SO(3)-invariant G_2-geometry

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Abstract

In this thesis, we study G_2-manifolds with SO(3)-symmetry by generalizing the celebrated examples due to Bryant and Salamon. As a main result, we give a Hamiltonian function on the cotangent bundle of the space of all Riemannian metrics on a 3-manifold M, and prove the orbits of the constrained Hamiltonian dynamical system describe G_2-manifolds foliated by hypersurfaces diffeomorphic to $M \times \mathrm{SO}(3)$. Interestingly, the Hamiltonian system is very similar to the Hamiltonian formulation of general relativity. Several properties of the dynamical systems are proved, and examples reduced into explicit ODEs are given. Moreover, we apply our method to G_2-cobordisms with SO(3)-symmetry. This cobordisms define a binary relation on the space of closed SO(3)-invariant $\mathrm{SL}(3;\mathbb{C})$-structures on $M \times \mathrm{SO}(3)$ that vanish along the fibers SO(3). Among other properties, we prove that the relation is irreflexive.

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Contents

Chapter 1

Introduction

1.1 Motivation

The holonomy group of a Riemannian manifold is one of the most fundamental concepts in Riemannian geometry. Berger classified the possible holonomy groups of simply connected, irreducible and nonsymmetric Riemannian manifolds. As in the case of simple Lie groups, Berger's classification includes exceptional cases of the holonomy group G_2 in dimension 7 and Spin(7) in dimension 8. Such Riemannian manifolds have torsion-free G_2- and Spin(7)-structures. Manifolds with these torsion-free structures are called G_2- and Spin(7)-manifolds, respectively. They have been attracting many geometers. Bryant and Salamon [BS89] gave many explicit examples including the first examples of complete examples with exceptional holonomy. Among these examples, they gave G_2-manifolds naturally regarded as one-parameter families of almost special Lagrangian SO(3)-fibrations over 3-manifolds. The purpose of the present thesis is to generalize the examples and to investigate G_2-manifolds with SO(3)-symmetry from a unified viewpoint.

Hitchin [Hit01] described G_2-manifolds as orbits of a constrained Hamiltonian dynamical system on the direct product of a fixed cohomology class of 3- and 4-forms on a 6-manifold. Then each point of the orbits corresponds to an SU(3)-structure on the manifold. This formulation is reminiscent of the Hamiltonian formalism of general relativity, called the ADM formalism named after the authors [ADM59]. Now let us briefly explain the ADM formalism. Let M be a closed oriented 3-manifold. Denote by $\mathbf{M}$ the space of all Riemannian metrics on M and by $T^*\mathbf{M}$ the cotangent bundle of $\mathbf{M}$.

The cotangent bundle $T^*\mathbf{M}$ has the standard symplectic structure. Arnowitt, Deser and Misner reformulated the vacuum Einstein equations as constrained Hamiltonian dynamical systems on $T^*\mathbf{M}$. By choosing specific time-slices, their Hamiltonian is given by

$$H_{GR}(\gamma, \pi) = -\int_M \mathrm{R}(\gamma)\mathrm{vol}(\gamma) + \int_M \left(\mathrm{tr}(\pi^2) - \frac{1}{2}\mathrm{tr}(\pi)^2\right)\mathrm{vol}(\gamma)^{-1}.$$

Here $\gamma \in \mathbf{M}$ and π is a symmetric $(2,0)$-tensor field tensored with a volume form on M. Also, $\mathrm{R}(\gamma)$, $\mathrm{vol}(\gamma)$, $\mathrm{tr}(\pi)$ and $\mathrm{tr}(\pi^2)$ denote the scalar curvature, volume element of γ and traces of π and π^2, respectively. Let $(\gamma(t), \pi(t))$ be a curve in $T^*\mathbf{M}$ parameterized by an interval (t_1, t_2). Then they proved that the Lorentzian metric $-(dt)^2 + \gamma_{ij}dx^i dx^j$ on $M \times (t_1, t_2)$ is Ricci-flat if and only if $(\gamma(t), \pi(t))$ is an orbit of the Hamiltonian dynamical system of H_{GR} satisfying the following pointwise constraint conditions:

$$R(\gamma)\mathrm{vol}(\gamma)^2 + \frac{1}{2}\mathrm{tr}(\pi)^2 - \mathrm{tr}(\pi^2) = 0 \quad \text{and} \quad \sum_{j=1}^{3} \nabla_j \pi^{ij} = 0$$

for $i = 1, 2, 3$, where ∇ denotes the Levi-Civita connection for each $\gamma(t)$. See e.g. [ADM59, BM95, HE73] for more details. We search for an explicit similarity between the ADM and Hitchin's formalism in the case of G_2-manifolds with SO(3)-symmetry.

Moreover, Donaldson [Don18] introduced the concept of G_2-cobordisms between 6-manifolds with $\mathrm{SL}(3;\mathbb{C})$-structures in his study of boundary value problems in G_2-geometry. Such cobordisms naturally define a binary relation on the space of closed $\mathrm{SL}(3;\mathbb{C})$-structures on a 6-manifold. Basic properties of the relation seem to be non-trivial and related to almost complex and symplectic geometries in dimension 6. We study the relation under SO(3)-symmetry.

1.2 Main results

In our previous work [Chi19], we studied special Lagrangian fibrations of 6-dimentional SU(3)-manifolds, not necessarily torsion-free. Such SU(3)-structures are a generalization of that of Bryant and Salamon's examples. In the case where the fiber is a unimodular Lie group G, we decomposed such G-invariant SU(3)-structures into triples of solder 1-forms, connection

1-forms and equivariant 3×3 positive-definite symmetric matrix-valued functions on principal G-bundles over 3-manifolds. Using this decomposition, we described regular parts of G_2-manifolds that admit Lagrangian-type 3-dimensional group actions as constrained dynamical systems on the spaces of the triples above in the cases of T^3- and SO(3)-fibrations.

In this thesis, we reduce the dynamical system in [Chi19] into a constrained Hamiltonian dynamical system on $T^*\mathbf{M}$. Let us define the Hamiltonian function H_{G_2} on $T^*\mathbf{M}$ by

$$H_{G_2}(\gamma,\pi) = -\int_M \mathrm{R}(\gamma)\mathrm{vol}(\gamma) + \int_M \det(\pi)\mathrm{vol}(\gamma)^{-2},$$

where $\det(\pi)$ denotes the determinant of π proportional to the third tensor power of $\mathrm{vol}(\gamma)$. This Hamiltonian is very similar to that of the ADM formalism of relativity. In this reduction, we use a principal bundle $\mathcal{P}\to T^*\mathbf{M}$, where $\mathcal{P}$ is isomorphic to the direct product of the spaces of solder 1-forms compatible with the orientation on M and that of equivariant symmetric 3×3 matrix-valued functions on $M\times \mathrm{SO}(3)$. Then the structure group of $\mathcal{P}$ is the gauge group of the trivial bundle $M\times\mathrm{SO}(3)$. The principal bundle $\mathcal{P}$ has a natural connection, and by this connection, any curve $\underline{c}(t)$ in $T^*\mathbf{M}$ parameterized by an interval (t_1,t_2) is lifted to a curve $c(t)$ in $\mathcal{P}$. Then the lift $c(t)$ gives a G_2-structure on $M\times\mathrm{SO}(3)\times(t_1,t_2)$. Our main results are as follows.

Theorem A. *Let $\underline{c}(t)=(\gamma(t),\pi(t))$ be a curve in $T^*\mathbf{M}$. Then a lift $c(t)$ of $\underline{c}(t)$ gives a torsion-free G_2-structure if and only if the curve $\underline{c}(t)$ satisfies the following*

1. *$\underline{c}(t)$ is a solution of the Hamiltonian dynamical system of H_{G_2},*
2. *$\pi/\mathrm{vol}(\gamma)$ is positive-definite, and $\sum_{j=1}^3 \nabla_j\pi^{ij}=0$ holds for $i=1,2,3$.*

Here the constraint conditions above are all pointwise, and satisfied for all t.

Theorem A is deduced from Theorem 3.4 combined with Remark 3.11. In addition, the orbits of the Hamiltonian dynamical system of H_{G_2} preserve the divergence-free conditions for π as in the case of that of the ADM formalism.

Proposition B. *Let $\underline{c}(t)$ be an orbit of the Hamiltonian dynamical system of H_{G_2}. Suppose that $\sum_{j=1}^3 \nabla_j\pi^{ij}(t_0)=0$ holds pointwisely for $i=1,2,3$ at some $t_0\in(t_1,t_2)$. Then $\sum_{j=1}^3 \nabla_j\pi^{ij}(t)=0$ holds for $i=1,2,3$ on the whole orbit.*

Proposition B is deduced from Proposition 3.5 combined with Remark 3.11. Finally, we can conclude

Theorem C. *All torsion-free* SO(3)*-invariant Lagrangian fibered* G_2*-structures are locally given by orbits of the systems in Theorem A.*

Theorem C is induced from Theorem A and Proposition 2.17. See Definition 2.7 for the precise definition of SO(3)-invariant Lagrangian fibered G_2-structures.

Moreover, we consider G_2-cobordisms under SO(3)-symmetry and coassociative conditions. Namely, we study closed SO(3)-invariant coassociative fibrations $M \times \mathrm{SO}(3) \times [t_1, t_2]$ over a 3-manifold M. Then closed SO(3)-invariant $\mathrm{SL}(3;\mathbb{C})$-structures vanishing along the fibers SO(3) are induced on the two connected components of the boundary. This gives a binary relation on the space of such $\mathrm{SL}(3;\mathbb{C})$-structures on $M \times \mathrm{SO}(3)$. As an application of our method, we describe the relation explicitly, and prove that the relation is irreflexive (Theorem 4.12). This supports Problem 4.1 for a relation between G_2-cobordisms and the existence of symplectic structures on 6-manifolds.

1.3 Outlook

In general relativity, the Hamiltonian formalism was very useful to exploit conserved quantities such as the ADM mass and singularities of solutions of the vacuum Einstein equations and to construct explicit solutions of that. Thus we might expect such applications of our Hamiltonian formulation to G_2-manifolds with SO(3)-symmetry. From our viewpoint, Bryant and Salamon's examples of G_2-manifolds are very similar to the FLRW metrics [Fri22, Lem31, Rob35, Wal37] in relativity (see Section 3.6.1). As this analogy, the Schwartzschild-type examples in Section 3.6.3 might have new explicit solutions of G_2-metrics similar to the Schwartzschild metrics.

Although we studied only SO(3)-fibrations in the thesis, the decomposition of SU(3)-structures in [Chi19] can be applied to other 3-dimensional unimodular Lie groups. The case of T^3-fibrations is already studied by several authors, but the case of the 3-dimensional Heisenberg group seems to be hardly studied. Extending our results to this case might be interesting.

Of course, it is interesting to generalize our results of G_2-manifolds with SO(3)-symmetry to Spin(7)-manifolds and Spin(7)-cobordisms with symmetries.

1.4 Organization of this thesis

This thesis is organized as follows. In Chapter 2, we review SU(3)- and G_2-structures, and summarize some results in [Chi19]. Chapter 3 is the main part of the thesis. In Section 3.1, we fix our notation for the phase spaces and introduce the principal bundle $\mathcal{P} \to T^*\mathbf{M}$. Section 3.2 states our main results, Theorem 3.4 and Proposition 3.5 (corresponding to Theorem A and Proposition B, respectively). In Section 3.3, we prove some elementary variation formulas used in the proof of the main results. In Section 3.4, we prove the main results. The behavior of the Hamiltonian dynamical system for scaling of H_{G_2} and variations of some functionals along the orbits of the Hamiltonian system are studied in Section 3.5. Section 3.6 demonstrates examples of our Hamiltonian dynamical systems. We observe Bryant-Salamon's examples in our formulation, and discuss smooth continuation of the orbits to regions corresponding to the indefinite counterparts of G_2-manifolds. Moreover we give left-invariant and Schwartzschild-type examples. In Chapter 4, we apply our method used the above sections to G_2-cobordisms under SO(3)-symmetry and coassociative conditions. In Section 4.2 we review definite 3-forms and G_2-cobordisms. Section 4.3 is devoted to the study of SO(3)-invariant objects. Main results in this chapter are stated and proved in Section 4.3.3.

1.5 Notation and conventions

We omit the symbol of summation adopting Einstein's convention, and often abbreviate $a \wedge b = ab$ and $c^i \wedge c^j = c^{ij}$. Also, we use the Levi-Civita symbol ϵ_{ijk} and write $\hat{c}^i = (1/2)\epsilon_{ijk}c^{jk}$ for a triple of 1-forms $\{c^1, c^2, c^3\}$. Denote by $\tilde{A}$ the adjugate matrix of an $n \times n$ matrix A, which satisfies $\tilde{A}A = \det(A)I$. Here $I = (\delta_{ij})$ is the identity matrix. Let G be a connected 3-dimensional Lie group with Lie algebra $\mathfrak{g}$, and $P \to M$ a principal G-bundle over a 3-manifold M. A tensorial $\mathfrak{g}$-valued 1-form e with respect to the adjoint action on $\mathfrak{g}$ is called a *solder 1-form* if $e = e^i X_i$ satisfies $e^{123} \neq 0$ at each $u \in P$ for a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}$. Let V be a vector space with a representation $\rho : G \to \mathrm{GL}(V)$. Denote the sets of equivariant and tensorial V-valued p-forms on P by $\Omega^p(P;V)^G$ and $\Omega^p(P;V)^G_{hor}$, respectively, and moreover the set of E-valued p-forms on M for a vector bundle $E \to M$ by $\Omega^p(M,E)$. Throughout this thesis, we assume all objects are of class C^∞.

Chapter 2

Preliminaries

In this chapter we review SU(3)-structures on 6-manifolds and G_2-structures on 7-manifolds, and summarize concepts and results in our previous work [Chi19].

2.1 SU(3)- and G_2-structures

2.1.1 SU(3)-structures

Let X be a 6-manifold and $\mathrm{Fr}(X)$ the frame bundle over X. We have the inclusion $\mathrm{SU}(3) \subset \mathrm{GL}(6;\mathbb{R})$ by the identification $\mathbb{R}^6 \cong \mathbb{C}^3$, where $z^i = x^i + \sqrt{-1}y^i$ for $i = 1, 2, 3$. A subbundle of $\mathrm{Fr}(X)$ is called an SU(3)-*structure* on X if the structure group is contained in SU(3). For an SU(3)-structure on X, we have the associated real 2-form ω and real 3-form ψ pointwisely isomorphic to $\omega_0 = \sum_{i=1}^{3} dx^i \wedge dy^i$ and $\psi_0 = \mathrm{Im}(dz^1 dz^2 dz^3)$ on $\mathbb{C}^3$. Here $\mathrm{Im}(*)$ denotes the imaginary part of $*$. We can identify an SU(3)-structure with such a pair (ω, ψ). Under this identification, we denote by $\psi^\#$ the real 3-form on X pointwisely corresponding to $\mathrm{Re}(dz^1 \wedge dz^2 \wedge dz^3)$. Here $\mathrm{Re}(*)$ denotes the real part of $*$.

An SU(3)-structure (ω, ψ) is called *torsion-free* if the holonomy group of the metric induced by the inclusion $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ is contained in SU(3). This condition is equivalent to $d\omega = 0$ and $d\psi = d\psi^\# = 0$. Then the metric is Ricci-flat. A 6-manifold with a torsion-free SU(3)-structure is called a *Calabi-Yau 3-fold.*

2.1.2 G_2-structures

Let Y be a 7-manifold and $\mathrm{Fr}(Y)$ the frame bundle over Y. Let us define G_2-structures on Y. The Lie group G_2 is defined as the linear automorphism group of the standard definite 3-form $\phi_0 = \omega_0 \wedge dx^0 + \psi_0$ on $\mathbb{R}^7 \cong \mathbb{R} \times \mathbb{R}^6$. It is known that this group coincides with the linear automorphism group of the cross product structure on $\mathrm{Im}\mathbb{O}$, where $\mathrm{Im}\mathbb{O}$ denotes the 7-dimensional imaginary part of the octonion algebra $\mathbb{O}$. A subbundle of $\mathrm{Fr}(Y)$ is called a *G_2-structure* on Y if the structure group is contained in G_2. For a G_2-structure on Y, we have the associated real 3-form ϕ on Y pointwisely isomorphic to ϕ_0 on $\mathbb{R}^7$. Such a 3-form is called a *definite 3-form*. We can identify a definite 3-form ϕ with a G_2-structure on Y. Since $G_2 \subset \mathrm{SO}(7)$, we have the Riemannian metric g_ϕ and orientation induced by a G_2-structure ϕ on Y. We denote the Hodge star compatible with ϕ by $\star_\phi$, and write $\star$ simply in situations without confusion.

A G_2-structure ϕ is called *torsion-free* if the Riemannian metric g_ϕ has the holonomy group contained in G_2. The structure ϕ is torsion-free if and only if $d\phi = 0$ and $d\star\phi = 0$. Then the metric on Y is Ricci-flat. A 7-manifold Y with a torsion-free G_2-structure ϕ is called a *G_2-manifold*.

2.1.3 Interpretation as a constraint dynamical system

G_2-structures are related to SU(3)-structures as follows. Let X be a 6-manifold and $(\omega(t), \psi(t))$ a one-parameter family of SU(3)-structures on X parameterized by an interval (t_1, t_2). Then the 3-form $\phi = \omega(t) \wedge dt + \psi(t)$ on the 7-manifold $X \times (t_1, t_2)$ is a G_2-structure. The Hodge dual $\star_\phi \phi$ is $-\psi(t)^\# \wedge dt + \frac{1}{2}\omega(t) \wedge \omega(t)$. The torsion-free conditions for ϕ are equivalent to the following constrained dynamical system on the space of SU(3)-structures on X.

Proposition 2.1 ([Hit01], Theorem 8)**.** *The G_2-structure ϕ is torsion-free if and only if $(\omega(t), \psi(t))$ satisfies the following four equations at all $t \in (t_1, t_2)$:*

1. *constraint conditions*

$$d\,(\omega \wedge \omega) = 0 \quad \textit{and} \quad d\psi = 0;$$

2. *equations of motion*

$$\frac{\partial \psi}{\partial t} = d\omega \quad \textit{and} \quad \frac{\partial}{\partial t}\left(\frac{1}{2}\omega \wedge \omega\right) = d\psi^\#.$$

Remark 2.2. It is clear that the solutions of the equations of motion preserve the constraint conditions. Hitchin [Hit01] found a Hamiltonian formulation of these equations as a constrained Hamiltonian flow on the direct product of a fixed cohomology class of 3- and 4-forms on the 6-manifold X.

2.2 Review of the previous work

2.2.1 G-invariant special Lagrangian fibrations

Let G be a 3-dimensional connected Lie group with Lie algebra $\mathfrak{g}$ and P the total space of a principal G-bundle $\pi : P \to M$ over a 3-manifold M.

Definition 2.3 ([Chi19], Definition 3.1)**.** An SU(3)-structure (ω, ψ) on P is called a *G-invariant special Lagrangian fibered* SU(3)*-structure* if it satisfies

1. ω and ψ are invariant under the right action of G on P;
2. the restrictions of ω and ψ to the fibers $F_m \subset P$ vanish: $\omega|_{F_m} = 0$ and $\psi|_{F_m} = 0$ for all $m \in M$.

Here we contain no integrability conditions in the definition.

Denote by $\Omega^p(P;V)^G$ and $\Omega^p(P;V)^G_{hor}$ the space of G-equivariant V-valued p-forms and horizontal (or tensorial) p-forms on P. Here V is a vector space with a representation $\rho : G \to \mathrm{GL}(V)$. Let $\mathrm{Sym}(3;\mathbb{R})$ and $\mathrm{Sym}^+(3;\mathbb{R})$ be the spaces of 3×3 symmetric and positive-definite matrices.

Example 2.4. Suppose that G is unimodular. A Lie group G is called *unimodular* if $\det(\mathrm{Ad}(g)) = \pm 1$ for all $g \in G$, where $\mathrm{Ad}(g)$ is the adjoint action of $g \in G$. Choosing a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}$, we have the identifications $\mathfrak{g} \cong \mathbb{R}^3$ and $\mathrm{Ad}(g) \in \mathrm{GL}(3;\mathbb{R})$. Let G act on $\mathrm{Sym}^+(3;\mathbb{R})$ by $g \cdot S = \mathrm{Ad}(g) \cdot S \cdot \mathrm{Ad}(g)^{\mathrm{T}}$ for $g \in G$ and $S \in \mathrm{Sym}^+(3;\mathbb{R})$. Here $\mathrm{Ad}(g)^{\mathrm{T}}$ is the transverse matrix of $\mathrm{Ad}(g)$. Take a triple (e, a, S) of a solder 1-form $e = e^k X_k \in \Omega^1(P;\mathfrak{g})^G_{hor}$, a connection 1-form $a = a^k X_k \in \Omega^1(P;\mathfrak{g})^G$ and an equivariant $\mathrm{Sym}^+(3;\mathbb{R})$-valued function $S = (S_{ij}) \in \Omega^0(P;\mathrm{Sym}^+(3;\mathbb{R}))^G$. Then the following 2- and 3-forms

$$\omega = (\det S)^{-\frac{1}{2}} \sum_{i,j=1}^{3} \tilde{S}_{ij} a^i e^j, \tag{2.1}$$

$$\psi = -(\det S) e^{123} + \sum_{k=1}^{3} e^k \hat{a}^k \tag{2.2}$$

are a G-invariant special Lagrangian fibered SU(3)-structure on P, where $\tilde{S} = (\det S) \cdot S^{-1}$.

The following is the key proposition in [Chi19].

Proposition 2.5 ([Chi19], Theorem 3.5)**.** *Suppose that G is unimodular. Fix a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}$. Then there exists a one-to-one correspondence between the G-invariant special Lagrangian fibered* SU(3)*-structures and the triples (e, a, S) in Example 2.4.*

Remark 2.6. Let (ω, ψ) be a G-invariant special Lagrangian fibered SU(3)-structure on P, and (e, a, S) the triple corresponding to (ω, ψ). Then the following 1-forms

$$\sum_{j=1}^{3} Q^{ij} a^j \quad \text{and} \quad \det(Q) \sum_{j=1}^{3} Q^{ij} e^j \quad \text{for} \quad i = 1, 2, 3$$

are an orthonormal coframe for the metric induced by (ω, ψ), where $S = Q \cdot Q^{\mathrm{T}}$ and $Q^{-1} = (Q^{ij})$.

2.2.2 Reduction of G-invariant G_2-manifolds

Let G be a 3-dimensional connected Lie group with Lie algebra $\mathfrak{g}$ and $Q \to N$ a principal G-bundle over a 4-manifold N.

Definition 2.7 ([Chi19], Definition 4.1)**.** A G_2-structure ϕ on Q is called a *G-invariant Lagrangian fibered G_2-structure* if it satisfies

1. ϕ is invariant under the right action of G on Q;

2. the restriction of ϕ to the fiber F_n vanishes: $\phi|_{F_n} = 0$ for all $n \in N$.

Example 2.8. Let $P \to M$ be a principal G-bundle over a 3-manifold M, and $(\omega(t), \psi(t))$ a one-parameter family of G-invariant special Lagrangian fibered SU(3)-structures on P parameterized by an interval (t_1, t_2). Let $f(u, t)$ be a G-invariant positive function on $P \times (t_1, t_2)$. Then the 3-form $\omega(t) \wedge f(u, t) dt + \psi(t)$ is a G-invariant Lagrangian fibered G_2-structure on $P \times (t_1, t_2)$.

Let ϕ be a G-invariant Lagrangian fibered G_2-structure on $Q \to N$ and $\{X_1, X_2, X_2\}$ a basis of $\mathfrak{g}$. Denote by X^* the infinitesimal vector field on Q of $X \in \mathfrak{g}$. The following gives a decomposition of a G-invariant Lagrangian fibered G_2-structure into a one-parameter family $(\omega(t), \psi(t), f(t))$ as above.

Proposition 2.9 ([Chi19], Proposition 4.3)**.** *Suppose that G is unimodular and that the 1-form $(\star_\phi \phi)(X_1^*, X_2^*, X_3^*, *)$ is closed. Then, for each $n \in N$, there exists a triple of a 3-dimensional submanifold $(n \in) D$ of N, a one-parameter family of G-invariant special Lgarangian fibered* $\mathrm{SU}(3)$*-structures on $\pi^{-1}(D)$ and a G-invariant positive function $f(u,t)$ on $\pi^{-1}(D) \times (t_1, t_2)$ such that ϕ is isomorphic to $\omega(t) \wedge f dt + \psi(t)$ on some neighborhood of $\pi^{-1}(D)$ in Q, where (t_1, t_2) denotes an interval by which the one-parameter family is parameterized.*

Remark 2.10. Goldstein [Gol01] remarked the following properties. Soppose that G is unimodular. Let $\{X_1, X_2, X_3\}$ be a basis of $\mathfrak{g}$, and Y a smooth 7-manifold with a G_2-structure ϕ invariant under a smooth action of G. Then

1. if ϕ is closed, then the function $\phi(X_1^*, X_2^*, X_3^*)$ is constant;
2. if $\star_\phi \phi$ is closed, then the 1-form $(\star_\phi \phi)(X_1^*, X_2^*, X_3^*, *)$ is closed.

The former follows from the following equation

$$\begin{aligned} d(\phi(X_1^*, X_2^*, X_3^*)) = {} & \phi([X_1^*, X_2^*], X_3^*, *) + \phi([X_2^*, X_3^*], X_1^*, *) + \phi([X_3^*, X_1^*], X_2^*, *) \\ & - (d\phi)(X_1^*, X_2^*, X_3^*, *). \end{aligned}$$

The latter is also proved in the same way as the former.

The following follows immediately from Proposition 2.9 and Remark 2.10.

Proposition 2.11. *If an invariant G_2-structure ϕ on Y is torsion-free and if G acts not freely on some part of Y, then ϕ is a G-invariant Lagrangian fibered G_2-structure on the part on which G acts freely.*

2.2.3 The case of $\mathrm{SO}(3)$-fibrations

Let $G = \mathrm{SO}(3)$ and $P \to M$ a principal $\mathrm{SO}(3)$-bundle over a 3-manifold M. Fix a basis $\{Y_1, Y_2, Y_3\}$ of $\mathfrak{so}(3)$ satisfying $[Y_i, Y_j] = \epsilon_{\alpha ij} Y_\alpha$ for $i, j = 1, 2, 3$. Any $\mathrm{SO}(3)$-invariant special Lagrangian fibered $\mathrm{SU}(3)$-structure (ω, ψ) on P decomposes uniquely into a triple (e, a, S) of a solder 1-form $e = e^i Y_i \in \Omega^1(P; \mathfrak{so}(3))_{hor}^{\mathrm{SO}(3)}$, a connection 1-form $a = a^i Y_i \in \Omega^1(P; \mathfrak{so}(3))^{\mathrm{SO}(3)}$ and an equivariant $\mathrm{Sym}^+(3;\mathbb{R})$-valued function $S = (S_{ij}) \in \Omega^0(P; \mathrm{Sym}^+(3;\mathbb{R}))^{\mathrm{SO}(3)}$.

Remark 2.12. If there exists an $\mathrm{SO}(3)$-invariant special Lagrangian fibered $\mathrm{SU}(3)$-structure, then P is a trivial $\mathrm{SO}(3)$-bundle over M. This is because P is isomorphic to the orthonormal frame bundle over M by the solder 1-form e, and because all orientable 3-manifolds are parallelizable.

Denote by $T = (T_{ij})$ and $\Omega = (\Omega_{ij})$ the torsion and curvature forms of (e, a). These are defined by $d_H e := de + [a \wedge e] = T_{ij}\hat{e}^j Y_i$ and $d_H a := da + (1/2)[a \wedge a] = \Omega_{ij}\hat{e}^j Y_i$. For $A \in \Omega^0(P; \mathrm{M}(k; \mathbb{R}))^{\mathrm{SO}(3)}$, denote by $A_{ij;k}$ the covariant derivative: $(d_H A)_{ij} = (dA + a^k[Y_k, A])_{ij} = A_{ij;k}e^k$.

Proposition 2.13 ([Chi19], Corollary 6.5)**.** *Let (e, a, S) be the triple corresponding to a* SO(3)*-invariant special Lagrangian fibered* SU(3)*-structure (ω, ψ) on P. Then*

1. *$d\omega \wedge \omega = 0$ and $d\psi = 0$ if and only if $T = 0$ and $S_{ik;k} = 0$ for $i = 1, 2, 3$;*
2. *if $d\psi = d\psi^{\#} = 0$ holds, then the Riemannian metric on M given by the solder 1-form e is constant negative curvature.*

Remark 2.14. An SO(3)-invariant special Lagrangian fibered SU(3)-structure is never symplectic because of the Liouville - Arnold theorem, which implies compact fibers of Lagrangian fibrations of symplectic manifolds are tori.

Remark 2.15. If $d_H e = T_{ij}\hat{e}^j Y_i = 0$, then the connection a is the Levi-Civita connection of the solder 1-form e. Thus the Bianchi identity $[d_H a \wedge e] = d_H d_H e = 0$ holds. Then $\Omega_{ij} = \Omega_{ji}$ holds for $i, j = 1, 2, 3$. In fact, Ω_{ij} coincides with the orthonormal representation of the Einstein-tensor of the Riemannian metric with respect to the local coframe $\{e^1, e^2, e^3\}$.

All torsion-free SO(3)-invariant Lagrangian fibered G_2-structures are locally described by orbits of a constrained dynamical system on the space of triples (e, a, S) as follows. Let $(\omega(t), \psi(t))$ be a one-parameter family of such SU(3)-structures parameterized by (t_1, t_2), and $(e(t), a(t), S(t))$ the triples corresponding to $(\omega(t), \psi(t))$. Then the 3-form $\omega(t) \wedge (\det S)^{\frac{1}{2}} dt + \psi(t)$ is a G_2-structure on $P \times (t_1, t_2)$.

Proposition 2.16 ([Chi19], Theorem 1.4)**.** *The G_2-structure $\omega(t) \wedge (\det S)^{\frac{1}{2}} dt + \psi(t)$ is torsion-free if and only if the triple $(e(t), a(t), S(t))$ is an orbit of the following constrained dynamical system on the space of triples (e, a, S):*

$$d_H e = 0, \quad S_{i\alpha;\alpha} = 0 \quad (\textit{constraint conditions}); \tag{2.3}$$

$$\frac{\partial e^i}{\partial t} = \tilde{S}_{ij} e^j, \quad \frac{\partial S}{\partial t} = -\Omega - \mathrm{tr}(\tilde{S})S + 2(\det S)I \quad (\textit{equations of motion}) \tag{2.4}$$

for $i = 1, 2, 3$, where $I = (\delta_{ij})$.

By scaling of the parameter t, combining Proposition 2.9 and 2.16 deduces the following.

Proposition 2.17 ([Chi19], Theorem 1.6). *All torsion-free* SO(3)*-invariant Lagrangian fibered* G_2*-structures are locally given by orbits of the constrained dynamical system in Proposition 2.16.*

2.2.4 Notes on other cases

We studied the case of T^3-fibrations together with that of SO(3)-fibrations in [Chi19]. This case was studied by Goldstein [Gol01] and more explicitly by Madsen and Swann [MS18] before our work. On the other hand, the case of the 3-dimensional Heisenberg group fibrations seems not to be studied much yet. This case is between the cases of T^3- and SO(3)-fibrations. The work [CCG⁺02] by physicists is pioneering in this case. Our approach might be applicable to make their work more systematic.

Chapter 3

G_2-manifolds and the ADM formalism

In this chapter we describe SO(3)-invariant G_2-manifolds as a constrained Hamiltonian dynamical system on the cotangent bundle over the space of all Riemannian metrics on a closed oriented 3-manifold.

3.1 Phase spaces

Let $\otimes^p E$, $S^p E$ and $\wedge^p E$ be the p-th tensor, symmetric, and anti-symmetric power of a vector bundle E over a manifold N. Denote by $\Omega^p(N, E)$ the space of E-valued p-forms on N. The symbol of summation is often omitted as in the chapters above.

Let M be a closed oriented 3-manifold. The space of all Riemannian metrics on M is denoted by

$$\mathbf{M} = \Omega^0(M, S^2_+ T^* M),$$

where T^*M is its cotangent bundle over M, and $S^2_+ T^* M$ is the positive-definite part of $S^2 T^* M$. For a metric $\gamma \in \mathbf{M}$, the tangent space at γ is $T_\gamma \mathbf{M} = \Omega^0(M, S^2 T^* M)$. Then $T\mathbf{M} = \mathbf{M} \times \Omega^0(M, S^2 T^* M)$. The cotangent space at γ is $T^*_\gamma \mathbf{M} = \Omega^3(M, S^2 TM)$ via $T^*_\gamma \mathbf{M} \otimes T_\gamma \mathbf{M} \to \mathbb{R}$ given by

$$\langle \pi, h \rangle = \int_M \pi^{ij} h_{ij} \quad \text{for} \quad \pi \in \Omega^3(M, S^2 TM) \text{ and } h \in \Omega^0(M, S^2 T^* M).$$

Then we have

$$T^*\mathbf{M} = \mathbf{M} \times \Omega^3(M, S^2TM).$$

The space $T^*\mathbf{M}$ has the standard symplectic form Θ given by

$$\Theta_{(\gamma,\pi)}((h_1,\omega_1),(h_2,\omega_2)) = \int_M (h_1\omega_2 - h_2\omega_1) \quad \text{for} \quad (h_1,\omega_1),(h_2,\omega_2) \in T_{(\gamma,\pi)}T^*\mathbf{M}.$$

We can see that $\Theta = -d\theta$ for the tautological Liouville 1-form θ on $T^*\mathbf{M}$ as in cases of finite-dimensional cotangent bundles.

Let $P = M \times \mathrm{SO}(3)$. Since any orientable 3-manifold M is parallelizable, P is regarded as a fixed oriented orthonormal frame bundle over M. Denote by $\Omega^p(P;V)$ the space of V-valued p-forms on P for a vector space V. For a representation $\rho : \mathrm{SO}(3) \to GL(V)$, denote by $\Omega^p(P;V)^{\mathrm{SO}(3)}_{hor}$ the space of $\alpha \in \Omega^p(P;V)$ satisfying $R_g^*\alpha = \rho(g^{-1})\alpha$ and $\iota(A^*)\alpha = 0$ for all $g \in \mathrm{SO}(3)$ and $A \in \mathfrak{so}(3)$. Here $R_g : P \to P$ denotes the right action, ι the inner product, and A^* the infinitesimal vector field of A on P. If $p = 0$, then $\Omega^0(P;V)^{\mathrm{SO}(3)}_{hor} = \Omega^0(P;V)^{\mathrm{SO}(3)}$ is the space of SO(3)-equivariant V-valued functions on P. Take $\mathbb{R}^3$ with the usual representation of SO(3). A 1-form $e = (e^1, e^2, e^3) \in \Omega^1(P;\mathbb{R}^3)^{\mathrm{SO}(3)}_{hor}$ is called a *solder 1-form* if $e^1_u \wedge e^2_u \wedge e^3_u \neq 0$ for all $u \in P$. Put $\mathrm{vol}(e) = e^1 \wedge e^2 \wedge e^3$, which is a volume form on M because of the SO(3)-equivariance of e.

Let $\mathcal{M}$ be the space of all solder 1-forms on P, compatible with the orientation on M, that is, satisfying $\mathrm{vol}(e) > 0$ on M. Denote by $\mathcal{G}$ the gauge group of P. Each element $\tau \in \mathcal{G}$ acts freely on $\mathcal{M}$ by $e \cdot \tau = \tau^* e$ for $e \in \mathcal{M}$. For any solder 1-form $e \in \mathcal{M}$, a Riemannian metric γ on M is uniquely given by the condition that (s^*e^1, s^*e^2, s^*e^3) is a local orthonormal coframe for any local section $s : U \to P$ on any domain U of M. Thereby this map

$$i : \mathcal{M} \to \mathbf{M}, \quad i(e) = \gamma$$

defines a natural principal $\mathcal{G}$-bundle over $\mathbf{M}$. Let $\mathrm{M}(n;\mathbb{R})$, $\mathrm{Sym}(n;\mathbb{R})$ and $\mathrm{Ant}(n;\mathbb{R})$ be the space of real $n \times n$ matrices, symmetric and anti-symmetric ones, respectively, on which $g \in \mathrm{SO}(n)$ acts by $g \cdot B = gBg^{-1}$, where B is an element of the spaces. For any solder 1-form $e \in \mathcal{M}$, the tangent space at e is

$$\begin{aligned} T_e\mathcal{M} &= \Omega^0(P;\mathrm{M}(3;\mathbb{R}))^{\mathrm{SO}(3)} \\ &= \Omega^0(P;\mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)} \oplus \Omega^0(P;\mathrm{Ant}(3;\mathbb{R}))^{\mathrm{SO}(3)}, \end{aligned}$$

by decomposition into symmetric and anti-symmetric parts and by the following identification

$$T_e\mathcal{M} \ni (\dot{e}^1, \dot{e}^2, \dot{e}^3) = (T_{1j}e^j, T_{2j}e^j, T_{3j}e^j) \mapsto (T_{ij}) \in \Omega^0(P; \mathrm{M}(3;\mathbb{R}))^{\mathrm{SO}(3)}.$$

Here the SO(3)-equivariance of e and $\dot{e}$ induces that of $T = (T_{ij})$. The infinitesimal action of $\mathcal{G}$ at $e \in \mathcal{M}$ maps $\mathrm{Lie}(\mathcal{G})$ onto $\Omega^0(P; \mathrm{Ant}(3;\mathbb{R}))^{\mathrm{SO}(3)} \subset T_e\mathcal{M}$, where $\mathrm{Lie}(\mathcal{G})$ is the Lie algebra of $\mathcal{G}$. Hence the horizontal distribution

$$H_e\mathcal{M} := \Omega^0(P; \mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)} \quad \text{for all } e \in \mathcal{M}$$

gives a non-flat connection in the principal $\mathcal{G}$-bundle $i : \mathcal{M} \to \mathbf{M}$.

Remark 3.1. The connection above is not flat. To see this, let us consider the local model of $i : \mathcal{M} \to \mathbf{M}$. Let $\mathrm{Fr}^+(\mathbb{V})$ and $\mathrm{S}^2_+(\mathbb{V}^*)$ be the spaces of basises compatible with the orientation and positive-definite symmetric 2-forms on an oriented real n-dimensional vector space $\mathbb{V}$, respectively. Then we have a principal $\mathrm{SO}(n)$-bundle $\mathrm{Fr}^+(\mathbb{V}) \to \mathrm{S}^2_+(\mathbb{V}^*)$. If we fix a basis on $\mathbb{V}$, then this bundle is identified with $\mathrm{GL}^+(n;\mathbb{R}) \to \mathrm{Sym}^+(n;\mathbb{R})$. Here $\mathrm{GL}^+(n;\mathbb{R})$ and $\mathrm{Sym}^+(n;\mathbb{R})$ are the identity component of a general linear group and the positive-definite subset of $\mathrm{Sym}(n;\mathbb{R})$, respectively. The decomposition $\mathrm{M}(n;\mathbb{R}) = \mathrm{Sym}(n;\mathbb{R}) \oplus \mathrm{Ant}(n;\mathbb{R})$ gives a non-flat connection on the bundle, since $[S_1, S_2] \in \mathrm{Ant}(n;\mathbb{R})$ for any $S_1, S_2 \in \mathrm{Sym}(n;\mathbb{R})$.

The differential $i^* : T\mathcal{M} \to T\mathbf{M}$ of $i : \mathcal{M} \to \mathbf{M}$ restricts to the following principal $\mathcal{G}$-bundle over $T\mathbf{M}$:

$$i_* : H\mathcal{M} \to T\mathbf{M}, \quad i_*(e, T) = (\gamma, \rho),$$

where $H\mathcal{M} = \mathcal{M} \times \Omega^0(P; \mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)}$ and $T\mathbf{M} = \mathbf{M} \times \Omega^0(M, S^2T^*M)$. Here $\tau \in \mathcal{G}$ acts on $H\mathcal{M}$ by $(e, T) \cdot \tau = (\tau^* e, \tau^* T)$. The dual of i_* defines a principal $\mathcal{G}$-bundle over $T^*\mathbf{M}$ as follows:

$$j : \mathcal{M} \times \Omega^3(P; \mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)}_{hor} \to T^*\mathbf{M}, \quad j(e, U) := (i(e), \pi) \tag{3.1}$$

given by

$$\langle \pi, \rho \rangle = \int_M \mathrm{tr}(TU) = \int_M T_{ij}U_{ij}$$

for all $\rho \in \Omega^0(M, S^2T^*M)$, which gives T uniquely by $(i(e), \rho) = i_*(e, T) \in T_\gamma\mathbf{M}$. Also, the trace $\mathrm{tr}(TU)$ is a volume form on M by SO(3)-equivariance of T and U. Let us denote this principal bundle by

$$\mathcal{P} := \mathcal{M} \times \Omega^3(P; \mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)}_{hor}.$$

Here $\tau \in \mathcal{G}$ acts freely on $\mathcal{P}$ by $(e, U) \cdot \tau = (\tau^* e, \tau^* U)$. Then we define a horizontal distribution on $\mathcal{P}$ by

$$H_{(e,U)}\mathcal{P} := H_e\mathcal{M} \oplus \Omega^3(P; \mathrm{Sym}(3; \mathbb{R}))^{\mathrm{SO}(3)}_{hor} \quad \text{for all } (e, U) \in \mathcal{P},$$

which gives a connection in the principal $\mathcal{G}$-bundle $j : \mathcal{P} \to T^*\mathbf{M}$. On the horizontal subbundle $H\mathcal{P} \to \mathcal{P}$, let us define the non-degenerate 2-form $\Theta' \in \Omega^0(\mathcal{P}, \wedge^2(H\mathcal{P})^*)$ by

$$\Theta'_{(e,U)}((T_1, W_1), (T_2, W_2)) = \int_M \mathrm{tr}(T_1W_2 - T_2W_1)$$

for any $(T_1, W_1), (T_2, W_2) \in H_{(e,U)}\mathcal{P}$. By Lemma 3.2 stated below, we can directly see that $\Theta' = j^*\Theta$, where $j^*\Theta$ is the pull-back of the standard symplectic form Θ on $T^*\mathbf{M}$ by $j : \mathcal{P} \to T^*\mathbf{M}$.

Take a coordinate neighborhood $(D; x^1, x^2, x^3)$ of M and a local section $f : D \to P$. For each $e \in \mathcal{M}$, we set an $\mathrm{M}(3; \mathbb{R})$-valued function $C = (C_{ij})$ on D by $f^*e^i = C_{ij}dx^j$ for $i = 1, 2, 3$. Denote the transvese matrix of C by C^{T} . For a metric $\gamma \in \mathbf{M}$, define $\mathrm{vol}(\gamma) =$ (the volume element of γ). Then by definition, we see $\mathrm{vol}(e) = \mathrm{vol}(\gamma)$ for any solder 1-form e with $\gamma = i(e)$.

Lemma 3.2. *On any neighborhood $(D; x^1, x^2, x^3)$, the map*

$$j : \mathcal{P} \to T^*\mathbf{M}, \quad j(e, S\mathrm{vol}(e)) = (\gamma, \pi)$$

is given by

$$(\gamma_{ij}) = C^{\mathrm{T}}C \quad \text{and} \quad (\pi^{ij}) = \frac{1}{2}C^{-1}(f^*S)(C^{\mathrm{T}})^{-1}\mathrm{vol}(e),$$

where $\gamma = \gamma_{ij}dx^idx^j$, $\pi = \pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$ and $S \in \Omega^0(P; \mathrm{Sym}(3; \mathbb{R}))^{\mathrm{SO}(3)}$.

Proof. By the derivation of $s^*e^i = C_{ij}dx^j$, we can see that

$$i_* : H\mathcal{M} \to T\mathbf{M}, \quad i_*(e, T) = (\gamma, \rho)$$

is given by $(\gamma_{ij}) = C^{\mathrm{T}}C$ and $(\rho_{ij}) = 2C^{\mathrm{T}}(f^*T)C$. Here $\gamma = \gamma_{ij}dx^idx^j$ and $\rho = \rho_{ij}dx^idx^j$ on $(D; x^1, x^2, x^3)$. Thus by the definition of j (see 3.1),

$$\langle \pi, \rho \rangle = \int_M \mathrm{tr}(TS)\mathrm{vol}(e) = \int_M \mathrm{tr}\left(\frac{1}{2}C^{-1}(f^*S)(C^{\mathrm{T}})^{-1}(\rho_{ij})\right)\mathrm{vol}(e)$$

for every $(\gamma, \rho) = i_*(e, T)$. Hence $(\pi_{ij}) = (1/2)C^{-1}(f^*S)(C^{\mathrm{T}})^{-1}\mathrm{vol}(e)$. $\square$

3.2 Constrained Hamiltonian dynamical systems

In this section we formulate our main results. Let $\mathrm{Sym}^+(3;\mathbb{R})$ be the positive-definite subsets of $\mathrm{Sym}(3;\mathbb{R})$, and let $\Omega^3(P;\mathrm{Sym}^+(3;\mathbb{R}))^{\mathrm{SO}(3)}_{hor}$ be that of $\Omega^3(P;\mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)}_{hor}$ with respect to the orientation of M. Let us define

$$\mathcal{P}^+ := \mathcal{M} \times \Omega^3(P;\mathrm{Sym}^+(3;\mathbb{R}))^{\mathrm{SO}(3)}_{hor} \subset \mathcal{P}.$$

Denote by $\mathcal{A}$ be the space of connection 1-forms on the trivial principle bundle $P = M \times \mathrm{SO}(3)$. Fix a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{so}(3)$ with $[X_i, X_j] = \epsilon_{ijk}X_k$ for $i, j = 1, 2, 3$. By this basis, we write a connection 1-form $a \in \mathcal{A}$ as $a = a^i X_i$. Let $(e, S\mathrm{vol}(e), a) \in \mathcal{P}^+ \times \mathcal{A}$, where $S \in \Omega^0(P;\mathrm{Sym}^+(3;\mathbb{R}))^{\mathrm{SO}(3)}$. As seen in Section 2.2.3, we have a one-to-one correspondence between the space $\mathcal{P}^+ \times \mathcal{A}$ and the space of SU(3)-structures (ω, ψ) on P satisfying the conditions in Definition 2.3 given by

$$\begin{aligned}
&\omega = (\det S)^{-\frac{1}{2}} \tilde{S}_{ij} a^i \wedge e^j, \\
&\psi = -(\det S) e^1 \wedge e^2 \wedge e^3 + e^1 \wedge a^2 \wedge a^3 + e^2 \wedge a^3 \wedge a^1 + e^3 \wedge a^1 \wedge a^2,
\end{aligned}$$

where $\tilde{S}$ is the adjugate matrix of S. Here note that $\tilde{B}_{ij} = (1/2)\epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}B_{\gamma\alpha}B_{\delta\beta}$ and $B\tilde{B} = \det B \cdot I$ for any $B \in \mathrm{M}(3;\mathbb{R})$. For a given $S \in \Omega^0(P;\mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)}$, define $S_{ij;k}$ by $(d_H S)_{ij} := (dS + [a, S])_{ij} = S_{ij;\alpha}e^\alpha$ for $i, j = 1, 2, 3$. A connection a is called the *Levi-Civita connection* of e if $d_H e := de + a \wedge e = 0$, and denoted by a_{LC}. Let $\mathcal{P}^+$ embed into $\mathcal{P}^+ \times \mathcal{A}$ by

$$\mathcal{P}^+ \ni (e, S\mathrm{vol}(e)) \mapsto (e, S\mathrm{vol}(e), a_{LC}) \in \mathcal{P}^+ \times \mathcal{A}.$$

Now take a curve $c(t) = (e(t), S(t)\mathrm{vol}(e(t)), a_{LC}(t))$ in $\mathcal{P}^+ \subset \mathcal{P}^+ \times \mathcal{A}$ parameterized by an interval (t_1, t_2). Then $c(t)$ defines a G_2-structure ϕ on $P \times (t_1, t_2)$ by

$$\phi = (\det S)^{\frac{1}{2}} \omega(t) \wedge dt + \psi(t), \tag{3.2}$$

where $(\omega(t), \psi(t))$ is the SU(3)-structure determined by $c(t)$ for each t. Define $G = (G_{ij})$ by $d_H a_{LC} = G_{\alpha\beta}\hat{e}^\beta X_\alpha$, where $d_H a_{LC} := a_{LC} + (1/2)[a_{LC} \wedge a_{LC}]$ and $\hat{e}^i = (1/2)\epsilon_{i\alpha\beta}e^\alpha \wedge e^\beta$ for $i = 1, 2, 3$. Recall that G_{ij} coincides with the orthonormal representation of the Einstein tensor of $\gamma = i(e)$ by local coframes $\{e^1, e^2, e^3\}$. From the embedding $\mathcal{P}^+ \hookrightarrow \mathcal{P}^+ \times \mathcal{A}$ and Proposition 2.16, the torsion-free conditions for the G_2-structure ϕ of (3.2) are described by the following constrained dynamical system on $\mathcal{P}^+$.

Proposition 3.3. *The G_2-structure ϕ of (3.2) is torsion-free if and only if the curve $c(t) = (e, S\mathrm{vol}(e))(t)$ in $\mathcal{P}^+$ satisfies*

$$S_{i\alpha;\alpha} = 0; \tag{3.3}$$

$$\frac{\partial e^i}{\partial t} = \tilde{S}_{i\alpha} e^\alpha, \quad \frac{\partial S}{\partial t} = -G - \mathrm{tr}(\tilde{S})S + 2(\det S)I \tag{3.4}$$

for all $t \in (t_1, t_2)$ and for $i = 1, 2, 3$.

Here note that $\mathcal{G}$-invariance of the equations in Proposition 3.3. We can restrict the action of $\mathcal{G}$ on $\mathcal{P}$ to $\mathcal{P}^+$. Then by the form of equations in Proposition 3.3, we can see that if $c(t) = (e(t), S(t)\mathrm{vol}(e(t)))$ is a solution, then so is $c(t) \cdot \tau = (\tau^* e(t), \tau^* S(t)\mathrm{vol}(\tau^* e(t)))$ for any $\tau \in \mathcal{G}$.

Let $\Omega^3(M, S^2_+ TM)$ be the subspace of positive-definite elements of $\Omega^3(M, S^2 TM)$ with respect to the orientation of M. Define

$$T^*\mathbf{M}^+ := \mathbf{M} \times \Omega^3(M, S^2_+ TM) \subset T^*\mathbf{M}.$$

Then the principal $\mathcal{G}$-bundle $j : \mathcal{P} \to T^*\mathbf{M}$ restricts to $j : \mathcal{P}^+ \to T^*\mathbf{M}^+$. Moreover, the connection on $\mathcal{P}$ also restricts to $\mathcal{P}^+$. For a curve $\underline{c}(t) = (\gamma(t), \pi(t))$ in $T^*\mathbf{M}^+$, a curve $c(t)$ in $\mathcal{P}^+$ is called a *lift* of $\underline{c}(t)$ if it satisfies $j(c(t)) = \underline{c}(t)$ and $\dot{c}(t) \in H_{c(t)}\mathcal{P}^+$. Here $\dot{c}(t)$ denotes the derivative of $c(t)$ at t, and $H\mathcal{P}^+$ the horizontal distribution on $\mathcal{P}^+$. We can construct a lift $c(t)$ of $\underline{c}(t)$ for any curve $\underline{c}(t)$ in $T^*\mathbf{M}$. (For example, by a fixed global coframe on M, the Gram-Schmidt process and Lemma 3.2, we construct a curve $c'(t)$ in $\mathcal{P}$ with $j(c'(t)) = \underline{c}(t)$, and determine $\tau(t) \in \mathcal{G}$ such that $c'(t) \cdot \tau(t)$ is horizontal by solving an ODE on $\mathcal{G}$. The ODE is a smooth family of ODEs on $\mathrm{SO}(3)$ in form $\dot{h}h^{-1} = A(t)$, where $h \in \mathrm{SO}(3)$ and $A(t) \in \mathfrak{so}(3)$.) A lift $c(t)$ gives a G_2-structure ϕ by (3.2). Note that any two lifts $c(t)$ and $c'(t)$ of $\underline{c}(t)$ satisfy $c'(t) = c(t) \cdot \tau$ for some $\tau \in \mathcal{G}$. Then $\mathrm{SU}(3)$-structures (ω, ψ) and G_2-structures ϕ corresponding to $c(t)$ and $c'(t)$ are isomorphic by $\tau : P \to P$.

Define a Hamiltonian function H on $(T^*\mathbf{M}, \Phi)$ by

$$H(\gamma, \pi) = -\frac{1}{2}\int_M \mathrm{R}(\gamma)\mathrm{vol}(\gamma) + 8\int_M \det(\pi^i_j)\mathrm{vol}(\gamma)^{-2} \tag{3.5}$$

for $(\gamma, \pi) \in T^*\mathbf{M}$. Here $\mathrm{R}(\gamma)$ denotes the scalar curvature of $\gamma \in \mathbf{M}$. We formulate our result by H, which is different from H_{G_2} mentioned in Introduction. This simplifies the proof of our main result in Section 3.4. The dynamical systems of H and H_{G_2} are mutually transformed by scaling (See Remark 3.11). Let ∇ be the Levi-Civita connection for γ. The main results in this chapter are as follows.

Theorem 3.4. *Let $\underline{c}(t) = (\gamma(t), \pi(t))$ be a curve in $T^*\mathbf{M}^+$ parameterized by (t_1, t_2). Then a lift $c(t)$ of $\underline{c}(t)$ gives a torsion-free G_2-structure ϕ by (3.2) if and only if $\underline{c}(t)$ satisfies the following:*

- *$\underline{c}(t)$ is a solution of the Hamiltonian dynamical system of H;*
- *$\nabla_\alpha \pi^{i\alpha}(t) = 0$ holds for $i = 1, 2, 3$ and for all $t \in (t_1, t_2)$*

Proposition 3.5. *Let $\underline{c}(t)$ be an orbit of the Hamiltonian dynamical system of H. Suppose that $\nabla_j \pi^{ij}(t_0) = 0$ holds for $i = 1, 2, 3$ at some t_0. Then $\nabla_j \pi^{ij}(t) = 0$ holds for $i = 1, 2, 3$ on the whole orbit.*

Remark 3.6. Let $\underline{c}(t)$ be a curve in $T^*\mathbf{M}$ satisfying the conditions in Theorem 3.4, defined on (t_1, t_2). If $\pi(t)$ is not positive-definite and $\det(\pi(t)) \neq 0$ for all t, then a G_2^*-structure is defined on $P \times (t_1, t_2)$ from a lift $c(t)$ of $\underline{c}(t)$ by (3.2). We conjecture that this G_2^*-structure is also torsion-free. In general, a torsion-free G_2^*-structure on a 7-manifold Y defines a Ricci-flat pseudo-Riemannian metric of signature $(3, 4)$ on Y. See ([CLSSH11], p. 122 and Theorem 2.3.) for more details.

3.3 Variation formulas

In this section we prove some formulas used in the proofs of Theorem 3.4 and Proposition 3.5. Let us identify $\Omega^p(P; \mathbb{R}^3)$ with $\Omega^p(P; \mathfrak{so}(3))$ taking a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{so}(3)$ satisfying $[X_i, X_j] = \epsilon_{ijk} X_k$ for $i, j = 1, 2, 3$. For given $e \in \mathcal{M}$, $a \in \mathcal{A}$ and $A = (A_{ij}) \in \Omega^0(P; \mathrm{M}(3; \mathbb{R}))^{\mathrm{SO}(3)}$, let us define $A_{ij;k}$ by $(d_H A)_{ij} := (dA + [a, A])_{ij} = A_{ij;\alpha} e^\alpha$ for $i, j = 1, 2, 3$. Denote by $\dot{\sigma}$ the t-derivative $\partial \sigma / \partial t$ of a differential form σ. Moreover, put $\hat{e}^i = (1/2) \epsilon_{i\alpha\beta} e^\alpha \wedge e^\beta$ for $i = 1, 2, 3$.

Lemma 3.7. *Let $e(t)$ be a curve in $\mathcal{M}$. Suppose that*

$$\frac{\partial e^i}{\partial t} = P_{i\alpha} e^\alpha \quad \textit{and} \quad P_{ij} = P_{ji} \quad \textit{for} \quad i, j = 1, 2, 3.$$

Then we have

$$\frac{\partial a^i_{LC}}{\partial t} = -\epsilon_{i\alpha\beta} P_{\gamma\alpha;\beta} e^\gamma \quad \textit{for} \quad i = 1, 2, 3.$$

Proof. Put $\dot{a}^i_{LC} = Q_{ij}e^j$ for $i = 1, 2, 3$. Then

$$\begin{aligned} 0 = \frac{\partial(d_H e)}{\partial t} &= \frac{\partial}{\partial t}(de + [a_{LC} \wedge e]) \\ &= d\dot{e} + [\dot{a}_{LC} \wedge e] + [a_{LC} \wedge \dot{e}] \\ &= d_H\dot{e} + [\dot{a}_{LC} \wedge e] \\ &= P_{ij;k}e^{kj}Y_i + [Q_{\alpha k}e^k Y_\alpha \wedge e^\beta Y_\beta] \\ &= (\epsilon_{\gamma kj}P_{ij;k} + \epsilon_{\gamma k\beta}\epsilon_{i\alpha\beta}Q_{\alpha k})\hat{e}^\gamma Y_i. \end{aligned}$$

Thus $\epsilon_{\gamma kj}P_{ij;k} + \epsilon_{\gamma k\beta}\epsilon_{i\alpha\beta}Q_{\alpha k} = 0$ for $\gamma, i = 1, 2, 3$. Since (P_{ij}) is symmetric, we can see $Q_{ij} = -\epsilon_{i\alpha\beta}P_{j\alpha;\beta}$ for $i, j = 1, 2, 3$. □

Lemma 3.8. *Let (e, a, A) be a curve in $\mathcal{M} \times \mathcal{A} \times \Omega^0(P; \mathrm{M}(3; \mathbb{R}))^{\mathrm{SO}(3)}$. Suppose that*

$$\frac{\partial e^i}{\partial t} = P_{ij}e^j, \quad \frac{\partial a^i}{\partial t} = Q_{ij}e^j \quad \textit{and} \quad \frac{\partial A}{\partial t} = B \quad \textit{for} \quad i = 1, 2, 3.$$

Then we have

$$\frac{\partial}{\partial t}(A_{ij;k}) = B_{ij;k} - A_{ij;\alpha}P_{\alpha k} + \epsilon_{i\alpha\gamma}A_{\gamma j}Q_{\alpha k} + \epsilon_{j\alpha\gamma}A_{i\gamma}Q_{\alpha k} \quad \textit{for} \quad i, j, k = 1, 2, 3.$$

In particular,

$$\frac{\partial}{\partial t}(A_{i\alpha;\alpha}) = B_{i\alpha;\alpha} - A_{i\alpha;\beta}P_{\beta\alpha} + \epsilon_{i\beta\gamma}A_{\gamma\alpha}Q_{\beta\alpha} + \epsilon_{\alpha\beta\gamma}A_{i\gamma}Q_{\beta\alpha} \quad \textit{for} \quad i = 1, 2, 3.$$

Proof. By definition, we have

$$\frac{\partial}{\partial t}(d_H A) = d_H B + (Q_{kj}e^j)[Y_k, A] \quad \text{and} \quad \frac{\partial}{\partial t}(d_H A)_{ij} = \frac{\partial}{\partial t}(A_{ij;k})e^k + A_{ij;k}(P_{k\alpha}e^\alpha)$$

for $i, j = 1, 2, 3$. By these identities, we obtain the equations in Lemma 3.8. □

For a given $e \in \mathcal{M}$, denote by $R(e)$ the scalar curvature of the metric $\gamma = i(e)$.

Lemma 3.9. *Let $(e(t), S(t))$ be a curve in $\mathcal{M} \times \Omega^0(P; \mathrm{Sym}(3;\mathbb{R})))^{\mathrm{SO}(3)}$, and set $\partial S/\partial t = \dot{S}$ and $\partial e^i/\partial t = P_{ij}e^j$ for $i = 1, 2, 3$. Then we have*

$$\frac{\partial}{\partial t}\int_M \mathrm{R}(e)\mathrm{vol}(e) = -2\int_M \mathrm{tr}(GP)\mathrm{vol}(e),$$
$$\frac{\partial}{\partial t}(\mathrm{vol}(e)) = \mathrm{tr}(T)\mathrm{vol}(e) \quad \textit{and}$$
$$\frac{\partial}{\partial t}(\det S) = \mathrm{tr}(\tilde{S}\dot{S}),$$

where G is the lift of the Einstein tensor of the metric $\gamma = i(e)$.

Proof. The last two are straightforward. Let $\langle A, B\rangle := -(1/2)\mathrm{tr}(AB)$ for any $A, B \in \mathfrak{so}(3)$. Since G is the Einstein tensor for $\gamma = i(e)$ on a 3-manifold M, we have $\mathrm{tr}(G) = -(1/2)R(e)$. Thus

$$\begin{aligned}\frac{\partial}{\partial t}\int_M \mathrm{R}(e)\mathrm{vol}(e) &= -2\frac{\partial}{\partial t}\int_M \mathrm{tr}(G)\mathrm{vol}(e) = -2\int_M \frac{\partial}{\partial t}\langle d_H a, e\rangle \\ &= -2\int_M \left(\langle d_H\dot{a}, e\rangle + \langle G_{ij}Y_i\hat{e}^j, \dot{e}\rangle\right) = -2\int_M \mathrm{tr}(GP)\mathrm{vol}(e).\end{aligned}$$

□

3.4 Proof of the main results

3.4.1 Proof of Theorem 3.4

Let $\underline{c}(t) = (\gamma(t), \pi(t))$ be a curve in $T^*\mathbf{M}$, and $c(t) \subset \mathcal{P}$ a lift curve of $\underline{c}(t)$. We can easily see that $\pi(t)$ is positive-definite and $\nabla_\alpha \pi^{i\alpha} = 0$ for $i = 1, 2, 3$ if and only if $S(t)$ of the lift $c(t) = (e(t), S(t)\mathrm{vol}(e(t)))$ is positive-definite and $S_{i\alpha;\alpha} = 0$ for $i = 1, 2, 3$. Here note that by Lemma 3.2, we have

$$\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{1}{2}(f^*S)_{ij}e_ie_j$$

for any local section $f : (D; x^1, x^2, x^3) \to P$, where (e_1, e_2, e_3) is the dual orthonormal basis of (f^*e^1, f^*e^2, f^*e^3).

Since the 2-form Φ' on $H\mathcal{P}$ is non-degenerate, for any function F' on $\mathcal{P}$, we can define uniquely the horizontal vector field $X^{F'} \in \Omega^0(\mathcal{P}, H\mathcal{P})$ by $dF'(Z) = \Phi'(X^{F'}, Z)$ for all $Z \in H\mathcal{P}$. Also for a function F on $(T^*\mathbf{M}, \Phi)$, denote by X^F the Hamiltonian vector field of F. From now, for a function F on $T^*\mathbf{M}$, put $F' = j^*F$, where $j : \mathcal{P} \to T^*\mathbf{M}$. Then by $j^*\Phi = \Phi'$, the vector field $X^{F'}$ is the lift of X^F satisfying $j_* X^{F'} = X^F$. Thus it suffices to check $X^{H'} = X^s$, where X^s is the horizontal vector field on $\mathcal{P}$ defined by solution curves $c(t) = (e(t), S(t)\mathrm{vol}(e(t))) \subset \mathcal{P}$ of (3.4) in Proposition 3.3.

For a horizontal curve $c(t) = (e(t), S(t)\mathrm{vol}(e(t))) \subset \mathcal{P}$, set $T(t)$ and $V(t)$ by

$$\frac{\partial c(t)}{\partial t} = (T(t), V(t)\mathrm{vol}(e(t))) \in H_{c(t)}\mathcal{P},$$

where

$$H_{c(t)}\mathcal{P} = \Omega^0(P, \mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)} \oplus \Omega^3(P;\mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)}_{hor}.$$

Then applying Leibniz rule gives

$$\begin{aligned} \frac{\partial e^i}{\partial t} &= T_{ij}e^j \quad \text{for} \quad i = 1, 2, 3, \\ \frac{\partial S}{\partial t} &= V - \mathrm{tr}(T)S. \end{aligned} \tag{3.6}$$

Define $T^s, V^s \in \Omega^0(P, \mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)}$ by

$$X^s = (T^s, V^s\mathrm{vol}(e)) \in H_{(e,S\mathrm{vol}(e))}\mathcal{P}$$

for each $(e, S\mathrm{vol}(e)) \in \mathcal{P}$. (3.4) and (3.6) give

$$\begin{aligned} T^s &= \tilde{S}, \\ V^s &= -G + 2(\det S) \cdot I. \end{aligned} \tag{3.7}$$

Put

$$\begin{aligned} H_1(\gamma, \pi) &= \int_M \mathrm{R}(\gamma)\mathrm{vol}(\gamma), \\ H_2(\gamma, \pi) &= 8\int_M \det(\pi)\mathrm{vol}(\gamma)^{-2} \end{aligned} \tag{3.8}$$

for $(\gamma, \pi) \in T^*\mathbf{M}$. Then $H = -\frac{1}{2}H_1 + H_2$. Lemma 3.2 gives

$$\begin{aligned} H_1'(e, S\mathrm{vol}(e)) &= \int_M \mathrm{R}(e)\mathrm{vol}(e), \\ H_2'(e, S\mathrm{vol}(e)) &= \int_M \det(S)\mathrm{vol}(e) \end{aligned}$$

for $(e, S\mathrm{vol}(e)) \in \mathcal{P}$. Here $\det(\pi) = \frac{1}{8}\det(S)\mathrm{vol}(e)^3$ for $j(e, S\mathrm{vol}(e)) = (\gamma, \pi) \in T^*\mathbf{M}$. Take any $Z = (T, V\mathrm{vol}(e)) \in H_{(e,S\mathrm{vol}(e))}\mathcal{P}$. Lemma 3.9 gives

$$\begin{aligned} dH_1'(Z) &= -2\int_M \mathrm{tr}(GT)\mathrm{vol}(e), \\ dH_2'(Z) &= \int_M \Big(\mathrm{tr}(\tilde{S}V) - 2(\det S)\mathrm{tr}(I \cdot T)\Big)\,\mathrm{vol}(e). \end{aligned} \tag{3.9}$$

On the other hand, we have

$$\Phi'(X^{H_i'}, Z) = \int_M \mathrm{tr}(T^{H_i'}V - TV^{H_i'})\mathrm{vol}(e) \tag{3.10}$$

for $X^{H_i'} = (T^{H_i'}, V^{H_i'}\mathrm{vol}(e)) \in H_{(e,S\mathrm{vol}(e))}\mathcal{P}$. Comparing (3.9) with (3.10) gives

$$\begin{aligned} T^{H_1'} &= 0, \quad V^{H_1'} = 2G, \\ T^{H_2'} &= \tilde{S}, \quad V^{H_2'} = 2(\det S)\cdot I. \end{aligned} \tag{3.11}$$

Thus $X^{H'} = -\frac{1}{2}X^{H_1'} + X^{H_2'} = X^s$. □

3.4.2 Proof of Proposition 3.5

Let $c(t) \subset \mathcal{P}$ be a lift curve of an orbit $\underline{c}(t)$ of the Hamiltonian dynamical system of H. Then by Theorem 3.4, $c(t)$ is a solution of (3.4) in Proposition 3.3. It suffices to prove that if the solution $c(t)$ satisfies (3.3) at some t_0, then $c(t)$ satisfies (3.3) at all t.

By (3.4), Lemma 3.7 and 3.8, we have

$$\begin{aligned} \frac{\partial}{\partial t}(S_{i\alpha;\alpha}) &= (-\mathrm{tr}(\tilde{S})S - G + 2(\det S)I)_{i\alpha;\alpha} - S_{i\alpha;\beta}\tilde{S}_{\beta\alpha} \\ &\quad + \epsilon_{i\beta\gamma}S_{\gamma\alpha}(-\epsilon_{\beta kl}\tilde{S}_{\alpha k;l}) + \epsilon_{\alpha\beta\gamma}S_{i\gamma}(-\epsilon_{\beta kl}\tilde{S}_{\alpha k;l}) \\ &= -\tilde{S}_{\beta\beta}S_{i\alpha;\alpha} - S_{\alpha\gamma;\gamma}\tilde{S}_{\alpha i} + S_{\gamma\alpha;i}\tilde{S}_{\alpha\gamma} - (\det S)_{;i} \end{aligned}$$

for $i = 1, 2, 3$,where $d(\det S) = (\det S)_{;k}e^k$. Moreover we have $\tilde{S}_{\alpha\beta}S_{\alpha\beta;i} = \mathrm{tr}(()\tilde{S}S_{;i}) = (\det S)_{;i}$ for $i = 1, 2, 3$. Hence we obtain

$$\frac{\partial}{\partial t}(S_{i\alpha;\alpha}) = -\tilde{S}_{\alpha\alpha}S_{i\beta;\beta} - \tilde{S}_{i\alpha}S_{\alpha\beta;\beta} \quad \text{for} \quad i = 1, 2, 3. \tag{3.12}$$

Here (3.12) is a first-order homogeneous linear ODE for $(S_{1\alpha;\alpha}(t), S_{2\alpha;\alpha}(t), S_{3\alpha;\alpha}(t))$. Thus by the uniqueness of the solution, we can conclude that if the solution $c(t)$ satisfies (3.3) at some t_0, then $c(t)$ satisfies (3.3) at all t. □

3.5 Properties of the Hamiltonian dynamical system

3.5.1 Scaling of the dynamical system

Let $c(t) = (e(t), S(t)\mathrm{vol}(e(t)))$ be a horizontal curve in $\mathcal{P}$. Fix $\alpha, \beta, \kappa > 0$. Define $c'(t') = (e'(t'), S'(t')\mathrm{vol}(e'(t'))) \subset \mathcal{P}$ by $e'(t') = \alpha e(\kappa t')$ and $S'(t') = \beta S(\kappa t')$. Recall that H, H_1 and H_2 defined by (3.5) and (3.8).

Proposition 3.10. *Take $a, b > 0$ with $\alpha = 2^{\frac{1}{2}} a^{\frac{1}{2}} b^{\frac{1}{4}} \kappa^{-\frac{1}{4}}$ and $\beta = b^{-\frac{1}{2}} \kappa^{\frac{1}{2}}$. Then $c(t)$ is a lift of an orbit of the Hamiltonian dynamical system of H if and only if $c'(t')$ is a lift of that of $-aH_1 + bH_2$.*

Proof. Assume that $c(t)$ is a lift of an orbit of the Hamiltonian dynamical ststem of H. Put

$$\frac{\partial c(t)}{\partial t} = (T(t), V(t)\mathrm{vol}(e(t))) \quad \text{and} \quad \frac{\partial c'(t')}{\partial t'} = (T'(t'), V'(t')\mathrm{vol}(e'(t'))).$$

Then by (3.6), we have

$$\begin{aligned}
\frac{\partial (e')^i}{\partial t'} &= \kappa T(\kappa t')_{ij} (e')^j \quad \text{for} \quad i = 1, 2, 3, \\
\frac{\partial S'}{\partial t'} &= \beta\kappa \frac{\partial S}{\partial t}(\kappa t') = \beta\kappa V(\kappa t') - \mathrm{tr}(()T'(t'))S'(t').
\end{aligned}$$

Thus $T'(t') = \kappa T(\kappa t')$ and $V'(t') = \beta\kappa V(\kappa t')$. Moreover, by assumption and (3.7),

$$\begin{aligned}
T(\kappa t') &= \tilde{S}(\kappa t'), \\
V(\kappa t') &= -G(\kappa t') + 2\det(S(\kappa t')) \cdot I.
\end{aligned}$$

Using $\tilde{S}(\kappa t') = \beta^{-2} \tilde{S'(t')}$, $\det(S(\kappa t')) = \beta^{-3}\det(S'(t'))$ and $G(\kappa t') = \alpha^2 G'(t')$, where $G'(t')$ denotes the orthonormal representation of the Einstein tensor defined by $e'(t')$, we have

$$\begin{aligned}
T'(t') &= \kappa\beta^{-2}\tilde{S}'(t'), \\
V'(t') &= -\kappa\alpha^2\beta G'(t') + 2\kappa\beta^{-2}\det(S'(t')) \cdot I.
\end{aligned} \tag{3.13}$$

On the other hand, by (3.11), we have

$$\begin{aligned} &- aT^{H_1'} + bT^{H_2'} = b\tilde{S}', \\ &- aV^{H_1'} + bV^{H_2'} = -2aG' + 2b\det(S')\cdot I. \end{aligned} \tag{3.14}$$

By assumption, (3.13) coincides with (3.14). Hence $c'(t')$ is a lift of $-aH_1 + bH_2$. The inverse is the same as above. □

Remark 3.11. Let $\underline{c}(t)$ be a curve in $T^*\mathbf{M}$, and $c(t) = (e(t), S(t)\mathrm{vol}(e(t)))$ a lift curve of $\underline{c}(t)$ in $\mathcal{P}$. In order to apply Proposition 3.10 to $H_{G_2} = -H_1 + (1/8)H_2$, take constants $\kappa > 0$, $\alpha = 2^{-\frac{1}{4}}\kappa^{-\frac{1}{4}}$ and $\beta = 2^{\frac{3}{4}}\kappa^{\frac{1}{2}}$. Then by Proposition 3.10, we see that $\underline{c}(t)$ is an orbit of the Hamiltonian dynamical system of H_{G_2} if and only if $c'(t') := (\alpha^{-1}e(\kappa^{-1}t'), \beta^{-1}S(\kappa^{-1}t')\mathrm{vol}(\alpha^{-1}e(\kappa^{-1}t')))$ is a lift of an orbit of the Hamiltonian dynamical system of H defined by (3.5). Thus by Theorem 3.4, we see that $\underline{c}(t)$ satisfies the conditions in Theorem A if and only if $c'(t')$ yields a torsion-free G_2-structure by (3.2). This is the statement of Theorem A. Here note that the positive-definiteness and the divergence-free conditions are preserved by the scaling with positive constants. Thus we can also deduce Proposition B form Proposition 3.5.

3.5.2 Variations of some functionals

Let $\underline{c}(t) = (\gamma(t), \pi(t))$ be an orbit of the Hamiltonian dynamical system of H, and $c(t) = (e(t), S(t)\mathrm{vol}(e(t)))$ be a lift of $\underline{c}(t)$ in $\mathcal{P}$. Put $H_3(\gamma, \pi) = \int_M \mathrm{vol}(\gamma)$ for $(\gamma, \pi) \in T^*\mathbf{M}$. By (3.7) and (3.9), we can easily deduce variation formulas of H_1, H_2 and H_3 along the orbit $\underline{c}(t)$:

$$\begin{aligned} \frac{\partial H_1}{\partial t} &= \frac{\partial}{\partial t}\int_M \mathrm{R}(\gamma)\mathrm{vol}(\gamma) \\ &= -2\int_M \mathrm{tr}((G\tilde{S}))\mathrm{vol}(e) = \int_M \left(\mathrm{R}(e)\mathrm{tr}(\tilde{S}) - 2\mathrm{tr}(\mathrm{Ric}\cdot\tilde{S})\right)\mathrm{vol}(e), \\ \frac{\partial H_2}{\partial t} &= 8\frac{\partial}{\partial t}\int_M \det(\pi)\mathrm{vol}(\gamma)^{-2} = \frac{1}{2}\frac{\partial H_1}{\partial t}, \\ \frac{\partial H_3}{\partial t} &= \frac{\partial}{\partial t}\int_M \mathrm{vol}(\gamma) = \int_M \mathrm{tr}(\tilde{S})\mathrm{vol}(e). \end{aligned}$$

Here, Ric denotes the Ricci tensor $\mathrm{Ric} = G - \mathrm{tr}(G)I$. For example, by the third formula, we can see that if $\pi(t_0)$ is positive- or negative-definite at any point on M, then

$$\frac{\partial H_3}{\partial t} = \frac{\partial}{\partial t} \int_M \mathrm{vol}(\gamma(t)) \Big|_{t=t_0} > 0.$$

Here note that $\tilde{S} = \det(S) S^{-1}$ and $\mathrm{tr}(\tilde{S}) = \det(S)\mathrm{tr}(S^{-1})$.

3.6 Examples

3.6.1 Bryant-Salamon G_2-metrics

Take $\sigma \in \mathbb{R}$ and $e_0 \in \mathcal{M}$ satisfying $G = -(1/6)\sigma \cdot I$. Here we identify I with the identity matrix-valued function in $\Omega^0(P, \mathrm{Sym}(3;\mathbb{R}))^{\mathrm{SO}(3)}$. Since M is a 3-manifold, $\mathrm{tr}(G) = -(1/2)\mathrm{R}(e)$. Then $\gamma = i(e_0) \in \mathbf{M}$ is a constant curvature Riemannian metric with scalar curvature σ. Define an enbedding $i_{e_0} : \mathbb{R}_{>0} \times \mathbb{R} \to \mathcal{P}$ by

$$\mathbb{R}_{>0} \times \mathbb{R} \ni (a, b) \mapsto (ae_0, bI\mathrm{vol}(ae_0)) \in \mathcal{P}.$$

Then by Proposition 3.3 and Theorem 3.4, we can easily see that the integral curves of the horizontal vector fields $X^{H'}$ on the image $i_{e_0}(\mathbb{R}_{>0} \times \mathbb{R})$ are restricted in $i_{e_0}(\mathbb{R}_{>0} \times \mathbb{R})$ and are solutions of the following dynamical system:

$$\begin{aligned} \frac{da}{dt} &= ab^2, \\ \frac{db}{dt} &= \sigma a^{-2} - b^3 \quad \text{for} \quad (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}. \end{aligned} \tag{3.15}$$

Here we remark about Bryant-Salamon's examples in ([BS89], Section 3). The examples are diffeomorphic to $M \times S^3 \times (t_1, t_2)$. The torsion-free G_2-structures of the examples are constructed by one-parameter families of SU(3)-structures on $M \times S^3$ satisfying the conditions in Definition 2.3. Moreover the SU(3)-structures are constructed by constant curvature metrics on M and on S^3. Thus the examples are locally isomorphic to torsion-free G_2-structures constructed by (3.2) form solution curves $(a(t)e_0, b(t)I\mathrm{vol}(a(t)e_0))$ of (3.15) satisfying $b(t) > 0$.

Put $x(t) = a(t)^2$ and $y(t) = a(t)b(t)$. Then (3.15) is equivalent to the following dynamical system:

$$\begin{aligned} \frac{dx}{dt} &= 2y^2, \\ \frac{dy}{dt} &= \sigma x^{-\frac{1}{2}} \quad \text{for} \quad (x, y) \in \mathbb{R}_{>0} \times \mathbb{R}. \end{aligned} \tag{3.16}$$

We can draw the vector field $(dx/dt, dy/dt)$ on $\mathbb{R}_{>0} \times \mathbb{R}$. Observing this, we see that the integral curve $(x(t), y(t))$ of (3.16) extends to $[0, +\infty)$ for any initial value $(x(0), y(0)) = (x_0, y_0) \in \mathbb{R}_{>0} \times \mathbb{R}$. Assume $\sigma > 0$. Then any integral curve $(x(t), y(t))$ defined on $[0, +\infty)$ has $y(t) > 0$ for all t more than sufficiently large t_1. Hence the curve $(a(t)e_0, b(t)I\mathrm{vol}(a(t)e_0)) \subset \mathcal{P}$ corresponding to $(x(t), y(t))$ yields torsion-free G_2-structure on $M \times \mathrm{SO}(3) \times (t_1, +\infty)$ by (3.2). Next, assume $\sigma < 0$. Then any integral curve $(x(t), y(t))$ defined on $[0, +\infty)$ has $y(t) < 0$ for all t more than sufficiently large t_1. Hence by Remark 3.6, the curve in $\mathcal{P}$ corresponding to $(x(t), y(t))$ yields indefinite G_2^*-structure on $M \times \mathrm{SO}(3) \times (t_1, +\infty)$ by (3.2). Finally, assume $\sigma = 0$. Then any integral curve $(x(t), y(t))$ satisfy $y(t) = y_0$ for all $t \in [0, +\infty)$. Hence the curve in $\mathcal{P}$ corresponding to $(x(t), y(t))$ is always contained in $\mathcal{P}^+$, or otherwise never contained.

For example, let us consider the case of $\sigma > 0$. If we start from an initial value (x_0, y_0) with $y_0 < 0$, the smooth integral curve $(x(t), y(t))$ defined on $[0, +\infty)$ yields indefinite G_2^*-structure on $M \times \mathrm{SO}(3) \times (0, t_1)$, and on the other hand yields torsion-free positive-definite G_2-structure on $M \times \mathrm{SO}(3) \times (t_1, +\infty)$ for some $t_1 > 0$. The SU(3)-structure defined by $(x(t_1,), y(t_1))$ breaks down at t_1. In this case, G_2^*- and G_2-structures are smoothly continued as an orbit of the Hamiltonian dynamical system of H on $T^*\mathbf{M}$.

3.6.2 Left-invariant examples

Let $M = \mathrm{SO}(3)$ and $P = \mathrm{SO}(3) \times \mathrm{SO}(3)$. Fix a global section of P, and by the pull-back, identify a triple $(e(t), a(t), S(t))$ in Proposition 2.16 with 1-forms and functions on M. Suppose that $(e(t), a(t), S(t))$ is left-invariant for the group structure on M. The equations in Proposition 2.16 are reduced to the following ordinary differential system. This situation is contained in that of [MS18]. Let $\theta^1, \theta^2, \theta^3$ be left-invariant 1-forms on M satisfying $d\theta^i = -\hat{\theta}^i$ for $i = 1, 2, 3$. Using $A(t) = (A(t)_{ij}), B(t) = (B(t)_{ij})$ and $C(t) = (C(t)_{ij}) \in \mathrm{M}(3; \mathbb{R})$, put $e^i = A_{ij}e^j$, $a^i = B_{ij}e^j$ and $de^i = C_{ij}\hat{e}^j$ for $i = 1, 2, 3$. Then $C = -\det(A) \cdot A \cdot A^{\mathrm{T}}$, and by simple computation, we can see that (2.4) in Proposition 2.16 is equivalent to the following:

$$\frac{dA}{dt} = \tilde{S}A \quad \textit{and} \quad \frac{dS}{dt} = -\mathrm{tr}(\tilde{S}) \cdot S + 2(\det S) \cdot I - C^2 - \tilde{C} + \frac{1}{4}(\mathrm{tr}(C))^2 \cdot I. \tag{3.17}$$

Here by the Levi-Civita condition, $B = C - (1/2)\mathrm{tr}(C) \cdot I$. Moreover, the condition $S_{ij;j} = 0$, for $i = 1, 2, 3$, is equivalent to $[S, B]$=0, and this is preserved by solutions of (3.17).

3.6.3 Schwarzschild-type examples

Let $M = \mathbb{R}^3$ and $P = \mathbb{R}^3 \times \mathrm{SO}(3)$. Take 1-forms $\theta^1 = dr$, $\theta^2 = d\rho$ and $\theta^3 = \sin(\rho) d\xi$ on $\mathbb{R}^3 \setminus \{0\}$, where (r, ρ, ξ) is the polar coordinate on $\mathbb{R}^3$. Using positive functions $f(r,t), g(r,t), k(r,t), l(r,t)$ for r and t, put $(e(t)^1, e(t)^2, e(t)^3) = (f(r,t)\theta^1, g(r,t)\theta^2, g(r,t)\theta^3)$ and $S = ((k(r,t), 0, 0), (0, l(r,t), 0), (0, 0, l(r,t)))$. Then (2.3) and (2.4) in Proposition 2.16 are equivalent to the following:

$$\begin{aligned}
&\frac{\partial f}{\partial t} = fl^2 \quad \textit{and} \quad \frac{\partial k}{\partial t} = kl^2 - 2k^2 l - (\frac{1}{fg} \cdot \frac{\partial g}{\partial r})^2 + \frac{1}{g^2}, \\
&\frac{\partial g}{\partial t} = gkl \quad \textit{and} \quad \frac{\partial l}{\partial t} = -l^3 - \frac{1}{f^2 g} \cdot \frac{\partial^2 g}{\partial r^2} + \frac{1}{f^3 g} \cdot \frac{\partial f}{\partial r} \cdot \frac{\partial g}{\partial r}, \\
&\frac{\partial k}{\partial r} - \frac{2}{g} \cdot \frac{\partial g}{\partial r} \cdot (l - k) = 0.
\end{aligned}$$

The last condition is preserved by solutions of the above four equations. Then the trivial solutions corresponding to flat metrics are the following:

$$f(r,t) = \alpha(2t + \beta)^{\frac{1}{2}}, \quad g(r,t) = \alpha(2t + \beta)^{\frac{1}{2}} r, \quad l(r,t)^2 = k(r,t)^2 = (2t + \beta)^{-1},$$

where $\alpha, \beta \in \mathbb{R}$.

Chapter 4

SO(3)-invariant G_2-cobordisms

In this chapter we study G_2-cobordisms under SO(3)-symmetry and coassociative conditions.

4.1 Problems

Donaldson introduced the concept of G_2-cobordisms between 6-manifolds with $\mathrm{SL}(3;\mathbb{C})$-structures in his study of boundary value problems in G_2-geometry ([Don18], Section 4). Let X be a fixed oriented spin 6-manifold, and let ψ_1, ψ_2 be closed definite 3-forms that define $\mathrm{SL}(3;\mathbb{C})$-structures on X. The forms ψ_1 and ψ_2 are called to be *G_2-cobordant* if there exists a closed definite 3-form ϕ defining a G_2-structure on $X \times [t_1, t_2]$ compatible with the orientation and satisfying $\phi|_{X\times\{t_i\}} = \psi_i$ for $i = 1, 2$. This gives a binary relation $\psi_1 \prec \psi_2$ on the set $\mathcal{C}(X)$ of closed definite 3-forms on X. Moreover, let us define a stronger binary relation $\psi_1 \ll \psi_2$ on $\mathcal{C}(X)$ by the existence of such a 3-form ϕ that also induces a product Riemannian metric on $X \times [t_1, t_2]$. See Section 4.2 for more details. In addition to Donaldson's Torelli-type problems ([Don18], Question 1), various basic properties of the relations are still open. We are now interested in the following problems:

Problem 4.1. *Suppose that X is closed. The spin 6-manifold X admits no symplectic structure if and only if the binary relation $\prec$ on $\mathcal{C}(X)$ is irreflexive, that is, $\psi \not\prec \psi$ for any $\psi \in \mathcal{C}(X)$.*

Problem 4.2. *Let $\psi_1, \psi_2 \in \mathcal{C}(X)$. If $\psi_1 \prec \psi_2$, then for any $\epsilon > 0$, there exists a diffeomorphism F on X such that $\psi_1 \ll F^*\psi_2$ and $\|F^*\psi_2 - \psi_1\| < \epsilon$, where $\| * \|$ is some appropriate norm.*

In this chapter we prove several results supporting the above problems under SO(3)-symmetry and coassociative conditions. More precisely, we study G_2-cobordisms by SO(3)-invariant coassociative fibrations $M \times \mathrm{SO}(3) \times [t_1, t_2]$ over an oriented 3-manifold M. This gives a binary relation on the set of SO(3)-invariant closed definite 3-forms on $M \times \mathrm{SO}(3)$ vanishing along the fibers SO(3). We describe the relation explicitly (Lemma 4.10 and Theorem 4.11), and prove that the relation is irreflexive (Theorem 4.12), which supports Problem 4.1. Moreover, Theorem 4.14 is proved, which is related to Problem 4.2.

4.2 Definite 3-forms and G_2-cobordisms

In this section we recall the notions of definite 3-forms and G_2-cobordisms. The former was studied in detail in [Hit00, Hit01], and the latter introduced in [Don18].

4.2.1 Definite 3-forms in dimension 6 and 7

Let V be a 6-dimensional real vector space and V^* its dual. Take a basis $\{v^1, w^1, v^2, w^2, v^3, w^3\}$ of V^*. Let us set a normal form

$$\begin{aligned}\psi_0 &= \mathrm{Im}\{(v^1 + \sqrt{-1}w^1) \wedge (v^2 + \sqrt{-1}w^2) \wedge (v^3 + \sqrt{-1}w^3)\} \\ &= -w^{123} + w^1 v^{23} + w^2 v^{31} + w^3 v^{12},\end{aligned}$$

where e.g. $v^{12} = v^1 \wedge v^2$. We call a 3-form ψ on V to be *definite* if ψ is contained in the orbit $\mathrm{GL}(V) \cdot \psi_0 \subset \wedge^3 V^*$. This condition is equivalent to saying that $\iota(v)\psi \in \wedge^2 V^*$ has rank 4 for any non-zero $v \in V$, where ι denotes inner product (see e.g. [Don18], Section 2.1). The definiteness of a 3-form is an open condition. Let X be an oriented 6-manifold. If X has an $\mathrm{SL}(3;\mathbb{C})$-structure, then X is spin. A 3-form $\psi \in \Omega^3(X)$ is called to be *definite* if $\psi_x \in \wedge^3 T^*_x X$ is definite at each point $x \in X$. We know that the group $\{g \in \mathrm{GL}^+(V) \mid g^*\psi_0 = \psi_0\}$ coincides with $\mathrm{SL}(3;\mathbb{C}) \subset \mathrm{GL}(6;\mathbb{R})$. Thus a definite 3-form ψ naturally gives an $\mathrm{SL}(3;\mathbb{C})$-structure on X. In paticular, ψ gives an almost complex structure J_ψ on X.

Let W be a 7-dimensional real vector space and W^* its dual. Take a basis

$\{v^0, v^1, w^1, v^2, w^2, v^3, w^3\}$, and set a normal form

$$\begin{aligned}\phi_0 &= (v^1w^1 + v^2w^2 + v^3w^3)v^0 + \psi_0 \\ &= -w^{123} + w^1(v^{01} + v^{23}) + w^2(v^{02} + v^{31}) + w^3(v^{03} + v^{12}).\end{aligned}$$

A 3-form ϕ on W is called to be *definite* if ϕ is contained in the orbit $\mathrm{GL}(W)\cdot\phi_0 \subset \wedge^3 W^*$. This condition is equivalent to saying that $\iota(w)\phi$ has rank 6 for any non-zero $w \in W$ (see e.g. [Don18], Section 2.1). Moreover, the restriction of a definite 3-form to any 6-dimensional subspace is also definite. Let Y be a 7-manifold. If Y has a G_2-structure, then Y is spin. A 3-form $\phi \in \Omega^3(Y)$ is called to be *definite* if $\phi_y \in \wedge^3 T^*_y Y$ is definite for each $y \in Y$. The isotropy group of ϕ_0 is known to coincide with $G_2 \subset \mathrm{SO}(7)$. Thus a definite 3-form ϕ gives a G_2-structure, which induces an orientation and a Riemannian metric on Y. Let us denote by $\mathrm{vol}(\phi)$ the volume form on Y induced by ϕ. See e.g. [Don18, Hit00, Hit01, Joy00] for more details on $\mathrm{SL}(3;\mathbb{C})$- and G_2-structures.

4.2.2 G_2-cobordisms

Let X be an oriented spin 6-manifold. Define

$$\mathcal{C}(X) := \{\psi \in \Omega^3(X) \mid \psi \text{ is closed and definite}\}.$$

This set is a subset of the $\mathrm{SL}(3;\mathbb{C})$-structures compatible with the orientation on X. Let us define a binary relation on $\mathcal{C}(X)$ using closed G_2-structures on $X \times [t_1, t_2]$ for a closed interval $[t_1, t_2]$.

Definition 4.3. Let $\psi_1, \psi_2 \in \mathcal{C}(X)$. Then $\psi_1 \prec \psi_2$ if there exists a closed definite 3-form ϕ on $X \times [t_1, t_2]$ satisfying the conditions: $\mathrm{vol}(\phi) > 0$ for the product orientation, $\phi|_{X\times\{t_1\}} = \psi_1$ and $\phi|_{X\times\{t_2\}} = \psi_2$.

This binary relation is linked to nondegenerate 2-forms on X by the following elementary proposition.

Proposition 4.4 ([Don18], the second paragraph in Section 4)**.** *Let $\psi_1, \psi_2 \in X$. Then $\psi_1 \prec \psi_2$ if and only if there exist $\omega_t \in \Omega^2(X)$ and $\psi_t \in \Omega^3(X)$ parameterized by $t \in [t_1, t_2]$ satisfying the following conditions:*

1. *$\psi_{t_1} = \psi_1$, $\psi_{t_2} = \psi_2$, and ψ_t is definite for each t;*
2. *$\partial\psi_t/\partial t = d\omega_t$ for each t; and*

3. *the (1,1) part of ω_t is positive with respect to the almost complex structure on X induced by each ψ_t.*

Proof. Suppose that $\psi_1 \prec \psi_2$. Then we have a closed definite 3-form ψ on $X \times [t_1, t_2]$ satisfying the conditions in Definition 4.3. Setting $\omega_t := \iota(\partial/\partial t)\phi|_{X\times\{t\}}$ and $\psi_t := \phi|_{X\times\{t\}}$, we can obtain the desired forms. Conversely, suppose that we have ω_t and ψ_t satisfying the conditions in Proposition 4.4. Then $\phi = \omega_t \wedge dt + \psi_t$ is the desired closed definite 3-form on $X \times [t_1, t_2]$. □

Donaldson remarked that the relation $\prec$ on $\mathcal{C}(X)$ is transitive ([Don18], Proposition 2). For Problem 4.1, by Proposition 4.4, we can evidently see that if $\psi \prec \psi$ by some one-parameter family (ω_t, ψ_t) with $\psi_t \equiv \psi$, then ω_t is a symplectic form on X for each $t \in [t_1, t_2]$.

Let us define a stronger version of the relation $\prec$.

Definition 4.5. Let $\psi_1, \psi_2 \in \mathcal{C}(X)$. Define $\psi_1 \ll \psi_2$ by that there exists a closed definite 3-form ϕ on $X \times [t_1, t_2]$ satisfying the conditions in Definition 4.3 and also $\iota(\partial/\partial t)\phi \wedge \phi|_{X\times\{t\}} = 0$ for each t.

The last condition is equivalent to saying that the 2-form $\iota(\partial/\partial t)\phi|_{X\times\{t\}}$ is a $(1,1)$ form on X with respect to the almost complex structure induced by each $\phi|_{X\times\{t\}}$. We expect that Problem 4.2 has a flavor of sub-Riemannian geometry.

4.3 Restriction to SO(3)-invariant structures

In this section we prove several results related to Problems 4.1 and 4.2 under SO(3)-symmetry and coassociative conditions.

4.3.1 Settings

Let M be an oriented 3-manifold. Fix a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{so}(3)$ satisfying $[X_i, X_j] = \epsilon_{ijk} X_k$ for $i, j = 1, 2, 3$, where ϵ_{ijk} is the 3-rd order Levi-Civita symbol. An orientation on SO(3) is given by $X_1 \wedge X_2 \wedge X_3$, and that on $M \times \mathrm{SO}(3)$ by direct product. Let us define $\psi \in \mathcal{C}^{\mathrm{inv}}(M)$ by the following conditions:

1. $\psi \in \mathcal{C}(M \times \mathrm{SO}(3))$;

2. ψ is SO(3)-invariant under the right action; and

3. ψ vanishes along each fiber SO(3).

Refinements of the relations $\prec$ and $\ll$ in Definition 4.3 and 4.5 are defined as follows.

Definition 4.6. Let $\psi_1, \psi_2 \in \mathcal{C}^{\mathrm{inv}}(M)$. Then $\psi_1 \prec^{\mathrm{inv}} \psi_2$ if there exists a closed definite 3-form ϕ on $M \times \mathrm{SO}(3) \times [t_1, t_2]$ satisfying the conditions in Definition 4.3 and also that ϕ is SO(3)-invariant and vanishes along each fiber $\mathrm{SO}(3) \times [t_1, t_2]$.

Namely, we suppose that each fiber $\mathrm{SO}(3) \times [t_1, t_2]$ is an almost coassociative submanifolds of $M \times \mathrm{SO}(3) \times [t_1, t_2]$ for the G_2-structure induced by the closed definite 3-form ϕ.

Definition 4.7. Let $\psi_1, \psi_2 \in \mathcal{C}^{\mathrm{inv}}(M)$. Then $\psi_1 \ll^{[\mathrm{inv}} \psi_2$ if there exists a closed definite 3-form ϕ satisfying the conditions in Definition 4.6 and also that $\iota(\partial/\partial t)\phi \wedge \phi|_{M \times \mathrm{SO}(3) \times \{t\}} = 0$ for each t.

Evidently, $\psi_1 \prec^{\mathrm{inv}} \psi_2$ implies $\psi_1 \prec \psi_2$, and so does for $\ll^{\mathrm{inv}}$ and $\ll$.

4.3.2 Lemmas

Let us prove lemmas used in the proofs of main results. Let $C^\infty(M)$ and $\mathcal{A}$ be the set of positive functions on M and that of connection 1-forms on the trivial bundle $M \times \mathrm{SO}(3)$, respectively. Here, a tensorial (or horizontal) $\mathfrak{so}(3)$-valued 1-form e is called a *solder 1-form* if $e = e^i X_i$ satisfies $e^{123} \neq 0$ at each $u \in P$ for the basis $\{X_1, X_2, X_3\} \subset \mathfrak{so}(3)$. By this basis, we write $a = a^i X_i \in \mathcal{A}$, and ofen follow Einstein's convention.

Lemma 4.8. *Let ψ be an* SO(3)*-invariant definite 3-form on $M \times \mathrm{SO}(3)$ vanishing along each fiber* SO(3)*. Then there exists a unique triple of $f \in C^\infty_+(M)$, $a \in \mathcal{A}$, and a solder 1-form e such that*

$$\psi = -fe^{123} + e^1 a^{23} + e^2 a^{31} + e^3 a^{12}.$$

Proof. We can easily see that there exists an SO(3)-invariant positive $(1,1)$ form ω on $M \times \mathrm{SO}(3)$ vanishing along each fiber. In fact, let V_1, V_2, V_3 be SO(3)-invariant linear-independent vector fields along the fibers, and let J_ψ be the almost complex structure induced by ψ. Put $W_i = J_\psi V_i$ for $i = 1, 2, 3$. Let $\{V^i, W^i \mid i = 1, 2, 3\}$ be the dual 1-forms of $\{V_i, W_i \mid i = 1, 2, 3\}$. Then we can see that $\sum_{i=1}^3 V^i \wedge W^i$ is the desired 2-form. Hence, by Proposition 2.5, there exists a triple (f, a, e) satisfying the desired conditions. We can easily see the uniqueness. □

Lemma 4.9. *Let ψ be a 3-form as in Lemma 4.8, and ω an $\mathrm{SO}(3)$-invariant 2-form on $M \times \mathrm{SO}(3)$ vanishing along each fiber. Then there exists a unique $\mathrm{SO}(3)$-equivariant $\mathrm{M}(3;\mathbb{R})$-valued function K_{ij} on $M \times \mathrm{SO}(3)$ such that*

$$\omega = \sum_{i,j=1,2,3} K_{ij} a^i \wedge e^j.$$

Proof. Using the basis $\{X_1, X_2, X_3\}$ of $\mathfrak{so}(3)$, let us define K_{ij} by $\iota(X_i^*)\omega = \sum_{j=1}^{3} K_{ij} e^j$ for $i = 1, 2, 3$, where A^* denotes the infinitesimal vector field of $A \in \mathfrak{so}(3)$. Then K_{ij} satisfies the conditions in Lemma 4.9. □

Let ψ_t and ω_t be one-parameter families of 3- and 2-forms satisfying the conditions in Lemma 4.8 and 4.9, and parameterized by an interval $[t_1, t_2]$. Then we have the family $(f_t, a_t, e_t, (K_{ij})_t)$ corresponding to (ω_t, ψ_t) by Lemma 4.8 and 4.9.

Lemma 4.10. *The 3-form $\phi = \omega_t \wedge dt + \psi_t$ on $M \times \mathrm{SO}(3) \times [t_1, t_2]$ is closed, definite and $\mathrm{vol}(\phi) > 0$ if and only if $(f_t, a_t, e_t, (K_{ij})_t)$ satisfies the following conditions:*

1. *$(1/2)\{(K_{ij})_t + (K_{ji})_t\}$ is positive-definite for each t;*
2. *a_t is the Levi-Civita connection for each e_t;*
3. *$\partial e_t^i / \partial t = \sum_{j=1}^{3} (K_{ij})_t e_t^j$ for $i = 1, 2, 3$ and for each t; and*
4. *$\partial f_t / \partial t = -f_t \mathrm{tr}(K_t) - \mathrm{tr}((K_t G_t))$ for each t, where G_t is the Einstein tensor for each coframe $e_t = (e_t^1, e_t^2, e_t^3)$.*

Proof. We can prove this lemma by direct computation as the proof of ([Chi19], Proposition 6.8). First, we can easily see that ϕ is definite and $\mathrm{vol}(\phi) > 0$ if and only if the first condition holds. Thus, all we have to do is to compute $d\phi$. We have

$$\begin{aligned} d\phi &= d(\omega_t \wedge dt + \psi_t) \\ &= (d\omega_t - \partial\psi_t/\partial t) \wedge dt + d\psi_t, \end{aligned}$$

thereby $d\phi = 0$ is equivalent to $d\psi_t = 0$ and $\partial\psi_t/\partial t = d\omega_t$. By Proposition 2.13, we see that $d\psi_t = 0$ if and only if a_t is Levi-Civita for e_t. From now on, suppose that $d\psi_t = 0$. Put $\partial e_t^i/\partial t = P_{ij} e_t^j$ and $\partial a_t^i/\partial t = Q_{ij} e_t^j$ for $i = 1, 2, 3$. We have

$$\frac{\partial \psi_t}{\partial t} = -\{\frac{\partial f}{\partial t} + f\mathrm{tr}(P)\}e^{123} + \{\delta_{ij}\mathrm{tr}(Q) - Q_{ji}\}a^i \hat{e}^j + P_{ij}\hat{a}^i e^j,$$
$$d\omega_t = \mathrm{tr}((KG))e^{123} + \epsilon_{j\alpha\beta}K_{i\alpha;\beta}a^i\hat{e}^j + K_{ij}\hat{a}^i e^j,$$

where we omit the subscript t of the right-hand sides, denote by $K_{i\alpha;\beta}e^\beta$ the covariant derivative of $(K_{ij})_t$ for the connection a_t, and e.g. $\hat{e}^i = (1/2)\epsilon_{ijk}e^{ij}$. Also, δ_{ij} is Kronecker's delta. As seen in Lemma 3.7, the Levi-Civita condition $de_t + [a_t \wedge e_t] = 0$ implies $\delta_{ij}\mathrm{tr}(Q) - Q_{ji} = \epsilon_{j\alpha\beta}K_{i\alpha;\beta}a^i$ for $i, j = 1, 2, 3$. Hence, by comparing the equations above, we obtain the conditions in Lemma 4.10. □

4.3.3 Results

Let $\mathbf{M}$ be the set of all Riemannian metrics on M. Using Lemma 4.8, we can define the projection $\pi : \mathcal{C}^{\mathrm{inv}}(M) \to C_+^\infty(M) \times \mathbf{M}$ by $\pi(\psi) = (f, \gamma)$ for each $\psi \in \mathcal{C}^{\mathrm{inv}}(M)$, where we take f and e as in Lemma 4.8, and γ is the Riemannian metric on M naturally induced by the solder 1-form e. Let us define a binary relation on $C_+^\infty(M) \times \mathbf{M}$. Let $(f_1, \gamma_1), (f_2, \gamma_2) \in C_+^\infty(M) \times \mathbf{M}$. Then $(f_1, \gamma_1) \prec (f_2, \gamma_2)$ if there exists a one-parameter family $(f_t, \gamma_t) \in C_+^\infty(M) \times \mathbf{M}$ parameterized by $t \in [t_1, t_2]$ satisfying the following conditions:

1. $(f_{t_1}, \gamma_{t_1}) = (f_1, \gamma_1)$ and $(f_{t_2}, \gamma_{t_2}) = (f_2, \gamma_2)$;
2. $\partial\gamma/\partial t$ is positive-definite covariant symmetric tensor for each t; and
3. $\partial f/\partial t = -A(f, \gamma)^{ij}\dot{g}_{ij}$ for each t. Here, $A(f, \gamma)^{ij} = 1/2(G_t^{ij} + f\gamma_t^{ij})$, and $(G_{ij})_t$ is the Einstein tensor for each metric γ_t.

Let us state and prove our main results. Let $\mathcal{G}$ be the gauge group of the trivial princpal bundle $M \times \mathrm{SO}(3)$.

Theorem 4.11. *Let $\psi_1, \psi_2 \in \mathcal{C}^{\mathrm{inv}}(M)$. Then $\psi_1 \prec^{\mathrm{inv}} \psi_2$ implies $\pi(\psi_1) \prec \pi(\psi_2)$.*

Proof. Suppose that $\psi_1 \prec^{\mathrm{inv}} \psi_2$. By Lemma 4.8–4.10, we can decompose a closed definite 3-form ϕ that gives the relation $\psi_1 \prec^{\mathrm{inv}} \psi_2$ into a one-parameter family (f_t, a_t, e_t, K_t) parameterized by $[t_1, t_2]$ and satisfying the conditions in Lemma 4.10. Then we can see that the projection $\pi(-f_t e_t^{123} + e_t^1 a_t^{23} + e_t^2 a_t^{31} + e_t^3 a_t^{12})$ gives $\pi(\psi_1) \prec \pi(\psi_2)$. □

Theorem 4.12. *The relation $\prec^{\mathrm{inv}}$ on $\mathcal{C}^{\mathrm{inv}}(M)$ is irreflexive, that is, $\psi \not\prec^{\mathrm{inv}} \psi$ for any $\psi \in \mathcal{C}^{\mathrm{inv}}(M)$.*

Proof. From Lemma 4.10, if $\psi_1 \prec^{\mathrm{inv}} \psi_2$, then we have $\partial(e_t^1 \wedge e_t^2 \wedge e_t^3)/\partial t = \mathrm{tr}(K_t) e_t^1 \wedge e_t^2 \wedge e_t^3$ and $\mathrm{tr}(K_t) > 0$. Hence $\psi_1 \neq \psi_2$. □

If an oriented 3-manifold M is closed, then $M \times \mathrm{SO}(3)$ admits no symplectic structure. Thus Theorem 4.12 supports Problem 4.1 stated in Introduction.

Proposition 4.13. *Let $\psi_1, \psi_2 \in \mathcal{C}^{\mathrm{inv}}(M)$. If $\pi(\psi_1) \prec \pi(\psi_2)$, then there exists $\tau \in \mathcal{G}$ such that $\psi_1 \ll^{\mathrm{inv}} \tau^*\psi_2$.*

Proof. Suppose that $\pi(\psi_1) \prec \pi(\psi_2)$. Then we have a one-parameter family (f_t, γ_t) parameterized by $[t_1, t_2]$ that gives the relation $\pi(\psi_1) \prec \pi(\psi_2)$. We can lift this curve (f_t, γ_t) by the natural connection in the principal $\mathcal{G}$-bundle $\mathcal{M} \to \mathbf{M}$ used in Section 3.1, where $\mathcal{M}$ denotes a connected component of the space of solder 1-forms on $M \times \mathrm{SO}(3)$. Note that the set of solder 1-forms has two connected component isomorphic to each other. Here, we choose the one containing the solder 1-form given by ψ_1 as in Lemma 4.8. The connection in $\mathcal{M} \to \mathbf{M}$ is defined by the decomposition $\mathrm{M}(3;\mathbb{R}) = \mathrm{Sym}(3;\mathbb{R}) \oplus \mathrm{Ant}(3;\mathbb{R})$ at each $T_e\mathcal{M}$, where $\mathrm{Sym}(3;\mathbb{R})$ and $\mathrm{Ant}(3;\mathbb{R})$ are the symmetric and anti-symmetric matrices. Let (f_t, e_t) be a horizontally lifted curve in $C^\infty_+ \times \mathcal{M}$, and define $(T_{ij})_t$ by $\partial e_t^i/\partial t = (T_{ij})_t e_t^j$ for each t. Here, by the definition of the connection in $\mathcal{M}$, $(T_{ij})_t$ is an $\mathrm{SO}(3)$-equivariant $\mathrm{Sym}(3;\mathbb{R})$-valued funcion on $M \times \mathrm{SO}(3)$ for each t. Then a pair of $\psi_t = -f_t e_t^{123} + e_t^1 a_t^{23} + e_t^2 a_t^{31} + e_t^3 a_t^{12}$ and $\omega_t = (T_{ij})_t a_t^i \wedge e_t^j$ gives $\psi_1 \ll^{\mathrm{inv}} \psi_{t_2}$, where a_t is the Levi-Civita connection for each e_t. Since $\pi(\psi_{t_2}) = \psi_2$, there exists a unique $\tau \in \mathcal{G}$ such that $\psi_{t_2} = \tau^*\psi_2$. □

By Theorem 4.11 and Proposition 4.13 combined, we have

Theorem 4.14. *Let $\psi_1, \psi_2 \in \mathcal{C}^{\mathrm{inv}}(M)$. If $\psi_1 \prec^{\mathrm{inv}} \psi_2$, then there exists $\tau \in \mathcal{G}$ such that $\psi_1 \ll^{\mathrm{inv}} \tau^*\psi_2$.*

Theorem 4.14 is related to Problem 4.2. Moreover, considering that how close we can take $\tau^*\psi_2$ to ψ_2 in Theorem 4.14 seems to be related to the isoholonomic problem (see e.g. [Mon90, Mon02]), which is a typical problem of sub-Riemannian geometry, for the infinite-dimensional principle $\mathcal{G}$-bundle $\mathcal{M} \to \mathbf{M}$ over the space $\mathbf{M}$ of all Riemannian metrics on the 3-manifold M.

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