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Bohr-Sommerfeld Quantization Rules Revisited: The Method of Positive Commutators (Erratum)

By Abdelwaheb Ifa, Hanen Louati and Michel Rouleux

Abstract. We supply some corrections to our previous paper [IfaLouRo].

0. Introduction

In our paper [IfaLouRo] formulae (3.26) and (3.27) (WKB solutions in the position representation) have to be corrected at order h. This is irrelevant in Sect.3 since calculations leading to Bohr-Sommerfeld quantization rule are carried over from Fourier representation, and the WKB solution (in position representation) at the focal point can be recovered by stationary phase in Lemma 3.4. However, formula (3.27) has been used in Sect.4 in case the phase function $\varphi_\pm$ is $S(t;E) = f_0^{-1}(E)t$. This entails the expressions (4.6) of the second term for the asymptotics of the semi-classical action $\mathcal{S}_h(E)$ in action-angle coordinates. The fact that $S(t;E)$ is linear in t simplifies considerably the computation leading in general to the correct formula (3.27). But the direct computation of WKB solutions in the position representation would also lead to a variant of our proof of Bohr-Sommerfeld quantization condition [Ifa].

1. Correction to Formulae (3.26) and (3.27)

Away from x_E, we use standard WKB theory extending (3.17), with Ansatz

$$u_\pm^a(x) = a_\pm(x;h)e^{i\varphi_\pm(x)/h} \tag{3.25}$$

We apply stationary phase formula (2.3) at second order in h, i.e. including operators L_0, L_1, L_2. Omitting indices $\pm$ and reference to the turning point

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$a = a(E)$, we look for

$$a(x;h) = a_0(x) + ha_1(x) + \cdots$$

Let $\beta(x;h) = \frac{\partial p}{\partial \xi}(x, \varphi'(x);h) = \beta_0(x) + h\beta_1(x) + \cdots$. The first (homogeneous) transport equation

$$\beta_0(x)\, a_0'(x) + (i\, p_1(x, \varphi'(x)) + \frac{1}{2}\,\beta_0'(x))\, a_0(x) = 0$$

gives the usual half-density

$$a_0(x) = \widetilde{C}_0 |\beta_0(x))|^{-1/2} \exp(-i \int_{x_E}^{x} \frac{p_1(y, \varphi'(y))}{\beta_0(y)}\, dy)$$

with a constant $\widetilde{C}_0 \in \mathbf{R}$ (that we take eventually equal to $C_0 = 1/\sqrt{2}$). The second (inhomogeneous) transport equation gives $a_1(x)$ of the form

$$a_1(x) = (\widetilde{C}_1 + \widetilde{D}_1(x)) |\beta_0(x))|^{-1/2} \exp(-i \int_{x_E}^{x} \frac{p_1(y, \varphi'(y))}{\beta_0(y)}\, dy)$$

where $\widetilde{D}_1$ is a complex function, vanishing at $x = x_E$, with

$$\frac{1}{\widetilde{C}_0} \operatorname{Re} \widetilde{D}_1(x) = -\frac{1}{2} \Big[\partial_\xi (\frac{p_1}{\partial_\xi p_0})(y, \varphi'(y))\Big]_{x_E}^{x}$$

and

$$\frac{1}{\widetilde{C}_0} \operatorname{Im} \widetilde{D}_1(x) = \int_{x_E}^{x} \frac{1}{\beta_0} \left(-p_2 + \frac{1}{8} \frac{\partial^4 p_0}{\partial y^2 \partial \xi^2} + \frac{\varphi''}{12} \frac{\partial^4 p_0}{\partial y \partial \xi^3} - \frac{(\varphi'')^2}{24} \frac{\partial^4 p_0}{\partial \xi^4}\right) dy$$
$$-\frac{1}{8} \int_{x_E}^{x} \frac{(\beta_0')^2}{\beta_0^3} \frac{\partial^2 p_0}{\partial \xi^2}\, dy$$
$$+\frac{1}{6} \int_{x_E}^{x} \varphi'' \frac{\beta_0'}{\beta_0^2} \frac{\partial^3 p_0}{\partial \xi^3}\, dy + \int_{x_E}^{x} \frac{p_1}{\beta_0^2} \left(\partial_\xi p_1 - \frac{p_1}{2\beta_0} \frac{\partial^2 p_0}{\partial \xi^2}\right) dy + \Big[\frac{\varphi''}{6\,\beta_0} \frac{\partial^3 p_0}{\partial \xi^3}\Big]_{x_E}^{x}$$
$$-\Big[\frac{\beta_0'}{4\beta_0^2} \frac{\partial^2 p_0}{\partial \xi^2}\Big]_{x_E}^{x}$$

which is the correct expression for (3.26). Normalization with respect to the "flux norm" consists as above in computing $F_\pm^a = \frac{i}{h}\,[P, \chi^a]_\pm u_\pm$ by stationary phase mod $\mathcal{O}(h^2)$, making use of the expression of $\operatorname{Re} \widetilde{D}_1^\pm(x)$. Assuming already $\widetilde{C}_0$, $\widetilde{C}_1$ to be real, a simple calculation using integration by parts yields $\widetilde{C}_0 = C_0 = 1/\sqrt{2}$, and

$$\widetilde{C}_1 = \widetilde{C}_1(a_E) = -\frac{1}{2\sqrt{2}}\,\partial_\xi\big(\frac{p_1}{\partial_\xi p_0}\big)(a_E)$$

As a result, outside any neighborhood of x_E, we have

$$(*)\qquad \begin{aligned} u_\pm(x;h) \;=\; & |\beta_0^\pm(x)|^{-1/2}\exp[iS_\pm(x_E,x;h)/h] \\ & \times(\widetilde{C}_0 + h\,\widetilde{C}_1 + h\,\widetilde{D}_1^\pm(x) + \mathcal{O}(h^2)) \end{aligned}$$

with

$$S_\pm(x_E,x;h) = \varphi_\pm(x) - h\int_{x_E}^{x}\frac{p_1(y,\xi_\pm(y))}{\beta_0^\pm(y)}\,dy$$

which is the correct expression for (3.27), where the term $h\,\widetilde{D}_1^\pm(x)$ was missing.

Remark. From $(*)$ we can recover the homology class of generalized action as in Proposition 3.3, considering the superposition $u(x;h) = e^{i\pi/4}u_+(x;h) + e^{-i\pi/4}u_-(x;h)$ near $a(E)$. The argument then is similar to this of Proposition 3.1 and 3.3, see [Ifa].

2. Correction to Semi-Classical Action in Action-Angle Variables

Here we correct the terms $\widetilde{S}_1(E), \widetilde{S}_2(E)$ given in (4.6). Our computation made use of the fact that in these coordinates, P is simply the h-pseudodifferential operator with constant coefficients $f(hD_t;h)$, and the phase $S(t,E) = f_0^{-1}(E)t = \tau t$. WKB solution satisfies

$$\begin{aligned} & e^{-iS(t,E)/h}\,(f(hD_t;h) - E)(a(t,E;h)\,e^{iS(t,E)/h}) \\ & = (f(\tau;h) - E)\,a(t,E;h) + \frac{h}{i}\,\partial_\tau f(\tau;h)\,\partial_t a(t,E;h) \\ & -\frac{h^2}{2}\,\frac{\partial^2 f}{\partial\tau^2}(\tau;h)\,\frac{\partial^2 a}{\partial t^2}(t,E;h) + \mathcal{O}(h^3) \end{aligned}$$

Here $f(\tau;h) = f_0(\tau) + hf_1(\tau) + \cdots$ is such that P is unitarily equivalent to $f(\frac{1}{2}((hD_y) + y^2);h)$. The amplitudes $a_0, a_1, \cdots$ are determined as before, as well as the normalisation constants C_0, C_1. We provide below all corrections implied by formula (3.27).

(1) Second formula (4.5) takes the form

$$a_1(t,E) = (C_1(E) + itC_0\widetilde{S}_2(E)))((f_0^{-1})'(E))^{1/2}e^{it\widetilde{S}_1(E)}$$

(the term $\beta(E)$ disappears) and it is convenient to write $(f_0^{-1})'(E) = 1/f_0'(\tau)$.

(2) Formula (4.6) takes the form

$$\begin{aligned} \widetilde{S}_1(E) &= -\frac{f_1(\tau)}{f_0'(\tau)} \\ \widetilde{S}_2(E) &= \frac{1}{f_0'(\tau)}(\frac{1}{2}(\frac{f_1^2}{f_0'})'(\tau) - f_2(\tau)) \end{aligned} \tag{4.6}$$

(3) In formula (4.7) one has to remove the term $hC_0\beta(E)$.

(4) Formula giving $C_1(E)$ after (4.9) takes the form

$$C_1(E) = -\frac{1}{2}((f_0'(\tau))^{-1/2}(\frac{f_1}{f_0'})'(\tau)$$

Then following the argument given in Sect.4 leads to:

Proposition. *In action-angle variables, Bohr-Sommerfeld quantization rule takes the form*

$$S_0(E) + h\,S_1(E) + h^2\,S_2(E) + \mathcal{O}(h^4) = 2\,\pi\,n\,h, \quad n \in \mathbf{Z}$$

where $S_0(E) = \oint_{\gamma_E} \xi\,dx$ *is the classical action along* γ_E *and*

$$S_1(E) = -2\,\pi\,(\frac{f_1}{f_0'})(\tau); \quad S_2(E) = \frac{2\pi}{f_0'(\tau)}\Big(\frac{1}{2}\,(\frac{f_1^2}{f_0'})'(\tau) - f_2(\tau)\Big)$$

References

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Abdelwaheb Ifa
Université de Tunis El-Manar
Département de Mathématiques
1091 Tunis, Tunisia
E-mail: abdelwaheb.ifa@fsm.rnu.tn

Hanen Louati
Université de Tunis El-Manar
Département de Mathématiques
1091 Tunis, Tunisia
and
Aix Marseille Univ.
Université de Toulon
CNRS, CPT, Marseille, France
E-mail: louatihanen42@yahoo.fr

Michel Rouleux
Aix Marseille Univ.
Université de Toulon
CNRS, CPT, Marseille, France
E-mail: rouleux@univ-tln.fr