（要約）

博士論文

論文題目　On induction for twisted representations of conformal nets
(共形ネットの捩れ表現の誘導について)

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On Induction for Twisted Representations of Conformal Nets

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Summary

In the Haag-Kastler framework of quantum field theory, a chiral components of 2D conformal field theory is described by a conformal net on the unit circle S^1. A conformal net $\mathcal{A}$ is defined to be a map $I \mapsto \mathcal{A}(I)$ from the set of open intervals of S^1 to that of von Neumann algebras. These von Neumann algebras are considered as algebras of observables and required to satisfy certain axioms. We have a natural notion of representations of $\mathcal{A}$ and the representation theory plays an important role in the study of conformal nets. Moreover, the study of representation categories of conformal nets themselves is the one of the most interesting topics in this area. By the Doplicher-Haag-Roberts theory [5, 6], it turns out that every representation is equivalent to a localized transportable endomorphism (called a DHR endomorphism) of $\mathcal{A}$ and $\mathrm{Rep}(\mathcal{A})$ has a structure of a braided C*-tensor category.

In the study of conformal nets, their inclusions arise in several ways. Let $\mathcal{B}$ a conformal net and $\mathcal{A}$ an extension of $\mathcal{B}$. This inclusion gives us a net of subfactors $\{\mathcal{B}(I) \subset \mathcal{A}(I)\}_{I \in \mathcal{I}}$. In the article [8], general theory of nets of subfactors has been developed. If the index of the subfactor $\mathcal{B}(I) \subset \mathcal{A}(I)$ is finite for some I, it has shown that the extension $\mathcal{A}$ is completely characterized by a commutative Q-system (or a standard C*-Frobenius algebra objects) $\Theta = (\theta, w, x)$ in $\mathrm{Rep}(\mathcal{B})$. For a finite index inclusion of conformal nets $\mathcal{B} \subset \mathcal{A}$, we can consider induction and restriction procedures for DHR endomorphisms of $\mathcal{B}$ and $\mathcal{A}$. The induction procedure is called the α-induction and the restriction procedure is called the σ-restriction, respectively. The notions of α-induction and σ-restriction were first introduced in [8]. Their properties have been studied with examples in [11], and then further developed in [2, 3, 4]. For the later explanation, we briefly review the definition of α-induction. For a given DHR endomorpshim λ of $\mathcal{B}$, $\alpha_\lambda^\pm$ is given as an extension of λ to $\mathcal{A}$. The endomorphism $\alpha_\lambda^\pm$ is defined with the Q-system Θ and the braiding on $\mathrm{Rep}(\mathcal{B})$. Note that the superscript $\pm$ in the notation of $\alpha_\lambda^\pm$ represents the choice of braiding. (We have two canonical choices of braiding on $\mathrm{Rep}(\mathcal{B})$.) Although an α-induced endomorphism does not necessarily give a DHR endomorphism of $\mathcal{A}$, it is known that a subobject of both α_λ^+ and α_λ^- is a DHR endomorphism. The α-induction construction is a powerful tool for studying representation categories of conformal nets.

We now consider the subnets obtained from subgroups of automorphism group of $\mathcal{A}$. Let $G < \mathrm{Aut}(\mathcal{A})$ be a subgroup. Then we can construct a subnet of $\mathcal{A}$ by taking its fixed

point net $\mathcal{A}^G$. If G is finite, the net obtained in this way is called an orbifold. Orbifolds of conformal nets and their representations have been studied in [12] and [9, 7]. To study the categorical structure of $\mathrm{Rep}(\mathcal{A}^G)$ more systematically, Müger introduced the category of G-twisted representation $G-\mathrm{Loc}\mathcal{A}$ in [10]. In addition to DHR endomorphisms of $\mathcal{A}$, this category contains g-localized transportable endomorphisms for all $g \in G$ as its objects. In the same article [10], it has been shown that $G-\mathrm{Loc}\mathcal{A}$ has a structure of braided G-crossed category. Roughly speaking, a G-crossed category is a tensor category with a G-graiding, a group action of G and a certain kind of braiding (called a G-crossed braiding). Also, the relation between $G-\mathrm{Loc}\mathcal{A}$ and $\mathrm{Rep}(\mathcal{A}^G)$ was clarified: There exists a braided equivalence $(G-\mathrm{Loc}\mathcal{A})^G \cong \mathrm{Rep}(\mathcal{A}^G)$. Moreover, there exists a equivalence of braided G-crossed categories $\mathrm{Rep}(\mathcal{A}^G) \rtimes \mathrm{Rep}(G) \cong G-\mathrm{Loc}\mathcal{A}$ (see [10], for notations and terminology which are not explained here). Thus the study of $G-\mathrm{Loc}\mathcal{A}$ leads to the study of $\mathrm{Rep}(\mathcal{A}^G)$.

In this thesis, we consider a situation that we have a given finite index inclusion of conformal nets $\mathcal{B} \subset \mathcal{A}$ and a group $G < \mathrm{Aut}(\mathcal{A})$ which preserves $\mathcal{B}$ globally. Let us denote by $G' < \mathrm{Aut}(\mathcal{B})$ the group obtained by restricting each element of G to $\mathcal{B}$. For such a situation, it is natural to study the relation between the categories $G-\mathrm{Loc}\mathcal{A}$ and $G'-\mathrm{Loc}\mathcal{B}$. Our question is how to capture the braided G-crossed category $G-\mathrm{Loc}\mathcal{A}$ in terms of the algebraic structure on $G'-\mathrm{Loc}\mathcal{B}$. More precisely, we consider the problem of generalizing the α-induction procedure to $G'-\mathrm{Loc}\mathcal{B}$. This question is motivated as follows. In many concrete examples, the determination of the category of G-twisted representations needs hard work. If we have a clear understanding of the category $G'-\mathrm{Loc}\mathcal{B}$, it is userful to have a way to capture the category $G-\mathrm{Loc}\mathcal{A}$ in terms of $G'-\mathrm{Loc}\mathcal{B}$.

We now explain how to generalize α-induction procedure to $G'-\mathrm{Loc}\mathcal{B}$. The main idea is to use the G'-crossed braiding of $G'-\mathrm{Loc}\mathcal{B}$ and the Q-system $\Theta = (\theta, w, x)$. But they are not enough to capture the category $G-\mathrm{Loc}\mathcal{A}$ by the following reaseon. In general G' does not remember the original group G. Even if the restriction map $G \to G'$ induces an isomorphism of groups, one cannot determine the position of G in $\mathrm{Aut}(\mathcal{A})$. Hence it is desirable to describe G and its position in $\mathrm{Aut}(\mathcal{A})$ by some algebraic structure on $\mathcal{B}$. This task is achieved by the notion of G-equivariant Q-system structures. Let us explain this in detail. Since G also acts on $\mathcal{B}$ by our assumption, we have the induced action of G on $\mathrm{Rep}(\mathcal{B})$. We denote by γ the action of G on $\mathrm{Rep}(\mathcal{B})$. Then one can construct the canonical G-equivariant Q-system structure on Θ. This is a family of unitary intertwiners $z = \{z_g \colon \gamma(\theta) \to \theta\}_{g \in G}$ satisfying certain algebraic relations. The G-equivariant structure z and $G' < \mathrm{Aut}(\mathcal{B})$ completely remember the group $G < \mathrm{Aut}(\mathcal{A})$. If $G \cong G'$ by the restriction map, the G-equivariant structure on the Q-system describe the extension of group action of G from $\mathcal{B}$ to $\mathcal{A}$. The correspondence between extensions of action of G and G-equivariant structure on Θ has been established in [1, Section 6] with a slightly abstract manner. For our pupose, we first summarize the propeties of G-equivariant structure on Θ arising from the group action of G on $\mathcal{B} \subset \mathcal{A}$ without assuming $G \cong G'$. (More precisely, we treat the subfactor setting, since it is enough to consider a single subfactor $\mathcal{B}(I) \subset \mathcal{A}(I)$ rather than net of subfactors.)

Using the G-equivariant Q-system structure z and the G-crossed braiding, we introduce two types of induced endomorphisms for objects in G'–Loc$\mathcal{B}$. These induced endomorphisms are defined as follows. Let us fix $g \in G$, and let g' be a restriction of $g \in G$ on $\mathcal{B}$ and λ a g'-localized transportable endomorphism of $\mathcal{B}$. In this setting, we introduce two extensions α_λ^- and $\alpha_\lambda^{g;+}$ of λ to $\mathcal{A}$. They are defined by a similar formulas as the α-induction for ordinary DHR endomorphisms. The endomorphism α_λ^- is defined with the opposite G-crossed braiding on $G'-$Loc$\mathcal{B}$ and the Q-system Θ, but $\alpha_\lambda^{g;+}$ is defined with the braiding on $G'-$Loc$\mathcal{B}$, the Q-system Θ and the unitary z_g in the G-equivariant structure on Θ as explained above.

After introducing α_λ^- and $\alpha_\lambda^{g;+}$, we study their basic properties and derive some formulas as in the case of α-induction. We see that many statements for α-induction have natural translations for our setting. In particular, we see that a subobject of both α_λ^- and $\alpha_\lambda^{g;+}$ is a g-localized endomorphism of $\mathcal{A}$. This result is one of the main result of this thesis and indicates that our definitions for α^- and $\alpha^{g;+}$ are correct generalizations of α-induction.

Also, we consider the relation between these induced endomorphisms and σ-restriction procedures. For the case of ordinary α-induction, it has been shown that we have the $\alpha\sigma$-reciprocity formula for α-inductions and σ-restrictions [2], which is a kind of the Frobenius reciprocity formula for the group representations. Generalizing this result, we show that the $\alpha\sigma$-reciprocity formula also hold for our two induction procedures α^- and $\alpha^{g;+}$. As a corollary, we show that every g-localized transportable endomorphism of $\mathcal{A}$ is a subobject of both α_λ^- and $\alpha_\lambda^{g;+}$ for some g'-localized transportable endomorphism λ of $\mathcal{B}$.

Finally, we consider the G-crossed braiding of $G-$Loc$\mathcal{A}$. We show that one can recover the G-crossed braiding of $G-$Loc$\mathcal{A}$ from the G'-crossed braiding of $G'-$Loc$\mathcal{B}$.

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