博士論文

On asymptotic behavior and maximal regularity of the Navier-Stokes equations and related problems

(ナヴィエ・ストークス方程式とそれに関連する問題に対する漸近挙動と最大正則性について)

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Abstract

In this dissertation, well-posedness, asymptotic stability and derivation for partial differential equations (PDE for short) related to the Navier-Stokes equations is considered. The Navier-Stokes equation is one of a most fundamental equation in fluid dynamics, and there is a large number of literature on the mathematical analysis. PDEs considered in this dissertation are deeply involved with fluid phenomena familiar to us, e.g. vortex, weather forecasts, and free boundary problems.

In Chapter 1 asymptotic stability of the three-dimensional Oseen vortex is considered. The three-dimensional Oseen vortex is one of a fundamental model for three-dimensional vortex and also an exact solution to the Navier-Stokes equations. The main result in this chapter is L^2-asymptotic stability of the three-dimensional Oseen vortex under the large perturbation on a periodic layer.

Mathematical studies for the Oseen vortex have been basically intended for the two-dimensional case and there are few mathematical results on the three-dimensional Oseen vortex. In the two-dimensional case, there are many researches on the Oseen vortex on stability. Especially, asymptotic stability of two-dimensional Oseen vortex on $\mathbb{R}^2$ with arbitrary total circulation has been established. However, asymptotic stability of three-dimensional Oseen vortex on $\mathbb{R}^3$ is still open even if total circulation is small. Difficulties of mathematical analysis of the three-dimensional Oseen vortex are that no spatial decay in the vertical direction is obtained, three-dimensional Oseen vortex has strong singularity, and inequalities that hold for the mild solutions of the three-dimensional Navier-Stokes equations are not applicable for the three-dimensional Oseen vortex.

The main result in this section is L^2-asymptotic stability of the three-dimensional Oseen vortex for large initial perturbation on an infinite layer $\mathbb{R}^2 \times \mathbb{T}$. Idea of the proof is based on consideration that methods used in the two-dimensional case are applicable when the domain is close to $\mathbb{R}^2$. Strategy to show the asymptotic stability is based on energy estimate for the perturbed equation. First, the weak solution to the perturbed equation is constructed. We first show the existence of the weak solution to the perturbed equations with logarithmic energy estimate. Finiteness of vertical length of the domain is essentially used in the construction. In fact, we construct a local-in-time strong solution that belongs to a subspace of L^2 using finiteness of vertical length and extend this solution in L^2 to get global in time weak solution. This procedure needed to avoid strong singularity of the initial data to the three-dimensional Oseen vortex. To show L^2 decay at infinity, the solution is divided

into two parts; the averaged part on the vertical variable and the average-free part. The Poincaré inequality is applicable to the average-free part to get the decay. The former part is independent of the vertical variable. We derive the equations that the averaged part satisfies and show the decay by two-dimensional arguments.

In Chapter 2, justification of the derivation of primitive equations is considered. Compared with the Navier-Stokes equations, the equation of the vertical component of the vector field w in the Navier-Stokes equations is replaced by the hydrostatic approximation in the primitive equations. The primitive equations describe the motion of the fluid filled in a thin domain, e.g. ocean and atmosphere, and is used to the prediction of the motion of the atmosphere and weather. Global well-posedness of the primitive equations in Sobolev space H^1 is known. Inspired by this fact, the primitive equations have been actively studied from a mathematical point of view in recent years.

The primitive equations is formally derived from the Navier-Stokes equations with anisotropic viscosity; the horizontal viscosity is $O(1)$ and the vertical one is $O(\varepsilon^2)$. Applying change of variable to this equations, we find the scaled Navier-Stokes equations (SNS)

$$\begin{aligned} \partial_t v - \Delta v + u \cdot \nabla v + \nabla_H p &= 0, \\ \partial_t w - \Delta w + u \cdot \nabla w + \frac{\partial_z p}{\varepsilon^2} &= 0, \\ \operatorname{div} u &= 0, \end{aligned}$$

where $u = (v, w)$, v and w are horizontal velocity and vertical velocity, respectively, and p is the pressure. ∇_H is the horizontal gradient,div is the divergence, Δ is the Laplace operator. Taking $\varepsilon \to 0$ for (SNS), we can formally derive the primitive equation. Justification of the above formal derivation from a mathematical point of view gives us information on the relationship between the primitive equations and the Navier-Stokes equations and new insights into the global well-posedness of the Navier-Stokes equations.

Our aim of this chapter is to justify the derivation of the primitive equations from (SNS) and, at the same time, prove the global well-posedness of the scaled Navier-Stokes equations under L^p-L^q maximal-regularity settings on the flat torus $\mathbb{T}^3$. Recently, justification of the derivation in L^2-framework with precise convergence rate $O(\varepsilon)$ was obtained. Global well-posedness of (SNS) for H^2-initial data was also proved. The main result of this section is the mathematical justification of the derivation of primitive equations in the maximal regularity class $\mathbb{E}_1(T) = W^{1,p}(0, T; L^q(\mathbb{T}^3)) \cap L^p(0, T; W^{2,q}(\mathbb{T}^3))$ $(T > 0)$ for $1 < p, q < \infty$ with appropriate conditions due to Sobolev embeddings, where L^r is the Lebesgue space and $W^{m,r}$ is the m-th order Sobolev space for a positive integer m and $r \in (1, \infty)$. Another important result of this section is global well-posedness of (SNS) in $\mathbb{E}_1(T)$ for small ε. Note that global well-posedness of the primitive equations in the maximal regularity class is known.

We derive the equation that difference between the solution of the primitive equation and the solution of (SNS) satisfies, and estimate the solution to the equations of difference by $O(\varepsilon)$. The basic idea is based on what small data implies global existence of the solution. Two problems need to be solved: the maximal regularity of the linearization operator and the improved regularity result for the vertical velocity of the solution to the primitive equations w. The former is shown by using scaling, L^q-boundedness of the Riesz operator, and maximal regularity of the Laplace operator. The latter is needed to estimate error terms in the equations of the difference. Since the primitive equations have no equation describing time evolution of w, less regularity for w has been only known. This difficulty is solved by deriving the non-linear parabolic equations that w satisfies from the time evolution of v and divergence-free condition, and estimate this equation by maximal regularity of the Laplace operator. Combining with and the Fujita-Kato's iteration, we construct the solution to the equation of the difference and, at the same time, show the justification of the derivation. This chapter is based on joint work with Professor Yoshikazu Giga, Professor Matthias Hieber, Professor Amru Hussein, Professor Takahito Kashiwabara and Doctor Marc Wrona.

In Chapter 3, we extend the result of Chapter 2 into the Dirichlet boundary condition. Namely, we justify the derivation of the primitive equations and prove the global well-posedness of the scaled Navier-Stokes equations(SNS) in maximal-regularity class $\mathbb{E}_1(T)$ on $\mathbb{T}^2 \times (-1,1)$ under the two boundary condition. Note that the case of the Dirichlet-Neumann boundary condition is included by reflection argument. It is rather important to consider the case of such boundary conditions than the periodic boundary condition since the Dirichlet boundary condition is physically more reasonable. Note that there is no result on the justification of the derivation even in L^2-settings under the Dirichlet boundary condition. Moreover, we also obtain global well-posedness of (SNS) in maximal regularity class for small ε.

The main difficulty is to prove the maximal regularity of the anisotropic Stokes operator, which is the linearized operator for (SNS), with uniform estimate on ε and the improved regularity result for w. To show maximal regularity of the anisotropic Stokes operator is substantially difficult and complicated compared with periodic case by the effect of the boundary. Moreover, the scaling argument used in Section 2 is not applicable. Sufficient conditions to show maximal regularity for linearized operators have beed studied by many researchers. For instance, bounded imaginary power (BIP), bounded H^∞-calculus and $\mathcal{R}$ sectoriality of the semigroup are typical sufficient conditions. In this chapter BIP of the anisotropic Stokes operator is proved. If we try to check other sufficient conditions, it is difficult to find ε-dependence of the estimate. The proof of BIP of the anisotropic Stokes operator is based on a concrete calculation of symbol and boundedness of Fourier multiplier operator and singular integral operator. It enables us to find the contribution of ε clearly and to establish uniform estimate on ε. To prove improved regularity is also harder than the case of periodic boundary condition since the method used in the periodic case does not work directly by the effect of boundary. If we apply the method used in Chapter 2 directly, second order derivative at the boundary appears. It is impossible to bound this term from above by interior $W^{2,p}$-norm. Thus, we use a cut-off technique to avoid this difficulty. We introduce a cut-off function ϕ to eliminate the effect near the boundary, multiply this function to vertical component of vector fields w and seek the solution that ϕw satisfies. Applying maximal regularity to the equation and summing up each function with cut-off, we have improved regularity of w. The result of this chapter is joint work with Professor Yoshikazu Giga and Professor Takahito Kashiwabara.

In Chapter 4 we consider global well-posedness of higher order linear elliptic problem with dynamic boundary conditions in L^p-L^q maximal regularity settings on bounded domain and exterior domain. Time derivative is included in boundary conditions in the case of dynamic boundary conditions. These types of problems are considered as the linearized equations for the various non-linear equations related to free-boundary problems. In recent years, free boundary problems have been actively studied in various types of problems, including those related to fluid mechanics. Free boundary problems are not always second-order and semi-linear equations, there are many quasi-linear problems with higher-order terms. It is known that approaches using maximal regularity are more effective to obtain local well-posedness for free-boundary problems rather than semi-group approaches. Therefore, considering the well-posedness of general elliptic problems with dynamic boundary conditions in maximal regularity settings leads to a comprehensive understanding of various free boundary problems. Well-posedness of higher-order linear parabolic problems with dynamic boundary conditions in general settings have already known. On the other hand, for the elliptic problem with dynamic boundary condition with general settings, well-posedness is not known.

The main result in this chapter is that, for bounded domains, or exterior domain, the above linear elliptic equation with dynamic boundary conditions is well-posed in the class of maximal regularity under conditions appropriate for each coefficient. We first establish the maximal regularity result in the half-space. Then extend this results into bounded and exterior domains with canonical way. In the case of the half space, we first seek the solution formula of Laplace-Fourier multiplier type on the half space with constant coefficients. To get the solution formula of the solution, so-called Lopatinskii-Shapiro condition is needed. This condition ensures solvability of the ordinal differential equations which is obtained by applying partial Fourier transform with respect to $x' = (x_1, \cdot, x_{n-1})$-variable. To show the boundedness of the Laplace-Fourier multiplier, asymptotic Lopatinskii-Shapiro condition is needed. This condition is used to control boundedness the multiplier at infinity. Combining with the boundedness of the Laplace-Fourier multiplier and operator valued Fourier multiplier theorem and H^∞-calculus, we obtained the boundedness solution operator. The result of this chapter is joint work with Doctor Naoto Kajiwara.

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Contents

Chapter 1

Asymptotic Stability of Small Oseen Type Vortex under Three-Dimensional Large Perturbation

We consider the three-dimensional Navier-Stokes equations whose initial data may have infinite kinetic energy. We establish unique existence of the mild solution to the Navier-Stokes equations for small initial data in the whole space $\mathbb{R}^3$ and a vertically periodic product space $\mathbb{R}^2 \times \mathbb{T}^1$ which may be constant in vertical direction so that it includes the Oseen vortex. We further discuss its asymptotic stability under arbitrarily large three dimensional perturbation in $\mathbb{R}^2 \times \mathbb{T}^1$.

1.1 Introduction

Let Ω be $\mathbb{R}^3$ or $\mathbb{R}^2 \times \mathbb{T}^1$, where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ is one dimensional flat torus. We consider the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \Omega \times (0, \infty), \\ \mathrm{div} u = 0 & \text{in} \quad \Omega \times (0, \infty), \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1.1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p(x,t)$ respectively stand for an unknown velocity field and a pressure. The functions u_0 denote a given initial velocity. ∂_t, Δ denotes partial derivative in time and Laplace operator on the Euclidean space respectively. The differential operator $u \cdot \nabla$ denotes $\sum_{1 \le j \le 3} u_j \partial_j$.

Let us recall a special self-similar solution called the three dimensional Oseen vortex or Lamb-Oseen vortex:

$$\mathrm{Os}(x_h, x_v, t) = \frac{\Gamma}{2\pi} \frac{(-x_2, x_1, 0)}{|x_h|^2} (1 - \mathrm{e}^{-\frac{|x_h|^2}{4t}}), \quad x_h = (x_1, x_2),\ x_v = x_3, \tag{1.1.2}$$

where Γ is the total circulations. The two-dimensional Oseen vortex is a solution to the Navier-Stokes equations whose initial vorticity is a Dirac measure supported at the origin, and it is one of the simplest vortex. The three-dimensional Oseen vortex is an extension of two-dimensional one.

The goals of this chapter are summarized as follows;

(1) We construct a unique solutions with non-smooth and singular initial data so that the Oseen vortex is included as a three-dimensional flow,

(2) We discuss its asymptotic stability under large three-dimensional perturbation periodic in vertical direction.

There are many results on the existence of the solution to (1.1.1). It is well known that Leray [17] showed the existence of a global-in-time weak solution u in $\mathbb{R}^n$ to (1.1.1) satisfying the following energy estimate:

$$\|u(\tau)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2$$

for initial data $u_0 \in L^2$. Unfortunately, the Oseen vortex is not a Leray's weak solution since the energy of the Oseen vortex is infinite.

For non-L^2-initial data, Kato [11] proved that (1.1.1) is globally well-posed for small L^m-initial data in $\mathbb{R}^m$ with $m \geq 2$ by using iteration to the integral formulation of (1.1.1):

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}P(u(\tau)\cdot\nabla u(\tau))d\tau, \tag{1.1.3}$$

where $e^{t\Delta}$ and P are the heat kernel and the Helmholtz projection respectively. The choice of function space is related to the scaling transformation:

$$v(x,t) \to \lambda v(\lambda x, \lambda^2 t), \quad p(x,t) \to \lambda^2 p(\lambda x, \lambda^2 t),$$

which dose not change the equation. Scale-invariant function spaces are critical ones that iteration method works. In this case $L^m(\mathbb{R}^m)$ and $L_t^\infty L_x^m(\mathbb{R}^m \times (0,\infty))$ are scale-invariant function space under the above scaling transformation. Independently, Giga and Miyakawa [7] proved the existence of the solutions in $L^r(\mathbb{R}^r)$ in bounded domains with the Dirichlet boundary condition. The result of this paper was obtained even before [11] but it took long time to be published after the paper was accepted.

In three-dimensional case, $L^3(\mathbb{R}^3)$ is one of scale-critical function spaces, but it does not include homogeneous functions like $\frac{1}{|x|}$. This means that $L^3(\mathbb{R}^3)$ is too restrictive to construct a self-similar solution. In this direction, Giga and Miyakawa [6] proved that the vorticity equations is well-posed for small initial data and there is a unique self-similar solution by taking initial vorticity in the Morrey space $M^{\frac{3}{2}}(\mathbb{R}^3)$. The Morrey space is scale-invariant and include homogeneous functions. Moreover, since $\mathrm{rotOs}(\cdot,0) \in M^{\frac{3}{2}}$, the result of [6] provides a class of function space of solution to the Navier-Stokes equations which includes the three dimensional Oseen vortex provided that Γ is sufficiently small. However, in [6], smoothness for initial data is needed to define $\mathrm{rot} u_0$. For instance, for a bounded function $\Theta(x)$ on the two dimensional unit sphere whose derivative is not a Radon measure, $\mathrm{rot}(\Theta(\frac{x}{|x|})\mathrm{Os}(\mathrm{x}, 0))$ is not in $M^{\frac{3}{2}}$. On the other hand, Kozono and Yamazaki [14] proved well-posedness for small initial data in weak-L^2 space in two-dimensional exterior domains. Since the two-dimensional Oseen vortex is in weak-L^2 space, the Oseen vortex belongs to the class of solutions provied in [14] Moreover, there is no restriction on smoothness of initial data in [14]. In Cannone [2] and Koch and Tataru [12], it was showed that (1.1.1) is globally well-posed for small initial data in the Besov spaces $B_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ $(1 < p < \infty)$ and $BMO^{-1}(\mathbb{R}^n)$ space respectively. The result of [12] is the most general on the well-posedness to (1.1.1).

Our second aim is to show L^2-asymptotic stability to the solution that is constructed in the first aim under large three-dimensional perturbation. We call a basic flow b, which is a solution to the Navier-Stokes equations, is L^2-asymptotically stable if we perturb its initial data b_0 by each solenoidal vector field v_0 belonging to the appropriate function space, then there exists a solution to the Navier-Stokes equations u with initial data $u_0 = b_0 + v_0$ such that the difference of u between b or sometimes the difference of u between b (or $b + e^{t\Delta}v_0$ if necessary) goes to zero as $t \to \infty$ in L^2 topology. Asymptotic stability for the Navier-Stokes equations has been widely studied. However, there are few the results on the asymptotic stability under large perturbation. In three-dimensional case, Schonbek [20] proved that 0 is asymptotically stable for $L^2 \cap L^1$-perturbation on $\mathbb{R}^3$. Subsequently, Miyakawa and Schonbek [19] study optimal decay rate. On the other hand, Kozono [13] proved asymptotic stability for the Leray's weak solution $u \in L_t^p L_x^q$ satisfying Serrin's condition [21] ($\frac{2}{p} + \frac{3}{q} = 1$ for $2 \leq p < \infty$ and $3 < q \leq \infty$) on uniformly C^3 domains. This result allows unbounded domains such as a exterior domain or a domain with non-compact boundbary. Karch, Pilarczyk and Schonbek [10] proved L^2-asymptotic stability for small mild solution $V \in \mathcal{X}_\sigma$, where $\mathcal{X}_\sigma$ is a function space of solenoidal vector fields satisfying $|\langle v \cdot \nabla V, w\rangle| \leq C(\sup_{t>0}\|V(t)\|_{\mathcal{X}_\sigma})\|\nabla v\|_{L^2}\|\nabla w\|_{L^2}$ for all $v, w \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$. This result allows many function spaces. For instance, weak L^3 space satisfies above estimate, and then it is a subspace of $\mathcal{X}_\sigma$. The decay rate to $L^{3,\infty}$-mild solutions was also studied by [8]. Although [10] is the most comprehensive result for the asymptotic stability of small mild solutions to (1.1.1), the three dimensional Oseen vortex is not included in this result.

In the two-dimensional case, Maekawa [18] proved asymptotic stability for the solutions obtained by [14] under $\overline{C_0^\infty}^{L^{2,\infty}}$-large perturbation in the whole space and the exterior domain. This result give us asymptotic stability to the small two-dimensional Oseen vortex.

Let us consider our two problems in more detail. For the first problem, since the two-dimensional Oseen vortex is in $L^{2,\infty}$ and three dimensional Oseen vortex is independent of x_v variable, it is good idea to construct mild solution in an anisotropic function space $Y^2 := L_v^\infty L_h^{2,\infty}$ with the norm $\|f\|_{Y^2} = \|\|f(x_h, x_v)\|_{L_h^{2,\infty}}\|_{L_v^\infty}$. Note the three dimensional Oseen vortex is in Y^2 at fixed time. Moreover, Y^2 is scale-invariant under the natural scaling and does not require any smoothness. In fact, we are able to construct a mild solution in this space for small initial data by using iteration. To this end it is needed to establish some L^p-L^q-like estimates for the heat kernel and the composite operator $e^{t\Delta}P\,\mathrm{div}$. L^p-L^q estimate for the heat kernel and the composite operator is well-known, but L^p-L^q-like estimate for their operators in anisotropic spaces something like Y^2 are not yet well studied. For that reason we first show L^p-L^q-like estimates, after that, we construct mild solution to (1.1.1). Althought the method is almost the same as [6] and [14], the choice of function space is new. Moreover, it is possible to construct mild solution to initial data which is not covered by [6] such as highly oscillating one.

Our second aim is to show asymptotic stability of mild solutions obtained in the first aim under arbitrarily large initial perturbation. Let b be a solution to (1.1.1) with initial data b_0 constructed in our first aim, which is a basic flow, and $v_0 \in L^\infty_v \overline{C^\infty_{0,h}}^{L^{2,\infty}}(\mathbb{R}^2 \times \mathbb{T}^1)$ be a arbitrary large initial perturbation. By choice of v_0, it can be decomposed as

$$v_0 = \tilde{v}_0 + \tilde{b}_{0,\epsilon},$$

where $v_0 \in L^\infty_v C^\infty_{0,h}(\mathbb{R}^2 \times \mathbb{T}^1)$ and $\|\tilde{b}_{0,\epsilon}\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq \epsilon$ for arbitrary small $\epsilon > 0$. Let u is a solution to (1.1.1) with initial data $b_0 + v_0$. Under this condition, we can show $\lim_{t\to\infty}\|u(t) - b(t) - e^{t\Delta}v_0\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\epsilon$, where $C > 0$ is independent of ϵ. If initial perturbation belongs to better space, taking $\epsilon = 0$, we can also show $\lim_{t\to\infty}\|u(t) - b(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} = 0$ without restriction on size of initial perturbation. This means asymptotic stability of b. Let us introduce our strategy to this end. We need several steps. For the basic flow b with initial data b_0, we can construct a new basic flow $\tilde{b}$ with initial data $b_0 + \tilde{b}_{0,\epsilon}$ so that the difference $\|\tilde{b}(t) - b(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}$ can be estimated small enough since the difference of b_0 and $\tilde{b}_0$ is sufficiently small. $v := u - \tilde{b}$ satisfies the following perturbed Navier-Stokes equations;

$$\begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + \tilde{b} \cdot \nabla v + v \cdot \nabla \tilde{b} + \nabla q = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{T}^1 \times (0,\infty), \\ \mathrm{div}\, v = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{T}^1 \times (0,\infty), \\ v(0) = \tilde{v}_0 \quad \text{on} \quad \mathbb{R}^2 \times \mathbb{T}^1. \end{cases} \tag{1.1.4}$$

We have to show the existence of a weak solution to this equation and its decay. Since the fifth term of the left-hand side of the above equation $v \cdot \nabla\tilde{b}$ has singularity at $t = 0$, it is difficult to get the energy inequality by integration by parts on $\mathbb{R}^2 \times \mathbb{T}^1 \times (0, T)$ for some $T > 0$ and show the existence of a weak solution to (1.1.4) directly. To avoid this, we construct a unique local-in-time mild solution v to (1.1.4) on $(0, T]$ for some $T > 0$ with initial data $\tilde{v}_0$ in a subspace of $L^2(\mathbb{R}^2 \times \mathbb{T}^1)$, after that, we show the existence of global-in-time weak solution with initial data $v(T)$. The local-in-time mild solution is constructed as [18]. We follow his approach. To show the existence of a weak solution with initial data $v(T)$, we first construct a unique solution to approximated equations to (1.1.4) with energy inequality that is independent of approximation parameter. Next, taking limit to the approximated solution, we obtain a weak solution to (1.1.4).

Finally, we prove the decay of $\|v(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}$ as $t \to \infty$. To prove this, since the domain is vertically periodic, we can apply the Fourier expansion to v with respect to x_v variable:

$$v(x_h, x_v, t) = v^0(x_h, t) + \sum_{j\neq 0} v^j(x_h, t)\ \mathrm{e}^{2\pi ij}$$
$$=: v^0 + v_{os}.$$

Using orthogonality of the Fourier series, we can derive the equation that v^0 satisfies. Since the averaged term v^0 is independent of x_v, we can apply two-dimensional argument as in [18] to get the decay of $\|v^0(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}$ as $t \to \infty$. Unfortunately, because of the non-linearity of (1.1.4) and dependence of v_{os} on x_v variable, it is difficult to show the decay to the oscillating term by using same way as the averaged term. However, we can avoid this difficulty using Poincáre-type inequality and get the decay of $\|v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}$. It is worth to mention that there was no result on asymptotic stability to the three-dimensional Oseen vortex under three-dimensional perturbation, even if basic flows or initial perturbation are small, and domain has no boundary. Our result is somewhat restrictive in terms of domain. We hope to get similar result on $\mathbb{R}^3$ under large L^2-initial perturbation in future work.

This chapter is organized as follows. In first section, we define notations and notions and state our main theorem. In section 2 the solutions to NS that contain the three dimensional Oseen vortex are constructed by using the Fujita-Kato iteration method. We state Maekawa's decomposition to the Oseen type flows in section 3. The existence of the solutions to the perturbed Navier-Stokes equations with logarithmic energy estimate is proved in section 4. In section 5 we establish energy estimate for the low-frequency part to the zero Fourier mode. In this section some lemmas that leads the energy decay to the oscillating part are shown. The final section we establish the energy decay which implies the the asymptotic stability for the solution that constructed in second section.

1.2 Notations and Main results

In this section, we introduce some notations to state our two main theorem and introduce them. In three-dimensional case, we write a variable x of the form $x = (x_h, x_v) \in \mathbb{R}^2 \times \mathbb{R}$, where x_h is a horizontal variable and x_v is a vertical variable. ∂_j is a partial derivative with respect to x_j. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and $x = (x_1, \dots, x_n)$, we write $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. For $x = (x_1, \dots, x_n)$, $1 \le m \le n$ and $1 \le k_1, k_2, \dots, k_m \le n$, we write $\partial^m_{k_1 k_2 \cdots k_m} = \partial_{k_1}\partial_{k_2}\cdots\partial_{k_m}$. We write $\nabla_h = (\partial_1, \partial_2)^T$ and $div_h = \nabla_h\cdot$. The norm in a Banach space B is denoted by $\|\cdot\|_B$. B_σ denotes space of solenoidal vector fields belonging to B. $C_0^\infty(M)$ denotes the set of all smooth and compactly supported functions in a manifold M. $\mathcal{S}$ denotes the space of all rapidly decreasing functions in the sense of Schwartz. $\mathcal{S}'$ denotes its topological dual, i.e. the space of tempered distribution. $\mathcal{F}f$ and $\hat{f}$ denote the Fourier transform

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-x\cdot\xi} f(x)dx.$$

$L^p(\mathbb{R}^n)$ denotes the Lebesgue spaces for $1 \le p \le \infty$ with the standard norm. L^p_{loc} is locally L^p space. $L^{p,q}(\mathbb{R}^n)$ denotes the Lorentz spaces for $1 < p < \infty$ and $1 \le q \le \infty$

with the quasi-norm

$$\|f\|_{L^{p,q}} = p^{\frac{1}{q}}\Big(\int_0^\infty t^q \,|\{x\in\mathbb{R}^n\,;\,|f(x)|>t\}|^{\frac{q}{p}}\Big)\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\|f\|_{L^{p,\infty}} = \sup t\,|\{x\in\mathbb{R}^n\,;\,|f(x)|>t\}|^{\frac{1}{p}}\,.$$

For $s\in\mathbb{R}$, $H^s(\mathbb{R}^n)$ denotes the Bessel potential spaces $H^s(\mathbb{R}^n) := \{f\in\mathcal{S}'\,;\,\|f\|_{H^s} := \|(1+|\xi|)^s\hat{f}\|_{L^2}<\infty\}$ and the Riesz potential space $\dot{H}^s := \{f\in\mathcal{S}'\,;\,\|f\|_{\dot{H}^s} := \||\xi|^s\hat{f}\|_{L^2}<\infty\}$. For a Banach space B, a domain U and $t\in U$, we write L^p_tB ($1\le p\le\infty$), $L^{q,r}_tB$ ($1<q<\infty$ and $1\le r\le\infty$) and H^s_tB ($s\in\mathbb{R}$) as the B-valued Lebesgue space, the B-valued Lorentz space and he B-valued Sobolev space, respectively. When B is a Lebesgue space, a Lorentz space or a Sobolev space in a domain Ω with variable x, we add a subscript x to B as B_x. In three-dimensional case, for $1\le p,r<\infty$ and $1<q<\infty$, we abbreviate $L^p_{x_v}L^{q,r}_{x_h}$ as $L^p_vL^{q,r}_h$. For $T>0$, let us denote $Q_{vper,T}$ the anisotropic space-time product space $(\mathbb{R}^2\times\mathbb{T}^1)\times(0,T)$. We denote by $e^{t\Delta}$ the heat semi-group which is written by convolution form $e^{t\Delta}f = G^n_t*f$ for the n-dimensional Gaussian kernel $G^n_t(x) = e^{-|x|^2/4t}/(4\pi t)^{n/2}$. We denote by P the Helmholtz projection.

We define vertically anisotropic function spaces to define the mild solutions to (1.2.3) that include the three dimensional Oseen vortex.

Definition 1.2.1. Let $\Omega = \mathbb{R}^3$ *or* $\mathbb{R}^2\times\mathbb{T}^1$ and $1\le p,q\le\infty$. The vertically anisotropic space $X^p(\Omega)$, $X_p(\Omega)$ $Y^q(\Omega)$ and $Y_q(\Omega)$ are the space of functions that are locally integrable and satisfy

$$\|f\|_{X^p} := \sup_{x_v}\Big(\int_{\mathbb{R}^2}|f(x_h,x_v)|^p\,dx_h\Big)^{\frac{1}{p}}<\infty,$$
$$\|f\|_{X_q} := \Big(\int_{\mathbb{R}^2}(\sup_{x_v}|f(x_h,x_v)|)^p dx_h\Big)^{\frac{1}{p}}<\infty,$$
$$\|f\|_{Y^q} := \sup_{x_v}\sup_{\lambda>0}\lambda\Big(\big|\{x_h\in\mathbb{R}^2\,;\,|f(x_h,x_v)|>\lambda\}\big|\Big)^{\frac{1}{q}}<\infty,$$
$$\|f\|_{Y_q} := \sup_{\lambda>0}\lambda\Big(\Big|\{x_h\in\mathbb{R}^2\,;\,\sup_{x_h}|f(x_h,x_v)|>\lambda\}\Big|\Big)^{\frac{1}{q}}<\infty$$

respectively, where $|S|$ denotes the Lebesgue measure of S.

Remark 1.2.2. Y^q is larger than Y_q. Indeed, for $x_v\in\mathbb{R}$ and $\lambda>0$, we find

$$\{x_h\in\mathbb{R}^2\,;\,\sup_{x_v\in\mathbb{R}}|f(x_h,x_v)|>\lambda\}\supset\{x_h\in\mathbb{R}^2\,;\,|f(x_h,x_v)|>\lambda\}.$$

This implies

$$\|f\|_{Y^q} = \sup_{x_v\in\mathbb{R}}\sup_{\lambda>0}\lambda\Big(\big|\{x_h\in\mathbb{R}^2\,;\,|f(x_h,x_v)|>\lambda\}\big|\Big)^{\frac{1}{q}}$$
$$\le \sup_{x_v\in\mathbb{R}}\sup_{\lambda>0}\lambda\Big(\Big|\{x_h\in\mathbb{R}^2\,;\,\sup_{x\in\mathbb{R}}|f(x_h,x_v)|>\lambda\}\Big|\Big)^{\frac{1}{q}}$$
$$= \|f\|_{Y_q}.$$

Definition 1.2.3. Let $T > 0$. Let

$$v_0 \in L^2_\sigma(\mathbb{R}^2 \times \mathbb{T}^1), \quad b \in C^\infty_{w,t} Y^2_x((\mathbb{R}^2 \times \mathbb{T}^1) \times (0,T))$$

be a solution to (1.1.1) with initial data $b_0 \in Y^2_\sigma(\mathbb{R}^2 \times \mathbb{T}^1)$ satisfying following estimates

$$\sup_{0 \le \tau \le T} \|b(\tau)\|_{Y^2(\mathbb{R}^2 \times \mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2 \times \mathbb{T}^1)} \tag{1.2.1}$$

$$\sup_{0 \le \tau \le T} \tau^{\frac{1}{4}} \|b(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2 \times \mathbb{T}^1)}. \tag{1.2.2}$$

A vector field $v \in C^\infty_{w,t} L^2_x \cap L^2_t H^1_x((\mathbb{R}^2 \times \mathbb{T}^1) \times (0,T))$ is called a weak solution to the perturbed Navier-Stokes equations by $\tilde{b}$ with initial data $v_0 \in L^2(\mathbb{R}^2 \times \mathbb{T}^1)$ if v satisfies

$$\begin{cases} \partial_t v - \Delta v + \operatorname{div}(v \otimes v + v \otimes b + b \otimes v) + \nabla q = 0 \quad \text{in} \quad (\mathbb{R}^2 \times \mathbb{T}^1) \times (0,T) \\ \operatorname{div} v = 0 \quad \text{in} \quad (\mathbb{R}^2 \times \mathbb{T}^1) \times (0,T) \\ v(0) = v_0 \quad \text{on} \quad \mathbb{R}^2 \times \mathbb{T}^1 \end{cases} \tag{1.2.3}$$

in the sense of distribution with $q \in L^1_t L^1_{x,loc}((0,T) \times (\mathbb{R}^2 \times \mathbb{T}^1))$ with energy inequality;

$$\begin{aligned} &\|v(t)\|^2_{L^2(\mathbb{R}^2 \times \mathbb{T}^1)} + 2\int_1^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2 \times \mathbb{T}^1)} d\tau \\ &\le C_1 \|v_0\|^2_{L^2(\mathbb{R}^2 \times \mathbb{T}^1)} (1+t)^{C_2 \|b_0\|^4_{Y^2(\mathbb{R}^2 \times \mathbb{T}^1)}} \end{aligned}$$

for all $t > 1$, where $C_1, C_2 > 0$ is independent of t, and continuity of initial data;

$$\|v(t) - v_0\|_{L^2(\mathbb{R}^2 \times \mathbb{T}^1)} \to 0 \tag{1.2.4}$$

as $t \to +0$.

Now, we state the main results in this chapter which the existence of the Oseen type vortex and its asymptotic stability.

Theorem 1.2.4. *Let $\Omega = \mathbb{R}^3$ or $\mathbb{R}^2 \times \mathbb{T}^1$. Let $u_0 \in Y^2_\sigma(\Omega)$. Then there exists a positive number $\delta > 0$, if $\|u_0\|_{Y^2(\Omega)} \le \delta$, there exists a unique mild solutions $u \in L^\infty_t Y^2_x(\Omega \times (0,\infty))$ of (1.1.1) such that*

$$u(x,t) = e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} P \operatorname{div} u(\tau) \otimes u(\tau)) d\tau \quad \text{in} \quad Y^2(\Omega)$$

for all $t \in (0,T)$, where $e^{t\Delta}$ and P are the heat kernel and the Helmholtz projection respectively, and

$$\sup_{0<t<T} \|u(t)\|_{Y^2(\Omega)} \le C\|u_0\|_{Y^2(\Omega)}, \tag{1.2.5}$$

$$\sup_{0<t<T} t^{\frac{1}{4}}\|u(t)\|_{X^4(\Omega)} \le C\|u_0\|_{Y^2(\Omega)}. \tag{1.2.6}$$

Moreover, u is weakly continuous with initial data u_0 in the sense following sense;

$$\lim_{t\to 0} |\langle u(t) - u_0, \phi\rangle| = 0,$$

for all $\phi \in (L^1_v L^p_h)_\sigma \cap \left(L^1_v L^{2,1}_h\right)_\sigma$, where $\frac{1}{p} = \frac{1}{r} + \frac{1}{4}$ for all $\frac{1}{2} < \frac{1}{r} < \frac{3}{4}$.

Remark 1.2.5. For $u_0 \in Y_{2,\sigma}(\Omega)$, unique existence of the unique mild solution to (1.2.3) can be proved in Theorem 1.2.4

The following corollary is the direct consequence of Theorem1.2.4.

Corollary 1.2.6. *Let $u_0 \in Y^2_\sigma(\mathbb{R}^3)$ satisfying $\lambda u_0(\lambda x) = u_0(x)$ for all $\lambda > 0$. Then there exists $\delta > 0$ such that, if $\|u_0\|_{Y^2(\mathbb{R}^3)} < \delta$, there exists a unique self-similar mild solution $u \in L^\infty_t Y^2_x(\mathbb{R}^3 \times (0,\infty))$ to (1.1.1) satisfying (1.2.7) and $u(x,t) = \lambda u(\lambda x, \lambda^2 t)$.*

Theorem 1.2.7. *Let $\delta > 0$ sufficiently small, $b_0 \in Y^2(\mathbb{R}^2\times\mathbb{T}^1)$ be a solenoidal vector fields satisfying $\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \le \delta$ and b is a mild solution to (1.1.1) constructed by Theorem 1.2.4 with initial data b_0. Let $v_0 \in L^\infty_v \overline{C^\infty_{0,h}}^{L^{2,\infty}_h}(\mathbb{R}^2 \times \mathbb{T}^1)$ be a solenoidal initial perturbation, which can be written as*

$$v_0 = \tilde{v}_0 + \tilde{b}_0, \tag{1.2.7}$$

for $\tilde{v}_0$ and $\tilde{b}_0$ are solenoidal vector fields satisfying $\tilde{v}_0 \in L^\infty_v C^\infty_{0,h}(\mathbb{R}^2 \times \mathbb{T}^1)$ and $\tilde{b}_0 \in Y^2(\mathbb{R}^2 \times \mathbb{T}^1)$ with $\|\tilde{b}_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \le \epsilon$ for some small $\epsilon > 0$. Let $\tilde{b}$ be a mild solution to (1.1.1) constructed by Theorem 1.2.4 with initial data $b_0+\tilde{b}_0$. Then there exits a weak solution $\tilde{v} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x\left((\mathbb{R}^2 \times \mathbb{T}^1) \times (0,\infty)\right)$ to (1.2.3) perturbed by $\tilde{b}$ such that

$$\lim_{t\to\infty} ||\tilde{v}(t)||_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} = 0, \tag{1.2.8}$$

Moreover, $u := \tilde{v} + \tilde{b}$ is a weak solution to (1.1.1) with initial data $v_0 + b_0$ such that

$$\lim_{t\to\infty} \left|\left|u(t) - b(t) - e^{t\Delta}v_0\right|\right|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} = C\epsilon, \tag{1.2.9}$$

where $C > 0$ is independent of small δ and small ϵ.

Remark 1.2.8. 1. In (1.2.9), $-e^{t\Delta}v_0$ is needed. $||u(t) - b(t)||_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}$ does not always decays at infinity since its initial data belongs to $Y^2(\mathbb{R}^2 \times \mathbb{T}^1)$. Role of $-e^{t\Delta}v_0$ is to remove this singularity of b_0 and v_0. Actually, we find from definition of u and v_0 that

$$\begin{aligned} &u(t) - b(t) - e^{t\Delta}v_0 \\ &= \tilde{v}(t) + (\tilde{b}(t) - b(t) - e^{t\Delta}\tilde{b}_0) - e^{t\Delta}\tilde{v}_0 \end{aligned}$$

First and third terms decays in L^2-sense as $t \to \infty$ since their initial data belong to $L^2(\mathbb{R}^2 \times \mathbb{T}^1)$ or much better space. Second term also belongs to $L^\infty_t L^2_x(\mathbb{R}^2 \times \mathbb{T}^1)$ since its initial data have no singularity.

2. ϵ can be taken arbitrarily small by choice of v_0, $\tilde{v}_0$ and $\tilde{b}_0$.

3. In our proof, if $v_0 \in X_\sigma^{4/3}(\mathbb{R}^2 \times \mathbb{T}^1) \cap X_\sigma^4(\mathbb{R}^2 \times \mathbb{T}^1)$, we can take $\epsilon = 0$. Thus, left hand side of (1.2.9) equals to zero.

1.3 Construction of the Oseen type vortex

In this section, we prove Theorem 1.2.4. The next estimates for the heat semigroup on our anisotropic spaces play a key role in this chapter.

Proposition 1.3.1. *1. Let $1 \le q \le r \le \infty$, $\alpha = (\alpha_1, \alpha_2)$ be a multi-index and $x = (x', x_n) = (x_1, \dots, x_{n-1})$. Then*

$$\|\partial_{x'}^{\alpha_1} \partial_n^{\alpha_2} \mathrm{e}^{t\Delta} f\|_{X^r(\mathbb{R}^n)} \le Ct^{-\frac{n-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\alpha|}{2}} \|f\|_{X^q(\mathbb{R}^n)} \tag{1.3.1}$$

for all $t > 0$ and $f \in X^q(\mathbb{R}^n)$, where the constant $C > 0$ depends only on n, r, q and α.

2. Let $1 < q < r < \infty$ and $\alpha = (\alpha_1, \alpha_2)$ be a multi-index and $x = (x', x_n) = (x_1, \dots, x_{n-1})$. Then

$$\|\partial_{x'}^{\alpha_1} \partial_n^{\alpha_2} \mathrm{e}^{t\Delta} f\|_{Y^r(\mathbb{R}^n)} \le Ct^{-\frac{n-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\alpha|}{2}} \|f\|_{Y^q(\mathbb{R}^n)} \tag{1.3.2}$$

for all $t > 0$ and $f \in Y^q(\mathbb{R}^3)$, where the constant $C > 0$ depends only on n, r, q and α.

3. Let $1 \le q \le r \le \infty$. Then

$$\|(e^{t\Delta} - e^{s\Delta}) f\|_{X^r(\mathbb{R}^n)} \le C(t-s)^\theta t^{-\theta-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} \|f\|_{X^q(\mathbb{R}^n)} \tag{1.3.3}$$

for all $0 < s < t$ and $f \in X^q(\mathbb{R}^n)$, where the constant $C > 0$ depends only on n, r and q.

4. Let $1 < q \le r < \infty$. Then the composite operator $\mathrm{e}^{t\Delta} P\mathrm{div}$ extends to a bounded operator from $X^q(\mathbb{R}^3)$ to $X^r(\mathbb{R}^n)$ with

$$\|\mathrm{e}^{t\Delta} P\mathrm{div} F\|_{X^r(\mathbb{R}^n)} \le Ct^{\frac{n-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}} \|F\|_{X^q(\mathbb{R}^n)} \tag{1.3.4}$$

for all $t > 0$ and $F \in X^q(\mathbb{R}^n)$, where the constant $C > 0$ depends only on n, r and q.

5. Let $1 < q \le r < \infty$ and $0 < \theta < 1$. Then

$$\|(e^{s\Delta} - \mathrm{id}) e^{t\Delta} P\mathrm{div} F\|_{X^r(\mathbb{R}^n)} \le Ct^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})-\theta-\frac{1}{2}} s^\theta \|F\|_{X^q(\mathbb{R}^n)}. \tag{1.3.5}$$

for all $s, t > 0$ and $F \in X^r(\mathbb{R}^n)$, where the constant $C > 0$ depends only on n, r and q.

Proof. Let G_t^n be the n-dimensional Gaussian kernel. Then we find from the Hölder inequality

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_n^{\alpha_2} G_t^1\left(x_n-\xi_n\right) g\left(\xi_n\right) d\xi_n \\
&\leq \int_{\mathbb{R}}\left[\partial_n^{\alpha_2} G_t^1\left(x_n-\xi_n\right)\right]^{\frac{1}{r'}} \times\left[\partial_n^{\alpha_2} G_t^1\left(x_n-\xi_n\right)\right]^{\frac{1}{r}} g\left(\xi_n\right) d\xi_n \\
&\leq C t^{-\frac{\alpha_2 r'}{2}}\left(\int_{\mathbb{R}} \partial_n^{\alpha_2} G_t^1\left(x_n-\xi_n\right)|g|^r\left(\xi_n\right) d\xi_n\right)^{1/r},
\end{aligned}$$

for all $g \in L^q(\mathbb{R})$, and thus

$$\begin{aligned}
&\left|\partial_n^{\alpha_2} \partial_{x'}^{\alpha_1} e^{t\Delta} f\right|^r\left(x', x_n\right) \\
&\leq C t^{-\frac{\alpha_2 r}{2r'}} \int_{\mathbb{R}} \partial_n^{\alpha_2} G_t^1\left(x_n-\xi_n\right)\left|\int_{\mathbb{R}^{n-1}} \partial_{x'}^{\alpha_1} G_t^{n-1}\left(x'-\xi'\right) f\left(\xi', \xi_n\right) d\xi'\right|^r d\xi_n .
\end{aligned}$$

We use the Fubini theorem and the Young inequality to get

$$\begin{aligned}
&\|\partial_n^{\alpha_2} \partial_{x'}^{\alpha_1} e^{t\Delta} f(\cdot, x_n)\|_{L_{x'}^r}^r \\
&\leq C t^{-\frac{\alpha_2 r}{2r'}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \partial_n^{\alpha_2} G_t^1(x_n-\xi_n)\left|\int_{\mathbb{R}^{n-1}} \partial_{x'}^{\alpha_1} G_t^{n-1}(x'-\xi')\left|f(\xi', \xi_n)\right| d\xi'\right|^r d\xi_n dx' \\
&\leq C t^{-\frac{\alpha_2 r}{2r'}} \int_{\mathbb{R}} \partial_n^{\alpha_2} G_t^1(x_n-\xi_n) t^{-\frac{r(n-1)}{2}(\frac{1}{q}-\frac{1}{r})-\frac{\alpha_1 r}{2}}\|f(\cdot, \xi_n)\|_{L_{x'}^q}^r d\zeta \\
&= C t^{-\frac{r(n-1)}{2}(\frac{1}{q}-\frac{1}{r})-\frac{r\alpha_2}{2r'}-\frac{\alpha_1 r}{2}} \partial_n^{\alpha_2} G_t^1 * \|f(\cdot, \xi_n)\|_{L_{x'}^q}^r .
\end{aligned}$$

Applying the Young inequality again, we have (1.3.1) .(1.3.2) follows from interpolation. Let us prove (1.3.3). Since

$$\begin{aligned}
(e^{t\Delta}-e^{s\Delta}) f &= \int_s^t \frac{d}{d\tau} e^{\tau\Delta} f d\tau \\
&= \int_s^t \Delta e^{\tau\Delta} f d\tau,
\end{aligned}$$

then we find from (1.3.1) that

$$\begin{aligned}
&\|(e^{t\Delta}-e^{s\Delta}) f\|_{X^r(\mathbb{R}^n)} \leq C \int_s^t \|\Delta e^{\tau\Delta} f\|_{X^r} d\tau \\
&\leq C\|f\|_{X^q(\mathbb{R}^n)} \int_s^t \tau^{-1-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} d\tau \\
&\leq C s^{-\theta-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_{X^q(\mathbb{R}^n)} \int_s^t \tau^{-1+\theta} d\tau \\
&\leq C s^{-\theta-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})}(t-s)^\theta\|f\|_{X^q(\mathbb{R}^n)} .
\end{aligned}$$

We write the composite operator as convolution form

$$(e^{t\Delta} P \operatorname{div} F)_j=\sum_{1 \leq k, l \leq 3} K_{j, k, l, t} * F_{k, l}$$

where

$$K_{j,k,l,t}(x) = \partial_l G_t^n(x)\delta_{j,k} + \int_t^\infty \partial_{jkl}^3 G_\tau^n(x) d\tau.$$

Let $\alpha = (\alpha_1, \alpha_2)$ be a multi-index with length three. Then we find from (1.3.1) that

$$\begin{aligned}
&\|\int_t^\infty \partial_{jkl}^3 G_\tau^n(x) d\tau * F_{k,l}\|_{L_{x'}^r} \\
&\leq \int_t^\infty \|\partial_{jkl}^3 G_\tau^n * F_{kl}\|_{L_{x'}^r} d\tau \\
&= \int_t^\infty \|\partial_{x'}^{\alpha_1}\partial_{x_n}^{\alpha_2} G_\tau^n * F_{kl}\|_{L_{x'}^r} d\tau \\
&\leq C\int_t^\infty \tau^{-\frac{|\alpha_2|}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} (\partial_{x'}^{\alpha_1} G_\tau^1(x_n) * \|F_{k,l}(\cdot, x_n)\|_{L_{x'}^q}^r)^{\frac{1}{r}} d\tau.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&\|\int_t^\infty \partial_{jkl}^3 G_\tau^n(x) d\tau * F_{k,l}\|_{X^r(\mathbb{R}^n)} \\
&\leq C\|F_{k,l}\|_{X^q(\mathbb{R}^n)} \int_t^\infty \tau^{-\frac{3}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} d\tau \\
&\leq Ct^{-\frac{1}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})}\|F_{j,k}\|_{X^q(\mathbb{R}^n)}.
\end{aligned}$$

This implies (1.3.4). Since

$$\begin{aligned}
(e^{s\Delta} - \mathrm{id})e^{t\Delta} P\mathrm{div}F &= \int_t^{s+t} \frac{d}{d\tau} e^{\tau\Delta} P\mathrm{div}F d\tau \\
&= \int_t^{s+t} \Delta e^{\frac{\tau}{2}\Delta} e^{\frac{\tau}{2}\Delta} P\mathrm{div}F d\tau,
\end{aligned}$$

we find from (1.3.4)

$$\begin{aligned}
&\|\int_t^{s+t} \Delta e^{\frac{\tau}{2}\Delta} e^{\frac{\tau}{2}\Delta} P\mathrm{div}F d\tau\|_{X^r(\mathbb{R}^n)} \\
&\leq C\int_t^{s+t} \tau^{-1}\|e^{\frac{\tau}{2}\Delta} P\mathrm{div}F\|_{X^r(\mathbb{R}^n)} d\tau \\
&\leq C\|F\|_{X^q(\mathbb{R}^n)} \int_t^{s+t} \tau^{-\frac{3}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} d\tau \\
&\leq C\|F\|_{X^q(\mathbb{R}^n)} t^{-\theta-\frac{1}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} \int_t^{s+t} \tau^{\theta-1} d\tau \\
&\leq C\|F\|_{X^q(\mathbb{R}^n)} t^{-\theta-\frac{1}{2}-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{r})} s^\theta.
\end{aligned}$$

This implies (1.3.5) □

Let $T > 0$ and $u_0 \in {Y^2}_\sigma(\mathbb{R}^3)$. We inductively define the function u_j as follows.

$$u_1 = \mathrm{e}^{t\Delta} u_0 \tag{1.3.6}$$

$$u_{j+1} = \mathrm{e}^{t\Delta} u_1 - \int_0^t \mathrm{e}^{(t-\tau)\Delta} P \mathrm{div}(u_j(\tau) \otimes u_j(\tau)) d\tau \tag{1.3.7}$$

for all $t \in (0,T)$ and positive integer j. Firstof all, we have to show uniform boundedness of $t^{\frac{1}{4}} \|u_j(t)\|_{X^4(\mathbb{R}^3)}$ and $\|u_j(t)\|_{Y^2(\mathbb{R}^3)}$ on j to prove Theorem1.2.4,.

Lemma 1.3.2. *Let j be any positive integer. Then there exists a positive constant C, C_1 and C_2 such that*

$$\sup_{t>0} t^{\frac{1}{4}} \|u_1(t)\|_{X^4(\mathbb{R}^3)} \leq C \|u_0\|_{Y^2(\mathbb{R}^3)} \tag{1.3.8}$$

$$\sup_{t>0} t^{\frac{1}{4}} \|u_{j+1}(t)\|_{X^4(\mathbb{R}^3)} \leq C_1 \sup_{t>0} t^{\frac{1}{4}} \|u_1(t)\|_{X^4(\mathbb{R}^3)} + C_2 (\sup_{t>0} t^{\frac{1}{4}} \|u_j(t)\|_{X^4(\mathbb{R}^3)})^2. \tag{1.3.9}$$

Proof. (1.3.8) is the direct consequence of (1.3.2). By definition of u_{j+1}, we find

$$\begin{aligned} &\|u_{j+1}(t)\|_{X^4(\mathbb{R}^3)} \\ &\leq C \|u_1(t)\|_{X^4(\mathbb{R}^3)} + \int_0^t \|\mathrm{e}^{(t-\tau)\Delta} P \mathrm{div}(u_j(\tau) \otimes u_j(\tau))\|_{X^4(\mathbb{R}^3)} d\tau. \end{aligned}$$

Using (1.3.4), we get

$$\begin{aligned} ||u_{j+1}||_{X^4(\mathbb{R}^3)} &\leq C \|u_1(t)\|_{X^4(\mathbb{R}^3)} + C \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{4}} \|u_j(\tau) \otimes u_j(\tau)\|_{X^2(\mathbb{R}^3)} d\tau \\ &\leq C \|u_1(t)\|_{X^4(\mathbb{R}^3)} + C \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{4}} \|u_j(\tau)\|^2_{X^4(\mathbb{R}^3)} d\tau \\ &\leq C \|u_1\|_{X^4(\mathbb{R}^3)} + C (\sup \tau^{\frac{1}{4}} \|u_j(\tau)\|_{X^4(\mathbb{R}^3)})^2 \int_0^t (t-\tau)^{-\frac{3}{4}} \tau^{-\frac{1}{2}} d\tau \\ &\leq C \|u_1\|_{X^4(\mathbb{R}^3)} + C (\sup \tau^{\frac{1}{4}} \|u_j(\tau)\|_{X^4(\mathbb{R}^3)})^2 t^{-\frac{1}{4}} \end{aligned}$$

for all $t \in (0,\infty)$. This implies the lemma. □

From Lemma 1.3.2, there exists some $0 < \alpha < \frac{1}{4C_1C_2}$, if $\sup_{t>0} t^{\frac{1}{4}} \|u_1(t)\|_{X^4(\mathbb{R}^3)} < \alpha$, we obtain

$$\sup_{t>0} t^{\frac{1}{4}} \|u_j(t)\|_{X^4(\mathbb{R}^3)} \leq A_\infty := \frac{1 - \sqrt{1 - 4C_1C_2\alpha}}{2C_2}. \tag{1.3.10}$$

Lemma 1.3.3. *Let j be any positive integer. Then there exists positive constants C, C_1 and C_2 such that*

$$\sup_{t>0} \|u_1(t)\|_{Y^2(\mathbb{R}^3)} \leq C \|u_0\|_{Y^2(\mathbb{R}^3)} \tag{1.3.11}$$

$$\sup_{t>0} \|u_{j+1}(t)\|_{Y^2(\mathbb{R}^3)} \leq C_1 \sup_{t>0} \|u_1(t)\|_{Y^2(\mathbb{R}^3)} + C_2 (\sup_{t>0} t^{\frac{1}{4}} \|u_j(t)\|_{X^4(\mathbb{R}^3)})^2. \tag{1.3.12}$$

Proof. (1.3.11) is the direct consequence of (1.3.2). Let us show (1.3.12). We use duality argument. Let $\phi \in C_{0,\sigma}^{\infty}(\mathbb{R}^3)$, then we find

$$\begin{aligned}
&|\langle u_{j+1}(t), \phi\rangle| \\
&\le |\langle u_1(t), \phi\rangle| + \int_0^t \left|\langle \mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(u_j(\tau) \otimes u_j(\tau)), \phi\rangle\right| d\tau \\
&\le |\langle u_1(t), \phi\rangle| + C\|\phi\|_{L_v^1 L_h^{2,1}(\mathbb{R}^3)} \int_0^t \|e^{(t-\tau)\Delta} P\mathrm{div} u_j(\tau) \otimes u_j(\tau)\|_{X^4(\mathbb{R}^3)} d\tau. \qquad (1.3.13)
\end{aligned}$$

Using (1.3.4), we get

$$\begin{aligned}
|\langle u_{j+1}(t), \phi\rangle| \le{}& \|u_1(t)\|_{Y^2(\mathbb{R}^3)} \|\phi\|_{L_v^1 L_h^{2,1}(\mathbb{R}^3)} \\
&+ C(\sup_{t>0} \tau^{\frac{1}{4}} \|u_j(\tau)\|_{X^4(\mathbb{R}^3)})^2 \|\phi\|_{L_v^1 L_h^2(\mathbb{R}^3)} \int_0^t (t-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau \\
&\le \|u_1(t)\|_{Y^2(\mathbb{R}^3)} \|\phi\|_{L_v^1 L_h^{2,1}(\mathbb{R}^3)} + C(\sup_{t>0} \tau^{\frac{1}{4}} \|u_j(\tau)\|_{X^4(\mathbb{R}^3)})^2 \|\phi\|_{L_v^1 L_h^2(\mathbb{R}^3)}
\end{aligned} \qquad (1.3.14)$$

for all $t > 0$. Since $C_{0,\sigma}^{\infty}(\mathbb{R}^3)$ is dense in $\left(L_v^1 L_h^{2,1}\right)_\sigma(\mathbb{R}^3)$, the above estimate leads (1.3.12). ☐

Next, we show the uniform bound for $t^{\frac{1}{4}} \|u_{j+1}(t) - u_j(t)\|_{X^4(\mathbb{R}^3)}$ and $\|u_{j+1}(t) - u_j(t)\|_{Y^2(\mathbb{R}^3)}$ for all $j \ge 1$.

Proposition 1.3.4. *Let j be any positive integer. Then there exists positive constants C and C_1 such that*

$$\sup_{t>0} t^{\frac{1}{4}} \|u_2(t) - u_1(t)\|_{X^4(\mathbb{R}^3)} \le C(\sup_{t>0} t^{\frac{1}{4}} \|u_1(t)\|_{X^4(\mathbb{R}^3)})^2, \qquad (1.3.15)$$

$$\begin{aligned}
&\sup_{t>0} t^{\frac{1}{4}} \|u_{j+2}(t) - u_{j+1}(t)\|_{X^4(\mathbb{R}^3)} \\
&\le C_1(\sup_{t>0} t^{\frac{1}{4}} \|u_j(t)\|_{X^4(\mathbb{R}^3)} + \sup_{t>0} t^{\frac{1}{4}} \|u_{j+1}(t)\|_{X^4(\mathbb{R}^3)}) \sup_{t>0} t^{\frac{1}{4}} \|u_{j+1}(t) - u_j(t)\|_{X^4(\mathbb{R}^3)}
\end{aligned} \qquad (1.3.16)$$

Proof. We find from definition of u_2 and ((1.3.4)

$$\begin{aligned}
&\|u_2(t) - u_1(t)\|_{X^4(\mathbb{R}^3)} \\
&\le \int_0^t \|\mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(u_1(\tau) \otimes u_1(\tau))\|_{X^4(\mathbb{R}^3)} d\tau \\
&\le \int_0^t (t-\tau)^{-\frac{3}{4}} \|u_1(\tau)\|_{X^4(\mathbb{R}^3)}^2 d\tau \\
&\le C(\sup_{\tau>0} \tau^{\frac{1}{4}} \|u_1(\tau)\|_{X^4(\mathbb{R}^3)})^2 \int_0^t (t-\tau)^{-\frac{3}{4}} \tau^{-\frac{1}{2}} d\tau \\
&\le C(\sup_{\tau>0} \tau^{\frac{1}{4}} \|u_1(\tau)\|_{X^4(\mathbb{R}^3)})^2 t^{-\frac{1}{4}}.
\end{aligned}$$

This lead (1.3.15). Similarly, we see from (1.3.4)

$$
\begin{aligned}
&\|u_{j+2}(t)-u_{j+1}(t)\|_{X^4(\mathbb{R}^3)}\\
&\le \int_0^t \|\mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}(u_{j+1}(\tau)\otimes u_{j+1}(\tau)-u_j(\tau)\otimes u_j(\tau))\|_{X^4(\mathbb{R}^3)}d\tau\\
&\le C\int_0^t \|\mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}((u_{j+1}(\tau)-u_j(\tau))\otimes u_{j+1}(\tau)\\
&\qquad\qquad -u_j(\tau)\otimes(u_{j+1}(\tau)-u_j(\tau))\|_{X^4(\mathbb{R}^3)}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{3}{4}}(\|u_{j+1}(\tau)\|_{X^4(\mathbb{R}^3)}+\|u_j(\tau)\|_{X^4(\mathbb{R}^3)})\|u_{j+1}(\tau)-u_j(\tau)\|_{X^4(\mathbb{R}^3)}d\tau\\
&\le C(\sup_{\tau>0}\tau^{\frac{1}{4}}\|u_j\|_{X^4(\mathbb{R}^3)})\sup_{\tau>0}(\tau^{\frac{1}{4}}\|u_{j+1}(\tau)-u_j(\tau)\|_{X^4(\mathbb{R}^3)})\int_0^t(t-\tau)^{-\frac{3}{4}}\tau^{-\frac{1}{2}}d\tau.\\
&\le Ct^{-\frac{1}{4}}(\sup_{\tau>0}\tau^{\frac{1}{4}}\|u_j\|_{X^4(\mathbb{R}^3)})\sup_{\tau>0}(\tau^{\frac{1}{4}}\|u_{j+1}(\tau)-u_j(\tau)\|_{X^4(\mathbb{R}^3)}). \qquad (1.3.17)
\end{aligned}
$$

This estimate implies (1.3.16). □

Proposition 1.3.5. *Let j be any positive integer. Then there exists a positive constant C and C_1 such that*

$$
\sup_{t>0}\|u_2(t)-u_1(t)\|_{Y^2(\mathbb{R}^3)}\le C(\sup_{t>0}t^{\frac{1}{4}}\|u_1(t)\|_{X^4(\mathbb{R}^3)})^2, \qquad (1.3.18)
$$

$$
\begin{aligned}
&\sup_{t>0}\|u_{j+2}(t)-u_{j+1}(t)\|_{Y^2(\mathbb{R}^3)}\\
&\le C_1(\sup_{t>0}t^{\frac{1}{4}}\|u_j(t)\|_{X^4(\mathbb{R}^3)}+\sup_{t>0}t^{\frac{1}{4}}\|u_{j+1}(t)\|_{X^4(\mathbb{R}^3)})\\
&\qquad\times\sup_{t>0}t^{\frac{1}{4}}\|u_{j+1}(t)-u_j(t)\|_{X^4(\mathbb{R}^3)}. \qquad (1.3.19)
\end{aligned}
$$

Proof. We use duality argument. Let $\phi\in C^\infty_{0,\sigma}(\mathbb{R}^3)$. Then (1.3.4) implies

$$
\begin{aligned}
&|\langle u_2(t)-u_1(t),\phi\rangle|\\
&\le\left|\int_0^t\langle \mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}(u_1(\tau)\otimes u_1(\tau)),\phi\rangle\right|d\tau\\
&\le C\int_0^t(t-\tau)^{-\frac{1}{2}}\|u_1(\tau)\|^2_{X^4(\mathbb{R}^3)}\|\phi\|_{L^1_vL^2_h(\mathbb{R}^3)}d\tau\\
&\le C\sup_{\tau>0}\tau^{\frac{1}{4}}(\|u_1(\tau)\|_{X^4(\mathbb{R}^3)})^2\|\phi\|_{L^1_vL^{2,1}_h(\mathbb{R}^3)}\int_0^t(t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}d\tau.
\end{aligned}
$$

This implies (1.3.18). Using (1.3.4) again, we get

$$
\begin{aligned}
&|\langle u_{j+2}(t)-u_{j+1}(t),\phi\rangle|\\
&\le\int_0^t\left|\langle \mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}(u_{j+1}(\tau)\otimes u_{j+1}(\tau)-u_j(\tau)\otimes u_j(\tau)),\phi\rangle\right|d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^t (t-\tau)^{-\frac{1}{2}}(\|u_{j+1}(\tau)\|_{X^4(\mathbb{R}^3)} + \|u_j(\tau)\|_{X^4(\mathbb{R}^3)}) \\
&\qquad \times \|u_{j+1}(\tau) - u_j(\tau)\|_{X^2(\mathbb{R}^3)} \|\phi\|_{L^1_v L^2_h(\mathbb{R}^3)} d\tau \\
&\leq C A_\infty \sup_{t>0}(\tau^{\frac{1}{4}} \|u_{j+1}(\tau) - u_j(\tau)\|_{X^4(\mathbb{R}^3)}) \|\phi\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)} \int_0^t (t-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau \\
&\leq C A_\infty \|\phi\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)} \sup_{t>0}(\tau^{\frac{1}{4}} \|u_{j+1}(\tau) - u_j(\tau)\|_{X^4(\mathbb{R}^3)})
\end{aligned}
$$

for all $t > 0$. Since $C^\infty_{0,\sigma}(\mathbb{R}^3)$ is dense in $\left(L^1_v L^{2,1}_h\right)_\sigma(\mathbb{R}^3)$, we have (1.3.19). □

Take $\|b_0\|_{Y^2}$ so small that $2C_1 A_\infty < 1$, where C_1 is the constant appearing in Proposition 1.3.4 and Proposition 1.3.5, then we find from Proposition1.3.4 and Proposition 1.3.5 that

$$
\begin{aligned}
&\sum_{j \geq 0} \sup_{t>0} t^{\frac{1}{4}} \|u_{j+1}(t) - u_j(t)\|_{X^4(\mathbb{R}^3)} < \infty, \\
&\sum_{j \geq 0} \sup_{t>0} \|u_{j+1}(t) - u_j(t)\|_{Y^2(\mathbb{R}^3)} < \infty.
\end{aligned}
$$

Thus $u_j = u_0 + \sum_{j=0}^{j-1}(u_{j+1} - u_j)$ converge in $\mathcal{A}$ and $L^\infty_t Y^2_x((\mathbb{R}^3) \times (0,\infty))$. where $\mathcal{A}$ is a vector valued measureable functions of $f(x,t)$ in $\mathbb{R}^3 \times (0,\infty)$ such that $\|f\|_{\mathcal{A}} = \sup_{t>0} t^{\frac{1}{4}} \|f(t)\|_{X^4(\mathbb{R}^3)} < \infty$. We denote $\lim_{j\to\infty} u_j$ as u. Let us show continuity of $\|u(t)\|_{X^4(\mathbb{R}^3)}$ and $\|u(t)\|_{Y^2(\mathbb{R}^3)}$.

Proposition 1.3.6. *Let u be a mild solutions to (1.1.1) satisfying*

$$
\sup_{t>0} \|u(t)\|_{Y^2} + \sup_{t>0} t^{\frac{1}{4}} \|u(t)\|_{X^4} < \infty.
$$

Then $u(t)$ is continuous in $X^4(\mathbb{R}^3)$ for $t \in (0,\infty)$.

Proof. It suffices to show $\lim_{s\to t-0} \|u(t) - u(s)\|_{X^4(\mathbb{R}^3)} = 0$. Let $0 < s < t < \infty$. Then we find

$$
\begin{aligned}
&\|u(t) - u(s)\|_{X^4(\mathbb{R}^3)} \\
&\leq \|\mathrm{e}^{t\Delta} u_0 - \mathrm{e}^{s\Delta} u_0\|_{X^4} \\
&\quad + \int_s^t \|\mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(u(\tau) \otimes u(\tau))\|_{X^4(\mathbb{R}^3)} d\tau \\
&\quad + \int_0^s \|\mathrm{e}^{(t-\tau)\Delta} P\mathrm{div}(u(\tau) \otimes u(\tau)) - \mathrm{e}^{(s-\tau)\Delta} P\mathrm{div}(u(\tau) \otimes u(\tau))\|_{X^4(\mathbb{R}^3)} d\tau \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

First, using (1.3.3), we find

$$
\begin{aligned}
I_1 &= \|(\mathrm{e}^{(t-s)\Delta} - \mathrm{id})\mathrm{e}^{s\Delta} u_0\|_{X^4(\mathbb{R}^3)} \\
&\leq C(t-s)^\theta s^{-\theta} \|u_0\|_{X^4(\mathbb{R}^3)}.
\end{aligned}
$$

Second, we see from (1.3.4) that

$$\begin{aligned}
I_2 &\le C\int_s^t (t-\tau)^{\frac14}\|u(\tau)\|^2_{X^4(\mathbb{R}^3)}d\tau\\
&\le C(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4(\mathbb{R}^3)})^2\int_s^t (t-\tau)^{-\frac34}\tau^{-\frac12}d\tau\\
&\le C(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4(\mathbb{R}^3)})^2 s^{-\frac12}\int_s^t (t-\tau)^{-\frac34}d\tau\\
&\le C(\sup_{\tau>0}(\tau^{\frac14}\|u(\tau)\|_{X^4(\mathbb{R}^3)}))^2 s^{-\frac12}(t-s)^{\frac14}
\end{aligned}$$

.

Finally, using (1.3.5), we obtain

$$\begin{aligned}
I_3 &\le \int_0^s \|(e^{(t-s)\Delta}-\mathrm{id})e^{(s-\tau)\Delta}P\mathrm{div}(u(\tau)\otimes u(\tau))\|_{X^4(\mathbb{R}^3)}d\tau\\
&\le C(t-s)^{\theta}\int_0^s (s-\tau)^{-\frac{\theta}{2}-\frac34}\|u(\tau)\|^2_{X^4(\mathbb{R}^3)}d\tau\\
&\le C(t-s)^{\theta}(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4(\mathbb{R}^3)})^2\int_0^s (s-\tau)^{-\frac{\theta}{2}-\frac34}\tau^{-\frac12}d\tau\\
&\le C(t-\tau)^{\theta}(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4(\mathbb{R}^3)})^2 s^{-\frac{\theta}{2}-\frac14}.
\end{aligned}$$

Therefore, $\|u(t)-u(s)\|_{X^4\mathbb{R}^3}\to 0$ as $s\to t-0$. The proposition is proved. □

Proposition 1.3.7. *Let u be a mild solution for (1.1.1) satisfying*

$$\sup_{t>0}\|u(t)\|_{Y^2(\mathbb{R}^3)}+\sup_{t>0}t^{\frac14}\|u(t)\|_{X^4(\mathbb{R}^3)}<\infty.$$

Then $u(t)$ is weakly continuous in $Y^2(\mathbb{R}^3)$ on $t\in(0,\infty)$.*

Proof. We use duality argument. Let $\phi\in C_0^\infty(\mathbb{R}^3)$ and $0<s<t<\infty$. It suffices to $\lim_{s\to t-0}|\langle u(t)-u(s),\phi\rangle|=0$. It follows

$$\begin{aligned}
&|\langle u(t)-u(s),\phi\rangle|\\
&\le \left|\langle e^{t\Delta}u_0-e^{s\Delta}u_0,\phi\rangle\right|\\
&+\int_s^t \left|\langle e^{(t-\tau)\Delta}P\mathrm{div}(u(\tau)\otimes u(\tau)),\phi\rangle\right|d\tau\\
&+\int_0^s \left|\langle (e^{(t-s)\Delta}-\mathrm{id})e^{(s-\tau)\Delta}P\mathrm{div}(u(\tau)\otimes u(\tau)),\phi\rangle\right|d\tau.\\
&=: I_1+I_2+I_3.
\end{aligned}$$

Set $\tilde\phi=e^{s\Delta}\phi$. Then we find

$$I_1\le\left|\langle e^{(t-s)\Delta}u_0-u_0,\tilde\phi\rangle\right|=\left|\langle u_0,e^{(t-s)\Delta}\tilde\phi-\tilde\phi\rangle\right|$$

$$
\begin{aligned}
&\le \left|\langle \int_{\mathbb{R}} G^1_{t-s}\left(x_3-y_3\right)u_0\left(x',y_3\right)dy_3, \int_{\mathbb{R}^2} G^2_{t-s}\left(x'-y'\right)\tilde{\phi}\left(y',x_3\right)dy' - \tilde{\phi}\rangle\right| \\
&+ \left|\langle u_0, \int_{\mathbb{R}} G^1_{t-s}\left(x_3-y_3\right)\tilde{\phi}\left(x',y_3\right)dy_3 - \tilde{\phi}\rangle\right| \\
&\le \left\|\int_{\mathbb{R}} G^1_{t-s}\left(x_3-y_3\right)u_0\left(x',y_3\right)dy_3\right\|_{Y^2(\mathbb{R}^3)} \\
&\quad \times \left\|\int_{\mathbb{R}^2} G^2_{t-s}\left(x'-y'\right)\tilde{\phi}\left(y',x_3\right)dy' - \tilde{\phi}\right\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)} \\
&+ \|u_0\|_{Y^2(\mathbb{R}^3)}\left\|\int_{\mathbb{R}} G^1_{t-s}\left(x_3-y_3\right)\tilde{\phi}\left(x',y_3\right)dy_3 - \tilde{\phi}\right\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)}.
\end{aligned}
$$

Thus, we find from continuity of heat semi-group and Lebesgue's dominated convergence theorem, $I_1 \to 0$ as $s \to t-0$. It follows from (1.3.4) that

$$
\begin{aligned}
|I_2| &\le C\int_s^t \|e^{(t-\tau)\Delta}P\mathrm{div}(u(\tau)\otimes u(\tau))\|_{X^2(\mathbb{R}^3)}\|\phi\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)}d\tau \\
&\le C\int_s^t (t-\tau)^{-\frac{1}{2}}\|u(\tau)\|^2_{X^4(\mathbb{R}^3)}\|\phi\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)}d\tau \\
&\le C(\sup_{\tau>0}\tau^{\frac{1}{4}}\|u(\tau)\|_{X^4(\mathbb{R}^3)})^2\|\phi\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)}\int_s^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}d\tau \\
&\le Cs^{-\frac{1}{2}}(\sup_{\tau>0}\tau^{\frac{1}{4}}\|u(\tau)\|_{X^4(\mathbb{R}^3)})^2\|\phi\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)}\int_s^t (t-\tau)^{-\frac{1}{2}}d\tau \\
&\le Cs^{-\frac{1}{2}}(t-s)^{\frac{1}{2}}(\sup_{\tau>0}\tau^{\frac{1}{4}}\|u(\tau)\|_{X^4(\mathbb{R}^3)})^2\|\phi\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)}.
\end{aligned}
$$

This implies $I_2 \to 0$ as $s \to t-0$. Let $0<\theta<\frac{1}{4}$. Using (1.3.3), we find

$$
\begin{aligned}
|I_3| &\le \int_0^s \|(e^{(t-s)\Delta}-\mathrm{id})e^{(s-\tau)\Delta}P\mathrm{div}(u(\tau)\otimes u(\tau))\|_{X^2(\mathbb{R}^3)}\|\phi\|_{L^1_v L^2_h(\mathbb{R}^3)}d\tau \\
&\le C\int_0^s (s-\tau)^{-\frac{3}{4}-\theta}(t-s)^{\theta}\|u(\tau)\|^2_{X^4(\mathbb{R}^3)}d\tau \\
&\le C(t-s)^{\theta}(\sup_{\tau>0}\tau^{\frac{1}{4}}\|u(\tau)\|_{X^4(\mathbb{R}^3)})^2\|\phi\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)}\int_0^s (s-\tau)^{-\frac{3}{4}}\tau^{-\frac{1}{2}}d\tau \\
&\le C(t-s)^{\theta}s^{-\frac{1}{4}}(\sup_{\tau>0}\tau^{\frac{1}{4}}\|u(\tau)\|_{X^4(\mathbb{R}^3)})^2\|\phi\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)}.
\end{aligned}
$$

This implies $I_3 \to 0$ as $s \to t-0$. We have required continuity on $(0,\infty)$. □

The following Lemma implies the continuity to the initial data.

Lemma 1.3.8. *Let $\frac{4}{3}<r<2$ and $\frac{1}{p}=\frac{1}{r}+\frac{1}{4}$. Let u be a mild solution for (1.1.1) satisfying*

$$
\sup_{t>0}\|u(t)\|_{Y^2(\mathbb{R}^3)} + \sup_{t>0} t^{\frac{1}{4}}\|u(t)\|_{X^4(\mathbb{R}^3)} < \infty.
$$

Then

$$\lim_{t\to 0} |\langle u(t) - u_0, \phi\rangle| = 0,$$

for all $\phi \in (L^1_v L^p_h)_\sigma (\mathbb{R}^3) \cap \left(L^1_v L^{2,1}_h\right)_\sigma (\mathbb{R}^3)$

Proof. We use duality argument. Let $\phi \in C^\infty_{0,\sigma}(\mathbb{R}^3)$ and $t > 0$. Then

$$\begin{aligned}
&|\langle u(t) - u, \phi\rangle| \\
&\leq \left|\langle e^{t\Delta} u_0 - u_0, \phi\rangle\right| \\
&+ \int_0^s \left|\langle e^{(t-\tau)\Delta} P\mathrm{div}(u(\tau) \otimes u(\tau)), \phi\rangle\right| d\tau \\
&=: I_1 + I_2.
\end{aligned}$$

We find from the Hölder inequality

$$\begin{aligned}
&\left|\langle e^{t\Delta} u_0 - u_0, \phi\rangle\right| = \left|\langle u_0, e^{t\Delta}\phi - \phi\rangle\right| \\
&\leq \left|\langle \int_{\mathbb{R}} G^1_t (x_3 - y_3)\, u_0 (x', y_3)\, dy_3, \int_{\mathbb{R}^2} G^2_t (x' - y')\, \phi (y', x_3)\, dy' - \phi\rangle\right| \\
&+ \left|\langle u_0, \int_{\mathbb{R}} G^1_t (x_3 - y_3)\, \phi (x', y_3)\, dy_3 - \phi\rangle\right| \\
&\leq \left\|\int_{\mathbb{R}} G^1_t (x_3 - y_3)\, u_0 (x', y_3)\, dy_3\right\|_{Y^2(\mathbb{R}^3)} \left\|\int_{\mathbb{R}^2} G^2_t (x' - y')\, \phi (y', x_3)\, dy' - \phi\right\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)} \\
&+ \|u_0\|_{Y^2(\mathbb{R}^3)} \left\|\int_{\mathbb{R}} G^1_t (x_3 - y_3)\, \phi (x', y_3)\, dy_3 - \phi\right\|_{L^1_v L^{2,1}_h(\mathbb{R}^3)}.
\end{aligned}$$

Thus, we find from continuity of heat semi-group and Lebesgue's dominated convergence theorem

$$|I_1| \to 0 \tag{1.3.20}$$

as $t \to 0$. Next, let $\frac{4}{3} < r < 2$ and $\frac{1}{p} = \frac{1}{r} + \frac{1}{4}$. Applying (1.3.4), we have

$$\begin{aligned}
I_2 &\leq \int_0^t \|e^{(t-\tau)\Delta} P \,\mathrm{div}\,(u(\tau) \otimes u(\tau))\|_{X^p + X^2} \|\phi\|_{L^1_v L^{p'}(\mathbb{R}^3) \cap L^1_v L^2(\mathbb{R}^3)} d\tau \\
&\leq C \int_0^t (t-\tau)^{-\frac{1}{2}} \|u(\tau) \otimes u(\tau)\|_{X^p(\mathbb{R}^3) + X^2(\mathbb{R}^3)} \|\phi\|_{L^1_v L^{p'}(\mathbb{R}^3) \cap L^1_v L^2(\mathbb{R}^3)(\mathbb{R}^3)} d\tau.
\end{aligned}$$

Using the Hölder inequality, duality argument and interpolation inequality

$$\|f\|_{L^{2,1}(\mathbb{R}^2)} \leq C \|f\|_{L^r(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)},$$

we find

$$I_2 \leq C \int_0^t (t-\tau)^{-\frac{1}{2}} \|u(\tau)\|_{X^4(\mathbb{R}^3) + X^r(\mathbb{R}^3)} \|u(\tau)\|_{X^4(\mathbb{R}^3)} \|\phi\|_{L^1_v L^{p'}(\mathbb{R}^3) \cap L^1_v L^2(\mathbb{R}^3)} d\tau$$

$$\le C\int_0^t (t-\tau)^{-\frac12}\|u(\tau)\|_{Y^2(\mathbb{R}^3)}\|u(\tau)\|_{X^4(\mathbb{R}^3)}\|\phi\|_{L^1_vL^{p'}(\mathbb{R}^3)\cap L^1_vL^2(\mathbb{R}^3)}d\tau$$
$$\le C(\sup_{\tau>0}\|(\tau)\|_{Y^2(\mathbb{R}^3)})(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4(\mathbb{R}^3)})\|\phi\|_{L^1_vL^{p'}(\mathbb{R}^3)\cap L^1_vL^2(\mathbb{R}^3)}\int_0^t (t-\tau)^{-\frac12}\tau^{-\frac14}d\tau$$
$$\le Ct^{\frac14}(\sup_{\tau>0}\|(\tau)\|_{Y^2(\mathbb{R}^3)})(\sup_{\tau>0}\tau^{\frac14}\|u(\tau)\|_{X^4(\mathbb{R}^3)})\|\phi\|_{L^1_vL^{p'}(\mathbb{R}^3)\cap L^1_vL^2(\mathbb{R}^3)}.$$

This implies $I_2 \to 0$ as $t \to 0$. The lemma is proved. □

The following proposition implies the uniqueness of u.

Lemma 1.3.9. *Let $\delta > 0$ sufficiently small and $u_0 \in Y^2_\sigma(\mathbb{R}^3)$ satisfying $\|u_0\|_{Y^2(\mathbb{R}^3)} \le \delta$. Then there exists at most one solutions u to (1.1.1) with initial data $u_0 \in Y^2(\mathbb{R}^3)$ satisfying*

$$\sup_{t>0} t^{\frac14}\|u(t)\|_{X^4(\mathbb{R}^3)} \le C\|u_0\|_{Y^2(\mathbb{R}^3)} \tag{1.3.21}$$

Proof. Let u_1 and u_2 be two solution to the Navier-Stokes equations satisfying

$$\sup_{t>0} t^{\frac14}\|u_j(t)\|_{X^4(\mathbb{R}^3)} \le C\|u_0\|_{Y^2(\mathbb{R}^3)}, \quad j=1,2.$$

Then we obtain

$$\begin{aligned}\|u_1(t)-u_2(t)\|_{X^4(\mathbb{R}^3)} &\le \int_0^t \|e^{(t-\tau)\Delta}P\mathrm{div}(u_1(\tau)\otimes u_1(\tau)-u_2(\tau)\otimes u_2(\tau))\|_{X^4(\mathbb{R}^3)}d\tau\\ &\le C(\sup_{\tau>0}\tau^{\frac14}\|u_1(\tau)\|_{X^4(\mathbb{R}^3)}+\sup_{\tau>0}\tau^{\frac14}\|u_2(\tau)\|_{X^4(\mathbb{R}^3)})\\ &\quad\times\sup_{\tau>0}\tau^{\frac14}\|u_1(\tau)-u_2(\tau)\|_{X^4(\mathbb{R}^3)}\int_0^t (t-\tau)^{-\frac34}\tau^{-\frac12}d\tau\\ &\le C't^{-\frac14}\|u_0\|_{Y^2(\mathbb{R}^3)}\sup_{\tau>0}\tau^{\frac14}\|u_1(\tau)-u_2(\tau)\|_{X^4(\mathbb{R}^3)}.\end{aligned}$$

If we take δ so small that $C'\|u_0\|_{Y^2(\mathbb{R}^3)} < 1$, we find

$$\sup_{\tau>0}\tau^{\frac14}\|u_1(\tau)-u_2(\tau)\|_{X^4(\mathbb{R}^3)} \equiv 0.$$

The lemma is proved. □

When $\Omega = \mathbb{R}^2 \times \mathbb{T}$, the mild solution u with initial data $u_0 \in Y^2(\Omega)$ is periodic with respect to vertical variable. Actually, if u_0 is periodic with respect vertical variable, then by definition u_1 is also periodic with respect to same variable. Thus we find u_j is also periodic with respect to vertical variable inductively, and limit functions also is periodic in same same variable. We complete the proof of Theorem1.2.4.

Now, we prove (1.1.1) is locally-in-time well-posed for large initial data if its singularity is sufficiently small.

Theorem 1.3.10. *Let $\epsilon > 0$ sufficiently small and $u_0 \in Y^2_\sigma(\Omega)$ such that*

$$\limsup_{\lambda\to\infty} \lambda \left|\{x \in \Omega\,;\, |u_0(x)| > \lambda\}\right|^{1/2} < \epsilon. \tag{1.3.22}$$

there exists $T > 0$ and a unique mild solution u to (1.1.1) satisfying (1.2.5), (1.2.6) and (1.2.7) for $1/p = 1/r + 1/4$ for all $1/2 < 1/r < 3/4$.

Proof. It is sufficient to show that there exists $T > 0$ such that

$$\sup_{0<t<T} t^{\frac{1}{4}} \|e^{t\Delta}u_0\|_{X^4(\Omega)} \le \delta$$

for some small $\delta > 0$. Actually, if we obtain this estimate, we find approximation solutions u_j $(j = 1, 2, ...)$ satisfy

$$\sup_{0<t<T} ||u_j(t)||_{Y^2(\Omega)} + \sup_{0<t<T} t^{1/4}||u_j(t)||_{X^4(\Omega)} \le C\delta. \tag{1.3.23}$$

for some $C > 0$. Passing their limit as $j \to \infty$, limit vector field is a mild solution to (1.1.1) satisfying (1.2.7). By (1.3.22), u_0 can be decomposed as

$$u_0 = u_{0,1} + u_{0,2}, \quad \text{where } u_{0,1} \in Y^2(\Omega),\ \|u_{0,1}\|_{Y^2(\Omega)} < \epsilon \text{ and } u_{0,2} \in X^4(\Omega).$$

(1.3.4) implies

$$\left\|e^{t\Delta}u_0\right\|_{X^4(\Omega)} \le \left\|e^{t\Delta}u_{0.1}\right\|_{X^4(\Omega)} + \left|\left|e^{t\Delta}u_{0.2}\right|\right|_{X^4(\Omega)} \tag{1.3.24}$$

$$\le Ct^{-\frac{1}{4}} \|u_{0,1}\|_{Y^2(\Omega)} + C\|u_{0,2}\|_{X^4(\Omega)}\,. \tag{1.3.25}$$

Thus we have

$$\sup_{0<t<T} t^{1/4} \left\|e^{t\Delta}u_0\right\|_{X^4(\Omega)} \le C\epsilon + CT^{1/4}\|u_{0,2}\|_{X^4(\Omega)}\,. \tag{1.3.26}$$

If we take $T > 0$ sufficiently small, we (1.3.23). We complete the proof. □

1.4 Maekawa's decompotion of basic flow and their estimate

In this section, we decompose the basic flow as the Maekawa's paper [18] to show the asymptotic stability of the Oseen type vortex. Let us recall Maekawa's decomposition of basic flows in [18].

Proposition 1.4.1. *(Maekawa's decomposition of basic flow and their estimate in two-dimensional case [18]) There exists a constant $\delta > 0$ such that, for any $b_0 \in L^{2,\infty}(\mathbb{R}^2)$ with $\|b_0\|_{L^{2,\infty}(\mathbb{R}^2)} \le \delta$ and $T > 1$, the solution b to two dimensional Navier-Stokes equation (1.1.1) with initial data b_0 is decomposed as $b = b_T + b^T$, for $b_T, b^T \in L^\infty_t L^{2,\infty}_x(\mathbb{R}^2 \times (0,\infty))$ satisfy*

$$\sup_{t>0}\|b_T(t)\|_{L^{2,\infty}(\mathbb{R}^2)} + \sup_{t>0}(t+T)^{\frac{1}{4}}\|b_T(t)\|_{L^4(\mathbb{R}^2)} \le C\|b_0\|_{L^{2,\infty}(\mathbb{R}^2)} \tag{1.4.1}$$

$$\sup_{t>0}\|b^T(t)\|_{L^{2,\infty}(\mathbb{R}^2)} + \sup_{t>0} t^{\frac{1}{4}}\|b^T(t)\|_{L^4(\mathbb{R}^2)} \le C\|b_0\|_{L^{2,\infty}(\mathbb{R}^2)} \tag{1.4.2}$$

and b^T also satisfies the energy estimate

$$\|b^T(t)\|^2_{L^2(\mathbb{R}^2)} + \int_1^t \|\nabla b^T(\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \le C\|b_0(\mathbb{R}^2)\|^2_{L^{2,\infty}(\mathbb{R}^2)} \log(1+T) \tag{1.4.3}$$

for all $t > 1$.

The following proposition is the Maekawa's decomposition to the three dimensional Oseen type vortex.

Proposition 1.4.2. *(Maekawa's decomposition of the Oseen type basic flow and its estimate) There exists a constant $\delta > 0$ such that, for any $b_0 \in Y^2(\mathbb{R}^2 \times \mathbb{T}^1)$ with $\|b_0\|_{Y^2_\sigma(\mathbb{R}^2\times\mathbb{T}^1)} \le \delta$ and $T > 1$, the solution b to (1.1.1) with initial data b_0 is decomposed as $b = b_T + b^T$, where b_T and b^T with $b_T, b^T \in C_{w^*,t}Y^2_x((\mathbb{R}^2\times\mathbb{T}^1)\times(0,\infty))$ satisfy*

$$\sup_{t>0}\|b_T(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0}(t+T)^{\frac{1}{4}}\|b_T(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{1.4.4}$$

$$\sup_{t>0}\|b^T(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0} t^{\frac{1}{4}}\|b^T(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{1.4.5}$$

and b^T also satisfies the energy estimate

$$\|b^T(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|\nabla b^T(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau \le C\|b_0\|^2_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \log(1+T) \tag{1.4.6}$$

for all $t > 1$.

To show the Proposition 1.4.2, we have to decompose the initial data to the basic flow b.

Proposition 1.4.3. *Let $T > 1$ and $b \in Y^2(\mathbb{R}^2 \times \mathbb{T}^1)$. Then there exists a positive constant C such that b_0 can be decomposed as $b_0 = b_{0,T} + b_0^T$ satisfying*

$$\|b_{0,T}\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + T^{\frac{1}{4}}\|b_{0,T}\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{1.4.7}$$

$$\|b_0^T\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \frac{(2-q)^{\frac{1}{2}}}{T^{\frac{1}{q}-\frac{1}{2}}}\|b_0^T\|_{X^q(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}, \tag{1.4.8}$$

for all $q \in [\frac{4}{3}, 2)$.

Proof. It follows from Lemma 3.2 in [18] that

$$\|b_{0,T}(\cdot,x_v)\|_{L^{2,\infty}_h(\mathbb{R}^2)} + T^{\frac{1}{4}}\|b_{0,T}(\cdot,x_v)\|_{L^4_h} \le C\|b_0(\cdot,x_v)\|_{L^{2,\infty}_h(\mathbb{R}^2)}$$
$$\|b_0^T(\cdot,x_v)\|_{L^{2,\infty}(\mathbb{R}^2)} \le C\|b_0(\cdot,x_v)\|_{L^{2,\infty}_h(\mathbb{R}^2)}$$
$$\|b_0^T(\cdot,x_v)\|_{L^q_h(\mathbb{R}^2)} \le C\frac{T^{\frac{1}{q}-\frac{1}{2}}}{(2-q)^{\frac{1}{2}}}\|b_0(\cdot,x_v)\|_{L^{2,\infty}(\mathbb{R}^2)}.$$

This inequalities imply the proposition. ∎

proof of Proposition1.4.2. Let $\delta > 0$ be sufficient small. We find $\|b_{0,T}\|_{Y^2}, \|b_0^T\|_{Y^2} \leq \delta$ by definition. Using contraction principle as[18], we can construct a unique mild solution to the following integral equation with initial data $b_{0,T}$

$$b_T(t) = e^{t\Delta}b_{0,T} - \int_0^t e^{(t-\tau)\Delta} P\mathrm{div}(b_T(\tau) \otimes b(\tau))d\tau, \tag{1.4.9}$$

where $e^{t\Delta}$ and P are the heat semigroup and the Helmholtz projection on $\mathbb{R}^2 \times \mathbb{T}^1$ respectively. Moreover, the solution b_T satisfies

$$\sup_{t>0}\|b_T(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0}(t+T)^{\frac{1}{4}}\|b_T(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\|b_{0,T}\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}.$$

in $\mathrm{Q}_{vper,T}$, where $Q_{vper,T} = \mathbb{R}^2 \times \mathbb{T}^1 \times (0,T)$. Similarly, there exits a functions b^T satisfying

$$b^T(t) = e^{t\Delta}b_0^T - \int_0^t e^{t-\tau\Delta} P\mathrm{div}(b^T(\tau) \otimes b(\tau))d\tau,$$

and

$$\sup_{t>0}\|b_T(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0} t^{\frac{1}{4}}\|b^T(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}.$$

Note that b_T and b^T satisfies $b = b_T + b^T$. Now, we prove the energy estimate (1.4.6). First, we have to check $b^T(t) \in L^2(\mathbb{R}^2 \times \mathbb{T}^1)$ for all $t \geq 1$. Indeed, it follows from (1.3.4) that

$$\begin{aligned} &\|e^{t\Delta}b_0^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq \|e^{t\Delta}b_0^T\|_{X^2((\mathbb{R}^2\times\mathbb{T}^1))} \\ &\leq Ct^{-(\frac{1}{q}-\frac{1}{2})}\|b_0^T\|_{X^q(\mathbb{R}^2\times\mathbb{T}^1)}, \quad \text{for all } q \in [\frac{4}{3}, 2), \end{aligned} \tag{1.4.10}$$

and

$$\begin{aligned} &\|\int_0^t e^{(t-\tau)\Delta} P\mathrm{div}(b^T(\tau) \otimes b(\tau))d\tau\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \\ &\leq C\int_0^t \|e^{(t-\tau)\Delta} P\mathrm{div}(b^T(\tau) \otimes b(\tau))\|_{X^2(\mathbb{R}^2\times\mathbb{T}^1)}d\tau \\ &\leq C(\sup_{t>0}\tau^{\frac{1}{4}}\|b^T(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)})(\sup_{\tau>0}\tau^{\frac{1}{4}}\|b(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)})\int_0^t (T-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}d\tau \\ &\leq C\|b_0\|^2_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}. \end{aligned} \tag{1.4.11}$$

Thus, we get $b^T(t) \in L^2(\mathbb{R}^2 \times \mathbb{T}^1)$. Next, since b^T satisfies

$$\partial_t b^T - \Delta b^T + b\cdot\nabla b^T + \nabla q = 0, \text{ and } \mathrm{div}\, b^T = 0,$$

it follows by testing with b^T and integration by parts

$$\|b^T(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|\nabla b(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}d\tau \leq \|b^T(1)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{1.4.12}$$

for all $t \geq 1$. From (1.4.10) and (1.4.11), the right hand side of (1.4.12) satisfies

$$\|b(1)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 \leq C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}\left(\frac{T^{\frac{1}{q}-\frac{1}{2}}}{(2-q)^{\frac{1}{2}}} + \|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}\right) \tag{1.4.13}$$

for all $q \in [\frac{4}{3}, 2)$. Taking q so that $2 - q = \frac{1}{4\log(1+T)}$, we finally obtain

$$\begin{aligned}&\|b^T(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 + \int_1^t \|\nabla b^T(\tau)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 d\tau \\ &\leq C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}(\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \log(1+T)).\end{aligned}$$

□

1.5 Logarithmic energy estimates for perturbed equations with their construction

In this section, we construct a weak solution to the perturbed Navier-Stoks equations v defined in the second section with initial data $v_0 \in L^\infty_v C^\infty_{0,h}(\mathbb{R}^2 \times \mathbb{T}^1)$. Firstly, we construct a local-in-time mild solution on $(0, T_*)$. Secondly, we establish the global-in-time weak solution with initial data $v(T_*)$.

Proposition 1.5.1. *Let $\delta > 0$ be sufficiently small and $v_0 \in X_\sigma^{\frac{4}{3}}(\mathbb{R}^2\times\mathbb{T}^1)\cap X_\sigma^4(\mathbb{R}^2\times\mathbb{T}^1)$. Let us assume that $b \in L_t^\infty Y_x^2(Q_{vper,\infty})$ be a solution to (1.1.1) with initial data $b_0 \in Y_\sigma^2(\mathbb{R}^2 \times \mathbb{T}^1)$ with $\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq \delta$ obtained in Theorem 1.2.4 satisfying*

$$\sup_{t>0}\|b(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0} t^{\frac{1}{4}}\|b(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \leq \delta.$$

Then there exist $T_* > 0$ and a unique mild solution $v \in Y_\sigma^{\frac{4}{3}}(\mathbb{R}^2 \times \mathbb{T}^1 \times (0, T_*)) \cap X_\sigma^4(\mathbb{R}^2 \times \mathbb{T}^1 \times (0, T_*))$ to (1.1.4) satisfying

$$v(t) = e^{t\Delta}b_0 - \int_0^t e^{(t-\tau)\Delta}P\mathrm{div}(v(\tau)\otimes v(\tau) + v(\tau)\otimes b(\tau) + b(\tau)\otimes v(\tau))d\tau \tag{1.5.1}$$

and

$$\sup_{0<\tau<T_*}\|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\|v_0\|_{X^{\frac{4}{3}}(\mathbb{R}^2\times\mathbb{T}^1)} \tag{1.5.2}$$

$$\sup_{0<\tau<T_*}\|v(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\|v_0\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}. \tag{1.5.3}$$

Proof. Put

$$N(v, w, t) := \int_0^t e^{(t-\tau)\Delta}P\mathrm{div}(v(\tau)\otimes w(\tau) + v(\tau)\otimes b(\tau) + b(\tau)\otimes v(\tau))d\tau,$$

where $v, w \in L_t^\infty(X^{\frac{4}{3}} \cap X^4)_x$. It is sufficient to show that there exist constants C_1 and C_2 such that

$$\|N(v,w,t)\|_{(L_t^\infty X^{\frac{4}{3}} \cap L_t^\infty X_x^4)(Q_{vper,T_*})} \tag{1.5.4}$$
$$\leq C_1 T_*^{\frac{1}{4}} \|v\|_{(L_t^\infty X^{\frac{4}{3}} \cap L_t^\infty X_x^4)(Q_{vper,T_*})} \|w\|_{(L_t^\infty X^{\frac{4}{3}} \cap L_t^\infty X_x^4)(Q_{vper,T_*})}$$
$$+ C_2 \delta \|v\|_{(L_t^\infty X^{\frac{4}{3}} \cap L_t^\infty X_x^4)(Q_{vper,T_*})} \tag{1.5.5}$$

for $v, w \in L_t^\infty(X^{\frac{4}{3}} \cap X^4)_x(Q_{T_*,per})$. Actually, if it holds, taking T_* small and using the Picard contraction principle, we obtain the mild solution. Using (1.3.4), we find

$$\|N(v,w,t)\|_{X^{\frac{4}{3}}(\mathbb{R}^2 \times \mathbb{T}^1)} \tag{1.5.6}$$
$$\leq \int_0^t \|e^{(t-\tau)\Delta} P\mathrm{div}(v(\tau) \otimes w(\tau) + v(\tau) \otimes b(\tau) + b(\tau) \otimes v(\tau))\|_{X^{\frac{4}{3}}(\mathbb{R}^2 \times \mathbb{T}^1)} d\tau$$
$$\leq C \int_0^t (t-\tau)^{-\frac{3}{4}} (\|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}^2 \times \mathbb{T}^1)} \|w(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)} \tag{1.5.7}$$
$$+ 2\|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}^2 \times \mathbb{T}^1)} \|b(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)}) d\tau$$
$$\leq C_1 t^{\frac{1}{4}} (\sup_{0<\tau<t} \|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}^2 \times \mathbb{T}^1)}) (\sup_{0<\tau<t} \|w(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)})$$
$$+ C_2 (\sup_{0<\tau<t} \|v(\tau)\|_{X^{\frac{4}{3}}(\mathbb{R}^2 \times \mathbb{T}^1)}) (\sup_{\tau>0} \tau^{\frac{1}{4}} \|b(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)}). \tag{1.5.8}$$

Similarly, we find

$$\|N(v,w,t)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)} \leq \int_0^t \|e^{(t-\tau)\Delta} P\mathrm{div}(v(\tau) \otimes w(\tau) + v(\tau) \otimes b(\tau) \tag{1.5.9}$$
$$+ b(\tau) \otimes v(\tau))\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)} d\tau$$
$$\leq C \int_0^t (t-\tau)^{-\frac{3}{4}} (\|v(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)} \|w(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)} \tag{1.5.10}$$
$$+ \|v(\tau)\|_{X^4} \|b(\tau)\|_{X^4}) d\tau$$
$$\leq C_1 t^{\frac{1}{4}} (\sup_{0<\tau<t} \|v(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)}) (\sup_{0<\tau<t} \|w(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)}) \tag{1.5.11}$$
$$+ C_2 (\sup_{0<\tau<t} \|v(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)}) (\sup_{\tau>0} \tau^{\frac{1}{4}} \|b(\tau)\|_{X^4(\mathbb{R}^2 \times \mathbb{T}^1)}). \tag{1.5.12}$$

□

We construct a global-in-time weak solution to the perturbed Navier-Stokes equations on $(0, \infty)$ with initial data $v(T_*) \in L^2(\mathbb{R}^2 \times \mathbb{T}^1)$. Firstly, we construct a solution to the mollified perturbed Navier-Stokes equations. Secondly, taking limit for it, we get a solution to the perturbed Navier-Stokes equations.

Let ψ be the standard mollifier and $(f)_\rho(x)$ denote $\frac{1}{\rho^3}\psi(\frac{\cdot}{\rho}) * f$. The following proposition assert that there exist a weak solutions to the mollified perturbed Navier-Stokes equations with initial data $v_0 \in L^2(\mathbb{R}^2 \times \mathbb{T}^1)$.

Proposition 1.5.2. *Let $0 < \rho < 1$ and $T > 0$. aLet $b \in L_t^\infty Y_x^2(Q_{vper,T})$ and be a mild solution to (1.1.1) with non-zero initial data $b_0 \in Y_\sigma^2$ satisfying*

$$\sup_{t>0}\|b(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0}(t+1)^{\frac{1}{4}}\|b(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}. \tag{1.5.13}$$

Then there exists a unique weak solution $v^\rho \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(Q_{vper,T})$ to the mollified perturbed Navier-Stokes equation

$$\partial_t v^\rho - \Delta v^\rho + (v^\rho)_\rho \cdot \nabla v^\rho + b\cdot\nabla v^\rho + v^\rho\cdot\nabla b + \nabla q = 0, \tag{1.5.14}$$

$$\mathrm{div}\ v = 0, \tag{1.5.15}$$

with initial data $v_0 \in L_\sigma^2(\mathbb{R}^2\times\mathbb{T}^1)$ satisfying

$$\int_0^t -\langle v^\rho, \partial_t\phi\rangle + \langle\nabla v^\rho : \nabla\phi\rangle - \langle v^\rho\otimes(v^\rho)_\rho + (b)_\rho\otimes v^\rho + v^\rho\otimes(b)_\rho : \nabla\phi\rangle d\tau$$
$$= \langle v_0, \phi\rangle \tag{1.5.16}$$

for any $\phi \in C_{0,\sigma}^\infty(Q_{vper,T})$. Moreover, v^ρ satisfies the energy estimate

$$\|v^\rho(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_0^t \|\nabla v^\rho(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}d\tau$$
$$\le C_1(1+t)^{C_2\|b_0\|^4_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}}\|v_0\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \tag{1.5.17}$$

for all $t \in (0,T)$, where constants C_1 and C_2 are independent of ρ.

Proof. Let $v, w \in L_t^\infty L_x^2(Q_{vper,T})$. We define N_ρ as

$$N_\rho(v,w,t)$$
$$:= \int_0^t e^{(t-\tau)\Delta}P\mathrm{div}(v(\tau)\otimes(w)_\rho(\tau) + v(\tau)\otimes(b)_\rho(\tau) + (b)_\rho(\tau)\otimes v(\tau))d\tau. \tag{1.5.18}$$

First, we show that there exits a positive constant $0 < T_* < 1$ and

$$v^\rho \in L_t^\infty L_x^2(Q_{vper,T_*})$$

$\cap L_t^2 H_x^1(Q_{vper,T_*})$ such that

$$v^\rho(t) = e^{t\Delta}v_0 - N_\rho(v^\rho, v^\rho, t), \quad in \quad (L_t^\infty L_x^2 \cap L_t^2 \dot{H}^1{}_x)(Q_{vper,T_*}). \tag{1.5.19}$$

It follows from integration by parts that

$$\|e^{t\Delta}v_0\|_{L_t^\infty L_x^2(Q_{vper,T_*})} + \|e^{t\Delta}v_0\|_{L_t^2\dot{H}_x^1(Q_{vper,T_*})} \le \|v_0\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}.$$

Since

$$(\int_0^t \|v(\tau)\otimes(w)_\rho(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \|(b)_\rho(\tau)\otimes v(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}$$
$$+ \|v(\tau)\otimes(b)_\rho(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}d\tau)^{\frac{1}{2}}$$

$$\le C_1\Big(\int_0^t \|v(\tau)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|(w)_\rho(\tau)\|_{L^\infty(\mathbb{R}^2\times\mathbb{T}^1)}$$
$$+ \|(b)_\rho(\tau)\|_{L^\infty(\mathbb{R}^2\times\mathbb{T}^1)}\|v(\tau)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}d\tau\Big)^{\frac{1}{2}}$$
$$\le C_1\rho^{-\frac{3}{2}}T_*^{\frac{1}{2}}(\|v\|_{L^\infty_t L^2_x(Q_{vper,T_*})}\|w\|_{L^\infty_t L^2_x(Q_{vper,T_*})}$$
$$+ \|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})}\|v\|_{L^\infty_t L^2_x(Q_{vper,T_*})}),$$

it follows from energy estimate that

$$\|N_\rho(v,w,t)\|_{L^\infty_t L^2_x(Q_{vper,T_*})} + \|N(v,w,t)\|_{L^2_t \dot{H}^1_x(Q_{vper,T_*})}$$
$$\le C_1\rho^{-\frac{3}{2}}T_*^{\frac{1}{2}}(\|v\|_{L^\infty_t L^2_x(Q_{vper,T_*})}\|w\|_{L^\infty_t L^2_x(Q_{vper,T_*})}$$
$$+ \|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})}\|v\|_{L^\infty_t L^2_x(Q_{vper,T_*})}).$$

Thus, if we take T_* so small that

$$T_*^{\frac{1}{2}} < \min(1, \rho^{\frac{3}{2}}\frac{\|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})} + 2\|v_0\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}}{4C_1\|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})}}$$
$$- \frac{\sqrt{(\|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})} + 2\|v_0\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)})^2 - \|b\|_{L^\infty_t X^4_x(Q_{vper,T_*})}}}{4C_1\|b\|^2_{L^\infty_t X^4_x(Q_{vper,T_*})}}),$$

there exists a unique mild solution v to

$$v^\rho(t)$$
$$= e^{t\Delta}v_0 - \int_0^t e^{(t-\tau)\Delta}P\mathrm{div}(v^\rho(\tau)\otimes(v^\rho)_\rho(\tau) + v^\rho(\tau)\otimes(b)_\rho(\tau) + (b)_\rho(\tau)\otimes v^\rho(\tau))d\tau$$

on $t\in(0,T_*)$.

Next, we show the a priori bound for v. This leads the existence of global-in-time weak solution to (1.5.17). Integration by parts to (1.5.14) yields

$$\frac{1}{2}\partial_t\|v^\rho(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \|\nabla v^\rho(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}$$
$$\le |\langle b(t)\otimes v^\rho(t) : \nabla v^\rho(t)\rangle|$$
$$\le C\|b(t)\otimes v^\rho(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|\nabla v^\rho(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}. \tag{1.5.20}$$

Using interpolation inequality and the Young inequality, we get

$$(1.5.20) \le C\|b(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}\|v^\rho(t)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|\nabla v^\rho(t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}$$
$$\le C\|b(t)\|^4_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}\|v^\rho(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \frac{1}{2}\|\nabla v^\rho(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}. \tag{1.5.21}$$

Applying the Gronwall inequality to (1.5.20) and (1.5.21), we obtain

$$\|v^\rho(t)\|^2_{L^2} + \int_0^t \|\nabla v^\rho(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}d\tau$$

$$\leq \exp(C\int_0^t \|b(\tau)\|^4_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}d\tau)\|v_0\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq C_2(1+t)^{C_3\|b_0\|^4_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}}\|v_0\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}.$$

Thus, we get a priori estimate $\|u(t)\|_{L^2} \leq C_2(1+T)^{C_3\|b_0\|^4_{Y^2}}\|v_0\|_{L^2}$. Using this estimate, we can extend the maximal existence time by

$$\min(1, \rho^3(\frac{\|b\|_{L^\infty_t X^4_x(Q_{vper,T})} + 2C_2(1+T)^{C_3\|b_0\|^4_{Y^2}}\|v_0\|_{L^2}}{4C_1\|b\|_{L^\infty_t X^4_x(Q_{vper,T})}}$$
$$-\frac{\sqrt{(\|b\|_{L^\infty_t X^4_x(Q_{vper,T})} + 2C_2(1+T)^{C_3\|b_0\|^4_{Y^2}}\|v_0\|_{L^2})^2 - \|b\|^2_{L^\infty_t X^4_x(Q_{vper,T})}}}{4C_1\|b\|^2_{L^\infty_t X^4_x(Q_{vper,T})}})).$$

Since T is finite, we can use same argument until the existence time become greater than T. The proposition is proved. □

Now, let us prove the existence of the perturbed Navier-Stokes equation for L^2-initial data.

Proposition 1.5.3. *Let $T > 0$, $v_0 \in L^2_\sigma(\mathbb{R}^2 \times \mathbb{T}^1)$ and $b \in L^\infty_t Y^2_\sigma(Q_{vper,T})$ be a mild solution to (1.1.1) with initial data $b_0 \in Y^2_\sigma$ satisfying*

$$\sup_{t>0}\|b(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0}(t+1)^{\frac{1}{4}}\|b(t)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}. \tag{1.5.22}$$

Then there exists a weak solution v to the perturbed Navier-Stokes equation

$$\begin{cases} \partial_t v - \Delta v + \operatorname{div}(v\otimes v + v\otimes b + b\otimes v) + \nabla q = 0 \text{ in } \mathbb{R}^2\times\mathbb{T}^1\times(0,T), \\ \operatorname{div}\, v = 0 \text{ in } \mathbb{R}^2\times\mathbb{T}^1\times(0,T), \\ v(0) = v_0 \quad \text{on}\, \mathbb{R}^2\times\mathbb{T}^1 \end{cases} \tag{1.5.23}$$

in the sense of distribution with $q \in L^1_t L^1_{x,loc}((0,T)\times(\mathbb{R}^2\times\mathbb{T}^1))$ with energy inequality;

$$\|v(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + 2\int_1^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}d\tau$$
$$\leq C_1\|v_0\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}(1+t)^{C_2\|b_0\|^4_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}}$$

for all $t > 1$, where $C_1, C_2 > 0$ is independent of t, and continuity of initial data;

$$\|v(t) - v_0\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \to 0 \tag{1.5.24}$$

as $t \to +0$

Proof. We have shown the existence of a solution to the mollified equations with energy estimate, which is uniform on ρ. We have to get uniform estimate in ρ to $\|\partial_t v^\rho\|_{L^2_t H^{-3}_x}$ to take limit to the mollified equations. Let $\phi \in L^2_t H^3_x(Q_{vper,T})$. Applying the Hölder inequality, the Sobolev embedding $H^s \hookrightarrow L^\infty$ for $s > \frac{3}{2}$, the Hölder inequality and the Gagliardo-Nirenberg ineqality, we have

$$\int_0^T |\langle \partial_t v^\rho(\tau), \phi(\tau)\rangle|\, d\tau$$

$$
\begin{aligned}
&\leq \int_0^T |\langle \nabla v^\rho(\tau), \nabla\phi(\tau)\rangle|\, d\tau \\
&+ \int_0^T \left|\langle < v^\rho(\tau)\otimes v^\rho(\tau) + v^\rho(\tau)\otimes (b)_\rho(\tau) + (b)_\rho(\tau)\otimes v^\rho(\tau); \nabla\phi(\tau)\rangle\right| d\tau \\
&\leq C\,\|\nabla v^\rho\|_{L^2_tL^2_x(Q_{vper,T})}\|\nabla\phi\|_{L^2_tL^2_x(Q_{vper,T})} \\
&+ C\int_0^T \|v^\rho(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|\nabla\phi(\tau)\|_{L^\infty(\mathbb{R}^2\times\mathbb{T}^1)}\,d\tau \\
&+ C\int_0^T \|b(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}\|v^\rho(\tau)\|^{\frac12}_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|\nabla v^\rho(\tau)\|^{\frac12}_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|\nabla\phi\|_{L^2_tL^2_x(Q_{vper,T})}\,d\tau \\
&\leq C\,\|\nabla v^\rho\|_{L^2_tL^2_x(Q_{vper,T})}\|\nabla\phi\|_{L^2_tL^2_x(Q_{vper,T})} \\
&+ C\,\|v^\rho\|^2_{L^\infty_tL^2_x(Q_{vper,T})}\|\nabla\phi\|_{L^2_tH^s_x(Q_{vper,T})} \\
&+ C\,\|v^\rho\|^{\frac12}_{L^\infty_tL^2_x(Q_{vper,T})}\|\nabla v^\rho\|^{\frac12}_{L^2_tL^2_x(Q_{vper,T})}\|\nabla\phi\|_{L^2_tL^2_x(Q_{vper,T})}
\end{aligned}
$$

for some $C > 0$ which is depend on T and $\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}$. Thus we have

$$||v^\rho||_{L^2_tH^{-3}_x(Q_{vper,T})} < \infty,$$

which is uniform in ρ. Therefore, from (1.5.17), the above estimate and the Aubin-Lions theorem, there exit a subsequence $\{v^{\rho_j}\}_{\rho_j} \subset \{v^\rho\}_\rho$ and a vector field v such that

$$
\begin{aligned}
&v^{\rho_j} \to v \quad \textit{weakly}^* \quad \text{in} \quad L^\infty_tL^2_x(Q_{vper,T}) && (1.5.25)\\
&\nabla v^{\rho_j} \to \nabla v \quad \textit{weakly} \quad \text{in} \quad L^2_tL^2_x(Q_{vper,T}) && (1.5.26)\\
&v^{\rho_j} \to v \quad \text{in} \quad L^2_tL^2_{loc,x}(Q_{vper,T}), && (1.5.27)
\end{aligned}
$$

as $j \to \infty$. v satisfies (1.5.23) in the sense of distribution. Moreover, the limit functions v satisfies the energy estimate

$$
\begin{aligned}
&\|v(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_0^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}d\tau \\
&\leq C_1(1+t)^{C_2\|b_0\|^4_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}}\|v_0\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}, \qquad (1.5.28)
\end{aligned}
$$

and the perturbed Navier-Stokes equation. From the estimates above,

$$t \to \langle v(t), \phi\rangle \qquad (1.5.29)$$

is continuous on $[0, T)$ for all $\phi \in L^2(\mathbb{R}^2\times\mathbb{T}^1)$, and

$$\|v(t) - v_0\|_{L^2} \to 0 \quad \text{as} \quad t \to 0. \qquad (1.5.30)$$

The proposition is proved. □

Fix $T > 0$. Then, from Proposition 1.5.1 and Proposition 1.5.3, we have a global weak solutions $v \in L^\infty_tL^2_x(Q_{T,vper})$ to the perturbed Navier-Stokes equations with initial data $v_0 \in (X^{\frac{4}{3}} \cap X^4)(\mathbb{R}^2 \times \mathbb{T}^1)$. Moreover, since $v(T_1) \in (X^{\frac{4}{3}} \cap X^4)(\mathbb{R}^2 \times \mathbb{T}^1)$ for all $0 < T_1 < T_*$, it follows that

$$\|v(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 + \int_{T_1}^t \|\nabla v(\tau)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 d\tau \le C_1(1+t)^{C_2\|b_0\|_{Y^2}^4}\|v(T_1)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 \tag{1.5.31}$$

for all $T_1 < t < T$, where $C > 0$ is independent of t. Hereafter, we denote T_1 as 1 for simplicity. The following proposition is the logarithmic energy estimate for v.

Proposition 1.5.4. *Let $\delta > 0$ sufficiently small. Let $b_0 \in Y(\mathbb{R}^2 \times \mathbb{T}^1)$ satisfy $\|b_0\|_{Y^2_\sigma(\mathbb{R}^2\times\mathbb{T}^1)} < \delta$ and $b \in L^\infty_t Y^2(\mathbb{R}^2 \times \mathbb{T}^1)$ be the mild solution with initial data b_0 such that*

$$\sup_{t>0}\|b(t)\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{t>0} t^{\frac{1}{4}}\|\sup_{t>0}\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \le C\|b_0\|_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)}$$

for some constant C. Then the solution v to the perturbed Navier-Stokes with b obtained by Proposition 1.5.1 *and* Proposition 1.5.3 *with initial data v_0 satisfies*

$$\|v(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 + \int_1^t \|\nabla v(\tau)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 d\tau \le C_\epsilon + C\delta^2\log(1+t) \tag{1.5.32}$$

for $t > 1$ where C_ϵ and C are independent of t.

Proof. First, we find from Proposition 1.4.2 there exist b_T and b^T such that $b = b_T + b^T$ satisfying (1.4.4), (1.4.5) and (1.4.6). Put $v^T := v - b^T$, then we find that v^T satisfies

$$\begin{aligned}
&\partial_t v^T - \Delta v^T + \mathrm{div}(v^T \otimes v^T + v^T \otimes b_T + b_T \otimes v^T - b_T \otimes b^T)\\
&+ \nabla q = 0, \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{T}^1 \times (0,\infty) &(1.5.33)\\
&\mathrm{div}\ v^T = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{T}^1 \times (0,\infty), &(1.5.34)
\end{aligned}$$

for some $q \in L^1_{loc,t}L^1_{loc,x}(\mathbb{R}^2 \times \mathbb{T}^1 \times (0,\infty))$. It follows from integration by parts that

$$\frac{1}{2}\partial_t\|v^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 + \|\nabla v^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 = \langle v^T \otimes b_T - b^T \otimes b_T : \nabla v\rangle. \tag{1.5.35}$$

Using the Hölder inequality and the Young inequality, we find

$$\begin{aligned}
\left|\langle v^T \otimes b_T : \nabla v^T\rangle\right| &\le C\|v^T \otimes b_T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|\nabla v^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\\
&\le C\|b_T\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}^4\|v^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 + \frac{1}{4}\|\nabla v^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 &(1.5.36)
\end{aligned}$$

and

$$\begin{aligned}
\left|\langle b^T \otimes b_T : \nabla v\rangle\right| &\le \|b_T \otimes b^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|v^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\\
&\le C\|b^T\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}^2\|b_T\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}^2 + \frac{1}{4}\|v^T\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2, &(1.5.37)
\end{aligned}$$

Using the Gronwall inequality, we find for $t \in (1, T]$ that

$$
\begin{aligned}
&\|v^T(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|\nabla v^T(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau \\
&\le C\exp(\int_1^t \|b_T(\tau)\|^4_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} d\tau)(\|v^T(1)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \\
&\qquad + \int_1^t \|b^T(\tau)\|^2_{X^4}\|b_T(\tau)\|^2_{X^4} d\tau) \\
&\le C\exp(C_1\int_1^t (T+\tau)^{-1} d\tau)(\|v^T(1)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \\
&\qquad + \|b^T\|^4_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \int_1^t \tau^{-\frac{1}{2}}(T+\tau)^{-\frac{1}{2}} d\tau) \\
&\le C(\|v^T(1)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \delta^4)
\end{aligned} \tag{1.5.38}
$$

Since $v = v^T + b^T$, it follows from energy inequality (1.4.6) that

$$
\begin{aligned}
&\|v^T(1)\|^2_{L^2} \le 2(\|v(1)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \|b^T(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}) \\
&\le C + C\|b_0\|^4_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \log(1+T).
\end{aligned} \tag{1.5.39}
$$

Then we obtain

$$
\begin{aligned}
&\|v(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau \\
&\le C(\|v^T(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|\nabla v^T(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau \\
&+ \|b^T(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|b^T(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau) \\
&\le C + C\delta^2 \log(1+T).
\end{aligned} \tag{1.5.40}
$$

If we teke $t = T$, then we have (1.5.32). □

1.6 Estimates for vertically averaged part

In this section, we show some lemmas that enable us to get the L^2-decay for the weak solution to the perturbed Navier-Stokes equations. The decay estimate of v in this section is possible for any v that is constructed as the limit function of solutions v^ρ obtained by Proposition 1.5.2.

Applying the Fourier expansion to v with respect to x_v, we can decompose v into averaged part v_a and oscillating part v_{os};

$$
\begin{aligned}
v(x_h, x_v, t) &= \sum_{k\in\mathbb{Z}} v_k(x_h, t)\mathrm{e}^{2\pi i x_v k} = v_0(x_h, t) + \sum_{k\neq 0} v_k(x_h, t)\mathrm{e}^{2\pi x_v k} \\
&=: v_a(x_h, t) + v_{os}(x_h, x_v, t).
\end{aligned}
$$

Because of orthogonality of the Fourier series, it follows from (1.5.32) that

$$\|v_a(t)\|^2_{L^2(\mathbb{R}^2)} + \int_1^t \|\nabla_h v_a\|^2_{L^2(\mathbb{R}^2)} \le C + C\delta^2 \log(1+t) \tag{1.6.1}$$

$$\|v_{os}(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_1^t \|\nabla v_{os}\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \le C + C\delta^2 \log(1+t). \tag{1.6.2}$$

We first show the following proposition to prove the decay of averaged part.

Proposition 1.6.1. *Let $T > 0$. Put $w_a := (-\Delta_h)^{-\frac{1}{4}} v_a$, where*

$$(-\Delta_h)^s f = \mathcal{F}^{-1}(|\xi_h|^s \mathcal{F} f)$$

for $s \in \mathbb{R}$. Then there exist constants $C > 0$ and $M > 0$ such that

$$\begin{aligned}
&\|w_a(t)\|^2_{L^2(\mathbb{R}^2)} + \int_1^t \|\nabla_h w_a(t)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&\le C(1+t)^{M\delta^2}(1 + \log(1+t) + \sup_{1\le\tau\le t} \|v_{os}(\tau)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \log(1+t))
\end{aligned} \tag{1.6.3}$$

for all $1 < t \le T$.

Proof. Integrating (1.2.3) with respect to x_v over $\mathbb{T}^1$, then we find

$$\partial_t v_a^1 - \Delta_h v_a^1 + \operatorname{div} \int_{\mathbb{T}^1} (v^1 v + b^1 v + v^1 b) dx_v + \partial_1 q = 0 \tag{1.6.4}$$

$$\partial_t v_a^2 - \Delta_h v_a^2 + \operatorname{div} \int_{\mathbb{T}^1} (v^2 v + b^2 v + v^2 b) dx_v + \partial_2 q = 0 \tag{1.6.5}$$

$$\partial_t v_a^3 - \Delta_h v_a^3 + \operatorname{div} \int_{\mathbb{T}^1} (v^3 v + b^3 v + v^3 b) dx_v = 0. \tag{1.6.6}$$

(1.6.4) (1.6.5) are the two dimensional perturbed Navier-Stokes system and (1.6.6) is two dimensional heat equation respectively. It follows from integration by parts

$$\begin{aligned}
&\frac{1}{2}\partial_t \|w_a\|^2_{L^2(\mathbb{R}^2)} + \|\nabla w_a\|^2_{L^2(\mathbb{R}^2)} \\
&\le \left|\int_{\mathbb{R}^2}\int_{\mathbb{T}^1} (v\otimes v + b\otimes v + v\otimes b) dx_v : \nabla_h (-\Delta_h)^{-\frac{1}{4}} w_a dx_h\right| \\
&= \left|\int_{\mathbb{R}^2}\int_{\mathbb{T}^1} ((v_a + v_{os})\otimes(v_a+v_{os}) + b\otimes(v_a+v_{os}) \right.\\
&\left. + (v_a+v_{os})\otimes b) dx_v : \nabla(-\Delta_h)^{-\frac{1}{4}} w_a dx_h\right| \\
&= \left|\int_{\mathbb{R}^2}\int_{\mathbb{T}^1} (v_a\otimes v_a + v_{os}\otimes v_{os} + b\otimes v_a + b\otimes v_{os} + v_a\otimes b \right.\\
&\left. + v_{os}\otimes b) dx_v : \nabla(-\Delta_h)^{-\frac{1}{4}} w_a dx_h\right| \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{1.6.7}$$

<u>Estimate for I_1</u> The Sobolev embedding

$$\|v_a\|_{L^4(\mathbb{R}^2)} \le C\|(-\Delta_h)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)} \tag{1.6.8}$$

and the interpolation inequality

$$\|(-\Delta_h)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)} \leq C\|v_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h v_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \tag{1.6.9}$$

yield

$$\begin{aligned}
|I_1| &\leq C\|v_a\|_{L^4(\mathbb{R}^2)}^2 \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\leq C\|(-\Delta_h)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)}^2 \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\leq C\|v_a\|_{L^2(\mathbb{R}^2)} \|(-\Delta_h)^{\frac{1}{2}} v_a\|_{L^2(\mathbb{R}^2)} \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\leq C\|\nabla_h v_a\|_{L^2(\mathbb{R}^2)} \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)}^2 \\
&\leq C\|\nabla_h v_a\|_{L^2(\mathbb{R}^2)} \|w_a\|_{L^2(\mathbb{R}^2)} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Applying the Young inequality to the last inequality, we find

$$|I_1| \leq C\|\nabla_h v_a\|_{L^2(\mathbb{R}^2)}^2 \|w_a\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8}\|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^2$$

Estimate for I_2 Using the Schwarz inequality, (1.6.8), (1.6.9) and the Young inequality, we find

$$\begin{aligned}
|I_2| &\leq C\|\int_{\mathbb{T}^1} v_{os} \otimes v_{os} dx_v\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\leq C\int_{\mathbb{T}} \|v_{os}\|_{L_h^4(\mathbb{R}^2)} dx_v \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \\
&\leq C\int_{\mathbb{T}} \|(-\Delta_h)^{\frac{1}{4}} v_{os}\|_{L_h^2(\mathbb{R}^2)}^2 dx_v \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\
&\leq C\int_{\mathbb{T}} \|v_{os}\|_{L_h^2(\mathbb{R}^2)} dx_v \|\nabla_h v_{os}\|_{L_h^2(\mathbb{R}^2)} dx_v \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\
&\leq C\|v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \|\nabla v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\
&\leq C_1\|v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 \|\nabla v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \\
&\quad + C_2\|\nabla v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 \|w_a\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8}\|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^2 \\
&\leq C_1\|v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \|\nabla v\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 \\
&\quad + C_2\|\nabla v\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 \|w_a\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8}\|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned}$$

Estimate for I_3 and I_5. Using the Hölder inequality,(1.6.8), (1.6.9) and the Young inequality, we find

$$\begin{aligned}
|I_3| + |I_5| &\leq C\int_{\mathbb{T}^1} \|b\|_{L_h^4(\mathbb{R}^2)} \|v_a\|_{L_h^4(\mathbb{R}^2)} dx_v \|(\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\leq C\|b\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \|v_a\|_{L^4(\mathbb{R}^2)} \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \\
&\leq C\|b\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \|(-\Delta)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)} \|w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}
\end{aligned}$$

$$\leq C\|b\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}\|w_a\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)}$$
$$\leq C\|b\|^4_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}\|w_a\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{8}\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}.$$

<u>Estimate for I_4 and I_6</u>. Using the Hölder inequality, (1.6.8), (1.6.9) and the Poincaré inequality, we find

$$|I_4| + |I_6| \leq C\int_{\mathbb{T}^1}\|b\|_{L^4_h(\mathbb{R}^2)}\|v_{os}\|_{L^4_h(\mathbb{R}^2)}dx_v\|(-\Delta_h)^{\frac{1}{4}}w_a\|_{L^2(\mathbb{R}^2)}$$
$$\leq C\int_{\mathbb{T}^1}\|b\|_{L^4_h(\mathbb{R}^2)}\|(-\Delta_h)^{\frac{1}{4}}v_{os}\|_{L^2_h(\mathbb{R}^2)}dx_v\|(-\Delta_h)^{\frac{1}{4}}w_a\|_{L^2(\mathbb{R}^2)}$$
$$\leq C\|b\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}\int_{\mathbb{T}^1}\|v_{os}\|^{\frac{1}{2}}_{L^2_h(\mathbb{R}^2)}\|\nabla_h v_{os}\|^{\frac{1}{2}}_{L^2_h(\mathbb{R}^2)}dx_v\|(-\Delta_h)^{\frac{1}{4}}w_a\|_{L^2(\mathbb{R}^2)}$$
$$\leq C\|b\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}\|v_{os}\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|\nabla v_{os}\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|w_a\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|b^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}$$
$$\leq C_1\|b\|^2_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}\|v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}$$
$$+ C_2\|\nabla v_{os}\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|w_a\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{8}\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}$$
$$\leq C_1\|b\|^2_{X^4(\mathbb{R}^2\times\mathbb{T}^1)}\|\nabla v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}$$
$$+ C_2\|\nabla v_{os}\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\|w_a\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{8}\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}.$$

Thus, from (1.6.7), above estimates and the Gronwall inequality, we get

$$\|w_a(t)\|^2_{L^2_h} + \int_1^t\|\nabla w_a(\tau)\|^2_{L^2_h}d\tau \leq \exp(\Phi(t))\|w_a(1)\|^2_{L^2_h} + \int_1^t\Psi(\tau)d\tau \tag{1.6.10}$$

where

$$\Phi(t) = C_1\int_1^t(\|\nabla v(\tau)\|^2_{L^2} + \|b(\tau)\|^4_{X^4})d\tau$$
$$\Psi(t) = C_2\exp(\int_\tau^t\Phi(s)ds)(\|v_{os}(t)\|_{L^2}\|\nabla v_{os}(t)\|^2_{L^2} + \|b(t)\|^2_{X^4}\|\nabla v_{os}(t)\|_{L^2}).$$

Using (1.6.2) and (1.2.6), we find

$$\Phi(t) \leq C_1(1 + \delta^2\log(1+t)).$$

and

$$\int_1^t\Psi(t)d\tau$$
$$\leq C_2(1+t)^{C_1\delta^2}(\sup_{1\leq\tau\leq t}\|v_{os}(\tau)\|_{L^2}\int_1^t\|\nabla v_{os}(\tau)\|^2d\tau + \int_1^t\|b(\tau)\|^2_{X^4}\|\nabla v_{os}(\tau)\|_{L^2}d\tau)$$
$$\leq C_2(1+t)^{C_1\delta^2}(\sup_{1\leq\tau\leq t}\|v_{os}(\tau)\|_{L^2}\int_1^t\|\nabla v_{os}(\tau)\|^2d\tau$$

$$
\begin{aligned}
&+ (\int_1^t \|b(\tau)\|_{X^4}^4 d\tau)^{\frac12} (\int_1^t \|\nabla v_{os}(\tau)\|_{L^2}^2 d\tau)^{\frac12}) \\
&\le C_2(1+t)^{C_1\delta^2}(1+\log(1+t) + \sup_{1\le\tau\le t} \|v_{os}(\tau)\|_{L^2} \log(1+t)).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\|w_a(t)\|_{L^2(\mathbb{R}^2)}^2 + \int_1^t \|\nabla w_a(t)\|_{L^2(\mathbb{R}^2)}^2 d\tau \\
&\le C(1+t)^{M\delta^2}(1+\log(1+t) + \sup_{1\le\tau\le t} \|v_{os}(\tau)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \log(1+t)) \qquad (1.6.11)
\end{aligned}
$$

□

1.7 Decay estimates for perturbation

In this section, we show the decay of $\|v(t)\|_{L^2} \to 0$ as $t \to \infty$. The Poincaré inequality is useful to derive the decay to the oscillating part.

Proposition 1.7.1. *Let $\delta > 0$ sufficient small, $b_0 \in Y_\sigma^2(\mathbb{R}^2\times\mathbb{T}^1)$ with $||b_0||_{Y^2(\mathbb{R}^2\times\mathbb{T}^1)} \le \delta$ and $v_0 \in \left(L_v^\infty C_{0,h}^\infty\right)_\sigma(\mathbb{R}^2\times\mathbb{T}^1)$. Let b be a mild solution to (1.1.1) obtained in Theorem 1.2.4 with initial data b_0 and v is a weak solution to the perturbed Navier-Stokes equations with initial data v_0 obtained by* Proposition 1.5.4. *Then there exists a constant C_δ and C which are independent of t such that*

$$
\|v(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \le Ct^{-\frac12}\{C + C_1(1+t)^{M\delta^2}(1+\log(1+t) + \log^{\frac32}(1+t)\}.
$$

for $t \ge 1$.

Proof. Let $t \ge 1$. From (1.6.2) and (1.6.11), there exists $t_0 \in [\frac{t}{2}, t]$ such that

$$
\begin{aligned}
&\|w_a(t_0)\|_{L_h^2(\mathbb{R}^2)}^2 + \|v_{os}(t_0)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 + t_0(\|\nabla w_a(t_0)\|_{L_h^2(\mathbb{R}^2)}^2 + \|v_{os}(t_0)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2) \\
&\le C + C_1(1+t_0)^{M\delta^2}(1+\log(1+2t_0) + \log^{\frac32}(1+2t_0)). \qquad (1.7.1)
\end{aligned}
$$

Therefore using interpolation the inequality, (1.7.1) and the Poincaré inequality, we have

$$
\begin{aligned}
&\|v(t_0)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2 \le 2(\|v_a(t_0)\|_{L_h^2(\mathbb{R}^2)}^2 + \|v_{os}(t_0)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}^2) \\
&\le \|w_a(t_0)\|_{L_h^2(\mathbb{R}^2)} \|\nabla_h w_a(t_0)\|_{L_h^2(\mathbb{R}^2)} + \|v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \|\nabla v_{os}\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \\
&\le t_0^{-\frac12}\{C + C_1(1+t_0)^{M\delta^2}(1+\log(1+2t_0) + \log^{\frac32}(1+2t_0)\}. \qquad (1.7.2)
\end{aligned}
$$

Since v solves the perturbed Navier-Stokes equations

$$
\begin{aligned}
&\partial_t v - \Delta v + \operatorname{div}(v\otimes v + v\otimes b + b\otimes v) + \nabla q = 0, \qquad (1.7.3) \\
&\operatorname{div}\, v = 0, \qquad (1.7.4)
\end{aligned}
$$

for some $q \in L^1_{loc,t}L^1_{loc,x}(\mathbb{R}^2 \times \mathbb{T}^1 \times (0,\infty))$. Then it follows from integration by parts and the Gronwall inequality that

$$\|v(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} + \int_{t_0}^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau \tag{1.7.5}$$

$$\leq \exp(\int_{t_0}^t C\|b(s)\|^4_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} ds)\|v(t_0)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}. \tag{1.7.6}$$

Since

$$\int_{t_0}^t \|b(s)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} ds \leq C\log\frac{t}{t_0} \leq C\log 2, \quad t_0 \in (\frac{t}{2}, t),$$

we finally obtain from (1.7.2) and (1.7.5) that

$$\begin{aligned}&\|v(t)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\|v_0(t_0)\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\\ &\leq Ct^{-\frac{1}{2}}\{C + C_1(1+t)^{M\delta^2}(1+\log(1+t)+\log^{\frac{3}{2}}(1+t)\}.\end{aligned}$$

□

Proof of theorem1.2.7. Fix arbitrarily small $\eta > 0$. Set

$$\begin{aligned}u(t) &= \tilde{v}(t) + (\tilde{b}(t) - b(t) - e^{t\Delta}\tilde{b}_0) - e^{t\Delta}\tilde{v}_0\\ &= \tilde{v}(t) + w_{0,\epsilon}(t) - e^{t\Delta}\tilde{v}_0\end{aligned}$$

Let $T >$ sufficiently large. By Proposition 1.7.1, we have $||\tilde{v}(t)||_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \to 0$ as $t \to \infty$. Since $\tilde{v}_0 \in L^\infty_v C^\infty_{0,h}(\mathbb{R}^2 \times \mathbb{T}^1) \subset L^1(\mathbb{R}^2 \times \mathbb{T}^1)$, we find $\left|\left|e^{t\Delta}\tilde{v}_0\right|\right|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq Ct^{-1/2}||\tilde{v}_0||_{L^1(\mathbb{R}^2\times\mathbb{T}^1)} \to 0$ as $t \to \infty$. Let us show $||w_\epsilon||_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\epsilon$. Since $\tilde{b}(t)$ and $b(t)$ are mild solutions to (1.1.1), it follows that

$$\begin{aligned}&\|w_\epsilon(t)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)}\\ &\leq C\int_0^t (t-\tau)^{-\frac{1}{2}}\|b(\tau)\otimes(\tilde{b}(\tau)-b(\tau)) + (\tilde{b}(\tau)-b(\tau))\otimes b(\tau)\|_{L^2(\mathbb{R}^2\times\mathbb{T}^1)} d\tau\\ &\leq C(\sup_{\tau>0}\tau^{\frac{1}{4}}\|b(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} + \sup_{\tau>0}\tau^{\frac{1}{4}}\|\tilde{b}(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)})\\ &\qquad \times (\sup_{\tau>0}\tau^{\frac{1}{4}}\|\tilde{b}(\tau)-b(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)})\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}} d\tau\\ &\leq C(\delta+\delta+\epsilon)\epsilon\\ &\leq C\epsilon.\end{aligned}$$

In the third inequality, we use $\sup_{\tau>0}\tau^{1/4}\|\tilde{b}(\tau) - b(\tau)\|_{X^4(\mathbb{R}^2\times\mathbb{T}^1)} \leq C\epsilon$, which is follows from construction of $\tilde{b}$ and b. □

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Chapter 2

Rigorous Justification of the Hydrostatic Approximation for the Primitive Equations by scaled Navier-Stokes equations - the Case of Perfectly Slip Boundary Condition

Considering the anisotropic Navier-Stokes equations as well as the primitive equations, it is shown that the horizontal velocity of the solution to the anisotropic Navier-Stokes equations in a cylindrical domain of height ε with initial data $u_0 = (v_0, w_0) \in B^{2-2/p}_{q,p}$, $1/q+1/p \leq 1$ if $q \geq 2$ and $4/3q+2/3p \leq 1$ if $q \leq 2$, converges as $\varepsilon \to 0$ with convergence rate $\mathcal{O}(\varepsilon)$ to the horizontal velocity of the solution to the primitive equations with initial data v_0 with respect to the maximal-L^p-L^q-regularity norm. Since the difference of the corresponding vertical velocities remains bounded with respect to that norm, the convergence result yields a rigorous justification of the hydrostatic approximation in the primitive equations in this setting. It generalizes in particular a result by Li and Titi for the L^2-L^2-setting. The approach presented here does not rely on second order energy estimates but on maximal L^p-L^q-estimates which allow us to conclude that local in-time convergence already implies global in-time convergence, where moreover the convergence rate is independent of p and q.

2.1 Introduction

The primitive equations for the ocean and atmosphere are considered to be a fundamental model for geophysical flows, see e.g. the survey article [15]. The mathematical analysis of these equations has been pioneered by Lions, Teman and Wang in their articles [16, 18, 34], where they proved the existence of global, weak solutions to the primitive equations. Their uniqueness remains an open problem until today. Global strong well-posedness of the primitive equations for initial data in H^1 was shown by Cao and Titi in [4] using energy methods. A different approach, based on the theory of evolution equations, was introduced by Hieber and Kashiwabara in [12] and subsequent works [7–9, 24].

It is the aim of this chapter to show that the primitive equations can be obtained as the limit of anisotropically scaled Navier-Stokes equations. The scaling parameter $\varepsilon > 0$ represents the ratio of the depth to the horizontal width. Such an approximation is motivated by the fact that for large-scale oceanic dynamics, this aspect ratio ε is rather small and implies anisotropic viscosity coefficients (see e.g. [19]). For an aspect ratio ε, i.e., in the case where the spacial domain can be represented as $\Omega_\varepsilon = G \times (-\varepsilon, +\varepsilon)$ for some $G \subset \mathbb{R}^2$, and a horizontal and vertical eddy viscosity 1 and ε^2, respectively, the system can be rescaled into the form

$$\begin{cases} \partial_t v_\varepsilon + u_\varepsilon \cdot \nabla v_\varepsilon - \Delta v_\varepsilon + \nabla_H p_\varepsilon = & 0, \\ \varepsilon(\partial_t w_\varepsilon + u_\varepsilon \cdot \nabla w_\varepsilon - \Delta w_\varepsilon) + \frac{1}{\varepsilon}\partial_z p_\varepsilon = & 0, \\ \operatorname{div} u_\varepsilon = & 0. \end{cases} \tag{2.1.1}$$

in the time-space domain $(0, T) \times \Omega_1$, which is *independent* of the aspect ratio. We refer to [14] for more details on this rescaling procedure. Here the horizontal and vertical velocities v_ε and w_ε describe the three-dimensional velocity $u_\varepsilon = (v_\varepsilon, w_\varepsilon)$, while p_ε denotes the pressure of the fluid. Here ∂_z denotes the vertical-derivative, ∇_H and div_H the horizontal gradient and divergence, whereas div, ∇ and Δ stand for the usual three-dimensional spatial divergence, gradient, and Laplacian.

First convergence results for the above system in the steady state case go back to Besson and Laidy [3]. The convergence of the above system has been studied first by Azérad and Guillén in [2] in the setting of weak convergence, where no uniform convergence rate was given.

Recently, Li and Titi [14] investigated the strong convergence of the above system within the L^2-L^2-setting for horizontal initial velocities belonging to H^1 and H^2. In addition, they showed a convergence rate of order $\mathcal{O}(\varepsilon)$.

There are two aims in this chapter; to show convergence results of the above system (justification of the hydrostatic approximation) in the strong sense and to give global well-posedness of the scaled Navier-Stokes equations (NS_ε) within the L^p-L^q-setting. Our method is very different from the one introduced by [14], whereas they rely on second order energy estimates, our approach is based on maximal L^p-L^q-regularity estimates for the heat equation and the non-linear terms. This allows us to give a very short proof of the convergence result in the more general L^p-L^q-setting, which even in the L^2-L^2-setting allows for a slightly larger class of initial data compared to the one introduced by Li and Titi in [14] by using energy estimates. Details is provided in Section 3.

Our methodology for the proofs of the convergence and global well-posedness of (NS_ε) is quite different from that of [14]. There they derived several a priori estimates for a difference system (see (2.3.1) below) obtained by subtracting (PE) from (NS_ε), which are then combined with the local well-posedness of (NS_ε) to conclude its global-in-time solvability. However, in this chapter we focus only on

(2.3.1) and directly derive its global well-posedness, together with a solution upper bound of $\mathcal{O}(\varepsilon)$, using maximal regularity results for a linearized problem. As a result, global well-posedness of (NS_ε) and a convergence result are obtained immediately. This is achieved by subdiving the whole time interval into small pieces and solving (2.3.1) on each subinterval. We emphasize that this strategy is based not on usual construction of a local-in-time solution with large data but on that of a global-in-time solution with small data, where the smallness of the data is provided by that of ε. Since our proof is straightforward, concise and short, we believe that it deserves to be presented as another approach to justification of hydrostatic approximation.

2.2 Preliminaries

Consider the cylindrical domain $\Omega := (0,1)^2 \times (-1,1)$. Let $u = (v,w)$ be the solution of the primitive equations

$$\begin{cases} \partial_t v + u\cdot\nabla v - \Delta v + \nabla_H p = & 0 & \text{in } (0,T)\times\Omega, \\ \partial_z p = & 0 & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u = & 0 & \text{in } (0,T)\times\Omega, \\ p \text{ periodic in } x,y & & \\ v,w \text{ periodic in } x,y,z, & \text{even and odd} & \text{in } z, \\ u(0) = & u_0 & \text{in } \Omega, \end{cases} \tag{PE}$$

and $u_\varepsilon = (v_\varepsilon, w_\varepsilon)$ be the solution of the anisotropic Navier-Stokes equations

$$\begin{cases} \partial_t v_\varepsilon + u_\varepsilon\cdot\nabla v_\varepsilon - \Delta v_\varepsilon + \nabla_H p_\varepsilon = & 0 & \text{in } (0,T)\times\Omega, \\ \partial_t w_\varepsilon + u_\varepsilon\cdot\nabla w_\varepsilon - \Delta w_\varepsilon + \frac{1}{\varepsilon^2}\partial_z p_\varepsilon = & 0 & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u_\varepsilon = & 0 & \text{in } (0,T)\times\Omega, \\ p_\varepsilon \text{ periodic in } x,y,z, & \text{even} & \text{in } z, \\ v_\varepsilon, w_\varepsilon \text{ periodic in } x,y,z, & \text{even and odd} & \text{in } z, \\ u_\varepsilon(0) = & u_0 & \text{in } \Omega. \end{cases} \tag{NS_ε}$$

Here v and v_ε denote the (two-dimensional) horizontal velocities, w and w_ε the vertical velocities, and p and p_ε denote the pressure term for the primitive equations as well as the Navier-Stokes equations, respectively. These are functions of three space variables $x,y \in (0,1)$, $z \in (-1,1)$. The vertical periodicity and parity conditions correspond to an equivalent set of equations with vertical Neumann boundary conditions for the horizontal velocity and vertical Dirichlet boundary conditions for the vertical velocity (cf. e.g. [5]). Since w is odd, the divergence free condition for the primitive equation translates into $\operatorname{div}_H \overline{v} = 0$, where $\overline{v}(x,y) = \frac{1}{2}\int_{-1}^{1} v(x,y,z)\,\mathrm{d}z$, and

$$w(\cdot,\cdot,z) = -\int_{-1}^{z} \operatorname{div}_H v(\cdot,\cdot,\zeta)\,\mathrm{d}\zeta. \tag{2.2.1}$$

For $p,q \in (1,\infty)$ and $s \in [0,\infty)$ we define the Bessel potential and Besov spaces

$$H^{s,p}_{per}(\Omega) = \overline{C^\infty_{per}(\overline{\Omega})}^{\|\cdot\|_{H^{s,p}}} \quad \text{and} \quad B^s_{p,q,per}(\Omega) = \overline{C^\infty_{per}(\overline{\Omega})}^{\|\cdot\|_{B^s_{p,q}}},$$

where $C^\infty_{per}(\overline{\Omega})$ denotes the space of smooth functions that are periodic of any order (cf. [12, Section 2]) in all three directions on $\partial\Omega$. The space $H^{s,p}(\Omega)$ denotes the Bessel potential space of order s, with norm $\|\cdot\|_{H^{s,p}}$ defined via the restriction of the corresponding space defined on the whole space to Ω (cf. [23, Definition 3.2.2.]). Moreover, $B^s_{p,q}(\Omega)$ denotes a Besov space on Ω, which is defined by restrictions of functions on the whole space to Ω, see e.g. [23, Definition 3.2.2.]. Note that $L^p(\Omega) = H^{0,p}_{per}(\Omega)$ and $B^s_{p,2,per}(\Omega) = H^s_{p,per}(\Omega)$. The anisotropic structure of the primitive equations motivates the definition of the Bessel potential spaces $H^{s,p}_{xy} := H^{s,p}((0,1)^2)$ and $H^{s,p}_z := H^{s,p}(-1,1)$ for the horizontal and vertical variables, respectively. Similarly as above we write $L^p_{xy} := H^{0,p}_{xy}$ and $L^p_z := H^{0,p}_z$ and set $H^{s,p}_{xy}H^{r,q}_z := H^{s,p}((0,1)^2; H^{r,q}_z)$.

The divergence free conditions in the above sets of equations can be encoded into the space of solenoidal functions

$$L^p_\sigma(\Omega) = \overline{\{u \in C^\infty_{per}(\overline{\Omega})^3 : \operatorname{div} u = 0\}}^{\|\cdot\|_{L^p}}$$

and

$$L^p_{\overline{\sigma}}(\Omega) = \overline{\{v \in C^\infty_{per}(\overline{\Omega})^2 : \operatorname{div}_H \overline{v} = 0\}}^{\|\cdot\|_{L^p}}.$$

For given $p, q \in (1,\infty)$ we set

$$\begin{aligned}
&X_0 := L^q(\Omega), \quad X_1 := H^{2,q}_{per}(\Omega),\\
&X^v_0 := \{v \in L^q_{\overline{\sigma}}(\Omega) : v \text{ even in } z\}, \quad X^v_1 := \{v \in H^{2,q}_{per}(\Omega)^2 \cap L^q_{\overline{\sigma}}(\Omega) : v \text{ even in } z\},\\
&X^u_0 := \{(v_1, v_2, w) \in L^q_\sigma(\Omega) : v_1, v_2 \text{ even } w \text{ odd in } z\},\\
&X^u_1 := \{(v_1, v_2, w) \in H^{2,q}_{per}(\Omega)^3 \cap L^q_\sigma(\Omega) : v_1, v_2 \text{ even } w \text{ odd in } z\},
\end{aligned}$$

and consider the traces spaces

$$X^u_\gamma = (X^u_0, X^u_1)_{1-1/p,p} \quad \text{and} \quad X^v_\gamma = (X^v_0, X^v_1)_{1-1/p,p},$$

where when there is no ambiguity we set $X_\gamma := X^u_\gamma$. Here $(\cdot,\cdot)_{1-1/p,p}$ denotes the real interpolation functor. Following the lines of [24, Section 4] and [16] the trace space X_γ can be characterized as follows.

Lemma 2.2.1 (Characterization of the trace space). *Let $p, q \in (1,\infty)$. Then*

$$X_\gamma = \begin{cases}
\{(v_1, v_2, w) \in B^{2-2/p}_{q,p,per}(\Omega)^3 \cap L^q_\sigma(\Omega) : & v = (v_1, v_2) \text{ even, } w \text{ odd in } z,\\
& (\partial_z v, w) = 0 \text{ at } z = -1, 0, 1\},\\
& 1 > \frac{2}{p} + \frac{1}{q},\\
\{(v_1, v_2, w) \in B^{2-2/p}_{q,p,per}(\Omega)^3 \cap L^q_\sigma(\Omega) : & v = (v_1, v_2) \text{ even, } w \text{ odd in } z,\\
& w = 0 \text{ at } z = -1, 0, 1\},\\
& 1 < \frac{2}{p} + \frac{1}{q}.
\end{cases}$$

For $p, q \in (1,\infty)$ and $t, T \in [0,\infty]$ we also define the maximal regularity spaces

$$\mathbb{E}_0(t,T) := L^p(t,T; X_0), \quad \mathbb{E}_1(t,T) := L^p(t,T; X_1) \cap H^{1,p}(t,T; X_0),$$

and analogously $\mathbb{E}_i^v(t,T)$ and $\mathbb{E}_i^u(t,T)$ with respect to X_i^v and X_i^u, respectively ($i = 0,1$). In order to simplify our notation, we sometimes write only $\mathbb{E}_i(t,T)$ without superscripts and $\mathbb{E}_i(T)$ when $t=0$. For a function space to describe pressure, we introduce $\mathbb{E}_1^\pi(t,T) := \{\pi \in L^p(t,T;H^{1,q}(\Omega)) \,:\, \int_\Omega \pi(s)\,dx = 0 \text{ for a.e. } s \in (t,T)\}$.

Finally, we say that $u=(v,w)$ is a *strong solution to the primitive equations* (in the L^p-L^q-setting), if $v \in \mathbb{E}_1^v$ and (PE) holds almost everywhere. We say that u_ε is a *strong solution to the Navier-Stokes equations*, if $u \in \mathbb{E}_1^u$ and (NS_ε) holds almost everywhere.

2.3 Main Result

Roughly speaking, the idea of our approach consists of controlling the maximal regularity norm of the differences $(v_\varepsilon - v, \varepsilon(w_\varepsilon - w))$ by the aspect ratio ε. To this end, we introduce the difference equations of (NS_ε) and (PE): setting $V_\varepsilon := v_\varepsilon - v$, $W_\varepsilon := w_\varepsilon - w$, $U_\varepsilon := (V_\varepsilon, W_\varepsilon)$ and $P_\varepsilon = p_\varepsilon - p$, we obtain

$$\left\{\begin{array}{rll} \partial_t V_\varepsilon - \Delta V_\varepsilon + \nabla_H P_\varepsilon = & F_H(U_\varepsilon,u) & \text{in } (0,T)\times\Omega, \\ \partial_t(\varepsilon W_\varepsilon) - \Delta(\varepsilon W_\varepsilon) + \frac{1}{\varepsilon}\partial_z P_\varepsilon = & \varepsilon F_z(U_\varepsilon,u) & \text{in } (0,T)\times\Omega, \\ \operatorname{div} U_\varepsilon = & 0 & \text{in } (0,T)\times\Omega, \\ P_\varepsilon \text{ periodic in } x,y,z, & \text{even} & \text{in } z, \\ V_\varepsilon, W_\varepsilon \text{ periodic in } x,y,z, & \text{even and odd} & \text{in } z, \\ U_\varepsilon(0) = & 0 & \text{in } \Omega, \end{array}\right. \tag{2.3.1}$$

where the forcing terms F_H and F_z are given by

$$\begin{aligned} F_H(U_\varepsilon,u) &:= -U_\varepsilon\cdot\nabla v - u\cdot\nabla V_\varepsilon - U_\varepsilon\cdot\nabla V_\varepsilon, \\ F_z(U_\varepsilon,u) &:= -U_\varepsilon\cdot\nabla w - u\cdot\nabla W_\varepsilon - U_\varepsilon\cdot\nabla W_\varepsilon - \partial_t w - u\cdot\nabla w + \Delta w. \end{aligned}$$

Applying the maximal regularity estimate given in (2.5.1) to (2.3.1), we are able to estimate $\|(V_\varepsilon, \varepsilon W_\varepsilon)\|_{\mathbb{E}_1}$ in terms of the right hand sides. The latter will be estimated in a series of lemmas in Section 2.4. Like this we obtain a quadratic inequality for the norm of the differences and we need to ensure that the constant term as well as the coefficient in front of the linear term are sufficiently small. This can be achieved provided the aspect ratio ε is small enough and provided both the vertical and horizontal solution of the primitive equations exist globally in the maximal regularity class (cf. Theorem 2.3.2 and Proposition 2.4.5). As a first step, the following global existence and uniqueness result on the vertical velocity of the primitive equations is needed.

Proposition 2.3.1. *Let $p,q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q} \le 1$ and $T>0$. Let $v_0 \in X_\gamma^v$. Then there exists a unique global strong solution $u=(v,w)$ of (PE), i.e, $v \in \mathbb{E}_1^v(\mathcal{T})$.*

Assumption (A). Let $q \in \left(\frac{4}{3},\infty\right)$ and $p \ge \max\left\{\frac{q}{q-1}; \frac{2q}{3q-4}\right\}$, i.e.,

$$1 \ge \left\{\begin{array}{ll} \frac{1}{p}+\frac{1}{q}, & \text{if } q \ge 2, \\ \frac{2}{3p}+\frac{4}{3q}, & \text{if } q \le 2. \end{array}\right.$$

Here, we introduce strategy to get the global well-posedness of (NS_ε) for $\varepsilon << 1$. As already mentioned, the solution to (NS_ε) is constructed as (the solution to (PE)) + (small perturbation). The small perturbation is the solution to (2.3.1). This is constructed with maximal L^p-L^q-regularity of linearized equations and iteration. The method is different from Li and Titi's strategy. Their proof of relies on the existence of the weak solution to the primitive equation and the scaled Navier-Stokes equations. On the other hand, our proof relies only on the existence of the primitive equation and the difference equation. Although initial data of (2.3.1) is zero, it is unable to obtain global solution to (2.3.1) directly since our non-linear estimate depends on time. Therefore, we construct the solution on small time interval and repeat this up to any finite time T. The time interval that iteration works is independent of ε. Note that if $p = q = 2$, the existence time of the solution to (2.3.1) admits $T = \infty$. Whether the existence time admits infinity in the case of $p = q = 2$ seems more clear than Li and Titi's paper. Uniqueness of our solution to (NS_ε) is rather clear since our maximal regularity space is super critical to the scale-invariant space. We are now in the position to state our main result.

Theorem 2.3.2. *Let p, q fulfill Assumption (A), $u_0 = (v_0, w_0) \in X_\gamma^u$ and $T > 0$. Let $u = (v, w) \in \mathbb{E}_1^u(T)$ be the solution of (PE) given by Proposition 2.3.1, where $w = -\int_{-1}^{x_3} \operatorname{div}_H v(\cdot, \cdot, \zeta) d\zeta$. Then $(v, w) \in \mathbb{E}_1^u(T)$.*

For later convenience, we employ the notation $\mathcal{T}$ in the next theorem to mean T appearing in (2.3.1) (T will be used in a different context in Subsection (2.5.2)).

Theorem 2.3.3. *Under the same assumptions as in Theorem 2.3.2, for sufficiently small $\varepsilon > 0$ there exists a unique solution $(U_\varepsilon, P_\varepsilon) \in \mathbb{E}_1^u(\mathcal{T}) \times \mathbb{E}_1^\pi(\mathcal{T})$ of* (2.3.1) *such that*

$$\|(V_\varepsilon, \varepsilon W_\varepsilon)\|_{\mathbb{E}_1(\mathcal{T})} + \|\nabla_\varepsilon P_\varepsilon\|_{\mathbb{E}_0(\mathcal{T})} \leq C\varepsilon,$$

where $\nabla_\varepsilon := (\nabla_H, \varepsilon^{-1}\partial_z)$ and the constant C depends only on p, q, T, u. Moreover, $(u_\varepsilon, p_\varepsilon) := (u + U_\varepsilon, p + P_\varepsilon)$ constitutes a unique solution of (NS_ε) *in $\mathbb{E}_1^u(\mathcal{T}) \times \mathbb{E}_1^\pi(\mathcal{T})$.*

Remarks 2.3.4. a) If the solution $u = (v, w)$ of the primitive equations exists globally in time, the convergence rate is uniform for all $\mathcal{T} \in (0, \infty]$. For example, if $p = q = 2$ and the initial data are mean value free, one can show that the solution to the primitive equations exists globally in $\mathbb{E}_1(\mathcal{T})$ with $\mathcal{T} = \infty$.
b) We note that the case $p = q = 2$, investigated before in [14], is covered by our result. More specifically, they assumed $v_0 \in H^2$ whereas for our purposes $v_0, \operatorname{div}_H v_0 \in H^1$ suffices. Also, it is remarkable that the convergence rate is independent of p and q, though the constant C in Theorem 2.3.3 may depend on p and q.
c) Our method can be adjusted to the case with perturbed initial data. That is, given initial data $(u_{0,\varepsilon})_{\varepsilon>0} \subset X_\gamma$ converging to u_0 in X_γ as $\varepsilon \to 0$ of order $\mathcal{O}(\delta_\varepsilon)$ for some null-sequence $(\delta_\varepsilon)_{\varepsilon>0}$, then Theorem 2.3.3 holds with $(v_\varepsilon, w_\varepsilon)$ replaced by the solution of (NS_ε) with initial data $v_{0,\varepsilon}$. In that case the maximal regularity norm of the differences is bounded by $C \max\{\varepsilon, \delta_\varepsilon\}$ and consequently the convergence rate is of order $\mathcal{O}(\max\{\varepsilon, \delta_\varepsilon\})$.

2.4 Nonlinear estimates and maximal regularity of (PE)

The proof of Theorem 2.3.3 relies upon estimates on the terms F_H and F_z in equations (2.3.1) within the L^p-L^q-framework. These estimates imply eventually a quadratic inequality for the difference of the velocities. In order to establish these estimates we need to ensure that the solution of the primitive equations belongs to the maximal regularity class (see Proposition 2.4.5) and that the nonlinear terms can be estimated in $\mathbb{E}_0(T)$, see Lemma 2.4.2 and 2.4.4. We hence subdivide our proof in three steps. Throughout this section let $T < \infty$. We first estimate the bilinear terms and keep track of the T-dependence of the norms involved.

We also prove prove that the vertical and horizontal solution of the primitive equations belong to the maximal regularity class $\mathbb{E}_1^u(T)$, which is Theorem 2.3.3.

2.4.1 Nonlinear estimates

We will make use of the following classical Mixed Derivative Theorem, see e.g. [20, Corollary 4.5.10].

Proposition 2.4.1 (Mixed Derivative Theorem)**.** *If $\theta \in [0,1]$, then*

$$\mathbb{E}_1(T) \hookrightarrow H^{\theta,p}(0,T; H^{2-2\theta,q}(\Omega)).$$

Lemma 2.4.2. *Let $p, q \in (1,\infty)$ such that $2/3p + 1/q \leq 1$. Then for all $v_1, v_2 \in \mathbb{E}_1(T)$ and $\partial \in \{\partial_x, \partial_y, \partial_z\}$, there exists a constant $C > 0$ such that*

$$\|v_1 \partial v_2\|_{\mathbb{E}_0(T)} \leq C \|v_1\|_{\mathbb{E}_1(T)} \|v_2\|_{\mathbb{E}_1(T)}, \quad C > 0. \tag{2.4.1}$$

Proof. Set $\theta_1 = \frac{2}{3p}$ and $\theta_2 = \frac{1}{2}\theta_1$. The Mixed Derivative Theorem and Sobolev's embeddings $H^{2/3p,p}(0,T) \hookrightarrow L^{3p}(0,T)$ and $H^{2-2\theta_1,q}(\mathbb{T}^3) \hookrightarrow L^{3q}(\mathbb{T}^3)$ yield

$$\begin{aligned}
\mathbb{E}_1(T) &\hookrightarrow H^{\theta_1,p}(0,T; H^{2-2\theta_1,q}) \hookrightarrow H^{2/3p,p}(0,T; H^{2-2\theta_1,q}) \hookrightarrow L^{3p}(0,T; L^{3q}(\Omega)),\\
\mathbb{E}_1(T) &\hookrightarrow H^{\theta_2,p}(0,T; H^{2-2\theta_2,q}) \hookrightarrow H^{1/3p,p}(0,T; H^{2-2\theta_2,q}) \hookrightarrow L^{3p/2}(0,T; H^{1,3q/2}(\Omega)).
\end{aligned}$$

Hölder's inequality thus implies

$$\begin{aligned}
\|v_1 \partial v_2\|_{L^p(L^q)} \leq\ & \| \|v_1\|_{L^{3q}} \|\partial v_2\|_{L^{3q/2}} \|_{L^p} \leq \|v_1\|_{L^{3p}(L^{3q})} \|v_2\|_{L^{3p/2}(H^{1,3q/2})}\\
\leq\ & C \|v_1\|_{\mathbb{E}_1} \|v_2\|_{\mathbb{E}_1}.
\end{aligned}$$ □

Lemma 2.4.3. *Let $q \in (1,\infty)$, $v_1, v_2 \in H^{1+1/q,q}(\Omega)$ and $w_1 := \int_{-1}^{z} \operatorname{div}_H v_1$. Then there exists a constant $C > 0$ such that*

$$\|w_1 \partial_z v_2\|_{L^q} \leq C \|v_1\|_{H^{1+1/q,q}} \|v_2\|_{H^{1+1/q,q}}.$$

Proof. Similarly as in [12, Lemma 5.1] we obtain by anisotropic Hölder's inequality and Sobolev inequalities

$$\begin{aligned}\|w_1\partial_z v_2\|_{L^q} &\leq \|w_1\|_{L^{2q}_{xy}L^\infty_z}\|\partial_z v_2\|_{L^{2q}_{xy}L^q_z}\\ &\leq C\|\operatorname{div}_H v_1\|_{L^{2q}_{xy}L^1_z}\|\partial_z v_2\|_{L^{2q}_{xy}L^q_z}\\ &\leq C\|v_1\|_{H^{1+1/q}_{xy}L^1_z}\|v_2\|_{H^{1/q,q}_{xy}H^{1,q}_z}.\end{aligned}$$

□

Lemma 2.4.4. *Let $p,q \in (1,\infty)$ such that $1/p+1/q \leq 1$. Then for all $v_1, v_2 \in \mathbb{E}_1(T)$ and w_1 given by $w_1 := \int_{-1}^z \operatorname{div}_H v_1$ there exists a constant $C > 0$ such that*

$$\|w_1\partial_z v_2\|_{\mathbb{E}_0(T)} \leq C\|v_1\|_{\mathbb{E}_1(T)}\|v_2\|_{\mathbb{E}_1(T)}. \tag{2.4.2}$$

Proof. Set $\theta = \frac{1}{p}$. Applying the Mixed Derivative Theorem, Proposition 2.4.1 and Sobolev's embeddings in the same way as Lemma 2.4.2 yield

$$\mathbb{E}_1(T) \hookrightarrow H^{\theta,p}(0,T;H^{2-2\theta,q}) \hookrightarrow H^{1/2p,p}(0,T;H^{2-2\theta,q}) \hookrightarrow L^{2p}(0,T;H^{1+1/q,q}(\Omega)),$$

Putting $X := H^{1+1/q,q}$ and $L^p(L^q) := \mathbb{E}_0(T)$, Lemma 2.4.3 and the above embeddings imply

$$\begin{aligned}\|w_1\partial_z v_2\|_{L^p(L^q)} &\leq C\,\|\|v_1\|_X\|v_2\|_X\|_{L^p} \leq C\|v_1\|_{L^{2p}(X)}\|v_2\|_{L^{2p}(X)}\\ &\leq C\|v_1\|_{\mathbb{E}_1(T)}\|v_2\|_{\mathbb{E}_1(T)}.\end{aligned}$$

□

2.4.2 Maximal regularity of (PE) including the vertical component

Together with non-linear estimates in Section 2.4.1, we prove that the solution $u = (v,w)$ of the primitive equations belongs to the maximal regularity class $\mathbb{E}_1^u(T)$.

Proposition 2.4.5. *Let p,q fulfill Assumption (A) and let v be the strong solution of the primitive equations associated to v_0 satisfying $(v_0,w_0) \in X_\gamma$. Then*

$$u = (v,w) \in \mathbb{E}_1^u(T) \textit{ for all } T > 0.$$

Proof. It was shown in [16, Theorem 3.3c] that the primitive equations admit a unique solution $v \in \mathbb{E}_1^v(T)$, which satisfies in addition $v \in C^\infty((0,T), C^\infty(\overline{\Omega})^2)$ and hence $w \in C^\infty((0,T), C^\infty(\overline{\Omega}))$ for any $T > 0$. It remains to show that w belongs to the maximal regularity class $\mathbb{E}_1(T^*)$ for some $T^* > 0$. Applying $\int_{-1}^z \operatorname{div}_H(\cdot)$ to (PE) yields

$$\partial_t w - \Delta w = f(v,w) \text{ in } (0,\infty)\times\Omega,$$

where $f(v,w) = -\int_{-1}^z \operatorname{div}_H(\nabla_H p + u\cdot\nabla v)$. Note that the trace of the second derivative with respect to z of w at $z = -1$ vanish since w is odd with respect to z. Using $\operatorname{div}_H \overline{v} = 0$, for $z = 1$ we obtain $2\Delta_H p = -\operatorname{div}_H \int_{-1}^1 u\cdot\nabla v$ and thus

$$f(v,w) = \frac{1}{2}\int_z \operatorname{div}_H u\cdot\nabla v = \frac{1}{2}\int_z \operatorname{div}_H \operatorname{div} u\otimes v,$$

where $\int_z := \int_z^1 - \int_{-1}^z + z\int_{-1}^1$. Observe that

$$\operatorname{div}_H \operatorname{div} u \otimes v = \partial_z(w \operatorname{div}_H v + v\cdot\nabla_H w) + (\operatorname{div}_H v)^2 + 2v\cdot\nabla_H \operatorname{div}_H v + \nabla_H v\cdot(\nabla_H v)^T.$$

Hence, $f(v,w) =: f_1(v,w) + f_2(v) + f_3(v,w)$ with

$$f_1 = (w\operatorname{div}_H v - v\cdot\nabla_H w)|_z^1, \quad f_2 = \frac{1}{2}\int_z \left(\nabla_H v\cdot(\nabla_H v)^T + (\operatorname{div}_H v)^2\right),$$
$$f_3 = \int_z \partial_z v\cdot\nabla_H w.$$

Here we used the fact that $\int_z v\cdot\nabla_H \operatorname{div}_H v = -2(v\cdot\nabla_H w)|_z^1 + \int_z \partial_z v\cdot\nabla_H w$, which follows by integration by parts. By Lemma 2.4.2 we obtain $\|f_1\|_{\mathbb{E}_0(T)} \le C\|v\|_{\mathbb{E}_1(T)}\|w\|_{\mathbb{E}_1(T)}$ and moreover

$$\|f_2\|_{L^p(L^q)} \le C\|\partial_z f_2\|_{L^p(L^q_{xy}L^1_z)} \le C\|v\|^2_{L^{2p}(H^{1,2q}_{xy}L^2_z)},$$
$$\|f_3\|_{L^p(L^q)} \le C\|\partial_z f_3\|_{L^p(L^q_{xy}L^1_z)} \le C\|v\|_{L^{2p}(L^{2q}_{xy}H^{1,2}_z)}\|w\|_{L^{2p}(H^{1,2q}_{xy}L^2_z)}.$$

For $1 \ge 1/p + 1/q$ we find by the Mixed Derivative Theorem and Sobolev's embeddings

$$\mathbb{E}_1 \hookrightarrow H^{1/2p,p}(H^{2-1/p,q}) \hookrightarrow H^{1/2p,p}(H^{1+1/q,q}_{xy}L^q_z \cap H^{1/q,q}_{xy}H^{1,q}_z)$$
$$\hookrightarrow L^{2p}(H^{1,2q}_{xy}L^q_z \cap L^{2q}_{xy}H^{1,q}_z).$$

If additionally $q \ge 2$, the above embedding implies $\mathbb{E}_1 \hookrightarrow L^{2p}(H^{1,2q}_{xy}L^2_z \cap L^{2q}_{xy}H^{1,2}_z)$. Similarly, for $q < 2$ and $1 \ge 4/3q + 2/3p$ we find

$$\mathbb{E}_1 \hookrightarrow H^{\frac{1}{2p},p}\left(H^{2-\frac{1}{p},q}\right) \hookrightarrow H^{\frac{1}{2p},p}\left(H^{1+\frac{1}{q},q}_{xy}H^{\frac{1}{q}-\frac{1}{2},q}_z \cap H^{\frac{1}{q},q}_{xy}H^{1+\frac{1}{q}-\frac{1}{2},q}_z\right)$$
$$\hookrightarrow L^{2p}\big(H^{1,2q}_{xy}L^2_z \cap L^{2q}_{xy}H^{1,2}_z\big).$$

The above embeddings imply

$$\|f_2(v)\|_{\mathbb{E}_0(T)} \le C\|v\|^2_{\mathbb{E}_1(T)} \text{ and } \|f_1(v,w)\|_{\mathbb{E}_0(T)} + \|f_3(v,w)\|_{\mathbb{E}_0(T)} \le C\|v\|_{\mathbb{E}_1(T)}\|w\|_{\mathbb{E}_1(T)}.$$

In particular $f_2(v) \in \mathbb{E}_0(T)$. By maximal regularity there exists a solution operator $\mathcal{S} : X_\gamma \times \mathbb{E}_0(T) \to \mathbb{E}_1(T)$ such that $u := \mathcal{S}(w_0, f_2(v))$ satisfies

$$\partial_t u - \Delta u = f_2 \text{ in } (0,\infty)\times\Omega, \quad u(0) = w_0.$$

Setting now $B_v = -f_1(v,\cdot) - f_3(v,\cdot) \in \mathcal{L}(\mathbb{E}_1(T), \mathbb{E}_0(T))$, i.e. it is a bounded linear operator from $\mathbb{E}_1(T)$ to $\mathbb{E}_0(T)$, and adding $B_v u$ on both sides we see that

$$\partial_t u - \Delta u + B_v u = f_2 + B_v u = [\mathbb{I} + B_v\mathcal{S}(0,\cdot)]f_2 + B_v\mathcal{S}(w_0, 0) \text{ in } (0,\infty)\times\Omega.$$

Next, note that $\|\mathcal{S}(0,\cdot)\|_{\mathcal{L}(\mathbb{E}_0(T),\mathbb{E}_1(T))}$ can be bounded uniformly for $T \le 1$. Moreover $\|B_v\|_{\mathcal{L}(\mathbb{E}_1(T),\mathbb{E}_0(T))} \le C\|v\|_{\mathbb{E}_1(T)}$ by the previous estimates on f_1 and f_3. Choosing now T^* small enough such that $\|v\|_{\mathbb{E}_1(T^*)} < \left(C\|\mathcal{S}(0,\cdot)\|_{\mathcal{L}(\mathbb{E}_0(T^*),\mathbb{E}_1(T^*))}\right)^{-1}$, we see that $\|B_v\mathcal{S}(0,\cdot)\|_{\mathcal{L}(\mathbb{E}_0(T^*))} < 1$. A Neumann series argument yields $[\mathbb{I}+B_v\mathcal{S}(0,\cdot)]^{-1} \in \mathcal{L}(\mathbb{E}_0(T^*))$, and thus

$$\tilde{u} := \mathcal{S}\big(w_0, [\mathbb{I}+B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0))\big) \in \mathbb{E}_1(T^*)$$

solves

$$\begin{aligned}
&\partial_t\tilde{u} - \Delta\tilde{u} + B_v\tilde{u} = \ [\mathbb{I}+B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0)) + B_v\tilde{u} \\
&= [\mathbb{I}+B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0)) + B_v\mathcal{S}\big(0, [\mathbb{I}+B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0))\big) \\
&\quad + B_v\mathcal{S}(w_0,0) \\
&= [\mathbb{I}+B_v\mathcal{S}(0,\cdot)][\mathbb{I}+B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0)) + B_v\mathcal{S}(w_0,0) = f_2
\end{aligned}$$

in $(0,T^*)\times\Omega$ with $\tilde{u}(0) = w_0$. Since $f(v,\cdot) = f_2(v) - B_v$ and since the heat equation is uniquely solvable, we finally obtain $w = \tilde{u} \in \mathbb{E}_1(T^*)$. Summing up, $w \in \mathbb{E}_1(T)$ for any $T > 0$. □

2.5 Proof of the main result

2.5.1 Maximal regularity for the anisotropic Stokes equations

Proposition 2.5.1. *For $p,q \in (1,\infty)$, $\varepsilon > 0$, $T > 0$, $F = (F_H, F_z) \in \mathbb{E}_0(T)$, and $U_0 \in X_\gamma$, there exists a unique solution $(U,P) = \big((V,W),P\big) \in \mathbb{E}_1^u(T) \times \mathbb{E}_1^\pi(T)$ of*

$$\begin{cases}
(\partial_t - \Delta)\left(\begin{smallmatrix} V \\ \varepsilon W\end{smallmatrix}\right) + \nabla_\varepsilon P = F & \textit{in } (0,T)\times\Omega, \\
\operatorname{div} U = 0 & \textit{in } (0,T)\times\Omega, \\
P \textit{ periodic in } x,y,z, \textit{ even} & \textit{in } z, \\
V,W \textit{ periodic in } x,y,z, \textit{ even and odd} & \textit{in } z, \\
U|_{t=0} = U_0 =: (V_0, W_0) & \textit{in } \Omega,
\end{cases}$$

satisfying

$$\|(V,\varepsilon W)\|_{\mathbb{E}_1(T)} + \|\nabla_\varepsilon P\|_{\mathbb{E}_0(T)} \le C\|F\|_{\mathbb{E}_0(T)} + C_T\|(V_0,\varepsilon W_0)\|_{X_\gamma}. \tag{2.5.1}$$

Here, the constant C depends only on p,q and C_T depends only on p,q,T.

Proof. Since uniqueness easily follows from (2.5.1), we prove existence of a solution. First, for a.e. $t \in (0,T)$ we construct $P = P(t)$ by solving the following anisotropic Poisson equation in a periodic setting:

$$\begin{aligned}
&\Delta_\varepsilon P := (\Delta_H + \varepsilon^{-2}\partial_z^2)P = \operatorname{div}_H F_H + \varepsilon^{-1}\partial_z F_z =: \operatorname{div}_\varepsilon F \text{ in } \ \Omega, \\
&\int_\Omega P\,dx = 0.
\end{aligned} \tag{2.5.2}$$

Denoting by $\mathcal{F}$ the Fourier transform and setting $k_\varepsilon = (k_1, k_2, \varepsilon^{-1}k_3)^T$ and $m_\varepsilon(k) = m(k_\varepsilon)$ with $m(k) = -\frac{k\otimes k}{|k|^2} \in \mathbb{R}^{3\times 3}$, we obtain $\nabla_\varepsilon P = \mathcal{F}^{-1}m_\varepsilon \mathcal{F}F$. Since $k_\varepsilon \nabla m_\varepsilon(k) = k_\varepsilon \nabla m(k_\varepsilon)$ and

$$\sup_{\gamma\in\{0,1\}^3} \sup_{k\neq 0} |k^\gamma D^\gamma m_\varepsilon(k)| = \sup_{\gamma\in\{0,1\}^3} \sup_{k_\varepsilon\neq 0} |k_\varepsilon^\gamma D^\gamma m(k_\varepsilon)| = 1,$$

Mikhlin's theorem in the period setting, see e.g. [11, Proposition 4.5], implies that m_ε is an L^p-Fourier multiplier satisfying $\|\mathcal{F}^{-1}m_\varepsilon\mathcal{F}\|_{\mathcal{L}(L^q(\Omega))} \leq C$ for some $C = C_q > 0$. Consequently,

$$\|\nabla_\varepsilon P\|_{L^p(0,T;L^q(\Omega))} \leq C\|F\|_{L^p(0,T;L^q(\Omega))}. \tag{2.5.3}$$

Next we define $U \in \mathbb{E}_1(T)$ by solving the following heat equation:

$$(\partial_t - \Delta)\left(\begin{smallmatrix} V \\ \varepsilon W\end{smallmatrix}\right) = F - \nabla_\varepsilon P, \quad U|_{t=0} = U_0, \tag{2.5.4}$$

with periodic boundary conditions and parity conditions. Such a solution does exist by the well-known maximal regularity result for the laplace operator, which combined with (2.5.3) also yields

$$\begin{aligned} \|(V,\varepsilon W)\|_{\mathbb{E}_1(T)} &\leq C\|F - \nabla_\varepsilon P\|_{\mathbb{E}_0(T)} + C_T\|(V_0, \varepsilon W_0)\|_{X_\gamma} \\ &\leq C\|F\|_{\mathbb{E}_0(T)} + C_T\|(V_0,\varepsilon W_0)\|_{X_\gamma}. \end{aligned} \tag{2.5.5}$$

It remains to prove that $G := \operatorname{div} U = \operatorname{div}_\varepsilon(V, \varepsilon W) \equiv 0$. In fact, by (2.5.2) and (2.5.4) we deduce $\partial_t G - \Delta G = 0$ in $(0,T)\times\Omega$. We also find that G is periodic in Ω and $G|_{t=0} = 0$. Therefore, G must vanish by the uniqueness of the heat equation. This completes the proof of Proposition 2.5.1. □

2.5.2 Proof of Theorem 2.3.3

It suffices for us to find an integer $M \geq 0$ and $\varepsilon_0 > 0$, which depend only on $p, q, u, \mathcal{T}$, such that the following assertion (A_m) holds true for all $m = 0, 1, \ldots, M$ recursively and for all $\varepsilon \in (0, \varepsilon_0]$. For this purpose we introduce $T := \mathcal{T}/(M+1)$ and a sequence of increasing positive numbers $\{C_{m,T}\}_{m=0}^M$ given by the recursive formula

$$C_{0,T} = 4C_*(C_u + C_u^2), \qquad C_{m,T} = 4\big(C_*(C_u + C_u^2) + C_T C_{tr} C_{m-1,T}\big),$$

where C_* is to be defined below (see the argument after (2.5.9)), $C_u = \|u\|_{\mathbb{E}_1(0,\mathcal{T})}$, C_T is the constant in (2.5.1), and C_{tr} is the embedding constant of $\mathbb{E}_1(mT,(m+1)T) \hookrightarrow C([mT,(m+1)T]; X_\gamma)$ that is independent of m.

Assertion $(\mathbf{A}_m)$. *The time trace of U_ε at $t = mT$, denoted by $U_\varepsilon(mT)$, is well defined in X_γ. In addition, there exists a unique solution $(U_\varepsilon, P_\varepsilon) \in \mathbb{E}_1^u(mT,(m+1)T) \times \mathbb{E}_1^\pi(mT,(m+1)T)$ of*

$$\begin{cases} (\partial_t - \Delta)\begin{pmatrix} V_\varepsilon \\ \varepsilon W_\varepsilon\end{pmatrix} + \nabla_\varepsilon P_\varepsilon = \begin{pmatrix} F_H(U_\varepsilon, u) \\ \varepsilon F_z(U_\varepsilon,u)\end{pmatrix} & \text{in} \quad (mT,(m+1)T)\times\Omega, \\ \operatorname{div} U_\varepsilon = 0 & \text{in} \quad (mT,(m+1)T)\times\Omega, \\ P_\varepsilon \text{ periodic in } x,y,z, \text{ even} & \text{in} \quad z, \\ V_\varepsilon, W_\varepsilon \text{ periodic in } x,y,z, \text{ even and odd} & \text{in} \quad z, \\ U_\varepsilon|_{t=mT} = U_\varepsilon(mT) & \text{in} \quad \Omega, \end{cases} \tag{2.5.6}$$

satisfying

$$X_{\varepsilon,m,T} := \|(V_\varepsilon, \varepsilon W_\varepsilon)\|_{\mathbb{E}_1(mT,(m+1)T)} + \|\nabla_\varepsilon P_\varepsilon\|_{\mathbb{E}_0(mT,(m+1)T)} \le C_{m,T}\,\varepsilon. \tag{2.5.7}$$

Let us establish this assertion by induction with respect to m. We only prove (A_m) assuming (A_{m-1}) because the proof for (A_0) can be done in the same way if we note that $U_\varepsilon(0) = 0$.

We construct a successive approximation $(U_\varepsilon^{(j)}, P_\varepsilon^{(j)}) =: \big((V_\varepsilon^{(j)}, W_\varepsilon^{(j)}), P_\varepsilon^{(j)}\big)$ to (2.5.6) by solving the following anisotropic Stokes equations:

$$\begin{cases} (\partial_t - \Delta)\begin{pmatrix} V_\varepsilon^{(j)} \\ \varepsilon W_\varepsilon^{(j)} \end{pmatrix} + \nabla_\varepsilon P_\varepsilon^{(j)} = \begin{pmatrix} F_H(U_\varepsilon^{(j-1)}, u) \\ \varepsilon F_z(U_\varepsilon^{(j-1)}, u) \end{pmatrix} & \text{in} \quad (mT, (m+1)T) \times \Omega, \\ \operatorname{div} U_\varepsilon^{(j)} = 0 & \text{in} \quad (mT, (m+1)T) \times \Omega, \\ P_\varepsilon^{(j)} \text{ periodic in } x, y, z, \text{ even} & \text{in} \quad z, \\ V_\varepsilon^{(j)}, W_\varepsilon^{(j)} \text{ periodic in } x, y, z, \text{ even and odd} & \text{in} \quad z, \\ U_\varepsilon^{(j)}|_{t=mT} = U_\varepsilon(mT), & \text{in} \quad \Omega, \end{cases} \tag{2.5.8}$$

whose existence is guaranteed by Proposition 2.5.1. When $j = 0$, we understand $U_\varepsilon^{(j-1)} \equiv 0$. We will derive a uniform bound for

$$X_{\varepsilon,m,T}^{(j)} := \|(V_\varepsilon^{(j)}, \varepsilon W_\varepsilon^{(j)})\|_{\mathbb{E}_1(mT,(m+1)T)} + \|\nabla_\varepsilon P_\varepsilon^{(j)}\|_{\mathbb{E}_0(mT,(m+1)T)}.$$

By Lemmas 2.4.2 and 2.4.4, the forcing terms

$$\begin{aligned} F_H(U_\varepsilon^{(j-1)}, u) &= -V_\varepsilon^{(j-1)} \cdot \nabla_H v - W_\varepsilon^{(j-1)} \partial_z v - v \cdot \nabla_H V_\varepsilon^{(j-1)} - w \partial_z V_\varepsilon^{(j-1)} \\ &\quad - V_\varepsilon^{(j-1)} \cdot \nabla_H V_\varepsilon^{(j-1)} - W_\varepsilon^{(j-1)} \partial_z V_\varepsilon^{(j-1)}, \\ \varepsilon F_z(U_\varepsilon^{(j-1)}, u) &= \varepsilon(-V_\varepsilon^{(j-1)} \cdot \nabla_H w - w \operatorname{div}_H V_\varepsilon^{(j-1)}) - \varepsilon W_\varepsilon^{(j-1)} \operatorname{div}_H(v + V_\varepsilon^{(j-1)}) \\ &\quad - (v + V_\varepsilon^{(j-1)}) \cdot \nabla_H(\varepsilon W_\varepsilon^{(j-1)}) - \varepsilon(\partial_t w + u \cdot \nabla w - \Delta w), \end{aligned}$$

are estimated as

$$\begin{aligned} &\|F_H(U_\varepsilon^{(j-1)}, u)\|_{\mathbb{E}_0(mT,(m+1)T)} \\ &\le C\|V_\varepsilon^{(j-1)}\|_{\mathbb{E}_1(mT,(m+1)T)}(\|V_\varepsilon^{(j-1)}\|_{\mathbb{E}_1(mT,(m+1)T)} + \|v\|_{\mathbb{E}_1(mT,(m+1)T)}), \\ &\|\varepsilon F_z(U_\varepsilon^{(j-1)}, u)\|_{\mathbb{E}_0(mT,(m+1)T)} \\ &\le C\|V_\varepsilon^{(j-1)}\|_{\mathbb{E}_1(mT,(m+1)T)}\|w\|_{\mathbb{E}_1(mT,(m+1)T)} \\ &\qquad + C\|\varepsilon W_\varepsilon^{(j-1)}\|_{\mathbb{E}_1(mT,(m+1)T)}(\|V_\varepsilon^{(j-1)}\|_{\mathbb{E}_1(mT,(m+1)T)} + \|v\|_{\mathbb{E}_1(mT,(m+1)T)}) \\ &\qquad + C\varepsilon(\|w\|_{\mathbb{E}_1(mT,(m+1)T)} + \|w\|^2_{\mathbb{E}_1(mT,(m+1)T)}). \end{aligned}$$

By the induction assumption (A_{m-1}) we have $U_\varepsilon \in \mathbb{E}_1((m-1)T, mT) \hookrightarrow C([(m-1)T, mT]; X_\gamma)$, which implies that $U_\varepsilon(mT) \in X_\gamma$ is well defined and that

$$\|(V_\varepsilon(mT), \varepsilon W_\varepsilon(mT))\|_{X_\gamma} \le C_{tr}\|(V_\varepsilon, \varepsilon W_\varepsilon)\|_{\mathbb{E}_1((m-1)T,mT)} \le (C_{tr}C_{m-1,T})\varepsilon.$$

We are in the position to apply the maximal regularity estimate (2.5.1) to (2.5.8), which gives

$$\begin{aligned} X^{(j)}_{\varepsilon,m,T} &\le C\big(\|u\|_{\mathbb{E}_1(mT,(m+1)T)} X^{(j-1)}_{\varepsilon,m,T} + (X^{(j-1)}_{\varepsilon,m,T})^2 + (C_u + C_u^2)\varepsilon\big) \\ &+ (C_T C_{tr} C_{m-1,T})\varepsilon \end{aligned} \tag{2.5.9}$$

for some constant C depending only on p, q. Let us designate this constant by C_* in the subsequent argument.

Now choose a sufficiently large M such that (recall $T = \mathcal{T}/(M+1)$)

$$C_*\|u\|_{\mathbb{E}_1(t,t+T)} \le 1/4 \quad \forall t \in [0, \mathcal{T} - T].$$

Then such M and T depend only on $p, q, \mathcal{T}, u$, and it follows that

$$X^{(j)}_{\varepsilon,m,T} \le \frac{1}{4} X^{(j-1)}_{\varepsilon,m,T} + C_*(X^{(j-1)}_{\varepsilon,m,T})^2 + \frac{C_{m,T}}{4}\varepsilon, \qquad j \ge 1.$$

Similarly we obtain $X^{(0)}_{\varepsilon,m,T} \le (C_{m,T}/4)\varepsilon$. This yields, for $\varepsilon \le \varepsilon_0 := (4C_* C_{M,T})^{-1}$, the upper bound

$$X^{(j)}_{\varepsilon,m,T} \le \frac{3/4 - \sqrt{9/16 - C_* C_{m,T}\,\varepsilon}}{2C_*} = \frac{C_* C_{m,T}\,\varepsilon}{2C_*\sqrt{9/16 + C_* C_{m,T}\,\varepsilon}} \le C_{m,T}\,\varepsilon, \tag{2.5.10}$$

which is uniform for $j = 0, 1, \dots$ By considering the equations that $\tilde{U}^{(j)}_\varepsilon := U^{(j)}_\varepsilon - U^{(j-1)}_\varepsilon$ and $\tilde{P}^{(j)}_\varepsilon := P^{(j)}_\varepsilon - P^{(j-1)}_\varepsilon$ $(j \ge 1)$ satisfy and applying the maximal regularity estimate (2.5.1), we deduce that

$$\begin{aligned} &\|(\tilde{V}^{(j)}_\varepsilon, \varepsilon\tilde{W}^{(j)}_\varepsilon)\|_{\mathbb{E}_1(mT,(m+1)T)} + \|\nabla_\varepsilon \tilde{P}^{(j)}_\varepsilon\|_{\mathbb{E}_0(mT,(m+1)T)} \\ \le\ &\frac{3}{4}\big(\|(\tilde{V}^{(j-1)}_\varepsilon, \varepsilon\tilde{W}^{(j-1)}_\varepsilon)\|_{\mathbb{E}_1(mT,(m+1)T)} + \|\nabla_\varepsilon \tilde{P}^{(j-1)}_\varepsilon\|_{\mathbb{E}_0(mT,(m+1)T)}\big), \qquad j \ge 2. \end{aligned}$$

From this the existence of $(U_\varepsilon, P_\varepsilon) := \lim_{j\to\infty}(U^{(j)}_\varepsilon, P^{(j)}_\varepsilon)$ in $\mathbb{E}^u_1(mT,(m+1)T) \times \mathbb{E}^\pi_1(mT,(m+1)T)$ follows. We now take the limit $j \to \infty$ in (2.5.8) and (2.5.10) to conclude the solvability of (2.5.6) and the estimate (2.5.7), respectively. The uniqueness proof is standard, so we omit it. This completes the proof of (A_m) and, consequently, that of Theorem 2.3.3.

Remark 2.5.2. If the solution of (PE) exists in the maximal regularity class with the time interval $(0, \infty)$, we can find some $\mathcal{T}_1$ such that $\|u\|_{\mathbb{E}_1(\mathcal{T}_1,\infty)} \le \frac{1}{4C_*}$. In this case our construction of the solution above can be applied to extend the result of Theorem 2.3.3 up to $\mathcal{T} = \infty$ as well.

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Chapter 3

Justification of the Hydrostatic Approximation - the Case of Non-Slip Boundary Condition

In this paper, justification of the hydrostatic approximation in the primitive equations in the maximal L^p-L^q-settings is considered under the non-slip boundary condition. We show the solution to the scaled Navier-Stokes equations with initial data $u_0 \in B^{2-2/p}_{q,p}(\Omega)$ converges to the solution to the primitive equations with the same initial data in $\mathbb{E}_1(T) = W^{1,p}(0,T;L^q(\Omega)) \cap L^p(0,T;W^{2,q}(\Omega))$ with order $O(\epsilon)$ where $(p,q) \in (1,\infty)^2$ satisfies $\frac{1}{p} \leq \min(1-1/q, 3/2-2/q)$. The global well-posedness of the scaled Navier-Stokes equations in $\mathbb{E}_1(T)$ is also proved for sufficiently small $\epsilon > 0$. Note that $T = \infty$ is included.

3.1 Introduction

The primitive equations with the non-slip boundary condition is

$$(PE)\begin{cases} \partial_t u - \Delta u + u \cdot \nabla v + \nabla_H \pi &= 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ \partial_z \pi &= 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,\infty) \end{cases}$$

where $u = (v, w) \in \mathbb{R}^2 \times \mathbb{R}$ and π are a velocity field and a pressure, respectively, $\nabla_H = (\partial_x, \partial_y)^T$, and $\Omega = \mathbb{T}^2 \times (-1,1)$. Hereafter, we say the non-slip boundary condition as the Dirichlet boundary condition. By divergence-free condition and the Dirichlet boundary condition, w is given by the formula

$$w(x', x_3, t) = -\int_{-1}^{x_3} \operatorname{div}_H v(x', x_3, t) d\zeta d\zeta = \int_{x_3}^{1} \operatorname{div}_H v(x', x_3, t) d\zeta d\zeta.$$

The primitive equation is a fundamental model for geographic flow. Existence of weak solution to the primitive equations with L^2-initial data was proved by Lions, Temam and Wang [17]. Local-in-time well-posedness was proved by [18]. Although global well-posedness of the 3-dimensional Navier-Stokes equations is the well-known open problem, for the primitive equation this problem have been solved by Cao and Titi [4]. Hieber and Kashiwabara [23] extended this result to prove global well-posedness for the primitive equation in L^p-settings. Recently, Giga, Gries, Hieber, Hussein and Kashiwabara [6] obtained global-in-time well-posedness in the maximal regularity space (mixed Lebesgue-Sobolev space) $W^{1,p}(0,T;L^q(\Omega))\cap L^p(0,T;W^{2,q}(\Omega))$ for $T>0$ under various boundary conditions.

Our first aim in this paper is to give a rigorous justification of derivation of the primitive equations under the Dirichlet boundary condition. We first give a explanation of its derivation. Let us consider the following anisotropic viscous Navier-Stokes equations in thin domain

$$(ANS)\begin{cases} \partial_t u-(\Delta_H+\epsilon^2\partial_z^2)u+u\cdot\nabla u+\nabla\pi = 0 & \text{in}\quad \Omega_\epsilon\times(0,\infty), \\ \operatorname{div} u = 0 & \text{in}\quad \Omega_\epsilon\times(0,\infty), \end{cases} \tag{3.1.1}$$

where $\Omega_\epsilon=(-\epsilon,\epsilon)\times\mathbb{T}^2$. If $\epsilon=1$, (ANS) is the usual Navier-Stokes equations. (ANS) valid for incompressible viscous fluid filled with thin domains. Actually, if we put the Reynolds number 1, since length and velocity is ϵ-order, apparent viscosity for vertical direction must be ϵ^2-order in the Reynolds number point of view. The primitive equations is formally derived from above equations. Set new unknowns;

- $u_\epsilon:=(v_\epsilon,w_\epsilon)$
- $v_\epsilon(x,y,z,t):=v(x,y,z/\epsilon,t)$
- $w_\epsilon(x,y,z,t):=w(x,y,z/\epsilon,t)/\epsilon$
- $\pi_\epsilon(x,y,z,t):=\pi(x,y,z/\epsilon,t)$.

Then, $(u_\epsilon,\pi_\epsilon)$ satisfy the scaled Navier-Stokes equations in a fixed domain

$$(SNS)\begin{cases} \partial_t v_\epsilon-\Delta v_\epsilon+u_\epsilon\cdot\nabla v_\epsilon+\nabla_H\pi_\epsilon = 0 & \text{in}\quad \Omega\times(0,\infty), \\ \epsilon^2\left(\partial_t w_\epsilon-\Delta w_\epsilon+u_\epsilon\cdot\nabla w_\epsilon\right)+\partial_z\pi_\epsilon = 0 & \text{in}\quad \Omega\times(0,\infty), \\ \operatorname{div}_\epsilon u = 0 & \text{in}\quad \Omega\times(0,\infty). \end{cases} \tag{3.1.2}$$

Taking formally $\epsilon\to 0$ for the above equations, we get the primitive equations.

The existence of weak solution to the Navier-Stoke equations first proved by Leray [29]. Its uniqueness is stile open problem. Global-in-time well-posedness first proved by Fujita and Kato [11] for $H^{1/2}$-initial data. After that, the solution space was extended by many researchers. For instance, Kato [26], Cannone [5] and Koch and Tataru [27] proved global-in-time well-posedness of small data in L^q for $q\geq n$, $B^{-1+n/q}_{q,p}$ for $1<q<\infty$ and BMO^{-1}, respectively, where n is the space dimension. See Lemarié-Rieusset's book [30] for further previous works.

Rigorous justification of the primitive equations from the scaled Navier-Stokes equations was studied by Aezérad and Guillén [2]. They obtained weak* convergence in the natural energy space $L^\infty(0,T;L^2(\mathbb{T}^3))\cap L^2(0,T;H^1(\mathbb{T}^3))$ for $T>0$. Recently, Li and Titi [14] improved their result to get strong convergence with the aid of regularity of the solution to the primitive equations. Furukawa, Giga, Hieber, Hussein, Kashiwabara and Wrona [12] extended Li and Titi's result in maximal-regularity space $H^{1,p}(0,T;L^q(\Omega))\cap L^p(0,T;H^{2,q}(\Omega))$ for $\Omega=\mathbb{T}^3$ and $1/p\le \min(1-1/q,3/2-2/q)$ with a different strategy. Note that the case of $p=q=2$ is corresponding to Li and Titi's result. As we already mentioned, the primitive equation is a model for geographic flow. Although, it is more physically natural to consider the case of Dirichlet-Neumann and Dirichlet boundary conditions, there was no result of justification of derivation to the primitive equation from the Navier-Stokes equations.

Set

$$\begin{aligned}
\mathbb{E}_1(T) &= \{u\in H^{1,p}(0,T;L^q(\Omega))\cap L^p(0,T;H^{2,q}(\Omega))\,;\,\operatorname{div} u=0,\ u|_{x=\pm1}=0\}\\
\mathbb{E}_0(T) &= \{u\in L^p(0,T;L^q(\Omega))\,;\,\operatorname{div} u=0,\ u|_{x=\pm1}=0\}\\
\mathbb{E}_1^\pi(T) &= \{\pi\in L^p(0,T;W^{1,q}(\Omega))\,;\,\int_\Omega \pi\,dx\equiv 0\}\\
X_\gamma &= \{u\in B^{2-2/p}_{q,p}\,;\,\operatorname{div} u=0,\ u|_{x=\pm1}=0\},
\end{aligned}$$

where X_γ is the trace space of $\mathbb{E}_1(T)$. Let us seek the solution $U_\epsilon=(V_\epsilon,W_\epsilon)$ to

$$\begin{cases}
\partial_t V_\epsilon-\Delta V_\epsilon+\nabla_H P_\epsilon = F_H & \text{in } \Omega\times(0,T),\\
\partial_t(\epsilon W_\epsilon)-\Delta(\epsilon W_\epsilon)+\frac{\partial_z}{\epsilon}P_\epsilon = \epsilon F_z+\epsilon F & \text{in } \Omega\times(0,T),\\
\operatorname{div} U_\epsilon = 0 & \text{in } \Omega\times(0,T),\\
U_\epsilon = 0 & \text{on } \partial\Omega\times(0,T),\\
U_\epsilon(0) = 0 & \text{on } \Omega,
\end{cases}\tag{3.1.3}$$

where

- $F_H=-\left(U_\epsilon\cdot\nabla V_\epsilon+u\cdot\nabla V_\epsilon+U_\epsilon\cdot\nabla v\right)$
- $F_z=-\left(U_\epsilon\cdot\nabla W_\epsilon+u\cdot\nabla W_\epsilon+U_\epsilon\cdot\nabla w\right)$
- $F=-\left(\partial_t w-\Delta w+u\cdot\nabla w\right)$.

(3.1.3) is the equation of the difference between the solution to the (PE) and (SNS).

Theorem 3.1.1. *Let $T>0$. Suppose (p,q) satisfies $\frac{1}{p}\le\min(1-1/q,3/2-2/q)$ and $u_0\in X_\gamma$. Let $u\in\mathbb{E}_1(T)$ be a solution of (PE) with initial data $u_0\in X_\gamma$. Then there exists constant $C=C(p,q,\|u\|_{\mathbb{E}_1(T)})$ and a unique solution $U_\epsilon=(V_\epsilon,W_\epsilon)$ to (3.1.3) such that*

$$\|\,(V_\epsilon,\epsilon W_\epsilon)\,\|_{\mathbb{E}_1(T)}\le\epsilon C.\tag{3.1.4}$$

Moreover, $u_\epsilon=(v_\epsilon,w_\epsilon):=(v+V_\epsilon,w+W_\epsilon)$ is the unique solution to (SNS) in $\mathbb{E}_1(T)$.

This theorem implies the justification of the hydrostatic approximation.

Corollary 3.1.2. *Let $T>0$. Suppose (p,q) satisfies $\frac{1}{p}\le\min(1-1/q,3/2-2/q)$ and $u_0\in X_\gamma$. Let u and u_ϵ be a solution of (PE) and (SNS) in $\mathbb{E}_1$ under the Dirichlet boundary condition with initial data u_0, respectively, such that*

$$||u||_{\mathbb{E}_1(T)}+||(v_\epsilon,\epsilon w_\epsilon)||_{\mathbb{E}_1(T)}\le C_0\tag{3.1.5}$$

for some $C_0 = C_0(u_0, p, q, T)$. Then there exists a positive $C = C(u_0, p, q, C_0)$ such that

$$||(v_\epsilon - v, \epsilon(w_\epsilon - w))||_{\mathbb{E}_1(T)} \leq \epsilon C.$$

Our strategy to show Theorem 3.1.1 is based on the estimate for $(V_\epsilon, \epsilon W_\epsilon)$. To explain the strategy some key lemma are needed; maximal regularity result of the anisotropic Stokes operator and improved regularity result for vertical component of the solution to the primitive equation. Consider the linearized problem of (SNS);

$$\begin{cases} \partial_t u - \Delta u + \nabla_\epsilon \pi &= f \quad \text{in} \quad \Omega \times (0,T), \\ \operatorname{div}_\epsilon u &= 0 \quad \text{in} \quad \Omega \times (0,T), \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(0) &= u_0 \quad \text{on} \quad \Omega, \end{cases} \tag{3.1.6}$$

where $\nabla_\epsilon = (\partial_1, \partial_2, \partial_3/\epsilon)$ and $\operatorname{div}_\epsilon = \nabla_\epsilon \cdot$. We introduce some known results on maximal regularity of the Stokes operator, which is corresponding to the case $\epsilon = 1$. Solonnikov [40] first proved L^q-L^q maximal regularity for the Stokes operator. Giga [13] proved bounded imaginary power in bounded domain, this implies maximal L^p-L^q regularity via Dore-Venni theory [7]. Giga and Sohr [19] proved maximal regularity in an exterior domain. Abels [1] proved bounded imaginary power of the Stokes operator in an infinite layer domain. Further studies on maximal regularity were done by many researchers, for instance, Dore and Veni [7] and Weis [42]. See Denk, Hieber and Prüss [8] for further comprehensive research. In our case, it is needed to clarify the dependence of ϵ to maximal regularity. This is a key point. We show the following

Lemma 3.1.3. *Let $1 < p, q < \infty$, $0 < \epsilon \leq 1$ and $T > 0$. Let $f \in \mathbb{E}_0(T)$ and $u_0 \in X_\gamma$. Then there there exist constants $C = C(p,q) > 0$ and $C' = C'(p,q) > 0$, which are independent of ϵ, and (u, π) satisfying (3.1.6) such that*

$$||\partial_t u||_{\mathbb{E}_0} + \left|\left|\nabla^2 u\right|\right|_{\mathbb{E}_0} + ||\nabla_\epsilon \pi||_{\mathbb{E}_0} \leq C||f||_{\mathbb{E}_0} + C'||u_0||_{X_\gamma}. \tag{3.1.7}$$

The proof of Lemma 3.1.3 is based on Abels's paper [1]. It is needed to clarify independence of ϵ of C and C_T. Unfortunately, it is not clear in [1]. However, the strategy in [1] works on our problem. We construct the anisotropic Stokes operator by the method in [1] and show the boundedness of imaginary power. Note that, in our previous paper [12], maximal regularity of the anisotropic Stokes operator is much easier since the corresponding Stokes operator is essentially the same as the Laplace operator on $\mathbb{T}^3$. In the case of the Dirichlet boundary condition, the corresponding Stokes operator becomes to be much more difficult by the effect of boundaries, which is essentially the different point to in the case of the periodic boundary conditions.

$F = \partial_t w - \Delta w + u \cdot \nabla w$ appears in the right hand side of (3.1.3). Thus, we need to improve the regularity of w and estimate this term in $L^p(0,T; L^q(\Omega))$.

Lemma 3.1.4. *Let $T > 0$ and $u_0 = (v_0, w_0) \in X_\gamma$ with $w_0 = -\int_{-1}^{x_3} \operatorname{div}_H v_0 \, d\zeta$ and $u = (v, w)$ be the solution to (PE). Assume $v \in \mathbb{E}_1(T)$. Then there exist a constant $C = C(p, q, \|u_0\|_{X_\gamma}, \|v\|_{\mathbb{E}_1(T)})$ such that*

$$\|w\|_{\mathbb{E}_1(T)} \leq C. \tag{3.1.8}$$

Since $v \in \mathbb{E}_1(T)$, which is the vertical component of the solution to the primitive equations, has already proved, it follows $w(\cdot, x_3) = -\int_{-1}^{x_3} \mathrm{div}_H v(\cdot, \zeta) d\zeta \in H^{1,p}(0,T; H^{-1,q}(\Omega)) \cap L^p(0,T; H^{1,p}(\Omega))$. This derivative loss is due to the absence of the equation of time-evolution of w in the primitive equation. In our previous paper [12], which is the case of periodic boundary condition, we recover the regularity of w by deriving the equation which w satisfies and applying maximal regularity of the Laplace operator to the equation. However, in the case of the Dirichlet boundary condition, this method is not applicable directly because of the second-order derivative term at the boundary, which banishes in the case of periodic boundary condition. Therefore, it is needed to escape this difficulty. Actually, we can escape this by so-called cut-off technique. We multiply w by the cut-off function $\phi \in C_0^\infty(\mathbb{T})$, which is equal to 0 near the one side of the boundary, and seek the equations that ϕw satisfies. In this equation, second-order derivative at the boundary does not appear, and thus applying maximal regularity of the Laplace operator, we can recover the regularity of w. Note that the remainder terms derived from the effect of cut-off can be treated as a low-order term.

Let us introduce our strategy to show Theorem 3.1.1. First, boundedness of nonlinear terms F_H and F_z in (3.1.3) on $\mathbb{E}_0$ is shown. F is also bounded $\mathbb{E}_0(T)$ by Lemma 3.1.3. Second, we apply Lemma 3.1.4 to (3.1.3) to get a quadratic inequality, which leads $||(V_\epsilon, \epsilon W_\epsilon)||_{\mathbb{E}_1(T^\star)} \leq C\epsilon$ for some short time $T^\star$ and ϵ-independent constant $C > 0$. Since C depends only on p, q, $\|u_0\|_{X_\gamma}$, $\|u\|_{\mathbb{E}_1(T)}$ and T, if we take ϵ small, we can extended the time to all finite time T by finite step.

In this paper, $||\cdot||_{X\to Y}$ denotes operator norm from a Banach space X to a Banach space Y. We define the Fourier transform by $\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$, the Fourier inverse transform by $\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\, d\xi$. The Fourier transform on the torus $\mathbb{T}^d$ and its inverse trance form are denoted by $\mathcal{F}_d$ and $\mathcal{F}_d$, respectively. $\mathcal{F}_{x'}$ means the partial Fourier transform with respect to $x' \in \mathbb{R}^2$ and the partial Fourier inverse transform with respect to ξ' by $\mathcal{F}_{\xi'}^{-1}$. Define $\Sigma_\theta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \pi - \theta\}$. For a Fourier multiplier operator $\mathcal{F}_\xi^{-1} m(\xi) \mathcal{F}_x$ in $\mathbb{R}^3$, we denote by $[m]_{\mathcal{M}}$ the Mikhlin constant. $\mathcal{F}_{\xi'}^{-1} m(\xi) \mathcal{F}_x'$ is a Fourier multiplier operator in $\mathbb{R}^2$ with Mikhlin constant $[m]_{\mathcal{M}'}$. For $0 < \epsilon \leq 1$, $\Delta_\epsilon = \partial_1^2 + \partial_2^2 + \partial_3^2/\epsilon^2$ denotes the anisotropic Laplace operator. $E_0 f$ is 0-extension onto $\mathbb{R}^3$ for a f supported in Ω. $R_0 f$ is the restriction on Ω for a function f defined on $\mathbb{R}^3$. For a integrable function f defined on Ω, we write its vertical average by $\overline{f} = \frac{1}{2} \int_{-1}^1 f(\cdot, \cdot, \zeta)\, d\zeta$

3.2 BIP of the Anisotropic Stokes Operator

This section is devoted to the proof of BIP of the anisotropic Stokes operator along with [1].

3.2.1 Boundedness of Fourier multipliers

Although, the case of infinite layer $\mathbb{R}^2 \times (-1.1)$ is considered in [1], his method also works in a case of periodic layer $\Omega = \mathbb{T}^2 \times (-1, 1)$ thanks to Fourier multiplier theorem on the torus, e.g.Proposition Proposition 4.5 in [22] and [20].

Proposition 3.2.1 ([22]). *Let $1 < p < \infty$ and $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ satisfies the Mikhlin condition:*

$$[m] := \sup_{\alpha \in \{0,1\}^d} \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} \left| \xi^\alpha \partial_\xi^\alpha m(\xi) \right| < \infty. \tag{3.2.1}$$

Let $a_k = m(k)$ for $k \in \mathbb{Z}^d \setminus \{0\}$ and $a_0 \in \mathbb{C}$. Then there exits a constant $C = C(p,d) > 0$, for $f = \sum_{n \in \mathbb{Z}^d} \widehat{f}_n e^{in\cdot} \in L^p(\mathbb{T}^d)$, the Fourier multiplier operator of discrete type

$$T : f \mapsto \sum_{n \in \mathbb{Z}^d} a_n \hat{f}_n e^{in\cdot} \tag{3.2.2}$$

is bounded such that

$$\|Tf\|_{L^p(\mathbb{T}^d)} \leq C \max([m], a_0) \|f\|_{L^p(\mathbb{T}^d)}. \tag{3.2.3}$$

Let us consider the resolvent problem to (3.1.6) ;

$$\begin{cases} \lambda u - \Delta u + \nabla_\epsilon \pi &= f \quad \text{in} \quad \Omega, \\ \operatorname{div}_\epsilon u &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.2.4}$$

for $\lambda \in \Sigma_\theta$ $(0 < \theta < \pi/2)$ and $f \in L^q(\Omega)$. Let $H_\epsilon : L^p(\Omega) \to L^p_{\sigma,\epsilon}(\Omega) = \{u \in L^p(\Omega)\,;\, \mathrm{div}_\epsilon u = 0, u|_{x_3 = \pm 1} = 0\}$ $(1 < p < \infty)$ be the anisotropic Helmholtz projection on Ω, its L^p-boundedness is proved later. Let $A_\epsilon = H_\epsilon(-\Delta)$ be the Stokes operator with the domain $D(A_\epsilon) = L^p_{\sigma,\epsilon}(\Omega) \cap W^{2,p}(\Omega)$. For $a > 0$ and $0 < z < 1/2$, fractional power of A_ϵ is defined via a Dunford calculus

$$A_\epsilon^z = \frac{1}{2\pi i} \int_{\Gamma_\theta} (-\lambda)^z (\lambda + A_\epsilon)^{-1} \, d\lambda,$$

where $0 < \theta < \pi/2$ and $\Gamma_\epsilon = \mathbb{R}e^{i(-\pi+\theta)} \cup \mathbb{R}e^{i(\pi-\theta)}$. Our aim in this section is to prove

Lemma 3.2.2. *Let $1 < q < \infty$, $0 < a < 1/2$, $z \in \mathbb{C}$ satisfying $-a < \operatorname{Re} z < 0$ and $0 < \theta < \pi/2$. Then there exist a constant $C = C(q, a, \theta)$ such that*

$$||A_\epsilon^z||_{L^q(\Omega) \to L^q(\Omega)} \leq C e^{\theta |\operatorname{Im} z|}. \tag{3.2.5}$$

If the above lemma is proved, then we obtain maximal of the anisotropic Stokes operator via the formula

$$\left(\frac{d}{dt} + A_\epsilon \right)^{-1} = \int_{c+i\infty}^{c-i\infty} \frac{(d/dt)^z A_\epsilon^{1-z}}{\sin \pi z} dz \tag{3.2.6}$$

for $0 < c < 1$ and the Dore-Venni theory [7].

To show Lemma 3.2.2, we decompose the solution (u, π) for (3.2.4) into three parts;

$$u = R_0 v_1 - v_2 + \nabla_\epsilon \pi_3, \tag{3.2.7}$$
$$\nabla_\epsilon \pi = \nabla_\epsilon \pi_1 + \nabla_\epsilon \pi_2, \tag{3.2.8}$$

where v_j and π_j are solutions to

$$(I)\begin{cases} \lambda v_1 - \Delta v_1 + \nabla_\epsilon \pi_1 = E_0 f & \text{in } \mathbb{T}^2 \times \mathbb{R}, \\ \operatorname{div}_\epsilon v_1 = 0 & \text{in } \mathbb{T}^2 \times \mathbb{R}, \end{cases}$$

$$(II)\begin{cases} \lambda v_2 - \Delta v_2 + \nabla_\epsilon \pi_2 = 0 & \text{in } \Omega, \\ \operatorname{div}_\epsilon v_2 = 0 & \text{in } \Omega, \\ v_2 = \gamma v_1 - (\gamma v_1 \cdot \nu)\nu & \text{on } \partial\Omega, \end{cases}$$

and

$$(III)\begin{cases} \Delta_\epsilon \pi_3 = 0 & \text{in } \Omega, \\ \nabla_\epsilon \pi_3 \cdot \nu = (\gamma v_1 \cdot \nu)\nu & \text{on } \partial\Omega, \end{cases}$$

respectively, where $\gamma = \gamma_\pm$ is the trace operator to the upper and lower boundary, respectively, and ν is the unit outer normal. To show Lemma 3.2.2, we need to obtain

$$\begin{aligned} &\left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z R_0 v_1 \, d\lambda\right|\right|_{L^q(\Omega)} + \left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z v_2 \, d\lambda\right|\right|_{L^q(\Omega)} \\ &+ \left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z \nabla_\epsilon \pi_3 \, d\lambda\right|\right|_{L^q(\Omega)} \le Ce^{\theta|\operatorname{Im} z|}||f||_{L^q(\Omega)}. \end{aligned}$$

We focus on the effect of ϵ to the solution of above three equations. Throughout this section we frequently use partial Fourier transform to construct solutions and estimate these partial Fourier multipliers.

Proposition 3.2.3 ([1])**.** *Let $m'(\xi', z, \zeta)$ such that*

$$[m'(\cdot, z, \zeta)]_{\mathcal{M}'} \le C^\star \left(|z - a| + |\zeta - b|\right)^{-1} \tag{3.2.9}$$

for some $a, b \in \{-1, 1\}$. Then there exists a constant $C = C(q)$, for the partial Fourier multiplier operator given by

$$Mf = \mathcal{F}^{-1}_{\xi'} \int_{-1}^{1} m'(\xi', z, \zeta)\mathcal{F}_{x'} f(\xi', \zeta) \, d\zeta, \tag{3.2.10}$$

it holds that

$$||Mf||_{L^q} \le CC^\star ||f||_{L^q}. \tag{3.2.11}$$

Rescaled L^p-Fourier multipliers are also bounded L^p multiplier by the direct consequence of the Mikhlin theorem.

Proposition 3.2.4. *Let $m \in C^{d/2+1}(\mathbb{R}^d \setminus \{0\})$ be a L^p-Fourier multiplier with the Mikhlin constant $[m]_{\mathcal{M}} \le C$ for some $C > 0$. Then rescaled one $m_\epsilon(\xi) := m(\epsilon\xi)$ is also bounded from L^p into itself such that*

$$[m_\epsilon] \le C.$$

The above proposition is frequently used in this section to get ϵ-independent estimate for scaled multipliers. We show boundedness of some Fourier multiplier operators in advance. We set

$$s_\lambda = (\lambda + |\xi'|^2)^{1/2}$$

for $\xi' \in \mathbb{R}^2$.

Proposition 3.2.5.

- *Let $0 < \theta < \pi/2$, $\lambda \in \Sigma_\theta$, $t > 0$ and α be a positive integer. Then there exist constants $c > 0$ and $C > 0$ such that*

$$\left[|\xi'|^\alpha e^{-ts_\lambda}\right]_{\mathcal{M}'} \le C\frac{e^{-ct|\lambda|^{1/2}}}{t^\alpha}, \quad \left[\frac{e^{-s_\lambda}}{s_\lambda}\right]_{\mathcal{M}'} \le C\,|\lambda|^{-1/2}\,e^{-c|\lambda|^{1/2}}. \tag{3.2.12}$$

- *Let $-1 \le x_3 \le 1$. Then there exists a constant $C > 0$ which is independent of ϵ, such that*

$$\begin{aligned} \left[\frac{\sinh(\epsilon\,|\xi'|\,x_3)}{\sinh(\epsilon\,|\xi'|)}\frac{\epsilon\,|\xi'|}{1+\epsilon\,|\xi'|}\right]_{\mathcal{M}'} &\le C, \\ \left[\frac{\cosh(\epsilon\,|\xi'|\,x_3)}{\sinh(\epsilon\,|\xi'|)}\frac{\epsilon\,|\xi'|}{1+\epsilon\,|\xi'|}\right]_{\mathcal{M}'} &\le C. \end{aligned} \tag{3.2.13}$$

  for all $0 < \epsilon \le 1$.

- *Let $-1 \le x_3 \le 1$. Then there exists a constant $C > 0$, which is independent of ϵ, such that*

$$\left[\frac{\sinh(\epsilon\,|\xi'|\,x_3)}{\cosh(\epsilon\,|\xi'|)}\right]_{\mathcal{M}'} \le C, \quad \left[\frac{\sinh(\epsilon\,|\xi'|\,x_3)}{\cosh(\epsilon\,|\xi'|)}\right]_{\mathcal{M}'} \le C \tag{3.2.14}$$

  for all $0 < \epsilon \le 1$.

Proof. (**??**) is the direct consequence of the Mikhlin theorem. (3.2.12) is a direct consequence of Lemma3.5 in [1] and the Mikhlin theorem. The Mikhlin theorem implies there exit a constant $C > 0$ such that

$$\left[|\xi'|^\alpha e^{-t|\xi'|}\right]_{\mathcal{M}'} \le \frac{C}{t^\alpha}, \quad \left[\frac{1}{\lambda + (1-\epsilon^2)\,|\xi'|^2}\right]_{\mathcal{M}'} \le \frac{C}{\lambda} \tag{3.2.15}$$

for all $t \in \mathbb{R}$ and $\alpha > 0$. By definition of sinh and cosh, we find the formula

$$\frac{\sinh(\epsilon\,|\xi'|\,x_3)}{\sinh(\epsilon\,|\xi'|)} = \frac{e^{\epsilon|\xi'|x_3} - e^{-\epsilon|\xi'|x_3}}{e^{\epsilon|\xi'|} - e^{-\epsilon|\xi'|}} = \frac{e^{-\epsilon|\xi'|(x_3-1)}}{1-e^{-2\epsilon|\xi'|}} - \frac{e^{-\epsilon|\xi'|(x_3+1)}}{1-e^{-2\epsilon|\xi'|}} \tag{3.2.16}$$

and

$$\frac{\sinh(\epsilon\,|\xi'|\,x_3)}{\cosh(\epsilon\,|\xi'|)} = \frac{e^{-\epsilon|\xi'|(x_3-1)}}{1-e^{-2\epsilon|\xi'|}} + \frac{e^{-\epsilon|\xi'|(x_3+1)}}{1-e^{-2\epsilon|\xi'|}} \tag{3.2.17}$$

Thus, multiplying $\frac{|\xi'|}{1+|\xi'|}$ by both sides of (3.2.16), we obtain

$$\begin{aligned}
&\left[\frac{\sinh(\epsilon\,|\xi'|\,x_3)}{\sinh(\epsilon\,|\xi'|)}\frac{\epsilon\,|\xi'|}{1+\epsilon\,|\xi'|}\right]_{\mathcal{M}'} \\
&\le C\left[e^{-\epsilon|\xi'|(x_3-1)}\frac{\epsilon\,|\xi'|}{(1-e^{-2\epsilon|\xi'|})}\frac{1}{(1+\epsilon\,|\xi'|)}\right]_{\mathcal{M}'} \\
&+\left[e^{-\epsilon|\xi'|(x_3+1)}\frac{\epsilon\,|\xi'|}{(1-e^{-2\epsilon|\xi'|})}\frac{1}{(1+\epsilon\,|\xi'|)}\right]_{\mathcal{M}'} \\
&\le C.
\end{aligned} \tag{3.2.18}$$

Second inequality of (3.2.13) is proved by the same as above using (3.2.17). By definition of sinh and cosh, the Mikhlin theorem implies (3.2.14). □

3.2.2 Estimate for v_1

Let us consider the equations (I). For $a \in \mathbb{C}$ we denote by $\tau_a f = f(a\cdot)$ the rescaling operator by a. The anisotropic Helmholtz projection $\mathbb{P}_\epsilon^{\mathbb{R}^3}$ on $\mathbb{R}^3$ with symbols

$$\mathcal{F}\mathbb{P}_\epsilon^{\mathbb{R}^3} = I_3 - \xi_\epsilon \otimes \xi_\epsilon, \quad \xi_\epsilon = \left(\xi_1, \xi_2, \frac{\xi_3}{\epsilon}\right) \in \mathbb{R}^3,$$

is bounded in $L^q(\mathbb{R}^3)$ by boundedness of the Riesz operator and the formula

$$\mathcal{F}_\xi^{-1} m(a\xi)\mathcal{F}_x f = \tau_{a^{-1}}\left[\mathcal{F}_\xi^{-1} m(\xi)\mathcal{F}_x \tau_a f\right], \tag{3.2.19}$$

Actually apply (3.2.19) with respect to third variable, then, the symbol is nolonger independent of ϵ. For $a \in \mathbb{C}$, we denote by τ_a^3 the rescaling operator with respect to third variable. Changing the variable with respect to and using boundness of the Riesz operator, we find

$$\|\mathbb{P}_\epsilon^{\mathbb{R}^3} f\|_{L^p(\mathbb{R}^3)} = \epsilon^{-1}\|\mathbb{P}_1\left[\tau_{1/\epsilon}^3 f\right]\|_{L^p(\mathbb{R}^3)} \le C\|f\|_{L^p(\mathbb{R}^3)},$$

where τ_a^3 is the rescaled operator with respect to third variable for $a > 0$. We define $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$ be the anisotropic Helmholtz projection on $\mathbb{T}^2 \times \mathbb{R}$ with symbols

$$\mathcal{F}_{x_3}\mathcal{F}_{d,x'}\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} = I_3 - \begin{pmatrix} n_1 \\ n_2 \\ \xi_3/\epsilon \end{pmatrix} \otimes \begin{pmatrix} n_1 \\ n_2 \\ \xi_3/\epsilon \end{pmatrix}, \quad n_1, n_2 \in \mathbb{Z},\ \xi_3 \in \mathbb{R}.$$

We find $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$ is bounded from $L^q(\mathbb{T}^2 \times \mathbb{R})$ into itself by boundedness of $\mathbb{P}_\epsilon$ and Proposition 3.2.1.

Proposition 3.2.6. *Let $1 < q < \infty$, $0 < a < 1/2$, $z \in \mathbb{C}$ satisfying $-a < \mathrm{Re} z < 0$ and $0 < \theta < \pi/2$. Then there exists a constant $C = C(q, a, \theta)$ such that*

$$\begin{aligned} &\left\| \frac{1}{2\pi i} R_0 \int_{\Gamma_\theta} (-\lambda)^z (\lambda - \Delta_{\mathbb{T}^2\times\mathbb{R}})^{-1} \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} E_0 f \; d\lambda \right\|_{L^q(\mathbb{T}^2\times\mathbb{R})} \\ &\leq C e^{\theta|\mathrm{Im}\, z|} ||f||_{L^q(\Omega)} \end{aligned} \tag{3.2.20}$$

for all $f \in L^q(\Omega)$.

Proof. It is known that the Laplace operator on a cylinder $\mathbb{T}^2 \times \mathbb{R}$ has BIP. Combining with this fact and L^q-boundedness of $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$, we have (3.2.20). □

Let us calculate the partial Fourier transform for v_1 with respect to the horizontal variable, which is needed to obtain representation formula for v_2 later. We temporary consider on $\mathbb{R}^2 \times \mathbb{R}$. Let $g \in L^q(\mathbb{R}^2 \times (-1, 1))$. The solution to the equation

$$\begin{cases} \lambda\tilde{v} - \Delta\tilde{v} + \nabla_\epsilon\tilde{\pi} &= g \quad \text{in} \quad \mathbb{R}^3, \\ \mathrm{div}_\epsilon\, \tilde{v} &= 0 \quad \text{in} \quad \mathbb{R}^3, \end{cases}$$

is given by

$$\tilde{v} = (\lambda - \Delta_{\mathbb{R}^3})^{-1} \mathbb{P}_\epsilon^{\mathbb{R}^3} E_0 f.$$

Moreover,

$$\begin{aligned} &(\lambda - \Delta_{\mathbb{R}^3})^{-1} \mathbb{P}_\epsilon^{\mathbb{R}^3} g \qquad (3.2.21) \\ &= \mathcal{F}^{-1} (\lambda + |\xi|^2)^{-1} \left(I_3 - \frac{\xi_\epsilon \otimes \xi_\epsilon}{|\xi_\epsilon|^2} \right) \mathcal{F} g \\ &= \mathcal{F}_{\xi'}^{-1} \int_{\mathbb{R}} k_{\lambda,\epsilon}(\xi', x_3 - \zeta) \mathcal{F}_{x'} g(\xi', \zeta) \; d\zeta, \qquad (3.2.22) \end{aligned}$$

where

$$\begin{aligned} &k'_{\lambda,\epsilon}(\xi', x_3) \\ &= \mathcal{F}_{\xi_3}^{-1} \left[(\lambda + |\xi|^2)^{-1} \left(I_3 - \frac{\xi_\epsilon \otimes \xi_\epsilon}{|\xi_\epsilon|^2} \right) \right] \\ &= \frac{e^{-s_\lambda}}{2s_\lambda} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\ &- \begin{pmatrix} \xi' \otimes \xi' \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|\xi'|^2} \frac{-\epsilon|\xi|' e^{-|x_3|s_\lambda} + s_\lambda e^{-|x_3|\epsilon|\xi'|}}{2s_\lambda \epsilon|\xi'|} & -i\xi' \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|\xi'|^2} \frac{e^{-|x_3|s_\lambda} - e^{-|x_3|\epsilon|\xi'|}}{2} \\ -i\xi'^T \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|\xi'|^2} \frac{e^{-|x_3|s_\lambda} - e^{-|x_3|\epsilon|\xi'|}}{2} & -|\xi'|^2 \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|\xi'|^2} \frac{-\epsilon|\xi|' e^{-|x_3|s_\lambda} + s_\lambda e^{-|x_3|\epsilon|\xi'|}}{2s_\lambda \epsilon|\xi'|} \end{pmatrix} \\ &=: \frac{e^{-s_\lambda}}{2s_\lambda} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \xi' \otimes \xi' \eta'_{\lambda,\epsilon}(\xi', x_3) & -i\xi' \partial_3 \eta'_{\lambda,\epsilon}(\xi', x_3) \\ -i\xi'^T \partial_3 \eta'_{\lambda,\epsilon}(\xi', x_3) & -|\xi'|^2 \eta'_{\lambda,\epsilon}(\xi', x_3) \end{pmatrix}. \qquad (3.2.23) \end{aligned}$$

$k_{\lambda,\epsilon}(\xi', x_3)$ is calculated by the residue theorem. Actually, since poles of $\left(\lambda + |\xi|^2\right)^{-1}$ are $\xi_3 = \pm i s_\lambda$, the residue theorem implies the partial Fourier inverse transform of

$\left(\lambda+|\xi|^2\right)^{-1}$ with respect to ξ_3 is given by inserting $\xi_3 = is_\lambda$ or $-is_\lambda$ into $e^{ix_3\xi_3}$ so that the real part become to be negative. Thus, we have

$$e'_\lambda(\xi', x_3) := \mathcal{F}_{\xi_3}^{-1}\left(\lambda+|\xi|^2\right)^{-1} = \frac{e^{-|x_3|s_\lambda}}{s_\lambda} \tag{3.2.24}$$

Moreover, this formula leads

$$\mathcal{F}_{\xi_3}^{-1}\left[|\xi_\epsilon|^2\right]^{-1} = \mathcal{F}_{\xi_3}^{-1}\left[\frac{\epsilon^2}{\epsilon^2|\xi'|^2+\xi_3^2}\right] = \frac{\epsilon e^{-|x_3|\epsilon|\xi'|}}{|\xi'|}.$$

Combining with the above two calculations and the formula

$$I_3 - \frac{\xi_\epsilon\otimes\xi_\epsilon}{|\xi_\epsilon|^2} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{\xi'\otimes\xi'}{|\xi'_\epsilon|^2} & \frac{\xi_3\xi'/\epsilon}{|\xi'_\epsilon|^2} \\ \frac{\xi_3\xi'^T/\epsilon}{|\xi'_\epsilon|^2} & -\frac{|\xi'_\epsilon|^2}{|\xi'_\epsilon|^2} \end{pmatrix},$$

we have $k_{\lambda,\epsilon}$. Thus, the solution v_1 to (I) with external force $g \in L^q(\mathbb{T}^2\times\mathbb{R})$ is given by

$$v_1 = K_{\lambda,\epsilon}g := \mathcal{F}_{d,n'}^{-1}\int_{\mathbb{R}} k_{\lambda,\epsilon}(n', x_3-\zeta)\mathcal{F}_{d,x'}g(n',\zeta)\, d\zeta, \tag{3.2.25}$$

3.2.3 Boundedness of the anisotropic Helmholtz projection

Next, we consider the equation (III) with boundary data $\phi = (\phi_+, \phi_-)$. Applying the partial Fourier transform to (III), we have

$$\begin{cases} \left(\frac{\partial_z^2}{\epsilon^2} - |n'|^2\right)\mathcal{F}_{d,x'}\pi_3(n', x_3) &= 0, \\ \frac{\partial_z}{\epsilon}\mathcal{F}_{d,x'}\pi_3(n', \pm 1) &= \mathcal{F}_{d,x'}\phi_\pm(n'). \end{cases} \tag{3.2.26}$$

for $n' \in \mathbb{Z}^2$ and $x_3 \in (-1, 1)$. The solution to (3.2.26) is of the form

$$\mathcal{F}_{d,x'}\pi_3(n', x_3) = C_1 e^{\epsilon x_3|n'|} + C_2 e^{-\epsilon x_3|n'|}$$

for some constant C_1 and C_2. Take the constants so that (3.2.26) satisfied, namely

$$C_1 = \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{4|n'|\cosh(\epsilon|n'|)} + \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{4|n'|\sinh(\epsilon|n'|)},$$
$$C_2 = -\frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{4|\xi'|\cosh(\epsilon|n'|)} + \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{4|\xi'|\sinh(\epsilon|n'|)},$$

then the solution to (3.2.26) is given by

$$\begin{aligned} &\pi_3(x', x_3) \\ &= \mathcal{F}_{d,n'}^{-1}\left(\frac{\sinh(\epsilon x_3|n'|)}{|n'|\cosh(\epsilon|n'|)}\frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{x'}\phi_-}{2} + \frac{\cosh(\epsilon x_3|n'|)}{|n'|\sinh(\epsilon|n'|)}\frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2}\right). \end{aligned}$$

Moreover, its anisotropic gradient given by

$$\nabla_\epsilon \pi_3 = \mathcal{F}^{-1}_{d,n'} \begin{pmatrix} \frac{in'\sinh(\epsilon x_3|n'|)}{|n'|\cosh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{2} + \frac{in'\cosh(\epsilon x_3|n'|)}{|n'|\sinh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2} \\ \frac{\cosh(\epsilon x_3|n'|)}{\cosh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{2} + \frac{\sinh(\epsilon x_3|n'|)}{\sinh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2} \end{pmatrix}$$
$$=: \mathcal{F}^{-1}_{d,n'}\alpha_{\epsilon,+}(n',x_3)\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}^{-1}_{d,n'}\alpha_{\epsilon,-}(n',x_3)\mathcal{F}_{d,x'}\phi_-. \tag{3.2.27}$$

We take trace to (3.2.27) to get

$$\gamma_\pm \nabla_\epsilon \pi_3 = \mathcal{F}^{-1}_{d,n'} \begin{pmatrix} \frac{\pm in'\sinh(\epsilon|n'|)}{|n'|\cosh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{2} + \frac{in'\cosh(\epsilon|n'|)}{|n'|\sinh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2} \\ \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{2} \pm \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2} \end{pmatrix}.$$

We take $\phi_\pm = \gamma_\pm \mathbb{P}_\epsilon^{\mathbb{T}^2\times(-1,1)} E_0 f$ for $f \in L^q(\Omega)$ and set

$$\Pi_\epsilon f := \mathcal{F}^{-1}_{d,n'}\left[\alpha_{\epsilon,+}(n',x_3)\gamma_+\mathcal{F}_{d,x'}\left(e_3 \cdot \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R})} E_0 f\right)\right]$$
$$+ \mathcal{F}^{-1}_{d,n'}\left[\alpha_{\epsilon,-}(n',x_3)\gamma_-\mathcal{F}_{d,x'}\left(e_3 \cdot \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} E_0 f\right)\right].$$

Lemma 3.2.7. *Let $1 < q < \infty$ and $0 < \epsilon \le 1$. Then there exist a constant $C = C(q)$ such that*

$$||\Pi_\epsilon f||_{L^q(\Omega)} \le C||f||_{L^q(\Omega)},$$

for all $f \in L^q(\Omega)$.

Proof. We seek the multiplier of Π_ϵ by a direct calculation. Recall that the symbol of $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$ is of the form

$$\mathcal{F}_{x_3}\mathcal{F}_{d,x'}\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{n'\otimes n'}{|n'_\epsilon|^2} & \frac{n'\xi_3/\epsilon}{|n'_\epsilon|^2} \\ \frac{n'^T\xi_3/\epsilon}{\epsilon|n'_\epsilon|^2} & -\frac{|n'|^2}{|n'_\epsilon|^2} \end{pmatrix}. \tag{3.2.28}$$

Since the symbol of $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$ have poles at $\xi_3 = \pm i\epsilon\,|n'|$, we apply $e_3\cdot$ to (3.2.28) by the left hand side and use the residue theorem so that the power of e is negative to get

$$\mathcal{F}_{d,x'}(e_3 \cdot \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} E_0 f) = -\int_{-1}^{1} \frac{ie^{-|x_3-\zeta|\epsilon|n'|}}{2}\epsilon n' \cdot \mathcal{F}_{d,x'}f'(n',\zeta)\,d\zeta$$
$$+ \int_{-1}^{1} \frac{e^{-|x_3-\zeta|\epsilon|n'|}}{2}\epsilon\,|n'|\,\mathcal{F}_{d,x'}f_3(n',\zeta)\,d\zeta. \tag{3.2.29}$$

Note that the integration is due to the relationship between the Fourier transform and convolution. Applying trace operators $\gamma_\pm$ and $\alpha_{\epsilon,\pm}(n',x_3)$, respectively, and taking Fourier inverse transform with respect to n', we find

$$\Pi_\epsilon f(x',x_3)$$
$$= -\mathcal{F}^{-1}_{d,n'}\int_{-1}^{1}\alpha_{\epsilon,+}(n',x_3)\frac{ie^{-|1-\zeta|\epsilon|n\prime|}}{2}\epsilon n' \cdot \mathcal{F}_{x'}f'(n',\zeta)\,d\zeta$$

$$
\begin{aligned}
&+ \mathcal{F}^{-1}_{d,n'} \int_{-1}^{1} \alpha_{\epsilon,+}(n',x_3) \frac{e^{-|1-\zeta|\epsilon|n'|}}{2} \epsilon\,|n'|\, \mathcal{F}_{d,x'} f_3(n',\zeta)\ d\zeta \\
&- \mathcal{F}^{-1}_{d,n'} \int_{-1}^{1} \alpha_{\epsilon,-}(n',x_3) \frac{ie^{-|-1-\zeta|\epsilon|n'|}}{2} \epsilon n' \cdot \mathcal{F}_{d,x'} f'(n',\zeta)\ d\zeta \\
&+ \mathcal{F}^{-1}_{d,n'} \int_{-1}^{1} \alpha_{\epsilon,-}(n',x_3) \frac{e^{-|-1-\zeta|\epsilon|n'|}}{2} \epsilon\,|n'|\, \mathcal{F}_{d,x'} f_3(n',\zeta)\ d\zeta \\
&=: I_1 + I_2 + I_3 + I_4. \qquad (3.2.30)
\end{aligned}
$$

and, by the definition of $\alpha_{\pm,\epsilon}$,

$$
I_1 = -\mathcal{F}^{-1}_{d,n'} \int_{-1}^{1} \frac{1}{2} \left(\frac{\frac{in' \sinh(x_3|n'|)}{|n'|\cosh(\epsilon|n'|)} + \frac{in' \cosh(\epsilon x_3|n'|)}{|n'|\sinh(\epsilon|n'|)}}{\frac{\cosh(\epsilon x_3|n'|)}{\cosh(\epsilon|n'|)} + \frac{\sinh(\epsilon x_3|n'|)}{\sinh(\epsilon|n'|)}} \right) \frac{ie^{-|1-\zeta|\epsilon|n'|}}{2} \epsilon n' \cdot \mathcal{F}_{d,x'} f'(n',\zeta)\ d\zeta.
$$

Symbols in the integral can be written by $A(\epsilon n')(1+\epsilon\,|n'|)e^{-\epsilon|n'|(|x_3\pm1|+|\zeta\pm1|)}$ for a symbol A with a ϵ-independent Mikhlin constant by Propositions 3.2.13 and 3.2.14. The same argument is valid for I_j $(j = 2,3,4)$. Thus, we find from Propositions 3.2.1, 3.2.3,3.2.4 and 3.2.5 that

$$
\begin{aligned}
||\Pi_\epsilon f||_{L^q(\Omega)} &\le C||f||_{L^p(\Omega)} + C\left|\left| \int_{-1}^{1} \frac{||f(\cdot,\zeta)||_{L^q(\mathbb{T}^2)}}{|x_3 \pm 1| + |\zeta \pm 1|}\ d\zeta \right|\right|_{L^q(-1,1)} \\
&\le C||f||_{L^q(\Omega)},
\end{aligned}
$$

for all $f \in L^q(\Omega)$, where the constant C is independent of ϵ. □

Let $P_{N,\epsilon} := R_0\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon E_0 - \Pi_\epsilon\right)$. Then lemma 3.2.7 implies $P_{N,\epsilon}$ is bounded from $L^p(\Omega)$ into itself.

Corollary 3.2.8. *Let $1 < q < \infty$ and $0 < \epsilon \le 1$. Then there exist a constant $C > 0$, which is independent of ϵ, such that*

$$
||P_{N,\epsilon} f||_{L^p(\Omega)} \le C||f||_{L^p(\Omega)}, \qquad (3.2.31)
$$

for all $f \in L^p(\Omega)$.

Remark 3.2.9. $P_{N,\epsilon}$ is not the anisotropic Helmholtz projection on Ω. $P_{N,\epsilon}$ is the operator which maps from the $L^p(\Omega)$-vector fields into $L^p(\Omega)$-divergence-free vector fields with tangential trace. We find that the anisotropic Helmholtz projection is bounded from $L^p(\Omega)$ into itself by the same method of Lemma 3.2.7. Let $u \in L^p(\Omega)$. Then, we obtain the solution π_ϵ to the Neumann problem

$$
\left\{
\begin{array}{lll}
\Delta_\epsilon \pi_\epsilon &= \mathrm{div}_\epsilon\, u & \text{in} \quad \Omega, \\
\gamma_\pm \frac{\partial_3 \pi_\epsilon}{\epsilon} &= u \cdot \nu_\pm & \text{on} \quad \partial\Omega.
\end{array}
\right. \qquad (3.2.32)
$$

The anisotropic Helmholtz projection H_ϵ is defined by

$$
H_\epsilon u = u - \nabla_\epsilon \pi_\epsilon.
$$

In the case of the Dirichlet boundary condition, i.e. $\gamma_\pm u = 0$, the right hand side of the second equality of (3.2.32) is zero. Let us consider the L^p-boundedness of $\nabla_\epsilon \pi_\epsilon$, which implies the boundedness of the anisotropic Helmholtz projection. Let π_ϵ^1 and π_ϵ^2 be the solutions to

$$\Delta_\epsilon \pi_\epsilon^1 = E_0 \mathrm{div}_\epsilon u \quad \text{in} \quad \mathbb{T}^2 \times \mathbb{R}, \tag{3.2.33}$$

and

$$\begin{cases} \Delta_\epsilon \pi_\epsilon^2 & = 0 & \text{in} \quad \Omega \\ \gamma_\pm \frac{\partial_3 \pi_\epsilon}{\epsilon} & = -\gamma_\pm \nu_\pm \cdot \nabla_\epsilon \Delta_\epsilon^{-1} E_0 \mathrm{div}_\epsilon\, u & \text{on} \quad \partial\Omega, \end{cases}$$

respectively. Then, we find

$$\pi_\epsilon = R_0 \pi_\epsilon^1 + \pi_\epsilon^2.$$

Let us first consider (3.2.33). It follows from integration by parts

$$\begin{aligned} \mathcal{F}_{x_3} \mathcal{F}_{d,x'} E_0 \mathrm{div}_\epsilon u &= \mathcal{F}_{d,x'} \int_{-1}^{1} e^{-ix_3\xi_3} \left(\mathrm{div}_H u'(\cdot, x_3) + \frac{\partial_3 u_3(\cdot, x_3)}{\epsilon} \right) dx_3 \\ &= i \begin{pmatrix} n' & \xi_3/\epsilon \end{pmatrix}^T \cdot \mathcal{F}_{d,x'}(E_0 u) \\ &= \mathcal{F}_{x_3} \mathcal{F}_{d,x'} \mathrm{div}_\epsilon (E_0 u). \end{aligned}$$

This formula, the Mikhlin theorem and Proposition 3.2.1 imply

$$\left|\left|\nabla_\epsilon \pi_\epsilon^1\right|\right|_{L^p(\Omega)} \leq C ||u||_{L^p(\Omega)}, \tag{3.2.34}$$

where $C > 0$ is independent of ϵ. Moreover, since $e_3 \cdot \nabla_\epsilon \Delta_\epsilon^{-1} \mathrm{div}_\epsilon$ is given by the left hand side of (3.2.29), we can use the same method as Lemma 3.2.7 to estimate

$$\left|\left|\nabla_\epsilon \pi_\epsilon^2\right|\right|_{L^p(\Omega)} \leq C ||u||_{L^p(\Omega)}, \tag{3.2.35}$$

where $C > 0$ is also independent of ϵ. (3.2.34) and (3.2.35) implies L^p-boundedness of the anisotropic Helmholtz projection on Ω.

Proposition 3.2.10. *Let $0 < \epsilon \leq 1$, $1 < q < \infty$, $0 < a < 1/2$, $z \in \mathbb{C}$ satisfying $-a < \mathrm{Re} z < 0$ and $0 < \theta < \pi/2$. Then there exists a constant $C = C(q, a, \theta)$, for each $f \in L^q(\Omega)$, the solution π_3 to (III) with boundary data $(\gamma K_{\lambda,\epsilon} f \cdot \nu)\nu$ satisfies*

$$\left|\left| \frac{1}{2\pi i} \int_{\Gamma_\theta} (-\lambda)^z \nabla_\epsilon \pi_3 \, d\lambda \right|\right|_{L^q(\Omega)} \leq C e^{\theta |\mathrm{Im}\, z|} ||f||_{L^q(\Omega)},$$

where $C > 0$ is independent of ϵ.

Proof. Since

$$\nabla_\epsilon \pi_3 = \Pi_\epsilon K_{\lambda,\epsilon} E_0 f \tag{3.2.36}$$

and the Cauchy integral commutes with Π_ϵ, combining with Proposition 3.2.6 and Lemma 3.2.7, we obtain the conclusion. □

3.2.4 Estimate for v_2

Let us consider the equation (II) with tangential boundary data $g = (g_+, g_-)$. Set

$$y'_{\lambda,\epsilon}(n') = 2s_\lambda \left(I_2 + \frac{\epsilon\,|n'|}{s_\lambda} \frac{n' \otimes n'}{|n'|^2} \right)$$

and

$$y_{\lambda,\epsilon}(n') = \begin{pmatrix} y'_{\lambda,\epsilon}(n') & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.2.37}$$

Then, $y_{\lambda,\epsilon}$ satisfies

$$k'_{\lambda,\epsilon}(n', 0) y_{\lambda,\epsilon}(n') = J_2 := \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.2.38}$$

Let us define a multiplier operator $L_{\lambda,\epsilon}$ as

$$\begin{aligned} L_{\lambda,\epsilon} g(n', x_3) = {} & \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} \mathcal{F}^{-1}_{d,n'} \left[e'_\lambda(n', 1 - x_3) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_+(n') \right] \\ & + \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} \mathcal{F}^{-1}_{d,n'} \left[e'_\lambda(n', -1 - x_3) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_-(n') \right], \end{aligned} \tag{3.2.39}$$

where e'_λ is defined by (3.2.24). Let $p'_{\lambda,\epsilon}(n', x_3)$ be a partial Fourier transform of the symbol of $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$ with respect to ξ_3 for $0 \le x_3 \le 1$. Then

$$\begin{aligned} L_{\lambda,\epsilon} g(n', \cdot) = {} & \mathcal{F}^{-1}_{d,n'} \left[p'_\epsilon(n', \cdot) *_3 e'_\lambda(n', 1 - \cdot) y'_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_+(n') \right] \\ & + \mathcal{F}^{-1}_{d,n'} \left[p'_\epsilon(n', \cdot) *_3 e'_\lambda(n', -1 - \cdot) y'_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_-(n') \right], \end{aligned} \tag{3.2.40}$$

where $\cdot *_3 \cdot$ is convolution with respect to x_3. We set

$$W_{\lambda,\epsilon} = P_{N,\epsilon} L_{\lambda,\epsilon}. \tag{3.2.41}$$

Then, $W_{\lambda,\epsilon} g$ is a solution to (II) with boundary data $\gamma W_{\lambda,\epsilon} g$.

We first get a Fourier multiplier of $\gamma W_{\lambda,\epsilon}$. Next, we show the map $S_{\lambda,\epsilon} : g \mapsto \gamma W_{\lambda,\epsilon} g$ has bounded inverse for large λ. Put

$$V_{\lambda,\epsilon} g = W_{\lambda,\epsilon} S^{-1}_{\lambda,\epsilon} g, \tag{3.2.42}$$

then, $V_{\lambda,\epsilon} g$ gives the solution to (II) with boundary data g.

Proposition 3.2.11. *Let $0 < \epsilon < 1$, $0 < \theta < \pi/2$, $1 < q < \infty$ and $\lambda \in \Sigma_\theta$. Let $r > 0$ be sufficiently large, which is independent of ϵ. Then, for $|\lambda| > r$, there exists a bounded operator $R_{\lambda,\epsilon}$ from $L^q(\Omega)$ into itself satisfying*

$$||R_{\lambda,\epsilon}||_{L^q(\Omega)\to L^q(\Omega)} \le \frac{C}{|\lambda|^{1/2}}, \tag{3.2.43}$$

where $C > 0$ is independent of ϵ, such that

$$-S^{-1}_{\lambda,\epsilon} = I + R_{\lambda,\epsilon}. \tag{3.2.44}$$

Proof. Let $g \in L^q(\mathbb{T}^2)$. Since e'_λ is an even function with respect to x_3, we find from a change of variable that

$$\begin{aligned} p'_\epsilon(n',\cdot) *_3 e'_\lambda(n',1-\cdot) &= \int_{\mathbb{R}} p'_\epsilon(n',\cdot-\zeta)e'_\lambda(n',1-\zeta)\; d\zeta \\ &= -\int_{\mathbb{R}} p'_\epsilon(n',\eta)e'_\lambda(n',-1+\cdot-\eta)\; d\eta \\ &= -k'_{\lambda,\epsilon}(n',-1+\cdot), \end{aligned}$$

and similarly

$$p'_\epsilon(n',\cdot) *_3 e'_\lambda(n',-1-\cdot) = -k'_{\lambda,\epsilon}(n',1+\cdot)$$

Thus, we find from (3.2.40) that

$$\begin{aligned} L_{\lambda,\epsilon}g(n',\cdot) = \mathcal{F}^{-1}_{d,n'}&\left[-k'_{\lambda,\epsilon}(\xi',-1+\cdot)y'_{\lambda,\epsilon}(\xi')\mathcal{F}_{d,x'}g_+(\xi')\right] \\ &+ \mathcal{F}^{-1}_{d,\xi'}\left[-k'_{\lambda,\epsilon}(\xi',1+\cdot)y'_{\lambda,\epsilon}(\xi')\mathcal{F}_{d,x'}g_-(\xi')\right], \end{aligned} \tag{3.2.45}$$

We apply $P_{N,\epsilon}$ to (3.2.45) to get

$$\begin{aligned} S_{\lambda,\epsilon}g &= \gamma_\pm W_{\lambda,\epsilon}g \\ &= -\mathcal{F}^{-1}_{d,n'}\left[k'_{\lambda,\epsilon}(n',-1\pm 1)y_{\lambda,\epsilon}(\xi')\mathcal{F}_{d,x'}g_+(n')\right] \\ &- \mathcal{F}^{-1}_{d,n'}\left[\alpha_{+,\epsilon}(n',\pm 1)e_3\cdot k'_{\lambda,\epsilon}(n',2)y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'}g_-(n')\right] \\ &- \mathcal{F}^{-1}_{d,n'}\left[k'_{\lambda,\epsilon}(n',1\pm 1)y_{\lambda,\epsilon}(\xi')\mathcal{F}_{d,x'}g_-(n')\right] \\ &- \mathcal{F}^{-1}_{d,n'}\left[\alpha_{-,\epsilon}(n',\pm 1)e_3\cdot k'_{\lambda,\epsilon}(n',-2)y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'}g_+(n')\right] \\ &= I_1+I_2+I_3+I_4. \end{aligned} \tag{3.2.46}$$

Let us estimate I_1 and I_3. (3.2.38) implies

$$\mathcal{F}^{-1}_{d,n'}\left[k'_{\lambda,\epsilon}(n',0)y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'}g_\pm(n')\right] = g_\pm. \tag{3.2.47}$$

We need to show the other terms are $O(1/|\lambda|^{1/2})$. By (3.2.23) and (3.2.37), we have

$$\begin{aligned} &k'_{\lambda,\epsilon}(\xi',\pm 2)y_{\lambda,\epsilon}(n') \\ &= e^{-s_\lambda}\left(J_2+\frac{\epsilon|n'|}{s_\lambda}\frac{J_2n\otimes J_2n}{|n'|^2}\right) \\ &- J_2n\otimes J_2n\frac{\epsilon^2}{\lambda+(1-\epsilon^2)|n'|^2}e^{-2s_\lambda}\left(J_2+\frac{\epsilon|n'|}{s_\lambda}\frac{J_2n\otimes J_2n}{|n'|^2}\right) \\ &- J_2n\otimes J_2n\frac{\epsilon^2}{\lambda+(1-\epsilon^2)|n'|^2}\frac{e^{-2\epsilon|n'|}}{\epsilon|n'|}s_\lambda\left(J_2+\frac{\epsilon|n'|}{s_\lambda}\frac{J_2n\otimes J_2n}{|n'|^2}\right) \\ &=: II_1+II_2+II_3. \end{aligned}$$

We find from (3.2.12) and (3.2.13) in Proposition 3.2.5 and

$$\left[\frac{1}{\lambda+(1-\epsilon^2)|\xi'|^2}\right]_{\mathcal{M}'} \le \frac{C}{|\lambda|}, \quad \left[\frac{|\xi'|}{s_\lambda}\right]_{\mathcal{M}'} + \left[\frac{J_2\xi\otimes J_2\xi}{|\xi'|^2}\right]_{\mathcal{M}'} \le C \tag{3.2.48}$$

that

$$[II_1]_{\mathcal{M}'} \le Ce^{-c|\lambda|^{\frac{1}{2}}} \tag{3.2.49}$$

and

$$[II_2]_{\mathcal{M}'} \le \frac{Ce^{-c|\lambda|^{1/2}}}{|\lambda|} \tag{3.2.50}$$

for large λ, where constants $c, C > 0$ are independent of ϵ. Note that I_3 have little bit problem near $\epsilon = 0$ since, at this point, we can not use decay of $e^{-2\epsilon|\xi'|}$ to obtain uniform boundedness of the Mikhlin constant. However, we can use decay of $1/(\lambda + (1-\epsilon^2)|\xi'|^2)$ around $\epsilon = 0$. On the other hand, when ϵ is away from 0, we have no problem. Thus, combining with these observation, we can conclude by the Mikhlin theorem that

$$[II_3]_{\mathcal{M}'} \le \frac{C}{|\lambda|^{\frac{1}{2}}}, \tag{3.2.51}$$

where $C > 0$ is independent of ϵ. Thus we find from (3.2.49), (3.2.50) and (3.2.50) that

$$\left|\left|\mathcal{F}^{-1}_{d,n'}\left[k'_{\lambda,\epsilon}(n', \pm 2)y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'}g(n')\right]\right|\right| \le \frac{C}{|\lambda|^{\frac{1}{2}}}||g||_{L^p(\Omega)}. \tag{3.2.52}$$

Next, we estimate I_2 and I_4. It follows from (3.2.23) that

$$\begin{aligned}
&e_3 \cdot k'_{\lambda,\epsilon}(n', \pm 2)y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'}g_{\mp}(n') \\
&= e_3 \cdot \left[\frac{e^{-s_\lambda}}{2s_\lambda}y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'}g_{\mp} - \begin{pmatrix} \eta'_{\lambda,\epsilon}(n', \pm 2)n' \otimes n' y'_{\lambda,\epsilon}(n') & 0 \\ -\partial_3\eta'_{\lambda,\epsilon}(n', \pm 2)in'^T y'_{\lambda,\epsilon}(n') & 0 \end{pmatrix}\mathcal{F}_{d,x'}g_{\mp}\right] \\
&= -\begin{pmatrix} -\partial_3\eta'_{\lambda,\epsilon}(n', \pm 2)in'^T y'_{\lambda,\epsilon}(n') & 0 \end{pmatrix}\mathcal{F}_{d,x'}g_{\mp}.
\end{aligned}$$

Recall $\partial_3\eta_{\lambda,\epsilon}(n', \pm 2) = \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|n'|^2}\frac{e^{-2s_\lambda} - e^{-2\epsilon|n'|}}{2}$. Then, we find from (3.2.12) in Proposition 3.2.5 and (3.2.48) that

$$\begin{aligned}
&[(1+\epsilon|\xi'|)\,\partial_3\eta_{\lambda,\epsilon}(\xi', \pm 2)y'_\lambda(\xi')]_{\mathcal{M}'} \\
&= \left[(1+\epsilon|\xi'|)\frac{\epsilon^2}{\lambda + (1-\epsilon^2)|\xi'|^2}\left(e^{-2s_\lambda} - e^{-2\epsilon|\xi'|}\right)s_\lambda\left(I_2 + \frac{\epsilon|\xi'|}{s_\lambda}\frac{\xi' \otimes \xi'}{|\xi'|^2}\right)\right]_{\mathcal{M}'} \\
&\le \frac{C}{|\lambda|^{\frac{1}{2}}},
\end{aligned}$$

where $C > 0$ is independent of ϵ. Since 3.2.13) and (3.2.14) in Proposition 3.2.5 imply

$$\left[\frac{\alpha_{\pm,\epsilon}(\xi', \pm 2)\epsilon|\xi'|}{(1+\epsilon|\xi'|}\right]_{\mathcal{M}'} < \infty$$

uniformly on ϵ, combining with Proposition 3.2.1, we finally obtain

$$\begin{aligned} &\left|\left|\mathcal{F}^{-1}_{d,n'}\alpha_{+,\epsilon}(n',\pm1)e_3\cdot k'_{\lambda,\epsilon}(n',2)y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'}g_-\right|\right|_{L^q(\mathbb{T}^2)} \\ &+\left|\left|\mathcal{F}^{-1}_{d,n'}\alpha_{-,\epsilon}(n',\pm1)e_3\cdot k'_{\lambda,\epsilon}(n',-2)y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'}g_+\right|\right|_{L^q(\mathbb{T}^2)} \\ &\leq \frac{C}{|\lambda|^{\frac{1}{2}}}||g||_{L^q(\mathbb{T}^2)}, \end{aligned} \tag{3.2.53}$$

where C is independent of ϵ. Thus, taking $|\lambda|$ sufficiently large, choice of λ is also independent of ϵ, we can conclude by (3.2.46), (3.2.47), (3.2.46), (3.2.52) and (3.2.53) that

$$-S_{\lambda,\epsilon} = I + O(|\lambda|^{-1/2}).$$

□

Proposition 3.2.12. *Let $0<\theta<\pi/2$, $\lambda\in\Sigma_\theta$, $1<q<\infty$ and $0<\epsilon\leq1$. Then there exist $r>0$ and a constant $C>0$, which is independent of ϵ and λ, if $|\lambda|\geq r$, $V_{\lambda,\epsilon}$ defined by (3.2.42) satisfies*

$$||V_{\lambda,\epsilon}g||_{L^q(\Omega)} \leq C\,|\lambda|^{-1/2q}\,||g||_{L^q(\partial\Omega)} \tag{3.2.54}$$

for all $g\in L^q(\partial\Omega)$.

Proof. We take $r>0$ so that $R_{\lambda,\epsilon}$ exists. Then $S^{-1}_{\lambda,\epsilon}$ is bounded on $L^q(\Omega)$. Together with boundedness of $\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon$ on $L^q(\mathbb{T}^2\times\mathbb{R})$ and $P_{N,\epsilon}$ on $L^q(\Omega)$ and the resolvent estimate for the Dirichlet Laplacian on Ω, see Lemma 5.3 in [1], we find

$$\begin{aligned} ||V_{\lambda,\epsilon}g||_{L^q(\Omega} &= \left|\left|P_{N,\epsilon}L_{\lambda,\epsilon}S^{-1}_{\lambda,\epsilon}g\right|\right|_{L^q(\Omega)} \\ &\leq C\left|\left|L_{\lambda,\epsilon}S^{-1}_{\lambda,\epsilon}g\right|\right|_{L^q(\Omega} \\ &\leq C\,|\lambda|^{-1/2q}\left|\left|S^{-1}_{\lambda,\epsilon}g\right|\right|_{L^q(\Omega} \\ &\leq C\,|\lambda|^{-1/2q}\,||g||_{L^q(\Omega}, \end{aligned}$$

where $C>0$ is independent of ϵ. □

Proposition 3.2.13. *Let $1<q<\infty$ and $0<\epsilon\leq1$. Then there exists a constant $C>0$, which is independent of ϵ, such that*

$$\left|\left|\mathcal{F}^{-1}_{d,n'}\frac{1+\epsilon\,|n'|}{\epsilon\,|n'|}\left(e_3\cdot\mathcal{F}_{x',d}\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon E_0 f\right)\right)\right|\right|_{L^q(\Omega)} \leq C||f||_{L^q(\Omega)}, \tag{3.2.55}$$

for all $f\in L^q(\Omega)$.

Proof. Since the symbol have poles at $\xi_3=\pm i\epsilon\,|n'|$, we obtain its partial Fourier transform with respect to ξ_3 by the residue theorem. Thus, we have

$$\mathcal{F}^{-1}_{d,n'}\frac{1}{\epsilon\,|n'|}\left(e_3\cdot\mathcal{F}_{x'}\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon E_0 f\right)\right)$$

$$
\begin{aligned}
=\frac{1}{2}\mathcal{F}^{-1}_{d,n'}\int_{-1}^{1}\frac{1}{\epsilon\,|n'|}&\Big[e^{-|x_3-\zeta|\epsilon|n'|}i\epsilon n'\cdot\mathcal{F}_{d,x'}f'(n',\zeta)\\
&+e^{-|x_3-\zeta|\epsilon|n'|}\epsilon\,|n'|\,\mathcal{F}_{d,x'}f_3(n',\zeta)\Big]\,d\zeta.
\end{aligned}
$$

This formula, the Mikhlin theorem and Proposition 3.2.1 imply

$$
\left|\left|\mathcal{F}^{-1}_{d,n'}\frac{1}{\epsilon\,|n'|}\left(e_3\cdot\mathcal{F}_{d,x'}\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_{\epsilon}E_0f\right)\right)\right|\right|_{L^q(\mathbb{T}^2)}\leq C||f||_{L^q(\mathbb{R}^2)},
$$

where C is independent of ϵ. □

Let us show BIP for the solution operator for v_2.

Proposition 3.2.14. *Let* $0<\theta<\pi/2$, $\lambda\in\Sigma_\theta$, $1<q<\infty$, $0<\epsilon<1$, $0<a<1/2$, *z satisfying* $-1/2<\mathrm{Re}z<0$ *and* $0<\theta<\pi$. *Then there exists a constant* $C=C(q,a,\theta)$, *it holds that*

$$
\left|\left|\frac{1}{2\pi i}\int_{\Gamma_\theta}(-\lambda)^z V_{\lambda,\epsilon}\left[\gamma v_1-(\gamma v_1\cdot\nu)\nu\right]\;d\lambda\right|\right|_{L^q(\Omega)}\leq Ce^{|\mathrm{Im}z|\theta}||f||_{L^q(\Omega)},\tag{3.2.56}
$$

where $v_1=K_{\lambda,\epsilon}f$ *for* $f\in L^q(\Omega)$.

Proof. It holds that

$$
\gamma v_1-(\gamma v_1\cdot\nu)\nu=\gamma K_{\lambda,\epsilon}E_0f-\gamma\Pi_\epsilon K_{\lambda,\epsilon}E_0f.
$$

We find from this formula, (3.2.41), (3.2.42), (3.2.39), Corollary 3.2.11 and (3.2.27) that the integrand of the left hand side of (3.2.56) can be essentially written as

$$
\begin{aligned}
&P_{N,\epsilon}\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_{\epsilon}\mathcal{F}^{-1}_{d,n'}\int_{-1}^{1}e'_\lambda(\xi',\pm1-x_3)y_{\lambda,\epsilon}(n')e'_\lambda(\xi',\pm1-\zeta)\\
&\qquad\times\mathcal{F}_{d,x'}\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_{\epsilon}E_0f\right)(n',\zeta)d\zeta\\
&+P_{N,\epsilon}\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_{\epsilon}\mathcal{F}^{-1}_{d,n'}\int_{-1}^{1}e'_\lambda(n',\pm1-x_3)y_{\lambda,\epsilon}(n')\alpha_{\epsilon,\pm}(n',\pm1)e'_\lambda(n',\pm1-\zeta)\\
&\qquad\qquad\times\frac{\epsilon\,|n'|}{1+\epsilon\,|n'|}\frac{1+\epsilon\,|n'|}{\epsilon\,|n'|}\mathcal{F}_{d,x'}\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_{\epsilon}E_0f\right)(n',\zeta)d\zeta\\
&+W_{\lambda,\epsilon}R_{\lambda,\epsilon}\left[\gamma K_{\lambda,\epsilon}E_0f-\gamma\Pi_\epsilon K_{\lambda,\epsilon}E_0f\right]\\
&=:I_1+I_2+I_3,
\end{aligned}
$$

where $\pm$ should be take properly. It follows from Proposition 3.2.5 that

$$
\begin{aligned}
&\left[e'_\lambda(\xi',\pm1-x_3)y_{\lambda,\epsilon}(\xi')e'_\lambda(\xi',\pm1-\zeta)\right]_{\mathcal{M}'}\\
&\leq C\left[e^{|\pm1-x_3|s_\lambda}\left(I_2+\frac{\epsilon\,|\xi'|}{s_\lambda}\frac{\xi'\otimes\xi'}{|\xi'|^2}\right)\frac{e^{|\pm1-\zeta|s_\lambda}}{s_\lambda}\right]_{\mathcal{M}'}\\
&\leq C\frac{e^{-c|\lambda|^{1/2}(|\pm1-x_3|+|\pm1-\zeta|)}}{|\lambda|^{1/2}}
\end{aligned}\tag{3.2.57}
$$

Thus, we find from change of integral line so that $|\lambda| > R$, where R is large enough to ensure existence of $S_{\lambda,\epsilon}^{-1}$ in Proposition 3.2.11, that

$$\begin{aligned}
&\left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z I_1 \; d\lambda\right|\right|_{L^q(\Omega}\\
&\leq C\left|\left|\int_{-1}^1\int_\Gamma |\lambda^z|\, C\frac{e^{-c|\lambda|^{1/2}(|x_3-a|+|\zeta-b|)}}{|\lambda|^{1/2}}||\mathbb{P}_\epsilon E_0 f(\cdot,\zeta)||_{L^q(\mathbb{T}^2)}\; d\lambda \; d\zeta\right|\right|_{L^q(-1,1)}\\
&\leq C||f||_{L^q(\Omega)}\\
&+C\left|\left|\int_{-1}^1\int_R^\infty e^{\theta|\mathrm{Im} z|}Cr^{\mathrm{Re} z-1/2}e^{-cr^{1/2}(|x_3-a|+|\zeta-b|)}||f(\cdot,\zeta)||_{L^q(\mathbb{T}^2)}\; dr\; d\zeta\right|\right|_{L^q(-1,1)}\\
&\leq C||f||_{L^q(\Omega)}+C_R e^{\theta|\mathrm{Im} z|}\left|\left|\int_{-1}^1\frac{||f(\cdot,\zeta)||_{L^q(\mathbb{T}^2)}}{|x_3-a|+|\zeta-b|}dr\right|\right|_{L^q(\Omega)}
\end{aligned}$$

where C is independent of ϵ. Applying Lemma 3.2.3, we obtain

$$\left|\left|\int_\Gamma (-\lambda)^z I_1 \; d\lambda\right|\right|_{L^q(\Omega)} \leq Ce^{\theta|\mathrm{Im} z|}||f||_{L^q(\Omega)} \tag{3.2.58}$$

Since $\alpha_{\pm,\epsilon}$ is definde by (3.2.27), it follows from (3.2.13), (3.2.37) and Proposition 3.2.5 that

$$\begin{aligned}
&\left[e'_\lambda(\xi',\pm 1-x_3)y_{\lambda,\epsilon}(\xi')\alpha_{\epsilon,\pm}(\xi',\pm 1)e'_\lambda(\xi',\pm 1-\zeta)\frac{\epsilon\,|\xi'|}{1+\epsilon\,|\xi'|}\right]_{\mathcal{M}'}\\
&\leq C\frac{e^{-c|\lambda|^{1/2}(|\pm 1-x_3|+|\pm 1-\zeta|)}}{|\lambda|^{1/2}}.
\end{aligned}$$

Thus we find from Proposition 3.2.13

$$\begin{aligned}
&\left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z I_2 \; d\lambda\right|\right|_{L^q(\Omega)}\\
&\leq Ce^{\theta|\mathrm{Im} z|}\left|\left|\mathcal{F}_{n'}^{-1}\frac{1+\epsilon\,|\xi'|}{\epsilon\,|n'|}\mathcal{F}_{x'}\left(\mathbb{P}_\epsilon E_0 f\right)(n',\zeta)\right|\right|_{L^q(\Omega)}\\
&\leq Ce^{\theta|\mathrm{Im} z|}||f||_{L^q(\Omega)}.
\end{aligned}$$

Applying $L^q(\Omega)$-boundedness of $P_{N,\epsilon}$, the resolvent estimate for the Dirichlet Laplacian, ee Lemma 5.3 in [1],, Proposition 3.2.11, the resolvent estimate for the Dirichlet Laplacian on $\mathbb{T}^2\times\mathbb{R}$, $L^q(\Omega)$-boundedness of Π_ϵ, the trace theorem and the interpolation inequality, we have

$$\begin{aligned}
||I_3||_{L^q(\Omega)} &= ||P_{N,\epsilon}L_{\lambda,\epsilon}R_{\lambda,\epsilon}\gamma\left[K_{\lambda,\epsilon}E_0 f-\gamma\Pi_\epsilon K_{\lambda,\epsilon}E_0 f\right]||_{L^q(\Omega)}\\
&\leq C\,|\lambda|^{-1/2q}\,||R_{\lambda,\epsilon}\gamma\left[K_{\lambda,\epsilon}E_0 f-\gamma\Pi_\epsilon K_{\lambda,\epsilon}E_0 f\right]||_{L^q(\Omega)}\\
&\leq C\,|\lambda|^{-1/2q-1/2}\,||\gamma\left[K_{\lambda,\epsilon}E_0 f-\gamma\Pi_\epsilon K_{\lambda,\epsilon}E_0 f\right]||_{L^q(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\,|\lambda|^{-1/2q-1/2-1/q'-1/2+\delta}\,||f||_{L^q(\Omega)} \\
&= C\,|\lambda|^{-3/2+\delta}\,||f||_{L^q(\Omega)}
\end{aligned}
$$

for small $\delta > 0$. Thus we find by the change of integral curve

$$
\left|\left|\frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^z I_3\, d\lambda\right|\right|_{L^q(\Omega)} \leq C||f||_{L^q(\Omega)}. \tag{3.2.59}
$$

□

Proof of Lemma 3.2.2. Lemma 3.2.2 is the direct consequence of Proposition 3.2.6, Proposition 3.2.10 and Proposition 3.2.14. □

Corollary 3.2.15. *Let $p, q \in (1, \infty)$, $T > 0$, $F = (f_H, f_z) \in \mathbb{E}_0(T)$, $U_0 \in X_\gamma$ and $\epsilon > 0$. Then there is a unique solution $(U_\epsilon, P_\epsilon) \in \mathbb{E}_1(T) \times \mathbb{E}_0(T)$ to the equation*

$$
\left\{
\begin{array}{rcll}
\partial_t V - \Delta V + \nabla_H P &=& f_H & \text{in} \quad (0,T)\times\Omega, \\
\partial_t(\epsilon W) - \Delta(\epsilon W) + \frac{\partial_3 P}{\epsilon} &=& f_z & \text{in} \quad \Omega\times(0,T), \\
\operatorname{div}_H V + \frac{\partial_z}{\epsilon}(\varepsilon W) &=& 0 & \text{in} \quad \Omega\times(0,T), \\
U &=& 0 & \text{on} \quad \partial\Omega\times(0,T) \\
U(0) &=& U_0 & \text{in} \quad \Omega,
\end{array}
\right. \tag{3.2.60}
$$

where P is unique up to a constant. Moreover, there exist constants $C > 0$ and $C_T > 0$, which is independent of ϵ, such that

$$
\|\,(V, \varepsilon W)\,\|_{\mathbb{E}_1(T)} + \|\nabla_\varepsilon P\|_{\mathbb{E}_0(T)} \leq C\|F\|_{\mathbb{E}_0(T)} + C_T\|\,(V_0, \epsilon W_0)\,\|_{X_\gamma}. \tag{3.2.61}
$$

Proof. Lemma 3.1.2 implies there exists a solution $(\tilde{U}, \tilde{P})$ to (3.1.6) with initial data U_0 such that

$$
\|\widetilde{U}\|_{\mathbb{E}_1(T)} + \|\nabla_\varepsilon \widetilde{P}\|_{\mathbb{E}_0(T)} \leq C\|F\|_{\mathbb{E}_0(T)} + C_T\|U_0\|_{X_\gamma}.
$$

Set

$$
V = \tilde{V}, \quad W = \epsilon\tilde{W}, \quad P = \tilde{P}.
$$

Then (U, P) is the desired solution satisfying (3.2.61). □

3.3 Nonlinear Estimates and Regularity of w

In this section we introduce some Propositions on nonlinear estimates to estimate F_H, F_z and F and on regularity of w, which is vertial component of the solution to the primitive equations. Although, the following Propositions have already proved in [12], we introduce them to explain our restriction for p and q and for reader's convenience.

Proposition 3.3.1 ([12])**.** *Let $T > 0$, $p, q \in (1, \infty)$ such that $2/3p + 1/q \leq 1$. Then for any $v_1, v_2 \in \mathbb{E}_1(T)$ there exist a constant $C = C(p, q) > 0$ such that*

$$
||v_1\partial_x v_2||_{\mathbb{E}_0(T)} \leq C||v_1||_{\mathbb{E}_1(T)}||v_2||_{\mathbb{E}_1(T)}
$$

Proposition 3.3.2 ([12]). *Let $T > 0$ and $z \in (-1,1)$. Let $p, q \in (1,\infty)$ such that $1/p + 1/q \leq 1$. Then for any $v_1, v_2 \in \mathbb{E}_2(T)$ and $w_1 := -\int_z^{-1} \mathrm{div}_H v_1 \, d\zeta$ there exist a constant $C = C(p,q) > 0$ such that*

$$||w_1 \partial_3 v_2||_{\mathbb{E}_0(T)} \leq C ||v_1||_{\mathbb{E}_1(T)} ||v_2||_{\mathbb{E}_1(T)}. \tag{3.3.1}$$

Restriction for p and q in our theorem is due to Proposition 3.3.1 and Proposition 3.3.2.

Proposition 3.3.3. *Let $T > 0$, $1 < p, q < \infty$, $v_0 \in X_\gamma, w_0 = \int_{-1}^{x_3} \mathrm{div}_H \, v_0 \, d\zeta \in X_\gamma(\Omega)$, f, $\partial_3 f \in \mathbb{E}_0(T)$ and $\phi \in C^\infty[-1,1]$ be a cut-off function*

$$\phi = \begin{cases} 0 & (-1 \leq x_3 \leq -\frac{3}{4}) \\ 1 & (0 \leq x_3 \leq 1) \end{cases}$$

Let $u^* = (v^*, w^*) \in \mathbb{E}_1(T) \times W^{1,p}(0,T; W^{-1,q}(\Omega)) \cap L^p(0,T; W^{1,q}(\Omega))$ be the solution of (PE). Then, there exists a solution $u = (v, w)$ to

$$\begin{array}{rcll} \partial_t v - \Delta v & = & f & \text{in} \quad \Omega \times (0,T), \\ \mathrm{div}_H \, v + \partial_3 w & = & \phi' w^* & \text{in} \quad \Omega \times (0,T), \\ u & = & 0 & \text{on} \quad \partial\Omega \times (0,T), \\ v(0) & = & \phi(v_0 - \overline{v_0}) & \text{in} \quad \Omega, \end{array} \tag{3.3.2}$$

where $w(0) = \phi w_0$ by compatibility condition, such that

$$\begin{aligned} & ||v||_{\mathbb{E}_1(T)} + ||\partial_3 v||_{\mathbb{E}_1(T)} + ||w||_{\mathbb{E}_1(T)} \\ & \leq C \left(||g||_{H^{1,p}(0,T;H^{-1,q}(\mathbb{T}^3)) \cap L^p(0,T;H^{1,q})(\mathbb{T}^3)} + ||\partial_3 g||_{H^{1,p}(0,T;H^{-1,q}(\mathbb{T}^3)) \cap L^p(0,T;H^{1,q}(\mathbb{T}^3))} \right) \\ & + C \left(||f||_{\mathbb{E}_0(T)} + ||\partial_3 f||_{\mathbb{E}_0(T)} \right), \end{aligned} \tag{3.3.3}$$

where $g := \phi' w^*$.

Proof. Let $u^1 = (v^1, w^1)$ be the solution to

$$\begin{array}{rcll} \partial_t v^1 - \Delta v^1 & = & f & \text{in} \quad \Omega \times (0,T) \\ v & = & 0 & \text{in} \quad \Omega \times (0,T) \\ v^1(0) & = & 0 & \text{on} \quad \Omega \end{array},$$

The maximal regularity of the Laplace operator implies

$$\|v^1\|_{\mathbb{E}_1(T)} \leq C \|f\|_{\mathbb{E}_0(T)}. \tag{3.3.4}$$

Moreover we find

$$\|\partial_3 v^1\|_{\mathbb{E}_1(T)} \leq C \|\partial_3\|_{\mathbb{E}_0(T)}. \tag{3.3.5}$$

Put $u^1 = (v^1, 0)^T$. By the Bogovskii lemma, see Section 2 of [39] for instance, there exists a vector $u^2 = (v^2, w^2) \in \mathbb{E}_1$ such that

$$\begin{array}{rcll} \mathrm{div} \, u^2 & = & g - \mathrm{div} \, u^1 & \text{in} \quad \Omega \times (0,T), \\ u^2 & = & 0 & \text{in} \quad \Omega \times (0,T), \end{array} \tag{3.3.6}$$

and ,by (3.3.4)

$$||u^2||_{\mathbb{E}_1(T)} \le C||g||_{H^{1,p}(0,T;H^{-1,q}(\mathbb{T}^3))\cap L^p(0,T;H^{1,q})(\mathbb{T}^3)} + ||f||_{\mathbb{E}_1(T)}. \tag{3.3.7}$$

Moreover, $\partial_3 g \in H^{1,p}(0,T;H^{-1,q}(\Omega)) \cap L^p(0,T;H^{1,q}(\Omega))$ and (3.3.5) implies

$$||\partial_3 u^2||_{\mathbb{E}_1(T)} \le C||\partial_3 g||_{H^{1,p}(0,T;H^{-1,q}(\mathbb{T}^3))\cap L^p(0,T;H^{1,q}(\mathbb{T}^3))} + ||\partial_3 f||_{\mathbb{E}_1(T)} \tag{3.3.8}$$

Set $\tilde{u} = (\tilde{v}, \tilde{w}) := u - u^1 - u^2$, then $\tilde{u}$ satisfies

$$\begin{array}{rcll} \partial_t \tilde{v} - \Delta \tilde{v} &=& -\partial_t v^2 + \Delta v^2 & \text{in} \quad \Omega \times (0,T) \\ \operatorname{div}_H \tilde{v} + \partial_3 \tilde{w} &=& 0 & \text{in} \quad \Omega \times (0,T), \\ w(\cdot,\cdot,-1) &=& 0 & \text{on} \quad \Omega \times (0,T), \\ \tilde{u}(0) &=& (\phi(v_0 - \overline{v_0}), \phi w_0)^T - u^2(0) & \text{on} \quad \Omega. \end{array} \tag{3.3.9}$$

It follows from $f \in \mathbb{E}_0(T)$, (3.3.7), the trace theorem and the maximal regularity of the Laplace operator that

$$||\tilde{v}||_{\mathbb{E}_1(T)} \le C\left(||g||_{H^{1,p}(0,T;H^{-1,q}(\mathbb{T}^3))\cap L^p(0,T;H^{1,q})(\mathbb{T}^3)} + ||f||_{\mathbb{E}_0(T)}\right). \tag{3.3.10}$$

Applying ∂_3 both sides of (3.3.9), similarly we have

$$||\partial_3 \tilde{v}||_{\mathbb{E}_1(T)} \le C\left(||\partial_3 g||_{H^{1,p}(0,T;H^{-1,q}(\mathbb{T}^3))\cap L^p(0,T;H^{1,q})(\mathbb{T}^3)} + ||\partial_3 f||_{\mathbb{E}_0(T)}\right). \tag{3.3.11}$$

Thus, by divergence-free condition and (3.3.8), $\tilde{w}$ satisfies

$$\begin{array}{rcll} \partial_t \tilde{w} - \Delta \tilde{w} &=& \operatorname{div}_H \partial_3 \tilde{v}|_{x_3=-1} - \partial_t w^2 + \Delta w^* - \operatorname{div}_H \partial_3 v^2|_{x_3=-1} & \text{in} \quad \Omega \times (0,T) \\ v &=& 0 & \text{on} \quad \Omega \times (0,T) \\ \tilde{w}(0) &=& \phi w_0 - w^2(0) & \text{in} \quad \Omega \end{array}$$

Thus we find from (3.3.10), (3.3.11), the trace theorem and the maximal regularity of the heat equation that

$$\begin{aligned} &||\tilde{w}||_{\mathbb{E}_1(T)} \\ &\le C\left(||\partial_3 g||_{H^{1,p}(0,T;H^{-1,q}(\mathbb{T}^3))\cap L^p(0,T;H^{1,q})(\mathbb{T}^3)} + ||\partial_3 f||_{\mathbb{E}_0(T)}\right) + C_T||\phi w_0 - w^2(0)||_{X_\gamma(\Omega)}. \end{aligned}$$

We completed the proof. □

Let us show $w \in \mathbb{E}_1(T)$. In our previous paper [12], we first derive the equation which w satisfies by applying $\int_{-1}^{x_3} \operatorname{div}_H \cdot \, d\zeta$ to the equations v satisfies Then, estimating the corresponding nonlinear terms and applying the maximal regularity principle, we obtain $w \in \mathbb{E}_1(T)$. However, it is hard to apply this method directly to the case of the Dirichlet boundary condition because of the term with the second derivative at the boundary. This term is due to integration over $(-1, x_3)$. Note that this difficulty does not appear under the periodic boundary condition since the boundary term vanishes by periodicity. We can avoid this difficulty by using some cut-off technique which eliminate the effect of the boundary condition. Lemma 3.1.4 is used in the proof to deal with reminder term coming from effect of cut-off.

Remark 3.3.4. It have already known that $v \in \mathbb{E}_1$ with initial data $v_0 \in X_\gamma$ by Giga, et al. [16].

Proof of Lemma 3.1.4. Integrating (PE) both sides over $(-1,1)$, we find $(\overline{v}, \overline{\pi})$ satisfies

$$\begin{array}{rcll} \partial_t \overline{v} - \Delta \overline{v} + \nabla_H \overline{\pi} & = & -\int_{-1}^{1} v \cdot \nabla_H v + w \partial_3 v \, d\zeta + (\partial_3 v)|_{x_3=-1}^{x_3=1} & \text{in} \quad \Omega \times (0,T) \\ \mathrm{div}_H \overline{v} & = & 0 & \text{in} \quad \Omega \times (0,T) \\ \overline{v} & = & 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \overline{v}(0) & = & \overline{v_0} & \text{in} \quad \Omega. \end{array} \tag{3.3.12}$$

Put $\tilde{v} = v - \overline{v}$. Then, $\tilde{u} = (\tilde{v}, w)$ satisfies

$$\begin{array}{rcll} \partial_t \tilde{v} - \Delta \tilde{v} & = & -\tilde{v} \cdot \nabla_H \tilde{v} - w \partial_z \tilde{v} - \overline{v} \cdot \nabla_H \tilde{v} - \tilde{v} \cdot \nabla_H \overline{v} & \\ & & -\frac{1}{2} \int_{-1}^{1} \tilde{v} \cdot \nabla_H \tilde{v} - (\mathrm{div}_H \, \tilde{v}) \, d\zeta & \\ & & +\frac{1}{2} (\partial_3 v)|_{x_3=-1}^{x_3=1} & \text{in} \quad \Omega \times (0,T), \\ \mathrm{div}_H \, \tilde{v} + \partial_z w & = & 0 & \text{in} \quad \Omega \times (0,T) \\ \tilde{v} & = & 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \tilde{v}(0) & = & v(0) - \overline{v_0} & \text{on} \quad \Omega \end{array} \tag{3.3.13}$$

Note that the pressure term no longer appears in the above equations. Let us introduce a cut-off function ϕ_1, $\phi_2 \in C^\infty(-1,1)$ such that $\phi_1 \equiv 0$ on $[-1,-3/4)$ and $\phi_1 \equiv 1$ on $(0,1]$ and $\phi_2 \equiv 1$ on $[-1,0)$ and $\phi_2 \equiv 0$ on $(3/4,1]$. Put $\phi_3 = 1 - \phi_1 - \phi_2$. Then obviously

$$w = \phi_1 w + \phi_3 w + \phi_2 w (=: w_1 + w_3 + w_2). \tag{3.3.14}$$

Now we show $w_j \in \mathbb{E}_1$ for $j = 1,2,3$. To this end, we first derive the equation that w_1 satisfies. Multiply ϕ_1 to (3.3.13) to get

$$\begin{array}{rll} \partial_t(\phi_1 \tilde{v}) - \Delta(\phi_1 \tilde{v}) & = 2\phi_1' \partial_z \tilde{v} + \phi_1'' \tilde{v} - \phi_1 \tilde{v} \cdot \nabla_H \tilde{v} & \\ & -\phi_1 w \partial_z \tilde{v} - \phi_1 \overline{v} \cdot \nabla_H \tilde{v} - \phi_1 \tilde{v} \cdot \nabla_H \overline{v} & \\ & -\frac{1}{2} \phi_1 \int_{-1}^{1} \tilde{v} \cdot \nabla_H \tilde{v} - (\mathrm{div}_H \, \tilde{v}) \tilde{v} \, d\zeta & \\ & +\frac{1}{2} \phi_1 (\partial_3 v)|_{x_3=-1}^{x_3=1} & \text{in} \quad \Omega \times (0,T), \\ \mathrm{div}_H(\phi_1 \tilde{v}) + \partial_z(\phi_1 w) & = \phi_1' w & \text{in} \quad \Omega \times (0,T) \\ \phi_1 \tilde{v}(0) & = \phi_1 (v(0) - \overline{v_0}) & \text{on} \quad \Omega \end{array} \tag{3.3.15}$$

Let $u^* = (v^*, w^*)$ be a solution to the equation

$$\begin{array}{rcll} \partial_t v^* - \Delta v^* & = & -\frac{1}{2} \phi (\partial_3 v)|_{x_3=-1}^{x_3=1} & \text{in} \quad \Omega \times (0,T), \\ \mathrm{div} \, u^* & = & \phi' w & \text{in} \quad \Omega \times (0,T) \\ u & = & 0 & \text{on} \quad \partial\Omega \times (0,T), \\ v^1(0) & = & \phi(v_0 - \overline{v_0}) & \text{in} \quad \Omega \end{array} \tag{3.3.16}$$

We find from Proposition 3.3.3 that u^* satisfies

$$\|u^*\|_{\mathbb{E}_1(T)} + \|\partial_3 v^*\|_{\mathbb{E}_1(T)} \leq C \|v\|_{\mathbb{E}_1(T)}.$$

Put $\tilde{U} = (\tilde{V}, \tilde{W}) := (\phi_1\tilde{v} - v^*, \phi_1 w - w^*)$. Since $\operatorname{div}\tilde{U} = \operatorname{div}_H \tilde{V} + \partial_3\tilde{W} = 0$, applying $-\operatorname{div}_H$ to the equation that $\tilde{U}$ satisfies and integrating over $(-1, x_3)$ with respect to vertical variable, we find

$$\begin{aligned}
&\partial_t\tilde{W} - \Delta\tilde{W}\\
&= \partial_z \mathrm{div}_H v^1|_{x_3=-1}\\
&+ \int_{-1}^{x_3} 2\phi_1' \mathrm{div}_H \partial_\zeta \tilde{v} + \phi_1'' \mathrm{div}_H \tilde{v}\\
&\qquad + \mathrm{div}_H(-\phi_1\tilde{v}\cdot\nabla_H\tilde{v} - (\tilde{W}+w^*)\partial_\zeta\tilde{v} - \phi_1\overline{v}\cdot\nabla_H\tilde{v} - \phi_1\tilde{v}\cdot\nabla_H\overline{v}) d\zeta\\
&- \frac{1}{2}\int_{-1}^{1}\phi_1(\zeta)\; d\zeta \mathrm{div}_H \int_{-1}^{1} \tilde{v}\cdot\nabla_H\tilde{v} - (\mathrm{div}_H\,\tilde{v})\tilde{v}\; d\zeta\\
&=: I_1 + I_2 + I_3,
\end{aligned} \tag{3.3.17}$$

with initial data $\phi_1 w_0$. By the choice of v^* and the trace theorem, we have $\|I_1\|_{\mathbb{E}_0(T)} \leq C$ for some $C > 0$. We use integration by parts to get

$$\begin{aligned}
I_2 &= (\mathrm{div}_H v)\phi_1' - \int_{-1}^{x_3} \mathrm{div}_H v\phi''\; d\zeta\\
&+ \tilde{v}\cdot\nabla_H(\tilde{W}+w^1) - (\tilde{W}+w^*)\mathrm{div}_H v + \overline{v}\cdot\nabla_H(\tilde{W}+w^*)\\
&+ \int_{-1}^{x_3} (\phi_1\partial_j\tilde{v}\cdot\nabla_H\tilde{v} - (\partial_\zeta\tilde{v})\cdot\nabla_H(\tilde{W}+w^*))\\
&+ \nabla_H(\tilde{W}+w^*)\cdot\partial_\zeta\tilde{v} - (\partial_\zeta(\tilde{W}+w^*)\mathrm{div}_H\tilde{v})d\zeta\\
&+ \int_{-1}^{x_3} \phi_1\partial_j\overline{v}\cdot\nabla_H\tilde{v} + \phi_1'\overline{v}\cdot\nabla_H(\tilde{W}+w^*) + \phi_1\partial_j\tilde{v}\cdot\nabla_H\overline{v}_j\; d\zeta.
\end{aligned} \tag{3.3.18}$$

We can apply Proposition 3.3.1 and 3.3.2 to I_2 to get

$$||I_2||_{\mathbb{E}_0(T)} \leq C\left(\|\tilde{W}\|_{\mathbb{E}_1(T)}||\tilde{v}||_{\mathbb{E}_1(T)} + ||\tilde{v}||^2_{\mathbb{E}_1(T)}\right). \tag{3.3.19}$$

Similarly, we have

$$||I_3||_{\mathbb{E}_0(T)} \leq C\left(\|\tilde{W}\|_{\mathbb{E}_1(T)}||\tilde{v}||_{\mathbb{E}_1(T)} + ||\tilde{v}||^2_{\mathbb{E}_1(T)}\right). \tag{3.3.20}$$

Note that constants in (3.3.19) and (3.3.20) are independent of T since constants in Proposition 3.3.1 and 3.3.2 are independent of ϵ. Thus we find from the maximal regularity of the heat equation, implicit function theorem and Neumann series argument, which is the same way as in Proposition 4.8 in [12], that

$$\|\tilde{W}\|_{\mathbb{E}_1(T)} \leq C$$

for some $C = C(p, q, ||v||_{\mathbb{E}_1(T)}, ||w_0||_{X_\gamma})$. Thus, we obtain

$$\|w_1\|_{\mathbb{E}_1(T)} \leq \|\tilde{W}\|_{\mathbb{E}_1(T)} + \|w^*\|_{\mathbb{E}_1(T)} \leq C$$

for some $C = C(p, q, ||v||_{\mathbb{E}_1(T)}, ||w_0||_{X_\gamma})$. Same argument is valid for w_2, w_3 and thus

$$\|\tilde{w_2}\|_{\mathbb{E}_1(T)} + \|\tilde{w_3}\|_{\mathbb{E}_1(T)} \leq C$$

Note that for w_3 need to take integral interval as $[x_3, 1]$ when we use divergence free condition to get the solution formula of vertical component of vector field from one of horizontal component. We complete the proof. □

3.4 Justification of the Hydrostatic approximation and Global-wellposedness of the anisotropic Navier-Stokes Equations

Let us prove our main theorem.

Proof of Theorem 3.1.1. Let $\mathcal{T} > 0$. Let $C_1/10$ be the maximum of constants C in Proposition 3.3.1 and 3.3.2, (3.2.61) and the constant in the trace theorem. Let us construct a solution $(V_\varepsilon, \varepsilon W_\varepsilon)$ to (3.1.3) with zero initial data on $[0, \mathcal{T}]$. Set $(u_\varepsilon, p_\varepsilon) := (v + V_\varepsilon, w + W_\varepsilon, p + P_\varepsilon)$, then this is the desired solution to (SNS). We denote by $\|\cdot\|_{\mathbb{E}_1(mT,(m+1)T)}$ and $\|\cdot\|_{\mathbb{E}_0(mT,(m+1)T)}$ the $\mathbb{E}_1$-norm and $\mathbb{E}_0$-norm on the time interval $[mT, (m+1)T]$, respectively. Let us take $0 < T \leq 1$ so small that

$$C_1 \|u\|_{\mathbb{E}_1(mT,(m+1)T)} \leq \frac{1}{4}, \tag{3.4.1}$$

for all positive integer m. This choice of T is clearly independent of ε. We divide the time interval $[0, \mathcal{T}]$ into $\cup_{m=0}^{N} [mT, (m+1)T]$. Put $F = F(V_\varepsilon, W_\varepsilon, u) := (F_H(V_\varepsilon, W_\varepsilon, u)), F_z(V_\varepsilon, W_\varepsilon, u))$ be the left hand side of (3.1.3). We denote by

$$\mathcal{R}(F, U_0) = (\mathcal{R}^u(F, U_0), \mathcal{R}^p(F, U_0)) = (U, P)$$

the solution to (3.2.60) with initial data U_0. Set inductively

$$U_{\varepsilon,1} = \mathcal{R}^u(F(0,u), 0), \quad P_{\varepsilon,1} = \mathcal{R}^p(F(0,u), 0), \tag{3.4.2}$$

$$U_{\varepsilon,j+1} = \mathcal{R}^u(F(U_j,u), 0), \quad P_{\varepsilon,j+1} = \mathcal{R}^p(F(U_j,u), 0) \tag{3.4.3}$$

Proposition 3.3.1, 3.3.2 and 3.2.15 lead

$$\begin{aligned} &\| (V_{\varepsilon,j+1}, \varepsilon W_{\varepsilon,j+1}) \|_{\mathbb{E}_1(T)} + \|\nabla_\varepsilon P_{\varepsilon,j+1}\|_{\mathbb{E}_0(T)} \\ &\leq C_1 T^\eta \left(\|u\|_{\mathbb{E}_1(T)} \| (V_{\varepsilon,j}, \varepsilon W_{\varepsilon,j}) \|_{\mathbb{E}_1(T)} + \| (V_{\varepsilon,j}, \varepsilon W_{\varepsilon,j}) \|_{\mathbb{E}_1(T)}^2 \right) \\ &+ \varepsilon C_1 T^\eta \left(\|u\|_{\mathbb{E}_1(T)} + \|u\|_{\mathbb{E}_1(T)}^2 \right), \end{aligned} \tag{3.4.4}$$

This quadratic inequality and (3.4.1) implies

$$\| (V_{\varepsilon,j}, \varepsilon W_{\varepsilon,j}) \|_{\mathbb{E}_1(T)} + \|\nabla_\varepsilon P_{\varepsilon,j}\|_{\mathbb{E}_0(T)} \leq 2\varepsilon C^*, \tag{3.4.5}$$

for $C^* = (1/4C_1 + 1/16C_1^2)$ and small $\varepsilon > 0$. Put

$$\widetilde{U}_{\varepsilon,j} = U_{\varepsilon,j+1} - U_{\varepsilon,j} \; (j \geq 1), \quad \widetilde{U}_{\varepsilon,0} = U_{\varepsilon,1} \tag{3.4.6}$$

$$\widetilde{P}_{\varepsilon,j} = P_{\varepsilon,j+1} - P_{\varepsilon,j} \; (j \geq 1), \quad \widetilde{P}_{\varepsilon,0} = P_{\varepsilon,1}. \tag{3.4.7}$$

Then seeking the equation which $(\widetilde{U}_{\varepsilon,j}, \widetilde{P}_{\varepsilon,j})$ satisfies and applying Proposition 3.3.1, 3.3.2 and 3.2.15, we have

$$\begin{aligned} &\|(\widetilde{V}_{\varepsilon,j+1}, \varepsilon \widetilde{W}_{\varepsilon,j+1})\|_{\mathbb{E}_1(T)} + \|\nabla_\varepsilon \tilde{P}_{j+1}\|_{\mathbb{E}_0(T)} \\ &\leq C_1 T^\eta \left(\| (V_{\varepsilon,j}, \varepsilon W_{\varepsilon,j}) \|_{\mathbb{E}_1(T)} + \| (V_{\varepsilon,j+1}, \varepsilon W_{\varepsilon,j+1}) \|_{\mathbb{E}_1(T)} \right. \end{aligned}$$

$$
\begin{aligned}
&+2\|u\|_{\mathbb{E}_1(T)}\big)\,\|(\widetilde{V}_{\varepsilon,j},\varepsilon\widetilde{W}_{\varepsilon,j})\|_{\mathbb{E}_1(T)}\\
&\le \frac{3}{4}\left(\|(\widetilde{V}_{\varepsilon,j},\varepsilon\widetilde{W}_{\varepsilon,j})\|_{\mathbb{E}_1(T)}+\|\nabla_\varepsilon\tilde{P}_j\|_{\mathbb{E}_0(T)}\right).
\end{aligned}
$$

Thus $(U_\varepsilon,P_\varepsilon):=(\lim_{j\to\infty}U_j,\lim_{j\to\infty}P_j)=(\sum_{j=0}\tilde{U}_{\varepsilon,j},\sum_{j=0}\tilde{P}_{\varepsilon,j})$ exists on $[0,T]$ and satisfies

$$
\|\,(V_\varepsilon,\varepsilon W_\varepsilon)\,\|_{\mathbb{E}_1(T)}+\|\nabla_\varepsilon P_\varepsilon\|_{\mathbb{E}_0(T)}\le 2\varepsilon C^*. \tag{3.4.8}
$$

By construction $(U_\varepsilon,P_\varepsilon)$ satisfies (3.1.3) on $[0,T]$. Moreover, by trace theorem there exists a constant $C_{tr}>0$ such that

$$
\|\,(V_\varepsilon(T),\varepsilon W_\varepsilon(T))\,\|_{X_\gamma}\le C_{tr}\|\,(V_\varepsilon,\varepsilon W_\varepsilon)\,\|_{E_1(0,T)}\le 2\varepsilon C^*C_{tr}. \tag{3.4.9}
$$

Next let us construct the solution to (3.1.3) on $[T,2T]$ with initial data $U_\varepsilon(T)$. By the trace theorem, $\|U_\varepsilon(T)\|_{X_\gamma}\le C_{tr}\|U_\varepsilon\|_{\mathbb{E}_1(T)}\le 2\varepsilon C^*C_{tr}$ for some constant $C_{tr}>0$. Put $a_{\varepsilon,1}=(b_{\varepsilon,1},c_{\varepsilon,1})=\mathcal{R}^u(0,U_\varepsilon(T))$ and $\pi_{\varepsilon,1}=\mathcal{R}^p(0,U_\varepsilon(T))$. Corollary 3.2.15 implies

$$
\|\,(b_{\varepsilon,1},\varepsilon c_{\varepsilon,1})\,\|_{\mathbb{E}_1(T,2T)}+\|\nabla_\varepsilon\pi_{\varepsilon,1}\|_{\mathbb{E}_0(T,2T)}\le 2\varepsilon C^*C_{tr}C_T. \tag{3.4.10}
$$

Let the vector field $a_\varepsilon=(b_\varepsilon,c_\varepsilon)$ be the solution to

$$
\begin{cases}
\partial_t b_\varepsilon-\Delta b_\varepsilon+\nabla_H\pi_\varepsilon &= F_H(b_{1,\varepsilon}+b_\varepsilon,c_{\varepsilon,1}+c_\varepsilon,u)\\
\partial_t(\varepsilon c_\varepsilon)-\Delta(\varepsilon c_\varepsilon)+\frac{\partial_3}{\varepsilon}\pi_\varepsilon &= \varepsilon F_z(b_{1,\varepsilon}+b_\varepsilon,c_{\varepsilon,1}+c_\varepsilon,u)\\
\operatorname{div}a_\varepsilon &= 0\\
a_\varepsilon(T) &= 0
\end{cases} \tag{3.4.11}
$$

Then $U_\varepsilon=a_{\varepsilon,1}+a_\varepsilon$ and $P_\varepsilon=\pi_{\varepsilon,1}+\pi_\varepsilon$ is a solution to ((3.1.3) with initial data $U_\varepsilon(T)$. Let us construct the solution to (3.2.60). Let $F(b_{1,\varepsilon}+b_\varepsilon,c_{\varepsilon,1}+c_\varepsilon,u)=(F_H(b_{1,\varepsilon}+b_\varepsilon,c_{\varepsilon,1}+c_\varepsilon,u),\varepsilon F_z(b_{1,\varepsilon}+b_\varepsilon,c_{\varepsilon,1}+c_\varepsilon,u))$ Set inductively

$$
a_{\varepsilon,j+1}=a_{\varepsilon,1}+\mathcal{R}^u(F(b_{1,\varepsilon}+b_{\varepsilon+j},c_{\varepsilon,1}+c_{\varepsilon+j},u),0), \tag{3.4.12}
$$

$$
\pi_{\varepsilon,j+1}=\mathcal{R}^p(F(b_{1,\varepsilon}+b_{\varepsilon+j},c_{\varepsilon,1}+c_{\varepsilon+j},u),0) \tag{3.4.13}
$$

for $j\ge 1$. Applying Proposition 3.3.1, 3.3.2 and 3.2.15 to (3.4.11), we find

$$
\begin{aligned}
&\|\,(b_{\varepsilon,j+1},\varepsilon c_{\varepsilon,j+1})\,\|_{\mathbb{E}_1(T,2T)}+\|\nabla_\varepsilon\pi_{\varepsilon,j+1}\|_{\mathbb{E}_0(T,2T)}\\
&\le C_1T^\eta\|u\|_{\mathbb{E}_1(T,2T)}\|\,(b_{\varepsilon,1}+b_{\varepsilon,j},\varepsilon(c_{\varepsilon,1}+c_{\varepsilon,j}))\,\|_{\mathbb{E}_1(T,2T)}\\
&+C_1T^\eta\|\,(b_{\varepsilon,1}+b_{\varepsilon,j},\varepsilon(c_{\varepsilon,1}+c_{\varepsilon,j}))\,\|^2_{\mathbb{E}_1(T,2T)}\\
&+\varepsilon C_1T^\eta\left[\|u\|_{\mathbb{E}_1(T,2T)}+\|u\|^2_{\mathbb{E}_1(T,2T)}\right]\\
&\le C_1T^\eta\|\,(b_{\varepsilon,j},\varepsilon c_{\varepsilon,j})\,\|^2_{\mathbb{E}_1(T,2T)}\\
&+C_1T^\eta\left(\|u\|_{\mathbb{E}_1(T,2T)}+2\|\,(b_{\varepsilon,1},\varepsilon c_{\varepsilon,1})\,\|_{\mathbb{E}_1(T,2T)}\right)\|\,(b_{\varepsilon,j},\varepsilon c_{\varepsilon,j})\,\|_{\mathbb{E}_1(T,2T)}\\
&+\varepsilon C_1T^\eta\left[\|u\|_{\mathbb{E}_1(T,2T)}\|\,(b_{\varepsilon,1},\varepsilon c_{\varepsilon,1})\,\|_{\mathbb{E}_1(T,2T)}+\|\,(b_{\varepsilon,1},\varepsilon c_{\varepsilon,1})\,\|^2_{\mathbb{E}_1(T,2T)}\right]\\
&+\varepsilon C_1T^\eta\left[\|u\|_{\mathbb{E}_1(T,2T)}+\|u\|^2_{\mathbb{E}_1(T,2T)}\right]
\end{aligned}
$$

If we take ε so small that

$$\|a_{\varepsilon,1}\|_{\mathbb{E}_1(T,2T)} \le 2\varepsilon C^* C_{tr} C_T \le \frac{1}{8C_1} \tag{3.4.14}$$

we have

$$\begin{aligned}
&\| \left(b_{\varepsilon,j+1}, \varepsilon c_{\varepsilon,j+1}\right) \|_{\mathbb{E}_1(T,2T)} + \|\nabla_\varepsilon \pi_{\varepsilon,j+1}\|_{\mathbb{E}_0(T,2T)} \\
&\le C_1 \| \left(b_{\varepsilon,j}, \varepsilon c_{\varepsilon,j}\right) \|^2_{\mathbb{E}_1(T,2T)} + \frac{1}{2} \| \left(b_{\varepsilon,j}, \varepsilon c_{\varepsilon,j}\right) \|_{\mathbb{E}_1(T,2T)} + \varepsilon C^* C_T C_{tr} + \varepsilon C^* \\
&\le C_1 \| \left(b_{\varepsilon,j}, \varepsilon c_{\varepsilon,j}\right) \|^2_{\mathbb{E}_1(T,2T)} + \frac{1}{2} \| \left(b_{\varepsilon,j}, \varepsilon c_{\varepsilon,j}\right) \|_{\mathbb{E}_1(T,2T)} + \varepsilon C^* (1 + C_T C_{tr}).
\end{aligned}$$

Thus, we have by induction

$$\| \left(b_{\varepsilon,j}, \varepsilon c_{\varepsilon,j}\right) \|_{\mathbb{E}_1(T,2T)} + \|\nabla_\varepsilon \pi_{\varepsilon,j}\|_{\mathbb{E}_0(T,2T)} \le 2\varepsilon C^* (1 + C_T C_{tr}). \tag{3.4.15}$$

for all $j \ge 1$. Set

$$\begin{aligned}
\tilde{a}_{\varepsilon,j} &= a_{\varepsilon,j+1} - a_{\varepsilon,j} \quad (j \ge 1), \quad \tilde{a}_{\varepsilon,0} = a_{\varepsilon,0} \\
\tilde{\pi}_{\varepsilon,j} &= \pi_{\varepsilon,j+1} - \pi_{\varepsilon,j} \quad (j \ge 1), \quad \tilde{\pi}_{\varepsilon,0} = \pi_{\varepsilon,0}
\end{aligned}$$

Applying Proposition 3.3.1, 3.3.2 and 3.2.15 to the equations that $(\tilde{a}_{\varepsilon,j+1}, \tilde{\pi}_{\varepsilon,j+1})$ satisfies, we find

$$\begin{aligned}
&\|(\tilde{b}_{\varepsilon,j+1}, \varepsilon \tilde{c}_{\varepsilon,j+1})\|_{\mathbb{E}_1(T,2T)} + \|\nabla_\varepsilon \tilde{\pi}_{\varepsilon,j+1}\|_{\mathbb{E}_0(T,2T)} \\
&\le C_1 T^\eta \left(\| \left(b_{\varepsilon,j}, \varepsilon c_{\varepsilon,j}\right) \|_{\mathbb{E}_1(T,2T)} + \| \left(b_{\varepsilon,j+1}, \varepsilon c_{\varepsilon,j+1}\right) \|_{\mathbb{E}_1(T,2T)} + 2\|u\|_{\mathbb{E}_1(T,2T)} \right) \\
&\qquad \times \|(\tilde{b}_{\varepsilon,j}, \varepsilon \tilde{c}_{\varepsilon,j},)\|_{\mathbb{E}_1(T,2T)} \\
&\le \left[C_1 \left(\| \left(b_{\varepsilon,j}, \varepsilon c_{\varepsilon,j}\right) \|_{\mathbb{E}_1(T,2T)} + \| \left(b_{\varepsilon,j+1}, \varepsilon c_{\varepsilon,j+1}\right) \|_{\mathbb{E}_1(T,2T)} \right) + \frac{1}{2} \right] \\
&\qquad \times \|(\tilde{b}_{\varepsilon,j}, \varepsilon \tilde{c}_{\varepsilon,j},)\|_{\mathbb{E}_1(T,2T)} \\
&\le \frac{3}{4} \|(\tilde{b}_{\varepsilon,j}, \varepsilon \tilde{c}_{\varepsilon,j},)\|_{\mathbb{E}_1(T,2T)}.
\end{aligned}$$

The last inequality holds if ε is sufficiently small. Thus,

$$(a_\varepsilon, \pi_\varepsilon) := (\lim_{j\to\infty} a_{\varepsilon,j}, \lim_{j\to\infty} \pi_{\varepsilon,j}) = (\sum_{j=0} \tilde{a}_{\varepsilon,j}, \sum_{j=0} \tilde{\pi}_{\varepsilon,j})$$

exists and satisfies (3.4.11) such that

$$\| \left(b_\varepsilon, \varepsilon c_\varepsilon\right) \|_{\mathbb{E}_1(T,2T)} + \|\nabla_\varepsilon \pi_\varepsilon\|_{\mathbb{E}_0(T,2T)} \le 2\varepsilon C^* (1 + C_T C_{tr}). \tag{3.4.16}$$

$(U_\varepsilon, P_\varepsilon)$ solves (3.1.3) on the time interval $[T, 2T]$ with initial data $U_\varepsilon(T)$ such that

$$\begin{aligned}
&\| \left(V_\varepsilon, \varepsilon W_\varepsilon\right) \|_{\mathbb{E}_1(T)} + \|\nabla_\varepsilon P_\varepsilon\|_{\mathbb{E}_0(T)} \\
&\le \| \left(b_{\varepsilon,1}, \varepsilon c_{\varepsilon,1}\right) \|_{\mathbb{E}_1(T,2T)} + \| \left(b_\varepsilon, \varepsilon c_\varepsilon\right) \|_{\mathbb{E}_1(T,2T)} + \|\nabla_\varepsilon \pi_{\varepsilon,1}\|_{\mathbb{E}_0(T,2T)} + \|\nabla_\varepsilon \pi_\varepsilon\|_{\mathbb{E}_0(T,2T)} \\
&\le C_T \|U_\varepsilon(T)\|_{X_\gamma} + 2\varepsilon C^* (1 + C_{tr} C_T) \le 2\varepsilon C^* (1 + 2C_{tr} C T)
\end{aligned}$$

By induction, the solution $(U_\varepsilon, P_\varepsilon)$ constructed by the same way on the time interval $[mT, (m+1)T]$ satisfies

$$\begin{aligned}
&\|\,(V_\varepsilon, \varepsilon W_\varepsilon)\,\|_{\mathbb{E}_1(mT,(m+1)T)} + \|\nabla_\varepsilon P_\varepsilon\|_{\mathbb{E}_0(mT,(m+1)T)} \\
&\le 2\varepsilon C^* \left[1 + 3C_T C_{tr}\left(1 + 3C_T C_{tr}(\cdots)\right)\right] =: 2\varepsilon\alpha_j
\end{aligned}$$

Since $\mathcal{T}$ is finite, this induction ends in finite steps. Thus we conclude

$$\|\,(V_\varepsilon, \varepsilon W_\varepsilon)\,\|_{\mathbb{E}_1(\mathcal{T})} + \|\nabla_\varepsilon P_\varepsilon\|_{\mathbb{E}_1(\mathcal{T})} \le 2\varepsilon \sum_{1\le j\le n} \alpha_j. \tag{3.4.17}$$

□

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Chapter 4

Solvability of the higher-order elliptic problem in L^p-L^q settings under dynamic boundary conditions

In this chapter, the higher-order elliptic problem under dynamic boundary conditions is considered. We give a sufficient condition of solvability of this problem in the maximal L_p-L_q settings on bounded and exterior domains. Our method is based on vector-valued harmonic analysis and abstract functional calculus.

4.1 Introduction

We consider the vector-valued quasi-steady problems of the following

$$\begin{cases} \eta u + \mathcal{A}(t,x,D)u &= f(t,x) \qquad (t \in J,\ x \in G), \\ \partial_t \rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_\Gamma)\rho &= g_0(t,x) \qquad (t \in J,\ x \in \Gamma), \\ \mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_\Gamma)\rho &= g_j(t,x) \qquad (t \in J,\ x \in \Gamma,\ j = 1, \cdots, m), \\ \rho(0,x) &= \rho_0(x) \qquad (x \in \Gamma), \end{cases} \tag{4.1.1}$$

where $\eta > 0$, $J \subset [0,T]$ is a finite interval or $G \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded and exterior domain with the boundary Γ. The functions $f, \{g_j\}_{j=0}^m, \rho_0$ are given data and the functions u and ρ are unknown functions. $(\mathcal{A}, \mathcal{B}_j, \mathcal{C}_j)$ are differential operators with order $(2m, m_j, k_j)$, respectively. The aim of this paper is to obtain maximal L_p-L_q regularity of these equations. More precisely we characterize the data space $X \times \prod_{j=0}^m Y_j \times \pi Z_\rho$ and the solution space $Z_u \times Z_\rho$ such that these spaces are isomorphism.

This quasi-steady problems are considered as the linearized equations for the various non-linear equations, e.g. free boundary problems. One of the successful methods to solve the free boundary problems is the transformation from time-varying domain to fixed domain. After we use this transformation, the equation has an unknown function called a height function on the boundary, and the equation on the boundary has time derivative of order one. If the original equation has a time derivative in an interior domain, the transformed equation also has a time derivative in a domain. On the other hand, if there is no time derivative in the original equation in the domain, the transformed equation does not have a time derivative. Usually, the derived equation is also non-linear, but the linearized equation corresponds to the relaxation type or the quasi-steady type. The first one corresponds to the first derivative in the interior equation and it has already considered in the paper [8]. As far as we know, the second one has not considered yet. Therefore we consider these problems in this paper.

The paper is organized as follows. In Section 4.2, basic function spaces and assumptions for $\mathcal{A}$, $\mathcal{B}_j$ and $\mathcal{C}_j$ including smoothness are introduced. Then our main result is stated. In Section 4.3, basic notions of operator theory, e.g. operator-valued multiplier theorems and H^∞-calculus, are introduced for the reader's convenience. In Section 4.4, we first consider (4.1.1) under $G = \mathbb{R}^n_+$ with the differential operators having no lower order terms and constant coefficients. The problem is first reduced into the case of $f = 0$ and $\rho_0 = 0$. Then the partial Laplace-Fourier transform is applied to get the solution formula of Fourier multiplier type. In this step, the Lopatinskii-Shapiro condition (LS) is frequently used. Operator-valued Fourier multiplier theorem due to Weis [42] and the operator-valued H^∞-functional calculus due to Kalton-Weis [25] are applied to the solution operator to obtain its maximal regularity of the solutions. Here, the asymptotic Lopatinskii-Shapiro (ALS) conditions are also needed. By perturbation and localization procedure, our maximal regularity result for the full problem of (4.1.1) is proved.

4.2 Main results

Let us introduce notation to give our main results and state our theorem. Let $\mathbb{N}$ is a set of positive integer and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Differential operators in (4.1.1) are given by

$$\begin{aligned} \mathcal{A}(t,x,D) &:= \sum_{|\alpha| \le 2m} a_\alpha(t,x) D^\alpha, \\ \mathcal{B}_j(t,x,D) &:= \sum_{|\beta| \le m_j} b_{j\beta}(t,x) D^\beta, \\ \mathcal{C}_j(t,x,D_\Gamma) &:= \sum_{|\gamma| \le k_j} c_{j\gamma}(t,x) D_\Gamma^\gamma, \end{aligned}$$

where m is a positive integer, $m_j \in \mathbb{N}_0 \cap [0, 2m)$, $k_j \in \mathbb{N}_0$ for $j = 0, \cdots, m$. The symbols D, respectively D_Γ mean $-i\nabla$, respectively $-i\nabla_\Gamma$, where ∇ denotes the gradient in G and ∇_Γ the surface gradient on Γ. We assume that all boundary operators $\mathcal{B}_j$ and at least one $\mathcal{C}_j$ are non-trivial. The order k_j is defined by $-\infty$ when $\mathcal{C}_j = 0$. The unknown functions $u(t,x), \rho(t,x)$ belongs to Hilbert spaces E and F. Note that the case $E = F = \mathbb{C}^N (N \in \mathbb{N})$ is allowed. For the coefficients of the above differential operators, $a_\alpha(t,x), b_{j\beta}(t,x) \in \mathcal{B}(E)$, $c_{j\gamma}(t,x) \in \mathcal{B}(F,E)$ for $j = 1, \cdots, m$, and $b_{0\beta}(t,x) \in \mathcal{B}(E,F)$ and $c_{0\gamma}(t,x) \in \mathcal{B}(F)$.

Let $1 < p, q < \infty$. We would like to find the maximal L_p-L_q regularity solutions, i.e.

$$u \in Z_u := L_p(J; W_q^{2m}(G;E)),$$

then we should assume

$$f \in X := L_p(J; L_q(G;E)).$$

Since we expect the regularity of g_j is the same as $\mathcal{B}_j u$,

$$\begin{aligned} &g_0 \in Y_0 := L_p(J; W_q^{2m\kappa_0}(\Gamma;F)), \\ &g_j \in Y_j := L_p(J; W_q^{2m\kappa_j}(\Gamma;E)) \quad (j = 1, \cdots, m) \end{aligned}$$

with

$$\kappa_j := 1 - \frac{m_j}{2m} - \frac{1}{2mq} \quad (j = 0, \cdots, m)$$

from the trace theorem. Thus, the solution class which ρ belongs to should be

$$\begin{aligned} &\rho \in W_p^1(J; W_q^{2m\kappa_0}(\Gamma;F)) \cap \bigcap_{j=0}^{m} L_p(J; W_q^{k_j + 2m\kappa_j}(\Gamma;F)) \\ &= W_p^1(J; W_q^{2m\kappa_0}(\Gamma;F)) \cap L_p(J; W_q^{l+2m\kappa_0}(\Gamma;F)) \\ &=: Z_\rho, \end{aligned}$$

from the differential structure of the equation (4.1.1), where $l := \max_{j=0,\cdots,m}$ with $l_j = k_j - m_j + m_0$. We always assume $l \geq 0$ in this paper. It can be expected by the trace theorem that

$$\rho_0 \in \pi Z_\rho := B_{qp}^{l(1-1/p)+2m\kappa_0}(\Gamma;F).$$

Under these settings and assumptions (E), (SA), (SB), (SC), (LS) and (ALS) introduced later, we shall show the solution operator is an isomorphism between the data $(f, \{g_j\}_{j=0}^m, \rho_0) \in X \times \prod_{j=0}^m Y_j \times \pi Z_\rho$ and the solution $(u, \rho) \in Z_u \times Z_\rho$.

First we assume normal ellipticity of $\mathcal{A}$ as usual. The subscript # denotes the principal part of the corresponding operator, e.g. $\mathcal{A}_\#(t,x,D) = \sum_{|\alpha|=2m} a_\alpha(t,x) D^\alpha$.

(E) (Ellipticity of the interior symbol) For all $t \in J$, $x \in \overline{G}$ in the case G is a bounded domain, $x \in \overline{G} \cup \{\infty\}$ in the case G is an exterior domain, and for all $\xi \in \mathbb{R}^n$ satisfying $|\xi| = 1$, we assume normal ellipticity for $\mathcal{A}(t,x,\xi)$ with an angle less than $\pi/2$, and thus

$$\sigma(\mathcal{A}_\#(t,x,\xi)) \subset \mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}.$$

Here $\sigma(\mathcal{A}_\#(t,x,\xi))$ denotes the spectrum of the bounded operator $\mathcal{A}_\#(t,x,\xi) \in \mathcal{B}(E)$.
Next, we introduce conditions of smoothness to the coefficients of $\mathcal{A}$, $\mathcal{B}_j$ and $\mathcal{C}_j$. These conditions allow us to use localization and perturbation argument.

(SA) For $|\alpha| = k \leq 2m-1$, there exists $r_\alpha \geq q$ with $\frac{1}{r_\alpha} \leq \frac{2m-k}{n}$ such that

$$a_\alpha \in L_\infty(J; L_{r_\alpha}(G; \mathcal{B}(E))).$$

For $|\alpha| = 2m$, assume

$$a_\alpha \in BUC(J \times \overline{G}; \mathcal{B}(E)).$$

In the case G is exterior domain, we impose the condition that the asymptotic state at infinity $a_\alpha(t,\infty) := \lim_{|x|\to\infty, x\in G} a_\alpha(t,x)$ exists and is bounded uniformly with respect to $t \in J$ for all $|\alpha| = 2m$.

(SB) Let $\mathcal{E}_0 := \mathcal{B}(E,F)$ and $\mathcal{E}_j := \mathcal{B}(E)$ for $j = 1, \cdots, m$. For each $j = 0, \cdots, m$ and $|\beta| = k \leq m_j - 1$, there exist $s_{j\beta}, r_{j\beta} \geq q$ with $\frac{1}{s_{j\beta}} \leq \frac{m_j-k}{n-1}$, $\frac{1}{r_{j\beta}} \leq \frac{2m-k-1/p}{n-1}$ such that

$$b_{j\beta} \in L_\infty(J; (L_{s_{j\beta}} \cap B^{2m\kappa_j}_{r_{j\beta},q})(\Gamma; \mathcal{E}_j)).$$

For $|\beta| = m_j$, assume

$$b_\beta \in BUC(J \times \Gamma; \mathcal{E}_j).$$

(SC) Let $\mathcal{F}_0 := \mathcal{B}(F)$ and $\mathcal{F}_j := \mathcal{B}(F,E)$ for $j = 1, \cdots, m$. For each $j = 0, 1, \cdots, m$ and $|\gamma| = k \leq k_j - 1$, there exist $t_{j\gamma}, \tau_{j\gamma} \geq p$ and $s^c_{j\gamma}, r^c_{j\gamma} \geq q$ with $\frac{l}{t_{j\gamma}} + \frac{n-1}{s^c_{j\gamma}} \leq l - k + m_j - m_0$ and $\frac{l}{\tau_{j\gamma}} + \frac{n-1}{r^c_{j\gamma}} \leq l - k + 2m\kappa_0$ such that

$$c_{j\gamma} \in L_{t_{j\gamma}}(J; L_{s^c_{j\gamma}}(\Gamma; \mathcal{F}_j)) \cap L_{\tau_{j\gamma}}(J; B^{2m\kappa_j}_{r^c_{j\gamma},q}(\Gamma; \mathcal{F}_j))$$

For $|\beta| = k_j$, assume

$$c_{j,\gamma} \in BUC(J \times \Gamma; \mathcal{F}_j).$$

The following two conditions are needed to get the formula of solution operator and ensure their boundedness.

(LS)(Lopatinskii-Shapiro conditions) For each fixed $t \in J$ and $x \in \Gamma$, we freeze the coefficients of differential operator at (t,x). We rewrite the equations (4.1.1) in coordinates associated with x so that the positive part of x_n-axis has the direction of the inner normal at x after a transformation and a rotation. For all $\eta > 0$, $(\lambda, \xi') \in (\overline{\mathbb{C}_+} \times \mathbb{R}^{n-1}) \setminus \{(0,0)\}$ and $\{h_j\}_{j=0}^m \in F \times E^m$, the ODEs on the half line $\mathbb{R}_+ = (0,\infty)$ given by

$$\begin{cases} \eta v(y) + \mathcal{A}_\#(t,x,\xi',D_y)v(y) &= 0 \qquad (y>0), \\ \mathcal{B}_{0\#}(t,x,\xi',D_y)v(0) + (\lambda + \mathcal{C}_{0\#}(t,x,\xi'))\sigma &= h_0, \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) + \mathcal{C}_{j\#}(t,x,\xi')\sigma &= h_j \qquad (j=1,\cdots,m) \end{cases} \tag{4.2.1}$$

admit a unique solution $(v,\sigma) \in C_0^{2m}(\mathbb{R}_+; E) \times F$, where

$$C_0^{2m}(\mathbb{R}_+; E) = \left\{ v \in C^{2m}(\mathbb{R}_+; E)\,;\, \lim_{y\to\infty} v(y) = 0 \right\}.$$

To obtain the maximal L_p-L_q regularity, we need another type of Lopatinskii-Shapiro condition which ensures boundedness of the symbol of the solution operator.

(ALS) (Asymptotic Lopatinskii-Shapiro conditions) For each fixed $t \in J$ and $x \in \Gamma$ we rewrite the equations (4.1.1) by the same way as above. For all $\eta > 0$, $\xi \in \mathbb{R}^{n-1}$ and $\{h_j\}_{j=1}^m \in F \times E^m$,

$$\begin{aligned} \eta v(y) + \mathcal{A}_{\#}(t,x,\xi',D_y)v(y) &= 0 \qquad (y>0), \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) &= h_j \qquad (j=1,\cdots,m) \end{aligned}$$

admit a unique solution $v \in C_0^{2m}(\overline{\mathbb{R}_+};E)$. For all $(\lambda,\xi') \in \overline{\mathbb{C}_+} \times \mathbb{S}^{n-2}$, all $\{h_j\}_{j=0}^m \in F \times E^m$ the ordinary differential equations in $\mathbb{R}_+ = [0,\infty)$ given by

$$\begin{aligned} \mathcal{A}_{\#}(t,x,\xi',D_y)v(y) &= 0 \qquad (y>0), \\ \mathcal{B}_{0\#}(t,x,\xi',D_y)v(0) + (\lambda + \delta_{l,l_0}\mathcal{C}_{0\#}(t,x,\xi'))\sigma &= h_0, \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) + \delta_{l,l_j}\mathcal{C}_{j\#}(t,x,\xi')\sigma &= h_j \qquad (j=1,\cdots,m) \end{aligned}$$

admit a unique solution $(v,\sigma) \in C_0^{2m}(\overline{\mathbb{R}_+};E) \times F$. Here $\mathbb{S}^{n-2} := \{\xi' \in \mathbb{R}^{n-1}; |\xi'| = 1\}$ and $\delta_{i,j}$ is the Kronecker delta, $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. Moreover, we assume the following elliptic equations. For $\xi' \in \mathbb{S}^{n-2}$,

$$\begin{aligned} \mathcal{A}_{\#}(t,x,\xi',D_y)v(y) = 0 \quad (y>0), \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) = h_j \quad (j=1,\cdots m) \end{aligned}$$

admit a unique solution $v \in C_0^{2m}(\overline{\mathbb{R}_+};E)$, respectively. We are now in the position to state our main results.

Theorem 4.2.1. *Let $J = [0,T]$, $G \subset \mathbb{R}^n$ be a domain with a compact boundary $\Gamma = \partial G$ of class C^{2m+l-m_0}, $1 < p,q < \infty$ and E and F be Hilbert spaces. Assume assumptions (E), (SA), (SB), (SC), (LS) and (ALS) hold. Then, there exist positive constants η_0, C and C_T, if $\eta \geq \eta_0$, for*

$$(f, \{g_j\}_{j=0}^m, \rho_0) \in X \times \prod_{j=0}^m Y_j \times \pi Z_\rho,$$

(4.1.1) admits a unique solution $(u,\rho) \in Z_u \times Z_\rho$ such that

$$\|u\|_{Z_u} + \|\rho\|_{Z_\rho} \leq C\|f\|_X + C\sum_{j=0}^m \|g_j\|_{Y_j} + C_T\|\rho_0\|_{\pi Z_\rho}.$$

4.3 Preliminaries

In this section, notation, notion, basic tools of vector valued harmonic analysis are introduced. For Banach spaces X and Y, $\mathcal{B}(X;Y)$ denotes the set of bounded linear operators from X to Y. $H^\infty(\Sigma_\phi)$ denotes the set of bounded holomorphic functions on a sector

$$\Sigma_\phi := \left\{ re^{i\theta} \in \mathbb{C} \setminus \{0\}\,; r>0,\, |\theta| < \phi \right\}.$$

For a Banach space X, $H^\infty(\Sigma_\phi; X)$ is the set of X-valued bounded holomorphic functions on Σ_ϕ for $0 < \phi < \pi$ equipped with the norm

$$\|f\|_{H^\infty(\Sigma_\phi)} := \sup_{\lambda \in \Sigma_\phi} |f(\lambda)| .$$

$L_p(\Omega; X)$ $(1 \le p \le \infty)$ and $W_q^s(\Omega; X)$ $(s \in \mathbb{R}, 1 \le q \le \infty)$ are the X-valued Lebesgue space and the Sobolev space on Ω. $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Fourier transform and its inverse transform, respectively. Especially, we denote $\mathcal{F}_{x'}$ by the partial Fourier transform with respect to x'-variable. $\mathcal{L}$ and $\mathcal{L}^{-1}$ are Laplace transform and its inverse transform, respectively.

Definition 4.3.1. A Banach space X is said to be of class $\mathcal{HT}$ if the Hilbert transform H defined by

$$Hf(t) := \frac{1}{\pi} \lim_{R\to\infty} \int_{R^{-1} \le |s| \le R} f(t-s) \frac{ds}{s}$$

is bounded on $L_p(\mathbb{R}; X)$ for some $p \in (1, \infty)$. When X is of the class $\mathcal{HT}$, then $L_p(J; X)$ is also of class $\mathcal{HT}$.

Definition 4.3.2. Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is said to be $\mathcal{R}$-bounded, if there exists a constant $C > 0$ and $p \in [1, \infty)$ such that, for each positive integer N, $T_i \in \mathcal{T}$, $x_i \in X$ and for all independent symmetric $\{-1, 1\}$-valued random variables ε_i on a probability space $(\Omega, \mathcal{A}, \mu)$, the inequality

$$\left| \sum_{i=1}^N \varepsilon_i T_i x_i \right|_{L_p(\Omega; Y)} \le C \left| \sum_{i=1}^N \varepsilon_i x_i \right|_{L_p(\Omega; X)} \tag{4.3.1}$$

holds. We denote by $\mathcal{RT}$ the infimum constant of C which (4.3.1) holds.

It is known that if (4.3.1) holds for some $p \in [1, \infty)$, then (4.3.1) holds for all $p \in [1, \infty)$. Note that uniformly bounded family of operators on Hilbert spaces is always $\mathcal{R}$-bounded.

Definition 4.3.3. A Banach space X is said to have property (α) if there exists a constants $C > 0$ such that

$$\left| \sum_{i,j=1}^N \alpha_{ij} \varepsilon_i \varepsilon'_j x_{ij} \right|_{L_2(\Omega \times \Omega'; X)} \le C \left| \sum_{i,j=1}^N \varepsilon_i \varepsilon'_j x_{ij} \right|_{L_2(\Omega \times \Omega'; X)}$$

for all $\alpha_{ij} \in \{-1, 1\}$, $x_{ij} \in X$, positive integer N, and all symmetric independent $\{-1, 1\}$-valued random variables ε_i (respectively ε'_j) on a probability space $(\Omega, \mathcal{A}, \mu)$ (respectively $(\Omega', \mathcal{A}', \mu')$). $\mathcal{HT}(\alpha)$ denotes the class of Banach spaces which belong to $\mathcal{HT}$ and have property (α).

Note that Hilbert space is of the class $\mathcal{HT}(\alpha)$ and all closed subspaces of $L_p(G)$ have property (α).

Proposition 4.3.4 (Operator-valued Fourier multiplier theorem of Lizorkin type, see [37]). *Let $1 < p < \infty$, X and Y be Banach spaces of the class $\mathcal{HT}(\alpha)$. Let $\mathcal{M} \subset C^n(\mathbb{R}^n \setminus \{0\}; \mathcal{B}(X;Y))$ be a family of multipliers such that*

$$\mathcal{R}\left\{\xi^\alpha \partial_\xi^\alpha m(\xi) \,:\, \xi \in \mathbb{R}^n \setminus \{0\},\, \alpha \in \{0,1\}^n,\, m \in \mathcal{M}\right\} =: C_L < \infty.$$

Then $\mathcal{F}^{-1} m \mathcal{F} \in \mathcal{B}\left(L^p(\mathbb{R}^n; X)\,;\, L^p(\mathbb{R}^n; Y)\right)$. Moreover,

$$\mathcal{R}\left\{\mathcal{F}^{-1} m \mathcal{F}\,;\, m \in \mathcal{M}\right\} \leq C C_L,$$

for some constant $C = C(p, n, X, Y)$.

We define a class of holomorphic functions vanishing at the origin and infinity by

$$H_0^\infty(\Sigma_\theta) = \left\{f \in H^\infty(\Sigma_\theta)\,;\, |f(\lambda)| \leq C\,|\chi(\lambda)|^\varepsilon \text{ for some } C > 0,\ \varepsilon > 0\right\},$$

where $1 < \theta < \pi$ and $\chi(\lambda) = \lambda/(1+\lambda)^2$. Let $0 < \theta_A < \theta < \pi$ and A be a sectorial operator with spectral angle θ_A and $f \in H_0^\infty(\Sigma_\theta)$. We define $f(A)$ via the Cauchy formula

$$f(A) := \frac{1}{2\pi i} \int_{\partial \Sigma_\theta} (\lambda - A)^{-1} f(\lambda) d\lambda.$$

It is called that a sectorial operator $A : D(A) \subset X \to Y$ with spectral angle θ_A have a bounded H^∞-calculus if there exists a constant $C > 0$

$$\|f(A)\|_{\mathcal{B}(X;Y)} \leq C \|f\|_{H^\infty(\Sigma_\theta)} \tag{4.3.2}$$

holds for all $f \in H_0^\infty(\Sigma_\theta)$, $(\theta > \theta_A)$. Sectrial operators satisfying (4.3.2) have an extended calculus $f(A)$ for $f \in H^\infty(\Sigma_\theta)$ by the canonical way, and this extension is uniquely determined.

Definition 4.3.5. Let X be a Banach space. Let $0 < \phi_A < \pi$ and A be a sectorial operator on X with spectral angle ϕ_A admitting a bounded H^∞-calculus. A is said to have a $\mathcal{R}$-bounded H^∞- calculus if

$$\left\{h(A) : h \in H^\infty(\Sigma_\phi), |h|_{H^\infty(\Sigma_\phi)} \leq 1\right\} \tag{4.3.3}$$

is $\mathcal{R}$-bounded for some $\phi \geq \phi_A$. Such an operator is denoted by $A \in \mathcal{R}H^\infty(X)$. We denote by $\phi_{\mathcal{R}H^\infty}$ the infimum of ϕ which (4.3.3) holds.

Let us introduce the Kalton-Weis theorem, which gives a sufficient condition for boundedness of joint functional calculus and is used to show boundedness of solution operator in this paper, see e.g. [10], [25], [28] and [37].

Lemma 4.3.6 (Kalton-Weis Theorem, see [25],[10], [28] and [37]). *Let X be a Banach space of the class $\mathcal{HT}(\alpha)$, A be a sectorial operator with spectral angle ϕ_A admitting a bounded H^∞-calculus and $\mathcal{F}$ be a family of operators satisfying $\mathcal{F} \subset H^\infty(\Sigma_\phi; \mathcal{B}(X))$ for $\phi > \phi_A$. Assume each $F \in \mathcal{F}$ commute with the resolvent of A, i.e. $F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1} F(\lambda)$, and*

$$\{F(z) : z \in \Sigma_\phi, F \in \mathcal{F}\}$$

is $\mathcal{R}$-bounded. Then there exist a constant $C > 0$ such that

$$\mathcal{R}\left(\mathcal{F}(A)\right) \leq C \mathcal{R}\left\{F(z) : z \in \Sigma_\phi, F \in \mathcal{F}\right\}.$$

Lemma 4.3.6 also implies each operator admitting bounded H^∞-calculus belongs to $\mathcal{R}H^\infty$ provided that X is of class $\mathcal{HT}(\alpha)$.

4.4 Solvability in the Maximal Regularity Space

4.4.1 Reduction to $f = 0$ and $\rho_0 = 0$

We first consider our problem on the half space $\mathbb{R}^n_+$ and assume the differential operators have constant coefficients without lower order. Let $E_{\mathbb{R}^n}$ be the zero extension operator from $L_p(J; L_q(\mathbb{R}^n_+; E))$ to $L_p(J; L_q(\mathbb{R}^n; E))$. It follows from the Mikhlin theorem that there exist a unique solution u_* to

$$\eta u + \mathcal{A}u = E_{\mathbb{R}^n} f \quad (t \in J,\ x \in \mathbb{R}^n),$$

for $f \in L_p(J; L_q(\mathbb{R}^n_+; E))$ such that

$$\eta \|u_*\|_{L_p(J;L_q(\mathbb{R}^n;E))} + \|u_*\|_{L_p(J;W^{2m}_q(\mathbb{R}^n;E))} \leq C\|f\|_{L_p(J;L_q(\mathbb{R}^n_+;E))}.$$

Let $\rho_0 \in B^{2m\kappa_0 + l(1-1/p)}_{q,p}(\mathbb{R}^{n-1}; F)$. Then we also find a unique solution ρ_* via maximal regularity of to $(\eta - \Delta)^{l/2}$ that

$$\begin{aligned} \partial_t \rho + (\eta - \Delta)^{l/2}\rho &= 0 \qquad (t \in \mathbb{R}_+, x \in \mathbb{R}^{n-1}), \\ \rho(0) &= \rho_0 \qquad (x \in \mathbb{R}^{n-1}) \end{aligned}$$

such that

$$\|\rho_*\|_{W^1_p(\mathbb{R}_+;W^{2m\kappa_0}_q(\mathbb{R}^{n-1};F)) \cap L_p(\mathbb{R}_+;W^{l+2m\kappa_0}_q(\mathbb{R}^{n-1};F))} \leq C\|\rho_0\|_{B^{2m\kappa_0+l(1-1/p)}_{q,p}(\mathbb{R}^{n-1};F)}.$$

For the solution (u, ρ) to (4.1.1), if we put $(\tilde{u}, \tilde{\rho}) := (u - u_*, \rho - \rho_*)$, then $(\tilde{u}, \tilde{\rho})$ satisfies

$$\begin{aligned} \eta\tilde{u} + \mathcal{A}\tilde{u} &= 0 && (t \in J,\ x \in G), \\ \partial_t\tilde{\rho} + \mathcal{B}_0\tilde{u} + \mathcal{C}_0\tilde{\rho} &= g_0 - (\partial_t\rho_* + \mathcal{B}_0 u_* + \mathcal{C}_0\rho_*) && (t \in J,\ x \in \Gamma), \\ \mathcal{B}_j\tilde{u} + \mathcal{C}_j\tilde{\rho} &= g_j - (\mathcal{B}_j u_* + \mathcal{C}_j\rho_*) && (t \in J,\ x \in \Gamma,\ j = 1, \cdots, m), \\ \tilde{\rho}(0, x) &= 0 && (x \in \Gamma). \end{aligned} \tag{4.4.1}$$

Note that $g_0 - (\partial_t\rho_* + \mathcal{B}_0 u_* + \mathcal{C}_0\rho_*) \in Y_0$ and $g_j - (\mathcal{B}_j u_* + \mathcal{C}_j\rho_*) \in Y_j$. Conversely, the solution of the original equations is given by $(u, \rho) := (\tilde{u} + u_*, \tilde{\rho} + \rho_*)$. Thus, it suffice to consider the case of $f = 0$ and $\rho_0 = 0$ from now on.

4.4.2 Partial Fourier transform and solution formula on the half space

We continue to consider the case of the half space and assume differential operators having constant coefficients without lower order terms. Assume that (u, ρ) are solutions to (4.1.1) with $f = 0$ and $\rho_0 = 0$. Put

$$v = \mathcal{L}_t\mathcal{F}_{x'}u, \quad \sigma = \mathcal{L}_t\mathcal{F}_{x'}\rho.$$

Then (v, ρ) satisfy

$$\begin{cases} \eta v + \mathcal{A}(\xi', D_y)v = 0, \\ \mathcal{B}_0(\xi', D_y)v(0) + (\lambda + \mathcal{C}_0(\xi'))\sigma = h_0, \\ \mathcal{B}_j(\xi', D_y)v(0) + \mathcal{C}_j(\xi')\sigma = h_j \qquad (j = 1, \cdots, m), \end{cases} \tag{4.4.2}$$

where $h_j = \mathcal{L}_t \mathcal{F}_{x'} g_j$ for $j = 0, \cdots, m$. The Lopatinskii–Shapiro condition (LS) ensures, for each $(\xi', \lambda) \in (\mathbb{R}^{n-1} \times \overline{\mathbb{C}_+}) \setminus \{(0, 0)\}$ and for any $\{h_j\}_{j=0}^m \in F \times E^m$, there exists a unique solution

$$v \in C_0^{2m}([0, \infty); E), \quad \sigma \in F$$

to (4.4.2). We derive the solution operator of Fourier multiplier type and show its boundedness. As [2, 8, 37], we construct the solution formula. By definition of $\mathcal{A}$ and $\mathcal{B}_j$,

$$\mathcal{A}(\xi', D_y) = \sum_{k=0}^{2m} a_k(\xi') D_y^{2m-k}, \quad \mathcal{B}_j(\xi', D_y) = \sum_{k=0}^{m_j} b_{jk}(\xi') D_y^{m_j - k},$$

where $a_k(\xi')$ and $b_{jk}(\xi')$ are homogeneous of degree k. Set

$$\mu = (\eta + |\xi'|^{2m})^{1/2m}, \quad b = |\xi'|/\mu, \quad \zeta = \xi'/\mu, \quad a = \eta/\mu^{2m}$$

and $w := (w_1, \cdots, w_{2m})^T$ for

$$w_k := \left(\frac{1}{\mu} D_y\right)^{k-1} v \quad (k = 1, \cdots, 2m).$$

Note that $(\mu, \zeta, a) \in [\eta^{1/2m}, \infty) \times B_{\mathbb{R}^{n-1}}(0\,; 1) \times (0, 1]$, where $B_{\mathbb{R}^d}(c; r)$ is the d-dimensional open ball with center c and radius r. Then the first equation of (4.4.2) is equivalent to

$$D_y w = \mu A_0(\zeta, a) w, \tag{4.4.3}$$

where

$$A_0(\zeta, a) := \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & I \\ c_{2m} & c_{2m-1} & \cdots & c_2 & c_1 \end{pmatrix},$$

and

$$\begin{aligned} c_j &:= c_j(\zeta) := -a_0^{-1} a_j(\zeta) \qquad (j = 1, \cdots, 2m-1), \\ c_{2m} &:= c_{2m}(\zeta, a) := -a_0^{-1}(a_{2m}(\zeta) + a). \end{aligned}$$

Actually, it follows from the first equation of (4.4.2) and definition of w_j that

$$(1/\mu)D_y w_{2m} = -a_0^{-1}\left(a_{2m}(\xi'/\mu) + \eta/\mu^{2m}\right) w_1 - \sum_{k=2}^{2m} a_0^{-1} a_{2m-k+1}(\xi'/\mu) w_k.$$

Thus, we find (4.4.3) from the definition of w. Moreover, (4.4.3) implies

$$w(y) := e^{\mu i A_0(\zeta,a)y} w(0) \quad (y \geq 0).$$

We write $w(0) = w|_{y=0}$ for simplicity. The functions $w(y)$ have to be determined so that tends to zero at infinity. This is guaranteed by

$$P_+(\zeta, a) w_0 = 0,$$

where $P_+(\zeta, a) \in \mathcal{B}(E^{2m})$ is the associated positive spectral projection with $iA_0(\zeta, a)$. Note that each spectrum of $iA_0(\zeta, a))$ do not lie on the imaginary axis and P_+ is holomorphic and bounded uniformly in (ζ, a) by the Lopatinskii-Shapiro condition since (ζ, a) run a compact set away from $(0, 0)$. Supremum of real part of negative spectrum of $iA(\zeta, a)$ is bounded by above and infimum of real part of positive spectrum is bounded by below, this facts imply $e^{\mu i A(\zeta,a)y} w(0) \to 0$ for $w(0)$ satisfying $P_+(\zeta, a)w(0) = 0$ as $y \to \infty$. See [8] and [37] for details of the above discussion.
Let $w := \mathcal{F}_{x'}\mathcal{L}_t h$ for $h \in L_p(J; W_q^{2m-1/p}(\mathbb{R}^{n-1}; E^{2m}))$. Define the canonical extension of functions from the boundary to the half space by

$$\mathcal{T}h := \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\mu^{2m} e^{\mu i A_0(\zeta,a)y}\left(I - P_+(\zeta, a)\right) w\right]. \tag{4.4.4}$$

Boundedness of this extension operator is ensured by the following

Proposition 4.4.1. *Let $1 < p, q < \infty$, $\eta \in \Sigma_\varphi$ for small $\varphi > 0$ and $J = \mathbb{R}_+$. Then there exists a constant $C > 0$ such that*

$$\|\mathcal{T}h\|_{L_p(J;L_q(\mathbb{R}^n_+;E^{2m}))} \leq C\|h\|_{L_p(J;W_q^{2m-1/p}(\mathbb{R}^{n-1};E^{2m}))}. \tag{4.4.5}$$

for $h \in L_p(J; W_q^{2m-1/p}(\mathbb{R}^{n-1}; E^{2m}))$.

Proof. See [8], section 7. □

Put $w^0 = w(0)$. Let us continue to seek the solution formula of (w^0, σ). Since

$$\mathcal{B}_j v = \sum_{k=0}^{m_j} b_{jk}(\xi')\mu^{m_j-k} w_{m_j-k+1} = \sum_{k=0}^{m_j} b_{jk}(\zeta)\mu^{m_j} w_{m_j-k+1}$$

and

$$\mathcal{C}_j(\xi')\sigma = \mathcal{C}_j(\zeta)\mu^{k_j}\sigma$$

the second and the third equations of (4.4.2) are equivalent to

$$B_0(\zeta)w^0 + \left\{\lambda\mu^{-m_0} + \mathcal{C}_0(\zeta)\mu^{-m_0+k_0}\right\}\sigma = \mu^{-m_0} h_0,$$

$$B_j(\zeta)w^0 + \mathcal{C}_j(\zeta)\mu^{-m_j+k_j}\sigma = \mu^{-m_j}h_j \quad (j = 1, \cdots, m), \qquad (4.4.6)$$
$$P_+(\zeta, a)w^0 = 0,$$

where $B_j(\zeta) := (b_{jm_j}(\zeta), \cdots, b_{j0}, 0, \cdots, 0)$ for $j = 0, \cdots, m$. Note that by the assumptions on (E) and (LS) the above equations (4.4.6) admit a unique solution

$$(w^0, \sigma) \in E^{2m} \times F$$

for each $(\xi', \lambda) \in (\mathbb{R}^{n-1} \times \overline{\mathbb{C}_+}) \setminus \{(0, 0)\}$ and $\{h_j\}_{j=0}^m \in F \times E^m$. Introducing $\sigma^0 := (\lambda + \mu^l)\mu^{-m_0}\sigma$ and $h := (h_j^0)_{j=0}^m := (\mu^{-m_j}h_j)_{j=0}^m$, we rewrite (4.4.6) into

$$\begin{aligned} B_0(\zeta)w^0 + \frac{\lambda + \mathcal{C}_0(\zeta)\mu^{l_0}}{\lambda + \mu^l}\sigma^0 &= h_0^0, \\ B_j(\zeta)w^0 + \frac{\mathcal{C}_j(\zeta)\mu^{l_j}}{\lambda + \mu^l}\sigma^0 &= h_j^0 \quad (j = 1, \cdots, m), \\ P_+(\zeta, a)w^0 &= 0. \end{aligned} \qquad (4.4.7)$$

Thus, it follows

$$\begin{aligned} B_0(\zeta)w^0 + \frac{\nu + \mathcal{C}_0(\zeta)\eta^{-(l_0-l)/2m}\tilde{a}^{l-l_0}}{\nu + 1}\sigma^0 &= h_0^0, \\ B_j(\zeta)w^0 + \frac{\mathcal{C}_j(\zeta)\eta^{-(l_j-l)/2m}\tilde{a}^{l-l_j}}{\nu + 1}\sigma^0 &= h_j^0 \quad (j = 1, \cdots, m), \\ P_+(\zeta, \tilde{a}^{2m})w^0 &= 0. \end{aligned} \qquad (4.4.8)$$

for $\nu = \lambda/\mu^l$ and $\tilde{a} = a^{1/2m}$. We write the solution to (4.4.8) as

$$w^0 := M_w^0(\zeta, \tilde{a}, \nu)h, \quad \sigma^0 := M_\sigma^0(\zeta, \tilde{a}, \nu)h.$$

Set $Y_E := L_p(J; W_q^{2m-1/p}(\mathbb{R}^{n-1}; E))$ and $Y_F := L_p(J; W_q^{2m-1/p}(\mathbb{R}^{n-1}; F))$.

Proposition 4.4.2. *Let $1 < p, q < \infty$, $\eta > 0$ and $G = \mathbb{R}_+^n$. Assume assumptions (E), (SA), (SB), (SC), (LS) and (ALS) hold. Then, there exist a positive constant $C > 0$, it holds that*

$$\|\mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\left(M_w^0, M_\sigma^0\right)\mathcal{F}_{x'}\mathcal{L}_t h\right]\|_{Y_E^{2m} \times Y_F} \le C\|h\|_{Y_F \times Y_E^m}$$

for all $h \in Y_F \times Y_E^{2m}$.

Proof. Analyticity of (M_w^0, M_σ^0) on some open set $D_\zeta \times D_a \times \mathbb{C}_+ \subset \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}_+$ including $B_{\mathbb{R}^{n-1}}(0\,; 1) \times [0, 1] \times \mathbb{C}_+$ has been already proved in [2]. Boundedness of (M_w^0, M_σ^0) is equivalent to the solvability of $\zeta \in D_\zeta$, $\tilde{a} \in D_a$ and $\nu \in \overline{\mathbb{C}_+} \cup \{\infty\}$. The solvability of (M_w^0, M_σ^0) in the case of $\mu \neq \infty$ and $\lambda \neq \infty$ is guaranteed by (LS). We need to control behaviour of (M_w^0, M_σ^0) on ν at infinity. Let us consider the case of $|\mu| \to \infty$ or $|\lambda| \to \infty$. We find

$$\frac{\eta^{-(l_0-l)/2m}\tilde{a}^{l-l_j}}{\nu + 1} \to \begin{cases} 0 & \text{if} \quad |\lambda|/|\mu|^l \to \infty, \\ \frac{\delta_{l,l_j}}{c+1} & \text{if} \quad \lambda/\mu^l \to c, \end{cases}$$

and

$$\frac{\eta^{-(l_0-l)/2m}\tilde{a}^{l-l_j}}{\nu+1} \to \begin{cases} 0 & \text{if} \quad l_j < l \quad \text{and} \quad |\lambda|/|\mu|^l \to \infty, \\ \frac{1}{c+1} & \text{if} \quad l_j = l \quad \text{and} \quad \lambda/\mu^l \to c, \end{cases}$$

and

$$\frac{\nu}{\nu+1} \to \begin{cases} 1 & \text{if} \quad |\lambda|/|\mu|^l \to \infty, \\ \frac{c}{c+1} & \text{if} \quad \lambda/\mu^l \to c, \end{cases}$$

for some $c \in \overline{\mathbb{C}_+}$. Let us consider the case (i) ($|\lambda|/|\mu|^l \to \infty$). The limit problem of this case is

$$\begin{aligned} B_0(\zeta)w^0 + \sigma^0 &= h_0^0, \\ B_j(\zeta)w^0 &= h_j^0 \quad (j = 1, \cdots, m), \\ P_+(\zeta, a_*)w^0 &= 0. \end{aligned} \tag{4.4.9}$$

for some $a_* > 0$ which is the limit of $\tilde{a}^{2m}$. If μ tend to infinity at the same time, this system corresponds to the following problem; for all $\{h_j\}_{j=1}^m \in E^m$ and for any $\xi' \in \mathbb{S}^{n-2}$,

$$\begin{aligned} \mathcal{A}(\xi', D_y)v(y) = 0 \quad (y > 0), \\ \mathcal{B}_0(\xi', D_y)v^0 + \sigma^0 = h_0^0, \\ \mathcal{B}_j(\xi', D_y)v^0 = h_j \quad (j = 1, \cdots, m), \end{aligned}$$

admits a unique solution $v \in C_0^{2m}([0,\infty); E)$, which is guaranteed by the third asymptotic Lopatinskii-Shapiro condition. On the other hand, if μ stile finite, the corresponding problem is given by

$$\begin{aligned} \eta v(y) + \mathcal{A}(\xi', D_y)v(y) = 0 \quad (y > 0), \\ \mathcal{B}_0(\xi', D_y)v^0 + \sigma^0 = h_0^0, \\ \mathcal{B}_j(\xi', D_y)v^0 = h_j \quad (j = 1, \cdots, m), \end{aligned}$$

for all $\xi' \in \mathbb{R}^{n-1}$ and $\{h_j\}_{j=1}^m \in E^m$. This problem is solvable by the first asymptotic Lopatinskii- Shapiro condition. Next we consider case (ii) ($\lambda/\mu^l \to c \in \overline{\mathbb{C}_+}$). In these cases, the limit problem is

$$\begin{aligned} \mathcal{B}_0(\zeta)w^0 + \frac{c + \delta_{l,l_0}\mathcal{C}_0(\zeta)}{c+1}\sigma^0 &= h_0^0, \\ \mathcal{B}_j(\zeta)w^0 + \frac{\delta_{l,l_j}\mathcal{C}_j(\zeta)}{c+1}\sigma^0 &= h_j^0 \quad (j = 1, \cdots, m), \\ P_+(\zeta, 0)w^0 &= 0, \end{aligned} \tag{4.4.10}$$

where $c = 0$ is included. To ensure solvability of this problem, it is enough to impose the following the condition; for all $\{h_j\}_{j=0}^m \in F \times E^m$and for any $\xi' \in \mathbb{S}^{n-2}$ and $\lambda \in \overline{\mathbb{C}_+}$,

$$\mathcal{A}(\xi', D_y)v(y) = 0 \quad (y > 0),$$

$$\mathcal{B}_{0\#}(\xi', D_y)v_0 + (\lambda + \delta_{l,l_0}\mathcal{C}_{0\#}(\xi'))\sigma = h_0,$$
$$\mathcal{B}_{j\#}(\xi', D_y)v_0 + \delta_{l,l_j}\mathcal{C}_{j\#}(\xi')\sigma = h_j \quad (j = 1, \cdots, m)$$

admits a unique solution $(v, \sigma) \in C_0^{2m}([0, \infty); E) \times F$. This condition is nothing but the second asymptotic Lopatinskii-Shapiro condition. We find from the above discussion that M_w^0 and M_σ^0 is bounded holomorphic on a open set $D_\zeta \times D_a \times \overline{\mathbb{C}_+}$. Moreover,

$$\left\{(M_w^0, M_\sigma^0)(\zeta, \tilde{a}, \nu) : (\zeta, \tilde{a}, \nu) \in \overline{B_{\mathbb{R}^{n-1}}(0\,; 1)} \times [0, 1] \times \overline{\mathbb{C}_+}\right\}$$

is $\mathcal{R}$-bounded since E and F are Hilbert spaces. Set $M^0 = (M_w^0, M_\sigma^0)$ and

$$L_1 = -i\nabla'(\eta + (-\Delta')^m)^{-1/2m},$$
$$L_2 = \eta^{1/2m}(\eta + (-\Delta')^m)^{-1/2m},$$
$$L_3 = \partial_t(\eta + (-\Delta')^m))^{-l/2m}.$$

Then, boundedness and analyticity of M^0 leads

$$\mathcal{R}\left\{\xi'^{\alpha}\partial_{\xi'}^{\alpha} M^0\left(\frac{\xi'}{(\eta + |\xi'|^{2m})^{1/2m}}, \frac{1}{(\eta + |\xi'|^{2m})^{1/2m}}, \nu\right)\right.$$
$$\left. : \xi' \in \mathbb{R}^{n-1} \setminus \{0\},\ \alpha \in \{0, 1\}^{n-1},\ \nu \in \overline{\mathbb{C}_+}\right\} < \infty.$$

Thus, the operator valued Fourier multiplier theorem implies

$$M^0(L_1, L_2, \nu) \in \mathcal{B}(Y_F \times Y_E^m; Y_E^{2m} \times Y_F)$$

and $\left\{M^0(L_1, \mu, \nu) : \nu \in \overline{\mathbb{C}_+}\right\}$ is $\mathcal{R}$-bounded on $\mathcal{B}(Y_F \times Y_E^m; Y_E^{2m} \times Y_F)$ such that

$$\mathcal{R}\left\{M^0(L_1, L_2, \nu) : \nu \in \overline{\mathbb{C}_+}\right\} < \infty.$$

Finally, we use the Kalton-Weis theorem to find
$M^0(L_1, L_2, L_3) \in \mathcal{B}(Y_F \times Y_E^m; Y_E^{2m} \times Y_F)$.

□

We find from Proposition 4.4.1, Proposition 4.4.2 and

$$\left(\frac{d}{dt} + (-\Delta')^m\right)^{l/2m}$$
$$\in \mathrm{Isom}(W_p^1(J; W_q^{2m-1/p}(\mathbb{R}^{n-1}; F)) \cap L^p(J; W_q^{l+2m-1/p}(\mathbb{R}^{n-1}; F)); Y_F),$$
$$(\eta + (-\Delta')^m)^{m_0/2m}$$
$$\in \mathrm{Isom}(W_p^1(J; W_q^{2m-1/p}(\mathbb{R}^{n-1}; F)) \cap L^p(J; W_q^{l+2m-1/p}(\mathbb{R}^{n-1}; F)); Z_\rho),$$
$$(\eta + (-\Delta')^m)^{m_0/2m} \in \mathrm{Isom}(Y_F; Y_0)$$
$$(\eta + (-\Delta')^m)^{m_j/2m} \in \mathrm{Isom}(Y_F; Y_j) \quad (j = 1, \cdots, m),$$

that

$$\|u(0)\|_{Y_E} + \left\| \left(\frac{d}{dt} + (-\Delta')^m \right)^{l/2m} (\eta + (-\Delta')^m)^{m_0/2m} \rho \right\|_{Y_F}$$
$$\leq C \left(\| (\eta + (-\Delta')^m)^{m_0/2m} g_0\|_{Y_F} + \sum_{j=1}^{m} \| (\eta + (-\Delta')^m)^{m_j/2m} g_j\|_{Y_E} \right),$$

where $u(0) := u|_{y=0}$ This leads the desired maximal L_p regularity

$$\eta\|u\|_X + \|u\|_{Z_u} + \|\rho\|_{Z_\rho} \leq \; C \left(\|g_0\|_{Y_0} + \sum_{j=1}^{m} \|g_j\|_{Y_j} \right)$$

for some $C > 0$. We finish proving Theorem 4.2.1 in the case of the half space.

4.4.3 The case of bounded domain

Let us consider the case of a bounded domain G and a exterior. The proof is based on (i) solving the case of variable coefficient with lower order terms, (ii) localization procedure and coordinate transform. Since this method is well-known, we do not give a detail of the proof, see [2, 8, 28, 37] for example. We show only outline of the proof. Note that conditions (E), (LS) and (ALS) are invariant under the coordinate transform. First we give estimates for lower-order terms.

Proposition 4.4.3. *Let $a_\alpha, b_{j\beta}, c_{j\gamma}$ satisfy (SA), (SB) and (SC), then there exists $C > 0$ such that*

$$\|a_\alpha D^\alpha u\|_X \leq C\|a_\alpha\|_{L_\infty(J;L_{r_\alpha}(G))}\|u\|_{Z_u}, \quad (|\alpha| \leq 2m-1)$$
$$\|b_{j\beta} D^\beta u\|_{Y_j} \leq C\|b_{j\beta}\|_{L_\infty(J;(L_{s_{j\beta}}\cap B^{2m\kappa_j}_{r_{j\beta},q})(\Gamma))}\|u\|_{Z_u}, \quad (|\beta| \leq m_j - 1)$$
$$\|c_{j\gamma} D^\gamma_\Gamma \rho\|_{Y_j} \leq C\|c_{j\gamma}\|_{L_{t_{j\gamma}}(J;L_{s_{j\gamma}}(\Gamma))\cap L_{\tau_{j\gamma}}(J;B^{2m\kappa_j}_{r_{j\gamma},q}(\Gamma))}\|\rho\|_{Z_\rho}, \quad (|\gamma| \leq k_j - 1)$$

Proof. First, for each $|\alpha| = k \leq 2m-1$, the assumption (SA) derives

$$\begin{aligned}\|a_\alpha D^\alpha u\|_{L_q(G)} &\leq \|a_\alpha\|_{L_{r_\alpha}(G)}\|D^\alpha u\|_{L_{r'_\alpha}(G)}\\ &\leq \|a_\alpha\|_{L_{r_\alpha}(G)}\|u\|_{W^{2m}_q(G)},\end{aligned}$$

where $1/q = 1/r_\alpha + 1/r'_\alpha$ and we use the embedding $W^{2m}_q(G) \hookrightarrow W^k_{r'_\alpha}(G)$. This means

$$\|a_\alpha D^\alpha u\|_X \leq \|a_\alpha\|_{L_\infty(J;L_{r_\alpha}(G))}\|u\|_{Z_u}$$

Second, for each $|\beta| = k \leq m_j - 1$, we find from paraproduct formula, definition of Besov spaces on a domain and the assumption (SB) that

$$\|b_{j\beta} D^\beta u\|_{W^{2m\kappa_j}_q(\Gamma)} \leq C(\|b_{j\beta}\|_{L_{s_{j\beta}}(\Gamma)}\|D^\beta u\|_{B^{2m\kappa_j}_{s'_{j\beta},q}(\Gamma)} + \|b_{j\beta}\|_{B^{2m\kappa_j}_{r_{j\beta},q}(\Gamma)}\|D^\beta u\|_{L_{r'_{j\beta}}(\Gamma)})$$

$$\leq C\|b_{j\beta}\|_{(L_{s_{j\beta}}\cap B^{2m\kappa_j}_{r_{j\beta},q})(\Gamma)}\|u\|_{W^{2m-1/p}_q(\Gamma)}, \tag{4.4.11}$$

where $1/q = 1/s_{j\beta} + 1/s'_{j\beta} = 1/r_{j\beta} + 1/r'_{j\beta}$ and we used the embeddings

$$\mathrm{tr}|_\Gamma u \in W^{2m-1/p}_q(\Gamma) \hookrightarrow (B^{k+2m\kappa_j}_{s'_{j\beta},q} \cap W^k_{r'_{j\beta}})(\Gamma).$$

Moreover, we have $\mathrm{tr}|_\gamma u \in L_p(J; W^{2m-1/p}(\Gamma))$ for $u \in Z_u$. This means

$$\|b_{j\beta}D^\beta u\|_{Y_j} \leq C\|b_{j\beta}\|_{L_\infty(J;(L_{s_{j\beta}}\cap W^k_{r'_{j\beta}})(\Gamma))}\|u\|_{Z_u}.$$

At last, for each $|\gamma| = k \leq k_j - 1$, it follows from the same way as for (4.4.11) that

$$\begin{aligned}&\|c_{j\gamma}D^\gamma_\Gamma\rho\|_{W^{2m\kappa_j}_q(\Gamma)}\\ &\leq C\left(\|c_{j\gamma}\|_{L_{s^c_{j\gamma}}(\Gamma)}\|D^\gamma_\Gamma\rho\|_{B^{2m\kappa_j}_{s^{c'}_{j\gamma},q}(\Gamma)} + \|c_{j\gamma}\|_{B^{2m\kappa_j}_{r^c_{j\gamma},q}(\Gamma)}\|D^\gamma_\Gamma\rho\|_{L_{r^{c'}_{j\gamma}}(\Gamma)}\right)\end{aligned}$$

where $1/q = 1/s^c_{j\gamma} + 1/s^{c'}_{j\gamma} = 1/r^c_{j\gamma} + 1/r^{c'}_{j\gamma}$. Integral in time and use Hölder's inequality,

$$\begin{aligned}\|c_{j\gamma}D^\gamma_\Gamma\rho\|_{Y_j} \leq{}& C\|c_{j\gamma}\|_{L_{t_{j\gamma}}(J;L_{s^c_{j\gamma}}(\Gamma))}\|D^\gamma_\Gamma\rho\|_{L_{t'_{j\gamma}}(J;B^{2m\kappa_j}_{s^{c'}_{j\gamma},q}(\Gamma))}\\ &+ C\|c_{j\gamma}\|_{L_{\tau_{j\gamma}}(J;B^{2m\kappa_j}_{r^c_{j\gamma},q}(\Gamma))}\|D^\gamma_\Gamma\rho\|_{L_{\tau'_{j\gamma}}(J;L_{r^{c'}_{j\gamma}}(\Gamma))})\end{aligned}$$

where $1/p = 1/t_{j\gamma} + 1/t_{j\gamma} = 1/\tau_{j\gamma} + 1/\tau'_{j\gamma}$. Here we use the mixed derivative theorems

$$Z_\rho = W^1_p(J; W^{2m\kappa_0}_q(\Gamma)) \cap L_p(J; W^{l+2m\kappa_0}_q(\Gamma)) = \bigcap_{0\leq\theta\leq1} W^\theta_p(J; W^{l(1-\theta)+2m\kappa_0}_q(\Gamma)).$$

The assumption (SC) ensures the existence of $\theta \in [0,1]$ such that

$$W^\theta_p(J; W^{l(1-\theta)+2m\kappa_0}_q(\Gamma)) \hookrightarrow L_{t'_{j\gamma}}(J; B^{k+2m\kappa_j}_{s^{c'}_{j\gamma},q}(\Gamma)), L_{\tau'_{j\gamma}}(J; W^k_{r^{c'}_{j\gamma}}(\Gamma)),$$

respectively. This means

$$\|c_{j\gamma}D^\gamma_\Gamma\rho\|_{Y_j} \leq C\|c_{j\gamma}\|_{L_{t_{j\gamma}}(J;L_{s^c_{j\gamma}}(\Gamma))\cap L_{\tau_{j\gamma}}(J;B^{2m\kappa_j}_{r^c_{j\gamma},q}(\Gamma))}\|\rho\|_{Z_\rho}.$$

□

Proposition 4.4.4. *Let $J = [0,T]$, $G = \mathbb{R}^n_+$ and $\Gamma = \partial G$, $1 < p, q < \infty$ and E and F be separable Hilbert spaces. Let assumptions (E), (SA), (SB), (SC), (LS) and (ALS) hold. Assume $\mathcal{A}$, $\mathcal{B}_j$ and $\mathcal{C}_j$ are given by*

$$\mathcal{A}(t,x,D) = \mathcal{A}_\#(D) + \mathcal{A}^{small}(t,x,D) + \mathcal{A}^{low}(t,x,D),$$
$$\mathcal{B}_j(t,x,D) = \mathcal{B}_{j\#}(D) + \mathcal{B}^{small}_j(t,x,D) + \mathcal{B}^{low}_j(t,x,D),$$

$$\mathcal{C}_j(t,x',D_\Gamma) = \mathcal{C}_{j\#}(D_\Gamma) + \mathcal{C}_j^{small}(t,x',D_\Gamma) + \mathcal{C}_j^{low}(t,x',D_\Gamma),$$

where $\mathcal{A}_\#$, $\mathcal{B}_{j\#}$ and $\mathcal{C}_{j\#}$ satisfy (LS) and (ALS), $\mathcal{A}^{low}$, $\mathcal{B}_j^{low}$ $(j = 0,\cdots,m)$ and $\mathcal{C}_j^{low}$ are lower order terms and

$$\begin{aligned} \|\mathcal{A}^{small}(x,D)u\|_X &\le \delta\|u\|_{Z_u}, \\ \|\mathcal{B}_j^{small}(t,x,D)u\|_{Y_j} &\le \delta\|u\|_{Z_u} \quad (j = 0,\cdots,m), \\ \|\mathcal{C}_j^{small}(t,x,D)\rho\|_{Y_j} &\le \delta\|\rho\|_{Z_\rho} \quad (j = 0,\cdots,m), \end{aligned}$$

for sufficiently small $\delta > 0$. Then, there exist positive constants $\eta_0 > 0$, C and C_T, for

$$(f, \{g_j\}_{j=0}^m, \rho_0) \in X \times \prod_{j=0}^m Y_j \times \pi Z_\rho,$$

if $\eta \ge \eta_0$, the equations (4.1.1) admits a unique solution $(u,\rho) \in Z_u \times Z_\rho$ such that

$$\eta\|u\|_X + \|u\|_{Z_u} + \|\rho\|_{Z_\rho} \le C\|f\|_X + C\sum_{j=0}^m \|g_j\|_{Y_j} + C_T\|\rho_0\|_{\pi Z_\rho}. \tag{4.4.12}$$

Proof. Assume $|J|$ be small, where $|J|$ is the length of J. Clearly, $\|u\|_{D(\mathcal{A}^{low})} \le \|u\|_{D(\mathcal{A}_\#)}$, $\|u\|_{D(\mathcal{B}_j^{low})} \le \|u\|_{D(\mathcal{B}_{j\#})}$ and $\|u\|_{D(\mathcal{C}_j^{low})} \le \|u\|_{D(\mathcal{C}_{j\#})}$. Thus, if we take $\eta > 0$ sufficiently large, we find from the space-time Sobolev embedding, which enable us to estimate lower-order terms as small perturbation since $|J|$ is small, and the Neumann series argument that can be estimated $\mathcal{A}^{small}$, $\mathcal{A}^{low}$, $\mathcal{B}_j^{small}$, $\mathcal{B}_j^{low}$, $\mathcal{C}_j^{small}$ and $\mathcal{C}_j^{low}$ as relatively small perturbations. For J with arbitrary finite length, we can divide J into finite short intervals. For these short intervals, we can apply the same argument as above step by step to get (4.4.12). □

Now we prove Theorem 4.2.1. For the sake of simplicity, we consider the case of bounded domains. The case of exterior domains is treated by similar way. Temporarily, we assume $|J|$ to be small. Let $\delta > 0$ be small. Let us introduce an open covering of G such that

$$\begin{aligned} &G \subset \cup_{k=0}^N U_k \\ &U_k = B(x_k,\delta), \quad x_k \in G \quad (k = 0,\cdots,M) \\ &U_k = B(x_k,\delta), \quad x_k \in \partial G \quad (k = M+1,\cdots,N) \end{aligned}$$

for some M and N. We also introduce a partition of unity $\{\varphi_j\}_{j=0}^N$ satisfying

$$\varphi_j \in C_0^\infty(\mathbb{R}^n), \quad 0 \le \varphi_j \le 1, \quad \operatorname{spt}\varphi_j \subset U_j, \quad \sum_{j=0}^N \varphi_j = 1 \text{ on } G$$

Suppose

$$(u,\rho) \in Z_u \times Z_\rho$$

be a solution to (4.1.1). For $k \geq M+1$, we apply the canonical coordinate transform, which is denoted by Φ_k, from U_k to local neighborhood of the half space so that $U_k \cap \Gamma$ is flat. Since coefficients are continuous, if we take $\delta > 0$ be sufficiently small beforehand, we can extend coefficients to the half space and write these extended coefficients as a_α^k, $b_{j\beta}^k$ and $c_{j\gamma}^k$, to the half space so that

$$\begin{aligned}
&\|a_\alpha^k - a_\alpha(0, x_k)\|_{L_\infty(J\times\mathbb{R}^n_+;E)},\\
&\|b_{j\beta}^k - b_{j\beta}(0, x_k)\|_{L_\infty(J\times\mathbb{R}^{n-1};E)},\\
&\|c_{j\gamma}^k - c_{j\gamma}(0, x_k)\|_{L_\infty(J\times\mathbb{R}^{n-1};F)}
\end{aligned}$$

are sufficiently small. Put $(u_k, \rho_k, f_k, g_{jk}) = (\phi_k u, \phi_k\rho, \phi_k f, \phi_k g_j)$ for $k = 1, \cdots, N$. Then, for $k = 0, \cdots, N$, (u_k, ρ_k) satisfies

$$\begin{cases}
\eta u_k + \mathcal{A}_\# u_k = F_k(f_k, u) & (t \in J,\ x \in G),\\
\partial_t \rho_k + \mathcal{B}_{0\#} u_k + \mathcal{C}_{0\#}\rho_k = G_{0,k}(g_{0,k}, u, \rho) & (t \in J,\ x \in \Gamma),\\
\mathcal{B}_{j\#} u_k + \mathcal{C}_{j\#}\rho_k = G_{j,k}(g_{j,k}, u, \rho) & (t \in J,\ x \in \Gamma) \quad (j = 1, \cdots, m),\\
\rho_k(0) = 0 & (x \in \Gamma),
\end{cases} \tag{4.4.13}$$

for

$$\begin{aligned}
F_k(f_k, u) &= f_k - \varphi_k(\mathcal{A} - \mathcal{A}_\#)u + [\mathcal{A}, \varphi_k]u\\
&G_{0,k}(g_{0,k}, u, \rho)\\
&= g_{0,k} - \varphi_k(\mathcal{B}_{0\#} - \mathcal{B}_0)u + [\mathcal{B}_0, \varphi_k]u - \varphi_k(\mathcal{C}_{0\#} - \mathcal{C}_0)\rho + [\mathcal{C}_0, \varphi_k]\rho\\
&G_{j,k}(g_{j,k}, u, \rho)\\
&= g_{j,k} - \varphi_k(\mathcal{B}_{j\#} - \mathcal{B}_j)u + [\mathcal{B}_j, \varphi_k]u - \varphi_k(\mathcal{C}_{j\#} - \mathcal{C}_j)\rho + [\mathcal{C}_j, \varphi_k]\rho\\
&\quad (j = 1, \cdots, m),
\end{aligned}$$

where $[\cdot,\cdot]$ is the commuter. For $k = 0, \cdots, M$, we can solve

$$\eta u_k + \mathcal{A}_\# u_k = F_k(u) \quad (t \in J,\ x \in \mathbb{R}^n)$$

and find from Proposition 4.4.3 and the regularity estimate of elliptic operators that

$$\eta\|u_k\|_X + \|u_k\|_{Z_u} \leq C_1\|f\|_X + \varepsilon\|u\|_{Z_u} + C_\varepsilon\|u\|_X. \tag{4.4.14}$$

for sufficiently small $\varepsilon > 0$. Put $\rho_k = 0$ for $k = 0, \cdots, M$. In the case of $k = M+1, \cdots, N$, we apply coordinate transform by Φ_k to (4.4.13). The transformed problem is solvable by Proposition 4.4.4. Pulling buck the solution to we obtain the solution to (4.4.13) such that

$$\begin{aligned}
&\eta\|u_k\|_X + \|u_k\|_{Z_u} + \|\rho_k\|_{Z_\rho}\\
&\leq C\|f\|_X + C\sum_{j=0}^m \|g_j\|_{Y_j} + \varepsilon\|u\|_{Z_u} + C_\varepsilon\|u\|_X + C\,|J|^\alpha\,\|\rho\|_{Z_\rho},
\end{aligned} \tag{4.4.15}$$

for some $\alpha > 0$. $|J|^\alpha$ appears because $F_k(u) - f_k$ has only lower-order differential terms. We denote by $\mathcal{S}^k : X \times Y_0 \times \cdots \times Y_m \to Z_u \times Z_\rho$ the solution operator of (4.4.13) with $\rho_0 = 0$, i.e. $(u_k, \rho_k) = \mathcal{S}^k(F_k, G_{0,k}, \cdots, G_{m,k})$. On the other hand, it follow that

$$
\begin{aligned}
(u,\rho) &= \sum_{k=0}^{N} (u_k, \rho_k) = \sum_{k=0}^{N} \mathcal{S}^k \left(F_k(f_k, u), G_{0,k}(g_{0,k}, u, \rho), \cdots, G_{m,k}(g_{m,k}, u, \rho)\right) \\
&= \sum_{k=0}^{N} \mathcal{S}^k \left(F_k(f_k, 0), G_{0,k}(g_{0,k}, 0, 0), \cdots, G_{m,k}(g_{m,k}, 0, 0)\right) \\
&+ \sum_{k=0}^{N} \mathcal{S}^k \left(F_k(0, u), G_{0,k}(0, u, \rho), \cdots, G_{m,k}(0, u, \rho)\right) \\
&=: \mathcal{T}(f, g_0, \cdots, g_m) - R(u, \rho), \qquad (4.4.16)
\end{aligned}
$$

where we write restriction of (u_k, ρ_k) on U_k as (u_k, ρ_k) for simplicity. (4.4.14) and (4.4.15) imply

$$\|R(u,\rho)\|_{Z_u \times Z_\rho} \leq \frac{1}{2}\|u\|_{Z_u} + C\,|J|^\gamma\,\|\rho\|_{Z_\rho}.$$

for some $\gamma > 0$. Let $\mathcal{S}_0 : X \times Y_0 \times \cdots \times Y_m \to Z_u \times Z_\rho$ be the solution operator of (4.1.1) with $\rho_0 = 0$, i.e. $(u, \rho) = \mathcal{S}_0(f, g_0, \cdots, g_m)$. It follows from (4.4.16)

$$\mathcal{S}_0 = \mathcal{T}(f, g_0, \cdots, g_m) - R\left(\mathcal{S}_0(f, g_0, \cdots, g_m)\right). \qquad (4.4.17)$$

Then, if we take $|J|$ small and $\eta > 0$ large, we can use the Neumann series argument to get $\mathcal{S}_0 = (Id + R)^{-1}\mathcal{T}$ and

$$\|\mathcal{S}_0\|_{\mathcal{B}(X \times Y_0 \times \cdots \times Y_m; Z_u \times Z_\rho)} \leq C.$$

For J with arbitrary finite length, we can divide J into finite short interval. For this short intervals, the same argument as above also works.

4.5 Examples

In this section we give some examples for our problems. We especially focus on checking the Lopatinskii-Shapiro and asymptotic Lopatinskii-Shapiro conditions. Throughout this section, we assume $E = F = \mathbb{C}$ and write the outer unit normal on the boundary by ν.
Example 4.5.1

$$
\begin{cases}
\eta u - \Delta u &= f \qquad (t \in J,\ x \in G), \\
\partial_\nu u + \partial_t \rho &= g_0 \qquad (t \in J,\ x \in \Gamma), \\
u - \rho &= 0 \qquad (t \in J,\ x \in \Gamma) \\
\rho(0, x) &= \rho_0(x) \qquad (x \in \Gamma).
\end{cases} \qquad (4.5.1)
$$

The equation of the Lopatinskii–Shapiro condition is

$$\begin{cases} (\eta + |\xi'|^2 - \partial_y^2)v(y) &= 0 \qquad (y > 0), \\ -\partial_y v(0) + \lambda\sigma &= h_0, \\ v(0) - \sigma &= h_1. \end{cases}$$

The solution of the first equation $C_0(\mathbb{R}_+; E)$ is given by
$v(y) = e^{-\sqrt{\eta+|\xi'|^2}}v(0) = e^{-\mu y}v(0)$ for $\mu = (\eta + |\xi'|^2)^{1/2}$. The boundary conditions lead to the equation

$$\begin{pmatrix} \sqrt{\eta + |\xi'|^2} & \lambda \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v(0) \\ \sigma \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}.$$

We see that the determinant of the matrix is $-\lambda - \sqrt{\eta + |\xi'|^2} \neq 0$ for $\eta > 0$, $(\lambda, \xi') \in (\overline{\mathbb{C}_+} \times \mathbb{R}^{n-1}) \setminus \{(0,0)\}$. Therefore, the Lopatinskii–Shapiro condition is satisfied.
The equation of the first asymptotic Lopatinskii–Shapiro condition is

$$\begin{cases} (\eta + |\xi'|^2 - \partial_y^2)v(y) &= 0 \qquad (y > 0), \\ v(0) &= h_1. \end{cases}$$

for $\eta > 0$ and $\xi \in \mathbb{R}^{n-1}$ satisfying $\eta + |\xi'| \neq 0$. The solution to this ODE is uniquely determined by $v(y) = e^{-\mu y}h_1$.
The equation of the second asymptotic Lopatinskii-Shapiro condition is

$$\begin{cases} (|\xi'|^2 - \partial_y^2)v(y) &= 0 \qquad (y > 0), \\ -\partial_y v(0) + \lambda\sigma &= h_0, \\ v(0) - \sigma &= h_1. \end{cases}$$

for $(\lambda, \xi') \in \overline{\mathbb{C}_+} \times \mathbb{S}^{n-2}$. The equation of the first equation implies $v(y) = e^{-|\xi'|y}v(0)$, and thus $-\partial_y v(0) = |\xi'|\, v(0)$. Since the determinant of the matrix

$$\begin{pmatrix} |\xi'| & \lambda \\ 1 & -1 \end{pmatrix}$$

is never zero by the choice of (λ, ξ'). The equation of the third asymptotic Lopatinskii-Shapiro condition is

$$\begin{cases} (|\xi'|^2 - \partial_y^2)v(y) &= 0 \qquad (y > 0), \\ v(0) &= h_1. \end{cases}$$

for $\xi \in \mathbb{S}^{n-2}$. This equation is uniquely determined by $v(y) = e^{-|\xi'|y}h_1$. Thus, the Lopatinskii- Shapiro and asymptotic Lopatinskii-Shapiro conditions are satisfied, and (4.5.1) is solvable in the maximal regularity space.
Example 4.5.2

$$\begin{cases} \eta u - \Delta u &= f \qquad (t \in J,\ x \in G), \\ \partial_\nu u + \partial_t \rho - \Delta_\Gamma \rho &= g_0 \qquad (t \in J,\ x \in \Gamma), \\ u - \rho &= g_0 \qquad (t \in J,\ x \in \Gamma), \\ \rho(0,x) &= \rho_0(x) \qquad (x \in \Gamma). \end{cases} \tag{4.5.2}$$

The equation of the Lopatinskii-Shapiro condition is

$$\begin{cases} (\eta + |\xi'|^2 - \partial_y^2)v(y) &= 0 \qquad (y>0) \\ -\partial_y v(0) + \lambda\sigma + |\xi'|^2 v &= h_0 \\ v(0) - \sigma &= h_1. \end{cases}$$

We find $v(y) = e^{-\mu y} v(0)$ for $\mu = (\eta + |\xi'|^2)^{1/2}$ and

$$\det \begin{pmatrix} \mu & \lambda + |\xi'|^2 \\ 1 & -1 \end{pmatrix} \neq 0$$

for $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \overline{\mathbb{C}_+}$ satisfying $|\xi'| + |\lambda| \neq 0$. Thus Lopatinskii-Shapiro condition is satisfied. Let us check asymptotic Lopatinskii-Shapiro conditions. The equation of the first and third asymptotic Lopatinskii-Shapiro conditions are

$$\begin{cases} (\eta + |\xi'|^2 - \partial_y^2)v(y) &= 0 \qquad (y>0) \\ v(0) - \sigma &= h_1. \end{cases}$$

for $\xi \in \mathbb{R}^{n-1}$ and

$$\begin{cases} (|\xi'|^2 - \partial_y^2)v(y) &= 0 \qquad (y>0) \\ v(0) - \sigma &= h_1. \end{cases}$$

for $\xi \in \mathbb{S}^{n-2}$. By the same way as above, we find this equation is uniquely solvable. The equation of the second asymptotic Lopatinskii-Shapiro condition is

$$\begin{cases} (|\xi'|^2 - \partial_y^2)v(y) &= 0 \qquad (y>0) \\ -\partial_y v(0) + \lambda\sigma + |\xi'|^2 v &= h_0 \\ v(0) &= h_1. \end{cases}$$

$v(y)$ is determined by the first and third equations, and σ is uniquely determined by the second equation for $|\xi'| + |\lambda| \neq 0$.

Example 4.5.3 The third example is the Cahn–Hilliard equations with the dynamic boundary condition and surface diffusion

$$\begin{cases} \eta u + \Delta^2 u &= f \qquad (t \in J,\ x \in G), \\ \partial_t u + \partial_\nu \rho - \Delta_\Gamma \rho &= g_0 \qquad (t \in J,\ x \in \Gamma), \\ \partial_\nu \Delta u &= g_1 \qquad (t \in J,\ x \in \Gamma), \\ u - \rho &= g_2 \qquad (t \in J,\ x \in \Gamma), \\ \rho(0,x) &= \rho_0(x) \quad (x \in \Gamma). \end{cases} \tag{4.5.3}$$

The equation of the Lopatinskii–Shapiro condition is

$$\begin{cases} (\eta + (|\xi'|^2 - \partial_y^2)^2)v(y) &= 0 \qquad (y>0), \\ -\partial_y v(0) + (\lambda + |\xi'|^2)\sigma &= h_0, \\ -\partial_y (|\xi'|^2 - \partial_y^2)v(0) &= h_1, \\ v(0) - \sigma &= h_2. \end{cases}$$

The solution of the first equation which belongs to $C_0(\mathbb{R}_+; E)$ is
$v(y) = C_1 e^{-z_1 y} + C_2 e^{-z_2 y}$ with $z_{1,2} := \sqrt{|\xi|^2 \pm \eta^{1/2} i}$ and $C_{1,2} \in \mathbb{C}$. Note that the real parts of z_1 and z_2 are non-negative. The boundary conditions lead

$$\begin{pmatrix} z_1 & z_2 & \lambda + |\xi'|^2 \\ z_1(z_1^2 - |\xi'|^2) & z_2(z_2^2 - |\xi'|^2) & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \sigma \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix}.$$

We see that the determinant of the matrix is $i\eta^{1/2} 2 z_1 z_2 + (\lambda + |\xi'|^2 (z_1 + z_2))$, and it never be zero for $\eta > 0$, $(\lambda, \xi') \in \overline{\mathbb{C}_+} \times \mathbb{R}^{n-1}$ satisfying $|\xi'| + |\lambda| \neq 0$. Therefore Lopatinskii–Shapiro condition is satisfied. Let us check asymptotic Lopatinskii-Shapiro conditions.

$$\begin{cases} \eta v(y) + (|\xi'|^2 - \partial_y^2) v(y) &= 0 \quad (y > 0) \\ \partial_y (|\xi'|^2 - \partial_y^2) v(0) &= h_1 \\ v(0) &= h_2 \end{cases}$$

The solution is of the form $v(y) = C_1 e^{-z_1 y} + C_2 e^{-z_2 y}$. Since

$$\det \begin{pmatrix} -|\xi'|^2 - z_1^3 & -|\xi'|^2 - z_2^3 \\ 1 & 1 \end{pmatrix} = -i\eta^{1/2} \left(\frac{2|\xi'|^2}{z_1 + z_2} + z_1 + z_2 \right) \neq 0,$$

for $\xi' \in \mathbb{R}^{n-1}$, the first asymptotic Lopatinskii-Shapiro condition is satisfied. The equation of the second asymptotic Lopatinskii-Shapiro condition is

$$\begin{cases} (|\xi'|^2 - \partial_y^2) v(y) &= 0 \quad (y > 0) \\ -\partial_y v(0) + \lambda\sigma + |\xi'|^2 \sigma &= h_0 \\ \partial_y (|\xi'|^2 - \partial_y^2) v(0) &= h_1 \\ v(0) &= h_2 \end{cases}$$

The solution is of the form $v(y) = C_1 e^{-|\xi'| y} + C_2 y e^{-|\xi'| y}$. Since

$$\det \begin{pmatrix} |\xi'| & -1 & \lambda + |\xi'|^2 \\ 0 & 2|\xi'|^2 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -2|\xi'| (\lambda + |\xi'|^2)$$

for $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \overline{\mathbb{C}_+}$ satisfying $|\xi'| + |\lambda| \neq 0$. The equation of the third asymptotic Lopatinskii-Shapiro condition

$$\begin{cases} (|\xi'|^2 - \partial_y^2) v(y) &= 0 \quad (y > 0) \\ \partial_y (|\xi'|^2 - \partial_y^2) v(0) &= h_1 \\ v(0) &= h_2 \end{cases}$$

We find from the same way as above this equation admits a unique solution for $\mathbb{S}^{n-2}$.

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