{1}, b, X_1, X_2, Y, Z_1, Z_2, Z_3$ respectively. The twists are as follows: $\theta_{\mathbf{1}} = \theta_b = \theta_{X_i} = 1$, $\theta_Y = i$, $\theta_{Z_1} = \theta_{Z_2} = e^{i5\pi/4}$, and $\theta_{Z_3} = -1$.

In order to finish, we must determine the action of b on this rank 8 category. Since $\dim(b \otimes X) = \dim(X)$ we immediately see that: $b \otimes Y \cong Y$ and

$b \otimes X_1 \cong X_2$ and of course $b \otimes b \cong \mathbf{1}$. The usual yoga of de-equivariantization implies that $F_b(Y) = Y_1 \oplus Y_2$ where Y_i are simple objects of dimension $\dim(Y)/2 = 2 + \sqrt{2}$ and $F_b(X_1) = F_b(X_2) = W$ is a simple object of dimension $3 + 2\sqrt{2}$. We deduce from the twists of Z_1, Z_2 and Z_3 above that $b \otimes Z_3 \cong Z_3$, so that $F_b(Z_3) = U_1 \oplus U_2$ with each U_i simple of dimension $\dim(Z_3)/2 = 1 + \sqrt{2}$. Finally, we use the Verlinde formula to show that $b \otimes Z_1 \cong Z_2$, which implies that $F_b(Z_1) = F_b(Z_2) = V$ is a simple object of dimension $2 + 2\sqrt{2}$. Thus $\mathcal{B}$ is a rank 7 modular category, with simple objects and data as in Table 1.

Table 1. $\mathcal{B}$ data.

X	d_X	θ_X	J_{ad}	$\mathcal{C}_{ad}$
$\mathbf{1}$	1	1	$(0,0),(2,2)$	$\mathbf{1}, b$
Y_i	$2+\sqrt{2}$	i	$(3,1)_g$	Y
W	$3+2\sqrt{2}$	1	$(0,0,\pi),(2,2,\pi)$	X_1, X_2
U_i	$1+\sqrt{2}$	-1	$(4,0)_\gamma$	Z_3
V	$2+2\sqrt{2}$	$e^{i5\pi/4}$	$(2,1)_\gamma,(6,1)_\gamma$	Z_1, Z_2

We must work out the fusion rules for $\mathcal{B}$. For this we use the Verlinde formula to determine the fusion rules of $\mathcal{C}_{ad}$, and then use the fact that $\mathrm{Hom}_{\mathcal{B}}(F(x_1), F(x_2)) \cong \mathrm{Hom}_{\mathcal{C}_{ad}}(x_1, x_2 \otimes (\mathbf{1} \oplus b))$ from [19] and some multiplicity/dimension arguments to determine the fusion rules. One can calculate that the fusion rules for $\mathcal{C}_{ad}$ are identical for the two choices of q_1^ϵ, so we may consider both cases simultaneously (see Remark 1 below). Table 2 show some of the relevant calculations.

Table 2. Fusion rules of $\mathcal{B}$ from condensation of $\mathcal{C}_{ad}$.

$\mathcal{C}_{ad}$	$\mathcal{B}$
$Z_1 \otimes X_1 \cong X_1 \oplus X_2 \oplus Y \oplus Z_2 \oplus Z_3$	$V \otimes W \cong 2W \oplus V \oplus Y_1 \oplus Y_2 \oplus U_1 \oplus U_2$
$Z_3^{\otimes 2} \cong \mathbf{1} \oplus b \oplus Y \oplus Z_1 \oplus Z_2 \oplus Z_3$	$(U_1 \oplus U_2)^{\otimes 2} \cong 2\mathbf{1} \oplus Y_1 \oplus Y_2 \oplus 2V \oplus U_1 \oplus U_2$
$Y^{\otimes 2} \cong \mathbf{1} \oplus b \oplus 2X_1 \oplus 2X_2 \oplus Y \oplus Z_1 \oplus Z_2 \oplus Z_3$	$(Y_1 \oplus Y_2)^{\otimes 2} \cong 2\mathbf{1} \oplus 4W \oplus Y_1 \oplus Y_2 \oplus 2V \oplus U_1 \oplus U_2$
$Z_3 \otimes Z_1 \cong X_1 \oplus X_2 \oplus Y \oplus Z_3$	$(U_1 \oplus U_2) \otimes V \cong 2W \oplus Y_1 \oplus Y_2 \oplus U_1 \oplus U_2$

Determining the fusion rules is a somewhat tedious sequence of calculations, essentially working out sufficiently many such rules until all are determined. To illustrate this, we will show that the Y_i and U_i are non-self-dual and determine $Y_1 \otimes Y_2$ and $U_1 \otimes U_2$.

By the 2nd row of Table 2 we see that $U_1 \otimes U_2 \cong \mathbf{1} \oplus V$ by multiplicity/dimension counting, hence $U_1 \cong U_2^*$. Moreover $V \subset U_1 \otimes U_2$ implies that $U_i \subset U_i \otimes V$ so the 4th and 1st rows of the table gives us $U_1 \otimes V \cong W \oplus U_1 \oplus Y_j$ and $U_2 \otimes V \cong W \oplus U_2 \oplus Y_k$ where $j \neq k$. (Since there is labeling ambiguity between Y_1 and Y_2 we may choose $Y_j = Y_1$ and $Y_k = Y_2$.) On the other hand, since $V^* \cong V$, we have $(U_1 \otimes V)^* \cong (U_2 \otimes V)$, so that $Y_1 \cong Y_2^*$. A similar multiplicity/dimension calculation using the 4th row of the table now implies that $Y_1 \otimes Y_2 \cong \mathbf{1} \oplus W \oplus V$.

Continuing in this way we eventually arrive at a complete set of fusion rules. Ordering the simple objects in $\mathcal{B}$ as $[\mathbf{1}, W, Y_1, Y_2, V, U_1, U_2]$ the fusion rules can be determined from the fusion matrix for Y_1:

$$N_{Y_1} := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now we claim that these fusion rules are the same as those of $PSU(3)_5$. To see this one simply observes that the matrix N_{Y_1} can be obtained from N_Λ below by a permutation of columns/rows.

We can now adapt the main theorem of [16] to show that this is enough to show that $\mathcal{B}$ is obtained from $PSU(3)_5$ by Galois conjugation. The argument is as follows: $PSU(3)_5$ is the trivial component of the $\mathbb{Z}_3$-grading on $SU(3)_5$, so that $\mathcal{B} \boxtimes \mathcal{P}(\mathbb{Z}_3)$ and $SU(3)_5$ have the same fusion rules, where $\mathcal{P}(\mathbb{Z}_3)$ is a pointed modular category with fusion rules like $\mathbb{Z}_3$. The main result of [16] now implies that $\mathcal{B} \boxtimes \mathcal{C}(\mathbb{Z}_3, q)$ is equivalent to a category $\mathcal{F}$ that is obtained from $SU(3)_5$ by twisting the associativity (by a 3rd root of unity, in this case) and applying a Galois automorphism of $\mathcal{Q}(e^{i\pi/8})$. Now the associativity twisting is trivial on the trivial component of $SU(3)_5$, so $\mathcal{F}_0 \cong \mathcal{B}$ is obtained from $PSU(3)_5$ by Galois conjugation.

Only 4 of the 8 Galois conjugates of $PSU(3)_5$ have positive dimensions, two of which are known to be unitary. Further observe that the modular category $PSU(3)_5$ and its Galois conjugates have no fusion subcategories, and are *unpointed* in the terminology of [20] and therefore have exactly two braidings by *loc. cit.* Corollary 4.8, which are reverses of each other. Since there are no invertible objects, there is a unique spherical structure. Comparing twists, we see that only the two choices $q = e^{-i\pi/8}$ and $q = e^{i7\pi/8}$ are possible. □

REMARK 1.

(i) Although we did not use it in our proof, the $\mathbb{Z}_4$-grading on $\mathcal{C}$ can be determined. We have $\mathcal{C}_0 = \mathcal{C}_{ad}$ so that $J_{ad} = J_0$ while the simple objects in $\mathcal{C}_2$ are those in $\mathcal{D} \setminus \mathcal{B}$ i.e.,

$$J_2 = \{(1,1),(3,3),(1,1,\pi),(3,3,\pi),(2,0)_g,(0,1)_\gamma,(2,0)_\gamma,(10,0)_\gamma\}.$$

The other two components $\mathcal{C}_1$ and $\mathcal{C}_3$ are dual to each other, and so we have (where duals are vertically aligned):

$$J_1 = \{(1,0)_g,(3,2)_g,(1,0)_\gamma,(5,0)_\gamma,(9,0)_\gamma,(7,1)_\gamma\},$$

$$J_3 = \{(3,0)_g,(2,1)_g,(13,1)_\gamma,(1,1)_\gamma,(5,1)_\gamma,(3,0)_\gamma\}.$$

(ii) We have uniquely determined the category $\mathcal{B}$ as obtained from the quantum group $U_q\mathfrak{sl}_3$ up to two choices of q: $e^{-i\pi/8}$ and $e^{i7\pi/8}$. It is conceivable that these two choices lead to equivalent categories, as $PSU(3)_5$ may only depend on q^2. The first choice is known to be unitary, while the second is pseudo-unitary but unitarity is open.

(iii) Observe that q_1^e and q_1^h take the same values on $K \times G$ (and $G \times K$), so that the S and T matrices for the two choices q_1^e and q_1^h are identical except for the objects with labels in G_*. This does not have any effect on the proof above as q_1^h and q_1^e coincide for $(3,1)_g$ and $(2,0)_g$.

The unitary modular category $SU(3)_5$ contains a pointed subcategory conjugate to $SU(3)_1$ whose complement is $PSU(3)_5$. The non-trivial simple objects in $PSU(3)_5$ have highest weights

$$\{(1,0,-1),(2,-1,-1),(1,1,-2),(2,0,-2),(3,-1,-2),(2,1,-3)\}.$$

In terms of the fundamental weights $\varpi_1 = \frac{1}{3}(2,-1,-1)$ and $\varpi_2 = \frac{1}{3}(1,1,-2)$ the coordinates are $[1,1],[3,0],[0,3],[2,2],[4,1]$ and $[1,4]$. Let us label them $\Upsilon,\Lambda,\Lambda^*,\Omega,\Xi,\Xi^*$ respectively. The basic data for $PSU(3)_5$ are found in Table 3.

Table 3. Basic data.

	$x_1\varpi_1 + x_1\varpi_2$	d_A	θ_A
$\mathbf{1}$	$[0,0]$	1	1
Λ,Λ^*	$[3,0],[0,3]$	$2+\sqrt{2}$	$-i$
Υ	$[1,1]$	$2+2\sqrt{2}$	$e^{i3\pi/4}$
Ξ,Ξ^*	$[4,1],[1,4]$	$1+\sqrt{2}$	-1
Ω	$[2,2]$	$3+2\sqrt{2}$	1

If we order the objects: $[\mathbf{1},\Lambda,\Lambda^*,\Upsilon,\Xi,\Xi^*,\Omega]$ the fusion rules are easily determined using standard Lie theory. We find that the fusion matrix for Λ is:

$$N_\Lambda := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

The full set of fusion rules can be derived from those of Λ. This can be seen as follows: since N_Λ has 7 distinct eigenvalues, any matrix that commutes with N_Λ is a polynomial in N_Λ. In particular each fusion matrix N_x is a polynomial in N_Λ, and one column/row of N_x is determined.

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