博士論文

論文題目　Spectral and scattering theory for generalized Schrödinger operators
(一般化シュレディンガー作用素に関するスペクトル理論と散乱理論)

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Abstract

In this thesis, we study spectral and scattering properties for generalized Schrödigner operators. In particular, we investigate essential self-adjointness and limiting absorption principle for some differential operators on $\mathbb{R}^d$ and $\mathbb{Z}^d$.

In Chapter 3, we show the essential self-adjointness and the limiting absorption principle for a d'Alembert operator on a Lorentzian space. Unlike the elliptic case, its proof is non-trivial. Moreover, we need geometric conditions even for operators on the Euclidean space with asymptotically constant coefficients.

In Chapter 4, we study the spectral properties of a repulsive Schrödinger operator. We give a micolocal proof for the classical result on its essential self-adjointness. A spectral property of its self-adjoint extensions is also studied.

In Chapter 5, we study the uniform bound of a Birman-Schwinger operator on a square lattice. For uniformly decaying potentials, we obtain the same bound as in the continuous setting. However, for non-uniformly decaying potential, our results are weaker than in the continuous setting.

In Chapter 6, we investigate L^p-mapping properties and the Carleman estimate of a Fourier multiplier operator and its resolvent. As an application, we prove existence and completeness of wave operators for a Dirac operator and a fractional Laplacian.

In Chapter 7, we address the precise asymptotic expansions and non-existence of resonant states for a discrete Schrödinger operator near its threshold energy.

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Contents

Chapter 1

Introduction

In this thesis, we study spectral properties of generalized Schrödinger operators. In particular, we study

- The essential self-adjointness for Schrödinger operators,
- The limiting absorption principle and an application to scattering theory,
- An asymptotic behavior of threshold states at elliptic thresholds and non-existence of thresholds states at hyperbolic thresholds for discrete Schrödinger operators.

The above topics and the method of these proofs are closely related to the scattering theory. Main technical tools are harmonic analysis and microlocal analysis.

In Chapter 3 and Chapter 4, we study the essential self-adjointness and spectral properties of Schrödinger operators. The notion of essential self-adjointness for a Schrödinger operator P is important since it is equivalent to existence and uniqueness of solutions to the following time-dependent Schrödinger equation:

$$i\partial_t u(t,x) - Pu(t,x) = 0,\ u(0,x) = u_0(x) \in L^2(\mathbb{R}^n). \tag{1.0.1}$$

When the operator P has a form $-\Delta + V(x)$, the essential self-adjointness has beed widely studied since Kato proved the essential self-adjointness for Schrödinger operator with the Coulomb potential $-\Delta - \frac{e}{|x|}$ [45]. It is believed that the completeness of the Hamilton vector field generated by the corrsponding symbol p is closely related to the essential self-adjointness of P, since they means well-posedness of fundamental equations in the classical mechanics and the quantum mechanics respectively. A natural question is whether for a differential operator P and its symbol p

Question 1. Is the completeness of H_p equivalent to the essential self-adjointness for P?.

It is known that the Laplace-Beltrami operator Δ_g is essentially self-adjoint on $C_c^\infty(M)$ if a Riemannian manifold (M, g) is geodesically complete. The converse is not true, in fact, a Riemmanian manifold $M = \mathbb{R}^n \setminus \{0\}$ with the Euclidean metric is not geodesically complete, but the Laplacian is essential self-adjoint on $C_c^\infty(M)$ with $n \geq 4$, which means that Question 1 is not always true. In addition, there exists examples of one dimensional Schrödinger operators for which Question 1 is not hold [61]. Thus a next natural question is to find a sufficient condition for a class of Schrödinger operators for which Question 1 is true. We expect that microanalysis is very effective for attacking Question 1, since our approach reveals the connection between the classical and quantum mechanics.

In Chapter 3, we prove the essential self-adjointness of real-principal type operators on $\mathbb{R}^n$ under a non-trapping condition on the characteristic set. Moreover, we prove the limiting absorption principle (a resolvent bound) of its self-adjoint extension. A typical example of our operators is the d'Alembert operator on the asymptotically Minkowski spacetime. In the quantum field theory, it is important to consider a special solution to the wave equation, say, the Feynman propagator. In the exact Minkowski spacetime, the Feynman propagator coincides with the outgoing resolvent of the d'Alembert operator. A natural question is whether this correspondence holds even on curved spacetimes. In this chapter, we prove the essential self-adjointness (which is needed for defining its resolvent) and existence of the outgoing resolvent in spacetimes where the structure near infinity is similar to the exact Minkowski spacetime. It remains still an open question that our outgoing resolvent coincides with the original definition of the Feynman propagator.

In Chapter 4, we consider the repulsive Schrödinger operators

$$P = -\Delta - (1 + |x|^2)^\alpha \ \alpha > 1$$

and show that P is not essential self-adjoint on $C_c^\infty(\mathbb{R}^n)$. This result is classically known, however, we give another proof of it via the microlocal or scattering technique. In addition, as byproducts of our analysis, we can prove the following:

- The repulsive Schrödinger operators P has many L^2 eigenfunctions associated with almost all spectral parameter $z \in \mathbb{C}$ in the distributional sense.
- Every self-adjoint extension of P has the discrete spectrum when the space dimension is one.

These results show that the repulsive Schrödinger operator with $\alpha > 1$ has spectral properties similar to the Laplacian in a bounded domain $\Omega \subset \mathbb{R}^n$.

In Chapters 5, 6 and 7, we mainly study the spectral properties of the discrete Schrödinger operator. The discrete Schrödinger operator is a natural discretization of the Laplacian on the Euclidean space. Moreover, the discrete Schrödinger operator appears in the model of an electron under the tight-binding approximation in the condensed matter physics and in the Anderson model of the random Schrödinger operators. Some mathematical structure of the discrete Schrödinger operator is different from that of the Schrödinger operator on the Euclidean space. For example,

- The discrete Schrödinger operator has thresholds strictly inside its spectrum.

- The discrete space $\mathbb{Z}^d$ has less symmetry than the continuous space $\mathbb{R}^d$.

We study the above apply this with the spectral theory and the scattering theory for the discrete Schrödinger operator.

In Chapter 5, we study a kind of limiting absorption principle for discrete Schrödinger operators:

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \||V|^{\frac{1}{2}}(H_0-z)^{-1}|V|^{\frac{1}{2}}\|_{B(l^2(\mathbb{Z}^d))}<\infty, \tag{1.0.2}$$

where the above operator is also called the Birman-Schwinger operator. A question which we consider is the following:

Question 2. What is the conditions on V for which (1.0.2) hold?

In the case of $H_0=-\Delta$ on $\mathbb{R}^d$, similar results are known as

- (1.0.2) hold if $d\geq 3$ and $V(x)=\langle x\rangle^{-2}$; $d\geq 3$ and $V\in L^{\frac{d}{2},\infty}(\mathbb{R}^d)$ ([50], [47]),
- (1.0.2) does not always hold if $d=1,2$; $d\geq 3$ and $V(x)=\langle x\rangle^{-k}$ with $k<2$; $d\geq 3$ and $V\in L^{p,\infty}(\mathbb{R}^d)$ with $p\neq \frac{d}{2}$,

where $L^{p,r}(\mathbb{R}^d)$ denotes the Lorentz space. For discrete Schrödinger operators, we show the following:

- (1.0.2) hold if $d\geq 3$ and $V(x)=\langle x\rangle^{-2}$; $d\geq 4$ and $V\in l^{\frac{d}{3},\infty}(\mathbb{Z}^d)$,
- (1.0.2) does not always hold if $d=1,2$; $d\geq 3$ and $V(x)=\langle x\rangle^{-k}$ with $k<2$; $d\geq 5$ and $V\in l^{p,\infty}(\mathbb{Z}^d)$ with $p=\frac{d}{2}$.

Compared with the results on $\mathbb{R}^d$, if V has the form $\langle x\rangle^{-k}$, our results are same as that on $\mathbb{R}^d$. However, in the case of a general class $V\in l^p$, our results are very different from that on $\mathbb{R}^d$. This seems to reflect a lack of symmetry of the discrete space $\mathbb{Z}^d$. Moreover, as a byproduct of our analysis, we show existence of the Feynman propagator on the exact Minkowski spacetime and its mapping property.

In Chapter 6, we study the L^p-L^q-mapping properties of resolvents of Fourier multipliers. We extend the results by [11] to (p,q) which is not Hölder exponent. Moreover, using the same method, we show the Hölder continuity and a Calreman type estimate which are quite useful for scattering theory. As an application, we show existence and completeness of the wave operators for the Dirac operators and the fractional Schrödinger operators.

In Chapter 7, we study thresholds properties for discrete Schrödinger operators. More precisely, we prove

- Threshold resonances (or eigenfunctions) at the elliptic thresholds have the same properties as threshold resonances for continuous Schrödinger operators
- There is no threshold resonances (or eigenfunctions) at the hyperbolic thresholds.

Due to the celebrated work by Jensen and Kato [42], it is known that properties and existence of threshold resonances (or eigenfunctions) are closely related to time decay properties of the time propagators. From our results, in the discrete setting, we expect

- Near the elliptic thresholds, time decay properties are the same as that of the continuous time propagators
- The hyperbolic thresholds are harmless for time decay properties of the time propagator.

We would like to leave these justification in future work.

A part of the results in Chapter 3 are from joint work with Shu Nakamura [58], the results in Chapter 5 are from joint work with Yukihide Tadano [68] and the results in Chapter 7 are from joint work with Yuji Nomura [59].

Chapter 2

Microlocal analysis

2.1 Definition of pseudodifferential operators

First of all, we recall the standard theory of microlocal analysis: the definition of the pseudodifferential operators, some basic calculus, symbol classes, and the Gårding type inequalities. The symbol classes which we mainly use in this paper, are not the standard Kohn-Nirenberg classes S^k, but so-called scattering symbol classes $S^{k,l}$. Symbols of the scattering classes have many better properties than the Kohn-Nirenberg symbols and play important roles for our analysis on the real principal type operators and the repulsive Schrödinger operators.

For $a \in \mathcal{S}'(\mathbb{R}^n)$, we define the Weyl quantization of a by

$$\mathrm{Op}(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(\frac{x+y}{2}, \xi) u(y) dy d\xi, \ u \in \mathcal{S}(\mathbb{R}^n),$$

which maps $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ continuously. This definition of quantization is a bit different from one of the standard quantization:

$$a(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(x, \xi) u(y) dy d\xi, \ u \in \mathcal{S}(\mathbb{R}^n).$$

2.2 Symbol class, symbol calculus

Let $k, l \in \mathbb{R}$. For $a \in C^\infty(\mathbb{R}^{2n})$, we call $a \in S^{k,l}$ if

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C_{\alpha\beta} \langle x \rangle^{l-|\alpha|} \langle \xi \rangle^{k-|\beta|}$$

where $C_{\alpha\beta}$ is independent of $(x, \xi) \in \mathbb{R}^{2n}$. Moreover, we denote

$$S^{k,-\infty} = \bigcap_{l \in \mathbb{R}} S^{k,l}, \ S^{-\infty,l} = \bigcap_{k \in \mathbb{R}} S^{k,l}, \ S^{-\infty,-\infty} = \bigcap_{k,l \in \mathbb{R}} S^{k,l}.$$

For $a \in S^{k,l}(\mathbb{R}^n)$ with $k, l \in \mathbb{R}$, the range of $\mathrm{Op}(a)$ in $\mathcal{S}(\mathbb{R}^n)$ is contained in $\mathcal{S}(\mathbb{R}^n)$ and hence

$$\mathrm{Op}(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is continous. Moreover, if $a \in S^{-\infty,-\infty}$, then $\mathrm{Op}(a)$ can be uniquely extended to a continuous linear map

$$\mathrm{Op}(a) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

We define the Poisson product of a and b by

$$\{a, b\} := H_a b := \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b.$$

It is known that the Weyl quantization has the following symmetric property: Let $a \in S^{k,l}$ be a real-valued symbol with some $k, l \in \mathbb{R}$. Then it follows that $\mathrm{Op}(a)$ is formally self-adjoint on $L^2(\mathbb{R}^n)$, that is,

$$(u, \mathrm{Op}(a)w)_{L^2(\mathbb{R}^n)} = (\mathrm{Op}(a)u, w)_{L^2(\mathbb{R}^n)} \text{ for } u, w \in \mathcal{S}(\mathbb{R}^n). \tag{2.2.1}$$

Lemma 2.2.1.

(i) (L^2*-boundedness) Let* $k, l \leq 0$ *and* $a \in S^{k,l}$. *Then a continuous linear map* $\mathrm{Op}(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *can be uniquely extended to a bounded linear operator on* $L^2(\mathbb{R}^n)$. *Moreover, there exists* $C, M > 0$ *which depend only on the dimension* n *such that for* $a \in S^{0,0}$, *we have*

$$\|\mathrm{Op}(a)\|_{B(L^2)} \leq C \sum_{|\alpha| \leq M} \sup_{z \in \mathbb{R}^{2n}} |\partial_z^\alpha a(z)|.$$

(ii) *(Compactness) If* $k, l < 0$ *and* $a \in S^{k,l}$, *then* $\mathrm{Op}(a)$ *is a compact operator on* $L^2(\mathbb{R}^n)$.

(iii) *(Composition) Let* $k_j, l_j \in \mathbb{R}$ *for* $j = 1, 2$, $a \in S^{k_1,l_1}$ *and* $b \in S^{k_2,l_2}$. *Then there exists* $c \in S^{k_1+k_2,l_1+l_2}$ *such that*

$$\mathrm{Op}(c) = \mathrm{Op}(a)\mathrm{Op}(b), \text{ we denote } a\#b := c.$$

In addition, we have

$$\begin{aligned} a\#b(x,\xi) =& e^{iQ(D)}(a(x,\xi)b(y,\eta))|_{x=y,\xi=\eta} \\ =& \frac{1}{\pi^n} \int_{\mathbb{R}^{4n}} e^{-2i\sigma(w_1,w_2)} a(z+w_1)b(z+w_2)dw_1dw_2, \end{aligned}$$

where $z = (x, \xi)$ *and*

$$\sigma((x,\xi),(y,\eta)) = \xi \cdot y - x \cdot \eta, \ Q(D) = \frac{1}{2}\sigma(D)$$
$$Q = \begin{pmatrix} \mathbf{0} & -\boldsymbol{J} \\ \boldsymbol{J} & \mathbf{0} \end{pmatrix} \in GL(4n, \mathbb{R}), \ J = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix} \in GL(2n, \mathbb{R}).$$

Moreover, we have

$$[\mathrm{Op}(a), i\mathrm{Op}(b)] = \mathrm{Op}(H_a b) + \mathrm{Op}S^{k_1+k_2-2, l_1+l_2-2},$$

where we note

$$H_a b \in \mathrm{Op}S^{k_1+k_2-1, l_1+l_2-1}.$$

(iv) (*Disjoint support property*) *Let $k_j, l_j \in \mathbb{R}$ for $j = 1, 2$, $a \in S^{k_1,l_1}$ and $b \in S^{k_2,l_2}$. Suppose*

$$\mathrm{dist}(\mathrm{supp}\ a, \mathrm{supp}\ b) > 0.$$

Then we have $a\#b \in S^{-\infty,-\infty}$.

(v) (*Sharp Gårding ineqality*) *Let $a \in S^{k,l}$ with $k, l \in \mathbb{R}$. Suppose $a(x,\xi) \geq 0$ for $(x,\xi) \in \mathbb{R}^{2n}$. Then there exists $C > 0$ such that for $u \in \mathcal{S}(\mathbb{R}^n)$*

$$(u, \mathrm{Op}(a)u)_{L^2(\mathbb{R}^n)} \geq -C\|u\|_{H^{k-1,l-1}(\mathbb{R}^n)}.$$

In chapter 3, we will use more general symbol $S(m, g)$. See [30, §18.4, §18.5] for more detail.

2.3 Auxiliary lemmas

The following lemma implies that the microlocal wavefront set is characterized by the semiclassical wavefront set.

Lemma 2.3.1. *Let $(x_0, \xi_0) \in \mathbb{R}^{2n}$ with $\xi_0 \neq 0$ and $u \in L^2(\mathbb{R}^n)$. Suppose that there exists $a_0 \in S$ such that $a_0(x_0,\xi_0) > 0$ and $\|\mathrm{Op}(a_{0,h})u\|_{L^2} = O(h^{k+\varepsilon})$ for some $\varepsilon > 0$, where $a_h(x,\xi) = a(x, h\xi)$. Then it follows that $u \in H^k$ microlocally at (x_0, ξ_0).*

Proof. First, we prove that there exists a neighborhood U of (x_0, ξ_0) such that for any $a \in S$ supported in U, we have $\|\mathrm{Op}(a_h)u\|_{L^2} = O(h^{k+\varepsilon})$. Take a relatively compact open set U such that $\inf_U a_0 > 0$ and $\bar{U} \cap \{\xi = 0\} = \emptyset$. Let $a \in S$ satisfying $\mathrm{supp}\ a \subset U$. By the standard parametrix construction, we can find $b \in S$ such that

$$a_h = b_h \# a_{0,h} + O_S(h^\infty).$$

Thus we have

$$\|\mathrm{Op}(a_h)u\|_{L^2} \leq \|\mathrm{Op}(b_h)\|_{B(L^2)}\|\mathrm{Op}(a_{0,h})u\|_{L^2} + O(h^\infty\|u\|_{L^2}) = O(h^{k+\varepsilon}).$$

We may assume that u is supported around x_0. Let $\chi \in C^\infty(\mathbb{R}^n; [0,1])$ such that $\chi(x,\xi) = 1$ near a conic neighborhood of ξ_0 and is supported in a conic neighborhood of ξ_0. It suffices to prove $\langle D\rangle^k\chi(D)u \in L^2$. We note

$$\|\langle D\rangle^k\chi(D)u\|_{L^2}^2 = \|\langle\xi\rangle^k\chi(\xi)\hat{u}\|_{L^2(|\xi|\le 1)}^2 + \sum_{j=1}^{\infty}\|\langle\xi\rangle^k\chi(\xi)\hat{u}\|_{L^2(2^j\le|\xi|\le 2^{j+1})}^2$$
$$\le C\|\chi(\xi)\hat{u}\|_{L^2(|\xi|\le 1)}^2 + \sum_{j=1}^{\infty}2^{2jk}\|\chi(\xi)\hat{u}\|_{L^2(2^j\le|\xi|\le 2^{j+1})}^2$$

Thus we only need to prove

$$\|\chi(\xi)\hat{u}\|_{L^2(2^j\le|\xi|\le 2^{j+1})}^2 = O(2^{-2j(k+\varepsilon)}).$$

This follows from the scaling and the first half part of the proof. □

The following lemma is useful for justifying the regularizing argument.

Lemma 2.3.2. *Let $k, \ell \in \mathbb{R}$, $a_j \in S^{k,\ell}$ be a unifornly bounded sequence in $S^{k,\ell}$ and $a \in S^{k,\ell}$. Suppose $a_j \to a$ in $S^{k+\delta,\ell+\delta}$ for some $\delta > 0$. Then, for each $s, t \in \mathbb{R}$ and $u \in H^{s,t}(\mathbb{R}^n)$, we have*

$$\|(\mathrm{Op}(a_j) - \mathrm{Op}(a))u\|_{H^{s-k,t-\ell}(\mathbb{R}^n)} \to 0 \quad \text{as } j \to \infty.$$

Proof. Let $u \in L^2(\mathbb{R}^n)$ and $\varepsilon > 0$. Set

$$C = \|\mathrm{Op}(a)\|_{B(H^{s,t},H^{s-k,t-\ell})} + \sup_j \|\mathrm{Op}(a_j)\|_{B(H^{s,t},H^{s-k,t-\ell})}.$$

Take $w \in \mathcal{S}(\mathbb{R}^n)$ such that $\|u - w\|_{H^{s,t}} < \varepsilon/C$.

$$\begin{aligned}
&\|\mathrm{Op}(a_j)u - \mathrm{Op}(a)u\|_{H^{s-k,t-\ell}}\\
&\quad\le \|\mathrm{Op}(a_j)u - \mathrm{Op}(a_j)w\|_{H^{s-k,t-\ell}} + \|\mathrm{Op}(a_j)w - \mathrm{Op}(a)w\|_{H^{s-k,t-\ell}}\\
&\quad\quad + \|\mathrm{Op}(a)w - \mathrm{Op}(a)u\|_{H^{s-k,t-\ell}}\\
&\quad\le C\|u - w\|_{H^{s,t}} + \|\mathrm{Op}(a_j)w - \mathrm{Op}(a)w\|_{H^{s-k,t-\ell}}\\
&\quad< \varepsilon + \|\mathrm{Op}(a_j)w - \mathrm{Op}(a)w\|_{H^{s-k,t-\ell}}.
\end{aligned}$$

Taking $j \to \infty$, we obtain the desired result. □

Chapter 3

Essential self-adjointness of real-principal type operators

3.1 Introduction

In this chapter, we consider formally self-adjoint real principal type operator $P = \mathrm{Op}(p)$ on the Euclidean space $\mathbb{R}^n$ with $n \geq 1$, where $\mathrm{Op}(\cdot)$ denotes the Weyl quantization. A typical example is the Klein-Gordon operator with variable coefficients (see Remark 3.1.2), and the propagation of singularities plays an essential role in the proof of the essential self-adjointness.

We suppose the symbol $p(x, \xi)$ is real principal type with asymptotically constant coefficients in the following sense:

Assumption A. Let $m \geq 2$, $p, p_m \in C^\infty(\mathbb{R}^{2n})$ and $p_0 \in C^\infty(\mathbb{R}^n)$ be real-valued functions of the form

$$p(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha,\ p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha,\ p_0(\xi) = \sum_{|\alpha|=m} b_\alpha \xi^\alpha$$

where $b_\alpha \in \mathbb{R}$ and $a_\alpha \in C^\infty(\mathbb{R}^n)$ such that for any multi-index $\alpha \in \mathbb{Z}_+^n$,

$$|\partial_x^\beta (a_\alpha(x) - b_\alpha)| \leq C_\beta \langle x \rangle^{-\mu - |\beta|}, \quad x \in \mathbb{R}^n$$

with some $\mu > 0$, where we set $b_\alpha = 0$ for $|\alpha| \leq m-1$. Moreover, there exists $C > 0$ such that

$$C^{-1}|\xi|^{m-1} \leq |\partial_\xi p_0(\xi)| \leq C|\xi|^{m-1},\ C^{-1}|\xi|^{m-1} \leq |\partial_\xi p_m(x,\xi)| \leq C|\xi|^{m-1}$$

for $(x, \xi) \in \mathbb{R}^{2n}$.

Let $(y(t), \eta(t)) = (y(t, x_0, \xi_0), \eta(t, x_0, \xi_0)) \in C^1(\mathbb{R} \times \mathbb{R}^{2n}; \mathbb{R}^{2n})$ be the solution to the Hamilton equation:

$$\frac{d}{dt} y(t) = \frac{\partial p_m}{\partial \xi}(y(t), \eta(t)), \quad \frac{d}{dt}\eta(t) = -\frac{\partial p_m}{\partial x}(y(t), \eta(t)), \quad t \in \mathbb{R},$$

with the initial condition: $(y(0), \eta(0)) = (x_0, \xi_0) \in \mathbb{R}^{2n}$. We suppose the following null non-trapping condition:

Assumption B. For any $(x_0, \xi_0) \in p_m^{-1}(0)$ with $\xi_0 \neq 0$, $|y(t, x_0, \xi_0)| \to \infty$ as $|t| \to \infty$.

Our main theorem is the following:

Theorem 3.1.1. *Suppose Assumption A and B. Then $P = \mathrm{Op}(p)$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^n)$.*

Remark 3.1.2. (Klein-Gordon operators on asymptotically Minkowski spaces) Let g_0 be the Minkowski metric on $\mathbb{R}^n$: $g_0 = dx_1^2 - dx_2^2 - \ldots - dx_n^2$ and $g_0^{-1} = (g_0^{ij})_{i,j=1}^n$ be its dual metric. A Lorentzian metric g on $\mathbb{R}^n$ is called asymptotically Minkowski if $g^{-1}(x) = (g^{ij}(x))_{i,j=1}^n$ satisfies, for any $\alpha \in \mathbb{Z}_+^n$ there is $C_\alpha > 0$ such that

$$|\partial_x^\alpha (g^{ij}(x) - g_0^{ij})| \leq C_\alpha \langle x \rangle^{-\mu - |\alpha|}, \quad x \in \mathbb{R}^n,$$

with some $\mu > 0$. Suppose $V(x), A_j(x) \in C^\infty(\mathbb{R}^n; \mathbb{R})$, $j = 1, \ldots, n$, such that

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\mu - |\alpha|}, \quad |\partial_x^\alpha A_j(x)| \leq C_\alpha \langle x \rangle^{-\mu - |\alpha|}, \quad x \in \mathbb{R}^n,$$

for any $\alpha \in \mathbb{Z}_+^n$. Then the symbol

$$p(x, \xi) = \sum_{j,k=1}^n g^{jk}(x)(\xi_j - A_j(x))(\xi_k - A_k(x)) + V(x)$$

satisfies Assumption A. The essential self-adjointness for this model is studied by Vasy [75].

Remark 3.1.3. In this paper, we only deal with operators with order greater than 1. The essential self-adjointness of first order operators on $C_c^\infty(\mathbb{R}^n)$ can be proved by Nelson's commutator theorem with its conjugate operator $N = -\Delta + |x|^2 + 1$ ([61, Theorem X.36]). We also note that if P commutes with the complex conjugation: $P\overline{u} = \overline{Pu}$, then, it is enough to assume the forward null non-trapping condition only instead of null non-trapping condition (cf. [61, Theorem X.3]).

The study of essential self-adjointness has a long history but mostly on operators of elliptic type (see [61] Chapter X and reference therein). For the construction of solutions to evolution equation with real principal type operators, we refer the classical paper [13] by Duistermaat and Hörmander, and the textbook by Hörmander [30]. Chihara [9] studies the well-posedness and the local smoothing effects of the Schrödinger-type equations : $\partial_t u(t, x) = -iPu(t, x)$ under the globally non-trapping condition. The well-posedness implies essential self-adjointness of P if the operator P is symmetric. We assume the non-trapping condition only for null trajectories, since the microlocally elliptic region should not be relevant.

Recently, the scattering theory for Klein-Gordon operators on Lorenzian manifolds has been studied by several authors (see, e.g., [3, 20, 75] and references therein). We also mention related work on Strichartz estimates for Lorenzian manifolds ([23, 55, 69]), nonlinear Schrödinger-type equations with Minkowski metric ([22, 66, 77]), and quantum field theory on Minkowski spaces ([76, 21]). In order to study spectral properties of such equations or operators, self-adjointness is fundamental. We note a sufficient condition for the essential self-adjointness is discussed in Taira [69]. The essential self-adjointness for Klein-Gordon operators on scattering Lorentzian manifolds is proved by Vasy [75] under the same null non-trapping condition. We had independently found a proof of the essential self-adjointness using different method for compactly supported perturbations (we discuss the basic idea in Section 3.5). Inspired by discussions with Vasy during 2017, we generalized the model to include long-range perturbations, and also to higher order real principal type operators. Our proof is considerably different from [75], relatively self-contained, and hopefully simpler even though our result is more general than [75] for the $\mathbb{R}^n$ case.

This chapter is constructed as follows: Our main result is proved in Section 3.2. In Subsection 3.2.1 we show that $(P-i)u = 0$ implies u is smooth. The basic idea of the proof is analogous to Nakamura [57] on microlocal smoothing estimates, and relies on the construction of time-global escaping functions (see also Ito, Nakamura [41] for related results for scattering manifolds). The technical detail is given in Section 3.4. In Subsection 3.2.2, we show the local smoothness implies an weighted Sobolev estimate, which is sufficient for the proof of the essential self-adjointness. The idea is analogous to the *radial point estimates* of Melrose [53], and also related to the positive commutators method of Mourre. Here we construct weight functions explicitly to show necessary operator inequalities. The proof relies on the standard pseudodifferential operator calculus. In Section 3.3, we prove non-trapping estimates for the classical trajectories generated by $p_m(x,\xi)$, which are necessary in Section 3.4. The main lemma (Lemma 3.3.2) is a generalization of a result by Kenig, Ponce, Rolvung and Vega [49], though the proof is significantly simplified. In Section 3.5, we give a simplified proof of the essential self-adjintness for the compactly supported perturbation case. In this case the relatively involved argument of Subsection 3.2.2 is not necessarily.

3.2 Proof of Theorem 3.1.1

By the basic criterion for the essential self-adjointness ([61, Theorem VII.3]), it is sufficient to show

$$\mathrm{Ker}\ (P^* \pm i) = \{0\}$$

to prove Theorem 3.1.1. Since $D(P) = C_c^\infty(\mathbb{R}^n)$, we have $D(P^*) = \{u \in L^2(\mathbb{R}^n) \,|\, Pu \in L^2(\mathbb{R}^n)\}$ where P acts on u in the distribution sense. We hence show:

$$(P \pm i)u = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ for } u \in L^2(\mathbb{R}^n) \text{ implies } u = 0.$$

We only consider "$-$" case. The "+" case is similarly handled. Moreover, we note if u satisfies $(P-i)u = 0$ and $u \in H^{\frac{m-1}{2},-\frac{1}{2}}(\mathbb{R}^n)$, then $u = 0$ follows from a simple argument in [75]. Namely, we take a real-valued function $\psi \in C_c^\infty(\{t \in \mathbb{R} \,|\, t \leq 2\})$ such that $\psi(t) = 1$ for $t \leq 1$ and set $\psi_R(x,\xi) = \psi(\langle x\rangle/R)\psi(\langle \xi\rangle/R)$. Then we have

$$-2i\|u\|_{L^2}^2 = (Pu,u)_{L^2} - (u,Pu)_{L^2} = \lim_{R\to\infty}([\mathrm{Op}(\psi_R),P]u,u)_{L^2}.$$

We note that $[\mathrm{Op}(\psi_R),P]$ is uniformly bounded in $\mathrm{Op}S^{m-1,-1}$ and converges to 0 in $\mathrm{Op}S^{m-1+\delta,-1+\delta}$ as $R \to \infty$ for any $\delta > 0$. We obtain $u = 0$ by using Lemma 2.3.2. Thus, in order to prove Theorem 3.1.1, it suffices to prove

Proposition 3.2.1. *If $u \in L^2(\mathbb{R}^n)$ satisfies $(P-i)u = 0$, then $u \in H^{\frac{m-1}{2},-\frac{1}{2}}$.*

The proof of Proposition 3.2.1 is divided into two parts. In Subsection 3.2.1, we prove the local smoothness of u. In Subsection 3.2.2, using the local smoothness of u, we prove weighted Sobolev properties of u.

3.2.1 Local regularity

The main result of this subsection is the following proposition. We note that we need the null non-trapping condition only for this proposition.

Proposition 3.2.2. *If $u \in L^2(\mathbb{R}^n)$ satisfies $(P-i)u = 0$, then $u \in C^\infty(\mathbb{R}^n)$.*

Proof. It suffices to prove $u \in H^k_{\mathrm{loc}}(\mathbb{R}^n)$ for any $k > 0$. We use the contradiction argument. Suppose $u \notin H^k_{\mathrm{loc}}(\mathbb{R}^n)$ with some k. By Lemma 2.3.1, there exist $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\xi_0 \neq 0$, $C > 0$, and a sequence $\{h_\ell\} \subset (0,1]$ such that for any $a \in C_0^\infty(\mathbb{R}^n)$ with $a(x_0,\xi_0) = 1$,

$$h_\ell \to 0 \text{ as } \ell \to \infty, \text{ and } \|\mathrm{Op}(a_{h_\ell,m})u\| \geq C h_\ell^{\frac{k}{m-1}+1},$$

where $a_{h,m}(x,\xi) = a(x, h^{\frac{1}{m-1}}\xi)$. We may assume $(x_0,\xi_0) \in p_m^{-1}(0)$ since u is smooth microlocally in $\mathbb{R}^{2n} \setminus p_m^{-1}(\{0\})$. Now we use the following proposition.

Proposition 3.2.3. *There exists a family of bounded operators $\{F(h,t)\}_{0<h\leq 1, t\geq 0}$ on $L^2(\mathbb{R}^n)$ such that*

(i) $F(h,0) = \mathrm{Op}(\psi_h)^2 = \mathrm{Op}(\psi_h)^*\mathrm{Op}(\psi_h)$, where ψ_h satisfies $\psi_h(x_0,\xi_0) \geq 1$ and for any $\alpha,\beta \in \mathbb{Z}_+^n$,

$$|\partial_x^\alpha \partial_\xi^\beta \psi_h(x,\xi)| \leq C_{\alpha\beta} h^{\frac{|\beta|}{m-1}} \langle x\rangle^{-|\alpha|}.$$

(ii) There exists $C > 0$ such that for $0 < h \leq 1$,

$$\|F(h,t)\|_{B(L^2)} \leq C\langle t\rangle h^{(-m+2)/(m-1)}, \quad t \geq 0.$$

(iii) There exists $R(h,t) \in B(L^2(\mathbb{R}^n))$ such that

$$\frac{d}{dt}F(h,t) + i[P,F(h,t)] \geq -R(h,t), \quad t \geq 0,$$
$$\sup_{t\geq 0}\langle t\rangle^{-1}\|R(h,t)\|_{B(L^2)} = O(h^\infty) \quad \text{as } h \to 0.$$

Proposition 3.2.3 can be proved similarly as [57, Lemma 9]. For the completeness, we give a proof of Proposition 3.2.3 in Section 3.4. Now we set $u(t,x) := e^{-t}u(x)$. Then $u(t,x)$ satisfies

$$i\partial_t u(t,x) - Pu(t,x) = 0,\ \|u(t)\|_{L^2(\mathbb{R}^n)} \le e^{-t}\|u\|_{L^2(\mathbb{R}^n)},$$

where the first equality is in the distributional sense. We set $F_\ell(t) = F(h_\ell, t)$. Then, we have

$$\begin{aligned}
Ch_\ell^{\frac{2k}{m-1}+2} \le& \|\mathrm{Op}(\psi_{h_\ell})u\|^2 = (u, F_\ell(0)u)\\
=&(u(t), F_\ell(t)u(t)) - \int_0^t \frac{d}{ds}\big(u(s), F_\ell(s)u(s)\big)ds\\
=&(u(t), F_\ell(t)u(t)) - \int_0^t \Big(u(s), \Big(\frac{dF_\ell}{ds}(s) + i[P, F_\ell(s)]\Big)u(s)\Big)ds\\
\le& Ch_\ell^{\frac{-m+2}{m-1}}\langle t\rangle e^{-2t}\|u\|^2 + O(h_\ell^\infty)\cdot\|u\|^2\int_0^t e^{-2s}\langle s\rangle ds,
\end{aligned}$$

where all the inner products and norms here are in $L^2(\mathbb{R}^n)$, and $O(h_\ell^\infty)$ is uniformly in t. Now, we take $t = h_\ell^{-1}$ then we conclude a contradiction. Thus, we obtain $u \in H^k_{\mathrm{loc}}(\mathbb{R}^n)$ for any $k > 0$. This completes the proof of Proposition 3.2.2 □

3.2.2 Uniform regularity outside a compact set

In this subsection, we prove a priori sub-elliptic estimates near infinity. The following estimates are based on the radial points estimates in [53], where the radial points estimates are used for scattering theory on scattering manifolds. By the classical propagation of singularities, the singularities of a solution to $Pu = 0$ (provided P is real-valued real principal type) propagate along the Hamilton flow associated with p. At points where the Hamilton vector filed vanishes, we may use the so-called radial points, which implies u is rapidly decaying at a radial source if u has a threshold regularity at the radial source.

In our case, the radial points estimates are analogous to the Mourre estimate microlocally near outgoing or incoming regions, which is used commonly in scattering theory. We give a self-contained proof of the radial point estimate based on an explicit construction of escaping functions. We note the operator theoretical framework of the Mourre theory is not applicable here since we do not have the self-adjointness of P at this point.

We set

$$P = P_0 + Q, \quad P_0 = p_0(D_x), \quad Q = \mathrm{Op}(q),$$

where

$$q(x,\xi) = p(x,\xi) - p_0(\xi) \in S^{m,-\mu},\ V(x,\xi) = p(x,\xi) - p_m(x,\xi) \in S^{m-1,-\mu}.$$

We use the following smooth cut-off functions: Let $\chi \in C^\infty(\mathbb{R})$ be such that

$$\chi(t) = \begin{cases} 1 & \text{if } t \le 1, \\ 0 & \text{if } t \ge 2, \end{cases} \quad 0 \le \chi(t) \le 1, \quad \chi'(t) \le 0 \ \text{ for } t \in \mathbb{R},$$

and $\operatorname{supp} \chi' \Subset (1,2)$. We write $\overline{\chi}(t) = 1 - \chi(t)$, and

$$\chi_M(x) = \chi(|x|/M), \quad \overline{\chi}_M(x) = \overline{\chi}(|x|/M), \quad x \in \mathbb{R}^n,$$

with $M > 0$. A main result of this subsection is the following theorem.

Theorem 3.2.4. *Let $\gamma > 0$ and $z \in \mathbb{C} \setminus \mathbb{R}$. There is $M > 0$ such that if $\varphi \in L^2(\mathbb{R}^n)$, $(P - z)\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\chi_M(x)\varphi \in C^\infty(\mathbb{R}^n)$, then $\varphi \in H^{k+1-m/2,-\gamma} \cap H^{k+1/2,-\gamma-1/2}$ for any $k \in \mathbb{R}$.*

Now we show Proposition 3.2.1 follows from Theorem 3.2.4.

Proof of Proposition 3.2.1. Suppose that $u \in L^2(\mathbb{R}^n)$ satisfies $(P - i)u = 0$. By Proposition 3.2.2, we have $u \in C^\infty(\mathbb{R}^n)$. In particular, we have $\chi_M(x)\varphi \in C^\infty(\mathbb{R}^n)$ for any $M \ge 1$. Taking $\gamma = 1/2$ and $k = m - 1$, we obtain $\varphi \in H^{m/2,-1/2} \subset H^{(m-1)/2,-1/2}$. This completes the proof of Proposition 3.2.1. □

Thus it remains to prove Theorem 3.2.4. In the following, we assume $\operatorname{Im} z > 0$ without loss of generality. We may also assume $0 < \gamma < \min(1/4, \mu/2)$.

Weight functions

We choose $\rho(t) \in C^\infty(\mathbb{R})$ such that

$$\rho(t) = \begin{cases} 0 & \text{if } t \le 0, \\ 1 & \text{if } t \ge 1/8, \end{cases} \quad 0 \le \rho(t) \le 1, \quad \rho'(t) \ge 0 \ \text{ for } t \in \mathbb{R}.$$

For $\delta \in (1/2, 7/8)$, we set

$$\rho_+^\delta(t) = \rho(t - \delta), \quad \rho_-^\delta(t) = 1 - \rho(t + 1 - \delta), \quad \rho_0^\delta(t) = 1 - \rho_+^\delta(t) - \rho_-^\delta(t),$$

for $t \in \mathbb{R}$. We use the following notation:

$$\hat{x} = \frac{x}{|x|}, \quad v(\xi) = \partial_\xi p_0(\xi), \quad \hat{v}(\xi) = \frac{v(\xi)}{|v(\xi)|}, \quad \eta = \eta(x,\xi) = \hat{x} \cdot \hat{v}(\xi).$$

Then we set

$$b^\delta(x,\xi) = \big(\rho_-^\delta(\eta)|x|^\gamma + \rho_0^\delta(\eta) + \rho_+^\delta(\eta)|x|^{-\gamma}\big)e^{-\gamma\eta},$$

which is defined for $x, \xi \in \mathbb{R}^n \setminus \{0\}$. We introduce cut-off functions and set

$$b_{M,\nu}^\delta(x,\xi) = b^\delta(x,\xi)\overline{\chi}_M(x)\overline{\chi}_\nu(\xi), \quad x, \xi \in \mathbb{R}^n.$$

with $M,\nu>0$. We also write

$$\Omega_1(M,\nu)=\{(x,\xi)\mid M\le|x|\le 2M, |\xi|\ge\nu\},$$
$$\Omega_2(M,\nu)=\{(x,\xi)\mid |x|\ge M, \nu\le|\xi|\le 2\nu\}.$$

The next lemma is a key of the proof of Theorem 3.2.4.

Lemma 3.2.5. *Let $1/2<\delta<\tilde\delta<7/8$, $k\in\mathbb{R}$, $0<\tilde M<M$, $0<\tilde\nu<\nu$, and write*

$$B=\mathrm{Op}(b^{\delta}_{M,\nu}),\quad \tilde B=\mathrm{Op}(b^{\tilde\delta}_{\tilde M,\tilde\nu}).$$

If $\tilde M$ is sufficiently large, then: There are pseudodifferential operators $S=\mathrm{Op}(f_1)$, $T=\mathrm{Op}(f_2)$ such that $f_1,f_2\in S(1,g)$ and supp $[f_1]\subset\Omega_1(M,\nu)$, supp $[f_2]\subset\Omega_2(M,\nu)$*; If $\varphi\in\mathcal{S}'$, $\tilde B\varphi\in H^{k-1+m/2,-1/2}$, $B(P-z)\varphi\in H^{k-(m-1)/2,1/2}$, $S\varphi\in H^{k+(m-1)/2}$ and $T\varphi\in L^2$ then*

$$B\varphi\in H^k\cap H^{k+(m-1)/2,-1/2}.$$

Moreover, For any $N>0$ and $k\ge0$ there is $C>0$ such that

$$\begin{aligned}&\|B\varphi\|^2_{H^{k+(m-1)/2,-1/2}}+(\mathrm{Im}\,z)\|B\varphi\|^2_{H^k}\\&\quad\le C\big(\|B(P-z)\varphi\|^2_{H^{k-(m-1)/2,1/2}}+\|\tilde B\varphi\|^2_{H^{k-1+m/2,-1}}\\&\qquad+\|S\varphi\|^2_{H^{k+(m-1)/2}}+\|T\varphi\|^2_{L^2}+\|\varphi\|^2_{H^{-N,-N}}\big).\end{aligned}\tag{3.2.1}$$

Remark 3.2.6. The constant C in the lemma is independent of φ and $z\in\mathbb{C}\setminus\mathbb{R}$. We note we assume $\tilde B\varphi\in H^{k+(m-1)/2,-1/2}$ for technical reasons, though only the norm of $\tilde B\varphi$ in $H^{k,-1}$ appears in the RHS of (3.2.1).

Theorem 3.2.4 follows from Lemma 3.2.5.

Proof of Theorem 3.2.4. For $j=0,1,2,\dots$, we choose ν_j and $\tilde\nu_j$ so that

$$0<\tilde\nu_0<\nu_0=\tilde\nu_1<\nu_1=\tilde\nu_2<\nu_2=\cdots<\delta_0<\infty$$

with an arbitrarily fixed $\delta_0>0$. We then choose M_j and $\tilde M_j$ so that the claim of Lemma 3.2.5 holds with $k=j/2$, $M=M_j$, $\tilde M=\tilde M_j$ and

$$0<\tilde M_0<M_0=\tilde M_1<M_1=\tilde M_2<M_2=\cdots.$$

We also set $\delta_j=(1+2^{-j})/4$ and $\tilde\delta_j=\delta_{j-1}=(1+2\cdot2^{-j})/4$ for $j=0,1,2,\dots$. We write $B_j=\mathrm{Op}(b^{\delta_j}_{M_j,\nu_j})$, $\tilde B_j=\mathrm{Op}(b^{\tilde\delta_j}_{\tilde M_j,\tilde\nu_j})=B_{j-1}$.

Suppose $\varphi\in L^2$ and $(P-z)\varphi\in\mathcal{S}(\mathbb{R}^n)$. Then we note

$$B_j(P-z)\varphi\in\mathcal{S}(\mathbb{R}^n).$$

At first, we have $\tilde B_0\varphi\in H^{0,-\gamma}\subset H^{0,-1/2}$. By Lemma 3.2.5 with $k=1-m/2$, we learn $\tilde B_1\varphi=B_0\varphi\in H^{1-m/2}\cap H^{1/2,-1/2}$, provided $S\varphi\in H^{1/2}$ and $T\varphi\in L^2$, which are satisfied under the assumptions of Theorem 3.2.4 (with $M_0 \le M$). Then we use Lemma 3.2.5 again with $k = (3-m)/2$ to learn $\tilde{B}_2\varphi = B_1\varphi \in H^{(3-m)/2} \cap H^{1,-1/2}$. Iterating this procedure $2k$-times, we arrive at

$$B_{2k}\varphi \in H^{k+1-m/2} \cap H^{k+1/2,-1/2}.$$

Note that conditions $S\varphi \in H^{k/2+1/2}$ and $T\varphi \in L^2$ are satisfied since $\chi_M(x)\varphi \in C^\infty(\mathbb{R}^n)$. Now we use the first inclusion $B_{2k}\varphi \in H^{k+1-m/2}$. We recall, by the assumption, $\chi_M\varphi \in H^{k+1-m/2}$, and this implies

$$B_{2k}\varphi + \chi_M(x)\varphi \in H^{k+1-m/2}.$$

Since

$$b_{M,\nu} + \chi_M(x) \ge c_0\langle x\rangle^{-\gamma}, \quad |\xi| \ge 2\nu,$$

by the elliptic estimates (or the sharp Gårding inequality), we have $\varphi \in H^{k+1-m/2,-\gamma}$. $\varphi \in H^{k+1/2,-\gamma-1/2}$ follows from $B_{2k}\varphi \in H^{k+1/2,-1/2}$ by the same argument. □

For the proof of Lemma 3.2.5, we compute the commutator of B and P, and then use a commutator inequality. We write $b = b^{\delta}_{M,\nu}$, $\tilde{b} = b^{\tilde{\delta}}_{\tilde{M},\tilde{\nu}}$, $\rho^{\delta}_* = \rho_*$ and $\tilde{\rho}_* = \rho^{\tilde{\delta}}_*$, where $* = +, -$, or 0. The following lower bound for the Poisson bracket is crucial in the proof of Lemma 3.2.5.

Lemma 3.2.7. *Let k, M and ν be as in Lemma 3.2.5. If M is sufficiently large, there are symbols $f_1, f_2 \in S(1, g)$ such that* supp $[f_1] \subset \Omega_1(M,\nu)$, supp $[f_2] \subset \Omega_2(M,\nu)$, *$f_1, f_2 \ge 0$, $f_2 \le C\langle x\rangle^{-(1+\mu-2\gamma)/2} b$, and $\delta_4 > 0$ such that*

$$\{p, \langle\xi\rangle^{2k}b^2\} \ge \delta_4 \frac{|v|}{|x|}\langle\xi\rangle^{2k}b^2 - \langle\xi\rangle^{2k+m-1}f_1^2 - f_2^2.$$

Proof. We first note

$$v\cdot\partial_x\eta = v\cdot\frac{\partial\hat{x}}{\partial x}\hat{v} = |v|\Big\langle \hat{v}, \Big(\frac{E}{|x|} - \frac{x\otimes x}{|x|^3}\Big)\hat{v}\Big\rangle = \frac{|v|}{|x|}(1-\eta^2),$$

where E denotes the identity matrix. We also note

$$\rho_0' = -\rho_+' - \rho_-', \quad \partial_x|x| = \hat{x}, \quad v\cdot(\partial_x|x|) = |v|\hat{v}\cdot\hat{x} = |v|\eta.$$

Using these, we compute:

$$\begin{aligned}
\{p_0, b\} &= v\cdot\partial_x b \\
&= (v\cdot\partial_x\eta)\big\{\rho_-'|x|^\gamma + \rho_0' + \rho_+'|x|^{-\gamma} - \gamma\big(\rho_-|x|^\gamma + \rho_0 + \rho_+|x|^{-\gamma}\big)\big\}\times \\
&\qquad \times \overline{\chi}_M(x)\overline{\chi}_\nu(\xi)e^{-\gamma\eta} \\
&\quad + (v\cdot\partial_x|x|)\big(\gamma\rho_-|x|^{\gamma-1} - \gamma\rho_+|x|^{-\gamma-1}\big)\overline{\chi}_M(x)\overline{\chi}_\nu(\xi)e^{-\gamma\eta} \\
&\quad + (v\cdot\partial_x|x|)\big(\rho_-|x|^\gamma + \rho_0 + \rho_+|x|^{-\gamma}\big)M^{-1}\overline{\chi}'(|x|/M)\overline{\chi}_\nu(\xi)e^{-\gamma\eta} \\
&= \frac{|v|}{|x|}(1-\eta^2)\big\{\rho_-'(|x|^\gamma - 1) + \rho_+'(|x|^{-\gamma}-1) - \gamma(\rho_-|x|^\gamma + \rho_0 + \rho_+|x|^{-\gamma})\big\}\times \\
&\qquad \times \overline{\chi}_M(x)\overline{\chi}_\nu(\xi)e^{-\gamma\eta} + \gamma\frac{|v|}{|x|}\big(\eta\rho_-|x|^\gamma - \eta\rho_+|x|^{-\gamma}\big)\overline{\chi}_M(x)\overline{\chi}_\nu(\xi)e^{-\gamma\eta} + r_0,
\end{aligned}$$

where

$$r_0(x,\xi) = |v(\xi)|\eta(x,\xi)b^\delta(x,\xi)M^{-1}\overline{\chi}'(|x|/M)\overline{\chi}_\nu(\xi),$$

which is supported in $\Omega_1(M,\nu)$. We may suppose $M \geq 1$, and then

$$\rho'_-(|x|^\gamma - 1) \leq 0, \quad \rho'_+(|x|^{-\gamma} - 1) \leq 0$$

on the support of b. We also note

$$\eta\rho_-(\eta) \leq (-7/8+\delta)\rho_-(\eta), \ -\eta\rho_+(\eta) \leq -\delta\rho_+(\eta),$$

and

$$(1-\eta^2)\rho_0(\eta) \geq \min(1-(\delta-1)^2, 1-(\delta+1/8)^2)\rho_0(\eta).$$

We set

$$\delta_3 = \min(7/8-\delta, \delta, 1-(\delta-1)^2, 1-(\delta+1/8)^2) > 0.$$

We substitute these inequality to the above formula on $\{p_0, b\}$ to learn

$$\begin{aligned}\{p_0, b\} &\leq -\gamma\delta_3\frac{|v|}{|x|}\{\rho_0 + \rho_-|x|^\gamma + \rho_+|x|^{-\gamma}\}\overline{\chi}_M(x)\overline{\chi}_\nu(\xi)e^{-\gamma\eta} + r_0 \\ &\leq -\delta_3\gamma\frac{|v|}{|x|}b(x,\xi) + r_0(x,\xi).\end{aligned}$$

Then we have

$$-\{p_0, b^2\} = -2b\{p_0, b\} \geq 2\delta_3\gamma\frac{|v|}{|x|}b^2 + 2br_0.$$

This also implies

$$-\{p_0, \langle\xi\rangle^{2k}b^2\} = -2b\{p_0, b\}\langle\xi\rangle^{2k} \geq 2\delta_3\gamma\frac{|v|}{|x|}\langle\xi\rangle^{2k}b^2 + 2\langle\xi\rangle^{2k}br_0. \tag{3.2.2}$$

On the other hand, we have $\{q, \langle\xi\rangle^{2k}b^2\} \in S(\langle x\rangle^{-\mu+2\gamma-1}\langle\xi\rangle^{2k+m-1}, g)$. We consider this function in more detail. We note, for any $\alpha, \beta \in \mathbb{Z}_+^n$,

$$\left|\partial_x^\alpha\partial_\xi^\beta b^\delta(x,\xi)\right| \leq C_{\alpha\beta}|x|^{\gamma-|\alpha|}|\xi|^{-|\beta|}, \quad x, \xi \neq 0, \tag{3.2.3}$$

with some $C_{\alpha\beta} > 0$. We also note

$$\begin{aligned}\{q, b\} &= \{q, b^\delta\}\overline{\chi}_M(x)\overline{\chi}_\nu(\xi) + b^\delta\{q, \overline{\chi}_M(x)\overline{\chi}_\nu(\xi)\} \\ &= \{q, b^\delta\}\overline{\chi}_M(x)\overline{\chi}_\nu(\xi) + r_1 + r_2,\end{aligned}$$

where

$$r_1 = b^\delta(\partial_\xi q)\cdot(\partial_x\overline{\chi}_M)\overline{\chi}_\nu(\xi), \quad r_2 = -b^\delta\overline{\chi}_M(x)(\partial_x q)\cdot(\partial_\xi\overline{\chi}_\nu).$$

We observe that r_1 is supported in $\Omega_1(M,\nu)$, and $r_1 \in S(\langle\xi\rangle^{m-1}, g)$; r_2 is supported in $\Omega_2(M,\nu)$ and $r_2 \in S(\langle x\rangle^{-1-\mu+\gamma}, g)$. Using (3.2.3), we have

$$\begin{aligned}\left|\{q, b^\delta\}\overline{\chi}_M(x)\overline{\chi}_\nu(\xi)\right| &\le C\langle x\rangle^{-\mu+\gamma-1}\langle\xi\rangle^{m-1}\overline{\chi}_M(x)\overline{\chi}_\nu(\xi)\\ &\le C'M^{-(\mu-2\gamma)}\frac{|v(\xi)|}{|x|}b(x,\xi)\end{aligned}$$

with some $C, C' > 0$. Hence we learn

$$\{q, \langle\xi\rangle^{2k}b^2\} \ge -2C'M^{-(\mu-2\gamma)}\frac{|v(\xi)|}{|x|}\langle\xi\rangle^{2k}b^2 + 2\langle\xi\rangle^{2k}br_1 + 2\langle\xi\rangle^{2k}br_2,$$

uniformly in $M \ge 1$. Combining this with (3.2.2), we learn

$$\{p, \langle\xi\rangle^{2k}b^2\} \ge (2\delta_3\gamma - 2C'M^{-(\mu-2\gamma)})\langle\xi\rangle^{2k}\frac{|v|}{|x|}b^2 + 2\langle\xi\rangle^{2k}b(r_0 + r_1 + r_2).$$

We recall $\gamma < \mu/2$. We now choose M so large that $2C'M^{-(\mu-2\gamma)} \le \delta_3\gamma$, and we obtain

$$\{p, \langle\xi\rangle^{2k}b^2\} \ge \delta_3\gamma\frac{|v|}{|x|}\langle\xi\rangle^{2k}b^2 + 2\langle\xi\rangle^{2k}b(r_0 + r_1 + r_2).$$

We note supp $[r_0 + r_1] \Subset \Omega_1(M,\nu)$ and $r_0 + r_1 \in S(\langle\xi\rangle^{m-1}, g)$, hence we can find $f_1 \in S(1,g)$, $f_1 \ge 0$, supp $[f_1] \subset \Omega_1(M,\nu)$ such that

$$2\langle\xi\rangle^{2k}b(r_0 + r_1) \ge -\langle\xi\rangle^{2k+m-1}f_1^2.$$

Similarly, since supp $[r_2] \Subset \Omega_2(M,\nu)$, $r_2 \in S(\langle x\rangle^{\gamma-\mu-1}, g)$, we can find $f_2 \in S(1,g)$, $f_2 \ge 0$, supp $[f_2] \subset \Omega_2(M,\nu)$ such that

$$2\langle\xi\rangle^{2k}br_2 \ge -f_2^2 \quad \text{and} \quad 0 \le f_2 \le C\langle x\rangle^{-(1+\mu-2\gamma)/2}b.$$

By setting $\delta_4 = \delta_3\gamma$, we arrive at the conclusion of the lemma. □

We write

$$B = \mathrm{Op}(b), \quad \tilde{B} = \mathrm{Op}(\tilde{b}), \quad \Lambda = \langle D_x\rangle^{(m-1)/2}\langle x\rangle^{-1/2}.$$

Lemma 3.2.8. *Under the above assumptions, there are pseudodifferential operators S, T, U, V and a constant $\delta_4 > 0$ such that*

$$-i[P, B\langle D_x\rangle^{2k}B] \ge \delta_4 B\langle D_x\rangle^k|\Lambda|^2\langle D_x\rangle^k B - S^*\langle D_x\rangle^{2k+m-1}S - T^*T - U - V,$$

where

(i) $S \in \mathrm{Op}S(1,g)$ and the symbol is supported in $\Omega_1(M,\nu)$;

(ii) $T \in \mathrm{Op}S(1,g)$ and the symbol is supported in $\Omega_2(M,\nu)$;

(iii) $U = \mathrm{Op}(u)$ *with* $u \in S(\langle x\rangle^{2\gamma-2}\langle\xi\rangle^{2k+m-2}, g)$ *and or any* $\alpha, \beta \in \mathbb{Z}_+^n$,

$$|\partial_x^\alpha \partial_\xi^\beta u(x,\xi)| \le C\langle x\rangle^{-2-|\alpha|}\langle\xi\rangle^{2k+m-2-|\beta|}\tilde{b}(x,\xi)^2;$$

(iv) $V \in \mathrm{Op}S(\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty}, g)$.

In the proof of Lemma 3.2.8, we use the following estimate:

Lemma 3.2.9. *Suppose* a *be a symbol such that* $\operatorname{supp}[a] \subset \Omega'$, *where* $\Omega' = \{(x,\xi) \mid |x| \ge M', |\xi| \ge \nu'\}$ *with* $M' > \tilde{M}$, $\nu' > \tilde{\nu}$, *and for any* $\alpha, \beta \in \mathbb{Z}_+^n$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha\beta}\langle x\rangle^{2\ell-|\alpha|}\langle\xi\rangle^{2s-|\beta|}\tilde{b}(x,\xi)^2,$$

where $s, \ell \in \mathbb{R}$. *Then for any* N, *there is* $C, C_N > 0$ *such that*

$$|\langle\varphi, \mathrm{Op}(a)\varphi\rangle| \le C\|\tilde{B}\varphi\|_{H^{s,\ell}}^2 + C_N\|\varphi\|_{H^{-N,-N}}^2, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Proof. We note, for any $\alpha, \beta \in \mathbb{Z}_+^n$,

$$\left|\partial_x^\alpha \partial_\xi^\beta \tilde{b}(x,\xi)\right| \le C'_{\alpha\beta}\langle x\rangle^{-|\alpha|+2\gamma(|\alpha+\beta|)}\langle\xi\rangle^{-|\beta|}\tilde{b}(x,\xi), \quad (x,\xi) \in \Omega'.$$

We write $\tilde{g} = \langle x\rangle^{-2+4\gamma}dx^2 + \langle x\rangle^{4\gamma}\langle\xi\rangle^{-2}d\xi^2$. Using the above estimate and the assumption on a, and following the construction of parametrices for elliptic operators, we can construct a symbol $h(x,\xi) \in S(1,\tilde{g})$ such that

$$\mathrm{Op}(a) = \tilde{B}\langle x\rangle^\ell\langle D_x\rangle^s\mathrm{Op}(h)\langle D_x\rangle^s\langle x\rangle^\ell\tilde{B} + R,$$

where $R \in S(\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty}, \tilde{g})$. The assertion follows from this since $\mathrm{Op}(h)$ is bounded in $L^2(\mathbb{R}^n)$. □

Proof of Lemma 3.2.8. By the standard pseudodifferential operator calculus, we can find $\tilde{f}_1$, $\tilde{f}_2$ such that $\tilde{f}_j \in S(1,g)$, $\operatorname{supp}[\tilde{f}_j] \subset \Omega_j(M,\nu)$, $j = 1, 2$, and

$$\begin{aligned}
&\mathrm{Op}(\langle\xi\rangle^{2k+m-1}f_1^2) \le \mathrm{Op}(\tilde{f}_1)^*\langle D_x\rangle^{2k+m-1}\mathrm{Op}(\tilde{f}_1) + R_1,\\
&\mathrm{Op}(\langle x\rangle^{2\gamma-1-\mu}f_2^2) \le \mathrm{Op}(\tilde{f}_2)^*\mathrm{Op}(\tilde{f}_2) + R_2,
\end{aligned}$$

where R_j are smoothing operators. We set $S = \mathrm{Op}(\tilde{f}_1)$ and $T = \mathrm{Op}(\tilde{f}_2)$. If we write

$$\zeta(x,\xi) = \{p, \langle\xi\rangle^{2k}b^2\} - \delta_4\frac{|v|}{|x|}\langle\xi\rangle^{2k}b^2 + \langle\xi\rangle^{2k+m-1}f_1^2 + f_2^2 \ge 0.$$

We note, by the construction, $\zeta(x,\xi)b'(x,\xi)^{-2} \in S(\langle x\rangle^{-1}\langle\xi\rangle^{2k+m-1}, g)$, where $b' = b_{M',\nu'}$ with $\tilde{M} < M' < M$, $\tilde{\nu} < \nu' < \nu$. Hence by the sharp Gårding inequality, we have

$$\mathrm{Op}(\zeta(b')^{-2}) \ge -C\langle D_x\rangle^{k-1+m/2}\langle x\rangle^{-2}\langle D_x\rangle^{k-1+m/2}$$

with some $C>0$. Then by the asymptotic expansion, we learn

$$\mathrm{Op}(\zeta) \geq -CB'\langle D_x\rangle^{k-1+m/2}\langle x\rangle^{-2}\langle D_x\rangle^{k-1+m/2}B' - R_3,$$

where $R_3 \in S(\langle x\rangle^{-3}\langle\xi\rangle^{2k+m-3}, g)$, and the symbol is supported in supp $[b']$ modulo $\mathcal{S}(\mathbb{R}^{2d})$. Using Lemma 3.2.7, we can estimate R_3 and other error terms from below by $-C\tilde{B}\langle D_x\rangle^{k-1+m/2}\langle x\rangle^{-2}\langle D_x\rangle^{k-1+m/2}\tilde{B}$, modulo smoothing operators, and these will be included in U to complete the proof. □

Lemma 3.2.10. *For* $\varphi \in \mathcal{S}(\mathbb{R}^n)$*, the inequality* (3.2.1) *holds, where* $S = \mathrm{Op}(f_1)$*,* $T = \mathrm{Op}(f_2)$*,* $f_1, f_2 \in S(1, g)$*, and* supp $[f_1] \subset \Omega_1(M,\nu)$, supp $[f_2] \subset \Omega_2(M,\nu)$.

Proof. We compute the commutator to obtain quadratic inequalities. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\big\langle \varphi, -i[P, B\langle D_x\rangle^{2k}B]\varphi\big\rangle = \big\langle \varphi, -i[(P-z), B\langle D_x\rangle^{2k}B]\varphi\big\rangle \\
&= -i\big(\big\langle \langle D_x\rangle^k B(P-\bar{z})\varphi, \langle D_x\rangle^k B\varphi\big\rangle - \big\langle \langle D_x\rangle^k B\varphi, \langle D_x\rangle^k B(P-z)\varphi\big\rangle\big) \\
&= -i\big(\big\langle (\Lambda^{-1})^*\langle D_x\rangle^k B(P-z)\varphi, \Lambda\langle D_x\rangle^k B\varphi\big\rangle \\
&\qquad - \big\langle \Lambda\langle D_x\rangle^k B\varphi, (\Lambda^{-1})^*\langle D_x\rangle^k B(P-z)\varphi\big\rangle\big) - 2(\mathrm{Im}\, z)\big\|\langle D_x\rangle^k B\varphi\big\|^2 \\
&\leq 2\big\|(\Lambda^{-1})^*\langle D_x\rangle^k B(P-z)\varphi\big\| \cdot \big\|\Lambda\langle D_x\rangle^k B\varphi\big\| - 2(\mathrm{Im}\, z)\big\|\langle D_x\rangle^k B\varphi\big\|^2.
\end{aligned}$$

Combining this with Lemma 3.2.8, we have

$$\begin{aligned}
&\delta_4\big\|\Lambda\langle D_x\rangle^k B\varphi\big\|^2 + 2(\mathrm{Im}\, z)\big\|\langle D_x\rangle^k B\varphi\big\|^2 \\
&\qquad - \langle \varphi, (S^*\langle D_x\rangle^{2k+m-1}S + T^*T + U + V)\varphi\rangle \\
&\quad\leq 2\big\|(\Lambda^{-1})^*\langle D_x\rangle^k B(P-z)\varphi\big\| \cdot \big\|\Lambda\langle D_x\rangle^k B\varphi\big\| \\
&\quad\leq \frac{\delta_4}{2}\big\|\Lambda\langle D_x\rangle^k B\varphi\big\|^2 + \frac{4}{\delta_4}\big\|(\Lambda^{-1})^*\langle D_x\rangle^k B(P-z)\varphi\big\|^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\frac{\delta_4}{2}\big\|\Lambda\langle D_x\rangle^k B\varphi\big\|^2 + 2(\mathrm{Im}\, z)\big\|\langle D_x\rangle^k B\varphi\big\|^2 \\
&\leq \frac{4}{\delta_4}\big\|(\Lambda^{-1})^*\langle D_x\rangle^k B(P-z)\varphi\big\|^2 \\
&\qquad + \langle \varphi, (S^*\langle D_x\rangle^{2k+m-1}S + T^*T + U + V)\varphi\rangle.
\end{aligned}$$

Now we note, by Lemma 3.2.9,

$$\langle \varphi, U\varphi\rangle \leq C\|\tilde{B}\varphi\|^2_{H^{k-1+m/2,-1}} + C\|\varphi\|^2_{H^{-N,-N}}$$

with any N. These imply (3.2.1) for $\varphi \in \mathcal{S}(\mathbb{R}^n)$. □

We now extend Lemma 3.2.10 to more general φ to prove Lemma 3.2.5. We choose M' and ν' so that $\tilde{M} < M' < M$, $\tilde{\nu} < \nu' < \nu$, $\delta < \delta' < \tilde{\delta}$, and set

$$b'(x,\xi) = b^{\delta'}_{M',\nu'}(x,\xi), \quad B' = \mathrm{Op}(b').$$

We write

$$A_\varepsilon = \langle \varepsilon D_x \rangle^{-1} B, \quad \tilde{A}_\varepsilon = \langle \varepsilon D_x \rangle^{-1} \tilde{B}, \quad A'_\varepsilon = \langle \varepsilon D_x \rangle^{-1} B'$$

and we denote their symbols by a_ε, $\tilde{a}_\varepsilon$ and a'_ε, respectively.

By the same computation as in the proof of Lemma 3.2.7, we have

$$\{p, \langle \xi \rangle^{2k} |a_\varepsilon|^2\} \geq \delta_4 \frac{|v|}{|x|} \langle \xi \rangle^{2k} |a_\varepsilon|^2 - \langle \xi \rangle^{2k+m-1} f_1^2 - \langle x \rangle^{2\gamma-1-\mu} f_2^2,$$

modulo $\mathcal{S}(\mathbb{R}^n)$-terms, where constants are independent of ε, and f_1 and f_2 are independent of ε. Then, as well as Lemma 3.2.8, we have

$$\begin{aligned} &- i[P, A^*_\varepsilon \langle D_x \rangle^{2k} A_\varepsilon] \\ &\quad \geq \delta_4 A^*_\varepsilon \langle D_x \rangle^k \Lambda^2 \langle D_x \rangle^k A_\varepsilon - S^* \langle D_x \rangle^{2k+m-1} S - T^* T - U_\varepsilon - V_\varepsilon, \end{aligned}$$

where the symbol of U_ε has the property:

$$|u_\varepsilon(x,\xi)| \leq C \langle x \rangle^{-2} \langle \xi \rangle^{2k+m-2} |a'_\varepsilon(x,\xi)|^2, \tag{3.2.4}$$

and symbols of U_ε and V_ε are bounded in the respective symbol classes. It follows that

$$|\langle \varphi, U_\varepsilon \varphi \rangle| \leq C \|A'_\varepsilon \varphi\|^2_{H^{k-1+m/2,-1}} + C \|\varphi\|^2_{H^{-N,-N}}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where the constant is independent of ε. Thus we have, as well as Lemma 3.2.10, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} &\|A_\varepsilon \varphi\|^2_{H^{k+(m-1)/2,-1/2}} + (\mathrm{Im}\, z) \|A_\varepsilon \varphi\|^2_{H^k} \\ &\quad \leq C\big(\|A_\varepsilon (P-z)\varphi\|^2_{H^{k-(m-1)/2,1/2}} + \|A'_\varepsilon \varphi\|^2_{H^{k-1+m/2,-1}} \\ &\qquad + \|S\varphi\|^2_{H^{k+(m-1)/2}} + \|T\varphi\|^2_{L^2} \big) + C_N \|\varphi\|^2_{H^{-N,-N}}, \end{aligned} \tag{3.2.5}$$

with any N, where C and C_N are independent of $\varepsilon \in (0,1]$.

Lemma 3.2.11. *Suppose that $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $\tilde{B}\varphi \in H^{k-1+m/2,-1/2}$,*

$$A_\varepsilon (P-z)\varphi \in H^{k-(m-1)/2,1/2}, \ S\varphi \in H^{k+(m-1)/2} \text{ and } T\varphi \in L^2.$$

Then $A_\varepsilon \varphi \in H^{k+(m-1)/2,-1/2} \cap H^m$ and (3.2.5) holds.

Proof. We set, for $L \gg 0$,

$$X_L = \chi_L(x) \chi_L(D_x).$$

We first note $\|X_L \psi - \psi\|_{H^{s,\ell}} \to 0$ as $L \to \infty$, provided $\psi \in H^{s,\ell}$. We also note $\psi \in H^{s,\ell}$ if and only if $\lim_{L\to\infty} \|X_L \psi\|_{H^{s,\ell}} < \infty$.

We observe that the symbol of $[X_L, A_\varepsilon]$ is bounded by $C\langle x\rangle^{-1}\langle\xi\rangle^{-1}a'_\varepsilon(x,\xi)$, modulo $\mathcal{S}(\mathbb{R}^{2d})$-terms, uniformly in L, and also it converges to 0 locally uniformly as $L \to \infty$. These imply

$$\begin{aligned}\lim_{L\to\infty} \|X_L A_\varepsilon \psi\|_{H^{s,\ell}} &\le \varliminf_{L\to\infty} (\|A_\varepsilon X_L\psi\|_{H^{s,\ell}} + \|[X_L, A_\varepsilon]\psi\|_{H^{s,\ell}}) \\ &\le \varliminf_{L\to\infty} \|A_\varepsilon X_L\psi\|_{H^{s,\ell}}\end{aligned}$$

with any N, provided $\tilde{B}\psi \in H^{s-1,\ell-1}$. In particular, since we assume $\tilde{B}\varphi \in H^{k-1+m/2,-1/2}$,

$$\begin{aligned}&\lim_{L\to\infty} (\|X_L A_\varepsilon\varphi\|^2_{H^{k+(m-1)/2,-1/2}} + \|X_L A_\varepsilon\varphi\|^2_{H^k}) \\ &\qquad\qquad \le \varliminf_{L\to\infty} (\|A_\varepsilon X_L\varphi\|^2_{H^{k+(m-1)/2,-1/2}} + \|A_\varepsilon X_L\varphi\|^2_{H^k}).\end{aligned}$$

By the same argument, using $\tilde{B}\varphi \in H^{k-1+m/2,-1/2}$, we learn

$$\varlimsup_{L\to\infty} \|A_\varepsilon(P-z)X_L\varphi\|^2_{H^{k-(m-1)/2,1/2}} \le \|A_\varepsilon(P-z)\varphi\|^2_{H^{k-(m-1)/2,1/2}}.$$

We have similar estimates for $\|S\varphi\|_{H^{k+(m-1)/2}}$ and $\|T\varphi\|_{L^2}$. Concerning the estimate for $\|A'_\varepsilon\varphi\|_{H^{k-1+m/2,-1}}$, we use the fact that $\tilde{B}\varphi \in H^{k-1+m/2,-1/2}$ to obtain

$$\varlimsup_{L\to\infty} \|A'_\varepsilon X_L\varphi\|^2_{H^{k-1+m/2,-1}} \le \|A'_\varepsilon\varphi\|^2_{H^{k-1+m/2,-1}}.$$

Combining these with (3.2.5) for $X_L\varphi$, we learn

$$\begin{aligned}&\lim_{L\to\infty} (\|X_L A_\varepsilon\varphi\|^2_{H^{k+(m-1)/2,-1/2}} + \|X_L A_\varepsilon\varphi\|^2_{H^k}) \\ &\quad \le \varlimsup_{L\to\infty} \big(C(\|A_\varepsilon(P-z)X_L\varphi\|^2_{H^{k-(m-1)/2,1/2}} + \|A'_\varepsilon X_L\varphi\|^2_{H^{k-1+m/2,-1}} \\ &\qquad\quad + \|SX_L\varphi\|^2_{H^{k+(m-1)/2}} + \|TX_L\varphi\|^2_{L^2}) + C_N\|X_L\varphi\|^2_{H^{-N,-N}}\big) \\ &\quad \le C\big(\|A_\varepsilon(P-z)\varphi\|^2_{H^{k-(m-1)/2,1/2}} + \|A'_\varepsilon\varphi\|^2_{H^{k-1+m/2,-1}} \\ &\qquad\quad + \|S\varphi\|^2_{H^{k+(m-1)/2}} + \|T\varphi\|^2_{L^2}\big) + C'_N\|\varphi\|^2_{H^{-N,-N}},\end{aligned}$$

and this implies the assertion. □

Proof of Lemma 3.2.5. It remains to take the limit $\varepsilon \to 0$ in (3.2.5). We note

$$\begin{aligned}\|A_\varepsilon\varphi\|_{H^{s,\ell}} &= \|\langle D_x\rangle^s\langle x\rangle^\ell\langle\varepsilon D_x\rangle^{-1}B\varphi\|_{L^2} \\ &= \|\langle\varepsilon D_x\rangle^{-1}\langle D_x\rangle^s\langle x\rangle^\ell B\varphi + \langle D_x\rangle^s[\langle x\rangle^\ell, \langle\varepsilon D_x\rangle^{-1}]B\varphi\|_{L^2},\end{aligned}$$

and hence

$$\|\langle\varepsilon D_x\rangle^{-1}\langle D_x\rangle^s\langle x\rangle^\ell B\varphi\|_{L^2} \le \|A_\varepsilon\varphi\|_{H^{s,\ell}} + C\|B\varphi\|_{H^{s-1,\ell-1}}.$$

Thus we have

$$\|B\varphi\|_{H^{s,\ell}} \le \varliminf_{\varepsilon\to 0} \|A_\varepsilon\varphi\|_{H^{s,\ell}} + C\|B\varphi\|_{H^{s-1,\ell-1}}.$$

We note this holds without assuming $B\varphi \in H^{s,\ell}$, and if the right hand side is finite, we obtain $B\varphi \in H^{s,\ell}$.

By the same argument, we also have

$$\begin{aligned}
&\varlimsup_{\varepsilon\to 0} \|A_\varepsilon(P-z)\varphi\|_{H^{k-(m-1)/2,1/2}} \\
&\quad \le \|B(P-z)\varphi\|_{H^{k-(m-1)/2,1/2}} + C\|B(P-z)\varphi\|_{H^{k-(m-3)/2,-1/2}} \\
&\quad \le (1+C)\|B(P-z)\varphi\|_{H^{k-(m-1)/2,1/2}}
\end{aligned}$$

and similarly,

$$\varlimsup_{\varepsilon\to 0} \|A'_\varepsilon\varphi\|_{H^{k-1+m/2,-1}} \le C'\|B'\varphi\|_{H^{k-1+m/2,-1}}.$$

Substituting these to (3.2.5), we have

$$\begin{aligned}
&\|B\varphi\|^2_{H^{k+(m-1)/2,-1/2}} + (\mathrm{Im}\, z)\|B\varphi\|^2_{H^k} \\
&\quad \le \varlimsup_{\varepsilon\to 0}\big(\|A_\varepsilon\varphi\|^2_{H^{k+(m-1)/2,-1/2}} + C\|B\varphi\|^2_{H^{k+(m-3)/2,-3/2}} \\
&\qquad + (\mathrm{Im}\, z)(\|A_\varepsilon\varphi\|^2_{H^k} + C\|B\varphi\|^2_{H^{k-1,-1}})\big) \\
&\quad \le \varlimsup_{\varepsilon\to 0} C\big(\|A_\varepsilon(P-z)\varphi\|^2_{H^{k-(m-1)/2,1/2}} + \|\tilde{A}_\varepsilon\varphi\|^2_{H^{k,-1}} + \|B\varphi\|^2_{H^{k-1/2,-1}}\big) \\
&\qquad + C(\|S\varphi\|^2_{H^{k+(m-1)/2}} + \|T\varphi\|^2_{L^2}) + C_N\|\varphi\|^2_{H^{-N,-N}} \\
&\quad \le C'\big(\|B(P-z)\varphi\|^2_{H^{k-(m-1)/2,1/2}} + \|\tilde{B}\varphi\|^2_{H^{k,-1}} \\
&\qquad + \|S\varphi\|^2_{H^{k+(m-1)/2}} + \|T\varphi\|^2_{L^2}\big) + C_N\|\varphi\|^2_{H^{-N,-N}},
\end{aligned}$$

and this completes the proof of Lemma 3.2.5. □

3.3 Estimates for the classical trajectories

In this section, we prove estimates on the classical trajectories which are used in the proof of Proposition 3.2.3. First, we show a classical Mourre estimate which implies the peudo-convexity of $\mathbb{R}^n$ with respect to P. We note

$$(y(t,x,\lambda\xi),\eta(t,x,\lambda\xi)) = (y(\lambda^{m-1}t,x,\xi),\lambda\eta(\lambda^{m-1}t,x,\xi)) \quad \text{for } \lambda > 0,$$

since p_m is homogeneous of degree m.

Lemma 3.3.1. *There exist $M > 0$ and $R_0 > 1$ such that*

$$H^2_{p_m}(|x|^2) \ge M|\xi|^{2(m-1)}$$

for any $(x,\xi) \in \{(y,\eta) \in T^*\mathbb{R}^n \mid |y| > R_0,\ |\eta| \ne 0\}$.

Proof. We have

$$\begin{aligned}
H^2_{p_m}(|x|^2) &= 2H_{p_m}(x\cdot\partial_\xi p_m) \\
&= 2|\partial_\xi p_m|^2 + 2\sum_{j,k=1}^n x_j(\partial_{x_k}\partial_{\xi_j}p_m)\partial_{\xi_k}p_m - 2\sum_{j,k=1}^n x_j(\partial_{\xi_j}\partial_{\xi_k}p_m)\partial_{x_k}p_m.
\end{aligned}$$

On the other hand, by Assumption A, there exists $C > 0$ such that

$$\left|2 \sum_{j,k=1}^{n} x_j(\partial_{x_k}\partial_{\xi_j}p_m)\partial_{\xi_k}p_m - 2 \sum_{j,k=1}^{n} x_j(\partial_{\xi_j}\partial_{\xi_k}p_m)\partial_{x_k}p_m\right| \leq C\langle R_0\rangle^{-\mu}|\xi|^{2(m-1)}.$$

Combining this with the non-degeneracy condition of $\partial_\xi p_0(\xi)$ in Assumption A, we conclude the assertion. □

Next, we observe that an energy bound on classical trajectories holds, even if p is not elliptic. We note an analogous result is proved in [49], though our proof is simpler.

Lemma 3.3.2. *Fix $(x_0, \xi_0) \in T^*\mathbb{R}^n$ with $\xi_0 \neq 0$ and suppose that (x_0, ξ_0) is forward non-trapping in the sense that $|y(t, x_0, \xi_0)| \to \infty$ as $t \to \infty$. Then, there exist $C_1, C_2 > 0$ such that*

$$C_1 \leq |\eta(t, x_0, \xi_0)|^{m-1} \leq C_2,$$

for $t \geq 0$.

Proof. Let R_0 be as in Lemma 3.3.1, and we let $R_1 \geq R_0$ which is determined later. We first note that by the forward non-trapping condition and Lemma 3.3.1, there exits $t_0 \geq 0$ such that for $t \geq t_0$, we have

$$|y(t, x_0, \xi_0)| \geq R_1, \quad \frac{d}{dt}|y(t, x_0, \xi_0)|^2 \geq 0. \tag{3.3.1}$$

By Lemma 3.3.1 and the non-trapping condition, it is easy to observe that there is $t_0 > 0$ such that $\frac{d^2}{dt^2}|y(t, x_0, \xi_0)|^2 > 0$ for $t \geq s_0$, and $\frac{d}{dt}|y(t_0, x_0, \xi_0)|^2 > 0$. Then for all $t \geq t_0$, the condition (3.3.1) is satisfied.

Let $C_0 > 0$ be a constant such that

$$|\partial_x p_m(x, \xi)| \leq C_0|x|^{-1-\mu}|\xi|^m,$$

and we write $\eta_0 = |\eta(t_0, x_0, \xi_0)| > 0$. We set

$$T = \sup\bigl\{s \geq t_0 \bigm| \eta_0/2 \leq |\eta(t, x_0, \xi_0)| \text{ for } t \in [t_0, s]\bigr\} \in (t_0, \infty].$$

By Lemma 3.3.1, we have

$$|y(t, x_0, \xi_0)|^2 \geq R_1^2 + \frac{M\eta_0^{2(m-1)}}{2^{2m-1}}(t - t_0)^2, \quad t_0 \leq t \leq T.$$

Now we note

$$\left|\frac{d}{dt}|\eta(t, x_0, \xi_0)|\right| \leq C_0|y(t, x_0, \xi_0)|^{-1-\mu}|\eta(t, x_0, \xi_0)|^m$$

and hence

$$\left|\frac{d}{dt}|\eta(t, x_0, \xi_0)|^{-(m-1)}\right| \leq (m-1)C_0\biggl(R_1^2 + \frac{M\eta_0^{2(m-1)}}{2^{2m-1}}(t - t_0)^2\biggr)^{-(1+\mu)/2}$$

for $t_0 \le t \le T$. Thus we have

$$\begin{aligned}
&\left|\eta_0^{-(m-1)} - |\eta(T, x_0, \xi_0)|^{-(m-1)}\right| \\
&\quad \le \int_{t_0}^{T} (m-1)C_0 \left(R_1^2 + \frac{M\eta_0^{2(m-1)}}{2^{2m-1}}(t-t_0)^2 \right)^{-(1+\mu)/2} dt \\
&\quad \le \frac{C_0 2^{(2m-1)/2} R_1^{-\mu}}{(1+\mu)\sqrt{M}} \eta_0^{-(m-1)}.
\end{aligned}$$

We now choose $R_1 > 0$ so large that

$$\frac{C_0 2^{(2m-1)/2} R_1^{-\mu}}{(1+\mu)\sqrt{M}} < 1/2, \quad \text{i.e.,} \quad R_1 > \left(\frac{C_0 2^{(2m+1)/2}}{(1+\mu)\sqrt{M}} \right)^{1/\mu},$$

then

$$|\eta(T, x_0, \xi_0)|^{-(m-1)} < (3/2)\eta_0^{-(m-1)},$$

i.e., $|\eta(T, x_0, \xi_0)| > (2/3)^{1/(m-1)}\eta_0 > (1/2)\eta_0$. If $T < \infty$, this is a contradiction, and hence $T = \infty$. Thus we also learn

$$2^{-1}\eta_0 \le |\eta(t, x_0, \xi_0)| \le 2^{1/(m-1)}\eta_0, \quad t \ge t_0.$$

□

Corollary 3.3.3. *Suppose the same assumptions as in Lemma 3.3.2 hold. Moreover, suppose* $|\xi_0| = 1$. *Then, we have*

$$C_1\lambda \le |\eta(t, x_0, \lambda\xi_0)| \le C_2\lambda$$

for any $\lambda > 0$ *and* $t \ge 0$.

Corollary 3.3.4. *Under the same assumptions as in Lemma 3.3.2 with* $|\xi_0| = 1$, *there exist* $C, C', K, K' > 0$ *such that*

$$C\lambda t - K \le |y(t, x_0, \xi_0)| \le C'\lambda t + K'$$

for $\lambda > 0$ *and* $t \ge 0$.

Combining with the estimate $|\partial_x p_m(x, \xi)| \le C\langle x\rangle^{-1-\mu}|\xi|^{m-1}$, we obtain:

Corollary 3.3.5. *Suppose that* $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ *is non-trapping. Then,*

$$\begin{aligned}
\eta_+ &= \lim_{t\to\infty} \eta(t, x_0, \xi_0) \ne 0, \\
v_+ &= \lim_{t\to\infty} \partial_\xi p_m(y(t, x_0, \xi_0), \eta(t, x_0, \xi_0)) = \lim_{t\to\infty} \partial_\xi p_0(\eta(t, x_0, \xi_0)) \ne 0
\end{aligned}$$

exist.

3.4 Construction of the conjugate operator

Let $(x_0, \xi_0) \in p_m^{-1}(0) \setminus \{\xi \neq 0\}$. By Assumption B, (x_0, ξ_0) is forward non-trapping. We denote $y(t) = y(t, x_0, \xi_0)$, $\eta(t) = \eta(t, x_0, \xi_0)$. We note that

$$\lim_{j\to\infty} \eta(t, x_0, \xi_0) = \eta_+ \neq 0, \ \lim_{t\to\infty} \partial_\xi p_m(y(t), \eta(t)) = v_+ \neq 0,$$

exist by Corollary 3.3.5. Moreover, there exist $M_1, M_2 > 0$ such that

$$\begin{aligned} &|y(t)/t - v_+|, \ |\eta(t) - \eta_+| = O(\langle t\rangle^{-\mu}) \quad \text{as } t \to \infty, \\ &M_1 \leq |\eta(t)| \leq M_2, \quad t \geq 0. \end{aligned} \tag{3.4.1}$$

We denote $B(r, s, z, \zeta) = \{(x, \xi) \in \mathbb{R}^{2n} | |z - x| < r, |\zeta - \xi| < s\} \subset \mathbb{R}^{2n}$. In order to prove Proposition 3.2.3, it suffices to prove the following theorem. We set an h-dependent metric g_h by

$$g_h = dx^2/\langle x\rangle^2 + h^{2/(m-1)} d\xi^2.$$

Theorem 3.4.1. *There exist $\psi_h \in C_c^\infty(\mathbb{R}^{2n})$ and $\varphi_{h,t} \in C^\infty(\mathbb{R}_{\geq 0}, C_c^\infty(\mathbb{R}^{2n}))$ such that $F(h, t) = \mathrm{Op}(\varphi_{h,t})$ and:*

(i) $F(h, 0) = |\mathrm{Op}(\psi_h)|^2$ *with* $\psi_h(x_0, \xi_0) \geq 1$.

(ii) $\varphi_{h,t}$ *satisfies*

$$\mathrm{supp}\ \varphi_{h,t} \subset B(4h^{-1}t\delta_1, 4h^{-1/(m-1)}\delta_2, h^{-1}tv_+, h^{-\frac{1}{m-1}}\eta_+)$$

modulo $S(h^\infty, g_h)$ if t/h is sufficiently large.

(iii) For any $\alpha, \beta \in \mathbb{N}_{\geq 0}^n$, there exists $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \varphi_{h,t}(x, \xi)| \leq C_{\alpha\beta} \langle t\rangle h^{(|\beta|+1)/(m-1)-1} \langle x\rangle^{-|\alpha|}.$$

(iv) There exists a family of bounded operator $R(h, t)$ in $L^2(\mathbb{R}^n)$ such that

$$\frac{\partial F}{\partial t} + i[P, F] \geq -R(h, t),$$

where $\sup_{\geq 0} \langle t\rangle^{-1} \|R(h, t)\|_{L^2\to L^2} = O(h^\infty)$.

The proof of Theorem 3.4.1 is based on the fact that any classical trajectory of H_p behave as straight lines even if p is not elliptic. We follow the argument in [57].

Lemma 3.4.2. *There exist constants $\delta_1, \delta_2 > 0$ with $|\eta_+| > 4\delta_1$ such that the following holds:*

There exists a smooth function $\psi \in C^\infty(\mathbb{R}_{\geq 0}, C_c^\infty(\mathbb{R}^{2n}))$ such that

(i) $\psi \geq 0$, *and* $\psi(0, x_0, \xi_0) \geq 1$.

(ii) $\operatorname{supp}\psi(t,\cdot,\cdot) \subset B(2t\delta_1, 2\delta_2, tv_+, \eta_+)$ *for* $t \geq T_0$, *where* $T_0 > 0$ *depends only on* (x_0, ξ_0), p_m *and* δ_1.

(iii) *For any* $\alpha, \beta \in \mathbb{N}^n$, *there exists* $C_{\alpha\beta} > 0$ *such that*

$$|\partial_x^\alpha \partial_\xi^\beta \psi(t,x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|},\ |\partial_x^\alpha \partial_\xi^\beta \partial_t\psi(t,x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{-1-|\alpha|}$$

for $t \geq 0$ *and* $x, \xi \in \mathbb{R}^n$.

(iv) ψ *satisfies*

$$\left(\frac{\partial\psi}{\partial t} + \{p_m, \psi\}\right)(t,x,\xi) \geq 0$$

for $t \geq 0$, $x, \xi \in \mathbb{R}^n$.

Proof. Let $\Psi \in C^\infty(\mathbb{R})$ such that $0 \leq \Psi \leq 1$, $\Psi' \leq 0$, $\Psi = 1$ for $r \leq \frac{1}{2}$, $\Psi = 0$ for $r \geq 1$, $\Psi(r) > 0$ if $\frac{1}{2} < r < 1$. We define

$$\psi_0(t,x,\xi) := \Psi\left(\frac{|x-y(t)|}{\delta_1\langle t\rangle}\right)\Psi\left(\frac{|\xi-\eta(t)|}{\gamma(t)}\right)$$

where we set $\gamma(t) = \delta_2 - C_1\langle t\rangle^{-\mu}$ and let $C_1 > 0$ be determined later. We set

$$\begin{aligned}
&L(t,x,\xi) = \partial_\xi p_m(x,\xi) - \partial_\xi p_m(y(t),\eta(t)),\\
&A_0(t,x,\xi) = \frac{1}{\delta_1\langle t\rangle}\left(L(t,x,\xi)\cdot\frac{x-y(t)}{|x-y(t)|} - \frac{t|x-y(t)|}{\langle t\rangle^2}\right),\\
&A_1(t,x,\xi) = \frac{1}{\gamma(t)}\left(-\frac{\gamma'(t)|\xi-\eta(t)|}{\gamma(t)} + \big(\partial_x p(y(t),\eta(t)) - \partial_x p(x,\xi)\big)\cdot\frac{\xi-\eta(t)}{|\xi-\eta(t)|}\right).
\end{aligned}$$

For $t > 0$, we have

$$\begin{aligned}
\left(\frac{\partial\psi_0}{\partial t} + \{p_m, \psi_0\}\right)(t,x,\xi) =& A_0(t,x,\xi)\Psi'\left(\frac{|x-y(t)|}{\delta_1 t}\right)\Psi\left(\frac{|\xi-\eta(t)|}{\gamma(t)}\right)\\
&+ A_1(t,x,\xi)\Psi\left(\frac{|x-y(t)|}{\delta_1 t}\right)\Psi'\left(\frac{|\xi-\eta(t)|}{\gamma(t)}\right).
\end{aligned} \tag{3.4.2}$$

Using $|\partial_\xi p(x,\xi) - \partial_\xi p(y(t),\eta(t))| \leq C_0|\xi-\eta(t)|$ with a constant $C > 0$, we have

$$\delta_1\langle t\rangle A_0(t,x,\xi) \leq -\frac{\delta_1 t}{2\langle t\rangle} + C_0\gamma(t) \leq -\frac{\delta_1 t}{2\langle t\rangle} + C_0\delta_2 - C_0 C_1\langle t\rangle^{-\mu} \tag{3.4.3}$$

for $(x,\xi) \in \operatorname{supp}\Psi'(|x-y(t)|/\delta_1\langle t\rangle)\Psi(|\xi-\eta(t)|/\gamma(t))$. By Assumption A and (3.4.1), there exists $C, T_{00} > 0$ such that for $(x,\xi) \in \operatorname{supp}\psi_0(t,x,\xi)$, we have

$$|\partial_x p_m(y(t),\eta(t)) - \partial_x p_m(x,\xi)| \leq C\langle t\rangle^{-1-\mu}$$

for $t \geq T_{00}$. Here, we can choose $C, T_{00} >$ independently of C_1. We note that and $\gamma(t)/2 \leq |\xi - \eta(t)|$ holds on the support of $\Psi'(|\xi - \eta(t)|/\gamma(t))$. Using these observations, we learn

$$\begin{aligned} A_1(t,x,\xi) &\leq -\frac{\gamma'(t)}{\gamma(t)^2}|\xi-\eta(t)| + \frac{C\langle t\rangle^{-1-\mu}}{\gamma(t)} \\ &= \frac{1}{\gamma(t)}\left(-\frac{C_1\mu t}{\langle t\rangle^{2+\mu}}\cdot\frac{|\xi-\eta(t)|}{\gamma(t)} + \frac{C}{\langle t\rangle^{1+\mu}}\right) \leq -\frac{1}{\gamma(t)}\left(\frac{C_1\mu t}{2\langle t\rangle^{2+\mu}} - \frac{C}{\langle t\rangle^{1+\mu}}\right) \end{aligned} \tag{3.4.4}$$

for $(x, \xi) \in \operatorname{supp} \Psi(|x - y(t)|/\delta_1\langle t\rangle)\Psi'(|\xi - \eta(t)|/\gamma(t))$ with $t \geq T_{00}$. By (3.4.2), (3.4.3) and (3.4.4) with $\Psi' \leq 0$ and $\delta_1 >> \delta_2$, we can select $T_{00} > 0$ and $C_1 > 0$ such that for $t \geq T_{00}$,

$$\left(\frac{\partial\psi_0}{\partial t} + \{p_m, \psi_0\}\right)(t,x,\xi) \geq 0. \tag{3.4.5}$$

Now we define $\psi(t, x, \xi)$ by the solution to

$$\begin{aligned} \left(\frac{\partial\psi}{\partial t} + \{p_m, \psi\}\right)(t,x,\xi) =& \rho(t)\left(\frac{\partial\psi_0}{\partial t} + \{p_m, \psi_0\}\right)(t,x,\xi), \quad 0 \leq t \leq T_{00}+1, \\ \psi(T_{00}+1, x, \xi) =& \psi_0(T_{00}+1, x, \xi), \end{aligned} \tag{3.4.6}$$

where $\rho \in C^\infty(\mathbb{R}, [0, 1])$ such that $\rho(t) = 1$ for $t \geq T_{00} + 1$, $\rho(t) = 0$ for $t \leq T_{00}$. Then we can extend ψ smoothly to $t \geq T_{00} + 1$ by $\psi(t, x, \xi) = \psi_0(t, x, \xi)$ for $t \geq T_{00} + 1$. For $(x, \xi) \in \mathbb{R}^{2n}$, by using $\rho(t) \leq 1$, we obtain

$$\frac{d\psi}{dt}(t, y(t,x,\xi), \eta(t,x,\xi)) \leq \frac{d\psi_0}{dt}(t, y(t,x,\xi), \eta(t,x,\xi)).$$

Let $0 \leq s \leq T_{00} + 1$. Integrating this inequality over $[s, T_{00} + 1]$ with $(x, \xi) = (x_0, \xi_0)$ and using $\psi(t, x, \xi) = \psi_0(t, x, \xi)$ with $(t, x, \xi) = (T_{00}+1, y(T_{00}+1), \eta(T_{00}+1))$, we have

$$\psi(s, y(s), \eta(s)) \geq \psi_0(s, y(s), \eta(s)) \geq 0.$$

Substituting this inequality with $s = 0$, we have $\psi(0, x_0, \xi_0) \geq \psi_0(0, x_0, \xi_0) = 1$. This implies that ψ satisfies (i). We set $T_0 = T_{00} + 1$. Now (ii) follows from (3.4.1) and the relation $\psi(t, x, \xi) = \psi_0(t, x, \xi)$ for $t \geq T_0$. (iv) follows from (3.4.5) and (3.4.6). Furthermore, (iii) follows from (3.4.1), (3.4.6), the relation $\psi(t, x, \xi) = \psi_0(t, x, \xi)$ for $t \geq T_0$ and the definition of ψ_0. □

We set

$$\psi_{h,t}(x,\xi) = \psi(t/h, x, h^{\frac{1}{m-1}}\xi), \quad \varphi_{0,h,t}(x,\xi) = \psi_{h,t}\#\psi_{h,t}(x,\xi), \tag{3.4.7}$$

and $F_0(h, t) = \operatorname{Op}(\varphi_0(h, t, \cdot, \cdot)) = |\operatorname{Op}(\psi_{h,t})|^2$, where # denotes the composition of the Weyl quantization ([79, (4.3.6)] with $h = 1$) and $|A|^2 = A^*A$ for an operator A.

Lemma 3.4.3. *(i) $F_0(0) = |\operatorname{Op}(\psi_{h,0})|^2$ with $\psi_{h,0}(x_0, \xi_0) \geq 1$.*

(ii) We have

$$\operatorname{supp}\varphi_{0,h,t}\subset B\big(2h^{-1}t\delta_1,2h^{-\frac{1}{m-1}}\delta_2,h^{-1}tv_+,h^{-\frac{1}{m-1}}\eta_+\big)$$

modulo $S(h^\infty,g_h)$ if $t/h\geq T_1$.

(iii) For any $\alpha,\beta\in\mathbb{N}^n_{\geq 0}$, there exists $C_{\alpha\beta}>0$ such that

$$|\partial_x^\alpha\partial_\xi^\beta\varphi_{0,h,t}(x,\xi)|\leq C_{\alpha\beta}h^{\frac{|\beta|}{m-1}}\langle x\rangle^{-|\alpha|},$$
$$|\partial_x^\alpha\partial_\xi^\beta\partial_t\varphi_{0,h,t}(x,\xi)|\leq C_{\alpha\beta}h^{\frac{|\beta|}{m-1}-1}\langle x\rangle^{-|\alpha|-1}.$$

(iv) There exists $r_0(t,x,\xi)\in C^\infty(\mathbb{R}_{\geq 0}\times\mathbb{R}^{2n})$ such that

$$\frac{\partial}{\partial t}F_0(h,t)+i[P,F_0(h,t)]\geq -\mathrm{Op}(r_{0,h,t}),$$

and $\operatorname{supp} r_{0,h,t}\subset\operatorname{supp}\varphi_{0,h,t}$ *modulo $S\big(h^\infty\langle x\rangle^{-\infty},g_h\big)$. Moreover, for any $\alpha,\beta\in\mathbb{N}^n_{\geq 0}$, there exists $C_{\alpha\beta}>0$ such that*

$$|\partial_x^\alpha\partial_\xi^\beta r_{0,h,t}(x,\xi)|\leq C_{\alpha\beta}h^{\frac{|\beta|-(m-2)}{m-1}}\langle x\rangle^{-|\alpha|-1-\mu}.$$

Proof. Propeties (i)–(iii) follow from (3.4.1) and Lemma 3.4.2. We prove (iv). Since $|x|\sim t/h$ holds on $\operatorname{supp}\psi_{h,t}$, we learn $\partial_t\varphi_{0,h,t}(\cdot,\cdot)\in S(h^{-1}\langle x\rangle^{-1},g_h)$. Moreover, we have $[P,F_0(h,t)]\in\mathrm{Op}S(\langle x\rangle^{-1}\langle\xi\rangle^{m-1},g_h)$. By its support property, $[P,F_0(h,t)]\in\mathrm{Op}S(h^{-1}\langle x\rangle^{-1},g_h)$ follows. We obtain

$$\frac{\partial}{\partial t}|\psi_{h,t}(h,t,x,\xi)|^2+\{p_m,|\psi_{h,t}(\cdot,\cdot)|^2\}(x,\xi)\geq 0$$

by Lemma 3.4.2 *(iv)*. We note $p=p_m+V$ with $V\in S^{m-1,-\mu}$ and

$$[V,F_0(h,t)]\in\mathrm{Op}S\big(h^{-\frac{m-2}{m-1}}\langle x\rangle^{-1-\mu},g_h\big).$$

By the sharp Gårding inequality, there exists $r_{0,h,t}\in S\big(h^{\frac{-(m-2)}{m-1}}\langle x\rangle^{-1-\mu},g_h\big)$ such that (iv) holds. □

Proof of Theorem 3.4.1. We choose $\lambda_0,\lambda_1,\lambda_2,...\in[1,2)$ such that

$$1=\lambda_0<\lambda_1<\lambda_2<\cdots<2,$$

and take $\psi_{k,h,t}(x,\xi)$ as $\psi_{h,t}(x,\xi)$ and T_k as T_0 with δ_j replaced by $\lambda_k\delta_j$ in Lemma 3.4.2 and (3.4.7). By the choice of Ψ, we note

$$\psi_{k+1,h,t}(x,\xi)\geq L_k \tag{3.4.8}$$

on $\operatorname{supp}\psi_{k,h,t}(\cdot,\cdot)$ for some $L_k>0$. For $k\geq 1$, set

$$\varphi_{k,h,t}(x,\xi)=h^{\frac{k-m+1}{m-1}}tC_k\psi_{k,h,t}\#\psi_{k,h,t}\in S(h^{\frac{k-m+1}{m-1}}t,g_h)$$

where $C_k > 0$ is determined later. By Lemma 3.4.3 (iv), we can write $r_{0,h,t} = r_{01,h,t} + r_{02,h,t}$, where

$$r_{01,h,t} \in S\big(h^{\frac{-(m-2)}{m-1}} \langle x\rangle^{-1-\mu}, g_h\big) \tag{3.4.9}$$

satisfies supp $r_{01,h,t}(t,\cdot,\cdot) \subset$ supp $\varphi_0(t,\cdot,\cdot)$ and $r_{02,h,t} \in S(h^\infty \langle x\rangle^{-\infty}, g_h)$. By (3.4.8), we can find $C_1 > 0$ such that

$$r_{01,h,t}(x,\xi) \leq C_1 h^{\frac{-m+2}{m-1}} |\psi_{1,h,t}(x,\xi)|^2.$$

This inequality with Lemma 3.4.2 (iv) implies

$$\begin{aligned} &C_1 h^{\frac{-m+2}{m-1}} \Big(\frac{\partial}{\partial t}(t|\psi_{1,h,t}|^2) + t\{p_m, |\psi_{1,h,t}|^2\}\Big)(x,\xi) \qquad (3.4.10)\\ &= C_1 h^{\frac{-m+2}{m-1}} t \Big(\frac{\partial}{\partial t}|\psi_{1,h,t}|^2 + \{p_m, |\psi_{1,h,t}|^2\}\Big)(x,\xi) + C_1 h^{\frac{-m+2}{m-1}} |\psi_{1,h,t}(x,\xi)|^2 \\ &\geq r_{01,h,t}(x,\xi). \end{aligned}$$

Taking $M_k = \max(T_k, ||v_+| - 2\lambda_k\delta_1|) > 0$, we have

$$t \leq M_k h \langle x\rangle, \text{ for } (t,x,\xi) \in \text{supp } \psi_{k,h,t} \tag{3.4.11}$$

by Lemma 3.4.3 (ii). Lemma 3.4.3 (iii) with (3.4.11) implies

$$h^{\frac{-m+2}{m-1}} t \Big(\frac{\partial |\psi_{1,h,t}|^2}{\partial t} + \{p_m, |\psi_{1,h,t}|^2\}\Big) \in S\big(h^{\frac{-m+2}{m-1}}, g_h)\big). \tag{3.4.12}$$

By (3.4.9), (3.4.10) and (3.4.12), it follows that the both sides in (3.4.10) belong to $S(h^{\frac{-m+2}{m-1}}, dx^2/\langle x\rangle^2 + h^{2/(m-1)} d\xi^2)$. The sharp Gårding inequality implies that there exists

$$r_{1,h,t} \in S(h^{\frac{-m+3}{m-1}} \langle x\rangle^{-1}, g_h)$$

which is supported in supp $\varphi_{1,h,t}$ modulo $S(h^\infty\langle x\rangle^{-\infty}, g_h)$ such that

$$\frac{\partial}{\partial t}\mathrm{Op}(\varphi_{1,h,t}) + i[P, \mathrm{Op}(\varphi_{1,h,t})] \geq \mathrm{Op}(r_{0,h,t}) - \mathrm{Op}(r_{1,h,t}).$$

We set $F_1(h,t) = F_0(h,t) + \mathrm{Op}(\varphi_{1,h,t})$, then we have

$$\frac{\partial}{\partial t}F_1(h,t) + i[P, F_1(h,t)] \geq -\mathrm{Op}(r_{1,t,h}).$$

Iterating the above argument, we can construct $C_k > 0$, $F_k(t)$ and

$$r_{k,h,t} \in S\big(h^{\frac{k-m+2}{m-1}} \langle x\rangle^{-1}, g_h\big)$$

such that $\mathrm{supp}\, r_{k,h,t} \subset \mathrm{supp}\, \varphi_{k,h,t}(\cdot,\cdot)$ modulo $S(h^\infty\langle x\rangle^{-\infty}, g_h)$ and

$$\frac{\partial}{\partial t}F_k(h,t) + i[P, F_k(h,t)] \geq -\mathrm{Op}(r_k(h,t,\cdot,\cdot)),$$
$$F_{k+1}(h,t) = F_k(h,t) + \mathrm{Op}(\varphi_{k,h,t}),$$
$$r_{k,h,t}(x,\xi) \leq C_{k+1}h^{\frac{k-m+2}{m-1}}\psi_{k+1,h,t}(x,\xi) \quad \text{modulo } S(h^\infty\langle x\rangle^{-\infty}, g_h).$$

By the Borel's Theorem (see [79] Theorem 4.15), we can define

$$\varphi_{h,t}(x,\xi) \sim \sum_{n=0}^{\infty}\varphi_{j,h,t}(x,\xi)$$

and $F(h,t) = \mathrm{Op}(\varphi_{h,t})$. Then, $F(h,t)$ satisfies the properties in Theorem 3.4.1. This completes the proof of Theorem 3.4.1. □

3.5 Compactly supported perturbation

The proof is considerably simpler if the perturbation is compactly supported, since we do not need the argument of Subsection 3.2.2. Here we discuss the simpler argument for this case. We assume that there exists $R > 0$ such that $\mathrm{supp}\, q \subset B_R(0) \times \mathbb{R}^n$, where $B_R(0) = \{x \in \mathbb{R}^n \mid |x| < R\}$. We note still the local regularity argument (Subsection 3.2.1 and Appendices 5.5, 3.4). Let $\psi \in C^\infty(\mathbb{R}^n)$ be a real-valued function such that $\psi = 1$ on $\mathbb{R}^n \setminus B_{R+1}(0)$ and $\psi = 0$ on $B_R(0)$.

Proposition 3.5.1. *Let $k \geq 0$ and $u \in L^2(\mathbb{R}^n) \cap H^{k+m-1}_{\mathrm{loc}}(\mathbb{R}^n)$ be a distributional solution to $(P - i)u = 0$. Then we have $\psi u \in H^k$. In particular, $u \in H^k$ follows.*

Proof. Set $N = I - \Delta$ and $N_\varepsilon = (I-\Delta)(I-\varepsilon\Delta)^{-1}$ and define $L = p_0(D)$ where Δ denotes the standard Laplacian on $\mathbb{R}^n$. By virtue of the support property of ψ, we compute

$$L(\psi u) = P(\psi u) = \psi P u + [P,\psi]u = i\psi u + Ku,$$

where $K := [P,\psi]$ is compactly supported coefficients differential operator with order $m-1$. We note $Ku \in H^1$ since $u \in H^m_{\mathrm{loc}}(\mathbb{R}^n)$. Hence, we have

$$\begin{aligned} 2i\mathrm{Im}\,(N_\varepsilon^{2k}(\psi u), L(\psi u))_{L^2} =& 2i\mathrm{Im}\,(N_\varepsilon^{2k}(\psi u), i\psi u + Ku)_{L^2} \\ =& 2i\|N_\varepsilon^k(\psi u)\|_{L^2}^2 + 2i\mathrm{Im}\,(N_\varepsilon^{2k}(\psi u), Ku)_{L^2}. \end{aligned}$$

On the other hand, by the Plancherel theorem, we have

$$2i\mathrm{Im}\,(N_\varepsilon^{2k}(\psi u), L(\psi u))_{L^2} = (N_\varepsilon^{2k}(\psi u), L(\psi u))_{L^2} - (L(\psi u), N_\varepsilon^{2k}(\psi u))_{L^2} = 0.$$

Thus, we have

$$\|N_\varepsilon^k(\psi u)\|_{L^2}^2 \leq |\mathrm{Im}\,(N_\varepsilon^{2k}(\psi u), Ku)| \leq \|N_\varepsilon^k(\psi u)\|_{L^2}\|N_\varepsilon^k Ku\|_{L^2}$$

Consequently, take $\varepsilon \to 0$ and we obtain $\|N^k(\psi u)\|_{L^2} \leq \|N^k K u\|_{L^2} < \infty$, by using the monotone convergence theorem and $Ku \in H^k$. This implies $\psi u \in H^k$. □

Proof of Proposition 3.2.1. Suppose that $u \in L^2(\mathbb{R}^n)$ satisfies $(P-i)u = 0$. By Proposition 3.2.2, we have $u \in C^\infty(\mathbb{R}^n) \subset H^{3(m-1)/2}_{\mathrm{loc}}(\mathbb{R}^n)$. By Proposition 3.5.1, we conclude $u \in H^{(m-1)/2} \subset H^{(m-1)/2,-1/2}$. □

3.6 Dynamical property, completeness of the Hamilton flow

In this short section, we show that the null non-trapping condition (Assumption B) implies the completeness of the flow. We mention that we do not this fact in the other part of this paper. We note that every non-trapped integral curve is complete by the estimate of classical trajectory (essentially due to the classical Mourre estimate), see the proof of Lemma 3.3.2.

Proposition 3.6.1. *Let $p_m(x,\xi)$ be a homogeneous of degree $m \geq 1$. Under the null non-trapping condition, it follows that every trapped integral curve of H_p on $\{p_m \neq 0\}$ is complete.*

Proof. Suppose that there exists a maximal trapped integral curve $(z(t), \zeta(t))$, $t \in [0,T)$ of H_p such that $p_m(z(t),\zeta(t)) \neq 0$. We note

$$\sup_{t\in[0,T)} |z(t)| < \infty, \quad \lim_{t\to T,\ t<T} |\zeta(t)| = \infty$$

since this trajectory is trapped. Since p_m is homogeneous of degree m, we have $\zeta(t,x,\xi) \neq 0$. Since $z(t)$ is trapped, it follows that a set

$$\{(z(t), \frac{\zeta(t)}{|\zeta(t)|}) \,|\, 0 \leq t < T\} \subset \mathbb{R}^{2n} \tag{3.6.1}$$

is compact. Hence there exist a sequence $\{t_j\}_{j=1}^\infty$ and $(x,\xi) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$ such that

$$t_j \to T,\ (z(t_j), \frac{\zeta(t_j)}{|\zeta(t_j)|}) \to (x,\xi)$$

as $j \to \infty$.

Next, we show

$$p_m(x,\xi) \neq 0.$$

To see this, we use a contradiction argument. Suppose $p_m(z,\zeta) = 0$. By the null non-trapping assumption, we have $|z(t,x,\xi)| \to \infty$ as $|t| \to \infty$. Since p_m is homogeneous of degree m, we have

$$\begin{aligned} z(t, z(t_j), \frac{\zeta(t_j)}{|\zeta(t_j)|}) =& z(\frac{t}{|\zeta(t_j)|^{m-1}}, z(t_j), \zeta(t_j)) \\ =& z(t_j + \frac{t}{|\zeta(t_j)|^{m-1}}) \end{aligned} \tag{3.6.2}$$

for $t_j + t/|\zeta(t_j)|^{m-1} < T$. Since $|\zeta(t_j)| \to \infty$, the right hand side of (3.6.2) (and hence also the left hand side) is well-defined for large t. Now we take $T_1 > 0$ such that

$$\inf_{T_1 \le t < \infty} |z(t, x, \xi)| > 2 \sup_{0 \le t < T} |z(t)| \tag{3.6.3}$$

and take j large enough such that $T_1 < |\zeta(t_j)|^{m-1} T$. Then we can substitute $t = T_1$ into (3.6.2) and obtain

$$|z(T_1, z(t_j), \frac{\zeta(t_j)}{|\zeta(t_j)|})| = |z(t_j + \frac{T_1}{|\zeta(t_j)|^{m-1}})| < \frac{1}{2} \inf_{T_1 \le t < \infty} |z(t, x, \xi)|$$

by (3.6.3). Taking $j \to \infty$, we have a contradiction. Thus $p_m(x, \xi) \neq 0$ follows.

Since p_m is homogeneous of degree m, we obtain

$$0 \neq p_m(x, \xi) = \lim_{j \to \infty} p_m(z(t_j), \frac{\zeta(t_j)}{|\zeta(t_j)|}) = \lim_{j \to \infty} \frac{p(z(t_j), \zeta(t_j))}{|\zeta(t_j)|^m} = 0,$$

where recall $p(z(t_j), \zeta(t_j))$ is conserved along the flow and $|\zeta(t_j)| \to \infty$ as $j \to \infty$. This is a contradiction. □

3.7 Limiting absorption principle for the wave operator

The results of this section is expected to be true for the real principal type operators considered in the above sections. However, for simplicity, we restrict our attention to the case of the wave operator on an asymptotically Minkowski space.

Let g_0 be a Minkowski metric on $\mathbb{R}^n$: $g_0 = dx_1^2 - dx_2^2 - ... - dx_n^2$ and $g_0^{-1} = \partial_{x_1}^2 - \partial_{x_2}^2 - ... - \partial_{x_n}^2 = (g_0^{ij})_{i,j=1}^n$ be its dual metric. A Lorentzian metric g on $\mathbb{R}^n$ is called asymptotically Minkowski if the inverse matrix $g^{-1}(x) = (g^{jk}(x))_{j,k=1}^n$ of $g(x)$ satisfies

$$|\partial_x^\alpha (g^{jk}(x) - g_0^{jk})| \le C_\alpha \langle x \rangle^{-\mu - |\alpha|},\ \mu > 0.$$

We set

$$P = \sum_{j,k=1}^n \partial_{x_j}(g^{jk}(x)\partial_{x_k}),\ p(x, \xi) = \sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k,\ p_0(\xi) = \sum_{j,k=1}^n g_0^{jk}\xi_j\xi_k.$$

Moreover, we write

$$\tilde{\xi} = \frac{1}{2}\partial_\xi p_0(\xi).$$

Let $(z(t, x, \xi), \zeta(t, x, \xi))$ denote the solution to the Hamilton equation:

$$\frac{d}{dt} z(t, x, \xi) = \frac{\partial p_m}{\partial \xi}(z(t, x, \xi), \zeta(t, x, \xi)), \quad \frac{d}{dt}\zeta(t) = -\frac{\partial p_m}{\partial x}(z(t, x, \xi), \zeta(t, x, \xi)), \quad t \in \mathbb{R}$$

with $z(0,x,\xi)=x$, $\zeta(0,x,\xi)=\xi$. Moreover, we introduce the conjugate operator A:

$$a(x,\xi)=\frac{x\cdot\tilde{\xi}}{1+|\xi|^2}\in S^{-1,1},\ A=\mathrm{Op}(a).$$

In this section, we always assume the null non-trapping condition (Assumption B). By using Nelson's commutator theorem and the result of the above section, it follows that $A|_{C_c^\infty(\mathbb{R}^n)}$ and $P|_{C_c^\infty(\mathbb{R}^n)}$ are essentially self-adjoint. We denote their unique self-adjoint extensions by the same symbols A and P respectively. We write the domain of the self-adjoint extension P by $D(P)$, then we have

$$D(P)=\{u\in L^2(\mathbb{R}^n)\mid Pu\in L^2(\mathbb{R}^n)\} \tag{3.7.1}$$

by essential self-adjointness of $P|_{C_c^\infty(\mathbb{R}^n)}$. We note that the essential self-adjointness of $P|_{C_c^\infty(\mathbb{R}^n)}$ implies

$$C_c^\infty(\mathbb{R}^n) \text{ is dense in } D(P) \text{ equipped with the graph norm of } P. \tag{3.7.2}$$

In this section, we prove the limiting absorption principle for P away from the zero energy. Our main theorem of this section is the following.

Theorem 3.7.1. *Assume the null non-trapping condition. Let $s>1/2$ and $I\Subset\mathbb{R}\setminus\{0\}$ be an open interval. Then it follows that $\#I\cap\sigma_{pp}(P)$ is finite and that for $I'\Subset I\setminus\sigma_{pp}(P)$, we have*

$$\sup_{z\in I'_\pm}\|\langle x\rangle^{-s}(P-z)^{-1}\langle x\rangle^{-s}\|_{B(L^2)}<\infty,$$

where

$$I'_\pm=\{z\in\mathbb{C}\mid \mathrm{Re}\, z\in I,\ \pm\mathrm{Im}\, z>0\}.$$

In particular, P has absolutely continuous spectrum on I'. Moreover, $z\in I'_\pm\mapsto\langle x\rangle^{-s}(P-z)^{-1}\langle x\rangle^{-s}$ is Hölder continuous in $B(L^2(\mathbb{R}^n))$ and the limits

$$\langle x\rangle^{-s}(P-\lambda\mp i0)^{-1}\langle x\rangle^{-s}:=\lim_{\varepsilon\to0,\,\varepsilon>0}\langle x\rangle^{-s}(P-\lambda\mp i\varepsilon)^{-1}\langle x\rangle^{-s}$$

exist in $B(L^2(\mathbb{R}^n))$.

Remark 3.7.2. In this theorem, we can replace P by $P+V$, where V is a real-valued long-range potential since

- our conjugate operator A belongs to $\mathrm{Op}S^{-1,1}$ and hence $[V,A]$ is a compact operator,
- $D(P)=D(P+V)$ due to $V\in B(L^2(\mathbb{R}^n))$,

- a difference $\varphi(P+V)-\varphi(P)$ is a compact operator for $\varphi \in C_c^\infty(\mathbb{R})$ by Corollary 3.7.9 below.

Remark 3.7.3. The above theorem for the interval I is proved by Vasy [75] under the non-trapping assumption on energy level I. We remove this additional assumption by using the local compactness of P (Proposition 3.7.9).

Remark 3.7.4. When $p = p_0$, then the limiting absorption principle holds even near the 0-energy with an additional weight, see Proposition 5.2.5 and Proposition 5.2.8.

To prove this theorem, we need some results of a pseudodifferential calculus of spectral cut-off functions for P (Proposition 3.7.7) and local compactness for P (Corollary 3.7.9). We prove these results in the subsections later and deduce Theorem 3.7.1 here.

Proof. We may assume $0 < \mu \leq 1$. Moreover, using a standard argument explained in subsection 3.7.1 (however, it is non-trivial in our case), we only have to prove the above statement replacing $\langle x \rangle$ by $\langle A \rangle$. We note $P \in C^2(A)$ which will be proved also in the subsection 3.7.1. Since $p - p_0 \in S^{2,-\mu}$ and $a \in S^{-1,1}$, we have

$$\{p, a\} = \frac{4|\xi|^2}{1+|\xi|^2} + S^{0,-\mu}.$$

This implies

$$[P, iA] = 4(I-\Delta)^{-1/2}(-\Delta)(I-\Delta)^{-1/2} + R,$$

where $R \in \mathrm{Op}S^{0,-\mu}$ (we note $0 < \mu \leq 1$). Let $J \Subset I$ be an open interval. Let $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\}; [0,1])$ which is supported in I and is equal to 1 on J. Moreover, take $\psi_1 \in C_c^\infty(\mathbb{R}^n; [0,1])$ which support is close to 0 such that

$$\mathrm{supp}\ \varphi \circ p \cap \mathrm{supp}\ \psi_1 = \emptyset.$$

We observe

$$(I-\Delta)^{-1/2}(-\Delta)(I-\Delta)^{-1/2} \geq c + \psi(D),$$

where

$$c = \inf_{\xi \in \mathrm{supp}\ (1-\psi_1)} \frac{|\xi|^2}{1+|\xi|^2} > 0,\ \ \psi(\xi) = \frac{|\xi|^2 \psi_1(\xi)}{1+|\xi|^2} - c\psi_1(\xi) \in C_c^\infty(\mathbb{R}^n).$$

Thus we have

$$\varphi(P)[P, iA]\varphi(P) \geq 4c\varphi(P)^2 + 4\varphi(P)\psi(D)\varphi(P) + \varphi(P)R\varphi(P) \tag{3.7.3}$$

Lemma 3.7.7 with a support propety $\mathrm{supp}\ \varphi \circ p \cap \mathrm{supp}\ \psi = \emptyset$ implies that the second term of the right hand side is a compact operator on $L^2(\mathbb{R}^n)$. Moreover, it follows that the third term $\varphi(P)R\varphi(P)$ is also compact by using the Helffer-Sjöstrand formula and the local compactness for P (Corollary 3.7.9). From the Mourre theory [53], we obtain the desired results.

□

3.7.1 A-regularity of P, $\langle A\rangle$-weight to $\langle x\rangle$-weight

In this subsection, we prove $P \in C^2(A)$ which is needed to apply the Mourre theory. First, we recall the definition of $C^1(A)$ and $C^2(A)$.

Definition 1. Let A be a self-adjoint operator and B be a bounded operator on a Hilbert space $\mathcal{H}$. We call $B \in C^1(A)$ if a quadratic form $[A, B]$ on $D(A) \times D(A)$ can be extended to a bounded operator on $\mathcal{H}$. We call $B \in C^2(A)$ if $B \in C^1(A)$ and $[A, B] \in C^1(A)$.

Let P be a self-adjoint operator. For $k = 1, 2$, we call $P \in C^k(A)$ if $(P - i)^{-1} \in C^2(A)$.

Proposition 3.7.5. *We have $P \in C^2(A)$ and*

$$[A, (P - i)^{-1}] = (P - i)^{-1}[P, A](P - i)^{-1} \tag{3.7.4}$$

as a bounded operator on $L^2(\mathbb{R}^n)$.

Proof. First, we show $P \in C^1(A)$. Setting $A_\varepsilon = \langle \varepsilon x\rangle^{-1/2} A \langle \varepsilon x\rangle^{-1/2} \in \mathrm{Op}S^{0,0}$ for $0 < \varepsilon \leq 1$, we have

$$[A_\varepsilon, (P - i)^{-1}] = (P - i)^{-1}[P, A_\varepsilon](P - i)^{-1}. \tag{3.7.5}$$

as a bounded operator on $L^2(\mathbb{R}^n)$. For $u, w \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\lim_{\varepsilon\to 0}(u, [A_\varepsilon, (P - i)^{-1}]w)_{L^2} = (Au, (P - i)^{-1}w)_{L^2} - ((P + i)^{-1}u, Aw)_{L^2}, \tag{3.7.6}$$

for $u, w \in \mathcal{S}(\mathbb{R}^n)$. On the other hands, since $[P, A_\varepsilon] \to [P, A]$ in the strong operator topology in $B(L^2(\mathbb{R}^n))$ which follows from Lemma 2.3.2, we have

$$\lim_{\varepsilon\to 0}(P - i)^{-1}[P, A_\varepsilon](P - i)^{-1} = (P - i)^{-1}[P, A](P - i)^{-1} \tag{3.7.7}$$

in the strong operator topology of $B(L^2(\mathbb{R}^n))$. From (3.7.5), (3.7.6) and (3.7.7), we obtain

$$(Au, (P - i)^{-1}w)_{L^2} - ((P + i)^{-1}u, Aw)_{L^2} = (u, (P - i)^{-1}[P, A](P - i)^{-1}w)_{L^2} \tag{3.7.8}$$

for $u, w \in \mathcal{S}(\mathbb{R}^n)$. Since $A|_{\mathcal{S}(\mathbb{R}^n)}$ is essentially self-adjoint, the equation (3.7.8) holds for $u, w \in D(A)$. This implies $P \in C^1(A)$ and (3.7.4).

Next, we show $P \in C^2(A)$. It suffices to show $[A, [A, (P - i)^{-1}]]$ can be extended to a bounded operator on $L^2(\mathbb{R}^n)$. Since $(P - i)^{-1}$ and $[P, A]$ are bounded in $L^2(\mathbb{R}^n)$, we observe

$$\begin{aligned}[A_\varepsilon, (P - i)^{-1}[P, A](P - i)^{-1}] =&[A_\varepsilon, (P - i)^{-1}][P, A](P - i)^{-1}] \\ &+ (P - i)^{-1}[A_\varepsilon, [P, A]](P - i)^{-1} \\ &+ (P - i)^{-1}[P, A][A_\varepsilon, (P - i)^{-1}].\end{aligned} \tag{3.7.9}$$

Moreover, from Lemma 2.3.2, we have

$$[A_\varepsilon, [P, A]] \to [A, [P, A]] \quad \text{in the strong operator topology.} \tag{3.7.10}$$

It follows from the equation (3.7.5), (3.7.7) and (3.7.10) that the right hand side of (3.7.9) converges to a bounded operator in the strong operator topology on $B(L^2(\mathbb{R}^n))$. On the other hand, we have

$$\begin{aligned}(u, [A_\varepsilon, (P-i)^{-1}[P, A](P-i)^{-1}]w)_{L^2} \to & (Au, (P-i)^{-1}[P, A](P-i)^{-1}w)_{L^2} \\ & - ((P-i)^{-1}[P, A](P-i)^{-1}u, Aw)_{L^2}\end{aligned} \tag{3.7.11}$$

for $u, w \in \mathcal{S}(\mathbb{R}^n)$. This implies $P \in C^2(A)$. □

The next Corollary is standard for experts of scattering theory, however, we give these proofs for the completeness of this thesis. We remark that the key point in the proof below is the equation (3.7.13) which follows from Proposition 3.7.5. We also note that the resolvent equation implies

$$(P-z)^{-1} = (P-i)^{-1} + (z-i)(P-i)^{-1} + (z-i)^2(P-i)^{-1}(P-z)^{-1}(P-i)^{-1}.$$

From this, in order to prove Theorem 3.7.1 from the same statement where $\langle x \rangle$ replaced $\langle A \rangle$, it suffices to prove the following lemma.

Corollary 3.7.6. *For* $0 \leq s \leq 1$, *we have*

$$\langle A \rangle^s (P-i)^{-1} \langle x \rangle^{-s} \in B(L^2(\mathbb{R}^n)).$$

Proof. The case of $s = 0$ follows from $(P-i)^{-1} \in B(L^2(\mathbb{R}^n))$. Next, we consider the case of $s = 1$. By the spectral theorem for A, we observe

$$\|\langle A \rangle (P-i)^{-1} \langle x \rangle^{-1}\|_{B(L^2)} \leq \|(P-i)^{-1} \langle x \rangle^{-1}\|_{B(L^2)} + \|A(P-i)^{-1} \langle x \rangle^{-1}\|_{B(L^2)}.$$

Thus it suffices to prove

$$\|A(P-i)^{-1} \langle x \rangle^{-1}\|_{B(L^2)} < \infty. \tag{3.7.12}$$

From Proposition 3.7.5, we have

$$A(P-i)^{-1}\langle x \rangle^{-1} = (P-i)^{-1} A \langle x \rangle^{-1} + (P-i)^{-1}[P, A](P-i)^{-1}\langle x \rangle^{-1}. \tag{3.7.13}$$

Since $A\langle x \rangle^{-1} \in \mathrm{Op}S^{-1,0}$ and $[P, A] \in \mathrm{Op}S^{0,0}$, we obtain (3.7.12).

Next, we prove the lemma in the cases of $0 < s < 1$. To do this, we use the standard interpolation argument. For $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$, we consider the function

$$f(z) = (\varphi, \langle A \rangle^z (P-i)^{-1} \langle x \rangle^{-z} \psi)_{L^2}.$$

We note that f is a holomorphic function inside the region

$$S = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}\, z \leq 1\}.$$

Moreover, it follows that f is continuous and bounded on S. In addition, from the above argument, there exists $C > 0$ such that

$$|f(z)| \leq C\|\varphi\|_{L^2}\|\psi\|_{L^2} \text{ for } \mathrm{Re}\, z = 0, 1. \tag{3.7.14}$$

By the Hadamard three line theorem [61, Appendix to IX.4 Lemma after Proposition 1], we obtain (3.7.14) for $z \in S$. By the density argument, we conclude $\langle A\rangle^s(P-i)^{-1}\langle x\rangle^{-s} \in B(L^2(\mathbb{R}^n))$.

□

3.7.2 Pseudodifferential calculus of spectral cut-off functions

In this subsection, we prove that $\varphi(D)\varphi(P)$ is a pseudodifferential operator plus a negligible term although $\varphi(P)$ itself cannot be written by such a form (even in the constant coefficient case $g = g_0$). The following proposition is stimulated by the construction in [44] for the Stark Hamiltonian (although the argument itself is standard). It is expected that $\mathrm{Op}(\psi \cdot \varphi \circ p)$ is actually a pseudodifferential operator of class $\mathrm{Op}S^{-\infty,0}$ (by using Beal's theorem), however, we only show a weaker result which is needed for the Mourre estimate.

Proposition 3.7.7. *Let $\varphi, \psi \in C_c^\infty(\mathbb{R})$. Then we can write*

$$\psi(D)\varphi(P) = \mathrm{Op}(\psi \cdot \varphi \circ p) + K,$$

where K is a compact operator on $L^2(\mathbb{R}^n)$.

Proof. First, we construct a parametrix of $\psi(D)(P-z)^{-1}$ for $\mathrm{Im}\, z \neq 0$. We note that for any integer $N \geq 0$, we have

$$|\partial_x^\alpha \partial_\xi^\beta \left(\psi(\xi)(p(x,\xi)-z)^{-1}\right)| \leq C_{N\alpha\beta}|\mathrm{Im}\, z|^{-|\alpha|-|\beta|-1}\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-N}$$

with a constant $C_{N\alpha\beta} > 0$ independent of z and $(x,\xi) \in \mathbb{R}^{2n}$. This implies

$$\psi(\xi) = \frac{\psi}{p-z}\#(p-z)(x,\xi) + r_z(x,\xi),$$

where $r_0 \in S^{-\infty,-1}$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta r_z(x,\xi)| \leq C_{N\alpha\beta}|\mathrm{Im}\, z|^{-N_{\alpha\beta}}\langle x\rangle^{-1-|\alpha|}\langle\xi\rangle^{-N} \tag{3.7.15}$$

with a constant $C_{N\alpha\beta} > 0$ and $N_{\alpha\beta} \geq 0$ independent of z and $(x,\xi) \in \mathbb{R}^{2n}$. Weyl quantizing this equation and multiplying $(P-z)^{-1}$ from left, we have

$$\psi(D)(P-z)^{-1} = \mathrm{Op}(\frac{\psi}{p-z}) + \mathrm{Op}(r_z)(P-z)^{-1}$$

as a bounded operator on $L^2(\mathbb{R}^n)$.

Now we denote the almost analytic extension [79, Theorem 3.6] of φ by $\tilde{\varphi}$. By the Helffer-Sjöstrand formula [79, Theorem 14.8], we have

$$\psi(D)\varphi(P) = \mathrm{Op}(\psi\cdot\varphi\circ p) + \frac{1}{\pi i}\int_{\mathbb{C}} \bar{\partial}_z\tilde{\varphi}(z)\mathrm{Op}(r_z)(P-z)^{-1}dz.$$

Lemma 2.2.1 (i), (3.7.15), $\|(P-z)^{-1}\|_{B(L^2)} \leq |\mathrm{Im}\, z|^{-1}$ and $\bar{\partial}_z\tilde{\varphi}(z) = O(|\mathrm{Im}\, z|^{\infty})$ as $\mathrm{Im}\, z \to 0$ imply that the second term of the right hand side is a bounded operator from $L^2(\mathbb{R}^n)$ to $H^{1,1}$. Since the natural injection $H^{1,1} \hookrightarrow L^2(\mathbb{R}^n)$ is compact, we obtain the desired result.

□

3.7.3 Local compactness

In this subsection, we prove the local compactness for P. The main result of this subsection is the following proposition.

Proposition 3.7.8. *Let $\delta > 0$. Then there exists $C > 0$ such that*

$$\|u\|_{H^{\frac{1}{2},-\frac{1+\delta}{2}}} \leq C\|Pu\|_{L^2} + C\|u\|_{L^2}, \tag{3.7.16}$$

for $u \in D(P)$, where we recall that $D(P)$ is as in (3.7.1)*. In particular, we have a continuous inclusion*

$$D(P) \hookrightarrow H^{\frac{1}{2},-\frac{1+\delta}{2}}.$$

where we regard $D(P)$ as a Banach space equipped with the graph norm of P.

Corollary 3.7.9. *Let $V \in C(\mathbb{R}^n)$ satisfying $|V(x)| \to 0$ as $|x| \to 0$. Then it follows that $V(P-i)^{-1}$ is a compact operator on $L^2(\mathbb{R}^n)$.*

Proof of Corollary 3.7.9. Let $V \in C(\mathbb{R}^n)$ satisfying $|V(x)| \to 0$ as $|x| \to 0$. Then there exists $V_k \in C_c^{\infty}(\mathbb{R}^n)$ such that $\|V_k - V\|_{L^{\infty}(\mathbb{R}^n)} \to 0$ as $k \to \infty$. Since the multiplication operator V_k is continuous from $H^{\frac{1}{2},-\frac{1+\delta}{2}}$ to $H^{\frac{1}{2},1}$ and the natural inclusion $H^{\frac{1}{2},1} \hookrightarrow L^2(\mathbb{R}^n)$ is compact, then Proposition 3.7.8 implies that $V_k(P-i)^{-1}$ is also compact in $L^2(\mathbb{R}^n)$. Since a limit of compact operators is also compact, then we conclude the compactness of $V(P-i)^{-1}$. □

In the following, we show Proposition 3.7.8. Now Proposition 3.7.8 follows from existence of the following escape function.

Lemma 3.7.10 (Escape function under null non-trapping condition)**.** *Let $0 < 2\delta < \mu$. There exist $\lambda_0 > 0$, $C_1 > 0$ and $a \in S^{0,0}$ such that*

$$H_p a(x,\xi) \geq C_1\langle x\rangle^{-1-\delta}\langle\xi\rangle - r(x,\xi),$$

where $r \in S^{1,-1}$ satisfies

$$\mathrm{supp}\, r \subset \{(x,\xi) \in \mathbb{R}^{2n} \mid |\xi| \leq 2\} \cup \{(x,\xi) \in \mathbb{R}^{2n} \mid |p(\xi)| \geq \frac{\lambda_0}{2}|\xi|^2\}.$$

Proof of Proposition 3.7.8 assuming Lemma 3.7.10. We may assume $0 < 2\delta < \mu$. By (3.7.2), it suffices to prove (3.7.16) for $u \in \mathcal{S}(\mathbb{R}^n)$. By using the sharp Gårding inequality (Lemma 2.2.1 (v)) and using $A \in \mathrm{Op}S^{0,0}$, for $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\|u\|^2_{H^{\frac{1}{2},-\frac{1+\delta}{2}}} \leq C\|Pu\|^2_{L^2} + C\|u\|^2_{L^2} + C|(u, \mathrm{Op}(r)u)_{L^2}| \tag{3.7.17}$$

with a constant $C > 0$. Now we write

$$r = r_1 + r_2,\ r_1 \in S^{-\infty,-1},\ r_2 \in S^{1,-1},$$
$$\mathrm{supp}\, r_1 \subset \{(x,\xi) \in \mathbb{R}^{2n} \mid |\xi| \leq 4\},\ \mathrm{supp}\, r_2 \subset \{(x,\xi) \in \mathbb{R}^{2n} \mid |\xi| \geq 3,\ |p(\xi)| \geq \frac{\lambda_0}{4}|\xi|^2\}.$$

By the standard elliptic parametrix construction, we have

$$|(u, \mathrm{Op}(r_1)u)_{L^2}| \leq C\|u\|^2_{L^2},\ |(u, \mathrm{Op}(r_2)u)_{L^2}| \leq C\|Pu\|^2_{L^2} + C\|u\|^2_{L^2} \tag{3.7.18}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. Combining (3.7.17) with (3.7.18), we obtain (3.7.16) for $u \in \mathcal{S}(\mathbb{R}^n)$. □

To prove Lemma 3.7.10, we need some preliminary lemmas.

Lemma 3.7.11 (Convexity at infinity 1)**.** *There exists $R_0 > 0$ such that for $(x,\xi) \in T^*\mathbb{R}^n$ with $|x| \geq R_0$, we have*

$$H_p^2|x|^2 \geq C|\xi|^2.$$

Proof. This lemma follows from an easy calculation. □

Lemma 3.7.12 (Convexity at infinity 2)**.** *Let $R \geq R_0$, where R_0 be same as that of Lemma 3.7.11. If $t_0 < t_1$ and $(x,\xi) \in T^*\mathbb{R}^n$ satisfy*

$$|z(t_j, x, \xi)| \leq R.$$

for $j = 1, 2$. Then for $t \in [t_1, t_2]$, we have

$$|z(t, x, \xi)| \leq R.$$

Proof. This lemma immediately follows from Lemma 3.7.11, where we note

$$\frac{dt}{dt^2}|z(t,x,\xi)|^2 = (H_p|x|^2)|_{x=z(t,x,\xi),\,\xi=\zeta(t,x,\xi)}.$$

□

We denote

$$D_R = \{x \in \mathbb{R}^n \mid |x| \leq R\},\ S^*D_R = \{(x,\xi) \in T^*\mathbb{R}^n \mid |x| \leq R,\ |\xi| = 1\}.$$

Lemma 3.7.13 (Stability of non-trapping orbit)**.** *We assume that for $(x, \xi) \in p^{-1}(\{0\}) \setminus \{\xi = 0\}$, we have $|z(t, x, \xi)| \to \infty$ as $|t| \to \infty$. Let $R \geq R_0$, where R_0 be same as that of Lemma 3.7.11. Then there exists $\lambda_0 > 0$ and $T > 1$ such that we have*

$$|z(t, x, \xi)| > R \text{ for } |t| \geq T,\ (x, \xi) \in p^{-1}([-\lambda_0, \lambda_0]) \cap S^* D_R(0).$$

Proof. By the assumption and Lemma 3.7.12, for any $(x, \xi) \in p^{-1}(\{0\}) \cap S^* D_R$ there exist $T(x, \xi) > 0$ and a neighborhood $U(x, \xi) \subset T^*\mathbb{R}^n$ of (x, ξ) such that

$$|z(t, y, \eta)| > R \text{ for } |t| \geq T(x, \xi),\ (y, \eta) \in U(x, \xi). \tag{3.7.19}$$

We prove this for $t \geq 0$. By the non-trapping assumption there exists $T(x, \xi)$ such that

$$|z(T(x, \xi), x, \xi)| > R + 1.$$

Since $\{(y, \eta) \in T^*\mathbb{R}^n \setminus \{\eta = 0\} \mid (|z(T(x, \xi), y, \eta)| > R + 1\}$ is open, there exists a neighborhood $U(x, \xi) \subset T^* D_{R+1}$ of (x, ξ) such that

$$|z(T_0(x, \xi), y, \eta)| > R + 1 \text{ for } (y, \eta) \in U(x, \xi).$$

This implies

$$|z(t, y, \eta)| > R + 1 \text{ for } t \geq T(x, \xi),\ (y, \eta) \in U(x, \xi).$$

This proves (3.7.19).

Since $p^{-1}(\{0\}) \cap S^* D_R$ is compact, there are finite many point $\{(x_j, \xi_j)\}_{j=1}^N \subset p^{-1}(\{0\}) \cap S^* D_R$ such that

$$p^{-1}(\{0\}) \cap S^* D_R \subset \bigcup_{j=1}^N U(x_j, \xi_j) =: U. \tag{3.7.20}$$

We set $T = \max_{1 \leq j \leq N} T(x_j, \xi_j)$. Then we have

$$|z(t, x, \xi)| > R \text{ for } |t| \geq T,\ (x, \xi) \in U.$$

Thus it suffices to prove that there is $\lambda_0 > 0$ such that

$$p^{-1}([-\lambda_0, \lambda_0]) \cap S^* D_R \subset U.$$

To prove this, we suppose that for any $k \in \mathbb{N}$, there exists $\rho_k \in p^{-1}([-1/k, 1/k]) \cap S^* D_R$ such that $\rho_k \in U^c$. Since $p^{-1}([-1, 1]) \cap S^* D_R$ is compact, there exist a subsequence ρ_{k_l} and $\rho \in p^{-1}([-1, 1]) \cap S^* D_R$ such that $\rho_{k_l} \to \rho$. However, this concludes $\rho \in p^{-1}(\{0\}) \cap S^* D_R \cap U^c$ since U^c is closed. This contradicts to (3.7.20). □

Lemma 3.7.14 (Escape function on a compact set)**.** *Let $\lambda_0 > 0$ be as in Lemma 3.7.13. Let $\chi \in C_c^\infty(\mathbb{R};\mathbb{R})$ and set*

$$a_0(x,\xi) = \int_0^\infty \chi(z(t,x,\xi))|\zeta(t,x,\xi)|dt.$$

Let $R \geq R_0$, where R_0 be same as that of Lemma 3.7.11. Then a_0 is well defined smooth function on the set

$$C_{\lambda_0} := \{(x,\xi) \in T^*\mathbb{R}^n \mid |x| \leq R,\ |\xi| \geq 1,\ \ |p(\xi)| < \lambda_0|\xi|^2\}$$

and a_0 satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a_0(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{-|\beta|} \text{ for } (x,\xi) \in C_{\lambda_0}. \tag{3.7.21}$$

Proof. We take $T > 0$ same as that of Lemma 3.7.13. We note $(z(t,x,\xi),\zeta(t,x,\xi)) = (z(|\xi|t,x,\frac{\xi}{|\xi|}),|\xi|\zeta(|\xi|t,x,\frac{\xi}{|\xi|}))$ and

$$\begin{aligned} a_0(x,\xi) =& |\xi|\int_0^\infty \chi(z(|\xi|t,x,\frac{\xi}{|\xi|}))|\zeta(|\xi|t,x,\frac{\xi}{|\xi|})|dt \\ =& \int_0^T \chi(z(t,x,\frac{\xi}{|\xi|}))|\zeta(t,x,\frac{\xi}{|\xi|})|dt \end{aligned}$$

for $(x,\xi) \in C_{\lambda_0}$. Thus it follows that a_0 is a well-defined smooth function. We note

$$|\partial_x^\alpha \partial_\xi^\beta k(t,x,\xi)| \leq C_{\alpha\beta},\ k \in \{z,\zeta\}$$

uniformly in $|t| \leq T$, $|x| \leq R$ and $|\xi| = 1$, and

$$|\partial_\xi^\beta \frac{\xi}{|\xi|}| \leq C_\beta \langle\xi\rangle^{-\beta},\ |\xi| \geq 1.$$

These inequalities give (3.7.21). □

Now we prove Lemma 3.7.10.

Proof of Lemma 3.7.10. Let $\lambda_0 > 0$ be as in Lemma 3.7.13. We fix some notations which works only in this subsection. For $\lambda_0 > 0$, let $\psi_{\lambda_0} \in C_c^\infty(\mathbb{R};[0,1])$ such that

$$\text{supp}\ \tilde{\psi}_{\lambda_0} \subset (-\lambda_0,\lambda_0),\ \tilde{\psi}_{\lambda_0}(t) = 1 \text{ for } t \in (-\frac{\lambda_0}{2},\frac{\lambda_0}{2}). \tag{3.7.22}$$

Moreover, we take $\chi \in C_c^\infty(\mathbb{R},[0,1])$ such that

$$\chi(t) = \begin{cases} 1, & t \leq 1, \\ 0, & t \geq 2, \end{cases} \qquad \chi'(t) \leq 0.$$

We set $\chi_R(t) = \chi(t/R)$ and $\bar{\chi}_R(t) = 1 - \chi_R(t)$ for $R > 0$.

$$\psi_{\lambda_0}(x,\xi) = \bar{\chi}(|\xi|)\tilde{\psi}_{\lambda_0}(\frac{p(x,\xi)}{|\xi|^2}).$$

We define

$$C_{\lambda_0}(P) = \{(x,\xi) \in \mathbb{R}^{2n} \mid |\xi| \geq 1,\ |p(\xi)| < \lambda_0|\xi|^2\},$$

then we note supp $\psi_{\lambda_0} \subset C_{\lambda_0}(P)$ and $\psi_{\lambda_0} \in S^{0.0}$.

It suffices to construct a smooth real-valued function q such that for $(x,\xi) \in C_{\lambda_0}(P)$, we have

$$|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|},\ H_p q(x,\xi) \geq C\langle x\rangle^{-1-\delta}\langle\xi\rangle. \tag{3.7.23}$$

In fact, setting $a(x,\xi) = \psi_{\lambda_0}(x,\xi)^2 q(x,\xi)$ and $r(x,\xi) = C\langle x\rangle^{-1-\delta}\langle\xi\rangle(1-\psi_{\lambda_0}(x,\xi)^2) + qH_p\psi_{\lambda_0}^2(x,\xi)$, then a and r satisfy the desired property.

For $R, M_1, M_2, L > 0$ which are large enough and determined later, we set

$$\begin{aligned} q_1(x,\xi) =& \frac{x\cdot\tilde{\xi}}{|x||\xi|}\int_1^{\frac{2|x|}{M}} \frac{1}{s^{1+\delta}}ds\bar{\chi}_M(|x|),\ q_2(x,\xi) = \chi_{4M}(|x|)\int_\infty^0 \chi_{2M}(z(t,x,\xi))|\zeta(t,x,\xi)|dt, \\ q(x,\xi) =& Lq_1(x,\xi) + q_2(x,\xi), \end{aligned}$$

where q_2 is well-defined for $(x,\xi) \in C_{\lambda_0}(P)$ by Lemma 3.7.14. We claim

$$H_{p_0}(\frac{x\cdot\tilde{\xi}}{|x||\xi|}\int_1^{\frac{2|x|}{M}}\frac{1}{s^{1+\delta}}ds) \geq C\langle x\rangle^{-1-\delta}\langle\xi\rangle,\ |x| \geq M,\ |\xi| \geq 1,$$

where $C > 0$ is independent of $M \geq 1$. In fact, we have

$$H_{p_0}(\frac{x\cdot\tilde{\xi}}{|x||\xi|}\int_1^{\frac{2|x|}{M}}\frac{1}{s^{1+\delta}}ds) = 2\frac{|x|^2|\xi|^2 - (x\cdot\tilde{\xi})^2}{|x|^3|\xi|}\int_1^{\frac{2|x|}{M}}\frac{1}{s^{1+\delta}}ds + M^\delta\frac{(x\cdot\tilde{\xi})^2}{2^{\delta-1}|x|^{3+\delta}|\xi|}.$$

This gives the above inequalities. We write

$$p(x,\xi) = p_0(\xi) + V(x,\xi),\ V \in S^{2,-\mu}.$$

Since $\delta < \mu$, there is $C > 0$ independent of M such that

$$|H_V q_1(x,\xi)| \leq C\langle M\rangle^{-(\mu-\delta)}\langle x\rangle^{-1-\delta}\langle\xi\rangle \text{ for } (x,\xi) \in C_{\lambda_0}(P).$$

Then there exist $C_3, C_4 > 0$ such that for $|\xi| \geq 1$ and $M \geq 1$, we have

$$H_p q_1(x,\xi) \geq C_3\langle\xi\rangle\langle x\rangle^{-1-\delta}\bar{\chi}_M(|x|) - C_4\langle M\rangle^{-(\mu-\delta)}\langle x\rangle^{-1-\delta}\langle\xi\rangle.$$

where we use $(x \cdot \tilde{\xi})H_{p_0}\chi(|x|) \geq 0$. Moreover, there exist $C_5, C_6 > 0$ such that for $(x,\xi) \in C_{\lambda_0}(P)$ and $M \geq 1$, we have

$$\begin{aligned} H_p q_2(x,\xi) \geq & C_5\langle \xi \rangle \chi_{2M}(|x|) - C_6 \langle x \rangle^{-1} \langle \xi \rangle \bar{\chi}_M(|x|)\chi_{8M}(|x|) \\ \geq & C_5\langle \xi \rangle \chi_{2M}(|x|) - C_6 \langle M \rangle^{\delta} \langle x \rangle^{-1-\delta} \langle \xi \rangle \bar{\chi}_M(|x|) \end{aligned}$$

Thus we have

$$\begin{aligned} H_p q(x,\xi) \geq & (C_3 L - C_6 \langle M \rangle^{\delta}) \langle \xi \rangle \langle x \rangle^{-1-\delta} \bar{\chi}_M(|x|) + C_5 \langle \xi \rangle \chi_{2M}(|x|) \\ & - C_4 L \langle M \rangle^{-(\mu-\delta)} \langle x \rangle^{-1-\delta} \langle \xi \rangle. \end{aligned}$$

Using $0 < 2\delta < \mu$, we can take $L, M > 0$ large enough such that

$$M^{\delta} << L << M^{\mu-\delta}.$$

and we obtain (3.7.23).

□

Remark 3.7.15. When we assume the globally non-trapping assumption, a more stronger equality holds: There exist $C_1, C_2 > 0$ and $a \in S^{0,0}$ such that

$$H_p a(x,\xi) \geq C_1 \langle x \rangle^{-1-\delta} \langle \xi \rangle - C_2.$$

In fact, in the proof above, we only have to replace ψ_{λ_0} by $\bar{\chi}(|\xi|)$. For a bit different proof, see [9, Lemma 7.1].

3.8 Mapping properties

Let P be as in the last section. In this section, we prove a good mapping property of the resolvent $(P-i)^{-1}$, which does not use in the other part of this thesis. We note that the operator $(P-i)^{-1}$ maps from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Next proposition claims that the range of this operator is contained in $\mathcal{S}(\mathbb{R}^n)$.

Proposition 3.8.1. *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then the resolvent $(P-z)^{-1}$ is a linear continuous operator on $\mathcal{S}(\mathbb{R}^n)$. In particular, $(P-z)^{-1}$ can be extended to a linear continuous operator on $\mathcal{S}'(\mathbb{R}^n)$.*

As in [44, Appendix] (for the Stark Schrödinger operator), we have the following commutator relation which is a generalization of the equation (3.7.4).

Corollary 3.8.2. *Let $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a continuous linear operator, then we have*

$$[(P-i)^{-1}, T] = (P-i)^{-1}[T, P](P-i)^{-1}.$$

In the following, we only deal with the case of $\mathrm{Im}\, z > 0$. In order to prove Proposition 3.8.1, we need some preliminary lemmas. We frequently use the following symbol:

$$\tilde{\eta}(x,\xi) = \frac{x \cdot \partial_\xi p(x,\xi)}{|x||\partial_\xi p(x,\xi)|}.$$

Moreover, for symbols a, b, we denote $a \Subset b$ if we have

$$\inf_{(x,\xi)\in \mathrm{supp}\, a} |b(x,\xi)| > 0,$$

and we denote $\mathrm{Op}(a) =: A \Subset B := \mathrm{Op}(b)$ if $a \Subset b$.

3.8.1 Radial source estimate

Proposition 3.8.3. *Let $A \in \mathrm{Op}S^{0,0}$ be supported in*

$$\{(x,\xi) \in \mathbb{R}^{2n} \mid |x| > R,\ |\xi| > r,\ \tilde{\eta}(x,\xi) \leq -1+\varepsilon\}. \tag{3.8.1}$$

with $R > 0$ large enough, $r > 0$ and $0 < \varepsilon < 1$. Let

$$z \in \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\} \cup \mathbb{R} \setminus \{0\}. \tag{3.8.2}$$

Let $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $A_1 u \in H^{k,-\frac{1}{2}+0} \cap H^{k,l-1+0}$, $A_1(P-z)u \in H^{k-\frac{1}{2},l+\frac{1}{2}}$ with some $k \in \mathbb{R}$, $l > 0$ and $A_1 = \mathrm{Op}(a_1) \in \mathrm{Op}S^{0,0}$ such that $A \Subset A_1$. Then we have $Au \in H^{k+\frac{1}{2},l-\frac{1}{2}}$.

Corollary 3.8.4. *Let $k \in \mathbb{R}$ and z as in* (3.8.2)*. Let $A_1 u \in H^{k,-\frac{1}{2}+0}$ satisfying $(P-z)u \in \mathcal{S}(\mathbb{R}^n)$, where A, A_1 are as in Proposition 3.8.3. Then we have $Au \in \mathcal{S}(\mathbb{R}^n)$.*

Corollary 3.8.4 immediately follows from Proposition 3.8.3 and the standard bootstrap argument.

In the proof of Proposition 3.8.3, the following commutator calculus has an important role: For pseudodifferential operators A, Λ, where A is formally self-adjoint and $\mathrm{Im}\, z \geq 0$, we have

$$\mathrm{Im}\,((P-z)u, A\Lambda^*\Lambda A u)_{L^2} = -\,(u, [P, iA\Lambda^*\Lambda A]u)_{L^2} + \mathrm{Im}\, z \|\Lambda A u\|_{L^2}^2 \tag{3.8.3}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. Moreover, the equation (3.8.3) with the Cauchy-Schwartz inequality implies that for any small $\varepsilon_1 > 0$, there exists $C > 0$ such that

$$-(u, [P, iA\Lambda^2 A]u)_{L^2} \leq C\|\Lambda A(P-z)u\|_{H^{-\frac{1}{2},\frac{1}{2}}}^2 + \varepsilon_1 \|\Lambda A u\|_{H^{\frac{1}{2},-\frac{1}{2}}}^2. \tag{3.8.4}$$

First, we construct the escape function near the incoming region.

Lemma 3.8.5 (Escape function near the incoming region)**.** *Let* $0 < \varepsilon < 1/8$ *and* $\rho, \chi \in C^\infty(\mathbb{R};[0,1])$ *such that*

$$\chi(t) = \begin{cases} 1 & \text{for } t \le 1, \\ 0 & \text{for } t \ge 2, \end{cases} \qquad \chi'(t) \le 0, \quad \rho(t) = \chi(\frac{2(t+1)}{\varepsilon})$$

Moreover, we set $\bar{\chi}_R(x) = 1 - \chi(|x|/R)$ *for* $R \ge 1$*. For* $k \in 0$*,* $l > 0$ *and* $0 < \delta \le 1$*, we set*

$$\begin{aligned} &\lambda_{k,l} = \langle\xi\rangle^k \langle x\rangle^l \ \lambda_{k,l,\delta} = \langle\xi\rangle^k \langle\delta\xi\rangle^{-|k|-1} \langle x\rangle^l \langle\delta x\rangle^{-l+0} \\ &b_\delta(x,\xi) = \lambda_{k,l,\delta} a(x,\xi), \ a(x,\xi) = \rho(\tilde{\eta}(x,\xi))\chi_R(x)\chi_r(\xi). \end{aligned}$$

For $R \ge 1$ *large enough and any* $r > 0$*, we have*

$$H_p b_\delta^2(x,\xi) \le -C\langle x\rangle^{-1}\langle\xi\rangle b_\delta(x,\xi)^2 + \lambda_{k,l,\delta}^2 r_1, \tag{3.8.5}$$

where $r_0 \in S^{-\infty,-\frac{1}{2}}$ *which support has close to* supp a*.*

Proof. Take $R \ge 1$ such that

$$\begin{aligned} H_p\tilde{\eta}(x,\xi) =& \frac{|x|^2|\partial_\xi p(x,\xi)|^2 - (x\cdot\partial_\xi p(x,\xi))^2}{|x|^3|\partial_\xi p(x,\xi)|} + S^{1,-1-\mu} \\ \ge& \begin{cases} C\langle x\rangle^{-1}\langle\xi\rangle, \text{ for } |x| \ge R, \ |\xi| \ge r, \ \tilde{\eta}(x,\xi) \in (-1+\frac{\varepsilon}{2}, -1+\varepsilon), \\ -C\langle x\rangle^{-1-\mu}\langle\xi\rangle, \text{ for } |x| > R, \ |\xi| \ge r. \end{cases} \end{aligned}$$

This implies $H_p(\rho(\tilde{\eta})) \le 0$. Since $\tilde{\eta}(x,\xi) \le 0$ for $(x,\xi) \in \text{supp}\, a$, we have $H_p(\bar{\chi}_R(x)) \le 0$. Thus we have

$$H_p a \le r_0^2, \ r_0 \in S^{-\infty,-\frac{1}{2}}.$$

Moreover, we have

$$H_p\langle\xi\rangle^k\langle\delta\xi\rangle^{-|k|-1} \le C\langle x\rangle^{-1-\mu}\langle\xi\rangle^{k+1}\langle\delta\xi\rangle^{-|k|-1}, \ H_p\langle x\rangle^l\langle\delta x\rangle^{-l} \le -C\langle\xi\rangle\langle x\rangle^{l-1}\langle\delta x\rangle^{-l}$$

for $(x,\xi) \in \text{supp}\, a$, where the constant $C > 0$ is independent of δ. Thus, we obtain (3.8.5).

□

Set $\Lambda_{k,l,\delta} = \text{Op}(\lambda_{k,l,\delta})$, $\Lambda_{k,l} = \text{Op}(\lambda_{k,l})$, $A = \text{Op}(a)$. Moreover, we set

$$\Lambda = \Lambda_{k,l,\delta}, \ \Theta = \langle x\rangle^{-\frac{1}{2}}\langle D\rangle^{\frac{1}{2}}.$$

Now we show Proposition 3.8.3. We take $r > 0$ small enough such that (3.8.6) below holds. Note that $r > 0$ depends only on z.

Proof of Proposition 3.8.3. By the standard elliptic estimate, it suffices to replace A in Proposition 3.8.3 by $A = \mathrm{Op}(a)$, where a is as in Lemma 3.8.5.

Let $N > 0$. Take $A_1 = \mathrm{Op}(a_1), A_2 = \mathrm{Op}(a_2) \in \mathrm{Op}S^{0,0}$ such that $a \Subset a_2 \Subset a_1$. Set $A = \mathrm{Op}(a)$. We observe that from (3.8.2), there exists $r > 0$ such that

$$P - z \quad \text{is elliptic in the region} \quad \{(x,\xi) \in \mathbb{R}^{2n} \mid |\xi| \leq 2r\}. \tag{3.8.6}$$

Then the standard elliptic parametrix construction implies

$$\|\Lambda A(P-z)u\|^2_{H^{-\frac{1}{2},\frac{1}{2}}} + \|\Lambda \mathrm{Op}(r_1)u\|^2_{L^2} \leq C\|\Lambda A_2(P-z)u\|^2_{H^{-\frac{1}{2},\frac{1}{2}}} + C\|u\|^2_{H^{-N,-N}},$$

where we use the support property of $r_1 \in S^{-\infty,-1/2}$. The sharp Gårding inequality with (3.8.3) and (3.8.4) implies that we have

$$\|\Lambda Au\|^2_{H^{\frac{1}{2},-\frac{1}{2}}} \leq C\|\Lambda A_2(P-z)u\|^2_{H^{-\frac{1}{2},\frac{1}{2}}} + C\|\Lambda_{k,l}A_2u\|^2_{H^{0,-1+0}} + C\|u\|^2_{H^{-N,-N}} \tag{3.8.7}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. Now we suppose $u \in H^{-N,-N}$, $A_1u \in H^{k,-\frac{1}{2}+0} \cap H^{k,l-1+0}$ with $A_1(P-z)u \in H^{k-\frac{1}{2},l+\frac{1}{2}}$. Since $\mathrm{Op}(\Lambda_{k,l,\delta})$ and A_2 belong to $\mathrm{Op}S^{0,0}$, by the standard limiting procedure, we substitute u into (3.8.7). This implies $Au \in H^{k+\frac{1}{2},l-\frac{1}{2}}$ by taking $\delta \to 0$, where we recall $\Lambda = \Lambda_{k,l,\delta}$.

□

3.8.2 Propagation to the radial source in the past infinity

To use the standard propagation of singularity, we need the following dynamical lemma.

Lemma 3.8.6. *Let $(x_0,\xi_0) \in T^*\mathbb{R}^n$ with $\xi \neq 0$ and $p(x,\xi) = 0$. We denote $z(t) = z(t,x_0,\xi_0)$, $\zeta(t) = \zeta(t,x_0,\xi_0)$ and $\eta(t) = \tilde{\eta}(z(t),\zeta(t))$. Then for any $0 < \varepsilon < 1$ and $R \geq 1$, there exists $T > 0$ such that $|z(-T)| > R$*

$$\eta(-T) < (-1+\varepsilon). \tag{3.8.8}$$

Proof. Let $0 < \varepsilon < 1$ and $R \geq 1$. Take $R_0 \geq R$ such that $H_p^2|x|^2 \geq C|\xi|^2$ for $|x| \geq R_0$ and

$$H_p\tilde{\eta}(x,\xi) \geq C\langle x\rangle^{-1}|\xi|. \tag{3.8.9}$$

for $|x| \geq R_0$ with $(-1+\varepsilon) \leq \eta(x,\xi) \leq 0$. By the null non-trapping condition, we can choose $T_0 > 0$ such that

$$|z(-T_0)| \geq 2R_0, \ \frac{d}{dt}|z(t)|^2|_{t=-T_0} \leq 0.$$

This with the convexity of H_p implies $|z(-t)| \geq R_0$ and $\eta(-t) \leq 0$ for $t \geq T_0$. Now suppose that (3.8.8) fails. Then, by Lemma 3.3.2, Corollary 3.3.4 and the inequality (3.8.9), we obtain

$$\eta(-T_0) = \eta(-t) + \int_{-t}^{-T_0} \eta'(s)ds \geq C\int_{-t}^{-T_0} \langle s\rangle^{-1}ds = \infty \quad \text{as} \quad t \to \infty$$

which is a contradiction.

□

This corollary implies that for any $(x_0, \xi_0) \in \mathbb{R}^n \setminus 0$, large enough $R \geq 1$, and $0 < \varepsilon < 1$, there exists $(x_1, \xi_1) \in T^*\mathbb{R}^n \setminus 0$ such that (x_1, ξ_1) lies in the same integral curve of H_p and

$$|x_1| > R, \ \tilde{\eta}(x_1, \xi_1) < -1 + \varepsilon.$$

Then the standard propagation of singularities theorem implies that u is smooth microlocally near (x_0, ξ_0) if and only if u is smooth microlocally near (x_1, ξ_1), for $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfying $(P - i)u \in C^\infty(\mathbb{R}^n)$. On the other hand, Corollary 3.8.4 implies that for $R >> 1$ large enough and $0 < \varepsilon < 1$, we have $u \in C^\infty(\mathbb{R}^n)$ microlocally on the region (3.8.1) if $u \in H^{k, -\frac{1}{2}+0}(\mathbb{R}^n)$ (for some $k \in \mathbb{R}$) satisfying $(P - z)u \in \mathcal{S}(\mathbb{R}^n)$, where z satisfies (3.8.2). Thus we obtain

Corollary 3.8.7. *Let $k \in \mathbb{R}$ and $z \in \mathbb{C}$ satisfying* (3.8.2). *Suppose that $u \in H^{k, -\frac{1}{2}+0}(\mathbb{R}^n)$ satisfies $(P - z)u \in \mathcal{S}(\mathbb{R}^n)$. Then we have $u \in C^\infty(\mathbb{R}^n)$.*

This corollary is a generalization of Proposition 3.2.2.

3.8.3 Subellipticity on the spectral parameter

The proof of Lemma 3.8.8 below looks the standard positive commutator argument at a first glance, however, it is very different from the usual positive commutator argument. In fact, in the proof below, we do not use the dynamical property for H_p. We only need to use the symbol calculus and the symbol class $p \in S^{2,0}$.

Lemma 3.8.8. *Let $k, l \in \mathbb{R}$, $z \in \mathbb{C} \setminus \mathbb{R}$. If $u \in H^{k+1/2, l-1/2}$ satisfies $(P - z)u \in H^{k,l}$, then we have $u \in H^{k,l}$.*

Remark 3.8.9. In this lemma, the assumption $z \notin \mathbb{R}$ is necessary.

Proof. Take $\chi \in C_c^\infty(\mathbb{R}; \mathbb{R})$ such that $\chi(t) = 1$ on $|t| \leq 1$ and set

$$a_R(x, \xi) = \chi(\frac{|(x, \xi)|}{R}), \ \Lambda_R = \langle x \rangle^l \langle D \rangle^k \mathrm{Op}(a_R) \in \mathrm{Op}S^{-\infty, -\infty}$$

for $R \geq 1$. We note Λ_R is uniformly bounded in $\mathrm{Op}S^{k,l}$. Since $\Lambda_R \in \mathrm{Op}S^{-\infty,-\infty}$, we have

$$((P - z)u, \Lambda_R^2 u)_{L^2} - (\Lambda_R^2 u, (P - z)u)_{L^2} = 2i\mathrm{Im}\, z \|\Lambda_R u\|_{L^2}^2 + (u, [P, \Lambda_R^2]u)_{L^2}$$

for $u \in \mathcal{S}'(\mathbb{R}^n)$. Now we let $u \in H^{k+1/2, l-1/2}$ satisfying $(P - z)u \in H^{k,l}$. Then we have

$$|(u, [P, \Lambda_R^2]u)_{L^2}| \leq C\|u\|_{H^{k+1/2, l-1/2}}^2$$

with a constant $C > 0$ independent of $R \geq 1$. Moreover, by the Cauchy-Schwartz inequality, we have

$$|((P - z)u, \Lambda_R^2 u)_{L^2} - (\Lambda_R^2 u, (P - z)u)_{L^2}| \leq \frac{C}{|\mathrm{Im}\, z|}\|(P - z)u\|_{H^{k,l}}^2 + |\mathrm{Im}\, z| \|\Lambda_R u\|_{L^2}^2$$

with a constant $C > 0$ independent of $R \geq 1$. Thus we have

$$|\mathrm{Im}\, z| \|\Lambda_R u\|_{L^2}^2 \leq \frac{C}{|\mathrm{Im}\, z|} \|(P-z)u\|_{H^{k,l}}^2 + C\|u\|_{H^{k+1/2,l-1/2}}^2.$$

Using $\mathrm{Im}\, z \neq 0$ and a limiting procedure, we obtain $u \in H^{k,l}$.

□

3.8.4 Proof of Proposition 3.8.1

Now we prove Proposition 3.8.1.

Lemma 3.8.10. *Let* $u \in L^2(\mathbb{R}^n)$ *and* $z \in \mathbb{C}$ *satisfying* $\mathrm{Im}\, z > 0$ *and* $(P-z)u \in \mathcal{S}(\mathbb{R}^n)$. *Then there is* $0 < \varepsilon < \frac{1}{2}$ *such that for any* $k \in \mathbb{R}$, *we have* $u \in H^{k,-\varepsilon}$.

Proof. Corollary 3.8.7 implies $u \in C^\infty(\mathbb{R}^n)$. By virtue of Theorem 3.2.4, we have $u \in \cup_{k\in\mathbb{R}} H^{k,-\varepsilon}$ with some $\varepsilon > 0$. □

Remark 3.8.11. When $z \in \mathbb{R}\setminus\{0\}$, the same conclusion holds with $\varepsilon > 1/2$. To see this, we only need to prove the same conclusion of Theorem (3.2.4) holds for $z \in \mathbb{R}\setminus\{0\}$ when we replace the conclusion $\varphi \in H^{k,-\gamma} \cap H^{k+\frac{1}{2},-\gamma-\frac{1}{2}}$ by $\varphi \in H^{k+\frac{1}{2},-\gamma-\frac{1}{2}}$. To prove this, it suffices to remove the assumption $\varphi \in L^2(\mathbb{R}^n)$ in Theorem 3.2.4 in view of its proof (we also observe the conclusion $\varphi \in H^{k+\frac{1}{2},-\gamma-\frac{1}{2}}$ comes from the first term of the left hand side of (3.2.1)). We note that $\nu > 0$ in the statement of Lemma (3.2.5) can be chosen as $P-z$ is elliptic on $\{|\xi| \leq 2\nu\}$. This implies that $T\varphi \in L^2(\mathbb{R}^n)$ if $(P-z)\varphi \in L^2(\mathbb{R}^n)$. Thus we can remove the assumption $u \in L^2(\mathbb{R}^n)$. Now we may take $\varepsilon = \gamma + 1/2$ where $\gamma > 0$.

Proof of Proposition 3.8.1. Suppose $z \in \mathbb{C}$ satisfying $\mathrm{Im}\, z > 0$. By duality, we only need to prove $P - i : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a homeomorphism. Since the operator $P-i$ maps between $\mathcal{S}(\mathbb{R}^n)$ continuously and is injective in $\mathcal{S}(\mathbb{R}^n)$ (this follows from its essential self-adjointness on $\mathcal{S}(\mathbb{R}^n)$), it suffices to prove that $P-i$ is surjective in $\mathcal{S}(\mathbb{R}^n)$ by the open mapping theorem.

Let $f \in \mathcal{S}(\mathbb{R}^n)$. Setting $u = (P-i)^{-1}f \in L^2(\mathbb{R}^n)$, we have $(P-i)u = f$. By Lemmas 3.8.8 and 3.8.10, we have $u \in \mathcal{S}(\mathbb{R}^n)$. This means $P-i$ is surjective in $\mathcal{S}(\mathbb{R}^n)$. □

Chapter 4

Repulsive Schrödinger operators

4.1 Introduction

In this chapter, we consider the following repulsive Schrödinger operator on $\mathbb{R}^n$:

$$P = P_\alpha = -\Delta - \langle x \rangle^{2\alpha} + \mathrm{Op}(V), \quad \alpha > 1, \tag{4.1.1}$$

where $\langle x \rangle = (1+|x|^2)^{1/2}$ and $\mathrm{Op}(V)$ is the Weyl quantization of a symbol $V : \mathbb{R}^{2n} \to \mathbb{R}$. We set

$$P_0 = P_{0,\alpha} = -\Delta - \langle x \rangle^{2\alpha}, \quad \alpha > 1.$$

Let $p_0(x,\xi) = |\xi|^2 - \langle x \rangle^{2\alpha}$ and $p(x,\xi) = p_0(x,\xi) + V(x,\xi)$. In this chapter, we always assume the following assumptions.

Assumption C. We set Suppose that V is of the form

$$V(x,\xi) = \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k + \sum_{j=1}^{n} b_j(x)\xi_j + c(x),$$

where $a_{jk} = a_{kj}$, b_j and c are real-valued smooth functions on $\mathbb{R}^n$ and satisfy

$$|\partial_x^\beta a_{jk}(x)| \le C_\beta(1+|x|)^{-\mu-|\beta|},\ |\partial_x^\beta b_j(x)| \le C_\beta(1+|x|)^{\alpha-\mu-|\beta|},$$
$$|\partial_x^\beta c(x)| \le C_\beta(1+|x|)^{2\alpha-\mu-|\beta|}.$$

with some $0 < \mu < 1/2$ and $C_\beta > 0$.

In particular, $\mathrm{Op}(V)$ is a symmetric differential operator and

$$V(x,\xi) = \sum_{j=0}^{2} V_j(x,\xi), \quad V_j \in S^{j,\alpha(2-j)-\mu}.$$

Assumption D. For any $M > 0$

$$|p(x,\xi)| \geq C\langle\xi\rangle^2, \quad |x| \leq M, |\xi| \geq R_0$$

with some $C > 0$ and $R_0 > 0$.

We study stationary scattering theory of P and give an application to limit circle problem. The usual scattering theory is based on the limiting absorption principle: the resolvent bound

$$\sup_{\mathrm{Re}\, z\in I, \mathrm{Im}\, z\neq 0} \|\langle x\rangle^{-1/2-0}(-\Delta + V - z)^{-1}\langle x\rangle^{-1/2-0}\|_{L^2\to L^2} < \infty \tag{4.1.2}$$

and existence of the boundary values of the resolvent

$$\lim_{\pm\mathrm{Im}\, z\to 0} \langle x\rangle^{-1/2-0}(-\Delta + V - z)^{-1}\langle x\rangle^{-1/2-0}. \tag{4.1.3}$$

(4.1.2) is used in order to prove existence and completeness of the wave operators. Existence of the boundary values (4.1.3) is used for a construction of generalized eigenfunctions of the stationary Schrödinger equation:

$$(-\Delta + V - z)u = 0.$$

The difficulty in the case of P with $\alpha > 1$ lies in the lack of essential self-adjointness of P on $\mathcal{S}(\mathbb{R}^n)$. Since P may have many self-adjoint extensions, "the boundary value of the resolvent" seems meaningless. The recent progress in the microlocal analysis gives another definition of the outgoing/incoming resolvents of pseudodifferential operators under some dynamical conditions. See [16] for the Anosov vector fields, [3] and [75] for the d'Alembertians in the scattering Lorentzian spaces. We apply this technique to the repulsive Schrödinger operator P even for $\alpha > 1$ and prove existence of the outgoing/incoming resolvents. Moreover, we show that P has many eigenfunctions associated with the eigenvalues $\lambda \in \mathbb{C}$ except for a discrete set. As a corollary, we give another proof of that P is not essentially self-adjoint for $\alpha > 1$ in view of scattering and microlocal theory. This is a classical result which is known as a typical limit circle case (for example, see [61]) when $\mathrm{Op}(V)$ is a multiplication operator. It seems to be new result when $\mathrm{Op}(V)$ is not a multiplication operator.

The repulsive Schrödinger operator is studied by several authors when $\mathrm{Op}(V)$ is a multiplication operator. Time-dependent scattering theory of the operator (4.1.1) for $0 < \alpha \leq 1$ is studied in [4] in the short-range case. The authors prove the existence and completeness of the wave operator and existence of the asymptotic velocity. They also study that existence of the outgoing/incoming resolvent and the absence of L^2-eigenvalues. The recent works in [38] and [39] extend some results in [4] for the long-range case. Moreover, in [38], the author of these papers proves the absence of eigenvalues in Besov spaces, where the order of Besov space is $\frac{\alpha-1}{2}$. This result is an extension of well-known results for the usual Schrödinger operators ($\alpha = 0$) to the repulsive Schrödinger operators ($0 < \alpha \leq 1$).

From the usual stationary scattering theory of $-\Delta$, we know that:

- Eigenfunctions of $-\Delta$ associated with positive eigenvalues do not exist in the threshold weighted L^2-space: $L^{2,-\frac{1}{2}}$.
- There are many eigenfunctions right above $L^{2,-\frac{1}{2}}$:

$$-\Delta u = \lambda u, \quad u \in \bigcap_{s>1/2} L^{2,-s}$$

  for each $\lambda > 0$.

The result in [38] and [39] suggests that the above results hold for the repulsive Schrödinger operator with $0 < \alpha \leq 1$ with threshold weight $\frac{\alpha-1}{2}$. It is expected that these results also hold for $\alpha > 1$. In this chapter, we almost justify these and we prove the existence of non-trivial L^2-solution to

$$(P - z)u = 0$$

for $z \in \mathbb{C}$ except for a discrete subset of $\mathbb{C}$.

We introduce the variable order weighted L^2-space $L^{2,k+tm(x,\xi)}$, where $k, t \in \mathbb{R}$ and m is a real-valued function on the phase space $\mathbb{R}^{2n}$. Though we give a precise definition of $L^{2,k+tm(x,\xi)}$ in Section 4.6, we state properties of $L^{2,k+tm(x,\xi)}$ here: If $u \in L^{2,k+tm(x,\xi)}$, then

$$u \in L^{2,k-t}, \text{microlocally near} \{|x|, |\xi| > R, |\xi| \sim |x|^\alpha, x \cdot \xi \sim |x||\xi|\} \tag{4.1.4}$$
$$u \in L^{2,k+t}, \text{microlocally near} \{|x|, |\xi| > R, |\xi| \sim |x|^\alpha, x \cdot \xi \sim -|x||\xi|\} \tag{4.1.5}$$

for large $R > 0$. The following theorem is an analog of [16, Theorem 1.4].

Theorem 4.1.1.

(i) *Let $t \neq 0$ and $z \in \mathbb{C}$. We define*

$$D_{tm} = \{u \in L^{2,\frac{\alpha-1}{2}+tm(x,\xi)} \mid (P - z)u \in L^{2,\frac{1-\alpha}{2}+tm(x,\xi)}\}.$$

Then

$$P - z : D_{tm} \to L^{2,\frac{1-\alpha}{2}+tm(x,\xi)} \tag{4.1.6}$$

is a Fredholm operator and coincides with the closure of $(P - z)$ with domain $\mathcal{S}(\mathbb{R}^n)$ with respect to its graph norm.

(ii) *There exists a discrete subset $T_{\alpha,t} \subset \mathbb{C}$ such that* (4.1.6) *is invertible for $\mathbb{C} \setminus T_{\alpha,t}$.*

Remark 4.1.2. By the standard radial point estimates and the propagation of singularities, it follows that $T_{\alpha,t} = T_{\alpha,\mathrm{sgn}\, t}$ is independent of $|t|$ and $T_{\alpha,t} \subset \mathbb{C}_{-\mathrm{sgn}\, t} = \{-(\mathrm{sgn}\, t)\mathrm{Im}\, z \geq 0\}$. Moreover, this theorem is true for $0 < \alpha \leq 1$ if we replace $z \in \mathbb{C}$ above by $z \in \mathbb{C}_{\mathrm{sgn}\, t}$ (though D_{tm} depends on z). We leave these proofs to future work.

This theorem also gives the bijectivity of $P - z$ in the usual weighted L^2-spaces: Suppose $z \in \mathbb{C} \setminus T_{\alpha,t}$. For any $f \in L^{(1-\alpha)/2+\varepsilon}$ with $\varepsilon > 0$, there exists a unique solution $u \in L^{2,(\alpha-1)/2-\varepsilon}$ to the equation

$$\begin{cases} (P-z)u = f, \text{ in the distributional sense,} \\ u \text{ is outgoing if the signature of } t \text{ is } + \text{ and incoming if the signature of } t \text{ is } -, \end{cases}$$

where "u is outgoing" says that (4.1.5) holds with $k = (\alpha-1)/2$ and $t = \varepsilon$ and "u is incoming" says that (4.1.4) holds with $k = (\alpha-1)/2$ and $t = -\varepsilon$.

Moreover, we construct non-trivial L^2 solutions to $Pu = zu$.

Theorem 4.1.3. *Let $\alpha > 1$ and $t \neq 0$. For $z \in \mathbb{C} \setminus T_{\alpha,t}$, there exists $u \in L^2 \setminus \{0\}$ such that $Pu = zu$.*

Remark 4.1.4. As is proved in Proposition 4.4.9, it follows that there are many eigenfunctions associated with $z \in \mathbb{C} \setminus T_{\alpha,t}$.

From Theorem 4.1.3 and the standard criterion for essential self-adjointness [61, Corollary after Theorem VIII.3], we conclude that P is not essentially self-adjoint if $\alpha > 1$.

Corollary 4.1.5. *Suppose $\alpha > 1$. Then $P = P_\alpha$ is not essentially self-adjoint both on $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$.*

The repulsive Schrödinger operator $P = P_\alpha$ for large α is expected to have the same structure as the Laplace operator on a bounded open set in $\mathbb{R}^n$. For a bounded open set Ω, it is well-known that the inclusion $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Here we note that $H_0^2(\Omega)$ is the minimum domain of $-\Delta|_{C_c^\infty(\Omega)}$. For the repulsive Schrödinger operator, we prove a similar result.

Theorem 4.1.6. *Define the Banach space*

$$D^\alpha_{\min} = \{u \in L^2(\mathbb{R}^n) \mid Pu \in L^2(\mathbb{R}^n), \exists u_k \in C_c^\infty(\mathbb{R}^n)\ u_k \to u, Pu_k \to Pu \, in \, L^2(\mathbb{R}^n)\}$$

with its graph norm. Then the inclusion $D^\alpha_{\min} \hookrightarrow L^2$ is compact.

Remark 4.1.7. $D^\alpha_{\min}$ coincides with the minimal domain of $P|_{C_c^\infty(\mathbb{R}^n)}$, that is the domain of the closure of $P|_{C_c^\infty(\mathbb{R}^n)}$.

Remark 4.1.8. Note that $D^\alpha_{\min} = \{u \in L^2 \mid Pu \in L^2\}$ for $0 < \alpha \leq 1$ since P is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$ for $\alpha \leq 1$. However, it follows that $D^\alpha_{\min} \neq \{u \in L^2 \mid Pu \in L^2\}$ for $\alpha > 1$.

Corollary 4.1.9. *Let $n = 1$ and P_U be a self-adjoint extension of P. Then there exists $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{R}$ such that $\sigma(P_U) = \sigma_d(P_U) = \{\lambda_k\}_{k=1}^\infty$ and $|\lambda_k| \to \infty$ as $k \to \infty$, where $\sigma(P_U)$ is the spectrum of P_U and $\sigma_d(P_U)$ is the discrete spectrum of P_U.*

Remark 4.1.10. For a relatively bounded open interval $I \subset \mathbb{R}$, it is proved that each self-adjoint extension of $-\Delta|_{C_c^\infty(I)}$ has a discrete spectrum by mimicking the proof of Corollary 4.1.9. However, in the case of $n \geq 2$, the situation is dramatically different. In fact, we consider the Klein Laplacian ($-\Delta$ with domain $\{u \in L^2(\Omega) \mid \Delta u = 0\} + H_0^2(\Omega)$) for the bounded domain with smooth boundary $\partial\Omega$. The Klein Laplacian has a nonempty essential spectrum for $n \geq 2$. In fact, we note that any L^2 harmonic functions on Ω lies in the domain of the Klein Laplacian. Since restrictions of harmonic functions on $\mathbb{R}^n$ to Ω are L^2 harmonic functions on Ω and since the dimension of the set of all harmonic functions for $n \geq 2$ is infinite, we conclude that 0 is the eigenvalue with infinite multiplicity. In this way, it follows that the essential spectrum is not empty.

Remark 4.1.11. As an analogy to $-\Delta$ on Ω, we naturally propose the following problems:

- Does there exist a distinguish self-adjoint extension of P (such as the Friedrichs extension of $-\Delta|_{C_c^\infty(\Omega)}$ in the case of $-\Delta$ on Ω)?
- How is the structure of the self-adjoint extension of P? (More concretely, does there exist a self-adjoint extension of P which has a discrete spectrum?)

We fix some notations. $\mathcal{S}(\mathbb{R}^n)$ denotes the set of all rapidly decreasing functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ denotes the set of all tempered distributions on $\mathbb{R}^n$. We use the weighted Sobolev space: $L^{2,l} = \langle x\rangle^{-l}L^2(\mathbb{R}^n)$, $H^k = \langle D\rangle^{-k}L^2(\mathbb{R}^n)$ and $H^{k,l} = \langle x\rangle^{-l}\langle D\rangle^{-k}L^2(\mathbb{R}^n)$ for $k, l \in \mathbb{R}$. For Banach spaces X, Y, $B(X, Y)$ denotes the set of all linear bounded operators form X to Y. For a Banach space X, we denote the norm of X by $\|\cdot\|_X$. If X is a Hilbert space, we write the inner metric of X by $(\cdot,\cdot)_X$, where $(\cdot,\cdot)_X$ is linear with respect to the right variable. We also denote $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(\mathbb{R}^n)}$ and $(\cdot,\cdot)_{L^2} = (\cdot,\cdot)_{L^2(\mathbb{R}^n)}$. We denote the distribution pairing by $< \cdot,\cdot >$. For $I \subset \mathbb{R}$, we denote $I_\pm = \{z \in \mathbb{C} \mid \mathrm{Re}\, z \in I, \pm\mathrm{Im}\, z \geq 0\}$. We denote $\langle x\rangle = (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^n$. Set

$$\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm\mathrm{Im}\, z \geq 0\}.$$

4.2 Notations, cut-off functions and elliptic estimates

In this subsection, we fix some notations and define cut-off functions which are used in this chapter many times.

Let $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ such that

$$\chi(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| \geq 2. \end{cases}$$

For $R, L \geq 1$ and $0 < r \leq 1$, set $\bar{\chi} = 1 - \chi$ and

$$a_{r,R}(x, \xi) = \bar{\chi}(|x|/R)\bar{\chi}(|\xi|/R)\chi((|\xi|^2 - |x|^{2\alpha})/r(|\xi|^2 + |x|^{2\alpha})), \quad (4.2.1)$$

$$a_R(x, \xi) = a_{R^{-1},R}(x, \xi), \; b_L(x, \xi) = \chi(|x|/L)\chi(|\xi|/L). \quad (4.2.2)$$

We often use the symbol

$$\eta(x,\xi) = \frac{x\cdot\xi}{|x||\xi|}.$$

We state the elliptic estimate of the repulsive Schrödinger operator P.

Proposition 4.2.1 (Elliptic estimate)**.** *Let $z\in\mathbb{C}$, $k,l\in\mathbb{R}$, $N>0$ and $k_1,l_1\geq 0$ with $k_1+l_1\leq 2$. For $R,M\geq 1$ and $\gamma>1$, set*

$$\begin{aligned}\Omega_{loc} =&\{(x,\xi)\in\mathbb{R}^{2n} \mid |x|<M\}\cup\{(x,\xi)\in\mathbb{R}^{2n}\mid |\xi|<M\},\\ \Omega_{R,\gamma,1} =&\{(x,\xi)\in\mathbb{R}^{2n}\mid |x|>R, |\xi|>R, |\xi|>\gamma|x|^{\alpha}\},\\ \Omega_{R,\gamma,2} =&\{(x,\xi)\in\mathbb{R}^{2n}\mid |x|>R, |\xi|>R, |x|^{\alpha}>\gamma|\xi|\}.\end{aligned}$$

Let $\gamma>1$. There exists $R_1>0$ such that if $R\geq R_1$ and $a,a_1\in S^{0,0}$ are supported in $\Omega_{loc}\cup\Omega_{R,\gamma,1}\cup\Omega_{R,\gamma,2}$ and $\inf_{\operatorname{supp} a}|a_1|>0$, then there exists $C>0$ such that for $u\in H^{-N,-N}$ with $\mathrm{Op}(a_1)Pu\in H^{k,l}$, we have $\mathrm{Op}(a)u\in H^{k+k_1,l+\alpha l_1}$ and

$$\|\mathrm{Op}(a)u\|_{H^{k+k_1,l+\alpha l_1}}\leq C\|\mathrm{Op}(a_1)(P-z)u\|_{H^{k,l}}+C\|u\|_{H^{-N,-N}}.$$

Here the constant $C>0$ is locally uniformly in $\mathrm{Re}\, z\in\mathbb{R}$.

This elliptic estimate follows from a standard parametrix construction.

Lemma 4.2.2. *Let b_L as in above and $Q\in S^{k,l}$ for some $k,l\in\mathbb{R}$. Then the symbol of $[Q,\mathrm{Op}(b_L)]$ is uniformly bounded in $S^{k-1,l-1}$ with respect to $L\geq 1$ and converges to 0 in $S^{k-1+\varepsilon,l-1+\varepsilon}$ as $L\to\infty$ for any $\varepsilon>0$.*

4.3 Proof of Theorem 4.1.1 (i)

For $k\in\mathbb{R}$, we set

$$S^k_{\alpha}=\bigcup_{l\in\mathbb{R}}S^{l,k-\alpha l}.$$

4.3.1 Construction of an escape function

Take $\rho\in C^{\infty}(\mathbb{R},[0,1])$ such that

$$\rho(t)=\begin{cases}1, \text{ if } t\geq 1/2,\\ -1 \text{ if } t\leq -1/2,\end{cases}\quad \inf_{|t|\geq 1/4}|\rho(t)|>0,\ t\rho(t)\geq 0, \tag{4.3.1}$$

$$\rho'(t)\leq C_3\leq C_4|\rho(t)|,\ \text{if } |t|\geq 1/4,\ \rho'(t)\geq 0,\ \rho'(t)\geq C_1\geq C_2|\rho(t)|,\ \text{if } |t|\leq 1/4. \tag{4.3.2}$$

We define

$$m(x,\xi) = m_R(x,\xi) = -\rho(\eta(x,\xi))a_R(x,\xi)^2,$$

where we recall $\eta(x,\xi) = x\cdot\xi/|x||\xi|$ and a_R is as in (4.2.2). Moreover, we set

$$\Omega_R = \{(x,\xi)\in\mathbb{R}^{2n} \mid |x|>R,\, |\xi|>R,\, -\frac{1}{2R} < \frac{|\xi|^2-|x|^{2\alpha}}{|\xi|^2+|x|^{2\alpha}} < \frac{1}{2R}\}.$$

Lemma 4.3.1. *There exists $R_0\geq 1$ such that if $R\geq R_0$, then*

$$H_p(m\log\langle x\rangle)(x,\xi) \leq -C\langle x\rangle^{\alpha-1}a_R(x,\xi)^2 - e(x,\xi),$$

where $e(x,\xi) = \rho(\eta(x,\xi))(H_p a_R^2)(x,\xi)\log\langle x\rangle \in S_\alpha^{\alpha-1+0}$.

Proof. We learn

$$\begin{aligned} H_p(\rho(\eta)\log\langle x\rangle) \geq& 2(\eta\rho(\eta))|x||\xi|\langle x\rangle^{-2} + (H_{p_0}\eta)\rho'(\eta)\log\langle x\rangle \\ &- C|\rho(\eta)|\langle x\rangle^{\alpha-1-\mu} - C|\rho'(\eta)|\langle x\rangle^{\alpha-1-\mu}\log\langle x\rangle. \end{aligned}$$

Note that the first line of the right hand side is positive for $(x,\xi)\in\Omega_R$. Moreover, we observe that $|\xi|\sim|x|^\alpha$ on Ω_R if R is large enough. For $|\eta(x,\xi)|\geq 1/4$, it follows

$$\begin{aligned} H_p(\rho(\eta)\log\langle x\rangle) \geq& 2(\eta\rho(\eta))|x||\xi|\langle x\rangle^{-2} - C|\rho(\eta)|\langle x\rangle^{\alpha-1-\mu} \\ &- C|\rho'(\eta)|\langle x\rangle^{\alpha-1-\mu}\log\langle x\rangle \\ \geq& C\langle x\rangle^{\alpha-1}|\rho(\eta)| - C\langle x\rangle^{\alpha-1-\mu}\log\langle x\rangle|\rho(\eta)| \geq C\langle x\rangle^{\alpha-1}. \end{aligned}$$

by (4.3.1) and (4.3.2). For $|\eta(x,\xi)|\leq 1/4$, we have

$$\begin{aligned} H_p(\rho(\eta)\log\langle x\rangle) \geq& (H_{p_0}\eta)\rho'(\eta)\log\langle x\rangle - C|\rho(\eta)|\langle x\rangle^{\alpha-1-\mu} \\ &- C|\rho'(\eta)|\langle x\rangle^{\alpha-1-\mu}\log\langle x\rangle \\ \geq& C\rho'(\eta)\langle x\rangle^{\alpha-1}\log\langle x\rangle - C|\rho'(\eta)|\langle x\rangle^{\alpha-1-\mu}\log\langle x\rangle \geq C\langle x\rangle^{\alpha-1}. \end{aligned}$$

Thus we complete the proof. □

4.3.2 Fredholm properties

Let $m = m_{R_0}$ be as in subsection 4.3.1, where R_0 is as in Lemma 4.3.1. Moreover, we set $k_\alpha = (\alpha-1)/2$. Let $S^{k,tm(x,\xi)+l}$ be as in Definition 2 and let $\tilde{G}_{k_\alpha,tm}(x,\xi) = \langle x\rangle^{k_\alpha+tm(x,\xi)} + S^{-\infty,-\infty}$ such that $\mathrm{Op}(\tilde{G}_{k_\alpha,tm}) : \mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ is invertible. Existence of such $\tilde{G}_{k_\alpha,tm}$ is proved in Lemma 4.6.2 (see also (4.6.1)). Moreover, the variable order weighted L^2-space $L^{2,k_\alpha+tm(x,\xi)}$ is defined by

$$L^{2,k_\alpha+tm(x,\xi)} = \mathrm{Op}(\tilde{G}_{k_\alpha,tm})^{-1}L^2.$$

By Lemma 4.6.3 (ii), we have

$$L^{2,k_\alpha+tm(x,\xi)} = \mathrm{Op}(\tilde{G}_{0,tm})^{-1}L^{2,k_\alpha}.$$

For $t \neq 0$ and $z \in \mathbb{C}_\pm$, we set

$$P_{tm}(z) = \mathrm{Op}(\tilde{G}_{0,tm})(P-z)\mathrm{Op}(\tilde{G}_{0,tm})^{-1}. \tag{4.3.3}$$

We note that the operator P on $L^{2,k_\alpha+tm(x,\xi)}$ is unitary equivalent to P_{tm} on L^{2,k_α}. This is why we study the Fredholm property of $P_{tm}(z)$ instead of P in order to prove Theorem 4.1.1. By the asymptotic expansion, we have

$$P_{tm}(z) = P - z + it\mathrm{Op}(H_p(m\log\langle x\rangle)) + \mathrm{Op}S^{0,-2+0}$$

since $|\xi| \sim |x|^\alpha$ on $\operatorname{supp} m$ and $\tilde{G}_{0,tm} = \langle x\rangle^{tm(x,\xi)} + S^{-\infty,-\infty}$.

Lemma 4.3.2. *We have*

$$-(u, \mathrm{Op}(H_p(m\log\langle x\rangle))u)_{L^2} \geq C\|\mathrm{Op}(a_R)u\|^2_{L^{2,\frac{\alpha-1}{2}}} - C\|u\|^2_{L^{2,-1+0}} + (u,\mathrm{Op}(e)u)_{L^2}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. By the construction, m is supported in $\operatorname{supp} a_R$. Hence we have

$$-H_p(m\log\langle x\rangle) - C\langle x\rangle^{\alpha-1}a_R(x,\xi)^2 - e(x,\xi) \in S^{1,-1+0}.$$

By Lemma 4.3.1 and the sharp Gårding inequality (Lemma 2.2.1 (v)), we obtain the above inequality. □

Lemma 4.3.3. *Set* $\tilde{D}_{tm}(z) = \{u \in L^{2,(\alpha-1)/2} \mid P_{tm}(z)u \in L^{2,(1-\alpha)/2}\}$. *We consider* $\tilde{D}_{tm}(z)$ *as a Banach space with its graph norm. Then* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $\tilde{D}_{tm}(z)$.

Proof. Let $u \in \tilde{D}_{tm}(z)$. We recall that $b_L(x,\xi) = \chi(|x|/L)\chi(|\xi|/L)$ is as in (4.2.2). Since $\mathrm{Op}(b_L)u \to u$ in $L^{2,(\alpha-1)/2}$ and $\mathrm{Op}(b_L)P_{tm}(z)u \to P_{tm}(z)u$ in $L^{2,(1-\alpha)/2}$, it suffices to prove that $[P_{tm}, \mathrm{Op}(b_L)]u \to 0$ in $L^{2,(1-\alpha)/2}$. We learn

$$\begin{aligned}\|[P_{tm}(z),\mathrm{Op}(b_L)]u\|_{L^{2,(1-\alpha)/2}} \leq &\|[P_{tm}(z),\mathrm{Op}(b_L)]\mathrm{Op}(a_R)u\|_{L^{2,(1-\alpha)/2}}\\ &+ \|[P_{tm}(z),\mathrm{Op}(b_L)](1-\mathrm{Op}(a_R))u\|_{L^{2,(1-\alpha)/2}}.\end{aligned}$$

Since $|\xi| \sim |x|^\alpha$ on a_R, it follows that $[P_{tm}(z),\mathrm{Op}(b_L)]\mathrm{Op}(a_R)$ is uniformly bounded in $S^{0,\alpha-1}$ and converges to 0 in $S^{0,(\alpha-1)/2+0}$. Lemma 4.2.2 and $u \in L^{2,(\alpha-1)/2}$ imply

$$\limsup_{L\to\infty} \|[P_{tm}(z),\mathrm{Op}(b_L)]\mathrm{Op}(a_R)u\|_{L^{2,(1-\alpha)/2}} = 0.$$

Moreover, since $u \in L^{2,(\alpha-1)/2}$ with $P_{tm}(z)u \in L^{2,(1-\alpha)/2}$, then the elliptic estimates (Proposition 4.2.1) implies $(1-\mathrm{Op}(a_R))u \in H^{k_1,(1-\alpha)/2+\alpha l_1}$ for $k_1, l_1 \geq 0$ with $k_1+l_1 \leq 2$. In particular,

$$(1-\mathrm{Op}(a_R))u \in \bigcap_{j=1}^{2} H^{j,\frac{\alpha+1}{2}+(j-1)\alpha}.$$

Since $[P_{tm}(z), \mathrm{Op}(b_L)]$ is uniformly bounded in $\sum_{j=0}^{2} S^{1-j, j\alpha-1}$ and converges to 0 in $\sum_{j=0}^{2} S^{1-j+\varepsilon, j\alpha-1+\varepsilon}$ for any $\varepsilon > 0$, then Lemma 4.2.2 gives

$$\limsup_{L\to\infty} \|[P_{tm}(z), \mathrm{Op}(b_L)](1-\mathrm{Op}(a_R))u\|_{L^{2,(1-\alpha)/2}} = 0.$$

This completes the proof. □

Proposition 4.3.4. *Let $I \subset \mathbb{R}$ be a relativity compact interval. Then there exists $C > 0$ such that for $z \in I_{\mathrm{sgn}\, t}$ we have*

$$\|u\|_{L^{2,(\alpha-1)/2}} \leq C\|P_{tm}(z)u\|_{L^{2,(1-\alpha)/2}} + C\|u\|_{H^{-N,-N}},\ u \in \tilde{D}_{tm}(z), \tag{4.3.4}$$

$$\|u\|_{L^{2,(\alpha-1)/2}} \leq C\|P_{tm}(z)^*u\|_{L^{2,(1-\alpha)/2}} + C\|u\|_{H^{-N,-N}},\ u \in \tilde{D}_{tm}(\bar{z}). \tag{4.3.5}$$

Moreover, (4.3.4) *and* (4.3.5) *hold for $z \in I_{-\mathrm{sgn}\, t}$ though the constant $C > 0$ depends on* $\mathrm{Im}\, z$.

Proof. First, we assume $z \in I_{\mathrm{sgn}\, t}$. We prove (4.3.4) only. Since $P_{tm}(z)^* = (P-z)^* - it\mathrm{Op}(H_p(m\log\langle x\rangle)) + \mathrm{Op}S^{0,-2+0}$ holds, (4.3.5) is similarly proved. By Lemma 4.3.3, we may assume $u \in \mathcal{S}(\mathbb{R}^n)$. By Lemma 4.3.2 and $t\mathrm{Im}\, z \geq 0$, then

$$-(\mathrm{sgn}\, t)\mathrm{Im}\, (u, P_{tm}(z)u)_{L^2} \geq C\|\mathrm{Op}(a_R)u\|^2_{L^{2,(\alpha-1)/2}} - C\|u\|^2_{L^{2,-1+0}} + (u, \mathrm{Op}(e)u)_{L^2}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. Since $t\mathrm{Im}\, z \geq 0$, then we have

$$\begin{aligned}\|\mathrm{Op}(a_R)u\|^2_{L^{2,(\alpha-1)/2}} \leq& C\|P_{tm}(z)u\|_{L^{2,(1-\alpha)/2}}\|u\|_{L^{2,(\alpha-1)/2}} + C\|u\|^2_{L^{2,-1+0}} \\ &+ |(u, \mathrm{Op}(e)u)_{L^2}|.\end{aligned} \tag{4.3.6}$$

By the elliptic estimate (Proposition 4.2.1) and the interpolation estimate, we have

$$\begin{aligned}&\|(1-\mathrm{Op}(a_R))u\|^2_{L^{2,(\alpha-1)/2}} + \|u\|^2_{L^{2,-1+0}} + |(u, \mathrm{Op}(e)u)_{L^2}| \\ &\leq C\|\mathrm{Op}(a_R)u\|^2_{L^{2,-1+0}} + C\|P_{tm}(z)u\|^2_{L^{2,(1-\alpha)/2}} + C\|u\|^2_{H^{-N,-N}} \\ &\leq \frac{1}{2}\|\mathrm{Op}(a_R)u\|^2_{L^{2,(\alpha-1)/2}} + C\|P_{tm}(z)u\|^2_{L^{2,(1-\alpha)/2}} + C\|u\|^2_{H^{-N,-N}}.\end{aligned} \tag{4.3.7}$$

By using (4.3.6), (4.3.7) and the Cauchy-Schwarz inequality, we obtain (4.3.4) for $u \in \mathcal{S}(\mathbb{R}^n)$.

Next, we prove that (4.3.4) and (4.3.5) hold for $z \in I_{-\mathrm{sgn}\, t}$ though the constant $C > 0$ depends on $\mathrm{Im}\, z$. In fact, since $(\alpha-1)/2 > (1-\alpha)/2$, then the elliptic estimate and the interpolation inequality implies that for any $\varepsilon_1 > 0$,

$$\begin{aligned}|\mathrm{Im}\, z|\|u\|^2_{L^{2,(1-\alpha)/2}} \leq& \varepsilon_1\|u\|^2_{L^{2,(\alpha-1)/2}} + C\|u\|^2_{L^{2,-N-N\alpha}} \\ \leq& \varepsilon_1\|u\|^2_{L^{2,(\alpha-1)/2}} + C\|P_{tm}(z)\|^2_{L^{2,(1-\alpha)/2}} + C\|u\|^2_{H^{-N,-N}}.\end{aligned}$$

Taking $\varepsilon_1 > 0$ small enough and use (4.3.4) and (4.3.5) for $\bar{z}$, we obtain (4.3.4) for $z \in I_{-\mathrm{sgn}\, t}$. □

Remark 4.3.5. Suppose $t \geq 0$. If $\operatorname{Im} z$ is large enough, then

$$\|u\|_{L^{2,(\alpha-1)/2}} \leq C\|P_{tm}(z)u\|_{L^{2,(\alpha-1)/2}}. \tag{4.3.8}$$

In fact, in (4.3.6), we have a stronger bound:

$$\|\mathrm{Op}(a_R)u\|^2_{L^{2,(\alpha-1)/2}} + \operatorname{Im} z\|u\|^2_{L^2} \leq (\text{RHS of } (4.3.6)).$$

Hence the argument after (4.3.6) implies

$$(1+\varepsilon)\|u\|^2_{L^{2,(\alpha-1)/2}} + \operatorname{Im} z\|u\|^2_{L^2} \leq C\|P_{tm}(z)u\|^2_{L^{2,(\alpha-1)/2}} + C\|u\|^2_{H^{-N,-N}}.$$

We use the trivial bounds $\|u\|_{H^{-N,-N}} \leq \|u\|_{L^2}$ $\|u\|_{H^{-N,-N}} \leq \|u\|_{L^{2,(\alpha-1)/2}}$ and we obtain (4.3.8). Similarly, for $t \leq 0$, (4.3.8) holds if $-\operatorname{Im} z$ is large enough.

Corollary 4.3.6. *The map*

$$P_{tm}(z) : \tilde{D}_{tm}(z) \to L^{2,(1-\alpha)/2} \tag{4.3.9}$$

is a Fredholm operator. Moreover, if $t\operatorname{Im} z \geq 0$ *holds and* $|\operatorname{Im} z|$ *is large enough, then* $P - z$ *is invertible. Furthermore,* (4.3.9) *is an analytic family of Fredholm operators with index zero. Moreover, there exists a discrete set* $T_{\alpha,t} \subset \mathbb{C}$ *such that* (4.3.9) *is invertible for* $z \in \mathbb{C} \setminus T_{\alpha,t}$.

Remark 4.3.7. Remark 4.3.5 implies that $P_{tm}(z)$ is invertible for $t \geq 0$ and for large $\operatorname{Im} z > 0$. In fact, the injectivity of $P_{tm}(z)$ follows from (4.3.8) and the surjectivity follows from the injectivity of $P_{tm}(z)^*$.

Proof. First, we prove that $\dim \operatorname{Ker} P_{tm}(z) < \infty$ is of finite dimension and $\operatorname{Ran} P_{tm}(z)$ is closed. Let a bounded sequence $u_k \in \tilde{D}_{tm}(z)$ such that $P_{tm}(z)u_k$ is convergent in $L^{2,(1-\alpha)/2}$. Due to [30, Proposition 19.1.3], it suffices to prove that u_k has a convergent subsequence in $\tilde{D}_{tm}(z)$. It easily follows from (4.3.4) and the compactness of the inclusion $L^{2,(\alpha-1)/2} \subset H^{-N,-N}$.

Next, we prove that the cokernel of $P_{tm}(z)$ is of finite dimension. To do this, it suffices to prove that the kernel of $P_{tm}(z)^* : L^{2,(\alpha-1)/2} \to \tilde{D}_{tm}(z)^*$ is of finite dimension. By definition, we have

$$\begin{aligned}\operatorname{Ker} P_{tm}(z)^* =& \{u \in L^{2,(\alpha-1)/2} \mid (u, P_{tm}(z)w)_{L^2} = 0,\ \forall w \in \tilde{D}_{tm}(z)\}\\ =& \{u \in L^{2,(\alpha-1)/2} \mid (u, P_{tm}(z)w)_{L^2} = 0,\ \forall w \in \mathcal{S}(\mathbb{R}^n)\},\end{aligned}$$

where we use Lemma 4.3.3 in the second line. If $u \in L^{2,(\alpha-1)/2}$ satisfies $P_{tm}(z)^*u = 0$, then this equality holds in the distributional sense. The claim follows same as in the first half part of this proof.

The invertibility of (4.3.9) when $t\operatorname{Im} z \geq 0$ and when $|\operatorname{Im} z|$ is large follows from Remark 4.3.5 and its dual statement. The analytic Fredholm theorem [79, Theorem D.4] imply existence of $T_{\alpha,t}$ as above.

□

Proof of Theorem 4.1.1. Theorem 4.1.1 follows from (4.3.3) and Corollary 4.3.6.

□

4.4 Proof of Theorem 4.1.3

4.4.1 Outgoing/incoming parametrices

In this subsection, we construct outgoing/incoming parametrices of a solution to $Pu = zu$. Set

$$S^k(\mathbb{R}^n) = \{a \in C^\infty(\mathbb{R}^n \setminus \{0\}) \mid |\partial_x^\beta a(x)| \le C_\beta \langle x\rangle^{k-|\beta|},\ \text{for } |x| > 1\}.$$

Moreover, we frequently use the following notation:

$$\hat{x} = x/|x|.$$

The main result of this subsection is the following theorem.

Theorem 4.4.1. *Fix a signature $\pm$ and $a \in C^\infty(\mathbb{S}^{n-1})$. Then there exists $\varphi_\pm \in S^{1+\alpha}(\mathbb{R}^n)$ such that*

$$\varphi_\pm \mp \frac{|x|^{1+\alpha}}{1+\alpha} \mp z\frac{|x|^{1-\alpha}}{2(1-\alpha)} \in S^{1+\alpha-\mu}(\mathbb{R}^n),\ \mathrm{Im}\,(\varphi_\pm \mp z\frac{|x|^{1-\alpha}}{2(1-\alpha)}) \in S^0(\mathbb{R}^n),$$
$$e^{-i\varphi}(-\Delta - |x|^{2\alpha} + \mathrm{Op}(V) - z)(e^{i\varphi}b) \in S^{-\frac{n+1-\alpha}{2}-\mu}(\mathbb{R}^n),$$

where $b(x) = |x|^{-\frac{n-1+\alpha}{2}}\bar{\chi}(|x|/R)a(\hat{x}) \in S^{-\frac{n-1+\alpha}{2}}(\mathbb{R}^n)$ and $\hat{x} = x/|x|$.

Theorem 4.4.1 is proved by Propositions 4.4.2 and 4.4.5 below.

Proposition 4.4.2. *Fix a signature $\pm$, $z \in \mathbb{C}$ and $a \in C^\infty(\mathbb{S}^{n-1})$. Set $b(x) = |x|^{-\frac{n-1+\alpha}{2}}\bar{\chi}(|x|/R)a(\hat{x}) \in S^{-\frac{n-1+\alpha}{2}}(\mathbb{R}^n)$. Let $\varphi_{\pm,z} \in S^{1+\alpha}(\mathbb{R}^n)$ be satisfying*

$$\varphi_{\pm,z} \mp \frac{|x|^{1+\alpha}}{1+\alpha} \mp z\frac{|x|^{1-\alpha}}{2(1-\alpha)} \in S^{1+\alpha-\mu}(\mathbb{R}^n),\ \mathrm{Im}\,(\varphi_{\pm,z} \mp z\frac{|x|^{1-\alpha}}{2(1-\alpha)}) \in S^0(\mathbb{R}^n).$$

Then we have

$$\begin{aligned} &e^{-i\varphi_{\pm,z}}(-\Delta - |x|^{2\alpha} + \mathrm{Op}(V) - z)(e^{i\varphi_{\pm,z}}b) \\ &= ((\nabla\varphi)^2 - |x|^{2\alpha} + V(x, \nabla\varphi_{\pm.z}(x)) - z)b(x) + S^{-\frac{n+1-\alpha}{2}-\mu}(\mathbb{R}^n). \end{aligned}$$

Proposition 4.4.2 directly follows from Lemmas 4.4.3 and 4.4.4 below.

Lemma 4.4.3. *Fix a signature $\pm$ and $z \in \mathbb{C}$. Let $\varphi_{\pm,z}$ and b be as in the above proposition. Then*

$$e^{-i\varphi_{\pm,z}}(-\Delta - |x|^{2\alpha} - z)(e^{i\varphi_{\pm,z}}b) = ((\nabla\varphi_{\pm,z})^2 - |x|^{2\alpha} - z)b + S^{-\frac{n+1-\alpha}{2}-\mu}(\mathbb{R}^n).$$

Proof. Set $k = -\frac{n-1+\alpha}{2}$, then we note $k + \alpha - \mu - 1 = -\frac{n+1-\alpha}{2} - \mu$. We write $\varphi = \varphi_\pm$ and $\varphi_0 = \varphi_{0,\pm} = \pm|x|^{1+\alpha}/(1+\alpha)$. By a simple calculation, we have

$$e^{-i\varphi}(-\Delta - |x|^{2\alpha} - z)(e^{i\varphi}b) = ((\nabla\varphi)^2 - |x|^{2\alpha} - z)b - i(2\nabla\varphi\cdot\nabla b + (\Delta\varphi)b) - (\Delta b).$$

Due to $b \in S^k$ and $\varphi - \varphi_0 \in S^{1+\alpha-\mu}$, we observe

$$\Delta b,\ 2\nabla(\varphi - \varphi_0)\cdot\nabla b + \Delta(\varphi - \varphi_0)b \in S^{k+\alpha-\mu-1}(\mathbb{R}^n).$$

Thus, it suffices to prove

$$2\nabla\varphi_0\cdot\nabla b + (\Delta\varphi_0)b \in S^{k+\alpha-\mu-1}(\mathbb{R}^n).$$

Since $\nabla\varphi_0 = \pm|x|^{\alpha-1}x,\ \Delta\varphi_0 = \pm(n-1+\alpha)|x|^{\alpha-1}$, we obtain

$$\begin{aligned}2\nabla\varphi_0\cdot\nabla b + (\Delta\varphi_0)b &= \pm\,(2|x|^\alpha\partial_r b(x) + (n-1+\alpha)|x|^{n-1+\alpha}b)\\ &= \pm\,\frac{2}{R}|x|^{k+\alpha}a(\hat{x})(\bar{\chi})'(|x|/R) \in C_c^\infty(\mathbb{R}^n) \subset S^{k+\alpha-\mu-1}(\mathbb{R}^n).\end{aligned}$$

□

Lemma 4.4.4. *Let $k \in \mathbb{R}$, $\varphi \in S^{1+\alpha}(\mathbb{R}^n)$ and $b \in S^k(\mathbb{R}^n)$. Set $\psi(x,y) = \int_0^1 \nabla\varphi(tx + (1-t)y)dt$. Then*

$$e^{-i\varphi(x)}\mathrm{Op}(V)e^{i\varphi}b(x) = V(x, \nabla\varphi(x))b(x) + L(x),$$

where $L \in S^{k+\alpha-\mu-1}(\mathbb{R}^n)$ is defined by

$$L(x) = D_y(\partial_\xi V(x, \psi(x,y))b(x))|_{x=y} + (D_y^2(\partial_\xi^2 V(\frac{x+y}{2}, 0)b(y))|_{x=y}.$$

Proof. By a simple calculation, we have

$$\begin{aligned}e^{-i\varphi(x)}\mathrm{Op}(V)(e^{i\varphi}b)(x) =& \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi - i(\varphi(x)-\varphi(y))}V(\frac{x+y}{2},\xi)b(y)dyd\xi\\ =& \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}V(\frac{x+y}{2},\xi+\psi(x,y))b(y)dyd\xi\\ =& V(x,\nabla\varphi(x))b(x) + L(x),\end{aligned}$$

where

$$L(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}(V(\frac{x+y}{2},\xi+\psi(x,y)) - V(\frac{x+y}{2},\psi(x,y)))b(y)dyd\xi.$$

Thus it suffices to compute L. Since V is a polynomial of degree 2 with respect to ξ-varibble, we have

$$\begin{aligned}V(\frac{x+y}{2},\xi+\psi(x,y)) =& V(\frac{x+y}{2},\psi(x,y)) + \xi\cdot\partial_\xi V(\frac{x+y}{2},\psi(x,y))\\ &+ \frac{1}{2}\xi\cdot\partial_\xi^2 V(\frac{x+y}{2},\psi(x,y))\cdot\xi.\end{aligned}$$

Note that $\partial_\xi^2 V(\frac{x+y}{2}, \psi(x,y)) = \partial_\xi^2 V(\frac{x+y}{2}, 0)$ since V is a second order differential operator. By integrating by parts, $L(x)$ is written as

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}(\xi\cdot\partial_\xi V(\frac{x+y}{2},\psi(x,y)) + \xi\cdot\partial_\xi^2 V(\frac{x+y}{2},\psi(x,y))\cdot\xi)b(y)dyd\xi\\ &= D_y(\partial_\xi V(x,\psi(x,y))b(x))|_{x=y} + (D_y^2(\partial_\xi^2 V(\frac{x+y}{2},0)b(y))|_{x=y} \in S^{k+\alpha-\mu-1}(\mathbb{R}^n).\end{aligned}$$

This completes the proof.

□

Now we find approximate solutions to the eikonal equations:

$$(\nabla\varphi(x))^2 - |x|^{2\alpha} + V(x,\nabla\varphi(x)) - z = 0. \tag{4.4.1}$$

In [19], solutions to eikonal equations is used for constructing eigenfunctions of a usual Schrödinger operator $-\Delta + V$ with a long range perturbation. Isozaki [33] proved the existence of solutions to eikonal equations for $-\Delta + V$ by using the estimates for the classical trajectories. In our case, we cannot directly apply this strategy since the classical trajectories may blow up at finite time. Instead, we use iteration and construct the approximate solutions to (4.4.1) even for $z \notin \mathbb{R}$.

Proposition 4.4.5. *Set* $\varphi_{0,\pm}(x) = \varphi_{0,\pm}(x,z) = \pm\frac{|x|^{\alpha+1}}{1+\alpha} \pm z\frac{|x|^{1-\alpha}}{2(1-\alpha)}$. *Let* $R \geq 1$ *be large enough. Then for any integer* $N > 0$, *there exists* $\varphi_{N,\pm} \in S^{1+\alpha}(\mathbb{R}^n)$ *such that* $\varphi_{N,\pm} - \varphi_{N-1,\pm} \in S^{1+\alpha-N\mu}(\mathbb{R}^n)$, $\mathrm{Im}\,(\varphi_{N,\pm} - \varphi_{0,\pm}) \in S^0(\mathbb{R}^n)$, $\varphi_{N,\pm} - \varphi_{N-1,\pm}$ *is supported in* $|x| \geq R$ *and*

$$(\nabla\varphi_{N,\pm}(x))^2 - |x|^{2\alpha} + V(x,\nabla\varphi_{N,\pm}(x)) - z \in S^{2\alpha-(N+1)\mu}(\mathbb{R}^n). \tag{4.4.2}$$

Remark 4.4.6. Such construction of φ_N succeeds for $0 < \alpha < 1$ and $z \in \mathbb{R}$. For $\alpha = 1$ and $z \in \mathbb{R}$, we have to replace $\varphi_{0,\pm}(x,z) = \pm\frac{|x|^2}{2} \pm \frac{z}{2}\log|x|$.

Proof. We find $\varphi_{N,\pm} \in S^{1+\alpha}(\mathbb{R}^n)$ of the form

$$\varphi_{N,\pm}(x) = \varphi_{0,\pm}(x) + \sum_{j=1}^N e_{j,\pm}(x),\ \ e_{j,\pm} \in S^{1+\alpha-j\mu}.$$

By a simple calculation, we have

$$\begin{aligned}&(\nabla\varphi_{N,\pm}(x))^2 - |x|^{2\alpha} + V(x,\nabla\varphi_{N,\pm}(x)) - z\\ =&\frac{z^2}{4}|x|^{-2\alpha} + 2\sum_{j=1}^N \nabla\varphi_{0,\pm}\cdot\nabla e_{j,\pm} + \sum_{j,k=1}^N \nabla e_{j,\pm}\cdot\nabla e_{k,\pm} + V(x,\nabla\varphi_{N,\pm}(x)).\end{aligned}$$

We set

$$\begin{aligned}
e_{1,\pm}(x) =& \mp \int_{\frac{R}{2}}^{|x|} \frac{1}{2s^\alpha}(V(s\hat{x}, \nabla\varphi_{0,\pm}(s\hat{x})) - \frac{z^2}{4}s^{-2\alpha})ds\bar{\chi}_R(x) \in S^{\alpha+1-\mu}(\mathbb{R}^n) \\
\varphi_{1,\pm}(x) =& \varphi_{0,\pm}(x) + e_{1,\pm}(x).
\end{aligned}$$

Note Im $e_{1,\pm} \in S^{1-\alpha-\mu}(\mathbb{R}^n)$. Then $(\nabla\varphi_{1,\pm}(x))^2 - |x|^{2\alpha} + V(x, \nabla\varphi_{1,\pm}(x)) - z$ is equal to

$$\begin{aligned}
&(\text{Im } \nabla\varphi_{0,\pm}) \cdot \nabla e_{1,\pm} + (\nabla e_{1,\pm})^2 + V(x, \nabla\varphi_{1,\pm}) - V(x, \nabla\varphi_{0,\pm}) \\
&= (\text{Im } \nabla\varphi_{0,\pm}) \cdot \nabla e_{1,\pm} + (\nabla e_{1,\pm})^2 + \int_0^1 \nabla e_{1,\pm} \cdot (\partial_\xi V)(x, \nabla\varphi_{0,\pm} + t\nabla e_{1,\pm})dt,
\end{aligned}$$

and this term belongs to $S^{2\alpha-2\mu}(\mathbb{R}^n)$. In fact, $\nabla\varphi_{0,\pm} + t\nabla e_{1,\pm}(x) = |x|^{\alpha-1}x + O(|x|^{\alpha-\mu})$ and hence $\partial_\xi V(x, \nabla\varphi_{0,\pm} + t\nabla e_{1,\pm}) = O(|x|^{\alpha-\mu})$ uniformly in $0 \le t \le 1$.

For $N \ge 1$, we define $\varphi_N \in S^{\alpha+1}$ and $e_N \in S^{\alpha+1-N\mu}$ inductively as follows:

$$\begin{aligned}
\varphi_{N+1,\pm}(x) =& \varphi_{N,\pm}(x) + e_{N+1,\pm}(x),\ e_{N+1,\pm}(x) = \mp \int_{\frac{R}{2}}^{|x|} \frac{E_{N+1}(s\hat{x})}{2s^\alpha} ds\bar{\chi}_R(|x|), \\
E_{N+1,\pm} =& \sum_{\substack{j+k=N+1,\\ 1\le j,k\le N}} \nabla e_{j,\pm} \cdot \nabla e_{k,\pm} + V(x, \nabla\varphi_{N,\pm}) - V(x, \nabla\varphi_{N-1,\pm}) \\
&- 2(\text{Im } \nabla\varphi_{0,\pm}) \cdot \nabla e_{N,\pm}.
\end{aligned}$$

We note Im $e_{N,\pm} \in S^{1-\alpha-N\mu}(\mathbb{R}^n)$. For $|x| \ge 2R$, we have

$$\begin{aligned}
(\nabla\varphi_{N+1,\pm})^2 - |x|^{2\alpha} - z =& (\nabla\varphi_{0,\pm}(x) + \sum_{j=1}^{N+1} \nabla e_{j,\pm})^2 - |x|^{2\alpha} - z \\
\equiv& 2\sum_{j=1}^{N+1} \nabla\varphi_{0,\pm} \cdot \nabla e_{j,\pm} + \sum_{m=2}^{N+1} \sum_{j+k=m} \nabla e_{j,\pm} \cdot \nabla e_{k,\pm} \\
=& -V(x, \nabla\varphi_{N,\pm}(x))
\end{aligned}$$

modulo $S^{2\alpha-(N+2)\mu}$. Hence

$$\begin{aligned}
|\nabla\varphi_{N+1,\pm}|^2 - |x|^{2\alpha} + V(x, \nabla\varphi_{N+1}(x)) \equiv& V(x, \nabla\varphi_{N+1}(x)) - V(x, \nabla\varphi_N(x)) \\
\equiv& 0
\end{aligned}$$

modulo $S^{2\alpha-(N+2)\mu}$. Moreover, we have Im $(\varphi_{N,\pm} \mp z\frac{|x|^{1-\alpha}}{2(1-\alpha)}) \in S^0(\mathbb{R}^n)$ since Im $e_{N,\pm} \in S^{1-\alpha-N\mu}(\mathbb{R}^n)$ and $\alpha > 1$. This completes the proof. □

Proof of Theorem 4.4.1. Fix a signature $\pm$. Let $N > 0$ be an integer such that

$$2\alpha - (N+1)\mu < -\frac{n+1-\alpha}{2} - \mu.$$

We take $\varphi = \varphi_\pm = \varphi_{\pm,N}$ as in Proposition 4.4.5. Then Proposition 4.4.2 gives Theorem 4.4.1. □

4.4.2 Construction of the L^2-solutions, proof of Theorem 4.1.3

Now we construct the L^2-solutions to

$$(P-z)u=0,$$

where u is of the form

$$u(x)=u_0(x)+u_1(x),\ u_0(x)=e^{i\varphi_-(x)}b(x),\ u_1\in L^{2,\frac{\alpha-1}{2}+tm(x,\xi)}. \tag{4.4.3}$$

Proof of Theorem 4.1.3. Set $\tilde{V}(x,\xi)=V(x,\xi)-(\langle x\rangle^{2\alpha}-|x|^{2\alpha})\bar{\chi}(2|x|/R)$ for $R>0$. Let $\varphi_-\in S^{1+\alpha}$ and $b=|x|^{-\frac{n-1+\alpha}{2}}\bar{\chi}(|x|/R)a(\hat{x})$ be as in Theorem 4.4.1 with $\tilde{V}$, where $a\in C^\infty(\mathbb{S}^{n-1})\setminus\{0\}$. Since $\bar{\chi}(2|x|/R)\bar{\chi}(|x|/R)=\bar{\chi}(|x|/R)$ and $S^{-\frac{n+1-\alpha}{2}-\mu}(\mathbb{R}^n)\subset L^{2,\frac{1-\alpha+\mu}{2}}$, we have

$$(P-z)(e^{i\varphi_-}b)\in L^{2,\frac{1-\alpha+\mu}{2}}. \tag{4.4.4}$$

Now we take $0<t<\min(\mu/2,(\alpha-1)/2)$ and $m=m_{R_0}$ be as in subsection 4.3.1, where R_0 is as in Lemma 4.3.1. Since

$$L^{2,(1-\alpha)/2+tm(x,\xi)}\subset L^{2,\frac{1-\alpha+\mu}{2}},\ z\in\mathbb{C}\setminus T_{\alpha,t},$$

then there exists $u_1\in L^{2,(\alpha-1)/2+tm(x,\xi)}$ such that

$$(P-z)u_1=-(P-z)(e^{i\varphi_-}b).$$

by Theorem 4.1.1. We set $u=u_1+e^{i\varphi_-}b\in L^2$, then u satisfies $(P-z)u=0$ since $t<(\alpha-1)/2$. Finally, we prove $u\neq 0$. In order to prove this, we use the wavefront condition of u_1 and $e^{i\varphi_-}b$.

Lemma 4.4.7. *Set $u_0=e^{i\varphi_-}b$, where $b(x)=|x|^{-\frac{n-1+\alpha}{2}}\bar{\chi}(|x|/R)a(\hat{x})$ and $a\in C^\infty(\mathbb{S}^{n-1})\setminus\{0\}$. Let $b_{R_1,\delta}(x,\xi)=\chi((\eta(x,\xi)+1)/\delta)a_{R_1}(x,\xi)$ and $A_{R_1,\delta}=\mathrm{Op}(b_{R_1,\delta})$ for $0<\delta<1$ small enough and $R_1\geq 1$ large enough. Then $A_{R_1,\delta}u_0\notin L^{2,\frac{\alpha-1}{2}}$.*

Proof. By (4.4.4), Proposition 4.2.1 implies that $(1-\mathrm{Op}(a_R))u_0\in L^{2,(\alpha-1)/2}$. Moreover, by a simple calculation, we have

$$|x|^{-\alpha-1}(x\cdot D_x-x\cdot\partial_x\varphi_-(x))u_0\in\bigcap_{\varepsilon>0}L^{2,\frac{\alpha-1}{2}+1-\varepsilon}\subset L^{2,\frac{\alpha-1}{2}}. \tag{4.4.5}$$

Note that if r_1,δ are small and R_1 is large, for $(x,\xi)\in\mathrm{supp}\ (a_{R_1}-b_{R_1,\delta})$

$$|x\cdot\xi-x\cdot\partial_x\varphi_-(x)|\geq C|x|^{1+\alpha}.$$

Since $u_0\notin L^{2,(\alpha-1)/2}$ and $u_0\in\cap_{\varepsilon>0}L^{2,(\alpha-1)/2-\varepsilon}$, we have

$$\begin{aligned}\mathrm{Op}(a_{R_1}-b_{R_1,\delta})u_0=&\mathrm{Op}\Big(\frac{a_{R_1}-b_{R_1,\delta}}{x\cdot\xi-x\cdot\partial_x\varphi_-(x)}|x|^{1+\alpha}\Big)\\&\cdot|x|^{-1-\alpha}(x\cdot D_x-x\cdot\partial_x\varphi_-(x))u_0+L^{2,\frac{\alpha-1}{2}}\in L^{2,\frac{\alpha-1}{2}}.\end{aligned}$$

by a symbol calculus and (4.4.5). Thus if we suppose $A_{R_1,\delta}u_0\in L^{2,\frac{\alpha-1}{2}}$, then $u_0\in L^{2,\frac{\alpha-1}{2}}$ follows. However, this is a contradiction since $u_0\notin L^{2,(\alpha-1)/2}$ by a simple calculation. □

Lemma 4.4.8. *For $0 < \delta < 1$ small enough and $R_1 \geq 1$ large enough, $A_{R_1,\delta}u_1 \in L^{2,\frac{\alpha-1}{2}}$.*

Proof. Note that $u_1 \in L^{2,(\alpha-1)/2-tm(x,\xi)} = \mathrm{Op}(\tilde{G}_{(\alpha-1)/2,-tm})^{-1}L^2$, $0 < t < (\alpha-1)/2$ and $\tilde{G}_{(\alpha-1)/2,-tm} = \langle x\rangle^{(\alpha-1)/2-tm(x,\xi)}$ by (4.6.1). Moreover, we note $m(x,\xi) = -1$ on supp $b_{R_1,\delta}$ if $0 < \delta < 1$ is small enough and $R_1 \geq 1$ is large enough. Thus $A_{R_1,\delta}u \in L^{2,(\alpha-1)/2}$. □

By the above two lemmas, we obtain $u = u_0 + u_1 \neq 0$. This completes the proof of Theorem 4.1.3.

□

Finally, we prove that there are many eigenfunctions associated with $\lambda \in \mathbb{C} \setminus T_{\alpha,t}$.

Proposition 4.4.9. *Suppose that $a, a' \in C^\infty(\mathbb{S}^{n-1})$ are linearly independent. Let $u, u' \in L^2 \setminus \{0\}$ be corresponding eigenfunctions as in* (4.4.3)*. Then u, u' are also linearly independent.*

Proof. By (4.4.3) and Lemma 4.4.8, we write

$$\begin{aligned} u(x) =& e^{i\varphi_-(x)}|x|^{-\frac{n-1+\alpha}{2}}\bar{\chi}(|x|/R)a(\hat{x}) + u_1(x), \\ u'(x) =& e^{i\varphi_-(x)}|x|^{-\frac{n-1+\alpha}{2}}\bar{\chi}(|x|/R)a'(\hat{x}) + u_1'(x), \end{aligned}$$

where $u_1, u_1' \in L^2$ satisfy $A_{R_1,\delta}u_1, A_{R_1,\delta}u_1' \in L^{2,\frac{\alpha-1}{2}}$, where $A_{R_1,\delta}$ is defined in Lemma 4.4.7. Suppose that $L, L' \in \mathbb{C}$ satisfy

$$Lu(x) + L'u'(x) = 0,\ x \in \mathbb{R}^n. \tag{4.4.6}$$

It suffices to prove that $La(\hat{x}) + L'a'(\hat{x}) = 0$ for $\hat{x} \in S^{n-1}$. Suppose $La(\hat{x}) + L'a'(\hat{x}) \neq 0$ for some $\hat{x} \in S^{n-1}$. By Lemma 4.4.7, we have

$$A_{R_1,\delta}(e^{i\varphi_-(x)}|x|^{-\frac{n-1+\alpha}{2}}\bar{\chi}(|x|/R)(La(\hat{x}) + L'a'(\hat{x}))) \notin L^{2,\frac{\alpha-1}{2}}. \tag{4.4.7}$$

(4.4.6) and (4.4.7) imply

$$A_{R_1,\delta}(Lu + L'u') \notin L^{2,\frac{\alpha-1}{2}}.$$

This is a contradiction. □

4.5 Proof of Theorem 4.1.6 and Corollary 4.1.9

4.5.1 Proof of Theorem 4.1.6

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Lemma 4.5.1. *Let $\alpha > 1$. For $\delta > 0$, there exists $C > 0$ such that*

$$\|\mathrm{Op}(a_{2R})u\|_{L^{2,\frac{\alpha-1-\delta}{2}}} \leq C\|Pu\|_{L^2} + C\|u\|_{L^2} \tag{4.5.1}$$

for $u \in D^{\alpha}_{\min}$, where we recall that a_{2R} is as in (4.2.2).

Proof. First, we prove (4.5.1) for $u \in \mathcal{S}(\mathbb{R}^n)$. We may assume $0 < \delta < \mu$. Set

$$b_R(x,\xi) = a_{2R}(x,\xi)^2 \frac{x \cdot \xi}{|x||\xi|} \int_1^{|x|/R} s^{-1-\delta} ds \in S^{0,0}.$$

We note that $|x| > 2R$, $|\xi| \geq 2R$ and $|x|^{\alpha} \sim |\xi|$ hold for $(x,\xi) \in \operatorname{supp} b_R$. For $(x,\xi) \in \operatorname{supp} b_R$, we have

$$\begin{aligned} H_{p_0}\left(\frac{x \cdot \xi}{|x||\xi|} \int_1^{|x|/R} s^{-1-\delta} ds\right) =& 2\frac{|x|^2|\xi|^2 - (x\cdot\xi)^2}{|x||\xi|}\left(\frac{1}{|x|^2} + \alpha\frac{|x|^{2\alpha-2}}{|\xi|^2}\right) \int_1^{\frac{|x|}{R}} \frac{1}{s^{1+\delta}} ds \\ &+ 2R^{\delta} \frac{(x\cdot\xi)^2}{|x|^{3+\delta}|\xi|} \\ \geq& C\langle x\rangle^{\alpha-1-\delta} \end{aligned}$$

with $C > 0$ if $R > 0$ is large enough. Since $H_V b_R \in S^{0,\alpha-1-\mu}$ and $0 < \delta < \mu$, we see

$$H_p b_R \geq C\langle x\rangle^{\alpha-1-\delta} a_{2R}^2 + e_R,$$

where $e_R \in S^{0,\alpha-1}$ is supported away from the elliptic set of P. By the sharp Gårding inequality (Lemma 2.2.1 (v)), we have

$$(u, [P, i\mathrm{Op}(b_R)]u)_{L^2} \geq C\|\mathrm{Op}(a_{2R})u\|^2_{L^{2,\frac{\alpha-1-\delta}{2}}} + (u, \mathrm{Op}(e_R)u)_{L^2} - C\|u\|^2_{H^{-\frac{1}{2},\frac{\alpha}{2}-1}} \tag{4.5.2}$$

for any $u \in \mathcal{S}(\mathbb{R}^n)$. Take $R_1 \geq 1$ such that $a_{2R}a_{R_1} = a_{2R}$. Substituting $\mathrm{Op}(a_{R_1})$ into (4.5.2) and using the disjoint support property and a support property of a_{R_1}, then we have

$$(u, [P, i\mathrm{Op}(b_R)]u)_{L^2} \geq C\|\mathrm{Op}(a_{2R})u\|^2_{L^{2,\frac{\alpha-1-\delta}{2}}} + (u, \mathrm{Op}(e_R)u)_{L^2} - C\|u\|^2_{L^2}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ with some $C > 0$. Using the elliptic estimate Proposition 4.2.1 in order to estimate the term $(u, \mathrm{Op}(e_R)u)_{L^2}$, we have

$$\|\mathrm{Op}(a_{2R})u\|_{L^{2,\frac{\alpha-1-\delta}{2}}} \leq C\|Pu\|_{L^2} + C\|u\|_{L^2}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ with some $C > 0$. Thus we obtain (4.5.1) for $u \in \mathcal{S}(\mathbb{R}^n)$.

In order to prove (4.5.1) for $u \in D^{\alpha}_{\min}$, it remains to use a standard density argument. Let $u \in D^{\alpha}_{\min}$. By definition of $D^{\alpha}_{\min}$, there exists $u_k \in C_c^{\infty}(\mathbb{R}^n)$ such that $u_k \to u$ and $Pu_k \to Pu$ in $L^2(\mathbb{R}^n)$. Substituting u_k into (4.5.1), we have

$$\sup_k \|\mathrm{Op}(a_{2R})u_k\|_{L^{2,\frac{\alpha-1-\delta}{2}}} < \infty.$$

Hence $\mathrm{Op}(a_{2R})u_k$ has a weak*-convergence subsequence in $L^{2,\frac{\alpha-1-\delta}{2}}$ and its accumulation point is $\mathrm{Op}(a_{2R})u$. Thus we obtain $\mathrm{Op}(a_{2R})u \in L^{2,\frac{\alpha-1-\delta}{2}}$ and

$$\|\mathrm{Op}(a_{2R})u\|_{L^{2,\frac{\alpha-1-\delta}{2}}} \leq \liminf_{k\to\infty} \|\mathrm{Op}(a_{2R})u_k\|_{L^{2,\frac{\alpha-1-\delta}{2}}} \leq C\|Pu\|_{L^2} + C\|u\|_{L^2}.$$

□

Combining this lemma with the elliptic estimate Proposition 4.2.1, we have the following proposition:

Proposition 4.5.2. *Let $\alpha > 1$ and $0 \leq \beta_1, \beta_2 \leq 4$ with $\beta_1 + \beta_2 = 1$. For $\delta > 0$, there exists $C > 0$ such that*

$$\|u\|_{H^{\frac{\alpha-1-\delta}{2\alpha}\beta_1, \frac{\alpha-1-\delta}{2}\beta_2}} \leq C\|Pu\|_{L^2} + C\|u\|_{L^2} \tag{4.5.3}$$

for $u \in D^\alpha_{\min}$. In particular, the natural embedding $D^\alpha_{\min} \hookrightarrow L^2(\mathbb{R}^n)$ is compact, where we regard $D^\alpha_{\min}$ as a Banach space equipped with its graph norm.

This proposition gives the proof of Theorem 4.1.6.

4.5.2 Proof of Corollary 4.1.9

Note that D^α_{min} is the domain of the closure of $P|_{C_c^\infty(\mathbb{R})}$. Set

$$D^\alpha = \{u \in L^2(\mathbb{R}^n) \mid Pu \in L^2(\mathbb{R}^n)\}.$$

We easily see that D^α is the domain of $(P|_{C_c^\infty(\mathbb{R})})^*$. Moreover, it follows that the action of $(P|_{C_c^\infty(\mathbb{R})})^*$ on D^α is in the distributional sense. In particular, we have

$$\mathrm{Ker}\ ((P|_{C_c^\infty(\mathbb{R})})^* \mp i) = \mathrm{Ker}\ _{L^2}(P \mp i).$$

We use the following von-Neumann theorem.

Lemma 4.5.3. *[61, Theorem X.2 and Corollary after Theorem X.2] Set $\mathcal{H}_\pm = \mathrm{Ker}\ _{L^2}(P \mp i)$. Then there is a one-to-one correspondence between self-adjoint extensions of $P|_{C_c^\infty(\mathbb{R})}$ and unitary operators from $\mathcal{H}_+$ to $\mathcal{H}_-$. Moreover, for $U \in B(\mathcal{H}_+, \mathcal{H}_-)$ be a unitary operator , we define*

$$D_U = \{v + w + Uw \mid v \in D^\alpha_{min}, w \in \mathcal{H}_+\}.$$

Then P is self-adjoint on D_U.

Now suppose $n = 1$. We prove that each self-adjoint extension of $P|_{C_c^\infty(\mathbb{R})}$ has a discrete spectrum.

Lemma 4.5.4. $\dim \mathcal{H}_+ = \dim \mathcal{H}_- = 2.$

Proof. By [61, Theorem X.1], it suffices to prove that

$$\dim \operatorname{Ker}_{L^2}(P - i\mu) = \dim \operatorname{Ker}_{L^2}(P + i\mu) = 2$$

for some $\mu > 0$. We note $\dim \operatorname{Ker}_{L^2}(P \pm i\mu) \leq 2$ by uniqueness of solutions to ODE. Hence it suffices to prove $\dim \operatorname{Ker}_{L^2}(P \pm i\mu) \geq 2$. We observe $\mathbb{S}^{n-1} = \mathbb{S}^0 = \{\pm 1\}$ and $\dim C^\infty(\{\pm 1\}) = 2$. By Proposition 4.4.9, the discreteness of $T_{\alpha,t}$ imply that for some $\mu \in \mathbb{C} \setminus \mathbb{R} \cup T_{\alpha,t}$ there exists linearly independent functions such that $u_\pm, u'_\pm \in \operatorname{Ker}_{L^2}(P \pm i\mu)$. This gives $\dim \operatorname{Ker}_{L^2}(P \pm i\mu) \geq 2$. □

The following proposition is a variant of [61, Theorem XIII.64]. We do not know whether a self-adjoint extension of $P|_{C_c^\infty(\mathbb{R}^n)}$ is bounded from below. Hence we cannot apply [61, Theorem XIII.64] with our case directly in order to prove Corollary 4.1.9.

Proposition 4.5.5. *Let $\mathcal{H}$ be a separable Hilbert space and A be a self-adjoint operator on $\mathcal{H}$. Suppose that $(A+i)^{-1}$ is a compact operator on $\mathcal{H}$. Then there exists $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{R}$ such that $|\lambda_k| \to \infty$ as $k \to \infty$ and $\sigma(A) = \sigma_d(A) = \{\lambda_j\}_{j=1}^\infty$, where $\sigma(A)$ is the spectrum of A and $\sigma_d(A)$ is the discrete spectrum of A.*

Proof. First, we prove existence of $\lambda \in \mathbb{R} \setminus \sigma(A)$. To prove this, we use a contradiction argument. Suppose $\sigma(A) = \mathbb{R}$. Set $B = (A - i)^{-1}(A + i)^{-1} = f(A)$, where $f(t) = 1/(t^2 + 1)$. By the spectrum mapping theorem, we have $\sigma(B) = [0, 1]$. On the other hand, by the assumption of the lemma, it follows that B is a compact self-adjoint operator on $\mathcal{H}$. This contradicts to $\sigma(B) = [0, 1]$.

We let $\lambda \in \mathbb{R} \setminus \sigma(A)$ and set $T = (A - \lambda)^{-1}$. Since $(A + i)^{-1}$ is compact and since $\lambda \in \mathbb{R}$, it easily follows that T is a compact self-adjoint operator. By the Hilbert-Schmidt theorem [61, Theorem VI.16], there exist a complete orthonormal basis $\varphi_k \in \mathcal{H}$ and a sequence $\mu_k \in \mathbb{R}$ such that

$$T\varphi_k = \mu_k \varphi_k, \ \mu_k \to 0 \text{ as } k \to \infty. \tag{4.5.4}$$

We note that φ_k belongs to the domain of A since $\varphi_k \in \operatorname{Ran} T$ and since $\operatorname{Ran} T$ is contained in the domain of A. Moreover, we observe $\mu_k \neq 0$. In fact, suppose $\mu_k = 0$ holds. Multiplying (4.5.4) by $A - \lambda$, we have $\varphi_k = 0$, which is a contradiction. By (4.5.4), we have

$$A\varphi_k = \lambda_k \varphi_k, \ \lambda_k = \lambda + 1/\mu_k.$$

Note $|\lambda_k| \to \infty$ as $k \to \infty$. Since λ_k has no accumulation point in $\mathbb{R}$, it suffices to prove $\sigma(A) = \{\lambda_k\}_{k=1}^\infty$. To see this, we prove that $A - z$ has a bounded inverse for $z \in \mathbb{R} \setminus \{\lambda_k\}_{k=1}^\infty$. We set

$$R(z)\psi = \sum_{k=1}^\infty \frac{1}{\lambda_k - z} (\varphi_k, \psi)\varphi_k, \ \psi \in \mathcal{H} \tag{4.5.5}$$

and $c = \inf_{k\geq 1} |\lambda_k - z|$. Since λ_k has no accumulation point in $\mathbb{R}$, we have $c > 0$. Thus we have

$$\sum_{k=1}^{\infty} \frac{1}{|\lambda_k - z|^2} |(\varphi_k, \psi)|^2 \leq c^{-2} \sum_{k=1}^{\infty} |(\varphi_k, \psi)|^2.$$

Hence $R(z)$ is a bounded operator on $\mathcal{H}$. Moreover, $(A-z)R(z)\psi = \psi$ holds by (4.5.5). These imply $z \notin \mathbb{R} \setminus \sigma(A)$. Thus we have $\sigma(A) = \{\lambda_j\}_{j=1}^{\infty}$. Moreover, it follows that $\sigma_d(A) = \sigma(A)$ holds since $\dim \mathrm{Ker}\,(A - \lambda_k) = \dim \mathrm{Ker}\,(T - \mu_k) < \infty$. □

By virtue of Lemma 4.5.4 and [61, Corollary after Theorem X.2], it follows that $P|_{C_c^\infty(\mathbb{R})}$ has a self-adjoint extension.

Proof of Corollary 4.1.9. Fix $U \in$ be a unitary operator and let D_U be as in Lemma 4.5.3. By virtue of Proposition 4.5.5, it suffices to prove that the inclusion $D_U \subset L^2$ is compact, where we regard D_U as a Hilbert space equipped with the graph norm of P. Let $\varphi_j \in D_U$ be a bounded sequence in D_U:

$$\sup_j (\|\varphi_j\|_{L^2} + \|P\varphi_j\|_{L^2}) < \infty.$$

We only need to prove that φ_j has a convergent subsequence in L^2. We write $\varphi_j = u_j + v_j + Uv_j$, where $u_j \in D_{min}^{\alpha}$ and $v_j \in \mathcal{H}_+$. By [61, Lemma before Theorem X.2], we see that

$$\begin{aligned} 0 = (u_j, v_j)_{L^2} + (Pu_j, Pv_j)_{L^2} =& (v_j, Uv_j)_{L^2} + (Pv_j, PUv_j)_{L^2} \\ =& (u_j, Uv_j)_{L^2} + (Pu_j, PUv_j)_{L^2}. \end{aligned}$$

Therefore, u_j and v_j are bounded in D_U. Since $u_j \in D_{min}^{\alpha}$, it follows that u_j has a convergent subsequence $\{u_{j_k}\}$ in L^2. Moreover, we see that $v_{j_k} \in \mathcal{H}_+$ has a convergent subsequence in L^2 due to the finiteness of the dimension of $\mathcal{H}_+$. Thus we conclude that φ_j has a convergent subsequence in L^2. □

4.6 Variable order spaces

In this section, we give a construction of variable order weighted L^2-spaces. Here, we follow the argument in [15]. See [3, Appendix A] for other ways of constructions.

Let $m \in S^{0,0}$ be real-valued and $k, t \in \mathbb{R}$. Suppose $|m(x,\xi)| \leq 1$ for $(x,\xi) \in \mathbb{R}^{2n}$. Set $G_{k,tm}(x,\xi) = \langle x\rangle^{k+tm(x,\xi)}$. Set $l(x) = \langle \log\langle x\rangle\rangle$.

Definition 2. For $a \in C^\infty(\mathbb{R}^{2n})$, we say that for $a \in S^{s,k+tm(x,\xi)}$ if

$$|\partial_x^{\gamma_1}\partial_\xi^{\gamma_2} a(x,\xi)| \leq C_{\gamma_1\gamma_2} l(x)^{|\gamma_1|+|\gamma_2|} \langle x\rangle^{k+tm(x,\xi)-|\gamma_1|} \langle\xi\rangle^{s-|\gamma_2|}$$

for $\gamma_1, \gamma_2 \in \mathbb{N}^n$.

Note that $G_{k,tm} \in S^{0,k+tm(x,\xi)}$.

Lemma 4.6.1. *An unbounded operator* $\mathrm{Op}(G_{k,tm})$ *on* $L^2(\mathbb{R}^n)$ *with domain* $\mathcal{S}(\mathbb{R}^n)$ *admits a self-adjoint extension.*

Proof. By virtue of [61, Theorem X.23], it suffices to prove that $\mathrm{Op}(G_{k,tm})$ is bounded below in $\mathcal{S}(\mathbb{R}^n)$. We note $G_{k,tm}(x,\xi) = G_{k/2,tm/2}(x,\xi)^2$. By the standard construction (see [15, Lemma 13]), there exists $R_j \in S^{-j,k/2-j+0+tm(x,\xi)/2}$ such that

$$\begin{aligned}&(G_{k/2,tm/2}(x,\xi)^2 + \sum_{j=1}^{N} R_j)^*(G_{k/2,tm/2}(x,\xi)^2 + \sum_{j=1}^{N} R_j)\\ &\in S^{-(N+1),k-(N+1)+0+tm(x,\xi)},\end{aligned}$$

where $(\cdot)^*$ denotes the adjoint symbol. By the Borel summation theorem, we have

$$G_{k,tm}(x,\xi) = b^*b + e,\ b \in S^{k/2+tm(x,\xi)/2},\ e \in S^{-\infty,-\infty}.$$

Thus we obtain

$$(u, \mathrm{Op}(G_{k,tm})u) \geq (u, \mathrm{Op}(e)u) \geq -C\|u\|_{L^2}^2,\ u \in \mathcal{S}(\mathbb{R}^n).$$

□

We denote a self-adjoint extension of $\mathrm{Op}(G_{k,tm})$ in $L^2(\mathbb{R}^n)$ by $G(t)$ and its domain by $D_{G(t)}$.

Lemma 4.6.2. *There exists* $R_1(t) \in \mathrm{Op}S^{-\infty,-\infty}$ *such that* $\mathrm{Op}(G_{k,tm}) + R_1(t)$ *is invertible in* $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$*. Moreover, its inverse is a pseudodifferential operator with its symbol in* $S^{0,-k-tm(x,\xi)}$*. Moreover, the symbol of its inverse is* $G_{-k,-tm} + S^{-1,-k-1-tm(x,\xi)+0}$.

Proof. We follow the argument as in [15, Appendix Lemma 12]. We decompose $L^2 = \overline{\mathrm{Ran}_{L^2} G(t)} \oplus \mathrm{Ker}_{L^2} G(t)$. We denote the orthogonal projection into $\mathrm{Ker}_{L^2} G(t)$ by $\pi(t) : L^2 \to \mathrm{Ker}_{L^2} G(t)$. By the standard parametrix construction of $G(t)$, we see that $\mathrm{Ker}_{L^2} G(t) \subset \mathcal{S}(\mathbb{R}^n)$ and $\mathrm{Ker}_{L^2} G(t)$ is of finite dimension. This implies $\pi(t) \in \mathrm{Op}S^{-\infty,-\infty}$. We define $\tilde{G}(t) = G(t)(I - \pi(t)) + \pi(t) \in \mathrm{Op}S^{0,k+tm(x,\xi)}$. We observe that $\tilde{G}(t) : D_{G(t)} \to L^2$ is invertible. We set $R_1(t) = (I - G(t))\pi(t) \in \mathrm{Op}S^{-\infty,-\infty}$, then $\tilde{G}(t) = G(t) + R_1(t)$.

We show that $\tilde{G}(t)$ is invertible in $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. This map is injective since $\tilde{G}(t)$ is injective in $D_{G(t)} \to L^2$. Next, we prove that $\tilde{G}(t) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is surjective. To see this, let $f \in \mathcal{S}(\mathbb{R}^n)$. Since $\tilde{G}(t) : D_{G(t)} \to L^2$ is invertible, there exists $u \in D_{G(t)}$ such that $G(t)u = f$. By using existence of the parametrix of $\tilde{G}(t)$, we obtain $u \in \mathcal{S}(\mathbb{R}^n)$.

Finally, we show that the inverse of $\tilde{G}(t)$ belongs to $\mathrm{Op}S^{0,-k-tm(x,\xi)}$ and its symbol is $G_{-k,-tm} + S^{-1,-k-1-tm(x,\xi)+0}$. Let $Q(t)$ is the parametrix of $\tilde{G}(t)$: $Q(t)\tilde{G}(t) = I + R_2(t)$, where $R_2(t) \in \mathrm{Op}S^{-\infty,-\infty}$. Then the symbol of $Q(t)$ is $G_{-k,-tm} + S^{-1,-k-1-tm(x,\xi)+0}$. Moreover, we observe

$$Q(t) = Q(t)\tilde{G}(t)\tilde{G}(t)^{-1} = \tilde{G}(t)^{-1} + R_2(t)\tilde{G}(t)^{-1}$$

in $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. By the open mapping theorem, $\tilde{G}(t)^{-1}$ is continuous in $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Thus we have $R_2(t)\tilde{G}(t)^{-1} \in \mathrm{Op}S^{-\infty,-\infty}$. We conclude that $\tilde{G}(t) = Q(t) - R_2(t)\tilde{G}(t) \in \mathrm{Op}S^{0,-k-tm(x,\xi)}$ and its symbol is $G_{-k,-tm(x,\xi)} + S^{-1,-k-1-tm(x,\xi)+0}$.

□

Let $\tilde{G}_{k,tm} \in S^{0,k+tm(x,\xi)}$ such that

$$\mathrm{Op}(\tilde{G}_{k,tm}) = \mathrm{Op}(G_{k,tm}) + R_1(t). \tag{4.6.1}$$

By Lemma 4.6.2 and duality, $\mathrm{Op}(\tilde{G}_{k,tm}) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is also invertible.

Now we define the variable order weighted L^2 space by

$$L^{2,k+tm(x,\xi)} = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathrm{Op}(\tilde{G}_{k,tm})u \in L^2(\mathbb{R}^n)\} \tag{4.6.2}$$

for $k \in \mathbb{R}$ and its inner metric by

$$(u, v)_{L^{2,k+tm(x,\xi)}} = (\mathrm{Op}(\tilde{G}_{k,tm})u, \mathrm{Op}(\tilde{G}_{k,tm})v)_{L^2}.$$

Then $L^{2,k+tm(x,\xi)}$ is a Hilbert space.

We state some properties of $L^{2,k+tm(x,\xi)}$.

Lemma 4.6.3. *(i)* $(L^{2,k+tm(x,\xi)})^* = L^{2,-k-tm(x,\xi)}$.

(ii) For $u \in \mathcal{S}'(\mathbb{R}^n)$, $u \in L^{2,k+tm(x,\xi)}$ if and only if $\langle x\rangle^k u \in L^{2,tm(x,\xi)}$. Moreover, there exists $C > 0$ such that $u \in L^{2,k+tm(x,\xi)}$

$$C^{-1}\|\langle x\rangle^k u\|_{L^{2,tm(x,\xi)}} \le \|u\|_{L^{2,k+tm(x,\xi)}} \le C\|\langle x\rangle^k u\|_{L^{2,tm(x,\xi)}}.$$

Proof. *(i)* This follows from the fact that the symbol of the inverse of $\mathrm{Op}(\tilde{G}_{k,tm})$ belongs to $S^{0,-k-tm(x,\xi)}$.

(ii) Note that $\mathrm{Op}(\tilde{G}_{0,tm})\langle x\rangle^k \mathrm{Op}(\tilde{G}_{k,tm})^{-1}$ and $\mathrm{Op}(\tilde{G}_{k,tm})\langle x\rangle^{-k}\mathrm{Op}(\tilde{G}_{0,tm})^{-1}$ is bounded in L^2 by Lemma 4.6.2. We are done.

□

Chapter 5

Discrete Schrödinger operators

5.1 Introduction

We consider the discrete Schrödinger operators:

$$H = H_0 + V(x) \quad \text{on} \quad \mathcal{H} = l^2(\mathbb{Z}^d),$$

where H_0 is the negative discrete Laplacian

$$H_0 u(x) = -\sum_{|x-y|=1} (u(y) - u(x)),$$

and V is a real-valued function on $\mathbb{Z}^d$. In this note, we study uniform bounds of the Birman-Schwinger operators:

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \left\| |V|^{\frac{1}{2}} (H_0 - z)^{-1} |V|^{\frac{1}{2}} \right\|_{B(\mathcal{H})} < \infty. \tag{5.1.1}$$

As an application, we give sufficient conditions for V that H_0 and H are unitarily equivalent. We also give examples of potentials for which (5.1.1) does not hold. Though this subject is studied in a recent preprint [1], their assumptions are stronger than ours and some proofs seem incomplete. One of the purposes of this note is to generalize their results and give an alternative proof.

We denote the Fourier expansion by $\mathcal{F}_d$:

$$\hat{u}(\xi) = \mathcal{F}_d u(\xi) = \sum_{x \in \mathbb{Z}^d} e^{-2\pi i x \cdot \xi} u(x), \quad \xi \in \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d.$$

Then it follows that

$$\mathcal{F}_d H_0 u(\xi) = h_0(\xi) \mathcal{F}_d u(\xi),$$

where $h_0(\xi) = 4 \sum_{j=1}^d \sin^2(\pi \xi_j)$, and hence $\sigma(H_0) = [0, 4d]$. We denote the set of the critical points of h_0 by Γ:

$$\Gamma = \{\xi \in \mathbb{T}^d \,|\, \nabla h_0(\xi) = 0\} = \{\xi \in \mathbb{T}^d \,|\, \xi_j \in \{0, 1/2\},\ j = 1, ..., d\}.$$

We call $\xi \in \Gamma$ an elliptic threshold if ξ attains maximum or minimum of h_0 and a hyperbolic threshold otherwise.

For a measure space (X, μ), $L^{p,r}(X, \mu)$ denotes the Lorentz space for $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$:

$$\|f\|_{L^{p,r}(X)} = \begin{cases} p^{\frac{1}{r}}(\int_0^\infty \mu(\{x \in X \,|\, |f(x)| > \alpha\})^{\frac{r}{p}} \alpha^{r-1} d\alpha)^{\frac{1}{r}}, & r < \infty, \\ \sup_{\alpha > 0} \alpha \mu(\{x \in X \,|\, |f(x)| > \alpha\})^{\frac{1}{p}}, & r = \infty, \end{cases}$$

$$L^{p,r}(X, \mu) = \{f : X \to \mathbb{C} \,|\, f : \text{measurable}, \|f\|_{L^{p,r}(X)} < \infty\}.$$

Moreover, we denote $L^{p,r}(\mathbb{R}^d) = L^{p,r}(\mathbb{R}^d, \mu_L)$ and $l^{p,r}(\mathbb{Z}^d) = L^{p,r}(\mathbb{Z}^d, \mu_c)$, where μ_L is the Lebesgue measure on $\mathbb{R}^d$ and μ_c is the counting measure on $\mathbb{Z}^d$. For a detail, see [25].

First, we state our positive results:

Theorem 5.1.1. *(i) Let $d \geq 4$. If $V \in l^{\frac{d}{3},\infty}(\mathbb{Z}^d)$, then (5.1.1) holds.*

(ii) Let $d \geq 3$. If $|V(x)| \leq C(1 + |x|)^{-2}$ for some $C > 0$, then (5.1.1) holds.

Corollary 5.1.2. *Under the condition of Theorem 5.1.1 (i) or (ii), $H = H_0 + \lambda V$ is unitarily equivalent to H_0 for small $\lambda \in \mathbb{R}$.*

Remark 5.1.3. For Theorem 5.1.1 (ii), we show stronger results in Proposition 5.3.4. For Theorem 5.1.1 (i), we also obtain stronger results: Uniform resolvent estimates in Lorentz spaces as in Proposition 5.3.3.

Remark 5.1.4. In [34] and [67], the authors prove the absence of eigenvalues of $H_0 + \lambda V$ for small $\lambda \in \mathbb{R}$ if $|V(x)| \leq C(1 + |x|)^{-2-\varepsilon}$ for some $C > 0$ and $\varepsilon > 0$ with $d = 3$ and $V \in l^{\frac{d}{3}}(\mathbb{Z}^d)$ with $d \geq 4$ respectively. In [1], (5.1.1) is proved under stronger assumptions: $|V(x)| \leq C\langle x\rangle^{-2(d+3)}$ with $d \geq 3$. Moreover, in [52], (5.1.1) is established for $V \in l^p(\mathbb{Z}^d)$ with $1 \leq p < 6/5$ if $d = 3$ and $1 \leq p < 3d/(2d + 1)$ if $d \geq 4$. The authors in [52] also study the scattering theory of $H_0 + V$.

Remark 5.1.5. Theorem 5.1.1 (ii) holds if H_0 is replaced by a Fourier multiplier $\mathcal{F}_d^{-1} e \mathcal{F}_d$, where e is a Morse function on $\mathbb{T}^d$. In fact, any Morse function can be deformed into ultrahyperbolic operators near its critical points. Thus we can apply the arguments in Section 3 directly. On the other hand, the authors are not confident whether we may replace H_0 by $\mathcal{F}_d^{-1} e \mathcal{F}_d$ in Theorem 5.1.1 (i) due to the difficulty of multidimensional versions of the van der Corput lemma.

Remark 5.1.6. Theorem 5.1.1 (ii) is optimal as is shown in Theorem 5.1.11 below. However, the authors expect that Theorem 5.1.1 (i) is far from optimal.

Corollary 5.1.2 follows from Theorem 5.1.1 and the following classical result due to T. Kato:

Lemma 5.1.7 ([61, Theorem XIII.26])**.** *Let H_0 be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ and let V be a bounded self-adjoint operator on $\mathcal{H}$. If*

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \left\| |V|^{\frac{1}{2}} (H_0 - z)^{-1} |V|^{\frac{1}{2}} \right\|_{B(\mathcal{H})} < \infty,$$

then H_0 and $H_0+\lambda V$ are unitarily equivalent for small $\lambda \in \mathbb{R}$.

Moreover, we state the existence of the boundary values of the free resolvent near Γ:

Theorem 5.1.8. *Suppose $s > 1$. Then, $\langle x\rangle^{-s}(H_0-z)^{-1}\langle x\rangle^{-s}$ is Hölder continuous in $B(\mathcal{H})$ with respect to $z \in \mathbb{C}_{\mp} = \{z \in \mathbb{C} \mid \mp \operatorname{Im} z > 0\}$. In particular, the incoming/outgoing resolvents*

$$\langle x\rangle^{-s}(H_0-\mu \pm i0)^{-1}\langle x\rangle^{-s}$$

exist in the operator norm topology of $B(\mathcal{H})$ for $\mu \in [0,4d]$.

As a corollary, we have upper bounds of the number of discrete eigenvalues of $H_0+\lambda V+W$ when W is finitely supported.

Corollary 5.1.9. *Let $H = H_0+\lambda V$, where V satisfies the condition of Theorem 5.1.1 (i) or (ii) and $\lambda \in \mathbb{R}$ is small. Let W be a real-valued finitely supported potential. Then*

$$\dim \operatorname{Ker}\,(H+W-\mu) \le \#\left\{x \in \mathbb{Z}^d \mid W(x) \neq 0\right\}$$

for any $\mu \in \mathbb{R}$ and

$$\begin{aligned}\dim \operatorname{Ran}\, E^{pp}_{H+W}\left((-\infty,0]\right) &\le \#\left\{x \in \mathbb{Z}^d \mid W(x) < 0\right\},\\ \dim \operatorname{Ran}\, E^{pp}_{H+W}\left([4d,\infty)\right) &\le \#\left\{x \in \mathbb{Z}^d \mid W(x) > 0\right\},\end{aligned}$$

where $\dim E^{pp}_{H+W}(I)$ denotes the projection onto the eigenspace of $H+W$ corresponding to the eigenvalues contained in $I \subset \mathbb{R}$.

Remark 5.1.10. This corollary appears in [1, Corollary 2.4] under different assumptions in a stronger form. However, their argument seems to be incomplete. Indeed, their proof of the positivity of the quadratic form $(\varphi,[V,iA]\varphi)_{\mathcal{H}}$ on the eigenspace of H_0+V does not work when H_0+V has at least two eigenvalues, where V is a real-valued function with finite support and A is the conjugate operator associated to H_0.

Next, we state our negative results:

Theorem 5.1.11. (i) *Suppose $d=1$ or 2. For a non positive potential $V \in l^\infty(\mathbb{Z}^d)$ which is not identically zero and vanishes at infinity, (5.1.1) does not hold. Moreover, $H_0+\lambda V$ has at least one eigenvalue for all $\lambda > 0$.*

(ii) *Suppose $d=2$. Let $\chi \in C^\infty(\mathbb{T}^d)$ be a non-negative function which is equal to 1 near $\{\xi_1,\xi_2 \in \{1/4,3/4\}\}$ and is supported near $\{\xi_1,\xi_2 \in \{1/4,3/4\}\}$. Then, there exists $w \in l^{2,\infty}(\mathbb{Z}^d)$ such that*

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \|w\chi(D)(H_0-z)^{-1}w\|_{B(\mathcal{H})} = \infty. \tag{5.1.2}$$

Moreover, $\chi(D)(H_0-z)^{-1}$ is not uniformly bounded in $B(l^p(\mathbb{Z}^d),l^q(\mathbb{Z}^d))$ for $1 \le p \le \infty$ and $q < \infty$.

(*iii*) *Let* $d \geq 3$ *and* $q > \frac{d+2}{3}$*. Then, there exists* $V \in l^{q,\infty}(\mathbb{Z}^d)$ *such that (5.1.1) does not hold. In particular, if* $d \geq 5$ *then there exists* $V \in l^{\frac{d}{2},\infty}(\mathbb{Z}^d)$ *such that (5.1.1) does not hold.*

(*iv*) *Let* $d \geq 3$ *and* $V(x) = (1+|x|)^{-\alpha}$ *for* $0 \leq \alpha < 2$*. Then (5.1.1) does not hold.*

Remark 5.1.12. Theorem 5.1.11 (*i*) for $d = 2$ was conjectured in [67].

Remark 5.1.13. Theorem 5.1.11 (*i*) and (*iv*) hold even when H_0 is a Fourier multiplier $\mathcal{F}_d^{-1} e \mathcal{F}_d$ with a Morse function e. We expect that (*iii*) also holds for such operators, however we have no proof for the moment.

Remark 5.1.14. The left hand side of (5.1.2) is finite for the continuous Schrödinger operator $H_0 = -\Delta$ on $L^2(\mathbb{R}^2)$, $w \in L^{q,\infty}(\mathbb{R}^2)$ $(2 < q \leq 3)$, and $\chi \in C_c^\infty(\mathbb{R}^d)$ which is supported away from the threshold 0. For a proof, we use [64, Theorem 5.8] (uniform resolvent estimates for the two dimensional case), a real interpolation argument, and Hölder's inequality as in the proof of [54, Corollary 2.3].

There are various works concerning bounds of Birman-Schwinger operators for the continuous Schrödinger operators (see [47], [50], [61]). For $H_0 = -\Delta$ on $L^2(\mathbb{R}^d)$ with $d \geq 3$, it is known that (5.1.1) holds for $V \in L^{\frac{d}{2},\infty}(\mathbb{R}^d)$ (see [50]). Moreover, this result is sharp in the sense that (5.1.1) does not hold for $V(x) = |x|^{-\frac{d}{q}} \in L^{q,\infty}(\mathbb{R}^d)$ if $q \neq \frac{d}{2}$. In fact, by scaling

$$\||x|^{-\frac{d}{2q}}(-\Delta - z)^{-1}|x|^{-\frac{d}{2q}}\|_{B(\mathcal{H})} = \varepsilon^{2-\frac{d}{q}}\||x|^{-\frac{d}{2q}}(-\Delta - \varepsilon^2 z)^{-1}|x|^{-\frac{d}{2q}}\|_{B(\mathcal{H})}$$

holds and we consider the limits as $\varepsilon \to 0$ and $\varepsilon \to \infty$. Cuenin's examples in [11, Remark 1.9] which are based on the examples by Frank and Simon ([18], see also [32]) show that there exists a sequence of real-valued potentials V_n which satisfy $|V_n(x)| \leq C(n+|x|)^{-1}$ and induce an embedded eigenvalue of $H_0 + V_n$.

We compare our results with the continuous case. For uniformly decaying potentials $V(x) = (1+|x|)^{-\alpha}$, the range of α where (5.1.1) holds is the same as in the case of continuous Schrödinger operators. However, for non-uniformly decaying potentials, for example $V \in l^{p,\infty}(\mathbb{Z}^d)$, the classes of potentials where (5.1.1) holds differ between the discrete case and the continuous case. It seems that this is a consequence of the anisotropy of the discrete Laplacian.

Our paper is organized as follows. In section 5.2, in order to study properties of the resolvent of H_0 near Γ, we investigate properties of the ultrahyperbolic operators. In section 5.3, we prove our positive results Theorems 5.1.1, 5.1.8 and Corollary 5.1.9. In section 5.4, we give the proofs of our negative results Theorem 5.1.11.

We use the following notations throughout this chapter. For Banach spaces X and Y, $B(X,Y)$ denotes a set of all bounded linear operators from X to Y. We denote the norm of a Banach space X by $\|\cdot\|_X$. We also denote $(\cdot,\cdot)_X$ by the inner product of a Hilbert space X. Moreover, we write $B(X) = B(X,X)$. We denote $\langle x \rangle = (1+|x|^2)^{\frac{1}{2}}$ and $D_x = (2\pi i)^{-1}\nabla_x$ for $x \in \mathbb{R}^d$. A symbol $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^d$:

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} u(x)dx, \quad \xi \in \mathbb{R}^d.$$

For $\chi \in C^\infty(\mathbb{T}^d)$ or $\chi \in C_c^\infty(\mathbb{R}^d)$, we denote $\chi(D) = \mathcal{F}_d^{-1}\chi\mathcal{F}_d$ or $\chi(D) = \mathcal{F}^{-1}\chi\mathcal{F}$ respectively. For $h \in C^\infty(\mathbb{T}^d)$ or $h \in C^\infty(\mathbb{R}^d)$, $a \in \mathbb{R}$ and a compactly supported smooth function f, if $\nabla h \neq 0$ on $\{h(\xi) = a\} \cap \operatorname{supp} f$, set

$$\delta(h(D) - a)f(x) = \int_{\{h=a\}} \hat{f}(\xi)e^{2\pi i x\cdot\xi}\frac{d\sigma(\xi)}{|\nabla h(\xi)|},$$

where $d\sigma$ is the induced surface measure.

We give a useful formula. Let $N \subset \mathbb{T}^d$ or $N \subset \mathbb{R}^d$ be a submanifold which has the following graph representation:

$$N = \{\xi \,|\, \xi_1 = f(\xi')\}$$

where we write $\xi = (\xi_1, \xi')$ and f is a $d-1$-variable smooth function. Then we have

$$d\sigma(\xi) = \sqrt{1 + |\nabla f(\xi')|^2}d\xi'. \tag{5.1.3}$$

5.2 Ultrahyperbolic operators

Let $d \geq 2$, $0 \leq k \leq d$ and let

$$p(\xi) = p_k(\xi) = \xi_1^2 + ... + \xi_k^2 - \xi_{k+1}^2 - ... - \xi_d^2$$

for $\xi \in \mathbb{R}^d$.

Definition 3. A differential operator P is called an ultrahyperbolic operator with index $0 \leq k \leq d$ if P has the following form:

$$P = \sum_{j=1}^k D_{x_j}^2 - \sum_{j=k+1}^d D_{x_j}^2 = -\frac{1}{(2\pi)^2}(\sum_{j=1}^k \partial_{x_j}^2 - \sum_{j=k+1}^d \partial_{x_j}^2).$$

Note that $P = \mathcal{F}^{-1}p\mathcal{F}$.

In this section, we study resolvent bounds of the ultrahyperbolic operators. Since h_0 is a Morse function on $\mathbb{T}^d$, near each critical value $q \in \Gamma$, h_0 can be expanded to as the following:

$$h_0(\xi) = h_0(q) + (2\pi)^2(\sum_{j=1}^k \eta_j^2 - \sum_{j=k+1}^d \eta_j^2) + O(|\eta|^3),$$

where $\eta = \xi - q$. Thus we study the ultrahyperbolic operators for analyzing the resolvent of the discrete Schrödinger operator near the thresholds.

5.2.1 Limiting absorption principle for ultrahyperbolic operators

In this subsection, we state a limiting absorption principle for the ultrahyperbolic operators. Let P be the ultrahyperbolic operator with index k. We define

$$A = \tilde{x} \cdot D_x (I - (2\pi)^{-2}\Delta)^{-1} + (I - (2\pi)^{-2}\Delta)^{-1} D_x \cdot \tilde{x}$$

on $C_c^\infty(\mathbb{R}^d)$, where $\tilde{x} = (x_1, ..., x_k, -x_{k+1}, ..., -x_d)$. Then, it follows that P and A are essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ and we also denote the unique self-adjoint extensions by P and A respectively. In fact, for the essential self-adjointness of P it is enough to prove the essential self-adjointness of the multiplication operator $p(\xi)$ on $L^2(\mathbb{R}^d)$ by the Fourier transform. However, this is shown since $(p(\xi) \pm i)u = 0$ and $u \in L^2(\mathbb{R}^d)$ imply $u = 0$. For the essential self-adjointness of A, we employ Nelson's commutator theorem (see [61, Theorem X.36]) with a conjugate operator $-\Delta + |x|^2 + 1$.

By a simple calculation, we have

$$[P, iA] = -\pi^{-2}\Delta(I - (2\pi)^{-2}\Delta)^{-1} = \mathcal{F}^{-1}\left(\frac{4|\xi|^2}{1 + |\xi|^2}\mathcal{F}\right).$$

In the following, we see that $[P, iA]$ satisfy the Mourre estimate except at 0. Note that $E_I(P) = \mathcal{F}^{-1}\chi_I \circ p\mathcal{F}$, where $E_I(P)$ is the spectral projection of P to I and χ_I is the characteristic function of $I \subset \mathbb{R}$. Fix $I \Subset \mathbb{R} \setminus \{0\}$ and set $a = \inf\{|\lambda| \mid \lambda \in I\} > 0$. Then for $\xi \in \mathrm{supp}\,(\chi_I(p(\cdot)))$, we learn

$$|\xi|^2 = \sum_{j=1}^{k} |\xi_j|^2 + \sum_{j=k+1}^{d} |\xi_j|^2 \geq a.$$

Thus we have

$$\chi_I(p(\xi))\frac{4|\xi|^2}{1 + |\xi|^2}\chi_I(p(\xi)) \geq \frac{4a}{1 + a}\chi_I(p(\xi))$$

and hence

$$E_I(P)[P, iA]E_I(P) \geq \frac{4a}{1 + a}E_I(P).$$

Moreover since $[P, iA]$ and $[[P, iA], iA]$ are bounded operators, it follows that $P \in C^2(A)$. Thus by the standard Mourre theory ([53]), we have the following proposition.

Proposition 5.2.1. *Let $I \Subset \mathbb{R} \setminus \{0\}$ be a bounded interval and $s > 1/2$. Then*

$$\sup_{z \in I_\pm} \|\langle A\rangle^{-s}(P - z)^{-1}\langle A\rangle^{-s}\|_{B(L^2(\mathbb{R}^d))} < \infty, \tag{5.2.1}$$

where $I_\pm = \{z \in \mathbb{C} \mid \mathrm{Re}\, z \in I, \pm\mathrm{Im}\, z > 0\}$. Moreover, the limits

$$\langle A\rangle^{-s}(P - \lambda \pm i0)^{-1}\langle A\rangle^{-s} = \lim_{\varepsilon \to +0}\langle A\rangle^{-s}(P - \lambda \pm i\varepsilon)^{-1}\langle A\rangle^{-s} \tag{5.2.2}$$

exist uniformly in $\lambda \in I$.

Remark 5.2.2. Since $\langle A\rangle^{s}(P-i)^{-1}\langle x\rangle^{-s}$ is a bounded operator, we can replace $\langle A\rangle^{-s}$ in (5.2.1) and (5.2.2) by $\langle x\rangle^{-s}$.

Remark 5.2.3. The Proposition 5.2.1 is possibly well-known. However, we cannot find a suitable reference and we give a self-contained proof.

Remark 5.2.4. By using a scaling argument and Proposition 5.2.1, we have a uniform estimate of the high energy limit

$$\sup_{|z|\geq 1}|z|^{1-s}\|\langle x\rangle^{-s}(P-z)^{-1}\langle x\rangle^{-s}\|_{B(L^2(\mathbb{R}^d))}<\infty$$

for $s>\frac{1}{2}$.

5.2.2 Uniform resolvent estimates for ultrahyperbolic operators

In this subsection, we assume $d\geq 3$.

Proposition 5.2.5. *Let P be an ultrahyperbolic operator. For $\alpha,\beta>\frac{1}{2}+\frac{1}{2(d-1)}$ with $\alpha+\beta\geq 2$,*

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}}\|\langle x\rangle^{-\alpha}(P-z)^{-1}\langle x\rangle^{-\beta}\|_{B(L^2(\mathbb{R}^d))}<\infty.$$

Proof. This follows from L^p-L^q resolvent estimates (see [43, Theorem 1.1]) and a real interpolation argument:

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}}\|(P-z)^{-1}\|_{B(L^{p,r}(\mathbb{R}^n),L^{q,r}(\mathbb{R}^n))}<\infty$$

for

$$\frac{1}{p}-\frac{1}{q}=\frac{2}{d},\quad \frac{2d(d-1)}{d^2+2d-4}<p<\frac{2(d-1)}{d},\quad 1\leq r\leq\infty.$$

By using Hölder's inequality, we can obtain the following: For $w_1\in L^{\frac{d}{s},\infty}(\mathbb{R}^d)$ and $w_2\in L^{\frac{d}{2-s},\infty}(\mathbb{R}^d)$ with $\frac{1}{2}+\frac{1}{2(d-1)}<s<\frac{3d-4}{2(d-1)}$,

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}}\|w_1(P-z)^{-1}w_2\|_{B(L^2(\mathbb{R}^d))}\leq C\|w_1\|_{L^{\frac{d}{s},\infty}(\mathbb{R}^d)}\|w_2\|_{L^{\frac{d}{2-s},\infty}(\mathbb{R}^d)}. \tag{5.2.3}$$

In particular, for $\alpha,\beta>\frac{d}{2(d-1)}$ with $\alpha+\beta\geq 2$,

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}}\|\langle x\rangle^{-\alpha}(P-z)^{-1}\langle x\rangle^{-\beta}\|_{B(L^2(\mathbb{R}^d))}\leq C_{\alpha\beta}<\infty. \tag{5.2.4}$$

□

Remark 5.2.6. In section 5.5, we give a self-contained proof of Proposition 5.2.5 with $\alpha = \beta = 1$.

Remark 5.2.7. Note that if P is elliptic (that is $k = 0$ or $k = d$), (5.2.3) holds for $\frac{1}{2} < s < \frac{3}{2}$ and (5.2.4) holds for $\alpha, \beta > \frac{1}{2}$ with $\alpha + \beta \geq 2$ (see [54, Corollary 2.3]).

Proposition 5.2.8. *For $\varepsilon > 0$, $\langle x\rangle^{-1-\varepsilon}(P-z)^{-1}\langle x\rangle^{-1-\varepsilon}$ is locally Hölder continuous on $B(L^2(\mathbb{R}^d))$ in $\mathbb{C}_{\mp}$. In particular, $\langle x\rangle^{-1-\varepsilon}(P-\lambda \pm i0)^{-1}\langle x\rangle^{-1-\varepsilon}$ exist in the operator norm topology of $B(L^2(\mathbb{R}^d))$ for $\lambda \in \mathbb{R}$.*

Proof. The proof is based on the argument in [62, Lemma 4.7]. Set $L^2_k(\mathbb{R}^d) = \langle x\rangle^{-k}L^2(\mathbb{R}^d)$ for $k \in \mathbb{R}$. Note that there are two continuous embeddings $L^{\frac{2d}{d-2}+\delta}(\mathbb{R}^d) \subset L^2_{-1-\varepsilon}(\mathbb{R}^d)$ and $L^2_{1+\varepsilon}(\mathbb{R}^d) \subset L^{\frac{2d}{d+2}-\delta}(\mathbb{R}^d)$ for small $\delta > 0$. By using (5.5.1) in section 5.5, there exists $\alpha_\delta > 0$ such that

$$\begin{aligned}
&\|(P-z)^{-1} - (P-z')^{-1}\|_{L^2_{1+\varepsilon}(\mathbb{R}^d)\to L^2_{-1-\varepsilon}(\mathbb{R}^d)} \\
&\quad \leq C\|(P-z)^{-1} - (P-z')^{-1}\|_{B(L^{\frac{2d}{d+2}-\delta}(\mathbb{R}^d), L^{\frac{2d}{d-2}+\delta}(\mathbb{R}^d))} \\
&\quad = C\left\|\int_0^\infty (e^{itz} - e^{itz'})e^{-itP}dt\right\|_{B(L^{\frac{2d}{d+2}-\delta}(\mathbb{R}^d), L^{\frac{2d}{d-2}+\delta}(\mathbb{R}^d))} \\
&\quad \leq C\int_0^\infty \min(2, |z-z'|t)\frac{1}{t^{1+\alpha_\delta}}dt \\
&\quad = C\int_0^{|z-z'|^{-1}} \frac{|z-z'|}{t^{1+\alpha_\delta}}dt + C\int_{|z-z'|^{-1}}^\infty \frac{1}{t^{1+\alpha_\delta}}dt \\
&\quad \leq C|z-z'|^{\alpha_\delta}.
\end{aligned}$$

The existence of the boundary values $\langle x\rangle^{-1-\varepsilon}(P-\lambda \pm i0)^{-1}\langle x\rangle^{-1-\varepsilon}$ directly follows from the Hölder continuity. $\square$

Next, we state the optimality of the estimate. For a preparation, we need the following lemma.

Lemma 5.2.9. *Let $d \geq 3$, $r > 0$ and $\varphi(x) = \chi(x)|x|^{-\frac{d-2}{2}}$, where $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi = 1$ on $|x| \leq r$. Then, $\varphi \in H^s(\mathbb{R}^d)$ for $0 \leq s < 1$ and $\varphi \notin H^1(\mathbb{R}^d)$.*

Proof. Note that $\varphi \in L^2(\mathbb{R}^d)$. We learn

$$\partial_{x_j}\varphi(x) = (\partial_{x_j}\chi(x))|x|^{-\frac{d-2}{2}} - \frac{d-2}{2}x_j|x|^{-\frac{d+2}{2}},$$

and hence

$$|\nabla\varphi(x)| \geq C|x|^{-\frac{d}{2}}$$

near $x = 0$. Thus, $|\nabla\varphi| \notin L^2(\mathbb{R}^d)$ and hence $\varphi \notin H^1(\mathbb{R}^d)$.

Next, we show that $\varphi \in H^s(\mathbb{R}^d)$ for $0 \le s < 1$. It suffices to prove that $\langle \xi \rangle^s \hat{\varphi}(\xi) \in L^2(\mathbb{R}^d)$ where $\hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi)$. Note that $\varphi \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $\widehat{|x|^{-\frac{d-2}{2}}}(\xi) = c_d |\xi|^{-\frac{d}{2}-1}$, where c_d is a constant depending only on d. We learn

$$\hat{\varphi}(\xi) = c_d \int_{\mathbb{R}^d} \hat{\chi}(\eta) |\xi - \eta|^{-\frac{d}{2}-1} d\eta.$$

Since

$$\begin{aligned} \left| \int_{|\xi-\eta| \le \frac{1}{2}|\xi|} \hat{\chi}(\eta)|\xi - \eta|^{-\frac{d}{2}-1} d\eta \right| \le & \left| \int_{\frac{1}{2}|\xi| \le |\eta| \le \frac{3}{2}|\xi|} \hat{\chi}(\eta)|\xi - \eta|^{-\frac{d}{2}-1} d\eta \right| \\ \le & C\langle \xi \rangle^{-N} \int_{\frac{1}{2}|\xi| \le |\eta| \le \frac{3}{2}|\xi|} |\xi - \eta|^{-\frac{d}{2}-1} d\eta \\ \le & C\langle \xi \rangle^{-N-1+\frac{d}{2}}, \end{aligned}$$

and

$$\begin{aligned} \left| \int_{|\xi-\eta| > \frac{1}{2}|\xi|} \hat{\chi}(\eta)|\xi - \eta|^{-\frac{d}{2}-1} d\eta \right| \le & C|\xi|^{-\frac{d}{2}-1} \int_{|\xi-\eta| > \frac{1}{2}|\xi|} \hat{\chi}(\eta) d\eta \\ \le & C|\xi|^{-\frac{d}{2}-1} \end{aligned}$$

for any $|\xi| \ge 1$ and any $N > 0$, we have $\langle \xi \rangle^s \hat{\varphi} \in L^2(\{|\xi| \ge 1\})$. This and $\varphi \in L^2(\mathbb{R}^d)$ imply $\langle \xi \rangle^s \hat{\varphi} \in L^2(\mathbb{R}^d)$. □

Using the above lemma, we obtain:

Proposition 5.2.10. *For* $0 \le s < 1$*, we have*

$$\sup_{z \in \mathbb{C} \setminus \sigma(P)} \| \langle x \rangle^{-s} (P - z)^{-1} \langle x \rangle^{-s} \|_{B(L^2(\mathbb{R}^d))} = \infty.$$

Proof. For simplicity, we deal with $k = 1$ only. We may assume $s > 1/2$. Note that by Proposition 5.2.1 and Remark 5.2.2, $\langle x \rangle^{-s}(P + \varepsilon \pm i0)^{-1} \langle x \rangle^{-s}$ exist in $B(L^2(\mathbb{R}^d))$ for $\varepsilon \ne 0$. Moreover, it follows that

$$\begin{aligned} &\langle x \rangle^{-s}(P + \varepsilon - i0)^{-1} \langle x \rangle^{-s} - \langle x \rangle^{-s}(P + \varepsilon + i0)^{-1} \langle x \rangle^{-s} \\ &= \langle x \rangle^{-s} \delta(P + \varepsilon) \langle x \rangle^{-s} \end{aligned}$$

due to Stone's theorem. Thus it suffices to prove that

$$\sup_{\varepsilon > 0} \| \langle x \rangle^{-s} \delta(P + \varepsilon) \langle x \rangle^{-s} \|_{B(L^2(\mathbb{R}^d))} = \infty.$$

By using the Fourier transform, it is sufficient to find $\varphi \in H^s(\mathbb{R}^d)$ such that

$$\sup_{\varepsilon > 0} \left| (\varphi, \delta(p + \varepsilon)\varphi)_{L^2(\mathbb{R}^d)} \right| = \infty.$$

Note that

$$(\varphi, \delta(p(\xi)+\varepsilon)\varphi) = \int_{p(\xi)=-\varepsilon} |\varphi(\xi)|^2 \frac{d\sigma(\xi)}{|\nabla_\xi p|} = \int_{p(\xi)=-\varepsilon} |\varphi(\xi)|^2 \frac{d\sigma(\xi)}{2|\xi|},$$

and

$$p(\xi) = -\varepsilon \Leftrightarrow \xi_1^2 = \sum_{j=2}^{d} \xi_j^2 + \varepsilon.$$

Using the formula (5.1.3), we learn

$$\begin{aligned}(\varphi, \delta(p(\xi)+\varepsilon)\varphi) =& \sum_{\pm} \int_{\mathbb{R}^{d-1}} |\varphi(\xi)|^2 \sqrt{1 + \left|\frac{\xi'}{\sqrt{|\xi'|^2+\varepsilon}}\right|^2} \frac{d\xi'}{2|\xi|} \\ =& \sum_{\pm} \int_{\mathbb{R}^{d-1}} |\varphi(\xi)|^2 \frac{d\xi'}{2\sqrt{|\xi'|^2+\varepsilon}},\end{aligned}$$

where $\xi' = (\xi_2, ..., \xi_d)$ and $\xi = (\pm\sqrt{|\xi'|^2+\varepsilon}, \xi')$ for $\xi \in \{p(\xi) = \varepsilon\}$. Thus we now take $\varphi(\xi) = \frac{1}{|\xi|^{(d-2)/2}}\chi(\xi)$, where $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi = 1$ on $|\xi| \leq 1$. Note that $\varphi \in H^s(\mathbb{R}^n)$ due to Lemma 5.2.9. Since

$$\int_{\xi' \in \mathbb{R}^{d-1}, |\xi'| \leq 1} \frac{1}{|\xi'|^{d-1}} d\xi' = \infty,$$

φ has the desired property. □

Remark 5.2.11. This proposition also follows from a scaling argument. In fact, for $\alpha, \beta > \frac{1}{2}$ we have

$$\begin{aligned}&\|(1+|x|)^{-\alpha}(P-z)^{-1}(1+|x|)^{-\beta}\|_{B(L^2(\mathbb{R}^d))} \\ =&\varepsilon^{2-(\alpha+\beta)}\|(\varepsilon^{-1}+|x|)^{-\alpha}(P-\varepsilon^2 z)^{-1}(\varepsilon^{-1}+|x|)^{-\beta}\|_{B(L^2(\mathbb{R}^d))} \\ \leq&\varepsilon^{2-(\alpha+\beta)}\|(1+|x|)^{-\alpha}(P-\varepsilon^2 z)^{-1}(1+|x|)^{-\beta}\|_{B(L^2(\mathbb{R}^d))}.\end{aligned}$$

for $0 < \varepsilon < 1$. If we take supremum of $z \in \mathbb{C} \setminus \mathbb{R}$ and take $\varepsilon \to 0$, then we obtain a contradiction unless $\alpha + \beta \geq 2$. For the Laplace operators, see [6]. However we give a more direct proof for a special case since the above argument can be applicable to the discrete Schrödinger operators near the hyperbolic thresholds. See Remark 5.4.4.

5.3 Proofs of positive results

In this section, we prove Theorems 5.1.1, 5.1.8 and Corollary 5.1.9.

5.3.1 Proof of Theorem 5.1.1 (i)

It is known that there is a deep connection between the time decay of the Schrödinger propagator e^{itP} and the threshold property of the resolvent of P. We refer [61, §XIII-A]. Here we employ a bit technical, but very strong tool due to Duyckaerts. His method allows us to deduce L^p-L^q uniform resolvent estimates from Strichartz estimates.

First, we state the dispersive estimates and the Strichartz estimates for the discrete Schrödinger operators.

Proposition 5.3.1 ([67]). *Let $d \geq 1$. Then, there exists $C > 0$ such that for any $t \in \mathbb{R}$*

$$\|e^{-itH_0}\|_{l^1(\mathbb{Z}^d)\to l^\infty(\mathbb{Z}^d)} \leq C\langle t\rangle^{-\frac{d}{3}}.$$

Corollary 5.3.2. *Let $d \geq 4$. Set $3^* = \frac{2d}{d-3}$ and $3_* = \frac{2d}{d+3}$. Then, we have the following Strichartz estimates: Suppose that $u \in C(\mathbb{R}, l^2(\mathbb{Z}^d))$ and $F \in L^2(\mathbb{R}, l^{3_*,2}(\mathbb{Z}^d))$ satisfy*

$$i\partial_t u(t) - H_0 u(t) = F, \quad u|_{t=0} = f \in l^2(\mathbb{Z}^d). \tag{5.3.1}$$

Then there exists $C > 0$ such that for $0 < T \leq \infty$ we have

$$\|u\|_{L^2(-T,T)l^{3^*,2}(\mathbb{Z}^d)} \leq C\|f\|_{l^2(\mathbb{Z}^d)} + C\|F\|_{L^2(-T,T)l^{3_*,2}(\mathbb{Z}^d)}.$$

Proof. We apply Theorem 10.1 in [48] with $\mathcal{H} = B_0 = l^2(\mathbb{Z}^d)$, $B_1 = l^1(\mathbb{Z}^d)$, $\sigma = \frac{d}{3}$ and $q = 2$. □

The next argument is due to T. Duyckaerts (see [5, Proposition 5.1]).

Proposition 5.3.3. *Let $d \geq 4$. Then, there exists $C > 0$ such that for $z \in \mathbb{C} \setminus \sigma(H_0)$*

$$\|(H_0 - z)^{-1}u\|_{l^{3^*,2}(\mathbb{Z}^d)} \leq C\|u\|_{l^{3_*,2}(\mathbb{Z}^d)}, \quad u \in l^2(\mathbb{Z}^d) \cap l^{3_*,2}(\mathbb{Z}^d).$$

Moreover, for $w \in l^{\frac{2d}{3},\infty}(\mathbb{R}^d)$,

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \|w(H_0 - z)^{-1}w\|_{B(\mathcal{H})} \leq C\|w\|^2_{l^{\frac{2d}{3},\infty}(\mathbb{Z}^d)}.$$

In particular, if $V \in l^{\frac{d}{3},\infty}(\mathbb{Z}^d)$, (5.1.1) holds.

Proof. Suppose that f is a finitely supported function on $\mathbb{Z}^d$. Let $z \in \mathbb{C} \setminus \sigma(H_0)$. We substitute $u(t) = e^{itz}f$ into (5.3.1) and then we have

$$\gamma(z, T)\|f\|_{l^{3^*,2}(\mathbb{Z}^d)} \leq C\|f\|_{l^2(\mathbb{Z}^d)} + C\gamma(z, T)\|(H_0 - z)f\|_{l^{3_*,2}(\mathbb{Z}^d)},$$

where $\gamma(z, T) = \|e^{izt}\|_{L^2(-T,T)}$. Since $\gamma(z, T) \geq \sqrt{T}$, by letting $T \to \infty$,

$$\|f\|_{l^{3^*,2}(\mathbb{Z}^d)} \leq C\|(H_0 - z)f\|_{l^{3_*,2}(\mathbb{Z}^d)}.$$

It remains to justify a density argument. □

5.3.2 Proof of Theorem 5.1.1 (ii)

In this subsection, we assume $d \geq 3$.

Proposition 5.3.4. *For $\alpha, \beta > \frac{1}{2} + \frac{1}{2(d-1)}$ with $\alpha + \beta \geq 2$, there exists $C > 0$ such that*

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \| \langle x \rangle^{-\alpha} (H_0 - z)^{-1} \langle x \rangle^{-\beta} \|_{B(\mathcal{H})} \leq C.$$

Proof. By using a partition of unity, it suffices to prove for each $\chi \in C^\infty(\mathbb{T}^d)$ with a small support, $f \in H^\alpha(\mathbb{T}^d)$ and $g \in H^\beta(\mathbb{T}^d)$

$$|(f, \chi^2 (h_0 - z)^{-1} g)_{L^2(\mathbb{T}^d)}| \leq C \|f\|_{H^\alpha(\mathbb{T}^d)} \|g\|_{H^\beta(\mathbb{T}^d)}, \tag{5.3.2}$$

where $C > 0$ is independent of f, g and z. We may suppose χ has one of the following properties: $\nabla h_0 \neq 0$ on $\operatorname{supp} \chi$ or $\operatorname{supp} \chi$ contains just one element of Γ. Since (5.3.2) follows from Proposition 5.6.5 in the former case, we may only deal with the latter case. We take a unique element $\xi_0 \in \Gamma \cap \operatorname{supp} \chi$. Then there exists a diffeomorphism κ from a neighborhood of $\operatorname{supp} \chi$ onto its image such that

$$h_0(\kappa^{-1}(\eta)) = h_0(\xi_0) + \eta_1^2 + ... + \eta_k - \eta_{k+1}^2 - ... - \eta_d^2, \quad \eta \in \kappa(\operatorname{supp} \chi) \subset \mathbb{R}^d$$

for some $0 \leq k \leq d$. Set $J(\eta) = |\det d\kappa^{-1}(\eta)|$ and $f_\kappa(\eta) = f(\kappa^{-1}(\eta))$. By using the change of variables and Proposition 5.2.5, we have

$$\begin{aligned} |(f, \chi^2 (h_0 - z)^{-1} g)_{L^2(\mathbb{T}^d)}| &= \left| \int_{\mathbb{T}^d} \frac{\chi(\xi)^2 \bar{f}(\xi) g(\xi)}{h_0(\xi) - z} d\xi \right| \\ &= \left| \int_{\mathbb{R}^d} \frac{\chi_\kappa(\eta)^2 \bar{f}_\kappa(\eta) g_\kappa(\eta)}{p_k(\eta) + h_0(\xi_0) - z} J(\eta) d\eta \right| \\ &\leq C \|\chi_\kappa f_\kappa\|_{H^\alpha(\mathbb{R}^d)} \|\chi_\kappa g_\kappa\|_{H^\beta(\mathbb{R}^d)} \\ &\leq C \|f\|_{H^\alpha(\mathbb{T}^d)} \|g\|_{H^\beta(\mathbb{T}^d)}, \end{aligned}$$

where we used Lemma 5.6.3 in the last inequality. Thus we obtain (5.3.2). □

5.3.3 Proof of Theorem 5.1.8

Let $s > 1$ and $\alpha_\delta > 0$ as in the proof of Proposition 5.2.8. Similarly to Subsection 5.3.2, it is enough to prove that

$$|(f, \chi^2((h_0 - z)^{-1} - (h_0 - z')^{-1}) g)| \leq C |z - z'|^{\alpha_\delta} \|f\|_{H^s(\mathbb{T}^d)} \|g\|_{H^s(\mathbb{T}^d)}$$

for $\chi \in C^\infty(\mathbb{T}^d)$ which is as in subsection 5.3.2. However, it is proved by changing variables, Proposition 5.2.8, Lemma 5.6.2 and Proposition 5.6.5.

5.3.4 Proof of Corollary 5.1.9

Corollary 5.1.9 follows from Lemma 5.3.5. The argument in the proof is due to [27, Lemma 2.1].

Lemma 5.3.5. *Let H be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ which has no eigenvalues and W be a finite rank self-adjoint operator on $\mathcal{H}$. Then:*

(i) For any $\mu \in \mathbb{R}$, $\dim \operatorname{Ker}(H + W - \mu) \leq \dim \operatorname{Ran} W$.

(ii) Suppose that $\sigma(H) = [a, b]$, $-\infty < a < b < \infty$ and $W = W_+ - W_-$, where $W_\pm$ are positive operators. Then

$$\dim(\operatorname{Ran} E^{pp}_{H+W}((-\infty, a])) \leq \dim \operatorname{Ran} W_-,$$
$$\dim(\operatorname{Ran} E^{pp}_{H+W}([b, \infty))) \leq \dim \operatorname{Ran} W_+.$$

Proof. (i) Suppose that the inequality fails. Let P be the projection onto $\operatorname{Ran} W = (\operatorname{Ker} W)^\perp$. Then $P|_{\operatorname{Ker}(H+W-\mu)}: \operatorname{Ker}(H + W - \mu) \to \operatorname{Ran} W$ has a non-trivial kernel, i.e. we can choose $u \in \operatorname{Ker}(H + W - \mu)$ such that $Wu = 0$ and $\|u\|_{\mathcal{H}} = 1$. Therefore

$$0 = (H + W - \mu)u = (H - \mu)u,$$

which contradicts the assumption that H has no eigenvalues.

(ii) Suppose that the first inequality fails. Then the same argument as in (i) implies that there exists $u \in \operatorname{Ran} E^{pp}_{H+W}((-\infty, a])$ such that $W_- u = 0$ and $\|u\|_{\mathcal{H}} = 1$. Therefore we have

$$a \geq (u, (H + W)u)_{\mathcal{H}} = (u, (H + W_+)u)_{\mathcal{H}} \geq (u, Hu)_{\mathcal{H}},$$

where the last inequality follows from the positivity of W_+. On the other hand, the assumption on H implies $(u, Hu)_{\mathcal{H}} \in (a, b)$, which is a contradiction. The other inequality is similarly proved. □

5.4 Proofs of negative results

5.4.1 Proof of Theorem 5.1.11 (i)

The following argument is similar to [61, Theorem XIII.11]. Note that $h_0(\xi) \sim 4\pi^2|\xi|^2$ near $\xi = 0$ and the operator H_0 is positive.

Set $K_\mu = |V|^{1/2}(H_0 + \mu^2)^{-1}|V|^{1/2}$ for $\mu \in \mathbb{R}$. First, we note that $H_0 + \lambda V$ has a negative eigenvalue if and only if there exists $\mu > 0$ such that $1/\lambda$ is an eigenvalue of K_μ. In fact, a direct calculus implies

$$\begin{aligned} H_0 u + \lambda V u = -\mu^2 u \Leftrightarrow &\lambda(H_0 + \mu^2)^{-1} V u = -u \\ \Rightarrow &\lambda K_\mu \psi = \psi, \end{aligned}$$

where $\psi = |V|^{\frac{1}{2}}u$. Conversely, if there exists $\psi \in \mathcal{H}$ such that $\lambda K_\mu \psi = \psi$, then

$$\lambda |V| u = |V|^{1/2}\psi = (H_0 + \mu^2)u,$$

where $u = (H_0 + \mu^2)^{-1}|V|^{1/2}\psi$.

Since V vanishes at infinity, K_μ is a positive compact operator. Then it suffices to prove that

$$\lim_{\mu^2 \to 0} \|K_\mu\|_{B(\mathcal{H})} = \infty.$$

In fact, the spectral radius of K_μ is equal to $\|K_\mu\|_{B(\mathcal{H})}$, $\sigma(K_\mu) = \sigma_{pp}(K_\mu)$ and $\sigma(K_\mu)$ has no accumulation point except at 0. Thus we need only find $\eta \in \mathcal{H}$ such that

$$\lim_{\mu^2 \to 0} \big(|V|^{1/2}\eta, (H_0 + \mu^2)^{-1}|V|^{1/2}\eta\big)_{\mathcal{H}} = \infty.$$

We choose a non negative finitely supported function $\eta \in \mathcal{H}$ which satisfies $\eta(x) > 0$ for some $x \in \operatorname{supp} V$ and set $\varphi = |V|^{1/2}\eta$. Then $\hat{\varphi}(0) = \sum_{x \in \mathbb{Z}^d} |V(x)|^{1/2}\eta(x) > 0$. Since φ is finitely supported, then $\hat{\varphi} \in L^2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$. Thus, $\hat{\varphi} \neq 0$ near zero. Note that $h_0(\xi) \sim 4\pi^2|\xi|^2$ near $\xi = 0$. Consequently,

$$\big(|V|^{1/2}\eta, (H_0 + \mu^2)^{-1}|V|^{1/2}\eta\big)_{\mathcal{H}} = \int_{\mathbb{T}^d} \frac{|\hat{\varphi}(\xi)|^2}{h_0(\xi) + \mu^2} d\xi$$

diverges as $\mu^2 \to 0$ if $d = 1$ or 2.

5.4.2 Proof of Theorem 5.1.11 *(ii)*

We consider near $\xi_1 = \xi_2 = \frac{1}{4}$ only, the other cases being similar.

Lemma 5.4.1. *In a neighborhood of* $(\frac{1}{4}, \frac{1}{4}) \in \mathbb{T}^2$, $h_0(\xi) = 4$ *is equivalent to* $\xi_1 + \xi_2 = \frac{1}{2}$.

Proof. Note that $h_0(\xi) = 4\sin^2 \pi\xi_1 + 4\sin^2 \pi\xi_2 = 4 - 2\cos 2\pi\xi_1 - 2\cos 2\pi\xi_2$. Thus, $h_0(\xi) = 4$ is equivalent to

$$\cos 2\pi\xi_1 + \cos 2\pi\xi_2 = \cos(\pi(\xi_1 + \xi_2))\cos(\pi(\xi_1 - \xi_2)) = 0.$$

Near $\xi_1 = \xi_2 = \frac{1}{4}$, this is equivalent to $\xi_1 + \xi_2 = \frac{1}{2}$. □

Proposition 5.4.2. *Let* $\chi \in C^\infty(\mathbb{T}^2)$ *be a non-negative function which is equal to* 1 *near* $\xi_1 = \xi_2 = 1/4$ *and is supported near* $\xi_1 = \xi_2 = 1/4$. *If* $q \neq \infty$,

$$(H_0 - 4 \pm i0)^{-1}\mathcal{F}_d^{-1}(\chi) \notin l^q(\mathbb{Z}^2).$$

Proof. Let us denote $\mathcal{H}_s = \langle x \rangle^{-s}\mathcal{H}$. If we take the support of χ small near $\xi_1 = \xi_2 = \frac{1}{4}$, then $\chi(D)(H_0 - 4 \pm i0)^{-1}$ exists in $B(\mathcal{H}_s, \mathcal{H}_{-s})$ for $s > \frac{1}{2}$ since $\nabla h_0(\xi) \neq 0$ on supp χ. Here we used Lemma 5.6.5. Then, it suffices to prove that

$$\big((H_0 - 4 - i0)^{-1} - (H_0 - 4 + i0)^{-1}\big)\,\mathcal{F}_d^{-1}\chi \notin l^q(\mathbb{Z}^2)$$

for $q \neq \infty$. Stone's theorem implies

$$\begin{aligned}
&\frac{1}{2\pi i}((H_0 - 4 - i0)^{-1} - (H_0 - 4 + i0)^{-1})\mathcal{F}_d^{-1}\chi \\
&= \delta(H_0 - 4)\mathcal{F}_d^{-1}\chi \\
&= \int_{h_0(\xi)=4} e^{2\pi i x_1\xi_1 + 2\pi i x_2\xi_2}\chi(\xi)\frac{d\sigma(\xi)}{|\nabla h_0(\xi)|}.
\end{aligned}$$

By using Lemma 5.4.1 and the formula (5.1.3),

$$\begin{aligned}
I(x_1, x_2) &:= \int_{h_0(\xi)=4} e^{2\pi i x_1\xi_1 + 2\pi i x_2\xi_2}\chi(\xi)\frac{d\sigma(\xi)}{|\nabla h_0(\xi)|} \\
&= \int_{\mathbb{R}} e^{2\pi i x_1\xi_1 + 2\pi i x_2(\frac{1}{2} - \xi_1)}\chi(\xi_1, \frac{1}{2} - \xi_1)\frac{d\xi_1}{4\pi\sin(2\pi\xi_1)}
\end{aligned}$$

is rapidly decreasing with respect to $|x_1 - x_2|$ since $\chi(1/4, 1/4) = 1$. However, we cannot obtain any decay with respect to $|x_1 + x_2|$. We write

$$I(x_1, x_2) = e^{\pi i x_2} J(x_1 - x_2),\ \ J(t) = \int_{\mathbb{R}} e^{2\pi i t\xi_1}\chi(\xi_1, \frac{1}{2} - \xi_1)\frac{d\xi_1}{4\pi\sin(2\pi\xi_1)}.$$

We employ the change of variables: $s = x_1 + x_2$ and $t = x_1 - x_2$ and write $I_1(s,t) = I(x_1, x_2)$. Since $|I_1(s,t)| = |J(t)|$ is independent of s, $|I_1(s,t)| \not\to 0$ as $|s| \to \infty$ unless $t \in \{J(t) = 0\}$. By the assumption of χ, we have $|I_1(s, 0)| = |J(0)| \neq 0$. Thus $|I|$ does not decay with respect to $s = x_1 + x_2$. As a consequence, $((H_0 - 4 - i0)^{-1} - (H_0 - 4 + i0)^{-1})\mathcal{F}_d^{-1}(\chi)$ does not belong to $l^q(\mathbb{Z}^2)$ unless $q = \infty$. □

The above proposition shows that if $d = 2$, $\chi(D)(H_0 - z)^{-1}$ is not bounded from $l^p(\mathbb{Z}^2)$ to $l^q(\mathbb{Z}^2)$ unless $q = \infty$. Thus, the proof of the second part of Theorem 5.1.11 (ii) is completed.

In the rest of the subsection, we prove the first part of Theorem 5.1.11 (ii). Let $w(x_1, x_2) = \langle x_1 + x_2\rangle^{-1/2}\langle x_1 - x_2\rangle^{-1} \in l^{2,\infty}(\mathbb{Z}^2)$ and ψ be a non zero finitely supported function such that $\psi \geq 0$. Let $u(x) = e^{-\pi i\frac{x_1+x_2}{2}}(w^{-1}\psi)(x)$. Note that

$$\mathcal{F}_d^{-1}(wu)(1/4, 1/4) = \sum_{x\in\mathbb{Z}^2} e^{2\pi i\times\frac{x_1+x_2}{4}} e^{-\pi i\frac{x_1+x_2}{2}}\psi(x) = \sum_{x\in\mathbb{Z}^2}\psi(x) > 0.$$

Thus $|w(x)\chi(D)\delta(H_0 - 4)wu(x)| \sim C(1 + |x_1 + x_2|)^{-1/2}(1 + |x_1 - x_2|)^{-\infty}$ as in the proof of Proposition 5.4.2 and the right hand side does not belong to $\mathcal{H}$.

5.4.3 Proof of Theorem 5.1.11 *(iii)*

In Proposition 5.3.3 and Theorem 5.3.4, we have seen uniform bounds of Birman-Schwinger operator for $V \in l^{d/3,\infty}(\mathbb{Z}^d)$ or $V(x) = \langle x \rangle^{-2}$. Since $\langle x \rangle^{-2} \in l^{d/2,\infty}(\mathbb{Z}^d)$, it is natural to ask whether it is true for general potentials $V \in l^{d/2,\infty}(\mathbb{Z}^d)$. However, the next proposition says that it is false at least if $d \geq 5$.

Proposition 5.4.3. *Let $d \geq 3$ and*

$$w(x) = w_p(x) = \langle x_d \rangle^{-1/p} \prod_{j=1}^{d-1} \langle x_j - x_d \rangle^{-1/p} \in l^{p,\infty}(\mathbb{Z}^d), \quad p > 0.$$

Suppose that

$$\sup_{\lambda \in \mathbb{R} \setminus \sigma(H_0)} \|w_p(H_0 - \lambda)^{-1} w_p\|_{B(\mathcal{H})} < \infty. \tag{5.4.1}$$

Then, $p \leq 2(d+2)/3$ holds. In particular, if $d \geq 5$ then w_d does not satisfy (5.4.1).

Proof. We construct a variant of the Knapp counter example near the energy surface $h_0(\xi) = 2d$. We denote the d-dimensional Fourier expansion $\mathcal{F}_d$ of u by $\hat{u}$ and the one dimensional Fourier transform by $\mathcal{F}^1$. We take a real valued function $\chi \in C_c^\infty\left((-\frac{1}{4}, \frac{1}{4})\right)$ such that $\chi = 1$ near 0. We can regard χ as a function on S^1. Let

$$\begin{aligned}\varphi_\varepsilon(x) =& e^{2\pi i(dx_d/4 + \sum_{j=1}^{d-1} x_j/4)} a\varepsilon^{d+2} w_p^{-1}(x)((\mathcal{F}^1)^{-1}\chi)(a\varepsilon^3 x_d) \\ &\times \prod_{j=1}^{d-1}((\mathcal{F}^1)^{-1}\chi)(\varepsilon(x_j - x_d))\end{aligned}$$

for $0 < \varepsilon \leq 1$, $a > 0$ and $x \in \mathbb{Z}^d$. Then,

$$\widehat{w\varphi_\varepsilon}(\xi) = \chi\left(\sum_{j=1}^{d} \frac{\xi_j - 1/4}{a\varepsilon^3}\right) \prod_{j=1}^{d-1} \chi\left(\frac{\xi_j - 1/4}{\varepsilon}\right) \in C^\infty(\mathbb{T}^d),$$

where $\xi \in [0,1)^d$ and we regard the function $\widehat{w\varphi_\varepsilon}$ on $[0,1)^d$ as a function on $\mathbb{T}^d$ by virtue of the support property of χ. Note that $w\varphi_\varepsilon$ is rapidly decreasing and $\widehat{w\varphi_\varepsilon}$ has a small support near $\{\xi_j = 1/4,\ j = 1, ..., d\}$ which does not contain critical points of $h_0(\xi) = 4\sum_{j=1}^{d} \sin^2(\pi\xi_j)$. Thus $(\varphi_\varepsilon, w\delta(H_0 - 2d)w\varphi_\varepsilon)_{\mathcal{H}\to\mathcal{H}}$ exists by Proposition 5.6.5. We observe that if $\xi' = (\xi_1, ..., \xi_{d-1}) \in \operatorname{supp}\left(\prod_{j=1}^{d-1} \chi(\frac{\xi_j - 1/4}{\varepsilon})\right)$ and $\xi \in h_0^{-1}(\{2d\})$, then

$$\sum_{j=1}^{d} (\xi_j - 1/4) = O(\varepsilon^3). \tag{5.4.2}$$

In fact, by using the Taylor expansion near $\{\xi_j = 1/4,\ j = 1, ..., d\}$, we have

$$\begin{aligned}0 = h_0(\xi) - 2d =& 4\sum_{j=1}^{d}(\xi_j - 1/4) + O\left(\sum_{j=1}^{d}(\xi_j - 1/4)^3\right)\\ =& 4\sum_{j=1}^{d}(\xi_j - 1/4) + O(\varepsilon^3) + O((\xi_d - 1/4)^3).\end{aligned}$$

This implies (5.4.2). Therefore, if we take $a > 0$ large enough (which remains to be independent of ε), it follows that

$$\mathrm{supp}\left(\prod_{j=1}^{d-1}\chi\left(\frac{\xi_j - 1/4}{\varepsilon}\right)\right) \cap h_0^{-1}(\{2d\}) \subset \mathrm{supp}\,\chi\left(\sum_{j=1}^{d}\frac{\xi_j - 1/4}{a\varepsilon^3}\right).$$

By using this, we obtain

$$\begin{aligned}(\varphi_\varepsilon, w\delta(H_0 - 2d)w\varphi_\varepsilon)_{\mathcal{H}} =& (\widehat{w\varphi_\varepsilon}, \delta(h_0 - 2d)\widehat{w\varphi_\varepsilon})_{L^2(\mathbb{T}^d)}\\ =& \int_{h_0(\{2d\})\cap(-\frac{1-\varepsilon}{4}, \frac{1+\varepsilon}{4})^d} |\widehat{w\varphi_\varepsilon}(\xi)|^2 \frac{d\sigma(\xi)}{|\nabla h_0(\xi)|}\\ \geq& C\varepsilon^{d-1}\end{aligned}$$

for some $C > 0$ which is independent of ε.

On the other hand, we observe that for $s > 2$

$$\begin{aligned}&\sum_{x_j\in\mathbb{Z}}\langle x_j - x_d\rangle^{2/p}|((\mathcal{F}_d^1)^{-1}\chi)(\varepsilon(x_j - x_d))|^2\\ &\leq C\sum_{x_j\in\mathbb{Z}}\langle x_j - x_d\rangle^{2/p}\langle\varepsilon(x_j - x_d)\rangle^{-2s}\\ &= C\left(\sum_{|x_j|<1/\varepsilon} + \sum_{|x_j|\geq 1/\varepsilon}\right)\langle x_j\rangle^{2/p}\langle\varepsilon x_j\rangle^{-2s} \leq C\varepsilon^{-1-2/p}.\end{aligned}$$

Then, we obtain

$$\|\varphi_\varepsilon\|_{\mathcal{H}}^2 \leq C\varepsilon^{2(d+2)} \cdot \varepsilon^{(d+2)(-1-2/p)} = C\varepsilon^{(d+2)(1-2/p)}.$$

By using (5.4.1), we have $\varepsilon^{d-1} \leq C\varepsilon^{(d+2)(1-2/p)}$. Since this holds for small $\varepsilon > 0$, we conclude $p \leq 2(d+2)/3$.

□

5.4.4 Proof of Theorem 5.1.11 (iv)

For $0 \leq s < 1$, we prove

$$\sup_{z\in\mathbb{C}\setminus\sigma(H_0)} \|\langle x\rangle^{-s}(H_0-z)^{-1}\langle x\rangle^{-s}\|_{B(\mathcal{H})} = \infty.$$

It suffices to prove that there exists $\varphi \in H^s(\mathbb{T}^d)$ such that

$$\lim_{\varepsilon\to 0,\varepsilon>0} |(\varphi, (h_0(\xi)+\varepsilon)^{-1}\varphi)_{L^2(\mathbb{T}^d)}| = \infty.$$

Fix $\xi_0 \in h_0^{-1}(\{0\})$. Then there exists a diffeomorphism f from a small neighborhood of ξ_0 to a small open ball in $\mathbb{R}^d$ such that $h_0(f^{-1}(\eta)) = |\eta|^2$. We take

$$\varphi(\xi) = \chi(\xi) f^*\left(\frac{1}{|\eta|^{\frac{d-2}{2}}}\right)(\xi),$$

where $\chi \in C^\infty(\mathbb{T}^d)$ has a small support near ξ_0. Note that $\varphi \in H^s(\mathbb{T}^d)$ for $0 \leq s < 1$ due to Lemma 5.2.9. Thus, we obtain

$$|(\varphi, (h_0(\xi)+\varepsilon)^{-1}\varphi)_{l^2}| \geq C \int_{\eta\in\mathbb{R}^d, |\eta|:\text{near } 0} \frac{1}{|\eta|^{d-2}(|\eta|^2+\varepsilon)} d\eta \to \infty$$

as $\varepsilon \to 0$.

Remark 5.4.4. In the above proof, we have constructed a function supported near an elliptic threshold. However, this argument is applicable to near a hyperbolic threshold. See the proof of Proposition 5.2.10.

5.5 Self-contained proof of Proposition 5.2.5 in a particular case

We can apply the argument in Subsection 5.3.1 to the ultrahyperbolic operators P: Note that

$$\begin{aligned} e^{4\pi^2 itP}u(x,y) =& e^{-it\Delta_x} e^{it\Delta_y} u(x,y) \\ =& \frac{1}{(-4\pi it)^{\frac{k}{2}}} \frac{1}{(4\pi it)^{\frac{d-k}{2}}} \int_{\mathbb{R}^k_{x'}} \int_{\mathbb{R}^{d-k}_{y'}} e^{\frac{|x-x'|^2}{4it}} e^{-\frac{|y-y'|^2}{4it}} u(x',y') dy' dy', \end{aligned}$$

where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{d-k}$. Thus, we obtain the following dispersive estimates:

$$\|e^{4\pi^2 itP}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq \frac{1}{(4\pi|t|)^{\frac{d}{2}}}, \quad t \in \mathbb{R}\setminus\{0\}.$$

Using a complex interpolation, we have

$$\|e^{4\pi^2 itP}\|_{L^p(\mathbb{R}^d)\to L^{p'}(\mathbb{R}^d)} \le \frac{1}{(4\pi|t|)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}} \tag{5.5.1}$$

for $1 \le p \le 2$ and $p' = (p-1)/p$. By using the unitarity of $e^{4\pi^2 itP}$ and [48, Theorem 10.1], we have the following:

Let $d \ge 3$ and P be an ultrahyperbolic operator. Let $2^* = \frac{2d}{d-2}$ and $2_* = \frac{2d}{d+2}$. Suppose that $u \in C(\mathbb{R}, L^2(\mathbb{R}^d))$ and $F \in L^2L^{2_*,2}$ satisfy

$$i\partial_t u(t) - Pu(t) = F, \quad u|_{t=0} = f \in L^2(\mathbb{R}^d). \tag{5.5.2}$$

Then there exists $C > 0$ such that for $0 < T \le \infty$ we have

$$\|u\|_{L^2(-T,T)L^{2^*,2}(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)} + C\|F\|_{L^2(-T,T)L^{2_*,2}(\mathbb{R}^d)}.$$

Replacing 3_*, 3^* in the arguments in Subsection 5.3.1 by 2_*, 2^* respectively, we have the following statements: Let $R_0(z) = (P-z)^{-1}$ for $z \in \mathbb{C}\setminus\sigma(P)$. Then there exists $C > 0$ such that for $z \in \mathbb{C}\setminus\sigma(P)$ and $f \in L^2(\mathbb{R}^d)\cap L^{2^*,2}(\mathbb{R}^d)$

$$\|R_0(z)f\|_{L^{2^*,2}(\mathbb{R}^d)} \le C\|f\|_{L^{2_*,2}(\mathbb{R}^d)}. \tag{5.5.3}$$

Moreover, for $w \in L^{d,\infty}(\mathbb{R}^d)$ we have

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \|wR_0(z)w\|_{B(L^2(\mathbb{R}^d))} \le C\|w\|^2_{L^{d,\infty}(\mathbb{R}^d)}.$$

In particular, $\|\langle x\rangle^{-1}R_0(z)\langle x\rangle^{-1}\|_{B(L^2(\mathbb{R}^d))}$ is bounded in $z \in \mathbb{C}\setminus\mathbb{R}$.

5.6 Resolvent near regular points

In this section, we study properties of the cut-off resolvent of H_0 near regular points of h_0.

Lemma 5.6.1. *Let $d \ge 1$ and $\varepsilon > 0$. Then,*

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \|\langle\eta_1\rangle^{-\frac{1}{2}-\varepsilon}(D_{\eta_1}-z)^{-1}\langle\eta_1\rangle^{-\frac{1}{2}-\varepsilon}\|_{B(L^2(\mathbb{R}^d))} < \infty. \tag{5.6.1}$$

If $0 < \varepsilon \le 1$, there exists $0 < \alpha_\varepsilon \le 1$ such that $\langle\eta_1\rangle^{-\frac{1}{2}-\varepsilon}(D_{\eta_1}-z)^{-1}\langle\eta_1\rangle^{-\frac{1}{2}-\varepsilon}$ is α_ε-Hölder continuous in the operator norm topology of $B(L^2(\mathbb{R}^d))$.

Proof. Suppose $\operatorname{Im} z > 0$. We denote $\eta = (\eta_1, \eta')$ for $\eta \in \mathbb{R}^d$, $\eta_1 \in \mathbb{R}$ and $\eta' \in \mathbb{R}^{d-1}$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|(D_{\eta_1}-z)^{-1}(\langle\eta\rangle^{-\frac{1}{2}-\varepsilon}u(\eta))| =& \left|2\pi i\int_{-\infty}^{\eta_1} e^{2\pi iz(\eta_1-s)}\langle s\rangle^{-\frac{1}{2}-\varepsilon}u(s,\eta')ds\right| \\ \le& C\|u(\cdot,\eta')\|_{L^2(\mathbb{R})}\end{aligned}$$

for $u \in L^2(\mathbb{R}^d)$, where $C > 0$ is independent of z, η and u. Thus, we have

$$\int_{\mathbb{R}} |\langle \eta_1 \rangle^{-\frac{1}{2}-\varepsilon}(D_{\eta_1} - z)^{-1}(\langle \eta_1 \rangle^{-\frac{1}{2}-\varepsilon} u(\eta))|^2 d\eta_1 \leq C\|u(\cdot, \eta')\|^2_{L^2(\mathbb{R})}$$

with some $C > 0$ which is independent of z and u. Integrating the above inequality with respect to $\eta' \in \mathbb{R}^{d-1}$, we obtain (5.6.1).

Suppose $\mathrm{Im}\, z, \mathrm{Im}\, z' > 0$. Set $w(\eta) = \langle \eta_1 \rangle^{-\frac{3}{2}-\varepsilon} u(\eta)$ for $\varepsilon > 0$. Using the Taylor theorem and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|\langle \eta_1 \rangle^{-\frac{3}{2}-\varepsilon}((D_{\eta_1} - z)^{-1} - (D_{\eta_1} - z')^{-1}) w(\eta)| \\
&= \left| 2\pi i \langle \eta_1 \rangle^{-\frac{3}{2}-\varepsilon} \int_{-\infty}^{\eta_1} (e^{2\pi i z(\eta_1 - s)} - e^{2\pi i z'(\eta_1 - s)}) w(s, \eta') ds \right| \\
&\leq 2\pi |z - z'| \langle \eta_1 \rangle^{-\frac{3}{2}-\varepsilon} \int_{-\infty}^{\eta_1} |(\eta_1 - s) w(s, \eta')| ds \\
&\leq C|z - z'| \langle \eta_1 \rangle^{-\frac{1}{2}-\varepsilon} \|u(\cdot, \eta')\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Integrating the square of the above inequality, we obtain

$$\|\langle \eta_1 \rangle^{-\frac{3}{2}-\varepsilon}((D_{\eta_1} - z)^{-1} - (D_{\eta_1} - z')^{-1})\langle \eta_1 \rangle^{-\frac{3}{2}-\varepsilon}\|_{B(L^2(\mathbb{R}^d))} \leq C|z - z'|. \tag{5.6.2}$$

Moreover, by (5.6.1), we have

$$\|\langle \eta_1 \rangle^{-\frac{1}{2}-\varepsilon}((D_{\eta_1} - z)^{-1} - (D_{\eta_1} - z')^{-1})\langle \eta_1 \rangle^{-\frac{1}{2}-\varepsilon}\|_{B(L^2(\mathbb{R}^n))} \leq C. \tag{5.6.3}$$

By using a complex interpolation between (5.6.2) and (5.6.3), we obtain the Hölder continuity of $\langle \eta_1 \rangle^{-\frac{1}{2}-\varepsilon}(D_{\eta_1} - z)^{-1}\langle \eta_1 \rangle^{-\frac{1}{2}-\varepsilon}$ in $B(L^2(\mathbb{R}^d))$ for $\varepsilon > 0$. The case $\mathrm{Im}\, z < 0$ is similarly proved.

□

For a proof of our main result in this section, we need the following two lemmas.

Lemma 5.6.2. *Let $\chi, \psi \in C_c^\infty(\mathbb{R}^d)$ satisfy* $\mathrm{supp}\, \chi \subset \{\psi = 1\}$. *Then, for $\alpha \in \mathbb{R}$ there exits $C > 0$ such that*

$$\|(1 - \psi)\langle D \rangle^\alpha \chi u\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{L^2(\mathbb{R}^d)}, \quad u \in L^2(\mathbb{R}^d).$$

Proof. This lemma follows from the disjoint support property of pseudodifferential operators. For the sake of the completeness of this paper, we give a self-contained proof. Considering the support property of χ and ψ, we observe that $c|x| \leq |x - y| \leq C|x|$ on $\mathrm{supp}\,(1 - \psi(x))\chi(y)$. Set $L = (1 + |x - y|^2)^{-1}(1 - (x - y) \cdot D_\xi)$, then note that $Le^{2\pi i(x-y)\cdot\xi} = e^{2\pi i(x-y)\cdot\xi}$. Integrating by parts, we have

$$\begin{aligned}
&|(1 - \psi(x))\langle D \rangle^\alpha \chi u(x)| \\
=&\left|(1 - \psi(x)) \int_{\mathbb{R}^{2d}} (L^*)^N(\langle \xi \rangle^\alpha) e^{2\pi i(x-y)\cdot\xi}(\chi u)(y) dy d\xi\right| \\
\leq&|1 - \psi(x)| \int_{\mathbb{R}^d} \frac{1}{|x - y|^{d+1}} |\chi(y) u(y)| dy \\
\leq&\langle x \rangle^{-2-2d}\|u\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

for any integer $N > \alpha + d + 1$. Integrating the square of the above inequality with respect to $x \in \mathbb{R}^d$, we obtain the desired result. $\square$

Lemma 5.6.3. *Let $U \subset \mathbb{T}^d$ be an open set and κ be a diffeomorphism from U onto an open set in $\mathbb{R}^d$. Set $u_\kappa(\eta) = u(\kappa^{-1}(\eta))$. Then, for $\chi \in C_c^\infty(U)$ and $\alpha \geq 0$, we have*

$$\|\chi_\kappa u_\kappa\|_{H^\alpha(\mathbb{R}^d)} \leq C\|u\|_{H^\alpha(\mathbb{T}^d)}, \quad u \in H^\alpha(\mathbb{T}^d)$$

for some $C > 0$.

Proof. We take $\varphi, \psi \in C_c^\infty(U)$ satisfying $\operatorname{supp}\chi \subset \{\varphi = 1\}$ and $\operatorname{supp}\varphi \subset \{\psi = 1\}$. Then we have

$$\|\chi_\kappa u_\kappa\|_{H^\alpha(\mathbb{R}^d)} \leq \|\psi_\kappa \langle D\rangle^\alpha \chi_\kappa u_\kappa\|_{L^2(\mathbb{R}^d)} + \|(1-\psi_\kappa)\langle D\rangle^\alpha \varphi_\kappa \chi_\kappa u_\kappa\|_{L^2(\mathbb{R}^d)}.$$

Using Lemma 5.6.2, we learn

$$\|(1-\psi_\kappa)\langle D\rangle^\alpha \varphi_\kappa \chi_\kappa u_\kappa\|_{L^2(\mathbb{R}^d)} \leq C\|\chi_\kappa u_\kappa\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{L^2(\mathbb{T}^d)}.$$

Due to the coordinate invariance of the Sobolev spaces and the support property of ψ_κ, we obtain

$$\|\psi_\kappa \langle D\rangle^\alpha \chi_\kappa u_\kappa\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{H^\alpha(\mathbb{T}^d)}.$$

This completes the proof. $\square$

Remark 5.6.4. The above lemma is trivial if 2α is an integer. The difficulty is due to the lack of the local property of the pseudodifferental operator $\langle D\rangle^{2\alpha}$ if 2α is not an integer.

We now state the main result of this section.

Proposition 5.6.5. *Suppose $d \geq 1$. Let $\chi \in C^\infty(\mathbb{T}^d)$ be a real-valued function satisfying* $\operatorname{supp}\chi \subset \{\nabla h_0 \neq 0\}$. *Then,*

$$\sup_{z \in \mathbb{C}\setminus\mathbb{R}} \|\langle x\rangle^{-\frac{1}{2}-\varepsilon}\chi(D)(H_0 - z)^{-1}\chi(D)\langle x\rangle^{-\frac{1}{2}-\varepsilon}\|_{B(\mathcal{H})} < \infty.$$

Moreover, $\langle x\rangle^{-\frac{1}{2}-\varepsilon}\chi(D)(H_0 - z)^{-1}\chi(D)\langle x\rangle^{-\frac{1}{2}-\varepsilon}$ is α_ε-Hölder continuous in the operator norm topology of $B(\mathcal{H})$, where α_ε is the constant in Lemma 5.6.1.

Proof. By using a partition of unity, we may suppose that $\operatorname{supp}\chi$ is small enough. Thus, we may suppose $\partial_{\xi_1} h_0(\xi) \neq 0$ on $\operatorname{supp}\chi$ without loss of generality. Set $\eta = \kappa(\xi) = (h_0(\xi), \xi')$. Then the inverse function theorem implies that κ is a diffeomorphism from a neighborhood of $\operatorname{supp}\chi$ onto its image. We denote $\kappa^{-1}(\eta) = (\xi_1(\eta), \eta')$ for

$\eta \in \kappa(\text{supp}\,\chi)$. We also denote $f_\kappa(\eta) = f(\kappa^{-1}(\eta))$. Using Lemma 5.6.1 and Lemma 5.6.3, we have

$$\begin{aligned}
&\left| \int_{\mathbb{T}^d} \bar{f}(\xi)\chi(\xi)^2 g(\xi)(h_0(\xi) - z)^{-1} d\xi \right| \\
&= \left| \int_{\mathbb{R}^d} \bar{f}_\kappa(\eta)\chi_\kappa(\eta)^2 g_\kappa(\eta)(\eta_1 - z)^{-1} \frac{d\eta}{|(\partial_{\xi_1} h_0)(\xi_1(\eta), \eta')|} \right| \\
&\leq C \|\chi_\kappa f_\kappa\|_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^d)} \|\chi_\kappa g_\kappa\|_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^d)} \\
&\leq C \|f\|_{H^{\frac{1}{2}+\varepsilon}(\mathbb{T}^d)} \|g\|_{H^{\frac{1}{2}+\varepsilon}(\mathbb{T}^d)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left| \int_{\mathbb{T}^d} \bar{f}(\xi)\chi(\xi)^2 g(\xi)((h_0(\xi) - z)^{-1} - (h_0(\xi) - z')^{-1}) d\xi \right| \\
&\leq C|z - z'|^{\alpha_\varepsilon} \|f\|_{H^{\frac{1}{2}+\varepsilon}(\mathbb{T}^d)} \|g\|_{H^{\frac{1}{2}+\varepsilon}(\mathbb{T}^d)}.
\end{aligned}$$

By using the Fourier transform, these imply the desired results. $\square$

Chapter 6

L^p-resolvent estimates

6.1 Introduction

In this chapter, we study L^p-estimates for resolvents of the Fourier multipliers and the scattering theory of the discrete Schrödinger operator, the fractional Schrödinger operators and the Dirac operators.

One of the interest in the scattering theory of the Schrödinger operator is to prove the asymptotic completeness of the wave operators:

$$W_\pm = s - \lim_{t\to\pm\infty} e^{it(-\Delta+V)}e^{-it(-\Delta)},$$

i.e. that $W_\pm$ are surjections onto the absolutely continuous subspace of $L^2(\mathbb{R}^d)$. Through the Kato's smooth perturbation theory, the asymptotic completeness of the wave operators is closely related to the limit absorption principle:

$$\sup_{z\in I_\pm\setminus I} \||V|^{\frac{1}{2}}(-\Delta - z)^{-1}|V|^{\frac{1}{2}}\|_{B(L^2(\mathbb{R}^d))} <\infty, \tag{6.1.1}$$

$$\sup_{z\in I_\pm\setminus I} \||V|^{\frac{1}{2}}(-\Delta + V - z)^{-1}|V|^{\frac{1}{2}}\|_{B(L^2(\mathbb{R}^d))} <\infty, \tag{6.1.2}$$

where $I \subset (0,\infty)$ is an interval and $I_\pm = \{z \in \mathbb{C} \mid \pm\mathrm{Im}\, z \geq 0\}$ and V is a real-valued function. A strong tool for proving (6.1.1) and (6.1.2) is the Mourre theory [53], which gives sufficient conditions that (6.1.1) and (6.1.2) hold.

On the other hands, Kenig, Ruiz and Sogge [50] establish the L^p-type limiting absorption principle for the free Schrödinger operator:

$$\|(-\Delta - z)^{-1}\|_{B(L^p(\mathbb{R}^d),L^q(\mathbb{R}^d))} \leq C_{p,q}|z|^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-1},\ z \in \mathbb{C}\setminus[0,\infty),\ d\geq 3 \tag{6.1.3}$$

where $C_{p,q} > 0$ is independent of $z \in \mathbb{C}\setminus[0,\infty)$ and $(1/p, 1/q) \in (0,1)\times(0,1)$ satisfies $2/(d+1) \leq 1/p - 1/q \leq 2/d$, $(d+1)/2d < 1/p$ and $1/q < (d-1)/(2d)$. (6.1.3) is also proved by Kato and Yajima [47] independently when $1/p + 1/q = 1$, and applied to the scattering theory of the Schrödinger operator $-\Delta + V$, where $V \in L^p(\mathbb{R}^d)$, $d/2 \leq$

$p < (d+1)/2$ is real-valued. Note that (6.1.1) for $V \in L^p(\mathbb{R}^d)$ for $d/2 \le p \le (d+1)/2$ follow from (6.1.3) and Hölder's inequality. Goldberg and Schlag [24] proved the L^p-type limiting absorption principle for Schrödinger operator $-\Delta + V$ with a real-valued potential $V \in L^r(\mathbb{R}^d) \cap L^{3/2}(\mathbb{R}^d)$, $r > 3/2$:

$$\sup_{\mathrm{Re}\, z \ge \lambda_0, 0 < \pm \mathrm{Im}\, z \le 1} \|(-\Delta + V - z)^{-1}\|_{B(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))} \le C(\mathrm{Re}\, z)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-1},$$

where $\lambda_0 > 0$, $d = 3$, $p = 4/3$ and $q = 4$. The strategy of the proof in [24] is to replace the L^2-trace theorem in the proof of the classical Agmon-Kato-Kuroda theorem [61, Theorem XIII. 33] by Stein-Tomas L^p-restriction theorem for the sphere [73]. Ionescu and Schlag [37] extends the result of [24] to a large class of potentials V, which contains $L^p(\mathbb{R}^d)$, $d/2 \le p \le (d+1)/2$, the global Kato class potentials and some perturbations of first order operators. See also the recent works by Huang, Yao, Zheng [31] and Mizutani [53]. Moreover, in [37], it is also proved that existence and asymptotic completeness of the wave operators. We note that there are no positive eigenvalues of $-\Delta + V$ when $V \in L^p(\mathbb{R}^d)$, $d/2 \le p \le (d+1)/2$ and it is false if $p > (d+1)/2$ ([32] and [51]).

In this paper, for a large class of operators $T(D)$ on X^d, we study uniform resolvent estimates, Hölder continuity of the resolvent and Carleman type inequalities for Fourier multipliers on X^d, where $X = \mathbb{R}$ or $X = \mathbb{Z}$. The uniform resolvent estimates for a Fourier multipliers are investigated in [11] and [12] in the duality line when $X = \mathbb{R}$ in order to study the Lieb-Thirring type bounds for fractional Schrödinger operators and Dirac operators. One of the purpose is to prove the uniform resolvent estimates away form the duality line and to extend to the case of $X = \mathbb{Z}$. To prove this, we follow the argument in [26, Appendix] for the Laplacian on the Euclidean space, however, the argument in [26] does not cover the general case since in the proof of [26, Theorem 6], the spherical symmetry and the Stein-Tomas theorem for the sphere are crucial. Moreover, we study the scattering theory of the discrete Schrödinger operator, the fractional Schrödinger operators and the Dirac operators. We note that the limiting absorption principle for free discrete Schrödinger operators is studied in [34], [52] and [68]. In [52], the scattering theory of the discrete Schrödinger operators perturbed by L^p-potentials are studied for a range of p. In [68], it is proved that the range of (p, q) which the uniform resolvent estimate holds for the discrete Schrödinger operators differs from the one for the continuous Schrödinger operators when $d \ge 5$.

We remark that almost all results in this paper can be extended to the Lorentz space $L^{p,r}$ by real interpolation. For simplicity we do not mention this below.

Throughout this paper, we denote $X^d = \mathbb{Z}^d$ or $\mathbb{R}^d$ for an integer $d \ge 2$. We denote μ by the Lebesgue measure if $X^d = \mathbb{R}^d$ by the counting measure if $X^d = \mathbb{Z}^d$. Moreover, we write $\widehat{X^d} = \mathbb{R}^d$ if $X^d = \mathbb{R}^d$ and $\widehat{X^d} = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ if $X^d = \mathbb{Z}^d$. We often use $[-1/2, 1/2)^d \subset \mathbb{R}^d$ as a fundamental domain of $\mathbb{T}^d$.

Let $T \in C^\infty(\widehat{X^d}, \mathbb{R})$. Moreover, we assume $T \in \mathcal{S}'(\mathbb{R}^d)$ if $X = \mathbb{R}$. We denote the set of all critical values of T by $\Lambda_c(T)$ and set $M_\lambda = \{\xi \in \widehat{X^d} \mid T(\xi) = \lambda\}$ for $\lambda \in \mathbb{R}$. We denote the induced surface measure by μ_λ away from the critical points of T. Moreover, for $I \subset \mathbb{R}$, we write $I_\pm = \{z \in \mathbb{C} \mid \mathrm{Re}\, z \in I,\ \pm \mathrm{Im}\, z \ge 0\}$.

Set

$$S_k = \{(\frac{1}{p}, \frac{1}{q}) \in [0,1] \times [0,1] \mid \frac{1}{q} \leq \frac{1}{p} - \frac{1}{k+1}, \frac{1+k}{1+2k} < \frac{1}{p}, \frac{1}{q} < \frac{k}{1+2k}\}. \tag{6.1.4}$$

Assumption E. Let $U \subset \widehat{X^d}$ be a relativity compact open set and $I \subset \mathbb{R}$ be an compact interval. Suppose $\partial_\xi T(\xi) \neq 0$ for $\xi \in \bar{U}$. The Fourier transform of the induced surface measure satisfies the following estimate: For any $\chi \in C_c^\infty(\widehat{X^d})$ supported in U, there exists $C > 0$ such that

$$|\int_{M_\lambda} e^{2\pi i x \cdot \xi} \chi(\xi) d\mu_\lambda(\xi)| \leq C(1+|x|)^{-k},\ x \in X^d, \lambda \in I. \tag{6.1.5}$$

Remark 6.1.1. If $\partial_{\xi_d} T \neq 0$ on $\operatorname{supp}\chi$ and $\operatorname{supp}\chi$ is small enough, (6.1.5) is rewritten as

$$|\int_{\widehat{X^{d-1}}} e^{2\pi i (x' \cdot \xi' + x_d h_\lambda(\xi'))} \chi(\xi', h_\lambda(\xi')) d\xi'| \leq C'(1+|x|)^{-k},\ x \in X^d, \lambda \in I$$

where $\xi = (\xi', \xi_d)$ and $M_\lambda = \{(\xi', \xi_d) \in \widehat{X^d} \mid \xi_d = h_\lambda(\xi')\}$. Moreover, if (6.1.5) holds, then there exits $N \geq 0$ such that

$$|\int_{\widehat{X^{d-1}}} e^{2\pi i (x' \cdot \xi' + x_d h_\lambda(\xi'))} b(\xi') d\xi'| \leq C \sum_{|\alpha| \leq N} \sup_{\xi' \in \widehat{X^{d-1}}} |\partial_{\xi'}^\alpha b(\xi')|$$

where $b \in C_c^\infty(\widehat{X^{d-1}})$ which is supported in $\{\xi' \mid (\xi', h_\lambda(\xi')) \in \operatorname{supp}\chi\}$ and C is independent of b.

Example 1. Suppose that $M_\lambda \cap \operatorname{supp}\chi$ has at least m nonvanishing principal curvature curvature at every point, then (6.1.5) holds for $k = m/2$ by the stationary phase theorem.

Set $R_0^\pm(z) = (T(D) - z)^{-1}$ for $z \in \{z \in \mathbb{C} \mid \pm \mathrm{Im}\, z > 0\}$. Moreover, for a signature $\pm$, we define $\chi(D)R_0^\pm(\lambda \pm i0)$ if $\partial_\xi T \neq 0$ on $\operatorname{supp}\chi$ by the Fourier multiplier with its symbol $\chi(\xi)(T(\xi) - \lambda \pm i0)^{-1}$. For $1 \leq p \leq \infty$, $L^p(X^d)$ denotes the Lebesgue space with the Lebesgue measure if $X = \mathbb{R}$ and with the counting measure if $X = \mathbb{Z}$.

Our first result is the following:

Theorem 6.1.2. *Let $T \in C^\infty(\widehat{X^d}, \mathbb{R})$ and let I be a compact interval of $\mathbb{R}$. Suppose that $T^{-1}(I)$ is compact. Fix a signature $\pm$ and let $U \subset \widehat{X^d}$ be an open set. Suppose that* (6.1.5) *holds for $\lambda \in I$ and $\chi \in C_c^\infty(\widehat{X^d})$ with* $\operatorname{supp}\chi \subset U$.

(*i*) *There exists such that*

$$\sup_{z \in I_\pm} \|\chi(D)R_0^\pm(z)\|_{B(L^p(X^d), L^q(X^d))} < \infty,$$

for $(1/p, 1/q) \in S_k$.

(ii) *Set* $k_\delta = k - \delta$ *for* $0 < \delta \le 1$ *and* $\beta_\delta = (2/p-1)\delta$. *Then*

$$\sup_{z,w\in I_\pm, |z-w|\le 1} |z-w|^{-\beta_\delta} \|\chi(D)(R_0^\pm(z) - R_0^\pm(w))\|_{B(L^p(X^d), L^{p^*}(X^d))} < \infty,$$

for $(1/p, 1/p^*) \in S_{k_\delta}$, *where* $p^* = p/(p-1)$.

(iii) *Suppose* $X = \mathbb{R}$. *Under Assumption E, for* $(1/p, 1/q) \in S_k$, *there exists* $C_{N,p,q} > 0$ *such that*

$$\|\mu_{N,\gamma}(x)\chi(D)u\|_{L^q(\mathbb{R}^d)\cap \mathcal{B}^*} \le C_{N,p,q} \|\mu_{N,\gamma}(x)(T(D)-\lambda)\chi(D)u\|_{L^p(\mathbb{R}^d)+\mathcal{B}}$$

for $u \in \mathcal{S}(\mathbb{R}^d)$.

6.1.1 Applications to the fractional Schrödinger operators and the Dirac operators

Let $n = 2^{d/2}$ if d is even and $n = 2^{(d+1)/2}$ if d is odd. We define the Dirac operators on $\mathbb{R}^d$:

$$\mathcal{D}_0 = \sum_{j=1}^d \alpha_j D_j,\ \ \mathcal{D}_1 = \sum_{j=1}^d \alpha_j D_j + \alpha_{d+1},$$

where α_j are $n\times n$ Hermitian matrix and satisfy the Clifford relations:

$$\alpha_j\alpha_k + \alpha_k\alpha_j = -2\delta_{jk} I_{n\times n}$$

and $D_j = \partial_{x_j}/(2\pi i)$. Note that if we define $D_{d+1} = mI_{n\times n}$, then

$$\mathcal{D}_0^2 = -(\sum_{j=1}^d I_{n\times n} D_j^2) = -\Delta\cdot I_{n\times n},\ \ \mathcal{D}_1^2 = (-\Delta+1)\cdot I_{n\times n},$$

where we denote $\Delta = (\sum_{j=1}^d \partial_{x_j}^2)/(4\pi^2)$. In this subsection, we suppose that $T(D)$ is the one of the following operators:

$$T(D) = (-\Delta)^{s/2},\ T(D) = (-\Delta+1)^{s/2} - 1,\ T(D) = \mathcal{D}_0,\ T(D) = \mathcal{D}_1,$$

where $0 < s < d$. We use the convention that $s = 1$ when $T(D) = \mathcal{D}_0$ or $T(D) = \mathcal{D}_1$. Moreover, we denote the product space Z^n for a function space Z by simply Z when $T(D) = \mathcal{D}_0$ or $T(D) = \mathcal{D}_1$. As is noted in [11, §2],

$$\Lambda_c((-\Delta)^{s/2}) = \begin{cases} \{0\} & \text{if } s > 1, \\ \emptyset & \text{if } s \le 1, \end{cases} \quad \Lambda_c((-\Delta+1)^{s/2} - 1) = \{0\},$$

and

$$\Lambda_c(\mathcal{D}_0) = \{0\}, \quad \Lambda_c(\mathcal{D}_1) = \{-1, 1\}.$$

Moreover, $T(D)$ is self-adjoint on its domain $H^s(\mathbb{R}^d)$ by the elliptic regularity.

Let Y_1, Y_2 be Banach spaces such that

$$(Y_1, Y_2) \in \bigcup_{(\frac{1}{p},\frac{1}{q})\in S_{\frac{d-1}{2}}} \{L^p(\mathbb{R}^d)\} \times \{L^q(\mathbb{R}^d)\}, \tag{6.1.6}$$

if $2d/(d+1) \leq s < d$ and

$$(Y_1, Y_2) \in \bigcup_{\substack{(\frac{1}{p_1},\frac{1}{q_1})\in S_{\frac{d-1}{2}}, \\ \frac{1}{p_2}-\frac{1}{q_2}\leq \frac{s}{d}}} \{L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d)\} \times \{L^{q_1}(\mathbb{R}^d) \cap L^{q_2}(\mathbb{R}^d)\}, \tag{6.1.7}$$

if $0 < s < \frac{2d}{d+1}$.

A part of the following estimate is a generalization of [11, Theorem3.1].

Theorem 6.1.3. *Let $I \subset \mathbb{R} \setminus \Lambda_c(T(D))$ be a compact interval. We define $R_0^{\pm}(\lambda)$ for $\lambda \in I$ by the Fourier multiplier of the distribution $(T(\xi) - (\lambda \pm i0))^{-1}$, where this distribution is well-defined since $T(\xi)$ has no critical points in $T^{-1}(I)$.*

(i) We have

$$\sup_{z\in I_{\pm}} \|R_0^{\pm}(z)\|_{B(Y_1,Y_2)} < \infty.$$

(ii) Let (Y_1, Y_2) be satisfying $p = q$ in (6.1.6) if $2d/(d+1) \leq s < d$ and $p_1 = q_1$ in (6.1.7) if $0 < s < 2d/(d+1)$. Let $0 < \delta \leq 1$ and $\beta_\delta = (2/p-1)\delta$. Then

$$\sup_{z,w\in I_{\pm}, |z-w|\leq 1} |z-w|^{-\beta_\delta} \|(R_0^{\pm}(z) - R_0^{\pm}(w))\|_{B(Y_1,Y_2)} < \infty.$$

(iii) Let $V \in L^{(d+1)/2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d, \mathbb{R})$. Set $H_0 = T(D)$ and $H = H_0 + V$ denotes the unique self-adjoint extensions of $T(D)|_{C_c^\infty(\mathbb{R}^d)}$ and $T(D) + V|_{C_c^\infty(\mathbb{R}^d)}$ respectively. Then the wave operators

$$W_{\pm} = s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}$$

exist and are complete, i.e. the ranges of $W_{\pm}$ are the absolutely continuous subspace $\mathcal{H}_{\rm ac}(H)$ of H.

(iv) Let $V \in L^{(d+1)/2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d, \mathbb{R})$. Assume $s > 1/2$ only when $T(D) = (-\Delta)^{s/2}$ with $2s \notin \mathbb{N}$. Then the set of nonzero eigenvalues $\sigma_{pp}(H) \setminus \{0\}$ is discrete in $\mathbb{R} \setminus \{0\}$. Moreover, each eigenvalue in $\sigma_{pp}(H) \setminus \{0\}$ has finite multiplicity.

Remark 6.1.4. (i) is proved in [11] if $1/p + 1/q = 1$. In [31], (i) is proved when $T(D) = (-\Delta)^{s/2}$ for $2d/(d+1) \leq s < d$.

Remark 6.1.5. In (iii) and (iv), the condition $V \in L^\infty(\mathbb{R}^d)$ is expected to be relaxed if we consider the appropriate selj-adjoint extension of $T(D) + V$. However, in order to avoid the technical difficulty, we assume $V \in L^\infty(\mathbb{R}^d)$.

Remark 6.1.6. When $T(D) = \mathcal{D}_0$ or $T(D) = (-\Delta)^{s/2}$, by a scaling argument as in [11, Remark 4.2], we have the uniform bound of $R_0^{\pm}(z)$ with $z \in \mathbb{C}_{\pm}$. Even when $T(D) = \mathcal{D}_1$ or $T(D) = (-\Delta+1)^{s/2} - 1$, the author expects to obtain the uniform bound of $R_0^{\pm}(z)$ with $z \in \mathbb{C}_{\pm}$ by further analysis.

Remark 6.1.7. When $T(D) = (-\Delta)^{s/2}$ or $T(D) = (-\Delta+1)^{s/2}-1$, under the assumption of part (iv), we can prove

$$\sup_{z \in I_{\pm}} \|(H-z)^{-1}\|_{B(X,X^*)} < \infty \tag{6.1.8}$$

for any compact set $I \subset \mathbb{R} \setminus (\sigma_{pp} \cup \{0\})$. In particular, the singular continuous spectrum of $T(D)$ is empty. For its proof, we may mimic the argument in [37, Section 4]. However, when $T(D) = \mathcal{D}_0$ or $T(D) = \mathcal{D}_1$, the author do not know whether (6.1.8) holds or not since the difference of the outgoing resolvent and incoming resolvent is not always positive definite:

$$\begin{aligned} R_0^+(\lambda) - R_0^-(\lambda) =& (\mathcal{D}_0 + \lambda)(\mathcal{R}_0^+(\lambda) - \mathcal{R}_0^-(\lambda)), \text{ if } T(D) = \mathcal{D}_0, \\ R_0^+(\lambda) - R_0^-(\lambda) =& (\mathcal{D}_1 + \lambda)(\mathcal{R}_1^+(\lambda) - \mathcal{R}_1^-(\lambda)), \text{ if } T(D) = \mathcal{D}_1, \end{aligned}$$

where $\mathcal{R}_0^{\pm}(\lambda) = (-\Delta - (\lambda \pm i0)^2)^{-1}$ and $\mathcal{R}_1^{\pm}(\lambda) = (-\Delta + 1 - (\lambda \pm i0)^2)^{-1}$. See the arguments in [37, Proof of Theorem 1.3 (d) and (e)] or [61, Lemma 8 in the proof of Theorem XIII.33].

Remark 6.1.8. Under the assumption of (iv), we can prove that each eigenfunction u of H associated with eigenvalue $\lambda \in \mathbb{R} \setminus \{0\}$ satisfies

$$(1+|x|)^N u \in H^1(\mathbb{R}^d), \ N \geq 0$$

and $N < s - 1/2$ only when $T(D) = (-\Delta)^{s/2}$ with $s \notin 2\mathbb{N}$. The restriction $N < s - 1/2$ when $T(D) = (-\Delta)^{s/2}$ with $s \notin 2\mathbb{N}$ is needed due to the singularity of the symbol $T(\xi) = |\xi|^s$ at $\xi = 0$.

6.1.2 Scattering theory for the discrete Schrödinger oeprators

The scattering theory of the discrete Schrödinger operators is studied in [52] for the potential $V \in L^p(\mathbb{Z}^d)$, with $1 \leq p < 6/5$ if $d = 3$ and $1 \leq p < 3d/(2d+1)$ if $d \geq 4$. In this subsection, we extend their results to when $V \in L^p(\mathbb{Z}^d)$ for $1 \leq p \leq d/3$ at the cost of the restriction of the dimension: $d \geq 4$.

We define the discrete Schrödinger operator:

$$H_0 u(x) = -\sum_{|x-y|=1, y \in \mathbb{Z}^d} (u(x) - u(y)), \quad x \in \mathbb{Z}^d.$$

Note that H_0 is a bounded self-adjoint operator on $L^2(\mathbb{Z}^d)$. We write

$$h_0(\xi) = 4\sum_{j=1}^{d} \sin^2 \pi\xi_j \text{ for } \xi \in \mathbb{T}^d, \ H_0 = h_0(D)$$

and hence the spectrum $\sigma(H_0)$ of H_0 is equal to $[0, 4d]$. Moreover, $\sigma_{ac}(H_0) = [0, 4d]$, where $\sigma_{ac}(H_0)$ is the absolutely continuous spectrum of H_0. Set $R_0^{\pm}(z) = (H_0 - z)^{-1}$ for $\pm\mathrm{Im}\, z > 0$. Note that $\Lambda_c(h_0(D)) = \{4k\}_{k=0}^{d}$, where we recall that $\Lambda_c(h_0(D))$ is the set of all critical values of $h_0(\xi)$. Moreover, if $V \in L^p(\mathbb{Z}^d, \mathbb{R})$ for some $1 \leq p < \infty$, $H = H_0 + V$ is a bounded self-adjoint operator and $\sigma_{ess}(H) = [0, 4d]$ since $V \in L^p(\mathbb{Z}^d) \subset L^\infty(\mathbb{Z}^d)$ and $V(x) \to \infty$ as $|x| \to \infty$. Here $\sigma_{ess}(H)$ denotes the essential spectrum of H.

We define $R_0^{\pm}(\lambda)$ for $\lambda \in I$ by the Fourier multiplier of the distribution $(h_0(\xi) - (\lambda \pm i0))^{-1}$, where this distribution is well-defined by virtue of [68, Theoerem 1.8]. Note that we may take λ as a critical value. We recall that

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|R_0^{\pm}(z)\|_{B(L^p(\mathbb{Z}^d), L^{p^*}(\mathbb{Z}^d))} < \infty,$$

holds for $1 \leq p \leq d/3$ ([68, Proposition 3.3]).

Theorem 6.1.9. *Fix a signature $\pm$ and let $d \geq 4$.*

(i) Let $1 \leq p \leq d/3$. Then

$$\sup_{z \in \mathbb{C}_{\pm}} \|R_0^{\pm}(z)\|_{B(L^p(\mathbb{Z}^d), L^{p^*}(\mathbb{Z}^d))} < \infty.$$

(ii) Let $1 \leq p < d/3$. Take $0 < \delta \leq 1$ such that $p < 2/(3\delta/d + (d+3)/d)$. Then

$$\sup_{z, w \in \mathbb{C}_{\pm}, |z-w| \leq 1} |z - w|^{-\beta_\delta} \|(R_0^{\pm}(z) - R_0^{\pm}(w))\|_{B(L^p(\mathbb{Z}^d), L^q(\mathbb{Z}^d))} < \infty.$$

(iii) Let $V \in L^p(\mathbb{Z}^d)$ for $1 \leq p < d/3$ and set $V^{1/2} = \mathrm{sgn}\, V |V|^{1/2}$. Then, a map $z \in I_{\pm} \mapsto |V|^{1/2} R_0^{\pm}(z) |V|^{1/2}$ is Hölder continuous. Moreover, for $V \in L^{d/3}(\mathbb{Z}^d)$, it follows that a map $z \in I_{\pm} \mapsto |V|^{1/2} R_0^{\pm}(z) |V|^{1/2}$ is continuous.

(iv) Let $V \in L^{d/3}(\mathbb{Z}^d, \mathbb{R})$ and set $H = H_0 + V$. Then the wave operators

$$W_{\pm} = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$$

exist and are complete, i.e. the ranges of $W_{\pm}$ are the absolutely continuous subspace $\mathcal{H}_{\mathrm{ac}}(H)$ of H.

Remark 6.1.10. In Proposition 6.4.10, we prove that the range of p can be extended in the low energy or the high energy.

We fix some notations. For an integer $k \geq 1$, $C_c^\infty(X^k)$ denotes $C_c^\infty(\mathbb{R}^k)$ if $X = \mathbb{R}$ and the set of all finitely supported functions if $X = \mathbb{Z}$. For $1 \leq p \leq \infty$, we write $p^* = p/(p-1)$. We denote $t_+ = \max(t, 0)$ for $t \in \mathbb{R}$. We define the Bezov space $\mathcal{B}$ and

$\mathcal{B}^*$ by

$$\|u\|_{\mathcal{B}} = \|u\|_{L^2(|x|\leq 1)} + \sum_{j=1}^{\infty} 2^{j/2}\|u\|_{L^2(2^{j-1}\leq |x|<2^j)},$$
$$\|u\|_{\mathcal{B}^*} = \|u\|_{L^2(|x|\leq 1)} + \sup_{j\geq 1} 2^{-j/2}\|u\|_{L^2(2^{j-1}\leq |x|<2^j)},$$
$$\mathcal{B} = \{u \in L^2_{loc}(X^d) \mid \|u\|_{\mathcal{B}} < \infty\},\ \mathcal{B}^* = \{u \in L^2_{loc}(X^d) \mid \|u\|_{\mathcal{B}^*} < \infty\},$$
$$\mathcal{B}^*_0 = \{u \in \mathcal{B}^* \mid \limsup_{R\to\infty} \frac{1}{R}\int_{|x|\leq R} |u(x)|^2 dx = 0\}.$$

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6.2 Abstract theorem

In this section, we state abstract theorems which give estimates for some integral operators. Let $K \in L^\infty(X^d \times X^d)$. For $x, y \in X^d$, we denote

$$K(x,y) = K(x', y', x_d, y_d) = K_{x_d,y_d}(x', y'),\ x = (x', x_d),\ y = (y', y_d),$$

where $x', y' \in X^{d-1}$ and $x_d, y_d \in X$. Moreover, we denote

$$Kf(x) = \int_{X^d} K(x,y)f(y)dy,\ T_{x_d,y_d}g(x') = \int_{X^{d-1}} K_{x_d,y_d}(x', y')f(y')dy'$$

for $f \in C_c^\infty(X^d)$ and $g \in C_c^\infty(X^{d-1})$.

6.2.1 Estimates for integral operators on duality line

We consider the following assumptions:

Assumption F. There exists $C_0, C_1 > 0$ such that for any $x_d, y_d \in X$ and $g \in C_c^\infty(X^{d-1})$

$$\|T_{x_d,y_d}g\|_{L^2(X^{d-1})} \leq C_0\|g\|_{L^2(X^{d-1})}, \tag{6.2.1}$$
$$\|T_{x_d,y_d}g\|_{L^\infty(X^{d-1})} \leq C_1(1 + |x_d - y_d|)^{-k}\|g\|_{L^1(X^{d-1})}. \tag{6.2.2}$$

Remark 6.2.1. Suppose that we can write $K(x,y) = K_1(x'-y', x_d, y_d)$ for some $K_1 \in L^\infty(X^{d+1})$. Then Assumption F directly follows from the following estimates:

$$\| \int_{X^{d-1}} K_1(x', x_d, y_d) e^{-2\pi i x'\cdot\xi'} dx' \|_{L^\infty(\widehat{X^{d-1}_{\xi'}})} \le C_0,$$
$$\sup_{x'\in X^{d-1}} |K_1(x', x_d, y_d)| \le C_1(1+|x_d-y_d|)^{-k}.$$

Remark 6.2.2. By the Riesz-Thorin interpolation theorem, (6.2.1) and (6.2.2) imply

$$\|T_{x_d,y_d} g\|_{L^{p^*}(X^{d-1})} \le C_0^{2-\frac{2}{p}} C_1^{\frac{2}{p}-1}(1+|x_d-y_d|)^{-k(\frac{2}{p}-1)}\|g\|_{L^p(X^{d-1})}, \tag{6.2.3}$$

for $1 \le p \le 2$.

Proposition 6.2.3. *Suppose Assumption F. Then there exists a universal constant $M_d > 0$ and $M_{p,k} > 0$ such that*

$$(\sup_{R>0, x_0\in\mathbb{R}^d} \frac{1}{R}\int_{|x-x_0|\le R} |Kf(x)|^2 dx)^{\frac{1}{2}} \le M_d C_0 \|f\|_{\mathcal{B}},\ f \in \mathcal{B}, \tag{6.2.4}$$
$$\|Kf\|_{L^{p^*}(X^d)} \le M_{p,k} C_0^{2-\frac{2}{p}} C_1^{\frac{2}{p}-1}\|f\|_{L^p(X^d)},\ f \in L^p(X^d) \tag{6.2.5}$$

for $1 \le p \le 2(k+1)/(k+2)$.

Remark 6.2.4. (6.2.5) follows from Proposition 6.2.8 below under the assumption of Proposition 6.2.8. However, the proof below is simpler than the proof of Proposition 6.2.8.

Proof. By a density argument, we may assume $f \in C_c^\infty(X^d)$. We observe

$$\sup_{R>0, x_0\in X^d} \frac{1}{R}\int_{|x-x_0|<R} |Kf(x)|^2 dx \le \sup_{x_d\in\mathbb{R}} \|Kf(\cdot, x_d)\|^2_{L^2(X^{d-1})}, \tag{6.2.6}$$
$$\int_{\mathbb{R}} \|Kf(\cdot, y_d)\|_{L^2(X^{d-1})} dy_d \le M_d \|Kf\|_B, \tag{6.2.7}$$

with some universal constant $M_d > 0$. Using the Minkowski inequality and (6.2.1), we obtain (6.2.4).

Next, we prove (6.2.5). We set $L_p = C_0^{2-\frac{2}{p}} C_1^{\frac{2}{p}-1}$. By the Minkowski inequality and (6.2.3), we have

$$\begin{aligned}
\|Kf\|_{L^{p^*}(X^d)} =& \|\| \int_X T_{x_d,y_d}(f(\cdot, y_d)) dy_d \|_{L^{p^*}(X^{d-1}_{x'})}\|_{L^{p^*}(X_{x_d})} \\
\le& L_p \| \int_X (1+|x_d-y_d|)^{-k(\frac{2}{p}-1)} \|f(\cdot, y_d)\|_{L^{p^*}(X^{d-1}_{y'})} dy_d \|_{L^{p^*}(X_{x_d})} \\
\le& M_{p,k} L_p \|f\|_{L^p(X^d)},
\end{aligned}$$

where we use the fractional integration theorem in the last line. This gives (6.2.5). □

6.2.2 Estimates for integral operators away from duality line

For $x_d \in X$, we define T_{x_d} and $T^*_{x_d}$ by

$$T_{x_d}f(x') = Kf(x', x_d) = \int_{X^d} K(x,y)f(y)dy,\ T^*_{x_d}g(y) = \int_{X^{d-1}} \bar{K}(x,y)g(x')dx'.$$

We define

$$S_{x_d}(y_d, z_d)g(y') = \int_{X^{d-1}}\int_{X^{d-1}} \bar{K}(x,y)K(x,z)g(z')dz'dx'.$$

Note that

$$T^*_{x_d}T_{x_d}f(y) = \int_X (S_{x_d}(y_d, z_d)f(\cdot, z_d))(y')dz_d.$$

Next, we consider the following assumption.

Assumption G. There exists $C_2, C_3 > 0$ such that for any $x_d, y_d, z_d \in X$

$$\|S_{x_d}(y_d, z_d)g\|_{L^2(X^{d-1})} \leq C_2^2 \|g\|_{L^2(X^{d-1})}, \tag{6.2.8}$$

$$\|S_{x_d}(y_d, z_d)g\|_{L^\infty(X^d)} \leq C_3^2 (1 + |y_d - z_d|)^{-k} \|g\|_{L^1(X^{d-1})}. \tag{6.2.9}$$

Remark 6.2.5. Suppose that we can write $K(x,y) = K_1(x'-y', x_d, y_d)$ for some $K_1 \in L^\infty(X^{d+1})$. Then Assumption G directly follows from the following estimates:

$$\| \int_{X^{d-1}}\int_{X^{d-1}} e^{2\pi i y'\cdot\xi'} \bar{K}_1(x', x_d, y_d)K_1(x'-y', x_d, z_d)dx'dy'\|_{L^\infty(\widehat{X^{d-1}})} \leq C_2^2,$$

$$\sup_{y', z' \in X^{d-1}} |\int_{X^{d-1}} \bar{K}_1(x'-y', x_d, y_d)K_1(x'-z', x_d, z_d)dx'| \leq C_3^2(1 + |y_d - z_d|)^{-k}.$$

Remark 6.2.6. By the Riesz-Thorin interpolation theorem, (6.2.8) and (6.2.9) imply

$$\|S_{x_d}(y_d, z_d)g\|_{L^{p^*}(X^{d-1})} \leq (C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1})^2 (1 + |y_d - z_d|)^{-k(\frac{2}{p}-1)} \|g\|_{L^p(X^{d-1})}, \tag{6.2.10}$$

for $1 \leq p \leq 2$.

Proposition 6.2.7. *Suppose that K satisfies Assumption G. Then there exists a universal constant $M'_{p,k} > 0$ such that*

$$(\sup_{R>0, x_0 \in \mathbb{R}^d} \frac{1}{R}\int_{|x-x_0|\leq R} |Kf(x)|^2 dx)^{\frac{1}{2}} \leq M'_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1} \|f\|_{L^p(X^d)},\ f \in L^p(X^d), \tag{6.2.11}$$

for $1 \leq p \leq 2(k+1)/(k+2)$. Moreover, if $K^*(x,y) = \bar{K}(y,x)$ satisfies Assumption G, then it follows that

$$\|K^* f\|_{L^q(X^d)} \leq M'_{q/(q-1),k} C_2^{\frac{2}{q}} C_3^{1-\frac{2}{q}} \|f\|_{\mathcal{B}},\ f \in \mathcal{B}, \tag{6.2.12}$$

for $2(k+1)/k \leq q \leq \infty$.

Proof. By a density argument, we may assume $f \in C_c^\infty(X^d)$. First, we prove (6.2.11). Due to (6.2.6), it suffices to prove

$$\|T_{x_d} f\|_{L^2(X^{d-1})} \le M'_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1} \|f\|_{L^p(X^d)},\ f \in C_c^\infty(X^d). \tag{6.2.13}$$

By the standard T^*T argument, this estimate is equivalent to

$$\|T_{x_d}^* T_{x_d} f\|_{L^{p^*}(X^d)} \le (M'_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1})^2 \|f\|_{L^p(X^d)}.$$

We set $L_p = (C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1})^2$. Using the Minkowski inequality and (6.2.10), we have

$$\begin{aligned}
\|T_{x_d}^* T_{x_d} f\|_{L^{p^*}(X^d)} =& \|\| \int_X (S_{x_d}(y_d, z_d) f(\cdot, z_d))(y') dz_d \|_{L^{p^*}(X_{y'}^{d-1})} \|_{L^{p^*}(X_{y_d})} \\
\le& L_p \| \int_X (1 + |y_d - z_d|)^{-k(\frac{2}{p}-1)} \|f(\cdot, y_d)\|_{L^{p^*}(X_{y'}^{d-1})} dy_d \|_{L^{p^*}(X_{y_d})} \\
\le& (M'_{p,k})^2 L_p \|f\|_{L^p(X^d)},
\end{aligned}$$

where we use the fractional integration theorem (the Hardy-Littlewood-Sobolev theorem) in the last line. This proves (6.2.11).

Next, we prove (6.2.12). Replacing K in (6.2.13) by K^*, we have

$$\| \int_{X^d} \bar{K}(y, x) f(y) dy \|_{L^2(X_{x'}^{d-1})} \le M'_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1} \|f\|_{L^p(X^d)},\ f \in C_c^\infty(X^d).$$

By duality, we have

$$\| \int_{X^{d-1}} K(y, x) g(x') dx' \|_{L^q(X_y^d)} \le M'_{q/(q-1),k} C_2^{\frac{2}{q}} C_3^{1-\frac{2}{q}} \|g\|_{L^2(X^{d-1})},\ x_d \in X,$$

where $q = p^*$. By (6.2.7) and the Minkowski inequality, we obtain

$$\begin{aligned}
\|Kf\|_{L^q(X^d)} \le& \int_X \| \int_{X^{d-1}} K(x, y) f(y) dy' \|_{L^q(X_x^d)} dy_d \\
\le& M'_{q/(q-1),k} C_2^{\frac{2}{q}} C_3^{1-\frac{2}{q}} \int_X \|f(\cdot, y_d)\|_{L^2(X_{y'}^{d-1})} dy_d \\
\le& M'_{q/(q-1),k} C_2^{\frac{2}{q}} C_3^{1-\frac{2}{q}} \|f\|_{\mathcal{B}}.
\end{aligned}$$

□

We impose the additional assumption.

Assumption H. There exists $C_4 > 0$ such that

$$|K(x, y)| \le C_4 (1 + |x - y|)^{-k},\ x \in X^d.$$

Under Assumption G and H, we obtain the estimates similar to (6.2.5) away from the Hölder exponent.

Proposition 6.2.8. *Suppose that K and $K^*(x,y) = \bar{K}(y,x)$ satisfy Assumption G and H. Then there exists a universal constant $L'_{p,,q,k} > 0$ such that*

$$\|Kf\|_{L^q(X^d)} \leq L'_{p,q,k} C_{p,q,k,l} \|f\|_{L^p(X^d)}, \; f \in L^p(X^d),$$

where $1/p - 1/q = 1/l$ and

$$C_{p,q,k,l} = \begin{cases} C_2^{\frac{2}{p^*}} C_3^{\frac{2}{p}-1} C_4^{1-\frac{2}{q}}, & \text{if } 1 \leq p \leq \frac{(k+1)(2k+1)}{k^2+3k+1}, q > \frac{1+2k}{k}, \frac{k+1}{p^*k} \leq \frac{1}{q}, \\ C_2^{\frac{2(k+1)}{2k+1}(1-\frac{1}{l})} C_3^{\frac{2(k+1)-l}{(2k+1)l}} C_4^{\frac{l+2k}{(2k+1)l}}, & \text{if } 1 \leq l \leq k+1, \frac{k}{(k+1)q} < \frac{1}{p^*} < \frac{k+1}{kq}, \\ C_2^{\frac{2}{q}} C_3^{\frac{2}{q^*}-1} C_4^{1-\frac{2}{p^*}}, & \text{if } 1 \leq p < \frac{1+2k}{1+k}, q \geq \frac{(2k+1)(k+1)}{k^2}, \frac{k+1}{kq} \leq \frac{1}{p^*}. \end{cases}$$

We prove this proposition by a series of lemmas.

Lemma 6.2.9. *Suppose that K satisfies Assumption G. Let $\psi \in C_c^\infty(\mathbb{R}^2)$. Define $K^j(x,y) = \psi((2x_d - z_d)/2^{j+1}, (2y_d - z_d)/2^{j+1}) K(x-y)$ for j and $z_d \in X$. Then for $1 \leq p \leq 2(k+1)/(k+2)$*

$$\|K^j f\|_{L^2(X^d)} \leq L' M'_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1} 2^{j/2} \|\psi\|_{L^\infty(X^2)} \|f\|_{L^p(X^d)}$$

with $L' > 0$ independent of $z_d \in X$ and j.

Proof. We take $L > 0$ such that $\operatorname{supp} \psi \subset B_L$, where $B_L \subset X^2$ is an open ball with radius L and with center 0. We observe

$$\|K^j f\|^2_{L^2(X^d)} = \int_{|x_d - z_d/2| \leq L2^j} \|K^j f(\cdot, x_d)\|^2_{L^2(X^{d-1})} dx_d$$

Replacing K in (6.2.13) with K^j, we have

$$\|K^j f(\cdot, x_d)\|_{L^2(X^{d-1})} \leq M'_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1} \|\psi\|_{L^\infty(X^2)} \|f\|_{L^p(X^d)}.$$

We note that there exists $L' > 0$ independent of z_d and j such that

$$(\int_{|x_d - z_d/2| \leq L2^j} dx_d)^{1/2} \leq L' 2^{j/2}.$$

Combining the above three inequality, we obtain the desired result. □

We need the following technical lemma in order to prove Lemma 6.2.11 below.

Lemma 6.2.10. *Let $F \in C_c^\infty(\mathbb{R})$. Then there exists $\psi \in C_c^\infty(\mathbb{R}^2)$ such that*

$$F(\frac{x_d - y_d}{2^j}) = L_j \int_X \psi(\frac{2x_d - z_d}{2^j}, \frac{2y_d - z_d}{2^j}) dz_d, \quad x_d, y_d \in \mathbb{R},$$

where $L_j = 2^{-j}$ if $X = \mathbb{R}$ and $2^{-j-2} \leq L_j \leq 2^{-j}$ if $X = \mathbb{Z}$.

Proof. We define $\psi \in C_c^\infty(\mathbb{R}^2)$ as follows: Take $\chi_2 \in C_c^\infty(\mathbb{R}, [0,1])$ such that $\int_{\mathbb{R}} \chi_2(x)dx = 2$ and $\operatorname{supp} \chi_2 \subset (-1/2, 1/2)$ if $X = \mathbb{R}$ and such that $\chi_2(t) = 1$ on $|t| \leq 1$ and $\chi(t) = 0$ on $|t| \geq 2$ if $X = \mathbb{Z}$. We define $\psi(z, z') = F(z - z')\chi_2(z + z')$, Then $F(x_d) = \int_X \psi(x_d + z, z)dz$ if $X = \mathbb{R}$ and

$$\begin{aligned}\int_X \psi(\frac{2x_d - z_d}{2^{j+1}}, \frac{2y_d - z_d}{2^{j+1}})dz_d =& \sum_{z_d \in \mathbb{Z}} \psi(\frac{2x_d - z_d}{2^{j+1}}, \frac{2y_d - z_d}{2^{j+1}}) \\ =& F(\frac{x_d - y_d}{2^j}) \sum_{z_d \in \mathbb{Z}} \chi_2(\frac{z_d}{2^j}).\end{aligned}$$

if $X = \mathbb{Z}$. We note

$$2^j \leq \sum_{z_d \in \mathbb{Z}} \chi_2(\frac{z_d}{2^j}) \leq 2^{j+2}.$$

We set $L_j = 1$ if $X = \mathbb{R}$ and $L_j = \sum_{z_d \in \mathbb{Z}} \chi_2(\frac{z_d}{2^j})$ if $X = \mathbb{Z}$ and we are done. □

The following lemma is a consequence of Lemma 6.2.9, however its proof is a bit technical due to the convolution type cut-off. The conclusion of the following lemma is same as [26, Lemma 1], where the uniform resolvent estimate of the Laplacian is studied. However, since their proof strongly depends on the spherical symmetry of the Laplacian and the Stein-Tomas theorem for the sphere, we cannot directly apply their argument to our cases. In order to overcome this difficulty, we borrow an idea from the proof of the Carleson-Sjölin theorem [29, Theorem 2.1].

Lemma 6.2.11. *Suppose that K satisfies Assumption G. Let $F \in C_c^\infty(\mathbb{R})$. Define $K^{j,\mathrm{conv}}(x, y) = F((x_d - y_d)/2^j)K(x, y)$ for non-negative integer j. Then for $1 \leq p \leq 2(k+1)/(k+2)$, there exists a universal constant $M''_{p,k}$ such that*

$$\|K^{j,\mathrm{conv}} f(x)\|_{L^2(X^d)} \leq M''_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1} 2^{\frac{1}{2}j} \|f\|_{L^p(X^d)} \tag{6.2.14}$$

Proof. By Lemma 6.2.10, we have

$$|K^{j,\mathrm{conv}} f(x)| \leq 2^{-2j} |\int_X K^{j,z_d}(x, y)dz_d|,$$

where we set $K^{j,z_d}(x, y) = K(x, y)\psi((2x_d - z_d)/2^{j+1}, (2y_d - z_d)/2^{j+1})$. Take $\varphi \in C_c^\infty(\mathbb{R})$ such that $\psi(x_d, y_d) = \psi(x_d, y_d)\varphi(y_d)$. We take $L > 0$ such that $\operatorname{supp} \psi \subset B_L$, where $B_L \subset X^2$ is an open ball with radius L and with center 0. We note

$$|\{z_d \in X \mid \psi(\frac{2x_d - z_d}{2^{j+1}}, \frac{2y_d - z_d}{2^{j+1}}) \neq 0\}| \leq L2^{j+1}.$$

Set $M = (M'_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1})^2$. Using the Cauchy-Schwarz inequality and Lemma 6.2.9, we have

$$\begin{aligned}\int_{X^d} |\int_X K^{j,z_d} f(x)dz_d|^2 dx \leq& L2^{j+1} \int_X \int_{X^d} |K^{j,z_d} f(x)|^2 dx dz_d \\ \leq& 2LL'M2^{2j} \int_X \|\varphi(\frac{2\cdot - z_d}{2^{j+1}}) f\|^2_{L^p(X^d)} dz_d.\end{aligned}$$

Since $p \leq 2$, by using the Minkowski inequality, we have

$$\begin{aligned}\int_X \|\varphi(\frac{2\cdot - z_d}{2^{j+1}})f\|_{L^p(X^d)}^2 dz_d \leq& \|\varphi(\frac{2\cdot - z_d}{2^{j+1}})\|_{L^2(X)}^2 \|f\|_{L^p(X^d)}^2 \\ \leq& L'' 2^j \|\varphi\|_{L^2(X)}^2 \|f\|_{L^p(X^d)}^2\end{aligned}$$

with L'' depends only on φ. Thus we obtain

$$\int_{X^d} |K^{j,conv} f(x)|^2 dx \leq (M''_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p}-1})^2 2^j \|f\|_{L^p(X^d)}^2,$$

where $(M''_{p,k})^2 = 2LL'L''(M'_{p,k})^2 \|\varphi\|_{L^2(X)}^2$. □

Corollary 6.2.12. *Suppose that K satisfies Assumption H. Then there exists a constant $L_1 > 0$ which depends only on F, d and k such that*

$$\|K^{j,conv} f\|_{L^\infty(X^d)} \leq L_1 C_4 2^{-jk} \|f\|_{L^1(X^d)}. \tag{6.2.15}$$

In addition, we suppose that K and $K^*(x,y) = \bar{K}(y,x)$ satisfy Assumption G. Set $1/p_1 = 1 - q/2p^*$ and $L_{2,p,q} = (M''_{p_1,k})^{2/q} L_1^{1-2/q}$. Then

$$\|K^{j,conv} f\|_{L^q(X^d)} \leq L_{2,p,q} C_2^{\frac{2}{p^*}} C_3^{\frac{2}{p}-1} C_4^{1-\frac{2}{q}} 2^{j\frac{1+2k}{q} - jk} \|f\|_{L^p(X^d)} \tag{6.2.16}$$

if $q \geq 2$ and $(k+1)(1-1/p)/k \leq 1/q$ and

$$\|K^{j,conv} f\|_{L^q(X^d)} \leq L_{2,q^*,p^*} C_2^{\frac{2}{q}} C_3^{\frac{2}{q^*}-1} C_4^{1-\frac{2}{p^*}} 2^{j\frac{(1+2k)}{p^*} - jk} \|f\|_{L^p(X^d)} \tag{6.2.17}$$

if $p \leq 2$ and $(k+1)/(kq) \leq 1 - 1/p$.

Proof. (6.2.15) follows from

$$\begin{aligned}\|K^{j,conv} f\|_{L^\infty(X^d)} \leq& \|F(\cdot/2^j)K\|_{L^\infty(X^d)} \|f\|_{L^1(X^d)} \\ \leq& L_1 C_4 2^{-jk} \|f\|_{L^1(X^d)}\end{aligned}$$

with some constant $L_1 > 0$ by Assumption H. By complex interpolating (6.2.14) and (6.2.15), we obtain (6.2.16). Since K^* also satisfies Assumption G and H, by duality, (6.2.17) holds.

□

Proof of Proposition 6.2.8. Take $\eta \in C_c^\infty(\mathbb{R}, [0,1])$ such that $\eta(t) = 1$ on $0 \leq t \leq 1$ and $\eta = 0$ on $t \geq 2$. Set $F(x) = \eta(|x|) - \eta(|x|/2)$. By Corollary 6.2.12, for $(k+1)(1 - 1/p)/k \leq 1/q$, $q > (1+2k)/k$, we have

$$\begin{aligned}\|Kf\|_{L^q(X^d)} =& \|\sum_{j=0}^\infty K^{j,conv} f\|_{L^q(X^d)} \leq \sum_{j=0}^\infty \|K^{j,conv} f\|_{L^q(X^d)} \\ \leq& L_{2,p,q} C_2^{\frac{2}{p^*}} C_3^{\frac{2}{p}-1} C_4^{1-\frac{2}{q}} \sum_{j=0}^\infty 2^{j/2 + jd(1/q-1/2)} \|f\|_{L^p(X^d)} \\ \leq& L'_{2,p,q} C_2^{\frac{2}{p^*}} C_3^{\frac{2}{p}-1} C_4^{1-\frac{2}{q}} \|f\|_{L^p(X^d)},\end{aligned}$$

where $L'_{2,p,q} = L_{2,p,q} \sum_{j=0}^{\infty} 2^{j/2+jd(1/q-1/2)}$. Similarly, for $(k+1)/(kq) \leq 1 - 1/p$, $p < (1+2k)/(1+k)$, we have

$$\|Kf\|_{L^q(X^d)} \leq L'_{2,q^*,p^*} C_2^{\frac{2}{p^*}} C_3^{\frac{2}{p}-1} C_4^{1-\frac{2}{p^*}} \|f\|_{L^p(X^d)}.$$

In order to prove the end point estimates, we use Bourgain's interpolation trick ([7], [8, §6.2], [43, Lemma 3.3]). This trick is also used in [2] for the Stein-Tomas theorem for a large class of measures in Euclidean space. See also [17] and [26]. We denote the Lorentz space for index $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$ by $L^{p,r}(X^d)$:

$$\|f\|_{L^{p,r}(X^d)} = \begin{cases} p^{\frac{1}{r}} (\int_0^\infty \mu(\{x \in X^d \mid |f(x)| > \alpha\})^{\frac{r}{p}} \alpha^{r-1} d\alpha)^{\frac{1}{r}}, & r < \infty, \\ \sup_{\alpha > 0} \alpha \mu(\{x \in X^d \mid |f(x)| > \alpha\})^{\frac{1}{p}}, & r = \infty, \end{cases}$$

$$L^{p,r}(X^d) = \{f : X^d \to \mathbb{C} \mid f : \text{measurable}, \|f\|_{L^{p,r}(X^d)} < \infty\}.$$

Bourgain's interpolation trick with (6.2.16) and (6.2.17) implies that for $1 \leq p \leq (k+1)(2k+1)/(k^2+3k+1), q = (1+2k)/k$, it follows that

$$\|Kf\|_{L^{q,\infty}(X^d)} \leq L'_{2,p,q} C_2^{\frac{2}{p^*}} C_3^{\frac{2}{p}-1} C_4^{1-\frac{2}{q}} \|f\|_{L^{p,1}(X^d)}$$

with a universal constant $L'_{2,p,q}$. Similarly, for $p = (1+2k)/(1+k)$, $q \geq (2k+1)(k+1)/k^2$, we have

$$\|Kf\|_{L^{q,\infty}(X^d)} \leq L'_{2,q^*,p^*} C_2^{\frac{2}{q}} C_3^{\frac{2}{q^*}-1} C_4^{1-\frac{2}{p^*}} \|f\|_{L^{p,1}(X^d)}.$$

By real interpolating above estimates, we complete the proof.

□

6.3 Uniform resolvent estimates

6.3.1 Proof of Theorem 7.1.1 (i) and (ii)

Proof of Theorem 7.1.1 (i) *and* (ii). We follows the argument as in [11, Lemma 3.3]. By using a partition of unity and a linear coordinate change, we may assume that $\partial_{\xi_d} T \neq 0$ on supp χ. Moreover, by the implicit function theorem, we may assume that for $\lambda \in I$, M_λ has the following graph representation:

$$M_\lambda \cap \operatorname{supp} \chi \subset \{(\xi', h_\lambda(\xi')) \in \widehat{X^d} \mid \xi' \in U\}$$

for some relativity compact open set $U \subset \widehat{X^{d-1}}$ and h_λ which is smooth with respect to $\xi' \in U$ and $\lambda \in I$ and

$$T(\xi) - \lambda = e(\xi, \lambda)(\xi_d - h_\lambda(\xi')), \tag{6.3.1}$$

where $e(\xi,\lambda) = \int_0^1 (\partial_{\xi_d} T)(\xi', t\xi_d + (1-t)h_\lambda(\xi'))dt$. Furthermore, we may assume $\min_{\xi \in \mathrm{supp}\,\chi, \lambda \in A} e(\xi,\lambda) > 0$ if necessary, we take supp χ small. Set

$$\begin{aligned} K_{z,\pm}(x) &= \int_{\widehat{X^d}} \frac{e^{2\pi i x\cdot\xi}\chi(\xi)}{T(\xi)-z} d\xi, \\ K_{z,w,\pm}(x) &= K_{z,\pm}(x) - K_{w,\pm}(x), \end{aligned}$$

where $\lambda = \mathrm{Re}\, z, \mu = \mathrm{Re}\, w \in I$ and $\pm\mathrm{Im}\, z, \pm\mathrm{Im}\, w \geq 0$. In order to prove Theorem 7.1.1, it suffices to show that $K_{z,\pm}$ satisfies Assumptions G and H, and that $K_{z,w,\pm}$ satisfies Assumption F.

Lemma 6.3.1. *Fix a signature $\pm$. For any $0 \leq \delta \leq 1$, there exists $C_0, C_1, C_{1,\delta} > 0$ such that for $x = (x', x_d) \in X^d$, $z, w \in I_\pm$ with $|z-w| \leq 1$, we have*

$$\begin{aligned} &\sup_{\xi' \in \widehat{X^{d-1}}} \left| \int_{X^{d-1}} K_{z,\pm}(y', x_d) e^{-2\pi i y'\cdot\xi'} dy' \right| \leq C_0, \; |K_{z,\pm}(x)| \leq C_1(1+|x|)^{-k} \\ &\sup_{\xi' \in \widehat{X^{d-1}}} \left| \int_{X^{d-1}} K_{z,w,\pm}(y', x_d) e^{-2\pi i y'\cdot\xi'} dy' \right| \leq 2C_0, \\ &|K_{z,w,\pm}(x)| \leq C_1' |z-w|^\delta (1+|x|)^{-k+\delta} \end{aligned}$$

Proof. Note that $\int_{X^{d-1}} K_{z,\pm}(y', x_d) e^{-2\pi i y'\cdot\xi'} dy' = \int_{\widehat{X}} \frac{e^{2\pi i x_d \xi_d}\chi(\xi)}{T(\xi)-z} d\xi_d$. If necessary we take supp χ is small, it suffices to replace the integration region by $\mathbb{R}$. Thus by (6.3.1), we have

$$\begin{aligned} \int_{\widehat{X}} \frac{e^{2\pi i x_d \xi_d}\chi(\xi)}{T(\xi)-z} d\xi_d &= \int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\chi(\xi)}{T(\xi)-z} d\xi_d \\ &= \int_{\mathbb{R}} \frac{e^{2\pi i x_d(\xi_d + h_\lambda(\xi'))}\chi(\xi', \xi_d + h_\lambda(\xi'))}{e(\xi', \xi_d + h_\lambda(\xi'), \xi_d)\xi_d - i\mathrm{Im}\, z} d\xi_d \\ &=: e^{2\pi i x_d h_\lambda(\xi')} \gamma_{z,\pm}(\xi', x_d). \end{aligned}$$

By using [11, (3.10)] for $\pm\mathrm{Im}\, z > 0$ and [12, (A.6)] for $\pm\mathrm{Im}\, z = 0$, we have

$$|\partial_{\xi'}^\alpha \gamma_{z,\pm}(\xi', x_d)| \leq C_\alpha \tag{6.3.2}$$

for $\alpha \in \mathbb{N}^{d-1}$. We will prove (6.3.2) in Lemma 6.5.3. Thus the first inequality holds. Moreover, we note that

$$K_{z,\pm}(x) = \int_{\widehat{X^{d-1}}} \gamma_{z,\pm}(\xi') e^{2\pi i(x'\cdot\xi' + x_d h_\lambda(\xi'))} d\xi'.$$

Since $\gamma_{z,\pm}$ is compactly supported in ξ'-variable, then (6.1.5) and (6.3.2) imply the second inequality. The estimates for $K_{z,w,\pm}(x)$ follow from the estimates

$$|\partial_{\xi'}^\alpha \gamma_{z,\pm}(\xi', x_d)| \leq C_\alpha' |z-w|^\delta (1+|x_d|)^\delta,$$

which is also proved after Lemma 6.5.3: (6.5.6). □

Lemma 6.3.2. *There exists $C_3 > 0$ such that*

$$|\int_{X^{d-1}}\int_{X^{d-1}} e^{2\pi i y'\cdot\xi'}\bar{K}_{z,\pm}(x', x_d - y_d)K_{z,\pm}(x'-y', x_d - z_d)dx'dy'| \leq C_0^2$$
$$|\int_{X^{d-1}} \bar{K}_{z,\pm}(x'-y', x_d - y_d)K_{z,\pm}(x'-z', x_d - z_d)dx'| \leq C_3^2(1+|y_d - z_d|)^{-k}$$

where $C_0 > 0$ is as in the proof of Lemma 6.3.1.

Proof. Note that

$$\int_{X^{d-1}}\int_{X^{d-1}} e^{2\pi i y'\cdot\xi'}\bar{K}_{z,\pm}(x', x_d - y_d)K_{z,\pm}(x'-y', x_d - z_d)dx'dy'$$
$$= e^{2\pi i(y_d - z_d)h_\lambda(\xi')}\gamma_{z,\pm}(\xi', x_d - z_d)\overline{\gamma_{z,\pm}(\xi', x_d - y_d)},$$

where $\gamma_{z,\pm}$ is as in the proof of Lemma 6.3.1. Moreover, we have

$$\int_{X^{d-1}} \bar{K}_{z,\pm}(x'-y', x_d - y_d)K_{z,\pm}(x'-z', x_d - z_d)dx'$$
$$= \int_{\widehat{X}^{d-1}} e^{2\pi i(y'-z')\cdot\xi' + 2\pi i(y_d - z_d)h_\lambda(\xi')}\gamma_{z,\pm}(\xi', x_d - z_d)\overline{\gamma_{z,\pm}(\xi', x_d - y_d)}d\xi'.$$

Thus (6.1.5) and (6.3.2) imply the conclusion.

□

Lemma 6.3.1 and 6.3.2 imply that $K_{z,\pm}$ satisfies Assumptions G and H and $K_{z,w,\pm}$ satisfies Assumption F. This completes the proof of Theorem 7.1.1.

□

Remark 6.3.3. In order to prove (i), it is sufficient to prove (i) for $\pm\mathrm{Im}\, z = 0$ by using the Phragmén-Lindelöf principle as in [64, Section 5.3]. See also [11, Appendix A] for the estimates of the Shatten norm of the resolvent. Here we avoid using the Phragmén-Lindelöf principle.

Corollary 6.3.4. *Let $r_1, r_2 \in (1, 4k+2]$ satisfying $1/r_1 + 1/r_2 \geq 1/(k+1)$. Then*

$$\sup_{z\in I_\pm} \|W_1\chi(D)R_0^\pm(z)W_2\|_{B(L^2(X^d))} \leq C\|W_1\|_{L^{r_1}(X^d)}\|W_2\|_{L^{r_2}(X^d)} \tag{6.3.3}$$

for $W_1 \in L^{r_1}(X^d)$ and $W_2 \in L^{r_2}(X^d)$. Moreover, let $W_1 \in L^{r_1}(X^d)$ and $W_2 \in L^{r_2}(X^d)$. Then it follows that $W_1\chi(D)R_0^\pm(z)W_2$ belongs to $B_\infty(L^2(X^d))$ and a map $z \in I_\pm \mapsto W_1\chi(D)R_0^\pm(z)W_2 \in B_\infty(L^2(X^d))$ is continuous in $z \in I_\pm$. In addition, for $r = r_1 = r_2 \in (1, 4k_\delta + 2)$, we have

$$\|W_1\chi(D)(R_0^\pm(z) - R_0^\pm(w))W_2\|_{B(L^2(X^d))} \leq C|z-w|^{\beta_\delta}\|W_1\|_{L^r(X^d)}\|W_2\|_{L^r(X^d)} \tag{6.3.4}$$

for $z, w \in I_\pm, |z-w| \leq 1$.

Proof. (6.3.3) and (6.3.4) follow from Theorem 7.1.1 and the Hölder inequality. For proving the other statements, we may assume $W_1, W_2 \in C_c^\infty(X^d)$ by $\varepsilon/3$-argument and (6.3.3). Since W_1 and W_2 are compactly supported and since the integral kernel of $\chi(D)R_0^\pm$ is in L^∞ by Lemma 6.3.1, then the integral kernel of $W_1\chi(D)R_0^\pm(z)W_2$ is square integrable and hence Hilbert-Schmidt. Thus it follows that $W_1\chi(D)R_0^\pm(z)W_2$ is compact. Moreover, by (6.3.4), we see that $W_1\chi(D)R_0^\pm(z)W_2$ is continuous in $z \in I_\pm$. The case of $W_1 \in L^{r_1}(X^d)$ and $W_2 \in L^{r_2}(X^d)$ follows from the $\varepsilon/3$-argument as in the proof of Lemma 6.4.9. □

6.3.2 Supersmoothing, Proof of Theorem 7.1.1 (iii)

In this subsection, we assume $X = \mathbb{R}$. The author expect that the following proposition with $X = \mathbb{Z}$ holds. However, we prove this with with $X = \mathbb{R}$ for possibly technical reason. We recall $\mu_{N,\gamma}(x) = (1+|x|^2)^N(1+\gamma|x|^2)^{-N}$. We restate Theorem 7.1.1 (iii):

Proposition 6.3.5. *Let $I \subset \mathbb{R}$ be a compact interval. Suppose $T^{-1}(I)$ is compact. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be supported in $T^{-1}(I)$. Under Assumption E, for $(1/p, 1/q) \in S_k$, there exists $C_{N,p,q} > 0$ such that*

$$\|\mu_{N,\gamma}(x)\chi(D)u\|_{L^q(\mathbb{R}^d)\cap\mathcal{B}^*} \leq C_{N,p,q}\|\mu_{N,\gamma}(x)(T(D)-\lambda)\chi(D)u\|_{L^p(\mathbb{R}^d)+\mathcal{B}} \tag{6.3.5}$$

for $u \in \mathcal{S}(\mathbb{R}^d)$.

Lemma 6.3.6. *Suppose that $m \in C^\infty(\mathbb{R}^d)$ satisfies*

$$|\partial_\xi^\alpha m(\xi)| \leq C_\alpha(1+|\xi|^2)^{-|\alpha|/2}$$

for $\alpha \in \mathbb{N}^d$. Let $1 < p < \infty$. We set $\tilde{\mu}_{N,\gamma}(x_d) = (1+|x_d|^2)^N(1+\gamma|x|^2)^{-N}$. Then we have

$$\|\mu(x)m(D)\mu(x)^{-1}\|_{B(L^p(\mathbb{R}^d))} \leq C_{N,m,p},\ \|\mu(x)m(D)\mu(x)^{-1}\|_{B(\mathcal{B}(\mathbb{R}^d))} \leq C_{N,m},$$
$$\|\mu(x)m(D)\mu(x)^{-1}\|_{B(\mathcal{B}^*(\mathbb{R}^d))} \leq C_{N,m}$$

if $\mu(x) \in \{\mu_{N,\gamma}(x), \mu_{N,\gamma}^{-1}(x), \tilde{\mu}_{N,\gamma}(x_d), \tilde{\mu}_{N,\gamma}^{-1}(x_d)\}$, where $C_{N,m,p}$ and $C_{N,m}$ are independent of $0 < \gamma \leq 1$ and depends only on d, N and finite number of C_M.

Proof. The proof is same as in the proof of [37, (3.7)]. In fact, though the range of p is restricted in [37], the proof succeeds even when $1 < p < \infty$. □

Lemma 6.3.7.

(i) *For $\alpha \in \mathbb{N}^d$, we have*

$$\partial_x^\alpha \mu_{N,\gamma}(x) = b_\alpha(x)\mu_{N,\gamma}(x) \tag{6.3.6}$$
$$\partial_x^\alpha \mu_{N,\gamma}(x)^{-1} = b'_\alpha(x)\mu_{N,\gamma}(x)^{-1} \tag{6.3.7}$$

for some functions $b_\alpha, b'_\alpha \in C^\infty(\mathbb{R}^d)$ such that for $\beta \in \mathbb{N}^d$,

$$|(1+|x|^2)^{(|\alpha|+|\beta|)/2}\partial_x^\beta b_\alpha(x)| \leq C_{\alpha,\beta,N},\ |(1+|x|^2)^{(|\alpha|+|\beta|)/2}\partial_x^\beta b'_\alpha(x)| \leq C_{\alpha,\beta,N}$$

with some constant $C_{\alpha,\beta,N}$ which is independent of $0 < \gamma \leq 1$.

(*ii*) *There exists* $C_N > 0$ *independent of* $0 < \gamma \le 1$ *such that*

$$\mu_{N,\gamma}(x)\mu_{N,\gamma}(y)^{-1} + \mu_{N,\gamma}(y)\mu_{N,\gamma}(x)^{-1} \le C_N(1 + |x-y|^2)^N,\ x, y \in \mathbb{R}^d.$$

Proof. (*i*) We prove (6.3.6) only. The proof of (6.3.7) is similar. We prove (6.3.6) by induction in $|\alpha|$. If $\alpha = 0$, then (6.3.6) is trivial. Let $M > 0$ be an integer. Suppose that (6.3.6) holds for $|\alpha| \le M$. If $|\alpha| = M$, by the induction hypothesis, we have

$$\begin{aligned}\partial_{x_j}\partial_x^\alpha \mu_{N,\alpha}(x) =&(\partial_{x_j} b_\alpha(x))\mu_{N,\gamma}(x) + b_\alpha(x)\partial_{x_j}\mu_{N,c}(x)\\ =&((\partial_{x_j} b_\alpha)(x) + b_\alpha(x) b_{e_j}(x))\mu_{N,\gamma}(x),\end{aligned}$$

where $(e_1, ..., e_d)$ is a standard basis in $\mathbb{R}^d$. Thus, if we set $b_{\alpha+e_j}(x) = (\partial_{x_j} b_\alpha)(x) + b_\alpha(x) b_{e_j}(x)$, then $|(1 + |x|^2)^{(|\alpha|+|\beta|)/2}\partial_x^\beta b_\alpha(x)| \le C_{\alpha,\beta,N}$ follows. This proves (6.3.6) for $|\alpha| = M + 1$. (*ii*) is easily proved. □

Corollary 6.3.8. *For* $k \in \mathbb{R}$ *we define* $\Lambda_k = (I - \Delta)^{k/2}$. *Then*

$$\begin{aligned}\|\mu\Lambda_k\mu^{-1}\Lambda_{-k}\|_{B(L^p(\mathbb{R}^d))} + \|\mu\Lambda_k\mu^{-1}L_{-k}\|_{B(\mathcal{B})} + \|\mu\Lambda_k\mu^{-1}L_{-k}\|_{B(\mathcal{B}^*)} \le& C_{N,k,p},\\ \|\Lambda_k\mu\Lambda_{-k}\mu^{-1}\|_{B(L^p(\mathbb{R}^d))} + \|\Lambda_k\mu\Lambda_{-k}\mu^{-1}\|_{B(\mathcal{B})} + \|\Lambda_k\mu\Lambda_{-k}\mu^{-1}\|_{B(\mathcal{B}^*)} \le& C_{N,k,p},\end{aligned}$$

with some $C_{N,k,p} > 0$ *independent of* $0 < \gamma \le 1$ *for* $\mu \in \{\mu_{N,\gamma}, \mu_{N,\gamma}^{-1}\}$ *and* $1 < p < \infty$.

Proof. The proof is same as in [37, Lemma 3.2] by virtue of Lemma 6.3.6 and 6.3.7. □

Proof of Proposition 6.3.5. Let $Y_1 \in \{L^p(\mathbb{R}^d), \mathcal{B}\}$ and $Y_2 \in \{L^q(\mathbb{R}^d), \mathcal{B}^*\}$. If necessary, we may assume supp χ is small enough. In fact, by using a partition of unity $\{\chi_j\}_{j=1}^M$ such that $\sum_{j=1}^M \chi_j = 1$ on supp χ, we have

$$\|\mu_{N,\gamma}(x)\chi(D)u\|_{Y_2} \le \sum_{j=1}^M \|\mu_{N,\gamma}(x)(\chi_j\chi)(D)u\|_{Y_2},$$

$$\sum_{j=1}^M \|\mu_{N,\gamma}(x)(T(D) - \lambda)(\chi_j\chi)(D)u\|_{Y_1} \le C_{N,m,p}\|\mu_{N,\gamma}(x)(T(D) - \lambda)\chi(D)u\|_{Y_1},$$

where we use the triangle inequality in the first line and Lemma 6.3.6 in the second line. Thus we may replace $\chi(D)$ by $(\chi_j\chi)(D)$ in (6.3.5).

We may suppose $\hat{u}$ and $\hat{f}$ are supported in supp χ and we may suppose $\partial_{\xi_d} T \ne 0$ on supp χ by rotating the coordinate and by taking supp χ small enough. We set $\xi_j^+ = \varepsilon_0 e_j + \sqrt{1 - \varepsilon_0^2} e_d$ for $j = 1, ..., d-1$ and $\xi_d^+ = \xi_d$, where $\varepsilon_0 > 0$ is a small constant and $(e_1, ..., e_d)$ is the standard basis of $\mathbb{R}^d$. Since $(\xi_1^+, ..., \xi_d^+)$ is the basis of $\mathbb{R}^d$, then

$$C^{-1}\sum_{j=1}^d \tilde{\mu}_{N,\gamma}(x \cdot \xi_j^+) \le \mu_{N,\gamma}(x) \le C\sum_{j=1}^d \tilde{\mu}_{N,\gamma}(x \cdot \xi_j^+)$$

with some constant $C > 0$ independent of γ, where

$$\tilde{\mu}_{N,\gamma}(t) = (1+t^2)^N(1+\gamma t^2)^{-N}.$$

Thus it suffices to prove that

$$\|\tilde{\mu}_{N,\gamma}(x\cdot\xi_j^+)u\|_{Y_2} \leq C_N\|\tilde{\mu}_{N,\gamma}(x\cdot\xi_j^+)(T(D)-\lambda)u\|_{Y_1}$$

for each $j = 1, ..., d$. If $\varepsilon_0 > 0$ is small, then $\partial_{\xi_d}T \neq 0$ implies $\xi_j^+ \cdot \nabla T(\xi) = \varepsilon_0\partial_{\xi_1}T + \sqrt{1-\varepsilon_0^2}\partial_{\xi_d}T \neq 0$ on supp χ. Thus by rotating the coordinate, we may reduce to prove

$$\|\tilde{\mu}_{N,\gamma}(x_d)u\|_{Y_2} \leq C_N\|\tilde{\mu}_{N,\gamma}(x_d)(T(D)-\lambda)u\|_{Y_1}.$$

We remark that this reduction is the only part to miss proving this Proposition when $X = \mathbb{Z}$. In fact, there are no basis containing the normal vector of $x\cdot\xi_j^+$-direction when $X = \mathbb{Z}$.

Set $f = (T(D)-\lambda)u$. By the implicit function theorem, we have $T(\xi)-\lambda = e(\xi,\lambda)(\xi_d - h_\lambda(\xi'))$ as in (6.3.1). Then we have $e(\xi,\lambda)^{-1}\hat{f}(\xi) = (\xi_d - h_\lambda(\xi'))\hat{u}(\xi)$ on supp χ. We denote $\tilde{f}(\xi', x_d)$ is the Fourier transform of f with respect to $\xi_1, ..., \xi_{d-1}$-variables and set $\hat{g}(\xi) = e(\xi,\lambda)^{-1}\hat{f}(\xi)$. Here $e(\xi,\lambda)^{-1}$ is well-defined on supp $\hat{f}$ since supp $f \subset$ supp χ. Then

$$(D_{x_d} - h_\lambda(\xi'))\tilde{u}(\xi', x_d) = \tilde{g}(\xi', x_d),$$

Since $\tilde{u}$ and $\tilde{g}$ are smooth, by using variation of parameters, we can write

$$\begin{aligned}\tilde{u}(\xi', x_d) &= \int_{-\infty}^{x_d} e^{2\pi i(x_d-y_d)h_\lambda(\xi')}\tilde{g}(\xi', y_d)dy_d \\ &= -\int_{x_d}^{\infty} e^{2\pi i(x_d-y_d)h_\lambda(\xi')}\tilde{g}(\xi', y_d)dy_d.\end{aligned}$$

Note that we use the first line of the above representation if $x_d \leq 0$ and the second line if $x_d \geq 0$. Taking the inverse Fourier transform and multiplying $\tilde{\mu}_{N,\gamma}(x_d)$, we have

$$\tilde{\mu}_{N,\gamma}(x_d)u(x) = \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}} K_{N,\gamma}(x'-y', x_d, y_d)\tilde{\mu}_{N,\gamma}(y_d)g(y)dy'dy_d$$

where

$$\begin{aligned}K_{N,\gamma}(x'-y', x_d, y_d) =& \frac{\tilde{\mu}_{N,\gamma}(x_d)}{\tilde{\mu}_{N,\gamma}(y_d)}(\chi_{x_d<0}\chi_{x_d\leq y_d} - \chi_{x_d>0}\chi_{x_d\leq y_d}) \\ &\times \int_{\widehat{\mathbb{R}^{d-1}}} e^{2\pi i(x'-y')\cdot\xi' + 2\pi i(x_d-y_d)h_\lambda(\xi')}\psi(\xi')d\xi'.\end{aligned}$$

Note that $\frac{\tilde{\mu}_{N,\gamma}(x_d)}{\tilde{\mu}_{N,\gamma}(y_d)}(\chi_{x_d<0}\chi_{x_d\leq y_d} - \chi_{x_d>0}\chi_{x_d\leq y_d}) \leq 1$. Let R be the linear operator on $\mathbb{R}^d$ with the integral kernel $K_{N,\gamma}$. We recall supp $\hat{f} \subset$ supp χ and $\hat{g} = e(\xi,\lambda)^{-1}\hat{f}(\xi)$. Hence we can write

$$\tilde{\mu}_{N,\gamma}(x_d)u(x) = K_{N,\gamma}(x'-y') * (\tilde{\mu}_{N,\gamma}(y_d)\varphi(D)e(D,\lambda)^{-1}\tilde{\mu}_{N,\gamma}^{-1}(y_d)\tilde{\mu}_{N,\gamma}(y_d)f)(x)$$

where $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi = 1$ on supp χ. By virtue of Lemma 6.3.6, it follows that the operator norms of $\tilde{\mu}_{N,\gamma}(y_d)\chi(D)e(D,\lambda)^{-1}\tilde{\mu}_{N,\gamma}(y_d)^{-1}$ on $L^p(\mathbb{R}^d)$ $(1 < p < \infty)$, $\mathcal{B}$ and $\mathcal{B}^*$ are uniformly bounded in $\lambda \in I$.

By virtue of Propositions 6.2.7 and 6.2.8, it suffices to $K_{N,\gamma}$ and $K_{N,\gamma}^*(x,y) = \bar{K}_{N,\gamma}(y,x)$ satisfies Assumptions G and H. To see this, we may mimic the proof of Lemma 6.3.2. We omit the detail.

□

6.4 Applications

6.4.1 Fractional Schrödinger operators and Dirac operators

In this subsection, we suppose that $T(D)$ is the one of the following operators:

$$T(D) = (-\Delta)^{s/2},\ T(D) = (-\Delta+1)^{s/2} - 1,\ T(D) = \mathcal{D}_0,\ T(D) = \mathcal{D}_1,$$

where $0 < s \leq d$.

Proof of Theorem 6.1.3. We consider the case when $T(D) = (-\Delta)^{s/2}$ or $T(D) = (1-\Delta)^{s/2}$ only. The case when $T(D) = \mathcal{D}_0$ or $T(D) = \mathcal{D}_1$ is similarly proved if we notice

$$\mathcal{D}_0^2 = -\Delta I_{n\times n},\ \ \mathcal{D}_0^2 = (-\Delta+1)I_{n\times n}$$

as in the proof of [11, Theorem 3.1]. We take a real-valued function $\chi \in C_c^\infty(\mathbb{R}^d, [0,1])$ such that $\chi = 1$ on $T^{-1}(I)$ and supp $\chi \subset \mathbb{R} \setminus \Lambda_c(T(D))$. Note that $M_\lambda = \{T(\xi) = \lambda\}$ is sphere and hence has non vanishing Gaussian curvature. if $\lambda \in \sigma(T(D)) \setminus \Lambda_c(T(D))$. Then we apply Theorem 7.1.1 with $k = (d-1)/2$ (see [66, Theorem 1.2.1]) and obtain

$$\sup_{z\in I_\pm} \|\chi(D)R_0^\pm(z)\|_{B(L^p(\mathbb{R}^d),L^q(\mathbb{R}^d))} < \infty \tag{6.4.1}$$

for $(p,q) \in S_{\frac{d-1}{2}}$. On the other hand, by the support property of χ and the Hardy-Littlewood-Sobolev inequality, we have

$$\sup_{z\in I_\pm} \|(1-\chi(D))R_0(z)\|_{B(L^p(\mathbb{R}^d),L^q(\mathbb{R}^d))} < \infty \tag{6.4.2}$$

if $1/p - 1/q \leq s/d$. In fact, if $2\alpha = -d/2 + d/p$ and $2\beta = -d/q + d/2$, then

$$\begin{aligned}
&\|(1-\chi(D))R_0(z)\|_{B(L^p(\mathbb{R}^d),L^q(\mathbb{R}^d))} \\
&\quad\leq \|(I-\Delta)^{-\alpha}\|_{B(L^p(\mathbb{R}^d),L^2(\mathbb{R}^d))}\|(1-\chi(D))(I-\Delta)^{\alpha+\beta}R_0(z)\|_{B(L^2(\mathbb{R}^d))} \\
&\qquad\times \|(I-\Delta)^{-\beta}\|_{B(L^2(\mathbb{R}^d),L^q(\mathbb{R}^d))}.
\end{aligned}$$

Thus (6.4.2) follows from the the Hardy-Littlewood-Sobolev inequality. Combining (6.4.1) with (6.4.2), we obtain (i). (ii) is similarly proved.

Lemma 6.4.1.

(i) *Suppose* $2d/(d+1) \le s < d$. *Let* $0 < \delta \le 1$, $r \in (2d/s, 2(d+1) - 4\delta]$ *and* $r_1, r_2 \in (1, 2(d+1)]$ *satisfying*

$$\frac{2}{d+1} \le \frac{1}{r_1} + \frac{1}{r_2} \le \frac{s}{d}.$$

Then

$$\sup_{z \in I_\pm} \|W_1 R_0^\pm(z) W_2\|_{B(L^2(\mathbb{R}^d))} \le C\|W_1\|_{L^{r_1}(\mathbb{R}^d)}\|W_2\|_{L^{r_2}(\mathbb{R}^d)}$$
$$\|W_3(R_0^\pm(z) - R_0^\pm(w))W_4\|_{B(L^2(\mathbb{R}^d))} \le C|z-w|^{\beta_\delta}\|W_3\|_{L^r(\mathbb{R}^d)}\|W_4\|_{L^r(\mathbb{R}^d)}$$

for $z, w \in I_\pm$ *with* $|z-w| \le 1$ *and* $W_1 \in L^{r_1}(\mathbb{R}^d)$, $W_2 \in L^{r_2}(\mathbb{R}^d)$, $W_3, W_4 \in L^r(\mathbb{R}^d)$.

Moreover, if $W_1 \in L^{r_1}(\mathbb{R}^d)$ *and* $W_2 \in L^{r_2}(\mathbb{R}^d)$, *then* $W_1 R_0^\pm(z) W_2 \in B_\infty(L^2(\mathbb{R}^d))$ *follows for* $z \in I_\pm$ *and a map* $z \in I_\pm \mapsto W_1 R_0^\pm(z) W_2$ *is continuous.*

(ii) *Suppose* $0 < s < 2d/(d+1)$. *Let* $0 < \delta \le 1$, $r \in (1, 2(d+1) - 4\delta]$, $r_1, r_2, \in (1, 2(d+1)]$ *and* $r_1', r_2', r' \in [2d/s, \infty)$ *satisfying*

$$\frac{2}{d+1} \le \frac{1}{r_1} + \frac{1}{r_2}, \quad \frac{1}{r_1'} + \frac{1}{r_2'} \le \frac{s}{d}.$$

The all results in Lemma 6.4.1 part (i) hold if we replace $L^{r_1}(\mathbb{R}^d)$, $L^{r_2}(\mathbb{R}^d)$ *and* $L^r(\mathbb{R}^d)$ *by* $L^{r_1}(\mathbb{R}^d) \cap L^{r_1'}(\mathbb{R}^d)$, $L^{r_2}(\mathbb{R}^d) \cap L^{r_2'}(\mathbb{R}^d)$ *and* $L^r(\mathbb{R}^d) \cap L^{r'}(\mathbb{R}^d)$ *respectively.*

Proof. Note that for $W_1, W_2 \in C_c^\infty(\mathbb{R}^d)$, it follows that $W_1(1-\chi(D))R_0^\pm(z)W_2$ is compact and smooth in $z \in I_\pm$ by using $dR_0(z)/dz = R_0(z)^2$ and the Rellich-Kondrachov theorem. The other parts of the proof are same as in the proof of Corollary 6.3.4. □

Part (*iii*): Existence and completeness of the wave operators are similarly proved as in the proof of Theorem 6.1.9 (*iv*) in subsection 6.4.3 by using Lemma 6.4.1.

Proof of Part (*iv*) is proved in subsection 6.4.2.

□

6.4.2 Carleman estimate, Proof of Theorem 6.1.3 (*iv*)

First, we give the Carleman estimate for $T(D)$. We recall $\mu_{N,\gamma}(x) = (1+|x|^2)^N(1+\gamma|x|^2)^{-N}$ and $\Lambda_l = (I-\Delta)^{l/2}$. For $1 < p < \infty$ and $l \in \mathbb{R}$, we introduce the standard Sobolev spaces

$$W^{l,p} = \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \Lambda_l u \in L^p(\mathbb{R}^d)\}, \ \|u\|_{W^{l,p}} = \|\Lambda_l u\|_{L^p(\mathbb{R}^d)}.$$

We set $p_d = 2(d+1)/(d+3)$, $p_d^* = 2(d+1)/(d-1)$, $l_d = s/2 - d/(d+1)$,

$$X_s = \begin{cases} W^{-l_d, p_d} + \Lambda_{s/2}\mathcal{B}, \text{ if } 2d/(d+1) \le s < d, \\ (L^{p_d}(\mathbb{R}^d) \cap L^{2d/(d+s)}(\mathbb{R}^d)) + \Lambda_{s/2}\mathcal{B}, \text{ if } 0 < s < 2d/(d+1), \end{cases}$$

and

$$X_s^* = \begin{cases} W^{l_d,p_d^*} \cap \Lambda_{-s/2}\mathcal{B}^*, & \text{if } 2d/(d+1) \le s < d, \\ (L^{p_d^*}(\mathbb{R}^d) + L^{2d/(d-s)}(\mathbb{R}^d)) \cap \Lambda_{-s/2}\mathcal{B}^*, & \text{if } 0 < s < 2d/(d+1). \end{cases}$$

By the Sobolev embedding theorem, we have

$$X_s \hookrightarrow W^{-s/2,2},\ W^{s/2,2} \hookrightarrow X_s^*. \tag{6.4.3}$$

Proposition 6.4.2. *Let $N \ge 0$ be a real number satisfying*

$$N < s/2,\ \textit{if } T(D) = (-\Delta)^{s/2} \textit{ with } s \notin 2\mathbb{N}. \tag{6.4.4}$$

Then there exists $C_{N,d} > 0$ independent of $0 < \gamma \le 1$ such that

$$\|\mu_{N,\gamma}(x)u\|_{X_s^*} \le C_{N,d}\|\mu_{N,\gamma}(x)(T(D) - \lambda)u\|_{X_s}$$

for $u \in \mathcal{B}_0^*$ and $|\lambda| \in I$.

Remark 6.4.3. The condition (6.4.4) is needed due to the singularity of the symbol $T(\xi) = |\xi|^s$ at $\xi = 0$.

Proof. First, we assume $u \in \mathcal{S}(\mathbb{R}^d)$. Let $\chi_0, \chi_1, \chi_2 \in C^\infty(\mathbb{R}^d)$ be smooth functions such that $\chi_0, \chi_1 \in C_c^\infty(\mathbb{R}^d)$ and

$$\chi_0 + \chi_1 + \chi_2 = 1,\ \chi_0(\xi) = 1 \text{ near } \xi = 0,\ \chi_1(\xi) = 1 \text{ on supp } T^{-1}(I).$$

By Lemma 6.3.6, it suffices to prove

$$\|\mu_{N,\gamma}(x)\psi(D)u\|_{X_s^*} \le C_{N,d}\|\mu_{N,\gamma}(x)\psi(D)(T(D) - \lambda)u\|_{X_s} \tag{6.4.5}$$

for $\psi \in \{\chi_0, \chi_1, \chi_2\}$. The case when $\psi = \chi_1$ directly follows from Proposition 6.3.5 and Corollary 6.3.8. The case when $\psi = \chi_2$ follows from Corollary 6.3.8 and (6.4.3):

$$\begin{aligned} \|\mu_{N,\gamma}(x)\chi_2(D)u\|_{X_s^*} \le& C\|\mu_{N,\gamma}(x)\chi_2(D)u\|_{W^{s/2,2}} \\ =& C\|\Lambda_{s/2}\mu_{N,\gamma}(x)u\|_{L^2(\mathbb{R}^d)}, \\ \Lambda_{s/2}\mu_{N,\gamma} =& (\Lambda_{s/2}\mu_{N,\gamma}\Lambda_{-s/2}\mu_{N,\gamma}^{-1}) \\ & \times (\mu_{N,\gamma}\Lambda_{s/2}\chi_3(D)(T(D)-\lambda)^{-1}\mu_{N,\gamma}^{-1}\Lambda_{s/2}) \\ & \times \Lambda_{-s/2}\mu_{N,\gamma}\chi_2(D)(T(D)-\lambda), \end{aligned}$$

where $\chi_3 \in C^\infty(\mathbb{R}^d)$ satisfies $\chi_3 = 1$ on supp χ_2 and supp $\chi_3 \cap T^{-1}(I) = \emptyset$. Moreover, the L^2-boundedness of $\Lambda_{s/2}\mu_{N,\gamma}\Lambda_{-s/2}\mu_{N,\gamma}^{-1}$ follows from Corollary 6.3.8 and L^2-boundedness of $\mu_{N,\gamma}\Lambda_{s/2}\chi_3(D)(T(D)-\lambda)^{-1}\mu_{N,\gamma}^{-1}\Lambda_{s/2}$ is proved by mimicking the proof of Corollary 6.3.8.

Finally, we deal with the case of $\psi = \chi_0$. (6.4.5) with $T(D) \neq (-\Delta)^{s/2}$ or $T(D) = (-\Delta)^{s/2}$ for $s \in 2\mathbb{N}$ is similarly proved as in the proof of (6.4.5) with $\psi = \chi_2$. Thus we may assume $T(D) = (-\Delta)^{s/2}$ with $s \notin 2\mathbb{N}$. For its proof, we need some lemmas.

Lemma 6.4.4. *Let $s > 0$ and $m \in C^\infty(\mathbb{R}^d \setminus \{0\}) \cap C_c(\mathbb{R}^d)$ satisfying*

$$|\partial_\xi^\alpha m(\xi)| \leq C_\alpha |\xi|^{M_\alpha}, \ M_\alpha = \begin{cases} 0, \ \text{if } \alpha = 0, \\ s - N, \ \text{if } |\alpha| \geq 1. \end{cases}$$

Then $m(D)(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} m(\xi) d\xi$ satisfies

$$|m(D)(x)| \leq C(1 + |x|)^{-s-d}.$$

Proof. Since m is compactly supported, we may assume $|x| \geq 1$. Let $\chi \in C_c^\infty(\mathbb{R})$ satisfying $\chi(t) = 1$ on $|t| \leq 1$ and $\chi(t) = 0$ on $|t| \geq 2$. Set $\bar{\chi} = 1 - \chi$. For $\delta > 0$, by integrating by parts, we have

$$\begin{aligned} m(D)(x) =& \frac{x}{|x|^2} \cdot \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} (-D_\xi m(\xi)) d\xi \\ =& \frac{x}{|x|^2} \cdot \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} (\chi(|\xi|/\delta) + \bar{\chi}(|\xi|/\delta))(-D_\xi m(\xi)) d\xi \\ =&: m_1(x) + m_2(x). \end{aligned}$$

We simply obtain

$$|m_1(x)| \leq C|x|^{-1} \int_{|\xi| \leq 2\delta} |\xi|^{s-1} d\xi \leq C|x|^{-1} \delta^{d+s-1}.$$

For $M \geq s + d + 2$, by integrating by parts, we have

$$\begin{aligned} |m_2(x)| \leq& C|x|^{-M-1} \sum_{|\alpha| \leq M} \int_{\mathbb{R}^d} |D_\xi^\alpha (\bar{\chi}(|\xi|/\delta) D_\xi m(\xi))| d\xi \\ \leq& C|x|^{-M-1} \delta^{d+s-1-M}. \end{aligned}$$

We set $\delta = |x|^{-1}$ and conclude $|m(D)(x)| \leq C|x|^{-d-s}$. □

Lemma 6.4.5. *Let m be as in Lemma 6.4.4 and $1 < p < \infty$. Moreover, let $0 \leq N < s/2$. Then we have*

$$\|\mu(x) m(D) \mu(x)^{-1}\|_{B(L^p(\mathbb{R}^d))} \leq C_{N,m,p}$$

for $\mu \in \{\mu_{N,\gamma}, \mu_{N,\gamma}^{-1}\}$, where $C_{N,m,p}$ and $C_{N,m}$ are independent of $0 < \gamma \leq 1$ and depends only on d, N and C in Lemma 6.4.4.

Proof. We note that the integral kernel of $\mu(x)m(D)\mu(x)^{-1}$ is $\mu(x)m(D)(x-y)\mu(y)^{-1}$ and satisfies

$$|\mu(x) m(D)(x-y) \mu(y)^{-1}| \leq C(1 + |x-y|)^{2N-d-s}$$

with $C > 0$ independent of $\gamma > 0$. Here we use Lemma 6.3.7 (*ii*) and Lemma 6.4.4. We note $2N - s < 0$ by the condition (6.4.4). Thus we have $(1 + |x|)^{2N-d-s} \in L^1(\mathbb{R}^d)$. By the Young inequality, we obtain the desired result. □

Remark 6.4.6. Replacing the Young inequality by the O'neil theorem (the Young inequality in the Lorentz spaces), we can relax the condition (6.4.4) as $2N \leq s$.

We return to the proof of (6.4.5) with $\psi = \chi_0$. We take $\chi \in C^\infty(\mathbb{R}^d)$ such that $\chi = 1$ on supp χ_0. We learn

$$\begin{aligned}\Lambda_{s/2}\mu_{N,\gamma} =& (\Lambda_{s/2}\mu_{N,\gamma}\Lambda_{-s/2}\mu_{N,\gamma}^{-1}) \times (\mu_{N,\gamma}\Lambda_{s/2}\chi(D)(T(D)-\lambda)^{-1}\Lambda_{s/2}\mu_{N,\gamma}^{-1}) \\ & \times (\mu_{N,\gamma}\Lambda_{-s/2}\mu_{N,\gamma}^{-1}\Lambda_{s/2}) \times \Lambda_{-s/2}\mu_{N,\gamma}\chi_0(D)(T(D)-\lambda).\end{aligned}$$

We set $m(D) = \mu_{N,\gamma}\Lambda_{s/2}\chi(D)(T(D)-\lambda)^{-1}\Lambda_{s/2}\mu_{N,\gamma}^{-1}$, then m satisfies the assumption of Lemma 6.4.4. Thus the inclusions (6.4.3), Corollary 6.3.8 and Lemma 6.4.4 imply (6.4.5) with $\psi = \chi_0$. This complete the proof of Proposition 6.4.2 with $u \in \mathcal{S}(\mathbb{R}^n)$.

In order to remove the condition $u \in \mathcal{S}(\mathbb{R}^n)$, we may use the Friedrichs modifier and a cut-off function as in [37, Proof of Theorem 1.2]. We omit the detail. □

The next lemma implies that the potential is "admissible".

Lemma 6.4.7. *Suppose $V \in L^p(\mathbb{R}^d)$ with $d/s \leq p \leq (d+1)/2$ for $2d/(d+1) \leq s < d$ and $V \in L^{(d+1)/2}(\mathbb{R}^d) \cap L^{d/s}(\mathbb{R}^d)$ for $0 < s < 2d/(d+1)$. Then we have $V \in B(X_s^*, X_s)$. Moreover, for each $\varepsilon > 0$ and $N \geq 0$ there exists $A_{N,\varepsilon}, R_{N,\varepsilon} \geq 1$ such that for $\gamma \in (0,1]$, we have*

$$\|\mu_{N,\gamma}Vu\|_{X_s} \leq \varepsilon\|\mu_{N,\gamma}u\|_{X_s^*} + A_{N,\varepsilon}\|u\|_{L^2(|x|\leq R_{N,\varepsilon})}. \tag{6.4.6}$$

Proof. First, we prove

$$\|Vu\|_{X_s} \leq \|V\|_{Y_s}\|u\|_{X_s^*}, \tag{6.4.7}$$

where $Y_s \in \{L^p(\mathbb{R}^d)\}_{d/s\leq p\leq(d+1)/2}$ for $2d/(d+1) \leq s < d$ and $Y_s = L^{(d+1)/2}(\mathbb{R}^d) \cap L^{d/s}(\mathbb{R}^d)$. By the Sobolev embedding theorem, we have

$$W^{l_d,p_d^*} \hookrightarrow L^{q^*}(\mathbb{R}^d),\ L^q(\mathbb{R}^d) \hookrightarrow W^{-l_d,p_d}$$

for $2d/(d+s) \leq q \leq p_d$. For $2d/(d+1) \leq s < d$ and $d/s \leq p \leq (d+1)/2$, we set $q_p = 2p/(p+1)$. We note $2d/(d+s) \leq q_p \leq p_d$. By the Hölder inequality, we have

$$\|Vu\|_{L^{q_p}(\mathbb{R}^d)} \leq \|V\|_{L^p(\mathbb{R}^d)}\|u\|_{L^{q_p^*}(\mathbb{R}^d)}.$$

We use $X_s^* \hookrightarrow W^{l_d,p_d^*}$ and $W^{-l_d,p_d} \hookrightarrow X_s$ and conclude $V \in B(X_s^*, X_s)$ and (6.4.7) for $2d/(d+1) \leq s < d$. In order to prove (6.4.7) with $0 < s < 2d/(d+1)$, it suffices to prove

$$\|Vu\|_{L^q(\mathbb{R}^d)} \leq \|V\|_{Y_s}\|u\|_{L^r(\mathbb{R}^d)},$$

where $q \in \{p_d, 2d/(d+s)\}$ and $r \in \{p_d^*, 2d/(d-s)\}$. This inequality follows from the fact $V \in Y_s = L^{(d+1)/2}(\mathbb{R}^d) \cap L^{d/s}(\mathbb{R}^d)$ and the complex interpolation.

Take $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi = 1$ on $|x| \leq 1/2$ and $\chi = 0$ on $|x| \geq 1$. For $R \geq 1$, we set $V_R = V\chi(x/R)$. Then we use the inclusion $\mathcal{B} \hookrightarrow X_s$ and have

$$\begin{aligned}\|\mu_{N,\gamma} V u\|_{X_s} &\leq \|V - V_R\|_{Y_s} \|\mu_{N,\gamma} u\|_{X_s^*} + \|\mu_{N,\gamma} V_R u\|_{X_s}\\ &\leq \|V - V_R\|_{Y_s} \|\mu_{N,\gamma} u\|_{X_s^*} + \|\mu_{N,\gamma} V_R u\|_{\mathcal{B}}.\end{aligned}$$

For each $\varepsilon > 0$, we take $R > 0$ large enough such $\|V - V_R\|_{Y_s} < \varepsilon$ and we obtain (6.4.6). $\square$

Proof of Theorem 6.1.3 (iv). We recall $H = T(D) + V$. Suppose that $\sigma_{pp}(H) \setminus \{0\}$ is not discrete in $\mathbb{R} \setminus \{0\}$. Then there exist an orthonormal system $\{u_j\}_{j=1}^\infty \subset L^2(\mathbb{R}^d)$, $\delta \geq 1$ and $\{\lambda_j\}_{j=1}^\infty \subset \{\lambda \in \mathbb{R} \mid \delta \leq |\lambda| \leq \delta^{-1}\}$ such that $Hu_j = \lambda_j u_j$. We note $u_j \in L^2(\mathbb{R}^d) \subset \mathcal{B}_0^*$. Let $N \geq 0$ satisfying (6.4.4). Applying Proposition 6.4.2 with u_j and Lemma 6.4.7 with small $\varepsilon > 0$, we have

$$\|\mu_{N,\gamma} u_j\|_{X_s^*} \leq C_{N,\varepsilon} \|u_j\|_{L^2(\mathbb{R}^d)}$$

with $C_{N,\varepsilon}$ independent of $\gamma \in (0, 1]$. The inclusion $\Lambda_{-s/2}(1 + |x|)^{1/2+\varepsilon_1} L^2(\mathbb{R}^d) \hookrightarrow X_s^*$ for $\varepsilon_1 > 0$ implies

$$\|(1 + |x|)^{-1/2-\varepsilon_1} \Lambda_{s/2} \mu_{N,\gamma} u_j\|_{L^2(\mathbb{R}^d)} \leq C_{N,\varepsilon} \|u_j\|_{L^2(\mathbb{R}^d)}.$$

Taking $\gamma \to 0$, we have

$$\|(1 + |x|)^{-1/2-\varepsilon_1} \Lambda_{s/2} (1 + |x|^2)^N u_j\|_{L^2(\mathbb{R}^d)} \leq C_{N,\varepsilon} \|u_j\|_{L^2(\mathbb{R}^d)} = C_{N,\varepsilon}. \tag{6.4.8}$$

We take ε_1 small enough and $N \geq 0$ satisfying (6.4.4) and $2N > 1/2 + \varepsilon_1$ when $T(D) = (-\Delta)^{s/2}$ with $2s \notin \mathbb{N}$. Then (6.4.8) implies that u_j is bounded in $(1 + |x|)^{1/2+\varepsilon_1-2N} \Lambda_{-s/2} L^2(\mathbb{R}^d)$. Since the inclusion $(1 + |x|)^{1/2+\varepsilon_1-2N} \Lambda_{-s/2} L^2(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ is compact, there exists a subsequence $\{u_{j_k}\}_k$ such that $u_{j_k} \to u$ in $L^2(\mathbb{R}^d)$ for some $u \in L^2(\mathbb{R}^d)$. On the other hand, since u_j converges to 0 in the weak topology of $L^2(\mathbb{R}^d)$, then we have $u = 0$. This contradicts to $\|u_j\|_{L^2(\mathbb{R}^d)} = 1$.

The same argument implies that the each eigenspace associated with eigenvalue $\lambda \in \mathbb{R} \setminus \{0\}$ is finite dimensional.

$\square$

6.4.3 Discrete Schrödinger operator

In this subsection, we consider the case of $X = \mathbb{Z}$ and consider the discrete Schrödinger operators.

Proof of Theorem 6.1.9. Part *(ii)* directly follows from the following lemma.

Lemma 6.4.8. *Let $d \geq 4$ and a signature $\pm$. Then maps $z \in \mathbb{C}_\pm \setminus \mathbb{R} \mapsto R_0^\pm(z)$ are Hölder continuous in $B(L^p(\mathbb{Z}^d), L^{p^*}(\mathbb{Z}^d))$ for $1 \leq p < 3_*$, where $3_* = 2d/(d+3)$.*

Proof. We follow the argument in [62, Lemma 4.7]. We prove the lemma in the case of $+$ only. The case of $-$ is similarly proved. For $1 \le p < 3_*$, there exists $0 < \delta \le 1$ such that $1 \le p < 3_{*,\delta}$, where

$$3_{*,\delta} = \frac{2}{3\delta/d + (3+d)/d}.$$

We use the following dispersive estimate ([67]):

$$\|e^{it\Delta_d}\|_{B(L^p(\mathbb{Z}^d), L^{p^*}(\mathbb{Z}^d))} \le C_p \langle t \rangle^{-\frac{d}{3}(\frac{2}{p}-1)}, \ 1 \le p \le 2. \tag{6.4.9}$$

Moreover,

$$|e^{itz} - e^{itz'}| \le 2^{1-\delta}|t|^\delta |z - z'|^\delta \tag{6.4.10}$$

holds for $t \ge 0$ and $z, z' \in \mathbb{C}_+$ since $|e^{itz} - e^{itz'}| \le 2$ and $|e^{itz} - e^{itz'}| \le |t||z - z'|$. By (6.4.9) and (6.4.10), we have

$$\begin{aligned} &\|R_0^+(z) - R_0^+(z')\|_{B(L^p(\mathbb{R}^d), L^{p^*}(\mathbb{R}^d))} \\ &= \left\| \int_0^\infty (e^{itz} - e^{itz'}) e^{it\Delta_d} dt \right\|_{B(L^p(\mathbb{R}^d), L^{p^*}(\mathbb{R}^d))} \\ &\le C_p 2^{1-\delta} |z - z'|^\delta \int_0^\infty |t|^\delta \langle t \rangle^{-\frac{d}{3}(\frac{2}{p}-1)} dt < \infty \end{aligned}$$

for $1 \le p < 3_{*,\delta}$. This completes the proof. $\square$

Now we prove part (i). The above lemma implies that

$$\lim_{\varepsilon \to 0, \varepsilon > 0} \|R_0^\pm(\lambda \pm i\varepsilon) - R_0^\pm(\lambda \pm i0)\|_{B(L^p(\mathbb{Z}^d), L^{p^*}(\mathbb{Z}^d))} = 0, \ \lambda \in \mathbb{R}, \ 1 \le p < 3_*, \tag{6.4.11}$$

where we recall $R_0^\pm(\lambda \pm i0)$ are Fourier multipliers of the distributions $(h_0(\xi) - (\lambda \pm i0))^{-1}$. We also use the uniform bounds ([68, Proposition 3.3]):

$$\sup_{z \in \mathbb{C}_\pm \setminus \mathbb{R}} \|R_0^\pm(z)\|_{B(L^{3_*}(\mathbb{Z}^d), L^{3^*}(\mathbb{Z}^d))} < \infty, \tag{6.4.12}$$

where $3^* = 2d/(d-3)$. By (6.4.11) and (6.4.12), taking a limiting argument, we have

$$\sup_{z \in \mathbb{C}_\pm} \|R_0^\pm(z)\|_{B(L^{3_*}(\mathbb{Z}^d), L^{3^*}(\mathbb{Z}^d))} < \infty.$$

This proves part (i).

Note that part (iii) with $V \in L^p(\mathbb{Z}^d)$ for $1 \le p < d/3$ follows from part (ii) and the Hölder inequality. Part (iii) with $V \in L^{d/3}(\mathbb{Z}^d)$ follows from the following lemma.

Lemma 6.4.9. *Let $d \ge 4$ and a signature $\pm$. For $W_1, W_2 \in L^{2d/3}(\mathbb{Z}^d)$, a map $z \in \mathbb{C}_\pm \mapsto W_1 R_0^\pm(z) W_2 \in B_\infty(L^2(\mathbb{Z}^d))$ is continuous.*

Proof. Take sequences of finitely supported potentials $W_{1,n}, W_{2,n}$ such that $W_{j,n} \to W_j$ in $L^{2d/3}(\mathbb{Z}^d)$ as $n \to \infty$ for $j = 1, 2$. For $z, z' \in \mathbb{C}_\pm$, the Hölder inequality implies

$$\begin{aligned}
&\|W_1(R_0^\pm(z) - R_0^\pm(z'))W_2\|_{B(L^2(\mathbb{Z}^d))} \\
\leq & 2\|W_1 - W_{1,n}\|_{L^{2d/3}(\mathbb{Z}^d)}\|W_2\|_{L^{2d/3}(\mathbb{Z}^d)} \sup_{z\in\mathbb{C}_\pm} \|R_0^\pm(z)\|_{B(L^{3_*}(\mathbb{Z}^d),L^{3^*}(\mathbb{Z}^d))} \\
&+2\|W_2 - W_{2,n}\|_{L^{2d/3}(\mathbb{Z}^d)} \sup_n(\|W_{1,n}\|_{L^{2d/3}(\mathbb{Z}^d)}) \sup_{z\in\mathbb{C}_\pm} \|R_0^\pm(z)\|_{B(L^{3_*}(\mathbb{Z}^d),L^{3^*}(\mathbb{Z}^d))} \\
&+\|W_{1,n}(R_0^\pm(z) - R_0^\pm(z'))W_{2,n}\|_{B(L^2(\mathbb{Z}^d))} \\
=:& I_1 + I_2 + I_3.
\end{aligned}$$

Now we let $\varepsilon > 0$. We fix a large n such that I_1+I_2 is smaller than $2\varepsilon/3$. Since $W_{1,n}$ and $W_{2,n}$ are finitely supported, the previous lemma implies that $W_{1,n}(R_0^\pm(z)-R_0^\pm(z'))W_{2,n}$ is Hölder continuous in $B(L^2(\mathbb{Z}^d))$. Thus there exists $\delta > 0$ such that $|z - z'| < \delta$ implies

$$I_3 = \|W_{1,n}(R_0^\pm(z) - R_0^\pm(z'))W_{2,n}\|_{B(L^2(\mathbb{Z}^d))} < \varepsilon/3.$$

Thus we conclude that maps $z \in \mathbb{C}_\pm \mapsto W_1R_0^\pm(z)W_2$ are continuous. □

It remains to prove (iv). We follow the argument as in [47] and [52]. Let $V \in L^{d/3}(\mathbb{Z}^d)$ be a real-valued function. Set $W_1 = (\text{sgn } V)|V|^{1/2} \in L^{2d/3}(\mathbb{Z}^d)$, $W_2 = |V|^{1/2} \in L^{2d/3}(\mathbb{Z}^d)$, $H = H_0+V$ and $R(z) = (H-z)^{-1}$ for $z \in \mathbb{C}\setminus\mathbb{R}$. We note that for $\pm\text{Im } z > 0$

$$W_1R_0^\pm(z)W_2 - W_1R(z)W_2 = W_1R(z)W_2W_1R_0^\pm(z)W_2. \tag{6.4.13}$$

By part (iii), it follows that $W_1R_0^\pm(z)W_2$ is continuous in $z \in I_\pm$ and hence is a compact operator . In addition, $I + W_1R_0^\pm(z)W_2$ is invertible in $B(L^2(\mathbb{Z}^d))$ for $z \in \mathbb{C} \setminus \mathbb{R}$ due to the Birman-Schwinger principle. In fact, if $I+W_1R_0^\pm(z)W_2$ is not invertible at $z \in \mathbb{C}\setminus\mathbb{R}$, then the compactness of $W_1R_0^\pm(z)W_2$ implies that $I + W_1R_0^\pm(z)W_2$ has a non-trivial kernel. Then it follows that $R(z)$ has a non-trivial kernel by the Birman-Schwinger principle. However, this contradicts to the self-adjointness of $H_0 + V$. Moreover, if we set

$$\sigma_{\text{BS}}(H) = \sigma_{\text{BS}}^\pm(H) = \{\lambda \in \mathbb{R} \mid \text{Ker }_{L^2(\mathbb{Z}^d)}(I + W_1R_0^\pm(z)W_2) \neq 0\},$$

we see that $\sigma_{\text{BS}}(H)$ is a closed set with Lebesgue measure zero by Proposition 6.6.3. Since $W_1R_0^\pm(z)W_2 \in B_\infty(L^2(\mathbb{Z}^d))$ for $z \in I_\pm$, $I + W_1R_0^\pm(z)W_2$ is a Fredholm operator with index 0. Thus (6.4.13) gives

$$W_2R(z)W_2 = W_2R_0^\pm(z)W_2(I + W_1R_0^\pm(z)W_2)^{-1},\ z \in I_\pm \setminus \sigma_{\text{BS}}(H_0).$$

Let $[a, b] \subset I\setminus\sigma_{\text{BS}}(H_0)$ with $a < b$. Since $(I+W_1R_0^\pm(z)W_2)^{-1}$ is continuous in $z \in [a, b]_\pm$, then

$$\sup_{z\in[a,b]_\pm} \|(I + W_1R_0^\pm(z)W_2)^{-1}\|_{B(L^2(\mathbb{Z}^d))} < \infty.$$

Combining this with the part (i) and Hölder's inequality, we obtain

$$\sup_{z\in[a,b]_\pm} \|W_2 R(z) W_2\|_{B(L^2(\mathbb{Z}^d))} < \infty.$$

Since $|W_1| = |W_2|$, then

$$\sup_{z\in[a,b]_\pm} \|W_{i_1} R(z) W_{i_2}\|_{B(L^2(\mathbb{Z}^d))} < \infty.$$

for $i_1, i_2 = 1, 2$. By [61, Theorem XIII. 30, 31], the local wave operators

$$s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0} E_{H_0}((a,b))$$

exist and are complete, where $E_{H_0}(J)$ is the spectral projection to the interval $J \subset \mathbb{R}$ associated with H_0. Since $[0,4d] \setminus \Lambda_c(H_0) \cup \sigma_{\mathrm{BS}}(H)$ is a countable union of such interval (a,b), the wave operators $W_\pm = s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}$ exist and are complete.

□

As an application of Theorem 7.1.1, we prove the further estimates of the uniform resolvent estimates for the discrete Schrödinger operators.

Proposition 6.4.10. *Suppose $I \subset (0,4) \cap (4(d-1), 4d)$ if $d = 2$ and $I \subset (0,2) \cap (4d-2, 4d)$ if $d \geq 3$. If $\operatorname{supp} \chi \subset h_0^{-1}(I)$, then*

$$\sup_{z\in I_\pm} \|\chi(D) R_0^\pm(z)\|_{B(L^p(\mathbb{Z}^d), L^q(\mathbb{Z}^d))} < \infty.$$

holds for $(1/p, 1, q) \in S_{(d-1)/2}$.

Proof. Let $\lambda \in I$. As is proved in [35, Lemma 4.3], all principal curvatures of $M_\lambda = \{h = \lambda\}$ are non-vanishing. By Example 1, we obtain the desired result. □

6.5 Some estimates for $\gamma_{z,\pm}$

In this section, we give proofs of the estimates for $\gamma_{z,\pm}$ which is needed for the proof of Theorem 7.1.1.

If necessary we take $\operatorname{supp} \chi$ small, we may assume $X = \mathbb{R}$. We recall the situation of the proof of Theorem 7.1.1. Set

$$\tilde{\chi}(\xi', \xi_d, \lambda) = \frac{\chi^2(\xi', \xi_d + h_\lambda(\xi'))}{e(\xi', \xi_d + h_\lambda(\xi'))},\ b(\xi', \xi_d, \lambda) = e(\xi', \xi_d + h_\lambda(\xi')))^{-1}.$$

Note that b is real-valued and $\min_{(\xi',\xi_d)\in\operatorname{supp}\chi(\cdot,\cdot,\lambda)\tilde{\chi},\lambda\in I} b(\xi', \xi_d, \lambda) > 0$. Recall that

$$\gamma_{z,\pm}(\xi', x_d) = \int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d} \tilde{\chi}(\xi', \xi_d, \lambda)}{\xi_d - i(\operatorname{Im} z) b(\xi', \xi_d, \lambda)} d\xi_d,\ \operatorname{Re} z = \lambda,\ \pm\operatorname{Im} z \geq 0.$$

Here if $\pm\mathrm{Im}\, z = 0$, we interpret $\gamma_{z,\pm}$ as

$$\begin{aligned}\gamma_{z,\pm}(\xi', x_d) &= \int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d} \chi^2(\xi', \xi_d + h_\lambda(\xi'))}{e(\xi', \xi_d + h_\lambda(\xi'))\xi_d \mp i0} d\xi_d \\ &= \int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d} \tilde{\chi}(\xi', \xi_d, \lambda))}{\xi_d \mp i0} d\xi_d,\end{aligned}$$

where $(\xi_d \mp i0)^{-1}$ denote the distributions $\lim_{\varepsilon>0,\varepsilon\to 0}(\xi_d \mp i\varepsilon)^{-1}$. In order to estimate $\gamma_{z,\pm}$, we need some lemmas.

Lemma 6.5.1. *Let $\psi, \psi_1 \in C_c^\infty(\mathbb{R})$ and $\mu_1, \mu_2 \in \mathbb{R} \setminus \{0\}$. Then*

$$\begin{aligned}&|\int_{\mathbb{R}} \psi(\mu_1 y_d)\mathrm{p.v.}\frac{e^{2\pi i y_d \xi_d}}{y_d} dy_d| \le \pi \|\hat{\psi}\|_{L^1(\mathbb{R})}, \ |\int_{\mathbb{R}} \mathrm{p.v.}\frac{e^{2\pi i y_d \xi_d}}{y_d} dy_d| = \pi \\ &|\int_{\mathbb{R}} \psi(\mu_1 y_d)\psi_1(\mu_2 y_d)\mathrm{p.v.}\frac{e^{2\pi i y_d \xi_d}}{y_d} dy_d| \le \pi \|\hat{\psi}\|_{L^1(\mathbb{R})}\|\hat{\psi}_1\|_{L^1(\mathbb{R})},\end{aligned}$$

Proof. We leran

$$\begin{aligned}|\int_{\mathbb{R}} \mathrm{p.v.}\frac{1}{y_d}\psi(y_d)e^{2\pi i y_d \xi_d} dy_d| =&\pi |\int_{\mathbb{R}} \mathrm{sgn}\ (\xi_d - \eta_d)\hat{\psi}(-\eta_d) d\eta_d| \\ \le&\pi \|\hat{\psi}\|_{L^1(\mathbb{R})}.\end{aligned}$$

By scaling, we obtain the first inequality. The second equality follows from $\mathcal{F}(\mathrm{p.v.}\frac{1}{y_d})(\xi_d) = -i\pi\mathrm{sgn}\ (\xi_d)$. The third inequality follows from the first inequality and the Young inequality:

$$\begin{aligned}\|\hat{\psi}\hat{\psi}_1\|_{L^1(\mathbb{R})} =&\|\hat{\psi} * \hat{\psi}_1\|_{L^1(\mathbb{R})} \\ \le&\|\hat{\psi}\|_{L^1(\mathbb{R})}\|\hat{\psi}_1\|_{L^1(\mathbb{R})}.\end{aligned}$$

□

Lemma 6.5.2. *Let $\mu \in \mathbb{R} \setminus \{0\}$ and $\varphi, a, a_1 \in C_c^\infty(\mathbb{R})$ such that a, a_1 are real-valued and $a, a_1 > 0$ on $\mathrm{supp}\ \varphi$.*

(i) There exists $C > 0$ independent of $x_d \in \mathbb{R}$, φ, a and $\mu \ne 0$ such that

$$|\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)}{\xi_d - ia(\mu\xi_d)} d\xi_d| \le C(\sup_{\xi_d\in\mathbb{R}} |\frac{\varphi(\xi_d)}{a(\xi_d)}| + \|\hat{\varphi}\|_{L^1(\mathbb{R})} + \sup_{\xi_d\in\mathbb{R}} |\varphi(\xi_d)a(\xi_d)|). \quad (6.5.1)$$

(ii) Let $l \ge 2$ be an integer. Then there exists $C' > 0$ independent of $x_d \in \mathbb{R}$, φ, a, l and $\mu \ne 0$ such that

$$|\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)}{(\xi_d - ia(\mu\xi_d))^l} d\xi_d| \le C'(\sup_{\xi_d\in\mathbb{R}} |\frac{|\varphi(\xi_d)|}{|a(\xi_d)|^l} + \|\varphi\|_{L^\infty(\mathbb{R})}). \quad (6.5.2)$$

(iii) *Let $l_1, l_2 \geq 1$ be an integer. Then there exists $C'' > 0$ independent of $x_d \in \mathbb{R}$, φ, a, l and $\mu \neq 0$ such that*

$$|\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)}{(\xi_d - ia(\mu\xi_d))^{l_1}(\xi_d - ia_1(\mu\xi_d))^{l_2}} d\xi_d| \leq C''(\sup_{\xi_d \in \mathbb{R}} |\frac{|\varphi(\xi_d)|}{|a(\xi_d)|^l} + \|\varphi\|_{L^\infty(\mathbb{R})}). \quad (6.5.3)$$

Proof. *(i)* Take $\psi \in C_c^\infty(\mathbb{R}, [0,1])$ such that $\psi = 1$ on $|t| \leq 1$ and $\psi = 0$ on $|t| \geq 2$. Since a is real-valued, then

$$\begin{aligned}|\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)\psi(\xi_d)}{\xi_d - ia(\mu\xi_d)} d\xi_d| &\leq \int_{\mathbb{R}} \frac{|\varphi(\mu\xi_d)\psi(\xi_d)|}{|a(\mu\xi_d)|} d\xi_d \\ &\leq \sup_{\xi_d \in \mathbb{R}} |\frac{\varphi(\xi_d)}{a(\xi_d)}| \|\psi\|_{L^1(\mathbb{R})}.\end{aligned}$$

We note that

$$\begin{aligned}\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)(1-\psi(\xi_d))}{\xi_d - ia(\mu\xi_d)} d\xi_d =& \int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)(1-\psi(\xi_d))}{\xi_d} d\xi_d \\ &+ i\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)a(\mu\xi_d)(1-\psi(\xi_d))}{\xi_d(\xi_d - ia(\mu\xi_d))} d\xi_d \\ =& : I_1 + I_2.\end{aligned}$$

By Lemma 6.5.1, we have

$$\begin{aligned}|I_1| =& |\int_{\mathbb{R}} \text{p.v.} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)}{\xi_d} d\xi_d - \int_{\mathbb{R}} \text{p.v.} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)\psi(\xi_d)}{\xi_d} d\xi_d| \\ \leq& \pi\|\hat{\varphi}\|_{L^1(\mathbb{R})}(1 + \|\hat{\psi}\|_{L^1(\mathbb{R})}).\end{aligned}$$

Moreover, since a is real-valued, we have

$$|I_2| \leq \sup_{\xi_d \in \mathbb{R}} |\varphi(\xi_d)a(\xi_d)| \int_{\mathbb{R}} \frac{1-\psi(\xi_d)}{\xi_d^2} d\xi_d.$$

Thus we set

$$C = \max(\|\psi\|_{L^1(\mathbb{R})}, \pi(1 + \|\hat{\psi}\|_{L^1(\mathbb{R})}), \int_{\mathbb{R}} \frac{1-\psi(\xi_d)}{\xi_d^2} d\xi_d),$$

and obtain (6.5.1).

(ii) follows from *(iii)*.

(iii) Let ψ be as above. Then

$$|\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)\psi(\xi_d)}{(\xi_d - ia(\mu\xi_d))^{l_1}(\xi_d - ia_1(\mu\xi_d))^{l_2}} d\xi_d| \leq \sup_{\xi_d \in \mathbb{R}} |\frac{|\varphi(\xi_d)|}{|a(\xi_d)|^{l_1}|a_1(\xi_d)|^{l_2}} \|\psi\|_{L^1(\mathbb{R})}.$$

Moreover, since a, a_1 is real-valued and $l_1 + l_2 \geq 2$, then

$$|\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}\varphi(\mu\xi_d)(1-\psi(\xi_d))}{(\xi_d - ia(\mu\xi_d))^{l_1}(\xi_d - ia_1(\mu\xi_d))^{l_2}} d\xi_d| \leq \|\varphi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \frac{1-\psi(\xi_d)}{|\xi_d|^{l_1+l_2}} d\xi_d \leq \|\varphi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \frac{1-\psi(\xi_d)}{|\xi_d|^2} d\xi_d.$$

Thus we set $C'' = \max(\|\psi\|_{L^1(\mathbb{R})}, \int_{\mathbb{R}} \frac{1-\psi(\xi_d)}{|\xi_d|^2} d\xi_d)$ and obtain (6.5.3).

□

The main result of this section is the following proposition.

Proposition 6.5.3. *Fix a signature* $\pm$.

(*i*) *For* $\alpha \in \mathbb{N}^{d-1}$, *there exists* $C_\alpha > 0$ *such that*

$$|\partial_{\xi'}^\alpha \gamma_{z,\pm}(\xi', x_d)| \leq C_\alpha \tag{6.5.4}$$

for $z \in I_\pm$, $x_d \in \mathbb{R}$ *and* $\xi' \in \mathbb{R}^{d-1}$.

(*ii*) *For* $\alpha \in \mathbb{N}^{d-1}$, *there exists* $C'_\alpha > 0$ *such that*

$$|\partial_{\xi'}^\alpha (\gamma_{z,\pm}(\xi', x_d) - \gamma_{w,\pm}(\xi', x_d))| \leq C'_\alpha (1 + |x_d|)|z - w| \tag{6.5.5}$$

for $z, w \in I_\pm$ *with* $|z - w| \leq 1$, $x_d \in \mathbb{R}$ *and* $\xi' \in \mathbb{R}^{d-1}$.

Remark 6.5.4. Let $0 \leq \delta \leq 1$. Combining (6.5.4) with (6.5.5), we have

$$|\partial_{\xi'}^\alpha (\gamma_{z,\pm}(\xi', x_d) - \gamma_{w,\pm}(\xi', x_d))| \leq C_\alpha^{1-s}(C'_\alpha)^s (1 + |x_d|)^\delta |z - w|^\delta. \tag{6.5.6}$$

Proof. (*i*) We follow the argument of the proof of [11, (3.10)]. We may assume $0 \leq \pm\mathrm{Im}\, z \leq 1$. First, we consider the case of $\pm\mathrm{Im}\, z = 0$. In this case, the claim follows from the fact that

$$\| \int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d}}{\xi_d \mp i0} d\xi_d \|_{L^\infty(\mathbb{R}_{x_d})} < \infty$$

and that $\tilde{\chi}$ is smooth with respect to $(\xi, \xi_d, \lambda) \in \mathbb{R}^d \times I$ and has a compact support with respect to (ξ', ξ_d)-variable which is bounded in $\lambda \in I$.

We take $\psi \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\psi(\xi_d) = 1$ on $|\xi_d| \leq 1$. We learn

$$\gamma_{z,\pm}(\xi', x_d) = \int_{\mathbb{R}} \frac{e^{2\pi i (\mathrm{Im}\, z) x_d \xi_d} \tilde{\chi}(\xi', (\mathrm{Im}\, z)\xi_d, \lambda)}{\xi_d - ib(\xi', (\mathrm{Im}\, z)\xi_d, \lambda)} d\xi_d.$$

We note that $\partial_{\xi'}^\alpha \gamma(\xi', x_d)$ is a linear combination of the form

$$\int_{\mathbb{R}} \frac{e^{2\pi i (\mathrm{Im}\, z) x_d \xi_d} (\partial_{\xi'}^{\alpha_0} \tilde{\chi})(\xi', (\mathrm{Im}\, z)\xi_d, \lambda) \prod_{j=1}^l (\partial_{\xi'}^{\alpha_j} b)(\xi', (\mathrm{Im}\, z)\xi_d, \lambda)}{(\xi_d - ib(\xi', (\mathrm{Im}\, z)\xi_d, \lambda))^l} d\xi_d,$$

where $l \geq 1$ is an integer and $\alpha_j \in \mathbb{N}^{d-1}$ for $j = 0, ..., l$. Applying Lemma 6.5.2 (i) if $l = 1$ and (ii) if $l > 1$ with $\varphi(\xi_d) = (\partial_{\xi'}^{\alpha_0}\tilde{\chi})(\xi', \xi_d, \lambda) \prod_{j=1}^{l} (\partial_{\xi'}^{\alpha_j} b)(\xi', \xi_d, \lambda)$, $a(\xi_d) = b(\xi', \xi_d, \lambda)$ and $\mu = \text{Im } z$, we obtain (6.5.4) with $|\alpha| \geq 1$.

(ii) We set $\lambda = \text{Re } z$ and $\sigma = \text{Re } w$. We take $0 < \varepsilon$ such that

$$\min_{(\xi',\xi_d)\in \text{supp } \chi(\cdot,\cdot,\lambda), |z-w|\leq\delta} |b(\xi', \xi_d, \sigma)| > 0.$$

Then we may assume $|z-w| < \varepsilon$. In fact, in order to prove (ii), we use (i) if $|z-w| \geq \varepsilon$. Note that

$$\gamma_{z,\pm}(\xi', x_d) - \gamma_{w,\pm}(\xi', x_d) = J_1(x_d) + J_2(x_d) + J_3(x_d),$$

where we set

$$\begin{aligned}
J_1(x_d) &= \int_{\mathbb{R}} e^{2\pi i x_d \xi_d} \Big(\frac{\tilde{\chi}(\xi', \xi_d, \lambda)}{\xi_d - i(\text{Im } z) b(\xi', \xi_d, \lambda)} - \frac{\tilde{\chi}(\xi', \xi_d, \lambda)}{\xi_d - i(\text{Im } w) b(\xi', \xi_d, \lambda)}\Big) d\xi_d \\
J_2(x_d) &= \int_{\mathbb{R}} e^{2\pi i x_d \xi_d} \frac{\tilde{\chi}(\xi', \xi_d, \lambda) - \tilde{\chi}(\xi', \xi_d, \sigma)}{\xi_d - i(\text{Im } w) b(\xi', \xi_d, \lambda)} d\xi_d \\
&= \int_{\mathbb{R}} e^{2\pi i (\text{Im } w) x_d \xi_d} \frac{\tilde{\chi}(\xi', (\text{Im } w)\xi_d, \lambda) - \tilde{\chi}(\xi', (\text{Im } w)\xi_d, \sigma)}{\xi_d - i b(\xi', (\text{Im } w)\xi_d, \lambda)} d\xi_d \\
J_3(x_d) &= \int_{\mathbb{R}} e^{2\pi i x_d \xi_d} \tilde{\chi}(\xi', \xi_d, \sigma) \Big(\frac{1}{\xi_d - i(\text{Im } w) b(\xi', \xi_d, \lambda)} - \frac{1}{\xi_d - i(\text{Im } w) b(\xi', \xi_d, \sigma)}\Big) d\xi_d \\
&= \int_{\mathbb{R}} e^{2\pi i (\text{Im } w) x_d \xi_d} \frac{i\tilde{\chi}(\xi', (\text{Im } w)\xi_d, \sigma)(b(\xi', (\text{Im } w)\xi_d, \lambda) - b(\xi', (\text{Im } w)\xi_d, \sigma))}{(\xi_d - i b(\xi', (\text{Im } w)\xi_d, \lambda))(\xi_d - i b(\xi', (\text{Im } w)\xi_d, \sigma))} d\xi_d.
\end{aligned}$$

First, we estimate J_2. Similarly to the proof of (i), $\partial_{\xi'}^{\alpha} J_2(\xi')$ is a finite sum of the form

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{e^{2\pi i (\text{Im } w) x_d \xi_d}((\partial_{\xi'}^{\alpha_0}\tilde{\chi})(\xi', (\text{Im } w)\xi_d, \lambda) - (\partial_{\xi'}^{\alpha_0}\tilde{\chi})(\xi', (\text{Im } w)\xi_d, \sigma))}{(\xi_d - i b(\xi', (\text{Im } w)\xi_d, \lambda))^l} \\
&\quad \times \prod_{j=1}^{l} (\partial_{\xi'}^{\alpha_j} b)(\xi', (\text{Im } w)\xi_d, \lambda) d\xi_d,
\end{aligned}$$

where $l \geq 1$ is an integer and $\alpha_j \in \mathbb{N}^{d-1}$ for $j = 0, ..., l$. We apply Lemma 6.5.2 (i) if $l = 1$ and (ii) $l \geq 2$ and obtain

$$|\partial_{\xi'}^{\alpha} J_2(\xi')| \leq C'_{\alpha} |z - w| \tag{6.5.7}$$

with $C'_{\alpha} > 0$ independent of $x_d \in \mathbb{R}$, $\xi_d \in \mathbb{R}^{d-1}$ and $z, w \in I_{\pm}$ with $|z - w| \leq \delta$.

Next, we estimate J_3. $\partial_{\xi'}^{\alpha} J_3$ is a linear combination of the form

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{e^{2\pi i (\text{Im } w) x_d \xi_d} (\partial_{\xi'}^{\alpha_0}\tilde{\chi})(\xi', (\text{Im } w)\xi_d, \sigma) \partial_{\xi'}^{\alpha_2}(b(\xi', (\text{Im } w)\xi_d, \lambda) - b(\xi', (\text{Im } w)\xi_d, \sigma))}{(\xi_d - i b(\xi', (\text{Im } w)\xi_d, \lambda))^{l_1} (\xi_d - i b(\xi', (\text{Im } w)\xi_d, \sigma))^{l_2}} \\
&\quad \times \prod_{j=2}^{l_1+l_2+1} (\partial_{\xi'}^{\alpha_j} b)(\xi', (\text{Im } w)\xi_d, \lambda) d\xi_d,
\end{aligned}$$

where $l_1, l_2 \geq 1$ are integers and $\alpha_j \in \mathbb{N}^{d-1}$ for $j = 0, ..., l_1 + l_2 + 1$. We apply Lemma 6.5.2 (iii) and obtain

$$|\partial_{\xi'}^{\alpha} J_3(\xi')| \leq C'_\alpha |z - w| \tag{6.5.8}$$

with $C'_\alpha > 0$ independent of $x_d \in \mathbb{R}$, $\xi_d \in \mathbb{R}^{d-1}$ and $z, w \in I_\pm$ with $|z - w| \leq \varepsilon$.

Finally, we estimate J_1. Note that $|\partial_{\xi'}^{\alpha} J_1(x_d)| \leq 2C_0$ by (i). Thus it suffices to prove that $|\partial_{\xi'}^{\alpha} J_1'(x_d)| \leq C'_\alpha |\text{Im } z - \text{Im } w|$. We learn

$$\begin{aligned}
\frac{J_1'(x_d)}{2\pi i} &= \int_{\mathbb{R}} e^{2\pi i x_d \xi_d} \Big(\frac{\xi_d \tilde{\chi}(\xi', \xi_d, \lambda)}{\xi_d - i(\text{Im } z) b(\xi', \xi_d, \lambda)} - \frac{\xi_d \tilde{\chi}(\xi', \xi_d, \lambda)}{\xi_d - i(\text{Im } w) b(\xi', \xi_d, \lambda)}\Big) d\xi_d \\
&= \int_{\mathbb{R}} e^{2\pi i x_d \xi_d} \frac{i(\text{Im } z - \text{Im } w) \xi_d \tilde{\chi}(\xi', \xi_d, \lambda) b(\xi', \xi_d, \lambda)}{(\xi_d - i(\text{Im } z) b(\xi', \xi_d, \lambda))(\xi_d - i(\text{Im } w) b(\xi', \xi_d, \lambda))} d\xi_d \\
&= \int_{\mathbb{R}} e^{2\pi i (\text{Im } w) x_d \xi_d} \frac{i(\text{Im } z - \text{Im } w) \xi_d \tilde{\chi}(\xi', (\text{Im } w)\xi_d, \lambda) b(\xi', (\text{Im } w)\xi_d, \lambda)}{(\xi_d - i\frac{\text{Im } z}{\text{Im } w} b(\xi', (\text{Im } w)\xi_d, \lambda))(\xi_d - i b(\xi', (\text{Im } w)\xi_d, \lambda))} d\xi_d.
\end{aligned}$$

Thus $\partial_{\xi'}^{\alpha} J_1'(x_1)/(-2\pi|\text{Im } z - \text{Im } w|)$ is a linear combination of the form

$$\begin{aligned}
&\Big(\frac{\text{Im } z}{\text{Im } w}\Big)^{l_1} \int_{\mathbb{R}} e^{2\pi i (\text{Im } w) x_d \xi_d} \frac{\xi_d \partial_{x'}^{\alpha_0} \tilde{\chi}(\xi', (\text{Im } w)\xi_d, \lambda) \partial_{\xi'}^{\alpha_2} b(\xi', (\text{Im } w)\xi_d, \lambda)}{(\xi_d - i\frac{\text{Im } z}{\text{Im } w} b(\xi', (\text{Im } w)\xi_d, \lambda))^{l_1} (\xi_d - i b(\xi', (\text{Im } w)\xi_d, \lambda))^{l_2}} \\
&\qquad \times \prod_{j=2}^{l_1+l_2+1} \partial_{\xi'}^{\alpha_j} b(\xi', \xi_d, \lambda) d\xi_d,
\end{aligned}$$

where $l_1, l_2 \geq 1$ are integers, $\alpha_j \in \mathbb{N}^{d-1}$ for $j = 1, ..., l_1 + l_2 + 1$. Applying Lemma 6.5.2 (i) and (ii) with

$$\varphi(\xi_d) = (\text{Im } z)^{l_1} \frac{\xi_d \partial_{x'}^{\alpha_0} \tilde{\chi}(\xi', (\text{Im } w)\xi_d, \lambda) \partial_{\xi'}^{\alpha_2} b(\xi', (\text{Im } w)\xi_d, \lambda)}{(\xi_d - i(\text{Im } z) b(\xi', \xi_d, \lambda))^{l_1}},$$

$a(\xi_d) = b(\xi', \xi_d, \lambda)$, $l = l_2$ and $\mu = \text{Im } w$, we have $|\partial_{\xi'}^{\alpha} J_1'(x_d)| \leq C'_\alpha |\text{Im } z - \text{Im } w|$. This completes the proof.

□

6.6 Complex analysis

We define $\log^+ t = \log t$ if $1 \leq t$, $\log^+ t = 0$ if $0 < t \leq 1$ and $\log^- t = \log t - \log^+ t$.

Lemma 6.6.1. *Let $f : \{z \in \mathbb{C} \mid |z| \leq 1\} \to \mathbb{C}$ be a continuous function which is holomorphic on $\{|z| < 1\}$ and has no zero on $\{|z| < 1\}$. Then $f(e^{i\theta}) \neq 0$ for almost everywhere $\theta \in [-\pi, \pi)$.*

Proof. We follow the argument of [63, Theorem 17.17]. By the mean value properties of the harmonic function, we have

$$\begin{aligned}\log|f(0)| =&\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|d\theta \\ =&\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}|f(re^{i\theta})|d\theta-\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{-}|f(re^{i\theta})|d\theta\end{aligned} \tag{6.6.1}$$

for $0<r<1$. On the other hand, by using $x\leq e^{x}$ for $x\in\mathbb{R}$ and Jensen's inequality, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}|f(re^{i\theta})|d\theta \leq&\exp(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}|f(re^{i\theta})|d\theta) \\ \leq&\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|d\theta.\end{aligned}$$

By Fatou's lemma and (6.6.1), we obtain $\log|f(e^{i\theta})|\in L^{1}([-\pi,\pi))$. In particular, $\log|f(e^{i\theta})|<\infty$ for almost everywhere $\theta\in[-\pi,\pi)$. Thus $f(e^{i\theta})\neq 0$ for almost everywhere $\theta\in[-\pi,\pi)$.

□

Corollary 6.6.2. *Let $J=(a,b)$ be an open interval and $r=(b-a)/2$. Let $f:\{z\in\mathbb{C}\mid |z-(a+b)/2|\leq r,\ \pm\mathrm{Im}\ z\geq 0\}\to\mathbb{C}$ be a continuous function which is holomorphic and has no zero on $\{|z-(a+b)/2|<r,\mathrm{Im}\ z>0\}$. Then $f(\lambda)\neq 0$ for almost everywhere $\lambda\in J$.*

Proof. For simplicity, we assume $a=-1$ and $b=1$. Define $\kappa_1: D=\{|z|<1,\mathrm{Im}\ z>0\}\to\{\mathrm{Im}\ z>0\}$ and $\kappa_2:\{\mathrm{Im}\ z>0\}\to\{|z|<1\}$ by $\kappa_1(z)=(1+z)^2/(1-z)^2$ and $\kappa_2(z)=(z-i)/(z+i)$. Then $\kappa=\kappa_2\circ\kappa_1$ is biholomorphic from $\{|z|<1,\mathrm{Im}\ z>0\}$ to $\{|z|<1\}$ and homeomorphic from $\{|z|\leq 1,\mathrm{Im}\ z\geq 0\}$ to $\{|z|\leq 1\}$. Moreover, since

$$\kappa^{-1}(w)=\frac{\sqrt{i\frac{1+w}{1-w}}-1}{\sqrt{i\frac{1+w}{1-w}}+1}$$

where we take a branch such that $\mathrm{Im}\sqrt{z}>0$, then $\kappa^{-1}|_{|z|=1}:\{|z|=1\}\to\bar{D}\setminus D$ is Hölder continuous. Thus $\kappa^{-1}|_{|z|=1}$ maps sets of Lebesgue measure zero to sets of Lebesgue measure zero. By Lemma 6.6.1, we obtain the desired result. □

Next proposition is a variant of [46, Lemma 4.20]. See also [52, Proposition 4.6].

Proposition 6.6.3. *Let Z be a Banach space and fix a sgnature. For $J\subset\mathbb{R}$ be an open set, we denote $J_{\pm}=\{z\in\mathbb{C}\mid \mathrm{Re}\ z\in J,\pm\mathrm{Im}\ z\geq 0\}$. Let $K:J_{\pm}\to B_{\infty}(Z)$ be continuous and holomorphic on $\{\pm\mathrm{Im}\ z>0\}$. If $I+K(z)$ has a inverse in $B(Z)$ for each $z\in\{\pm\mathrm{Im}\ z>0\}$, then $\Gamma_0=\{\lambda\in\mathbb{R}\mid I+K(\lambda)$ is not invertible$\}$ is a closed set with Lebesgue measure zero.*

Proof. Since the set of all invertible operators in $B(Z)$ is open and since K is continuous, then Γ_0 is closed. Thus it suffices to prove that the Lebesgue measure of Γ_0 is zero. Note that $I + K(\lambda)$ is not invertible if and only if -1 is in the spectrum of $K(\lambda)$ for $\lambda \in \Gamma_0$. Fix $\lambda \in \Gamma_0$. Since $K(\lambda)$ is compact, there exists a circle C_λ enclosing -1 such that C_λ is contained in the resolvent set of $K(\lambda)$. Since K is continuous, there exists $r_\lambda > 0$ such that C_λ is contained in the resolvent set of $K(z)$ for $z \in \overline{B^\pm_{r_\lambda}(\lambda)}$ where $B^\pm_{r_\lambda}(\lambda) = \{z \in \mathbb{C} \mid \pm\text{Im}\, z \geq 0,\, |z - \lambda| < r_\lambda\}$. We define

$$P_z = \frac{1}{2\pi i}\int_{C_\lambda}(w - K(z))^{-1}dw,$$

then $z \in \overline{B^\pm_{r_\lambda}(\lambda)} \mapsto P_z \in B(Z)$ is analytic in $B^\pm_{r_\lambda}(\lambda) \setminus \mathbb{R}$ and continuous in $B^\pm_{r_\lambda}(\lambda)$. Note that $n_0 = \dim \text{Ran}\, P_z < \infty$ is independent of $z \in \overline{B^\pm_{r_\lambda}(\lambda)}$. Set $Z_z = \text{Ran}\, P_z$ and fix a linear isomorphism $\Pi_\lambda : \mathbb{C}^{n_0} \to Z_\lambda$. We choose r_λ smaller such that $I + P_\lambda(P_z - P_\lambda)$ has an inverse in $B(Z_\lambda)$. Then $\Theta_z = P_z|_{Z_\lambda} : Z_\lambda \to Z_z$ is a linear isomorphism with its inverse

$$(I + P_\lambda(P_z - P_\lambda))^{-1}P_\lambda : Z_z \to Z_\lambda.$$

Now we set

$$X(z) = \Pi_\lambda^{-1}\Theta_z^{-1}(I + K(z))\Theta_z\Pi_\lambda$$

for $z \in \overline{B^\pm_{r_\lambda}(\lambda)}$. Then X is continuous on $\overline{B^\pm_{r_\lambda}(\lambda)}$ and analytic in $B^\pm_{r_\lambda}(\lambda)$. Moreover, $\det X(z)$ is also continuous on $\overline{B^\pm_{r_\lambda}(\lambda)}$ and analytic in $B^\pm_{r_\lambda}(\lambda)$. We note that $\det X(z) = 0$ if and only if -1 is in the spectrum of $K(z)$. By Corollary 6.6.2 and the compactness argument, we conclude that the Lebesgue measure of Γ_0 is zero.

□

Chapter 7

Some properties of threshold eigenstates and resonant states of discrete Schrödinger operators

7.1 Main results

We consider the discrete Schrödinger operators:

$$H = H_0 + V(x) \quad \text{on} \quad \mathcal{H} = l^2(\mathbb{Z}^d),$$

where H_0 is the negative discrete Laplacian

$$H_0 u(x) = -\sum_{|x-y|=1} (u(y) - u(x)),$$

and V is a real-valued function on $\mathbb{Z}^d$. We denote the Fourier expansion by $\mathcal{F}_d$:

$$\hat{u}(\xi) = \mathcal{F}_d u(\xi) = \sum_{x\in\mathbb{Z}^d} e^{-2\pi i x\cdot\xi} u(x), \quad \xi \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d.$$

Then it follows that

$$\mathcal{F}_d H_0 u(\xi) = h_0(\xi)\mathcal{F}_d u(\xi) \text{ for } u \in \bigcup_{s\in\mathbb{R}} l^{2,s}(\mathbb{Z}^d) \tag{7.1.1}$$

in the distributional sense, where $h_0(\xi) = 4\sum_{j=1}^d \sin^2(\pi\xi_j)$, and hence $\sigma(H_0) = [0, 4d]$. In this note, we often use $[-\frac{1}{2}, \frac{1}{2}]^d$ as a fundamental domain of $\mathbb{T}^d$. Moreover, we identify the integral over $\mathbb{T}^d$ with the integral over this fundamental domain $[-\frac{1}{2}, \frac{1}{2}]^d$. We denote $\langle x\rangle = (1 + |x|^2)^{1/2}$ and $l^{2,s}(\mathbb{Z}^d) = \langle x\rangle^{-s} l^2(\mathbb{Z}^d)$. It is known that $l^{2,s}(\mathbb{Z}^d)$ is isometric to the Sobolev space $H^s(\mathbb{T}^d)$ through the Fourier expansion $\mathcal{F}_d$.

Critical values of h_0 are called thresholds of H_0. We denote the set of all thresholds by Γ:

$$\Gamma = \{\lambda \in [0, 4d] \mid \lambda \text{ is a critical value of } h_0\} = \{4k\}_{k=0}^d.$$

Note that any critical points of h_0 is non-degenerate, that is, h_0 is a Morse function. We say that 0 and $4d$ are elliptic thresholds and $\lambda \in \{4k\}_{k=1}^{d-1}$ are hyperbolic thresholds. Near each critical point of h_0, we have the following Taylor expansion:

$$h_0(\xi) - \lambda \sim 4\pi^2 \Big(-\sum_{j=1}^{k} (\xi_{\sigma(j)} - \eta_{\sigma(j)})^2 + \sum_{j=k+1}^{d} (\xi_{\sigma(j)} - \eta_{\sigma(j)})^2\Big),$$

where $\eta \in h_0^{-1}(\{\lambda\})$, $\lambda \in \Gamma$, $k = k(\eta)$ is the Morse index at η and $\sigma : \{1, ..., d\} \to \{1, ..., d\}$ is a bijection. Moreover, it easily follows that $k(\eta) = 0, d$ if $h(\eta) \in \{0, 4d\}$ and $k(\eta) \neq 0, d$ if $h(\eta) \in \Gamma \setminus \{0, 4d\}$. This implies that h_0 behaves like the symbol $\pm|\xi|^2$ of the elliptic operator $\mp\Delta$ near critical points with the elliptic thresholds and behaves like the symbol $-|\xi'|^2 + |\xi''|^2$ ($\xi = (\xi', \xi'')$) of the ultrahyperbolic operator $\Delta_{x'} - \Delta_{x''}$ near critical points with the hyperbolic thresholds.

It is known that the behavior of the resolvent at thresholds is closely related to a time decay of the propagator and that existence of eigenstates and resonant states disturbs a decay property of the propagator [42]. Ito and Jensen obtain an analytic continuation near thresholds of the integral kernels for discrete Schrödinger operators [32]. The purpose of this note is to study some properties of resonant states: Resonant states at elliptic thresholds have same properties as continuous one's and resonances at hyperbolic thresholds are absent. From this, we expect that the hyperbolic thresholds is harmless for the decay property of the propagator.

First, we give a definition of resonances at elliptic thresholds.

Definition 4. Let $d \geq 3$ and $\lambda = 0$ or $4d$. Suppose that a real-valued function V satisfies $|V| \leq C\langle x\rangle^{-2-\delta}$ with $\delta > 0$. We say that $u \in l^{2,-3/2}(\mathbb{Z}^d) \setminus l^2(\mathbb{Z}^d)$ is a resonant state of $H = H_0 + V$ if u satisfies

$$Hu = \lambda u.$$

If such u exists, we say that λ is a resonance of H.

From now on, we concentrate to the case of $\lambda = 0$. Now we state our first theorem, which is an analogy of the continuous model (for example, see [78, Lemma 2.4]).

Theorem 7.1.1. *Let $d \geq 3$. Suppose that V is a real-valued function satisfying $|V(x)| \leq c\langle x\rangle^{-2-\varepsilon}$ for an $0 < \varepsilon \leq 1$ and $u \in l^{2,-3/2}(\mathbb{Z}^d)$ satisfies $(H_0 + V)u = 0$. Then there exists $C > 0$ such that*

$$\begin{aligned} &|u(x)| \leq C\langle x\rangle^{-d+2}, \\ &u(x) = -c_d |x|^{-d+2} \sum_{y \in \mathbb{Z}^d} Vu(y) + O(|x|^{-d+2-\varepsilon}) \end{aligned}$$

as $|x| \to \infty$, where

$$c_d = \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{\frac{d}{2}}}. \tag{7.1.2}$$

In particular, if $\sum_{x \in \mathbb{Z}^d} Vu(x) \neq 0$ holds, then $|u(x)| \geq C|x|^{-d+2}$ follows as $|x| \to \infty$.

Remark 7.1.2. This theorem implies that

(*i*) Set $N_s = \{u \in l^{2,-s}(\mathbb{Z}^d) \mid (H_0 + V)u = 0\}$ for $1/2 < s \leq 3/2$. Then $N_s = N_{s'}$ for $s, s' \in (1/2, 3/2]$.

(*ii*) Suppose that $d = 3$ with $\varepsilon > 1/2$ or $d = 4$ with $\varepsilon > 0$. Then it follows that the function u in Theorem 7.1.1 is an l^2-eigenfunction of H_0+V if and only if $\sum_{y\in\mathbb{Z}^d} Vu(y) = 0$.

(*iii*) There are no resonances at zero energy for $d \geq 5$.

Let $d \geq 3$. We recall some results from [68, Theorem 1.1, Theorem 1.8 and Proposition 3.4]. We have the following limiting absorption principle with the thresholds weight:

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \|\langle x\rangle^{-1+\delta}(H_0 - z)^{-1}\langle x\rangle^{-1-\delta}\|_{B(l^2(\mathbb{Z}^d))} < \infty \tag{7.1.3}$$

if $|\delta| \geq 0$ is small enough. Moreover, the following limits exist in $B(l^{2,s}(\mathbb{Z}^d), l^{2,-s}(\mathbb{Z}^d))$ for $s > 1$:

$$(H_0 - \lambda \mp i0)^{-1} := \lim_{\varepsilon\to 0,\, \varepsilon>0} (H_0 - \lambda \mp i\varepsilon)^{-1},\ \lambda \in [0, 4d]. \tag{7.1.4}$$

We note that (7.1.3) and (7.1.4) away from Γ directly follow from the Mourre theory or [68, Proposition B.5]. The novelty of (7.1.3) and (7.1.4) lie in the estimates near $z, \lambda \in \Gamma$. Furthermore, we have the following lemma which immediately follows from (7.1.3) and (7.1.4) by the density argument.

Lemma 7.1.3. *Let $d \geq 3$. The operators $(H_0 - \lambda \mp i0)^{-1} \in B(l^{2,s}(\mathbb{Z}^d), l^{2,-s}(\mathbb{Z}^d))$ for $s > 1$ and $\lambda \in [0, 4d]$ uniquely extend to bounded linear operators from $l^{2,1}(\mathbb{Z}^d)$ to $l^{2,-1}(\mathbb{Z}^d)$. Moreover, we have*

$$\sup_{\lambda\in\mathbb{R}} \|\langle x\rangle^{-1}(H_0 - \lambda \mp i0)^{-1}\langle x\rangle^{-1}\|_{B(l^2(\mathbb{Z}^d))} < \infty. \tag{7.1.5}$$

Remark 7.1.4. This lemma does not assert

$$(H_0 - \lambda \mp i0)^{-1} = \lim_{\varepsilon\to 0, \varepsilon>0} (H_0 - \lambda \mp i\varepsilon)^{-1} \text{ in } B(l^{2,1}(\mathbb{Z}^d), l^{2,-1}(\mathbb{Z}^d)).$$

Now we give a definition of resonance at hyperbolic thresholds.

Definition 5. Let $d \geq 3$. Suppose that a real-valued function V satisfies $|V| \leq C\langle x\rangle^{-2-\delta}$ with $\delta > 0$. Let $\lambda \in \Gamma \setminus \{0, 4d\}$, that is, λ is a hyperbolic threshold. We call $u \in l^{2,-1}(\mathbb{Z}^d) \setminus l^2(\mathbb{Z}^d)$ a resonant state of $H = H_0 + V$ if u satisfies

$$u + (H_0 - \lambda \mp i0)^{-1}Vu = 0.$$

If such u exists, we say that λ is a resonance of H.

Remark 7.1.5. The validity of this definition lies in Proposition 7.4.4: If λ is not an eigenvalue and not a resonance of H, then the outgoing/ incoming resolvent $(H - \lambda \mp i0)^{-1}$ exist.

Remark 7.1.6. As is shown in Lemma 7.4.3, we can replace $u \in l^{2,-1}(\mathbb{Z}^d)$ by $l^{2,-1-\delta}(\mathbb{Z}^d)$.

The following theorem implies that resonances of H at hyperbolic thresholds do not exist under a stronger assumption of V even when $d = 3$ or 4.

Theorem 7.1.7. *Let $d \geq 3$, $\lambda \in \Gamma \setminus \{0, 4d\}$ and V be a real-valued function satisfying $|V(x)| \leq C\langle x\rangle^{-\delta}$ with $\delta > d/2+2$. If $u \in l^{2,-1}(\mathbb{Z}^d)$ satisfies $u + (H_0 - \lambda \pm i0)^{-1}Vu = 0$, then $u \in l^2(\mathbb{Z}^n)$.*

We recall from [35] that for a finitely supported real-valued potential V, H has no eigenvalues in $(0, 4d)$. Combining this result with Theorem 7.1.7, we obtain the following corollary.

Corollary 7.1.8. *Let $d \geq 3$ and V be a finitely supported real-valued potential. Then $H_0 + V$ has no resonances and no eigenvalues in $(0, 4d)$.*

This corollary implies the limiting absorption principle for $H = H_0 + V$ near hyperbolic thresholds.

Theorem 7.1.9. *Let $d \geq 3$ and V be a finitely supported real-valued potential. Set*

$$\Omega_{\varepsilon_1,\pm} = \{z \in \mathbb{C} \mid \pm\mathrm{Im}\, z > 0,\ |z| > \varepsilon_1,\ |z - 4d| > \varepsilon_1\}$$

for $0 < \varepsilon_1 < 1$ and a signature $\pm$.

(i) We have

$$\sup_{z\in\Omega_{\varepsilon_1,\pm}} \|\langle x\rangle^{-1}(H - z)^{-1}\langle x\rangle^{-1}\|_{B(l^2(\mathbb{Z}^d))} < \infty. \tag{7.1.6}$$

(ii) For each $s > 1$, the operators $z \in \Omega_{\varepsilon_1,\pm} \mapsto (H - z)^{-1} \in B(l^{2,s}(\mathbb{Z}^d), l^{2,-s}(\mathbb{Z}^d))$ is Hölder continuous. In particular, limits

$$(H - \lambda \mp i0)^{-1} := \lim_{\varepsilon\to 0,\, \varepsilon>0}(H - \lambda \mp i\varepsilon)^{-1}$$

exist in the norm operator topology of $B(l^{2,s}(\mathbb{Z}^d), l^{2,-s}(\mathbb{Z}^d))$ for $\varepsilon_1 < \lambda < 4d - \varepsilon_1$.

(iii) Let $s > 1$ and $\varepsilon_1 < \lambda < 4d - \varepsilon_1$. The outgoing/incoming resolvents $(H - \lambda \mp i0)^{-1} \in B(l^{2,s}(\mathbb{Z}^d), l^{2,-s}(\mathbb{Z}^d))$ uniquely extend to bounded linear operators from $l^{2,1}(\mathbb{Z}^d)$ to $l^{2,-1}(\mathbb{Z}^d)$. Moreover, we have

$$\sup_{\varepsilon_1<\lambda<4d-\varepsilon_1} \|\langle x\rangle^{-1}(H - \lambda \mp i0)^{-1}\langle x\rangle^{-1}\|_{B(l^2(\mathbb{Z}^d))} < \infty. \tag{7.1.7}$$

Remark 7.1.10. Suppose that there are no resonances and no eigenvalues at $\{0, 4d\}$. Then the all results in the above theorem still hold if we replace $\Omega_{\varepsilon_1,\pm}$ by $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm\mathrm{Im}\, z > 0\}$. See Proposition 7.4.4.

As mentioned above, for the case of finitely supported potentials it is known that there are no eigenvalues in open interval $(0, 4d)$ (see [3]). However it is possible that the threshold 0 or $4d$ is an embedded eigenvalue. The persistent set (variety) P_S of embedded eigenvalue 0 is defined as the set of all potentials V supported on S such that $H = H_0 + V$ has the eigenvalue 0, that is

$$P_S = \{V \in \mathbb{R}^S \mid \mathrm{supp}V \subset S \text{ and } 0 \text{ is an eigenvalue of } H_0 + V\}.$$

Here S is a fixed finite subset of $\mathbb{Z}^d$. In [28], some geometrical structure and properties of P_S are considered. Moreover the notion of the threshold resonances is defined and non-existence of them for $d \geq 5$ and the persistent set of them for $d = 2, 3, 4$ are studied. The proof for many statements in [28], however, depends on the finiteness of potential support. So in our article we attempt to give an appropriate definition of threshold resonant states of more general potentials and investigate some properties of them by using a method of harmonic analysis. Furthermore we study the limiting absorption principle and resonances at hyperbolic thresholds.

We fix some notations. For Banach spaces X, Y, we denote the set of all bounded linear operators from X to Y by $B(X, Y)$ and set $B(X) := B(X, X)$.

We need the following useful representation. We assume $\nabla h_0 \neq 0$ on $\{h_0(\xi) = \lambda\} \cap U$ for a $\lambda \in \mathbb{R}$ and and an open set U. Moreover, we assume $\{h_0(\xi) = \lambda\} \cap U$ has the following graph representation:

$$\{h_0(\xi) = \lambda\} \cap U = \{\xi \mid \xi_d = g(\xi')\}, \ \xi = (\xi', \xi_d).$$

Then the induced surface measure $d\sigma$ on $\{h_0(\xi) = \lambda\} \cap U$ is written as

$$d\sigma(\xi) = \sqrt{1 + |\nabla g(\xi')|^2} d\xi' = \frac{|(\nabla_\xi h_0)(\xi', g(\xi'))|}{|(\partial_{\xi_d} h_0)(\xi', g(\xi'))|} d\xi'. \tag{7.1.8}$$

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7.2 Pointwise estimates, Proof of Theorem 7.1.1

7.2.1 Upper bounds

Let $d \geq 3$. We consider the solution to

$$(H_0 + V)u = 0. \tag{7.2.1}$$

First, we reduce the equation (7.2.1) to the integral equation, which is useful for estimating u:

$$u + H_0^{-1} V u = 0, \tag{7.2.2}$$

where

$$K_2(x) = \int_{\mathbb{T}^d} e^{2\pi i x\cdot\xi} h_0(\xi)^{-1} d\xi,\ H_0^{-1}w(x) = \sum_{y\in\mathbb{Z}^d} K_2(x-y)w(y),$$

for $w \in l^{2,1/2+\varepsilon}(\mathbb{Z}^d)$ with $\varepsilon > 0$. Here H_0^{-1} is the bounded operator from $l^{2,\alpha}(\mathbb{Z}^d)$ to $l^{2,-\beta}(\mathbb{Z}^d)$ for $\alpha,\beta > 1/2$ with $\alpha+\beta \geq 2$ (see section B, Corollary 7.6.3). Moreover, it also follows that the multiplication operator

$$h_0^{-1} : \bigcap_{s>0} H^s(\mathbb{T}^d) \to \bigcup_{s\in\mathbb{R}} H^s(\mathbb{T}^d)$$

can be uniquely extended to the operator

$$h_0^{-1} : H^\alpha(\mathbb{T}^d) \to H^{-\beta}(\mathbb{T}^d),\ \alpha,\beta > \frac{1}{2},\ \alpha+\beta \geq 2 \tag{7.2.3}$$

and that

$$h_0^{-1} = \mathcal{F}_d^{-1} H_0^{-1} \mathcal{F}_d : H^\alpha(\mathbb{T}^d) \to H^{-\beta}(\mathbb{T}^d),\ \alpha,\beta > \frac{1}{2},\ \alpha+\beta \geq 2.$$

Lemma 7.2.1. *We assume* $|V(x)| \leq C\langle x\rangle^{-2-\varepsilon}$ *for some* $\varepsilon > 0$. *For* $u \in l^{2,-3/2}(\mathbb{Z}^d)$, (7.2.1) *implies* (7.2.2).

Proof. The relations (7.1.1) and (7.2.1) implies

$$h_0(\xi)\hat{u}(\xi) = -\widehat{Vu}(\xi),\ \hat{u} \in H^{-\frac{3}{2}}(\mathbb{T}^d). \tag{7.2.4}$$

First, we note $\hat{u}(\xi) = -h_0(\xi)^{-1}\widehat{Vu}(\xi)$ in $\mathcal{D}'(\mathbb{T}^d \setminus \{0\})$. We note $h_0^{-1}\widehat{Vu} \in \mathcal{D}'(\mathbb{T}^d)$ by (7.2.3). These imply that $\hat{u} + h_0^{-1}\widehat{Vu}$ is supported in $\{0\}$ as an element of $\mathcal{D}'(\mathbb{T}^d)$ and can be written as a linear combination of the derivatives of the Dirac measure. Since $\partial_\xi^\gamma \delta \notin H^{-d/2}(\mathbb{T}^d)$ for any multi-index γ, it suffices to prove $\hat{u} + h_0^{-1}\widehat{Vu} \in H^{-d/2}(\mathbb{T}^d)$ in order to deduce $\hat{u} = -h_0^{-1}\widehat{Vu}$. Since $\hat{u} \in H^{-3/2}(\mathbb{T}^d) \subset H^{-d/2}(\mathbb{T}^d)$, we only need to prove $h_0^{-1}\widehat{Vu} \in H^{-d/2}(\mathbb{T}^d)$. Using $\widehat{Vu} \in H^{1/2+\varepsilon}(\mathbb{T}^d)$ and (7.2.3) with $\alpha = 1/2+\varepsilon$ and $\beta = 3/2$, we obtain $h_0^{-1}\widehat{Vu} \in H^{-3/2}(\mathbb{T}^d) \subset H^{-d/2}(\mathbb{T}^d)$. This completes the proof. □

The main result of this subsection is the following proposition.

Proposition 7.2.2. *Let* $u \in l^{2,-3/2}(\mathbb{Z}^d)$ *be a solution to* (7.2.2). *Then we have*

$$|u(x)| \leq C\langle x\rangle^{-d+2}.$$

The following lemma is useful.

Lemma 7.2.3. *Let* $d \geq 1$.

(i) Let $k, l < d$ with $k + l > d$. Then we have

$$I = \sum_{y \in \mathbb{Z}^d} \langle x - y \rangle^{-k} \langle y \rangle^{-l} \leq C \langle x \rangle^{d-k-l}.$$

(ii) Let $0 < k < d$ and $l = d$. For any $\delta > 0$, there exists $C_\delta > 0$ such that $I \leq C_\delta \langle x \rangle^{\delta - k}$.

(iii) Let $0 < k < d < l$. Then we have

$$I \leq C \langle x \rangle^{-k}.$$

(iv) Let $k = d$ and $l > d$. Then we have

$$I \leq C \langle x \rangle^{-d}.$$

Proof. (*i*) We decompose $I = I_1 + I_2 + I_3$ such that

$$I_1 = \sum_{|x-y| \leq |x|/2} \langle x - y \rangle^{-k} \langle y \rangle^{-l}, \quad I_2 = \sum_{\substack{|x-y| \geq |x|/2, \\ |y| \leq 2|x|}} \langle x - y \rangle^{-k} \langle y \rangle^{-l},$$

$$I_3 = \sum_{\substack{|x-y| \geq |x|/2, \\ |y| > 2|x|}} \langle x - y \rangle^{-k} \langle y \rangle^{-l}.$$

We note that $|x - y| \leq |x|/2$ implies $|x|/2 \leq |y| \leq 3|x|/2$. Using this and $k < d$, we have

$$I_1 \leq C \langle x \rangle^{-l} \sum_{|x-y| \leq |x|/2} \langle x - y \rangle^{-k} = C \langle x \rangle^{-l} \sum_{|y| \leq 1/2|x|} \langle y \rangle^{-k} \leq C \langle x \rangle^{d-k-l}.$$

Moreover, using $l < d$, we learn

$$I_2 \leq C \langle x \rangle^{-k} \sum_{\substack{|x-y| \geq |x|/2, \\ |y| \leq 2|x|}} \langle y \rangle^{-l} \leq C \langle x \rangle^{d-k-l}.$$

To estimate I_3, we observe that $|x - y| \geq |y|/2$ holds in $\{|y| > 2|x|\}$. Using this and $k + l > d$, we obtain

$$I_3 \leq C \sum_{|y| > 2|x|} \langle y \rangle^{-k-l} \leq C \langle x \rangle^{d-k-l}.$$

Thus we conclude $I \leq C \langle x \rangle^{d-k-l}$.

(*ii*) As in the proof of (*i*), using $k < d$ and $k + l > d$ with $l = d$, we have $I_1 + I_3 \leq C \langle x \rangle^{-k}$. We observe

$$I_2 \leq C \langle x \rangle^{-k} \sum_{|y| \leq 2|x|} \langle y \rangle^{-d} \leq C_\delta \langle x \rangle^{\delta - k}.$$

This proves (ii).

(iii) As in the proof of (i), using $k < d$, we have $I_1 \leq C\langle x\rangle^{d-k-l}$. The inequality $l > d$ implies $I_1 \leq C\langle x\rangle^{-k}$. On the other hand, using $l > d$, we observe

$$I_2 + I_3 \leq C\langle x\rangle^{-k} \sum_{y\in\mathbb{Z}^d} \langle y\rangle^{-l} \leq C\langle x\rangle^{-k}.$$

We conclude $I \leq C\langle x\rangle^{-k}$.

(iv) As in the proof of (iii), using $l > d$, we have $I_2 + I_3 \leq C\langle x\rangle^{-d}$. Since $|x - y| \leq |x|/2$ holds on $\{|x|/2 \leq |y| \leq 3|x|/2\}$, we have

$$I_1 \leq C\langle x\rangle^{-l} \sum_{|y|\leq|x|/2} \langle y\rangle^{-d} \leq C_\delta\langle x\rangle^{\delta-l}$$

for any $\delta > 0$. We take $\delta = l - d > 0$ and obtain $I_3 \leq C\langle x\rangle^{-d}$.

□

Proof of Proposition 7.2.2. We may assume $0 < \varepsilon < 1$. Using $u \in l^{2,-3/2}(\mathbb{Z}^d)$, $|V(x)| \leq C\langle x\rangle^{-2-\varepsilon}$ and Corollary 7.6.2 with $l = 2$, we have

$$\begin{aligned}|u(x)| = |H_0^{-1}Vu(x)| \leq& C \sum_{y\in\mathbb{Z}^d} \langle y\rangle^{-d+2}|Vu(x-y)| \\ \leq& C(\sum_{y\in\mathbb{Z}^d} \langle y\rangle^{-2d+4}\langle x-y\rangle^{-1-2\varepsilon})^{1/2}\|\langle x\rangle^{1/2+\varepsilon}Vu\|_{l^2(\mathbb{Z}^d)}.\end{aligned}$$

Applying Lemma 7.2.3 with $k = 1 + 2\varepsilon$ and $l = 2d - 4$, we have $|u(x)| \leq C\langle x\rangle^{-\varepsilon} \leq C\langle x\rangle^{-\varepsilon/2}$ for $d = 3$, $|u(x)| \leq C\langle x\rangle^{-1/2-\varepsilon/2}$ for $d \geq 4$.

The argument below is based on the standard bootstrap technique (for example, see [61, Lemma 8 in the proof of Theorem XIII.33]). Set $\alpha_d = 0$ for $d = 3$ and $\alpha_d = 1/2$ for $d \geq 4$. Let N be a real number such that $2 + \alpha_d + (N + 1)\varepsilon < d$. Suppose $|u(x)| \leq C\langle x\rangle^{-\alpha_d-N\varepsilon}$ holds. Then it follows that

$$|u(x)| \leq C \sum_{y\in\mathbb{Z}^d} \langle y\rangle^{-d+2}\langle x-y\rangle^{-2-\alpha_d-(N+1)\varepsilon}$$

Applying Lemma 7.2.3 with $k = 2 + \alpha_d + (N + 1)\varepsilon$ and $l = d - 2$, we have $|u(x)| \leq C\langle x\rangle^{-\alpha_d-(N+1)\varepsilon}$. By an induction argument, we obtain $|u(x)| \leq C\langle x\rangle^{-d+2}$.

□

7.2.2 Lower bounds, Proof of Theorem 7.1.1

We need some elementary lemmas.

Lemma 7.2.4. *[25, Theorem2.4.6] Let $c_d > 0$ be as in* (7.1.2). *Then we have*

$$\int_{\mathbb{R}^d} e^{2\pi ix\cdot\xi}\frac{1}{4\pi^2|\xi|^2}d\xi = c_d|x|^{-d+2}.$$

We omit the proof of this lemma.

Lemma 7.2.5. *There exists $C > 0$ such that*

$$||x-y|^{-d+2} - |x|^{-d+2}| \le C|x|^{-d+1}|y|$$

for $x, y \in \mathbb{R}^d$ with $|x|/2 > |y|$.

Proof. We note $(d/d\theta)(|x-\theta y|) \le |y|$ and $|x-\theta y| \ge |x|/2$ for $0 \le \theta \le 1$ and $|x|/2 > |y|$. Then we have

$$\begin{aligned}
||x-y|^{-d+2} - |x|^{-d+2}| \le & \int_0^1 |(d/d\theta)|x-\theta y|^{-d+2}| d\theta \\
=&(d-2)\int_0^1 \frac{|(d/d\theta)|x-\theta y||}{|x-\theta y|^{d-1}} d\theta \\
\le & 2^{d-1}(d-2)|x|^{-d+1}|y|.
\end{aligned}$$

□

Proof of Theorem 7.1.1. Note that $|Vu(x)| \le C\langle x\rangle^{-d-\varepsilon}$ and

$$u(x) = -\sum_{y\in\mathbb{Z}^d} G(x,y)Vu(y),\ G(x,y) = \int_{\mathbb{T}^d} e^{2\pi i(x-y)\cdot\xi}\frac{1}{h_0(\xi)}d\xi.$$

For small $r > 0$, take $\chi \in C^\infty(\mathbb{T}^d, [0,1])$ such that $\chi = 1$ on $|\xi| \le r$ and $\chi = 0$ outside $|\xi| \le 2r$. Then

$$u(x) = -\sum_{y\in\mathbb{Z}^d} G_1(x,y)Vu(y) + O(\langle x\rangle^{-d-\varepsilon}),\ G_1(x,y) = \int_{\mathbb{T}^d} e^{2\pi i(x-y)\cdot\xi}\frac{\chi(\xi)}{h_0(\xi)}d\xi.$$

We use the following lemmas.

Lemma 7.2.6. *We have*

$$u(x) = -\sum_{y\in\mathbb{Z}^d} G_2(x,y)Vu(y) + O(\langle x\rangle^{-d}),$$

where $G_2(x,y) = \int_{\mathbb{R}^d} e^{2\pi i(x-y)\cdot\xi}\frac{\chi(\xi)}{4\pi^2|\xi|^2}d\xi$.

Proof. If $|\xi| \le 2r$ for small $r > 0$, then we expand $h_0(\xi)^{-1} = 1/(4\pi|\xi|^2) + R(\xi)$, where $|\partial_\xi^\alpha R(\xi)| \le C_\alpha|\xi|^{-|\alpha|}$. Thus we have

$$u(x) = -\sum_{y\in\mathbb{Z}^d} G_2(x,y)Vu(y) - \sum_{y\in\mathbb{Z}^d} G_3(x,y)Vu(y) + O(\langle x\rangle^{-d-\varepsilon}),$$

where

$$G_3(x,y) = \int_{\mathbb{R}^d} e^{2\pi i(x-y)\cdot\xi}\chi(\xi)R(\xi)d\xi.$$

By Lemmas 7.6.1 and 7.2.3 (iv) with $k = d$ and $l = d+\varepsilon$, the second term is $O(|x|^{-d})$. This completes the proof. □

Lemma 7.2.7. *For $|x| \geq 1$, we have*

$$\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \frac{\chi(\xi)}{4\pi^2|\xi|^2} d\xi = c_d|x|^{-d+2} + O(\langle x\rangle^{-\infty}).$$

Proof. We have in the distribution sense

$$\begin{aligned}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \frac{\chi(\xi)}{4\pi^2|\xi|^2} d\xi =& \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \frac{1}{4\pi^2|\xi|^2} d\xi + \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \frac{\chi(\xi)-1}{4\pi^2|\xi|^2} d\xi \\ =& c_d|x|^{-d+2} + \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \frac{\chi(\xi)-1}{4\pi^2|\xi|^2} d\xi.\end{aligned}$$

The second term decays rapidly at infinity as can be shown by integration by parts. □

Lemma 7.2.8.

$$\sum_{|y|\geq |x|/2} G_2(x,y)Vu(y) = O(\langle x\rangle^{-d+2-\varepsilon}).$$

Proof. By Lemma 7.2.7, we have $G_2(x,y) = O(\langle x-y\rangle^{-d+2})$. Since $V(x) = O(\langle x\rangle^{-2-\varepsilon})$ holds, by Proposition 7.2.2, we have $Vu = O(\langle x\rangle^{-d-\varepsilon})$. Now the lemma is proved by an easy calculation using the condition $\{|y| \geq |x|/2\}$. □

Lemma 7.2.9.

$$\sum_{|y|<1/2|x|} G_2(x,y)Vu(y) = c_d|x|^{-d+2} \sum_{y\in\mathbb{Z}^d} Vu(y) + O(\langle x\rangle^{-d+2-\varepsilon}).$$

Proof. By Lemma 7.2.7 and $Vu = O(\langle x\rangle^{-d-\varepsilon})$, we have

$$\begin{aligned}\sum_{|y|<|x|/2} G_2(x,y)Vu(y) =& c_d \sum_{|y|<|x|/2} |x-y|^{-d+2} Vu(y) \\ &+ \sum_{|y|<|x|/2} O(\langle x-y\rangle^{-\infty}\langle y\rangle^{-d-\varepsilon}) \\ =& c_d \sum_{|y|<|x|/2} |x-y|^{-d+2} Vu(y) + O(\langle x\rangle^{-\infty}),\end{aligned}$$

By Lemma 7.2.5, we have

$$\begin{aligned}
c_d \sum_{|y|<|x|/2} |x-y|^{-d+2} Vu(y) =& c_d |x|^{-d+2} \sum_{|y|<|x|/2} Vu(y) \\
&+ \sum_{|y|<|x|/2} O(\langle x\rangle^{-d+1} \langle y\rangle^{-d+1-\varepsilon}) \\
=& c_d |x|^{-d+2} \sum_{|y|<|x|/2} Vu(y) + O(\langle x\rangle^{-d+2-\varepsilon}) \\
=& c_d |x|^{-d+2} \sum_{y\in\mathbb{Z}^d} Vu(y) + c_d |x|^{-d+2} \sum_{|y|\geq|x|/2} Vu(y) \\
&+ O(\langle x\rangle^{-d+2-\varepsilon}) \\
=& c_d |x|^{-d+2} \sum_{y\in\mathbb{Z}^d} Vu(y) + O(\langle x\rangle^{-d+2-\varepsilon}),
\end{aligned}$$

where we use $Vu = O(\langle x\rangle^{-d-\varepsilon})$. This completes the proof. □

We return to the proof of Theorem 7.1.1. By virtue of Lemmas 7.2.6 and 7.2.8, we write

$$u(x) = -\sum_{|y|<|x|/2} G_2(x,y) Vu(y) + O(|x|^{-d+2-\varepsilon}).$$

Note that $|x-y|$ is large if $|x|$ is large and $|y| < |x|/2$. Using Lemma 7.2.9, we complete the proof of Theorem 7.1.1.

□

7.3 Absence of embedded resonances, Proof of Theorem 7.1.7

7.3.1 Preliminary lemmas

Let $d \geq 3$ and $\lambda \in \{4k\}_{k=1}^{d-1}$. Set $M_\lambda = \{\xi \in \mathbb{T}^d \mid h_0(\xi) = \lambda\}$ and

$$\begin{aligned}
\Sigma_\lambda =& \{\xi \in M_\lambda \mid \nabla h_0(\xi) = 0\} = \{\xi \in M_\lambda \mid \sin 2\pi\xi_j = 0, \text{ for all } j = 1,...,d\} \\
=& \{\xi \in M_\lambda \mid \xi_j \in \{0, \frac{1}{2}\}, \text{ for all } j = 1,...,d\}.
\end{aligned}$$

We note $M_\lambda \setminus \Sigma_\lambda$ is an embedded submanifold of $\mathbb{T}^d$ with codimension 1 and M_λ is a Lipschitz submanifold in the sense that M_λ has a graph representation by a Lipschitz function. We denote the induced surface measure of M_λ by $d\sigma(\xi)$. Set

$$d\mu(\xi) = \frac{1}{|\nabla h_0(\xi)|} d\sigma(\xi)$$

We note that $|\nabla h_0(\xi)|^{-1} \sim |\xi - \xi_0|^{-1}$ near $\xi_0 \in \Sigma_\lambda$ implies that $d\mu$ is singular at any points of Σ_λ for $\lambda \in \Gamma$, though $|\nabla h_0(\xi)|^{-1}$ is harmless on M_λ with a regular value λ. Moreover, we denote $R_0(\lambda \pm i0) = (H_0 - \lambda \mp i0)^{-1} \in B(l^{2,1}(\mathbb{Z}^d), l^{2,-1}(\mathbb{Z}^d))$. First, we show Σ_λ is of measure zero with respect to $d\sigma$ and $d\mu$, which essentially follows from the fact that $d\sigma$ and $d\mu$ are finite sums of the absolutely continuous measures with respect to $d-1$-dimensional Lebesgue measure.

Lemma 7.3.1. *$d\sigma(\Sigma_\lambda) = 0$ and $d\mu(\Sigma_\lambda) = 0$.*

Proof. First, we note that the measure μ is absolutely continuous with respect to $d\sigma$. To see this, it suffices to show that $1/|\nabla h_0(\xi)|$ is integrable with respect to the measure σ. We note that for $\eta \in \Sigma_\lambda$ and $\xi = (\xi', \xi_d) \in M_\lambda$, we have $|\nabla h_0(\xi)| \sim 2\pi|\xi - \eta| \sim C|\xi' - \eta'|$ near $\xi = \eta$ and $\pm(\xi_d - \eta_d) \geq |\xi' - \eta'|/2d$. The integrability of $1/|\xi' - \eta'|$ over $\{\xi' \in \mathbb{R}^{d-1} \mid |\xi' - \eta'| : \text{small}\}$ which follows from the assumption $d \geq 3$ implies $1/|\nabla h_0(\xi)|$ is integrable over $\{\pm(\xi_d - \eta_d) \geq |\xi' - \eta'|\}$. By using a partition of unity, the integrability of $1/|\nabla h_0(\xi)|$ over M_λ follows.

Thus a proof of $d\mu(\Sigma_\lambda) = 0$ reduces to a proof of $d\sigma(\Sigma_\lambda) = 0$. Let $\eta \in \Sigma_\lambda$. Since Σ_λ is a finite set, it suffices to prove that $\{\eta\}$ has zero measure with respect to $\chi d\sigma$, where $\chi \in C^\infty(\mathbb{T}^d)$ is any function supported close to η. Set

$$A_{j,\pm} = \{\xi \in \text{supp}\ \chi \mid \pm(\xi_j - \eta_j) \geq |\xi - \eta|/2d\}.$$

Then we have

$$\chi(\xi)d\sigma(\xi) = \sum_{j=1\dots,d,\, a=\pm} \chi_{A_{j,a}}(\xi)\chi(\xi)d\sigma(\xi) =: \sum_{j=1\dots,d,\, a=\pm} d\sigma_{j,a}(\xi),$$

where χ_A is the characteristic function of $A \subset \mathbb{T}^d$. Thus it suffices to prove that $\{\eta\}$ is zero measure with respect to $d\sigma_{j,a}$ for any $j = 1, .., d$ and $a = \pm$.

By rotating and reflecting the coordinate, we may assume $j = d$ and $a = +$. If $\text{supp}\ \chi$ is small enough, we have the following graph representation:

$$M_\lambda \cap \text{supp}\ \chi \cap \{\pm(\xi_d - \eta_d) \geq |\xi - \eta|/2d\} = \{(\xi', g(\xi'))\}$$

where g is a Lipschitz function. On this coordinate, we write

$$d\sigma_{d,+}(\xi) = \chi_{A_{j,a}}(\xi)\chi(\xi)\sqrt{1 + |\nabla_{\xi'} g(\xi')|^2}d\xi'$$

by (7.1.8). This implies that $d\sigma_{d,+}$ is absolutely continuous with respect to the $d-1$-dimensional Lebesgue measure $d\xi'$. This completes the proof.

□

We recall the standard L^2-restriction theorem: For $f \in l^{2,s}(\mathbb{Z}^d)$ with $s > 1/2$, then

$$\hat{f}|_{M_\lambda} \in L^2_{loc}(M_\lambda, d\sigma).$$

For $f \in l^{2,1}(\mathbb{Z}^d)$, we have sharper integrability of $\hat{f}|_{M_\lambda}$ near Σ_λ with respect to $d\mu$.

Lemma 7.3.2. *For $f \in l^{2,1}(\mathbb{Z}^d)$, a restriction $\hat{f}|_{M_\lambda} \in L^2_{loc}(M_\lambda, d\sigma)$ satisfies $\hat{f}|_{M_\lambda} \in L^2(M_\lambda, d\mu)$. Moreover, we have*

$$\|\hat{f}\|_{L^2(M_\lambda, d\mu)} \le C\|f\|_{l^{2,1}(\mathbb{Z}^d)}. \tag{7.3.1}$$

Proof. Let $z \in \Sigma_\lambda$ and $\chi \in C^\infty(\mathbb{T}^d)$ with a sufficiently small support around z. For proving $\hat{f}|_{M_\lambda} \in L^2(M_\lambda, d\mu)$, it suffices to show

$$(\chi\hat{f})|_{M_\lambda} \in L^2(M_\lambda, d\mu). \tag{7.3.2}$$

Moreover, we take a partition of unity $\{(\tilde{\psi}_{j,a})^2\}_{j=1,...d,a=\pm}$ of $\mathbb{S}^{d-1}$ such that

$$\text{supp } \tilde{\psi}_{j,a} \subset \{x \in \mathbb{S}^{d-1} \subset \mathbb{R}^d \mid \pm x_d \ge \frac{|x|}{2d}\}.$$

We set $\psi_{j,a}(\xi) = \tilde{\psi}_{j,a}((\xi - z)/|\xi - z|)$.

First, for $j = 1, ...d$ and $a = \pm$, we shall prove

$$\int_{M_\lambda \setminus \Sigma_\lambda} |(\psi_{j,a}\chi\hat{f})|_{M_\lambda}(\xi)|^2 d\mu(\xi) \le C \int_{\mathbb{R}^{d-1}} \frac{|(\psi_{j,a}\chi\hat{f})(\xi', g(\xi'))|^2}{|\xi' - z'|} d\xi'. \tag{7.3.3}$$

We may assume $j = d$ and $a = +$. We define a real-valued function g by

$$\sin \pi g(\xi') = \sqrt{(\frac{\lambda}{4} - \sum_{j=1}^{d-1} \sin^2 \pi\xi_j)}, \; g(\xi') > 0.$$

Then g satisfies

$$h_0(\xi', g(\xi')) = \lambda \text{ for } \xi = (\xi', g(\xi')) \in \text{supp } (\psi_{d,+}\chi) \setminus \Sigma_\lambda.$$

Since $\partial_{\xi_d} h_0(z', z_d) = 0$ and h is a Morse function, it follows that

$$\begin{aligned} &|\partial_{\xi_d} h_0(\xi', g(\xi'))| \ge C|g(\xi') - z_d| \ge C|\xi' - z'|, \\ &|\partial_{\xi'} g(\xi')| = |-\frac{(\partial_{\xi'} h_0)(\xi', g(\xi'))}{(\partial_{\xi_d} h_0)(\xi', g(\xi'))}| \le C|\frac{\xi' - z'}{|g(\xi') - z_d|}| \le C. \end{aligned} \tag{7.3.4}$$

on supp $(\psi_{d,+}\chi) \setminus \Sigma_\lambda$. These inequalities with (7.1.8) implies

$$\begin{aligned} \int_{M_\lambda} |(\psi_{j,a}\chi\hat{f})|_{M_\lambda \setminus \Sigma_\lambda}(\xi)|^2 d\mu(\xi) = &\int_{\mathbb{R}^{d-1}} \frac{|(\psi_{j,a}\chi\hat{f})(\xi', g(\xi'))|^2}{|(\partial_{\xi_d} h_0)(\xi', g(\xi'))|} d\xi' \\ \le &C \int_{\mathbb{R}^{d-1}} \frac{|(\psi_{j,a}\chi\hat{f})(\xi', g(\xi'))|^2}{|\xi' - z'|} d\xi'. \end{aligned}$$

Summing (7.3.3) over $j = 1, ..., d$ and $a = \pm$, we obtain

$$\int_{M_\lambda} |(\chi\hat{f})|_{M_\lambda\setminus\Sigma_\lambda}(\xi)|^2 d\mu(\xi) \leq C\int_{\mathbb{R}^{d-1}} \frac{|(\chi\hat{f})(\xi', g(\xi'))|^2}{|\xi' - z'|} d\xi' \\ \leq C\|\langle D_{\xi'}\rangle^{1/2}((\chi\hat{f})(\xi', g(\xi')))\|^2_{L^2(\mathbb{R}^{d-1})} \\ \leq C\|\chi\hat{f}\|^2_{H^1(\mathbb{R}^d)}. \tag{7.3.5}$$

where we use the Hardy inequality in the second line and use Proposition 7.7.2 in the third line. We recall that supp χ is small enough and we identify the integral over $\mathbb{T}^d$ with the integral over this fundamental domain $[-\frac{1}{2}, \frac{1}{2}]^d$. This implies $\|\chi\hat{f}\|_{H^1(\mathbb{R}^d)} = \|\chi\hat{f}\|_{H^1(\mathbb{T}^d)}$. Since $f \in l^{2,1}(\mathbb{Z}^d)$, we have $\chi\hat{f} \in H^1(\mathbb{T}^d)$. Thus we conclude (7.3.2). The estimate (7.3.1) follows from (7.3.5) by using a partition of unity and the standard L^2 restriction theorem.

□

Remark 7.3.3. The assumption $d \geq 3$ is needed once more for using the Hardy inequality.

Now we prove a similar formula as [61, Lemma 7 in the proof of Theorem XIII.33] around the hyperbolic threshold.

Lemma 7.3.4. *For $f \in l^{2,1}(\mathbb{Z}^d)$, we have*

$$\frac{1}{2\pi}\text{Im }(f, R_0(\lambda \pm i0)f) = \int_{M_\lambda} |\hat{f}(\xi)|^2 d\mu(\xi). \tag{7.3.6}$$

Proof. For $\hat{f} \in C^\infty(\mathbb{T}^d)$, (7.3.6) follows from a simple calculation. Let $f \in l^{2,1}(\mathbb{Z}^d)$. Take a sequence $\widehat{f_k} \in C^\infty(\mathbb{T}^d)$ such that $\widehat{f_k} \to \hat{f}$ in $H^1(\mathbb{T}^d)$. Then (7.3.6) follows from (7.1.3) and (7.3.1). □

Lemma 7.3.5. *Let V be a real-valued function satisfying $|V| \leq C\langle x\rangle^{-2}$. If $u \in l^{2,-1}(\mathbb{Z}^d)$ satisfies $u + R_0(\lambda \pm i0)Vu = 0$, then $\widehat{Vu}|_{M_\lambda} = 0$.*

Proof. We note $\widehat{Vu}|_{M_\lambda}$ and (Vu, u) are both well-defined, which follow from $u \in l^{2,-1}(\mathbb{Z}^d)$ and $Vu \in l^{2,1}(\mathbb{Z}^d)$. Then we have

$$0 = -\text{Im }(Vu, u) = \text{Im }(Vu, R_0(\lambda \pm i0)Vu) = 2\pi\int_{M_\lambda} |\widehat{Vu}(\xi)|^2 d\mu(\xi).$$

Thus we obtain $\widehat{Vu}|_{M_\lambda} = 0$. □

7.3.2 No resonance in the interior of the spectrum

For $0 \leq k \leq d$, we define

$$p_k(\eta) = -\sum_{j=1}^{k} \eta_j^2 + \sum_{j=k+1}^{d} \eta_j^2.$$

The next lemma is a weaker version of [61, Theorem IX.41] (which theorem is for sphere) near the hyperbolic thresholds.

Lemma 7.3.6. *Suppose $d \geq 3$. Let $f \in C^1(\mathbb{T}^d)$ such that $f|_{M_\lambda} = 0$. Then we have $(h_0 - \lambda)^{-1} f \in L^2(\mathbb{T}^d)$.*

Remark 7.3.7. We regard $(h_0 - \lambda)^{-1} f$ as a principal value:

$$((h_0 - \lambda)^{-1} f, \varphi) = \lim_{\varepsilon \to 0} \int_{|h_0 - \lambda| > \varepsilon} \frac{f(\xi)\varphi(\xi)}{h_0(\xi) - \lambda} d\xi.$$

However, since $f|_{M_\lambda} = 0$, $(h_0 - \lambda \pm i0)^{-1} f$ coincides with $(h_0 - \lambda)^{-1} f$.

Proof. Take $\xi_0 \in \mathbb{T}^d$ such that $h_0(\xi_0) = \lambda$ and $dh_0(\xi_0) = 0$. By the Morse lemma, there exist an open neighborhood $U \subset \mathbb{T}^d$ and a diffeomorphism κ from U to its image such that $h_0(\kappa^{-1}(\eta)) - \lambda = p_k(\eta)$ for some $0 \leq k \leq d$. Set $J(\eta) = |\det d\kappa^{-1}(\eta)|$. Take a cut-off function $\chi \in C^\infty(\mathbb{T}^d, [0, 1])$ such that $\operatorname{supp} \chi \subset U$. We only show that $\chi(h_0 - \lambda)^{-1} f \in L^2(\mathbb{T}^d)$. Apart from the hyperbolic threshold, the proof is easier and omitted since f vanishes at the submanifold $h_0 = \lambda$.

We may assume that $\kappa(U) \subset \mathbb{R}^d$ is convex. We write $f_\kappa(\eta) = f(\kappa^{-1}(\eta))$ for $\eta \in \operatorname{supp} \kappa(U)$. Since $f|_{M_\lambda} = 0$ holds, we have $f_\kappa(|\eta''|\o_1, |\eta''|\o_2) = 0$, where we write $\eta = (|\eta'|\o_1, |\eta''|\o_2)$ with $\o_1 \in \mathbb{S}^{k-1}$, $\o_2 \in \mathbb{S}^{d-k-1}$, hence $p_k(\eta) = |\eta'|^2 - |\eta''|^2$. Set

$$a(\eta) = \int_0^1 \o_1 \cdot (\partial_{\eta'} f_\kappa)(((1-t)|\eta''| + t|\eta'|)\o_1, |\eta''|\o_2) dt.$$

By Taylor's formula, we see

$$\begin{aligned} f_\kappa(\eta) =& f_\kappa(|\eta'|\o_1, |\eta''|\o_2) \\ =& f_\kappa(|\eta''|\o_1, |\eta''|\o_2) + (|\eta'| - |\eta''|) \cdot a(\eta) \\ =& (|\eta'| - |\eta''|) \cdot a(\eta). \end{aligned}$$

Thus we have $|f_\kappa(\eta)| \leq C_{\eta_0} ||\eta'| - |\eta''||$ on $\eta \in \kappa(U)$. Hence we obtain

$$\int_{|\eta - \eta_0| \leq 1} \chi_\kappa(\eta) J(\eta) \frac{|f_\kappa(\eta)|^2}{p_k(\eta)^2} d\eta \leq C_{\eta_0}^2 \int_{|\eta - \eta_0| \leq 1} \frac{1}{(|\eta'| + |\eta''|)^2} d\eta < \infty.$$

This implies $\chi(h_0(\xi) - \lambda)^{-1} f \in L^2(\mathbb{T}^d)$.

□

Proof of Theorem 7.1.7. By the assumption, we note $\widehat{Vu} \in C^1(\mathbb{R}^n)$ by the Sobolev embedding theorem. By Lemma 7.3.5 and Lemma 7.3.6, we have $u \in l^2(\mathbb{Z}^n)$. □

7.4 Limiting absorption principle, Proof of Theorem 7.1.9

Suppose $d \geq 3$ and $|V(x)| \leq C\langle x\rangle^{-2-\delta}$ with $\delta > 0$. Fix a signature $\pm$. Set

$$\mathbb{C}_{\pm} = \{z \in \mathbb{C} \mid \pm\mathrm{Im}\, z > 0\},\ \overline{\mathbb{C}_{\pm}} = \{z \in \mathbb{C} \mid \pm\mathrm{Im}\, z \geq 0\}.$$

We define $R_{0,\pm}(z) \in B(l^{2,1}(\mathbb{Z}^d), l^{2,-1}(\mathbb{Z}^d))$ for $z \in \overline{\mathbb{C}_{\pm}}$ by

$$R_{0,\pm}(z) = \begin{cases} (H_0 - z)^{-1} \text{ for } \pm \mathrm{Im}\, z > 0, \\ (H_0 - z \mp i0)^{-1} \text{ for } z \in \mathbb{R}. \end{cases}$$

We recall from [68, Theorem 1.8] that

$$z \in \overline{\mathbb{C}_{\pm}} \mapsto R_{0,\pm}(z) \in B(l^{2,s}(\mathbb{Z}^d), l^{2,-s}(\mathbb{Z}^d)) \text{ is Hölder continuous} \tag{7.4.1}$$

for $s > 1$.

Lemma 7.4.1. *Let $1 \leq s < 1+\delta$. Then it follows that $R_{0,\pm}(z)V$ is a compact operator in $B(l^{2,-s}(\mathbb{Z}^d))$ for $z \in \overline{\mathbb{C}_{\pm}}$. Moreover, a map $z \in \overline{\mathbb{C}_{\pm}} \mapsto R_{0,\pm}(z)V \in B(l^{2,-s}(\mathbb{Z}^d))$ is continuous.*

Proof. In order to prove that $R_{0,\pm}(z)V$ is compact in $B(l^{2,-s}(\mathbb{Z}^d))$, it suffices to prove that $\langle x\rangle^{-1}R_{0,\pm}(z)V\langle x\rangle$ is compact in $B(l^2(\mathbb{Z}^d))$. We write

$$\langle x\rangle^{-s}R_{0,\pm}(z)V\langle x\rangle^{s} = \langle x\rangle^{-s}R_{0,\pm}(z)\langle x\rangle^{-1} \times V\langle x\rangle^{1+s}$$

From (7.1.3), we have $\langle x\rangle^{-s}R_{0,\pm}(z)\langle x\rangle^{-1} \in B(l^2(\mathbb{Z}^d))$. Moreover, $|V(x)| \leq C\langle x\rangle^{-2-\delta}$ with $\delta > 0$ implies that $V\langle x\rangle^{1+s}$ is a compact operator since each multiplication operator which vanishes at infinity is a compact operator on $l^2(\mathbb{Z}^d)$. Thus the compactness of $R_{0,\pm}(z)V$ follows.

Next, we prove that a map $z \in \overline{\mathbb{C}_{\pm}} \mapsto \langle x\rangle^{-s}R_{0,\pm}(z)\langle x\rangle^{-1-\delta} \in B(l^2(\mathbb{Z}^d))$ is continuous, which implies the continuity of $R_{0,\pm}(z)V \in B(l^{2,-s}(\mathbb{Z}^d))$. We may assume $\delta > 0$ is small enough. By (7.1.3) and a density argument, we have

$$\sup_{z\in\overline{\mathbb{C}_{\pm}}} \|\langle x\rangle^{-1+\delta}R_{0,\pm}(z)\langle x\rangle^{-1-\delta}\|_{B(l^2(\mathbb{Z}^d))} < \infty. \tag{7.4.2}$$

for $\delta > 0$ small enough. From (7.4.2), we see that there exists $M > 0$ such that

$$\sup_{z\in\overline{\mathbb{C}_{\pm}}} \|\langle x\rangle^{-s}R_{0,\pm}(z)\langle x\rangle^{-1-\delta}\|_{B(l^2(\mathbb{Z}^d), l^2(|x|\geq M))} < \frac{\varepsilon}{3}. \tag{7.4.3}$$

On the other hand, (7.4.1) implies that a map

$$z \in \overline{\mathbb{C}_{\pm}} \mapsto \chi_{\{|x|<M\}}\langle x\rangle^{-s}R_{0,\pm}(z)\langle x\rangle^{-1-\delta} \in B(l^2(\mathbb{Z}^d))$$

is continuous, where χ_A is the characteristic function of $A \subset \mathbb{R}^d$. Thus there exists $\delta_1 > 0$ such that $|z - z'| < \delta_1$ with $z, z' \in \overline{\mathbb{C}_\pm}$ implies

$$\|\chi_{\{|x|<M\}}\langle x\rangle^{-s} R_{0,\pm}(z')\langle x\rangle^{-1-\delta} - \chi_{\{|x|<M\}}\langle x\rangle^{-s} R_{0,\pm}(z')\langle x\rangle^{-1-\delta}\|_{B(l^2(\mathbb{Z}^d))} < \frac{\varepsilon}{3}.$$

This inequality with (7.4.3) gives

$$\|\langle x\rangle^{-s} R_{0,\pm}(z')\langle x\rangle^{-1-\delta} - \langle x\rangle^{-s} R_{0,\pm}(z')\langle x\rangle^{-1-\delta}\|_{B(l^2(\mathbb{Z}^d))} < \varepsilon$$

for $|z - z'| < \delta$. This completes the proof. □

Lemma 7.4.2. *Let $z \in \mathbb{C} \setminus \mathbb{R}$ and let $s \in \mathbb{R}$. Then H_0, H, $(H_0 - z)^{-1}$ and $(H - z)^{-1}$ preserve $l^{2,s}(\mathbb{Z}^d)$. In particular, $H_0 - z$ and $H - z$ are invertible on $l^{2,s}(\mathbb{Z}^d)$.*

Proof. By using relations $[V, \langle x\rangle^s] = 0$ and

$$[(P - z)^{-1}, \langle x\rangle^s] = (P - z)^{-1}[\langle x\rangle^s, P](P - z)^{-1}, \ P \in \{H_0, H\},$$

it suffices to prove $[H_0, \langle x\rangle^s]\langle x\rangle^{-s} \in B(l^2(\mathbb{Z}^d))$. This is easily proved since its Fourier conjugate $[h_0, \langle D_\xi\rangle^s]\langle D_\xi\rangle^{-s}$ of $[H_0, \langle x\rangle^s]\langle x\rangle^{-s}$ is a pseudodifferential operator of order -1 on $\mathbb{T}^d$. This completes the proof.

□

Lemma 7.4.3. *Let $z \in \overline{\mathbb{C}_\pm}$. Suppose that $u \in l^{2,-1-\delta}(\mathbb{Z}^d)$ satisfies $(I + R_{0,\pm}(z)V)u = 0$. Then we have $u \in l^{2,-1}(\mathbb{Z}^d)$.*

Proof. This lemma immediately follows from $|V| \le C\langle x\rangle^{-2-\delta}$ and (7.1.3). □

Proposition 7.4.4. *Let $U \subset \mathbb{C}_\pm$ be a bounded open set satisfying*

$$\{u \in l^{2,-1}(\mathbb{Z}^d) \mid (I + R_{0,\pm}(z)V)u = 0\} = \{0\}, \text{ for any } z \in \overline{U}. \tag{7.4.4}$$

(i) Let $1 \le s < 1 + \delta$. Then an inverse $(I + R_{0,\pm}(z)V)^{-1} \in B(l^{2,-s}(\mathbb{Z}^d))$ exists for $z \in \overline{U}$ and

$$\sup_{z\in\overline{U}} \|(I + R_{0,\pm}(z)V)^{-1}\|_{B(l^{2,-s}(\mathbb{Z}^d))} < \infty.$$

(ii) For $z \in \overline{U}$, we set

$$R_\pm(z) = (I + R_{0,\pm}(z)V)^{-1}R_{0,\pm}(z) \in B(l^{2,1}(\mathbb{Z}^d), l^{2,-1}(\mathbb{Z}^d)).$$

Then we have $R_\pm(z) = (H - z)^{-1}$ for $z \in \overline{U} \setminus \mathbb{R}$ and

$$\sup_{z\in\overline{U}} \|R_\pm(z)\|_{B(l^{2,1}(\mathbb{Z}^d), l^{2,-1}(\mathbb{Z}^d))} < \infty.$$

(iii) Let $1 < s \le 1 + \delta/2$. Then a map $z \in \overline{U} \mapsto R_\pm(z) \in B(l^{2,s}(\mathbb{Z}^d), l^{2,-s}(\mathbb{Z}^d))$ is Hölder continuous.

Proof. Lemma 7.4.1 implies that $\{I+R_{0,\pm}(z)V\}_{z\in\overline{U}}$ is a continuous family of Fredholm operators with index 0 on $B(l^{2,-s}(\mathbb{Z}^d))$. Thus the assumption (7.4.4) implies that $I+R_{0,\pm}(z)V$ is invertible for $z\in\overline{U}$ and that a map $z\mapsto (I+R_{0,\pm}(z)V)^{-1}\in B(l^{2,-s}(\mathbb{Z}^d))$ is continuous. This with the compactness of $\overline{U}$ gives the proof of (i).

The part (ii) follows from the part (i), (7.1.3) and the resolvent equation:

$$(I+(H_0-z)^{-1}V)(H-z)^{-1}=(H_0-z)^{-1},\ z\in\mathbb{C}\setminus\mathbb{R}.$$

To prove part (iii), we observe that $z\in\overline{U}\mapsto (I+R_{0,\pm}(z)V)^{-1}\in B(l^{2,-s}(\mathbb{Z}^d))$ is Hölder continuous. In fact, for $z,z'\in\overline{U}$, we have

$$\begin{aligned}&(I+R_{0,\pm}(z)V)^{-1}-(I+R_{0,\pm}(z')V)^{-1}\\&=(I+R_{0,\pm}(z)V)^{-1}(R_{0,\pm}(z')-R_{0,\pm}(z))V(I+R_{0,\pm}(z')V)^{-1}.\end{aligned}$$

Part (i), (7.4.1), and $V\in B(l^{2,-s}(\mathbb{Z}^d),l^{2,s}(\mathbb{Z}^d))$ imply the Hölder continuity of $(I+R_{0,\pm}(z)V)^{-1}$. This, (7.4.1) and the following representation:

$$\begin{aligned}R_\pm(z)-R_\pm(z')=&(I+R_{0,\pm}(z)V)^{-1}(R_{0,\pm}(z)-R_{0,\pm}(z'))\\&+((I+R_{0,\pm}(z)V)^{-1}-(I+R_{0,\pm}(z')V)^{-1})R_{0,\pm}(z'),\end{aligned}$$

finish the proof of part (iii).

□

Proof of Theorem 7.1.9. From now on, we assume that V is a finitely supported potential. We take $R>0$ such that $\sigma(H)\subset\{|z|<R\}$. Then (7.4.4) holds for

$$U=\{z\in\mathbb{C}\mid \pm\mathrm{Im}\,z\geq 0,\ |z|<2R,\ |z|>\varepsilon_1,\ |z-4d|>\varepsilon_1\}.$$

Moreover, we note $\sigma(H)\cap\Omega_{\varepsilon_1,\pm}\setminus U=\emptyset$. Now Theorem 7.1.9 follows from Corollary 7.1.8 and Proposition 7.4.4.

□

7.5 Lorentz space

For a measure space (X,μ), $L^{p,r}(X,\mu)$ denotes the Lorentz space for $1\leq p\leq\infty$ and $1\leq r\leq\infty$:

$$\begin{aligned}\|f\|_{L^{p,r}(X)}&=\begin{cases}p^{\frac{1}{r}}(\int_0^\infty \mu(\{x\in X\,|\,|f(x)|>\alpha\})^{\frac{r}{p}}\alpha^{r-1}d\alpha)^{\frac{1}{r}}, & r<\infty,\\ \sup_{\alpha>0}\alpha\mu(\{x\in X\mid |f(x)|>\alpha\})^{\frac{1}{p}}, & r=\infty,\end{cases}\\ L^{p,r}(X,\mu)&=\{f:X\to\mathbb{C}\mid f:\text{measurable},\ \|f\|_{L^{p,r}(X)}<\infty\}.\end{aligned}$$

Moreover, we denote $L^{p,r}(\mathbb{R}^d)=L^{p,r}(\mathbb{R}^d,\mu_L)$ and $l^p_r(\mathbb{Z}^d)=L^{p,r}(\mathbb{Z}^d,\mu_c)$, where μ_L is the Lebesgue measure on $\mathbb{R}^d$ and μ_c is the counting measure on $\mathbb{Z}^d$. For a detail, see [25]. In this section, we state some fundamental properties of the Lorentz spaces. Note that $L^{p,p}(X,\mu)=L^p(X)$.

Lemma 7.5.1 (The Young inequalities in the Lorentz spaces)**.** *Let* $1 < p_i < \infty$*,* $1 \le q_i \le \infty$ *with* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1 > 0$ *and* $s \ge 1$ *with* $\frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{s}$*. Then we have*

$$\|f * g\|_{l^r_s(\mathbb{Z}^d)} \le C\|f\|_{l^{p_1}_{q_1}(\mathbb{Z}^d)}\|g\|_{l^{p_2}_{q_2}(\mathbb{Z}^d)}.$$

Lemma 7.5.2 (The Hölder inequalities in the Lorentz spaces)**.** *If* $1 \le p_1, p_2, q_1, q_2 \le \infty$ *and* $1 \le r \le \infty$ *satisfy*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} < 1,$$

then

$$\|fg\|_{l^r_{\min(q_1,q_2)}(\mathbb{Z}^d)} \le \|f\|_{l^{p_1}_{q_1}(\mathbb{Z}^d)}\|g\|_{l^{p_2}_{q_2}(\mathbb{Z}^d)}.$$

For these proofs, see [60].

7.6 Harmonic analysis

Proposition 7.6.1. *Let* $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ *be a function on* $\mathbb{R}^d$ *which is compactly supported,* C^∞ *for* $\xi \ne 0$ *and satisfies for a* $0 \le k < d$ *that*

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-k-|\alpha|}, \quad x \in \mathbb{R}^d \tag{7.6.1}$$

for $|\alpha| \le d - k + 1$*. Then if we set*

$$I = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} m(\xi) d\xi,$$

then $|I| \le C\langle x\rangle^{-d+k}$*.*

Proof. Since m is compactly supported, we may assume $|x| \ge 1$. Take $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi = 1$ on $|\xi| \le 1$ and $\chi = 0$ on $|\xi| \ge 2$. Set $\bar\chi = 1 - \chi$. For $\delta > 0$, we have

$$I = \int_{\mathbb{R}^d} (\chi(\xi/\delta) + \bar\chi(\xi/\delta)) e^{-2\pi i x\cdot\xi} m(\xi) d\xi =: I_1 + I_2.$$

Since m is integrable on $\mathbb{R}^d$, we have

$$|I_1| \le \int_{|\xi|\le 2\delta} |\chi(\xi/\delta)||\xi|^{-k} d\xi \le C\delta^{d-k}.$$

By integrating by parts, for $N > d - k$ we have

$$\begin{aligned}
|I_2| \le C|x|^{-N} \sum_{|\alpha|=N} |&\int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} D_\xi^\alpha(\bar\chi(\xi/\delta) m(\xi)) d\xi| \\
\le& C|x|^{-N} \sum_{|\alpha|=N} |\sum_{\beta\le\alpha} \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} D_\xi^\beta(\bar\chi(\xi/\delta))\partial_\xi^{\alpha-\beta} m(\xi) d\xi| \\
\le& C|x|^{-N} \sum_{|\alpha|\le N}\sum_{\beta\le\alpha} \int_{\mathbb{R}^d} \delta^{-|\beta|}\bar\chi^{(\beta)}(\xi/\delta)|\xi|^{-k-(N-|\beta|)} d\xi.
\end{aligned}$$

For $\beta = 0$,

$$\int_{\mathbb{R}^d} \bar{\chi}(\xi/\delta)|\xi|^{-k-N} d\xi \le C\delta^{d-k-N}$$

follows and for $\beta \neq 0$,

$$\begin{aligned}\int_{\mathbb{R}^d} \delta^{-|\beta|}\bar{\chi}^{(\beta)}(\xi/\delta)|\xi|^{-k-(N-|\beta|)} d\xi \le& C\int_{\delta\le|\xi|\le 2\delta} \delta^{-|\beta|}|\xi|^{-k-N+|\beta|} d\xi \\ \le& C\delta^{d-k-N}.\end{aligned}$$

These imply $|I_2| \le C|x|^{-N}\delta^{d-k-N}$. We set $\delta = |x|^{-1}$ and obtain $|I| \le C|x|^{-d+k}$ for $|x| \ge 1$. □

Corollary 7.6.2. *Let $d \ge 1$, $0 < l < d$ and K_l be defined by*

$$K_l(x) = \int_{\mathbb{T}^d} e^{2\pi i x\dot{\xi}} h_0(\xi)^{-l/2} d\xi.$$

Then we have a pointwise bound $|K_l(x)| \le C\langle x\rangle^{-d+l}$.

Proof. By the Morse lemma, we have $|\partial_\xi^\alpha h_0(\xi)^{-l/2}| \le C_\alpha|\xi|^{-l-|\alpha|}$ near $\xi = 0$ for any multi-index α. Moreover, it follows that $h_0(\xi)^{-l/2}$ is smooth away from $\xi = 0$. Applying Proposition 7.6.1, we obtain $|K_l(x)| \le C\langle x\rangle^{-d+l}$. □

Now we define operators $H_0^{-l/2}$ for $0 < l < d$ by

$$H_0^{-l/2}u(x) = \sum_{y\in\mathbb{Z}^d} K_l(x-y)u(y),\ u \in \bigcap_{s>0} l^{2,s}(\mathbb{Z}^d).$$

It is easily seen that $H_0^{-l/2}$ is a continuous linear operator:

$$H_0^{-l/2} : \bigcap_{s>0} l^{2,s}(\mathbb{Z}^d) \to \bigcup_{s\in\mathbb{R}} l^{2,s}(\mathbb{Z}^d).$$

The next corollary implies that H_0^{-1} can be uniquely extended to the continuous linear operator from $l^{2,\alpha}(\mathbb{Z}^d)$ to $l^{2,-\beta}(\mathbb{Z}^d)$ for $\alpha, \beta > 1/2$ with $\alpha + \beta \ge 2$.

Corollary 7.6.3 (Discrete version of the HLS inequality)**.** *Let $d \ge 1$ and $0 < k < d$. Then $H_0^{-l/2}$ is bounded from $l_r^p(\mathbb{Z}^d)$ to $l_r^q(\mathbb{Z}^d)$ if $1 < p < q < \infty$ satisfies*

$$\frac{1}{p} - \frac{1}{q} = \frac{l}{d} \tag{7.6.2}$$

and $1 \le r \le \infty$.

Moreover, if $W_1 \in l_\infty^{r_1}(\mathbb{Z}^d)$ and $W_2 \in l_\infty^{r_2}(\mathbb{Z}^d)$ with $1/r_1 + 1/r_2 = l/d$ with $r_1, r_2 > 2$. Then we have

$$W_1 H_0^{-l/2} W_2 \in B(l^2(\mathbb{Z}^d))$$

In particular, $\langle x\rangle^{-\alpha} H_0^{-1}\langle x\rangle^{-\beta} \in B(l^2(\mathbb{Z}^d))$ if $\alpha + \beta \ge 2$ and $\alpha, \beta > 0$ if $d \ge 4$ and $\alpha + \beta \ge 2$ and $\alpha, \beta > 1/2$ if $d = 3$.

Remark 7.6.4. This corollary gives $H_0^{-l/2}\langle x\rangle^{-l} \in B(l^2(\mathbb{Z}^d))$ for $0 < l < d$. In fact,

$$\|H_0^{-l/2}\langle x\rangle^{-l} f\|_{l^2(\mathbb{Z}^d)} \le C\|H_0^{-l/2}\|_{B(l_\infty^{\frac{2l}{l+2d}}(\mathbb{Z}^d), l^2(\mathbb{Z}^d))}\|\langle x\rangle^{-l}\|_{l_\infty^{\frac{d}{l}}(\mathbb{Z}^d)}\|f\|_{l^2(\mathbb{Z}^d)}.$$

These are exactly the discrete Hardy inequalities.

7.7 Restriction theorem for a Lipschitz manifold

In this section, we prove the L^2-restriction theorem for a Lipschitz manifold. Its proof is standard, however, we give its proof for readers' convenience.

Lemma 7.7.1. *Let $f \in H^1(\mathbb{R}^d)$ and g be a real-valued Lipschitz function on $\mathbb{R}^{d-1}$. Then it follows that $k(\xi) = f(\xi', \xi_d + g(\xi'))$ belongs to $H^1(\mathbb{R}^d)$ and there exists $C > 0$ which depends only on the dimension d and $\|\partial_\xi g\|_{L^\infty(\mathbb{R}^d)}$ such that*

$$\|k\|_{H^1(\mathbb{R}^d)} \le C\|f\|_{H^1(\mathbb{R}^d)}. \tag{7.7.1}$$

Proof. It is evident that $\|k\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$. For $j = 1, ...d-1$, we have

$$\begin{aligned}\partial_{\xi_j}(k(\xi', \xi_d + g(\xi'))) =& (\partial_{\xi_j} k)(\xi', \xi_d + g(\xi')) + (\partial_{\xi_j} g)(\xi')(\partial_{\xi_d} k)(\xi', \xi_d + g(\xi')),\\ \partial_{\xi_d}(k(\xi', \xi_d + g(\xi'))) =& (\partial_{\xi_d} k)(\xi' + \xi_d + g(\xi')).\end{aligned}$$

Using this computation, we obtain (7.7.1). □

Proposition 7.7.2. *Under the assumption of Lemma 7.7.1, we have*

$$\|\langle D_{\xi'}\rangle^{1/2}(f(\xi', g(\xi')))\|_{L^2(\mathbb{R}^{d-1})} \le C\|f\|_{H^1(\mathbb{R}^d)}.$$

Proof. In the following, we denote the Fourier transform of f by $\hat{f}$. By using Fourier inversion formula and by using Schwarz's inequality, we have

$$\begin{aligned}|\int_{\mathbb{R}^{d-1}} f(\xi', g(\xi'))e^{-2\pi i x'\cdot\xi'}d\xi'| =& |\int_{\mathbb{R}^d} \hat{k}(x)dx_d|\\ \le& (\int_{\mathbb{R}} \langle x\rangle^{-2}dx_d)^{1/2}(\int_{\mathbb{R}} |\langle x\rangle\hat{k}(x)|^2dx_d)^{1/2}\\ \le& C\langle x'\rangle^{-1/2}(\int_{\mathbb{R}} |\langle x\rangle\hat{k}(x)|^2dx_d)^{1/2}.\end{aligned}$$

Thus we have

$$\begin{aligned}\|\langle D_{\xi'}\rangle^{1/2}(f(\xi', g(\xi')))\|^2_{L^2(\mathbb{R}^{d-1})} =& \|\langle x'\rangle^{1/2}\widehat{f(\xi', g(\xi'))}(x)\|^2_{L^2(\mathbb{R}^{d-1})}\\ \le& C^2\|\langle x\rangle\hat{k}\|^2_{L^2(\mathbb{R}^d)}\\ =& C^2\|k\|^2_{H^1(\mathbb{R}^d)}.\end{aligned}$$

This computation with Lemma 7.7.1 completes the proof. □

Bibliography

[1] V. Bach, W. de Siqueira Pedra, S. N. Lakaev, Bounds on the Pure Point Spectrum of Lattice Schrödinger Operators. preprint. https://arxiv.org/pdf/1709.09200.pdf.

[2] J.G. Bak, A. Seeger, Extensions of the Stein-Tomas theorem. Math. Res. Lett. 18 (2011), no. 4, 767–781.

[3] D. Baskin, A. Vasy, J. Wunsch, Asymptotics of scalar waves on long-range asymptotically Minkowski spaces. Adv. Math. 328 (2018), 160–216.

[4] J. F. Bony, R. Carles, D. Häfner, L. Michel, Scattering theory for the Schrödinger equation with repulsive potential. J. Math. Pures Appl. (9) 84 (2005), no. 5, 509–579.

[5] J.-M. Bouclet, H. Mizutani, Uniform resolvent and Strichartz estimates for Schrödinger equations with critical singularities. Trans. Amer. Math. Soc. 370 (2018), no. 10, 7293–7333.

[6] J.-M. Bouclet, J. Royer, Sharp low frequency resolvent estimates on asymptotically conical manifolds. Comm. Math. Phys. 335 (2015), no. 2, 809–850.

[7] J. Bourgain, Estimations de certaines fonctions maximales. (French) [Estimates of some maximal operators] C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), no. 10, 499–502.

[8] A. Carbery, A. Seeger, S. Wainger, J. Wright, Classes of singular integral operators along variable lines. J. Geom. Anal. 9 (1999), no. 4, 583–605.

[9] H. Chihara, Smoothing effects of dispersive pseudodifferential equations. Comm. Partial Differential Equations, Vol. 27, No. 9-10, pp. 1953-2005, 2002.

[10] J.-C. Cuenin, Embedded eigenvalues of generalized Schrödinger operators, to appear in J. Spectral Theory.

[11] J.C. Cuenin, Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials. J. Funct. Anal. 272 (2017), no. 7, 2987–3018.

[12] J.C. Cuenin, Eigenvalue Estimates for Bilayer Graphene. Ann. Henri Poincaré 13 (2019), 1-16.

[13] J. Duistermaat, L. Hörmander, Fourier integral operators. II. Acta Math. 128 (1972), no. 3-4, 183–269.

[14] S. Dyatlov, M. Zworski, *Mathematical theory of scattering resonances*, to appear in the AMS Graduate Studies in Mathematics series, (http://math.mit.edu/ dyatlov/res/).

[15] F. Faure, N. Roy, J. Sjöstrand, Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances. Open Math. J. 1 (2008), 35–81.

[16] F. Faure, J. Sjöstrand, Upper bound on the density of Ruelle resonances for Anosov flows. Comm. Math. Phys. 308 (2011), no. 2, 325–364.

[17] R. L. Frank, L. Schimmer, Endpoint resolvent estimates for compact Riemannian manifolds. J. Funct. Anal. 272 (2017), no. 9, 3904–3918.

[18] R. L. Frank, B. Simon, Eigenvalue bounds for Schrödinger operators with complex potentials. II. J. Spectr. Theory 7 (2017), no. 3, 633–658.

[19] Y. Gâtel, D. Yafaev, On solutions of the Schrödinger equation with radiation conditions at infinity: the long-range case. Ann. Inst. Fourier (Grenoble) 49 (1999), no. 5, 1581–1602.

[20] J. Gell-Redman, N. Haber, A. Vasy: The Feynman propagator on perturbations of Minkowski space. Comm. Math. Phys., Vol. 342, No. 1, 333–384 (2016).

[21] C. Gérard and M. Wrochna: The massive Feynman propagator on asymptotically Minkowski spacetimes. to appear in American J. Math.

[22] J. Ghidaglia, J. C. Saut: Nonelliptic Schrödinger equations. J. Nonlinear Sci., Vol. 3, No. 2, pp. 169–195 (1993).

[23] N. Godet, N. Tzvetkov: Strichartz estimates for the periodic non-elliptic Schrödinger equation. C. R. Math. Acad. Sci. Paris Vol. 350, no. 21–22, 955–958 (2012).

[24] M. Goldberg, W. Schlag, A limiting absorption principle for the three-dimensional Schrödinger equation with L^p potentials. Int. Math. Res. Not. (2004), no. 75, 4049–4071.

[25] L. Grafakos, Classical Fourier analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, (2008).

[26] Gutiérrez, Susana Non trivial L^q solutions to the Ginzburg-Landau equation. Math. Ann. 328 (2004), no. 1-2, 1–25.

[27] Y. Higuchi, T. Matsumoto, O. Ogurisu, On the spectrum of a discrete Laplacian on $\mathbb{Z}$ with finitely supported potential. Linear and Multilinear Algebra. 59 (2011), no. 8, 917–927.

[28] Y. Higuchi, Y. Nomura and O. Ogurisu, Persistent set of the threshold embedded eigenvalue for discrete Schrödinger operators with finite supported potentials, in preparation.

[29] L. Hörmander, Oscillatory integrals and multipliers on FLp. Ark. Mat. 11, 1–11. (1973).

[30] L. Hörmander, *Analysis of Linear Partial Differential Operators*, Vol. I-IV. Springer Verlag, 1983–1985.

[31] S. Huang, X. Yao, Q. Zheng, Remarks on L^p-limiting absorption principle of Schrödinger operators and applications to spectral multiplier theorems. Forum Math. 30 (2018), no. 1, 43–55.

[32] A. D. Ionescu, D. Jerison, On the absence of positive eigenvalues of Schrödinger operators with rough potentials. Geom. Funct. Anal. 13 (2003), no. 5, 1029–1081.

[33] H. Isozaki, Eikonal equations and spectral representations for long range Schrödinger Hmiltonians, J. Math. Kyoto Univ. 20 (1980), 243–261.

[34] H. Isozaki, E. Korotyaev, Inverse problems, trace formulae for discrete Schrödinger operators. Ann. Henri Poincaré 13 (2012), no. 4, 751-788.

[35] H. Isozaki, H. Morioka, Inverse scattering at a fixed energy for discrete Schrödinger operators on the square lattice. Ann. Inst. Fourier (Grenoble) 65 (2015), no. 3, 1153–1200.

[36] H. Isozaki, H. Morioka A Rellich type theorem for discrete Schrödinger operators, Inverse Problems and Imaging 8 (2014), 475–489.

[37] A. D. Ionescu, W. Schlag, Agmon-Kato-Kuroda theorems for a large class of perturbations. Duke Math. J. 131 (2006), no. 3, 397–440.

[38] K. Itakura, Rellich's theorem for spherically symmetric repulsive Hamiltonians, Math.Z. 291 (2019), no. 3, 1435–1449.

[39] K. Itakura. K, Limiting absorption principle and radiation condition for repulsive Hamiltonians, to appear in Funkcial. Ekvac.

[40] K. Ito, A. Jensen, Branching form of the resolvent at threshold for ultra-hyperbolic operators and discrete Laplacians, J. Funct. Anal. 277 (2019), no. 4, 965–993.

[41] K. Ito, S. Nakamura: Singularities of solutions to Schrödinger equation on scattering manifold. American J. Math. Vol. 131, No. 6, 1835–1865 (2009).

[42] A. Jensen, T. Kato, Spectral properties of Schrödinger operators and time-decay of wave functions. Duke Math J 46, 583–611 (1979).

[43] E. Jeong, Y. Kwon, S. Lee, Uniform Sobolev inequalities for second order non-elliptic differential operators. Adv. Math. 302 (2016), 323–350.

[44] K. Kameoka, Semiclassical study of shape resonances in the Stark effect. Preprint. https://arxiv.org/abs/1901.08315. (2019).

[45] T. Kato, Fundamental properties of Hamiltonian operators of Schrödinger type, Trans. Amer. Math. Soc. 70 (1951), 195–211.

[46] T. Kato, S. Kuroda, The abstract theory of scattering. Rocky Mountain J. Math. 1 (1971), no. 1, 127–171.

[47] T. Kato, K. Yajima, Some examples of smooth operators and the associated smoothing effect. Rev. Math. Phys. 1 (1989), no. 4, 481–496.

[48] M. Keel, T. Tao, Endpoint Strichartz estimates. Amer. J. Math. 120 (1998), no. 5, 955–980.

[49] C. Kenig, G. Ponce, C. Rolvung, L. Vega, Variable coefficient Schrödinger flows for ultrahyperbolic operators. Adv. Math. (2005) Vol. 196, No. 2, pp. 373–486.

[50] C. E. Kenig, A. Ruiz, C. D. Sogge, Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. Duke Math. J. 55 (1987), no. 2, 329–347.

[51] H. Koch, D. Tataru, Carleman estimates and absence of embedded eigenvalues. Comm. Math. Phys. 267 (2006), no. 2, 419–449.

[52] E. Korotyaev, J. Møller Weighted estimates for the discrete Laplacian on the cubic lattice. Preprint. https://arxiv.org/abs/1701.03605. (2017).

[53] R. B. Melrose, Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces. Marcel Dekker, (1994), pp. 85–130.

[54] H. Mizutani, Uniform Sobolev estimates for Schrödinger operators with scaling-critical potentials and applications, to appear in Analysis and PDE.

[55] H. Mizutani, N. Tzvetkov: Strichartz estimates for non-elliptic Schrödinger equations on compact manifolds. Comm. Partial Differential Equations, Vo. 40, no. 6, 1182–1195 (2015).

[56] E. Mourre, Absence of singular continuous spectrum for certain selfadjoint operators. Comm. Math. Phys. 78 (1980/81), no. 3, 391–408.

[57] S. Nakamura, Propagation of the homogeneous wave front set for Schrödinger equations. Duke Math. J. vol. 126 (2005), 349–367.

[58] S. Nakamura, K. Taira, Essential self-adjointness of real principal type operators, Preprint, (2019), (https://arxiv.org/abs/1912.05711).

[59] Y. Nomura, K. Taira, Some properties of threshold eigenstates and resonant states of discrete Schrödinger operators. Preprint, (2019), (https://arxiv.org/abs/1909.10014).

[60] R. O'Neil, Convolution operators and $L(p,q)$ spaces. Duke Math. J. 30 1963 129–142.

[61] Reed, M., Simon, B, *The Methods of Modern Mathematical Physics*, Vol. I–IV. Academic Press, 1972–1980.

[62] I. Rodnianski, W. Schlag, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. Invent. Math. 155 (2004), no. 3, 451–513.

[63] W. Rudin, *Real and complex analysis. Third edition*, McGraw-Hill Book Co., New York, (1987).

[64] A. Ruiz, Harmonic analysis and inverse problem. Lecture notes, (2002). https://www.uam.es/gruposinv/inversos/publicaciones/Inverseproblems.pdf.

[65] D. Salort: The Schrödinger equation type with a nonelliptic operator. Comm. Partial Differential Equations. Vol. 32 , no. 1–3, 209–228 (2007).

[66] C. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, 105. Cambridge University Press, Cambridge, (1993).

[67] A. Stefanov, P. G. Kevrekidis, Asymptotic behaviour of small solutions for the discrete nonlinear Schrödinger and Klein-Gordon equations. Nonlinearity 18 (2005), no. 4, 1841–1857.

[68] Y. Tadano, K. Taira, Uniform bounds of discrete Birman-Schwinger operators. Trans. Amer. Math. Soc. 372 (2019), 5243-5262.

[69] K. Taira: Strichartz estimates for non-degenerate Schrödinger equations. Preprint, (2017), (https://arxiv.org/abs/1708.01989).

[70] K. Taira, Limiting absorption principle on L^p-spaces and scattering theory. Preprint, (2019), (https://arxiv.org/abs/1904.00505).

[71] K. Taira, Scattering theory for repulsive Schrödinger operators and applications to limit circle problem, Preprint, (2019), (https://arxiv.org/abs/1904.04212).

[72] M. E. Taylor, *Partial differential equations. II.* Qualitative studies of linear equations. Applied Mathematical Sciences, 116. Springer-Verlag, New York, (1996).

[73] P.A. Tomas, A restriction theorem for the Fourier transform. Bull. Amer. Math. Soc. 81 (1975), 477–478.

[74] A. Vasy, Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov). Invent. Math. 194 (2013), no. 2, 381–513.

[75] A. Vasy, Essential self-adjointness of the wave operator and the limiting absorption principle on Lorentzian scattering spaces. Preprint, (2017), (https://arxiv.org/abs/1712.09650).

[76] A. Vasy, M. Wrochna: Quantum fields from global propagators on asymptotically Minkowski and extended de Sitter spacetime. Ann. Henri Poincaré 19 (2018), no. 5, 1529–1586.

[77] Y. Wang: Periodic cubic hyperbolic Schrödinger equation on $\mathbb{T}$. J. Funct. Anal. Vol. 265, no. 3, 424–434 (2013).

[78] K. Yajima, The Lp boundedness of wave operators for Schrödinger operators with threshold singularities I, Odd dimensional case, J. Math. Sci. Univ. Tokyo 7 (2006), 43–93.

[79] M. Zworski, *Semiclassical analysis. Graduate Studies in Mathematics*, 138. American Mathematical Society, Providence, RI, (2012).

[80] M. Zworski, Resonances for asymptotically hyperbolic manifolds: Vasy's method revisited. J. Spectr. Theory 6 (2016), no. 4, 1087–1114.