博士論文

論文題目 Studies of compactifications of affine homology 3-cells into del Pezzo fibrations
(Del Pezzo ファイブレーションへのアフィンホモロジー 3-胞体のコンパクト化の研究)

氏　　名 長岡　大

Studies of compactifications of affine homology 3-cells into del Pezzo fibrations

Masaru Nagaoka

Contents

Preface

This thesis concerns complex projective compactifications of smooth affine 3-folds with the same homology rings as that of the affine 3-space.

For an affine variety U, the pair (X, D) of a smooth proper variety X and its reduced effective divisor D is called *a compactification of* U when the complement $X \backslash D$ is algebraically isomorphic to U. F. Hirzebruch raised the problem to classify all the compactifications (X, D) of the affine n-space $\mathbb{A}^n$ with second Betti number $B_2(X) = 1$ in his problem list [**Hir54**]. Here we call this problem *the Hirzebruch problem*. This problem is trivial when $n = 1$ because the projective line $\mathbb{P}^1$ is the unique rational smooth proper curve, and when $n = 2$, it was solved by R. Remmert and T. Van de Ven [**RvdV60**]. Also by the contribution of M. Furushima, N. Nakayama, Th. Peternell, Y. Prokhorov and M. Schneider [**Fur86**, **Fur90**, **Fur93a**, **Fur93b**, **FN89a**, **FN89b**, **Pet89**, **PS88**, **Pro91**], this problem was solved in the projective case when $n = 3$. We note that the ambient space X is a Fano variety in the projective case since it is rational with $B_2 = 1$. There is also a generalization of the Hirzebruch problem with $B_2 \geq 2$, which is studied by several authors [**Mor73**, **MS90**, **Kis05**, **Nag18**].

In this thesis, we will study three problems which originated from the Hirzebruch problem. It is worth pointing out that up to the present all of them have been investigated only when the dimension is at most 2 or when the ambient spaces are Fano. For this reason, we will discuss three problems when ambient spaces are del Pezzo fibrations, which form a building block of the minimal model program for 3-folds as well as Fano 3-folds.

The first one concerns characterizations of $\mathbb{A}^n$. For $n \in \mathbb{Z}_{>0}$, *an affine homology n-cell* is a smooth affine variety of dimension n with the same homology ring as that of $\mathbb{A}^n$. By several authors [**Ram71**, **KR97**, **tDP90**], it is known that there are many affine homology n-cells not isomorphic to $\mathbb{A}^n$. One of natural questions about affine homology n-cells is how to characterize $\mathbb{A}^n$ among them. For this question, Furushima [**Fur00**] pointed out that $\mathbb{A}^3$ can be characterized as the affine homology 3-cell which is compactified into a smooth Fano 3-fold with $B_2 = 1$. Chapter 2 of this thesis gives another characterization of $\mathbb{A}^3$ via compactifications into quadric fibrations, i.e., del Pezzo fibrations of degree 8.

The second problem is a construction of standard maps preserving $\mathbb{A}^n$ from compactifications of $\mathbb{A}^n$ to a standard one. In [**Mor73**], S. Mori introduced three kinds of explicit birational transformations between Hirzebruch surfaces, and showed that any compactifications of $\mathbb{A}^2$ into Hirzebruch surfaces are constructed from the standard compactification $(\mathbb{P}^2, \mathbb{P}^1)$ of $\mathbb{A}^2$ with finite composition of these birational transformations. Chapter 2 of this thesis deals a construction of standard maps for a certain family of compactifications of $\mathbb{A}^3$ into quadric fibrations, where the standard compactification is the pair $(\mathbb{P}^3, \mathbb{P}^2)$.

The third problem is the Hirzebruch problem for the affine n-space $\mathbb{G}_a^n$ equipped with the additive group structure. *A* $\mathbb{G}_a^n$*-variety* is defined to be a variety with a $\mathbb{G}_a^n$-action whose dense orbit is isomorphic to $\mathbb{G}_a^n$. The study of smooth projective $\mathbb{G}_a^n$-varieties is started by B. Hassett and Y. Tschinkel [**HT12**], and they classified smooth projective $\mathbb{G}_a^n$-varieties with $B_2 = 1$ when $n \leq 3$, on which situation $\mathbb{G}_a^n$-varieties are Fano. After that, smooth Fano $\mathbb{G}_a^n$-varieties are studied by several authors [**HM18, FM19**]. Chapter 3 discusses the existences of $\mathbb{G}_a^3$*-structures*, i.e., $\mathbb{G}_a^3$-actions which give structures of $\mathbb{G}_a^3$-varieties, on del Pezzo fibrations.

This thesis consists of three chapters.

Chapter 1 is the preliminary chapter; we recall definitions and basic properties of del Pezzo fibrations and certain elementary links, which we will use throughout this thesis.

Chapter 2 deals with compactifications of affine homology 3-cells into quadric fibrations such that the boundary divisors contain fibers. In this chapter, we show that all such affine homology 3-cells are isomorphic to $\mathbb{A}^3$, and give explicit birational maps from these compactifications to $\mathbb{P}^3$ preserving $\mathbb{A}^3$ using the technique of elementary links.

Chapter 3 deals with $\mathbb{G}_a^3$-varieties with del Pezzo fibration structures. In this chapter, we show that del Pezzo fibrations admit $\mathbb{G}_a^3$-structures if and only if they are $\mathbb{P}^2$-bundles.

Chapter 2 and 3 are based on papers [**Nag19b**] and [**Nag19a**] respectively.

Acknowledgement

First of all, the author would like to express his sincere gratitude to my supervisor Professor Hiromichi Takagi for his encouragement, comments, and suggestions during the master course and doctoral course. The author also wishes to express his deep gratitude to Professor Keiji Oguiso for his encouragement and supervising in the author's last two years of doctoral course.

The author wishes to express his deep gratitude to Professor Takashi Kishimoto for his helpful comments and suggestions. The author also wishes to express his deep gratitude to Professor Adrien Dubouloz for his helpful comments, warm encouragement during my stay in University of Bourgogne, and for drawing his attention to [**HM18**].

The author is grateful to Takeru Fukuoka, Katsuhisa Furukawa, Akihiro Kanemitsu, Shugo Matsumoto and Takayuki Miura, for listening the author's talks in the weekly seminar and giving valuable comments. Especially, the author would like to express his gratitude to Takeru Fukuoka for several helpful discussions concerning the technique of elementary links.

This work was supported by JSPS KAKENHI Grant Number JP19J14397 and the Program for Leading Graduate Schools, MEXT, Japan.

Finally, the author would like to express his gratitude to his family for their support and warm encouragement.

Notation and Convention

This chapter is devoted to fixing the notation, which will be used throughout this thesis.

CONVENTIONS. We work over the field of complex numbers $\mathbb{C}$. For a surjective morphism $f\colon X \to Y$ and divisors D_1, D_2 on X, the notation $D_1 \sim_Y D_2$ means that $D_1 - D_2$ is linearly equivalent to the pullback of some divisor on Y.

NOTATION. We use the following notation:

- $\mathbb{Q}^3$: the smooth quadric hypersurface in $\mathbb{P}^4$. $\mathcal{O}_{\mathbb{Q}^3}(1) \coloneqq \mathcal{O}_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}$.
- $\mathbb{Q}^2_0$: the quadric cone in $\mathbb{P}^3$. $\mathcal{O}_{\mathbb{Q}^2_0}(1) \coloneqq \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{Q}^2_0}$.
- $\mathbb{F}_d$: the Hirzebruch surface of degree d.
- f_d: a fiber of $\mathbb{F}_d$.
- Σ_d: the minimal section of $\mathbb{F}_d$.
- $\mathbb{P}_X(\mathcal{E}) \coloneqq \mathbf{Proj}_{\mathcal{O}_X} \oplus_{m\geq 0} \operatorname{Sym}^m(\mathcal{E})$: the projectivization of a locally free sheaf on a variety X. We often write it $\mathbb{P}(\mathcal{E})$ for short.
- $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$: the tautological bundle of a projective bundle $\mathbb{P}(\mathcal{E})$.
- $\xi_{\mathbb{P}(\mathcal{E})}$: the tautological divisor of a projective bundle $\mathbb{P}(\mathcal{E})$.
- $\mathbb{F}(a,b,c) \coloneqq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c))$.
- E_f: the exceptional divisor of a birational morphism f.
- $\operatorname{Sing} X$: the singular locus of a variety X.
- $Y_{\widetilde{X}}$: the strict transformation of a closed subscheme Y of a normal variety X in a birational model $\widetilde{X}$ of X.
- $\chi_{\mathrm{top}}(X)$: the topological Euler number of a topological space X.
- $h^{i,j}(X)$: the dimension of $H^i(X, \wedge^j \Omega_X)$ of a smooth projective 3-fold X.
- $p_a(C)$: the arithmetic genus of a smooth projective curve C.
- $\operatorname{Supp} Y$: the support of a closed subscheme Y of an ambient variety.
- $N_Y X$: the normal bundle of a smooth subvariety Y of a smooth variety X.
- $\Lambda_{\mathrm{eff}}(X)$: the cone of effective Cartier divisors on a projective variety X.

CHAPTER 1

Preliminaries

In this chapter, we compile definitions and some facts on del Pezzo fibrations and elementary links, which will be needed in Chapter 2 and 3.

1.1. Del Pezzo fibrations

In this thesis, we employ the following definition for del Pezzo fibrations.

DEFINITION 1.1.1. *A del Pezzo fibration* is an extremal contraction of relative Picard number one from a smooth projective 3-fold to a smooth projective curve. *The degree of a del Pezzo fibration* is the anti-canonical volume of a general fiber, which is a del Pezzo surface. *A quadric fibration* is a del Pezzo fibration of degree 8.

We will use the following theorem without any mentions.

THEOREM 1.1.2 ([**Mor82**, Theorem 3.2, 3.5]). *Let $f\colon X \to C$ be a del Pezzo fibration of degree d. Then the following holds.*

(1) $d \leq 9$.
(2) *We have an exact sequence* $0 \to \operatorname{Pic} C \xrightarrow{f^*} \operatorname{Pic} X \xrightarrow{(\ \cdot l)} \mathbb{Z} \to 0$, *where l is a line in a general f-fiber, which is a smooth del Pezzo surface.*
(3) *Each f-fiber is irreducible and reduced.*
(4) *If $d = 9$, then f is a $\mathbb{P}^2$-bundle.*
(5) *If $d = 8$, then X is embedded in a $\mathbb{P}^3$-bundle $f\colon \mathbb{F} \to C$ as a member of $|2\xi_{\mathbb{F}} + f^*L|$ for some $L \in \operatorname{Pic} C$. In particular, any f-fiber is isomorphic to either $\mathbb{F}_0$ or $\mathbb{Q}_0^2$.*

1.2. Definition of elementary links

In this thesis, we define elementary links as follows.

DEFINITION 1.2.1. Let X be a smooth 3-fold and $\sigma\colon X \to C$ be an extremal contraction of relative Picard number one. Let $r \subset X$ be a smooth curve (or $x \in X$ be a point). Denote by $\varphi\colon \widetilde{X} \to X$ the blow-up of X with center r (resp. with center x). We assume that $-K_{\widetilde{X}}$ is $(\sigma \circ \varphi)$-ample. Then there exists the unique contraction $\psi\colon \widetilde{X} \to Y$ of the other $K_{\widetilde{X}}$-negative ray in $\overline{\mathrm{NE}}(\widetilde{X}/C)$. Let $\tau\colon Y \to C$ be the induced morphism.

(1.2.0.1)
$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[d, "\sigma"'] & & Y \arrow[d, "\tau"] \\ C \arrow[rr, equal] & & C. \end{tikzcd}$$

When ψ is birational, we call the diagram (1.2.0.1) *the elementary link with center along* r (resp. *at* x). In this thesis the pushforward of the φ-exceptional divisor by ψ is called *the exceptional divisor of the elementary link.* We write it $X \leftarrow \widetilde{X} \rightarrow Y$ or $X \dashrightarrow Y$ for short when the base variety C is obvious.

We note that this is a particular case of elementary links of type II in [**Cor95**, Definition 3.4] and that the exceptional divisor of the elementary link is actually a divisor by [**Cor95**, Proposition 3.5].

In the following situation, the assumption of Definition 1.2.1 is satisfied, and hence we can construct an elementary link. For the detail, see §1.3–1.6.

- σ is the blow-up at a point and τ is the blow-up along a curve.
- σ is a $\mathbb{P}^2$-bundle and r is a linear subspace of a fiber [**Mar73**].
- σ is a quadric fibration and r is a section [**D'S88**].
- σ is a quadric fibration and r is a ruling in a σ-fiber [**HT12**].

1.3. Elementary links between blow-ups

First we check that the change of the order of the blow-ups at a point and along a curve does not change the output.

Lemma 1.3.1. *Let X be a smooth 3-fold, $C \subset X$ a smooth irreducible curve and $p \in C$ a point. Denote by $\varphi_1\colon X_1 \to X$ the blow-up at p and by $\varphi_2\colon X_2 \to X$ the blow-up along C. Let C_1 be the strict transform of C in X_1 and $f_p \coloneqq \varphi_2^{-1}(p)$. Then the following holds:*

(1) $\mathrm{Bl}_{C_1}(\mathrm{Bl}_p X) \cong \mathrm{Bl}_{f_p}(\mathrm{Bl}_C X)$ *over X.*
(2) $N_C X \cong N_{C_1} X_1 \otimes \mathcal{O}_{C_1}(p_1)$, *where $p_1 \coloneqq E_{\varphi_1}|_{C_1}$.*

Proof. (1): Let $\psi_1\colon \widetilde{X} \to X_1$ be the blow-up along C_1 and $\chi \coloneqq \varphi_1 \circ \psi_1$. Let E_p be the strict transform of E_{φ_1} in $\widetilde{X}$. Then we have $-K_{\widetilde{X}} \sim_X -2E_p - E_{\psi_1}$.

Consider the divisor $-E_p - E_{\psi_1}$. Each irreducible curve $l \subset \widetilde{X}$ contracted by χ is either a fiber of $\psi_1|_{E_{\psi_1}}\colon E_{\psi_1} \to C_1$ or a curve in E_p. The former satisfies $(l \cdot -E_p - E_{\psi_1}) = 1$ and the latter satisfies $(l \cdot -E_p - E_{\psi_1}) = (l \cdot f_1)_{E_p} \geq 0$ regarding E_p as $\mathbb{F}_1$. Hence $-E_p - E_{\psi_1}$ is a χ-nef divisor and $R \coloneqq (-E_p - E_{\psi_1})^{\perp} \cap \overline{\mathrm{NE}}(\widetilde{X}/X)$ is generated by f_1 in $E_p \cong \mathbb{F}_1$.

Since $(-K_{\widetilde{X}} \cdot f_1) = (-E_p \cdot f_1) > 0$, there is the contraction morphism $\psi_2\colon \widetilde{X} \to X_2'$ of the extremal ray R. Let $\varphi_2'\colon X_2' \to X$ be the induced morphism. Since the centers of both ψ_2 and φ_2' is a curve, each of them is the blow-up along a smooth curve by [**Mor82**, Theorem 3.3]. Hence we have $\varphi_2' = \varphi_2$ and ψ_2 is the blow up of X_2 along f_p, which proves (1).
(2): It holds that $E_p = E_{\psi_2}$ and $\psi_{2*}E_{\psi_1} = E_{\varphi_2}$ by (1). Hence we have:

$$
\begin{aligned}
(\psi_2|_{E_{\psi_1}})^*\mathcal{O}_{\mathbb{P}(N_C X^\vee)}(1) &\cong \mathcal{O}_{E_{\psi_1}}(-\psi_2^* E_{\varphi_2}) \\
&\cong \mathcal{O}_{E_{\psi_1}}(-E_{\psi_1}) \otimes \mathcal{O}_{E_{\psi_1}}(-E_{\psi_2}) \\
&\cong \mathcal{O}_{\mathbb{P}(N_{C_1} X_1^\vee)}(1) \otimes \mathcal{O}_{E_{\psi_1}}(-\psi_1^* E_{\varphi_1}). \\
&\cong \mathcal{O}_{\mathbb{P}(N_{C_1} X_1^\vee)}(1) \otimes (\psi_{1|_{E_{\psi_1}}})^*\mathcal{O}_{C_1}(-E_{\varphi_1}).
\end{aligned}
\tag{1.3.0.1}
$$

Pushing forward (1.3.0.1) by $\chi|_{E_{\psi_1}}$, we get $N_C X \cong N_{C_1} X_1 \otimes \mathcal{O}_{C_1}(p_1)$. □

1.4. Elementary links between $\mathbb{P}^2$-bundles

Elementary links between projective bundles are considered by M. Maruyama [**Mar73**] in any dimension. Here we restrict our attention to $\mathbb{P}^2$-bundles.

LEMMA 1.4.1 ([**Mar73**, Theorem 1.3]). *Let $p\colon P \to C$ be a $\mathbb{P}^2$-bundle and $L \subset P$ a n-dimensional linear subspace of a p-fiber ($n \le 1$). Let $\varphi\colon \widetilde{P} = \mathrm{Bl}_L P \to P$ be the blow-up along L. Then*

(1) *There exists a divisorial contraction $\psi\colon \widetilde{P} \to P'$ over C such that the induced morphism $p'\colon P' \to C$ is a $\mathbb{P}^2$-bundle and ψ is the blow-up along a $(1-n)$-dimensional linear subspace L' of a p'-fiber.*
(2) *The exceptional divisor E_ψ is the strict transform of the p-fiber containing L.*
(3) *For an associated vector bundle $\mathcal{E}$ to $p\colon P \to C$, we can take a vector bundle $\mathcal{E}'$ associated to $p'\colon P' \to C$ such that $\deg \mathcal{E}' = \deg \mathcal{E} - (n+1)$.*

(1.4.0.1)
$$\begin{tikzcd} & \mathrm{Bl}_L P = \widetilde{P} = \mathrm{Bl}_{L'} P' \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ P \arrow[d, "p"'] & & P' \arrow[d, "p'"] \\ C \arrow[rr, equal] & & C. \end{tikzcd}$$

COROLLARY 1.4.2. *We follow the notation of Lemma 1.4.1. Suppose that $C \cong \mathbb{P}^1$. Let F be a p-fiber and D a sub $\mathbb{P}^1$-bundle of P. Take $a \in \mathbb{Z}$ such that $D \sim \xi_P + aF$. Then the following hold:*

$$\begin{cases} D_{P'} \sim \xi_{P'} + (a+1)F \text{ and } L' \subset D_{P'} & \text{if } L \not\subset D, \\ D_{P'} \sim \xi_{P'} + aF \text{ and } L' \not\subset D_{P'} & \text{if } L \subset D. \end{cases} \tag{1.4.0.2}$$

PROOF. By the canonical bundle formula, we have:

$$-K_P \sim 3\xi_P - (\deg \mathcal{E} - 2)F, \tag{1.4.0.3}$$
$$-K_{P'} \sim 3\xi_{P'} - (\deg \mathcal{E} - (3+n))F. \tag{1.4.0.4}$$

Also it holds that

$$-K_{\widetilde{P}} \sim \varphi^*(-K_P) - (2-n)E_\varphi \sim \psi^*(-K_{P'}) - (n+1)E_\psi. \tag{1.4.0.5}$$

Combining (1.4.0.3)–(1.4.0.5) and $E_\varphi \sim F - E_\psi$, we have $3\varphi^*\xi_P \sim 3\psi^*\xi_{P'} + 3(F - E_\psi)$. Since $\mathrm{Pic}\,\widetilde{P}$ is torsion-free, it holds that:

$$\varphi^*\xi_P \sim \psi^*\xi_{P'} + F - E_\psi. \tag{1.4.0.6}$$

On the other hand, we have:

$$D_{\widetilde{P}} \sim \begin{cases} \varphi^*\xi_P + aF & \text{if } L \not\subset D, \\ \varphi^*\xi_P + aF - E_\varphi & \text{if } L \subset D. \end{cases} \tag{1.4.0.7}$$

Combining (1.4.0.6), (1.4.0.7) and $E_\varphi \sim F - E_\psi$, we have:

$$D_{\widetilde{P}} \sim \begin{cases} \psi^*\xi_{P'} + (a+1)F - E_\psi & \text{if } L \not\subset D, \\ \psi^*\xi_{P'} + aF & \text{if } L \subset D. \end{cases} \tag{1.4.0.8}$$

By pushing forward (1.4.0.8) by ψ, we have the assertion. $\square$

1.5. Elementary links from quadric fibrations to $\mathbb{P}^2$-bundles

H. D'Souza [**D'S88**] showed the existence of elementary links from quadric fibrations to $\mathbb{P}^2$-bundles. The precise statement is as follows.

LEMMA 1.5.1 ([**D'S88**, (2.7.3)], [**Fuk18**, Proposition 3.1]). *Let $q\colon Q \to C$ be a quadric fibration and $s \subset Q$ a q-section. Let $\varphi\colon \widetilde{Q} = \mathrm{Bl}_s Q \to Q$ be the blow-up of Q along s. Then there exists a divisorial contraction $\psi\colon \widetilde{Q} \to P$ over C such that the induced morphism $p\colon P \to C$ is a $\mathbb{P}^2$-bundle and ψ is the blow-up along a smooth connected p-bisection $B \subset P$.*

$$\begin{tikzcd} & \mathrm{Bl}_s Q = \widetilde{Q} = \mathrm{Bl}_B P \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ Q \arrow[d, "q"'] & & P \arrow[d, "p"] \\ C \arrow[rr, equal] & & C. \end{tikzcd} \tag{1.5.0.1}$$

Moreover, let H_Q be a q-ample divisor with $2H_Q \sim_C -K_Q$ and H_P a p-ample divisor such that $3H_P \sim_C -K_P$. Then:

(1) *It holds that $E_\psi \sim_C \varphi^* H_Q - 2E_\varphi$ and $E_\varphi \sim_C \psi^* H_P - E_\psi$.*
(2) *The branched locus of $p|_B$ coincides with the closed set*

$$\Sigma := \{t \in C \mid q^{-1}(t) \textit{ is singular }\}. \tag{1.5.0.2}$$

(3) *It holds that $(-K_Q)^3 = 40 - (8p_a(B) + 32p_a(C))$.*

LEMMA 1.5.2. *We follow the notation of Lemma 1.5.1. Let $E := \psi_*(E_\varphi)$. Suppose that H_Q is a prime divisor containing s and assume that H_Q is normal. Then $(H_Q)_{\widetilde{Q}} \sim_C \psi^* H_P$.*

Moreover, when $H_P = (H_Q)_P$, the following holds for $t \in C$.

(1) *If $t \notin \Sigma$, then $(q|_{H_Q})^{-1}(t)$ is*

$$\begin{aligned} \text{smooth} &\iff p^{-1}(t) \cap B \cap H_P = \emptyset. \\ \text{reducible} &\iff p^{-1}(t) \cap B \cap H_P \neq \emptyset. \end{aligned}$$

(2) *If $t \in \Sigma$, then $(q|_{H_Q})^{-1}(t)$ is*

$$\begin{aligned} \text{smooth} &\iff p^{-1}(t) \cap B \cap H_P = \emptyset. \\ \text{reducible} &\iff p^{-1}(t) \cap B \cap H_P \neq \emptyset \text{ and } E|_{p^{-1}(t)} \neq H_P|_{p^{-1}(t)}. \\ \text{non-reduced} &\iff E|_{p^{-1}(t)} = H_P|_{p^{-1}(t)}. \end{aligned}$$

PROOF. The first assertion follows from Lemma 1.5.1 (1). we take the following diagram as the base change of (1.5.0.1) at $t \in C$:

$$\begin{tikzcd} & \mathrm{Bl}_{s_t} Q_t = \widetilde{Q_t} = \mathrm{Bl}_{B_t} P_t \arrow[dl, "\varphi_t"'] \arrow[dr, "\psi_t"] & \\ Q_t & & P_t. \end{tikzcd} \tag{1.5.0.3}$$

Write $G_t := (\varphi_* E_\psi)|_{Q_t}$, $H_t := H_Q|_{Q_t}$, $s_t := s|_{Q_t}$ and $B_t := B|_{P_t}$.

(1): In this case we have $Q_t \cong \mathbb{F}_0$. By lemma 1.5.1 (1), it holds that $H_t \sim G_t \sim \Sigma_0 + f_0$ and G_t is the union of two rulings containing s_t. Since H_t is smooth if and only if H_t is irreducible, we only have to show that H_t is smooth $\Rightarrow (H_t)_{P_t} \cap B_t = \emptyset \Rightarrow H_t$ is irreducible.

Let $G_t = G_1 + G_2$ be the irreducible decomposition. Note that $(H_t \cdot G_i)_{Q_t} = 1$ for $i = 1, 2$. Suppose that H_t is smooth. Then $H_t \cap G_i = s_t$ scheme-theoretically for $i = 1, 2$. Hence we have $(H_t)_{\widetilde{Q}_t} \cap E_{\psi_t} = \emptyset$ and $(H_t)_{P_t} \cap B_t = \emptyset$. On the other hand, if $(H_t)_{P_t} \cap B_t = \emptyset$, then $(H_t)_{\widetilde{Q}_t} \cong (H_t)_{P_t}$ is irreducible and so is H_t, and (1) is proved.

(2): In this case, ψ_t is a weighted blow-up and hence $Q_t \cong \mathbb{Q}_0^2$. Since $(s \cdot Q_t)_Q = 1$, the point s_t is not the vertex of $Q_t \cong \mathbb{Q}_0^2$. By lemma 1.5.1 (1), it holds that $H_t \sim G_t \sim \mathcal{O}_{\mathbb{Q}_0^2}(1)$, and we have $G_t = 2l'$, where l' is the unique ruling of $\mathbb{Q}_0^2$ containing s_t.

Suppose that H_t is smooth. Since $H_t \cap l' = s_t$ scheme-theoretically, we have $(H_t)_{\widetilde{Q}_t} \cap E_{\psi_t} = \emptyset$ and hence $(H_t)_{P_t} \cap B_t = \emptyset$.

Suppose that H_t is reducible. Then H_t is the union of two distinct ruling of $Q_t \cong \mathbb{Q}_0^2$. Since $s_t \in H_t$, there exists a ruling $l \neq l'$ of Q_t such that $H_t = l + l'$. Since H_t is smooth at s_t, it holds that $(H_t)_{\widetilde{Q}_t} = l_{\widetilde{Q}_t} + E_{\psi_t}$. Hence $(H_t)_{P_t} = l_{P_t}$ contains $\operatorname{Supp} B_t$ but $(H_P)|_{P_t} \neq E|_{P_t}$

Suppose that H_t is non-reduced. Then $\operatorname{Supp} H_t$ is a ruling of $Q_t \cong \mathbb{Q}_0^2$. Since $s_t \in H_t$, we have $H_t = G_t$ and hence $H_P|_{P_t} = E|_{P_t}$.

Combining these results, we complete the proof. $\square$

1.6. Elementary links between quadric fibrations

B. Hassett and Y. Tschinkel [**HT12**] considered elementary links between quadric fibrations with center a ruling in a smooth fiber. We can prove that a similar elementary link appears in the case of a singular fiber as follows.

LEMMA 1.6.1. *Let $q\colon Q \to C$ be a quadric fibration and l a ruling of a q-fiber. Let $\varphi\colon \widetilde{Q} = \operatorname{Bl}_l Q \to Q$ be the blow-up of Q along l. Then there exists a divisorial contraction $\psi\colon \widetilde{Q} \to Q'$ over C such that the induced morphism $q'\colon Q \to C$ is a quadric fibration and ψ is the blow-up along a ruling l' of a q'-fiber.*

(1.6.0.1)
$$\begin{tikzcd} & \operatorname{Bl}_l Q = \widetilde{Q} = \operatorname{Bl}_{l'} Q' \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ Q \arrow[d, "q"'] & & Q' \arrow[d, "q'"] \\ C \arrow[rr, equal] & & C. \end{tikzcd}$$

PROOF. Let $F \subset Q$ be the q-fiber containing l. When F is smooth, then the assertion is already shown by [**HT12**, §5]. Hence we may assume that $F \cong \mathbb{Q}_0^2$.

First we calculate $N_l Q$. Let $v \in F$ be the vertex of $\mathbb{Q}_0^2$ and $h\colon Q_1 \to Q$ the blow-up at v. Let F_1 (resp. l_1) be the strict transform of F (resp. l) in Q_1. Then l_1 is a fiber of $F_1 \cong \mathbb{F}_2$ and $F_1 \sim h^*F - 2E_h$. Since $N_{l_1} F_1 \cong \mathcal{O}_{l_1}$ and $(N_{F_1} Q_1)|_{l_1} \cong \mathcal{O}_{l_1}((F_1 \cdot l_1)) \cong \mathcal{O}_{l_1}(-2)$, we have $N_{l_1} Q_1 \cong \mathcal{O}_{l_1} \oplus \mathcal{O}_{l_1}(-2)$ by the normal bundle sequence. Hence $N_l Q = \mathcal{O}_l(1) \oplus \mathcal{O}_l(-1)$ by Lemma 1.3.1 (2)

Therefore we have $E_\varphi \cong \mathbb{F}_2$ and $E_\varphi|_{E_\varphi} \sim -(\Sigma_2 + f_2)$. Since $F_1 \cong \mathbb{F}_2$, it follows that $F_{\widetilde{Q}} \cong \mathbb{F}_2$ from Lemma 1.3.1 (1). Since $(E_\varphi + F_{\widetilde{Q}})|_{E_\varphi} = \varphi^* F|_{E_\varphi} \sim 0$, we have $F_{\widetilde{Q}}|_{E_\varphi} \sim \Sigma_2 + f_2$.

Hence $F_{\widetilde{Q}}|_{E_\varphi}$ is the sum of $C_1 \sim \Sigma_2$ and $C_2 \sim f_2$. On the other hand, in $F_{\widetilde{Q}}$, we have $C_1 \sim f_2$ and $C_2 \sim \Sigma_2$ because $F_{\widetilde{Q}}$ is the minimal resolution of F. By symmetry of E_φ and $F_{\widetilde{Q}}$, there is the blow-down of $F_{\widetilde{Q}}$ as desired. $\square$

The following is the key to proving Theorem 2.1.6 (1).

LEMMA 1.6.2. *Let $q\colon Q \to C$ be a quadric fibration and $D_h \subset Q$ a prime divisor such that $2D_h \sim_C -K_Q$. Suppose that D_h is non-normal. Let R be the 1-dimensional component of $\operatorname{Sing} D_h$. Then:*

(1) *R is a q-section.*

(2) *If we take the elementary link $Q \xleftarrow{\varphi} \widetilde{P} \xrightarrow{\psi} P$ with center along R, then we have $D_h = (E_\psi)_Q$. In particular, we have $R = \operatorname{Sing} D_h$.*

PROOF. (1): Let r be an irreducible component of R. To seek a contradiction, assume that $q(r)$ is a point. Take a q-fiber F containing r. Since D_h is singular along r, the restriction $D_h|_F$ is non-reduced along r. If $F \cong \mathbb{F}_0$, then $D_h|_F$ is reduced since $D_h|_F \sim \Sigma_0 + f_0$, a contradiction. Therefore $F \cong \mathbb{Q}_0^2$ and there is a ruling $r \subset F$ such that $D_h|_F = 2r$.

Let $\chi\colon \widetilde{Q} \to Q$ be the blow-up along r. In the proof of Lemma 1.6.1, we have shown that $F_{\widetilde{Q}} \cong \mathbb{F}_2$ and $F_{\widetilde{Q}}|_{F_{\widetilde{Q}}} \sim -E_\chi|_{F_{\widetilde{Q}}} \sim -(\Sigma_2 + f_2)$. Hence we have:

$$\begin{aligned}
(1.6.0.2) \qquad ((D_h)_{\widetilde{Q}}^2 \cdot F_{\widetilde{Q}}) &= ((\chi^* D_h - 2E_\chi)^2 \cdot (\chi^* F - E_\chi)) \\
&= (D_h^2 \cdot F)_Q - 4(D_h \cdot r)_Q - 4(F \cdot r)_Q - 4E_\chi^3 = -2, \\
(1.6.0.3) \qquad ((D_h)_{\widetilde{Q}} \cdot F_{\widetilde{Q}}^2) &= ((\chi^* D_h - 2E_\chi) \cdot (\chi^* F - E_\chi)^2) \\
&= (D_h \cdot F^2)_Q - (D_h \cdot r)_Q - 4(F \cdot r)_Q - 2E_\chi^3 = -1.
\end{aligned}$$

Take $a, b \in \mathbb{Z}$ such that $(D_h)_{\widetilde{Q}}|_{F_{\widetilde{Q}}} \sim a\Sigma_2 + bf_2$. By (1.6.0.2) and (1.6.0.3), we have $-2a^2 + 2ab = -2$ and $a - b = -1$. Hence $(a, b) = (-1, 0)$, which is absurd.

Therefore r dominates C. Let F be a smooth q-fiber. Then we have $\emptyset \neq \operatorname{Supp}(R \cap F) \subset \operatorname{Sing}(D_h|_F)$. Since $D_h|_F \sim \Sigma_0 + f_0$, it follows that $\operatorname{Supp}(R \cap F) = \operatorname{Sing}(D_h|_F)$ is a point and hence R is a q-section.

(2): By Lemma 1.5.1 (1), $(E_\psi)_Q$ is singular along R. For each smooth q-fiber F, there is the unique member of $|\Sigma_0 + f_0|$ singular at $\operatorname{Supp}(R \cap F)$. Hence $D_h|_F = (E_\psi)_Q|_F$, and the first assertion follows. Since φ is the blow-up along R and E_ψ is smooth, the last assertion follows. $\square$

CHAPTER 2

Compactifications of affine homology 3-cells into quadric fibrations

2.1. Introduction to Chapter 2

In this chapter we are interested in compactifications of affine homology n-cells into smooth projective n-fold. We recall that *a compactification of an affine variety U* is a pair (X, D) of smooth proper variety X and its reduced effective divisor D such that the complement $X \setminus D$ is algebraically isomorphic to U. Also by *an affine homology n-cell* we mean a smooth affine n-fold U such that $H_i(U, \mathbb{Z}) = 0$ for $i > 0$. The main problem is the following, which is based on the characterization of $\mathbb{A}^3$ among all the affine homology 3-cells via compactifications into Fano 3-folds by Furushima [**Fur00**].

Problem 2.1.1. *Let $f\colon X \to C$ be an extremal contraction of relative Picard number one from a smooth projective n-fold X to a smooth projective curve C. Let $U \subset X$ be an open subscheme.*

1. *If U is an affine homology n-cell, then is it isomorphic to $\mathbb{A}^n$?*
2. *If U is isomorphic to $\mathbb{A}^n$, then can we construct an explicit birational map from X to a compactification of $\mathbb{A}^n$ with $B_2 = 1$ preserving $U \cong \mathbb{A}^n$?*

In this problem, we set not only $\mathbb{P}^n$ but also all compactifications of $\mathbb{A}^n$ with $B_2 = 1$ as the target of birational maps preserving $\mathbb{A}^n$. It is because there is a copy of $\mathbb{A}^3$ in the quintic del Pezzo 3-fold which we can regard naturally as an affine modification (for the detail, see [**KZ99**]) of an another copy of $\mathbb{A}^3$ in $\mathbb{P}^3$ via the birational map constructed in [**Fur00**].

We note that even when $n = 2$, Problem 2.1.1 (1) have a negative answer in general. In fact, T. tom Dieck and T. Petrie [**tDP90**] showed that there are infinitely many contractible affine surfaces of logarithmic Kodaira dimension one in the blow-up of $\mathbb{P}^2$ at a point. However, if we assume the following condition, the problem have an affirmative answer in the case where $n = 2$.

Definition 2.1.1. Let $f\colon X \to C$ and U be as in Problem 2.1.1. Let $D \coloneqq X \setminus U$ be the boundary divisor. We say that (X, D, f) is *a compactification of U compatible with f* if D contains a f-fiber. When $D_f \subset D$ is a f-fiber and $D_h \subset D$ is the other components, we also call (X, D_h, D_f) *a compactification of U compatible with f*.

By [**vdV62**, Proposition 2.1] and the Poincare duality, D_h in the setting of Definition 2.1.1 is a prime divisor. Suppose that (X, D_h, D_f) is a compactification of homology 2-cell U compatible with $\mathbb{P}^1$-bundle. By [**Fuj82**, Corollary 1.20], it holds that D_h is a f-section. Hence we have $U \cong \mathbb{A}^2$ since $f|_U$ is an $\mathbb{A}^1$-bundle over $\mathbb{A}^1$.

Problem 2.1.1 (2) was solved when $n = 2$ by Mori [**Mor73**]. He introduced three kinds of explicit birational transformations preserving $\mathbb{A}^2$ between Hirzebruch surfaces, which are called J-, R-, and L-transform. He solved the problem as in the following theorem:

THEOREM 2.1.2. *Let $f\colon X \to \mathbb{P}^1$ be a $\mathbb{P}^1$-bundle and D a reduced effective divisor on X such that $X \setminus D \cong \mathbb{A}^2$.*

(1) *There exists a compactification (X_1, D_1, f_1) of $\mathbb{A}^2$ compatible with a $\mathbb{P}^1$-bundle $f_1\colon X_1 \to \mathbb{P}^1$ and a birational map $g_1\colon X \dashrightarrow X_1$ preserving $\mathbb{A}^2$ which is a finite composition of J-, R-, and L-transforms.*
(2) *Let $X_2 := \mathbb{F}_1$ be a Hirzebruch surface of degree 1 with the $\mathbb{P}^1$-bundle structure f_2. Let D_2 be the union of an f_2-fiber and the minimal section. Then there exists a birational map $g_2\colon X_1 \dashrightarrow X_2$ preserving $\mathbb{A}^2$ which is a finite composition of elementary transformations of $\mathbb{P}^1$-bundles.*

Summarizing, we have the following diagram of birational maps preserving $X \setminus D \cong \mathbb{A}^2$:

$$\begin{tikzcd}
(X, D) \arrow[r, dashed, "g_1"] \arrow[d, "f"'] & (X_1, D_1) \arrow[r, dashed, "g_2"] \arrow[d, "f_1"'] & (X_2, D_2) \arrow[r, "g_3"] \arrow[d, "f_2"'] & (\mathbb{P}^2, \mathbb{P}^1) \\
\mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1,
\end{tikzcd} \tag{2.1.0.1}$$

where $g_3\colon X_2 \to \mathbb{P}^2$ is the blow-down of the minimal section.

In this chapter we consider Problem 2.1.1 when $n = 3$ and (X, D_h, D_f) is compatible with f. In this case, f is a del Pezzo fibration. When f is a $\mathbb{P}^2$-bundle, then the problem is easy by the same reason as when $n = 2$ (see §2.6). However, if the degree of f is smaller than 9, then the problem is not obvious since a general $(f|_U)$-fiber often differs from $\mathbb{A}^2$.

The main purpose of this chapter is to give a solution to Problem 2.1.1 for compactifications compatible with a quadric fibration. Our main result consists of three theorems. One is the following theorem, which is the solution to Problem 2.1.1 (1).

THEOREM 2.1.3. *Let $q\colon Q \to C$ be a quadric fibration, D_h a reduced effective divisor on Q, and D_f a q-fiber. Then the following are equivalent.*

(1) *The complement $Q \setminus (D_h \cup D_f)$ is an affine homology 3-cell.*
(2) *It holds that $C \cong \mathbb{P}^1$ and $Q \setminus (D_h \cup D_f) \cong \mathbb{A}^3$.*

The others are Theorems 2.1.6 and 2.1.7, which give a solution to Problem 2.1.1 (2). Before stating the theorems, we introduce some examples of compactifications of $\mathbb{A}^3$ compatible with del Pezzo fibrations and explicit birational maps preserving $\mathbb{A}^3$ from them to $\mathbb{P}^3$.

EXAMPLE 2.1.4. Let $g_3\colon P' \to \mathbb{P}^3$ be the blow-up along a line and $D_{h,2}$ the exceptional divisor. Then the linear system $|g_3^*\mathcal{O}_{\mathbb{P}^3}(1) - D_{h,2}|$ defines a $\mathbb{P}^2$-bundle structure $p'\colon P' \to \mathbb{P}^1$. Let $D_{f,2}$ be a p'-fiber. Then $(P', D_{h,2}, D_{f,2})$ is a compactification of $\mathbb{A}^3$ compatible with p' because $P' \setminus (D_{h,2} \cup D_{f,2}) \cong \mathbb{P}^3 \setminus g_{3*}D_{f,2} \cong \mathbb{A}^3$. Hence $g_3\colon P' \to \mathbb{P}^3$ is a birational map preserving $\mathbb{A}^3$.

EXAMPLE 2.1.5. Let $h_2\colon Q' \to \mathbb{Q}^3$ be the blow-up of the smooth quadric $\mathbb{Q}^3 \subset \mathbb{P}^4$ along a smooth conic and D'_h the exceptional divisor. Then the linear system $|h_2^*\mathcal{O}_{\mathbb{Q}^3}(1) - D'_h|$ defines a quadric fibration structure $q'\colon Q' \to \mathbb{P}^1$. Let D'_f be a singular q'-fiber, which is isomorphic to the quadric cone $\mathbb{Q}^2_0 \subset \mathbb{P}^3$. Then h_2 induces an isomorphism $Q' \setminus (D'_h \cup D'_f) \cong \mathbb{Q}^3 \setminus \mathbb{Q}^2_0$. Let $h_3\colon \mathbb{Q}^3 \dashrightarrow \mathbb{P}^3$ be the projection from the vertex of $\mathbb{Q}^2_0$. Then, by the discussion in [**Fur00**, pp.117–119], h_3 induces an isomorphism $\mathbb{Q}^3 \setminus \mathbb{Q}^2_0 \cong \mathbb{P}^3 \setminus \mathbb{P}^2 \cong \mathbb{A}^3$. Hence (Q', D'_h, D'_f) is a compactification of $\mathbb{A}^3$ compatible with q' and $h_3 \circ h_2 \colon Q' \dashrightarrow \mathbb{P}^3$ is a birational map preserving $\mathbb{A}^3$.

With the above examples, the other main theorems are stated as follows.

THEOREM 2.1.6. *Let (Q, D_h, D_f) be a compactification of $\mathbb{A}^3$ compatible with a quadric fibration $q\colon Q \to \mathbb{P}^1$. Suppose that D_h is non-normal.*

(1) *Let $g_1\colon Q \dashrightarrow P$ be the elementary link with center along the singular locus of D_h, which is a q-section. Let $D_{f,1}$ be the strict transform of D_f in P and $D_{h,1}$ the exceptional divisor of the elementary link. Then P has a $\mathbb{P}^2$-bundle structure p over $\mathbb{P}^1$ and $(P, D_{h,1}, D_{f,1})$ is a compactification of $\mathbb{A}^3$ compatible with p.*
(2) *We follow the notation of Example 2.1.4. Regard $p(D_{f,1})$ and $p'(D_{f,2})$ as $\infty \in \mathbb{P}^1$. Then there is the composition $g_2\colon P \dashrightarrow P'$ of elementary links with center along linear subspaces in the fibers at ∞ such that $D_{h,2}$ is the strict transform of $D_{h,1}$ in P'.*

Summarizing, we have the following diagram of rational maps preserving $Q \setminus (D_h \cup D_f) \cong \mathbb{A}^3$:

$$\begin{tikzcd} (Q, D_h, D_f) \arrow[r, dotted, "g_1"] \arrow[d, "q"'] & (P, D_{h,1}, D_{f,1}) \arrow[r, dotted, "g_2"] \arrow[d, "p"'] & (P', D_{h,2}, D_{f,2}) \arrow[r, "g_3"] \arrow[d, "p'"'] & (\mathbb{P}^3, H) \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1, & \end{tikzcd} \tag{2.1.0.2}$$

where $H \coloneqq g_{3*}D_{f,2}$.

THEOREM 2.1.7. *Let (Q, D_h, D_f) be a compactification of $\mathbb{A}^3$ compatible with a quadric fibration $q\colon Q \to \mathbb{P}^1$. Suppose that D_h is normal. We follow the notation of Example 2.1.5. Regard $q(D_f)$ and $q'(D'_f)$ as $\infty \in \mathbb{P}^1$. Then there is the composition $h_1\colon Q \dashrightarrow Q'$ of elementary links with center along rulings in the fibers at ∞ such that D'_h is the strict transform of D_h in Q'. In particular, we have the following diagram of rational maps preserving $Q \setminus (D_h \cup D_f) \cong \mathbb{A}^3$:*

$$\begin{tikzcd} (Q, D_h, D_f) \arrow[r, dotted, "h_1"] \arrow[d, "q"'] & (Q', D'_h, D'_f) \arrow[r, "h_2"] \arrow[d, "q'"'] & (\mathbb{Q}^3, \mathbb{Q}^2_0) \arrow[r, dotted, "h_3"] & (\mathbb{P}^3, H) \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1, & & \end{tikzcd} \tag{2.1.0.3}$$

where we regard $h_{2*}D'_f$ as $\mathbb{Q}^2_0$ and $H \coloneqq \mathbb{P}^3 \setminus h_3(\mathbb{Q}^3 \setminus \mathbb{Q}^2_0)$.

In Example 2.5.4, we construct a compactification of an affine homology 3-cell into a quadric fibration, which gives a negative answer to Problem 2.1.1 (1) in the case where $n = 3$ without the assumption on the compatibility. Problem 2.1.1 (2) for general compactifications into del Pezzo fibrations is at present far from being solved.

2.2. Structure of Chapter 2

This article is structured as follows.

In §2.3, we determine the Hodge diamonds of del Pezzo fibrations containing affine homology 3-cells. We also show that the base curve must be $\mathbb{P}^1$.

In §2.4, we give precise statement of Theorem 2.1.3 as in Theorem 2.4.2 and prove it by using elementary links from quadric fibrations to $\mathbb{P}^2$-bundles.

In §2.5, we construct several examples of compactifications of $\mathbb{A}^3$ compatible with quadric fibrations as applications of Theorem 2.4.2. We note that these examples are erroneously omitted from [**Kis05**, Table 1] or [**MS90**, Table 1]. We also construct an example of compactifications of affine homology 3-cells. This gives a negative answer to Problem 2.1.1 (1) in the case where $n = 3$ and the compactification is not compatible with the extremal contraction.

In §2.6, we give a solution to Problem 2.1.1 for compactifications compatible with $\mathbb{P}^2$-bundles. Theorem 2.1.6 follows as a corollary.

In the rest of this chapter, we prove Theorem 2.1.7 as follows. Let (Q, D_h, D_f) be a compactification of $\mathbb{A}^3$ compatible with a quadric fibration such that D_h is normal.

First, in §2.7.1, we determine the singularities of D_h and $D_f|_{D_h}$. We also assign a non-negative integer to them, which we call the type of (Q, D_h, D_f). By definition, (Q, D_h, D_f) is of type 0 if and only if D_h is a Hirzebruch surface.

Next, in §2.7.2, we suppose that (Q, D_h, D_f) is of type $m > 0$. We construct a birational map preserving $\mathbb{A}^3$ from (Q, D_h, D_f) to another compactification of type $(m-1)$ via elementary links between quadric fibrations. Composing such maps, we get a birational map from (Q, D_h, D_f) to a compactification of $\mathbb{A}^3$ of type 0. Hence we reduce to proving Theorem 2.1.7 when (Q, D_h, D_f) is of type 0, i.e., when D_h is a Hirzebruch surface.

Finally, in §2.7.3, we suppose that D_h is a Hirzebruch surface of degree $d \in \mathbb{Z}_{\geq 0}$. When $d > 0$, we give a birational map preserving $\mathbb{A}^3$ from (Q, D_h, D_f) to another compactification (Q', D'_h, D'_f) of $\mathbb{A}^3$ of type 0 such that D'_h is a Hirzebruch surface of degree $(d-1)$. When $d = 0$, we show that (Q, D_h, D_f) is actually the same as (Q', D'_h, D'_f) as in Example 2.1.5. We have thus proved Theorem 2.1.7.

2.3. Topological invariants of the ambient space

In this section, we determine the Hodge diamonds of del Pezzo fibrations containing affine homology 3-cells, and that of the base curves.

Lemma 2.3.1. *Let $f\colon X \to C$ be a del Pezzo fibration and D a reduced effective divisor on X such that $X \setminus D$ is an affine homology 3-cell. Then the Hodge diamond of X is as follows:*

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 2 & & 0 & \\
0 & & h^{1,2}(X) & & h^{1,2}(X) & & 0 \\
 & 0 & & 2 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Moreover, It holds that $C \cong \mathbb{P}^1$.

Proof. By the Hodge symmetry, we only have to compute $h^{i,0}(X)$ for $1 \leq i \leq 3$ and $h^{1,1}(X)$. Since $-K_X$ is f-ample, we have the following by the relative Kawamata-Viehweg vanishing theorem:

$$h^i(X, \mathcal{O}_X) = h^i(C, \mathcal{O}_C) \text{ for } i \geq 0. \tag{2.3.0.1}$$

In particular, we have $h^{2,0}(X) = h^{3,0}(X) = 0$. Since the Picard number of X is two by assumption, we have $h^{1,1}(X) = h^{1,1}(X) + 2h^{2,0}(X) = 2$.

On the other hand, by [**vdV62**, Proposition 2.1], we have $H^5(X, \mathbb{Z}) \cong H^5(D, \mathbb{Z}) = 0$. Hence $H^1(X, \mathbb{Z}) = 0$ by the Poincare duality and $h^{1,0}(X) = 0$ by the Hodge decomposition, which proves the first assertion. The second assertion follows from (2.3.0.1). □

2.4. Proof of Theorem 2.1.3.

This section is devoted to the proof of Theorem 2.1.3. First we determine the linear equivalence class of the irreducible components of the boundary divisor. Then we give the precise statement of Theorem 2.1.3 as in Theorem 2.4.2 and prove it by using Lemma 1.5.1, i.e. elementary links from quadric fibrations to $\mathbb{P}^2$-bundles.

LEMMA 2.4.1. *Let $q\colon Q \to C$ be a quadric fibration, D_h a reduced effective divisor on Q, and D_f a q-fiber. If $U \coloneqq Q \setminus (D_h \cup D_f)$ is an affine homology 3-cell, then D_h is a prime divisor such that $2D_h \sim_C -K_Q$.*

PROOF. By Lemma 2.3.1, it follows that $\mathrm{Pic}_0(Q) = 0$. By [**Fuj82**, Corollary 1.20], we have $\mathrm{Pic}\, U = 0$ and the group of invertible functions on U coincides with non-zero constants $\mathbb{C}^*$. Hence D_h is a prime divisor such that $\mathrm{Pic}\, Q = \mathbb{Z}D_f \oplus \mathbb{Z}D_h$. By Theorem 1.1.2 (5) and the Grothendieck-Lefschetz theorem, there exists a divisor H_Q on Q such that $\mathrm{Pic}\, Q = \mathbb{Z}D_f \oplus \mathbb{Z}H_Q$ and $2H_Q \sim_C -K_Q$. Hence $D_h \sim_C H_Q$, which proves the lemma. □

The following is the precise statement of Theorem 2.1.3.

THEOREM 2.4.2. *Let $q\colon Q \to C$ be a quadric fibration, D_h a reduced effective divisor on Q, and D_f a q-fiber.*

- (A) *Suppose that D_h is non-normal. Then the following are equivalent.*
  - (1) *The complement $Q \setminus (D_h \cup D_f)$ is an affine homology 3-cell.*
  - (2) *$C \cong \mathbb{P}^1$ and D_h is a prime divisor such that $2D_h \sim_C -K_Q$.*
  - (3) *It holds that $Q \setminus (D_h \cup D_f) \cong \mathbb{A}^3$.*
- (B) *Suppose that D_h is normal. Then the following are equivalent.*
  - (1) *The complement $Q \setminus (D_h \cup D_f)$ is an affine homology 3-cell.*
  - (2) *$C \cong \mathbb{P}^1$ and D_h is a prime divisor such that $2D_h \sim_C -K_Q$. Also we have $D_f \cong \mathbb{Q}^2_0$ and $h^{1,2}(Q) = 0$. Moreover, each $(q|_{D_h})$-fiber is smooth except possibly $D_f|_{D_h}$.*
  - (3) *It holds that $Q \setminus (D_h \cup D_f) \cong \mathbb{A}^3$.*

PROOF. (A): Since (3) ⇒ (1) is trivial and (1) ⇒ (2) follows from Lemma 2.3.1 and Lemma 2.4.1, we only have to show (2) ⇒ (3).

Suppose that (2) holds. Let $s \coloneqq \mathrm{Sing}\, D_h$, which is a q-section by Lemma 1.6.2. Construct $\varphi, \psi, p, \widetilde{Q}$ and P as in Lemma 1.5.1. Then we have $(E_\psi)_Q = D_h$ by Lemma 1.6.2 (2). Therefore we have:

$$Q \setminus (D_h \cup D_f) \cong \widetilde{Q} \setminus ((D_h)_{\widetilde{Q}} \cup (D_f)_{\widetilde{Q}} \cup E_\varphi) \cong P \setminus ((D_f)_P \cup (E_\varphi)_P). \tag{2.4.0.1}$$

Since $(E_\varphi)_P$ is a sub $\mathbb{P}^1$-bundle by Lemma 1.5.1 (1) and $(D_f)_P$ is a p-fiber, we have $Q \setminus (D_h \cup D_f) \cong \mathbb{A}^3$ by [**Kis05**, Lemma 5.15].

(B): Since (3) ⇒ (1) is trivial, we only have to show (1) ⇒ (2) ⇒ (3). Let $U \coloneqq Q \setminus (D_h \cup D_f)$. We note that (1) implies $C \cong \mathbb{P}^1$ by Lemma 2.3.1. Hence we may assume that $C \cong \mathbb{P}^1$ throughout the proof. Let $\infty \coloneqq q(D_f)$ and regard $C \setminus \{\infty\}$ as $\mathbb{A}^1$.

(1) $\Rightarrow$ (2): Suppose that (1) holds. Then the second assertion follows from Lemma 2.3.1. Since U is an affine homology 3-cell, we have:

$$\begin{aligned} \chi_{\mathrm{top}}(Q) &= \chi_{\mathrm{top}}(U) + \chi_{\mathrm{top}}(D_h \setminus (D_f|_{D_h})) + \chi_{\mathrm{top}}(D_f) \\ &= 1 + \chi_{\mathrm{top}}(D_h \setminus (D_f|_{D_h})) + \chi_{\mathrm{top}}(D_f). \end{aligned} \tag{2.4.0.2}$$

Let $\sigma \coloneqq \{t \in \mathbb{A}^1 \mid (q|_{D_h})^*(t) \text{ is reducible}\}$. For $t \in \mathbb{A}^1$, the divisor $(q|_{D_h})^*(t)$ is a member of either $|\Sigma_0 + f_0|$ in $\mathbb{F}_0$ or $|\mathcal{O}_{\mathbb{Q}_0^2}(1)|$ in $\mathbb{Q}_0^2$. In particular, we have:

$$\begin{aligned} t \notin \sigma &\iff \mathrm{Supp}\ (q|_{D_h})^*(t) \cong \mathbb{P}^1 \\ &\iff \chi_{\mathrm{top}}((q|_{D_h})^*(t)) = 2. \\ t \in \sigma &\iff q^*(t) \cong \mathbb{F}_0 \text{ and } (q|_{D_h})^*(t) \text{ is reducible} \\ &\iff \chi_{\mathrm{top}}((q|_{D_h})^*(t)) = 3. \end{aligned}$$

Hence we have:

$$\chi_{\mathrm{top}}(D_h \setminus (D_f|_{D_h})) = 2\,\chi_{\mathrm{top}}(\mathbb{A}^1 \setminus \sigma) + 3\,\chi_{\mathrm{top}}(\sigma) = 2 + \sharp\sigma. \tag{2.4.0.3}$$

Also by Lemma 2.3.1, we have:

$$\chi_{\mathrm{top}}(Q) = 6 - 2h^{1,2}(Q). \tag{2.4.0.4}$$

Combining (2.4.0.2)–(2.4.0.4), we have:

$$6 - 2h^{1,2}(Q) \geq 3 + \sharp\sigma + \chi_{\mathrm{top}}(D_f). \tag{2.4.0.5}$$

we note that $\chi_{\mathrm{top}}(D_f) = 3$ when $D_f \cong \mathbb{Q}_0^2$ and $\chi_{\mathrm{top}}(D_f) = 4$ when $D_f \cong \mathbb{F}_0$. Hence (2.4.0.5) implies that $h^{1,2}(Q) = 0$, $\sigma = \emptyset$ and $D_f \cong \mathbb{Q}_0^2$. In particular, we get the third and fourth assertion of (2).

It remains to prove the last assertion. Take a q-section $s \subset D_h$ and construct $\varphi, \psi, p, \widetilde{Q}, P$ and B and as in Lemma 1.5.1. By (1.5.0.1), we have:

$$\chi_{\mathrm{top}}(Q) = 6 - 2p_a(B). \tag{2.4.0.6}$$

Combining (2.4.0.4) and (2.4.0.6), we have $p_a(B) = h^{1,2}(Q) = 0$. In particular, the branch locus of $p|_B$ consists of two points. By Lemma 1.5.1 (2), there is exactly two singular q-fibers. Since $q^*(\infty) = D_f \cong \mathbb{Q}_0^2$, we may assume that $q^*(0)$ is the other singular fiber. By Lemma 1.5.2 and the fact that $\sigma = \emptyset$, each $(q|_{D_h})$-fiber is smooth except possibly $D_f|_{D_h}$ and $(q|_{D_h})^*(0)$. Hence we only have to show that $(q|_{D_h})^*(0)$ is smooth.

Conversely, suppose that $(q|_{D_h})^*(0)$ is not smooth. Let $E \coloneqq \psi_*(E_\varphi)$ and $U' \coloneqq P \setminus ((D_h)_P \cup (D_f)_P) \cong \mathbb{A}^3$. Since $U \cong \widetilde{Q} \setminus ((D_h)_{\widetilde{Q}} \cup (D_f)_{\widetilde{Q}} \cup E_\varphi)$, we can regard U as the affine modification of U' with the locus $(B \cap U' \subset E \cap U')$ (see [**KZ99**] for the definition). By [**KZ99**, Theorem 3.1], the morphism between homologies $\tau \colon H_1(B \cap U', \mathbb{Z}) \to H_1(E \cap U', \mathbb{Z})$ induced by the inclusion $B \cap U' \hookrightarrow E \cap U'$ is an isomorphism of $\mathbb{Z}$-modules.

On the other hand, $(q|_{D_h})^*(0)$ is non-reduced because $\sigma = \emptyset$. Lemma 1.5.2 now shows that $E \cap U' \cong \mathbb{A}^1 \times \mathbb{C}^*$ and $B \cap U' \cong \mathbb{C}^*$, which is an unramified 2-section of the second projection of $E \cap U'$. Hence $H_1(B \cap U', \mathbb{Z}) \cong H_1(E \cap U', \mathbb{Z}) \cong \mathbb{Z}$, but $\tau = 2 \times \mathrm{id}_{\mathbb{Z}}$, a contradiction.

(2) $\Rightarrow$ (3): Suppose that (2) holds. Let $E \coloneqq \psi_*(E_\varphi)$ and $U' \coloneqq P \setminus ((D_h)_P \cup (D_f)_P) \cong \mathbb{A}^3$. Then we can regard U as the affine modification of U' with the locus $(B \cap U' \subset E \cap U')$. By Lemma 1.5.2, we have $E \cap U' \cong \mathbb{A}^2$ and $B \cap U' \cong \mathbb{A}^1$. By the Abhyanker-Mor theorem over Noetherian rings containing $\mathbb{Q}$ [**BD93**, Theorem B], there is a coordinate $\{x, y, z\}$ of $U' = \mathbb{A}^3$

such that $E \cap U' = \{x = 0\}$ and $B \cap U' = \{x = y = 0\}$. Hence U is isomorphic to the affine modification of $\mathbb{A}^3_{[x,y,z]}$ with the locus $(\{x = y = 0\} \subset \{x = 0\})$, which is isomorphic to $\mathbb{A}^3$ as desired. □

2.5. Examples

This section provides several examples of compactifications of affine homology 3-cells compatible with quadric fibrations. For the construction, we often use Theorem 2.4.2. Throughout this section, (Q, D_h, D_f) stands for a compactification of $\mathbb{A}^3$ compatible with a quadric fibration $q\colon Q \to \mathbb{P}^1$. We note that $K_Q + D_h + D_f$ is not nef since $(K_Q + D_h + D_f \cdot l) = -1$ for each ruling l of a q-fiber.

First suppose that Q is a Fano 3-fold and $D_h + D_f$ is ample. Let us mention that then in [**Kis05**, Lemma 5.9], D_h is erroneously claimed to be normal. In Example 2.5.1, we construct examples with non-normal D_h.

Example 2.5.1. Let $q\colon Q \to \mathbb{P}^1$ be a Fano quadric fibration, i.e. either No. 18, No. 25 or No. 29 in [**MM82**, Table 2]. Let D_f be a q-fiber. By [**Man66**, Theorem 4.2], we can take a q-section s. By Lemma 1.5.1 (1), there is a prime divisor D_h on Q such that $2D_h \sim_{\mathbb{P}^1} -K_Q$ and $\operatorname{Sing} D_h = s$. Theorem 2.4.2 (A) now shows that (Q, D_h, D_f) is a compactification of $\mathbb{A}^3$ compatible with q.

Assume that $D_h + D_f$ is not ample. Then by [**Kis05**, Lemma 2.2] there is a birational extremal contraction φ of Q such that $E_\varphi = D_h$ or D_f. Since D_f is a q-fiber, we have $E_\varphi = D_h$, which is impossible since D_h is non-normal and E_φ is normal by [**Mor82**, Theorem 3.3]. Hence $D_h + D_f$ is ample.

Secondly, suppose that D_h is normal and Q is No. 29 in [**MM82**, Table 2], i.e. the blow-up of $\mathbb{Q}^3$ along a smooth conic. Let us mention that then in [**Kis05**, Lemma 5.13], $D_h + D_f$ erroneously claimed to be not ample. In Example 2.5.2, we construct an example with $D_h + D_f$ ample.

Example 2.5.2. Take H, S and C in $\mathbb{Q}^3 = \{X_0X_1 + X_2^2 + X_3X_4 = 0\} \subset \mathbb{P}^4_{[X_0:\cdots:X_4]}$ as follows:

$$H := \{X_0X_1 + X_2^2 + X_3X_4 = X_0 = 0\}, \tag{2.5.0.1}$$
$$S := \{X_0X_1 + X_2^2 + X_3X_4 = X_1X_3 + X_0^2 = 0\}, \tag{2.5.0.2}$$
$$C := \{X_0X_1 + X_2^2 + X_3X_4 = X_0 = X_1 = 0\}. \tag{2.5.0.3}$$

Let $P := \{X_0Y_1 = X_1Y_0\} \subset \mathbb{P}^4_{[X_0:\cdots:X_4]} \times \mathbb{P}^1_{[Y_0:Y_1]}$, and $\Phi\colon P \to \mathbb{P}^4$ be the blow-up along $\{X_0 = X_1 = 0\}$. Set Q, D_f and D_h as the strict transformations of $\mathbb{Q}^3$, H and S in P respectively. Then $\Phi|_Q\colon Q \to \mathbb{Q}^3$ is the blow-up along C, and the second projection of $\mathbb{P}^4 \times \mathbb{P}^1$ induces a quadric fibration $q\colon Q \to \mathbb{P}^1_{[Y_0:Y_1]}$. The defining equations of Q, D_f and D_h in P are as follows:

$$Q = \{X_0X_1 + X_2^2 + X_3X_4 = 0\}, \tag{2.5.0.4}$$
$$D_f = \{X_0X_1 + X_2^2 + X_3X_4 = Y_0 = 0\}, \tag{2.5.0.5}$$
$$D_h = \{X_0X_1 + X_2^2 + X_3X_4 = Y_1X_3 + Y_0X_0 = 0\}. \tag{2.5.0.6}$$

Then D_f is a singular q-fiber and D_h has only one DuVal singularity of type D_4. Also D_h is a prime divisor with $2D_h \sim_{\mathbb{P}^1} -K_Q$. Since $C \cong \mathbb{P}^1$, we have $h^{1,2}(Q) = 0$. Also we have:

$$\begin{aligned} D_h \setminus (D_f|_{D_h}) &\cong \left\{ \begin{array}{l} X_0X_1 + X_2^2 + X_3X_4 = 0, \\ X_1 = X_0Y_1, Y_1X_3 + X_0 = 0 \end{array} \right\} \text{ in } \mathbb{P}^4_{[X_0:\cdots:X_4]} \times \mathbb{A}^1_{(Y_1)} \\ &\cong \{Y_1^3X_3^2 + X_2^2 + X_3X_4 = 0\} \text{ in } \mathbb{P}^2_{[X_2:X_3:X_4]} \times \mathbb{A}^1_{(Y_1)}. \end{aligned}$$

Hence each $(q|_{D_h})$-fiber is smooth except $D_f|_{D_h}$.

Theorem 2.4.2 (B) now shows that (Q, D_h, D_f) is a compactification of $\mathbb{A}^3$ compatible with q. Since both D_h and D_f differ from $E_{\Phi|_Q}$, the ampleness of $D_h + D_f$ follows from [**Kis05**, Lemma 2.2].

Thirdly, suppose that Q is an arbitrary quadric fibration and $D_f|_{D_h}$ is smooth. Then D_h is normal by Lemma 1.6.2. In fact, it holds that $D_h \cong \mathbb{F}_d$ for some $d \in \mathbb{Z}_{\geq 0}$ by Theorem 2.4.2 (B). Let us mention that in [**MS90**, §4.4, Lemma 2], it is erroneously claimed that $d = 0$. In Example 2.5.3, we construct an example with $D_h \cong \mathbb{F}_d$ for each $d \in \mathbb{Z}_{\geq 0}$.

EXAMPLE 2.5.3. Let $d \in \mathbb{Z}_{\geq 0}$ and $P \coloneqq \mathbb{F}(0, 1, d)$ with the $\mathbb{P}^2$-bundle structure $p\colon P \to \mathbb{P}^1$. For $i = 1, d$, let S_i be the sub $\mathbb{P}^1$-bundle of P associated with the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(d) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(i)$ and F a p-fiber. Then it holds that $S_i \cong \mathbb{F}_i$ and $S_i \sim \xi_P - (d+1-i)F$,. Also we have:

$$\begin{aligned} (S_i|_{S_{(d+1-i)}})^2 &= (\xi_P - (d+1-i)F)^2 \cdot (\xi_P - iF) \\ &= \xi_P^3 - (2d+2-i)\xi_P^2 \cdot F = -(d+1-i). \end{aligned} \tag{2.5.0.7}$$

Hence $S_d|_{S_1} = \Sigma_1$ and $S_1|_{S_d} = \Sigma_d$.

Now take $B \subset S_1$ as a smooth member of $|2(\Sigma_1 + f_1)|$, which is a p-bisection. Let $\psi\colon \widetilde{P} \to P$ be the blow-up along B. Then $-K_{\widetilde{P}}$ is $(p \circ \psi)$-ample. An easy computation shows that there is the elementary link with center along B:

(2.5.0.8)
$$\begin{tikzcd} & \widetilde{P} \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\ P \arrow[d, "p"'] & & Q \arrow[d, "q"] \\ \mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1 \end{tikzcd}$$

such that φ is the blow-up of a quadric fibration Q along a q-section. In fact, this is the inverse of an elementary link as in Lemma 1.5.1. Since $B \cong \mathbb{P}^1$, we have $h^{1,2}(Q) = 0$ by (2.4.0.4) and (2.4.0.6).

Let $D_h \coloneqq (S_d)_Q$ and D_f a singular q-fiber, which exists by Lemma 1.5.1 (2). Then $2D_h \sim_{\mathbb{P}^1} -K_Q$ by Lemma 1.5.1 (1). Since $B \cap S_d = \emptyset$ and $E_\varphi = (S_1)_{\widetilde{P}}$, it holds that $D_h \cong S_d \cong \mathbb{F}_d$. Theorem 2.4.2 (B) now shows that (Q, D_h, D_f) is a compactification of $\mathbb{A}^3$ compatible with q.

Finally, we give an example of compactifications of affine homology 3-cells into quadric fibrations which gives a negative answer to Problem 2.1.1 (1) without the assumption on the compatibility.

EXAMPLE 2.5.4. Take H, S and C in $\mathbb{Q}^3 = \{X_0X_1 + X_2^2 + X_3X_4 = 0\} \subset \mathbb{P}^4_{[X_0:\cdots:X_4]}$ as follows:

$$H := \{X_0X_1 + X_2^2 + X_3X_4 = X_0 = 0\}, \tag{2.5.0.9}$$
$$S := \{X_0X_1 + X_2^2 + X_3X_4 = 0, X_0X_3^2 = X_4^3\}, \tag{2.5.0.10}$$
$$C := \{X_0X_1 + X_2^2 + X_3X_4 = 0, X_3 = X_4 = X_0\}. \tag{2.5.0.11}$$

As in Example 2.5.2, the blow-up $Q := \mathrm{Bl}_C\,\mathbb{Q}^3$ has a quadric fibration structure $q\colon Q \to \mathbb{P}^1$. Since each q-fiber is the strict transform of a hyperplane section of $\mathbb{Q}^3$ containing C, both H_Q and S_Q are not q-fibers.

Now set $U := Q \setminus (H_Q \cup S_Q)$, $\mathbb{Q}^0 := \mathbb{Q}^3 \setminus H$, $S^0 := S \setminus (S \cap H)$ and $C^0 := C \setminus (C \cap H)$. Then U is the affine modification of $\mathbb{Q}^0$ with the locus $(C^0 \subset S^0)$. In $\mathbb{P}^4 \setminus H \cong \mathbb{A}^4_{[x_1,\ldots,x_4]}$, we have an isomorphism $\mathbb{Q}^0 \cong \{x_1 + x_2^2 + x_3x_4 = 0\} \cong \mathbb{A}^1_{[x_2]} \times \mathbb{A}^2_{[x_3,x_4]}$. This isomorphism sends S^0 and C^0 to $\mathbb{A}^1_{[x_2]} \times \{x_3^2 = x_4^3\}$ and $\mathbb{A}^1_{[x_2]} \times \{(1,1)\}$ respectively. By [**tDP90**], U is isomorphic to $\mathbb{A}^1_{[x_2]} \times V(3,2)$, where $V(3,2) = \{z^2x_4^3 + 3zx_4^2 + 3x_4 - zx_3^2 - 2x_3 = 1\} \subset \mathbb{A}^3_{[x_3,x_4,z]}$ is an affine homology 2-cell of logarithmic Kodaira dimension one. Hence $(Q, H_Q \cup S_Q)$ is a compactification of an affine homology 3-cell $\mathbb{A}^1 \times V(3,2)$. We note that $\mathbb{A}^1 \times V(3,2) \ncong \mathbb{A}^3$ by [**IF77**, Theorem 1].

2.6. Compactifications of affine homology 3-cells compatible with $\mathbb{P}^2$-bundles

In this section, we will give a solution of Problem 2.1.1 for compactifications compatible with $\mathbb{P}^2$-bundles. Theorem 2.1.6 follows as a corollary.

First we give the solution of Problem 2.1.1 (1) for such compactifications.

LEMMA 2.6.1. *Let $p\colon P \to C$ be a $\mathbb{P}^2$-bundle, D_h a reduced effective divisor on P, and D_f a p-fiber. Then the following are equivalent.*

1. *The complement $P \setminus (D_h \cup D_f)$ is an affine homology 3-cell.*
2. *$C \cong \mathbb{P}^1$ and D_h is a sub $\mathbb{P}^1$-bundle.*
3. *It holds that $P \setminus (D_h \cup D_f) \cong \mathbb{A}^3$.*

PROOF. Since (3) $\Rightarrow$ (1) is trivial and (2) $\Rightarrow$ (3) follows from [**Kis05**, Lemma 5.15], we only have to show (1) $\Rightarrow$ (2).

Suppose that (1) holds. The first assertion follows from Lemma 2.3.1. By the same argument as in the proof of Lemma 2.4.1, D_h is a prime divisor such that $\mathrm{Pic}\,P = \mathbb{Z}D_f \oplus \mathbb{Z}D_h$. On the other hand, we have $\mathrm{Pic}\,P = \mathbb{Z}D_f \oplus \mathbb{Z}\xi_P$. Hence $D_h \sim_C \xi_P$, which implies that D_h is a sub $\mathbb{P}^1$-bundle of P. □

Next we characterize $(P', D_{h,2}, D_{f,2})$ as in Example 2.1.4.

LEMMA 2.6.2. *Let $p\colon P \to \mathbb{P}^1$ be a $\mathbb{P}^2$-bundle with associated vector bundle $\mathcal{E}$. Let F be a p-fiber and $D \subset P$ a sub $\mathbb{P}^1$-bundle. Suppose that $\deg \mathcal{E} = 3n+1$, $D \cong \mathbb{F}_0$ and $D \sim \xi_P - (n+1)F$ for some $n \in \mathbb{Z}$. Then we have $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(n+1)$, and D is the exceptional divisor of the blow-up $f\colon P \cong \mathbb{F}(0,0,1) \to \mathbb{P}^3$ along a line.*

PROOF. By replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(-n)$, we may assume that $n = 0$. Let us show the ampleness of $-K_P$. It is obvious that $-K_P|_F$ is ample. By the canonical bundle formula and the adjunction formula, we have:

$$-K_P \sim 3\xi_P + F \sim 3D + 4F. \tag{2.6.0.1}$$

$$D|_D \sim -\frac{1}{2}(K_P + D)|_D - 2F|_D \sim -\frac{1}{2}K_D - 2f_0 \sim \Sigma_0 - f_0. \tag{2.6.0.2}$$

we thus get $-K_P|_D \sim (3D+4F)|_D \sim 3\Sigma_0 + f_0$, which is also ample.

Suppose that $(-K_P \cdot r) \leq 0$ holds for some curve $r \subset P$. Since both $-K_P|_F$ and $-K_P|_D$ are ample, (2.6.0.1) now shows that r must be disjoint from any p-fiber, a contradiction. Hence $-K_P$ is strictly nef. On the other hand, we have $(-K_P)^3 = 54$ since P is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. Hence $-K_P$ is big and semiample by the base-point free theorem. Since $-K_P$ is strictly nef and semiample, it is ample.

Therefore P is a Fano $\mathbb{P}^2$-bundle. By [**MM82**, Table 2], P is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{F}(0,0,1)$. Since $\deg \mathcal{E} = 1$, it holds that $P \cong \mathbb{F}(0,0,1)$ and $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, which is the first assertion.

Since $F \sim f^*\mathcal{O}_{\mathbb{P}^3}(1) - E_f$ and $-K_P \sim f^*\mathcal{O}_{\mathbb{P}^3}(4) - E_f \sim 3E_f + 4F$, the second assertion follows from (2.6.0.1). □

Now we can give a solution to Problem 2.1.1 (2) for compactification compatible with $\mathbb{P}^2$-bundles.

PROPOSITION 2.6.3. *Let (P, D_h, D_f) be a compactification of $\mathbb{A}^3$ compatible with a $\mathbb{P}^2$-bundle $p\colon P \to \mathbb{P}^1$. We follow the notation of Example 2.1.4. Regard $p(D_f)$ and $p'(D_{f,2})$ as $\infty \in \mathbb{P}^1$. Then there is the composition $g_2\colon P \dashrightarrow P'$ of elementary links with center along linear subspaces in the fibers at ∞ such that $D_{h,2} = (D_h)_{P'}$. In particular, there exists the following diagram of rational maps preserving $P \setminus (D_h \cup D_f) \cong \mathbb{A}^3$:*

$$\begin{tikzcd} (P, D_h, D_f) \arrow[r, dashed, "g_2"] \arrow[d, "p"] & (P', D_{h,2}, D_{f,2}) \arrow[r, "g_3"] \arrow[d, "p'"] & (\mathbb{P}^3, H) \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1, & \end{tikzcd} \tag{2.6.0.3}$$

where $H \coloneqq g_{3*}D_{f,2}$.

PROOF. Suppose that $D_h \cong \mathbb{F}_d$ for some $d > 0$. Take the elementary link $P \dashrightarrow P_1$ with center at a point $p \in D_h \cap D_f$ such that $p \notin \Sigma_d$. This elementary link preserves $P \setminus (D_h \cup D_f) \cong \mathbb{A}^3$. Also we have $(D_h)_{P_1} \cong \mathbb{F}_{d-1}$ because $P \dashrightarrow P_1$ induces an elementary transform of D_h with center at p. Taking such elementary links d times, we may assume that $D_h \cong \mathbb{F}_0$.

Let $\mathcal{E}$ be an associated vector bundle of P. Set $d \coloneqq \deg \mathcal{E}$ and take $e \in \mathbb{Z}$ such that $D_h \sim \xi_P + eD_f$.

Let us show that $d+e \in 2\mathbb{Z}$. By the canonical bundle formula and the adjunction formula, we have:

$$-K_X \sim 3\xi_P - (d-2)D_f \sim 3D_h - (d+3e-2)D_f. \tag{2.6.0.4}$$

$$\begin{aligned} -K_{D_h} &\sim (2D_h - (d+3e-2)D_f)|_{D_h} \\ &\sim 2(D_h - (e-1)D_f)|_{D_h} - (d+e)D_f|_{D_h}. \end{aligned} \tag{2.6.0.5}$$

This gives $d+e \in 2\mathbb{Z}$ because $-K_{D_h} \sim 2(\Sigma_0 + f_0)$ and $D_f|_{D_h} \sim f_0$.

Now let $L \subset D_f$ be a linear subspace and $P \dashrightarrow P_1$ the elementary link with center along L. This elementary link preserves $P \setminus (D_h \cup D_f) \cong \mathbb{A}^3$. Let F be a fiber of the induced $\mathbb{P}^2$-bundle $p_1\colon P_1 \to \mathbb{P}^1$. Take an associated vector bundle $\mathcal{E}'$ of P_1 as in Lemma 1.4.1.

Consider the case where L is a point outside D_h. Then we have $(D_h)_{P_1} \cong D_h \cong \mathbb{F}_0$. By Lemma 1.4.1 and Corollary 1.4.2, we have $\deg \mathcal{E}' = d-1$ and $(D_h)_{P_1} \sim \xi_{P_1} + (e+1)F$. For each $m \in \mathbb{Z}_{\geq 0}$, taking such elementary links m times, we can replace (d,e) with $(d-m, e+m)$.

Consider the case where $L = D_f \cap D_h$. Then we have $(D_h)_{P_1} \cong D_h \cong \mathbb{F}_0$. Replacing $\mathcal{E}'$ with $\mathcal{E}' \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(1)$, we have $\deg \mathcal{E}' = d+1$ and $(D_h)_{P_1} \sim \xi_{P_1} + (e-1)F$ by Lemma 1.4.1 and Corollary 1.4.2. For each $m \in \mathbb{Z}_{\geq 0}$, taking such elementary links m times, we can replace (d,e) with $(d+m, e-m)$.

Now set $m \coloneqq \frac{d+3e}{2} + 1 \in \mathbb{Z}$ and replace (d,e) with $(d+m, e-m) = (\frac{3(d+e)}{2}+1, -\frac{d+e}{2}-1)$. Applying Lemma 2.6.2 with $n = \frac{d+e}{2}$, we have the assertion. $\square$

Now we can prove Theorem 2.1.6.

Proof of Theorem 2.1.6. We have shown that $\operatorname{Sing} D_h$ is a q-section in Lemma 1.6.2. By Lemma 1.5.1, there is the elementary link $g_1\colon Q \dashrightarrow P$ with center along $\operatorname{Sing} D_h$ and the induced morphism $p\colon P \to \mathbb{P}^1$ is a $\mathbb{P}^2$-bundle. Let E be the exceptional divisor of the elementary link. As in the proof of Theorem 2.4.2 (A), we can show that g_1 induces an isomorphism $\mathbb{A}^3 \cong Q \setminus (D_h \cup D_f) \cong P \setminus (E \cup (D_f)_P)$. Hence $(P, D_{h,1}, D_{f,1}) \coloneqq (P, E, (D_f)_P)$ is a compactification of $\mathbb{A}^3$ compatible with p, which proves (1). The assertions (2) follow from Proposition 2.6.3. $\square$

2.7. Proof of Theorem 2.1.7

The remainder of this chapter will be devoted to the proof of Theorem 2.1.7. From now on, we assume that (Q, D_h, D_f) is a compactification of $\mathbb{A}^3$ compatible with a quadric fibration $q\colon Q \to C \cong \mathbb{P}^1$ such that D_h is normal. Also we use the following notation:

Notation 1. For $d \in \mathbb{Z}_{>0}$, we will denote by S_d the blow-up of $\mathbb{F}_d$ at a point outside Σ_d. We note that S_d is also the blow-up of $\mathbb{F}_{d-1}$ at a point in Σ_{d-1}.

2.7.1. Singularities of D_h and $D_f|_{D_h}$. First, we establish a relation between the singularity of $D_f|_{D_h}$ and that of D_h. Theorem 2.4.2 (B) shows that $D_f \cong \mathbb{Q}^2_0$ and $D_h|_{D_f} \sim \mathcal{O}_{\mathbb{Q}^2_0}(1)$. Hence $D_h|_{D_f}$ is either a smooth conic, the union of two distinct rulings, or a non-reduced curve supporting on a ruling of $\mathbb{Q}^2_0$.

Theorem 2.7.1. *We have the following correspondence.*

(1) *If $D_f|_{D_h}$ is smooth, then $D_h \cong \mathbb{F}_d$ for some $d \geq 0$.*
(2) *If $D_f|_{D_h}$ is reducible, then $D_h \cong S_d$ for some $d > 0$.*
(3) *If $D_f|_{D_h}$ is non-reduced, then D_h has either exactly two DuVal singularities of type A_1, or the unique DuVal singularity of type A_3 or D_m $(m \geq 4)$.*

Proof. Take a q-section $s \subset D_h$ and construct $\varphi, \psi, p, \widetilde{Q}, P$ and B as in Lemma 1.5.1. Write $G \coloneqq \psi_* E_\varphi$, $\infty \coloneqq q(D_f)$, $f_\infty \coloneqq (p|_G)^*(\infty)$ and $l_t \coloneqq (p|_{(D_h)_P})^*(t)$ for $t \in C$.

Recall that $B \cong \mathbb{P}^1$, $D_f \cong \mathbb{Q}^2_0$ and singular $(q|_{D_h})$-fibers are at most $D_f|_{D_h}$ by Theorem 2.4.2 (B). In particular, $p|_B$ is ramified over ∞. By Lemma 1.5.2, $(p|_G)$-fibers contained in $(D_h)_P$ are at most f_∞. Also $f_\infty \not\subset (D_h)_P$ if and only if $D_f|_{D_h}$ is reduced.

By Lemma 2.4.1, we have $2D_h \sim_C -K_Q$. Hence $(D_h)_P$ is a sub $\mathbb{P}^1$-bundle of P not containing B by Lemma 1.5.1 (1). Since G is also a sub $\mathbb{P}^1$-bundle of P, there exists a unique p-section s' and $a \in \mathbb{Z}_{\geq 0}$ such that $(D_h)_P|_G = s' + af_\infty$.

Let us show that $(B \cdot s')_G \leq 1$. For $t \in C$, it holds that $l_t \cap B = \emptyset$ if and only if $(q|_{D_h})^*(t)$ is smooth by Lemma 1.5.2. Since singular $(q|_{D_h})$-fibers are at most $D_f|_{D_h}$, we have $\mathrm{Supp}((D_h)_P \cap B) \subset f_\infty$. Hence $\mathrm{Supp}(s' \cap B) \subset \mathrm{Supp}(f_\infty \cap B)$. Since $p|_B$ is ramified over ∞, the support of $f_\infty \cap B$ is a point. By the same reason, B and f_∞ have the same tangent direction at $\mathrm{Supp}(f_\infty \cap B)$ in G. Since $(f_\infty \cdot s')_G = 1$, we have $(B \cdot s')_G \leq 1$ as desired.
(1): Suppose that $D_f|_{D_h}$ is smooth. Then we have $a = 0$ and $l_\infty \cap B = \emptyset$ by Lemma 1.5.2. The former implies that $(D_h)_P \cup G$ is a SNC divisor, and the latter implies that $B \cap (D_h)_P = \emptyset$. Hence ψ is an isomorphism along $(D_h)_{\widetilde{Q}}$. Since $(D_h)_{\widetilde{Q}} \cup E_\varphi = (D_h)_{\widetilde{Q}} \cup G_{\widetilde{Q}}$ is a SNC divisor, we have $D_h \cong (D_h)_{\widetilde{Q}} \cong (D_h)_P$, which is a Hirzebruch surface, and (1) is proved.
(2): Suppose that $D_f|_{D_h}$ is reducible. Then we have $a = 0$ and $l_\infty \cap B \neq \emptyset$ by Lemma 1.5.2. Hence $(D_h)_P \cup G$ is a SNC divisor. Also $(D_h)_{\widetilde{Q}}$ is the blow-up of $(D_h)_P$ at a point because $((D_h)_P \cdot B)_P = (e \cdot B)_G = 1$. Since $(D_h)_{\widetilde{Q}} \cap E_\varphi$ is the strict transform of s' in $\widetilde{Q}$, the divisor $(D_h)_{\widetilde{Q}} \cup E_\varphi$ is a SNC divisor and hence we have $D_h \cong (D_h)_{\widetilde{Q}}$, which is the blow-up of a Hirzebruch surface at a point, and (2) is proved.
(3): Suppose that $D_f|_{D_h}$ is non-reduced. Then we have $a \geq 1$ by Lemma 1.5.2. Set $m \coloneqq (B \cdot (D_h)_P)_P = (B \cdot s')_G + 2a \geq 2$.

For $0 \leq i \leq m-1$, we define P_i, $\widetilde{Q}_i$, x_i, h_i and ψ_i by induction as follows. Let $P_0 \coloneqq P$, $\widetilde{Q}_0 \coloneqq \widetilde{Q}$, $x_0 \coloneqq \mathrm{Supp}((D_h)_P \cap B)$, $h_0 \coloneqq \mathrm{id}_P$ and $\psi_0 \coloneqq \psi$. For $i > 0$, denote by $h_i \colon P_i \to P_{i-1}$ the blow-up at x_{i-1}. Let $x_i \coloneqq \mathrm{Supp}((D_h)_{P_i} \cap B_{P_i})$, which is a point. We also define $\psi_i \colon \widetilde{Q}_i \to P_i$ as the blow-up along B_{P_i}.

Then we have the following diagram by Lemma 1.3.1 (1), where $\varphi_i \colon \widetilde{Q}_i \to \widetilde{Q}_{i-1}$ is the blow-up along $(\psi_{i-1})^{-1}(x_{i-1})$ for $1 \leq i \leq m-1$.

(2.7.1.1)
$$\begin{tikzcd}
\widetilde{Q}_{m-1} \arrow[r, "\varphi_{m-1}"] \arrow[d, "\psi_{m-1}"] & \widetilde{Q}_{m-2} \arrow[r, "\varphi_{m-2}"] \arrow[d, "\psi_{m-2}"] & \cdots \arrow[r, "\varphi_2"] & \widetilde{Q}_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"] & \widetilde{Q}_0 \arrow[r, "\varphi"] \arrow[d, "\psi_0=\psi"] & Q \\
P_{m-1} \arrow[r, "h_{m-1}"'] & P_{m-2} \arrow[r, "h_{m-2}"'] & \cdots \arrow[r, "h_2"'] & P_1 \arrow[r, "h_1"'] & P_0. &
\end{tikzcd}$$

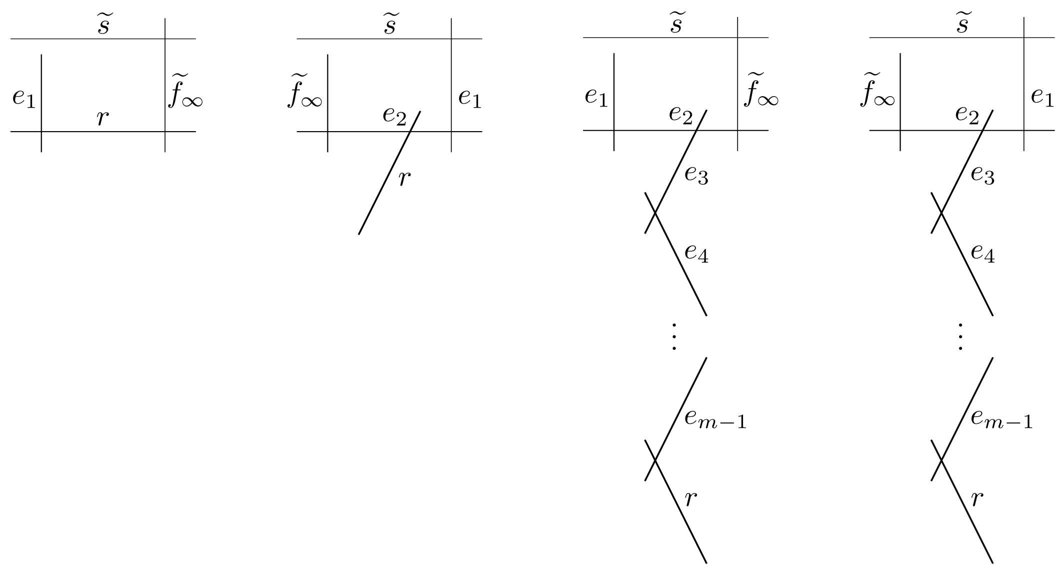
Let $\alpha \colon (D_h)_{\widetilde{Q}_{m-1}} \to D_h$ be the induced morphism. To know the singularities on D_h, it suffices to detect that of $(D_h)_{\widetilde{Q}_{m-1}}$ and the shape of E_α.

For $1 \leq i \leq m-1$, it holds that $(D_h)_{P_i}$ is smooth and $((D_h)_{P_i} \cdot B_{P_i})_{P_i} = m - i$ because h_i is the blow-up at the point x_{i-1}. Hence $(D_h)_{P_{m-1}}$ intersects with $B_{P_{m-1}}$ at x_{m-1} transversally and $(D_h)_{\widetilde{Q}_{m-1}}$ is the blow-up of $(D_h)_{P_{m-1}}$ at x_{m-1}, which is also smooth.

Let us reveal the precise location of $x_i \in (D_h)_{P_i}$ for $0 \leq i \leq m-1$ to detect the shape of E_α. Note that $x_i \in E_{h_i}$ by construction. We already showed that $x_0 = \mathrm{Supp}(B \cap f_\infty)$. Since $(B \cdot f_\infty)_G = 2$, we have $(B_{P_1} \cdot (f_\infty)_{P_1})_{G_{P_1}} = 1$. Hence we have $x_1 = \mathrm{Supp}(B_{P_1} \cap (f_\infty)_{P_1}) = \mathrm{Supp}(E_{h_1} \cap (f_\infty)_{P_1})$.

We now turn to the case $i \geq 2$. We have $x_2 \notin (f_\infty)_{P_2}$ since $(B_{P_2} \cdot (f_\infty)_{P_2})_{G_{P_2}} = 0$. Also we have $x_2 \notin (E_{h_1})_{P_2}$ since $(B_{P_2} \cdot (E_{h_1})_{P_2})_{P_2} = 0$. Hence $x_2 \in E_{h_2}$ and $x_2 \notin (E_{h_1})_{P_2} \cup (f_\infty)_{P_2}$. Similarly, for $i \geq 3$, we have $x_i \in E_{h_i}$ and $x_i \notin (E_{h_{i-1}})_{P_i}$.

Let e_i be the strict transform of $E_{h_i}|_{D_{h,i}}$ in $\widetilde{Q}_{m-1}$ for $1 \leq i \leq m-1$. Set $\widetilde{f}_\infty := (f_\infty)_{\widetilde{Q}_{m-1}}$, $\widetilde{s} := s'_{\widetilde{Q}_{m-1}}$ and $r := E_{\psi_{m-1}}|_{(D_h)_{\widetilde{Q}_{m-1}}}$. By the above observation on x_i, the configuration of e_i, $\widetilde{f}_\infty$, $\widetilde{s}$ and r in $(D_h)_{\widetilde{Q}_{m-1}}$ is as in FIGURE 1.



Case $m = 2$ Case $m = 3$ Case $m \geq 4$ and m is even Case $m \geq 4$ and m is odd

FIGURE 1. The configuration of e_i, $\widetilde{f}_\infty$, $\widetilde{s}$ and r in $(D_h)_{\widetilde{Q}_{m-1}}$

It is clear that $(f_\infty)_{\widetilde{Q}}$ is the exceptional divisor of $(D_h)_{\widetilde{Q}} \to D_h$. On the other hand, by Lemma 1.3.1 (1), e_{m-1} is the exceptional divisor of $(D_h)_{\widetilde{Q}_{m-1}} \to (D_h)_{\widetilde{Q}_{m-2}}$. Repeated application of Lemma 1.3.1 (1) shows us that $E_\alpha = \widetilde{f}_\infty \cup \bigcup_{i=1}^{m-1} e_i$. Since each of them is (-2)-curve in $(D_h)_{\widetilde{Q}_{m-1}}$, the singularity of D_h is the DuVal singularity of type $2A_1$ when $m = 2$, A_3 when $m = 3$ and D_m when $m \geq 4$, which completes the proof. □

The next aim is to construct explicit birational maps preserving $\mathbb{A}^3$ from (Q, D_h, D_f) to an another compactification (Q', D'_h, D'_f) compatible with quadric fibration such that the singularity of D'_h is milder than that of D_h. To do so, we define the type of (Q, D_h, D_f) as follows.

DEFINITION 2.7.2. Let $m \in \mathbb{Z}_{\geq 0}$. We call (Q, D_h, D_f) *a compactification of*

- *type* 0 when $D_h \cong \mathbb{F}_d$ for some $d \geq 0$.
- *type* 1 when $D_h \cong S_d$ for some $d > 0$.
- *type* 2 when D_h has two DuVal singularities of type A_1.
- *type* 3 when D_h has a DuVal singularity of type A_3
- *type* $m(\geq 4)$ when D_h has a DuVal singularity of type D_m.

We note that $D_f|_{D_h}$ contains a ruling of $\mathbb{Q}_0^2 \cong D_f$ if and only if $m > 0$. It is easy to check that the number of the type coincides with $(B \cdot (D_h)_P)_P$ as in the proof of Theorem 2.7.1. Hence we have the following.

Corollary 2.7.3. *Take any q-section $s \subset D_h$ and construct P and B as in Lemma 1.5.1. Then (Q, D_h, D_f) is of type $m \in \mathbb{Z}_{\geq 0}$ if and only if $(B \cdot (D_h)_P)_P = m$.*

2.7.2. The case of singular $D_f|_{D_h}$. Next we shall give an elementary link from each compactification of $\mathbb{A}^3$ of type $m > 0$ to that of type $(m-1)$. Composing such elementary links, we get a birational map from each compactification of $\mathbb{A}^3$ of type $m > 0$ to that of type 0.

Lemma 2.7.4. *Let $q\colon Q \to C$ be a quadric fibration, F a singular q-fiber, $s \subset Q$ a q-section and l the ruling of $F \cong \mathbb{Q}_0^2$ which intersects with s. We use the same letter l and s for their strict transformations by abuse of notation. Consider the following four elementary links:*

- $Q \xleftarrow{\varphi_{1,1}} Q_{1,1} \xrightarrow{\psi_{1,1}} Q'$*: the elementary link with center along l.*
- $Q' \xleftarrow{\varphi_{1,2}} Q_{1,2} \xrightarrow{\psi_{1,2}} P_1$*: the elementary link with center along s.*
- $Q \xleftarrow{\varphi_{2,1}} Q_{2,1} \xrightarrow{\psi_{2,1}} P$*: the elementary link with center along s.*
- $P \xleftarrow{\varphi_{2,2}} Q_{2,2} \xrightarrow{\psi_{2,2}} P_2$*: the elementary link with center at the point $x := \psi_{2,1}(l)$.*

Summarizing these notation, we have the following diagram:

(2.7.2.1)
$$\begin{tikzcd}
& Q_{1,2} \arrow[dl, "\psi_{1,2}"'] \arrow[dr, "\varphi_{1,2}"] & & Q_{1,1} \arrow[dl, "\psi_{1,1}"'] \arrow[dr, "\varphi_{1,1}"] & & Q_{2,1} \arrow[dl, "\varphi_{2,1}"'] \arrow[dr, "\psi_{2,1}"] & & Q_{2,2} \arrow[dl, "\varphi_{2,2}"'] \arrow[dr, "\psi_{2,2}"] & \\
P_1 \arrow[d] & & Q' \arrow[d] & & Q \arrow[d] & & P \arrow[d] & & P_2 \arrow[d] \\
C \arrow[rr, equal] & & C \arrow[rr, equal] & & C \arrow[rr, equal] & & C \arrow[rr, equal] & & C,
\end{tikzcd}$$

Then the birational map $\iota\colon P_1 \dashrightarrow P_2$ induced by (2.7.2.1) *is an isomorphism.*

Proof. By [**Cor95**, Proposition 3.5], we only have to show that ι is an isomorphism in codimension one.

Let l' be the strict transform of the center of $\psi_{1,1}$ in $Q_{1,2}$. Let $\chi_1\colon X_1 \to Q_{1,2}$ be the blow-up along l'. Since $s \subset Q_{1,1}$ is disjoint from $F_{Q_{1,1}} = E_{\psi_{1,1}}$, we have $X_1 \cong \mathrm{Bl}_s Q_{1,1}$. On the other hand, let B be the strict transform of the center of $\psi_{2,1}$ in $Q_{2,2}$. Let $\chi_2\colon X_2 \to Q_{2,2}$ be the blow-up along B. By Lemma 1.3.1 (1), we have $X_2 \cong \mathrm{Bl}_l Q_{2,1}$. Summarizing these arguments, we have the following diagram:

(2.7.2.2)
$$\begin{tikzcd}
& & X_1 \arrow[dl, "\chi_1"'] \arrow[dr, "\mathrm{Bl}_s"] & & & & X_2 \arrow[dl, "\mathrm{Bl}_l"'] \arrow[dr, "\chi_2"] & & \\
& Q_{1,2} \arrow[dl, "\psi_{1,2}"'] \arrow[dr, "\varphi_{1,2}"] & & Q_{1,1} \arrow[dl, "\psi_{1,1}"'] \arrow[dr, "\varphi_{1,1}"] & & Q_{2,1} \arrow[dl, "\varphi_{2,1}"'] \arrow[dr, "\psi_{2,1}"] & & Q_{2,2} \arrow[dl, "\varphi_{2,2}"'] \arrow[dr, "\psi_{2,2}"] & \\
P_1 & & Q' & & Q & & P & & P_2,
\end{tikzcd}$$

In Q, the curve s intersects with l transversally. Hence the induced map $X_1 \dashrightarrow X_2$ is the Atiyah flop. By construction both E_{χ_1} and $E_{\psi_{2,2}}$ are the strict transforms of F. By

Lemma 1.6.2 (2) both E_{χ_2} and $E_{\psi_{1,2}}$ are the strict transforms of $E_{\psi_{2,1}}$. Therefore ι is also an isomorphism in codimension one, which completes the proof. $\square$

THEOREM 2.7.5. *Suppose that (Q, D_h, D_f) is of type $m > 0$. Let l be an irreducible component of $\operatorname{Supp}(D_f|_{D_h})$ and take the elementary link $Q \leftarrow Q_{1,1} \rightarrow Q'$ with center along l. Let E be the exceptional divisor of the elementary link. Then $(Q', (D_h)_{Q'}, E)$ is a compactification of $\mathbb{A}^3$ compatible with a quadric fibration of type $(m-1)$.*

PROOF. By Lemma 1.6.1, we have $Q \setminus ((D_h)_{Q'} \cup E) \cong \mathbb{A}^3$. By Lemma 1.6.2, $(D_h)_{Q'}$ is normal. Hence it suffices to show that $(Q', (D_h)_{Q'}, E)$ is of type $(m-1)$.

By Theorem 2.7.1 we can take a q-section $s \subset D_h$ intersecting with l. Take elementary transformations as in Lemma 2.7.4. Let $B \subset P_1$ be the center of $\psi_{1,2}$. By Corollary 2.7.3, it suffices to show that $((D_h)_{P_1} \cdot B)_{P_1} = m-1$.

By Lemma 2.7.4 we have $P_1 \cong P_2$ and B_P is the center of $\psi_{2,1}$. Since $D_h|_{D_f}$ is not smooth, Lemma 1.5.2 now implies $x \in (D_h)_P \cap B_P$. Hence $(D_h)_{Q_{2,2}} \sim \varphi_{2,2}^*(D_h)_P - E_{\varphi_{2,2}} \sim \psi_{2,2}^*(D_h)_{P_1}$ by Corollary 1.4.2 and $((D_h)_{P_1} \cdot B)_{P_1} = ((D_h)_P \cdot B_P)_P - (E_{\varphi_{2,2}} \cdot B_{Q_{2,2}})_{Q_{2,2}} = m-1$ by Corollary 2.7.3. $\square$

An easy computation shows the following.

COROLLARY 2.7.6. *We follow the notation of Theorem 2.7.5. Suppose that $m = 1$ and take $d > 0$ such that $D_h \cong S_d$. Then $(D_h)_{Q'} \cong \mathbb{F}_d$ (resp. $\mathbb{F}_{d-1}$) when l intersects with (resp. is disjoint from) the strict transform of Σ_d in S_d.*

2.7.3. The case of smooth $D_f|_{D_h}$. By Theorem 2.7.5, we are reduced to prove Theorem 2.1.7 for the case where (Q, D_h, D_f) is of type 0, i.e. where D_h is a Hirzebruch surface. First we construct a birational map which decreases the degree of D_h as a Hirzebruch surface.

LEMMA 2.7.7. *Suppose that $D_h \cong \mathbb{F}_d$ for some $d > 0$. Set $\infty := q(D_h)$. Then there are an another compactification (Q', D'_h, D'_f) of $\mathbb{A}^3$ compatible with a quadric fibration q' and the composition $h \colon Q \dashrightarrow Q'$ of elementary links with center along rulings in the fibers at ∞ such that $(D_h)_{Q'} = D'_h \cong \mathbb{F}_{d-1}$ and $D'_f = (q')^*(\infty)$. In particular, h preserves $Q \setminus (D_f \cup D_h) \cong \mathbb{A}^3$.*

(2.7.3.1)
$$\begin{tikzcd}
(Q, D_h, D_f) \arrow[r, dotted, "h"] \arrow[d, "q"] & (Q', D'_h, D'_f) \arrow[d, "q'"] \\
\mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1,
\end{tikzcd}$$

PROOF. Take the elementary link $f_1 \colon Q \dashrightarrow Q_1$ with center along a ruling of $D_f \cong \mathbb{Q}_0$ which is disjoint from $\Sigma_d \subset D_h$. Let E be the exceptional divisor of the elementary link. Since f_1 induces the elementary transformation of D_h with center a point outside Σ_d, we get a compactification $(Q_1, (D_h)_{Q_1}, E)$ of $\mathbb{A}^3$ such that $(D_h)_{Q_1} \cong S_d$.

Now take the elementary link $f_2 \colon Q_1 \dashrightarrow Q'$ with center along the irreducible component of $\operatorname{Supp}(E|_{(D_h)_{Q_1}})$ which is disjoint from the strict transform of Σ_d in $(D_h)_{Q_1}$. Then by Corollary 2.7.6, we get a compactification (Q', D'_h, D'_f) of $\mathbb{A}^3$ such that $D'_h \cong \mathbb{F}_{d-1}$. Hence $h := f_2 \circ f_1$ is the desired birational map. $\square$

Repeated application of Lemma 2.7.7 enables us to assume that (Q, D_h, D_f) satisfies $D_h \cong \mathbb{F}_0$. Next we show that such a compactification is the same as (Q', D'_h, D'_f) as in Example 2.1.5.

Lemma 2.7.8. *Suppose that $D_h \cong \mathbb{F}_0$. Then Q is the blow-up of $\mathbb{Q}^3$ along a smooth conic and D_h is the exceptional divisor of the blow-up.*

Proof. First let us show the ampleness of $-K_Q$. By Lemma 2.4.1, we can take $a \in \mathbb{Z}$ such that $-K_Q \sim 2D_h + aD_f$. By the adjunction formula, we have $D_h|_{D_h} \sim -K_{D_h} - aD_f|_{D_h}$. Since $D_h \cong \mathbb{F}_0$, we have:

$$\begin{aligned} D_h^3 &= (K_{D_h} + aD_f|_{D_h})^2 \\ &= (K_{D_h})^2 + 2a(K_{D_h} \cdot D_f|_{D_h}) = 8 - 4a. \end{aligned} \tag{2.7.3.2}$$

$$\begin{aligned} (-K_Q)^3 &= (2D_h + aD_f)^3 \\ &= 8D_h^3 + 12a(D_h^2 \cdot D_f) = 64 - 8a. \end{aligned} \tag{2.7.3.3}$$

On the other hand, by Lemma 1.5.1 (3), it holds that $(-K_Q)^3 = 40 - (8p_a(B) + 32p_a(C))$. We have $C \cong \mathbb{P}^1$ by assumption. Combining Theorem 2.4.2 (B), (2.4.0.4) and (2.4.0.6), we get $p_a(B) = 0$. Hence $(-K_Q)^3 = 40$. Substituting this into (2.7.3.3), we have $a = 3$. Hence we have

$$-K_Q \sim 2D_h + 3D_f \tag{2.7.3.4}$$

and $-K_Q|_{D_h} \sim (2D_h + 3D_f)_{D_h} \sim -2K_{D_h} - 3D_f|_{D_h} \sim 4\Sigma_0 + f_0$, which is ample. Clearly $-K_Q|_{D_f}$ is also ample.

Suppose that $(-K_Q \cdot r) \leq 0$ holds for some curve $r \subset Q$. Since both $-K_Q|_{D_h}$ and $-K_Q|_{D_f}$ are ample, (2.7.3.4) now shows that r must be disjoint from any q-fiber, a contradiction. Hence $-K_Q$ is strictly nef. Also $-K_Q$ is big since $(-K_Q)^3 = 40$ and is semiample by the base-point free theorem. Since $-K_Q$ is strictly nef and semiample, it is ample.

Therefore Q is a Fano quadric fibration with $(-K_Q)^3 = 40$. By [**MM82**, Table 2], Q is the blow-up of $\mathbb{Q}^3$ along a smooth conic, which is the first assertion.

Let $h_2 \colon Q \to \mathbb{Q}^3$ be the blow-up morphism. Since $D_f \sim h_2^*\mathcal{O}_{\mathbb{Q}^3}(1) - E_{h_2}$ and $-K_Q \sim h_2^*\mathcal{O}_{\mathbb{Q}^3}(3) - E_{h_2} \sim 2E_{h_2} + 3D_f$, the second assertion follows from (2.7.3.4). □

Now we can prove Theorem 2.1.7.

Proof of Theorem 2.1.7. Suppose that (Q, D_h, D_f) is a compactification of $\mathbb{A}^3$ of type m. Taking elementary links m times as in Theorem 2.7.5, we may assume that $m = 0$. Repeated application of Lemma 2.7.7 enables us to assume that $D_h \cong \mathbb{F}_0$. Then $h_1 \coloneqq \mathrm{id}_Q$ and $(Q', D'_h, D'_f) \coloneqq (Q, D_h, D_f)$ satisfies all the assertion by Lemma 2.7.8. □

CHAPTER 3

$\mathbb{G}_a^3$-structures in del Pezzo fibrations

3.1. Introduction to Chapter 3

In this chapter, we are interested in compactifications of the affine n-space $\mathbb{G}_a^n$ with the additive group structure in the following sense.

Definition 3.1.1 ([**HT99**, Definition 2.1]). Let $\mathbb{G}$ be a connected linear algebraic group. *A $\mathbb{G}$-variety X* is a variety with a fixed (left) $\mathbb{G}$-action such that the stabilizer of a general point is trivial and the orbit of a general point is dense.

By *a $\mathbb{G}$-structure on X with the boundary divisor D*, we mean a $\mathbb{G}$-action on X which makes X a $\mathbb{G}$-variety whose dense open orbit is $X \setminus D$. We note that when $\mathbb{G} = \mathbb{G}_a^n$, we can reword a $\mathbb{G}_a^n$-variety as a variety with a fixed $\mathbb{G}_a^n$-action whose dense orbit is isomorphic to $\mathbb{G}_a^n$ because $\mathbb{G}_a^n$ is simply connected.

B. Hassett and Y. Tschinkel [**HT99**] considered $\mathbb{G}_a^n$-varieties originally, and classified all the smooth projective $\mathbb{G}_a^n$-varieties with the second Betti number $B_2 = 1$ when $n \leq 3$. Since smooth rational projective varieties with $B_2 = 1$ are Fano, we can rephrase their result as the classification of all the smooth Fano $\mathbb{G}_a^n$-varieties with $B_2 = 1$ when $n \leq 3$. After that, Z. Huang and P. Montero [**HM18**] classified all the smooth Fano $\mathbb{G}_a^3$-varieties with $B_2 \geq 2$. B. Fu and P. Montero [**FM19**] also classified all the smooth Fano $\mathbb{G}_a^n$-varieties with Fano index at least $n - 2$ for any dimension.

In this chapter, we consider smooth projective $\mathbb{G}_a^3$-varieties with $B_2 = 2$, which are not necessarily Fano. Take such a variety X, which is rational by definition. By virtue of the Mori theory, it has an extremal contraction $f \colon X \to C$ of relative Picard number one with $\dim C \geq 1$. The purpose of this chapter is to determine the structure of f when $\dim C = 1$, i.e., when f is a del Pezzo fibration. The main theorem of this chapter is the following,

Theorem 3.1.2. *Let X be a smooth projective 3-fold, D a reduced effective divisor on X and $f \colon X \to C$ a del Pezzo fibration. Then the following are equivalent.*

(1) *X has a $\mathbb{G}_a^3$-structure with the boundary divisor D.*
(2) *f is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ and D consists of a sub $\mathbb{P}^1$-bundle D_1 and a f-fiber D_2 which generate $\Lambda_{\mathrm{eff}}(X)$.*

3.2. Structure of Chapter 3

This chapter is structured as follows. In §3.3, we recall some facts on actions of algebraic groups on algebraic varieties. Using them, we prove that Theorem 3.1.2 (1) implies (2) in §3.4. The main step to prove this implication is Proposition 3.4.4, that is, the exclusion of the case of quadric fibrations. For this, we use the results in Chapter 2. Finally, we prove the opposite implication in §3.5. For that, we construct a $\mathbb{G}_a^3$-structure for each $\mathbb{P}^2$-bundle P over $\mathbb{P}^1$ via a sequence of elementary links from $\mathbb{P}^1 \times \mathbb{P}^2$ to P.

3.3. Preliminaries on group actions

In this section, we compile some facts on actions of algebraic groups on algebraic varieties, which will be needed in §3.4 and §3.5.

THEOREM 3.3.1 ([**HT99**, Theorem 2.5, 2.7]). *Let X be a normal proper $\mathbb{G}_a^3$-variety with the boundary divisor D and $D = \cup_{i=1}^n D_i$ the irreducible decomposition. Then we have the following:*

(1) $\mathrm{Pic}(X) = \bigoplus_{i=1}^n \mathbb{Z}D_i$.
(2) $-K_X \sim \sum_{i=1}^n a_i D_i$ *for some integers* $a_1, \ldots, a_n \geq 2$.
(3) $\Lambda_{\mathrm{eff}}(X) = \bigoplus_{i=1}^n \mathbb{R}_{\geq 0} D_i$.

THEOREM 3.3.2 ([**Bri17**, Theorem 7.2.1]). *Let G be a connected algebraic group, X a variety with G-action, Y a variety and $f\colon X \to Y$ a proper morphism such that $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. Then there exists the unique G-action on Y such that f is equivariant.*

3.4. Proof of Theorem 3.1.2 (1) $\Rightarrow$ (2)

In this section, we prove that Theorem 3.1.2 (1) implies (2). For this, we make the following assumption in this section:

ASSUMPTION 1. X is a smooth projective $\mathbb{G}_a^3$-variety with the boundary divisor D. $f\colon X \to C$ is a del Pezzo fibration of degree d.

By Theorem 3.3.1, D consists of two irreducible components, say $D_1 \cup D_2$.

LEMMA 3.4.1. *It holds that $C \cong \mathbb{P}^1$.*

PROOF. X is rational since it contains $\mathbb{G}_a^3$ as the dense open orbit. Since $H^0(C, \Omega_C) \hookrightarrow H^0(X, \Omega_X) \cong 0$, we have $H^0(C, \Omega_C) \cong 0$ and the assertion holds. □

PROPOSITION 3.4.2. *The boundary divisor D contains a f-fiber which is stable under $\mathbb{G}_a^3$-action.*

PROOF. By Theorem 3.3.2, there is the $\mathbb{G}_a^3$-action on C such that f is $\mathbb{G}_a^3$-equivariant. By the Borel fixed-point theorem [**Hum75**, §21.2], the action $\mathbb{G}_a^3 \curvearrowright C$ has a fixed point, say $\infty \in C$. Since the divisor $f^*(\infty)$ is stable under the $\mathbb{G}_a^3$-action, it is contained in D. □

In the remainder of this section we require D_2 to be a f-fiber.

PROPOSITION 3.4.3. *It holds that $d \geq 8$.*

PROOF. Conversely, suppose that $d \leq 7$. By Theorem 3.3.1 (1), we have $\mathrm{Pic}(X) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2$. On the other hand, take a (-1)-curve l in a general f-fiber. Combining $(-K_X \cdot l) = 1$ and [**Mor82**, Theorem 3.2] (2), we have $\mathrm{Pic}(X) = \mathbb{Z}(-K_X) \oplus \mathbb{Z}D_2$. Hence we can write $-K_X \sim a_1 D_1 + a_2 D_2$ with $a_1 = 1$ and $a_2 \in \mathbb{Z}$, a contradiction with Theorem 3.3.1 (2). □

PROPOSITION 3.4.4. *It holds that $d \neq 8$.*

Proof. Conversely, suppose that $d = 8$.
Step 1: First we show that we get a contradiction if there is a $\mathbb{G}_a^3$-stable f-section, say s. In this case, applying Lemma 1.5.1 with q replaced by f, we can obtain the following commutative diagram:

(3.4.0.1)
$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[d, "f"'] & & P \arrow[d, "p"] \\ C \arrow[rr, equal] & & C \end{tikzcd}$$

where φ is the blow-up along s, p is a $\mathbb{P}^2$-bundle and ψ is the blow-up along a smooth connected p-bisection, say B.

Since s is $\mathbb{G}_a^3$-stable, $\widetilde{X}$ admits the unique $\mathbb{G}_a^3$-action such that φ is equivariant. By Theorem 3.3.2, P and C also admit the unique $\mathbb{G}_a^3$-actions such that ψ and p are equivariant respectively. Since E_ψ is $\mathbb{G}_a^3$-stable, so is B. Hence $p|_B \colon B \to C$ is a $\mathbb{G}_a^3$-equivariant double covering. Since X has the dense open orbit, so does C. Since $p|_B$ is surjective, finite and $\mathbb{G}_a^3$-equivariant, B also has the dense open orbit. Since C and B have dominant maps from $\mathbb{G}_a^3$, we obtain $C \cong B \cong \mathbb{P}^1$.

Let us show that B has the unique $\mathbb{G}_a^3$-fixed point. By [**HM18**, Proposition 3.6], $\mathbb{G}_a^3$ contains a subgroup $G \cong \mathbb{G}_a^2$ such that the $\mathbb{G}_a^3$-action on B factorizes via $\mathbb{G}_a^3/G \cong \mathbb{G}_a^1$. Since $\mathbb{G}_a^1$ has no non-trivial algebraic subgroup, the stabilizer of a general point of this $\mathbb{G}_a^1$-action is trivial. Hence this action is a $\mathbb{G}_a^1$-structure of B. By [**HT99**, Proposition 3.1], B has the unique fixed point. By the same argument, C also has the unique $\mathbb{G}_a^3$-fixed point.

Let $b \in B$ and $c \in C$ are the $\mathbb{G}_a^3$-fixed points. Since $p|_B$ is equivariant, we have $p(b) = c$. If $p|_B$ is unramified at b, then the point in $(p|_B)^{-1}(c) \setminus \{b\}$ is also fixed, a contradiction. Hence $p|_B$ is ramified at b. Since $C \cong B \cong \mathbb{P}^1$, $p|_B$ has the other ramification point, which is also fixed, a contradiction.
Step 2: Now it suffices to find a $\mathbb{G}_a^3$-stable f-section. By Theorem 3.3.1 (2), there are integers $a_1, a_2 \geq 2$ such that $-K_X \sim a_1 D_1 + a_2 D_2$. For a smooth f-fiber $F \cong \mathbb{F}_0$, the restriction $-K_X|_F \sim a_1 D_1|_F$ is a divisor of bidegree $(2, 2)$. Hence $a_1 = 2$. On the other hand, by the choice of D_2, (X, D_1, D_2) is a compactification of $\mathbb{A}^3$ compatible with f (See Definition 2.1.1).

If D_1 is non-normal, then $s \coloneqq \operatorname{Sing} D_1$ forms a section by Lemma 1.6.2. Since D_1 is $\mathbb{G}_a^3$-stable, so is s. Therefore we derive a contradiction as in Step 1.

Hence D_1 is normal. By Theorem 2.4.2, we obtain $D_2 \cong \mathbb{Q}_0^2$. Suppose that (X, D_1, D_2) is of type $m > 0$ in the sense of Definition 2.7.2. Then $\operatorname{Supp}(D_1|_{D_2})$ contains a ruling of the quadric cone D_2 by Theorem 2.7.1, say l. applying Lemma 1.6.1 with q replaced by f, we can obtain the following commutative diagram:

(3.4.0.2)
$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[d, "f"'] & & X' \arrow[d, "f'"] \\ C \arrow[rr, equal] & & C \end{tikzcd}$$

where φ is the blow-up along l, f' is a quadric fibration and ψ is the blow-up along a ruling in a singular f'-fiber such that $E_\psi = (D_2)_{\widetilde{X}}$.

Since $\mathrm{Supp}(D_1|_{D_2})$ is $\mathbb{G}_a^3$-stable and $\mathbb{G}_a^3$ is irreducible, l is also $\mathbb{G}_a^3$-stable. Hence $\widetilde{X}$ admits a $\mathbb{G}_a^3$-structure with the boundary divisor $(D_1 \cup D_2)_{\widetilde{X}} \cup E_\varphi$. Theorem 3.3.2 now gives X' a $\mathbb{G}_a^3$-structure with the boundary divisor $(D_1)_{X'} \cup (E_\varphi)_{X'}$. By Theorem 2.7.5, $(X', (D_1)_{X'}, (E_\varphi)_{X'})$ is of type $m-1$.

By repeated application of the above construction, we only have to exclude the case when (X, D_1, D_2) is of type 0. Then D_1 is $\mathbb{G}_a^3$-stable and is isomorphic to $\mathbb{F}_n$ for some n by definition. If $n > 0$, then the negative section s in D_1 is a $\mathbb{G}_a^3$-stable f-section, and we derive a contradiction as in Step 1. Hence $n = 0$. There is the $\mathbb{P}^1$-bundle structure $h\colon D_1 \to \mathbb{P}^1$ other than $f|_{D_1}$. Combining Theorem 3.3.2 and the Borel fixed-point theorem, we get a $\mathbb{G}_a^3$-stable h-fiber s, which is a f-section. Therefore we derive a contradiction as in Step 1. □

Proof of Theorem 3.1.2 (1) ⇒ (2). Suppose that (1) holds, Combining Propositions 3.4.3 and 3.4.4, we get $d = 9$. By Theorem 3.3.1 (2), there are integers $a_1, a_2 \geq 2$ such that $-K_X \sim a_1 D_1 + a_2 D_2$. By the adjunction formula, we have $a_1 D_1|_{D_2} \sim -K_X|_{D_2} \sim -K_{D_2} \sim \mathcal{O}_{\mathbb{P}^2}(3)$. Hence $a_1 = 3$ and D_1 is a sub $\mathbb{P}^1$-bundle. The second assertion of (2) follows from Theorem 3.3.1 (3). □

3.5. Proof of Theorem 3.1.2 (2) ⇒ (1)

In this section, we prove that Theorem 3.1.2 (2) implies (1).

Notation 2. For this, we make the following notation in this section:

- p_{d_1,d_2}: the $\mathbb{P}^2$-bundle structure of $\mathbb{F}(-d_1, -d_2, 0)$.
- ξ_{d_1,d_2}: a tautological divisor of $\mathbb{F}(-d_1, -d_2, 0)$.

To complete the proof of Theorem 3.1.2, we prepare the following five lemmas.

Lemma 3.5.1. *Let $P := \mathbb{F}(-d_1, -d_2, 0)$ with $d_1 \geq d_2 \geq 0$, E a sub $\mathbb{P}^1$-bundle of P and F a p_{d_1,d_2}-fiber. Then E and F generate $\Lambda_{\mathrm{eff}}(P)$ if and only if $E \sim \xi_{d_1,d_2}$. Moreover, in this case, the pair (E, F) is unique up to* $\mathrm{Aut}(X)$.

Proof. Recall from [**Rei97**, Chapter 2] that $P = \mathbb{F}(-d_1, -d_2, 0)$ is defined as the quotient of $(\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\})$ by the following $(\mathbb{G}_m)^2$-action:

$$\begin{array}{rcl} (\mathbb{G}_m)^2 \times (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\}) & \to & (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\}) \\ ((\lambda, \mu), (t_1, t_2; x_1, x_2, x_3)) & \mapsto & (\lambda t_0, \lambda t_1; \lambda^{d_1} \mu x_1, \lambda^{d_2} \mu x_2, \mu x_3). \end{array}$$

We also have $\mathrm{Pic}\, P = \mathbb{Z}\xi_{d_1,d_2} \oplus \mathbb{Z}F$, and for each $a, b \in \mathbb{Z}$, the linear system $|a\xi_{d_1,d_2} + bF|$ is parametrized by the vector space of polynomials spanned by monomials $t_1^{b_1} t_2^{b_2} x_1^{a_1} x_2^{a_2} x_3^{a_3} \in \mathbb{C}[t_1, t_2, x_1, x_2, x_3]$ with $a_1 + a_2 + a_3 = a$ and $b_1 + b_2 = -d_1 a_1 - d_2 a_2 + b$. Hence $|a\xi_{d_1,d_2} + bF| \neq \emptyset$ if and only if $a \geq 0$ and $b \geq 0$, and the first assertion follows.

Now suppose that $E \sim \xi_{d_1,d_2}$. Then E is defined by $\sum_{i=1}^3 u_i x_i$ for some $u_i \in \mathbb{C}$ for $i = 1, 2, 3$ such that $u_i = 0$ unless $d_i = 0$ for $i = 1, 2$. Suppose that $u_3 = 0$. Then $u_i \neq 0$ for some $i = 1, 2$. Take $\widetilde{h} \in \mathrm{Aut}((\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\}))$ which interchanges x_i and x_3, which is $(\mathbb{G}_m)^2$-equivariant. Since P is the geometric quotient by [**MFK94**, Proposition 1.9], it descends to an element in $\mathrm{Aut}(P)$. Hence we may assume that $u_3 = 1$. By a similar argument, we also may assume that F is defined by $t_1 + v t_2$ for some $v \in \mathbb{C}$.

Now let E' and F' be divisors on P defined by x_3 and t_1 respectively. Take $\widetilde{h} \in \operatorname{Aut}((\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\}))$ such that

$$\widetilde{h}^*(x_1) = x_1, \widetilde{h}^*(x_2) = x_2, \widetilde{h}^*(x_3) = c_1x_1 + c_2x_2 + x_3, \tag{3.5.0.1}$$

$$\widetilde{h}^*(t_1) = t_1 + vt_2, \widetilde{h}^*(t_2) = t_2. \tag{3.5.0.2}$$

Since $\widetilde{h}$ is $(\mathbb{G}_m)^2$-equivariant, it descends to $h \in \operatorname{Aut}(P)$ such that $h(E) = E'$ and $h(F) = F'$, which complete the proof. □

Lemma 3.5.2. *We follow the situation of Lemma 1.4.1. Suppose that $P = \mathbb{F}(-d, -d, 0)$ with $d \geq 0$ and $n = 1$. If there exists $H \in |\xi_{d,d}|$ containing L, then $P' \cong \mathbb{F}(-d-1, -d-1, 0)$ and $H_{P'} \sim \xi_{d+1,d+1}$.*

Proof. Set $\mathcal{F} = p'_*\mathcal{O}_{P'}(H_{P'})$. It suffices to show that $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(-d-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}$. Pushing forward the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{P}}(\varphi^*H - E_\varphi) \longrightarrow \mathcal{O}_{\widetilde{P}}(\varphi^*H) \longrightarrow \mathcal{O}_{E_\varphi}(\varphi^*H|_{E_\varphi}) \longrightarrow 0 \tag{3.5.0.3}$$

by $p \circ \varphi$, we get the following exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathbb{C}^{\oplus 2} \longrightarrow 0 \tag{3.5.0.4}$$

since $\varphi^*H - E_\varphi \sim \psi^*(H_{P'})$ by Theorem 1.4.1 (2). On the other hand, we have $H_{P'} \cong \mathbb{F}_0$ because $L \subset H$ and $H \cong \mathbb{F}_0$. By the definition of $\mathcal{F}$, the inclusion $H_{P'} \subset P'$ corresponds to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-a)^{\oplus 2} \longrightarrow 0 \tag{3.5.0.5}$$

for some $a \in \mathbb{Z}$. Combining (3.5.0.4) and (3.5.0.5), we obtain $-2a = \deg \mathcal{F} = -2d-2$. Hence $a = d+1$ and (3.5.0.5) splits, which proves the lemma. □

Lemma 3.5.3. *We follow the situation of Lemma 3.5.2. Set $\infty := p(L) \in C$. If P admits a $\mathbb{G}_a^3$-structure with the boundary divisor $H \cup p^*(\infty)$, then so does P' with the boundary divisor $H_{P'} \cup p'^*(\infty)$.*

Proof. Since $L = H \cap p^*(\infty)$, this is $\mathbb{G}_a^3$-stable. Hence $\widetilde{P}$ admits a $\mathbb{G}_a^3$-structure with the boundary divisor $H_{\widetilde{P}} \cup (p \circ \varphi)^*(\infty)$. Applying Theorem 3.3.2 to $\psi \colon \widetilde{P} \to P'$, we obtain a desired $\mathbb{G}_a^3$-structure on P'. □

Lemma 3.5.4. *We follow the situation of Lemma 1.4.1. Suppose that $P = \mathbb{F}(-d_1, -d_2, 0)$ with $d_1 \geq d_2 \geq 0$ and $n = 0$. Assume that there exists $H \in |\xi_{d_1,d_2}|$ containing L, and when $d_1 > d_2$, assume that the negative section of $H \cong \mathbb{F}_{d_1-d_2}$ passes through L in addition. Then $P' \cong \mathbb{F}(-d_1-1, -d_2, 0)$ and $H_{P'} \sim \xi_{d_1+1,d_2}$.*

Proof. Set $\mathcal{F} = p'_*\mathcal{O}_{P'}(H_{P'})$. It suffices to show that $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(-d_1-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-d_2) \oplus \mathcal{O}_{\mathbb{P}^1}$. By similar arguments as in Lemma 3.5.2, we get the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-d_2) \oplus \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathbb{C} \longrightarrow 0. \tag{3.5.0.6}$$

Hence $\deg \mathcal{F} = -d_1 - d_2 - 1$. On the other hand, we have $H_{P'} \cong \mathbb{F}_{d_1-d_2+1}$ by the choice of L. By the definition of $\mathcal{F}$, the inclusion $H_{P'} \subset P'$ corresponds to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d_1-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-d_2) \longrightarrow 0. \tag{3.5.0.7}$$

Since (3.5.0.7) splits, we get the assertion. □

LEMMA 3.5.5. *We follow the situation of Lemma 3.5.4. Set $\infty \coloneqq p(L) \in C$. If P admits a $\mathbb{G}_a^3$-structure with the boundary divisor $H \cup p^*(\infty)$ such that L is a fixed point, then so does P' with the boundary divisor $H_{P'} \cup p'^*(\infty)$.*

PROOF. Since L is $\mathbb{G}_a^3$-stable by assumption, we can prove the assertion in much the same way as Lemma 3.5.3. □

Now we can prove that Theorem 3.1.2 (2) implies (1).

PROOF OF THEOREM 3.1.2 (2) ⇒ (1). In $\mathbb{P}^1_{[t_1:t_2]} \times \mathbb{P}^2_{[x_1:x_2:x_3]}$, set $E \coloneqq \{x_3 = 0\}$ and $F \coloneqq \{t_1 = 0\}$. Write $\infty \coloneqq [0:1] \in \mathbb{P}^1$. Then E and F generate $\Lambda_{\mathrm{eff}}(\mathbb{P}^1 \times \mathbb{P}^2)$. By [**HM18**, Lemma 3.7], $\mathbb{P}^1 \times \mathbb{P}^2$ admits a $\mathbb{G}_a^3$-structure with the boundary divisor $E \cup F$. Write this structure as $\rho\colon \mathbb{G}_a^3 \curvearrowright \mathbb{P}^1 \times \mathbb{P}^2$.

Now suppose that (2) follows. Then $X \cong \mathbb{F}(-d_1, -d_2, 0)$ for some $d_1 \geq d_2 \geq 0$ and $f = p_{d_1,d_2}$. By assumption and Lemma 3.5.1, it holds that $D_1 \sim \xi_{d_1,d_2}$ and D_2 is a p_{d_1,d_2}-fiber.

Suppose that $d_1 = d_2 = 0$. Then we may assume that $(D_1, D_2) = (E, F)$ by Lemma 3.5.1 and hence ρ is a desired structure.

Suppose that $d_1 = d_2 > 0$. Then by Lemma 3.5.2, we can inductively construct the sequence of the elementary links from $p_{0,0}\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$:

$$
\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^2 \arrow[r, dotted, "h_0"] \arrow[d, "p_{0,0}"'] & \mathbb{F}(-1,-1,0) \arrow[r, dotted, "h_1"] \arrow[d, "p_{1,1}"'] & \cdots \arrow[r, dotted, "h_{d_1-1}"] & \mathbb{F}(-d_1,-d_1,0) \arrow[r, equal] \arrow[d, "p_{d_1,d_1}=f"'] & X \\
\mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \arrow[r, equal] & \cdots \arrow[r, equal] & \mathbb{P}^1,
\end{tikzcd}
\tag{3.5.0.8}
$$

where the center of h_i is the intersection of $E_i \coloneqq E_{\mathbb{F}(-i,-i,0)}$ and $F_i \coloneqq p_{i,i}^*(\infty)$ for $0 \leq i \leq d_1 - 1$. Set $E_{d_1} \coloneqq E_X$ and $F_{d_1} \coloneqq f^*(\infty)$. Then $E_i \sim \xi_{i,i}$ for $0 \leq i \leq d_1$ by Lemma 3.5.2 and hence we may assume that $(D_1, D_2) = (E_{d_1}, F_{d_1})$ by Lemma 3.5.1.

For $0 \leq i \leq d_1 - 1$, suppose that $\mathbb{F}(-i,-i,0)$ admits a $\mathbb{G}_a^3$-structure with the boundary divisor $E_i \cup F_i$. Then so does $\mathbb{F}(-(i+1), -(i+1), 0)$ with the boundary divisor $E_{i+1} \cup F_{i+1}$ by Lemma 3.5.3. Thus ρ induces a desired $\mathbb{G}_a^3$-structure on X.

Suppose that $d_1 > d_2 \geq 0$. Set $d = d_1 - d_2$. Let ρ' be a $\mathbb{G}_a^3$-structure of $\mathbb{F}(-d_2, -d_2, 0)$, which we have already constructed. Write its boundary divisor as $E' \cup F'$ such that $E' \sim \xi_{d_2,d_2}$ and $F' = p_{d_2,d_2}^*(\infty)$. By the Borel fixed-point theorem, there is a $\mathbb{G}_a^3$-fixed point in $E' \cap F'$, say t_0. Then by Lemma 3.5.4, we can inductively construct the sequence of the elementary links from $p_{d_2,d_2}\colon \mathbb{F}(-d_2,-d_2,0) \to \mathbb{P}^1$:

$$
\begin{tikzcd}
\mathbb{F}(-d_2,-d_2,0) \arrow[r, dotted, "h_0"] \arrow[d, "p_{d_2,d_2}"'] & \mathbb{F}(-d_2-1,-d_2,0) \arrow[r, dotted, "h_1"] \arrow[d, "p_{d_2+1,d_2}"'] & \cdots \arrow[r, dotted, "h_{d-1}"] & \mathbb{F}(-d_1,-d_2,0) \arrow[r, equal] \arrow[d, "p_{d_1,d_2}=f"'] & X \\
\mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \arrow[r, equal] & \cdots \arrow[r, equal] & \mathbb{P}^1,
\end{tikzcd}
\tag{3.5.0.9}
$$

where the center of h_i is t_0 for $i = 0$ and the intersection of the negative section of $E'_i \coloneqq E'_{\mathbb{F}(-d_2-i,-d_2,0)} \cong \mathbb{F}_i$ and $F'_i \coloneqq p_{d_2+i,d_2}^*(\infty)$ for $1 \leq i \leq d-1$. Set $E'_d \coloneqq E'_X$ and $F'_d \coloneqq f^*(\infty)$. Then $E'_i \sim \xi_{d_2+i,d_2}$ for $0 \leq i \leq d$ by Lemma 3.5.4 and hence we may assume that $(D_1, D_2) = (E'_d, F'_d)$ by Lemma 3.5.1.

Since t_0 is a fixed point of the action ρ', $\mathbb{F}(-d_2-1,-d_2,0)$ admits a $\mathbb{G}_a^3$-structure with the boundary divisor $E_1' \cup F_1'$ by Lemma 3.5.5.

For $1 \leq i \leq d-1$, suppose that $\mathbb{F}(-d_2-i,-d_2,0)$ admits a $\mathbb{G}_a^3$-structure with the boundary divisor $E_i' \cup F_i'$. Then t_i is a $\mathbb{G}_a^3$-fixed point by construction. Hence $\mathbb{F}(-d_2-(i+1),-d_2,0)$ admits a $\mathbb{G}_a^3$-structure with the boundary divisor $E_{i+1}' \cup F_{i+1}'$ by Lemma 3.5.5. Thus ρ' induces a desired $\mathbb{G}_a^3$-structure on X. $\square$

Bibliography

[BD93] S. M. Bhatwadekar and Amartya K. Dutta. On residual variables and stably polynomial algebras. *Comm. Algebra*, 21(2):635–645, 1993.

[Bri17] Michel Brion. Some structure theorems for algebraic groups. In *Algebraic groups: structure and actions*, volume 94 of *Proc. Sympos. Pure Math.*, pages 53–126. Amer. Math. Soc., Providence, RI, 2017.

[Cor95] Alessio Corti. Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.*, 4(2):223–254, 1995.

[D'S88] Harry D'Souza. Threefolds whose hyperplane sections are elliptic surfaces. *Pacific J. Math.*, 134(1):57–78, 1988.

[FM19] Baohua Fu and Pedro Montero. Equivariant compactifications of vector groups with high index. *C. R. Math. Acad. Sci. Paris*, 357(5):455–461, 2019.

[FN89a] Mikio Furushima and Noboru Nakayama. The family of lines on the Fano threefold V_5. *Nagoya Math. J.*, 116:111–122, 1989.

[FN89b] Mikio Furushima and Noboru Nakayama. A new construction of a compactification of $\mathbf{C}^3$. *Tohoku Math. J. (2)*, 41(4):543–560, 1989.

[Fuj82] Takao Fujita. On the topology of noncomplete algebraic surfaces. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 29(3):503–566, 1982.

[Fuk18] Takeru Fukuoka. Relative linear extensions of sextic del pezzo fibrations. *arXiv preprint arXiv:1803.01264*, 2018.

[Fur86] Mikio Furushima. Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space $\mathbf{C}^3$. *Nagoya Math. J.*, 104:1–28, 1986.

[Fur90] Mikio Furushima. Complex analytic compactifications of $\mathbf{C}^3$. *Compositio Math.*, 76(1-2):163–196, 1990. Algebraic geometry (Berlin, 1988).

[Fur93a] Mikio Furushima. The complete classification of compactifications of $\mathbf{C}^3$ which are projective manifolds with the second Betti number one. *Math. Ann.*, 297(4):627–662, 1993.

[Fur93b] Mikio Furushima. A new example of a compactification of $\mathbf{C}^3$. *Math. Z.*, 212(3):395–399, 1993.

[Fur00] Mikio Furushima. A birational construction of projective compactifications of $\mathbf{C}^3$ with second Betti number equal to one. *Ann. Mat. Pura Appl. (4)*, 178:115–128, 2000.

[Hir54] Friedrich Hirzebruch. Some problems on differentiable and complex manifolds. *Ann. of Math. (2)*, 60:213–236, 1954.

[HM18] Zhizhong Huang and Pedro Montero. Fano threefolds as equivariant compactifications of the vector group. *arXiv preprint arXiv:1802.08090*, 2018.

[HT99] Brendan Hassett and Yuri Tschinkel. Geometry of equivariant compactifications of $\mathbf{G}_a^n$. *Internat. Math. Res. Notices*, (22):1211–1230, 1999.

[HT12] Brendan Hassett and Yuri Tschinkel. Spaces of sections of quadric surface fibrations over curves. In *Compact moduli spaces and vector bundles*, volume 564 of *Contemp. Math.*, pages 227–249. Amer. Math. Soc., Providence, RI, 2012.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.

[IF77] Shigeru Iitaka and Takao Fujita. Cancellation theorem for algebraic varieties. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 24(1):123–127, 1977.

[Kis05] Takashi Kishimoto. Compactifications of contractible affine 3-folds into smooth Fano 3-folds with $B_2 = 2$. *Math. Z.*, 251(4):783–820, 2005.

[KR97] M. Koras and Peter Russell. Contractible threefolds and $\mathbf{C}^*$-actions on $\mathbf{C}^3$. *J. Algebraic Geom.*, 6(4):671–695, 1997.

[KZ99] Sh. Kaliman and M. Zaidenberg. Affine modifications and affine hypersurfaces with a very transitive automorphism group. *Transform. Groups*, 4(1):53–95, 1999.

[Man66] Ju. I. Manin. Rational surfaces over perfect fields. *Inst. Hautes Études Sci. Publ. Math.*, (30):55–113, 1966.

[Mar73] Masaki Maruyama. On a family of algebraic vector bundles. In *Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki*, pages 95–146. Kinokuniya, Tokyo, 1973.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[MM82] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. *Manuscripta Math.*, 36(2):147–162, 1981/82.

[Mor73] Shigefumi Mori. $\mathbb{A}^2$ の rational ruled surfaces への埋め込みについて (An article about compactifications of $\mathbb{A}^2$ into Hirzebruch surfaces). *Sūrikaisekikenkyūsho Kōkyūroku*, 183:31–50, 1973.

[Mor82] Shigefumi Mori. Threefolds whose canonical bundles are not numerically effective. *Ann. of Math. (2)*, 116(1):133–176, 1982.

[MS90] Stefan Müller-Stach. Compactifications of $\mathbf{C}^3$ with reducible boundary divisor. *Math. Ann.*, 286(1-3):409–431, 1990.

[Nag18] Masaru Nagaoka. Fano compactifications of contractible affine 3-folds with trivial log canonical divisors. *Internat. J. Math.*, 29(6):1850042, 33, 2018.

[Nag19a] Masaru Nagaoka. $\mathbb{G}_a^3$-structures on del pezzo fibrations. *arXiv preprint arXiv:1912.02524*, 2019.

[Nag19b] Masaru Nagaoka. On compactifications of affine homology 3-cells into quadric fibrations. *arXiv preprint arXiv:1906.10626*, 2019.

[Pet89] Thomas Peternell. Compactifications of $\mathbf{C}^3$. II. *Math. Ann.*, 283(1):121–137, 1989.

[Pro91] Yu. G. Prokhorov. Fano threefolds of genus 12 and compactifications of $\mathbf{C}^3$. *Algebra i Analiz*, 3(4):162–170, 1991.

[PS88] Thomas Peternell and Michael Schneider. Compactifications of $\mathbf{C}^3$. I. *Math. Ann.*, 280(1):129–146, 1988.

[Ram71] C. P. Ramanujam. A topological characterisation of the affine plane as an algebraic variety. *Ann. of Math. (2)*, 94:69–88, 1971.

[Rei97] Miles Reid. Chapters on algebraic surfaces. In *Complex algebraic geometry (Park City, UT, 1993)*, volume 3 of *IAS/Park City Math. Ser.*, pages 3–159. Amer. Math. Soc., Providence, RI, 1997.

[RvdV60] Reinhold Remmert and Ton van de Ven. Zwei Sätze über die komplex-projektive Ebene. *Nieuw. Arch. Wisk. (3)*, 8:147–157, 1960.

[tDP90] Tammo tom Dieck and Ted Petrie. Contractible affine surfaces of Kodaira dimension one. *Japan. J. Math. (N.S.)*, 16(1):147–169, 1990.

[vdV62] A. van de Ven. Analytic compactifications of complex homology cells. *Math. Ann.*, 147:189–204, 1962.