博士論文

論文題目　Boundedness of Weak Fano Pairs with Alpha-invariants and Volumes Bounded Below

（体積とアルファ不変量が下に有界な弱 Fano 対の有界性）

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BOUNDEDNESS OF WEAK FANO PAIRS WITH ALPHA-INVARIANTS AND VOLUMES BOUNDED BELOW

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ABSTRACT. We show that fixed dimensional klt weak Fano pairs with alpha-invariants and volumes bounded away from 0 and the coefficients of the boundaries belonging to a fixed DCC set $\mathscr{S}$ form a bounded family. Moreover, such pairs admit a strong ϵ-lc $\mathbb{R}$-complement for some fixed $\epsilon > 0$. We also show $\alpha(X, B)^{d-1}\mathrm{vol}(-(K_X + B))$ are bounded from above for all klt weak Fano pairs (X, B) of a fixed dimension d.

1. Introduction

Throughout this paper, we work over an uncountable algebraically closed field of characteristic 0, for instance, the complex number field $\mathbb{C}$.

In the birational geometry, as the first step of moduli theory, it is interesting to consider whether a certain kind of family of varieties satisfy certain finiteness. For varieties of Fano type with bounded log discrepancies, Birkar shows in [3, Theorem 1.1] that

Theorem 1.1. *Fix a positive integer d and a positive real number ϵ. The projective varieties X satisfying*

(1) $\dim X = d$,
(2) there exists a boundary B such that (X, B) is ϵ-lc, and
(3) $-(K_X + B)$ *is nef and big,*

form a bounded family.

Theorem 1.1 was known as the Borisov-Alexeev-Borisov (BAB) Conjecture for decades before Birkar proved it. Equivalently, we can state Theorem 1.1 in the following form of boundedness of varieties of Calabi–Yau type.

Theorem 1.2. *Fix a positive integer d and a positive real number ϵ. The projective varieties X satisfying*

(1) $\dim X = d$,
(2) there exists a boundary B such that (X, B) is ϵ-lc, and
(3) $K_X + B \sim_{\mathbb{R}} 0$ *and B is big,*

form a bounded family.

In Theorem 1.1, it is necessary to take $\epsilon > 0$. In fact, klt Fano threefolds do not even form a birational family (see [18]). Nevertheless, Jiang shows in [12] that if we bound the alpha-invariants and the volumes from below, we have

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Theorem 1.3 ([12, Theorem1.6]). *Fix a positive integer d and a positive real number θ. The normal projective klt Fano (i.e. $\mathbb{Q}$-Fano in* [12]*) varieties X satisfying*

(1) $\dim X = d$, *and*
(2) $\alpha(X)^d(-K_X)^d > \theta$,

form a bounded family.

Inspired by Theorem 1.3, it is natural to ask if certain boundedness holds for varieties of Fano type or under more general setting. Thanks to boundedness of complements by Birkar (Theorem 3.10), if the coefficients of the boundaries are well controlled, then the boundedness, which is one of our main theorems, holds as follows.

Theorem 1.4. *Fix a positive integer d, positive real numbers θ and δ and a finite set $\mathscr{R}$ of rational numbers in* $[0, 1]$. *The set of all klt Fano pairs (X, B) satisfying*

(1) $\dim X = d$,
(2) the coefficients of $B \in \Phi(\mathscr{R})$,
(3) $\alpha(X, B) > \theta$, *and*
(4) $(-(K_X + B))^d > \theta$,

is log bounded. Moreover, there exists a natural number k, depending only on d, θ, δ and $\mathscr{S}$ such that there exists a strong klt k-complement $K_X + \Theta$ of $K_X + B$.

Combining Theorem 3.11 with boundedness of complements for DCC coefficients of boundaries by Han, Liu and Shokurov, we can weaken the assumption of the coefficients of boundaries as in the following theorem.

Theorem 1.5. *Fix a positive integer d, positive real numbers θ and δ and a DCC set $\mathscr{S}$ of real numbers in* $[0, 1]$. *The set of all klt Fano pairs (X, B) satisfying*

(1) $\dim X = d$,
(2) the coefficients of $B \in \mathscr{S}$,
(3) $\alpha(X, B) > \theta$, *and*
(4) $(-(K_X + B))^d > \theta$,

is log bounded. Moreover, there exists a natural number k, finite sets $\Gamma_1 = \{a_i\}_i \in [0, 1]$ with $\sum_i a_i = 1$ and $\Gamma_2 \in [0, 1] \cap \mathbb{Q}$ depending only on d, θ, δ and $\mathscr{S}$ such that there exists a strong klt (k, Γ_1, Γ_2)-complement $K_X + \Theta$ of $K_X + B$.

Letting $B = 0$ in Theorem 3.11, we have the following corollary, which answers the question asked by Jiang in [12, Remark 1.7].

Corollary 1.6. *Fix a positive integer d and two positive real numbers δ and θ. Then the set of klt Fano varieties X satisfying*

(1) $\dim X = d$, *and*
(2) $\alpha(X)^{d-1+\delta}(-K_X)^d > \theta$,

forms a bounded family.

Now we consider $\alpha(X, B)^d(-(K_X+B))^d$ as an invariant for d-dimensional klt Fano pairs (X, B). It is well known that this invariant has an upper bound, which can be given by the following lemma.

Lemma 1.7 ([15, Theorem 6.7.1])**.** *Let (X, B) be a klt pair of dimension d. Then we have*

$$\mathrm{lct}((X, B), |H|_{\mathbb{R}})^d H^d \leq d^d$$

for any nef and big $\mathbb{Q}$-Cartier divisor H on X.

We will show that for d-dimensional klt Fano pairs (X, B),

$$\alpha(X, B)^{d-1}(-(K_X + B))^d$$

is also bounded from above. In fact, we have the following theorem under a more general setting.

Theorem 1.8. *Fix a positive integer d. There exists a real number M, depending only on d, such that for any projective pair (X, B) and for any big and nef $\mathbb{Q}$-Cartier divisor H on X satisfying*

(1) $\dim X = d$, *and*
(2) (X, B) *is klt,*

we have

$$\mathrm{lct}((X, B), |H|_{\mathbb{R}})^{d-1}\overline{\tau}((X, B), H)H^d \leq M,$$

where $\overline{\tau}$ denotes the anti-pseudo-effective threshold (see Definition 2.13).

Remark 1.9. In contrast with the above, for d-dimensional klt Fano pairs (X, B), $\alpha(X, B)^{d'}(-(K_X+B))^d$ are not bounded from above if $d' < d-1$. To see this, consider the weighted projective spaces $X_n = \mathbb{P}(1^d, n)$, which are klt Fano varieties of dimension d with $(-K_{X_n})^d = \frac{(n+d)^d}{n}$ and $\alpha(K_{X_n}) = \frac{1}{n+d}$. Then we have, $\alpha(X)^{d-1-\delta}(-K_{X_n})^d = \frac{(n+d)^{(1+\delta)}}{n}$, which are not bounded from above for any positive real number δ.

Remark 1.10. We remark that for any klt Fano variety X of dimension d, a lower bound of $\alpha(X)$ provides an upper bound of $(-K_X)^d$ by Lemma 1.7. However, the set of klt Fano varieties with $(-K_X)^d$ both side bounded does not form a bounded family. As an example, consider the family of weighted projective spaces $\{X_{p,q,r} = \mathbb{P}(p, q, r)\}$ with (p, q, r) pairwisely coprime. Then $\{(-K_{X_{p,q,r}})^2\} = \{\frac{(p+q+r)^2}{pqr}\}$ is a dense subset of $\mathbb{R}_{>0}$. Therefore, for any two positive integers $a < b$, $\{X_{p,q,r} | \frac{(p+q+r)^2}{pqr} \in (a, b)\}$ is a family of klt Fano varieties which is not bounded.

2. Preliminaries

We adopt the standard notation and definitions in [13] and [16], and will freely use them.

2.1. **Pairs and singularities.** A *sub-pair* (X, B) consists of a normal projective variety X and an $\mathbb{R}$-divisor B on X such that $K_X + B$ is $\mathbb{R}$-Cartier. B is called the *sub-boundary* of this pair.

A *log pair* (X, B) is a sub-pair with $B \geq 0$. We call B a *boundary* in this case.

Let $f : Y \to X$ be a log resolution of the log pair (X, B), write

$$K_Y = f^*(K_X + B) + \sum a_i F_i,$$

where F_i are distinct prime divisors. For a non-negative real number ϵ, the log pair (X, B) is called

(a) *ϵ-kawamata log terminal* (*ϵ-klt*, for short) if $a_i > -1 + \epsilon$ for all i;
(b) *ϵ-log canonical* (*ϵ-lc*, for short) if $a_i \geq -1 + \epsilon$ for all i;

Usually we write X instead of $(X, 0)$ in the case when $B = 0$. Note that 0-klt (resp. 0-lc) is just klt (resp. lc) in the usual sense. Also note that ϵ-lc singularities only make sense if $\epsilon \in [0, 1]$, and ϵ-klt singularities only make sense if $\epsilon \in [0, 1)$.

Similarly, sub-ϵ-klt and sub-ϵ-lc sub-pairs can be defined.

The *log discrepancy* of the divisor F_i is defined to be $a(F_i, X, B) = 1 + a_i$. It does not depend on the choice of the log resolution f.

F_i is called a *non-klt place* of (X, B) if $a_i \leq -1$. A subvariety $V \subset X$ is called a *non-klt center* of (X, B) if it is the image of a non-klt place. The *non-klt locus* $\mathrm{Nklt}(X, B)$ is the union of all non-klt centers of (X, B). We recall the Kollár-Shokurov connectedness lemma.

Lemma 2.1 (cf. [19], [20] and [14, Theorem 17.4])**.** *Let (X, B) be a log pair, and let $\pi\colon X \to S$ be a proper morphism with connected fibers. Suppose $-(K_X + B)$ is π-nef and π-big. Then $\mathrm{Nklt}(X, B) \cap X_s$ is connected for any fiber X_s of π.*

2.2. **Fano pairs and Calabi–Yau pairs.** A projective pair (X, B) is a *Fano* (resp. *weak Fano*, resp. *Calabi–Yau*) pair if it is lc and $-(K_X + B)$ is ample (resp. $-(K_X + B)$ is nef and big, resp. $K_X + B \equiv 0$). A projective variety X is called Fano, (resp. Calabi–Yau) if $(X, 0)$ is Fano (resp. Calabi–Yau). It is called $\mathbb{Q}$-*Fano* if it is klt and Fano. It is called *of Fano type* if (X, B) is klt weak Fano for some boundary B.

2.3. **Bounded pairs.** A collection of varieties $\mathcal{D}$ is said to be *bounded* (resp. *birationally bounded*) if there exists $h\colon \mathcal{Z} \to S$ a projective morphism of schemes of finite type such that each $X \in \mathcal{D}$ is isomorphic (resp. birational) to $\mathcal{Z}_s$ for some closed point $s \in S$.

A couple (X, D) consists of a normal projective variety X and a reduced divisor D on X. Note that we do not require $K_X + D$ to be $\mathbb{Q}$-Cartier here.

We say that a collection of couples $\mathcal{D}$ is *log birationally bounded* (resp. *log bounded*) if there is a quasi-projective scheme $\mathcal{Z}$, a reduced divisor $\mathcal{E}$ on $\mathcal{Z}$, and a projective morphism $h\colon \mathcal{Z} \to S$, where S is of finite type and $\mathcal{E}$ does not contain any fiber, such that for every $(X, D) \in \mathcal{D}$, there is a closed point $s \in S$ and a birational (resp. isomorphic) map $f\colon \mathcal{Z}_s \dashrightarrow X$ such that $\mathcal{E}_s$ contains the support of $f_*^{-1}D$ and any f-exceptional divisor.

A set of log pairs $\mathcal{P}$ is *log birationally bounded* (resp. *log bounded*) if the set of the corresponding couples $\{(X, \mathrm{Supp}B) | (X, B) \in \mathcal{P}\}$ is log birationally bounded (resp. log bounded).

2.4. **Volumes.** Let X be a d-dimensional normal projective variety and D a Cartier divisor on X. The *volume* of D is the real number

$$\mathrm{vol}(X, D) = \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}.$$

For more backgrounds on the volume, see [17, 2.2.C]. By the homogenous property and continuity of the volume, we can extend the definition to $\mathbb{R}$-Cartier $\mathbb{R}$-divisors. Moreover, if D is a nef $\mathbb{R}$-divisor, then $\operatorname{vol}(X,D) = D^d$.

2.5. **Complements.**

Definition 2.2. Let (X,B) be a pair and n a positive integer. We write $B = \lfloor B \rfloor + \{B\}$. An *n-complement* of $K_X + B$ is a divisor of the form $K_X + B^+$ such that

(1) (X, B^+) is lc,
(2) $n(K_X + B^+) \sim 0$, and
(3) $nB^+ \geq n\lfloor B \rfloor + \lfloor (n+1)\{B\} \rfloor$.

If moreover, $B^+ \geq B$ (resp. (X,B^+) is klt, resp. (X,B^+) is ϵ-lc, where $\epsilon > 0$), we say that the complement is *strong* (resp. klt, resp. ϵ-lc).

We say that (X,B) has an $\mathbb{R}$-*complement* or is *complementary* if there exists $\overline{B} \geq B$ such that $(X,\overline{B})$ is lc. In this case, we call $K_X + \overline{B}$ an $\mathbb{R}$-complement of $K_X + B$.

Definition 2.3. For a subset $\mathscr{R}$ of $[0,1]$, we define the set of *hyperstandard multiplicities* associated to $\mathscr{R}$ to be

$$\Phi(\mathscr{R}) = \{1 - \frac{r}{m} | r \in \mathscr{R},\ m \in \mathbb{N}\}.$$

Note that the only possible accumulating point of $\Phi(\mathscr{R})$ is 1 if $\mathscr{R}$ is finite. Birkar shows the following boundedness of complements.

Theorem 2.4 ([2, Theorem 1.7])**.** *Fix a positive integer d and a finite set $\mathscr{R}$ of rational numbers in $[0,1]$. Then there exists a positive integer n depending only on d and $\mathscr{R}$, such that if (X,B) is a projective pair with*

(1) (X,B) is lc dimension d,
(2) the coefficients of $B \in \Phi(\mathscr{R})$,
(3) X is of Fano type, and
(4) $-(K_X+B)$ is nef,

then there is an n-complement $K_X + B^+$ of $K_X + B$ such that $B^+ \geq B$.

Based on Theorem 3.10 and ACC for log canonical threshold [[9, Theorem 1.1]], Filipazzi and Moraga showed the following existence of bounded complements for DCC coefficients of the boundaries.

Theorem 2.5 ([6, Theorem 1.2])**.** *Fix a positive integer d and a closed DCC set $\mathscr{S}$ of rational numbers in $[0,1]$. Then there exists a positive integer n depending only on d and $\mathscr{S}$, such that if (X,B) is a projective pair with*

(1) (X,B) is lc dimension d,
(2) the coefficients of $B \in \mathscr{S}$,
(3) X is of Fano type, and
(4) $-(K_X+B)$ is nef,

then there is an n-complement $K_X + B^+$ of $K_X + B$ such that $B^+ \geq B$.

Theorem 2.6 ([7, Theorem 5.22])**.** *Let d be a natural number and $\Gamma \subset [0,1]$ be a closed DCC set. Then there exists a finite subset Γ_1 of Γ and a projection $g : \Gamma \to \Gamma_1$ such that $g(g(x)) = g(x)$, $g(x) \geq x$ and $g(y) \geq g(x)$ for every $x \leq y \in \Gamma$, depending only on d and Γ satisfying the following. Suppose*

$X, B := \sum b_iB_i$ is lc of dimension d where $b_i \in \Gamma$, $B_i \geq 0$ is $\mathbb{Q}$-Cartier Weil divisor for any i, (X, B) has an $\mathbb{R}$-complement and X is of Fano type, then

(1) $(X, \sum g(b_i)B_i)$ is lc, and
(2) $-(K_X + \sum g(b_i)B_i)$ is pseudo-effective.

We recall the construction of (n, Γ_1, Γ_2)-complement by Han, Liu and Shokurov.

Theorem 2.7 ([7, Theorem 1.13])**.** *Let d be a natural number $\delta > 0$ be a positive real number and $\Gamma \subset [0, 1]$ be a closed DCC set. Then there exist integers $n > 0$ and $r > 0$, finite sets $\Gamma_1 \subset [0, 1]$, $\{a_i\}_{i=1}^r \in (0, 1]$, $\Gamma_2 = \cup_{i=1}^r \Gamma_i' \subset \mathbb{Q} \cap [0, 1]$, a projection $g : \Gamma \to \Gamma_1$ and bijections $g_i : \Gamma_1 \to \Gamma_i'$ for every $1 \leq i \leq r$ which only depend only on d and Γ satisfying the following.*

Γ_1 and g are given by Theorem 2.6. $g(x) = \sum_{i=1}^r a_i(g_i \circ g(x))$ and $\sum_{i=1}^r a_i = 1$. $|g_i(x) - x| < \delta$ for every $1 \leq i \leq r$ and $x \in \Gamma_1$. Assume X is a normal variety and B is an effective divisor on X with the following conditions:

(1) X is of Fano type,
(2) $\dim X \leq d$,
(3) the coefficients of $B \in \Gamma$,
(4) (X, B) has an $\mathbb{R}$-complement, and
(5) we can write $B := \sum b_jB_j$ where $b_j \in \Gamma$ and $B_j \geq 0$ is a $\mathbb{Q}$-Cartier Weil divisor for any j,

then $(X, \sum g(b_j)B_j)$ and $(X, \sum_j g_i \circ g(b_j)B_j)$ are lc for every and $K_X + \sum_j g_i \circ g(b_j)B_j$ has a strong n-complement for every $1 \leq i \leq r$. In particular, $(X, \sum g(b_j)B_j)$ has an $\mathbb{R}$-complement.

Definition 2.8 ([7, Definition 1.12])**.** Assume (X, B) is a normal pair, n is a positive integer, and $\Gamma_1 \subset [0, 1]$ and $\Gamma_2 \subset \mathbb{Q} \cap [0, 1]$ are two finite sets. We say that $K_X + B^+$ is an (n, Γ_1, Γ_2)-complement of (X, B) if the following conditions are satisfied.

(1) $B^+ \geq B$,
(2) there is a finite set $\{a_i\}_{i=1}^r \subset \Gamma_1$ with $\sum a_i = 1$,
(3) there are divisors $B_i \geq 0$ on X with coefficients of $B_i \in \Gamma_2$ for every $1 \leq i \leq r$,
(4) $\sum a_iB_i = B^+$, and
(5) $K_X + B_i$ is an n-complement of itself for each $1 \leq i \leq r$.

Definition 2.9. Under the notation of Definition 2.8, if (X, B_i) are klt, then we say that the (n, Γ_1, Γ_2)-complement $K_X + B^+$ is klt. Note that this implies that (X, B_i) and (X, B^+) are $\frac{1}{n}$-lc.

Theorem 2.10. *Under the notation of Theorem 2.7, for each $1 \leq i \leq r$ there exist a closed DCC set of rational numbers $\mathscr{S}_i$ and a function $g_i' : \Gamma \to \mathscr{S}_i$ such that $B \leq \sum_i(a_i \sum_j g_i'(b_j)B_j)$ and $\sum_j g_i'(b_j)B_j \leq \frac{B+3\sum_j g(b_j)B_j}{4}$ for every $1 \leq i \leq r$.*

Proof. By shrinking Γ_1, we may assume that it is the image of g and so g is identity on Γ_1. We may also assume that 0 is not in Γ_1. We define a finite rational set Γ^- and a function $g^- : \Gamma \to \Gamma^-$ as following. For $\gamma \in \Gamma_1$, let γ^- be any rational number in the inteval

$$(\max\{0, x|x \in \Gamma, g(x) < \gamma\}, \min\{x|x \in \Gamma, g(x) \geq \gamma\}).$$

Note that here the maximum and minimum exist and $\max\{0, x|x \in \Gamma, g(x) < \gamma\} < \min\{x|x \in \Gamma, g(x) \geq \gamma\}$. Let $g^-(x) = g(x)^-$ for every $x \in \Gamma$. It follows by construction that $g^-(x) \leq x < g(x)$ for every $x \in \Gamma$. By abuse of notations, we denote the functions $g_i \circ g$ as g_i. Since Γ_1 and Γ^- are finite, by possibly replacing δ, we may assume that $g(x) - g_i(x) \leq \frac{g(x)-g^-(x)}{2}$ for every $x \in \Gamma$. Therefore we have $2(g_i(x) - g^-(x)) \geq (g(x) - g^-(x))$ for every $x \in \Gamma$. Let $f : \mathbb{Q} \to \mathbb{N}$ be any bijection. We now define the functions g'_i as following. Assume that $x \in \Gamma$, let $x' = g^-(x) + \frac{(g_i(x)-g^-(x))(x-g^-(x))}{(g(x)-g^-(x))}$. Then

$$\begin{aligned} g_i(x) - x' =& g_i(x) - g^-(x) - \frac{(g_i(x) - g^-(x))(x - g^-(x))}{(g(x) - g^-(x))} \\ =& (g_i(x) - g^-(x))\frac{(g(x) - g^-(x)) - (x - g^-(x))}{(g(x) - g^-(x))} \\ =& (g_i(x) - g^-(x))\frac{(g(x) - x)}{(g(x) - g^-(x))} \\ \geq& 0, \end{aligned}$$

where the equality holds if and only if $g(x) = x$. Let $g'_i(x)$ be the rational number in $[x', \frac{x'+g_i(x)}{2}]$ with the smallest f value. Note that here, if $x' = \frac{x'+g_i(x)}{2}$, then it is a rational number in Γ_i. Let $\mathscr{S}_i$ be the closure or the image of g'_i.

We now argue that $\mathscr{S}_i$ is a rational DCC set. It is enough to show that the image of g'_i is a DCC set with rational accumulation points.

Fix i for each $1 \leq i \leq r$. Let $\{x_l\}_l$ be a sequence of elements in Γ such that $\{g'_i(x_l)\}_l$ is and strictly converges to some point $x_0 \in \mathscr{S}_i$. Suppose either $\{g'_i(x_l)\}_l$ is strictly decreasing or x_0 is irrational. By construction, $g'_i(x_l)$ is the element with the smallest f value in $\mathbb{Q} \cap [x'_l, \frac{x'_l+g_i(x_l)}{2}]$, where $x'_l = g^-(x_l) + \frac{(g_i(x_l)-g^-(x_l))(x_l-g^-(x_l))}{(g^+(x_l)-g^-(x_l))}$. Because the image of g^-, g_i and g^+ are finite, $\frac{(g_i(x)-g^-(x))}{(g^+(x)-g^-(x))} > 0$ for every $x \in \Gamma$, and Γ is DCC, it follows that $\{x'_l\}_l$ and $\{\frac{x'_l+g_i(x_l)}{2}\}_l$ are both DCC. Passing through a subsequence, we may assume that $\{x'_l\}_l$ and $\{\frac{x'_l+g_i(x_l)}{2}\}_l$ are non-decreasing. Passing to a subsequence, we may assume the sequence $\{g_i(x_l)\}_l$ is a constant sequence. Further, if $\{x'_l\}_l$ is a constant sequence, $\{\frac{x'_l+g_i(x_l)}{2}\}_l$ is also a constant sequence, and this implies $\{g'_i(x_l)\}_l$ is also constant, which is a contradiction. Now we may assume that $\{x'_l\}_l$ and $\{\frac{x'_l+g_i(x_l)}{2}\}_l$ are strictly increasing and $\{x'_l\}_l$ converges to $x'_0 \leq g_i(x_l) \in \Gamma'_i$. If $x'_0 = g_i(x_l)$, then neither $\{g'_i(x_l)\}_l$ is strictly decreasing nor $x_0 = g_i(x_l)$ is irrational, which is a contradiction. So we have $x'_0 < g_i(x_l)$. Let y be a rational number in $[x'_0, \frac{x'_0+g_i(x_l)}{2})$ Passing through a subsequence again, we may assume that $\frac{x'+g_i(x)}{2} > y$ for every l. This implies that $x'_l < x'_0 < y < \frac{x'_l+g_i(x_l)}{2}$ for every l. Then $f(g'_i(x_l)) \leq f(y)$

for every l because $g_i'(x_l)$ is the rational number in $[x_l', \frac{x'+g_i(x)}{2}]$ with the smallest f value. But the set $\{f(g_i'(x_l))\}$ is infinite because $\{g_i'(x_l)\}_l$ is strictly converging. This contradicts to our assumption that f is a bijection.

It then remains to show that $B \leq \sum_i (a_i \sum_j g_i'(b_j)B_j)$ and $\sum_j g_i'(b_j)B_j \leq \frac{B+3\sum_j g(b_j)B_j}{4}$ for every $1 \leq i \leq r$. We simply compute that

$$\begin{aligned}\sum_i (a_i \sum_j g_i'(b_j)B_j) &\geq \sum_i (a_i \sum_j (g^-(b_j) + \frac{(g_i(b_j) - g^-(b_j))(b_j - g^-(b_j))}{(g(b_j) - g^-(b_j))})B_j) \\ &= \sum_j (g^-(b_j) + \frac{(\sum_i (a_i g_i(b_j)) - g^-(b_j))(b_j - g^-(b_j))}{(g(b_j) - g^-(b_j))})B_j \\ &= \sum_j (g^-(b_j) + \frac{(g(b_j) - g^-(b_j))(b_j - g^-(b_j))}{(g(b_j) - g^-(b_j))})B_j \\ &= \sum_j b_j B_j = B.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_j g(b_j)B_j - \sum_j g_i'(b_j)B_j) &= \sum_i (a_i \sum_j g_i(b_j)B_j) - \sum_i (a_i \sum_j g_i'(b_j)B_j) \\ &= \sum_j \sum_i a_i (g_i(b_j) - g_i'(b_j))B_j.\end{aligned}$$

We compute that for each i,

$$\begin{aligned}g_i(b_j) - g_i'(b_j) &\geq g_i(b_j) - \frac{g_i(b_j) + g^-(b_j) + \frac{(g_i(b_j)-g^-(b_j))(b_j-g^-(b_j))}{(g(b_j)-g^-(b_j))}}{2} \\ &= \frac{g_i(b_j) - g^-(b_j) - \frac{(g_i(b_j)-g^-(b_j))(b_j-g^-(b_j))}{(g(b_j)-g^-(b_j))}}{2} \\ &= \frac{1}{2}(g_i(b_j) - g^-(b_j))\frac{(g(b_j) - g^-(b_j)) - (b_j - g^-(b_j))}{(g(b_j) - g^-(b_j))} \\ &= \frac{1}{2}(g_i(b_j) - g^-(b_j))\frac{(g(b_j) - b_j)}{(g(b_j) - g^-(b_j))} \\ &\geq \frac{(g(b_j) - b_j)}{4}.\end{aligned}$$

It follows that $\sum_j g(b_j)B_j - \sum_j g_i'(b_j)B_j) \geq \frac{\sum_j g(b_j)B_j - B}{4}$. Therefore, we have

$$\begin{aligned}\sum_j g_i'(b_j)B_j &\leq \sum_j g(b_j)B_j - \frac{\sum_j g(b_j)B_j - B}{4} \\ &= \frac{B + 3\sum_j g(b_j)B_j}{4}.\end{aligned}$$

□

2.6. **α-invariants, log canonical thresholds and anti-pseudo-effective thresholds.**

Definition 2.11. Let (X,B) be a projective lc pair and let D be an effective $\mathbb{R}$-Cartier divisor, we define the *log canonical threshold* of D with respect of (X,B) to be

$$\mathrm{lct}((X,B),D)=\sup\{t\in\mathbb{R}\mid (X,B+tD)\text{ is lc}\}.$$

The log canonical threshold of $|D|_{\mathbb{R}}$ with respect of (X,B) is defined to be

$$\mathrm{lct}((X,B),|D|_{\mathbb{R}})=\inf\{\mathrm{lct}((X,B),M)|M\in|D|_{\mathbb{R}}\}.$$

Definition 2.12. Let (X,B) be a projective lc pair such that $|-(K_X+B)|_{\mathbb{R}}$ is non-empty, we define the *α-invariant* of (X,B) to be

$$\alpha(X,B)=\mathrm{lct}((X,B),|-(K_X+B)|_{\mathbb{R}}).$$

In the case when $B=0$, we usually write $\alpha(X):=\alpha(X,0)$ for convenience.

Definition 2.13. Let (X,B) be a projective pair, and H a big $\mathbb{R}$-Cartier divisor. The *anti-pseudo-effective threshold* of H respect to (X,B) is defined by

$$\begin{aligned}\overline{\tau}((X,B),H)=&\sup\{t\in\mathbb{R}|\ -K_X-B-tH\text{ is pseudo-effective}\}\\ =&\sup\{t\in\mathbb{R}|\ K_X+B+tH\text{ is anti-pseudo-effective}\}.\end{aligned}$$

2.7. **Potentially birational divisors.**

Definition 2.14 (cf. [9, Difinition 3.5.3])**.** Let X be a projective normal variety, and D a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then, we say that D is *potentially birational* if for any two general points x and y of X, there is an effective $\mathbb{Q}$-divisor $\Delta\sim_{\mathbb{Q}}(1-\epsilon)D$ for some $0<\epsilon<1$, such that, after possibly switching x and y, (X,Δ) is not klt at y, lc at x and x is a non-klt center.

Lemma 2.15 ([8, Lemma 2.3.4])**.** *Let X be a projective normal variaty, and D a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. If D is potentially birational, then $|K_X+\lceil D\rceil|$ defines a birational map.*

Lemma 2.16. *Let X be a normal projective variety and let D be a potentially birational divisor on X. Let $\overline{D}$ and D^+ be $\mathbb{Q}$-divisors on X such that $\overline{D}$ is integral, $\overline{D}\sim_{\mathbb{Q}}D^+$ and $D^+\geq D$. Then $|K_X+\overline{D}|$ defines a birational map.*

Proof. By Lemma 2.15, $|K_X+\lceil\overline{D}-D^++D\rceil|$ defines a birational map. Since $\lceil\overline{D}-D^++D\rceil\leq\lceil\overline{D}\rceil=\overline{D}$, we are done. □

2.8. **Exceptional pairs.**

Definition 2.17. A projective klt pair (X,B) is called *exceptional* if

$$\mathrm{lct}((X,B),|-(K_X+B)|_{\mathbb{R}})>1.$$

In particular, if (X,B) is weak Fano, then it is exceptional if and only if $\alpha(X,B)>1$.

Theorem 2.18 ([2, Theorem 1.11])**.** *Fix a positive integer d and a finite set of rational numbers $\mathscr{R}$ in $[0,1]$. Then the set of all projective pairs (X,B) satisfying*

(1) (X,B) *is lc of dimension* d*,*
(2) the coefficients of $B \in \Phi(\mathscr{R})$*,*
(3) X *is of Fano type,*
(4) $-(K_X+B)$ *is nef, and*
(5) (X,B) *is exceptional,*

forms a log bounded family.

2.9. **Descending chain condition.**

Definition 2.19. A set of real numbers $\mathscr{S}$ is said to *satisfy descending chain condition (DCC for short)* if for every non-empty subset S of $\mathscr{S}$, there is a minimum element in S. $\mathscr{S}$ is called a DCC set if it satisfies DCC.

We recall and improve slightly the following proposition of Birkar. It is shown for ample divisors D and A in [2, Proposition2.31(2)]. We modify it for nef and big divisors A and D.

Proposition 2.20. *Let* (X,B) *be a log pair of dimension* d*. Let* D *and* A *be big and nef* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisors on* X*. Assume that* $D^d > (2d)^d$*. Then there is a bounded family* $\mathcal{P}$ *of subvarieties of* X *such that for each pair* x*,* $y \in X$ *of general closed points, there is a member* G *of* $\mathcal{P}$ *and an effective divisor* $\Delta \sim_{\mathbb{Q}} D+(d-1)A$ *such that*

(1) $(X,B+\Delta)$ *is lc near* x *with a unique non-klt place whose centre is* G*,*
(2) $(X,B+\Delta)$ *is not klt at* y*, and*
(3) either $\dim G = 0$ *or* $A^{\dim G}\cdot G \le d^d$*.*

Proof. First, by [9, Lemma 7.1], there is a bounded family $\mathcal{P}_0$ of subvarieties of X such that for each pair x, $y \in X$ of general closed points, there is a member G_0 of $\mathcal{P}_0$ and an effective divisor $\Delta_0 \sim_{\mathbb{Q}} D$ such that $(X,B+\Delta_0)$ is lc near x with a unique non-klt place whose centre is G_0 and $(X,B+\Delta_0)$ is not klt at y.

Now suppose for some $0 \le i \le d-2$, we are given a family $\mathcal{P}_i$ of subvarieties of X such that for each pair x, $y \in X$ of general closed points, there is a member G_i of $\mathcal{P}_i$ and an effective divisor $\Delta_i \sim_{\mathbb{Q}} D+iA$ such that

(1) $(X,B+\Delta_i)$ is lc near x with a unique non-klt place whose centre is G_i,
(2) $(X,B+\Delta_i)$ is not klt at y, and
(3) either $\dim G_i \le d-i-1$ or $A^{\dim G_i}\cdot G_i \le d^d$.

If either $\dim G_i = 0$ or $A^{\dim G_i}\cdot G_i \le d^d$, then we set $G_{i+1} = G_i$ and $\Delta_{i+1} = \Delta_i + A$.

On the other hand, if $\dim G_i > 0$ and $A^{\dim G_i}\cdot G_i > d^d$, then we can write $A = A_i + E_i$ for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A_i and effective $\mathbb{Q}$-divisor E_i such that $A_i^{\dim G_i}\cdot G_i > d^d$. Note that such a decomposition can be done independently of G_i because $G_i \in \mathcal{P}_i$ with $A_i^{\dim G_i}\cdot G_i > d^d$ form a bounded family. By [15, Theorem 6.8.1 and Theroem 6.8.1.3] and [2, Lemma 2.32], there are positive rational numbers $\delta \ll 1$ and $c < 1$, such that there is an effective $\mathbb{Q}$-divisor $H \sim_{\mathbb{Q}} A_i$ such that

(1) $(X, B+(1-\delta)\Delta_i + cH)$ is lc near x with a unique non-klt place whose centre is G_i',

(2) $(X, B + (1-\delta)\Delta_i + cH)$ is not klt at y, and
(3) $\dim G_i' < \dim G_i$.

We set $G_{i+1} = G_i'$ and $\Delta_{i+1} = (1-\delta)\Delta_i + cH + \delta(D+iA) + (1-c)A_i + E_i \sim \Delta + A$ in this case. Note that D, A, A_i and E_i are independent of x and y.

Set $\mathcal{P}_{i+1} = \{G_{i+1}\}$, then the proposition follows from induction on i. Note that either $\dim G_i \le d - i - 1$ or $A^{\dim G_i} \cdot G_i \le d^d$ implies $\mathcal{P} = \mathcal{P}_{d-1}$ is bounded. □

Next, we recall the following lemma by Jiang, which aims to cut down the dimension of G to 0 in the previous proposition. Jiang shows it in [12, Lemma 3.1] for $H = -K_X$ being ample. In fact, the proof works under the following setting.

Lemma 2.21. *Fix positive integers $d > k$. Let (X, B) be a projective klt pair of dimension d and H be a nef and big $\mathbb{Q}$-Cartier divisor on X. Assume there is a morphism $f : Y \to T$ of projective varieties with a surjective morphism $\phi : Y \to X$ such that a general fiber F of f is of dimension k and $\phi|_F : F \to \phi(F) = G$ is birational, then*

$$H^k \cdot G \ge \frac{\mathrm{lct}((X,B), |H|_{\mathbb{R}})^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d.$$

Proof. We follow the proof of [12, Lemma 3.1]. Taking normalizations and resolutions of Y and T, we may assume they are smooth. Cutting by general hyperplane sections of T, we may assume ϕ is generically finite. Therefore, it holds that $\dim Y = d$. Let $\mathcal{I}_G^{<m>}$ (resp. $\mathcal{I}_F^{<m>}$) be the sheaf of ideal of regular functions vanishing along G (resp. F) to order at least m. Then $\mathcal{I}_F^{<m>} = \mathcal{I}_F^m$, where $\mathcal{I}_F$ denotes the ideal sheaf of F. So we have a natural injection

$$\mathcal{O}_X/\mathcal{I}_G^{<m>} \to \phi_*(\mathcal{O}_Y/\mathcal{I}_F^m).$$

Now we consider a rational number $l > 0$ and a positive integer m such that lmH is Cartier. By the projection formula, we have

$$\begin{aligned} &h^0(X, \mathcal{O}_X(lmH) \otimes \mathcal{O}_X/\mathcal{I}_G^{<m>}) \\ \le& h^0(X, \mathcal{O}_X(lmH) \otimes \phi_*(\mathcal{O}_Y/\mathcal{I}_F^m)) \\ =& h^0(Y, \phi^*\mathcal{O}_X(lmH) \otimes \mathcal{O}_Y/\mathcal{I}_F^m). \end{aligned}$$

On the other hand, since F is a general fiber of f, the conormal sheaf of F is trivial. That is, we have $\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_F^{\oplus(d-k)}$. Furthermore, we have

$$\mathcal{I}^{i-1}/\mathcal{I}^i \simeq S^{i-1}(\mathcal{I}/\mathcal{I}^2) \simeq \mathcal{O}_F^{\oplus\binom{d-k+i-2}{d-k-1}}$$

for every $i \ge 1$ (see [11, II. Theorem 8.24]). Hence,

$$\begin{aligned} &h^0(Y, \phi^*\mathcal{O}_X(lmH) \otimes \mathcal{O}_Y/\mathcal{I}_F^m) \\ \le& \sum_{i=1}^m h^0(Y, \phi^*\mathcal{O}_X(lmH) \otimes \mathcal{I}_F^{i-1}/\mathcal{I}_F^i) \\ =& \sum_{i=1}^m \binom{d-k+i-2}{d-k-1} h^0(Y, \phi^*\mathcal{O}_X(lmH) \otimes \mathcal{O}_F) \end{aligned}$$

$$=\binom{d-k+m-1}{d-k}h^0(F,\phi^*\mathcal{O}_X(lmH)|_F).$$

Now we consider the exact sequence

$$0\to\mathcal{O}_X(lmH)\otimes\mathcal{I}_G^{<m>}\to\mathcal{O}_X(lmH)\to\mathcal{O}_X(lmH)\otimes\mathcal{O}_X/\mathcal{I}_G^{<m>}\to 0,$$

which implies

$$\begin{aligned}&h^0(X,\mathcal{O}_X(lmH)\otimes\mathcal{I}_G^{<m>})\\ \geq&h^0(X,\mathcal{O}_X(lmH))-h^0(X,\mathcal{O}_X(lmH)\otimes\mathcal{O}_X/\mathcal{I}_G^{<m>})\\ \geq&h^0(X,\mathcal{O}_X(lmH))-\binom{d-k+m-1}{d-k}h^0(F,\phi^*\mathcal{O}_X(lmH)|_F).\end{aligned}$$

We fix l and consider the asymptotic behavior of the last two terms as m goes to infinity. By the definition of volume,

$$\lim_{m\to\infty}\frac{d!}{m^d}h^0(X,\mathcal{O}_X(lmH))=\mathrm{vol}(\mathcal{O}_X(lH))=l^dH^d.$$

On the other hand,

$$\begin{aligned}&\lim_{m\to\infty}\frac{d!}{m^d}\binom{d-k+m-1}{d-k}h^0(F,\phi^*\mathcal{O}_X(lmH)|_F)\\ =&\lim_{m\to\infty}\frac{d!m^{d-k}}{m^d(d-k)!}\frac{\mathrm{vol}(\phi^*\mathcal{O}_X(lH)|_F)m^k}{k!}\\ =&\binom{d}{k}\mathrm{vol}(\phi^*\mathcal{O}_X(lH)|_F)\\ =&\binom{d}{k}(\phi^*(lH)|_F)^k\\ =&\binom{d}{k}(\phi^*(lH))^k\cdot F\\ =&\binom{d}{k}(lH)^k\cdot G\\ =&\binom{d}{k}l^k(H)^k\cdot G.\end{aligned}$$

Consequently, for $l>\sqrt[d-k]{\binom{d}{k}\frac{(H)^k\cdot G}{H^d}}$ and m sufficiently large, we have

$$h^0(X,\mathcal{O}_X(lmH))>\binom{d-k+m-1}{d-k}h^0(F,\phi^*\mathcal{O}_X(lmH)|_F).$$

Therefore $h^0(X,\mathcal{O}_X(lmH)\otimes\mathcal{I}_G^{<m>})>0$, so there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}lH$ such that $\mathrm{mult}_GD\geq 1$. Let X_{sm} be the smooth locus of X. Since ϕ is surjective and G is the image of a general fiber F of f, $G|_{X_{sm}}$ is not empty. By [16, Lemma 2.29], $(X_{sm},(d-k)D|_{X_{sm}})$ is not klt along $G|_{X_{sm}}$. So $(X,B+(d-k)D)$ is not klt. Hence,

$$(d-k)l\geq\mathrm{lct}((X,B),\frac{1}{l}D)\geq\mathrm{lct}((X,B),|H|_{\mathbb{R}}).$$

Since l is chosen arbitrarily such that $l > \sqrt[d-k]{\binom{d}{k}\frac{(H)^k \cdot G}{H^d}}$, we have

$$(d-k)\sqrt[d-k]{\binom{d}{k}\frac{(H)^k \cdot G}{H^d}} \geq \operatorname{lct}((X,B),|H|_{\mathbb{R}}).$$

That is,

$$H^k \cdot G \geq \frac{\operatorname{lct}((X,B),|H|_{\mathbb{R}})^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d$$

holds. □

Lemma 2.22. *Under the setting of Lemma 2.21, let $Z \subseteq T$ be the union of points in T over which the fiber F' of f is of dimension k and $\phi|_{F'} : F' \to \phi(F')$ is birational. Then there is a proper closed subset S of T, such that for every fiber F' of f over $Z - S$, we have*

(1) $H^k \cdot \phi(F') \geq \frac{\operatorname{lct}((X,B),|H|_{\mathbb{R}})^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d$, *and*
(2) $\phi|_{f^{-1}(S\cap Z)}$ *does not dominate X.*

Proof. We apply the Noetherian induction (see, for example, [11, II, exercise 3.16]) on Lemma 2.21.

As the first step, by Lemma 2.21, there is a closed set $S_0 \subset T$ such that (1) holds for every fiber F' of f over $Z - S_0$. If $\phi|_{f^{-1}(S_0\cap Z)}$ does not dominate X, we finish the proof by taking $S = S_0$. Otherwise, $\phi|_{f^{-1}(S_0\cap Z)}$ dominates X and we go to the next step.

As the second step, we replace Y with the closure of $f^{-1}(S_0 \cap Z)$ in Y and apply Lemma 2.21. Then there is a closed set $S_1 \subset T$ such that (1) holds for every fiber F' of f over $Z \cap S_0 - S_1$. Combining this with the previous step, (1) holds for every fiber F' of f over $Z - S_1$. If $\phi|_{f^{-1}(S_1\cap Z)}$ does not dominate X, we finish the proof by taking $S = S_1$. Otherwise, $\phi|_{f^{-1}(S_1\cap Z)}$ dominates X and we go to the next step.

As the third step, we replace S_0 by S_1 and repeat the second step.

Going on this procession, after finitely many steps, we can get a closed S of T we want.

Note that S must be a proper subset because ϕ is surjective and $f^{-1}Z$ is dense in Y.

□

3. Proofs of Theorems

Now we restate and prove the theorems in Section 1.

Theorem 3.1. *Fix a positive integer d and a positive real number θ. Then there is a number m depending only on d and θ such that if X is a projective normal variety satisfying*

(1) $\dim X = d$,
(2) there exists a boundary B such that (X,B) is klt,
(3) there is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on X with $\operatorname{lct}((X,B),|H|_{\mathbb{R}}) > \theta$, *and*
(4) $H^d > \theta$,

then $|K_X + \lceil mH \rceil|$ defines a birational map and mH is potentially birational.

Proof. Suppose we are given X, B and H as in the assumption.

Set

$$q = \lceil \frac{2d}{\sqrt[d]{\theta}} \rceil,$$

$$p = \max_{1 \le k \le d-1} \left\{ \lceil \sqrt[k]{\frac{\binom{d}{k}(d-k)^{d-k} d^d}{\theta^{d-k+1}}} \rceil \right\}.$$

Then by construction, $(qH)^d > (2d)^d$.

Now apply Proposition 2.20 with $D = qH$ and $A = pH$ and we have a bounded family $\mathcal{P}$ of subvarieties of X such that for each pair $x, y \in X$ of general closed points, there is a member G_{xy} of $\mathcal{P}$ and an effective divisor $\Delta_{xy} \sim_{\mathbb{Q}} D + (d-1)A$ such that

1. $(X, B + \Delta_{xy})$ is lc near x with a unique non-klt place whose centre is G_{xy},
2. $(X, B + \Delta_{xy})$ is not klt at y, and
3. either $\dim G_{xy} = 0$ or $A^{\dim G_{xy}} \cdot G_{xy} \le d^d$.

By [2, Lemma 2.21], this means that there is a finite set $\{\phi_j : V_j \to T_j\}$ of projective varieties with surjective morphisms $\pi_j : V_j \to X$ such that each member $G \in \mathcal{P}$ is isomorphic through π_j to a fiber of ϕ_j for some j. Let Z_j be the set of points in T_j over which the fiber of ϕ_j is isomorphic through π_j to an element of $\mathcal{P}$. By shrinking V_j if necessary, we may assume that Z_j is an open dense subset of T_j.

Now consider a general fiber G' of ϕ_j for each j. If $\dim G' > 0$. By Lemma 2.22, there is a closed subset S_j of T_j, such that

$$H^k \cdot G \ge \frac{\operatorname{lct}((X,B), |H|_{\mathbb{R}})^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d > \frac{\theta^{d-k+1}}{\binom{d}{k}(d-k)^{d-k}}$$

for every image $G \in \mathcal{P}$ of π_j of a fiber F' of ϕ_j over $Z_j - S_j$, and $\pi_j|_{\phi_j^{-1}(Z_j \cap S_j)}$ is not dominant. On the other hand, we set S_j to be the empty set if $\dim G' = 0$.

Now since x and y are general, they are not in $\pi_j(\phi_j^{-1}(Z_j \cap S_j))$ for each j. So G_{xy} is image of π_j of a fiber of ϕ_j over $Z_j - S_j$ for some j. Suppose $\dim G_{xy} > 0$, then by the definition of p, $A^k \cdot G' = (pH)^k \cdot G' > d^d$. This contradicts (3) of Proposition 2.20 and thus $\dim G' = 0$ for any general fiber G' of ϕ_j for each j. In particular, $\dim G_{xy} = 0$, and hence $G_{xy} = \{x\}$ by (1).

Since x and y are general, we have $B = 0$ in a neighborhood around x and y. So $B + \Delta_{xy} = \Delta_{xy} \sim_{\mathbb{Q}} D + (d-1)A$ in a neighborhood around x and y. Since $D + (d-1)A = (q + (d-1)p)H$, by the argument above, we know that $(q + (d-1)p + 1)H$ is potentially birational. By Lemma 2.15, $|K_X + \lceil (q+(d-1)p+1)H \rceil|$ defines a birational map. Let $m = q+(d-1)p+1$ and we are done.

□

Corollary 3.2. *Fix positive integers d, n, and a positive real number θ. Then there is a number m depending only on d, n and θ such that if (X, B) is a weak Fano pair satisfying*

(1) $\dim X = d$,
(2) $\alpha(X, B) > \theta$,
(3) $K_X + B$ *is a* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor,*
(4) $(-K_X - B)^d > \theta$*, and*
(5) there is an n*-complement* $K_X + B^+$ *of* $K_X + B$ *with* $B^+ \geq B$*,*

then $|\lfloor m(B^+ - B)\rfloor|$ *defines a birational map.*

Proof. Suppose we are given (X, B) and B^+ as in the assumption. By Theorem 3.1, there exists a number m' depending only on d and θ such that $m'(-K_X - B)$ is potentially birational. Let $m = nm'$. Then, by construction, $-K_X + \lfloor m(B^+ - B)\rfloor \sim_{\mathbb{Q}} \lfloor m(B^+ - B)\rfloor + B^+ \geq m'(B^+ - B)$. By Lemma 2.16, we are done. □

Theorem 3.3. *Fix a positive integer* d*. There exists a real number* M*, depending only on* d*, such that for any projective pair* (X, B) *and for any big and nef* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor* H *on* X *satisfying*

(1) $\dim X = d$*, and*
(2) (X, B) *is klt,*

we have

$$\mathrm{lct}((X, B), |H|_{\mathbb{R}})^{d-1}\overline{\tau}((X, B), H)H^d \leq M.$$

Proof. By the linearity of lct, $\overline{\tau}$ and the volume function with respect to H, we may assume that $\overline{\tau}((X, B), H) = 2$. So that $-(K_X + B) - H = -(K_X + B) - 2H + H$ is big and then we can write $-(K_X + B) - H = F + E$ for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor F and effective $\mathbb{Q}$-divisor E. Assume the theorem fail to hold. Then for any number M, there is a pair (X, B) a big and nef $\mathbb{Q}$-Cartier H on X satisfying the assumptions, such that

$$\mathrm{lct}((X, B), |H|_{\mathbb{R}})^{d-1}H^d = \mathrm{lct}((X, B), |H|_{\mathbb{R}})^{d-1}\frac{\overline{\tau}((X, B), H)}{2}H^d > \frac{M}{2}.$$

By Lemma 1.7, we have

$$\mathrm{lct}((X, B), |H|_{\mathbb{R}})^{d-1}H^d \leq \frac{d^d}{\mathrm{lct}((X, B), |H|_{\mathbb{R}})}.$$

So we may assume $\mathrm{lct}((X, B), |H|) \leq 1$ and $H^d > \frac{M}{2} \geq (4d)^d$. Moreover, for every $0 < k < d$, we may assume

$$M > 2\binom{d}{k}(d-k)^{d-k}(2(d-1)d)^d.$$

Now apply Proposition 2.20 with $D = \frac{1}{2}H$ and $A = \frac{1}{2(d-1)}H$. We have a bounded family $\mathcal{P}$ of subvarieties of X such that for each pair $x, y \in X$ of general closed points, there is a member G_{xy} of $\mathcal{P}$ and an effective divisor $\Delta \sim_{\mathbb{Q}} D + (d-1)A = H$ such that

(1) $(X, B + \Delta)$ is lc near x with a unique non-klt place whose centre is G_{xy},
(2) $(X, B + \Delta)$ is not klt at y, and
(3) either $\dim G_{xy} = 0$ or $A^{\dim G_{xy}} \cdot G_{xy} \leq d^d$.

By [2, Lemma 2.21], this means that there is a finite set $\{\phi_j : V_j \to T_j\}$ of projective varieties with surjective morphisms $\pi_j : V_j \to X$ such that each member $G \in \mathcal{P}$ is isomorphis through π_j to a fiber of ϕ_j for some j.

Now consider a general fiber G' of ϕ_j for each j. If $\dim G' > 0$. By Lemma 2.22, there is a closed subset S_j of T_j, such that

$$H^k \cdot G \geq \frac{\operatorname{lct}((X,B),|H|_{\mathbb{R}})^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d$$

for every image G of π_j of a fiber F' of ϕ_j over $T_j - S_j$, and $\pi_j|_{\phi_j^{-1}(S_j)}$ is not dominant. On the other hand, we set S_j to be the empty set if $\dim G' = 0$.

Now since x and y are general, they are not in $\pi_j(\phi_j^{-1}(S_j))$ for each j. So G_{xy} is image of π_j of a fiber of ϕ_j over $T_j - S_j$ for some j. Suppose $\dim G_{xy} > 0$, then we have

$$\begin{aligned} H^k \cdot G_{xy} &\geq \frac{\operatorname{lct}((X,B),|H|_{\mathbb{R}})^{d-k}}{\binom{d}{k}(d-k)^{d-k}} H^d \\ &\geq \frac{\operatorname{lct}((X,B),|H|_{\mathbb{R}})^{d-1}}{\binom{d}{k}(d-k)^{d-k}} H^d \\ &> \frac{M}{2\binom{d}{k}(d-k)^{d-k}} > (2(d-1)d)^d. \end{aligned}$$

By construction, $A^k \cdot G_{xy} = (\frac{1}{2(d-1)})^k H^k \cdot G_{xy} > d^d$. This contradicts (3) of the above and thus $\dim G_{xy} = 0$.

Now for general x and y, $\operatorname{Nklt}(X, B+\Delta)$ contains y and x and isolates x. Since x is general, $\operatorname{Nklt}(X, B+\Delta+E)$ also isolates x. By Lemma 2.1, $-K_X - (B+\Delta+E) \sim_{\mathbb{Q}} -(K_X+B) - H - E = F$ is not ample, which is a contradiction. □

We recall the following theorem by Hacon and Xu.

Theorem 3.4 ([10, Theorem 1.3]). *Fix a positive integer d and a DCC set $\mathscr{I}$ of rational numbers in $[0,1]$. The set of all projective pairs (X,B) satisfying*

(1) (X,B) is klt log Calabi–Yau of dimension d,
(2) B is big, and
(3) the coefficients of $B \in \mathscr{I}$,

forms a bounded family.

Then we show the following log-version of [2, Lemma 2.26].

Lemma 3.5. *Fix positive integers d, k and a non-negative real number ϵ. Let $\mathcal{P}$ be a set of klt weak Fano pairs of dimension d. Assume that for every element $(Y,B_Y) \in \mathcal{P}$, there is a k-complement (resp. $\mathbb{R}$-complement) $K_Y + B_Y^+$ of $K_Y + B_Y$ such that (Y, B_Y^+) is ϵ-lc and $B_Y^+ \geq B_Y$. Let $\mathcal{Q}$ be the set of projective pairs (X,B) such that*

(1) there is $(Y,B_Y) \in \mathcal{P}$ and a birational map $X \dashrightarrow Y$,
(2) there is a common resolution $\phi : W \to Y$ and $\psi : W \to X$, and
(3) $\phi^*(K_Y+B_Y) \geq \psi^*(K_X+B)$.

Then for every element $(X,B) \in \mathcal{Q}$, there is a k-complement $K_X + B^+$ of $K_X + B$ such that (X,B^+) is ϵ-lc and $B^+ \geq B$.

Proof. Let K_X+B^+ be the crepant pullback of $K_Y+B_Y^+$ to X. Then (X,B^+) is ϵ-lc. Since $k(K_Y+B_Y^+)\sim 0$ (resp. $K_Y+B_Y^+\sim_{\mathbb{R}} 0$) and $\phi^*(K_Y+B_Y)\geq \psi^*(K_X+B)$, $B^+-B\geq \psi_*\phi^*(B_Y^+-B_Y)\geq 0$ and K_X+B^+ is a strong k-complement (resp. $\mathbb{R}$-complement) of K_X+B. □

Lemma 3.6. *Let (X,B) be a projective lc weak Fano pair with $\alpha(X,B)\leq 1$. Suppose $\phi: X \dashrightarrow Y$ is a partial MMP (of any divisor). Let $B_Y=\phi_*(B)$. Then (Y,B_Y) is lc and $\alpha(Y,B_Y)\geq \alpha(X,B)$. If moreover (X,B) is klt, then (Y,B_Y) is also klt.*

Proof. Since (X,B) is weak Fano, there is an $\mathbb{R}$-complement $K_X+\overline{B}$ of K_X+B. Let $\overline{B}_Y=\phi_*(\overline{B})$, then $K_Y+\overline{B}_Y$ is an $\mathbb{R}$-complement of K_Y+B_Y. In particular, (Y,B_Y) is lc and $|-(K_Y+B_Y)|_{\mathbb{R}}$ is non-empty. Let $t=\alpha(X,B)$. Now we consider the following argument for any $M_Y\in|-(K_Y+B_Y)|_{\mathbb{R}}$. Let K_X+B+M be the crepant pullback of $K_Y+B_Y+M_Y$ to X. As in the proof of Lemma 3.5, we have $M\in|-(K_X+B)|_{\mathbb{R}}$ and $\phi_*(M)=M_Y$. By assumption, $(X,B+tM)$ is lc and $-(K_X+B+tM)\sim_{\mathbb{R}}-(1-t)(K_X+B)$ is nef and big if $t\neq 1$. So there is an effective divisor G on X such that $(X,B+tM+G)$ is lc Calabi–Yau. Let $G_Y=\phi_*(G)$. Then $(Y,B_Y+tM_Y+G_Y)$ is an $\mathbb{R}$-complement of $K_Y+B_Y+tM_Y$. In particular, (Y,B_Y+tM_Y) is lc. Therefore, we have $\alpha(Y,B_Y)\geq t$. If moreover (X,B) is klt, then we may assume $(X,\overline{B})$ is klt and therefore (Y,B_Y) is klt. □

Next, we recall the following proposition of Birkar with a small observation.

Proposition 3.7 ([2, Proposition 4.4])**.** *Fix positive integers d, v and a positive real number ϵ. Then there exists a bounded set of couples $\mathcal{P}$ and a positive real number c depending only on d, v and ϵ satisfies the following. Assume*

- *X is a normal projective variety of dimension d,*
- *B is an effective $\mathbb{R}$-divisor with coefficient at least ϵ,*
- *M is a $\mathbb{Q}$-divisor with $|M|:=|\lfloor M\rfloor|$ defining a birational map,*
- *$M-(K_X+B)$ is pseudo-effective,*
- *$\mathrm{vol}(M)<v$, and*
- *$\mu_D(B+M)\geq 1$ for every component D of M.*

Then there is a projective log smooth couple $(\overline{W},\Sigma_{\overline{W}})\in\mathcal{P}$, a birational map $\overline{W}\dashrightarrow X$ and a common resolution X' of this map such that

1. *Supp$\Sigma_{\overline{W}}$ contains the exceptional divisor of $\overline{W}\dashrightarrow X$ and the birational transform of Supp$(B+M)$, and*
2. *there is a resolution $\phi: W\to X$ such that $M_W:=\phi^*M\sim A_W+R_W$ where A_W is the movable part of $|M_W|$, $|A_W|$ is base point free, $\psi: X'\to X$ factors through W and $A_{X'}\sim 0/\overline{W}$, where $A_{X'}$ is the pullback of A_W on X'.*

Moreover, if M is nef and $M_{\overline{W}}$ is the pushdown of $M_{X'}:=\psi^*M$. Then each coefficient of $M_{\overline{W}}$ is at most c.

Note that in the original statement of [2, Proposition 4.4], M is assumed to be nef. We observe from Birkar's proof that the nefness of M is used only when showing the existence of c and is not necessary when showing (1) and (2) of proposition 3.7.

Now we are ready to show the main theorem of this paper. The idea is to follow the strategy of [2, Proposition 7.13], which is to construct a klt complement with coefficients in a finite set depending only on d, θ and $\mathscr{R}$, and then apply Theorem 3.4.

Theorem 3.8. *Fix a positive integer d, a positive real number θ and a closed DCC set $\mathscr{S}$ of rational numbers in $[0,1]$. The set $\mathcal{D}$ of all klt weak Fano pairs (X, B) satisfying*

(1) $\dim X = d$,
(2) the coefficients of $B \in \mathscr{S}$,
(3) $\alpha(X, B) > \theta$, *and*
(4) $(-(K_X + B))^d > \theta$

forms a log bounded family. Moreover, there is a rational number k depending only on d, θ and $\mathscr{S}$, such that every element $(X, B) \in \mathcal{D}$, there is a strong klt k-complement $K_X + \Theta$ of $K_X + B$.

Proof. By Theorem 3.4, it is enough to show the existence of k.

By Lemma 3.5, replacing by a small $\mathbb{Q}$-factorialisation of X, we may assume X is $\mathbb{Q}$-factorial.

By Theorem 2.5, there is an n-complement $K_X + B^+$ of $K_X + B$ such that $B^+ \geq B$.

Let m be given by Corollary 3.2 such that $|\lfloor m(B^+ - B)\rfloor|$ defines a birational map. Replacing n and m by $2mn$, we may assume $n = m > 1$. On the other hand, by Lemma 1.7, $\mathrm{vol}(\lfloor m(B^+ - B)\rfloor) \leq \mathrm{vol}(m(B^+ - B)) < v$ for some v depending only on d, m and θ.

Let M be a general element of $|\lfloor m(B^+ - B)\rfloor|$. By Proposition 3.7, there is a bounded set of couples $\mathcal{P}$ depending only on d, m and θ, such that there is a projective log smooth couple $(\overline{W}, \Sigma_{\overline{W}}) \in \mathcal{P}$, a birational map $\overline{W} \dashrightarrow X$ and a common resolution X' of this map such that

(1) $\mathrm{Supp}\Sigma_{\overline{W}}$ contains the exceptional divisor of $\overline{W} \dashrightarrow X$ and the birational transform of $\mathrm{Supp}(B^+ + M)$, and
(2) there is a resolution $\phi : W \to X$ such that $M_W := \phi^* M \sim A_W + R_W$ where A_W is the movable part of $|M_W|$, $|A_W|$ is base point free, $X' \to X$ factors through W and $A_{X'} \sim 0/\overline{W}$, where $A_{X'}$ is the pullback of A_W on X'.

Since M is a general element of $|\lfloor m(B^+ - B)\rfloor|$, we may assume $M_W = A_W + R_W$ and A_W is general in $|A_W|$. In particular, if $A_{\overline{W}}$ is the pushdown of $A_W|_{X'}$ to $\overline{W}$, then $A_{\overline{W}} \leq \Sigma_{\overline{W}}$. Let M, A, R be the pushdowns of M_W, A_W, R_W to X.

Since $|A_{\overline{W}}|$ defines a birational contraction and $A_{\overline{W}} \leq \Sigma_{\overline{W}}$, there exists $l \in \mathbb{N}$ depending only on $\mathcal{P}$ such that $lA_{\overline{W}} \sim G_{\overline{W}}$ for some $G_{\overline{W}} \geq 0$ whose support contains $\Sigma_{\overline{W}}$. Let $K_{\overline{W}} + B^+_{\overline{W}}$ be the crepant pullback of $K_X + B^+$ to $\overline{W}$. Then $(\overline{W}, B^+_{\overline{W}})$ is sub-lc and

$$\mathrm{Supp}B^+_{\overline{W}} \subseteq \mathrm{Supp}\Sigma_{\overline{W}} \subseteq \mathrm{Supp}G_{\overline{W}}.$$

Let G be the pushdown of $G_{X'} := G_{\overline{W}}|_{X'}$ to X. Since $A_{X'}$ is the pullback of $A_{\overline{W}}$, we obtain $lA_{X'} \sim G_{X'}$ and $lA \sim G$. Therefore, $G + lR + l\{m(B^+ - B)\} \sim_{\mathbb{Q}} m(B^+ - B)$.

Take a positive rational number $t \leq (lm)^{-d}\theta$, then

$$(X, B + t(G + lR + l\{m(B^+ - B)\}))$$

is klt. Moreover, we have

$$-K_X - B - t(G + lR + l\{m(B^+ - B)\}) \sim_{\mathbb{Q}} B^+ - B - t(lm(B^+ - B)).$$

By replacing t, we may assume $t < \frac{1}{lm}$. Since

$$B^+ - B - t(lm(B^+ - B)) = (1 - tlm)(B^+ - B) \geq 0,$$

$B^+ - B - t(lm(B^+ - B))$ is nef and big. Therefore

$$(X, B + t(G + lR + l\{m(B^+ - B)\}))$$

is klt weak Fano.

Now we argue that the coefficients of $B+t(G+lR+l\{m(B^+-B)\})$ are in a closed DCC set of rational numbers in $[0, 1]$. It is clear that the coefficients of $B + t(G + lR + l\{m(B^+ - B)\})$ are in a set of rational numbers in $[0, 1]$, so it is enough to show that they are in a set with rational accumulation points. Write

$$B + tl\{m(B^+ - B)\} = (1 - tlm)B + tlmB^+ - tl\lfloor m(B^+ - B)\rfloor.$$

Since m, l and t are determined by the fixed terms d, θ and $\mathscr{R}$, the coefficients of $tlmB^+ - tl\lfloor m(B^+ - B)\rfloor + t(G + lR)$ are in a fixed finite rational set. By assumption, the coefficients of B are in a DCC set with rational accumulation points $\mathscr{R}$. Therefore the coefficients of $(1-tlm)B+tlmB^+ - tl\lfloor m(B^+ - B)\rfloor + t(G + lR)$ are in a DCC set with rational accumulation points.

By Theorem 2.5, there is a positive integer n' depending only on d, $\mathscr{R}$ m, l and t such that there is an n'-complement $K_X + \Omega$ of $K_X + B + t(G+lR+l\{m(B^+ - B)\})$, such that

$$\Omega \geq B + t(G + lR + l\{m(B^+ - B)\}).$$

On the other hand, let

$$\Delta_{\overline{W}} := B^+_{\overline{W}} + \frac{t}{m}A_{\overline{W}} - \frac{t}{lm}G_{\overline{W}}.$$

Then $(\overline{W}, \Delta_{\overline{W}})$ is sub-2ϵ-klt for some $\epsilon > 0$ depending only on $\mathcal{P}$, t, l and m since $\mathrm{Supp}B^+_{\overline{W}} \subseteq \mathrm{Supp}\Sigma_{\overline{W}} \subseteq \mathrm{Supp}G_{\overline{W}}$, $(\overline{W}, \Sigma_{\overline{W}})$ is log smooth, $(\overline{W}, B^+_{\overline{W}})$ is sub-lc, and $A_{\overline{W}}$ is not a component of $\lceil B^+_{\overline{W}}\rceil$. Moreover, $K_{\overline{W}} + \Delta_{\overline{W}} \sim_{\mathbb{Q}} 0$.

Let

$$\Delta := B^+ + \frac{t}{m}A - \frac{t}{lm}G.$$

Then $K_X + \Delta \sim_{\mathbb{Q}} 0$. (X, Δ) is sub-klt since $K_X + \Delta$ is the crepant pullback of $K_{\overline{W}} + \Delta_{\overline{W}}$.

Let $\Theta = \frac{1}{2}\Delta + \frac{1}{2}\Omega$. Then

$$\Theta = \frac{1}{2}B^+ + \frac{t}{2m}A - \frac{t}{2lm}G + \frac{1}{2}\Omega$$

$$\geq \frac{1}{2}B^+ + \frac{t}{2m}A - \frac{t}{2lm}G + \frac{1}{2}B + \frac{t}{2}(G + lR) \geq -\frac{t}{2lm}G + \frac{t}{2}G \geq 0.$$

Since (X, Δ) is sub-ϵ-klt, $K_X + \Delta \sim_{\mathbb{Q}} 0$ and (X, Ω) is lc log Calabi–Yau, (X, Θ) is klt log Calabi–Yau. The coefficients of Θ belong to a fixed finite set depending only on t, l, m and n'. Moreover, $\Theta \geq \frac{1}{2}B^+ + \frac{1}{2}\Omega \geq \frac{1}{2}B + \frac{1}{2}B = B$. Recall that $n(K_X + B^+) \sim 0$, $lA \sim G$ and $n'(K_X + \Omega) \sim 0$. Let k be an integer such that $k\{\frac{1}{2n}, \frac{t}{2lm}, \frac{1}{2n'}\} \subseteq \mathbb{N}$, then $K_X + \Theta$ is a k-complement of $K_X + B$, with (X, Θ) ϵ-lc and $\Theta \geq B$. It follows that (X, Θ) is $\frac{1}{k}$-lc □

Theorem 3.9. *Fix a positive integer d, a positive real number θ and a DCC set $\mathscr{S}'$ of real numbers in $[0, 1]$. The set $\mathcal{D}'$ of all klt weak Fano pairs (X, B) satisfying*

(1) $\dim X = d$,
(2) the coefficients of $B \in \mathscr{S}'$,
(3) $\alpha(X, B) > \theta$, and
(4) $(-(K_X + B))^d > \theta$

forms a log bounded family. Moreover, there is a rational number k, finite sets $S_1 \subset [0, 1]$ and $S_2 \subset \mathbb{Q} \cap [0, 1]$, depending only on d, θ and $\mathscr{S}'$, such that for every element $(X, B) \in \mathcal{D}'$, there is a strong klt (S_1, S_2, k)-complement $K_X + \Theta$ of $K_X + B$.

Proof. As in the proof of Theorem 3.8, by Theorem 3.4 and by Lemma 3.5, we may assume X is $\mathbb{Q}$-factorial and it is enough to show the existence of k. Replacing $\mathscr{S}'$ by its closure, we may assume it is closed.

Let Γ_1, Γ_2, r, a_i, g, Γ'_i, g_i, $\mathscr{S}_i$ and g'_i be as given in Theorem 2.10. Let $(X, B) \in \mathcal{D}'$ and write $B = \sum_j b_j B_j$, where B_j are distinct Weil $\mathbb{Q}$-Cartier divisors. Then we have $B \leq \sum_i (a_i \sum_j g'_i(b_j) B_j)$ and $\sum_j g'_i(b_j) B_j \leq \frac{B + 3\sum_j g(b_j) B_j}{4}$ for every $1 \leq i \leq r$. Furthermore, $(X, \sum_j g(b_j) B_j)$ is lc and $-(K_X + \sum_j g(b_j) B_j)$ is pseudo-effective by Theorem 2.6. So $(X, \sum_j g'(b_j) B_j)$ is klt for every $1 \leq i \leq r$.

For each fixed i, we consider the following argument. Sine X is of Fano type and $-(K_X + \sum_j g'_i(b_j) B_j)$ is pseudo-effective, we may run a $-(K_X + \sum_j g'_i(b_j) B_j)$-MMP and reach a model Y_i such that $-(K_{Y_i} + \sum_j g'_i(b_j) B_{Y_i j})$ is nef, where $B_{Y_i j}$ is the strict transform of B_j on Y_i. Therefore, by Lemma 3.6, $(Y_i, \sum_j g'_i(b_j) B_{Y_i j})$ is a klt weak Fano pair. Since $(X, \sum_j g(b_j) B_j)$ has an $\mathbb{R}$-complement, so does $(Y_i, \sum_j g(b_j) B_{Y_i j})$. On the other hand, by Lemma 3.6 and the fact that volume is invariant under small birational maps and is increased by pushing-forward through morphisms, we have

(i) $\operatorname{vol}(Y_i, -(K_{Y_i} + \sum_j b_j B_{Y_i j})) \geq \operatorname{vol}(X, -(K_X + B)) > \theta$, and
(ii) $\alpha(Y_i, \sum_j b_j B_{Y_i j}) > \theta$.

By Lemma 3.5, we may replace X by Y and replace B_j by $B_{Y_i j}$ except that $(X, \sum_j g'_i(b_j) B_j)$ is klt weak Fano but (X, B) might not be weak Fano any more.

We estimate that

$$\operatorname{vol}(X, -(K_X + \sum_j g'_i(b_j) B_j))$$

$$
\begin{aligned}
&= \mathrm{vol}(-(K_X+\sum_j g(b_j)B_j)+\sum_j g(b_j)B_j-\sum_j g_i'(b_j)B_j)\\
&\geq \mathrm{vol}(-(K_X+\sum_j g(b_j)B_j)+\sum_j g(b_j)B_j-\frac{B+3\sum_j g(b_j)B_j}{4})\\
&\geq \mathrm{vol}(-\frac{1}{4}(K_X+\sum_j g(b_j)B_j)+\frac{\sum_j g(b_j)B_j-B}{4})\\
&= (\frac{1}{4})^d \mathrm{vol}(-(K_X+B)).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\alpha(X,\sum_j g_i'(b_j)B_j) &= \mathrm{lct}((X,\sum_j g_i'(b_j)B_j),|-(K_X+\sum_j g_i'(b_j)B_j)|_{\mathbb{R}})\\
&\geq \mathrm{lct}((X,\frac{B+3\sum_j g(b_j)B_j}{4}),|-(K_X+B)|_{\mathbb{R}})\\
&\geq \frac{1}{4}\mathrm{lct}((X,B),|-(K_X+B)|_{\mathbb{R}})\\
&= \frac{1}{4}\alpha(X,B),
\end{aligned}
$$

where that last inequality follows from the following argument. If $0<t\leq \frac{1}{4}\mathrm{lct}((X,B),|-(K_X+B)|_{\mathbb{R}})$, then for any $M\in|-(K_X+B)|_{\mathbb{R}}$, $(X,B+4tM)$ is lc. It follows that $(X,\frac{B+4tM+3\sum_j g(b_j)B_j}{4})$ is lc by the linearity of log discrepancies because $(X,\sum_j g(b_j)B_j)$ is lc. Therefore we have

$$
t\leq \mathrm{lct}((X,\frac{B+3\sum_j g(b_j)B_j}{4}),|-(K_X+B)|_{\mathbb{R}}).
$$

As a conclusion, $\mathrm{vol}(X,-(K_X+\sum_j g_i'(b_j)B_j))$ and $\alpha(X,\sum_j g_i'(b_j)B_j)$ are both bounded below away from 0 and $(X,\sum_j g_i'(b_j)B_j)$ is a klt weak Fano pair.

By Theorem 3.8, for each i, there is a rational number k_i depending only on d, θ and $\mathscr{S}'$ such that there is a strong klt k_i-complement $K_X+\Theta_i$ of $K_X+\sum_j g_i(b_j)B_j$. Replacing k_i by $k:=\prod_{h=1}^r k_h$ for each i, we may assume that $k_i=k$. Let $S_1=\{a_i\}_i$ and $S_2=\frac{1}{k}\mathbb{N}\cap[0,1]$. Let $\Theta:=\sum_{i=1}^r a_i\Theta_i$. Then $K_X+\Theta$ is the complement we want since $K_X+\Theta\geq K_X+\sum_i(a_i\sum_j g_i'(b_j)B_j)\geq K_X+B$.

□

Theorem 3.10. *Fix a positive integer d, and a positive real number θ. Let $\mathcal{P}$ be a log bounded set of all klt Fano pairs (X,B) satisfying*

(1) $\dim X=d$,
(2) $\alpha(X,B)>\theta$, *and*
(3) $(-(K_X+B))^d>\theta$.

Then there is a natural number k, depending only on d, θ and $\mathcal{P}$, such that for every $(X,B)\in\mathcal{P}$, there is a strong klt k-complement $K_X+\Theta$ of K_X+B.

Proof. By Lemma 3.5, we may assume X is $\mathbb{Q}$-factorial. Since $\mathcal{P}$ is log bounded and $(-(K_X+B))^d>\theta$, there is a positive real number a such that for every $(X,B)\in\mathcal{P}$, $-(K_X+B+a\mathrm{Supp}B)$ is effective. Since $\alpha(X,B)>\theta$, by replacing a, we may assume that $(X, B + a\text{Supp}B)$ is klt. Therefore, there is a natural number k such that for every $(X, B) \in \mathcal{P}$, there is a divisor $\overline{B} \geq B$ on X such that $\overline{B} \leq B + \frac{1}{2}a\text{Supp}B$ and $k\overline{B}$ is an integral divisor. Therefore, $(X, 2\overline{B} - B)$ is klt and $-K_X - 2\overline{B} + B$ is pseudo-effective. Replacing a further, we may assume that $(X, 2\overline{B} - B)$ has an $\mathbb{R}$-complement. We may run a $-(K_X + \overline{B})$-MMP and reach a model Y such that $-(K_Y + \overline{B}_Y)$ is nef, where $\overline{B}_Y$ and B_Y are the strict transforms of $\overline{B}$ and B on Y respectively. Therefore, by Lemma 3.6, $(Y, \overline{B}_Y)$ is a klt weak Fano pair, and $(Y, 2\overline{B}_Y - B_Y)$ has an $\mathbb{R}$-complement. By Lemma 3.6 and the fact that volume is invariant under small birational maps and is increased by pushing-forward through morphisms, we have

(i) $\text{vol}(Y, -(K_Y + B_Y)) \geq \text{vol}(X, -(K_X + B)) > \theta$, and
(ii) $\alpha(Y, B_Y) > \theta$.

By Lemma 3.5, we may replace X, B and $\overline{B}$ by Y, B_Y and $\overline{B}_Y$ respectively except that $(X, \overline{B})$ is klt weak Fano but (X, B) might not be weak Fano any more. By the same computation as in Theorem 3.9, we can bound $\alpha(X, \overline{B})$ and $(-(K_X + \overline{B}))^d$ below away from 0. Now we apply Theorem 3.8 and we are done. □

Theorem 3.11. *Fix a positive integer d, positive real numbers θ and δ and a closed DCC set $\mathscr{R}$ in $[0, 1]$. The set of all klt Fano pairs (X, B) satisfying*

(1) $\dim X = d$,
(2) the coefficients of $B \in \mathscr{R}$, *and*
(3) $\alpha(X, B)^{d-1+\delta}(-(K_X + B))^d > \theta$

forms a log bounded family.

Proof. This follows from Theorem 2.18, Theorem 3.3 and Theorem 3.8. □

Corollary 3.12. *Fix a positive integer d and two positive real numbers δ and θ. Then the set of klt-Fano varieties X satisfying*

(1) $\dim X = d$, *and*
(2) $\alpha(X)^{d-1+\delta}(-K_X)^d > \theta$

forms a bounded family.

Proof. This follows either from the previous theorem or from Theorem 2.18, Theorem 3.3 and Theorem 1.3. □

4. Other Results on Boundedness Problems: Boundedness of Rationally Connected Calabi–Yau 3-fold

In this section I list some results in my joint work with G. Di Cerbo, J. Han, C. Jiang, and R. Svaldi in [5]. In this work, we prove that for any $\epsilon > 0$, rationally connected ϵ-klt Calabi–Yau pairs of dimension 3 form a birationally boundedness. Moreover, if we further bound the coefficients of the boundaries below, such pairs are then log bounded modulo flops. As a corollary, the set of rationally connected klt Calabi–Yau 3-folds with mld bounded away from 1 are bounded modulo flops.

Definition 4.1. A variety is *rationally connected* if any two general points can be connected by a rational curve.

Definition 4.2. Let ϵ be a positive real number. A normal projective variety X is *of ϵ-Calabi–Yau type* (or ϵ-CY type for short) if there exists an effective $\mathbb{R}$-divisor B such that (X, B) is an ϵ-klt Calabi–Yau pair.

Theorem 4.3 ([5, Thorem 1.5])**.** *Fix a positive real number ϵ. The set of rationally connected 3-folds of ϵ-CY type forms a birationally bounded family.*

We recall the definition of boundedness modulo flops for Calabi–Yau varieties.

Definition 4.4 (c.f. [5, 2.4])**.** We say that a collection of log pairs $\mathcal{D}$ is *log bounded in codimension one* if there is a quasi-projective scheme $\mathcal{Z}$, a reduced divisor $\mathcal{E}$ on $\mathcal{Z}$, and a projective morphism $h\colon \mathcal{Z} \to S$, where S is of finite type and $\mathcal{E}$ does not contain any fiber, such that for every $(X, B) \in \mathcal{D}$, there is a closed point $s \in S$ and a birational map $f\colon \mathcal{Z}_s \dashrightarrow X$ which is isomophic in codimension one such that $\mathcal{E}_s$ coincides with the support of $f_*^{-1}B$.

Moreover, if $\mathcal{D}$ is a set of klt Calabi–Yau varieties (resp., klt Calabi–Yau pairs), then it is said to be *bounded modulo flops* (resp., *log bounded modulo flops*) if it is (log) bounded in codimension one, and each fiber $\mathcal{Z}_s$ corresponding to X in the definition is normal projective, and $K_{\mathcal{Z}_s}$ is $\mathbb{Q}$-Cartier (resp., $K_{\mathcal{Z}_s} + f_*^{-1}B$ is $\mathbb{R}$-Cartier).

Theorem 4.5 ([5, Thorem 1.6])**.** *Fix positive real numbers ϵ, δ. Then, the set of pairs (X, B) satisfying*

(1) (X, B) is an ϵ-lc log Calabi–Yau pair of dimension 3,
(2) X is rationally connected,
(3) $B > 0$, and the coefficients of B are at least δ

forms a log bounded family modulo flops.

Definition 4.6. Let (X, B) be an lc pair and $Z \subset X$ an irreducible closed subset with η_Z the generic point of Z. The *minimal log discrepancy* of (X, B) over Z is defined as

$$\mathrm{mld}_Z(X, B) = \inf\{a(E, X, B) \mid \mathrm{center}_X(E) \subset Z\},$$

and the minimal log discrepancy of (X, B) at η_Z is defined as

$$\mathrm{mld}_{\eta_Z}(X, B) = \inf\{a(E, X, B) \mid \mathrm{center}_X(E) = Z\}.$$

For simplicity, we just write $\mathrm{mld}(X, B)$ instead of $\mathrm{mld}_X(X, B)$.

Theorem 4.7 ([5, Thorem 1.7])**.** *Fix $0 < c < 1$. Let $\mathcal{D}$ be the set of varieties X such that*

(1) X is a rationally connected Calabi–Yau 3-fold, and
(2) $0 < \mathrm{mld}(X) < c$.

Then $\mathcal{D}$ is bounded modulo flops.

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