博士論文（要約）

論文題目　Classification and construction of minimal representations

（極小表現の分類と構成）

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The main results in this thesis consist of the following four parts.

- Classification of minimal representations ([Tam19]).
- Construction of minimal representations.
- Construction and classification of analogues of minimal representations for simple Lie groups of type A.
- New expression of unramified local L-functions by certain Hecke operators ([OST], joint work with Masao Oi and Ryotaro Sakamoto).

1. Classification of minimal representations

Let G be a connected simple real Lie group not of type A, and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0 := \mathrm{Lie}(G)$, and K the analytic subgroup with Lie algebra $\mathfrak{k}_0$. For simplicity, assume that the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ is simple. An irreducible admissible representation of G is called minimal if the annihilator of the underlying $(\mathfrak{g}, K)$-module is the Joseph ideal J_0 [Jos76], which is the unique completely prime two-sided ideal whose associated variety is the closure $\overline{\mathcal{O}^{\min}}$ of the minimal nilpotent orbit $\mathcal{O}^{\min}$ in $\mathfrak{g}$ [GS04, Theorem 3.1].

For $G = \mathrm{Mp}(n, \mathbb{R})$ (the connected double cover of the real symplectic group $\mathrm{Sp}(n, \mathbb{R})$), the two irreducible components of the Weil representation, which is also referred to as the harmonic, Segal–Shale–Weil, oscillator, or metaplectic representation, are classical examples of minimal representations.

Unitary minimal representations are the smallest "unipotent" representations, and are considered to be a part of building blocks of the unitary dual of G. From such qualities, minimal representations for various simple real Lie groups have been studied and constructed in various ways (see [BSZ06, BZ91, BJ98, Bry98, BK94a, BK94b, BK94c, BK95, EPWW85, GS05, Gon82, GW94, GW96, HKM14, HKMØ12, Hua95, HL99, Kaz90, KS90, Kob11, KM11, KØ98, KØ03a, KØ03b, KØ03c, Kos90, Li00, LS08, Sab96, Sal06, Tor97, Vog81, Vog94] for example).

An easy necessary condition for the existence of minimal $(\mathfrak{g}, K)$-modules comes from their associated varieties. That is, if the intersection of the minimal nilpotent orbit $\mathcal{O}^{\min}$ and $\mathfrak{p}$, as a subset of $\mathfrak{g}$, is empty, then there exist no minimal $(\mathfrak{g}, K)$-modules. Moreover, R. Howe and D. Vogan [Vog81, Theorem 2.13] proved that for $\mathfrak{g}_0 = \mathfrak{so}(p, q)$ with $p, q \geq 4$ and $p + q$ odd, there exists no irreducible $(\mathfrak{g}, K)$-module whose Gelfand–Kirillov dimension is $p + q - 3$, namely half the complex dimension of $\mathcal{O}^{\min}$.

The above previous studies on the construction of minimal representations imply that the converse to the non-existence statement holds: if $\mathfrak{g}_0$ is not of type A, not isomorphic to $\mathfrak{so}(p, q)$ with $p, q \geq 4$ and $p + q$ odd, and $\mathcal{O}^{\min} \cap \mathfrak{p} \neq \emptyset$, then there exists a minimal $(\mathfrak{g}, K)$-module for some covering K.

The first main theorem in this thesis argues that there are no new minimal representations up to infinitesimal equivalence.

Theorem 1 ([Tam19, Theorem 4.1])**.** *Let G be a connected simply connected simple Lie group not of type A. Then the number of infinitesimal equivalence classes of minimal representations of G is are given as in Table 1.*

As corollaries, we obtain the following

Corollary 2.

(1) *When the Riemannian symmetric space G/K is Hermitian, minimal representations are highest (or lowest) weight representations.*
(2) *Minimal representations are infinitesimally equivalent to unitary representations.*

TABLE 1. The number of infinitesimal equivalence classes of minimal representations for simply connected G

$\mathfrak{g}_0$	number
$\mathfrak{sp}(n,\mathbb{R})(n \geq 2)$	4
$\mathfrak{so}(p,2)(p \geq 5), \mathfrak{so}^*(2n)(n \geq 4), \mathfrak{e}_{6(-14)}, \mathfrak{e}_{7(-25)}$	2
$\mathfrak{so}(p,q)(p,q \geq 3, p+q \geq 8 \text{ even}), \mathfrak{so}(p,3)(p \geq 4 \text{ even}),$ $\mathfrak{e}_{6(6)}, \mathfrak{e}_{6(2)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{8(8)}, \mathfrak{e}_{8(-24)}, \mathfrak{f}_{4(4)}, \mathfrak{g}_{2(2)}$	1
$\mathfrak{sp}(n)(n \geq 2), \mathfrak{so}(n)(n \geq 7), \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2, \mathfrak{so}(n,1)(n \geq 6),$ $\mathfrak{sp}(p,q)(p,q \geq 1), \mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)}, \mathfrak{so}(p,q)(p,q \geq 4, p+q \text{ odd})$	0
$\mathfrak{sp}(n,\mathbb{C})(n \geq 2)$	2
$\mathfrak{so}(n,\mathbb{C})(n \geq 7), \mathfrak{e}_6(\mathbb{C}), \mathfrak{e}_7(\mathbb{C}), \mathfrak{e}_8(\mathbb{C}), \mathfrak{f}_4(\mathbb{C}), \mathfrak{g}_2(\mathbb{C})$	1

Our argument is based on the idea of W. T. Gan and G. Savin. They obtained a relation of Casimir elements for two simple components of $\mathfrak{k}$ modulo the Joseph ideal J_0, and considered K-types satisfying the relation. The key point of their proof is to apply a proposition by B. Kostant: if two minimal $(\mathfrak{g}, K)$-modules have a common K-type, then they are isomorphic when $\mathfrak{g}$ is simple not of type A, $\mathcal{O}^{\min} \cap \mathfrak{p} \neq \emptyset$ and G/K is a non-Hermitian symmetric space. As is implicit in [GS05], the proposition can be generalized to any connected simple real Lie group G not of type A similarly.

From an explicit description of homogeneous elements of degree two in J_0 in terms of weight vectors in $\mathfrak{g}$ with respect to a Cartan subalgebra of $\mathfrak{k}$, we see that highest weights of K-types in minimal representations belong to some one or two lines which depend only on the $\mathfrak{k}_0$-module $\mathfrak{p}_0$. Hence we obtain an upper bound for the number of infinitesimal equivalent classes of minimal representations. Moreover, we obtain a simple proof of the non-existence of minimal representations for $\mathfrak{g}_0 \cong \mathfrak{so}(p,q)(p,q \geq 4, p+q \text{ odd})$ as a corollary of the description K-types of minimal representations.

2. Construction of minimal representations

T. Kobayashi and B. Ørsted [KØ03a] constructed minimal representations of the indefinite orthogonal group $G = O(p,q)(p,q \geq 2, p+q \geq 8 \text{ even})$ via conformal geometry. They showed that the Yamabe operator on the pseudo-Riemannian manifold $S^{p-1} \times S^{q-1}$ of signature $(p-1, q-1)$, where G acts conformally, is a G-invariant differential operator between some degenerate principal series representations, and that the kernel is a minimal representation of G. Based on the construction, they described discrete branching laws of the minimal representation [KØ03b] and constructed conservative quantities of ultrahyperbolic equations [KØ03c]. In this way, researches on minimal representations from analytic viewpoints has been active since 2000's [Kob11, KM11].

Our next main theorem construct minimal representations as the kernel of intertwining differential operator between parabolically induced representations, as the construction by [KØ03a].

Let G be a connected split real simple Lie group of type D or E, and B a Borel subgroup of G. Write $B = MAN$ for a Langlands decomposition. Let ν be a character of the Lie algebra of A, and write triv_ν for the character of B where the action of M, N is trivial and the one of A agrees with $\exp(\nu)$. The group G acts the induced representation

$$C^\infty(G, \mathrm{triv}_\nu)^B := \{f \in C^\infty(G, \mathrm{triv}_\nu) \mid f(gb) = \mathrm{triv}_\nu(b^{-1})f(g) \text{ for } g \in G, b \in B\}$$

by the left transition. By the differentiation with respect to the right transition, we define a G-intertwining differential operator

$$D\colon C^\infty(G,\mathrm{triv}_\nu)^B \to C^\infty(G, W^\vee \otimes \mathrm{triv}_\nu)^B.$$

Here $W^\vee$ denotes the dual of a quotient B-module of the $\mathrm{ad}(\mathfrak{g})$-submodule of J_0 spanned by homogeneous elements of degree two in J_0.

Theorem 3. *Let G be a connected split real simple Lie group of type $D_n(n \geq 4), E_6, E_7$ or E_8. Then the following are equivalent:*

(i) *The subrepresentation* $\operatorname{Ker} D$ *is minimal.*
(ii) *The annihilator* $\operatorname{Ann} L(-\nu)$ *of the irreducible highest weight $\mathfrak{g}$-module with highest weight $-\nu$ equals J_0.*
(iii) *The weight ν is the unique element in $(\rho(\alpha_1,-1)+\mathbb{C}\alpha_1)\cap\alpha_2^\perp$ for some simple roots α_1,α_2 with $(\alpha_1,\check{\alpha_2})=-1$. Here $\check{\alpha}$ is the coroot of α, $\rho(\alpha_1,-1)$ denotes half the sum of positive roots ϕ with $(\phi,\check{\alpha_1})=-1$, and $\alpha_2^\perp$ denotes the set of weights orthogonal to $\mathbb{C}\alpha_2$.*

When one of the above conditions holds, the K-spherical functions in $C^\infty(G,\mathrm{triv}_\nu)^B$ are annihilated by $D(G,\mathrm{triv}_\nu)$.

Our classification of weights λ satisfying $\operatorname{Ann} L(\lambda) = J_0$ seems to include weights which cannot be obtained from the previous researches [Jos76, Gar82, GW94, BJ98] and the work on primitive ideals with a fixed infinitesimal character [Duf77, Jos77]. According to the classification, Theorem 3 gives $\mathrm{rank}(\mathfrak{g})$-ways of constructing a minimal representation of G as the kernel of a G-intertwining differential operator between parabolically induced representations.

3. Construction and classification of analogues of minimal representations for simple Lie groups of type A

Minimal representations are of interest in physics. For example, the oscillator representation of the metaplectic group (whose irreducible components are minimal) has a realization as the bound states of the quantum harmonic oscillator, and the minimal representation of the indefinite orthogonal group $\mathrm{O}(p,2)(p \geq 6)$ has a realization as some solution space of the wave equation in Minkowski space (see [KØ03c, Theorem 1.4] for example).

On the other hand, there exist representations called "minimal" even when $\mathfrak{g}$ is simple of type A, that is, when the definition of the Joseph ideal (hence the one of minimality) is not given. Here the vague term "minimal" means that there is an interest in physics as in the previous examples, the $\mathfrak{k}$-types are as simple as possible (called pencil), or there is a relation with the minimal nilpotent orbit. Such representations include the ladder representation of $\mathrm{O}(2,4)$ (which is locally isomorphic to $\mathrm{SU}(2,2)$) expressing the bound states of the Hydrogen atom [KØ03a, Remark 3.6.2 (3)] and the irreducible unitary representations of the double cover $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ of $\mathrm{SL}(3,\mathbb{R})$ given by Torasso (see [Tor83, Théorème VII.1], [RS82]).

We define minimality in terms of annihilators of $(\mathfrak{g},\mathfrak{k})$-modules so that the above representations are minimal, and to classify minimal representations for connected simple real Lie groups of type A_{n-1} $(n \geq 3)$. Hence this study can be regarded as a small step of classifying irreducible $(\mathfrak{g},\mathfrak{k})$-modules associated with a fixed completely prime primitive ideal.

Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})(n \geq 3)$. Set $T_{i,j} := E_{i,j} - \delta_{i,j}/nI_n$, where I_n denotes the unit n-by-n matrix, $E_{i,j}$ denotes the matrix whose (i,j)-th component is one and the others are zero, and $\delta_{i,j}$ denotes the Kronecker delta. For $a \in \mathbb{C}$, write J_a for the two-sided ideal of $U(\mathfrak{g})$ generated by

$$T_{1,n}T_{2,n-1} - T_{1,n-1}T_{2,n} (\text{if } n \geq 4),$$

$$\frac{1}{2}\sum_{i=1}^{n}(T_{1,i}T_{i,n}+T_{i,n}T_{1,i})-\frac{a(n-2)}{n}T_{1,n},$$

$$\sum_{1\le i,j\le n}T_{i,j}T_{j,i}-\frac{(n-1)(2a+n)(2a-n)}{4n}.$$

A condition characterizing the Joseph ideal for simple Lie algebra not of type A defines a family of two-sided ideals in $U(\mathfrak{g})$ parametrized by complex numbers:

Theorem 4. *Let J be a two-sided ideal of $U(\mathfrak{g})$. The following are equivalent:*

(i) *$J = J_a$ for some $a \in \mathbb{C}$.*
(ii) *the associated graded ideal $\operatorname{gr} J$ of the symmetric algebra $S(\mathfrak{g})$ is equal to the ideal $\mathcal{I}(\overline{\mathcal{O}^{\min}})$ defined by the closure $\overline{\mathcal{O}^{\min}}$ of the minimal nilpotent orbit in $\mathfrak{g}^\vee$.*
(iii) *J is completely prime, primitive and has $\overline{\mathcal{O}^{\min}}$ as its associated variety.*

Let G be a Lie group whose Lie algebra is isomorphic to $\mathfrak{g}_0$. Fix $a \in \mathbb{C}$. Let us call an irreducible admissible representations of G a-minimal if the annihilator of the underlying $(\mathfrak{g}, K)$-module is J_a. By applying methods in the proofs of Theorems 1 and 3, we classify a-minimal representations:

Theorem 5. *Let $a \in \mathbb{C}$ and G a connected simply connected simple Lie group of type $A_n (n \geq 2)$. Assume that the complexification $\mathfrak{g}$ is simple. Then the number of the infinitesimal equivalence classes of a-minimal representations of G is as in Table 2.*

Table 2. The number of the isomorphism classes of a-minimal $(\mathfrak{g}, K)$-modules for simply connected K

$\mathfrak{g}$	a	number
$\mathfrak{su}(p,1)(p \geq 2)$	$\mathbb{C}$	2
$\mathfrak{su}(p,q)(p,q \geq 2)$	$(p+q)/2+\mathbb{Z}$	2
	$\mathbb{C}\setminus((p+q)/2+\mathbb{Z})$	0
$\mathfrak{sl}(3,\mathbb{R})$	$\mathbb{Z}$	3
	$\mathbb{C}\setminus\mathbb{Z}$	2
$\mathfrak{sl}(n,\mathbb{R})(n \geq 4)$	$\mathbb{C}$	2
$\mathfrak{su}(n)(n \geq 3), \mathfrak{sl}(n,\mathbb{H})(n \geq 2)$	$\mathbb{C}$	0

We construct the underlying $(\mathfrak{g}, K)$-modules of a-minimal representations as highest weight modules $L(\lambda)$ when $\mathfrak{g}_0 \cong \mathfrak{su}(p, n-p)$, and construct a-minimal representations as a subquotient of the kernel of a G-intertwining differential operator. The construction of Torasso's genuine 0-minimal representation of the double cover $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ of $\mathrm{SL}(3,\mathbb{R})$ agrees with the one given by T. Kubo and B. Ørsted [KØ19]. Unlike the proof of Theorem 1, we need to calculate the kernel of G-intertwining differential operators for the non-existence of a-minimal representations of $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ with certain K-types.

We remark that many of minimal $(\mathfrak{g}, \mathfrak{k})$-modules do not admit nondegenerate invariant Hermitian form and are not unitarizable, which does not occur when $\mathfrak{g}_0$ is not of type A.

4. New expression of unramified local L-functions by certain Hecke operators.

This content is a joint work with Masao Oi and Ryotaro Sakamoto.

Let $\mathbf{G}$ be a split connected reductive group over a non-archimedean local field F. The unramified representations of $G := \mathbf{G}(F)$ are one of the most fundamental classes in representation theory of the group G. Their importance can be explained in relation to the global theory, that is, almost all local components of automorphic representations are unramified. Hence unramified representations have been investigated from the early days, and a lot of results have been obtained so far.

One of such accumulation is a construction of the local L-functions for unramified representations, while the existence of the local L-functions for irreducible smooth representations is conjectural in general. Since the existence of the local L-functions for unramified representations enables us to define the global (partial) L-functions for automorphic representations, local L-functions for unramified representations have an important meaning.

The local L-functions for unramified representations are defined by using a classification of unramified representations. More precisely, by using the Satake isomorphism, we can parametrize unramified representations of G via Satake parameters, which are semisimple conjugacy classes in the Langlands dual group $\widehat{G}$ of $\mathbf{G}$. Then we can attach the local L-function $L(s, \pi, r)$ to each unramified representation π of G and finite-dimensional representation r of $\widehat{G}$ by considering the image of the Satake parameter under r.

The aim of this paper is to give a new formula describing the local L-functions for unramified representations. Before we explain the main result of this paper, let us introduce some motivative examples.

The first example is the case of the standard L-function of GL_2. Let π be an irreducible unramified representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. We can take an unramified character χ of the diagonal maximal torus of GL_2 such that π is realized as a subquotient of the normalized parabolic induction (I_χ, V_χ) of χ. Consider the standard representation Std of the Langlands dual group $\mathrm{GL}_2(\mathbb{C})$ of GL_2. Then, by an easy computation, we can check the following equality:

$$L(s, \pi, \mathrm{Std}) = \det(1 - p^{-(s+1/2)} I_\chi(U_K) \mid V_\chi^K)^{-1}.$$

Here K is the open compact subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ defined by

$$K = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \;\middle|\; c \in p\mathbb{Z}_p \right\}$$

and U_K is the characteristic function of the open compact subset $K \operatorname{diag}(p,1) K$ normalized so that $U_K(\operatorname{diag}(p,1)) = \mathrm{vol}(K)^{-1}$.

The second example is the case of the spin L-function of GSp_4. We put

$$\mathrm{GSp}_4 := \left\{ g \in \mathrm{GL}_4 \;\middle|\; {}^t g \begin{pmatrix} & -J_2 \\ J_2 & \end{pmatrix} g = x \begin{pmatrix} & -J_2 \\ J_2 & \end{pmatrix} \text{ for some } x \in \mathbb{G}_{\mathrm{m}} \right\},$$

where J_2 denotes the anti-diagonal matrix whose anti-diagonal entries are one. We consider the spin representation Spin of $\mathrm{GSpin}_5(\mathbb{C}) = \widehat{\mathrm{GSp}_4}$. Let (π, V) be an irreducible unramified principal series representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$. Then, in [Tay88, Section 2.4] (see also [LSZ17, Section 3.4.2]), Taylor established a similar identity to above for the spin L-function $L(s, \pi, \mathrm{Spin})$ in his study of p-adic family of Siegel modular forms. More precisely, by using the Siegel parahoric subgroup K of $\mathrm{GSp}_4(\mathbb{Q}_p)$, which is defined by

$$K = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{Q}_p) \;\middle|\; A, D \in \mathrm{GL}_2(\mathbb{Z}_p), B \in M_2(\mathbb{Z}_p), C \in M_2(p\mathbb{Z}_p) \right\},$$

Taylor proved that

$$L(s, \pi, \mathrm{Spin}) = \det(1 - p^{-(s+3/2)}\pi(U_K) \mid V^K)^{-1},$$

where U_K is the characteristic function of the open compact subset $K \operatorname{diag}(p, p, 1, 1)K$ normalized so that $U_K(\operatorname{diag}(p, p, 1, 1)) = \operatorname{vol}(K)^{-1}$.

These formulas are, in addition to their original importance in a study of modular forms, also interesting from the purely representation-theoretic viewpoint as follows. In the original definition of the local L-functions for unramified representations via the Satake isomorphism, we consider the action of the whole spherical Hecke algebra on the subspace of spherical vectors, which is 1-dimensional. In contrast to this original definition, in the above examples, the local L-function is expressed by the characteristic polynomial of the action of some specific test function on the subspace whose dimension is the same as the degree of the local L-function.

In this paper, we establish these kind of formulas for split connected reductive groups and general finite-dimensional representations of the Langlands dual groups. For a dominant cocharacter $\check{\gamma}$ of a fixed maximal torus $\mathbf{T}$, we consider some open compact subgroup $K_{\check{\gamma}}$ of G and a normalized characteristic function $\mathbb{1}_{\check{\gamma}}$ of a certain $K_{\check{\gamma}}$-double coset. For a finite-dimensional representation r of the Langlands dual group $\widehat{G}$, we put $\mathcal{P}^+(r)$ to be the set of dominant weights in r with respect to a fixed maximal torus of $\widehat{G}$. Note that each element $\check{\gamma}$ of $\mathcal{P}^+(r)$ can be regarded as a dominant cocharacter of $\mathbf{T}$. For each $\check{\gamma} \in \mathcal{P}^+(r)$, we write $m_{\check{\gamma}}$ for the multiplicity of $\check{\gamma}$ in r. In this setting, the following is the last main theorem of this thesis.

Theorem 6 ([OST, Theorem 4.2])**.** *Let π be an irreducible unramified representation of G. We take an unramified character χ of $\mathbf{T}(F)$ such that π is realized as a subquotient of the normalized parabolic induction (I_χ, V_χ) of χ. Then we have an equality*

$$L(s, \pi, r) = \prod_{\check{\gamma} \in \mathcal{P}^+(r)} \det\bigl(1 - q^{-(s+\langle \rho, \check{\gamma} \rangle)} I_\chi(\mathbb{1}_{\check{\gamma}}) \bigm| V_\chi^{K_{\check{\gamma}}}\bigr)^{-m_{\check{\gamma}}},$$

where ρ is the half sum of the positive roots of $\mathbf{T}$ in $\mathbf{G}$.

Here note that if $(\mathbf{G}, r)$ is $(\mathrm{GL}_2, \mathrm{Std})$ or $(\mathrm{GSp}_4, \mathrm{Spin})$, then the set $\mathcal{P}^+(r)$ is a singleton and the formula in Theorem 6 is nothing but the identity in the above examples. More generally, when r is a quasi-minuscule representation, we get a similar formula to the above examples.

The proof of this theorem is given by computing the eigenvalues of the action $\mathbb{1}_{\check{\gamma}}$ on the space $V_\chi^{K_{\check{\gamma}}}$. First, by considering the generalized Iwasawa decomposition with respect to the open compact subgroup $K_{\check{\gamma}}$ of G, we take an explicit basis of the space $V_\chi^{K_{\check{\gamma}}}$ of $K_{\check{\gamma}}$-invariant functions on G. Second, to describe the action of $\mathbb{1}_{\check{\gamma}}$ on $V_\chi^{K_{\check{\gamma}}}$ in terms of the basis, we write the support of $\mathbb{1}_{\check{\gamma}}$, which is a $K_{\check{\gamma}}$-double coset, as the disjoint union of right $K_{\check{\gamma}}$-cosets. To carry it out, we prove several technical results on group-theoretic properties of our group $K_{\check{\gamma}}$. Third, we consider how the generalized Iwasawa decomposition behaves under the right multiplication by support of the function $\mathbb{1}_{\check{\gamma}}$, and show that the action of $\mathbb{1}_{\check{\gamma}}$ on $V_\chi^{K_{\check{\gamma}}}$ can be triangulated in an appropriate order on our basis. Once we achieve such a triangulation, we can compute the eigenvalues easily and our result follows. Here we note that most of the arguments in the second and third steps are based on the general results by Bruhat and Tits established in [BT72] and [BT84].

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