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修士論文

Hierarchical structure in the Vlasov-Ampere system characterized by Casimir invariants

― Casimir 不変量によって特徴づけられる Vlasov-Ampere 系の階層構造 ―

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Abstract. We elucidate the intermediate of the macroscopic fluid model and the microscopic kinetic model by studying the Poisson algebraic structure of the one-dimensional Vlasov–Poisson system. The water-bag model helps to formulate the hierarchy of sub-algebras that interpolate the gap between the fluid and kinetic models. By analyzing how the sub-manifold of an intermediate hierarchy is embedded in a more microscopic hierarchy, we characterize the microscopic effect as the symmetry breaking pertinent to a macroscopic invariant. Additionally, we will construct a numerical scheme of the two-dimensional Waterbag model for the application of the research results in the one-dimensional case.

Contents

1. Introduction

The purpose herein is to elucidate the hierarchical structure interpolating the macroscopic fluid model and the microscopic Vlasov model of collisionless plasmas, and we describe kinetic effects as a symmetry breaking that reduces a larger-scale hierarchy to a smaller-scale one. The central idea is to formulate a series of self-consistent subsystems (sub-algebras) of the Vlasov Lie–Poisson algebra, and to characterize each subsystem by Casimir invariants; conservation of a Casimir invariant is then the reflection of a particular symmetry in the distribution function, and hence the kinetic effect breaking such a symmetry manifests as non-conservation of the corresponding Casimir invariant.

Conventionally, the relation between the fluid and kinetic models is discussed by invoking the velocity-space moments of the distribution function f (with non-negative exponents s_1, s_2, and s_3)

$$P_{s_1 s_2 s_3} = \int v_1^{s_1} v_2^{s_2} v_3^{s_3} f(\boldsymbol{x}, \boldsymbol{v}, t) \mathrm{d}^3 v,$$

and imposing a *closure relation*. The lower-order moments constitute the fluid-mechanical variables (density, fluid velocity, pressure tensor), while the higher-order moments measure some deviation of f from Gaussian, so the moment hierarchy is useful for quantifying the probability distribution caused by collisions. A modern approach considers the Hamiltonian closure.[1, 2, 3]

However, herein we consider the problem from the perspective of the Poisson algebra that governs the collisionless dynamics of plasmas. Instead of the moment hierarchy, we construct a hierarchy of sub-algebras of the Poisson algebra, with the lowest-dimension system corresponding to the fluid model. Concretely, we consider the reduction of the Vlasov Lie–Poisson system by restricting the distribution function f to be a sum of step functions in the velocity space (to be denoted by V). Thanks to Liouville's theorem, the height of each flat top of f remains constant (while the area of the plateau in the V-space may change). Hence, such a system of distribution functions defines a sub-algebra (closed subsystem) of the Vlasov system[4, 5] (see Refs. [6, 7, 8] for the general idea of reduction of a Hamiltonian system). Interestingly, a single-plateau system corresponds to the fluid model; in fact, the fluid model allows only one velocity at each point of the configuration space (to be denoted by X). Increasing the number of plateaus, we formulate a hierarchy of sub-algebras; the limit of an infinite number of plateaus recovers the full Vlasov system. Each subsystem is a leaf of the Vlasov system, which is characterized by Casimir invariants. Generally, a Casimir invariant is a generator of some gauge symmetry pertinent to reduction[9, 10, 11]; when used as a Hamiltonian in a higher-order system, the corresponding Hamiltonian flow keeps the reduced variables constant. In other words, the breaking of the gauge symmetry and the variance of the Casimir imply the higher-order effect that violates the closedness of the lower-order (macroscopic) subsystem.

As a simple and analytically tractable example, we study the water-bag model of a one-dimensional charged-beam system.[12, 13, 14, 15] A series of water-bag models with different numbers of water bags defines the hierarchical Vlasov sub-algebra. In addition to this, preparing to application of the one-dimensional study, a numerical scheme for 2D water-bag model will be developed to extend it to 2D water-bag.

2. Preliminary details

2.1. Hamiltonian mechanics and Poisson manifold

In Hamiltonian mechanics, the equation of motion is written as

$$\dot{\boldsymbol{z}} = J\partial_{\boldsymbol{z}} H(\boldsymbol{z}), \tag{1}$$

where $\boldsymbol{z} \in M$ is the state vector (its totality M is the phase space), J is the Poisson operator, H is the Hamiltonian, and $\partial_{\boldsymbol{z}} H$ is the gradient of H. We define the Poisson bracket by

$$[G, H] = \langle \partial_{\boldsymbol{z}} G, J\partial_{\boldsymbol{z}} H \rangle, \tag{2}$$

and the Poisson operator J must be defined appropriately for this bracket to define a Poisson algebra (Lie algebra with Leibniz property). Endowing the function space $C^\infty(M)$ (a member $G(\boldsymbol{z}) \in C^\infty(M)$ represents a physical quantity and is called an observable) with the Poisson bracket, we call M the Poisson manifold.

In the *canonical* Hamiltonian system, $\boldsymbol{z}$ is given as a conjugate $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{p})^{\mathrm{T}}$, and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{3}$$

However, there are many possible *non-canonical* Hamiltonian systems (or degenerate Poisson algebras) that are defined by more-complicated J; in general, J may depend on $\boldsymbol{z}$ and may have nontrivial kernels (i.e., $\operatorname{rank} J < \dim M$). As we will show by an example (Sec. 2.2), such a *degeneracy* is often brought about by some *reduction* from a higher-dimensional canonical system; here, the reduction means that we admit observables with only a restricted dependence on $\boldsymbol{z}$, hence the effective degree of freedom (dimension of the actual phase space) is reduced. Physically, such reduction can be argued in the context of *macro-hierarchy*, which is the suppression of some microscopic degree of freedom.

The nullity of J implies that the vector $\dot{\boldsymbol{z}}$ has codimensions. If $\boldsymbol{z}_0 \in \operatorname{Ker} J$ can be *integrated* as

$$\boldsymbol{z}_0 = \partial_{\boldsymbol{z}} C, \tag{4}$$

then we call $C \in C^\infty(M)$ a *Casimir*. Then, the level sets of $C(\boldsymbol{z})$ foliate M so that $\boldsymbol{z}(t)$ is restricted to move on only a leaf $C(\boldsymbol{z}) = \text{constant}$. In fact,

$$\dot{C} = \{C, H\} = -\{H, C\} = -\langle \partial_{\boldsymbol{z}} H, J \partial_{\boldsymbol{z}} C \rangle = 0. \tag{5}$$

Notice that the invariance of a Casimir C is independent of any specific choice of the Hamiltonian H. By interpreting the degeneracy of J as the suppression of some microscopic degree of freedom, the leaf of C is the mathematical identification of a macro-hierarchy.

2.2. Reduction: an example

Because the Casimir plays the central role in this work, we explain how it is "created" by a *reduction* invoking a simple example. We start with the canonical Hamiltonian system of point mass moving in $\mathbb{R}^n$. The phase space is $M = \mathbb{R}^{2n}$, and the state vector is $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{p})^{\mathrm{T}}$ with position $\boldsymbol{q}$ and momentum $\boldsymbol{p}$. On $C^\infty(M)$, we define the canonical Poisson bracket

$$[G, H] = \sum_{j=1}^{n} (\partial_{q^j} G)\,(\partial_{p(j)} H) - (\partial_{q^j} H)\,(\partial_{p(j)} G). \tag{6}$$

Here, the canonical Poisson bracket is denoted by $[\ ,\ ]$, which will be used later in defining the Poisson bracket of the Vlasov system.

We set $n = 2$ and denote the corresponding Poisson manifold by M_4 $(= \mathbb{R}^4)$. As a trivial example of reduction, we assume that all observables are independent to q^2 and p_2, then the Poisson bracket evaluates as

$$[G, H] = (\partial_{q^1} G)\,(\partial_{p_1} H) - (\partial_{q^1} H)\,(\partial_{p_1} G), \tag{7}$$

which defines a canonical Poisson algebra on the submanifold $M_2 = \{\boldsymbol{z}_2 = (q^1, p_1)^{\mathrm{T}}\} = \mathbb{R}^2$, which is embedded in M_4 as a leaf $\{\boldsymbol{z} \in M_4;\ q^2 = c,\ p_2 = c'\}$ (c and c' are arbitrary constants).

An interesting reduction occurs if we suppress only q^2 in observables. The reduced phase space is the three-dimensional submanifold $M_3 = \{\boldsymbol{z}_3 = (q^1, p_1, p_2)^{\mathrm{T}}\}$. For G and H such that $\partial_{q^2} G = \partial_{q^2} H = 0$, the Poisson bracket evaluates the same as (7). The Poisson operator J may be written as

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

whose rank is two. Therefore, M_3 is a degenerate Poisson manifold. The kernel of this J includes the vector $(0, 0, 1)^{\mathrm{T}}$, which can be integrated to define a Casimir $C = p_2$. Therefore, the effective degree of freedom is reduced further to two; the state vector $\boldsymbol{z}$ can move only on the two-dimensional leaf M_2. Evidently, the "freezing" of $C = p_2$ is due to the absence of its conjugate variable q^2.

When we observe M_3 from M_4, the reduction (i.e., the suppression of the parameter q^2

in the observables) means the *symmetry* $\partial_{q^2} = 0$. As the usual manifestation of the integral of motion, p_2 in M_4 becomes invariant if the Hamiltonian has the symmetry $\partial_{q^2} H = 0$.

The conjugate variable q^2 corresponding to the Casimir $C = p_2$ can be regarded as the *gauge parameter*. The gauge group (denoted by Ad_C)—which does not change the submanifold M_3 embedded in M_4—is generated by the adjoint action

$$\mathrm{ad}_C = [\circ, C] = \partial_{q^2},$$

implying that the gauge symmetry is written as $\partial_{q^2} = 0$. This is evident because the state vector $\boldsymbol{z}_3 = (q^1, p_1, p_2)^{\mathrm{T}} \in M_3$ is "independent" of q^2. In the next subsection, we invoke another example to see a more nontrivial relation between the Casimir and gauge symmetry.

2.3. Gauge symmetry generated by Casimir

Here, we assume $n = 3$ and consider the six-dimensional phase space M_6. We define the angular momentum as

$$\omega = \boldsymbol{q} \times \boldsymbol{p}. \tag{8}$$

We consider a system where every observable is a function of ω; the Euler top is such an example (the Hamiltonian is $H(\omega) = \sum_j \omega_j^2/(2I_j)$, where I_1, I_2, I_3 are the three moments of inertia). For such a system, the effective phase space is reduced to

$$M_\omega = \{\omega = \boldsymbol{q} \times \boldsymbol{p};\ (\boldsymbol{q}, \boldsymbol{p})^{\mathrm{T}} \in M\} \cong \mathbb{R}^3.$$

Let us evaluate $[\ ,\ ]$ of (7) for the reduced class of observables $\in C^\infty(M_{\omega a})$. The gradient of a functional $G \in C^\infty(M_\omega)$ is given by (denoting by$(\boldsymbol{x}, \boldsymbol{y})$ the $\mathbb{R}^3$ Euclidean inner product)

$$\delta G = (\partial_{\boldsymbol{q}} G, \delta \boldsymbol{q}) + (\partial_{\boldsymbol{p}} G, \delta \boldsymbol{p}) = (\partial_\omega G, \delta\omega).$$

Inserting $\delta\omega = (\delta\boldsymbol{q}) \times \boldsymbol{p} + \boldsymbol{q} \times (\delta\boldsymbol{p})$, we find

$$\partial_{\boldsymbol{q}} G = \boldsymbol{p} \times \partial_\omega G, \quad \partial_{\boldsymbol{p}} G = -\boldsymbol{q} \times \partial_\omega G.$$

Therefore,

$$[G, H] = (\partial_\omega G, \partial_\omega H \times \omega) =: [G, H]_\omega.$$

We may rewrite

$$[G, H]_\omega = (\partial_\omega G, J(\omega)\partial_\omega H),$$

$$J(\omega) := -\omega \times \circ = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{9}$$

We find $\operatorname{rank} J(\omega) = 2$ (at $\omega = 0$, $\operatorname{rank} J(\omega) = 0$). Evidently,

$$C = |\omega|^2$$

is the Casimir ($J(\omega)\partial_\omega C = 0$). The reduction from $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{p})^{\mathrm{T}} \in \mathbb{R}^6$ to $\omega = \boldsymbol{q} \times \boldsymbol{p} \in \mathbb{R}^3$ yields another reduction of degree of freedom due to the Casimir C; the effective degree of freedom given to $\omega \in M_\omega$ is only two.

The Hamiltonian flow (the adjoint action) given by C generates the *gauge transformation* in M_6:

$$\begin{aligned} \mathrm{ad}_C = [\circ, C] &= \left(\sum_{j=1}^{3} \partial_{p_j} C \partial_{q^j} - \partial_{q^j} C \partial_{p_j} \right) \\ &= \omega \times \boldsymbol{q} \cdot \partial_{\boldsymbol{q}} + \omega \times \boldsymbol{p} \cdot \partial_{\boldsymbol{p}}. \end{aligned} \tag{10}$$

By direct calculation, we can show $[\omega_j, C] = 0$ ($j = 1, 2, 3$). This gauge transformation has the following geometrical meaning: by (10), the transformation $\boldsymbol{z} \mapsto \boldsymbol{z} + \epsilon\tilde{\boldsymbol{z}}$ ($\tilde{z}_j = [z_j, C]$) gives a co-rotation of $\boldsymbol{q}$ and $\boldsymbol{p}$ around the axis ω (note that this rotation is in the space M_6, not in the space M_ω), hence $\omega = \boldsymbol{q} \times \boldsymbol{p}$ does not change. The rotation angle can be written as

$$\theta = \frac{1}{2|\omega|} \tan^{-1}\left(\frac{(\omega \times \boldsymbol{q})_j}{q_j|\omega|}\right)$$

(we choose the coordinate $q_j \neq 0$). We find $[\theta, C] = 1$. Let us embed M_ω into a four-dimensional space $\widetilde{M_\omega} = \{(\omega, \theta);\ \omega \in M_\omega,\ \theta \in [0, 2\pi)\}$. For $G(\omega, \theta) \in C^\infty(\widetilde{M_\omega})$, we obtain

$$[G, C] = \sum_{j=1}^{3} \partial_{\omega_j} G[\omega_j, C] + \partial_\theta G[\theta, C] = \partial_\theta G.$$

Therefore, the gauge symmetry $[\circ, C]$ can be rewritten as $\partial_\theta = 0$. Reversing the perspective, for every Hamiltonian $H(\omega, \theta) \in C^\infty(\widetilde{M_\omega})$ that has the gauge symmetry $\partial_\theta H = 0$, C is invariant:

$$\dot{C} = [C, H] = -\partial_\theta G = 0.$$

3. Comparison of hierarchical structure using one-dimensional Water-bag model

3.1. Water-bag model of one-dimensional beam propagation

The aim of this section is elucidate the hierarchical structure encompassing the microscopic (kinetic) Vlasov model and the macroscopic fluid model. We use the *water-bag model* of one-dimensional charged-particle beam propagation (applicable for a nonlinear electron plasma[12] or the drift-kinetic model of plasma[13]). We start by reviewing the Hamiltonian formalism of the Vlasov system.

3.1.1. Vlasov system as an infinite-dimensional Poisson manifold Let $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{v}) = (x^1, \cdots, x^n, v_1, \cdots, v_n)$ denote the coordinates of $M = X \times V = \mathbb{T}^n \times \mathbb{R}^n$, the phase space of a particle. Assuming the non-relativistic limit, the particle mass is normalized to 1, so the velocity $\boldsymbol{v}$ parallels the momentum. We call a real-valued function $\psi(\boldsymbol{z}) \in C^\infty(M)$ an observable. We endow the space $C^\infty(M)$ with the canonical Poisson bracket

$$[\psi, \varphi] = \sum_{j=1}^{n} (\partial_{x^j}\psi)\,(\partial_{v_j}\varphi) - (\partial_{v_j}\psi)\,(\partial_{x^j}\varphi) \tag{11}$$

and denote $\mathfrak{g} = C^\infty(M)$; cf. (6). The dual space $\mathfrak{g}^*$ is the set of *distribution functions*; for an observable $\psi \in \mathfrak{g}$ and a distribution function $f \in \mathfrak{g}^*$,

$$\langle \psi, f \rangle = \int_M \psi(z) f(z)\, \mathrm{d}z \tag{12}$$

evaluates the mean value of ψ over the distribution function f.

On the space $\mathfrak{V} = C^\infty(\mathfrak{g}^*)$ of functionals, we define

$$\{G, H\} = \langle [\partial_f G, \partial_f H], f \rangle, \tag{13}$$

where $\partial_f H \in T^*\mathfrak{V} = \mathfrak{g}$ is the gradient of $H \in \mathfrak{V}$. The bracket $\{\ ,\ \}$ satisfies the conditions required for a Poisson bracket, hence $\mathfrak{g}^*$ is a Poisson manifold (infinite dimension). Integrating by parts, we may rewrite (13) as

$$\{G, H\} = \langle \partial_f G, [\partial_f H, f]^* \rangle = \langle \partial_f G, J(f) \partial_f H \rangle, \tag{14}$$

where $[\ ,\ ]^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ evaluates formally as $[a, b]^* = [a, b]$.

For $G(f) = \langle \delta(\boldsymbol{z}-\boldsymbol{\zeta}), f(\boldsymbol{z})\rangle$, $\dot{G} = \{G, H\}$ reads as the Vlasov equation

$$\dot{f} = [\partial_f H, f]^* \tag{15}$$

evaluated at every $\boldsymbol{z} = \boldsymbol{\zeta} \in M$, which describes the reaction of the distribution function $f(\boldsymbol{z})$ to the motion of particles dictated by the *mean-field Hamiltonian* $h = \partial_f H$.

Evidently, every $C(f) = \int_M g(f)\mathrm{d}z$ (g is a smooth function: $\mathbb{R} \to \mathbb{R}$) is a Casimir. Inserting $\partial_f C = g'(f)$, we find, for every H,

$$\{C, H\} = -\langle \partial_f H, [\partial_f C, f]^*\rangle = -\langle \partial_f H, [g'(f), f]^*\rangle = 0.$$

3.1.2. One-dimensional Vlasov–Poisson system We have to incorporate a dynamical electromagnetic field coupled with the dynamics of charged particles. Neglecting the magnetic field, we consider a simple one-dimensional system in which the longitudinal electric field E accelerates particles in the direction $x \in X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. By $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$ (where ρ is the charge density and ϵ_0 is the vacuum permittivity), we can relate $E(x,t)$ and $f(x,v,t)$:

$$\partial_x E = \frac{q}{\epsilon_0}\int_V f(x,v,t)\mathrm{d}v, \tag{16}$$

where q is the charge of the particle. Putting $E = -\partial_x \phi$ with the scalar potential $\phi(x,t)$, we obtain the Poisson equation

$$-\partial_x^2 \phi = \frac{q}{\epsilon_0}\int_V f(x,v,t)\mathrm{d}v. \tag{17}$$

With the periodic boundary conditions $\phi(0) = \phi(1)$ and $\phi'(0) = \phi'(1)$ (here, $'$ denotes ∂_x), we can solve (17) as

$$\phi(x,t) = \frac{q}{\epsilon_0}\mathcal{K} f(x,v,t) \tag{18}$$

with the integral operator

$$\mathcal{K} = (-\Delta)^{-1} \int_V \circ \, \mathrm{d}v, \tag{19}$$

where $(-\Delta)^{-1} : L^2(X) \to H^2(X)/\{c\}$ is the self-adjoint operator ($\{c\}$ is the one dimensional space of constant functions in X) such that

$$-\partial_x^2 (-\Delta)^{-1} \rho = \rho \quad (\rho \in L^2(X)). \tag{20}$$

We define the quadratic form

$$\Phi(f) = \frac{q^2}{2\epsilon_0} \langle \mathcal{K} f, f \rangle.$$

Using (17) and (18), we may rewrite

$$\Phi(f) = \frac{1}{2} \int_X \phi \left(q \int_V f \, \mathrm{d}v \right) \mathrm{d}x = \int \frac{\epsilon_0 E^2}{2} \mathrm{d}x.$$

By the symmetry $\langle \mathcal{K} f, g \rangle = \langle f, \mathcal{K} g \rangle$, we obtain

$$\partial_f \Phi(f) = \frac{q^2}{\epsilon_0} \mathcal{K} f = q\phi.$$

With the Hamiltonian

$$H(f) = \int_M \frac{v^2}{2} f \, \mathrm{d}z + \Phi(f), \tag{21}$$

the Vlasov equation (15) reads

$$\dot{f} + v \partial_x f + q E \partial_v f = 0. \tag{22}$$

Remark 1 (Vlasov–Ampère system) As an alternative formulation, we may include the electric field E as an independent variable spanning the Poisson manifold. The dynamics of E are determined by Ampère's law, which—neglecting the magnetic field—reads as

$$\dot{E} = -\frac{1}{\epsilon_0} j, \tag{23}$$

where j is the current density, which is related to the distribution function $f(x, v, t)$ by

$$j = q \int_V v f(x, v, t) \mathrm{d}v. \tag{24}$$

For $E = -\partial_x \phi$ with the electric potential ϕ in the periodic domain X, we have

$$E \in \mathfrak{E} = \{E \in L^2(X);\ \int_X E(x)\,\mathrm{d}x = 0\}. \tag{25}$$

The Poisson manifold of the Vlasov–Ampère system is the direct product

$$\mathfrak{G}_{VA} = \mathfrak{g}^* \times \mathfrak{E},$$

on which we define

$$\{G, H\}_{VA} = \left\langle \begin{pmatrix} \partial_f G \\ \partial_E G \end{pmatrix}, \begin{pmatrix} [\circ, f]^* & -\frac{q}{\epsilon_0}\partial_v(\circ f) \\ -\frac{q}{\epsilon_0}(\partial_v \circ) f & 0 \end{pmatrix} \begin{pmatrix} \partial_f H \\ \partial_E H \end{pmatrix} \right\rangle. \tag{26}$$

With the Hamiltonian

$$H(f, E) = \int_M \frac{v^2}{2} f(x, v, t)\,\mathrm{d}z + \int_X \frac{\epsilon_0 E^2}{2} \mathrm{d}x,$$

for $G = f(\boldsymbol{z}, t) = \int_M \delta(\boldsymbol{z} - \boldsymbol{\zeta}) f(\boldsymbol{\zeta}, t)\,\mathrm{d}\zeta$ we obtain

$$\dot{f}(\boldsymbol{z}, t) = [v^2/2, f]^* - qE\partial_v f = -v\partial_x f - qE\partial_v f,$$

and for $G = E(x, t) = \int_X \delta(x - \xi) E(\xi, t)\,\mathrm{d}\xi$ we obtain

$$\dot{E}(x, t) = -\frac{q}{\epsilon_0} \int_V v f\,\mathrm{d}v = -\frac{1}{\epsilon_0} j(x, t).$$

3.1.3. Water-bag distribution function In the water-bag model (see Fig. 1), we consider distribution functions that are linear combinations of V-space indicator functions:

$$f(x, v, t) = \sum_{j=1}^{N} A_j g_j(x.v.t), \tag{27}$$

$$
g_j(x.v.t) = \begin{cases} 0 & v < V_j(x,t), \\ 1 & V_j(x,t) \leq v \leq V_{j+1}(x,t), \\ 0 & V_{N+1}(x,t) < v, \end{cases} \tag{28}
$$

where N is the number of water bags, and $A_j \in \mathbb{R}$ $(j = 1, \cdots, N)$ are constants (being amenable to Liouville's theorem). For the convenience of later calculations, we put $A_0 = A_{N+1} = 0$ and define

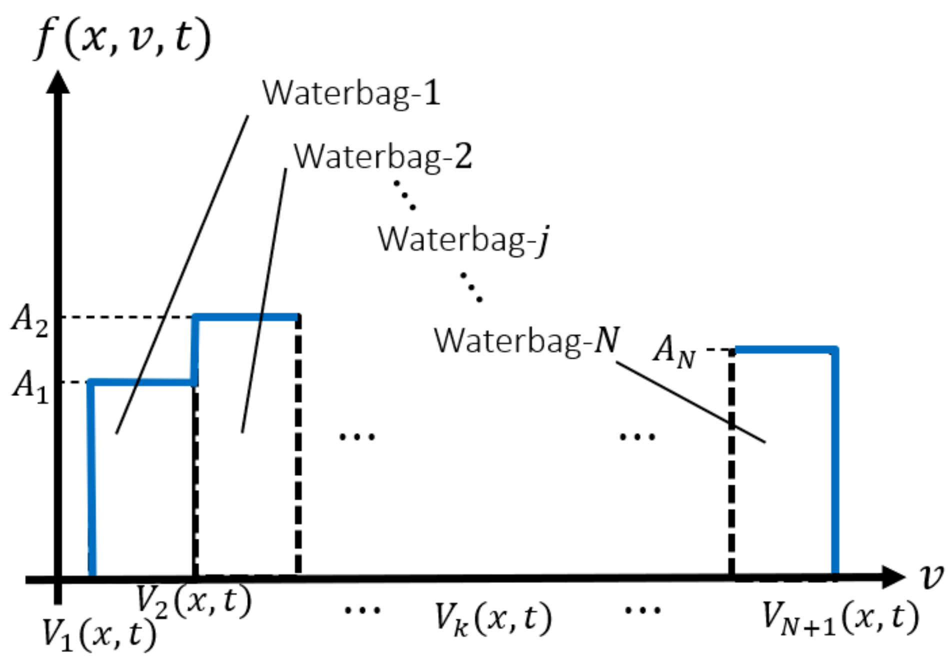

Figure 1. Graph of $f(x,v)$ on a cross section of x = constant. The degree of freedom of the distribution function is given by $\{V_k(x)\}$.

$$
a_k = A_{k-1} - A_k \quad (k = 1, \cdots, N+1). \tag{29}
$$

Each water bag is bounded by the velocities $V_j(x,t)$ and $V_{j+1}(x,t)$ $(j = 1, \cdots, N+1)$, which are assumed to be smooth functions of x and t. Using the step function, we may write

$$g_j(x,v,t) = Y(v - V_j) - Y(v - V_{j+1}).$$

For the convenience of later discussion, we fill the gap of the graph of the step function, i.e., we put $Y(0) = [0,1]$, allowing it to be multivalued.

The function space

$$\mathfrak{g}_N^* = \{V_k(x,t);\ k = 1, \cdots, N+1\}$$

is the phase space (Poisson manifold) of the N-bag system. We endow $\mathfrak{g}_N^*$ with the L^2 inner product

$$(u,v) = \int_X u(x) \cdot v(x)\, \mathrm{d}x.$$

We use the index "j" to address each water bag, and "k" for the boundaries; the latter runs over 1 to $N+1$.

Let us derive the reduction of the Poisson bracket (14) for observables $\in C^\infty(\mathfrak{g}_N^*)$. We may evaluate the perturbation of $f \in \mathfrak{g}_N^*$ as

$$\begin{aligned}\delta f = \sum_{j=1}^{N} A_j \delta g_j &= \sum_{j=1}^{N} A_j[-\delta(v - V_j)\delta V_j + \delta(v - V_{j+1})\delta V_{j+1}] \\ &= \sum_{k=1}^{N+1} a_k \delta(v - V_k)\delta V_k.\end{aligned}$$

For $G(V_1, \cdots, V_{N+1}) \in C^\infty(\mathfrak{g}_N^*)$, the chain rule reads

$$\begin{aligned}\delta G = \langle \partial_f G, \delta f\rangle &= \langle \partial_f G, \sum_{k=1}^{N+1} a_k \delta(v - V_k)\delta V_k\rangle \\ &= \sum_{k=1}^{N+1} (\partial_{V_k} G, \delta V_k).\end{aligned}$$

Therefore, denoting $\partial_{V_k} G = G_k$, we may write

$$\partial_f G|_{v=V_k} = \frac{1}{a_k} G_k \quad (k = 1, \cdots, N+1). \tag{30}$$

Inserting

$$\begin{aligned}
\partial_v f &= \sum_{j=1}^{N} A_j [\delta(v - V_j) - \delta(v - V_{j+1})], \\
\partial_x f &= -\sum_{j=1}^{N} A_j [\delta(v - V_j)\partial_x V_j - \delta(v - V_{j+1})\partial_x V_{j+1}],
\end{aligned}$$

for $G, H \in C^\infty(\mathfrak{g}_N^*)$ we obtain

$$\begin{aligned}
\{G, H\}_N &= \langle \partial_f G, [\partial_f H, f]^* \rangle \\
&= \sum_{j=1}^{N} A_j \int_X (\partial_f G)\partial_x(\partial_f H) + (\partial_f G) V_j \partial_v (\partial_f H)|_{v=V_j} \mathrm{d}x \\
&\qquad - A_j \int_X (\partial_f G)\partial_x(\partial_f H) + (\partial_f G) V_{j+1} \partial_v (\partial_f H)|_{v=V_{j+1}} \mathrm{d}x \\
&= \sum_{j=1}^{N} A_j \int_X \frac{1}{a_j^2} \left(G_j \partial_x H_j - \frac{1}{a_{j+1}^2} G_{j+1} \partial_x H_{j+1} \right) \mathrm{d}x \\
&= \sum_{k=1}^{N+1} - \int_X \frac{1}{a_k} G_k \partial_x H_k \, \mathrm{d}x,
\end{aligned}$$

where we have used $\partial_v(\partial_f G)|_{v=V_k} = \partial_v \frac{1}{a_k} G_k = 0$. In a more illuminating form, we may write

$$\{G, H\}_N = (\nabla_{\boldsymbol{V}} G, J_N \nabla_{\boldsymbol{V}} H), \tag{31}$$

where $\nabla_{\boldsymbol{V}} = (\partial_{V_1}, \cdots, \partial_{V_{N+1}})^{\mathrm{T}}$ and

$$J_N = \begin{pmatrix} \frac{-1}{a_1}\partial_x & & 0 \\ & \ddots & \\ 0 & & \frac{-1}{a_{N+1}}\partial_x \end{pmatrix}. \tag{32}$$

3.1.4. Hamiltonian The Hamiltonian of this system is represented as

$$H(V_1, \cdots, V_{N+1}) =$$

$$-\frac{1}{6}\int_{\mathbf{M}} \qquad a_j \;\; V_j^3 \mathrm{d}z + \Phi(V_1, \cdots, VN+1), \tag{33}$$

which is the Vlasov–Poisson Hamiltonian (21) whose distribution function is applied to the distribution function of the water-bag model.

3.2. Casimirs

Evidently,

$$\mathrm{Ker}\, J_N = \{\boldsymbol{c} = (c_1, \cdots, c_{N+1})^{\mathrm{T}};\, c_k \in \mathbb{R}\,(k = 1, \cdots, N+1)\}.$$

This is easily integrated to derive $N+1$ independent Casimirs

$$\overline{V}_k(V_1, \cdots, V_{N+1}) = (1, V_k) = \int_X V_k(x,t)\mathrm{d}x \quad (k = 1, \cdots, N+1). \tag{34}$$

Physically, each $\overline{V}_k$ means the average velocity (momentum) of the particles aligned along the contour (in the phase space M) of the distribution function $f(x, v, t)$. Because the periodic E yields no net acceleration for each particle, the average velocity $\overline{V}_k$ remains constant.

Any smooth function $G(\overline{V}_1, \cdots, \overline{V}_{N+1})$ is a Casimir of the N-bag system. The density of water-bag j is $\int_{V_j}^{V_{j+1}} f\,\mathrm{d}v$, which evaluates as

$$\rho_j = A_j(V_{j+1} - V_j) \quad (j = 1, \cdots, N), \tag{35}$$

which gives a more convenient representation of N independent Casimirs (representing the total particle number in water-bag j), i.e.,

$$\overline{\rho}_j = \int_X \rho_j\,\mathrm{d}x = A_j(\overline{V}_{j+1} - \overline{V}_j) \quad (j = 1, \cdots, N). \tag{36}$$

However, for the purpose of our analysis, these Casimirs $\overline{\rho}_j$ $(j = 1, \cdots, N)$ are not useful because they all remain constant in any higher-N system (that is a sub-algebra of the Vlasov system $\mathfrak{V}$). In the next subsection, we formulate the remaining one Casimir of the N-bag system, which ceases to be constant when we increase N.

3.3. Fragile Casimir characterizing hierarchy of sub-algebras

This subsection introduces an observable that is not Casimir invariant as the degree of freedom N of the water-bag model increases. U_j can be written as

$$U_j \equiv \int_X u_j \mathrm{d}x, \tag{37}$$

where

$$u_j \equiv \frac{\int_{V_j}^{V_{j+1}} v f \mathrm{d}v}{\int_{V_j}^{V_{j+1}} f \mathrm{d}v},$$

and in the N-bag model, U_j becomes

$$U_j = \int_X \frac{1}{2} \frac{-A_j (V_j^2 - V_{j+1}^2)}{A_j (V_{j+1} - V_j)} \mathrm{d}x = \int_X \frac{1}{2} (V_j + V_{j+1}) \mathrm{d}x. \tag{38}$$

Physically, U_j represents the fluid velocity of the particles contained in water-bag j. Now, we consider the case in which this observable U_j is obtained in the $N+1$-bag model in which the new function $V_{j+\frac{1}{2}}(x,t)$ is introduced into the function space $\mathfrak{g}_N$ as shown in Fig. 2 (new constants $A'_j, A'_{j+\frac{1}{2}}$ are also introduced, and the height of water-bag j is A'_j, where $\{A'_j\} = \{A_1, \cdots, A_{j-1}, A'_j, A'_{j+\frac{1}{2}}, A_{j+1}, \cdots, A_{N+1}\}$). Although its physical meaning has not changed,

$$U_j = \int_X \frac{1}{2} \frac{-A_j V_j^2 + a_{j+\frac{1}{2}} V_{j+\frac{1}{2}}^2 + A_{j+1} V_{j+1}^2}{-A_j V_j + a_{j+\frac{1}{2}} V_{j+\frac{1}{2}} + A_{j+1} V_{j+1}} \mathrm{d}x \tag{39}$$

is no longer invariant for the outbreak of cross terms. The fluid velocity of water-bag j is given by the reference [14].

The symmetry of the distribution function preserved by such a fragile invariant U_j that depends on the macroscopic hierarchy can be written as $\{\circ, U_j\}_{N+1} = 0$ in the $(N+1)$-bag model. Using Eq. (31) and

$$
\begin{aligned}
&\partial_{V_k} U_2(i) \\
&= \begin{cases}
\frac{1}{2}\frac{-A_i(-A_iV_i^2+a_{i+1}V_{i+1}^2+A_{i+1}V_{i+2}^2)}{(A_i(V_{i+1}-V_i)+A_{i+1}(V_{i+2}-V_{i+1}))^2} + \frac{1}{2}\frac{-2A_iV_i}{(A_i(V_{i+1}-V_i)+A_{i+1}(V_{i+2}-V_{i+1}))} & (k=i) \\
\frac{1}{2}\frac{a_{i+1}(-A_iV_i^2+a_{i+1}V_{i+1}^2+A_{i+1}V_{i+2}^2)}{(A_i(V_{i+1}-V_i)+A_{i+1}(V_{i+2}-V_{i+1}))^2} + \frac{1}{2}\frac{2a_{i+1}V_{i+1}}{(A_i(V_{i+1}-V_i)+A_{i+1}(V_{i+2}-V_{i+1}))} & (k=i+1) \\
\frac{1}{2}\frac{A_{i+1}(-A_iV_i^2+a_{i+1}V_{i+1}^2+A_{i+1}V_{i+2}^2)}{(A_i(V_{i+1}-V_i)+A_{i+1}(V_{i+2}-V_{i+1}))^2} + \frac{1}{2}\frac{2A_{i+1}V_{i+2}}{(A_i(V_{i+1}-V_i)+A_{i+1}(V_{i+2}-V_{i+1}))} & (k=i+2) \\
0 & (otherwise)
\end{cases} \\
&= \begin{cases}
-A_i\left(-\frac{u_2(i)}{\rho_i+\rho_{i+1}} + \frac{V_i}{\rho_i+\rho_{i+1}}\right) & (k=i) \\
a_{i+1}\left(-\frac{u_2(i)}{\rho_i+\rho_{i+1}} + \frac{V_{i+1}}{\rho_i+\rho_{i+1}}\right) & (k=i+1) \\
A_{i+1}\left(-\frac{u_2(i)}{\rho_i+\rho_{i+1}} + \frac{V_{i+2}}{\rho_i+\rho_{i+1}}\right) & (k=i+2) \\
0 & (otherwise)
\end{cases},
\end{aligned}
\tag{40}
$$

$\{\circ, U_j\}$ can be calculated as

$$
\{\circ, U_j\}_{N+1}
$$

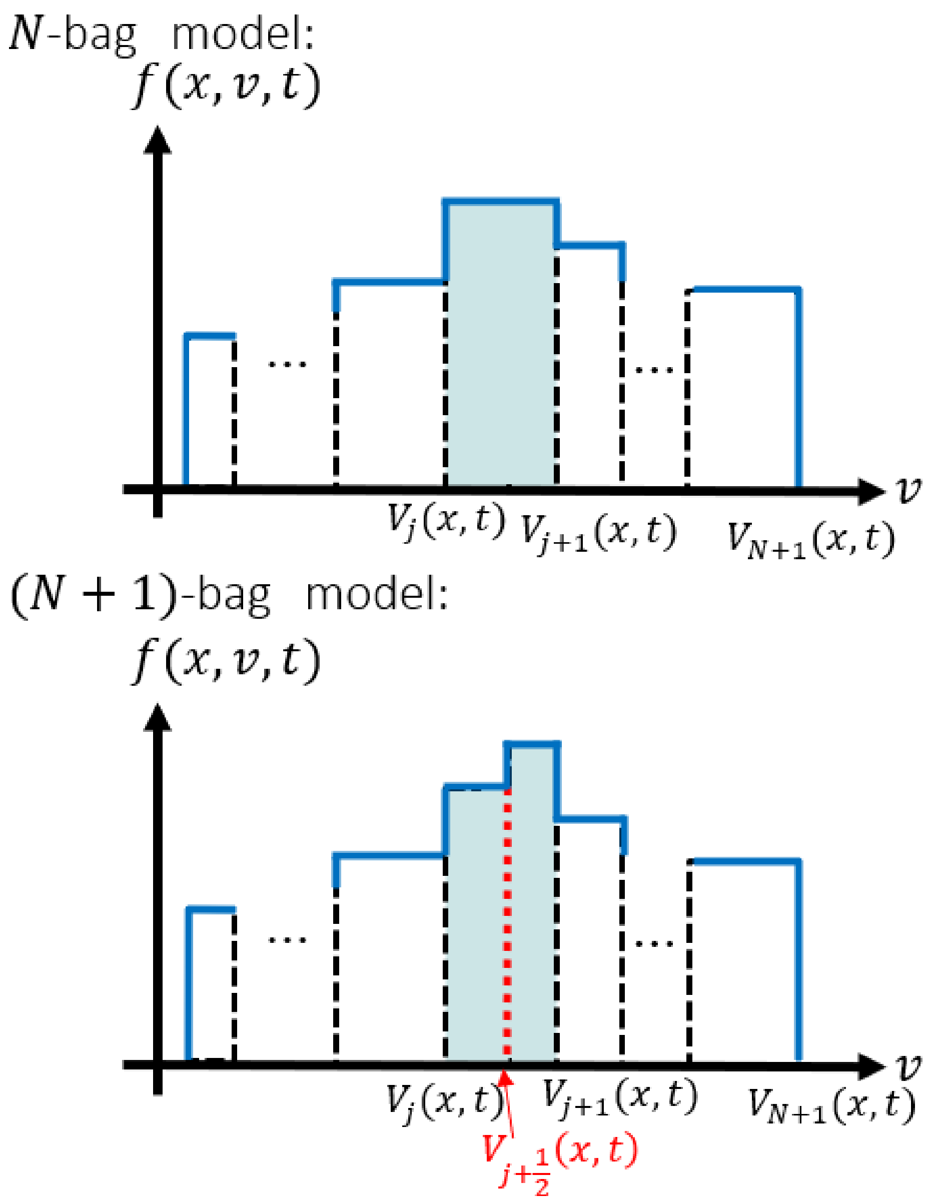

Figure 2. Upper: distribution function f of N-bag model with $\mathfrak{g}_N$ as the topological space. Lower: distribution function f of $N+1$-bag model with $\mathfrak{g}_{N+1} = \{V_1, \cdots, V_j, V_{j+\frac{1}{2}}, V_{j+1}, \cdots, V_{N+1}\}$ as the phase space. In both the upper and lower figures, the light-blue region is the integration range of U_j. Although U_j has the same physical meaning in both the N-bag and $N+1$-bag case, U_j is a Casimir invariant in the former but not in the latter.

$$= \left(\nabla_{\boldsymbol{V}} \circ, \partial_x \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{A_j}{a_j}\left(-\frac{u_j}{\rho_j+\rho_{j+\frac{1}{2}}}+\frac{V_j}{\rho_j+\rho_{j+\frac{1}{2}}}\right) \\ (\ j\text{th element}) \\ -\frac{a_{j+\frac{1}{2}}}{a_{j+\frac{1}{2}}}\left(-\frac{u_j}{\rho_j+\rho_{j+\frac{1}{2}}}+\frac{V_{j+\frac{1}{2}}}{\rho_j+\rho_{j+\frac{1}{2}}}\right) \\ (\ j+\frac{1}{2}\text{th element}) \\ -\frac{A_{j+\frac{1}{2}}}{a_{j+1}}\left(-\frac{u_j}{\rho_j+\rho_{j+\frac{1}{2}}}+\frac{V_{j+1}}{\rho_j+\rho_{j+\frac{1}{2}}}\right) \\ (\ j+1\text{th element}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right). \tag{41}$$

Because transformation of (41) is a variable transformation of the distribution function conserving U_2, $\{f, U_j\}_{N+1} = 0$ is a condition on f that conserves U_2. From (41), the condition on f (or that on $\{V_k\}$ in the $N+1$-bag model) is

$$\partial_x \frac{u_{j'}}{\rho_j + \rho_{j+1}} = \partial_x \frac{V_{j'}}{\rho_j + \rho_{j+1}} \quad (j' = j, j + \frac{1}{2}, j + 1), \tag{42}$$

which can be simplified by calculating (let $j'' = j, j + \frac{1}{2}, j + 1, j' \neq j''$)

$$\begin{aligned} &\partial_x \left(\frac{u_j}{\rho_j + \rho_{j+1}} - \frac{V_{j'}}{\rho_j + \rho_{j+1}} \right) - \partial_x \left(\frac{u_j}{\rho_j + \rho_{j+1}} - \frac{V_{j''}}{\rho_j + \rho_{j+1}} \right) \\ = &- (V_{j'} - V_{j''}) \frac{\partial_x (\rho_j + \rho_{j+1})}{(\rho_j + \rho_{j+1})^2} - \frac{\partial_x (V_j - V_{j'})}{(\rho_j + \rho_{j+1})} \end{aligned}$$

$$= -\frac{1}{\rho_j + \rho_{j+1}} \left(\frac{\partial_x (A_j (V_{j+1} - V_j) + A_{j+1}(V_{j+2} - V_{j+1}))}{A_j (V_{j+1} - V_j) + A_{j+1}(V_{j+2} - V_{j+1})} \right.$$
$$\left. - \frac{\partial_x (V_{j'} - V_j)}{V_{j'} - V_j} \right). \tag{43}$$

The symmetry condition $\{\circ, U_j\}_N = 0$ holds if

$$\frac{\partial_x (A_j (V_{j+\frac{1}{2}} - V_j) + A_{j+\frac{1}{2}}(V_{j+1} - V_{j+\frac{1}{2}}))}{A_j (V_{j+\frac{1}{2}} - V_j) + A_{j+\frac{1}{2}}(V_{j+1} - V_{j+\frac{1}{2}})} = \frac{\partial_x (V_{j'} - V_j)}{V_{j'} - V_j},$$

which demands

$$A_j (V_{j+\frac{1}{2}} - V_j) + A_{j+\frac{1}{2}}(V_{j+1} - V_{j+\frac{1}{2}})$$
$$= (V_{j'} - V_j) exp(C_{jj'}). \tag{44}$$

The symmetry condition reads

$$V_{j+\frac{1}{2}} = \alpha V_j + (1-\alpha) V_{j+1} \quad (\alpha = \text{const.},\ 0 < \alpha < 1), \tag{45}$$

implying that the new contour $V_{j+\frac{1}{2}}$ included in the inflated $N+1$ system must be an internally dividing point of the original contours with an (arbitrary) homogeneous ratio α. Every deformation of the contour V_{j+1} from this symmetry violates the conservation of U_j.

3.4. Numerical experiment

Now we show simple examples that support condition (45) in the case of $N = 6$. Unlike in the previous discussion, we set a specific Hamiltonian

$$H_N = -\int_X \left(\frac{1}{3} \Sigma_{j=1}^{N+1} \frac{1}{2} a_j V_j^3 + \Phi(f) \right) \mathrm{d}x, \tag{46}$$

constants A_j as

$$\{A_j\} = \{1, 4.5, 3.5, 1.5, 0.5, 2\}, \tag{47}$$

and the initial conditions of the distribution function as

$$
\begin{aligned}
V_k|_{t=0,k=1,4}(x) &= \frac{3}{14}k^2 + \frac{100.86}{175\cosh^2(4.1x)}, \\
V_k|_{t=0,k=2}(x) &= -\frac{6}{7} - \frac{100.86}{175\cosh^2(4.1x)} + 2.69, \\
V_k|_{t=0,k=3}(x) &= 2.62, \\
V_k|_{t=0,k=5,6,7}(x) &= \frac{3}{14}k^2 + \frac{100.86}{175\cosh^2(4.1(x+0.5))}
\end{aligned} \tag{48}
$$

(Fig. 3) so that (45) is not satisfied only when $j = 3$. Because the shape of the distribution function $(\{A_j\}, \{V_k|_{t=0}\})$ is arbitrary when deriving (45), the verification of (45) in this subsection does not require any statistical-mechanics justification for $(\{A_j\}, \{V_k|_{t=0}\})$. To simulate this system, we have to set some normalized parameters as

$$\hat{q} = 1, \tag{49}$$

$$\hat{\varepsilon}_0 = 368.32. \tag{50}$$

To demonstrate a practicality of condition (45), we show the time variation of U_2 (Fig. 4). Because condition (45) is not satisfied only for $j = 3$, the time variation of U at $t = 0$ is expected to be small for $j = 1, 2, 4, 5, 6$, and indeed the results in Fig. 4 meet that expectation.

4. Two-dimensionally extended Waterbag model(“mesa”-model)

To apply the conclusions drawn in the previous section, this section creates a numerical scheme for the two-dimensionally extended Waterbag model (“ mesa”-model). Here, the formulation of the 2D Waterbag model is based on the literature [16]. In this section, following subsections will be used to introduce the content of the relevant document.

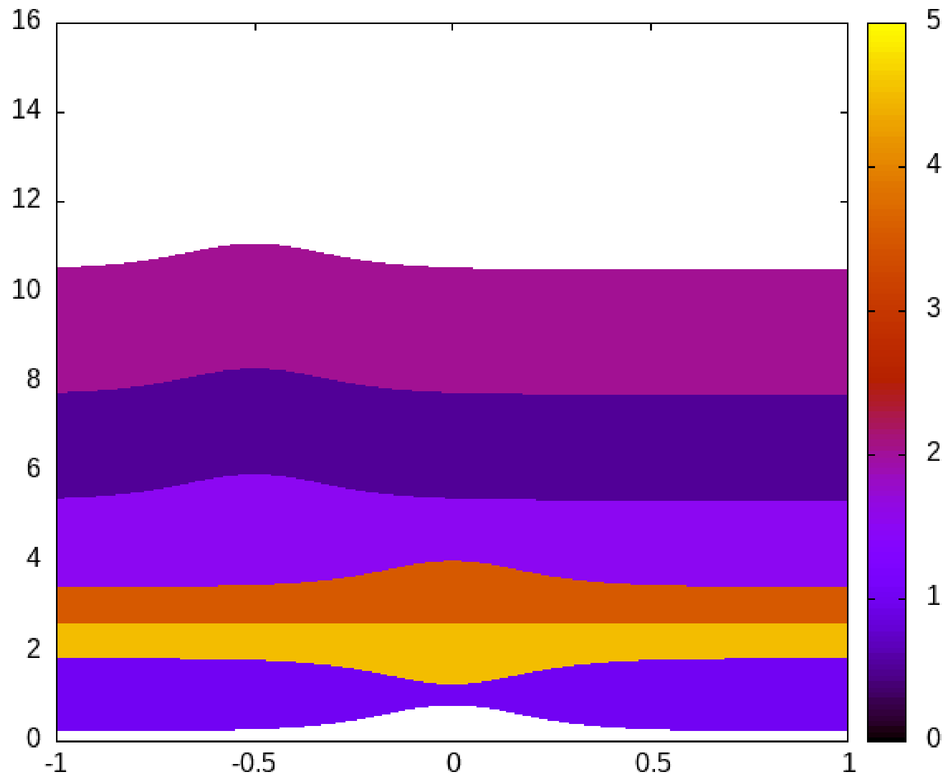

Figure 3. Initial conditions of simulation. The vertical axis represents the position x, and the horizontal axis represents the velocity v. The color bar shows the value of $f(x, v, t)$, and the white regions correspond to $f(x, v, t) = 0$. Relationship (45) is not satisfied only for $k = 3$.

4.1. Reduced kinetic system —mesas in the velocity space

We formulate a reduced kinetic system by restricting the distribution functions to be flat functions supported in simplexes (and chains) in the v-space V ; cartoon image of such a distribution function is mesas. We start by considering a single "mesa" in the v-space. Formally, we consider a subset $\mathfrak{g}^*_\Delta \subset \mathfrak{g}^*$ of distributions such that

$$g^*_\Delta = f_\Delta(\boldsymbol{x}, \boldsymbol{v}) = \mathbb{I}_\Delta(V0(x), \cdots, Vn(x)), \tag{51}$$

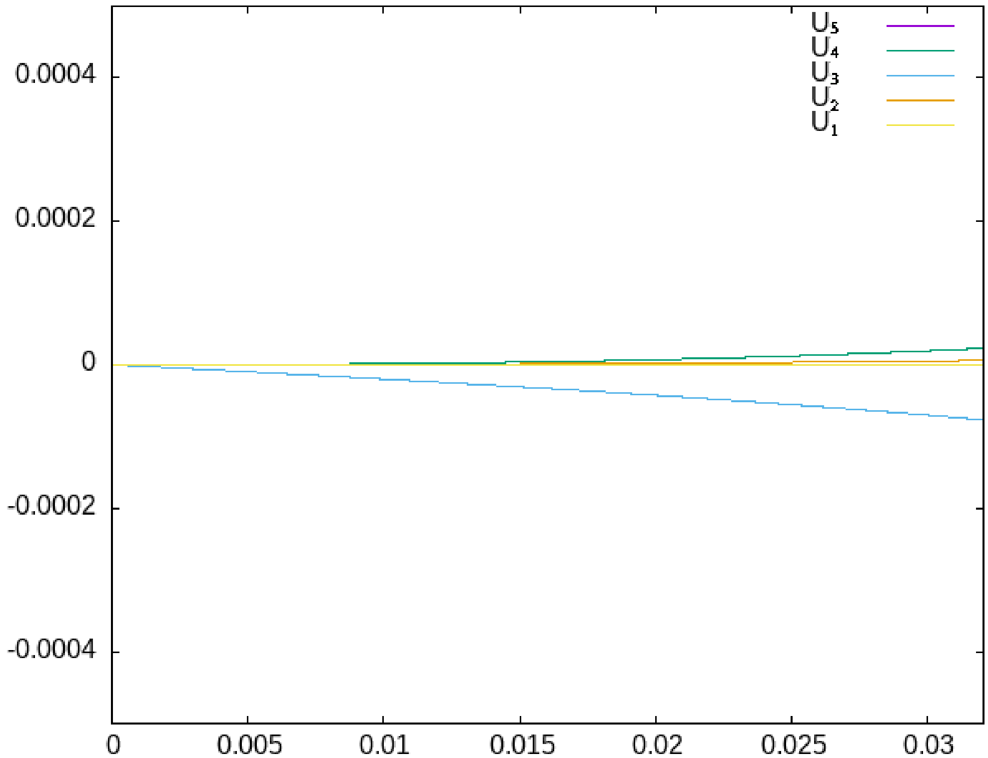

Figure 4. Horizontal axis represents time, vertical axis represents rate of change of U_j from its initial value. U_j changes significantly at $t=0$ only for $j=3$, for which condition (45) is not satisfied.

where $\mathbb{I}_\Delta$ is the indentificator function of a domain $\Delta \subset V$, and $\mathbb{I}_\Delta(V0(x), \cdots, Vn(x))$ is a "linear simplex" in the n-dimensional v-space with vertexes $V0(x), \cdots, Vn(x)$; by "linear simplex" we mean an affine map of the standard n-simplex (hence, faces of Δ are flat). Note that a general simplex is an image of any differentiable map, but here we restrict to affine maps. Hereafter we will abbreviate it as "simplex". For a distribution function $f_\Delta(\boldsymbol{x}, \boldsymbol{v})$ with a simplex support to be persistent in a Hamiltonian flow, the Hamiltonian must be restricted to some subset $\mathfrak{H}_L \subset \mathfrak{H} = C^\infty(\mathfrak{g}^*)$. In fact, we will find that the Hamiltonian flow must be "linear" with respect to $\boldsymbol{v}$ (or an affine map in v-space); otherwise the faces of Δ would be deformed to curved surfaces, and then, the vertexes fall short of characterizing f. The following fact may be well-known, but will play an essential role in the following discussions.

Lemma 1 (subalgebra)

Let us consider a subset of observables such that

$$\mathfrak{g}_L = \left\{ \sum_{k=0}^{n} \alpha_k(\boldsymbol{x}) v^k ; \alpha_k(\boldsymbol{x}) \in C^\infty(X) \right\}$$

. where $v^1, \cdots, v^n$ are the coordinates of the v-space, and $v^0 = 1$. This $\mathfrak{g}_L$ is a subalgebra of $\mathfrak{g}$, i.e.

$$[\psi, \phi] \in \mathfrak{g}_L \quad (\forall \psi, \phi \in \mathfrak{g}_L).$$

(proof) By direct calculation, we obtain, for $\psi = \sum_k \alpha_k(\boldsymbol{x}) v^k$ and $\phi = \sum_k \beta_k(\boldsymbol{x}) v^k$,

$$[\psi, \phi] = \sum_k \left(\sum_j \beta_j \partial_{x^j} \alpha_k - \alpha_j \partial_{x^j} \beta_k \right) v^k. \quad \blacksquare$$

Notice that $\psi \in \mathfrak{g}_L$ must be a linear function of v^j , while it may be an arbitrary (smooth) function of x^j.

For the convenience of latter calculations, we extend the idea of including "0" component for other expressions:

$$\begin{cases} v^0 = 1, \partial_{v^0} = \text{projection to } v\text{−independent terms}, \\ V_k^0 = 1 (k = 0, \cdots, n), \\ x^0 = 1, \partial_{x^0} = 0. \end{cases} \tag{52}$$

Both indexes for the coordinates and the vertexes range from 0 to n.

Remark 2 *(higher moments)If we include higher-order terms in v^j, such subsets are not closed. For example, let us consider a second-order class. For*

$$\psi = \sum_{kl} = \alpha_{kl}(\boldsymbol{x})v^k v^l, \phi = \sum_{kl} \beta_{kl}(\boldsymbol{x})v^k v^l,$$

(where α_{kl} and β_{kl} are symmetric tensors), we observe

$$[\psi, \phi] = 2\sum_{klm}\left(\sum_j \beta_{jm}\partial_{x^j}\alpha_{kl} - \alpha_{jm}\partial_{x^j}\beta_{kl}\right)v^k v^l v^m,$$

which becomes a third-order function of v^j.

The Vlasov system $\mathfrak{O} = C^\infty_{\{,\}}(\mathfrak{g}^*)$ will be reduced to a closed subsystem $\mathfrak{O}_L$ such that $T^*\mathfrak{O}_L = \mathfrak{g}_L$. The aim of the following study is to construct a consistent relation between $\mathfrak{g}_L$ and $\mathfrak{g}^*_\Delta$ so that $\mathfrak{O}_L = C^\infty_{\{,\}_L}(\mathfrak{g}^*_\Delta)$ with a reduced bracket $\{,\}_L$. $\{,\}_L$, we will find that it is equivalent to the fluid bracket $\{,\}_F$.

4.2. Closedness of $\mathfrak{g}^*_\Delta$ in Hamiltonian flows of $\mathfrak{g}_L$

We may write

$$\{G, H\} = \langle[\partial_f G, \partial_f H], f\rangle = \langle\partial_f G, [\partial_f H, f]^*\rangle$$

where $[,]^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$, which evaluates just as $[\ ,\ ]$ for functions $\in C^\infty(M)$. Denoting $\partial_f G = g$and $\partial_f H = h$, we write

$$\{G, H\} = \langle g, [h, f]^*\rangle,$$

and

$$\mathrm{ad}^*_h = [h, \circ]^*,$$

which is the co-adjoint action generated by a Hamiltonian $h \in \mathfrak{g}$. Explicitly, we may evaluate

$$\mathrm{ad}_h^* f = [h, f]^* = \sum_{j=1}^{n} (\partial_{x^j} h)(\partial_{v^j} f) - (\partial_{v^j} h)(\partial_{x^j} f). \tag{53}$$

Theorem 1 (Closedness)

The system $\mathfrak{g}_\Delta^*$ of mesa distribution functions is closed with respect to the action of ad_h^* *with* $h \in \mathfrak{g}_L$. (proof) First we note that $\mathfrak{g}^* = (\wedge^n T^* X) \wedge (\wedge^n T^* V)$, i.e., the distribution function f_Δ is a $2n$-form (n is the configuration space dimension). Therefore, $\partial_{\boldsymbol{v}} f_\Delta$ is defined as $\delta_x f_\Delta$, where $\delta_v = *\mathrm{d}v*$ is the co-differential in v-space (d_v is the exterior differential in v-space, and $*$ is the Hodge star operator with respect to the v-space volume $vol_v^n = \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^n$). Similarly, $\partial_{\boldsymbol{x}} f_\Delta$ is defined as $\delta_x f_\Delta$ with the x-space co-differential. By the Liouville theorem, the Hamiltonian flow ad_h^* is incompressible in the phase space $M = X \times V$. Hence, the hight of the mesa f_Δ is constant even when the v- space volume $|\Delta| = \int_\Delta \mathrm{vol}_v^n$ changes, i.e. $*f_\Delta = 1$ in Δ. Therefore, what we have to prove is that the faces of Δ remain flat when transported by the Hamiltonian flow ad_h^*. Notice that the distribution function f_Δ can be totally characterized by its vertexes $\boldsymbol{V}_l (l = 0, \cdots, n)$ only when the faces are straight; otherwise we need additional information about the structure of distribution function.

The co-adjoint action (53) consists of two terms, which respectively defines a vector in v-space and x-space. We look into the action of the first term $\sum_j (\partial_{x^j} h)\partial_{v^j}$, which transports the domain Δ in v-space. Let us write the boundary as

$$\partial_\Delta = \bigcup_{l=0}^{n} \sigma_l,$$

where each σ_l is the segment facing $\boldsymbol{V}_l$. We denote by i_{∂_Δ} the inclusion map of ∂_Δ into V , and its dual by $i^*_{\partial\Delta}$ that is the restriction map. We may write

$$*d * f_\Delta = i^*_{\partial_\Delta} = 1.$$

On each segment, $i^*_{\sigma_l}1$ is the unit tangential $(n-1)$-form. We may regard it as the δ-measure placed on the surface σ_l (to be written as δ_{σ_l}). We denote by $\boldsymbol{\sigma}_l$ the unit normal vector (inward) to Δ. Given a 1-form u, the inner-product $u \wedge i^*_{\sigma_l}1$ evaluates (at each point on σ_l) the normal component of u with respect to the surface σ_l. Identifying u as a co-vector $\boldsymbol{u}$, we may calculate $u \wedge i^*_{\sigma_l}1 = i^*_{\sigma_l}(\boldsymbol{u} \cdot \boldsymbol{\sigma}_l)$, which is the restriction of the n-form $(\boldsymbol{u} \cdot \boldsymbol{\sigma}_l)$ onto the surface σ_l. For a Hamiltonian co-vector $u = \sum_j (\partial_{x^j} h) \mathrm{d}v^j$, we obtain

$$\sum_j (\partial_{x^j} h)(\partial_{v^j} f_\Delta) = \sum_l i^*_{\sigma_l}(\nabla_x h \cdot \boldsymbol{\sigma}_l), \tag{54}$$

which evaluates the normal velocity of the surface σ_l driven by the Hamiltonian flow ad^*_h. If $h \in \mathfrak{g}_L$, $\nabla_x h$ is a linear function of v^j. On the other hand $\boldsymbol{\sigma}_l$ is a constant vector on each σ_l‡. Therefore, the normal velocity is a linear function of v^j , i.e. ad^*_h gives an affine map in v-space; hence, the surface σ_l remains flat.

4.3. The Poisson bracket of the mesa system

We formulate a reduced Lie-Poisson bracket such that

$$\{G, H\}_L = \langle [g, h], f_\Delta \rangle\rangle, \partial_f G = g, \partial_f H = h \in \mathfrak{g}_L. \tag{55}$$

‡ We may include in h a purely velocity term like $|\boldsymbol{v}|^2$, which does not appear in the v-space transport term $\partial_{x^j} * h \partial_{v^j} f_\Delta$. However, the corresponding x-space transport term is affected, by which the Lie- Poisson algebra may not be closed.

Before showing how $\{G,H\}_L$ works as a "reduction" of the Vlasov bracket $\{G,H\}$, we start by examining how the right-hand side of (55) evaluates.

4.4. Separation of v-space:

For $g,h,\in \mathfrak{g}_L$, we may write

$$g=\sum_{k=0}^{n}\alpha_k(\boldsymbol{x})v^k, h=\sum_{k=0}^{n}\beta_k(\boldsymbol{x})v^k. \tag{56}$$

Remember the notation (55); with $\partial_{x^0}=0$, we observe

$$\begin{aligned}\langle[g,h],f_\Delta\rangle &= \sum_{j,k=0}^{n}\langle[(\partial_{x^j}\alpha k)\beta j-(\partial_{x^j}\beta k)\alpha j]vk,f_\Delta\rangle \\ &= \sum_{j,k=0}^{n}\int_X[(\partial_{x^j}\alpha_k)\beta_j-(\partial_{x^j}\beta_k)\alpha_j]\left(\int_V vkf_\Delta \mathrm{d}^n v\right)\mathrm{d}^n x. \end{aligned}\tag{57}$$

Notice that, due to the closedness of $\mathfrak{g}_L$ (Lemma 1), only v^k appears in the integrand. Performing the v-space integral, we obtain

$$p^k:=\int_V v^k f_\Delta \mathrm{d^n v}=|\Delta|\frac{1}{\mathrm{n+1}}\sum_{\mathrm{k=0}}^{\mathrm{n}}\mathrm{V_l^k}, \tag{58}$$

with the volume

$$|\Delta|=\int_\Delta 1\mathrm{d}^n v=\frac{1}{n!}det(\dot{V}_k^j),(\dot{V}_k^j=V_k^j-V_0^j,j,k=1,\cdots,n).$$

Notice that $p^0=|\Delta|$ (because we put $V_0^k=1$). Using these parameters, we may write

$$\langle[g,h],f_\Delta\rangle=\sum_{j,k=0}^{n}\int_X(\partial_{x^j}\alpha_k)(p^k\beta_j)-(\partial_{x^j}\beta_k)(p^k\alpha_j)\mathrm{d}^n x \tag{59}$$

4.4.1. Parameterization of $\mathfrak{O}_L$: In order to restrict $\mathfrak{O}$ to $\mathfrak{O}_L$, the parameters characterizing a functional $H\in\mathfrak{O}_L$ are limited to $p^0,p^1,\cdots,p^n$. For $H(p^0,\cdots,p^n)$, the chain rule reads

$$\delta_H=\int_M\partial_f H\delta_f\mathrm{d}^n v\mathrm{d}^n x=\int_X\sum_{k=0}^{n}\partial_p kH\delta_p k\mathrm{d}^n x.$$

By $\delta_{p^k} = \int_V v^k \delta_f \mathrm{d}^n v$, we obtain

$$\partial_f H = \sum_{k=0}^{n} (\partial_{p^k} H) v^k (k = 0, \cdots, n). \tag{60}$$

Comparing this with (56), we may put $\alpha k = \partial_p k G, \beta k = \partial_p k H$ in (59), to obtain

$$\{G, H\}_L = \sum_{j,k=0}^{n} \int_X (\partial_{x^j} \partial_{p^k} G)(p^k \partial_{p^j} H) - (p^k \partial_{p^j} G)(\partial_{x^j} \partial_{p^k} H) \mathrm{d}^n x. \tag{61}$$

To rewrite (61) in a more transparent form, let us denote $\boldsymbol{P} = (p^0, p^1, \cdots, p^n)^T$, $\boldsymbol{p} = (p^1, \cdots, p^n)^T, \partial_{\boldsymbol{P}} = (\partial_{p^0}, \partial_{p^1}, \cdots, \partial_{p^n})^T$,and$\partial_{\boldsymbol{p}} = (\partial_{p^1}, \cdots, \partial_{p^n})^T$. In (61), the terms including $k = 0$ are summarized as

$$\int_X -(\partial_{p^0} G) \nabla \cdot (p^0 \partial_p H) - (\partial_p G) \cdot p^0 \nabla (\partial_{p^0} H) \mathrm{d}^n x, \tag{62}$$

and from $k = 1, \cdots, n$ (here we assume $n = 3$ for the convenience of vector notation)

$$\int_X -(\partial_p G) \cdot [(\nabla \times \boldsymbol{p}) \times (\partial_{\boldsymbol{p}} H) + \boldsymbol{p} \nabla \cdot (\partial_{\boldsymbol{p}} H) + \nabla (\boldsymbol{p} \cdot (\partial_{\boldsymbol{p}} H))] \mathrm{d}^3 x. \tag{63}$$

The bracket $\{G, H\}_L$ is the sum of (62) and (63):

$$\begin{aligned} &\{G, H\}_L = (\partial_{\boldsymbol{P}} G, J_L \partial_{\boldsymbol{P}} H), \\ &J_L = \begin{pmatrix} 0 & -\nabla \cdot (p^0 \circ) \\ -p^0 \nabla & -(\nabla \times \boldsymbol{p}) \times \circ - \boldsymbol{p}(\nabla \cdot \circ) - \nabla (\boldsymbol{p} \cdot \circ) \end{pmatrix} \end{aligned} \tag{64}$$

4.5. Vertex dynamics

4.5.1. Co-adjoint orbit of vertexes Since f_Δ is totally characterized by the vertexes $\boldsymbol{V}_l(\boldsymbol{x})(l = 0, \cdots, n)$, we can parameterize $H(f_\Delta)$ as $H(\boldsymbol{V}_0, \cdots, \boldsymbol{V}_n)$. Rewriting the Hamiltonian in terms of V_l^j, we obtain the system of equations that dictates the co-adjoint orbits of vertexes. However, the totality of possible functionals $G(\boldsymbol{V}_0, \cdots, \boldsymbol{V}_n)$ is larger than $\mathfrak{O}_L$ (if $n \geq 2$) .

4.5.2. Poisson operator In response to the motion of the vertexes, the simplex Δ moves to modify the distribution function f_Δ; its action, caused by a Hamiltonian $h = \partial_f H$, is represented by a Poisson operator given bellow. In (64) , we have already derived the Poisson operator (J_L) acting on the "fluid observables" based on the configurations space X. Here we write down the Poisson operator in a more naïve form (i.e. for kinetic observables) to see how it operates in the phase space M ; in rewriting

$$\{G, H\} = \langle g, [h, f_\Delta]^* \rangle, \partial_f G = g, \partial_f H = h, \tag{65}$$

the Poisson operator is

$$J = [\circ, f_\Delta]^* = \sum_{j=1}^{n} (\partial_{v^j} f_\Delta) \partial_{x^j} - (\partial_{x^j} f_\Delta) \partial_{v^j}. \tag{66}$$

For the first term of J, we have already derived an explicit representation (54), which now reads

$$(\partial_{v^j} f_\Delta) \partial_{x^j} = \sum_l i^*_{\sigma_l} \boldsymbol{\sigma}_l \cdot \partial_{x^j} = \sum_l \delta_{\sigma_l} \boldsymbol{\sigma}_l \cdot \partial_{x^j}. \tag{67}$$

The second term requires a more involved analysis, because we have to evaluate the reaction of f_Δ against the perturbation of the vertex points $\boldsymbol{V}_l(x)$ that have $\boldsymbol{x}$ dependences. In what follows, we will derive the following representation for $n = 2$:

$$(\partial_{x^j} f_\Delta) \partial_{v^j} = -\sum_l \delta_{\sigma_l} [(\boldsymbol{\sigma}_l \cdot \partial_{x^j} \boldsymbol{W}_{l-1}) \bar{\tau}_{l-1} + (\boldsymbol{\sigma}_l \cdot \partial_{x^j} \boldsymbol{V}_{l+1})] \partial_{v^j}, \tag{68}$$

where $\boldsymbol{W}_l = \boldsymbol{V}_l - \boldsymbol{V}_{l-1}$, and $\bar{\tau}_l$ is the coordinate (normalized) interpolating $\boldsymbol{V}_{l-1}$ and $\boldsymbol{V}_l$.

4.5.3. Preparation of symbols for 2D system: In 2D v-space, a simplex (triangle) has 3 vertexes V_0, V_1, V_2 (we give the indexes in anti-clockwise order, and indexes are mod 2). We denote

$$\boldsymbol{W}_l = \boldsymbol{V}_l - \boldsymbol{V}_{l-1}, L_l = |\boldsymbol{W}_l|.$$

The vector $\boldsymbol{W}_l$ spans the side σ_{l+1} (remember that σ_{l+1} faces $\boldsymbol{V}_{l+1}$). The inward unit normal vector on σ_{l+1} is

$$\boldsymbol{\sigma}_{l+1} = \frac{1}{L_l} \begin{pmatrix} -W_l^2 \\ \\ W_l^1 \end{pmatrix}. \tag{69}$$

The tangential coordinate on σ_{l+1} is

$$\tau_l = \frac{1}{L_l} \left(W_l^1 v^1 + W_l^2 v^2 \right), \tag{70}$$

which starts from $\boldsymbol{V}_{l-1}$ and ends at $\boldsymbol{V}_l$ when $\tau_l = L_l$. A point on σ_{l+1} is given by the interpolation formula:

$$\boldsymbol{v} = \bar{\tau}_l \boldsymbol{V}_l + (1 - \bar{\tau}_l)\boldsymbol{V}_{l-1} = \boldsymbol{V}_{l-1} + \bar{\tau}_l \boldsymbol{W}_l, \bar{\tau}_l = \frac{\tau_l}{L_l}. \tag{71}$$

This normalized coordinate $\bar{\tau}_l$ will be used to identify points on σ_{l+1}.

4.5.4. x-space vector: To calculate $\partial_x f_\Delta$, we have to evaluate the reaction of f_Δ to the displacement of the vertexes $\boldsymbol{V}_l(x)$. When a function $f(t)$ is shifted by δ_t on the t-coordinate, the local value of $f(t)$ will be changed by $\delta_f(t) = -(\partial_t f)\delta_t$. Applying this relation to the deformation of f_Δ by the shifts of vertexes $\boldsymbol{V}_l$, we obtain

$$\delta_{f_\Delta} = -\sum_l \left[\delta_{\sigma_{l+1}} (\boldsymbol{\sigma}_{l+1} \cdot \delta_{\boldsymbol{V}_l}) \bar{\tau}_l + \delta_{\sigma_{l-1}} (\sigma_{l-1} \cdot \delta_{\boldsymbol{V}_l})(1 - \bar{\tau}_{l+1}) \right].$$

Hence,

$$\begin{aligned}\partial_{x^j} f_\Delta = & -\sum_l \left[\delta_{\sigma_{l+1}}(\boldsymbol{\sigma}_{l+1}\cdot\partial_{x^j}\boldsymbol{V}_l)\bar{\tau}_l + \delta_{\sigma_{l-1}}(\boldsymbol{\sigma}_{l+1} - \partial_{x^j}\boldsymbol{V}_l)(1-\bar{\tau}_l+1)\right] \\ = & -\sum_l \delta_{\sigma_l}[(\sigma_l\cdot\partial_{x^j}\boldsymbol{W}_{l-1})\bar{\tau}_{l-1} + (\sigma_l\cdot\partial_{x^j}\boldsymbol{V}_{l+1})]. \end{aligned} \tag{72}$$

4.5.5. Lie-Poisson bracket (in terms of vertexes) Here we construct the Lie-Poisson bracket from the Poisson operator $[\circ, f_\Delta]^*$:

$$\begin{aligned}\langle g, [h, f_\Delta]^*\rangle = & \sum_{j=1}^n \langle g, (\partial_{v^j} f_\Delta)(\partial_{x^j} h) - (\partial_{x^j} f_\Delta)(\partial_{v^j} h)\rangle \\ = & \sum_{j=1,k=0,m=0}^n \langle \alpha_k v^k, (\partial_{v^j} f_\Delta)(\partial_{x^j}\beta_m v^m)\rangle - \langle \alpha_k v^k, (\partial_{x^j} f_\Delta)(\partial_{v^j}\beta_m v^m)\rangle.\end{aligned}$$

The first term under the summation on the right-hand side may be written as

$$\langle \alpha_k v^k, (\partial_{v^j} f_\Delta)(\partial_{x^j}\beta_m v^m)\rangle = \int \alpha k (\int v^k v^m \partial_{v^j} f_\Delta \mathrm{d}^n v)\partial_{x^j}\beta m \mathrm{d}^n x =: (\alpha k, \mathcal{K}^{jkm}\partial_{x^j}\beta m)X,$$

where $(a,b)X = \int X a(x) b(x) \mathrm{d}^n x$. On the other hand, the second term defines

$$\langle \alpha_k v^k, (\partial_{x^j} f_\Delta)(\partial_{v^j}\beta_m v^m)\rangle = \int \alpha k (\int v^k \partial_{x^j} f_\Delta \mathrm{d}^n v)\delta_{jm}\beta m \mathrm{d}^n x =: (\alpha k, \mathcal{L}^{jkm}\delta_{jm}\beta m)X.$$

Combining these two, we define the Poisson operator

$$J_{jkm} = \mathcal{K}^{jkm}\partial_{x^j} - \mathcal{L}^{jkm}\delta_j m, \tag{73}$$

by which (55) reads

$$\{G, H\} = \sum^n j = 1, k = 0, m = 0(\partial_{v^k} g, \mathcal{J} jkm \partial_{v^m} h). \tag{74}$$

Using (68), we obtain

$$\mathcal{K}^{jkm} = \int v^k v^m \partial_{v^j} f_\Delta \mathrm{d}^2 v = \sum_l \sigma^j_{l+1} \int_0^{L_l} v^k v^m \mathrm{d}\tau_l.$$

We evaluate, (71),

$$\chi_l^{km} := \frac{1}{L_l}\int_0^{L_l} v^k v^m \mathrm{d}\tau_l = \frac{1}{3}(V_l^k + V_{l-1}^k)(V_l^m + V_{l-1}^m) - \frac{1}{6}(V_l^k V_{l-1}^m + V_{l-1}^k V_l^m).$$

By our notation (52), we have put $V_l^0 = 1(\forall l)$. Using this coefficient and (69), we now have

$$\mathcal{K}^{jkm} = (-1)^j \sum_l (V_l^{j'} - V_{l-1}^{j'})\chi_l^{km},$$

where $j' = 2(1)$ for $j = 1(2)$. On the other hand, by (72) we evaluate $\int v^k \partial_{x^j} f_\Delta \mathrm{d}^2 v$:

$$\mathcal{L}^{jkm} = -\delta_j m \sum_l L_{l-1}\left[\boldsymbol{\sigma}_l \cdot \partial_{x^j}\boldsymbol{V}_{l-1}\left(\frac{1}{3}V_{l-1}^k + \frac{1}{6}V_{l+1}^k\right) + \boldsymbol{\sigma}_l \cdot \partial_{x^j}\boldsymbol{V}_{l+1}\left(\frac{1}{3}V_{l+1}^k + \frac{1}{6}V_{l-1}^k\right)\right].$$

Combining there expressions, the Poisson operator is now given as

$$\begin{aligned}\mathcal{J}^{jkm} &= (-1)^j \sum_l \left(V_l^{j'} - V_{l-1}^{j'}\right)\chi_l^{km}\partial_{x^j} \\ &+ \delta_{jm}\sum_l L_l\Big[\boldsymbol{\sigma}_l \cdot \partial_{x^j}\boldsymbol{V}_{l-1}\left(\frac{1}{3}V_{l-1}^k + \frac{1}{6}V_{l+1}^k\right) \\ &+ \boldsymbol{\sigma}_l \cdot \partial_{x^j}\boldsymbol{V}_{l+1}\left(\frac{1}{3}V_{l+1}^k + \frac{1}{6}V_{l-1}^k\right)\Big].\end{aligned} \tag{75}$$

5. Building simulation code for the two-dimensionally expanded Water-bag model

5.1. two-dimensional Vlasov-Ampére system

In the previous section, we introduced the formulation of "mesa" model based on the literature[16]. In this section, we will actually build a numerical calculation scheme based on the contents. The Vlasov-Amp ére Hamiltonian is expressed as follows [17].

$$H = \frac{1}{2}\int_M |v|^2 f \mathrm{d}z + H(E, B). \tag{76}$$

What is important here is that the Hamiltonian is represented by the sum of the momentum term and the energy term of the electromagnetic field in the absence of the electromagnetic field. Therefore, it is possible to calculate the effect when the electromagnetic field does not exist and the effect of the electromagnetic field individually, and then superimpose them to calculate the time change of the correct velocity contour line $\boldsymbol{V}_l$.

5.2. The Hamiltonian without an electromagnetic field

The Hamiltonian of the two-dimensional Water-bag model in the absence of an electromagnetic field can be written as follows.

$$H_K(\boldsymbol{V}_l) = H(f)|_{\boldsymbol{v}=\boldsymbol{V}_l} = -\frac{1}{6}|\boldsymbol{V}_l|^3 \tag{77}$$

$$\partial_f H_K = -\partial_{\boldsymbol{V}_l} H_K(\boldsymbol{V}_l) = -\frac{1}{2}|\boldsymbol{V}_l|^2$$

Therefore, the time change of the velocity contour line can be written as follows.

$$\partial_t V_l^j = -|\boldsymbol{V}_l|\partial_x V_l^j \tag{78}$$

5.3. Contribution of electromagnetic field

Next, we consider the Hamiltonian electromagnetic field term. In this case, the calculation can be facilitated by changing the degree of freedom of $f$from the contour line $\boldsymbol{V}_l$ of the distribution function to $\boldsymbol{p}$ of Eq.(58). The electric and magnetic fields in this case are

$$E^j = \nabla A^0 + \partial_t A^j$$

$$\boldsymbol{B} = \nabla \times (A^1 \cdots A^n)^T$$

with the potential

$$\begin{cases} A^0 & = -(-\Box)^{-1}\frac{q}{\epsilon_0}p^0 \\ \\ A^j & = -(-\Box)^{-1}q\mu_0 p^j \quad (j = 1\cdots n) \end{cases}$$

where $(-\Box)^{-1}$ is the operator such that

$$-\Box(-\Box)^{-1}\boldsymbol{p}' = \boldsymbol{p}'.$$

where $-\Box = -\Delta + \frac{1}{c^2}\partial_t^2$ and $\boldsymbol{p}'$ is the vector such that

$$\begin{cases} p'^0 & = \frac{1}{\epsilon_0}p^0 \\ \\ p'^j & = \mu_0 p^j \quad (j = 1\cdots n). \end{cases}$$

We define the quadratic form

$$H_{EMF}(\boldsymbol{V}_l, t) = q^2\int_{-\infty}^{\infty}\langle -(-\Box)^{-1}\boldsymbol{p}', \boldsymbol{p}'\rangle\delta(t-t')\mathrm{d}t' = \int_X \frac{\epsilon_0|\boldsymbol{E}|^2 + \frac{|\boldsymbol{B}|^2}{\mu_0}}{2}\mathrm{d}^n x.$$

For Eq.(60), $\partial_f H$ can be represented by following $\partial_{\boldsymbol{p}} H_{EMF}$:

$$\begin{aligned} \partial_{p^j} H_{EMF} &= \begin{cases} \frac{1}{\epsilon_0}\partial_{p'^j} H_{EMF} \quad (j = 0) \\ \\ \mu_0 \partial_{p'^j} H_{EMF} \quad (j > 0) \end{cases} \\ \partial_{p'^j} H_{EMF} &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{q^2\langle -(-\Box)^{-1}(p'^j+\varepsilon\delta), p'^j+\varepsilon\delta\rangle - q^2\langle -(-\Box)^{-1}p'^j, p'^j\rangle}{\varepsilon\delta}\delta_t \mathrm{d}t \\ &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{q^2\int_M (A^j+\varepsilon G)(p'^j+\varepsilon\delta)\mathrm{d}\boldsymbol{z} - q^2\int_M A^j p'^j \mathrm{d}\boldsymbol{z}}{\varepsilon\delta}\mathrm{d}t \\ &= 2q^2\langle p'^j, G\rangle, \end{aligned} \tag{79}$$

where $\delta_t = \delta(t-t')$ and G is the Green function of $-\Box$ such that $-\Box G = \delta\delta_t$. With Eq.(60), the time change of $\boldsymbol{V}_l$ due to the effect of H_{EMF} is,

$$\begin{aligned} \partial_t V_l^k(x'^k) &= \int_X \delta(\boldsymbol{x}-\boldsymbol{x}')q^2\left(\langle p'^k, G(\boldsymbol{x}-\boldsymbol{x}'')\rangle + \langle A^k, \delta(\boldsymbol{x}-\boldsymbol{x}'')\rangle\right)V_l^k \mathrm{d}x''^k \\ &= \int_X \delta(\boldsymbol{x}-\boldsymbol{x}')q^2\left(\langle p'^k, G(\boldsymbol{x}-\boldsymbol{x}'')\rangle + \langle p'^k G(\boldsymbol{x}-\boldsymbol{x}''), \delta(\boldsymbol{x}-\boldsymbol{x}'')\rangle\right)V_l^k \mathrm{d}x''^k \end{aligned}$$

$$= 2\int_X \delta(\boldsymbol{x}-\boldsymbol{x}')q^2\langle p'^k, G(\boldsymbol{x}-\boldsymbol{x}'')\rangle V_l^k \mathrm{d}x''^k \tag{80}$$

6. Verification of simulation code for two-dimensionally extended Water-bag model

Time plots are made for the two Casimirs ($\bar{\rho} = \int_X \rho \mathrm{d}^2x$, which is the total number of particles, and $C = \int_X \omega^2/\rho \mathrm{d}^2x$, which is the enstrophy when ρ is a consistant (the flow is incompressible).) to verify the validity of the simulation code, showing that the changes are small enough. From Eq.(77), Hamiltonian can be divided into a component that depends on the electromagnetic field and a component that does not depend on the electromagnetic field. In this section, the simulation is created and verified under the condition that the Hamiltonian does not include the electric field term§.

6.1. Initial conditions of the distribution function

The initial conditions of the distribution function as

$$\boldsymbol{V}_l = R_{\theta(x^1,x^2,l)} \begin{pmatrix} 5 + 0.1sin(2\pi x^1 x^2) \\ 0 \end{pmatrix} \tag{81}$$

$$\theta(x,y,l) = \frac{\pi l}{n+1}(x^2+3)^{\left(\frac{1}{3}\right)}(x^1+3)^{\left(\frac{1}{2}\right)} \tag{82}$$

where R_θ is the rotation matrix with the angle of rotation θ. Simple demonstration of the distribution function in this initial condition is shown in Fig. 5.

§ In this simulation, I gave up electromagnetic field calculation because the calculation of Green's function $G(x^1, x^2, x''^1, x''^2, t, t'')$ requires a large storage area when sufficient calculation accuracy is guaranteed.

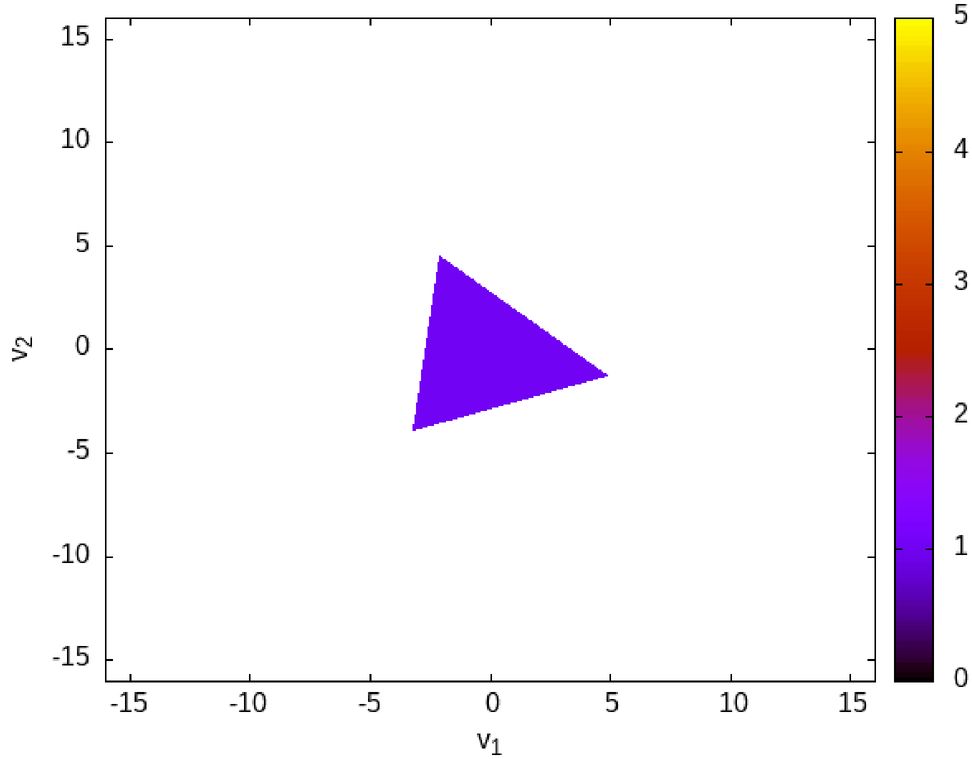

Figure 5. Initial conditions of two-dimensional simulation. The vertical axis represents the velocity v^1, and the horizontal axis represents the velocity v^2. The color bar shows the value of $f(\boldsymbol{x}, \boldsymbol{v}, t)|_{\boldsymbol{x}=(0,0)^T}$ and the white regions correspond to $f(\boldsymbol{x}, \boldsymbol{v}, t) = 0$.

6.2. Result

To verification, we show the time variation of $C, \bar{\rho}$, which are expected to be small, and indeed the results in Fig. 6 meet that expectation.

7. Conclusion

Caused by the limit of the state that the model can express, kinetic effects are usually discussed as physical phenomenon. However, they can be separated clearly into (i) the geometrical effect of (the degree of freedom of) the model and (ii) the effect of physical phenomena by expressing the former with the effect of the Poisson bracket $\{,\}$ and the latter with the effect of the Hamiltonian H. From the perspective of the degree of freedom, some observables can become Casimir invariants determined by the Poisson bracket of the model, which give us an index of model evaluation in another model that has more degrees of freedom.

Finally, we clarified the relationship between different levels of the hierarchy of the water-bag model—containing both the kinetic model and the fluid model—in terms of the degree of freedom. U_j is a Casimir invariant in the N-bag model but is not in the $N+1$-bag model, and the symmetry can be represented as whether one contour line can be expressed as a linear combination of the remaining contour lines in $k = i, i+\frac{1}{2}, i+1$ (when focusing on U_j). This means that such degeneracy of the degree of freedom in the $N+1$-bag model caused by the Hamiltonian constructs a leaf $U_j =$ constant, which corresponds to the space of the N-bag model.

Additionally, for the two-dimensionally extended Waterbag model, we were able to build a Casimir-preserving simulation of the fluid model (neglecting electromagnetic field) for 1-bag model (that is, with the same degree of freedom as the fluid model). Even when there are a plurality of waterbags, they can be similarly associated by the formulation shown in the reference [16].

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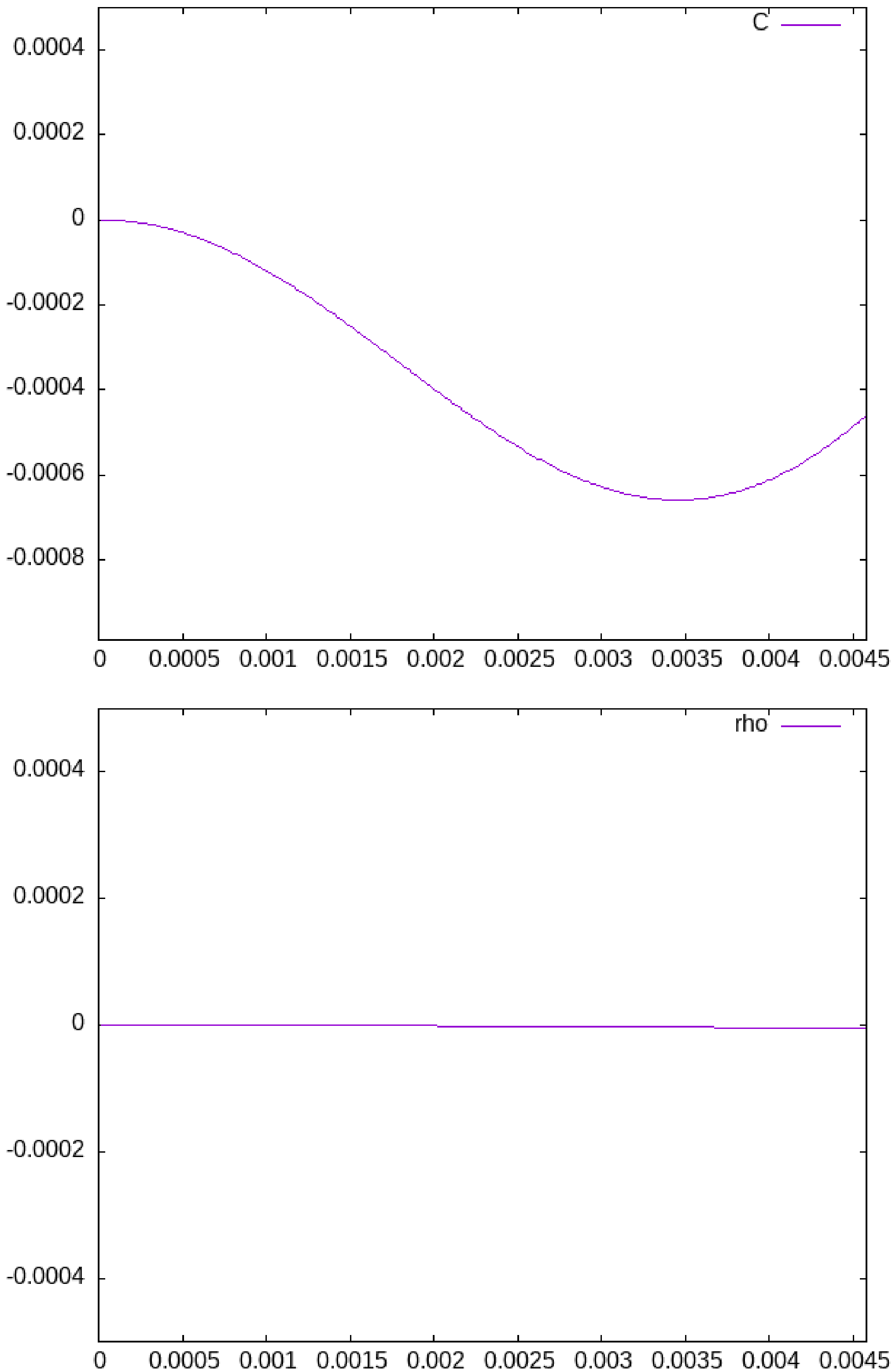

Figure 6. Horizontal axis represents time, vertical axis represents rate of change of C (upper figure) and $\bar{\rho}$ (upper figure) from its initial value. The conditions of this simulation do not include the effects of electromagnetic fields. The time variation of both variables is small enough.