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Superconducting Phase in the BCS Model with Imaginary Magnetic Field. III. Non-Vanishing Free Dispersion Relations

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Abstract. We analyze a class of the BCS model, whose free dispersion relation is non-vanishing, under the influence of imaginary magnetic field at positive temperature. The magnitude of the negative coupling constant must be small but is allowed to be independent of the temperature and the imaginary magnetic field. The infinite-volume limit of the free energy density is characterized. A spontaneous symmetry breaking and an off-diagonal long range order are proved to occur only in high temperatures. This is because the gap equation in this model has a positive solution only if the temperature is higher than a critical value. The proof is based on a double-scale integration of the Grassmann integral formulation. In this scheme we integrate with the infrared covariance first and with the ultra-violet covariance afterwards, which is opposite to the previous schemes in [Kashima, Y., J. Math. Sci. Univ. Tokyo **28** (2021), 1–179], [Kashima, Y., J. Math. Sci. Univ. Tokyo **28** (2021), 181–398] or [13], [14] in short. As the other focus, we study geometric properties of the phase boundaries, which are periodic copies of a closed curve in the two-dimensional space of the temperature and the real time variable. Here we adopt the real time variable in place of the temperature times the imaginary magnetic field by considering its relevance within contemporary physics of dynamical phase transition at positive temperature. As the main result, we show that for any choice of a non-vanishing free dispersion relation the representative curve of the phase boundaries has only one local minimum point, or in other words the phase boundaries do not oscillate with temperature, if and only if the minimum of the magnitude of the free dispersion relation over the maximum is larger than the critical value $\sqrt{17-12\sqrt{2}}$. Overall we use the same notational conventions as in [13], [14]. So this work is a continuation of these preceding papers.

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Contents

1. Introduction

1.1. Introductory remarks

Since the proposal in 1957 ([2]), the Bardeen-Cooper-Schrieffer (BCS) model of interacting electrons has been considered as a primal model to explain superconductivity from a microscopic principle. Apart from the conventional reduction of its quartic Fermionic interaction to a solvable quadratic one, we are still unable to make explicit the thermodynamic limit of the BCS model for full set of physical parameters. It is our longstanding desire to complete the rigorous derivation of the thermodynamics and acquire fully coherent applications of the BCS model.

It was shown in our previous works [13], [14] that the infinite-volume limit of the BCS model interacting with imaginary magnetic field can be rigorously derived. The main difference between these two constructions lies in properties of the free dispersion relation. In [13] we assumed the nearest-neighbor hopping and tuned the chemical potential in a way that the free Fermi surface does not degenerate. On the contrary, in [14] we considered a class of free dispersion relations which widely cover the ones with degenerate but not empty free Fermi surface. Our mission here is to achieve the same goal for non-vanishing free dispersion relations. We characterize the infinite-volume limit of the free energy density and the thermal expectation values of Cooper pair operators. The proof is based on a multi-scale analysis of the Grassmann integral formulation. As an illustration, let us summarize the applicability of the main theorems of this series to the typical free dispersion relation of nearest-neighbor hopping electron, $e(\mathbf{k}) = 2\sum_{j=1}^{d} \cos k_j - \mu : \mathbb{R}^d \to \mathbb{R}$, where d $(\in \mathbb{N})$ is the spatial dimension and μ $(\in \mathbb{R})$ is the chemical potential.

- [13, Theorem 1.3] applies to the case that d is arbitrary and $|\mu| < 2d$.
- [14, Theorem 1.3] applies to the case that $d \in \{3, 4\}$ and $|\mu| = 2d$.
- Theorem 1.3 of the present paper applies to the case that d is arbitrary and $|\mu| > 2d$.

Qualitative properties of the free dispersion relation around its zero points deeply affect the possible magnitude of interaction in this approach. Therefore, characteristics of each paper of this series can be explained in terms of dependency of the allowed magnitude of the coupling constant on the temperature and the imaginary magnetic field. In [13] the magnitude of the coupling constant must be smaller than some power of these parameters. Though the claimed dependency is most complicated in this series, we can actually choose the parameters so that they obey the necessary constraint and the gap equation has a positive solution at the same time. In [14] the magnitude of the coupling constant can be largely independent of the temperature and the imaginary magnetic field if the temperature is lower than a certain constant. As the result, we were able to prove phase transitions in arbitrarily small temperatures for a fixed coupling constant. In this paper the magnitude of negative coupling constant must be small but is independent of the temperature and the imaginary magnetic field. It turns out that the gap equation has a positive solution only if the temperature is higher than a critical value. Accordingly, the phase transitions characterized by spontaneous symmetry breaking (SSB) and off-diagonal long range order (ODLRO) are proved to occur in the high-temperature regions.

The gapped property of the free dispersion relation is one essential factor to make it possible to analyze the system independently of the temperature and the imaginary magnetic field. However, a direct combination of the non-vanishing free dispersion relation and the same strategy as the core part of the multi-scale integrations of [13], [14] does not lead to the desired result. We can see from the constraints on the coupling constant [13, (1.2)], [14, (1.18)] that the magnitude of the coupling constant must be arbitrarily small in high temperatures for some choices of the imaginary magnetic field in our previous constructions. The extra constraint in high temperatures stems from a determinant bound on the covariance of the last integration scale, which is tactically manipulated to be independent of (imaginary) time variables. This constraint remains regardless of the gapped property of the free dispersion relation as long as we follow the same strategy as in [13], [14].

Let us explain this issue more by using formulas in a simple way, as it also shows a novel aspect of the present construction. As usual, let β ($\in \mathbb{R}_{>0}$) denote the inverse temperature. Take an artificial parameter $h \in \frac{2}{\beta}\mathbb{N}$ and set

$$[0,\beta)_h := \left\{0, \frac{1}{h}, \frac{2}{h}, \cdots, \beta - \frac{1}{h}\right\},$$

which is a discrete analogue of the interval $[0,\beta)$. For a finite set S, which should be considered as a generalization of the product set of the spatial lattice points and the orbital index, let $C : (S\times[0,\beta)_h)^2 \to \mathbb{C}$ denote the full covariance of the Grassmann Gaussian integral formulation of our system. The main object to analyze is the Grassmann Gaussian integral

$$\int e^{V(\psi)} d\mu_C(\psi)$$

with a quartic Grassmann polynomial $V(\psi)$, which is as before a correction term left after extracting the reference Grassmann polynomial. The full covariance C can be decomposed as follows.

$$C(Xs,Yt) = e^{i\frac{\pi}{\beta}(s-t)}(C_0(Xs,Yt) + C_1(Xs,Yt)), \quad (\forall X,Y \in S,\ s,t \in [0,\beta)_h), \tag{1.1}$$

where the covariance $C_0 : (S \times [0,\beta)_h)^2 \to \mathbb{C}$ is in particular independent of the time variables.

$$C_0(Xs,Yt) = C_0(X0,Y0),\ (\forall X,Y \in S,\ s,t \in [0,\beta)_h).$$

In essence the Matsubara frequency is fixed to be π/β inside C_0 and C_1 sums over all the Matsubara frequencies but π/β. Due to the gapped property of the free dispersion relation and the partition of the Matsubara frequencies, the covariances C_0, C_1 satisfy the following bound properties.

$$
\begin{aligned}
&|\det(C_0(X_i s_i, Y_j t_j))_{1\le i,j\le n}| \le {c_{onst}}^n \beta^{-n},\\
&|\det(C_1(X_i s_i, Y_j t_j))_{1\le i,j\le n}| \le {c_{onst}}^n,\\
&(\forall n\in\mathbb{N},\ X_j, Y_j \in S,\ s_j, t_j \in [0,\beta)_h\ (j=1,\cdots,n)),\\
&\sup_{(Y,t)\in S\times[0,\beta)_h}\left(\frac{1}{h}\sum_{(X,s)\in S\times[0,\beta)_h}(|C_a(Xs,Yt)|+|C_a(Yt,Xs)|)\right)\le c_{onst},\\
&(\forall a\in\{0,1\}),
\end{aligned}
$$

where c_{onst} $(\in\mathbb{R}_{>0})$ is independent of β and the imaginary magnetic field, though it may depend on other parameters such as the spatial dimension or the minimum value of the magnitude of the free dispersion relation. By (1.1) and a gauge invariance we can transform as follows.

$$
\begin{aligned}
\int e^{V(\psi)}d\mu_C(\psi) &= \int\int e^{V(\psi^0+\psi^1)}d\mu_{C_0}(\psi^0)d\mu_{C_1}(\psi^1)\\
&= \int\int e^{V(\psi^0+\psi^1)}d\mu_{C_1}(\psi^1)d\mu_{C_0}(\psi^0).
\end{aligned}
$$

At this point we have two ways to proceed, either integrating with C_0 first or with C_1 first. Integrating with C_1 first is essentially the same strategy as in the previous papers and the determinant bound on C_0 remains to affect the possible magnitude of the coupling constant at the end. This is the reason why the coupling constant needed to be small even in high temperatures in [13], [14]. We can see from the β-dependent determinant bound on C_0 claimed above that this is not the way to achieve our goal. Interestingly we find that the determinant bound on C_0 does not affect the magnitude of the coupling constant at all if we integrate with C_0 first and make use of a vanishing property of the kernel function of $V(\psi)$. Since the other bounds on C_0, C_1 listed above are independent of β and the imaginary magnetic field, this way leads to the goal.

We can apply many of the general estimates established in [13], [14] and the Grassmann Gaussian integral formulation stated in [14] without any modification. At the same time we need some modified versions of the previous general estimates in order to implement the present double-scale integration scheme. However, the modification can be done in a systematic way so that it does not require a widespread reconstruction. Therefore, as far as it concerns the general estimation of the Grassmann integration, the present construction should not be longer than the previous ones. Moreover, the conclusive part of the derivation of the infinite-volume limit after building the general integration regime is essentially parallel to that of the previous papers. Not to disappoint the readers later, we should clearly mention at this stage that we will only explain which lemmas are necessary to complete the proof of each claim of the theorem in the final part of our construction (Subsection 3.4). On the other hand, estimation of the real covariance needs to be carefully performed so that it does not yield any extra dependency on the temperature and the imaginary magnetic field in the resulting theory. In particular the determinant bound on the ultra-violet covariance C_1 requires a complicated application of the useful general determinant bound by de Siqueira Pedra and Salmhofer [20, Theorem 1.3]. The parts making up the proof of the derivation of the infinite-volume limit are presented in the second half of the paper, namely Section 3.

As yet we cannot prove a superconducting order characterized by SSB and ODLRO by this method in the BCS model without imaginary magnetic field. In this approach we fail to take the coupling constant large enough to ensure the solvability of the gap equation without the imaginary magnetic field. The present class of free dispersion relations includes the non-zero constant ones, with which the Hamiltonian is called the strong coupling limit of the BCS model. We should remark that a totally different method based on characterization of equilibrium state on C^*-algebra applies to the strong coupling limit of the BCS model and proves SSB and ODLRO ([4]). However, the method is not known to be applicable to the BCS model with imaginary magnetic field, which is not hermitian, at present.

In the first half of the paper we analyze the free energy density, which is made explicit by the theorem proved in the second half of the paper, as a real-valued function of the temperature and the real time variable. Here let us introduce the free energy density at a formal level for illustrative purposes. The official definition will be given in the next subsection. Let H, S_z denote the BCS model Hamiltonian and the z-component of the spin operator respectively. For $\theta \in \mathbb{R}$ we consider the operator $\mathsf{H} + i\theta\mathsf{S}_z$ as the

BCS model interacting with the imaginary magnetic field. The infinite-volume limit of the free energy density is the following.

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left(-\frac{1}{\beta L^d}\log(\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)})\right),$$

where the parameter L ($\in \mathbb{N}$) controls the size of a d-dimensional spatial lattice. By admitting the explicit form of the limit we study regularity of the function

$$(1.2)\qquad (\beta,t)\mapsto \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left(-\frac{1}{\beta L^d}\log(\mathrm{Tr}\, e^{-\beta\mathsf{H}+it\mathsf{S}_z})\right):\mathbb{R}_{>0}\times\mathbb{R}\to\mathbb{R}$$

and geometric properties of the subset of $\mathbb{R}_{>0}\times\mathbb{R}$ where this function loses analyticity. The reason why we study the free energy density as a function of (β,t) rather than (β,θ) is that functions of the form

$$(1.3)\qquad (\beta,t)\mapsto \lim_{L\to\infty}\frac{1}{L^d}\log\left(\frac{\mathrm{Tr}\, e^{-\beta\mathsf{H}+it\mathsf{S}_z}}{\mathrm{Tr}\, e^{-\beta\mathsf{H}}}\right):\mathbb{R}_{>0}\times\mathbb{R}\to\mathbb{R}$$

are becoming relevant in contemporary physics of dynamical phase transition (DPT) at positive temperature ([3], [8], [1], [19], [18] and so on). In this context the function (1.3) is seen as a finite-temperature version of the infinite-volume limit of the overlap amplitude

$$\lim_{L\to\infty}\frac{1}{L^d}\log\langle\psi_0, e^{it\mathsf{S}_z}\psi_0\rangle,$$

where ψ_0 is a ground state of H. Since the function

$$\beta\mapsto\lim_{L\to\infty}\frac{1}{L^d}\log(\mathrm{Tr}\, e^{-\beta\mathsf{H}}):\mathbb{R}_{>0}\to\mathbb{R}$$

is real analytic in the weak coupling regime of this paper (see Proposition 2.5 (i)), the regularity of the function (1.3) is equivalent to that of (1.2). In fact the concept of dynamical quantum phase transition at zero temperature has become a notable topic of physics ([9], [7], [22]) and it recently reached a state of experimental confirmation (see e.g. [10], [21], [6]). As the term indicates, non-analyticity with the real time variable t defines an occurrence of DPT both at zero temperature and at positive temperature. DPTs at positive temperature have been shown in quantum many-body systems which can be mapped to Fermionic systems governed by quadratic Hamiltonians (see e.g. [3], [8]). To the author's knowledge, no rigorous result of DPT in the BCS model at positive temperature has been reported. In this situation we believe that we should push forward mathematical analysis of the function (1.2) for possible future physical applications.

It is advantageous that with the present class of free dispersion relations the characterization of the function (1.2) is justified for any $(\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R}$ as long as the coupling constant is fixed to be small. The contents of the first half of this paper, which is Section 2 plus Appendices A, B, are essentially independent of the second half. The readers who want to complete the proof of the characterization of the function (1.2) can read the second half first. We prove that the function is C^1-class in $\mathbb{R}_{>0} \times \mathbb{R}$ and the second order derivatives have jump discontinuity across a one-dimensional submanifold of $\mathbb{R}_{>0} \times \mathbb{R}$ which we call phase boundaries. Then we focus on describing geometric properties of the phase boundaries. We find that the phase boundaries consist of periodic copies of one closed curve (or more precisely periodic copies of the restriction of one closed curve in $\mathbb{R}^2$ to $\mathbb{R}_{>0} \times \mathbb{R}$) and the representative curve is axially symmetric with respect to the horizontal line $\{(\beta, 2\pi) \mid \beta \in \mathbb{R}_{>0}\}$. Therefore, letting β_c denote the critical inverse temperature, the problem is reduced to an analysis of graph of a function on $(0, \beta_c)$, which is the lower half of the representative curve. In particular we focus on determining when the function has only one local minimum point in $(0, \beta_c)$, or in other words, when the representative curve of the phase boundaries does not oscillate with temperature. It will turn out that answers to this question can be expressed in terms of the ratio of the maximum and the minimum of the magnitude of the free dispersion relation. The results are summarized in Theorem 2.19 as the second main result of this paper.

Overall we keep using the same notational conventions as in [13], [14]. We will often refer the readers to related parts of these papers for the meaning of notations rather than restating them. We provide a supplementary short list of notations which only contains new notations at the end of the paper. The readers should refer to the comprehensive lists presented at the end of [13], [14] for the other notations.

This paper is organized as follows. In the next subsection we state the theorem concerning the infinite-volume limit of the BCS model with imaginary magnetic field at positive temperature and outline the main results concerning the analysis of the free energy density and the phase boundaries. In Section 2 by admitting the explicit form of the free energy density we study its regularity and geometric properties of the phase boundaries. Moreover we analyze the phase boundaries for a couple of specific examples of the free Hamiltonian. In Section 3 we prove the theorem concerning the infinite-volume limit in the constructive manner. In Appendix A we prepare a lemma which is used to study the phase boundaries in Section 2. In Appendix B we give a formula of a definite integral which we need to analyze a specific model in Sub-subsection 2.3.2.

1.2. The main results

First let us state our main results on the derivation of the infinite-volume limit of the free energy density and the thermal expectation values. Let d ($\in \mathbb{N}$) denote the spatial dimension. Let $\{\mathbf{v}_j\}_{j=1}^d$, $\{\hat{\mathbf{v}}_j\}_{j=1}^d$ denote a basis of $\mathbb{R}^d$, its dual basis respectively. They satisfy that

$$\langle \mathbf{v}_i, \hat{\mathbf{v}}_j \rangle = \delta_{i,j}, \quad (\forall i, j \in \{1, \cdots, d\}),$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product of $\mathbb{R}^d$. With $L \in \mathbb{N}$ the spatial lattice Γ and the momentum lattice Γ^* are defined by

$$\Gamma := \left\{ \sum_{j=1}^{d} m_j \mathbf{v}_j \ \Big| \ m_j \in \{0, 1, \cdots, L-1\} \ (j = 1, \cdots, d) \right\},$$
$$\Gamma^* := \left\{ \sum_{j=1}^{d} \hat{m}_j \hat{\mathbf{v}}_j \ \Big| \ \hat{m}_j \in \left\{0, \frac{2\pi}{L}, \frac{4\pi}{L}, \cdots, 2\pi - \frac{2\pi}{L}\right\} \ (j = 1, \cdots, d) \right\}.$$

To formulate the infinite-volume limit of our interest, we use the infinite sets Γ_∞, Γ_∞^* defined by

$$\Gamma_\infty := \left\{ \sum_{j=1}^{d} m_j \mathbf{v}_j \ \Big| \ m_j \in \mathbb{Z} \ (j = 1, \cdots, d) \right\},$$
$$\Gamma_\infty^* := \left\{ \sum_{j=1}^{d} \hat{k}_j \hat{\mathbf{v}}_j \ \Big| \ \hat{k}_j \in [0, 2\pi] \ (j = 1, \cdots, d) \right\}.$$

Let us define a set of matrix-valued functions, which are one-particle Hamiltonians in momentum space. Using a function belonging to the set, we will define the free part of our Hamiltonian. For $b \in \mathbb{N}$ and $e_{min}, e_{max} \in \mathbb{R}_{>0}$ with $e_{min} \le e_{max}$ we define the subset $\mathcal{E}(e_{min}, e_{max})$ of $\mathrm{Map}(\mathbb{R}^d, \mathrm{Mat}(b, \mathbb{C}))$ as follows. E belongs to $\mathcal{E}(e_{min}, e_{max})$ if and only if

$$E \in C^\infty(\mathbb{R}^d, \mathrm{Mat}(b, \mathbb{C})),$$

$$E(\mathbf{k}) = E(\mathbf{k})^*, \quad (\forall \mathbf{k} \in \mathbb{R}^d), \tag{1.4}$$

$$E(\mathbf{k} + 2\pi \hat{\mathbf{v}}_j) = E(\mathbf{k}), \quad (\forall \mathbf{k} \in \mathbb{R}^d,\ j \in \{1, \cdots, d\}),$$

$$E(\mathbf{k}) = \overline{E(-\mathbf{k})}, \ (\forall \mathbf{k} \in \mathbb{R}^d), \tag{1.5}$$

$$\inf_{\mathbf{k} \in \mathbb{R}^d} \inf_{\substack{\mathbf{u} \in \mathbb{C}^b \\ \text{with } \|\mathbf{u}\|_{\mathbb{C}^b} = 1}} \|E(\mathbf{k})\mathbf{u}\|_{\mathbb{C}^b} = e_{min} (> 0), \tag{1.6}$$

$$\sup_{\mathbf{k} \in \mathbb{R}^d} \|E(\mathbf{k})\|_{b \times b} = e_{max}.$$

We remark that for $n \in \mathbb{N}$ $\mathrm{Mat}(n, \mathbb{C})$ denotes the set of $n \times n$ complex matrices, $\|\cdot\|_{n\times n}$ denotes the operator norm on $\mathrm{Mat}(n, \mathbb{C})$ and $\|\cdot\|_{\mathbb{C}^n}$ denotes the canonical norm of $\mathbb{C}^n$. Set $\mathcal{B} := \{1, 2, \cdots, b\}$. For $E \in \mathcal{E}(e_{min}, e_{max})$ we define the free Hamiltonian H_0 as follows.

$$\mathsf{H}_0 := \frac{1}{L^d} \sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y}) \in \mathcal{B}\times\Gamma} \sum_{\sigma \in \{\uparrow,\downarrow\}} \sum_{\mathbf{k} \in \Gamma^*} e^{i\langle \mathbf{x}-\mathbf{y}, \mathbf{k}\rangle} E(\mathbf{k})(\rho, \eta) \psi^*_{\rho\mathbf{x}\sigma} \psi_{\eta\mathbf{y}\sigma},$$

where $\psi_{\rho\mathbf{x}\sigma}$ ($\psi^*_{\rho\mathbf{x}\sigma}$) denotes the Fermionic annihilation (creation) operator for $(\rho, \mathbf{x}, \sigma) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}$. It follows from (1.4) that H_0 is a self-adjoint operator on the Fermionic Fock space $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$. With the negative coupling constant U ($\in \mathbb{R}_{<0}$) the interacting part V is defined by

$$\mathsf{V} := \frac{U}{L^d} \sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y}) \in \mathcal{B}\times\Gamma} \psi^*_{\rho\mathbf{x}\uparrow} \psi^*_{\rho\mathbf{x}\downarrow} \psi_{\eta\mathbf{y}\downarrow} \psi_{\eta\mathbf{y}\uparrow}.$$

The whole Hamiltonian H is then defined by $\mathsf{H} := \mathsf{H}_0 + \mathsf{V}$. As a common purpose of this series, we study the infinite-volume limit of the many-electron system governed by $\mathsf{H} + i\theta \mathsf{S}_z$ ($\theta \in \mathbb{R}$), where S_z is the z-component of the spin operator defined by

$$\mathsf{S}_z := \frac{1}{2} \sum_{(\rho,\mathbf{x}) \in \mathcal{B}\times\Gamma} (\psi^*_{\rho\mathbf{x}\uparrow} \psi_{\rho\mathbf{x}\uparrow} - \psi^*_{\rho\mathbf{x}\downarrow} \psi_{\rho\mathbf{x}\downarrow}).$$

To describe SSB, we need the symmetry breaking external field operator F defined by

$$\mathsf{F} := \gamma \sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma} (\psi^*_{\rho\mathbf{x}\uparrow}\psi^*_{\rho\mathbf{x}\downarrow} + \psi_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\uparrow}), \quad (\gamma \in \mathbb{R}).$$

One essential difference from the previous works [13], [14] is the solvability of the gap equation. Let us formulate the gap equation and see when it is solvable. Take $E \in \mathcal{E}(e_{min}, e_{max})$ and define the function $g_E : \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} &g_E(x,t,z) \\ &:= -\frac{2}{|U|} \\ &\quad + D_d \int_{\Gamma^*_\infty} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(x\sqrt{E(\mathbf{k})^2+z^2})}{(\cos(t/2)+\cosh(x\sqrt{E(\mathbf{k})^2+z^2}))\sqrt{E(\mathbf{k})^2+z^2}}\right), \end{aligned}$$

where

$$D_d := |\det(\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_d)|^{-1}(2\pi)^{-d}.$$

As in [14], throughout the paper we admit that for any function $f : \mathbb{R}\backslash\{0\} \to \mathbb{C}$ and $E \in \mathcal{E}(e_{min}, e_{max})$ the map $f(E(\cdot)) : \mathbb{R}^d \to \mathrm{Mat}(b, \mathbb{C})$ is defined via the spectral decomposition of $E(\mathbf{k})$ for each $\mathbf{k} \in \mathbb{R}^d$. We should remark that because of the property (1.6), $f(E(\mathbf{k}))$ is well-defined for any $\mathbf{k} \in \mathbb{R}^d$ even if $f(x)$ is not defined at $x = 0$. Our gap equation is to find $\Delta \in \mathbb{R}_{\geq 0}$ such that

$$g_E(\beta, \beta\theta, \Delta) = 0.$$

The following lemma can be proved by using the fact that for any $\varepsilon \in [-1, 1]$ the function

$$x \mapsto \frac{\sinh x}{(\varepsilon + \cosh x)x} : (0,\infty) \to \mathbb{R} \tag{1.7}$$

is strictly monotone decreasing in the same way as in the proof of [14, Lemma 1.2].

LEMMA 1.1. *The following statements hold for any $(\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}$. The equation $g_E(\beta, \beta\theta, \Delta) = 0$ has a solution Δ in $[0, \infty)$ if and only if $g_E(\beta, \beta\theta, 0) \geq 0$. Moreover, if a solution exists in $[0, \infty)$, it is unique.*

The next lemma tells us that if the interaction in the present model is weak, there is a critical temperature such that the gap equation has a positive solution if and only if the temperature is higher than the critical temperature.

LEMMA 1.2. *Assume that*

$$|U| < \frac{2e_{min}}{b}.$$

Then, there uniquely exists

$$\beta_c \in \left(0, \frac{2}{e_{min}} \tanh^{-1}\left(\frac{b|U|}{2e_{min}}\right)\right]$$

such that the following statements hold.

(i) For any $\beta \in \mathbb{R}_{>0}$ $g_E(\beta, \pi, 0) < 0$.

(ii) For any $\beta \in (0, \beta_c)$ $g_E(\beta, 2\pi, 0) > 0$ *and thus there exists* $\theta \in \mathbb{R}$ *such that the gap equation* $g_E(\beta, \beta\theta, \Delta) = 0$ *has a solution in* $(0, \infty)$.

(iii) $g_E(\beta_c, 2\pi, 0) = 0$ *and thus there exists* $\theta \in \mathbb{R}$ *such that* $g_E(\beta_c, \beta_c\theta, \Delta) = 0$ *has the solution* $\Delta = 0$.

(iv) For any $\beta \in (\beta_c, \infty)$ $g_E(\beta, 2\pi, 0) < 0$ *and thus for any* $\theta \in \mathbb{R}$ *the gap equation* $g_E(\beta, \beta\theta, \Delta) = 0$ *has no solution in* $[0, \infty)$.

PROOF. By the assumption, for any $\beta \in \mathbb{R}_{>0}$

$$g_E(\beta, \pi, 0) = -\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\tanh(\beta E(\mathbf{k}))}{E(\mathbf{k})}\right) \le -\frac{2}{|U|} + \frac{b}{e_{min}} < 0.$$

Thus (i) holds.

Observe that the function $\beta \mapsto g_E(\beta, 2\pi, 0) : \mathbb{R}_{>0} \to \mathbb{R}$ is monotone decreasing,

$$\lim_{\beta \searrow 0} g_E(\beta, 2\pi, 0) = \infty,$$

$$\lim_{\beta \nearrow \infty} g_E(\beta, 2\pi, 0) = -\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{1}{|E(\mathbf{k})|}\right) \le -\frac{2}{|U|} + \frac{b}{e_{min}} < 0.$$

Thus, there uniquely exists $\beta_c \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} &g_E(\beta, 2\pi, 0) > 0, \quad (\forall \beta \in (0, \beta_c)),\\ &g_E(\beta_c, 2\pi, 0) = 0,\\ &g_E(\beta, 2\pi, 0) < 0, \quad (\forall \beta \in (\beta_c, \infty)). \end{aligned}$$

Moreover,

$$0 = g_E(\beta_c, 2\pi, 0) \le -\frac{2}{|U|} + \frac{b}{\tanh(\beta_c e_{min}/2) e_{min}},$$

which implies that

$$\beta_c \le \frac{2}{e_{min}} \tanh^{-1}\left(\frac{b|U|}{2e_{min}}\right).$$

The claims (ii), (iii), (iv) follow from these properties. □

To shorten formulas, let us introduce the parameterized matrix-valued functions $G_{x,y,z} : \mathbb{R}^d \to \mathrm{Mat}(b, \mathbb{C})$ $((x, y, z) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R})$ by

$$G_{x,y,z}(\mathbf{k}) := \frac{\sinh(x\sqrt{E(\mathbf{k})^2 + z^2})}{(\cos(xy/2) + \cosh(x\sqrt{E(\mathbf{k})^2 + z^2}))\sqrt{E(\mathbf{k})^2 + z^2}}.$$

Also, for $E \in \mathcal{E}(e_{min}, e_{max})$ let us set

$$c_E := \sup_{\mathbf{k}\in\mathbb{R}^d} \sup_{\substack{m_j \in \mathbb{N}\cup\{0\} \\ (j=1,\cdots,d)}} \left\| \prod_{j=1}^{d} \frac{\partial^{m_j}}{\partial k_j^{m_j}} E(\mathbf{k}) \right\|_{b\times b} 1_{\sum_{j=1}^{d} m_j \le d+2}. \tag{1.8}$$

For any $\sum_{j=1}^d m_j \mathbf{v}_j \in \Gamma_\infty$ there uniquely exists $\sum_{j=1}^d m'_j \mathbf{v}_j \in \Gamma$ such that $m_j = m'_j \pmod L$ for any $j \in \{1, \cdots, d\}$. This rule defines the map $r_L : \Gamma_\infty \to \Gamma$. For any $(\rho, \mathbf{x}, \sigma) \in \mathcal{B} \times \Gamma_\infty \times \{\uparrow, \downarrow\}$ we identify $\psi^*_{\rho\mathbf{x}\sigma}$, $\psi_{\rho\mathbf{x}\sigma}$ with $\psi^*_{\rho r_L(\mathbf{x})\sigma}$, $\psi_{\rho r_L(\mathbf{x})\sigma}$ respectively. For clarity of the statements of the main results let us recall a few more notational rules. For a function $f : \Gamma_\infty \times \Gamma_\infty \to \mathbb{C}$ and $a \in \mathbb{C}$ we write $\lim_{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d}\to\infty} f(\mathbf{x}, \mathbf{y}) = a$ if for any $\varepsilon \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ such that for any $\mathbf{x}, \mathbf{y} \in \Gamma_\infty$ satisfying $\|\mathbf{x} - \mathbf{y}\|_{\mathbb{R}^d} > \delta$, $|f(\mathbf{x}, \mathbf{y}) - a| < \varepsilon$. Here $\| \cdot \|_{\mathbb{R}^d}$ denotes the Euclidean norm of $\mathbb{R}^d$.

THEOREM 1.3. *Let $E \in \mathcal{E}(e_{min}, e_{max})$. Let Δ $(\in \mathbb{R}_{\ge 0})$ be the solution of the gap equation $g_E(\beta, \beta\theta, \Delta) = 0$ if $g_E(\beta, \beta\theta, 0) \ge 0$. Let $\Delta := 0$ if $g_E(\beta, \beta\theta, 0) < 0$. Then, there exists $c' \in (0,1]$ depending only on $d, b, (\hat{\mathbf{v}}_j)_{j=1}^d, c_E$ such that the following statements hold for any*

$$U \in \left(-\frac{2c'}{b}\min\{e_{min}, e_{min}^{d+1}\}, 0\right),$$

$\beta \in \mathbb{R}_{>0}$, $\theta \in \mathbb{R}$.

(i) There exists $L_0 \in \mathbb{N}$ such that

$$\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})} \in \mathbb{R}_{>0}, \quad (\forall L \in \mathbb{N} \text{ with } L \ge L_0,\ \gamma \in [0,1]).$$

(ii)

$$\begin{aligned}
&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left(-\frac{1}{\beta L^d}\log(\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)})\right)\\
&= \frac{\Delta^2}{|U|} - \frac{D_d}{\beta}\int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\log\Bigg(2\cos\left(\frac{\beta\theta}{2}\right)e^{-\beta E(\mathbf{k})}\\
&\qquad\qquad + e^{\beta(\sqrt{E(\mathbf{k})^2+\Delta^2}-E(\mathbf{k}))} + e^{-\beta(\sqrt{E(\mathbf{k})^2+\Delta^2}+E(\mathbf{k}))}\Bigg).
\end{aligned}$$

(iii)

$$\begin{aligned}
&\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}\\
&= \lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}\\
&= -\frac{\Delta D_d}{2}\int_{\Gamma_\infty^*} d\mathbf{k} G_{\beta,\theta,\Delta}(\mathbf{k})(\hat{\rho},\hat{\rho}), \quad (\forall \hat{\rho} \in \mathcal{B},\ \hat{\mathbf{x}} \in \Gamma_\infty).
\end{aligned}$$

(iv) If $g_E(\beta, \beta\theta, 0) \ne 0$,

$$\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}$$

$$= \Delta^2 \prod_{\rho \in \{\hat{\rho},\hat{\eta}\}} \left(\frac{D_d}{2} \int_{\Gamma_\infty^*} d\mathbf{k} G_{\beta,\theta,\Delta}(\mathbf{k})(\rho,\rho) \right), \quad (\forall \hat{\rho}, \hat{\eta} \in \mathcal{B}).$$

If $g_E(\beta, \beta\theta, 0) = 0$,

$$\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty} \limsup_{\substack{L\to\infty \\ L\in\mathbb{N}}} \left| \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}} \right| = 0, \quad (\forall \hat{\rho}, \hat{\eta} \in \mathcal{B}).$$

(v)

$$\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \frac{1}{L^{2d}} \sum_{(\hat{\rho},\hat{\mathbf{x}}),(\hat{\eta},\hat{\mathbf{y}})\in\mathcal{B}\times\Gamma} \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}} = \frac{\Delta^2}{U^2}.$$

REMARK 1.4. We should emphasis that c' is independent of β, θ. Thus, once U is fixed, the infinite-volume limits are valid for all $(\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}$. This is a notable difference from [13, Theorem 1.3], [14, Theorem 1.3] where it is assumed that $\beta\theta/2 \notin \pi(2\mathbb{Z}+1)$ and U is not independent of (β, θ). Since

$$|U| < \frac{2c'}{b} \min\{e_{min}, e_{min}^{d+1}\} \le \frac{2e_{min}}{b},$$

Lemma 1.2 ensures that there exists $(\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that $g_E(\beta, \beta\theta, 0) > 0$ and $\Delta > 0$. Thus the claims (iii), (iv) in particular imply the existence of SSB, ODLRO respectively.

REMARK 1.5. The smoothness of $\mathbf{k} \mapsto E(\mathbf{k})$ is assumed only for simplicity. All the results in this paper can be reconstructed by assuming that $\mathbf{k} \mapsto E(\mathbf{k}) : \mathbb{R}^d \to \mathrm{Mat}(b, \mathbb{C})$ is continuously differentiable to some finite degree depending only on the spatial dimension. The symmetry (1.5) is assumed to adopt [14, Lemma 3.6] as our formulation. More precisely, we used the symmetry (1.5) to characterize the covariance "$C(\phi)$" in [14, Lemma 3.5 (ii)]. Since the Grassmann integral formulation [14, Lemma 3.6] contains the covariance "$C(\phi)$", accordingly we assume (1.5). The covariance "$C(\phi)$" will be explicitly written in Subsection 3.1 in the same form as in [14, Lemma 5.1], which was derived from [14, Lemma 3.5 (ii)]. However, the symmetry (1.5) itself plays no explicit role in this paper.

Remark 1.6. In [14, Corollary 1.11] we derived the zero-temperature limit of the free energy density and the thermal expectations. By arguing in parallel with the proof of [14, Corollary 1.11] presented at the end of [14, Subsection 5.2] we can derive the zero-temperature limit from Theorem 1.3. Not to lengthen the paper, let us state the results in an abbreviated form.

There exists $c'' \in (0,1]$ depending only on $d, b, (\hat{\mathbf{v}}_j)_{j=1}^d, c_E$ such that for any

$$U \in \left(-\frac{2c''}{b}\min\{e_{min}, e_{min}^{d+1}\}, 0\right)$$

and $\theta \in \mathbb{R}$ five claims which are same as the claims "(i), (ii), (iii), (iv), (v)" of [14, Corollary 1.11] without the constraint $\beta\theta/2 \notin \pi(2\mathbb{Z}+1)$ hold.

Here we can drop the constraint $\beta\theta/2 \notin \pi(2\mathbb{Z}+1)$ as we do not need it throughout this paper thanks to the assumption (1.6). After the inequality "(5.72)" in the proof of [14, Corollary 1.11] a spatial decay property of the infinite-volume, zero-temperature limit of the covariance was proved in order to study the zero-temperature limit of the 4-point correlation function. This part can be replaced by the decay property discussed in Remark 3.3 later. The property (1.6) also helps to shorten the derivation of the zero-temperature limit of the free energy density. Apart from these changes, the arguments close to the proof of [14, Corollary 1.11] yield the claims. Again the results imply no superconducting order in the zero-temperature limit. However, this time the results may not come as a surprise, since in low temperatures our gap equation has no solution at all as shown in Lemma 1.2 (iv).

Remark 1.7. Since we do not have any β-dependent constraint on U in Theorem 1.3, we can also study the infinite-temperature limit $\beta \searrow 0$ of the free energy density and the thermal expectations. If we set $\Delta \in \mathbb{R}_{\geq 0}$ by the same rule as in Theorem 1.3, it follows that for any $U \in \mathbb{R}_{<0}$, $\theta \in \mathbb{R}$ there exists $\beta'_c \in \mathbb{R}_{>0}$ such that $\Delta = 0$ for any $\beta \in (0, \beta'_c]$. This is because

$$\lim_{\beta \searrow 0} g_E(\beta, \beta\theta, 0) = -\frac{2}{|U|} < 0.$$

Let us take $U \in (-\frac{2c'}{b}\min\{e_{min}, e_{min}^{d+1}\}, 0)$ for the constant c' introduced in Theorem 1.3 and fix any $\theta \in \mathbb{R}$. Considering the above property of Δ, we can see from Theorem 1.3 (ii) that

$$\lim_{\beta\searrow 0}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left(-\frac{1}{\beta L^d}\log(\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)})\right)=-\infty.$$

Moreover, it is not difficult to modify the proof of [14, Corollary 1.11] to confirm that three claims which are same as the claims "(iii), (iv), (v)" of [14, Corollary 1.11] apart from having the notation $\lim_{\beta\searrow 0}$ in place of

$$\lim_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}}$$

hold. To prove the analogue of the claim "(iv)", we need a spatial decay property of the covariance in the limit $L\to\infty$, $\beta\searrow 0$ in particular. We can explicitly take the limit $L\to\infty$, $\beta\searrow 0$ in the characterization [14, Lemma 5.11] and observe that the covariance is in fact diagonal with the spatial variables in the limit. Again the results imply no superconducting order in the limit $\beta\searrow 0$.

Theorem 1.3 (ii) gives the exact formula for the function (1.2), provided $|U|$ is small as required in the theorem. Loss of analyticity of the function (1.2) with t is considered as an indication of DPT at positive temperature in contemporary physics (see e.g. [3], [8], [1]), though the function (1.2) with the BCS model has not been rigorously treated yet, to the author's knowledge. As one of the main themes of this paper, we focus on the following questions.

- At which $(\beta,t)\in\mathbb{R}_{>0}\times\mathbb{R}$ does the function (1.2) lose analyticity ?
- What is the regularity of the function (1.2) when it is not analytic ?
- What is the shape of the subset of $\mathbb{R}_{>0}\times\mathbb{R}$ where the function (1.2) is not analytic ?

We will study these questions in Section 2. Answers to the first and the second question can be found without much difficulty, since we have already studied similar questions in [14, Section 2]. After studying these two questions, we will know that the function (1.2) is C^1-class in $\mathbb{R}_{>0}\times\mathbb{R}$ and its 2nd order derivatives have jump discontinuities across a subset of $\mathbb{R}_{>0}\times\mathbb{R}$, which consists of periodic copies of one closed curve. To answer the last question, we need constructive arguments. It will turn out that the ratio e_{min}/e_{max} is the key parameter to classify the shape of the set of our interest. In particular we will show that the lower half of the representative curve of the set has only one local minimum point, in other words the representative curve does not oscillate with the temperature, for any $E \in \mathcal{E}(e_{min}, e_{max})$ if and only if e_{min}/e_{max} is larger than the critical value $\sqrt{17 - 12\sqrt{2}}$. The result will be officially stated in Theorem 2.19 as the second theorem of this paper.

2. Analysis of the Free Energy Density

We assume that $|U| < 2e_{min}/b$ throughout this section so that we can refer to the results of Lemma 1.2. Let $E \in \mathcal{E}(e_{min}, e_{max})$ and let us define the function $\Delta : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ as follows. Let $\Delta(\beta, t)$ be the solution of $g_E(\beta, t, \Delta) = 0$ if $g_E(\beta, t, 0) \geq 0$. Let $\Delta(\beta, t) := 0$ if $g_E(\beta, t, 0) < 0$. The well-definedness of the function $\Delta(\cdot)$ is guaranteed by Lemma 1.1. Then we define the function $F_E : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} &F_E(\beta, t) \\ &:= \frac{\Delta(\beta, t)^2}{|U|} - \frac{D_d}{\beta} \int_{\Gamma_\infty^*} d\mathbf{k} \operatorname{Tr} \log \Bigg(2\cos\left(\frac{t}{2}\right) e^{-\beta E(\mathbf{k})} \\ &\qquad + e^{\beta(\sqrt{E(\mathbf{k})^2 + \Delta(\beta,t)^2} - E(\mathbf{k}))} + e^{-\beta(\sqrt{E(\mathbf{k})^2 + \Delta(\beta,t)^2} + E(\mathbf{k}))} \Bigg). \end{aligned}$$

It follows from Theorem 1.3 (ii) that if $U \in (-\frac{2c'}{b} \min\{e_{min}, e_{min}^{d+1}\}, 0)$,

$$F_E(\beta, t) = \lim_{\substack{L \to \infty \\ L \in \mathbb{N}}} \left(-\frac{1}{\beta L^d} \log(\operatorname{Tr} e^{-\beta \mathsf{H} + it\mathsf{S}_z}) \right), \quad (\forall (\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R}).$$

Thus the function $F_E(\beta, t)$ can be seen as an extension of the free energy density with respect to the magnitude of the coupling constant. In this section we study the regularity of F_E with (β, t) and characterize the subset of $\mathbb{R}_{>0} \times \mathbb{R}$ where the analyticity is lost. The contents of this section are independent of Section 3, which is devoted to proving Theorem 1.3. The readers can read this section separately from Section 3.

2.1. Phase transitions

The domain $\mathbb{R}_{>0} \times \mathbb{R}$ can be decomposed as follows. $\mathbb{R}_{>0} \times \mathbb{R} = Q_+ \sqcup Q_- \sqcup Q_0$, where

$$\begin{aligned}
Q_+ &:= \{(\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R} \mid g_E(\beta, t, 0) > 0\}, \\
Q_- &:= \{(\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R} \mid g_E(\beta, t, 0) < 0\}, \\
Q_0 &:= \{(\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R} \mid g_E(\beta, t, 0) = 0\}.
\end{aligned}$$

In this subsection we will prove that the function F_E is C^1-class in $\mathbb{R}_{>0} \times \mathbb{R}$, real analytic in $Q_+ \cup Q_-$ and non-analytic at any point of Q_0 as a function of two variables. More specifically, we will prove that 2nd order derivatives of F_E have jump discontinuity across Q_0, which is a sign of 2nd order phase transition. Also, we will see that Q_0 consists of periodic copies of a restriction of a closed curve in $\mathbb{R}^2$. Let us call the curves making up Q_0 phase boundaries. In fact the regularity of F_E can be studied in a way similar to [14, Section 2]. However, we decide not to omit it, since it characterizes the nature of the phase transitions.

Let us start by describing universal properties of the phase boundaries, which hold regardless of e_{min}, $e_{max} (\in \mathbb{R}_{>0})$. We can deduce from Lemma 1.2 (i),(ii) that for any $\beta \in (0, \beta_c)$ there uniquely exists $\tau(\beta) \in (\pi, 2\pi)$ such that $g_E(\beta, \tau(\beta), 0) = 0$. This rule defines the function $\tau : (0, \beta_c) \to (\pi, 2\pi)$. The following lemma means little at this point. However, it will support conclusive parts of our construction later, or more specifically the proofs of Proposition 2.13 and Proposition 2.23. Also, it will implicitly support the proof of Proposition 2.16.

LEMMA 2.1. *Assume that $|U| < 2e_{min}/b$, $y \in (-1, 0)$, $\beta \in \mathbb{R}_{>0}$, $E \in \mathcal{E}(e_{min}, e_{max})$ and*

$$-\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(\beta E(\mathbf{k}))}{(y + \cosh(\beta E(\mathbf{k})))E(\mathbf{k})} \right) = 0.$$

Then $\beta \in (0, \beta_c)$ and $y = \cos(\tau(\beta)/2)$.

Basic properties of the function $\tau(\cdot)$ are summarized as follows. For an open set O of $\mathbb{R}^n$ let $C^\omega(O)$ denote the set of real analytic functions on O.

LEMMA 2.2.

(i)

$$\tau \in C^{\omega}((0,\beta_c)).$$

(ii)

$$\lim_{\beta \nearrow \beta_c} \tau(\beta) = \lim_{\beta \searrow 0} \tau(\beta) = 2\pi.$$

(iii)

$$\lim_{\beta \nearrow \beta_c} \frac{d\tau}{d\beta}(\beta) = +\infty, \quad \lim_{\beta \searrow 0} \frac{d\tau}{d\beta}(\beta) = -\infty.$$

REMARK 2.3. In the proofs of Lemma 2.2, Proposition 2.4, Proposition 2.5 and Proposition 2.10 we will apply the implicit function theorem, the inverse function theorem and the identity theorem for real analytic functions. These theorems are found in e.g. [15, Chapter 1, Chapter 2].

PROOF OF LEMMA 2.2. (i): One can see from the definition that the function $(x,t) \mapsto g_E(x,t,0) : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ is real analytic. Since $\frac{\partial g_E}{\partial t}(\beta,\tau(\beta),0) \neq 0$ for all $\beta \in (0,\beta_c)$, the analytic implicit function theorem ensures the claim.

(ii): Suppose that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that for any $\delta \in \mathbb{R}_{>0}$ there exists $\beta_\delta \in (\beta_c - \delta, \beta_c) \cap (0,\beta_c)$ such that $\tau(\beta_\delta) \leq 2\pi - \varepsilon$. Then for any $\delta \in \mathbb{R}_{>0}$

$$0 = g_E(\beta_\delta, \tau(\beta_\delta), 0) \leq g_E(\beta_\delta, 2\pi - \varepsilon, 0) \leq \sup_{\beta \in (\beta_c - \delta, \beta_c)} g_E(\beta, 2\pi - \varepsilon, 0).$$

By sending $\delta \searrow 0$, $0 \leq g_E(\beta_c, 2\pi - \varepsilon, 0) < g_E(\beta_c, 2\pi, 0) = 0$, which is a contradiction. Thus $\lim_{\beta \nearrow \beta_c} \tau(\beta) = 2\pi$.

Suppose that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that for any $\delta \in \mathbb{R}_{>0}$ there exists $\beta_\delta \in (0,\delta) \cap (0,\beta_c)$ such that $\tau(\beta_\delta) \leq 2\pi - \varepsilon$. Then for any $\delta \in \mathbb{R}_{>0}$

$$0 = g_E(\beta_\delta, \tau(\beta_\delta), 0) \leq \sup_{\beta \in (0,\delta)} g_E(\beta, 2\pi - \varepsilon, 0).$$

By sending $\delta \searrow 0$, $0 \leq g_E(0, 2\pi - \varepsilon, 0) = -2/|U| < 0$, which is a contradiction. Thus $\lim_{\beta \searrow 0} \tau(\beta) = 2\pi$.

(iii): For $\beta \in (0, \beta_c)$

$$\frac{d\tau}{d\beta}(\beta) = -\frac{\frac{\partial g_E}{\partial x}(\beta, \tau(\beta), 0)}{\frac{\partial g_E}{\partial t}(\beta, \tau(\beta), 0)} \tag{2.1}$$
$$= -\frac{2\int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{1+\cos(\tau(\beta)/2)\cosh(\beta E(\mathbf{k}))}{(\cos(\tau(\beta)/2)+\cosh(\beta E(\mathbf{k})))^2}\right)}{\sin\left(\frac{\tau(\beta)}{2}\right)\int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(\beta E(\mathbf{k}))}{(\cos(\tau(\beta)/2)+\cosh(\beta E(\mathbf{k})))^2 E(\mathbf{k})}\right)}.$$

Then by using the result of (ii),

$$\lim_{\beta \nearrow \beta_c} \frac{d\tau}{d\beta}(\beta) = \lim_{\beta \nearrow \beta_c} \frac{2\int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{1}{\cosh(\beta_c E(\mathbf{k}))-1}\right)}{\sin\left(\frac{\tau(\beta)}{2}\right)\int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(\beta_c E(\mathbf{k}))}{(\cosh(\beta_c E(\mathbf{k}))-1)^2 E(\mathbf{k})}\right)} = \infty.$$

To study the limit $\lim_{\beta \searrow 0} \frac{d\tau}{d\beta}(\beta)$, let us show that

$$\lim_{\beta \searrow 0} \frac{\cos(\tau(\beta)/2)+1}{\beta} = \frac{b|U|}{2}. \tag{2.2}$$

Suppose that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$\sup_{\beta \in (0,\delta)} \frac{\cos(\tau(\beta)/2)+1}{\beta} \ge \frac{b|U|}{2} + \varepsilon, \quad (\forall \delta \in (0, \beta_c)).$$

Take any $\delta \in (0, \beta_c)$. Then there exists $\beta_\delta \in (0, \delta)$ such that

$$\frac{\cos(\tau(\beta_\delta)/2)+1}{\beta_\delta} \ge \frac{b|U|}{2} + \frac{\varepsilon}{2},$$

and thus

$$\frac{2}{|U|} \le D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(\beta_\delta E(\mathbf{k}))}{\left(\frac{b|U|}{2} + \frac{\varepsilon}{2} + \frac{\cosh(\beta_\delta E(\mathbf{k}))-1}{\beta_\delta}\right)\beta_\delta E(\mathbf{k})}\right)$$
$$\le \frac{2}{b|U|+\varepsilon} D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(\delta E(\mathbf{k}))}{\delta E(\mathbf{k})}\right).$$

By sending $\delta \searrow 0$, $\frac{2}{|U|} \le \frac{2b}{b|U|+\varepsilon} < \frac{2}{|U|}$, which is a contradiction. Thus for any $\varepsilon \in \mathbb{R}_{>0}$ there exists $\delta \in (0, \beta_c)$ such that

$$\sup_{\beta \in (0,\delta)} \frac{\cos(\tau(\beta)/2)+1}{\beta} < \frac{b|U|}{2} + \varepsilon,$$

which implies that

$$\limsup_{\beta\searrow 0}\frac{\cos(\tau(\beta)/2)+1}{\beta}\le\frac{b|U|}{2}.$$

On the other hand, suppose that there exists $\varepsilon\in\mathbb{R}_{>0}$ such that

$$\inf_{\beta\in(0,\delta)}\frac{\cos(\tau(\beta)/2)+1}{\beta}\le\frac{b|U|}{2}-\varepsilon,\quad (\forall\delta\in(0,\beta_c)).$$

Take any $\delta\in(0,\beta_c)$. Then there exists $\beta_\delta\in(0,\delta)$ such that

$$\frac{\cos(\tau(\beta_\delta)/2)+1}{\beta_\delta}\le\frac{b|U|}{2}-\frac{\varepsilon}{2},$$

and thus

$$\begin{aligned}\frac{2}{|U|}&\ge D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(\beta_\delta E(\mathbf{k}))}{\left(\frac{b|U|}{2}-\frac{\varepsilon}{2}+\frac{\cosh(\beta_\delta E(\mathbf{k}))-1}{\beta_\delta}\right)\beta_\delta E(\mathbf{k})}\right)\\&\ge D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{\frac{b|U|}{2}-\frac{\varepsilon}{2}+\frac{\cosh(\delta E(\mathbf{k}))-1}{\delta}}\right).\end{aligned}$$

By sending $\delta\searrow 0$, $\frac{2}{|U|}\ge\frac{2b}{b|U|-\varepsilon}>\frac{2}{|U|}$, which is a contradiction. Thus for any $\varepsilon\in\mathbb{R}_{>0}$ there exists $\delta\in(0,\beta_c)$ such that

$$\inf_{\beta\in(0,\delta)}\frac{\cos(\tau(\beta)/2)+1}{\beta}>\frac{b|U|}{2}-\varepsilon,$$

which implies that

$$\liminf_{\beta\searrow 0}\frac{\cos(\tau(\beta)/2)+1}{\beta}\ge\frac{b|U|}{2}.$$

Therefore, the property (2.2) follows.

By applying (2.2) we can derive that

$$\lim_{\beta\searrow 0}\frac{\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{1+\cos(\tau(\beta)/2)\cosh(\beta E(\mathbf{k}))}{(\cos(\tau(\beta)/2)+\cosh(\beta E(\mathbf{k})))^2}\right)}{\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(\beta E(\mathbf{k}))}{(\cos(\tau(\beta)/2)+\cosh(\beta E(\mathbf{k})))^2E(\mathbf{k})}\right)}=\frac{b|U|}{2}.$$

By combining this with (2.1) and the result of (ii) we can deduce the claim on the limit $\lim_{\beta\searrow 0}\frac{d\tau}{d\beta}(\beta)$. □

By parity, periodicity and Lemma 1.2 the set Q_0 is characterized as follows.

$$\begin{aligned}(2.3)\qquad Q_0 =&\{(\beta,\delta\tau(\beta)+4\pi m)\mid \beta\in(0,\beta_c),\ \delta\in\{1,-1\},\ m\in\mathbb{Z}\}\\ &\cup\{(\beta_c,2\pi+4\pi m)\mid m\in\mathbb{Z}\}.\end{aligned}$$

Set

$$\widehat{Q}_0 := \{(\beta,\tau(\beta)),\ (\beta,4\pi-\tau(\beta))\mid \beta\in(0,\beta_c)\}\cup\{(0,2\pi),\ (\beta_c,2\pi)\},$$

which is a closed curve in $\mathbb{R}^2$ by Lemma 2.2 (ii). We can see that Q_0 consists of periodic copies of $\widehat{Q}_0\cap(\mathbb{R}_{>0}\times\mathbb{R})$. This fact motivates us to study the curve $\widehat{Q}_0$ as the representative of the phase boundaries.

PROPOSITION 2.4. *$\widehat{Q}_0$ is a 1-dimensional real analytic submanifold of $\mathbb{R}^2$.*

PROOF. By Lemma 2.2 (i) the maps

$$\begin{aligned}&\beta\mapsto(\beta,\tau(\beta)):(0,\beta_c)\to\{(\beta,\tau(\beta))\mid\beta\in(0,\beta_c)\},\\ &\beta\mapsto(\beta,4\pi-\tau(\beta)):(0,\beta_c)\to\{(\beta,4\pi-\tau(\beta))\mid\beta\in(0,\beta_c)\}\end{aligned}$$

are real analytic homeomorphism. Thus it suffices to prove that there exist open intervals I_1, I_2, an open neighborhood U_1 of $(0,2\pi)$, an open neighborhood U_2 of $(\beta_c,2\pi)$ in $\mathbb{R}^2$ and real analytic homeomorphisms $f_j:I_j\to U_j\cap\widehat{Q}_0$ $(j=1,2)$. We can see that $\frac{\partial g_E}{\partial x}(\beta_c,2\pi,0)<0$. Thus the analytic implicit function theorem ensures that there exists $\varepsilon_1\in(0,\pi)$ and $\hat{f}\in C^{\omega}((2\pi-\varepsilon_1,2\pi+\varepsilon_1))$ such that $\hat{f}(2\pi)=\beta_c$, $\hat{f}(t)>0$ and $g_E(\hat{f}(t),t,0)=0$ for any $t\in(2\pi-\varepsilon_1,2\pi+\varepsilon_1)$. Thus, $(\beta_c,2\pi)\in\{(\hat{f}(t),t)\mid t\in(2\pi-\varepsilon_1,2\pi+\varepsilon_1)\}\subset\widehat{Q}_0$. Since $\widehat{Q}_0$ is symmetric with respect to the line $\{(x,2\pi)\mid x\in\mathbb{R}\}$, there exists $\varepsilon_2\in\mathbb{R}_{>0}$ such that

$$\begin{aligned}&\{(\hat{f}(t),t)\mid t\in(2\pi-\varepsilon_1,2\pi+\varepsilon_1)\}\\ &=(\beta_c-\varepsilon_2,\beta_c+\varepsilon_2)\times(2\pi-\varepsilon_1,2\pi+\varepsilon_1)\cap\widehat{Q}_0.\end{aligned}$$

If we define the map $f_2 : (2\pi - \varepsilon_1, 2\pi + \varepsilon_1) \to (\beta_c - \varepsilon_2, \beta_c + \varepsilon_2) \times (2\pi - \varepsilon_1, 2\pi + \varepsilon_1) \cap \widehat{Q}_0$ by $f_2(t) := (\hat{f}(t), t)$, we see that the claim on $(\beta_c, 2\pi)$ holds.

Let us prove the claim on $(0, 2\pi)$. Observe that there exist ε_3, $\varepsilon_4 \in \mathbb{R}_{>0}$ such that the function

$$(x, y) \mapsto -\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(xE(\mathbf{k}))}{(y + \frac{\cosh(xE(\mathbf{k}))-1}{x})xE(\mathbf{k})} \right)$$

is real analytic in $(-\varepsilon_3, \varepsilon_3) \times (b|U|/2 - \varepsilon_4, b|U|/2 + \varepsilon_4)$. Let $\phi(x, y)$ denote this function. We can check that $\phi(0, b|U|/2) = 0$ and $\frac{\partial \phi}{\partial y}(0, b|U|/2) < 0$. Thus by the analytic implicit function theorem there exist $\varepsilon_5 \in (0, \varepsilon_3)$ and a real analytic function $\eta : (-\varepsilon_5, \varepsilon_5) \to \mathbb{R}_{>0}$ such that $\eta(0) = b|U|/2$, $\phi(x, \eta(x)) = 0$, $(\forall x \in (-\varepsilon_5, \varepsilon_5))$. Then let us define the function $\xi : (-\varepsilon_5, \varepsilon_5) \to \mathbb{R}$ by $\xi(x) := x\eta(x) - 1$. It follows that $\xi \in C^\omega((-\varepsilon_5, \varepsilon_5))$, $\xi(0) = -1$, $\frac{d\xi}{dx}(0) = b|U|/2 > 0$. Thus there exists $\varepsilon_6 \in (0, \varepsilon_5)$ such that $\xi(\cdot)$ is strictly monotone increasing in $(-\varepsilon_6, \varepsilon_6)$ and

$$-\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(xE(\mathbf{k}))}{(\xi(x) + \cosh(xE(\mathbf{k})))E(\mathbf{k})} \right) = 0,$$
$$(\forall x \in (-\varepsilon_6, \varepsilon_6) \backslash \{0\}).$$

Then by the inverse function theorem there exist $\varepsilon_7 \in \mathbb{R}_{>0}$ and a real analytic function $\lambda : (-1 - \varepsilon_7, -1 + \varepsilon_7) \to (-\varepsilon_6, \varepsilon_6)$ such that $\lambda(\cdot)$ is strictly monotone increasing, $\lambda(-1) = 0$, $\xi(\lambda(y)) = y$, $(\forall y \in (-1 - \varepsilon_7, -1 + \varepsilon_7))$. It follows that

$$-\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(\lambda(y)E(\mathbf{k}))}{(y + \cosh(\lambda(y)E(\mathbf{k})))E(\mathbf{k})} \right) = 0,$$
$$(\forall y \in (-1 - \varepsilon_7, -1 + \varepsilon_7) \backslash \{-1\}).$$

We can take $\varepsilon_8 \in (0, \pi)$ so that $\cos(t/2) \in [-1, -1 + \varepsilon_7)$, $(\forall t \in (2\pi - \varepsilon_8, 2\pi + \varepsilon_8))$. Let us define the function $\nu : (2\pi - \varepsilon_8, 2\pi + \varepsilon_8) \to \mathbb{R}$ by $\nu(t) := \lambda(\cos(t/2))$. Observe that $\nu \in C^\omega((2\pi - \varepsilon_8, 2\pi + \varepsilon_8))$, $\nu(2\pi) = 0$, $\nu(t) > 0$, $(\forall t \in (2\pi - \varepsilon_8, 2\pi + \varepsilon_8) \backslash \{2\pi\})$, $g_E(\nu(t), t, 0) = 0$, $(\forall t \in (2\pi - \varepsilon_8, 2\pi + \varepsilon_8) \backslash \{2\pi\})$. Thus $(0, 2\pi) \in \{(\nu(t), t) \mid t \in (2\pi - \varepsilon_8, 2\pi + \varepsilon_8)\} \subset \widehat{Q}_0$. Since $\widehat{Q}_0$ is symmetric with respect to the line $\{(x, 2\pi) \mid x \in x \in \mathbb{R}\}$, there exists $\varepsilon_9 \in \mathbb{R}_{>0}$ such that

$$\{(\nu(t), t) \mid t \in (2\pi - \varepsilon_8, 2\pi + \varepsilon_8)\} = (-\varepsilon_9, \varepsilon_9) \times (2\pi - \varepsilon_8, 2\pi + \varepsilon_8) \cap \widehat{Q}_0.$$

We can define the map $f_1 : (2\pi - \varepsilon_8, 2\pi + \varepsilon_8) \to (-\varepsilon_9, \varepsilon_9) \times (2\pi - \varepsilon_8, 2\pi + \varepsilon_8) \cap \widehat{Q}_0$ by $f_1(t) := (\nu(t), t)$ so that the claim on $(0, 2\pi)$ holds as well. The proof is now complete. □

By taking into account the definition of the function $\Delta(\cdot)$, Lemma 2.2 and Proposition 2.4 we can schematically draw a $\beta - t$ phase diagram restricted within the plane $\mathbb{R}_{>0} \times (0, 4\pi)$ as in Figure 1.

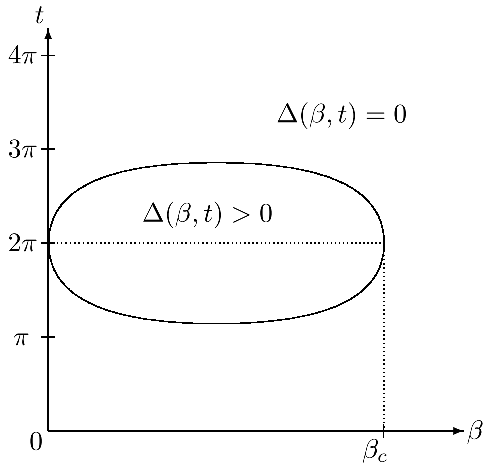

Fig. 1. The schematic $\beta - t$ phase diagram restricted within $\mathbb{R}_{>0} \times (0, 4\pi)$. The curve corresponds to $\widehat{Q}_0$.

Next let us study the regularity of $F_E(\cdot, \cdot)$. In particular let us show non-analyticity of $F_E(\cdot, \cdot)$ on Q_0.

PROPOSITION 2.5. *The following statements hold.*

(i)

$$F_E|_{Q_+ \cup Q_-} \in C^\omega(Q_+ \cup Q_-), \quad F_E \in C^1(\mathbb{R}_{>0} \times \mathbb{R}).$$

(ii) For any $(\beta_0, t_0) \in Q_0$ $\lim_{(\beta,t)\to(\beta_0,t_0),(\beta,t)\in Q_+} \frac{\partial^2 F_E}{\partial \beta^2}(\beta, t)$, $\lim_{(\beta,t)\to(\beta_0,t_0),(\beta,t)\in Q_-} \frac{\partial^2 F_E}{\partial \beta^2}(\beta, t)$ *converge to finite values. Moreover,*

if $\beta_0 \in (0, \beta_c)$ *and* $\frac{d\tau}{d\beta}(\beta_0) \neq 0$ *or* $\beta_0 = \beta_c$,

$$\lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\(\beta,t)\in Q_+}} \frac{\partial^2 F_E}{\partial \beta^2}(\beta,t) < \lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\(\beta,t)\in Q_-}} \frac{\partial^2 F_E}{\partial \beta^2}(\beta,t).$$

If $\beta_0 \in (0, \beta_c)$ *and* $\frac{d\tau}{d\beta}(\beta_0) = 0$,

$$\lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\(\beta,t)\in Q_+}} \frac{\partial^2 F_E}{\partial \beta^2}(\beta,t) = \lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\(\beta,t)\in Q_-}} \frac{\partial^2 F_E}{\partial \beta^2}(\beta,t).$$

(iii) For any $(\beta_0, t_0) \in Q_0$ $\lim_{(\beta,t)\to(\beta_0,t_0),(\beta,t)\in Q_+} \frac{\partial^2 F_E}{\partial t^2}(\beta,t)$, $\lim_{(\beta,t)\to(\beta_0,t_0),(\beta,t)\in Q_-} \frac{\partial^2 F_E}{\partial t^2}(\beta,t)$ *converge to finite values. Moreover, if* $\beta_0 \in (0, \beta_c)$,

$$\lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\(\beta,t)\in Q_+}} \frac{\partial^2 F_E}{\partial t^2}(\beta,t) < \lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\(\beta,t)\in Q_-}} \frac{\partial^2 F_E}{\partial t^2}(\beta,t).$$

If $\beta_0 = \beta_c$,

$$\lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\(\beta,t)\in Q_+}} \frac{\partial^2 F_E}{\partial t^2}(\beta,t) = \lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\(\beta,t)\in Q_-}} \frac{\partial^2 F_E}{\partial t^2}(\beta,t).$$

PROOF. The claims can be proved in a way similar to the proofs of "Lemma 2.2", "Proposition 2.6" of [14]. However, we do not significantly skip the explanations for the readers' convenience.

(i): By using the fact that for $\varepsilon \in [-1, 1]$ the function (1.7) is strictly monotone decreasing we can check that $\frac{\partial g_E}{\partial z}(\beta, t, \Delta(\beta, t)) < 0$ for any $(\beta, t) \in Q_+$. Thus by the analytic implicit function theorem $\Delta|_{Q_+} \in C^\omega(Q_+)$. Since $\Delta|_{Q_-} \in C^\omega(Q_-)$ trivially, $\Delta|_{Q_+ \cup Q_-} \in C^\omega(Q_+ \cup Q_-)$. Let us prove that $\Delta \in C(\mathbb{R}_{>0} \times \mathbb{R})$. Let $(\beta_0, t_0) \in Q_0$. Suppose that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that for any $\delta \in \mathbb{R}_{>0}$ there exists $(\beta_\delta, t_\delta) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that $\|(\beta_0, t_0) - (\beta_\delta, t_\delta)\|_{\mathbb{R}^2} < \delta$ and $\Delta(\beta_\delta, t_\delta) \geq \varepsilon$. Then,

$$0 = g_E(\beta_\delta, t_\delta, \Delta(\beta_\delta, t_\delta)) \leq \sup_{\substack{(\beta,t)\in\mathbb{R}_{>0}\times\mathbb{R}\\ \text{with } \|(\beta,t)-(\beta_0,t_0)\|_{\mathbb{R}^2}<\delta}} g_E(\beta, t, \varepsilon).$$

By sending $\delta \searrow 0$, $0 \le g_E(\beta_0, t_0, \varepsilon) < g_E(\beta_0, t_0, 0) = 0$, which is a contradiction. Thus, $\lim_{(\beta,t)\to(\beta_0,t_0)} \Delta(\beta,t) = 0 = \Delta(\beta_0, t_0)$. This implies that $\Delta \in C(\mathbb{R}_{>0} \times \mathbb{R})$. It readily follows from the confirmed regularity of Δ that $F_E|_{Q_+\cup Q_-} \in C^\omega(Q_+ \cup Q_-)$, $F_E \in C(\mathbb{R}_{>0} \times \mathbb{R})$. Let us define the function $\widehat{F}_E : \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

(2.4)

$$\widehat{F}_E(x,t,z) := \frac{z^2}{|U|} - \frac{D_d}{x} \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr} \log \left(\cos\left(\frac{t}{2}\right) + \cosh(x\sqrt{E(\mathbf{k})^2 + z^2}) \right).$$

Observe that the regularity of the function $(\beta,t) \mapsto \widehat{F}_E(\beta,t,\Delta(\beta,t)) : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ is same as that of $F_E(\beta,t)$. By considering the definition of $\Delta(\beta,t)$ we can derive that for any $(\beta,t) \in Q_+ \cup Q_-$

$$\frac{\partial}{\partial \beta} \widehat{F}_E(\beta,t,\Delta(\beta,t)) = \frac{\partial \widehat{F}_E}{\partial x}(\beta,t,\Delta(\beta,t)), \tag{2.5}$$

$$\frac{\partial}{\partial t} \widehat{F}_E(\beta,t,\Delta(\beta,t)) = \frac{\partial \widehat{F}_E}{\partial t}(\beta,t,\Delta(\beta,t)).$$

Take any $(\beta_0, t_0) \in Q_0$. The above equalities imply that

$$\lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\ (\beta,t)\in Q_+\cup Q_-}} \frac{\partial}{\partial \beta} \widehat{F}_E(\beta,t,\Delta(\beta,t)) = \frac{\partial \widehat{F}_E}{\partial x}(\beta_0,t_0,\Delta(\beta_0,t_0)), \tag{2.6}$$

$$\lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\ (\beta,t)\in Q_+\cup Q_-}} \frac{\partial}{\partial t} \widehat{F}_E(\beta,t,\Delta(\beta,t)) = \frac{\partial \widehat{F}_E}{\partial t}(\beta_0,t_0,\Delta(\beta_0,t_0)). \tag{2.7}$$

We remark that the function $\beta \mapsto g_E(\beta, t_0, 0)$ is real analytic in $\mathbb{R}_{>0}$. Since

$$\lim_{\beta\to\infty} g_E(\beta,t_0,0) \le -\frac{2}{|U|} + \frac{b}{e_{min}} < 0$$

by assumption, this function is not identically zero. Therefore, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that for any $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)\backslash\{\beta_0\}$ $g_E(\beta, t_0, 0) \neq 0$. Otherwise the identity theorem for real analytic functions yields a contradiction. This means that $(\beta, t_0) \in Q_+ \cup Q_-$ for any $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)\backslash\{\beta_0\}$. Thus, it follows from (2.6) that $\beta \mapsto \widehat{F}_E(\beta, t_0, \Delta(\beta, t_0))$ is differentiable at $\beta = \beta_0$ and

$$\frac{\partial}{\partial \beta} \widehat{F}_E(\beta,t_0,\Delta(\beta,t_0))\Big|_{\beta=\beta_0} = \frac{\partial \widehat{F}_E}{\partial x}(\beta_0,t_0,\Delta(\beta_0,t_0)).$$

By recalling Lemma 2.2 (ii),(iii) and (2.3) we see that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $(\beta_0, t) \in Q_+ \cup Q_-$ for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)\backslash\{t_0\}$. Thus by (2.7) $t \mapsto \widehat{F}_E(\beta_0, t, \Delta(\beta_0, t))$ is differentiable at $t = t_0$ and

$$\frac{\partial}{\partial t}\widehat{F}_E(\beta_0, t, \Delta(\beta_0, t))\Big|_{t=t_0} = \frac{\partial \widehat{F}_E}{\partial t}(\beta_0, t_0, \Delta(\beta_0, t_0)).$$

Since $(\beta, t) \mapsto (\frac{\partial \widehat{F}_E}{\partial x}(\beta, t, \Delta(\beta, t)), \frac{\partial \widehat{F}_E}{\partial t}(\beta, t, \Delta(\beta, t)))$ is continuous in $\mathbb{R}_{>0} \times \mathbb{R}$, it follows that $(\beta, t) \mapsto \widehat{F}_E(\beta, t, \Delta(\beta, t))$ is C^1-class in $\mathbb{R}_{>0} \times \mathbb{R}$ and so is $F_E(\cdot, \cdot)$.

(ii): We can derive from (2.5) and the gap equation $g_E(\beta, t, \Delta(\beta, t)) = 0$ $((\beta, t) \in Q_+)$ that

$$\begin{aligned}
&\frac{\partial^2}{\partial \beta^2}\widehat{F}_E(\beta, t, \Delta(\beta, t)) = \frac{\partial^2 \widehat{F}_E}{\partial x^2}(\beta, t, \Delta(\beta, t)), \quad (\forall(\beta, t) \in Q_-), \\
&\frac{\partial^2}{\partial \beta^2}\widehat{F}_E(\beta, t, \Delta(\beta, t)) \\
&= \frac{\partial^2 \widehat{F}_E}{\partial x^2}(\beta, t, \Delta(\beta, t)) + \frac{\partial^2 \widehat{F}_E}{\partial x \partial z}(\beta, t, \Delta(\beta, t))\frac{\partial \Delta}{\partial \beta}(\beta, t) \\
&= \frac{\partial^2 \widehat{F}_E}{\partial x^2}(\beta, t, \Delta(\beta, t)) - \frac{\partial g_E}{\partial x}(\beta, t, \Delta(\beta, t))\Delta(\beta, t)\frac{\partial \Delta}{\partial \beta}(\beta, t) \\
&= \frac{\partial^2 \widehat{F}_E}{\partial x^2}(\beta, t, \Delta(\beta, t)) + \Delta(\beta, t)\frac{\left(\frac{\partial g_E}{\partial x}(\beta, t, \Delta(\beta, t))\right)^2}{\frac{\partial g_E}{\partial z}(\beta, t, \Delta(\beta, t))}, \\
&(\forall(\beta, t) \in Q_+).
\end{aligned} \tag{2.8}$$

Let us define the function $\hat{g} : \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$\hat{g}(x, t, z) := \frac{\sinh(xz)}{(\cos(t/2) + \cosh(xz))\, z}.$$

Observe that for $(\beta, t) \in Q_+$

$$\begin{aligned}
&\frac{1}{\Delta(\beta, t)} \cdot \frac{\partial g_E}{\partial z}(\beta, t, \Delta(\beta, t)) \\
&= D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\partial \hat{g}}{\partial z}\big(\beta, t, \sqrt{E(\mathbf{k})^2 + \Delta(\beta, t)^2}\big) \cdot \frac{1}{\sqrt{E(\mathbf{k})^2 + \Delta(\beta, t)^2}}\right)
\end{aligned}$$

and

$$\begin{aligned}\text{(2.9)}\qquad &\lim_{\substack{(\beta,t)\to(\beta_0,t_0)\\ (\beta,t)\in Q_+}} \frac{1}{\Delta(\beta,t)}\cdot\frac{\partial g_E}{\partial z}(\beta,t,\Delta(\beta,t))\\ &= D_d\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\partial \hat{g}}{\partial z}(\beta_0,t_0,|E(\mathbf{k})|)\cdot\frac{1}{|E(\mathbf{k})|}\right)\\ &< 0.\end{aligned}$$

Here we again used the monotone decreasing property of the function (1.7). Let us study the term $\frac{\partial g_E}{\partial x}(\beta_0,t_0,0)$. If $\beta_0\in(0,\beta_c)$,

$$\frac{\partial g_E}{\partial x}(\beta_0,t_0,0)=\frac{\partial g_E}{\partial x}(\beta_0,\tau(\beta_0),0)=-\frac{d\tau}{d\beta}(\beta_0)\frac{\partial g_E}{\partial t}(\beta_0,\tau(\beta_0),0).$$

Since $\tau(\beta_0)\in(0,2\pi)$ and $t\mapsto g_E(\beta_0,t,0)$ is strictly monotone increasing in $(0,2\pi)$,

$$\text{(2.10)}\qquad \frac{\partial g_E}{\partial t}(\beta_0,\tau(\beta_0),0)\neq 0.$$

Thus, $\frac{\partial g_E}{\partial x}(\beta_0,t_0,0)=0$ if and only if $\frac{d\tau}{d\beta}(\beta_0)=0$. If $\beta_0=\beta_c$, $t_0=2\pi$ (mod 4π). In this case we can directly check that $\frac{\partial g_E}{\partial x}(\beta_0,t_0,0)<0$. We can conclude the claimed convergent properties by combining the above properties of $\frac{\partial g_E}{\partial x}(\beta_0,t_0,0)$ with (2.8), (2.9).

(iii): In the same way as in the proof of (ii) we have that

(2.11)

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}\widehat{F}_E(\beta,t,\Delta(\beta,t))=\frac{\partial^2\widehat{F}_E}{\partial t^2}(\beta,t,\Delta(\beta,t)),\quad(\forall(\beta,t)\in Q_-),\\ &\frac{\partial^2}{\partial t^2}\widehat{F}_E(\beta,t,\Delta(\beta,t))=\frac{\partial^2\widehat{F}_E}{\partial t^2}(\beta,t,\Delta(\beta,t))+\Delta(\beta,t)\frac{\left(\frac{\partial g_E}{\partial t}(\beta,t,\Delta(\beta,t))\right)^2}{\frac{\partial g_E}{\partial z}(\beta,t,\Delta(\beta,t))},\\ &(\forall(\beta,t)\in Q_+).\end{aligned}$$

If $\beta_0\in(0,\beta_c)$, $\tau(\beta_0)\in(0,2\pi)$. By periodicity and (2.10)

$$\left|\frac{\partial g_E}{\partial t}(\beta_0,t_0,0)\right|=\left|\frac{\partial g_E}{\partial t}(\beta_0,\tau(\beta_0),0)\right|>0.$$

If $\beta_0 = \beta_c$, $t_0 = 2\pi \pmod{4\pi}$, and thus $\frac{\partial g_E}{\partial t}(\beta_0, t_0, 0) = 0$. The claimed convergent properties follow from (2.9), (2.11) and the above properties of $\frac{\partial g_E}{\partial t}(\beta_0, t_0, 0)$. $\square$

REMARK 2.6. For $(\rho, \eta) = (+, -)$ or $(-, +)$ let us set

$$Q^{\beta}_{\rho,\eta} := \left\{ (\beta_0, t_0) \in Q_0 \ \middle| \ \exists \varepsilon \in \mathbb{R}_{>0} \text{ s.t. } \begin{array}{l} (\beta, t_0) \in Q_\rho, \ (\forall \beta \in (\beta_0 - \varepsilon, \beta_0)), \\ (\beta, t_0) \in Q_\eta, \ (\forall \beta \in (\beta_0, \beta_0 + \varepsilon)) \end{array} \right\}.$$

Proposition 2.5 (ii) implies that if $(\beta_0, t_0) \in Q^{\beta}_{\rho,\eta}$ satisfies $\beta_0 \neq \beta_c$ and $\frac{\partial \tau}{\partial \beta}(\beta_0) \neq 0$ or $\beta_0 = \beta_c$,

$$\lim_{\beta \nearrow \beta_0} \frac{\partial^2 F_E}{\partial \beta^2}(\beta, t_0) \neq \lim_{\beta \searrow \beta_0} \frac{\partial^2 F_E}{\partial \beta^2}(\beta, t_0).$$

This means that a 2nd order phase transition driven by β occurs at $(\beta, t) = (\beta_0, t_0)$. Assume that $\beta_0 \in (0, \beta_c)$, $\frac{d\tau}{d\beta}(\beta_0) = 0$ and $\beta \mapsto \tau(\beta)$ is monotone increasing or decreasing in a neighborhood of β_0. Then $(\beta_0, \tau(\beta_0)) \in Q^{\beta}_{+,-}$ or $(\beta_0, \tau(\beta_0)) \in Q^{\beta}_{-,+}$ respectively. In this case Proposition 2.5 (ii) implies that $\beta \mapsto \frac{\partial^2 F_E}{\partial \beta^2}(\beta, \tau(\beta_0))$ is continuous at $\beta = \beta_0$, even though the trajectory $\beta \mapsto (\beta, \tau(\beta_0))$ crosses Q_0 at $\beta = \beta_0$ from Q_+ to Q_- or from Q_- to Q_+. This interestingly suggests a possibility of higher order phase transition with β at $(\beta, t) = (\beta_0, \tau(\beta_0))$. However, as we will see in the following subsections, the monotonicity of $\tau(\cdot)$ is sensitive to individual characteristics of $E(\cdot)$ and we do not pursue the question whether $\tau(\cdot)$ can satisfy the above properties in this paper. On the other hand, if we set

$$Q^{t}_{\rho,\eta} := \left\{ (\beta_0, t_0) \in Q_0 \ \middle| \ \exists \varepsilon \in \mathbb{R}_{>0} \text{ s.t. } \begin{array}{l} (\beta_0, t) \in Q_\rho, \ (\forall t \in (t_0 - \varepsilon, t_0)), \\ (\beta_0, t) \in Q_\eta, \ (\forall t \in (t_0, t_0 + \varepsilon)) \end{array} \right\}$$

for $(\rho, \eta) = (+, -)$ or $(-, +)$, we can see from Lemma 2.2 and (2.3) that

$$Q^{t}_{+,-} \sqcup Q^{t}_{-,+} = \{(\beta_0, t_0) \in Q_0 \mid \beta_0 \neq \beta_c\}.$$

Thus by Proposition 2.5 (iii)

$$\lim_{t \nearrow t_0} \frac{\partial^2 F_E}{\partial t^2}(\beta_0, t) \neq \lim_{t \searrow t_0} \frac{\partial^2 F_E}{\partial t^2}(\beta_0, t), \quad (\forall (\beta_0, t_0) \in Q^{t}_{+,-} \cup Q^{t}_{-,+}).$$

In other words, $t \mapsto \frac{\partial^2 F_E}{\partial t^2}(\beta, t)$ has jump discontinuity whenever the trajectory $t \mapsto (\beta, t)$ crosses Q_0 from Q_+ to Q_- or from Q_- to Q_+. This means that the phase transitions driven by t in this system are of 2nd order.

REMARK 2.7. The free energy density characterized in Theorem 1.3 (ii) corresponds to $F_E(\beta, \beta\theta)$. In [14, Subsection 2.3] we focused on the properties of the function $(\beta, \theta) \mapsto F_E(\beta, \beta\theta) : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ under different assumptions on $E(\cdot)$. The reason why we treated the function $(\beta, t) \mapsto F_E(\beta, t)$ here is that it is considered as a dynamical free energy density studied in today's physics of DPT. At this point the function $F_E(\beta, \beta\theta)$ lacks physical interpretation and its phase boundaries are structurally more complicated to analyze than those of $F_E(\beta, t)$. Nonetheless, it is possible to study the regularity of $(\beta, \theta) \mapsto F_E(\beta, \beta\theta)$ in a manner similar to Proposition 2.5. In this case the set

$$\{(\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R} \mid g_E(\beta, \beta\theta, 0) = 0\}$$

defines the phase boundaries and it can be shown that 2nd order partial derivatives of the function $(\beta, \theta) \mapsto F_E(\beta, \beta\theta)$ have jump discontinuities on the phase boundaries. However, we do not explicitly present the results for conciseness of the paper.

2.2. Shape of the phase boundary

In view of the characterization (2.3), we notice that the graph of the function $\tau(\cdot)$ determines the shape of the phase boundaries. So let us study the profile of $\tau(\cdot)$ more deeply. Its universal properties have already been summarized in Lemma 2.2 and Proposition 2.4. As the next step, we should try to reveal geometric properties which may vary with details of $E(\cdot)$. It will turn out that the ratio e_{min}/e_{max} is a prime index to classify the shape of $\tau(\cdot)$. From now on we let c denote a generic positive constant independent of any parameter. The following proposition tells us when $\tau(\cdot) : (0, \beta_c) \to \mathbb{R}$ is strictly downward convex.

PROPOSITION 2.8. *There exists $e_0 \in (0,1)$ independent of any parameter such that if $e_{min}/e_{max} \ge e_0$, for any $U \in [-\frac{e_{min}}{\sinh(2)b}, 0)$, $E \in \mathcal{E}(e_{min}, e_{max})$ and $\beta \in (0, \beta_c)$,*

$$\frac{d^2\tau}{d\beta^2}(\beta) > 0.$$

Proof. First of all let us prepare a few quantitative bounds based on the assumption

$$|U| \le \frac{e_{min}}{\sinh(2)b}. \tag{2.12}$$

Observe that $1/\sinh(2) < 2\tanh(1) < 2$, which implies that $|U| < 2e_{min}/b$ and combined with Lemma 1.2 that

$$\beta_c \le \frac{2}{e_{min}} \tanh^{-1}\left(\frac{b|U|}{2e_{min}}\right) < \frac{2}{e_{min}} \tanh^{-1}(\tanh(1)) = \frac{2}{e_{min}}. \tag{2.13}$$

It follows from (2.13) and the equality

$$g_E(\beta, \tau(\beta), 0) = 0, \quad (\beta \in (0, \beta_c)) \tag{2.14}$$

that

$$\begin{aligned} \frac{2}{|U|} &\le \frac{b\sinh(\beta e_{min})}{e_{min}(\cos(\tau(\beta)/2) + \cosh(\beta e_{min}))} \\ &\le \frac{b\sinh(2)}{e_{min}(\cos(\tau(\beta)/2) + \cosh(\beta e_{min}))}, \end{aligned} \tag{2.15}$$

or by (2.12)

$$\cos\left(\frac{\tau(\beta)}{2}\right) + \cosh(\beta e_{min}) \le \frac{1}{2}, \tag{2.16}$$

$$-\cos\left(\frac{\tau(\beta)}{2}\right) \ge \frac{1}{2}, \quad (\forall \beta \in (0, \beta_c)). \tag{2.17}$$

By differentiating both sides of (2.14) twice and substituting the first equality of (2.1) we obtain that for any $\beta \in (0, \beta_c)$

(2.18)

$$\begin{aligned} &\frac{d^2\tau}{d\beta^2}(\beta) \\ &= \frac{1}{\left(\frac{\partial g_E}{\partial t}(\beta, \tau(\beta), 0)\right)^3} \Bigg(2\frac{\partial^2 g_E}{\partial x \partial t}(\beta, \tau(\beta), 0)\frac{\partial g_E}{\partial x}(\beta, \tau(\beta), 0)\frac{\partial g_E}{\partial t}(\beta, \tau(\beta), 0) \\ &\quad - \frac{\partial^2 g_E}{\partial x^2}(\beta, \tau(\beta), 0)\left(\frac{\partial g_E}{\partial t}(\beta, \tau(\beta), 0)\right)^2 \end{aligned}$$

$$- \frac{\partial^2 g_E}{\partial t^2}(\beta, \tau(\beta), 0) \left(\frac{\partial g_E}{\partial x}(\beta, \tau(\beta), 0) \right)^2 \Bigg).$$

Define the functions f_β^0, f_β^x, f_β^t, f_β^{xx}, f_β^{xt}, $f_\beta^{tt} : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}
&f_\beta^0(y) := \cos\left(\frac{\tau(\beta)}{2}\right) + \cosh(\beta y),\\
&f_\beta^x(y) := y\left(\cos\left(\frac{\tau(\beta)}{2}\right)\cosh(\beta y) + 1\right),\\
&f_\beta^t(y) := \frac{1}{2}\sin\left(\frac{\tau(\beta)}{2}\right)\sinh(\beta y),\\
&f_\beta^{xx}(y) := y^2\sinh(\beta y)\left(\cos^2\left(\frac{\tau(\beta)}{2}\right) - \cos\left(\frac{\tau(\beta)}{2}\right)\cosh(\beta y) - 2\right),\\
&f_\beta^{xt}(y) := \frac{y}{2}\sin\left(\frac{\tau(\beta)}{2}\right)\left(\cos\left(\frac{\tau(\beta)}{2}\right)\cosh(\beta y) + 1 - \sinh^2(\beta y)\right),\\
&f_\beta^{tt}(y) := \frac{1}{4}\sinh(\beta y)\left(1 + \sin^2\left(\frac{\tau(\beta)}{2}\right) + \cos\left(\frac{\tau(\beta)}{2}\right)\cosh(\beta y)\right).
\end{aligned}$$

Then the formula (2.18) can be rewritten as follows. For any $\beta \in (0, \beta_c)$

$$\begin{aligned}
\frac{d^2\tau}{d\beta^2}(\beta) = &\frac{1}{\left(\frac{\partial g_E}{\partial t}(\beta, \tau(\beta), 0)\right)^3} \prod_{j=1}^{3}\left(D_d \int_{\Gamma_\infty^*} d\mathbf{k}_j\right)\\
&\cdot \Bigg(2\,\mathrm{Tr}\left(\frac{f_\beta^{xt}(|E(\mathbf{k}_1)|)}{|E(\mathbf{k}_1)|f_\beta^0(E(\mathbf{k}_1))^3}\right)\mathrm{Tr}\left(\frac{f_\beta^{x}(|E(\mathbf{k}_2)|)}{|E(\mathbf{k}_2)|f_\beta^0(E(\mathbf{k}_2))^2}\right)\\
&\quad\cdot \mathrm{Tr}\left(\frac{f_\beta^{t}(|E(\mathbf{k}_3)|)}{|E(\mathbf{k}_3)|f_\beta^0(E(\mathbf{k}_3))^2}\right)\\
&\quad - \mathrm{Tr}\left(\frac{f_\beta^{xx}(|E(\mathbf{k}_1)|)}{|E(\mathbf{k}_1)|f_\beta^0(E(\mathbf{k}_1))^3}\right)\mathrm{Tr}\left(\frac{f_\beta^{t}(|E(\mathbf{k}_2)|)}{|E(\mathbf{k}_2)|f_\beta^0(E(\mathbf{k}_2))^2}\right)\\
&\quad\cdot \mathrm{Tr}\left(\frac{f_\beta^{t}(|E(\mathbf{k}_3)|)}{|E(\mathbf{k}_3)|f_\beta^0(E(\mathbf{k}_3))^2}\right)\\
&\quad - \mathrm{Tr}\left(\frac{f_\beta^{tt}(|E(\mathbf{k}_1)|)}{|E(\mathbf{k}_1)|f_\beta^0(E(\mathbf{k}_1))^3}\right)\mathrm{Tr}\left(\frac{f_\beta^{x}(|E(\mathbf{k}_2)|)}{|E(\mathbf{k}_2)|f_\beta^0(E(\mathbf{k}_2))^2}\right)
\end{aligned}$$

$$\cdot \mathrm{Tr}\left(\left(\frac{f_\beta^x(|E(\mathbf{k}_3)|)}{|E(\mathbf{k}_3)|f_\beta^0(E(\mathbf{k}_3))^2}\right)\right).$$

Let us define the function $f_\beta : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned}&f_\beta(y_1,y_2,y_3)\\&:= 2f_\beta^{xt}(y_1)f_\beta^x(y_2)f_\beta^t(y_3) - f_\beta^{xx}(y_1)f_\beta^t(y_2)f_\beta^t(y_3) - f_\beta^{tt}(y_1)f_\beta^x(y_2)f_\beta^x(y_3).\end{aligned}$$

We can see from above that if

$$\min\{f_\beta(y_1,y_2,y_3) \mid y_j \in [e_{min}, e_{max}]\ (j=1,2,3)\} > 0, \tag{2.19}$$

then $\frac{d^2\tau}{d\beta^2}(\beta) > 0$.

Let us prove (2.19). Observe that

$$\begin{aligned}&f_\beta(y,y,y)\\
&= \frac{y^2}{2}\sin^2\left(\frac{\tau(\beta)}{2}\right)\sinh(\beta y)\left(\cos\left(\frac{\tau(\beta)}{2}\right)\cosh(\beta y)+1\right)^2\\
&\quad - \frac{y^2}{4}\sin^2\left(\frac{\tau(\beta)}{2}\right)\sinh^3(\beta y)\left(\cos^2\left(\frac{\tau(\beta)}{2}\right)+\cos\left(\frac{\tau(\beta)}{2}\right)\cosh(\beta y)\right)\\
&\quad - f_\beta^{tt}(y)f_\beta^x(y)^2\\
&= -\frac{y^2}{4}\sinh(\beta y)\left(\cos^2\left(\frac{\tau(\beta)}{2}\right)+\cos\left(\frac{\tau(\beta)}{2}\right)\cosh(\beta y)\right)\\
&\qquad \cdot\left(\cos\left(\frac{\tau(\beta)}{2}\right)\cosh(\beta y)+1\right)^2\\
&\quad - \frac{y^2}{4}\sin^2\left(\frac{\tau(\beta)}{2}\right)\sinh^3(\beta y)\left(\cos^2\left(\frac{\tau(\beta)}{2}\right)+\cos\left(\frac{\tau(\beta)}{2}\right)\cosh(\beta y)\right)\\
&= -\frac{y^2}{4}\sinh(\beta y)\cos\left(\frac{\tau(\beta)}{2}\right)f_\beta^0(y)^3,\end{aligned}$$

which combined with (2.17) implies that

$$\min_{y\in[e_{min},e_{max}]} f_\beta(y,y,y) \ge \frac{e_{min}^2}{8}\sinh(\beta e_{min})f_\beta^0(e_{min})^3. \tag{2.20}$$

For a continuous function $f : [e_{min}, e_{max}] \to \mathbb{R}$ let $\|f\|_\infty$ denote $\sup_{y\in[e_{min},e_{max}]}|f(y)|$ in the following. For any $y_j \in [e_{min}, e_{max}]$ $(j=1,2,3)$

(2.21)

$$|f_\beta(y_1,y_1,y_1) - f_\beta(y_1,y_2,y_3)|$$

$$\le |f_\beta(y_1,y_1,y_1) - f_\beta(y_1,y_2,y_1)| + |f_\beta(y_1,y_2,y_1) - f_\beta(y_1,y_2,y_3)|$$
$$\le 2(e_{max} - e_{min})\Bigg(\|f_\beta^{xt}\|_\infty \left(\left\| \frac{d}{dy} f_\beta^x \right\|_\infty \|f_\beta^t\|_\infty + \|f_\beta^x\|_\infty \left\| \frac{d}{dy} f_\beta^t \right\|_\infty \right)$$
$$+ \|f_\beta^{xx}\|_\infty \|f_\beta^t\|_\infty \left\| \frac{d}{dy} f_\beta^t \right\|_\infty + \|f_\beta^{tt}\|_\infty \|f_\beta^x\|_\infty \left\| \frac{d}{dy} f_\beta^x \right\|_\infty \Bigg).$$

To estimate the right-hand side of the above inequality, let us prepare necessary bounds.

$$\|f_\beta^x\|_\infty \le c e_{max} f_\beta^0(e_{max}),$$
$$\left\| \frac{d}{dy} f_\beta^x \right\|_\infty \le c \left(f_\beta^0(e_{max}) + \sinh^2(\beta e_{max}) \right),$$
$$\|f_\beta^t\|_\infty \le c f_\beta^0(e_{max})^{\frac{1}{2}} \sinh(\beta e_{max}),$$
$$\left\| \frac{d}{dy} f_\beta^t \right\|_\infty \le \beta \left| \sin\left(\frac{\tau(\beta)}{2}\right) f_\beta^0(e_{max}) \right| + \beta \left| \sin\left(\frac{\tau(\beta)}{2}\right) \cos\left(\frac{\tau(\beta)}{2}\right) \right|$$
$$\le c\beta (f_\beta^0(e_{max})^{\frac{3}{2}} + f_\beta^0(e_{max})^{\frac{1}{2}}),$$
$$\|f_\beta^{xx}\|_\infty \le c e_{max}^2 \sinh(\beta e_{max}) f_\beta^0(e_{max}),$$
$$\|f_\beta^{xt}\|_\infty \le c e_{max} f_\beta^0(e_{max})^{\frac{1}{2}} \left(f_\beta^0(e_{max}) + \sinh^2(\beta e_{max}) \right),$$
$$\|f_\beta^{tt}\|_\infty \le c \sinh(\beta e_{max}) f_\beta^0(e_{max}),$$

which lead to that

$$\|f_\beta^{xt}\|_\infty \left\| \frac{d}{dy} f_\beta^x \right\|_\infty \|f_\beta^t\|_\infty$$
$$\le c e_{max} \sinh(\beta e_{max}) f_\beta^0(e_{max})^3 + c e_{max} \sinh^5(\beta e_{max}) f_\beta^0(e_{max}),$$
$$\|f_\beta^{xt}\|_\infty \|f_\beta^x\|_\infty \left\| \frac{d}{dy} f_\beta^t \right\|_\infty$$
$$\le c e_{max} \sinh(\beta e_{max}) \sum_{j=3}^{4} f_\beta^0(e_{max})^j + c e_{max} \sinh^3(\beta e_{max}) \sum_{j=2}^{3} f_\beta^0(e_{max})^j,$$
$$\|f_\beta^{xx}\|_\infty \|f_\beta^t\|_\infty \left\| \frac{d}{dy} f_\beta^t \right\|_\infty \le c e_{max} \sinh^3(\beta e_{max}) \sum_{j=2}^{3} f_\beta^0(e_{max})^j,$$
$$\|f_\beta^{tt}\|_\infty \|f_\beta^x\|_\infty \left\| \frac{d}{dy} f_\beta^x \right\|_\infty$$

$$\leq ce_{max}\sinh(\beta e_{max})f_\beta^0(e_{max})^3 + ce_{max}\sinh^3(\beta e_{max})f_\beta^0(e_{max})^2.$$

By combining these inequalities with (2.21) we obtain that

$$\begin{aligned}(2.22)\qquad &|f_\beta(y_1,y_1,y_1) - f_\beta(y_1,y_2,y_3)|\\ &\leq c(e_{max}-e_{min})e_{max}\sinh(\beta e_{max})\\ &\quad\cdot\Bigg(\sum_{j=3}^{4} f_\beta^0(e_{max})^j + \sinh^2(\beta e_{max})\sum_{j=2}^{3} f_\beta^0(e_{max})^j\\ &\qquad + \sinh^4(\beta e_{max})f_\beta^0(e_{max})\Bigg).\end{aligned}$$

Let us bound the right-hand side of this inequality by that of (2.20). Let us prepare a few more inequalities for this purpose. We can use (2.13) to derive that

(2.23)

$$\begin{aligned}\sinh(\beta e_{max}) &= \left(\frac{\sinh(\beta e_{max})-\sinh(\beta e_{min})}{\sinh(\beta e_{min})}+1\right)\sinh(\beta e_{min})\\ &\leq \left(\frac{\beta(e_{max}-e_{min})\cosh(\beta e_{max})}{\sinh(\beta e_{min})}+1\right)\sinh(\beta e_{min})\\ &\leq \left(\left(\frac{e_{max}}{e_{min}}-1\right)\cosh\left(\frac{2e_{max}}{e_{min}}\right)+1\right)\sinh(\beta e_{min}),\end{aligned}$$

(2.24)

$$\begin{aligned}f_\beta^0(e_{max}) &\leq \cos\left(\frac{\tau(\beta)}{2}\right) + 1 + (\beta e_{max})^2\cosh(\beta e_{max})\\ &\leq \cos\left(\frac{\tau(\beta)}{2}\right) + 1 + 2\left(\frac{e_{max}}{e_{min}}\right)^2\cosh(\beta e_{max})(\cosh(\beta e_{min})-1)\\ &\leq 2\left(\frac{e_{max}}{e_{min}}\right)^2\cosh\left(\frac{2e_{max}}{e_{min}}\right)f_\beta^0(e_{min}).\end{aligned}$$

Moreover, by (2.13) and (2.24)

(2.25)

$$\begin{aligned}\sinh^2(\beta e_{max}) &= \cosh^2(\beta e_{max}) - 1\\ &\leq (\cosh(\beta e_{max})+1)f_\beta^0(e_{max})\end{aligned}$$

$$\leq 2\left(\frac{e_{max}}{e_{min}}\right)^2\cosh\left(\frac{2e_{max}}{e_{min}}\right)\left(\cosh\left(\frac{2e_{max}}{e_{min}}\right)+1\right)f_\beta^0(e_{min}).$$

Substitution of (2.16), (2.23), (2.24), (2.25) into (2.22) yields the following inequality. We especially use (2.23) to bound $\sinh(\beta e_{max})$ in front of the large parenthesis and (2.25) to bound $\sinh^2(\beta e_{max})$, $\sinh^4(\beta e_{max})$ inside the large parenthesis.

$$\begin{aligned}
&|f_\beta(y_1,y_1,y_1)-f_\beta(y_1,y_2,y_3)|\\
&\leq c\left(\frac{e_{max}}{e_{min}}-1\right)\frac{e_{max}}{e_{min}}\left(\left(\frac{e_{max}}{e_{min}}-1\right)\cosh\left(\frac{2e_{max}}{e_{min}}\right)+1\right)\\
&\quad\cdot\Bigg(\left(\frac{e_{max}}{e_{min}}\right)^8\left(\cosh\left(\frac{2e_{max}}{e_{min}}\right)\right)^4\\
&\qquad+\left(\cosh\left(\frac{2e_{max}}{e_{min}}\right)+1\right)\left(\frac{e_{max}}{e_{min}}\right)^8\left(\cosh\left(\frac{2e_{max}}{e_{min}}\right)\right)^4\\
&\qquad+\left(\cosh\left(\frac{2e_{max}}{e_{min}}\right)+1\right)^2\left(\frac{e_{max}}{e_{min}}\right)^6\left(\cosh\left(\frac{2e_{max}}{e_{min}}\right)\right)^3\Bigg)\\
&\quad\cdot e_{min}^2\sinh(\beta e_{min})f_\beta^0(e_{min})^3,\\
&(\forall y_j\in[e_{min},e_{max}]\ (j=1,2,3)).
\end{aligned}$$

We can see that there exists $e_0\in(0,1)$ independent of any parameter such that if $e_{min}/e_{max}\geq e_0$,

$$\begin{aligned}
&|f_\beta(y_1,y_1,y_1)-f_\beta(y_1,y_2,y_3)|\leq\frac{e_{min}^2}{16}\sinh(\beta e_{min})f_\beta^0(e_{min})^3,\\
&(\forall y_j\in[e_{min},e_{max}]\ (j=1,2,3)).
\end{aligned}\tag{2.26}$$

The inequalities (2.20), (2.26) imply (2.19) and thus the claim holds true. □

Proposition 2.8 together with Lemma 2.2 means in particular that under the assumptions of Proposition 2.8 $\tau(\cdot)$ has one and only one local minimum point in $(0,\beta_c)$. We will see that this property does not always hold if e_{min}/e_{max} is small. To describe the profile of $\tau(\cdot)$ in terms of number of local minimum points, let us make clear the definition.

DEFINITION 2.9. Let f be a real-valued function on an open interval (a,b) and $c \in (a,b)$. The point c is said to be a local minimum point of f if there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $f(c) \le f(x)$ for any $x \in (c-\varepsilon, c+\varepsilon)$.

Our main goal in this section is to give a necessary and sufficient condition for $\tau(\cdot)$ to have only one local minimum point for any choice of $E \in \mathcal{E}(e_{min}, e_{max})$. The next proposition gives a sufficient condition.

PROPOSITION 2.10. *Assume that* $e_{min}/e_{max} > \sqrt{17-12\sqrt{2}}$. *Then there exists* $U_0(b, e_{min}, e_{max}) \in (0, \frac{e_{min}}{\sinh(2)b}]$ *depending only on* b, e_{min}, e_{max} *such that for any* $U \in [-U_0(b, e_{min}, e_{max}), 0)$ *and* $E \in \mathcal{E}(e_{min}, e_{max})$ $\tau(\cdot)$ *has one and only one local minimum point in* $(0, \beta_c)$.

REMARK 2.11. According to the proof of the proposition, $U_0(b, e_{min}, e_{max})$ is equal to

$$\frac{c' \frac{e_{min}^2}{e_{max}} \left(\left(\frac{e_{min}}{e_{max}} \right)^2 - 17 + 12\sqrt{2} \right)}{\sinh(2) b \cosh^2 \left(2c'' \frac{e_{max}}{e_{min}} \right) \cosh^2 \left(c'' \frac{e_{max}}{e_{min}} \right)}$$

with generic constants $c' \in (0,1]$, $c'' \in \mathbb{R}_{>0}$. More specifically, $U_0(b, e_{min}, e_{max})$ is given by the right-hand side of (2.40).

Let us prepare an essential part of the proof of Proposition 2.10 separately in the next lemma. Define the function $u : \mathbb{R}_{>0} \times [-1,1] \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$u(x,y,z) := \frac{\sinh(xz)}{(y+\cosh(xz))z}. \tag{2.27}$$

LEMMA 2.12. *Assume that* $\sqrt{17-12\sqrt{2}} < e_{min}/e_{max} < 1$. *Then there exists* $c_1 \in \mathbb{R}_{>0}$ *independent of any parameter such that for any* $(x,y) \in \mathbb{R}_{>0} \times (-1,0)$ *satisfying*

$$\frac{|y+1|}{1-|y+1|} < c_1 \frac{\frac{e_{min}}{e_{max}} \left(\left(\frac{e_{min}}{e_{max}} \right)^2 - 17 + 12\sqrt{2} \right)}{\cosh^2(2x)\cosh^2(x)}$$

and e_1, $e_2 \in \mathbb{R}_{>0}$ *satisfying* $e_{max} \ge e_1 > e_2 \ge e_{min}$,

$$\frac{\partial u}{\partial x} \left(\sqrt{y+1} \cdot \frac{x}{e_1}, y, e_1 \right) \frac{\partial^2 u}{\partial x^2} \left(\sqrt{y+1} \cdot \frac{x}{e_1}, y, e_2 \right) \tag{2.28}$$

$$- \frac{\partial^2 u}{\partial x^2}\left(\sqrt{y+1}\cdot\frac{x}{e_1}, y, e_1\right)\frac{\partial u}{\partial x}\left(\sqrt{y+1}\cdot\frac{x}{e_1}, y, e_2\right) > 0.$$

PROOF. Define the function $v : \mathbb{R}_{>0} \times (-1,0) \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$\begin{aligned} &v(x,y,z) \\ &:= \frac{1}{x(y+1)^{\frac{7}{2}}}\Bigg(z\sinh(\sqrt{y+1}\cdot xz)\big(y^2-\cosh(\sqrt{y+1}\cdot xz)y-2\big) \\ &\qquad\cdot\big(\cosh(\sqrt{y+1}\cdot x)y+1\big)\big(\cosh(\sqrt{y+1}\cdot x)+y\big) \\ &\qquad-\sinh(\sqrt{y+1}\cdot x)\big(y^2-\cosh(\sqrt{y+1}\cdot x)y-2\big) \\ &\qquad\cdot\big(\cosh(\sqrt{y+1}\cdot xz)y+1\big)\big(\cosh(\sqrt{y+1}\cdot xz)+y\big)\Bigg). \end{aligned}$$

Let us observe that for any $(x,y) \in \mathbb{R}_{>0} \times (-1,0)$

(2.29)

$$(\text{L.H.S of (2.28)}) = \frac{e_1 x(y+1)^{\frac{7}{2}}}{\prod_{j=1}^2\big(\cosh\big(\sqrt{y+1}\cdot x\frac{e_j}{e_1}\big)+y\big)^3}\cdot v\left(x,y,\frac{e_2}{e_1}\right).$$

We can also derive that

(2.30)

$$\begin{aligned} &v(x,y,z) \\ &= z\left(z+\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}(y+1)^n z^{2n+1}x^{2n}\right) \\ &\quad\cdot\left(y-2-y\sum_{n=1}^{\infty}\frac{1}{(2n)!}(y+1)^{n-1}z^{2n}x^{2n}\right) \\ &\quad\cdot\left(1+y\sum_{n=1}^{\infty}\frac{1}{(2n)!}(y+1)^{n-1}x^{2n}\right)\left(1+\sum_{n=1}^{\infty}\frac{1}{(2n)!}(y+1)^{n-1}x^{2n}\right) \\ &\quad-\left(1+\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}(y+1)^n x^{2n}\right)\left(y-2-y\sum_{n=1}^{\infty}\frac{1}{(2n)!}(y+1)^{n-1}x^{2n}\right) \end{aligned}$$

$$\cdot\left(1+y\sum_{n=1}^{\infty}\frac{1}{(2n)!}(y+1)^{n-1}z^{2n}x^{2n}\right)$$
$$\cdot\left(1+\sum_{n=1}^{\infty}\frac{1}{(2n)!}(y+1)^{n-1}z^{2n}x^{2n}\right).$$

This expansion implies that the function $v(\cdot,\cdot,\cdot)$ can be analytically continued into $\mathbb{C}^3$. By abusing notation we let $v(x,y,z)$ denote the entire function defined by the right-hand side of (2.30) as well. It follows from the assumption that for any $x\in\mathbb{R}$

$$\begin{aligned}
(2.31)\qquad & v\left(x,-1,\frac{e_2}{e_1}\right)\\
&=3\left(\frac{e_2}{e_1}\right)^2\left(1-\left(\frac{e_2}{e_1}\right)^2\right)\\
&\quad\cdot\left(\left(\frac{x^2}{2}-\frac{1+\left(\frac{e_2}{e_1}\right)^2}{6\left(\frac{e_2}{e_1}\right)^2}\right)^2+\frac{-\left(\frac{e_2}{e_1}\right)^4+34\left(\frac{e_2}{e_1}\right)^2-1}{36\left(\frac{e_2}{e_1}\right)^4}\right)\\
&\geq\frac{1-\left(\frac{e_2}{e_1}\right)^2}{12\left(\frac{e_2}{e_1}\right)^2}\left(17+12\sqrt{2}-\left(\frac{e_2}{e_1}\right)^2\right)\left(\left(\frac{e_2}{e_1}\right)^2-17+12\sqrt{2}\right)\\
&\geq\left(1-\frac{e_2}{e_1}\right)\left(\left(\frac{e_{min}}{e_{max}}\right)^2-17+12\sqrt{2}\right)>0.
\end{aligned}$$

Also, the Taylor expansion and the Cauchy formula yield that for any $x\in\mathbb{C}$, $y\in(-1,0)$

$$\begin{aligned}
& v\left(x,y,\frac{e_2}{e_1}\right)\\
&=v\left(x,-1,\frac{e_2}{e_1}\right)+\sum_{m=1}^{\infty}\frac{1}{2\pi i}\oint_{|\zeta+1|=1}d\zeta\frac{v\left(x,\zeta,\frac{e_2}{e_1}\right)}{(\zeta+1)^{m+1}}(y+1)^m\\
&=v\left(x,-1,\frac{e_2}{e_1}\right)\\
&\quad+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2\pi i)^2}\oint_{|\zeta+1|=1}d\zeta\oint_{|\xi-1|=1}d\xi\frac{v(x,\zeta,\xi)}{(\zeta+1)^{m+1}(\xi-1)^{n+1}}\\
&\qquad\qquad\cdot(y+1)^m\left(\frac{e_2}{e_1}-1\right)^n.
\end{aligned}$$

In the second equality we used the fact that $v(x,y,1)=0$ for any x, $y\in\mathbb{C}$. Moreover, by considering (2.30) we can see that for any $x\in\mathbb{R}_{>0}$, $y\in(-1,0)$

$$\begin{aligned}
&\left|v\left(x,y,\frac{e_2}{e_1}\right)-v\left(x,-1,\frac{e_2}{e_1}\right)\right|\\
&\le c\cosh^2(2x)\cosh^2(x)\sum_{m=1}^{\infty}|y+1|^m\sum_{n=1}^{\infty}\left|\frac{e_2}{e_1}-1\right|^n\\
&\le c\cosh^2(2x)\cosh^2(x)\frac{e_{max}}{e_{min}}\left|\frac{e_2}{e_1}-1\right|\frac{|y+1|}{1-|y+1|},
\end{aligned}$$

which combined with (2.31) implies that for any $x\in\mathbb{R}_{>0}$, $y\in(-1,0)$

$$\begin{aligned}
&v\left(x,y,\frac{e_2}{e_1}\right)\\
&\ge\left(1-\frac{e_2}{e_1}\right)\\
&\cdot\left(\left(\frac{e_{min}}{e_{max}}\right)^2-17+12\sqrt{2}-c\frac{e_{max}}{e_{min}}\cosh^2(2x)\cosh^2(x)\frac{|y+1|}{1-|y+1|}\right).
\end{aligned}\tag{2.32}$$

We can deduce the claim from (2.29), (2.32). □

In the following we let $\cosh^{-1}$ $(:\mathbb{R}_{\ge1}\to\mathbb{R}_{\ge0})$ denote the inverse function of $\cosh|_{\mathbb{R}_{\ge0}}:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge1}$.

Proof of Proposition 2.10. Let us fix $L\in\mathbb{N}$ and $y\in(-1,-1/2]$. Define the function $F_L:\mathbb{R}\to\mathbb{R}$ by

$$F_L(x):=\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\left(\frac{\sinh(xE(\mathbf{k}))}{(y+\cosh(xE(\mathbf{k})))E(\mathbf{k})}\right).$$

There are $e_j\in[e_{min},e_{max}]$ $(j=1,2,\cdots,bL^d)$ such that $e_{max}\ge e_1\ge e_2\ge\cdots\ge e_{bL^d}\ge e_{min}$ and

$$F_L(x)=\frac{1}{L^d}\sum_{j=1}^{bL^d}u(x,y,e_j),$$

where $u(\cdot)$ is the function defined in (2.27). Let us prove that

$$\exists x_0 \in \left[\frac{1}{e_{max}}\cosh^{-1}(|y|^{-1}), \frac{1}{e_{min}}\cosh^{-1}(|y|^{-1})\right] \tag{2.33}$$
$$\text{s.t.}\quad \begin{array}{l} \frac{d}{dx}F_L(x) > 0, \quad (\forall x \in (0, x_0)), \\ \frac{d}{dx}F_L(x_0) = 0, \\ \frac{d}{dx}F_L(x) < 0, \quad (\forall x \in (x_0, \infty)). \end{array}$$

We can check by calculation that for any $z \in \mathbb{R}_{>0}$

$$\begin{aligned} &\frac{\partial u}{\partial x}(x, y, z) > 0, \quad \left(\forall x \in \left(0, \frac{1}{z}\cosh^{-1}(|y|^{-1})\right)\right), \\ &\frac{\partial u}{\partial x}\left(\frac{1}{z}\cosh^{-1}(|y|^{-1}), y, z\right) = 0, \\ &\frac{\partial u}{\partial x}(x, y, z) < 0, \quad \left(\forall x \in \left(\frac{1}{z}\cosh^{-1}(|y|^{-1}), \infty\right)\right), \\ &\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{z}\cosh^{-1}(|y|^{-1}), y, z\right) < 0. \end{aligned} \tag{2.34}$$

Thus, if $e_1 = e_{bL^d}$, the claim (2.33) holds with $x_0 = \frac{1}{e_1}\cosh^{-1}(|y|^{-1})$. Let us assume that $e_1 > e_{bL^d}$. This obviously implies that $e_{max} > e_{min}$. We can deduce from (2.34) that

$$\begin{aligned} &\frac{d}{dx}F_L(x) > 0, \quad \left(\forall x \in \left(0, \frac{1}{e_1}\cosh^{-1}(|y|^{-1})\right]\right), \\ &\frac{d}{dx}F_L(x) < 0, \quad \left(\forall x \in \left[\frac{1}{e_{bL^d}}\cosh^{-1}(|y|^{-1}), \infty\right)\right). \end{aligned}$$

Thus there exists $x_0 \in (\frac{1}{e_1}\cosh^{-1}(|y|^{-1}), \frac{1}{e_{bL^d}}\cosh^{-1}(|y|^{-1}))$ such that $\frac{d}{dx}F_L(x_0) = 0$. Set

$$c_{max} := \sup_{y \in (-1, -\frac{1}{2}]} \frac{\cosh^{-1}(|y|^{-1})}{\sqrt{y+1}}. \tag{2.35}$$

By using the equality

$$\cosh^{-1}(|y|^{-1}) = \log(|y|^{-1} + \sqrt{|y|^{-2} - 1}), \tag{2.36}$$

one can confirm that $0 < c_{max} < \infty$. It follows that

$$\left|\frac{e_j}{\sqrt{y+1}}x_0\right| \le c_{max}\frac{e_{max}}{e_{min}}, \quad (\forall j \in \{1,\cdots,bL^d\}).$$

Then we can apply Lemma 2.12 to conclude that if

$$|y+1| < \frac{c_1}{2}\cdot\frac{\frac{e_{min}}{e_{max}}\left(\left(\frac{e_{min}}{e_{max}}\right)^2 - 17 + 12\sqrt{2}\right)}{\cosh^2\left(2c_{max}\frac{e_{max}}{e_{min}}\right)\cosh^2\left(c_{max}\frac{e_{max}}{e_{min}}\right)} \tag{2.37}$$

and $e_j > e_{bL^d}$,

$$\frac{\partial u}{\partial x}(x_0,y,e_j)\frac{\partial^2 u}{\partial x^2}(x_0,y,e_{bL^d}) - \frac{\partial^2 u}{\partial x^2}(x_0,y,e_j)\frac{\partial u}{\partial x}(x_0,y,e_{bL^d}) > 0.$$

Since $e_1 > e_{bL^d}$, this implies that

$$\frac{\partial u}{\partial x}(x_0,y,e_{bL^d})\frac{d^2}{dx^2}F_L(x_0) = \frac{1}{L^d}\sum_{j=1}^{bL^d}\frac{\partial^2 u}{\partial x^2}(x_0,y,e_j)\frac{\partial u}{\partial x}(x_0,y,e_{bL^d})$$
$$< \frac{1}{L^d}\sum_{j=1}^{bL^d}\frac{\partial u}{\partial x}(x_0,y,e_j)\frac{\partial^2 u}{\partial x^2}(x_0,y,e_{bL^d}) = \frac{d}{dx}F_L(x_0)\frac{\partial^2 u}{\partial x^2}(x_0,y,e_{bL^d}) = 0.$$

Since $x_0 \in (0, \frac{1}{e_{bL^d}}\cosh^{-1}(|y|^{-1}))$, $\frac{\partial u}{\partial x}(x_0,y,e_{bL^d}) > 0$ by (2.34). Thus we obtain that $\frac{d^2}{dx^2}F_L(x_0) < 0$. It follows from the above argument that if $e_1 > e_{bL^d}$ and (2.37) holds, the claim (2.33) holds. This can be confirmed as follows. Suppose that $x_1, x_2 \in [\frac{1}{e_{max}}\cosh^{-1}(|y|^{-1}), \frac{1}{e_{min}}\cosh^{-1}(|y|^{-1})]$, $x_1 < x_2$ and $\frac{d}{dx}F_L(x_j) = 0$ for $j = 1,2$. Since the function $\frac{d}{dx}F_L(\cdot)$ is non-constant and real analytic in $\mathbb{R}_{>0}$,

$$\sharp\left\{x \in [x_1,x_2] \ \Big|\ \frac{d}{dx}F_L(x) = 0\right\} < \infty.$$

Thus, there exists $x_3 \in (x_1, x_2]$ such that $\frac{d}{dx}F_L(x_3) = 0$ and $\frac{d}{dx}F_L(x) \neq 0$ for any $x \in (x_1, x_3)$. Since $\frac{d^2}{dx^2}F_L(x_j) < 0$ for $j = 1,3$, there exists $x_4 \in (x_1, x_3)$ such that $\frac{d}{dx}F_L(x_4) = 0$, which is a contradiction. Now we can conclude that under the assumption (2.37) the claim (2.33) holds.

Define the function $F_\infty : \mathbb{R} \times (-1,0) \to \mathbb{R}$ by

$$F_\infty(x,y) := D_d\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(xE(\mathbf{k}))}{(y+\cosh(xE(\mathbf{k})))E(\mathbf{k})}\right). \tag{2.38}$$

Since $E \in C^\infty(\mathbb{R}^d, \mathrm{Mat}(b,\mathbb{C}))$, for any $y \in (-1,-\frac{1}{2}]$ $\frac{d}{dx}F_L(\cdot)$ converges to $\frac{\partial F_\infty}{\partial x}(\cdot, y)$ locally uniformly as $L \to \infty$. Therefore if $y \in (-1,-\frac{1}{2}]$ satisfies (2.37), there exists $\hat{x} \in [\frac{1}{e_{max}} \cosh^{-1}(|y|^{-1}), \frac{1}{e_{min}} \cosh^{-1}(|y|^{-1})]$ such that

$$\begin{aligned} &\frac{\partial F_\infty}{\partial x}(x,y) \geq 0, \quad (\forall x \in (0,\hat{x})),\\ &\frac{\partial F_\infty}{\partial x}(\hat{x},y) = 0,\\ &\frac{\partial F_\infty}{\partial x}(x,y) \leq 0, \quad (\forall x \in (\hat{x},\infty)). \end{aligned} \tag{2.39}$$

Let us recall that the assumption (2.12) implies (2.15) and (2.17). If we assume that

$$|U| \leq \frac{\min\left\{1, \frac{c_1}{2}\right\} \frac{e_{min}^2}{e_{max}} \left(\left(\frac{e_{min}}{e_{max}}\right)^2 - 17 + 12\sqrt{2}\right)}{\sinh(2) b \cosh^2\left(2c_{max}\frac{e_{max}}{e_{min}}\right) \cosh^2\left(c_{max}\frac{e_{max}}{e_{min}}\right)}, \tag{2.40}$$

(2.12) holds. Thus, by (2.17) $\cos(\tau(\beta)/2) \in (-1,-1/2]$ for all $\beta \in (0,\beta_c)$. Moreover, (2.15) and (2.40) again ensure that (2.37) holds with $y = \cos(\tau(\beta)/2)$ for any $\beta \in (0,\beta_c)$. Let us note that the right-hand side of (2.40) does not depend on E ($\in \mathcal{E}(e_{min}, e_{max})$). These properties combined with (2.39) imply that on the assumption (2.40) for any $E \in \mathcal{E}(e_{min}, e_{max})$, $\beta \in (0,\beta_c)$ there exists $\tilde{x} \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} &\frac{\partial g_E}{\partial x}(x,\tau(\beta),0) \geq 0, \quad (\forall x \in (0,\tilde{x})),\\ &\frac{\partial g_E}{\partial x}(\tilde{x},\tau(\beta),0) = 0,\\ &\frac{\partial g_E}{\partial x}(x,\tau(\beta),0) \leq 0, \quad (\forall x \in (\tilde{x},\infty)). \end{aligned} \tag{2.41}$$

Finally let us prove that $\tau(\cdot)$ has one and only one local minimum point in $(0,\beta_c)$. Suppose that $0 < \beta_1 < \beta_2 < \beta_c$ and β_1, β_2 are local minimum points. If $\tau(\beta_1) \leq \tau(\beta_2)$, there exist β_1', β_2', $\beta_3' \in (0,\beta_2]$ such that $\beta_1' < \beta_2' < \beta_3'$ and $\tau(\beta_1') = \tau(\beta_2') = \tau(\beta_3')$. If $\tau(\beta_1) > \tau(\beta_2)$, we can take such β_1', β_2', β_3' from $[\beta_1,\beta_c)$. It follows that $g_E(\beta_j', \tau(\beta_1'), 0) = 0$ for all $j \in \{1,2,3\}$. By (2.41) there exists $\tilde{x} \in \mathbb{R}_{>0}$ such that

$$\frac{\partial g_E}{\partial x}(x,\tau(\beta_1'),0) \geq 0, \quad (\forall x \in (0,\tilde{x})),$$

$$\frac{\partial g_E}{\partial x}(\tilde{x},\tau(\beta_1'),0)=0,$$
$$\frac{\partial g_E}{\partial x}(x,\tau(\beta_1'),0)\le 0,\quad (\forall x\in(\tilde{x},\infty)).$$

If $\tilde{x}\in(0,\beta_2']$, the function $x\mapsto g_E(x,\tau(\beta_1'),0)$ must be identically zero in $[\beta_2',\beta_3']$. Since this function is real analytic in $\mathbb{R}_{>0}$, the identity theorem ensures that this function is identically zero in $\mathbb{R}_{>0}$, which is a contradiction. If $\tilde{x}\in(\beta_2',\infty)$, this function must be identically zero in $[\beta_1',\beta_2']$, which also leads to a contradiction. Therefore, if $\tau(\cdot)$ has a local minimum point in $(0,\beta_c)$, it must be unique. Let us define the function $\hat{\tau}(\cdot):[0,\beta_c]\to\mathbb{R}$ as follows. $\hat{\tau}(x):=2\pi$ for $x\in\{0,\beta_c\}$, $\hat{\tau}(x):=\tau(x)$ for $x\in(0,\beta_c)$. By Lemma 2.2, $\hat{\tau}\in C([0,\beta_c])$ and $\hat{\tau}(x)\le\hat{\tau}(0)=\hat{\tau}(\beta_c)$ for any $x\in[0,\beta_c]$. Thus $\hat{\tau}(\cdot)$ attains its global minimum in $(0,\beta_c)$, which implies that $\tau(\cdot)$ has a local minimum point in $(0,\beta_c)$. The proof is complete. □

Next we will prove that the conclusion of Proposition 2.10 does not hold if $e_{min}/e_{max}\le\sqrt{17-12\sqrt{2}}$. We divide the problem into two cases, $e_{min}/e_{max}=\sqrt{17-12\sqrt{2}}$ or $e_{min}/e_{max}<\sqrt{17-12\sqrt{2}}$. The following proposition states the result for the case that the equality holds.

PROPOSITION 2.13. *Assume that $e_{min}/e_{max}=\sqrt{17-12\sqrt{2}}$. Then for any d, $b\in\mathbb{N}$, basis $(\hat{\mathbf{v}}_j)_{j=1}^d$ of $\mathbb{R}^d$, $U_0\in(0,2e_{min}/b)$ there exist $U\in[-U_0,0)$ and $E\in\mathcal{E}(e_{min},\ e_{max})$ such that $\tau(\cdot)$ has more than one local minimum points in $(0,\beta_c)$.*

REMARK 2.14. We should stress that in our proof we construct such $E(\in\mathcal{E}(e_{min},\ e_{max}))$ depending on U_0. On the contrary, we will construct $E(\in\mathcal{E}(e_{min},e_{max}))$ independently of the magnitude of the coupling constant when we deal with the case $e_{min}/e_{max}<\sqrt{17-12\sqrt{2}}$ in Proposition 2.16.

Let us show a lemma which we need to prove the above proposition. Set

(2.42)
$$D:=\left\{(x,y,z)\in\mathbb{R}_{>0}\times(-1,0)\times\mathbb{R}_{>0}\ \middle|\ x<\frac{1}{2z(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right\}.$$

Define the function $w:D\to\mathbb{R}$ by

$$w(x,y,z):=-\frac{(1+y\cosh(\sqrt{y+1}\sqrt{2x}))(y+\cosh(\sqrt{y+1}\sqrt{2zx}))^2}{(1+y\cosh(\sqrt{y+1}\sqrt{2zx}))(y+\cosh(\sqrt{y+1}\sqrt{2x}))^2}. \quad (2.43)$$

The necessary lemma concerns properties of the function w. For $(x,y,z) \in D$ we can rewrite as follows.

(2.44)

$$w(x,y,z) = -\frac{\left(1 + y\sum_{m=1}^{\infty}\frac{(y+1)^{m-1}}{(2m)!}2^m x^m\right)\left(1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n z^n x^n\right)^2}{\left(1 + y\sum_{m=1}^{\infty}\frac{(y+1)^{m-1}}{(2m)!}2^m z^m x^m\right)\left(1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n x^n\right)^2}.$$

Define the open set $\tilde{D}$ of $\mathbb{C}^3$ by

(2.45)

$$\tilde{D} := \left\{ (x,y,z) \in \mathbb{C}^3 \;\middle|\; \left|1 + y\sum_{m=1}^{\infty}\frac{(y+1)^{m-1}}{(2m)!}2^m z^m x^m\right|\left|1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n x^n\right|^2 > 0 \right\}.$$

Then we can define the analytic function $\tilde{w} : \tilde{D} \to \mathbb{C}$ by the right-hand side of (2.44). It follows that $\tilde{w}|_D = w$. It will often be more convenient to deal with $\tilde{w}$ than w during our construction. Note that for $z \in \mathbb{R}_{>0}$ and $x \in (0, z^{-1})$, $(x,-1,z) \in \tilde{D}$. We will particularly use the following equalities. For $z \in \mathbb{R}_{>0}$ and $x \in (0,z^{-1})$

(2.46)

$$\tilde{w}(x,-1,z) = \frac{(x-1)(1+zx)^2}{(1-zx)(1+x)^2},$$

(2.47)

$$\frac{\partial \tilde{w}}{\partial x}(x,-1,z) = \frac{3z(1-z)(1+zx)}{(1-zx)^2(1+x)^3}\left(x^2 - \frac{z+1}{3z}x + \frac{1}{z}\right),$$

(2.48)

$$\frac{\partial \tilde{w}}{\partial y}(x,-1,z) = -\frac{x(1+zx)}{6(1-zx)^2(x+1)^3} \cdot ((6+3x+x^2)(1-z^2x^2) + z(x^2-1)(6+3zx+z^2x^2)).$$

To shorten subsequent formulas, let us set $a_0 := 3 + 2\sqrt{2}$, $\eta_0 := 17 - 12\sqrt{2}$.

LEMMA 2.15. *There exists $y_0 \in (-1,0)$ such that for any $y \in (-1, y_0]$*

$$\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2 < a_0 < \frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2, \tag{2.49}$$

(2.50) $\quad 0 < w(a_0, y, \eta_0) < 1.$

Moreover, there exist

$$x_1(y) \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, a_0\right),$$
$$x_2(y) \in \left(a_0, \frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right)$$

such that

$$w(x_1(y), y, \eta_0) = w(a_0, y, \eta_0) = w(x_2(y), y, \eta_0),$$
$$w(x, y, \eta_0) > w(a_0, y, \eta_0), \quad (\forall x \in (x_1(y), a_0)),$$
$$w(x, y, \eta_0) < w(a_0, y, \eta_0), \quad (\forall x \in (a_0, x_2(y))).$$

PROOF. The following equalities are useful.

$$\eta_0^2 - 34\eta_0 + 1 = 0, \tag{2.51}$$

$$a_0 = \frac{\eta_0 + 1}{6\eta_0}, \tag{2.52}$$

$$a_0(\eta_0 + 1) = 6, \tag{2.53}$$

$$a_0^2 = \frac{1}{\eta_0}, \tag{2.54}$$

$$a_0^2 - \frac{\eta_0 + 1}{3\eta_0} a_0 + \frac{1}{\eta_0} = 0. \tag{2.55}$$

We can deduce from (2.36) that

$$\begin{aligned} &\lim_{y \searrow -1} \frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2 = 1 < a_0 < \frac{1}{\eta_0} \\ &= \lim_{y \searrow -1} \frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2. \end{aligned} \tag{2.56}$$

This implies that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $(a_0, y, \eta_0) \in D$ for any $y \in (-1, -1+\varepsilon)$. Moreover, $(a_0, -1, \eta_0) \in \tilde{D}$. By multiplying both the denominator and the numerator of (2.46) by a_0^2 and using (2.54) we can derive that

$$\tilde{w}(a_0, -1, \eta_0) = \frac{1}{a_0}. \tag{2.57}$$

Thus, there exists $y_1 \in (-1, -1+\varepsilon)$ such that for any $y \in (-1, y_1]$ (2.49) and (2.50) hold. Also, by (2.47) and (2.55) $\frac{\partial \tilde{w}}{\partial x}(a_0, -1, \eta_0) = 0$. Next let us compute $\frac{\partial^2 \tilde{w}}{\partial x \partial y}(a_0, -1, \eta_0)$. The computation can be quite complicated if we follow a wrong way. Let us present right steps leading to a concise formula, though this would not be the only approach. Let us decompose the right-hand side of (2.48) as follows.

$$\begin{aligned}
&\frac{\partial \tilde{w}}{\partial y}(x, -1, \eta_0) = w_1(x) w_2(x),\\
&w_1(x) := -\frac{x(1+\eta_0 x)}{6(1-\eta_0 x)^2 (x+1)^3},\\
&w_2(x) := (6+3x+x^2)(1-\eta_0^2 x^2) + \eta_0(x^2-1)(6+3\eta_0 x + \eta_0^2 x^2).
\end{aligned}$$

Using (2.54), (2.52), (2.53), (2.54) in this order, we obtain that

(2.58)

$$\begin{aligned}
\frac{dw_1}{dx}(a_0) &= -\frac{1+(3\eta_0-2)a_0+3\eta_0 a_0^2+3\eta_0^2 a_0^3}{6(1-\eta_0 a_0)^3(a_0+1)^4} = -\frac{5+\eta_0-2a_0}{6(1-\eta_0 a_0)^3(a_0+1)^4}\\
&= \frac{1-\eta_0}{6(1-\eta_0 a_0)^3(a_0+1)^3} = -\frac{w_1(a_0)}{a_0}.
\end{aligned}$$

By using (2.51) and (2.54) repeatedly

$$w_2(a_0) = (1-\eta_0)(46+3(1+\eta_0)a_0). \tag{2.59}$$

By using (2.54) only,

$$\frac{dw_2}{dx}(a_0) = (1-\eta_0)(2(\eta_0^2+5\eta_0+1)a_0 + 3(1+\eta_0)).$$

Then by using (2.51) and (2.54) again

$$a_0 \frac{dw_2}{dx}(a_0) = (1-\eta_0)(78+3(1+\eta_0)a_0). \tag{2.60}$$

By combining (2.58), (2.59), (2.60) and using (2.54) once

$$\begin{aligned}
\frac{\partial^2 \tilde{w}}{\partial x \partial y}(a_0, -1, \eta_0) &= -\frac{w_1(a_0)}{a_0}\left(w_2(a_0) - a_0 \frac{dw_2}{dx}(a_0)\right) = 32(1-\eta_0)\frac{w_1(a_0)}{a_0}\\
&= -\frac{16(1-\eta_0)^2}{3(1-\eta_0 a_0)^3(a_0+1)^3}.
\end{aligned}$$

Since $\frac{\partial^2 \tilde{w}}{\partial x \partial y}(a_0, -1, \eta_0) < 0$, there exists $y_2 \in (-1, y_1]$ such that

$$\frac{\partial^2 \tilde{w}}{\partial x \partial y}(a_0, -1, \eta_0) + \sup_{t \in [-1, y_1]} \left| \frac{\partial^3 \tilde{w}}{\partial x \partial y^2}(a_0, t, \eta_0) \right| (y_2 + 1) < 0.$$

Since $\frac{\partial \tilde{w}}{\partial x}(a_0, -1, \eta_0) = 0$, this estimate ensures that for any $y \in (-1, y_2]$

$$\begin{aligned} &\frac{\partial \tilde{w}}{\partial x}(a_0, y, \eta_0) \\ &= \frac{\partial^2 \tilde{w}}{\partial x \partial y}(a_0, -1, \eta_0)(y+1) + \int_{-1}^{y} dt (y-t) \frac{\partial^3 \tilde{w}}{\partial x \partial y^2}(a_0, t, \eta_0) \\ &\le \left(\frac{\partial^2 \tilde{w}}{\partial x \partial y}(a_0, -1, \eta_0) + \sup_{t \in [-1, y_1]} \left| \frac{\partial^3 \tilde{w}}{\partial x \partial y^2}(a_0, t, \eta_0) \right| (y_2 + 1) \right) (y+1) \\ &< 0. \end{aligned} \tag{2.61}$$

Let us fix $y \in (-1, y_2]$. Observe that

$$\lim_{x \searrow \frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2} w(x, y, \eta_0) = 0,$$
$$\lim_{x \nearrow \frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2} w(x, y, \eta_0) = \infty,$$

which combined with the inequality (2.61) imply the existence of $x_1(y)$, $x_2(y)$ with the claimed properties. □

Define the function $W : \mathbb{R}_{>0} \times (-1, 0) \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$W(x, y, z, s) := \frac{\sinh(x)}{y + \cosh(x)} + s \frac{\sinh(zx)}{(y + \cosh(zx))z}. \tag{2.62}$$

We will use this function and the functions $w : D \to \mathbb{R}$, $\tilde{w} : \tilde{D} \to \mathbb{C}$ in the rest of this section mainly for organizing proofs.

Proof of Proposition 2.13. By Lemma 2.15 there exists $y_0 \in (-1, 0)$ such that for any $y \in (-1, y_0]$

$$\frac{1}{\sqrt{y+1}} \cosh^{-1}(|y|^{-1}) < \sqrt{2a_0} < \frac{1}{\sqrt{\eta_0 (y+1)}} \cosh^{-1}(|y|^{-1}), \tag{2.63}$$

$$0 < w(a_0, y, \eta_0) < 1.$$

Observe that by (2.63) and the inequality $\sinh(x) \ge x$ ($\forall x \in \mathbb{R}_{>0}$),

$$\begin{aligned}(2.64)\quad &\frac{b}{w(a_0,y,\eta_0)+1}W(\sqrt{2a_0(y+1)}, y, \sqrt{\eta_0}, w(a_0,y,\eta_0)) \\ &\ge \frac{b\cosh^{-1}(|y|^{-1})}{y+\cosh(\eta_0^{-\frac{1}{2}}\cosh^{-1}(|y|^{-1}))}\end{aligned}$$

for any $y \in (-1, y_0]$. Take any $U_0 \in (0, 2e_{min}/b)$. By using (2.36) one can check that the right-hand side of (2.64) diverges to $+\infty$ as $y \searrow -1$. Thus, there exists $y_1 \in (-1, y_0]$ such that for any $y \in (-1, y_1]$

$$(2.65)\quad \frac{b}{w(a_0,y,\eta_0)+1}W(\sqrt{2a_0(y+1)}, y, \sqrt{\eta_0}, w(a_0,y,\eta_0)) > \frac{2}{U_0}.$$

Note that for $(x, y, z) \in \mathbb{R}_{>0} \times (-1, 0) \times \mathbb{R}_{>0}$ satisfying $x < \frac{1}{\sqrt{z(y+1)}}\cosh^{-1}(|y|^{-1})$,

$$\begin{aligned}(2.66)\quad &\frac{\partial W}{\partial x}(\sqrt{y+1}\cdot x, y, \sqrt{z}, s) \\ &= \frac{1+y\cosh(\sqrt{z(y+1)}\cdot x)}{(y+\cosh(\sqrt{z(y+1)}\cdot x))^2}\left(s - w\left(\frac{x^2}{2}, y, z\right)\right).\end{aligned}$$

Let us fix $y \in (-1, y_1]$. Lemma 2.15 ensures that there exist

$$\begin{aligned}\hat{x}_1(y) &\in \left(\frac{1}{\sqrt{y+1}}\cosh^{-1}(|y|^{-1}), \sqrt{2a_0}\right), \\ \hat{x}_2(y) &\in \left(\sqrt{2a_0}, \frac{1}{\sqrt{\eta_0(y+1)}}\cosh^{-1}(|y|^{-1})\right)\end{aligned}$$

such that

$$\begin{aligned}&\frac{\partial W}{\partial x}(\sqrt{y+1}\cdot\hat{x}_j(y), y, \sqrt{\eta_0}, w(a_0,y,\eta_0)) \\ &= \frac{\partial W}{\partial x}(\sqrt{2a_0(y+1)}, y, \sqrt{\eta_0}, w(a_0,y,\eta_0)) = 0, \quad (j = 1, 2), \\ &\frac{\partial W}{\partial x}(\sqrt{y+1}\cdot x, y, \sqrt{\eta_0}, w(a_0,y,\eta_0)) < 0, \quad (\forall x \in (\hat{x}_1(y), \sqrt{2a_0})),\end{aligned}$$

$$\frac{\partial W}{\partial x}(\sqrt{y+1}\cdot x, y, \sqrt{\eta_0}, w(a_0,y,\eta_0)) > 0, \quad (\forall x \in (\sqrt{2a_0}, \hat{x}_2(y))).$$

These imply that

$$\min_{j\in\{1,2\}} W(\sqrt{y+1}\cdot \hat{x}_j(y), y, \sqrt{\eta_0}, w(a_0,y,\eta_0)) > W(\sqrt{2a_0(y+1)}, y, \sqrt{\eta_0}, w(a_0,y,\eta_0)). \tag{2.67}$$

By (2.65), (2.67) we can take small $\delta\in\mathbb{R}_{>0}$ so that

$$\frac{1}{w(a_0,y,\eta_0)+1}-\delta>0,$$

(2.68)
$$\begin{aligned}\frac{2}{U_0} &< b\left(\frac{1}{w(a_0,y,\eta_0)+1}-\delta\right)\\ &\quad\cdot W\left(\sqrt{2a_0(y+1)}, y, \sqrt{\eta_0}, \frac{w(a_0,y,\eta_0)+\delta(w(a_0,y,\eta_0)+1)}{1-\delta(w(a_0,y,\eta_0)+1)}\right)\\ &< \frac{b}{w(a_0,y,\eta_0)+1}\min_{j\in\{1,2\}} W(\sqrt{y+1}\cdot\hat{x}_j(y), y, \sqrt{\eta_0}, w(a_0,y,\eta_0)).\end{aligned}$$

Here let us apply Lemma A.1 proved in Appendix A with $e_{min}=\sqrt{\eta_0}$, $e_{max}=1$, $s=\frac{1}{w(a_0,y,\eta_0)+1}-\delta$, $t=\frac{1}{w(a_0,y,\eta_0)+1}$. By substituting the matrix-valued function E into the function (2.38) and recalling the monotone decreasing property of the function (1.7) we observe that for any $x\in\mathbb{R}_{>0}$

(2.69)

$$\begin{aligned}&F_\infty(x,y)\\ &\ge bs\frac{\sinh(x)}{y+\cosh(x)}+b(t-s)\frac{\sinh(x)}{y+\cosh(x)}+b(1-t)\frac{\sinh(x\sqrt{\eta_0})}{(y+\cosh(x\sqrt{\eta_0}))\sqrt{\eta_0}}\\ &=\frac{b}{w(a_0,y,\eta_0)+1}W(x,y,\sqrt{\eta_0},w(a_0,y,\eta_0)),\\ &F_\infty(x,y)\\ &\le bs\frac{\sinh(x)}{y+\cosh(x)}+b(t-s)\frac{\sinh(x\sqrt{\eta_0})}{(y+\cosh(x\sqrt{\eta_0}))\sqrt{\eta_0}}\\ &\quad+b(1-t)\frac{\sinh(x\sqrt{\eta_0})}{(y+\cosh(x\sqrt{\eta_0}))\sqrt{\eta_0}}\end{aligned}$$

$$= b\left(\frac{1}{w(a_0,y,\eta_0)+1} - \delta\right)$$
$$\cdot W\left(x, y, \sqrt{\eta_0}, \frac{w(a_0,y,\eta_0)+\delta(w(a_0,y,\eta_0)+1)}{1-\delta(w(a_0,y,\eta_0)+1)}\right).$$

By combining these inequalities with (2.68) we have that

$$F_\infty(\sqrt{2a_0(y+1)}, y) < \min_{j\in\{1,2\}} F_\infty(\sqrt{y+1}\cdot \hat{x}_j(y), y),$$
$$\frac{2}{U_0} < \min_{j\in\{1,2\}} F_\infty(\sqrt{y+1}\cdot \hat{x}_j(y), y).$$

This implies that there exists $U \in [-U_0, 0)$ such that

$$F_\infty(\sqrt{2a_0(y+1)}, y) < \frac{2}{|U|} < \min_{j\in\{1,2\}} F_\infty(\sqrt{y+1}\cdot \hat{x}_j(y), y).$$

Therefore, by taking into account the fact $F_\infty(0, y) = 0$ we see that there exist

$$\beta_1 \in \big(0, \sqrt{y+1}\cdot \hat{x}_1(y)\big),$$
$$\beta_2 \in \big(\sqrt{y+1}\cdot \hat{x}_1(y), \sqrt{2a_0(y+1)}\big),$$
$$\beta_3 \in \big(\sqrt{2a_0(y+1)}, \sqrt{y+1}\cdot \hat{x}_2(y)\big)$$

such that $-2/|U| + F_\infty(\beta_j, y) = 0$ for all $j \in \{1,2,3\}$. Moreover, it follows from Lemma 2.1 that $0 < \beta_1 < \beta_2 < \beta_3 < \beta_c$, $y = \cos(\tau(\beta_j)/2)$ for all $j \in \{1,2,3\}$, and thus $\tau(\beta_1) = \tau(\beta_2) = \tau(\beta_3)$.

Finally let us prove that there exist $\hat{\beta}_1$, $\hat{\beta}_2 \in (0, \beta_c)$ such that $\hat{\beta}_1 < \hat{\beta}_2$ and these are local minimum points of $\tau(\cdot)$. If $\tau(\beta) = \tau(\beta_2)$ $(\forall \beta \in (\beta_1, \beta_2))$ or $\tau(\beta) = \tau(\beta_2)$ $(\forall \beta \in (\beta_2, \beta_3))$, such $\hat{\beta}_1$, $\hat{\beta}_2$ obviously exist. If there exists $\beta' \in (\beta_1, \beta_2)$ such that $\tau(\beta') > \tau(\beta_2)$, since $\lim_{\beta\searrow 0}\tau(\beta) = \lim_{\beta\nearrow\beta_c}\tau(\beta) = 2\pi > \tau(\beta_1) = \tau(\beta_2)$, local minimum points $\hat{\beta}_1$, $\hat{\beta}_2$ exist in $(0, \beta')$, (β', β_c) respectively. The same conclusion holds if there exists $\beta' \in (\beta_2, \beta_3)$ such that $\tau(\beta') > \tau(\beta_2)$. It remains to study the case that there are $\beta_1' \in (\beta_1, \beta_2)$, $\beta_2' \in (\beta_2, \beta_3)$ such that $\tau(\beta_j') < \tau(\beta_2)$ for $j \in \{1, 2\}$. In this case local minimum points $\hat{\beta}_1$, $\hat{\beta}_2$ exist in (β_1, β_2), (β_2, β_3) respectively. The proposition has been proved. □

A stronger conclusion than Proposition 2.13 holds when $e_{min}/e_{max} < \sqrt{17 - 12\sqrt{2}}$.

PROPOSITION 2.16. *Assume that* $e_{min}/e_{max} < \sqrt{17-12\sqrt{2}}$. *Then for any* d, $b \in \mathbb{N}$, *basis* $(\hat{\mathbf{v}}_j)_{j=1}^d$ *of* $\mathbb{R}^d$ *there exist* $E \in \mathcal{E}(e_{min}, e_{max})$ *and* $U_0 \in (0, 2e_{min}/b)$ *such that for any* $U \in [-U_0, 0)$ $\tau(\cdot)$ *has more than one local minimum points in* $(0, \beta_c)$.

REMARK 2.17. The difference from the conclusion of Proposition 2.13 is that here E $(\in \mathcal{E}(e_{min}, e_{max}))$ is independent of the choice of small U. This conclusion implies the conclusion of Proposition 2.13.

Observe that for $\eta \in (0, 17-12\sqrt{2}]$, $(\frac{1+\eta}{6\eta})^2 - \frac{1}{\eta} \geq 0$. This allows us to define the real numbers $a_+(\eta)$, $a_-(\eta)$, $\hat{a}(\eta)$ by

$$
\begin{aligned}
a_+(\eta) &:= \frac{1+\eta}{6\eta} + \left(\left(\frac{1+\eta}{6\eta}\right)^2 - \frac{1}{\eta}\right)^{\frac{1}{2}}, \\
a_-(\eta) &:= \frac{1+\eta}{6\eta} - \left(\left(\frac{1+\eta}{6\eta}\right)^2 - \frac{1}{\eta}\right)^{\frac{1}{2}}, \\
\hat{a}(\eta) &:= a_-(\eta) + \frac{a_+(\eta) - a_-(\eta)}{2}.
\end{aligned} \tag{2.70}
$$

Let us summarize basic properties concerning these numbers, which can be deduced from (2.47), (2.52) and will be used not only in the proof of Proposition 2.16 but also in Sub-subsection 2.3.1.

LEMMA 2.18. *If* $\eta = 17 - 12\sqrt{2}$ $(= \eta_0)$,

$$
1 < a_+(\eta) = a_-(\eta) = \hat{a}(\eta) = a_0 = 3 + 2\sqrt{2} < \eta^{-1}. \tag{2.71}
$$

For any $\eta \in (0, 17 - 12\sqrt{2})$

$$
1 < a_-(\eta) < \hat{a}(\eta) < a_+(\eta) < \eta^{-1}, \tag{2.72}
$$

$$
\begin{aligned}
&\frac{\partial \tilde{w}}{\partial x}(x, -1, \eta) > 0, \quad (\forall x \in (0, a_-(\eta))), \\
&\frac{\partial \tilde{w}}{\partial x}(a_-(\eta), -1, \eta) = 0, \\
&\frac{\partial \tilde{w}}{\partial x}(x, -1, \eta) < 0, \quad (\forall x \in (a_-(\eta), a_+(\eta))), \\
&\frac{\partial \tilde{w}}{\partial x}(a_+(\eta), -1, \eta) = 0,
\end{aligned} \tag{2.73}
$$

$$\frac{\partial \tilde{w}}{\partial x}(x,-1,\eta) > 0, \quad (\forall x \in (a_+(\eta), \eta^{-1})),$$
$$0 < \tilde{w}(a_+(\eta),-1,\eta) < \tilde{w}(\hat{a}(\eta),-1,\eta) < \tilde{w}(a_-(\eta),-1,\eta). \tag{2.74}$$

PROOF OF PROPOSITION 2.16. Define the function $\widehat{W} : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$\widehat{W}(x,z,s) := \frac{x}{1+\frac{x^2}{2}} + s\frac{x}{1+z^2\frac{x^2}{2}}.$$

Let us observe that for $(x,z) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ satisfying $x < \sqrt{2/z}$

$$\frac{\partial \widehat{W}}{\partial x}(x,\sqrt{z},s) = \frac{1-z\frac{x^2}{2}}{(1+z\frac{x^2}{2})^2}\left(s - \tilde{w}\left(\frac{x^2}{2},-1,z\right)\right).$$

Fix $\eta \in (0, 17-12\sqrt{2})$. On the basis of (2.73) and the facts $\tilde{w}(1,-1,\eta) = 0$, $\lim_{x \nearrow \eta^{-1}} \tilde{w}(x,-1,\eta) = +\infty$, we conclude that there exist $x_1 \in (\sqrt{2}, \sqrt{2\hat{a}(\eta)})$, $x_2 \in (\sqrt{2\hat{a}(\eta)}, \sqrt{2\eta^{-1}})$ such that

$$\begin{aligned}
&\frac{\partial \widehat{W}}{\partial x}(x_j,\sqrt{\eta},\tilde{w}(\hat{a}(\eta),-1,\eta)) \\
&= \frac{\partial \widehat{W}}{\partial x}(\sqrt{2\hat{a}(\eta)},\sqrt{\eta},\tilde{w}(\hat{a}(\eta),-1,\eta)) = 0, \quad (\forall j \in \{1,2\}), \\
&\frac{\partial \widehat{W}}{\partial x}(x,\sqrt{\eta},\tilde{w}(\hat{a}(\eta),-1,\eta)) < 0, \quad (\forall x \in (x_1, \sqrt{2\hat{a}(\eta)})), \\
&\frac{\partial \widehat{W}}{\partial x}(x,\sqrt{\eta},\tilde{w}(\hat{a}(\eta),-1,\eta)) > 0, \quad (\forall x \in (\sqrt{2\hat{a}(\eta)}, x_2)).
\end{aligned}$$

These imply that

$$\min_{j\in\{1,2\}} \widehat{W}(x_j,\sqrt{\eta},\tilde{w}(\hat{a}(\eta),-1,\eta)) > \widehat{W}(\sqrt{2\hat{a}(\eta)},\sqrt{\eta},\tilde{w}(\hat{a}(\eta),-1,\eta)).$$

We can choose small $\delta \in \mathbb{R}_{>0}$ so that

$$\frac{1}{\tilde{w}(\hat{a}(\eta),-1,\eta)+1} - \delta > 0,$$
$$\frac{b}{\tilde{w}(\hat{a}(\eta),-1,\eta)+1}\min_{j\in\{1,2\}} \widehat{W}(x_j,\sqrt{\eta},\tilde{w}(\hat{a}(\eta),-1,\eta)) \tag{2.75}$$

$$> b\left(\frac{1}{\tilde{w}(\hat{a}(\eta),-1,\eta)+1}-\delta\right)$$
$$\cdot \widehat{W}\left(\sqrt{2\hat{a}(\eta)},\sqrt{\eta},\frac{\tilde{w}(\hat{a}(\eta),-1,\eta)+\delta(\tilde{w}(\hat{a}(\eta),-1,\eta)+1)}{1-\delta(\tilde{w}(\hat{a}(\eta),-1,\eta)+1)}\right).$$

Here we apply Lemma A.1 with $e_{min}=\sqrt{\eta}$, $e_{max}=1$, $s=\frac{1}{\tilde{w}(\hat{a}(\eta),-1,\eta)+1}-\delta$, $t=\frac{1}{\tilde{w}(\hat{a}(\eta),-1,\eta)+1}$. With the matrix-valued function E we define the function $\widehat{F}_\infty:\mathbb{R}\to\mathbb{R}$ by

$$\widehat{F}_\infty(x):=D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{x}{1+\frac{x^2}{2}E(\mathbf{k})^2}\right).$$

By arguing in the same way as in (2.69) we can derive from (2.75) that $\widehat{F}_\infty(\sqrt{2\hat{a}(\eta)})<\min_{j\in\{1,2\}}\widehat{F}_\infty(x_j)$. Check that $\lim_{y\searrow -1}\sqrt{y+1}F_\infty(\sqrt{y+1}\cdot x,y)=\widehat{F}_\infty(x)$ for all $x\in\mathbb{R}$, where $F_\infty(\cdot)$ is the function defined in (2.38). Thus, there exists $y_1(\eta)\in(-1,0)$ such that for any $y\in(-1,y_1(\eta)]$

$$F_\infty(\sqrt{2\hat{a}(\eta)(y+1)},y)<\min_{j\in\{1,2\}}F_\infty(\sqrt{y+1}\cdot x_j,y). \tag{2.76}$$

By recalling the monotone decreasing property of the function (1.7) we have that for any $y\in(-1,y_1(\eta)]$

$$\frac{b\sinh(\sqrt{2\hat{a}(\eta)(y+1)})}{y+\cosh(\sqrt{2\hat{a}(\eta)(y+1)})}\le F_\infty(\sqrt{2\hat{a}(\eta)(y+1)},y) \le \frac{b\sinh(\sqrt{2\hat{a}(\eta)(y+1)\eta})}{(y+\cosh(\sqrt{2\hat{a}(\eta)(y+1)\eta}))\sqrt{\eta}}. \tag{2.77}$$

Set

$$U_0:=\min\left\{\frac{e_{min}}{b},\frac{(y_1(\eta)+\cosh(\sqrt{2\hat{a}(\eta)(y_1(\eta)+1)\eta}))\sqrt{\eta}}{b\sinh(\sqrt{2\hat{a}(\eta)(y_1(\eta)+1)\eta})}\right\}.$$

It follows that $U_0\in(0,2e_{min}/b)$. Take any $U\in[-U_0,0)$. By (2.77)

$$F_\infty(\sqrt{2\hat{a}(\eta)(y_1(\eta)+1)},y_1(\eta))<\frac{2}{U_0}\le\frac{2}{|U|}.$$

Set

$$S := \left\{ y \in (-1, y_1(\eta)] \;\middle|\; F_\infty(\sqrt{2\hat{a}(\eta)(y+1)}, y) = \frac{2}{|U|} \right\}.$$

By considering the fact that the left-hand side of (2.77) diverges to $+\infty$ as $y \searrow -1$ we see that $S \neq \emptyset$. Set $y_2(\eta, U) := \sup S$. Then, $-1 < y_2(\eta, U) < y_1(\eta)$ and by (2.76)

$$\begin{aligned} &F_\infty(\sqrt{2\hat{a}(\eta)(y_2(\eta,U)+1)}, y_2(\eta,U)) = \frac{2}{|U|} \\ &< \min_{j\in\{1,2\}} F_\infty(\sqrt{y_2(\eta,U)+1}\cdot x_j, y_2(\eta,U)), \\ &F_\infty(\sqrt{2\hat{a}(\eta)(y+1)}, y) < \frac{2}{|U|}, \quad (\forall y \in (y_2(\eta,U), y_1(\eta)]). \end{aligned}$$

This implies that if we take $y_3(\eta, U) \in (y_2(\eta, U), y_1(\eta)]$ sufficiently close to $y_2(\eta, U)$,

$$\begin{aligned} &F_\infty(\sqrt{2\hat{a}(\eta)(y_3(\eta,U)+1)}, y_3(\eta,U)) < \frac{2}{|U|} \\ &< \min_{j\in\{1,2\}} F_\infty(\sqrt{y_3(\eta,U)+1}\cdot x_j, y_3(\eta,U)). \end{aligned}$$

Then we only need to repeat the same argument as in the last part of the proof of Proposition 2.13 to conclude that $\tau(\cdot)$ has at least two local minimum points. □

By combining Proposition 2.10, Proposition 2.13, Proposition 2.16 we reach the following theorem.

THEOREM 2.19. *For any $d, b \in \mathbb{N}$, basis $(\hat{\mathbf{v}}_j)_{j=1}^d$ of $\mathbb{R}^d$ and $e_{min}, e_{max} \in \mathbb{R}_{>0}$ satisfying $e_{min} \le e_{max}$ the following statements are equivalent to each other.*

(i) There exists $U_0 \in (0, 2e_{min}/b)$ such that for any $U \in [-U_0, 0)$ and $E \in \mathcal{E}(e_{min}, e_{max})$ $\tau(\cdot)$ has one and only one local minimum point in $(0, \beta_c)$.

(ii)

$$\frac{e_{min}}{e_{max}} > \sqrt{17 - 12\sqrt{2}}.$$

REMARK 2.20. According to Theorem 1.3, we have to take small U depending on $E \in \mathcal{E}(e_{min}, e_{max})$ in order to justify the derivation of the infinite-volume limit of the free energy density and the thermal expectations from the finite-volume lattice Fermion system. The graph $\{(\beta, \tau(\beta)) \mid \beta \in (0, \beta_c)\}$ can be rigorously considered as the representative curve of the phase boundaries of the phase transition happening in our system only if the derivation of the infinite-volume limit is justified. Here let us summarize what we can conclude by combining the results obtained in this section with the sufficient condition for justifying the derivation.

By Proposition 2.8 for any $E \in \mathcal{E}(e_{min}, e_{max})$ with $e_{min}/e_{max} \geq e_0$ there exists $U_0 \in (0, 2e_{min}/b)$ such that for any $U \in [-U_0, 0)$ the derivation is justified and $\frac{d^2\tau}{d\beta^2}(\beta) > 0$ for any $\beta \in (0, \beta_c)$.

By Proposition 2.10 for any $E \in \mathcal{E}(e_{min}, e_{max})$ with $e_{min}/e_{max} > \sqrt{17 - 12\sqrt{2}}$ there exists $U_0 \in (0, 2e_{min}/b)$ such that for any $U \in [-U_0, 0)$ the derivation is justified and $\tau(\cdot)$ has only one local minimum point in $(0, \beta_c)$.

By Proposition 2.16 for any e_{min}, $e_{max} \in \mathbb{R}_{>0}$ with $e_{min}/e_{max} < \sqrt{17 - 12\sqrt{2}}$ there exist $E \in \mathcal{E}(e_{min}, e_{max})$ and $U_0 \in (0, 2e_{min}/b)$ such that for any $U \in [-U_0, 0)$ the derivation is justified and $\tau(\cdot)$ has more than one local minimum points in $(0, \beta_c)$.

However, in Proposition 2.13 we do not have freedom to choose small U. The coupling constant U was chosen depending on E in the proof and it is not clear whether for such U the derivation is justified by Theorem 1.3. Thus, strictly speaking, in the case $e_{min}/e_{max} = \sqrt{17 - 12\sqrt{2}}$ we cannot claim that $\tau(\cdot)$ has more than one local minimum points while justifying the derivation.

REMARK 2.21. In view of Proposition 2.8, we can propose a problem to find a necessary and sufficient condition in terms of e_{min}/e_{max} for that $\tau(\cdot)$ is downward convex in $(0, \beta_c)$ for any $E \in \mathcal{E}(e_{min}, e_{max})$. However, we are unable to solve the problem at present.

2.3. Study of specific models

In the proofs of Proposition 2.13 and Proposition 2.16 we constructed particular examples of E ($\in \mathcal{E}(e_{min}, e_{max})$) for which $\tau(\cdot)$ has more than one local minimum points. However, these results do not tell us whether $\tau(\cdot)$ can have more than one local minimum points when we change the value e_{min}/e_{max} within a one-particle Hamiltonian explicitly parameterized by e_{min}, e_{max}, though we know that $\tau(\cdot)$ must have only one local minimum point for small U when $e_{min}/e_{max} > \sqrt{17-12\sqrt{2}}$ by Proposition 2.10. In this subsection we study this question for the following two models. Let I_n denote the $n \times n$ unit matrix for $n \in \mathbb{N}$.

(1) Let $d \in \mathbb{N}$, $b \in \mathbb{N}_{\ge 2}$, $b' \in \{1, 2, \cdots, b-1\}$ and $(\hat{\mathbf{v}}_j)_{j=1}^d$ be any basis of $\mathbb{R}^d$. Let us define $E_b \in \mathcal{E}(e_{min}, e_{max})$ by

$$E_b(\mathbf{k}) := ((e_{max}1_{i \le b'} + e_{min}1_{i>b'})1_{i=j})_{1 \le i,j \le b} = \begin{pmatrix} e_{max}I_{b'} & 0 \\ 0 & e_{min}I_{b-b'} \end{pmatrix},$$
$$(\mathbf{k} \in \mathbb{R}^d),$$

which is a b-orbital model without hopping.

(2) Let $d = b = 1$ and $\hat{\mathbf{v}}_1 = 1$. For $t \in \mathbb{R}_{\ge 0}$, $e_{min} \in \mathbb{R}_{>0}$ let us define $E_1 \in \mathcal{E}(e_{min}, 2t + e_{min})$ by $E_1(k) := t(\cos k + 1) + e_{min}$, $(k \in \mathbb{R})$. The function $E_1(\cdot)$ is the dispersion relation of nearest-neighbor hopping free electron on the 1-dimensional lattice $\mathbb{Z}$.

It will turn out that the uniqueness of local minimum points is sensitive to the ratios e_{min}/e_{max}, $(b-b')/b'$ in the model (1), while the uniqueness holds for any $t \in \mathbb{R}_{\ge 0}$, $e_{min} \in \mathbb{R}_{>0}$ in the model (2).

REMARK 2.22. For t, $\mu \in \mathbb{R}$ let us define the function $e_1 : \mathbb{R} \to \mathbb{R}$ by $e_1(k) := t \cos k + \mu$. The function $e_1(\cdot)$ satisfying the condition $\inf_{k \in \mathbb{R}} |e_1(k)| > 0$ is the most general form of a non-vanishing dispersion relation of nearest-neighbor hopping free electron on $\mathbb{Z}$. We can check that

$$\begin{aligned}&\int_0^{2\pi} dk \frac{\sinh(\beta e_1(k))}{(y + \cosh(\beta e_1(k)))e_1(k)} \\ &= \int_0^{2\pi} dk \frac{\sinh(\beta(|t| \cos k + |\mu|))}{(y + \cosh(\beta(|t| \cos k + |\mu|)))(|t| \cos k + |\mu|)}, \\ &(\forall \beta \in \mathbb{R}_{>0},\ y \in (-1, 0)).\end{aligned}$$

By using the above equality and the fact that $\inf_{k \in \mathbb{R}} |e_1(k)| > 0$ is equivalent to $|\mu| > |t|$ we can reduce the problem with $e_1(\cdot)$ to that with $E_1(\cdot)$ defined in (2). This means that the results we will obtain in Sub-subsection 2.3.2 for $E_1(\cdot)$ also hold for $e_1(\cdot)$ satisfying $\inf_{k \in \mathbb{R}} |e_1(k)| > 0$.

2.3.1 The multi-orbital model without hopping

Here let us study the profile of $\tau(\cdot)$ in the model defined in (1). Our central question is when $\tau(\cdot)$ has only one local minimum point. The answer is given in the next proposition.

PROPOSITION 2.23. *Set the condition ($\star$) as follows.*

($\star$) *There exists $U_0 \in (0, 2e_{min}/b)$ such that for any $U \in [-U_0, 0)$ $\tau(\cdot)$ has one and only one local minimum point in $(0, \beta_c)$.*

Then the following statements hold.

(i) *Assume that $\frac{b-b'}{b'} \in [3-2\sqrt{2}, \infty)$. Then for any e_{min}, $e_{max} \in \mathbb{R}_{>0}$ with $e_{min} \le e_{max}$ ($\star$) holds.*

(ii) *Assume that $\frac{b-b'}{b'} \in (1/8, 3-2\sqrt{2})$. Then there exist e_1, $e_2 \in (0, \sqrt{17-12\sqrt{2}})$ such that $e_1 < e_2$ and ($\star$) holds if $e_{min}/e_{max} \in (e_2, 1]$, ($\star$) does not hold if $e_{min}/e_{max} \in (e_1, e_2]$, ($\star$) holds if $e_{min}/e_{max} \in (0, e_1]$.*

(iii) *Assume that $\frac{b-b'}{b'} \in (0, 1/8]$. Then there exists $e_1 \in (0, \sqrt{17-12\sqrt{2}})$ such that ($\star$) holds if $e_{min}/e_{max} \in (e_1, 1]$, ($\star$) does not hold if $e_{min}/e_{max} \in (0, e_1]$.*

Again the proof of this proposition is based on some properties of the function w defined in (2.43). Let us set two conditions concerning the function w. Let $\eta \in (0,1)$, $s \in \mathbb{R}_{>0}$.

$(\mathrm{i})_{\eta,s}$ There exists $y_0 \in (-1, 0)$ such that for any $y \in (-1, y_0]$ there exists

$$x_0(y) \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right)$$

such that

$$\begin{aligned}
&w(x,y,\eta) < s, \quad \left(\forall x \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, x_0(y)\right)\right),\\
&w(x_0(y),y,\eta) = s,\\
&w(x,y,\eta) > s, \quad \left(\forall x \in \left(x_0(y), \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right)\right).
\end{aligned}$$

$(\mathrm{ii})_{\eta,s}$ There exists $y_0 \in (-1,0)$ such that for any $y \in (-1,y_0]$ there exist

$$x_j(y) \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right),$$
$$(j=1,2,3)$$

such that $x_1(y) < x_2(y) < x_3(y)$,

$$\begin{aligned} &w(x_j(y),y,\eta) = s, \quad (\forall j \in \{1,2,3\}),\\ &w(x,y,\eta) > s, \quad (\forall x \in (x_1(y),x_2(y))),\\ &w(x,y,\eta) < s, \quad (\forall x \in (x_2(y),x_3(y))). \end{aligned}$$

We summarize sufficient conditions for $(\mathrm{i})_{\eta,s}$ (or $(\mathrm{ii})_{\eta,s}$) to hold in the next lemma. To understand the statements, we should recall the inequalities (2.74).

LEMMA 2.24.

(i) Assume that $\eta = 17-12\sqrt{2}$. *Then for any* $s \in (0,\infty)$ $(i)_{\eta,s}$ *holds.*

(ii) Assume that $\eta \in (0, 17-12\sqrt{2})$. *Then for any* $s \in [\tilde{w}(a_-(\eta),-1,\eta), \infty)$ $(i)_{\eta,s}$ *holds. For any* $s \in [\tilde{w}(a_+(\eta),-1,\eta), \tilde{w}(a_-(\eta),-1,\eta))$ $(ii)_{\eta,s}$ *holds. For any* $s \in (0,\tilde{w}(a_+(\eta),-1,\eta))$ $(i)_{\eta,s}$ *holds.*

PROOF. Assume that $\eta \in (0,17-12\sqrt{2}]$. Let us prepare necessary basic properties related to the function w. The preparation continues until we prove the claim (2.86). Observe that there exists $y_0 \in (-1,0)$ such that

$$\frac{1}{\sqrt{2(y+1)}}\cosh^{-1}(|y|^{-1}) > 1, \quad (\forall y \in (-1,y_0]).$$

This claim can be proved efficiently by proving the equivalent statement that there exists $y_0 \in (-1,0)$ such that $|y|^{-1} > \cosh(\sqrt{2(y+1)})$ for any $y \in (-1,y_0]$. Moreover, by (2.71), (2.72) there exists $y_0(\eta) \in (-1,y_0]$ such that $|y|a_+(\eta) > 1$ for any $y \in (-1,y_0(\eta)]$. We can see from (2.44) and the limit in the left-hand side of (2.56) that for any $y \in (-1,y_0(\eta)]$, $\varepsilon \in (0,\eta^{-1}-a_+(\eta))$ and

$$x \in \left[\eta^{-1}-\varepsilon, \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right),$$

(2.78)

$$
\begin{aligned}
&w(x,y,\eta)\\
&\ge \frac{|y|(\eta^{-1}-\varepsilon)-1}{(1+y\eta(\eta^{-1}-\varepsilon))\left(1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n\left(\frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right)^n\right)^2},\\
&\lim_{y\searrow -1}(\text{R.H.S of (2.78)})=\frac{\eta^{-1}-\varepsilon-1}{\eta\varepsilon(1+\eta^{-1})^2}.
\end{aligned}
$$

Take any $s\in\mathbb{R}_{>0}$. Note that there exists $\varepsilon(s,\eta)\in(0,\eta^{-1}-a_+(\eta))$ such that

$$\frac{\eta^{-1}-\varepsilon(s,\eta)-1}{\eta\varepsilon(s,\eta)(1+\eta^{-1})^2}\ge 2s.$$

Then it follows from the above claims that there exists $y_1(s,\eta)\in(-1,y_0(\eta)]$ such that for any $y\in(-1,y_1(s,\eta)]$

$$
\begin{aligned}
1&<\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2<a_-(\eta)\le a_+(\eta)<\eta^{-1}-\varepsilon(s,\eta)\\
&<\frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2
\end{aligned}
\tag{2.79}
$$

and for any

$$x\in\left[\eta^{-1}-\varepsilon(s,\eta),\frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right),$$

$$w(x,y,\eta)>s. \tag{2.80}$$

Recall (2.45). To justify the subsequent argument, let us check that there exists $\delta(s,\eta)\in(0,\varepsilon(s,\eta))$ such that

$$
\begin{aligned}
&(1-\delta(s,\eta),\eta^{-1}-\varepsilon(s,\eta)+\delta(s,\eta))\times(-1-\delta(s,\eta),-1+\delta(s,\eta))\times\{\eta\}\\
&\subset\tilde{D}.
\end{aligned}
$$

We can deduce from (2.48) that $\sup_{x\in[1,\eta^{-1}-\varepsilon(s,\eta)]}\frac{\partial\tilde{w}}{\partial y}(x,-1,\eta)<0$. For $(x,y)\in[1,\eta^{-1}-\varepsilon(s,\eta)]\times(-1,-1+\delta(s,\eta)/2]$

$$\tilde{w}(x,y,\eta)$$

$$
\begin{aligned}
&= \tilde{w}(x,-1,\eta) + \frac{\partial \tilde{w}}{\partial y}(x,-1,\eta)(y+1) + \int_{-1}^{y} d\xi (y-\xi)\frac{\partial^2 \tilde{w}}{\partial y^2}(x,\xi,\eta)\\
&\le \tilde{w}(x,-1,\eta) + \sup_{\zeta\in[1,\eta^{-1}-\varepsilon(s,\eta)]}\frac{\partial \tilde{w}}{\partial y}(\zeta,-1,\eta)(y+1)\\
&\quad + \sup_{\substack{\zeta\in[1,\eta^{-1}-\varepsilon(s,\eta)]\\ \xi\in[-1,-1+\delta(s,\eta)/2]}}\left|\frac{\partial^2 \tilde{w}}{\partial y^2}(\zeta,\xi,\eta)\right|(y+1)^2.
\end{aligned}
$$

Recall (2.42). These imply that there exists $y_2(s,\eta)\in(-1,y_1(s,\eta)]$ such that $[1,\eta^{-1}-\varepsilon(s,\eta)]\times(-1,y_2(s,\eta)]\times\{\eta\}\subset D$ and

(2.81)

$$
w(x,y,\eta) < \tilde{w}(x,-1,\eta), \quad (\forall (x,y)\in[1,\eta^{-1}-\varepsilon(s,\eta)]\times(-1,y_2(s,\eta)]).
$$

We will also refer to the basic fact that

$$
w\left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, y, \eta\right) = 0, \quad (\forall y\in(-1,y_2(s,\eta)]). \tag{2.82}
$$

Let us define the function $\underline{w}:\mathbb{R}^3\to\mathbb{R}$ by

$$
\begin{aligned}
&\underline{w}(x,y,z)\\
&:= \left(1+y\sum_{m=1}^{\infty}\frac{(y+1)^{m-1}}{(2m)!}2^m x^m\right)\left(1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n z^n x^n\right)\\
&\quad\cdot\left(y\sum_{m=1}^{\infty}\frac{m(y+1)^{m-1}}{(2m)!}2^m z^m x^{m-1}\left(1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n x^n\right)\right.\\
&\qquad\left.+2\left(1+y\sum_{m=1}^{\infty}\frac{(y+1)^{m-1}}{(2m)!}2^m z^m x^m\right)\sum_{n=1}^{\infty}\frac{n(y+1)^{n-1}}{(2n)!}2^n x^{n-1}\right)\\
&\quad-\left(y\sum_{m=1}^{\infty}\frac{m(y+1)^{m-1}}{(2m)!}2^m x^{m-1}\left(1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n z^n x^n\right)\right.\\
&\qquad\left.+2\left(1+y\sum_{m=1}^{\infty}\frac{(y+1)^{m-1}}{(2m)!}2^m x^m\right)\sum_{n=1}^{\infty}\frac{n(y+1)^{n-1}}{(2n)!}2^n z^n x^{n-1}\right)\\
&\quad\cdot\left(1+y\sum_{m=1}^{\infty}\frac{(y+1)^{m-1}}{(2m)!}2^m z^m x^m\right)\left(1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n x^n\right).
\end{aligned}
$$

By differentiating (2.44) we can derive that

(2.83)

$$\frac{\partial w}{\partial x}(x,y,\eta) = \frac{1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n\eta^n x^n}{\left(1+y\sum_{m=1}^{\infty}\frac{(y+1)^{m-1}}{(2m)!}2^m\eta^m x^m\right)^2\left(1+\sum_{n=1}^{\infty}\frac{(y+1)^{n-1}}{(2n)!}2^n x^n\right)^3}\underline{w}(x,y,\eta),$$
$$(\forall (x,y)\in[1,\eta^{-1}-\varepsilon(s,\eta)]\times(-1,y_2(s,\eta)]).$$

Let us observe that

$$\underline{w}(x,-1,\eta) = 3\eta(1-\eta)(x-a_+(\eta))(x-a_-(\eta)),$$
$$\frac{\partial^2 \underline{w}}{\partial x^2}(x,-1,\eta) = 6\eta(1-\eta) > 0. \tag{2.84}$$

The above inequality implies that there exists $y_3(s,\eta)\in(-1,y_2(s,\eta)]$ such that

$$\frac{\partial^2 \underline{w}}{\partial x^2}(x,y,\eta) > 0, \quad (\forall (x,y)\in[1,\eta^{-1}-\varepsilon(s,\eta)]\times(-1,y_3(s,\eta)]). \tag{2.85}$$

Also, by (2.79) and (2.84) $\underline{w}(\eta^{-1}-\varepsilon(s,\eta),-1,\eta) > 0$. Thus, there exists $y_4(s,\eta)\in(-1,y_3(s,\eta)]$ such that

$$\underline{w}(\eta^{-1}-\varepsilon(s,\eta),y,\eta) > 0, \quad (\forall y\in(-1,y_4(s,\eta)]). \tag{2.86}$$

As we have prepared necessary tools, let us start proving the claims of the lemma case by case.

(i): Assume that $\eta = 17-12\sqrt{2}$. Recall the relation (2.71). Assume that $s\in(\tilde{w}(a_-(\eta),-1,\eta),\infty)$. We can deduce from (2.46), (2.47) that

$$x\mapsto \tilde{w}(x,-1,\eta):[1,\eta^{-1})\to\mathbb{R} \text{ is strictly monotone increasing,}$$
$$\tilde{w}(1,-1,\eta) = 0 \text{ and } \lim_{x\nearrow\eta^{-1}}\tilde{w}(x,-1,\eta) = \infty. \tag{2.87}$$

Thus there uniquely exists $a_1\in(a_-(\eta),\eta^{-1})$ such that $s=\tilde{w}(a_1,-1,\eta)$. If $a_1\in(\eta^{-1}-\varepsilon(s,\eta),\eta^{-1})$, by (2.81) and (2.87) $w(x,y,\eta) < s$, $(\forall (x,y)\in[1,\eta^{-1}-\varepsilon(s,\eta)]\times(-1,y_2(s,\eta)])$. This contradicts (2.80). Thus, $a_1\in$

$(a_-(\eta), \eta^{-1} - \varepsilon(s, \eta)]$. By (2.81) and (2.87) again $w(x, y, \eta) < s$ for all $(x, y) \in [1, a_1] \times (-1, y_2(s, \eta)]$. This property coupled with (2.80) ensures that for any $y \in (-1, y_4(s, \eta)]$

$$(2.88)\quad \emptyset \neq \left\{ x \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2 \right) \;\middle|\; w(x, y, \eta) = s \right\} \subset [a_1, \eta^{-1} - \varepsilon(s, \eta)].$$

By (2.84) $\underline{w}(x, -1, \eta) \geq 3\eta(1-\eta)(a_1 - a_-(\eta))^2 > 0$ for any $x \in [a_1, \eta^{-1} - \varepsilon(s, \eta)]$. Thus there exists $y_5(s, \eta) \in (-1, y_4(s, \eta)]$ such that for any $(x, y) \in [a_1, \eta^{-1} - \varepsilon(s, \eta)] \times (-1, y_5(s, \eta)]$ $\underline{w}(x, y, \eta) > 0$ and by (2.83) $\frac{\partial w}{\partial x}(x, y, \eta) > 0$. This property combined with (2.88) implies that $(\mathrm{i})_{\eta,s}$ holds.

Assume that $s \in (0, \tilde{w}(a_-(\eta), -1, \eta))$. By (2.87) there uniquely exists $a_1 \in (1, a_-(\eta))$ such that $\tilde{w}(a_1, -1, \eta) = s$. Since $\tilde{w}(x, -1, \eta) > s$ for any $x \in [a_1 + \frac{1}{2}(a_-(\eta) - a_1), \eta^{-1} - \varepsilon(s, \eta)]$, there exists $y_6(s, \eta) \in (-1, y_4(s, \eta)]$ such that

$$(2.89)\quad w(x, y, \eta) > s, \quad \left(\forall (x, y) \in \left[a_1 + \frac{1}{2}(a_-(\eta) - a_1), \eta^{-1} - \varepsilon(s, \eta) \right] \times (-1, y_6(s, \eta)] \right).$$

By (2.84) $\underline{w}(x, -1, \eta) > 0$ for any $x \in [1, a_1 + \frac{1}{2}(a_-(\eta) - a_1)]$. Thus there exists $y_7(s, \eta) \in (-1, y_6(s, \eta)]$ such that for any $(x, y) \in [1, a_1 + \frac{1}{2}(a_-(\eta) - a_1)] \times (-1, y_7(s, \eta)]$ $\underline{w}(x, y, \eta) > 0$ and thus $\frac{\partial w}{\partial x}(x, y, \eta) > 0$. This property together with (2.79), (2.80), (2.82), (2.89) implies that $(\mathrm{i})_{\eta,s}$ holds.

Assume that $s = \tilde{w}(a_-(\eta), -1, \eta)$. Since $\eta = \eta_0$, $a_-(\eta) = \frac{\eta_0+1}{6\eta_0} = a_0$ by (2.52), we can apply (2.61) to ensure that there exists $y_8(s, \eta) \in (-1, y_4(s, \eta)]$ such that for any $y \in (-1, y_8(s, \eta)]$ $\frac{\partial w}{\partial x}(a_-(\eta), y, \eta) < 0$. This combined with (2.83) implies that $\underline{w}(a_-(\eta), y, \eta) < 0$ for any $y \in (-1, y_8(s, \eta)]$. Therefore, by (2.85), (2.86) for any $y \in (-1, y_8(s, \eta)]$ there exists $x_1(y) \in (a_-(y), \eta^{-1} - \varepsilon(s, \eta))$ such that

$$\begin{aligned} &\underline{w}(x, y, \eta) < 0, \quad (\forall x \in [a_-(\eta), x_1(y))), \\ &\underline{w}(x_1(y), y, \eta) = 0, \\ &\underline{w}(x, y, \eta) > 0, \quad (\forall x \in (x_1(y), \eta^{-1} - \varepsilon(s, \eta)]), \end{aligned}$$

or by (2.83)

$$\begin{aligned}&\frac{\partial w}{\partial x}(x,y,\eta)<0,\quad (\forall x\in[a_-(\eta),x_1(y))),\\&\frac{\partial w}{\partial x}(x_1(y),y,\eta)=0,\\&\frac{\partial w}{\partial x}(x,y,\eta)>0,\quad (\forall x\in(x_1(y),\eta^{-1}-\varepsilon(s,\eta)]).\end{aligned}\tag{2.90}$$

We can see from (2.81) and (2.87) that

$$w(x,y,\eta)<s,\quad (\forall (x,y)\in[1,a_-(\eta)]\times(-1,y_8(s,\eta)]).\tag{2.91}$$

Considering (2.79), (2.80), (2.90) and (2.91), we can conclude that $(\mathrm{i})_{\eta,s}$ holds in this case.

(ii): Assume that $\eta\in(0,17-12\sqrt{2})$ and $s\in[\tilde{w}(a_-(\eta),-1,\eta),\infty)$. The properties (2.72), (2.73), (2.74) tell us the profile of the function $\tilde{w}(\cdot,-1,\eta)$, which together with (2.81) implies that

$$w(x,y,\eta)<s,\quad (\forall (x,y)\in[1,a_+(\eta)]\times(-1,y_2(s,\eta)]).\tag{2.92}$$

Since $\underline{w}(a_-(\eta)+(a_+(\eta)-a_-(\eta))/2,-1,\eta)<0$ by (2.84), there exists $y_9(s,\eta)\in(-1,y_4(s,\eta)]$ such that $\underline{w}(a_-(\eta)+(a_+(\eta)-a_-(\eta))/2,y,\eta)<0$ for any $y\in(-1,\ y_9(s,\eta)]$. By taking this property, (2.85) and (2.86) into account we can prove the following statement. For any $y\in(-1,y_9(s,\eta)]$ there exists $x_2(y)\in(a_-(\eta)+(a_+(\eta)-a_-(\eta))/2,\eta^{-1}-\varepsilon(s,\eta))$ such that

$$\begin{aligned}&\underline{w}(x,y,\eta)<0,\quad \left(\forall x\in\left(a_-(\eta)+\frac{1}{2}(a_+(\eta)-a_-(\eta)),x_2(y)\right)\right),\\&\underline{w}(x_2(y),y,\eta)=0,\\&\underline{w}(x,y,\eta)>0,\quad (\forall x\in(x_2(y),\eta^{-1}-\varepsilon(s,\eta)]),\end{aligned}$$

or by (2.83)

$$\begin{aligned}&\frac{\partial w}{\partial x}(x,y,\eta)<0,\quad \left(\forall x\in\left(a_-(\eta)+\frac{1}{2}(a_+(\eta)-a_-(\eta)),x_2(y)\right)\right),\\&\frac{\partial w}{\partial x}(x_2(y),y,\eta)=0,\\&\frac{\partial w}{\partial x}(x,y,\eta)>0,\quad (\forall x\in(x_2(y),\eta^{-1}-\varepsilon(s,\eta)]).\end{aligned}$$

By this property, (2.80) and (2.92) for any $y \in (-1, y_9(s,\eta)]$ there exists $x_3(y) \in (a_+(\eta), \eta^{-1} - \varepsilon(s,\eta))$ such that

$$\begin{aligned} &w(x,y,\eta) < s, \quad (\forall x \in [1, x_3(y))),\\ &w(x_3(y),y,\eta) = s,\\ &w(x,y,\eta) > s, \quad \left(\forall x \in \left(x_3(y), \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right)\right). \end{aligned}$$

Thus, the property $(\mathrm{i})_{\eta,s}$ holds.

Assume that $s \in [\tilde{w}(a_+(\eta), -1, \eta), \tilde{w}(a_-(\eta), -1, \eta))$. By (2.81) there exists $y_{10}(s,\eta) \in (-1, y_4(s,\eta)]$ such that

$$w(a_-(\eta), y, \eta) > s > w(a_+(\eta), y, \eta), \quad (\forall y \in (-1, y_{10}(s,\eta)]). \tag{2.93}$$

Since $\underline{w}(1+(a_-(\eta)-1)/2, -1, \eta) > 0$ and $\underline{w}(a_-(\eta)+(a_+(\eta)-a_-(\eta))/2, -1, \eta) < 0$ by (2.84), there exists $y_{11}(s,\eta) \in (-1, y_{10}(s,\eta)]$ such that $\underline{w}(1 + (a_-(\eta) - 1)/2, y, \eta) > 0$, $\underline{w}(a_-(\eta) + (a_+(\eta) - a_-(\eta))/2, y, \eta) < 0$ for any $y \in (-1, y_{11}(s,\eta)]$. This property combined with (2.85), (2.86) implies the following statement. For any $y \in (-1, y_{11}(s,\eta)]$ there exist $x_4(y) \in (1 + (a_-(y) - 1)/2, a_-(y) + (a_+(y) - a_-(y))/2)$, $x_5(y) \in (a_-(y) + (a_+(y) - a_-(y))/2, \eta^{-1} - \varepsilon(s,\eta))$ such that

$$\begin{aligned} &\underline{w}(x_4(y),y,\eta) = \underline{w}(x_5(y),y,\eta) = 0,\\ &\underline{w}(x,y,\eta) > 0, \quad (\forall x \in [1, x_4(y))),\\ &\underline{w}(x,y,\eta) < 0, \quad (\forall x \in (x_4(y), x_5(y))),\\ &\underline{w}(x,y,\eta) > 0, \quad (\forall x \in (x_5(y), \eta^{-1} - \varepsilon(s,\eta)]), \end{aligned}$$

or by (2.83)

$$\begin{aligned} &\frac{\partial w}{\partial x}(x_4(y),y,\eta) = \frac{\partial w}{\partial x}(x_5(y),y,\eta) = 0,\\ &\frac{\partial w}{\partial x}(x,y,\eta) > 0, \quad (\forall x \in [1, x_4(y))),\\ &\frac{\partial w}{\partial x}(x,y,\eta) < 0, \quad (\forall x \in (x_4(y), x_5(y))),\\ &\frac{\partial w}{\partial x}(x,y,\eta) > 0, \quad (\forall x \in (x_5(y), \eta^{-1} - \varepsilon(s,\eta)]). \end{aligned}$$

By considering these properties we can picture the profile of the function $w(\cdot, y, \eta)$. Take any $y \in (-1, y_{11}(s,\eta)]$. Suppose that $s \geq w(x_4(y), y, \eta)$.

Then, by the profile of $w(\cdot,y,\eta)$ in $[1,\eta^{-1}-\varepsilon(s,\eta)]$, if $a\in[1,\eta^{-1}-\varepsilon(s,\eta)]$ and $w(a,y,\eta)>s$, $w(a',y,\eta)>s$ for any $a'\in[a,\eta^{-1}-\varepsilon(s,\eta)]$. This claim contradicts (2.93). Suppose that $s\le w(x_5(y),y,\eta)$. Then, if $a\in[1,\eta^{-1}-\varepsilon(s,\eta)]$ and $w(a,y,\eta)>s$, $w(a',y,\eta)\ge s$ for any $a'\in[a,\eta^{-1}-\varepsilon(s,\eta)]$. This claim contradicts (2.93) as well. Therefore, $w(x_4(y),y,\eta)>s>w(x_5(y),y,\eta)$. Moreover, by (2.80), (2.82) and the profile of $w(\cdot,y,\eta)$, $x_4(y)>\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2$ and there exist

$$
\begin{aligned}
&x_6(y)\in\left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2,x_4(y)\right),\\
&x_7(y)\in(x_4(y),x_5(y)),\\
&x_8(y)\in(x_5(y),\eta^{-1}-\varepsilon(s,\eta))\left(\subset\left(x_5(y),\frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right)\right)
\end{aligned}
$$

such that

$$
\begin{aligned}
&w(x_6(y),y,\eta)=w(x_7(y),y,\eta)=w(x_8(y),y,\eta)=s,\\
&w(x,y,\eta)>s,\quad (\forall x\in(x_6(y),x_7(y))),\\
&w(x,y,\eta)<s,\quad (\forall x\in(x_7(y),x_8(y))).
\end{aligned}
$$

This means that $(\mathrm{ii})_{\eta,s}$ holds.

Finally let us assume that $s\in(0,\tilde{w}(a_+(\eta),-1,\eta))$. We can see from the profile of $\tilde{w}(\cdot,-1,\eta)$ that there uniquely exists $a_2\in(1,a_-(\eta))$ such that $s=\tilde{w}(a_2,-1,\eta)$. Moreover, there exists $a_3\in(a_2,a_-(\eta))$ such that $\tilde{w}(x,-1,\eta)\ge\tilde{w}(a_3,-1,\eta)>s$ for all $x\in[a_3,\eta^{-1})$. Thus we can take $y_{12}(s,\eta)\in(-1,y_4(s,\eta)]$ so that $w(x,y,\eta)>s$ for any $(x,y)\in[a_3,\eta^{-1}-\varepsilon(s,\eta)]\times(-1,y_{12}(s,\eta)]$. Since $\underline{w}(x,-1,\eta)\ge\underline{w}(a_3,-1,\eta)>0$ for any $x\in[1,a_3]$ by (2.84), there exists $y_{13}(s,\eta)\in(-1,y_{12}(s,\eta)]$ such that for any $(x,y)\in[1,a_3]\times(-1,y_{13}(s,\eta)]$ $\underline{w}(x,y,\eta)>0$, and thus by (2.83) $\frac{\partial w}{\partial x}(x,y,\eta)>0$. These combined with (2.79), (2.82) imply that for any $y\in(-1,y_{13}(s,\eta)]$ $a_3>\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2$ and there exists $x_9(y)\in(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2,a_3)$ such that

$$
\begin{aligned}
&w(x,y,\eta)<s,\quad \left(\forall x\in\left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2,x_9(y)\right)\right),\\
&w(x_9(y),y,\eta)=s,\\
&w(x,y,\eta)>s,\quad (\forall x\in(x_9(y),\eta^{-1}-\varepsilon(s,\eta)]).
\end{aligned}
$$

Taking into account (2.80), we conclude that $(\mathrm{i})_{\eta,s}$ holds. □

By applying Lemma 2.24 we can prove Proposition 2.23.

PROOF OF PROPOSITION 2.23. Recalling (2.62), we see that

(2.94)
$$g_{E_b}(x,t,0) = \frac{b'}{e_{max}}\left(-\frac{2e_{max}}{b'|U|} + W\left(e_{max}x, \cos\left(\frac{t}{2}\right), \frac{e_{min}}{e_{max}}, \frac{b-b'}{b'}\right)\right).$$

Theorem 2.19 implies that if $e_{min}/e_{max} > \sqrt{17-12\sqrt{2}}$, for any $b \in \mathbb{N}_{\geq 2}$, $b' \in \{1,\cdots,b-1\}$ $(\star)$ holds. Let us assume that $e_{min}/e_{max} \leq \sqrt{17-12\sqrt{2}}$ in the following. For $\eta = (e_{min}/e_{max})^2$, $s = (b-b')/b'$ let us prove the following statements.

- If the condition $(\mathrm{i})_{\eta,s}$ holds, $(\star)$ holds.
- If the condition $(\mathrm{ii})_{\eta,s}$ holds, $(\star)$ does not hold.

Assume that $(\mathrm{i})_{\eta,s}$ holds. Then by (2.66) there exists $y_0 \in (-1,0)$ such that for any $y \in (-1,y_0]$ there exists

$$x_0(y) \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right)$$

such that

$$\begin{aligned}
&\frac{\partial W}{\partial x}(\sqrt{y+1}\cdot x, y, \sqrt{\eta}, s) > 0,\\
&\left(\forall x \in \left(\frac{1}{\sqrt{y+1}}\cosh^{-1}(|y|^{-1}), \sqrt{2x_0(y)}\right)\right),\\
&\frac{\partial W}{\partial x}(\sqrt{2(y+1)x_0(y)}, y, \sqrt{\eta}, s) = 0,\\
&\frac{\partial W}{\partial x}(\sqrt{y+1}\cdot x, y, \sqrt{\eta}, s) < 0,\\
&\left(\forall x \in \left(\sqrt{2x_0(y)}, \frac{1}{\sqrt{\eta(y+1)}}\cosh^{-1}(|y|^{-1})\right)\right).
\end{aligned}$$

Let $\cos^{-1} : [-1,1] \to [0,\pi]$ denote the inverse function of $\cos|_{[0,\pi]}$. The above property and (2.94) ensure that for any $t \in [2\cos^{-1} y_0, 2\pi)(\subset (\pi, 2\pi))$ there exists

$$\hat{x}(t) \in \left(\frac{1}{e_{max}} \cosh^{-1}\left(\left|\cos\left(\frac{t}{2}\right)\right|^{-1}\right), \frac{1}{e_{min}} \cosh^{-1}\left(\left|\cos\left(\frac{t}{2}\right)\right|^{-1}\right)\right)$$

such that

$$\begin{aligned}
&\frac{\partial g_{E_b}}{\partial x}(x,t,0) > 0, \quad \left(\forall x \in \left(\frac{1}{e_{max}} \cosh^{-1}\left(\left|\cos\left(\frac{t}{2}\right)\right|^{-1}\right), \hat{x}(t)\right)\right),\\
&\frac{\partial g_{E_b}}{\partial x}(\hat{x}(t),t,0) = 0,\\
&\frac{\partial g_{E_b}}{\partial x}(x,t,0) < 0, \quad \left(\forall x \in \left(\hat{x}(t), \frac{1}{e_{min}} \cosh^{-1}\left(\left|\cos\left(\frac{t}{2}\right)\right|^{-1}\right)\right)\right),
\end{aligned}$$

or by taking into account (2.34)

$$\begin{aligned}
&\frac{\partial g_{E_b}}{\partial x}(x,t,0) > 0, \quad (\forall x \in (0, \hat{x}(t))),\\
&\frac{\partial g_{E_b}}{\partial x}(\hat{x}(t),t,0) = 0,\\
&\frac{\partial g_{E_b}}{\partial x}(x,t,0) < 0, \quad (\forall x \in (\hat{x}(t), \infty)).
\end{aligned} \tag{2.95}$$

Suppose that ($\star$) does not hold. Then for any $U_0 \in (0, 2e_{min}/b)$ there exists $U \in [-U_0, 0)$ such that $\tau(\cdot)$ has more than one local minimum points in $(0, \beta_c)$. By (2.15) $\cos(\tau(\beta)/2) + 1 \le \frac{\sinh(2)b}{2e_{min}} U_0$. Thus if we take U_0 sufficiently small, $\tau(\beta) \in [2\cos^{-1} y_0, 2\pi)$ for any $\beta \in (0, \beta_c)$. Now there are $\beta_j \in (0, \beta_c)$ $(j = 1, 2, 3)$ such that $\beta_1 < \beta_2 < \beta_3$ and $\tau(\beta_1) = \tau(\beta_2) = \tau(\beta_3)$. Thus, $g_{E_b}(\beta_j, \tau(\beta_1), 0) = 0$ for $j \in \{1, 2, 3\}$, which implies that there exist $x_1 \in (\beta_1, \beta_2)$, $x_2 \in (\beta_2, \beta_3)$ such that $\frac{\partial g_{E_b}}{\partial x}(x_j, \tau(\beta_1), 0) = 0$ for $j \in \{1, 2\}$. This contradicts (2.95) with $t = \tau(\beta_1)$. Therefore, ($\star$) must hold.

Assume that $(\mathrm{ii})_{\eta,s}$ holds. Take any $U_0 \in (0, 2e_{min}/b)$. The limit in the left-hand side of (2.56) tells us that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$\left(\frac{1}{\sqrt{y+1}} \cosh^{-1}(|y|^{-1}), \frac{1}{\sqrt{\eta(y+1)}} \cosh^{-1}(|y|^{-1})\right) \subset \left[1, \frac{2}{\sqrt{\eta}}\right]$$

for any $y \in (-1, -1+\varepsilon)$. Thus,

$$W(\sqrt{y+1}\cdot x, y, \sqrt{\eta}, s) \ge \inf_{\xi\in[1,\frac{2}{\sqrt{\eta}}]} \frac{\sinh(\sqrt{y+1}\cdot\xi)}{y+\cosh(\sqrt{y+1}\cdot\xi)},$$
$$\Bigg(\forall y \in (-1,-1+\varepsilon),$$
$$x \in \left(\frac{1}{\sqrt{y+1}}\cosh^{-1}(|y|^{-1}), \frac{1}{\sqrt{\eta(y+1)}}\cosh^{-1}(|y|^{-1})\right)\Bigg).$$

Since the right-hand side of the above inequality diverges to $+\infty$ as $y \searrow -1$, there exists $y_1 \in (-1, 0)$ such that

(2.96)

$$W(\sqrt{y+1}\cdot x, y, \sqrt{\eta}, s) > \frac{2e_{max}}{b'U_0},$$
$$\left(\forall y \in (-1, y_1],\ x \in \left(\frac{1}{\sqrt{y+1}}\cosh^{-1}(|y|^{-1}), \frac{1}{\sqrt{\eta(y+1)}}\cosh^{-1}(|y|^{-1})\right)\right).$$

By the assumption and (2.66) there exist $y \in (-1, y_1]$,

$$x_j(y) \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right),$$
$$(j = 1, 2, 3)$$

such that $x_1(y) < x_2(y) < x_3(y)$,

$$\frac{\partial W}{\partial x}\big(\sqrt{2(y+1)x_j(y)}, y, \sqrt{\eta}, s\big) = 0, \quad (\forall j \in \{1,2,3\}),$$
$$\frac{\partial W}{\partial x}(\sqrt{y+1}\cdot x, y, \sqrt{\eta}, s) < 0, \quad (\forall x \in (\sqrt{2x_1(y)}, \sqrt{2x_2(y)})),$$
$$\frac{\partial W}{\partial x}(\sqrt{y+1}\cdot x, y, \sqrt{\eta}, s) > 0, \quad (\forall x \in (\sqrt{2x_2(y)}, \sqrt{2x_3(y)})).$$

By combining this with (2.96) we have that

$$\min_{j\in\{1,3\}} W\big(\sqrt{2(y+1)x_j(y)}, y, \sqrt{\eta}, s\big) > W(\sqrt{2(y+1)x_2(y)}, y, \sqrt{\eta}, s)$$

$$> \frac{2e_{max}}{b'U_0}.$$

Thus, there exists $U \in [-U_0, 0)$ such that

$$\min_{j\in\{1,3\}} W(\sqrt{2(y+1)x_j(y)}, y, \sqrt{\eta}, s) > \frac{2e_{max}}{b'|U|}$$
$$> W(\sqrt{2(y+1)x_2(y)}, y, \sqrt{\eta}, s).$$

Therefore, there exist

$$\hat{\beta}_1 \in \left(0, \frac{1}{e_{max}}\sqrt{2(y+1)x_1(y)}\right),$$
$$\hat{\beta}_2 \in \left(\frac{1}{e_{max}}\sqrt{2(y+1)x_1(y)}, \frac{1}{e_{max}}\sqrt{2(y+1)x_2(y)}\right),$$
$$\hat{\beta}_3 \in \left(\frac{1}{e_{max}}\sqrt{2(y+1)x_2(y)}, \frac{1}{e_{max}}\sqrt{2(y+1)x_3(y)}\right)$$

such that $W(e_{max}\hat{\beta}_j, y, \sqrt{\eta}, s) = \frac{2e_{max}}{b'|U|}$ for $j \in \{1, 2, 3\}$, or by (2.94) and Lemma 2.1

$$\hat{\beta}_j \in (0, \beta_c), \quad \cos\left(\frac{\tau(\hat{\beta}_j)}{2}\right) = y, \quad (\forall j \in \{1, 2, 3\}).$$

Then by repeating the same argument as the final part of the proof of Proposition 2.13 we can reach the conclusion that $\tau(\cdot)$ has more than one local minimum points. This means that $(\star)$ does not hold.

Now we know that it suffices to determine for which (η, s) $(\mathrm{i})_{\eta,s}$ (or $(\mathrm{ii})_{\eta,s}$) holds. In fact for this purpose we have prepared Lemma 2.24. We still need more information about how the function $\tilde{w}(\cdot, -1, \eta)$ behaves when η varies. We can derive from (2.46) that for $z \in \mathbb{R}_{>0}$, $x \in (0, z^{-1})$

$$\frac{\partial \tilde{w}}{\partial z}(x, -1, z) = \frac{(x-1)x(1+zx)(3-zx)}{(x+1)^2(1-zx)^2}.$$

Then we can see from this equality and (2.72) that for $\eta \in (0, 17 - 12\sqrt{2})$, $\delta \in \{+, -\}$, $\frac{\partial \tilde{w}}{\partial z}(a_\delta(\eta), -1, \eta) > 0$. Combination of this inequality and (2.73) implies that for $\eta \in (0, 17 - 12\sqrt{2})$, $\delta \in \{+, -\}$

$$\frac{d}{d\eta}\tilde{w}(a_\delta(\eta), -1, \eta) = \frac{\partial \tilde{w}}{\partial x}(a_\delta(\eta), -1, \eta)\frac{da_\delta}{d\eta}(\eta) + \frac{\partial \tilde{w}}{\partial z}(a_\delta(\eta), -1, \eta) > 0.$$

By this and (2.73) again we can understand that both the local maximum and the local minimum of the function $x \mapsto \tilde{w}(x, -1, \eta) : (0, \eta^{-1}) \to \mathbb{R}$ are strictly monotone increasing with $\eta \in (0, 17 - 12\sqrt{2})$. Moreover, by (2.46), (2.57), (2.70), (2.71)

$$\begin{aligned}
&\lim_{\eta \nearrow 17-12\sqrt{2}} a_+(\eta) = \lim_{\eta \nearrow 17-12\sqrt{2}} a_-(\eta) = 3 + 2\sqrt{2},\\
&\lim_{\eta \nearrow 17-12\sqrt{2}} \tilde{w}(a_+(\eta), -1, \eta) = \lim_{\eta \nearrow 17-12\sqrt{2}} \tilde{w}(a_-(\eta), -1, \eta) = 3 - 2\sqrt{2},\\
&\lim_{\eta \searrow 0} a_-(\eta) = 3, \quad \lim_{\eta \searrow 0} a_+(\eta) = +\infty,\\
&\lim_{\eta \searrow 0} \tilde{w}(a_-(\eta), -1, \eta) = \frac{1}{8}, \quad \lim_{\eta \searrow 0} \tilde{w}(a_+(\eta), -1, \eta) = 0.
\end{aligned}$$

In the following we let $\eta = (e_{min}/e_{max})^2$, $s = (b - b')/b'$. If $e_{min}/e_{max} = \sqrt{17 - 12\sqrt{2}}$, by Lemma 2.24 (i) for any $b \in \mathbb{N}_{\ge 2}$, $b' \in \{1, \cdots, b-1\}$ the condition $(\mathrm{i})_{\eta,s}$ holds and thus $(\star)$ holds.

Assume that $e_{min}/e_{max} \in (0, \sqrt{17 - 12\sqrt{2}})$. In this situation Lemma 2.24 (ii) is applicable. If $\frac{b-b'}{b'} \in [3 - 2\sqrt{2}, \infty)$, $\frac{b-b'}{b'} \in [\tilde{w}(a_-(\eta), -1, \eta), \infty)$ and thus $(\mathrm{i})_{\eta,s}$ holds. Thus $(\star)$ holds. If $\frac{b-b'}{b'} \in (1/8, 3 - 2\sqrt{2})$, there exist e_1, $e_2 \in (0, \sqrt{17 - 12\sqrt{2}})$ such that $e_1 < e_2$,

$$\begin{aligned}
&\frac{b - b'}{b'} \in (0, \tilde{w}(a_+(\eta), -1, \eta)) \text{ if } \frac{e_{min}}{e_{max}} \in \left(e_2, \sqrt{17 - 12\sqrt{2}}\right),\\
&\frac{b - b'}{b'} \in [\tilde{w}(a_+(\eta), -1, \eta), \tilde{w}(a_-(\eta), -1, \eta)) \text{ if } \frac{e_{min}}{e_{max}} \in (e_1, e_2],\\
&\frac{b - b'}{b'} \in [\tilde{w}(a_-(\eta), -1, \eta), \infty) \text{ if } \frac{e_{min}}{e_{max}} \in (0, e_1],
\end{aligned}$$

or

$$\begin{aligned}
&(\mathrm{i})_{\eta,s} \text{ holds and thus } (\star) \text{ holds if } \frac{e_{min}}{e_{max}} \in \left(e_2, \sqrt{17 - 12\sqrt{2}}\right),\\
&(\mathrm{ii})_{\eta,s} \text{ holds and thus } (\star) \text{ does not hold if } \frac{e_{min}}{e_{max}} \in (e_1, e_2],\\
&(\mathrm{i})_{\eta,s} \text{ holds and thus } (\star) \text{ holds if } \frac{e_{min}}{e_{max}} \in (0, e_1].
\end{aligned}$$

If $\frac{b-b'}{b'} \in (0, 1/8]$, there exists $e_1 \in (0, \sqrt{17 - 12\sqrt{2}})$ such that

$$\frac{b - b'}{b'} \in (0, \tilde{w}(a_+(\eta), -1, \eta)) \text{ if } \frac{e_{min}}{e_{max}} \in \left(e_1, \sqrt{17 - 12\sqrt{2}}\right),$$

$$\frac{b-b'}{b'} \in [\tilde{w}(a_+(\eta),-1,\eta),\tilde{w}(a_-(\eta),-1,\eta)) \text{ if } \frac{e_{min}}{e_{max}} \in (0,e_1],$$

or

$$(\text{i})_{\eta,s} \text{ holds and thus } (\star) \text{ holds if } \frac{e_{min}}{e_{max}} \in \left(e_1,\sqrt{17-12\sqrt{2}}\right),$$
$$(\text{ii})_{\eta,s} \text{ holds and thus } (\star) \text{ does not hold if } \frac{e_{min}}{e_{max}} \in (0,e_1].$$

These can be summarized as in the statements of the proposition. □

In fact in this model $\tau(\beta)$ can be exactly computed. Remind us that $\cos^{-1}:[-1,1]\to[0,\pi]$ denotes the inverse function of $\cos|_{[0,\pi]}$.

PROPOSITION 2.25. *Set*

$$\begin{aligned} D_0 := \cosh(\beta e_{max})\cosh(\beta e_{min}) - \frac{|U|}{2}\Big(&\frac{b'}{e_{max}}\sinh(\beta e_{max})\cosh(\beta e_{min}) \\ &+\frac{b-b'}{e_{min}}\cosh(\beta e_{max})\sinh(\beta e_{min})\Big), \end{aligned}$$

$$\begin{aligned} D_1 :=\ &\cosh(\beta e_{max})+\cosh(\beta e_{min}) \\ &-\frac{|U|}{2}\left(\frac{b'}{e_{max}}\sinh(\beta e_{max})+\frac{b-b'}{e_{min}}\sinh(\beta e_{min})\right). \end{aligned}$$

Assume that $U\in(-2e_{min}/b,0)$. Then for any $\beta\in(0,\beta_c)$, $D_1^2-4D_0>0$, $\frac{1}{2}(-D_1+\sqrt{D_1^2-4D_0})\in(-1,0)$ and

$$\tau(\beta)=2\cos^{-1}\left(\frac{-D_1+\sqrt{D_1^2-4D_0}}{2}\right).$$

PROOF. The statements of Lemma 1.2 (i),(ii) imply the following basic fact. On the assumption $|U|<2e_{min}/b$ for any $\beta\in(0,\beta_c)$ there uniquely exists $y\in(-1,0)$ such that

(2.97)
$$-\frac{2}{|U|}+b'\frac{\sinh(\beta e_{max})}{(y+\cosh(\beta e_{max}))e_{max}}+(b-b')\frac{\sinh(\beta e_{min})}{(y+\cosh(\beta e_{min}))e_{min}}=0.$$

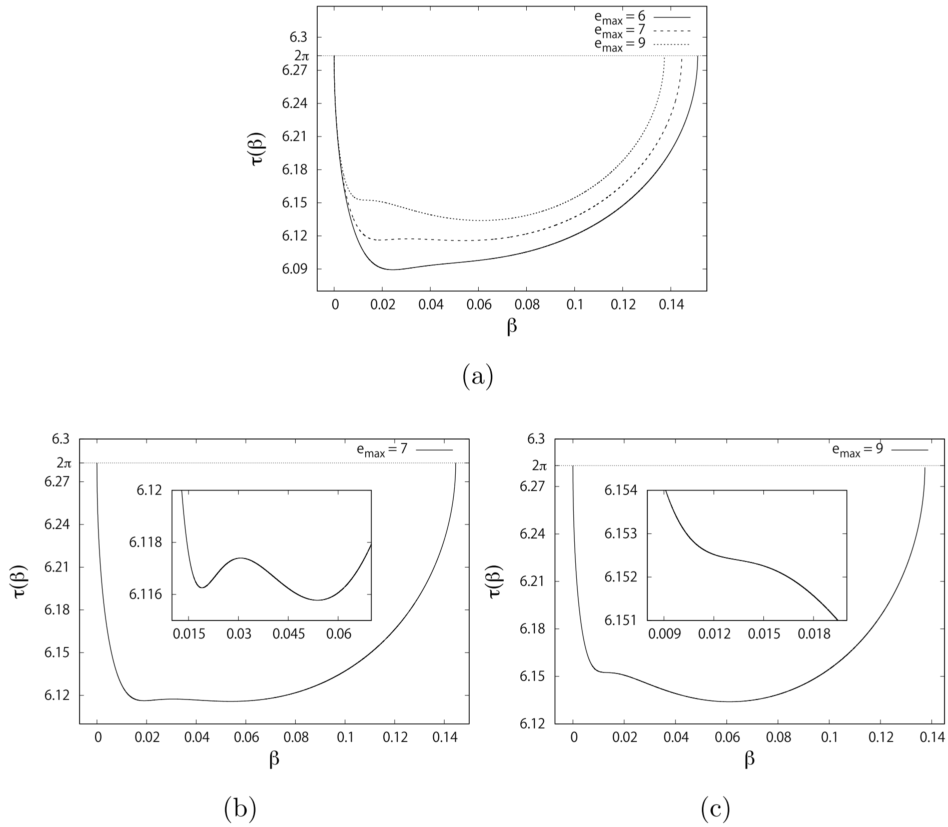

Fig. 2. The graph $\{(\beta, \tau(\beta)) \mid \beta \in (0, \beta_c)\}$ drawn by implementing the exact solution for $b = 8$, $b' = 7$, $U = -1/8$, $e_{min} = 1$ and $e_{max} = 6, 7, 9$. Picture (a) shows the graphs for $e_{max} = 6, 7, 9$. We can see that $\tau(\cdot)$ has only one local minimum point when $e_{max} = 6$. Picture (b) shows the graph for $e_{max} = 7$. By magnifying we can see that $\tau(\cdot)$ has two local minimum points. Picture (c) shows the graph for $e_{max} = 9$. By magnifying we can see that $\tau(\cdot)$ has only one local minimum point.

Moreover, for $y \in [0, \infty)$ (2.97) does not hold. Observe that $y \in (-1, 0)$ and y solves (2.97) if and only if $y \in (-1, 0)$ and y solves $y^2 + D_1 y + D_0 = 0$. Setting

$$X_1 := \cosh(\beta e_{max}), \quad X_2 := \cosh(\beta e_{min}),$$

$$Y_1 := \frac{|U|b'}{2e_{max}}\sinh(\beta e_{max}), \quad Y_2 := \frac{|U|(b-b')}{2e_{min}}\sinh(\beta e_{min}),$$

we can derive that

$$D_1^2 - 4D_0 = (X_1 - X_2 - Y_1 + Y_2)^2 + 4Y_1Y_2 > 0.$$

Set $y_+ := \frac{1}{2}(-D_1 + \sqrt{D_1^2 - 4D_0})$, $y_- := \frac{1}{2}(-D_1 - \sqrt{D_1^2 - 4D_0})$. These are the roots of $y^2 + D_1 y + D_0$. The unique solution to (2.97) in $(-1, 0)$ must be one of them. If $y_+ \geq 0$, (2.97) has a non-negative solution, which is a contradiction. Thus $y_+ < 0$. If $y_- > -1$, (2.97) has the 2 different solutions y_+, $y_- \in (-1, 0)$, which is again a contradiction. Thus $y_- \leq -1$. Therefore the solution to (2.97) in $(-1, 0)$ must be y_+, and thus the claims follow. □

Let $b = 8$, $b' = 7$, $e_{min} = 1$. In this case $\frac{b-b'}{b'} = 1/7 \in (1/8, 3 - 2\sqrt{2})$. Proposition 2.23 (ii) implies that there exist $U \in (-2e_{min}/b, 0)$ $(= (-1/4, 0))$ and $e_{max,1}$, $e_{max,2}$, $e_{max,3} \in (1/\sqrt{17 - 12\sqrt{2}}, \infty)$ $(\approx (5.83, \infty))$ such that $e_{max,1} < e_{max,2} < e_{max,3}$ and for U $\tau(\cdot)$ has only one local minimum point if $e_{max} = e_{max,1}$, $\tau(\cdot)$ has more than one local minimum points if $e_{max} = e_{max,2}$, $\tau(\cdot)$ has only one local minimum point if $e_{max} = e_{max,3}$. Figure 2 shows the graph $\{(\beta, \tau(\beta)) \mid \beta \in (0, \beta_c)\}$ for $U = -1/8$, $e_{max} = 6, 7, 9$. In these cases $U \in (-2e_{min}/b, 0)$, $e_{max} \in (1/\sqrt{17 - 12\sqrt{2}}, \infty)$. The figure demonstrates the properties described above. The graph was drawn by implementing the exact solution obtained in Proposition 2.25.

2.3.2 The one-dimensional model with nearest-neighbor hopping

As for the model defined in (2), we find a simpler result as follows.

PROPOSITION 2.26. *For any $t \in \mathbb{R}_{\geq 0}$, $e_{min} \in \mathbb{R}_{>0}$ there exists $U_0 \in (0, 2e_{min})$ such that for any $U \in [-U_0, 0)$ $\tau(\cdot)$ has one and only one local minimum point in $(0, \beta_c)$.*

PROOF. Let us assume that $e_{min} = 1$ for the moment. We will see that the other case can be deduced from this special case. It follows that $e_{max} = 2t + 1$. Define the open set $\mathcal{O}$ of $\mathbb{R}^2$ by

$$\mathcal{O} := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \sum_{n=2}^{\infty} \frac{x^{2n}}{(2n)!}|y+1|^{n-1}e_{max}^{2n} < 1 \text{ or } y > -1 \right\}.$$

We define the function $P:\mathcal{O}\to\mathbb{R}$ as follows.

$$P(x,y):=\frac{1}{2\pi}\int_0^{2\pi}dk\frac{x+\sum_{n=1}^{\infty}\frac{x^{2n+1}}{(2n+1)!}(y+1)^nE_1(k)^{2n}}{1+\frac{x^2}{2}E_1(k)^2+\sum_{n=2}^{\infty}\frac{x^{2n}}{(2n)!}(y+1)^{n-1}E_1(k)^{2n}}.$$

The function P is real analytic in $\mathcal{O}$. Let us observe that for $(x,y)\in\mathbb{R}_{>0}\times(-1,\infty)$

$$P(x,y)=\frac{\sqrt{y+1}}{2\pi}\int_0^{2\pi}dk\frac{\sinh(\sqrt{y+1}\cdot xE_1(k))}{(y+\cosh(\sqrt{y+1}\cdot xE_1(k)))E_1(k)}. \tag{2.98}$$

We can apply Lemma B.1 proved in Appendix B to derive that for any $x\in\mathbb{R}$

$$\begin{aligned}&P(x,-1)^2\\&=\frac{e_{max}^{-1}\left(\frac{x^2}{2}\right)}{\left(\frac{x^2}{2}+1\right)\left(\frac{x^2}{2}+e_{max}^{-2}\right)}\left(\left(\frac{x^2}{2}+1\right)^{\frac{1}{2}}\left(\frac{x^2}{2}+e_{max}^{-2}\right)^{\frac{1}{2}}-\frac{x^2}{2}+e_{max}^{-1}\right).\end{aligned}$$

To facilitate the derivation of the above equality from Lemma B.1, let us add that we multiplied both the numerator and the denominator of $P(x,-1)^2$ by

$$\left(\frac{x^2}{2}+1\right)^{\frac{1}{2}}\left(e_{max}^2\frac{x^2}{2}+1\right)^{\frac{1}{2}}-e_{max}\frac{x^2}{2}-1$$

at the beginning. Moreover, setting

$$\begin{aligned}P_1(x)&:=\left(\frac{x^2}{2}+1\right)^{\frac{1}{2}}\left(\frac{x^2}{2}+e_{max}^{-2}\right)^{\frac{1}{2}}\left((1+e_{max}^{-2})\frac{x^2}{2}+2e_{max}^{-2}\right),\\P_2(x)&:=2(e_{max}^{-2}+e_{max}^{-1}+1)\left(\frac{x^2}{2}\right)^2+4e_{max}^{-2}\left(\frac{x^2}{2}\right)-2e_{max}^{-3},\\P_3(x)&:=\frac{2e_{max}}{x}\left(\frac{x^2}{2}+1\right)^2\left(\frac{x^2}{2}+e_{max}^{-2}\right)^2,\end{aligned}$$

we see that for any $x\in\mathbb{R}_{>0}$

$$\frac{d}{dx}P(x,-1)^2=\frac{P_1(x)-P_2(x)}{P_3(x)}.$$

If we assume that $\hat{x} \in \mathbb{R}_{>0}$, $y \in (-1, -\frac{1}{2}]$ and $\frac{\partial P}{\partial x}(\hat{x}, y) = 0$, it follows from (2.34) that

$$\hat{x} \in \left[\frac{1}{e_{max}\sqrt{y+1}}\cosh^{-1}(|y|^{-1}), \frac{1}{\sqrt{y+1}}\cosh^{-1}(|y|^{-1})\right],$$
$$\frac{d}{dx}P(x,-1)^2\Big|_{x=\hat{x}} + \frac{\partial}{\partial x}(P(x,y)^2 - P(x,-1)^2)\Big|_{x=\hat{x}} = 0.$$

Let us recall the definition (2.35) of c_{max}. We can also deduce from (2.36) that if we set

$$c_{min} := \inf_{y\in(-1,-\frac{1}{2}]} \frac{\cosh^{-1}(|y|^{-1})}{\sqrt{y+1}},$$

$0 < c_{min} < \infty$. Then the above properties lead to that

$$\hat{x} \in \left[\frac{c_{min}}{e_{max}}, c_{max}\right],$$
$$P_1(\hat{x})^2 - P_2(\hat{x})^2 + 2P_2(\hat{x})P_3(\hat{x})\frac{\partial}{\partial x}(P(x,y)^2 - P(x,-1)^2)\Big|_{x=\hat{x}}$$
$$- P_3(\hat{x})^2\left(\frac{\partial}{\partial x}(P(x,y)^2 - P(x,-1)^2)\Big|_{x=\hat{x}}\right)^2 = 0.$$

Let us define the function $Q : \mathbb{R}_{>0} \times (-1, \infty) \to \mathbb{R}$ by

$$Q(x,y) := P_1(x)^2 - P_2(x)^2 + 2P_2(x)P_3(x)\frac{\partial}{\partial x}(P(x,y)^2 - P(x,-1)^2)$$
$$- P_3(x)^2\left(\frac{\partial}{\partial x}(P(x,y)^2 - P(x,-1)^2)\right)^2.$$

We will prove the following statement.

(2.99)

There exists $y_0(e_{max}) \in \left(-1, -\frac{1}{2}\right]$ depending only on e_{max} such that

if for $y \in (-1, y_0(e_{max})]$ a solution to $Q(x,y) = 0$ exists in $\left[\frac{c_{min}}{e_{max}}, c_{max}\right]$,

then it is unique.

We can expand $P_1(x)^2 - P_2(x)^2$ as follows.

$$P_1(x)^2 - P_2(x)^2 = \sum_{j=1}^{4} a_j(e_{max})\left(\frac{x^2}{2}\right)^j,$$

where $a_j(e_{max})$ $(j=1,\cdots,4)$ are real coefficients depending only on e_{max}. We can check that

$$a_1(e_{max})>0,\quad a_2(e_{max})>0,\quad a_4(e_{max})<0. \tag{2.100}$$

We do not need to deal with $a_3(e_{max})$, since the term involving $a_3(e_{max})$ will be subsequently canceled. Though it is not essential to make explicit, $a_2(e_{max})$ is computed as follows. $a_2(e_{max}) = 5e_{max}^{-6} + 8e_{max}^{-5} + 6e_{max}^{-4} + 8e_{max}^{-3} + 5e_{max}^{-2}$. Assume that $(x_0,y)\in[c_{min}/e_{max},c_{max}]\times(-1,-1/2]$ and $Q(x_0,y)=0$. We can derive that

$$\begin{aligned}
&x_0\frac{\partial Q}{\partial x}(x_0,y)\\
&=\sum_{j=1}^{4}2ja_j(e_{max})\left(\frac{x_0^2}{2}\right)^j\\
&\quad+x_0\frac{\partial}{\partial x}\Bigg(2P_2(x)P_3(x)\frac{\partial}{\partial x}(P(x,y)^2-P(x,-1)^2)\\
&\qquad\qquad-P_3(x)^2\left(\frac{\partial}{\partial x}(P(x,y)^2-P(x,-1)^2)\right)^2\Bigg)\Bigg|_{x=x_0}\\
&=\sum_{j\in\{1,2,4\}}(2j-6)a_j(e_{max})\left(\frac{x_0^2}{2}\right)^j\\
&\quad-12P_2(x_0)P_3(x_0)\frac{\partial}{\partial x}(P(x,y)^2-P(x,-1)^2)\Big|_{x=x_0}\\
&\quad+6P_3(x_0)^2\left(\frac{\partial}{\partial x}(P(x,y)^2-P(x,-1)^2)\Big|_{x=x_0}\right)^2\\
&\quad+x_0\frac{\partial}{\partial x}\Bigg(2P_2(x)P_3(x)\frac{\partial}{\partial x}(P(x,y)^2-P(x,-1)^2)\\
&\qquad\qquad-P_3(x)^2\left(\frac{\partial}{\partial x}(P(x,y)^2-P(x,-1)^2)\right)^2\Bigg)\Bigg|_{x=x_0}\\
&\le-2a_2(e_{max})\left(\frac{c_{min}^2}{2e_{max}^2}\right)^2\\
&\quad+c\sup_{x\in[\frac{c_{min}}{e_{max}},c_{max}]}\Bigg((1+c_{max})|P_2(x)P_3(x)|+(1+c_{max})|P_3(x)^2|
\end{aligned}$$

$$
+ c_{max}\left|\frac{dP_2}{dx}(x)P_3(x)\right| + c_{max}\left|P_2(x)\frac{dP_3}{dx}(x)\right| + c_{max}\left|\frac{dP_3}{dx}(x)P_3(x)\right|\Bigg)
$$
$$
\cdot\left(1+\sum_{i,j\in\{0,1,2\}}1_{1\le i+j\le 2}\sup_{x\in[\frac{c_{min}}{e_{max}},c_{max}]}\sup_{\eta\in[-1,-\frac{1}{2}]}\left|\frac{\partial^i P}{\partial x^i}(x,\eta)\frac{\partial^{j+1}P}{\partial x^j\partial y}(x,\eta)\right|\right)^2
$$
$$
\cdot(y+1),
$$

where c is a positive constant independent of any parameter. In the second equality we used the equality $Q(x_0,y)=0$ to erase the term $a_3(e_{max})(x_0^2/2)^3$. In the last inequality we took (2.100) into account. The above inequality implies that there exists $y_0(e_{max})\in(-1,-\frac{1}{2}]$ depending only on e_{max} such that if $y\in(-1,y_0(e_{max})]$, $\frac{\partial Q}{\partial x}(x_0,y)<0$. We can sum up the above arguments to conclude that if $(x_0,y)\in[c_{min}/e_{max},c_{max}]\times(-1,y_0(e_{max})]$ and $Q(x_0,y)=0$, then $\frac{\partial Q}{\partial x}(x_0,y)<0$. This ensures that the claim (2.99) holds true.

If for $y\in(-1,y_0(e_{max})]$ $\hat{x}$ is a solution to $\frac{\partial P}{\partial x}(x,y)=0$ in $\mathbb{R}_{>0}$, then $\hat{x}\in[c_{min}/e_{max},c_{max}]$ and $Q(\hat{x},y)=0$ and thus it must be unique by (2.99). We can deduce from (2.34) that

$$
\begin{aligned}
&\frac{\partial P}{\partial x}(x,y)>0, \quad \left(\forall x\in\left(0,\frac{1}{e_{max}\sqrt{y+1}}\cosh^{-1}(|y|^{-1})\right)\right),\\
&\frac{\partial P}{\partial x}(x,y)<0, \quad \left(\forall x\in\left(\frac{1}{\sqrt{y+1}}\cosh^{-1}(|y|^{-1}),\infty\right)\right),
\end{aligned}
$$

which means that a solution to $\frac{\partial P}{\partial x}(x,y)=0$ actually exists in $\mathbb{R}_{>0}$. Thus we have proved that for any $y\in(-1,y_0(e_{max})]$ a solution to $\frac{\partial P}{\partial x}(x,y)=0$ uniquely exists in $\mathbb{R}_{>0}$. Therefore, by (2.98) for any $y\in(-1,y_0(e_{max})]$ there uniquely exists $\tilde{x}\in\mathbb{R}_{>0}$ such that

$$
\frac{d}{dx}\left(\frac{1}{2\pi}\int_0^{2\pi}dk\frac{\sinh(xE_1(k))}{(y+\cosh(xE_1(k)))E_1(k)}\right)\Bigg|_{x=\tilde{x}}=0. \tag{2.101}
$$

Now let us lift the condition $e_{min}=1$. Since $E_1(k)=e_{min}(\frac{t}{e_{min}}(\cos k+1)+1)$, the above result implies that there exists $y_0(t/e_{min})\in(-1,-1/2]$ depending only on t/e_{min} such that for any $y\in(-1,y_0(t/e_{min})]$ there uniquely exists $\tilde{x}\in\mathbb{R}_{>0}$ such that (2.101) with this E_1 holds. This further implies that for any $y \in [2\cos^{-1}(y_0(t/e_{min})), 2\pi)$ there exists $\hat{x}(y) \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}
&\frac{\partial g_{E_1}}{\partial x}(x, y, 0) > 0, \quad (\forall x \in (0, \hat{x}(y))),\\
&\frac{\partial g_{E_1}}{\partial x}(\hat{x}(y), y, 0) = 0,\\
&\frac{\partial g_{E_1}}{\partial x}(x, y, 0) < 0, \quad (\forall x \in (\hat{x}(y), \infty)).
\end{aligned}$$

Then by repeating the same proof by contradiction as that after (2.95) in the proof of Proposition 2.23 we can conclude that the claim holds true. □

REMARK 2.27. One natural question is whether the same result holds for the model in higher spatial dimensions

$$E(\mathbf{k}) = t\left(\sum_{j=1}^{d} \cos k_j + d\right) + e_{min}, \quad (t, e_{min} \in \mathbb{R}_{>0},\ d \in \mathbb{N}). \tag{2.102}$$

In the above proof we relied on the exact formula Lemma B.1. Since we do not have a useful formula of the definite integral for the model (2.102) with $d \geq 2$, we cannot find an answer to this question by this approach at present.

3. Derivation of the Infinite-Volume Limit

In this section we will prove Theorem 1.3. As in the previous work [13], [14], the proof is based on multi-scale analysis of Grassmann integral formulations of the free energy density and the thermal expectations. In this approach qualitative bound properties of the covariance matrices are the essential ingredients. This time we decide to prepare them in the first subsection (Subsection 3.1). The focus of this part is to find optimal upper bounds on norms of the covariances with respect to dependency on the inverse temperature β and the magnitude of the imaginary magnetic field θ. Then in Subsections 3.2-3.3 we will develop a general double-scale integration scheme by assuming only generic bounds of the covariances. In Subsection 3.4 we combine the proved bound properties of the real covariances with the general integration scheme to complete the proof of Theorem 1.3. The index set of the finite-dimensional Grassmann algebra is exactly

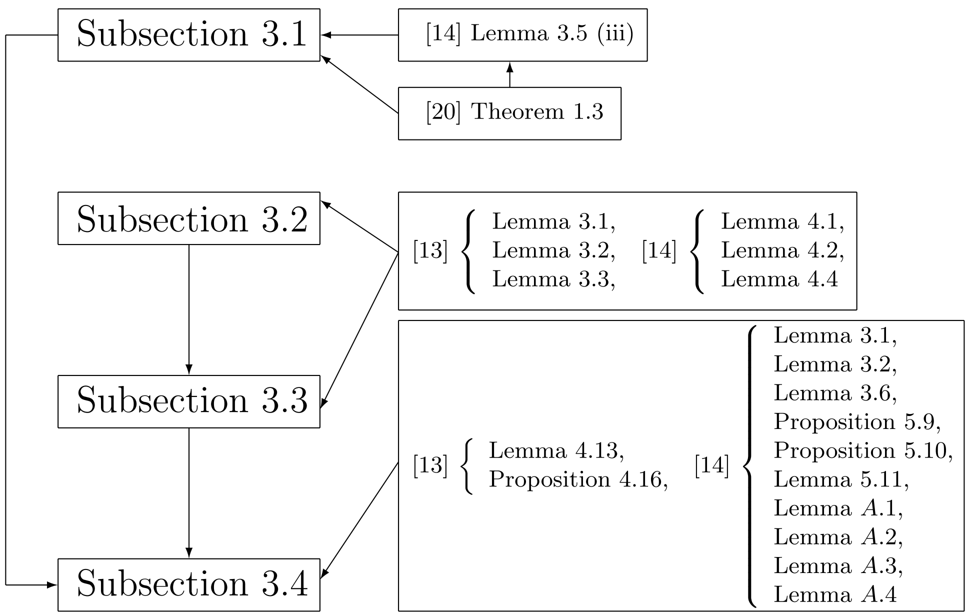

Fig. 3. Dependencies between Subsections 3.1-3.4, results of [13], [14] and [20, Theorem 1.3].

same as that in [14]. Accordingly, concerning the Grassmann integration, we can use the same notations as in [14]. We will sometimes refer to the definitions presented in [14] or [13] instead of restating them in order not to lengthen the paper. We will also skip proofs of lemmas if they straightforwardly follow from lemmas presented in [13], [14]. To support the readers, we illustrate the dependencies between the following subsections and the previous constructions in Figure 3.

One important difference from the previous construction is that here the parameter θ is allowed to take any real value thanks to the gapped property of band spectra (1.6), while it could not belong to $\frac{2\pi}{\beta}(2\mathbb{Z}+1)$ in [13], [14]. This affects the allowed value of $\theta(\beta)$ as well. To make clear, we should state the definition of $\theta(\beta)$ here. For any $\beta \in \mathbb{R}_{>0}$, $\theta \in \mathbb{R}$ there uniquely exists $\theta' \in (-2\pi/\beta, 2\pi/\beta]$ such that $\theta = \theta' \pmod{4\pi/\beta}$. We define the number $\theta(\beta) \in [0, 2\pi/\beta]$ by $\theta(\beta) := |\theta'|$.

3.1. Properties of covariances

With the artificial parameter $h \in \frac{2}{\beta}\mathbb{N}$, we set $[0,\beta)_h := \{0, 1/h, 2/h, \cdots, \beta - 1/h\}$ as already stated in Subsection 1.1. Define the sets I_0, I by

$$I_0 := \{1,2\} \times \mathcal{B} \times \Gamma \times [0,\beta)_h, \quad I := I_0 \times \{1,-1\}.$$

As we have seen in [14, Section 3], our many-electron system is formulated into the (imaginary) time-continuum limit $h \to \infty$ of the Grassmann Gaussian integral, which has the covariance $C(\phi) : I_0^2 \to \mathbb{C}$ ($\phi \in \mathbb{C}$) defined by

$$\begin{aligned}&C(\phi)(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t)\\ &:= \frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle + i\omega(s-t)}\\ &\quad \cdot h^{-1}(I_{2b} - e^{-\frac{i}{h}(\omega - \frac{\theta(\beta)}{2})I_{2b} + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1}((\overline{\rho}-1)b+\rho, (\overline{\eta}-1)b+\eta).\end{aligned}$$

Here $\mathcal{M}_h$ is the set of the Matsubara frequencies with cut-off

$$\left\{\omega \in \frac{\pi}{\beta}(2\mathbb{Z}+1) \;\middle|\; |\omega| < \pi h\right\}$$

and

$$E(\phi)(\mathbf{k}) := \begin{pmatrix} E(\mathbf{k}) & \overline{\phi} I_b \\ \phi I_b & -E(\mathbf{k}) \end{pmatrix} \in \mathrm{Mat}(2b, \mathbb{C})$$

for $\phi \in \mathbb{C}$. In fact $C(\phi)$ was originally defined as the free 2-point correlation function in [14, Section 3] and was rewritten in the above form in [14, Lemma 5.1]. As explained in Remark 1.5, the symmetry (1.5) was used in the derivation of $C(\phi)$. Apart from the necessity to adopt the previous derivation, we do not use the symmetry (1.5) in this paper. Our double-scale integration regime is based on the following decomposition of the covariance.

$$\begin{aligned}&e^{-i\frac{\pi}{\beta}(s-t)}C(\phi)(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t) = C_0(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t) + C_1(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t),\\ &((\overline{\rho},\rho,\mathbf{x},s), (\overline{\eta},\eta,\mathbf{y},t) \in I_0,\ \phi \in \mathbb{C}),\end{aligned} \tag{3.1}$$

where the covariances C_0, $C_1 : I_0^2 \to \mathbb{C}$ are defined by

$$C_0(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t)$$

$$:= \frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*} e^{i\langle \mathbf{k},\mathbf{x}-\mathbf{y}\rangle}$$
$$\cdot h^{-1}(I_{2b} - e^{-\frac{i}{h}(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}((\overline{\rho}-1)b+\rho,(\overline{\eta}-1)b+\eta),$$
$$C_1(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t)$$
$$:= \frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h\backslash\{\frac{\pi}{\beta}\}} e^{i\langle \mathbf{k},\mathbf{x}-\mathbf{y}\rangle+i(\omega-\frac{\pi}{\beta})(s-t)}$$
$$\cdot h^{-1}(I_{2b} - e^{-\frac{i}{h}(\omega-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}((\overline{\rho}-1)b+\rho,(\overline{\eta}-1)b+\eta).$$

Our aim here is to establish necessary bound properties of $C(\phi)$, C_0, C_1. The bounds must be so sharp that the resulting multi-scale analysis does not require any (β,θ)-dependent condition on the coupling constant U. First let us present bound properties which can be proved by standard arguments. In the following we use the norms $\|\cdot\|_{1,\infty}$, $\|\cdot\|'_{1,\infty}$ defined in [14, Subsection 4.1]. Let $\langle\cdot,\cdot\rangle_{\mathbb{C}^m}$ denote the canonical inner product of $\mathbb{C}^m$. More precisely, for $\mathbf{u}=(u_1,\cdots,u_m)$, $\mathbf{v}=(v_1,\cdots,v_m)\in\mathbb{C}^m$ $\langle\mathbf{u},\mathbf{v}\rangle_{\mathbb{C}^m} := \sum_{j=1}^m \overline{u_j}v_j$. Moreover, for any $f : I_0^2\to\mathbb{C}$ let $\tilde{f} : I^2\to\mathbb{C}$ denote the anti-symmetric extension of f defined by

$$\tilde{f}((X,\xi),(Y,\zeta)) := \frac{1}{2}(1_{(\xi,\zeta)=(1,-1)}f(X,Y) - 1_{(\xi,\zeta)=(-1,1)}f(Y,X)), \quad (\forall X,Y\in I_0,\ \xi,\zeta\in\{1,-1\}). \tag{3.2}$$

From here for any objects $\alpha_1,\cdots,\alpha_m$ we let $c(\alpha_1,\cdots,\alpha_m)$ denote a positive constant depending only on $\alpha_1,\cdots,\alpha_m$.

Lemma 3.1. *Assume that*

$$h \ge \max\{\sqrt{e_{max}^2+|\phi|^2},1\}. \tag{3.3}$$

Then there exists $c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E)\in\mathbb{R}_{>0}$ depending only on $d,b,(\hat{\mathbf{v}}_j)_{j=1}^d$, c_E such that the following statements hold.

(i)

$$|\det(\langle\mathbf{u}_i,\mathbf{w}_j\rangle_{\mathbb{C}^m}C(\phi)(X_i,Y_j))_{1\le i,j\le n}| \le (c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E)(1+\beta^{-1}e_{min}^{-1}))^n, \tag{3.4}$$
$$(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{w}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{w}_i\|_{\mathbb{C}^m}\le 1,\ X_i,Y_i\in I_0\ (i=1,\cdots,n)).$$

(ii)

$$|\det(\langle \mathbf{u}_i, \mathbf{w}_j\rangle_{\mathbb{C}^m} C_0(X_i,Y_j))_{1\le i,j\le n}| \le (c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d, c_E)\beta^{-1}e_{min}^{-1})^n,$$
$$(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{w}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m}, \|\mathbf{w}_i\|_{\mathbb{C}^m}\le 1,\ X_i,Y_i\in I_0\ (i=1,\cdots,n)).$$

(iii)

$$\|\tilde{C}_0\|_{1,\infty} \le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d, c_E)\max\{e_{min}^{-1}, e_{min}^{-d-1}\},$$
$$\|\tilde{C}_0\|'_{1,\infty} \le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d, c_E)\beta^{-1}\max\{e_{min}^{-1}, e_{min}^{-d-1}\}.$$

(iv)

$$\|\tilde{C}_1\|_{1,\infty} \le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d, c_E)\max\{e_{min}^{-1}, e_{min}^{-d-1}\},$$
$$\|\tilde{C}_1\|'_{1,\infty} \le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d, c_E)(e_{min}+\beta^{-1}+\beta^{-1}e_{min}^{-1}+1) \cdot \max\{e_{min}^{-1}, e_{min}^{-d-1}\}.$$

REMARK 3.2. The bound (3.4) is not directly used in our multi-scale integration process, so its dependency on β does not affect the magnitude of the coupling constant. The upper bounds on $\|\tilde{C}_0\|'_{1,\infty}$, $\|\tilde{C}_1\|'_{1,\infty}$ depend on β. However, they are to be multiplied by L^{-d} during the multi-scale integration and thus do not yield a β-dependent condition on the coupling constant. Our essential problem is to prevent the β-dependent determinant bound of C_0 from affecting the magnitude of the coupling constant. Solving this problem is the main novelty of the present double-scale integration scheme.

PROOF OF LEMMA 3.1. We fix $\phi\in\mathbb{C}$ during the proof. Resulting bounds will be independent of ϕ, mainly due to the assumption (3.3). First of all let us list useful estimates. For $(\omega,\mathbf{k})\in\mathbb{R}^{d+1}$, set

$$B(\omega,\mathbf{k}) := h(I_{2b} - e^{-\frac{i}{h}(\omega-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})}).$$

We should recall the definition (1.8) of c_E beforehand.

$$\inf_{\mathbf{k}\in\mathbb{R}^d}\ \inf_{\substack{\mathbf{u}\in\mathbb{C}^{2b}\\ \text{with } \|\mathbf{u}\|_{\mathbb{C}^{2b}}=1}} \|E(\phi)(\mathbf{k})\mathbf{u}\|_{\mathbb{C}^{2b}} = \sqrt{e_{min}^2+|\phi|^2}, \tag{3.5}$$

$$\sup_{\mathbf{k}\in\mathbb{R}^d} \|E(\phi)(\mathbf{k})\|_{2b\times 2b} = \sqrt{e_{max}^2 + |\phi|^2}, \tag{3.6}$$

$$\|B(\omega,\mathbf{k})^{-1}\|_{2b\times 2b} \le c\left(h^2\sin^2\left(\frac{1}{2h}\left(\omega - \frac{\theta(\beta)}{2}\right)\right) + e_{min}^2\right)^{-\frac{1}{2}}, \tag{3.7}$$

$$\left\|\left(\frac{\partial}{\partial\omega}\right)^m B(\omega,\mathbf{k})\right\|_{2b\times 2b} \le ch^{-m+1}, \tag{3.8}$$

$$\left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^m B\left(\omega, \sum_{i=1}^d \hat{k}_i\hat{\mathbf{v}}_i\right)\right\|_{2b\times 2b} \le c(d, (\hat{\mathbf{v}}_j)_{j=1}^d, c_E), \tag{3.9}$$

$$(\forall m \in \{1,\cdots,d+2\},\ j\in\{1,\cdots,d\},\ \omega\in\mathbb{R},\ \mathbf{k},\hat{\mathbf{k}}\in\mathbb{R}^d).$$

In the derivation of (3.7), (3.8), (3.9) we use (3.3), (3.5), (3.6). Also, to derive (3.9), one can repeatedly use the formula

$$\begin{aligned}&\frac{\partial}{\partial k_j} e^{\frac{1}{h}E(\phi)(\mathbf{k})}\\ &= \frac{1}{h}\int_0^1 ds e^{\frac{s}{h}E(\phi)(\mathbf{k})}\frac{\partial}{\partial k_j}E(\phi)(\mathbf{k})e^{\frac{1-s}{h}E(\phi)(\mathbf{k})}, \quad (j\in\{1,\cdots,d\}).\end{aligned}$$

(i): It was proved in [14, Lemma 3.5 (iii)], which is based on the general determinant bound [20, Theorem 1.3], that

$$\begin{aligned}&|\det(\langle\mathbf{u}_i,\mathbf{w}_j\rangle_{\mathbb{C}^m} C(\phi)(X_i,Y_j))_{1\le i,j\le n}|\\ &\le \left(\frac{2^4 b}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\left(1+2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}}\right.\right.\\ &\qquad\qquad\left.\left.+ e^{-2\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}}\right)^{-\frac{1}{2}}\right)^n,\end{aligned}$$

$$(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{w}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{w}_i\|_{\mathbb{C}^m}\le 1,\ X_i,Y_i\in I_0\ (i=1,\cdots,n)).$$

Observe that

$$\begin{aligned}&\mathrm{Tr}\left(1+2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}} + e^{-2\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}}\right)^{-\frac{1}{2}}\\ &\le b(1-e^{-\beta e_{min}})^{-1} \le cb(1+\beta^{-1}e_{min}^{-1}).\end{aligned}$$

Thus the claimed bound holds.

(ii): Let $L^2(\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathcal{M}_h)$ be the Hilbert space whose inner product is defined by

$$\langle f,g\rangle_{L^2}:=\frac{1}{\beta L^d}\sum_{K\in\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathcal{M}_h}\overline{f(K)}g(K).$$

We derive the claimed bound by applying the Gram inequality in the Hilbert space $\mathbb{C}^m\otimes L^2(\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathcal{M}_h)$. Let us define the vectors f_X, $g_X\in L^2(\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathcal{M}_h)$ $(X\in I_0)$ by

$$\begin{aligned}&f_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},\omega):=e^{-i\langle\mathbf{k},\mathbf{x}\rangle}1_{\omega=\frac{\pi}{\beta}}1_{(\overline{\rho},\rho)=(\overline{\tau},\tau)}e_{min}^{-\frac{1}{2}},\\&g_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},\omega)\\&:=e^{-i\langle\mathbf{k},\mathbf{x}\rangle}1_{\omega=\frac{\pi}{\beta}}e_{min}^{\frac{1}{2}}B\left(\frac{\pi}{\beta},\mathbf{k}\right)^{-1}((\overline{\tau}-1)b+\tau,(\overline{\rho}-1)b+\rho).\end{aligned}$$

It follows that $C_0(X,Y)=\langle f_X,g_Y\rangle_{L^2}$ for any X, $Y\in I_0$. We can apply (3.7) to verify that

$$\|f_X\|_{L^2}^2\le\beta^{-1}e_{min}^{-1},\quad\|g_X\|_{L^2}^2\le c(b)\beta^{-1}e_{min}^{-1},\quad(\forall X\in I_0).$$

Therefore by the Gram inequality

$$\begin{aligned}|\det(\langle\mathbf{u}_i,\mathbf{w}_j\rangle_{\mathbb{C}^m}C_0(X_i,Y_j))_{1\le i,j\le n}|&\le\prod_{i=1}^n\|\mathbf{u}_i\|_{\mathbb{C}^m}\|\mathbf{w}_i\|_{\mathbb{C}^m}\|f_{X_i}\|_{L^2}\|g_{Y_i}\|_{L^2}\\&\le(c(b)\beta^{-1}e_{min}^{-1})^n,\end{aligned}$$
$$(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{w}_i\in\mathbb{C}^m\text{ with }\|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{w}_i\|_{\mathbb{C}^m}\le1,\ X_i,Y_i\in I_0\ (i=1,\cdots,n)).$$

(iii): By applying e.g. the formula [12, (C.1)] we can derive the following inequality.

$$\begin{aligned}&\left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^nB\left(\omega,\sum_{i=1}^d\hat{k}_i\hat{\mathbf{v}}_i\right)^{-1}\right\|_{2b\times2b}\\&\le c(d)\sum_{m=1}^n\prod_{u=1}^m\left(\sum_{l_u=1}^n\right)1_{\sum_{u=1}^m l_u=n}\end{aligned}$$

$$\cdot \prod_{p=1}^{m} \left\| B\left(\omega, \sum_{i=1}^{d} \hat{k}_i \hat{\mathbf{v}}_i\right)^{-1} \left(\frac{\partial}{\partial \hat{k}_j}\right)^{l_p} B\left(\omega, \sum_{i=1}^{d} \hat{k}_i \hat{\mathbf{v}}_i\right) \right\|_{2b\times 2b}$$
$$\cdot \left\| B\left(\omega, \sum_{i=1}^{d} \hat{k}_i \hat{\mathbf{v}}_i\right)^{-1} \right\|_{2b\times 2b},$$
$$(\forall n \in \{1, \cdots, d+2\},\ j \in \{0, \cdots, d\},\ \omega \in \mathbb{R},\ \hat{\mathbf{k}} \in \mathbb{R}^d),$$

where $\frac{\partial}{\partial \hat{k}_0}$ denotes $\frac{\partial}{\partial \omega}$. Combination of this inequality and (3.7), (3.8), (3.9) yields that

(3.10)
$$\left\| \left(\frac{\partial}{\partial \hat{k}_j}\right)^{n} B\left(\omega, \sum_{i=1}^{d} \hat{k}_i \hat{\mathbf{v}}_i\right)^{-1} \right\|_{2b\times 2b}$$
$$\le c(d, (\hat{\mathbf{v}}_j)_{j=1}^{d}, c_E)$$
$$\cdot \sum_{m=1}^{n} \left(h^2 \sin^2\left(\frac{1}{2h}\left(\omega - \frac{\theta(\beta)}{2}\right)\right) + e_{min}^2\right)^{-\frac{m+1}{2}} (1_{j=0} h^{-n+m} + 1_{j\ge 1}),$$
$$(\forall n \in \{1, \cdots, d+2\},\ j \in \{0, \cdots, d\},\ \omega \in \mathbb{R},\ \hat{\mathbf{k}} \in \mathbb{R}^d).$$

By periodicity we can perform integration by parts to derive that for any $\mathbf{x}$, $\mathbf{y} \in \Gamma$, $s, t \in [0, \beta)_h$, $j \in \{1, \cdots, d\}$

$$\left(\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y}, \hat{\mathbf{v}}_j\rangle} - 1)\right)^{d+1} C_0(\cdot \mathbf{x} s, \cdot \mathbf{y} t)$$
$$= \frac{1}{\beta L^d} \sum_{\mathbf{k}\in \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y}\rangle}$$
$$\cdot \prod_{m=1}^{d+1} \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dp_m\right) \left(\frac{\partial}{\partial \hat{k}_j}\right)^{d+1} B\left(\frac{\pi}{\beta}, \mathbf{k} + \hat{k}_j \hat{\mathbf{v}}_j\right)^{-1} \Big|_{\hat{k}_j = \sum_{m=1}^{d+1} p_m}.$$

Substitution of (3.7), (3.10) gives that

$$\left| \left(\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y}, \hat{\mathbf{v}}_j\rangle} - 1)\right)^{d+1} \right| \| C_0(\cdot \mathbf{x} s, \cdot \mathbf{y} t) \|_{2b\times 2b}$$

$$\begin{aligned}
&\le c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E)\beta^{-1}\sum_{m=1}^{d+1}e_{min}^{-m-1}\\
&\le c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E)\beta^{-1}\max\{e_{min}^{-2},e_{min}^{-d-2}\},\\
&\|C_0(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\|_{2b\times 2b}\le c\beta^{-1}e_{min}^{-1},\\
&(\forall\mathbf{x},\mathbf{y}\in\Gamma,\ s,t\in[0,\beta)_h).
\end{aligned}$$

These bounds lead to that

$$\begin{aligned}
\|\tilde{C}_0\|_{1,\infty}&\le\sum_{\mathbf{x}\in\Gamma}\frac{c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E)e_{min}^{-1}}{1+(\max\{e_{min}^{-1},e_{min}^{-d-1}\})^{-1}\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\hat{\mathbf{v}}_j\rangle}-1)|^{d+1}}\\
&\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E)e_{min}^{-1}\left(\sum_{\mathbf{x}\in\Gamma}\frac{1_{e_{min}\ge 1}}{1+\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\hat{\mathbf{v}}_j\rangle}-1)|^{d+1}}\right.\\
&\qquad\left.+\sum_{\mathbf{x}\in\Gamma}\frac{1_{e_{min}<1}}{1+e_{min}^{d+1}\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\hat{\mathbf{v}}_j\rangle}-1)|^{d+1}}\right)\\
&\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E)\max\{e_{min}^{-1},e_{min}^{-d-1}\}.
\end{aligned}$$

The claimed bound on $\|\tilde{C}_0\|'_{1,\infty}$ is proved in the same way.

(iv): Let us apply a standard method of slicing the covariance. Let us take a function $\chi\in C^\infty(\mathbb{R},\mathbb{R})$ satisfying that

$$\begin{aligned}
&\chi(x)=1,\quad(\forall x\in(-\infty,1]),\\
&\chi(x)\in(0,1),\quad(\forall x\in(1,2)),\\
&\chi(x)=0,\quad(\forall x\in[2,\infty)),\\
&\frac{d}{dx}\chi(x)\le 0,\quad(\forall x\in\mathbb{R}).
\end{aligned}$$

Set

$$N_h:=\left\lfloor\frac{\log h}{\log 2}\right\rfloor+1,\quad N_0:=\left\lfloor\frac{\log(\max\{e_{min},\beta^{-1}\})}{\log 2}\right\rfloor,$$

where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x for $x\in\mathbb{R}$. By (3.3) and the definition of h, $h\ge\max\{e_{min},\beta^{-1}\}$ and thus $N_0<N_h$. Then we define the functions $\chi_l\in C^\infty(\mathbb{R})$ $(l=N_0,N_0+1,\cdots,N_h)$ by

$$\chi_{N_0}(\omega):=\chi\left(2^{-N_0}h\left|\sin\left(\frac{\omega-\theta(\beta)/2}{2h}\right)\right|\right),$$

$$\chi_l(\omega) := \chi\left(2^{-l}h\left|\sin\left(\frac{\omega-\theta(\beta)/2}{2h}\right)\right|\right) - \chi\left(2^{-(l-1)}h\left|\sin\left(\frac{\omega-\theta(\beta)/2}{2h}\right)\right|\right),$$
$$(l = N_0+1,\cdots,N_h).$$

These functions behave as follows.

$$\chi_{N_0}(\omega) = \begin{cases} 1 & \text{if } h\left|\sin\left(\frac{\omega-\theta(\beta)/2}{2h}\right)\right| \le 2^{N_0}, \\ \in (0,1) & \text{if } 2^{N_0} < h\left|\sin\left(\frac{\omega-\theta(\beta)/2}{2h}\right)\right| < 2^{N_0+1}, \\ 0 & \text{if } h\left|\sin\left(\frac{\omega-\theta(\beta)/2}{2h}\right)\right| \ge 2^{N_0+1}, \end{cases} \tag{3.11}$$

$$\chi_l(\omega) = \begin{cases} 0 & \text{if } h\left|\sin\left(\frac{\omega-\theta(\beta)/2}{2h}\right)\right| \le 2^{l-1}, \\ \in (0,1] & \text{if } 2^{l-1} < h\left|\sin\left(\frac{\omega-\theta(\beta)/2}{2h}\right)\right| < 2^{l+1}, \\ 0 & \text{if } h\left|\sin\left(\frac{\omega-\theta(\beta)/2}{2h}\right)\right| \ge 2^{l+1}, \end{cases}$$
$$(l = N_0+1,\cdots,N_h).$$

Moreover, there exists $c(d,\chi) \in \mathbb{R}_{>0}$ depending only on d, χ such that the following statements hold.

- $$\sum_{l=N_0}^{N_h} \chi_l(\omega) = 1, \quad (\forall \omega \in \mathbb{R}). \tag{3.12}$$

- $$\left|\left(\frac{\partial}{\partial\omega}\right)^n \chi_l(\omega)\right| \le c(d,\chi)2^{-nl}, \tag{3.13}$$
$$(\forall n \in \{1,\cdots,d+2\},\ l \in \{N_0,\cdots,N_h\},\ \omega \in \mathbb{R}).$$

- $$\frac{1}{\beta}\sup_{x\in\mathbb{R}} \sum_{\omega\in\mathcal{M}_h} 1_{\chi_l(\omega+x)\neq 0} \le c(d,\chi)2^l, \quad (\forall l \in \{N_0,\cdots,N_h\}). \tag{3.14}$$

To prove (3.12), (3.13), we use that

$$2^{N_h-1} \le h \le 2^{N_h}. \tag{3.15}$$

To prove (3.14), we use that $\beta^{-1} \le c2^{N_0}$. Then let us define the covariances $C_l' : I_0^2 \to \mathbb{C}$ $(l = N_0, N_0+1, \cdots, N_h)$ by

$$C_l'(\cdot\mathbf{x}s, \cdot\mathbf{y}t) := \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in\mathcal{M}_h} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle + i\omega(s-t)} \chi_l(\omega) B(\omega, \mathbf{k})^{-1}.$$

It follows from (3.12) that

$$\sum_{l=N_0}^{N_h} C_l'(\cdot\mathbf{x}s, \cdot\mathbf{y}t) = C(\phi)(\cdot\mathbf{x}s, \cdot\mathbf{y}t), \quad (\forall \mathbf{x}, \mathbf{y} \in \Gamma,\ s, t \in [0,\beta)_h). \tag{3.16}$$

Our strategy is as follows. We first find upper bounds on $\|\tilde{C}(\phi)\|_{1,\infty}$, $\|\tilde{C}(\phi)\|_{1,\infty}'$ by estimating each C_l' and summing up them. Then we derive the claimed bounds on $\|\tilde{C}_1\|_{1,\infty}$, $\|\tilde{C}_1\|_{1,\infty}'$ by using the relation (3.1) and the results of (iii). By (3.7), (3.11), (3.14)

$$\begin{aligned} &\|C_l'(\cdot\mathbf{x}s, \cdot\mathbf{y}t)\|_{2b\times 2b} \le c(d,\chi) 2^l (1_{l=N_0} e_{min}^{-1} + 1_{l\ge N_0+1}(2^l + e_{min})^{-1}), \\ &(\forall l \in \{N_0, \cdots, N_h\},\ \mathbf{x}, \mathbf{y} \in \Gamma,\ s, t \in [0,\beta)_h). \end{aligned} \tag{3.17}$$

Integrating by parts based on periodicity yields that

(3.18)
$$\begin{aligned} &\left(\frac{\beta}{2\pi}(e^{-i\frac{2\pi}{\beta}(s-t)} - 1)\right)^n C_l'(\cdot\mathbf{x}s, \cdot\mathbf{y}t) \\ &= \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in\mathcal{M}_h} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle + i\omega(s-t)} \\ &\quad \cdot \prod_{m=1}^n \left(\frac{\beta}{2\pi}\int_0^{\frac{2\pi}{\beta}} dr_m\right) \left(\frac{\partial}{\partial r}\right)^n \chi_l(r) B(r, \mathbf{k})^{-1}\Big|_{r=\omega+\sum_{m=1}^n r_m}, \end{aligned}$$

(3.19)
$$\begin{aligned} &\left(\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle} - 1)\right)^n C_l'(\cdot\mathbf{x}s, \cdot\mathbf{y}t) \\ &= \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in\mathcal{M}_h} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle + i\omega(s-t)} \\ &\quad \cdot \prod_{m=1}^n \left(\frac{L}{2\pi}\int_0^{\frac{2\pi}{L}} dp_m\right) \chi_l(\omega) \left(\frac{\partial}{\partial \hat{k}_j}\right)^n B(\omega, \mathbf{k} + \hat{k}_j\hat{\mathbf{v}}_j)^{-1}\Big|_{\hat{k}_j=\sum_{m=1}^n p_m}, \end{aligned}$$

$(\forall l \in \{N_0, \cdots, N_h\},\ \mathbf{x}, \mathbf{y} \in \Gamma,\ s, t \in [0, \beta)_h,\ j \in \{1, \cdots, d\},$
$n \in \{1, \cdots, d+2\}).$

Assume that $l \geq N_0 + 1$. By (3.7), (3.10), (3.11), (3.13), (3.14) and (3.18)

$$
\begin{aligned}
(3.20)\qquad & \left|\frac{\beta}{2\pi}(e^{-i\frac{2\pi}{\beta}(s-t)} - 1)\right|^{d+2} \|C_l'(\cdot\mathbf{x}s, \cdot\mathbf{y}t)\|_{2b\times 2b} \\
&\leq \prod_{m=1}^{d+2}\left(\frac{\beta}{2\pi}\int_0^{\frac{2\pi}{\beta}} dr_m\right) \\
&\quad \cdot \frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h} 1_{\chi_l(\omega+\sum_{m=1}^{n} r_m)\neq 0} \\
&\quad \cdot \sup_{r\in[-\pi h, \pi h]}\left\|\left(\frac{\partial}{\partial r}\right)^{d+2}\chi_l(r)B(r, \mathbf{k})^{-1}\right\|_{2b\times 2b} \\
&\leq c(d, (\hat{\mathbf{v}}_j)_{j=1}^d, c_E, \chi)2^l \\
&\quad \cdot \left(\sum_{p=0}^{d+1} 2^{-pl}\sum_{m=1}^{d+2-p} h^{-(d+2-p)+m}(2^{2l} + e_{min}^2)^{-\frac{m+1}{2}}\right. \\
&\qquad \left. + 2^{-(d+2)l}(2^{2l} + e_{min}^2)^{-\frac{1}{2}}\right) \\
&\leq c(d, (\hat{\mathbf{v}}_j)_{j=1}^d, c_E, \chi)2^{-(d+2)l}.
\end{aligned}
$$

In the last inequality we also used (3.15). On the other hand, by (3.10), (3.11), (3.14) and (3.19) for $j \in \{1, \cdots, d\}$, $n \in \{1, \cdots, d+2\}$

$$
\begin{aligned}
(3.21)\qquad & \left|\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y}, \hat{\mathbf{v}}_j\rangle} - 1)\right|^{n} \|C_l'(\cdot\mathbf{x}s, \cdot\mathbf{y}t)\|_{2b\times 2b} \\
&\leq c(d, (\hat{\mathbf{v}}_j)_{j=1}^d, c_E, \chi)2^l\sum_{m=1}^{n}(2^{2l} + e_{min}^2)^{-\frac{m+1}{2}} \\
&\leq c(d, (\hat{\mathbf{v}}_j)_{j=1}^d, c_E, \chi)\max\{e_{min}^{-1}, e_{min}^{-n}\}.
\end{aligned}
$$

By combining (3.17), (3.20) and (3.21) for $n = d + 2$

$\|C_l'(\cdot\mathbf{x}s, \cdot\mathbf{y}t)\|_{2b\times 2b}$

$$\leq c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\Bigg/\Bigg(1+2^{(d+2)l}\left|\frac{\beta}{2\pi}(e^{i\frac{2\pi}{\beta}(s-t)}-1)\right|^{d+2} + (\max\{e_{min}^{-1},e_{min}^{-d-2}\})^{-1}\sum_{j=1}^d\left|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)\right|^{d+2}\Bigg),$$

$(\forall\mathbf{x},\mathbf{y}\in\Gamma,\ s,t\in[0,\beta)_h)$,

which together with (3.15) implies that

$$\begin{aligned}\|\tilde{C}_l'\|_{1,\infty}&\leq c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)2^{-l}(1_{e_{min}\geq 1}+1_{e_{min}<1}e_{min}^{-d})\\&\leq c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)2^{-l}\max\{1,e_{min}^{-d}\}.\end{aligned}\tag{3.22}$$

Also by (3.21) for $n=d+1$

$$\begin{aligned}&\sum_{\substack{\mathbf{x}\in\Gamma\\ \mathbf{x}\neq\mathbf{0}}}(\|C_l'(\cdot\mathbf{x}s,\cdot\mathbf{0}t)\|_{2b\times 2b}+\|C_l'(\cdot\mathbf{0}t,\cdot\mathbf{x}s)\|_{2b\times 2b})\\&\leq c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\max\{2^{-l},2^{-(d+1)l}\}\sum_{\substack{\mathbf{x}\in\Gamma\\ \mathbf{x}\neq\mathbf{0}}}\frac{1}{\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\hat{\mathbf{v}}_j\rangle}-1)|^{d+1}}\\&\leq c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\max\{2^{-l},2^{-(d+1)l}\},\quad(\forall s,t\in[0,\beta)_h).\end{aligned}\tag{3.23}$$

Let us derive necessary bounds for $l=N_0$. By (3.10), (3.14), (3.19)

$$\begin{aligned}&\left|\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)\right|^n\|C_{N_0}'(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\|_{2b\times 2b}\\&\leq c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)2^{N_0}\max\{e_{min}^{-2},e_{min}^{-n-1}\},\\&(\forall\mathbf{x},\mathbf{y}\in\Gamma,\ s,t\in[0,\beta)_h,\ j\in\{1,\cdots,d\},\ n\in\{1,\cdots,d+2\}).\end{aligned}\tag{3.24}$$

Assume that $e_{min}\leq\beta^{-1}$. It follows from (3.17), (3.24) for $n=d+1$ that

$$\begin{aligned}&\|C_{N_0}'(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\|_{2b\times 2b}\\&\leq\frac{c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)2^{N_0}e_{min}^{-1}}{1+(\max\{e_{min}^{-1},e_{min}^{-d-1}\})^{-1}\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)|^{d+1}},\\&(\forall\mathbf{x},\mathbf{y}\in\Gamma,\ s,t\in[0,\beta)_h),\end{aligned}$$

and thus

$$\begin{aligned}\|\tilde{C}'_{N_0}\|_{1,\infty} &\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\beta 2^{N_0} e_{min}^{-1}(1_{e_{min}\ge 1}+1_{e_{min}<1}e_{min}^{-d})\\ &\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\max\{e_{min}^{-1},e_{min}^{-d-1}\},\end{aligned}$$

where we used that $2^{N_0}\le\beta^{-1}$. On the other hand, let us assume that $e_{min}>\beta^{-1}$. By (3.7), (3.10), (3.13), (3.14) and (3.18)

(3.25)

$$\begin{aligned}&\left|\frac{\beta}{2\pi}(e^{-i\frac{2\pi}{\beta}(s-t)}-1)\right|^{d+2}\|C'_{N_0}(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\|_{2b\times 2b}\\ &\le c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)2^{N_0}\\ &\quad\cdot\left(\sum_{p=0}^{d+1}2^{-pN_0}\sum_{m=1}^{d+2-p}h^{-(d+2-p)+m}e_{min}^{-m-1}+2^{-(d+2)N_0}e_{min}^{-1}\right)\\ &\le c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)2^{N_0}\left(\sum_{p=0}^{d+1}\sum_{m=1}^{d+2-p}2^{-N_0(d+2-m)}e_{min}^{-m-1}+2^{-(d+2)N_0}e_{min}^{-1}\right)\\ &\le c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)2^{-(d+1)N_0}e_{min}^{-1},\\ &(\forall\mathbf{x},\mathbf{y}\in\Gamma,\ s,t\in[0,\beta)_h).\end{aligned}$$

In the second inequality we used (3.15). In the last inequality we used that $2^{N_0}\le e_{min}$. By using (3.17), (3.24) for $n=d+2$ and (3.25) we have that

$$\begin{aligned}&\|C'_{N_0}(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\|_{2b\times 2b}\\ &\le c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)2^{N_0}e_{min}^{-1}\Bigg/\Bigg(1+2^{(d+2)N_0}\left|\frac{\beta}{2\pi}(e^{i\frac{2\pi}{\beta}(s-t)}-1)\right|^{d+2}\\ &\qquad+(\max\{e_{min}^{-1},e_{min}^{-d-2}\})^{-1}\sum_{j=1}^d\left|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)\right|^{d+2}\Bigg),\end{aligned}$$

and thus by using (3.15)

$$\begin{aligned}\|\tilde{C}'_{N_0}\|_{1,\infty} &\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)e_{min}^{-1}(1_{e_{min}\ge 1}+1_{e_{min}<1}e_{min}^{-d})\\ &\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\max\{e_{min}^{-1},e_{min}^{-d-1}\}.\end{aligned}$$

In both cases we have derived that

$$\|\tilde{C}'_{N_0}\|_{1,\infty} \le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\max\{e_{min}^{-1},e_{min}^{-d-1}\}. \tag{3.26}$$

Moreover, it follows from (3.24) for $n=d+1$ that

$$\begin{aligned}&\sum_{\substack{\mathbf{x}\in\Gamma\\ \mathbf{x}\neq\mathbf{0}}}(\|C'_{N_0}(\cdot\mathbf{x}s,\cdot\mathbf{0}t)\|_{2b\times 2b}+\|C'_{N_0}(\cdot\mathbf{0}t,\cdot\mathbf{x}s)\|_{2b\times 2b})\\ &\le c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)2^{N_0}\max\{e_{min}^{-2},e_{min}^{-d-2}\},\quad (\forall s,t\in[0,\beta)_h).\end{aligned} \tag{3.27}$$

Let us sum up the above estimates. By (3.16), (3.22) and (3.26)

$$\begin{aligned}\|\tilde{C}(\phi)\|_{1,\infty}&\le\sum_{l=N_0}^{N_h}\|\tilde{C}'_l\|_{1,\infty}\\ &\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)(\max\{e_{min}^{-1},e_{min}^{-d-1}\}+2^{-N_0}\max\{1,e_{min}^{-d}\})\\ &\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\max\{e_{min}^{-1},e_{min}^{-d-1}\}.\end{aligned} \tag{3.28}$$

Also, we can apply (3.4), (3.16), (3.23) and (3.27) to deduce that

$$\begin{aligned}&\|\tilde{C}(\phi)\|'_{1,\infty}\\ &\le c(b)\sup_{s,t\in[0,\beta)_h}\|C(\phi)(\cdot\mathbf{0}s,\cdot\mathbf{0}t)\|_{2b\times 2b}\\ &\quad+c(b)\sup_{s,t\in[0,\beta)_h}\sum_{l=N_0}^{N_h}\sum_{\substack{\mathbf{x}\in\Gamma\\ \mathbf{x}\neq\mathbf{0}}}(\|C'_l(\cdot\mathbf{x}s,\cdot\mathbf{0}t)\|_{2b\times 2b}+\|C'_l(\cdot\mathbf{0}t,\cdot\mathbf{x}s)\|_{2b\times 2b})\\ &\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\\ &\quad\cdot\left(1+\beta^{-1}e_{min}^{-1}+2^{N_0}\max\{e_{min}^{-2},e_{min}^{-d-2}\}+\sum_{l=N_0+1}^{N_h}\max\{2^{-l},2^{-(d+1)l}\}\right)\\ &\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)\\ &\quad\cdot(1+\beta^{-1}e_{min}^{-1}+(e_{min}+\beta^{-1})\max\{e_{min}^{-2},e_{min}^{-d-2}\}+e_{min}^{-1}+e_{min}^{-d-1})\\ &\le c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,c_E,\chi)(e_{min}+\beta^{-1}+\beta^{-1}e_{min}^{-1}+1)\max\{e_{min}^{-1},e_{min}^{-d-1}\}.\end{aligned} \tag{3.29}$$

Observe that by (3.1)

$$\|\tilde{C}_1\|_{1,\infty} \le \|\tilde{C}(\phi)\|_{1,\infty} + \|\tilde{C}_0\|_{1,\infty}, \quad \|\tilde{C}_1\|'_{1,\infty} \le \|\tilde{C}(\phi)\|'_{1,\infty} + \|\tilde{C}_0\|'_{1,\infty}.$$

Then, substitution of (3.28), (3.29) and the results of (iii) yields the claimed inequalities. □

REMARK 3.3. Assume that $\beta \ge e_{min}^{-1}$. Then it follows from (3.16), (3.21), (3.24) that

$$\sum_{j=1}^{d} \left| \frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle} - 1)\right| \|C(\phi)(\cdot\mathbf{x}0, \cdot\mathbf{y}0)\|_{2b\times 2b} \le c(d, (\hat{\mathbf{v}}_j)_{j=1}^{d}, c_E, \chi)e_{min}^{-1},$$

$$(\forall \mathbf{x}, \mathbf{y} \in \Gamma,\ \phi \in \mathbb{C}).$$

The above inequality holds for any $\phi \in \mathbb{C}$ due to the fact that $C(\phi)(\cdot\mathbf{x}0, \cdot\mathbf{y}0)$ is independent of h (see [14, (3.2)]). As explained in Remark 1.6, the above spatial decay property can be used to study the zero-temperature limit of the 4-point correlation function.

Lemma 3.1 does not include a determinant bound of C_1, which crucially affects the possible magnitude of the coupling constant in our double-scale integration scheme. A determinant bound of C_1 can be useful only if it is optimal with respect to the dependency on (β, θ). Let us derive a desirable bound in the next lemma. Again we will essentially apply not only the general bound [20, Theorem 1.3] but the representation techniques presented in [20, Subsection 4.1] by de Siqueira Pedra and Salmhofer as in our previous derivation of determinant bound [13, Proposition 4.2]. We should remark more specifically that the decompositions (3.36), (3.44) below are influenced by the techniques of [20, Subsection 4.1]. However, the choice of the Hilbert space, which will be denoted by $\mathcal{H}$, and the construction of necessary vectors belonging to the Hilbert space are much more complicated than the corresponding parts of the previous papers. The essential idea here is to replace the sum over $\mathcal{M}_h\backslash\{\pi/\beta\}$ by a contour integral plus an extra term by means of the residue theorem.

LEMMA 3.4. *Assume that*

$$h \ge \sqrt{e_{max}^2 + |\phi|^2} + \frac{1}{\beta}(3\pi + 2). \tag{3.30}$$

Then there exists $c(b) \in \mathbb{R}_{>0}$ *depending only on* b *such that*

$$|\det(\langle \mathbf{u}_i, \mathbf{w}_j\rangle_{\mathbb{C}^m} C_1(X_i, Y_j))_{1\le i,j\le n}| \le c(b)^n,$$
$$(\forall m, n \in \mathbb{N},\ \mathbf{u}_i, \mathbf{w}_i \in \mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m}, \|\mathbf{w}_i\|_{\mathbb{C}^m} \le 1,\ X_i, Y_i \in I_0$$
$$(i = 1, \cdots, n)).$$

Proof. Let us fix $\phi \in \mathbb{C}$ throughout the proof. We will need to assume that h is large depending on ϕ on several occasions. We will eventually see that the assumption (3.30) is sufficient. Let $\sigma(E(\mathbf{k}))$, $\sigma(E(\phi)(\mathbf{k}))$ denote the set of eigenvalues of $E(\mathbf{k})$, $E(\phi)(\mathbf{k})$ respectively. For any $\mathbf{k} \in \Gamma^*$ there exist $e_\rho(\mathbf{k}) \in \mathbb{R}$ $(\rho = 1, \cdots, b)$ such that $e_1(\mathbf{k}) \le e_2(\mathbf{k}) \le \cdots \le e_b(\mathbf{k})$ and $\sigma(E(\mathbf{k})) = \{e_\rho(\mathbf{k})\}_{\rho\in\mathcal{B}}$. Set

$$\hat{e}_\rho(\mathbf{k}) := \sqrt{e_\rho(\mathbf{k})^2 + |\phi|^2}$$

for $\rho \in \mathcal{B}$. Observe that $\sigma(E(\phi)(\mathbf{k})) = \{\pm\hat{e}_\rho(\mathbf{k})\}_{\rho\in\mathcal{B}}$. For any $\mathbf{k} \in \Gamma^*$ there exists $x_{\mathbf{k}} \in [1/\beta, 2/\beta]$ such that

$$\left[x_{\mathbf{k}} - \frac{1}{2(b+1)\beta}, x_{\mathbf{k}} + \frac{1}{2(b+1)\beta}\right) \cap \sigma(E(\phi)(\mathbf{k})) = \emptyset. \tag{3.31}$$

This claim can be proved as follows. Suppose that

$$\left[\frac{1}{\beta} + \frac{2m+1}{2(b+1)\beta}, \frac{1}{\beta} + \frac{2m+3}{2(b+1)\beta}\right) \cap \sigma(E(\phi)(\mathbf{k})) \neq \emptyset$$

for any $m \in \{0, 1, \cdots, b\}$. Since these $b+1$ intervals are disjoint, it implies that $\sharp\sigma(E(\phi)(\mathbf{k})) \cap \mathbb{R}_{\ge 0} \ge b+1$. However, $\sharp\sigma(E(\phi)(\mathbf{k})) \cap \mathbb{R}_{\ge 0} \le b$, which is a contradiction. Thus the claim holds with some $x_{\mathbf{k}} \in \{\frac{1}{\beta} + \frac{m+1}{(b+1)\beta}\}_{m=0}^{b}$. Fix such $\{x_{\mathbf{k}}\}_{\mathbf{k}\in\Gamma^*}$. For $\mathbf{k} \in \Gamma^*$ let us set

$$\mathcal{B}(\mathbf{k}) := \left\{\rho \in \mathcal{B}\ \middle|\ \hat{e}_\rho(\mathbf{k}) \ge x_{\mathbf{k}} + \frac{1}{2(b+1)\beta}\right\},$$
$$P_1 := \{z \in \mathbb{C} \mid |z| = \pi h\},$$
$$P_2(\mathbf{k}) := \left\{x + i\frac{2\pi}{\beta}\ \middle|\ -x_{\mathbf{k}} \le x \le x_{\mathbf{k}}\right\} \cup \left\{x_{\mathbf{k}} + iy\ \middle|\ -\frac{\pi}{2\beta} \le y \le \frac{2\pi}{\beta}\right\}$$

$$\cup \left\{ x - i\frac{\pi}{2\beta} \ \Big| \ -x_{\mathbf{k}} \le x \le x_{\mathbf{k}} \right\}$$
$$\cup \left\{ -x_{\mathbf{k}} + iy \ \Big| \ -\frac{\pi}{2\beta} \le y \le \frac{2\pi}{\beta} \right\}.$$

By the assumption (3.30), $\sqrt{x_{\mathbf{k}}^2 + (2\pi/\beta)^2} < \pi h$. This implies that $P_1 \cap P_2(\mathbf{k}) = \emptyset$. We consider P_1 as a contour oriented counter-clockwise and $P_2(\mathbf{k})$ as a contour oriented clockwise. Let us admit a convention that for $A, B \in \mathrm{Mat}(b, \mathbb{C})$ $A \oplus B$ denotes the $2b \times 2b$ matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

For any $\mathbf{k} \in \Gamma^*$ there exists a $2b \times 2b$ unitary matrix $U(\mathbf{k})$ such that

$$U(\mathbf{k})^* E(\phi)(\mathbf{k}) U(\mathbf{k}) = (\delta_{\rho,\eta}\hat{e}_\rho(\mathbf{k}))_{1\le\rho,\eta\le b} \oplus (-\delta_{\rho,\eta}\hat{e}_\rho(\mathbf{k}))_{1\le\rho,\eta\le b}. \tag{3.32}$$

It follows that

(3.33)

$$\begin{aligned}
&C_1(\cdot\mathbf{x}s, \cdot\mathbf{y}t)\\
&= \frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle - i\frac{\pi}{\beta}(s-t)}\\
&\quad \cdot U(\mathbf{k}) \left(\frac{\delta_{\rho,\eta}}{\beta} \sum_{\omega\in\mathcal{M}_h\backslash\{\frac{\pi}{\beta}\}} e^{i\omega(s-t)} h^{-1}(1 - e^{-\frac{i}{h}(\omega - \frac{\theta(\beta)}{2}) + \frac{1}{h}\hat{e}_\rho(\mathbf{k})})^{-1} \right)_{1\le\rho,\eta\le b}\\
&\quad \oplus \left(\frac{\delta_{\rho,\eta}}{\beta} \sum_{\omega\in\mathcal{M}_h\backslash\{\frac{\pi}{\beta}\}} e^{i\omega(s-t)} h^{-1}(1 - e^{-\frac{i}{h}(\omega - \frac{\theta(\beta)}{2}) - \frac{1}{h}\hat{e}_\rho(\mathbf{k})})^{-1} \right)_{1\le\rho,\eta\le b} U(\mathbf{k})^*,
\end{aligned}$$

$(\forall \mathbf{x}, \mathbf{y} \in \Gamma,\ s, t \in [0,\beta)_h)$.

The assumption (3.30) implies that $|i\theta(\beta)/2 + \delta\hat{e}_\rho(\mathbf{k})| < \pi h$ for any $\mathbf{k} \in \Gamma^*$, $\rho \in \mathcal{B}$, $\delta \in \{1,-1\}$. Based on this fact and the property (3.31), the residue theorem ensures that for any $r \in \mathbb{R}$, $\mathbf{k} \in \Gamma^*$, $\rho \in \mathcal{B}$, $\delta \in \{1,-1\}$

$$\frac{1}{2\pi i}\oint_{P_1\cup P_2(\mathbf{k})} dz \frac{e^{zr}}{1+e^{\beta z}} h^{-1}(1 - e^{-\frac{1}{h}(z - i\frac{\theta(\beta)}{2}) + \frac{\delta}{h}\hat{e}_\rho(\mathbf{k})})^{-1} \tag{3.34}$$

$$= -\frac{1}{\beta}\sum_{\omega\in\mathcal{M}_h\backslash\{\frac{\pi}{\beta}\}} e^{i\omega r}h^{-1}(1-e^{-\frac{i}{h}(\omega-\frac{\theta(\beta)}{2})+\frac{\delta}{h}\hat{e}_\rho(\mathbf{k})})^{-1}$$
$$+ 1_{\rho\in\mathcal{B}(\mathbf{k})}\frac{e^{(i\frac{\theta(\beta)}{2}+\delta\hat{e}_\rho(\mathbf{k}))r}}{1+e^{\beta(i\frac{\theta(\beta)}{2}+\delta\hat{e}_\rho(\mathbf{k}))}}.$$

Let us define the functions $C^{\ge}_{1-1}$, $C^{\ge}_{1-2}$, $C^{<}_{1-1}$, $C^{<}_{1-2}$, C_{1-1}, $C_{1-2} : (\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta))^2\to\mathbb{C}$ as follows.

$$C^{\ge}_{1-1}(\cdot\mathbf{x}s,\cdot\mathbf{y}t)$$
$$:= \frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\frac{-1}{2\pi i}$$
$$\cdot\oint_{P_1\cup P_2(\mathbf{k})}dz\frac{e^{z(s-t)}}{1+e^{\beta z}}h^{-1}(I_{2b}-e^{-\frac{1}{h}(z-i\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1},$$
$$C^{\ge}_{1-2}(\cdot\mathbf{x}s,\cdot\mathbf{y}t)$$
$$:= \frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}U(\mathbf{k})\left(\frac{\delta_{\rho,\eta}1_{\rho\in\mathcal{B}(\mathbf{k})}e^{(i\frac{\theta(\beta)}{2}+\hat{e}_\rho(\mathbf{k}))(s-t)}}{1+e^{\beta(i\frac{\theta(\beta)}{2}+\hat{e}_\rho(\mathbf{k}))}}\right)_{1\le\rho,\eta\le b}$$
$$\oplus\left(\frac{\delta_{\rho,\eta}1_{\rho\in\mathcal{B}(\mathbf{k})}e^{(i\frac{\theta(\beta)}{2}-\hat{e}_\rho(\mathbf{k}))(s-t)}}{1+e^{\beta(i\frac{\theta(\beta)}{2}-\hat{e}_\rho(\mathbf{k}))}}\right)_{1\le\rho,\eta\le b}U(\mathbf{k})^*,$$
$$C^{<}_{1-1}(\cdot\mathbf{x}s,\cdot\mathbf{y}t)$$
$$:= \frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\frac{-1}{2\pi i}$$
$$\cdot\oint_{P_1\cup P_2(\mathbf{k})}dz\frac{e^{z(s-t+\beta)}}{1+e^{\beta z}}h^{-1}(I_{2b}-e^{-\frac{1}{h}(z-i\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1},$$
$$C^{<}_{1-2}(\cdot\mathbf{x}s,\cdot\mathbf{y}t)$$
$$:= \frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}U(\mathbf{k})\left(\frac{\delta_{\rho,\eta}1_{\rho\in\mathcal{B}(\mathbf{k})}e^{(i\frac{\theta(\beta)}{2}+\hat{e}_\rho(\mathbf{k}))(s-t+\beta)}}{1+e^{\beta(i\frac{\theta(\beta)}{2}+\hat{e}_\rho(\mathbf{k}))}}\right)_{1\le\rho,\eta\le b}$$
$$\oplus\left(\frac{\delta_{\rho,\eta}1_{\rho\in\mathcal{B}(\mathbf{k})}e^{(i\frac{\theta(\beta)}{2}-\hat{e}_\rho(\mathbf{k}))(s-t+\beta)}}{1+e^{\beta(i\frac{\theta(\beta)}{2}-\hat{e}_\rho(\mathbf{k}))}}\right)_{1\le\rho,\eta\le b}U(\mathbf{k})^*,$$
$$C_{1-1}(\cdot\mathbf{x}s,\cdot\mathbf{y}t) := 1_{s\ge t}C^{\ge}_{1-1}(\cdot\mathbf{x}s,\cdot\mathbf{y}t)-1_{s<t}C^{<}_{1-1}(\cdot\mathbf{x}s,\cdot\mathbf{y}t),$$

$$C_{1-2}(\cdot\mathbf{x}s,\cdot\mathbf{y}t) := 1_{s\ge t}C_{1-2}^{\ge}(\cdot\mathbf{x}s,\cdot\mathbf{y}t) - 1_{s<t}C_{1-2}^{<}(\cdot\mathbf{x}s,\cdot\mathbf{y}t),$$
$$(\forall \mathbf{x},\mathbf{y}\in\Gamma,\ s,t\in[0,\beta)).$$

By combining these with (3.32), (3.33), (3.34) we have that

$$e^{i\frac{\pi}{\beta}(s-t)}C_1(\cdot\mathbf{x}s,\cdot\mathbf{y}t) = C_{1-1}(\cdot\mathbf{x}s,\cdot\mathbf{y}t) + C_{1-2}(\cdot\mathbf{x}s,\cdot\mathbf{y}t),$$
$$(\forall \mathbf{x},\mathbf{y}\in\Gamma,\ s,t\in[0,\beta)_h). \tag{3.35}$$

Let us find suitable determinant bounds of C_{1-1}, C_{1-2} so that the claimed determinant bound of C_1 can be derived from them.

Let us consider C_{1-1} first. Let $\mathcal{H}$ denote the Hilbert space $L^2(\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathbb{R}\times[0,1]\times\{1,2,3,4,5\})$ whose inner product is given by

$$\langle f,g\rangle_{\mathcal{H}}$$
$$:= \sum_{(\overline{\tau},\tau)\in\{1,2\}\times\mathcal{B}}\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\int_{-\infty}^{\infty}du\int_0^1 dv\sum_{j=1}^{5}\overline{f(\overline{\tau},\tau,\mathbf{k},u,v,j)}g(\overline{\tau},\tau,\mathbf{k},u,v,j).$$

Let us define the vectors f_X^a, $g_X^a\in\mathcal{H}$ ($X\in\{1,2\}\times\mathcal{B}\times\Gamma\times\mathbb{R},\ a\in\{1,-1\}$) in the following arguments. For $(\overline{\rho},\rho,\mathbf{x},s)\in\{1,2\}\times\mathcal{B}\times\Gamma\times\mathbb{R}$, $a\in\{1,-1\}$, $(\overline{\tau},\tau,\mathbf{k},u)\in\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathbb{R}$, $z\in P_1\cup P_2(\mathbf{k})$, set

$$f_{\overline{\rho}\rho\mathbf{x}s}^a(\overline{\tau},\tau,\mathbf{k},u)(z)$$
$$:= \frac{1}{\sqrt{2\pi}}1_{a\,\mathrm{Re}\,z>0}e^{-i\langle\mathbf{k},\mathbf{x}\rangle-is(a\,\mathrm{Im}\,z-u)}1_{(\overline{\tau},\tau)=(\overline{\rho},\rho)}\frac{1+e^{-\beta az}}{|1+e^{-\beta az}|^{\frac{3}{2}}}$$
$$\cdot\sqrt{\frac{|\mathrm{Re}\,z|}{\pi}}\frac{\|h^{-1}(I_{2b}-e^{-\frac{1}{h}(z-i\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\|_{2b\times 2b}^{\frac{1}{2}}}{iu+\mathrm{Re}\,z},$$
$$g_{\overline{\rho}\rho\mathbf{x}s}^a(\overline{\tau},\tau,\mathbf{k},u)(z)$$
$$:= \frac{1}{\sqrt{2\pi}i}1_{a\,\mathrm{Re}\,z>0}e^{-i\langle\mathbf{k},\mathbf{x}\rangle-is(a\,\mathrm{Im}\,z-u)}\frac{1}{|1+e^{-\beta az}|^{\frac{1}{2}}}$$
$$\cdot\sqrt{\frac{|\mathrm{Re}\,z|}{\pi}}\frac{\|h^{-1}(I_{2b}-e^{-\frac{1}{h}(z-i\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\|_{2b\times 2b}^{-\frac{1}{2}}}{iu+\mathrm{Re}\,z}$$
$$\cdot h^{-1}(I_{2b}-e^{-\frac{1}{h}(z-i\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}((\overline{\tau}-1)b+\tau,(\overline{\rho}-1)b+\rho).$$

Then, for $(\overline{\tau},\tau,\mathbf{k},u,v,j)\in\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathbb{R}\times[0,1]\times\{1,2,3,4,5\}$, set

$$f_{\overline{\rho}\rho\mathbf{x}s}^a(\overline{\tau},\tau,\mathbf{k},u,v,j)$$

$$
\begin{aligned}
&:= 1_{j=1}\sqrt{2h}\pi f^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)(\pi h e^{i2\pi v})\\
&\quad + 1_{j=2}\sqrt{2x_{\mathbf{k}}} f^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)\left(2x_{\mathbf{k}}v - x_{\mathbf{k}} + i\frac{2\pi}{\beta}\right)\\
&\quad + 1_{j=3}\sqrt{\frac{5\pi}{2\beta}} f^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)\left(x_{\mathbf{k}} - i\frac{5\pi}{2\beta}v + i\frac{2\pi}{\beta}\right)\\
&\quad + 1_{j=4}\sqrt{2x_{\mathbf{k}}} f^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)\left(-2x_{\mathbf{k}}v + x_{\mathbf{k}} - i\frac{\pi}{2\beta}\right)\\
&\quad + 1_{j=5}\sqrt{\frac{5\pi}{2\beta}} f^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)\left(-x_{\mathbf{k}} + i\frac{5\pi}{2\beta}v - i\frac{\pi}{2\beta}\right),\\
&g^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u,v,j)\\
&:= 1_{j=1} i\sqrt{2h}\pi e^{i2\pi v} g^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)(\pi h e^{i2\pi v})\\
&\quad + 1_{j=2}\sqrt{2x_{\mathbf{k}}} g^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)\left(2x_{\mathbf{k}}v - x_{\mathbf{k}} + i\frac{2\pi}{\beta}\right)\\
&\quad + 1_{j=3}\left(-i\sqrt{\frac{5\pi}{2\beta}}\right) g^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)\left(x_{\mathbf{k}} - i\frac{5\pi}{2\beta}v + i\frac{2\pi}{\beta}\right)\\
&\quad + 1_{j=4}(-\sqrt{2x_{\mathbf{k}}}) g^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)\left(-2x_{\mathbf{k}}v + x_{\mathbf{k}} - i\frac{\pi}{2\beta}\right)\\
&\quad + 1_{j=5} i\sqrt{\frac{5\pi}{2\beta}} g^{a}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)\left(-x_{\mathbf{k}} + i\frac{5\pi}{2\beta}v - i\frac{\pi}{2\beta}\right).
\end{aligned}
$$

Moreover, using the vectors f^{1}_{X}, f^{-1}_{X}, g^{1}_{X}, $g^{-1}_{X} \in \mathcal{H}$ defined above, we define the vectors $f^{\ge}_{X}$, $f^{<}_{X}$, $g^{\ge}_{X}$, $g^{<}_{X} \in \mathcal{H}$ ($X \in \{1,2\}\times\mathcal{B}\times\Gamma\times\mathbb{R}$) as follows. For $(\overline{\rho},\rho,\mathbf{x},s) \in \{1,2\}\times\mathcal{B}\times\Gamma\times\mathbb{R}$

$$
\begin{aligned}
(3.36)\qquad & f^{\ge}_{\overline{\rho}\rho\mathbf{x}s} = f^{<}_{\overline{\rho}\rho\mathbf{x}s} := f^{1}_{\overline{\rho}\rho\mathbf{x}s} + f^{-1}_{\overline{\rho}\rho\mathbf{x}(-s)},\\
& g^{\ge}_{\overline{\rho}\rho\mathbf{x}s} := -g^{1}_{\overline{\rho}\rho\mathbf{x}(\beta+s)} - g^{-1}_{\overline{\rho}\rho\mathbf{x}(-s)},\quad g^{<}_{\overline{\rho}\rho\mathbf{x}s} := -g^{1}_{\overline{\rho}\rho\mathbf{x}s} - g^{-1}_{\overline{\rho}\rho\mathbf{x}(\beta-s)}.
\end{aligned}
$$

By using the formula

$$
(3.37)\qquad e^{-tD} = \frac{D}{\pi}\int_{-\infty}^{\infty} du \frac{e^{itu}}{u^2+D^2},\quad (\forall t \in \mathbb{R}_{\ge 0},\ D \in \mathbb{R}_{>0})
$$

one can verify that for any $(\overline{\rho},\rho,\mathbf{x},s)$, $(\overline{\eta},\eta,\mathbf{y},t) \in \{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)$

$$
1_{s\ge t}\langle f^{\ge}_{\overline{\rho}\rho\mathbf{x}s}, g^{\ge}_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\mathcal{H}} = -1_{s\ge t}(\langle f^{1}_{\overline{\rho}\rho\mathbf{x}s}, g^{1}_{\overline{\eta}\eta\mathbf{y}(\beta+t)}\rangle_{\mathcal{H}} + \langle f^{-1}_{\overline{\rho}\rho\mathbf{x}(-s)}, g^{-1}_{\overline{\eta}\eta\mathbf{y}(-t)}\rangle_{\mathcal{H}})
$$

$$
\begin{aligned}
&= -1_{s\ge t}\sum_{(\overline{\tau},\tau)\in\{1,2\}\times\mathcal{B}}\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\int_{-\infty}^{\infty}du\oint_{P_1\cup P_2(\mathbf{k})}dz\\
&\quad\cdot\Big(\overline{f^1_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)(z)}g^1_{\overline{\eta}\eta\mathbf{y}(\beta+t)}(\overline{\tau},\tau,\mathbf{k},u)(z)\\
&\qquad+\overline{f^{-1}_{\overline{\rho}\rho\mathbf{x}(-s)}(\overline{\tau},\tau,\mathbf{k},u)(z)}g^{-1}_{\overline{\eta}\eta\mathbf{y}(-t)}(\overline{\tau},\tau,\mathbf{k},u)(z)\Big)\\
&= 1_{s\ge t}C^{\ge}_{1-1}(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t),\\
1_{s<t}\langle f^{<}_{\overline{\rho}\rho\mathbf{x}s},g^{<}_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\mathcal{H}} &= -1_{s<t}(\langle f^1_{\overline{\rho}\rho\mathbf{x}s},g^1_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\mathcal{H}}+\langle f^{-1}_{\overline{\rho}\rho\mathbf{x}(-s)},g^{-1}_{\overline{\eta}\eta\mathbf{y}(\beta-t)}\rangle_{\mathcal{H}})\\
&= 1_{s<t}C^{<}_{1-1}(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t),
\end{aligned}
$$

and thus

$$
C_{1-1}(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t)=1_{s\ge t}\langle f^{\ge}_{\overline{\rho}\rho\mathbf{x}s},g^{\ge}_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\mathcal{H}}-1_{s<t}\langle f^{<}_{\overline{\rho}\rho\mathbf{x}s},g^{<}_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\mathcal{H}}. \tag{3.38}
$$

To apply [20, Theorem 1.3], we need to estimate $\|f^{\ge}_X\|_{\mathcal{H}}$, $\|g^{\ge}_X\|_{\mathcal{H}}$, $\|f^{<}_X\|_{\mathcal{H}}$, $\|g^{<}_X\|_{\mathcal{H}}$, $(X\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta))$. These can be expanded as follows. For any $(\overline{\rho},\rho,\mathbf{x},s)\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)$ and $A\in\{f,g\}$

$$
\begin{aligned}
\|A^{\ge}_{\overline{\rho}\rho\mathbf{x}s}\|^2_{\mathcal{H}} &= \|A^{<}_{\overline{\rho}\rho\mathbf{x}s}\|^2_{\mathcal{H}}\\
&= \sum_{(\overline{\tau},\tau)\in\{1,2\}\times\mathcal{B}}\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\int_{-\infty}^{\infty}du\int_0^1dv\sum_{a\in\{1,-1\}}\\
&\quad\cdot\Bigg(2h\pi^2|A^a_{\overline{\rho}\rho\mathbf{0}0}(\overline{\tau},\tau,\mathbf{k},u)(\pi he^{i2\pi v})|^2\\
&\qquad+2x_{\mathbf{k}}\left|A^a_{\overline{\rho}\rho\mathbf{0}0}(\overline{\tau},\tau,\mathbf{k},u)\left(2x_{\mathbf{k}}v-x_{\mathbf{k}}+i\frac{2\pi}{\beta}\right)\right|^2\\
&\qquad+\frac{5\pi}{2\beta}\left|A^a_{\overline{\rho}\rho\mathbf{0}0}(\overline{\tau},\tau,\mathbf{k},u)\left(x_{\mathbf{k}}-i\frac{5\pi}{2\beta}v+i\frac{2\pi}{\beta}\right)\right|^2\\
&\qquad+2x_{\mathbf{k}}\left|A^a_{\overline{\rho}\rho\mathbf{0}0}(\overline{\tau},\tau,\mathbf{k},u)\left(-2x_{\mathbf{k}}v+x_{\mathbf{k}}-i\frac{\pi}{2\beta}\right)\right|^2\\
&\qquad+\frac{5\pi}{2\beta}\left|A^a_{\overline{\rho}\rho\mathbf{0}0}(\overline{\tau},\tau,\mathbf{k},u)\left(-x_{\mathbf{k}}+i\frac{5\pi}{2\beta}v-i\frac{\pi}{2\beta}\right)\right|^2\Bigg).
\end{aligned} \tag{3.39}
$$

As the next step, let us fix $\mathbf{k}\in\Gamma^*$ and estimate

$$
\inf_{z\in P_1\cup P_2(\mathbf{k})}|1+e^{\beta z}|,\qquad \inf_{z\in P_1\cup P_2(\mathbf{k})}|1+e^{-\beta z}|.
$$

For $z \in P_1$ there exists $t \in [-1,1]$ such that

$$|1+e^{\beta z}|^2 = 1 + 2\cos(\pi\beta h\sqrt{1-t^2})e^{\pi\beta ht} + e^{2\pi\beta ht}.$$

There exists $m \in \mathbb{N}$ such that $h = 2m/\beta$. Then there exist $n \in \{0,1,\cdots, m-1\}$, $\theta \in [0,2\pi]$ such that $\pi\beta h\sqrt{1-t^2} = \theta + 2n\pi$. If $\theta \in [0,\pi/2] \cup [3\pi/2, 2\pi]$, $|1+e^{\beta z}|^2 \geq 1 + e^{2\pi\beta ht} \geq 1$. If $\theta \in (\pi/2, 3\pi/2)$, $(\pi\beta ht)^2 = (2m\pi - \theta - 2n\pi)(2m\pi + \theta + 2n\pi) \geq \pi^2/4$, and thus $|1+e^{\beta z}|^2 \geq (1-e^{\pi\beta ht})^2 \geq (1-e^{-\frac{\pi}{2}})^2$. We have proved that

$$\inf_{z\in P_1} |1+e^{\beta z}| = \inf_{z\in P_1} |1+e^{-\beta z}| \geq 1 - e^{-\frac{\pi}{2}}.$$

If $z \in \{x_{\mathbf{k}} + iy, -x_{\mathbf{k}} + iy \mid -\frac{\pi}{2\beta} \leq y \leq \frac{2\pi}{\beta}\}$, $\min\{|1+e^{\beta z}|, |1+e^{-\beta z}|\} \geq 1 - e^{-\beta x_{\mathbf{k}}} \geq 1 - e^{-1}$, where we used that $x_{\mathbf{k}} \geq 1/\beta$. If $z \in \{x + i\frac{2\pi}{\beta}, x - i\frac{\pi}{2\beta} \mid -x_{\mathbf{k}} \leq x \leq x_{\mathbf{k}}\}$, $\min\{|1+e^{\beta z}|, |1+e^{-\beta z}|\} \geq 1$. Thus

$$\inf_{z\in P_2(\mathbf{k})} \min\{|1+e^{\beta z}|,\ |1+e^{-\beta z}|\} \geq 1 - e^{-1}.$$

Now we can see that

$$\inf_{z\in P_1\cup P_2(\mathbf{k})} |1+e^{a\beta z}| \geq 1 - e^{-1}, \quad (\forall a \in \{1,-1\}). \tag{3.40}$$

We also need to find upper bounds on

$$\sup_{z\in P_1} \|h^{-1}(I_{2b} - e^{-\frac{1}{h}(z - i\frac{\theta(\beta)}{2})I_{2b} + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\|_{2b\times 2b},$$
$$\sup_{z\in P_2(\mathbf{k})} \|h^{-1}(I_{2b} - e^{-\frac{1}{h}(z - i\frac{\theta(\beta)}{2})I_{2b} + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\|_{2b\times 2b}.$$

On the assumption (3.30)

$$\begin{aligned}\sup_{z\in P_1}\sup_{\alpha\in\sigma(E(\phi)(\mathbf{k}))}\left|-\frac{1}{h}\left(z - i\frac{\theta(\beta)}{2}\right) + \frac{1}{h}\alpha\right| &\leq \pi + \frac{1}{h}\left(\frac{\pi}{\beta} + \sqrt{e_{max}^2 + |\phi|^2}\right)\\ &\leq \frac{3\pi}{2},\\ \inf_{z\in P_1}\inf_{\alpha\in\sigma(E(\phi)(\mathbf{k}))}\left|-\frac{1}{h}\left(z - i\frac{\theta(\beta)}{2}\right) + \frac{1}{h}\alpha\right| &\geq \pi - \frac{1}{h}\left(\frac{\pi}{\beta} + \sqrt{e_{max}^2 + |\phi|^2}\right)\\ &\geq \frac{\pi}{2},\end{aligned}$$

which imply that

$$\inf_{z\in P_1}\inf_{\alpha\in\sigma(E(\phi)(\mathbf{k}))}|1-e^{-\frac{1}{h}(z-i\frac{\theta(\beta)}{2})+\frac{1}{h}\alpha}|\ge\inf_{\substack{z\in\mathbb{C}\\ \text{with }\frac{\pi}{2}\le|z|\le\frac{3\pi}{2}}}|1-e^{z}|>0,$$

and thus

$$\sup_{z\in P_1}\|h^{-1}(I_{2b}-e^{-\frac{1}{h}(z-i\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\|_{2b\times 2b}\le ch^{-1}. \tag{3.41}$$

On the other hand, since $x_{\mathbf{k}}\in[1/\beta,2/\beta]$,

$$\sup_{z\in P_2(\mathbf{k})}\sup_{\alpha\in\sigma(E(\phi)(\mathbf{k}))}\left|\mathrm{Re}\left(-\frac{z}{h}+i\frac{\theta(\beta)}{2h}+\frac{\alpha}{h}\right)\right|\le\frac{1}{h}\left(\frac{2}{\beta}+\sqrt{e_{max}^2+|\phi|^2}\right),$$
$$\sup_{z\in P_2(\mathbf{k})}\sup_{\alpha\in\sigma(E(\phi)(\mathbf{k}))}\left|\mathrm{Im}\left(-\frac{z}{h}+i\frac{\theta(\beta)}{2h}+\frac{\alpha}{h}\right)\right|\le\frac{3\pi}{\beta h}.$$

By the assumption (3.30)

$$\sup_{z\in P_2(\mathbf{k})}\sup_{\alpha\in\sigma(E(\phi)(\mathbf{k}))}\left|-\frac{z}{h}+i\frac{\theta(\beta)}{2h}+\frac{\alpha}{h}\right|\le 1,$$

and thus for any $z\in P_2(\mathbf{k})$, $\alpha\in\sigma(E(\phi)(\mathbf{k}))$

$$|1-e^{-\frac{z}{h}+i\frac{\theta(\beta)}{2h}+\frac{\alpha}{h}}|\ge\left(1-\sum_{n=2}^{\infty}\frac{1}{n!}\right)\left|-\frac{z}{h}+i\frac{\theta(\beta)}{2h}+\frac{\alpha}{h}\right|.$$

Therefore

(3.42)

$$\begin{aligned}&\sup_{z\in P_2(\mathbf{k})}\|h^{-1}(I_{2b}-e^{-\frac{1}{h}(z-i\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\|_{2b\times 2b}\\&\le c\sup_{z\in P_2(\mathbf{k})}\sup_{\alpha\in\sigma(E(\phi)(\mathbf{k}))}\left(|\mathrm{Re}\,z-\alpha|+\left|\mathrm{Im}\,z-\frac{\theta(\beta)}{2}\right|\right)^{-1}\\&\le c\max\left\{\sup_{\alpha\in\sigma(E(\phi)(\mathbf{k}))}|x_{\mathbf{k}}-\alpha|^{-1},\ \left|\frac{2\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1},\ \left|\frac{\pi}{2\beta}+\frac{\theta(\beta)}{2}\right|^{-1}\right\}\\&\le c(b)\beta,\end{aligned}$$

where we used that

$$\inf_{\alpha\in\sigma(E(\phi)(\mathbf{k}))}|x_{\mathbf{k}}-\alpha|\geq\frac{1}{2(b+1)\beta},$$

which is ensured by (3.31).

By substituting (3.40), (3.41), (3.42) into (3.39) and using (3.37) and $x_{\mathbf{k}}\leq 2/\beta$ ($\forall\mathbf{k}\in\Gamma^*$) we observe that for any $X\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)$ and $A\in\{f,g\}$

$$\begin{aligned}
&\|A_X^{\geq}\|_{\mathcal{H}}^2=\|A_X^{\leq}\|_{\mathcal{H}}^2\\
&\leq\frac{c(b)}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\int_{-\infty}^{\infty}du\int_0^1 dv\sum_{a\in\{1,-1\}}\\
&\quad\cdot\left(\frac{|\pi h\cos(2\pi v)|1_{a\pi h\cos(2\pi v)>0}}{u^2+(\pi h\cos(2\pi v))^2}+\frac{|2x_{\mathbf{k}}v-x_{\mathbf{k}}|1_{a(2x_{\mathbf{k}}v-x_{\mathbf{k}})>0}}{u^2+(2x_{\mathbf{k}}v-x_{\mathbf{k}})^2}+\frac{x_{\mathbf{k}}1_{ax_{\mathbf{k}}>0}}{u^2+x_{\mathbf{k}}^2}\right)\\
&\leq c(b).
\end{aligned}$$

Now we can apply the extended Gram inequality [20, Theorem 1.3] in the representation (3.38) to derive that

$$\begin{aligned}
(3.43)\quad &|\det(\langle\mathbf{u}_i,\mathbf{w}_j\rangle_{\mathbb{C}^m}C_{1-1}(X_i,Y_j))_{1\leq i,j\leq n}|\leq c(b)^n,\\
&(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{w}_i\in\mathbb{C}^m\text{ with }\|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{w}_i\|_{\mathbb{C}^m}\leq 1,\ X_i,Y_i\in I_0\\
&\quad(i=1,\cdots,n)).
\end{aligned}$$

The readers can refer to [11, Remark 5.2] for a minor necessary modification of [20, Theorem 1.3] concerning the factor $\langle\mathbf{u}_i,\mathbf{w}_j\rangle_{\mathbb{C}^m}$ $(i,j=1,\cdots,n)$, as it was originally claimed only for $m=n$ in [20, Theorem 1.3].

Let us treat C_{1-2}. In fact the procedure to find a determinant bound on C_{1-2} is simpler than that on C_{1-1}. Let $\widehat{\mathcal{H}}$ denote the Hilbert space $L^2(\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathbb{R})$ whose inner product is defined by

$$\langle f,g\rangle_{\widehat{\mathcal{H}}}:=\sum_{(\overline{\tau},\tau)\in\{1,2\}\times\mathcal{B}}\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\int_{-\infty}^{\infty}du\overline{f(\overline{\tau},\tau,\mathbf{k},u)}g(\overline{\tau},\tau,\mathbf{k},u).$$

Define the vectors $\hat{f}_X^{\bar{a}}$, $\hat{g}_X^{\bar{a}}\in\widehat{\mathcal{H}}$ ($X\in\{1,2\}\times\mathcal{B}\times\Gamma\times\mathbb{R}$, $\bar{a}\in\{1,2\}$) as follows.

$$\hat{f}_{\overline{\rho}\rho\mathbf{x}s}^{\bar{a}}(\overline{\tau},\tau,\mathbf{k},u)$$

$$
\begin{aligned}
&:= 1_{\overline{\tau}=\bar{a}} e^{-i\langle \mathbf{k},\mathbf{x}\rangle - is((-1)^{\bar{a}+1}\frac{\theta(\beta)}{2}-u)} 1_{\tau\in\mathcal{B}(\mathbf{k})} \overline{U(\mathbf{k})((\overline{\rho}-1)b+\rho,(\overline{\tau}-1)b+\tau)}\\
&\quad \cdot \frac{1+e^{-\beta((-1)^{\bar{a}+1}i\frac{\theta(\beta)}{2}+\hat{e}_\tau(\mathbf{k}))}}{|1+e^{-\beta((-1)^{\bar{a}+1}i\frac{\theta(\beta)}{2}+\hat{e}_\tau(\mathbf{k}))}|^{\frac{3}{2}}} \sqrt{\frac{\hat{e}_\tau(\mathbf{k})}{\pi}} \frac{1}{iu+\hat{e}_\tau(\mathbf{k})},\\
&\hat{g}^{\bar{a}}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\tau},\tau,\mathbf{k},u)\\
&:= 1_{\overline{\tau}=\bar{a}} e^{-i\langle \mathbf{k},\mathbf{x}\rangle - is((-1)^{\bar{a}+1}\frac{\theta(\beta)}{2}-u)} 1_{\tau\in\mathcal{B}(\mathbf{k})} U(\mathbf{k})^*((\overline{\tau}-1)b+\tau,(\overline{\rho}-1)b+\rho)\\
&\quad \cdot \frac{1}{|1+e^{-\beta((-1)^{\bar{a}+1}i\frac{\theta(\beta)}{2}+\hat{e}_\tau(\mathbf{k}))}|^{\frac{1}{2}}} \sqrt{\frac{\hat{e}_\tau(\mathbf{k})}{\pi}} \frac{1}{iu+\hat{e}_\tau(\mathbf{k})}.
\end{aligned}
$$

Then let us define $\hat{f}^{\geq}_X$, $\hat{f}^{<}_X$, $\hat{g}^{\geq}_X$, $\hat{g}^{<}_X \in \widehat{\mathcal{H}}$ $(X\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta))$ by

$$
\begin{aligned}
(3.44)\qquad &\hat{f}^{\geq}_{\overline{\rho}\rho\mathbf{x}s} = \hat{f}^{<}_{\overline{\rho}\rho\mathbf{x}s} := \hat{f}^{1}_{\overline{\rho}\rho\mathbf{x}s} + \hat{f}^{2}_{\overline{\rho}\rho\mathbf{x}(-s)},\\
&\hat{g}^{\geq}_{\overline{\rho}\rho\mathbf{x}s} := \hat{g}^{1}_{\overline{\rho}\rho\mathbf{x}(\beta+s)} + \hat{g}^{2}_{\overline{\rho}\rho\mathbf{x}(-s)},\quad \hat{g}^{<}_{\overline{\rho}\rho\mathbf{x}s} := \hat{g}^{1}_{\overline{\rho}\rho\mathbf{x}s} + \hat{g}^{2}_{\overline{\rho}\rho\mathbf{x}(\beta-s)}.
\end{aligned}
$$

By applying (3.37) repeatedly we can confirm that for any $(\overline{\rho},\rho,\mathbf{x},s)$, $(\overline{\eta},\eta,\mathbf{y},t)\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)$

$$
\begin{aligned}
1_{s\geq t}\langle \hat{f}^{\geq}_{\overline{\rho}\rho\mathbf{x}s},\hat{g}^{\geq}_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\widehat{\mathcal{H}}} &= 1_{s\geq t}(\langle \hat{f}^{1}_{\overline{\rho}\rho\mathbf{x}s},\hat{g}^{1}_{\overline{\eta}\eta\mathbf{y}(\beta+t)}\rangle_{\widehat{\mathcal{H}}} + \langle \hat{f}^{2}_{\overline{\rho}\rho\mathbf{x}(-s)},\hat{g}^{2}_{\overline{\eta}\eta\mathbf{y}(-t)}\rangle_{\widehat{\mathcal{H}}})\\
&= 1_{s\geq t} C^{\geq}_{1-2}(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t),\\
1_{s<t}\langle \hat{f}^{<}_{\overline{\rho}\rho\mathbf{x}s},\hat{g}^{<}_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\widehat{\mathcal{H}}} &= 1_{s<t}(\langle \hat{f}^{1}_{\overline{\rho}\rho\mathbf{x}s},\hat{g}^{1}_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\widehat{\mathcal{H}}} + \langle \hat{f}^{2}_{\overline{\rho}\rho\mathbf{x}(-s)},\hat{g}^{2}_{\overline{\eta}\eta\mathbf{y}(\beta-t)}\rangle_{\widehat{\mathcal{H}}})\\
&= 1_{s<t} C^{<}_{1-2}(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t),
\end{aligned}
$$

and thus

$$
(3.45)\qquad C_{1-2}(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t) = 1_{s\geq t}\langle \hat{f}^{\geq}_{\overline{\rho}\rho\mathbf{x}s},\hat{g}^{\geq}_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\widehat{\mathcal{H}}} - 1_{s<t}\langle \hat{f}^{<}_{\overline{\rho}\rho\mathbf{x}s},\hat{g}^{<}_{\overline{\eta}\eta\mathbf{y}t}\rangle_{\widehat{\mathcal{H}}}.
$$

To estimate the norms of $\hat{f}^{\geq}_X$, $\hat{f}^{<}_X$, $\hat{g}^{\geq}_X$, $\hat{g}^{<}_X$ $(X\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta))$, let us observe that for $\mathbf{k}\in\Gamma^*$, $\tau\in\mathcal{B}(\mathbf{k})$, $\bar{a}\in\{1,2\}$

$$
(3.46)\qquad |1+e^{-\beta((-1)^{\bar{a}+1}i\frac{\theta(\beta)}{2}+\hat{e}_\tau(\mathbf{k}))}|^2 \geq (1-e^{-\beta\hat{e}_\tau(\mathbf{k})})^2 \geq (1-e^{-1})^2,
$$

where we used the fact that $\hat{e}_\tau(\mathbf{k}) \geq x_{\mathbf{k}} + \frac{1}{2(b+1)\beta} \geq 1/\beta$. Taking into account (3.46) and the unitary property of $U(\mathbf{k})$ and using (3.37), we can derive that for any $(\overline{\rho},\rho,\mathbf{x},s)\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)$ and $A\in\{f,g\}$

$$
\|\hat{A}^{\geq}_{\overline{\rho}\rho\mathbf{x}s}\|^2_{\widehat{\mathcal{H}}} = \|\hat{A}^{<}_{\overline{\rho}\rho\mathbf{x}s}\|^2_{\widehat{\mathcal{H}}} = \|\hat{A}^{1}_{\overline{\rho}\rho\mathbf{0}0}\|^2_{\widehat{\mathcal{H}}} + \|\hat{A}^{2}_{\overline{\rho}\rho\mathbf{0}0}\|^2_{\widehat{\mathcal{H}}}
$$

$$\leq \frac{c}{L^d} \sum_{(\overline{\tau},\tau)\in\{1,2\}\times\mathcal{B}} \sum_{\mathbf{k}\in\Gamma^*} |U(\mathbf{k})((\overline{\rho}-1)b+\rho,(\overline{\tau}-1)b+\tau))|^2 = c.$$

With these bounds we can apply [20, Theorem 1.3] in (3.45) and conclude that

$$\begin{aligned}(3.47)\quad & |\det(\langle \mathbf{u}_i, \mathbf{w}_j\rangle_{\mathbb{C}^m} C_{1-2}(X_i,Y_j))_{1\le i,j\le n}| \le c^n,\\ & (\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{w}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{w}_i\|_{\mathbb{C}^m}\le 1,\ X_i,Y_i\in I_0\\ & \quad (i=1,\cdots,n)).\end{aligned}$$

Since we have (3.43) and (3.47), we can apply the Cauchy-Binet formula in a standard way (see e.g. [13, Lemma A.1]) in (3.35) to obtain the claimed determinant bound. □

3.2. General estimation

Let $\mathcal{V}$ denote the complex vector space spanned by the abstract basis $\{\psi_X\}_{X\in I}$. Then let $\bigwedge\mathcal{V}$ be the Grassmann algebra generated by $\{\psi_X\}_{X\in I}$ and $\bigwedge_{even}\mathcal{V}$ be the subspace of $\bigwedge\mathcal{V}$ spanned by even monomials. These Grassmann algebras are exactly same as those defined in [14]. The grand canonical partition function and the thermal expectations are formulated into a hybrid of Gaussian integral with real variables and Grassmann Gaussian integral over $\bigwedge\mathcal{V}$ in the same way as [14, Lemma 3.6]. As in the previous papers, the proof of Theorem 1.3 relies on analysis of the Grassmann Gaussian integral appearing in the hybrid formulation. The aim of this subsection is to summarize necessary estimates of the output of the Grassmann Gaussian integral in a generalized setting. Here we do not introduce concrete model-dependent Grassmann polynomials or covariances. We only assume generic properties of Grassmann polynomials and a covariance. The estimates can be used as tools to analyze the Grassmann integral formulation if the real Grassmann polynomials and the real covariances stemming from the model are substituted. In fact all the inequalities claimed below are straightforward variants of the results of [13, Subsection 3.2], [14, Subsection 4.2]. We only provide minimum sketches of the proofs rather than fully repeat parallel arguments. However, the resulting inequalities themselves will be stated without omission. We will see that seemingly subtle changes from the previous estimates constitute the essence of the proof of Theorem 1.3.

In this subsection we assume that the covariance $\mathcal{C} : I_0^2 \to \mathbb{C}$ satisfies with a constant $D \in \mathbb{R}_{>0}$ that

$$
\begin{aligned}
(3.48)\quad & \mathcal{C}(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t) = \mathcal{C}(\overline{\rho}\rho\mathbf{x}0, \overline{\eta}\eta\mathbf{y}0), \quad (\forall(\overline{\rho}, \rho, \mathbf{x}, s), (\overline{\eta}, \eta, \mathbf{y}, t) \in I_0),\\
& |\det(\langle \mathbf{u}_i, \mathbf{w}_j\rangle_{\mathbb{C}^m}\mathcal{C}(X_i, Y_j))_{1\le i,j\le n}| \le D^n,\\
& (\forall m, n \in \mathbb{N},\ \mathbf{u}_i, \mathbf{w}_i \in \mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m}, \|\mathbf{w}_i\|_{\mathbb{C}^m} \le 1,\ X_i, Y_i \in I_0\\
& \quad (i = 1, \cdots, n)).
\end{aligned}
$$

A common property satisfied by kernels of Grassmann polynomials in the following analysis is the invariance

$$
(3.49)\qquad F(\mathcal{R}_\beta(\mathbf{X} + s)) = F(\mathbf{X}), \quad \left(\forall \mathbf{X} \in I^m,\ s \in \frac{1}{h}\mathbb{Z}\right),
$$

where $F : I^m \to \mathbb{C}$. Let us refer to [14, Subsection 4.2] for the definition of the map $\mathcal{R}_\beta$. Also, the meaning of the notation $\mathbf{X} + s$ is explained in [13, Subsection 3.1] in a parallel situation. The property (3.48) implies that its extension $\tilde{\mathcal{C}} : I^2 \to \mathbb{C}$ defined as in (3.2) satisfies (3.49). In the following we assume that $F^j(\psi)$ $(j \in \mathbb{N})$, $F(\psi) \in \bigwedge_{even} \mathcal{V}$ and the anti-symmetric kernels $F_m^j : I^m \to \mathbb{C}$, $F_m : I^m \to \mathbb{C}$ $(m = 2, 4, \cdots, N)$ satisfy (3.49). Here N denotes $4b\beta hL^d$, the cardinality of I. We use these Grassmann polynomials as input to the tree expansions. As another input, we take $G \in \bigwedge_{even} \mathcal{V}$ having the form

$$
G(\psi) = \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{p+q} \sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}} G_{p,q}(\mathbf{X}, \mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}
$$

with the bi-anti-symmetric kernels $G_{p,q} : I^p \times I^q \to \mathbb{C}$ $(p, q = 2, 4, \cdots, N)$ satisfying (3.49) and the vanishing property

(3.50)

$$
\begin{aligned}
& \sum_{(s_1,\cdots,s_p)\in[0,\beta)_h^p} G_{p,q}((\overline{\rho}_1\rho_1\mathbf{x}_1s_1\xi_1, \cdots, \overline{\rho}_p\rho_p\mathbf{x}_ps_p\xi_p), \mathbf{Y})f(s_1, \cdots, s_p) = 0,\\
& (\forall(\overline{\rho}_1, \rho_1, \mathbf{x}_1, \xi_1), \cdots, (\overline{\rho}_p, \rho_p, \mathbf{x}_p, \xi_p) \in \{1, 2\} \times \mathcal{B} \times \Gamma \times \{1, -1\},\ \mathbf{Y} \in I^q),\\
& \sum_{(t_1,\cdots,t_q)\in[0,\beta)_h^q} G_{p,q}(\mathbf{X}, (\overline{\eta}_1\eta_1\mathbf{y}_1t_1\zeta_1, \cdots, \overline{\eta}_q\eta_q\mathbf{y}_qt_q\zeta_q))g(t_1, \cdots, t_q) = 0,
\end{aligned}
$$

$$(\forall \mathbf{X} \in I^p,\ (\overline{\eta}_1, \eta_1, \mathbf{y}_1, \zeta_1), \cdots, (\overline{\eta}_q, \eta_q, \mathbf{y}_q, \zeta_q) \in \{1,2\} \times \mathcal{B} \times \Gamma \times \{1,-1\}),$$

for any $f : [0,\beta)_h^p \to \mathbb{C}$, $g : [0,\beta)_h^q \to \mathbb{C}$ satisfying that

$$\begin{aligned}
&f(r_\beta(s_1+s), \cdots, r_\beta(s_p+s)) = f(s_1, \cdots, s_p),\\
&\left(\forall (s_1, \cdots, s_p) \in [0,\beta)_h^p,\ s \in \frac{1}{h}\mathbb{Z}\right),\\
&g(r_\beta(s_1+s), \cdots, r_\beta(s_q+s)) = g(s_1, \cdots, s_q),\\
&\left(\forall (s_1, \cdots, s_q) \in [0,\beta)_h^q,\ s \in \frac{1}{h}\mathbb{Z}\right).
\end{aligned}$$

Recall that for any $s \in \frac{1}{h}\mathbb{Z}$, $r_\beta(s) \in [0,\beta)_h$ and $r_\beta(s) = s$ in $\frac{1}{h}\mathbb{Z}/\beta\mathbb{Z}$. The definition of the map $r_\beta : \frac{1}{h}\mathbb{Z} \to [0,\beta)_h$ was originally given in [13, Subsection 3.2]. We also introduce $G^j \in \bigwedge_{even} \mathcal{V}$ ($j \in \mathbb{N}$), assuming that G^j has the bi-anti-symmetric kernels $G^j_{p,q} : I^p \times I^q \to \mathbb{C}$ ($p,q = 2,4,\cdots,N$) satisfying (3.49) and (3.50).

For $n \in \mathbb{N}_{\ge 2}$, $l \in \{0,1,\cdots,n\}$ we define $A^{(n,l)}(\psi) \in \bigwedge_{even} \mathcal{V}$ by

$$\begin{aligned}
A^{(n,l)}(\psi) := Tree(\{1,2,\cdots,n\}, \mathcal{C}) \prod_{j=1}^{l} F^j(\psi^j + \psi)\\
\cdot \prod_{k=l+1}^{n} G^k(\psi^k + \psi)\Bigg|_{\substack{\psi^i = 0\\ (\forall i \in \{1,\cdots,n\})}}.
\end{aligned}$$

The definition of the operator "$Tree(\{1,\cdots,n\}, \mathcal{C})$" is written in [13, Subsection 3.1]. It applies to the present case if we add the set $\mathcal{B}$ to the index set "I" of [13]. In fact the current version of $Tree(\{1,\cdots,n\}, \mathcal{C})$ is exactly same as that used in [14, Subsection 4.2]. In the first lemma we summarize necessary bound properties of the anti-symmetric kernels of $A^{(n,l)}(\psi)$. Let us refer to [14, Subsection 4.1] for the definition of the norm $\|\cdot\|_1$.

LEMMA 3.5. *For any $m \in \{2,4,\cdots,N\}$, $n \in \mathbb{N}_{\ge 2}$, $l \in \{0,1,\cdots,n\}$ the anti-symmetric kernel $A_m^{(n,l)}(\cdot)$ satisfies* (3.49). *Moreover, the following inequalities hold for any $m \in \{0,2,\cdots,N\}$, $n \in \mathbb{N}_{\ge 2}$, $l \in \{0,1,\cdots,n\}$, $l' \in \{1,2,\cdots,n\}$.*

(3.51) $\|A_m^{(n,l)}\|_{1,\infty}$

$$\le \left(\frac{N}{h}\right)^{1_{m=0}} (n-2)! D^{-n+1-\frac{m}{2}} 2^{-2m} \|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}$$
$$\cdot \prod_{j=1}^{l} \left(\sum_{p_j=2}^{N} 2^{3p_j} D^{\frac{p_j}{2}} \|F_{p_j}^j\|_{1,\infty} \right) \prod_{k=l+1}^{n} \left(\sum_{p_k=4}^{N} 2^{3p_k} D^{\frac{p_k}{2}} \|G_{p_k}^k\|_{1,\infty} \right)$$
$$\cdot 1_{\sum_{j=1}^{n} p_j - 2(n-1) \ge m \ge 2(n-l)}.$$

(3.52) $\|A_m^{(n,l')}\|_1$
$$\le (n-2)! D^{-n+1-\frac{m}{2}} 2^{-2m} \|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1} \sum_{p_1=2}^{N} 2^{3p_1} D^{\frac{p_1}{2}} \|F_{p_1}^1\|_1$$
$$\cdot \prod_{j=2}^{l'} \left(\sum_{p_j=2}^{N} 2^{3p_j} D^{\frac{p_j}{2}} \|F_{p_j}^j\|_{1,\infty} \right) \prod_{k=l'+1}^{n} \left(\sum_{p_k=4}^{N} 2^{3p_k} D^{\frac{p_k}{2}} \|G_{p_k}^k\|_{1,\infty} \right)$$
$$\cdot 1_{\sum_{j=1}^{n} p_j - 2(n-1) \ge m \ge 2(n-l')}.$$

Proof. The statement concerning the property (3.49) is essentially implied by [13, Lemma 3.1]. Let us define the map $P_0 : I^m \to I^m$ by

$$\begin{aligned}
&P_0((\overline{\rho}_1, \rho_1, \mathbf{x}_1, s_1, \xi_1), \cdots, (\overline{\rho}_m, \rho_m, \mathbf{x}_m, s_m, \xi_m)) \\
&:= ((\overline{\rho}_1, \rho_1, \mathbf{x}_1, 0, \xi_1), \cdots, (\overline{\rho}_m, \rho_m, \mathbf{x}_m, 0, \xi_m)), \\
&(\forall (\overline{\rho}_j, \rho_j, \mathbf{x}_j, s_j, \xi_j) \in I \ (j = 1, \cdots, m)).
\end{aligned}$$

Let us use the notation P_0 for different m for simplicity. Then by taking into account anti-symmetry and the time-independent property (3.48) we observe that for $m \in \{0, 2, \cdots, N\}$, $n \in \mathbb{N}_{\ge 2}$, $l \in \{0, 1, \cdots, n\}$

$$\begin{aligned}
A_m^{(n,l)}(\psi) &= Tree(\{1, \cdots, n\}, \mathcal{C}) \\
&\cdot \prod_{j=1}^{l} \left(\sum_{n_j=2}^{N} \sum_{m_j=0}^{n_j-1} \begin{pmatrix} n_j \\ m_j \end{pmatrix} \left(\frac{1}{h}\right)^{n_j} \right. \\
&\qquad \left. \cdot \sum_{\mathbf{X}_j \in I^{m_j}} \sum_{\mathbf{Y}_j \in I^{n_j - m_j}} F_{n_j}^j(\mathbf{Y}_j, \mathbf{X}_j) \psi_{P_0(\mathbf{Y}_j)}^j \psi_{\mathbf{X}_j} \right) \\
&\cdot \prod_{k=l+1}^{n} \left(\sum_{n_k=4}^{N} \sum_{m_k=0}^{n_k-1} \begin{pmatrix} n_k \\ m_k \end{pmatrix} \left(\frac{1}{h}\right)^{n_k} \right.
\end{aligned}$$

$$\cdot \sum_{\mathbf{X}_k \in I^{m_k}} \sum_{\mathbf{Y}_k \in I^{n_k - m_k}} G^k_{n_k}(\mathbf{Y}_k, \mathbf{X}_k) \psi^k_{P_0(\mathbf{Y}_k)} \psi_{\mathbf{X}_k} \Bigg)$$

$$\cdot \Bigg|_{\substack{\psi^i = 0 \\ (\forall i \in \{1,\cdots,n\})}} 1_{\sum_{j=1}^n m_j = m}.$$

By the uniqueness of an anti-symmetric kernel, for any $\mathbf{X} = (X_1, \cdots, X_{n_k}) \in I^{n_k}$

$$\begin{aligned} &G^k_{n_k}(\mathbf{X}) \\ &= \sum_{p,q=2}^{N} 1_{p,q \in 2\mathbb{N}} 1_{p+q=n_k} \\ &\quad \cdot \frac{1}{n_k!} \sum_{\sigma \in \mathbb{S}_{n_k}} \mathrm{sgn}(\sigma) G^k_{p,q}((X_{\sigma(1)}, \cdots, X_{\sigma(p)}), (X_{\sigma(p+1)}, \cdots, X_{\sigma(p+q)})), \end{aligned}$$

where $\mathbb{S}_{n_k}$ is the set of permutations of $\{1, \cdots, n_k\}$ and $\mathrm{sgn}(\sigma)$ is the sign of $\sigma \in \mathbb{S}_{n_k}$. If $m_k \le 1$, the property (3.50) implies that

$$\sum_{\mathbf{Y}_k \in I^{n_k - m_k}} G^k_{n_k}(\mathbf{Y}_k, \mathbf{X}_k) \psi^k_{P_0(\mathbf{Y}_k)} = 0$$

for any $\mathbf{X}_k \in I^{m_k}$. Therefore

$$\begin{aligned} &A^{(n,l)}_m(\psi) \\ &= Tree(\{1, \cdots, n\}, \mathcal{C}) \\ &\quad \cdot \prod_{j=1}^{l} \left(\sum_{n_j=2}^{N} \sum_{m_j=0}^{n_j-1} \begin{pmatrix} n_j \\ m_j \end{pmatrix} \left(\frac{1}{h}\right)^{n_j} \sum_{\mathbf{X}_j \in I^{m_j}} \sum_{\mathbf{Y}_j \in I^{n_j - m_j}} F^j_{n_j}(\mathbf{Y}_j, \mathbf{X}_j) \psi^j_{\mathbf{Y}_j} \psi_{\mathbf{X}_j} \right) \\ &\quad \cdot \prod_{k=l+1}^{n} \Bigg(\sum_{n_k=4}^{N} \sum_{m_k=0}^{n_k-1} \begin{pmatrix} n_k \\ m_k \end{pmatrix} \left(\frac{1}{h}\right)^{n_k} \\ &\qquad \cdot \sum_{\mathbf{X}_k \in I^{m_k}} \sum_{\mathbf{Y}_k \in I^{n_k - m_k}} G^k_{n_k}(\mathbf{Y}_k, \mathbf{X}_k) \psi^k_{\mathbf{Y}_k} \psi_{\mathbf{X}_k} \Bigg) \\ &\quad \cdot \Bigg|_{\substack{\psi^i = 0 \\ (\forall i \in \{1,\cdots,n\})}} 1_{\sum_{j=1}^n m_j = m \ge 2(n-l)} \end{aligned}$$

$$= 1_{m\ge 2(n-l)}A_m^{(n,l)}(\psi).$$

We can apply the inequality "(3.16)" of [13, Lemma 3.1] or "(4.8)" of [14, Lemma 4.1] to estimate the anti-symmetric kernel of $A_m^{(n,l)}(\psi)$. Multiplying the result by $1_{m\ge 2(n-l)}$ yields (3.51). Now we have $A_m^{(n,l')}(\psi) = 1_{m\ge 2(n-l')}A_m^{(n,l')}(\psi)$. We can apply "(3.17)" of [13, Lemma 3.1] or "(4.9)" of [14, Lemma 4.1] to bound $\|A_m^{(n,l')}\|_1$ and multiply the result by $1_{m\ge 2(n-l')}$ to obtain (3.52). □

Next we consider the Grassmann polynomials $B^{(n)}(\psi)$, $\widehat{B}^{(n')}(\psi) \in \bigwedge_{even}\mathcal{V}$ ($n\in\mathbb{N}$, $n'\in\mathbb{N}_{\ge 2}$) defined as below.

$$\begin{aligned}
B^{(n)}(\psi) :=& \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}} G_{p,q}(\mathbf{X},\mathbf{Y})Tree(\{1,\cdots,n+1\},\mathcal{C})\\
&\cdot(\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}\prod_{j=3}^{n+1}G^j(\psi^j+\psi)\Bigg|_{\substack{\psi^i=0\\(\forall i\in\{1,\cdots,n+1\})}},\\
\widehat{B}^{(n')}(\psi) :=& \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}} G_{p,q}(\mathbf{X},\mathbf{Y})Tree(\{1,\cdots,n'+1\},\mathcal{C})\\
&\cdot(\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}\\
&\cdot\prod_{j=3}^{n'}G^j(\psi^j+\psi)F(\psi^{n'+1}+\psi)\Bigg|_{\substack{\psi^i=0\\(\forall i\in\{1,\cdots,n'+1\})}}.
\end{aligned}$$

The anti-symmetric kernels of these polynomials can be estimated as follows. See [14, Subsection 4.1] for the definition of the measurement $[\cdot,\cdot]_{1,\infty}$.

Lemma 3.6. *For any $m\in\{2,4,\cdots,N\}$, $n\in\mathbb{N}$, $n'\in\mathbb{N}_{\ge 2}$ the anti-symmetric kernels $B_m^{(n)}(\cdot)$, $\widehat{B}_m^{(n')}(\cdot)$ satisfy* (3.49). *Moreover, the following inequalities hold for any $m\in\{0,2,\cdots,N\}$, $n\in\mathbb{N}_{\ge 2}$.*

(3.53)

$$\|B_m^{(1)}\|_{1,\infty}$$

$$\le D^{-1-\frac{m}{2}} \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} 2^{2p_1+2p_2} D^{\frac{p_1+p_2}{2}} [G_{p_1,p_2}, \tilde{\mathcal{C}}]_{1,\infty} 1_{p_1+p_2-2\ge m\ge 2}.$$

(3.54)

$$\begin{aligned}\|B_m^{(n)}\|_{1,\infty} \le (n-1)! D^{-n-\frac{m}{2}} 2^{-2m} \|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1} \\ \cdot \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} 2^{3p_1+3p_2} D^{\frac{p_1+p_2}{2}} [G_{p_1,p_2}, \tilde{\mathcal{C}}]_{1,\infty} \\ \cdot \prod_{j=3}^{n+1} \left(\sum_{p_j=4}^{N} 2^{3p_j} D^{\frac{p_j}{2}} \|G_{p_j}^j\|_{1,\infty} \right) 1_{\sum_{j=1}^{n+1} p_j - 2n \ge m \ge 2n}.\end{aligned}$$

(3.55)

$$\begin{aligned}\|\widehat{B}_m^{(n)}\|_{1} \le (n-1)! D^{-n-\frac{m}{2}} 2^{-2m} \|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1} \\ \cdot \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} 2^{3p_1+3p_2} D^{\frac{p_1+p_2}{2}} [G_{p_1,p_2}, \tilde{\mathcal{C}}]_{1,\infty} \\ \cdot \prod_{j=3}^{n} \left(\sum_{p_j=4}^{N} 2^{3p_j} D^{\frac{p_j}{2}} \|G_{p_j}^j\|_{1,\infty} \right) \\ \cdot \sum_{p_{n+1}=2}^{N} 2^{3p_{n+1}} D^{\frac{p_{n+1}}{2}} \|F_{p_{n+1}}^{n+1}\|_1 1_{\sum_{j=1}^{n+1} p_j - 2n \ge m \ge 2n-2}.\end{aligned}$$

Proof. The first statement of the lemma is essentially proved in [13, Lemma 3.2]. By the same consideration based on anti-symmetry and the properties (3.48), (3.50) as in the proof of Lemma 3.5 we can deduce that for any $n \in \mathbb{N}_{\ge 2}$, $m \in \{0, 2, \cdots, N\}$

$$\begin{aligned}&\widehat{B}_m^{(n)}(\psi) \\ &= \sum_{p_1,p_2=2}^{N} \left(\frac{1}{h}\right)^{p_1+p_2} \sum_{m_1=0}^{p_1-1} \sum_{m_2=0}^{p_2-1} \begin{pmatrix} p_1 \\ m_1 \end{pmatrix} \begin{pmatrix} p_2 \\ m_2 \end{pmatrix} \\ &\quad \cdot \sum_{\substack{\mathbf{X}_1\in I^{m_1} \\ \mathbf{Y}_1 \in I^{p_1-m_1}}} \sum_{\substack{\mathbf{X}_2\in I^{m_2} \\ \mathbf{Y}_2 \in I^{p_2-m_2}}} G_{p_1,p_2}((\mathbf{Y}_1,\mathbf{X}_1),(\mathbf{Y}_2,\mathbf{X}_2))\end{aligned}$$

$$
\begin{aligned}
&\cdot Tree(\{1,\cdots,n+1\},\mathcal{C})\psi^1_{\mathbf{Y}_1}\psi_{\mathbf{X}_1}\psi^2_{\mathbf{Y}_2}\psi_{\mathbf{X}_2}\\
&\cdot\prod_{j=3}^{n}\left(\sum_{p_j=4}^{N}\left(\frac{1}{h}\right)^{p_j}\sum_{m_j=0}^{p_j-1}\left(\begin{array}{c}p_j\\ m_j\end{array}\right)\sum_{\substack{\mathbf{X}_j\in I^{m_j}\\ \mathbf{Y}_j\in I^{p_j-m_j}}}G^j_{p_j}(\mathbf{Y}_j,\mathbf{X}_j)\psi^j_{\mathbf{Y}_j}\psi_{\mathbf{X}_j}\right)\\
&\cdot\sum_{p_{n+1}=2}^{N}\left(\frac{1}{h}\right)^{p_{n+1}}\sum_{m_{n+1}=0}^{p_{n+1}-1}\left(\begin{array}{c}p_{n+1}\\ m_{n+1}\end{array}\right)\\
&\cdot\sum_{\substack{\mathbf{X}_{n+1}\in I^{m_{n+1}}\\ \mathbf{Y}_{n+1}\in I^{p_{n+1}-m_{n+1}}}}F_{p_{n+1}}(\mathbf{Y}_{n+1},\mathbf{X}_{n+1})\psi^{n+1}_{\mathbf{Y}_{n+1}}\psi_{\mathbf{X}_{n+1}}\\
&\cdot\Bigg|_{\substack{\psi^i=0\\ (\forall i\in\{1,\cdots,n+1\})}}1_{\sum_{j=1}^{n+1}m_j=m\ge 2n-2}\\
&=1_{m\ge 2n-2}\widehat{B}^{(n)}_m(\psi).
\end{aligned}
$$

In the first equality we took into account the constraints $m_1\ge 1$, $m_2\ge 1$, $m_j\ge 2$ $(j=3,\cdots,n)$. Then we can apply "(3.27)" of [13, Lemma 3.2] or "(4.14)" of [14, Lemma 4.2] to derive (3.55). In the same way as above we have that for any $m\in\{0,2,\cdots,N\}$, $n\in\mathbb{N}_{\ge 2}$ $B^{(1)}_m(\psi)=1_{m\ge 2}B^{(1)}_m(\psi)$, $B^{(n)}_m(\psi)=1_{m\ge 2n}B^{(n)}_m(\psi)$. Then we can apply "(3.24)" of [13, Lemma 3.2] or "(4.11)" of [14, Lemma 4.2] to derive (3.53) and "(3.26)" of [13, Lemma 3.2] or "(4.13)" of [14, Lemma 4.2] to derive (3.54). □

Assume that $n\in\mathbb{N}$, $m\in\{0,1,\cdots,n-1\}$,

$$
\begin{aligned}
&1=s_1<s_2<\cdots<s_{m+1}\le n,\quad 1=t_1<t_2<\cdots<t_{n-m}\le n,\\
&\{s_j\}_{j=2}^{m+1}\cup\{t_k\}_{k=2}^{n-m}=\{2,3,\cdots,n\},\quad \{s_j\}_{j=2}^{m+1}\cap\{t_k\}_{k=2}^{n-m}=\emptyset.
\end{aligned}
$$

Finally let us study the Grassmann polynomials $E^{(n)}(\psi)$, $\widehat{E}^{(n)}(\psi)\in\bigwedge_{even}\mathcal{V}$ defined as follows.

$$
\begin{aligned}
E^{(n)}(\psi):=&\sum_{p,q=2}^{N}1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}G_{p,q}(\mathbf{X},\mathbf{Y})\\
&\cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C})(\psi^1+\psi)_{\mathbf{X}}\prod_{j=2}^{m+1}G^{s_j}(\psi^{s_j}+\psi)\Bigg|_{\substack{\psi^{s_j}=0\\ (\forall j\in\{1,2,\cdots,m+1\})}}
\end{aligned}
$$

$$\cdot Tree(\{t_k\}_{k=1}^{n-m}, \mathcal{C})(\psi^1 + \psi)_{\mathbf{Y}} \prod_{k=2}^{n-m} G^{t_k}(\psi^{t_k} + \psi)\Big|_{\substack{\psi^{t_k}=0 \\ (\forall k \in \{1,2,\cdots,n-m\})}},$$

$$\widehat{E}^{(n)}(\psi) := \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{p+q} \sum_{\substack{\mathbf{X}\in I^p \\ \mathbf{Y}\in I^q}} G_{p,q}(\mathbf{X},\mathbf{Y})$$
$$\cdot Tree(\{s_j\}_{j=1}^{m+1}, \mathcal{C})(\psi^1 + \psi)_{\mathbf{X}}$$
$$\cdot \prod_{j=2}^{m+1} (1_{s_j \neq n} G^{s_j}(\psi^{s_j} + \psi) + 1_{s_j = n} F(\psi^{s_j} + \psi))\Big|_{\substack{\psi^{s_j}=0 \\ (\forall j \in \{1,2,\cdots,m+1\})}}$$
$$\cdot Tree(\{t_k\}_{k=1}^{n-m}, \mathcal{C})(\psi^1 + \psi)_{\mathbf{Y}}$$
$$\cdot \prod_{k=2}^{n-m} (1_{t_k \neq n} G^{t_k}(\psi^{t_k} + \psi) + 1_{t_k = n} F(\psi^{t_k} + \psi))\Big|_{\substack{\psi^{t_k}=0 \\ (\forall k \in \{1,2,\cdots,n-m\})}}.$$

These Grassmann polynomials are special examples of those studied in [14, Lemma 4.4] and also close to those studied in [13, Lemma 3.3]. The properties we need for later application are summarized in the next lemma. The definition of the measurement $[\cdot,\cdot]_1$ is found in [14, Subsection 4.1].

Lemma 3.7. *For any $n \in \mathbb{N}$, $a, b \in \{2, 4, \cdots, N\}$ there exist functions $E_{a,b}^{(n)}$, $\widehat{E}_{a,b}^{(n)} : I^a \times I^b \to \mathbb{C}$ such that they are bi-anti-symmetric, satisfy* (3.49), (3.50) *and*

$$E^{(n)}(\psi) = \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{a+b} \sum_{\substack{\mathbf{X}\in I^a \\ \mathbf{Y}\in I^b}} E_{a,b}^{(n)}(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}},$$
$$\widehat{E}^{(n)}(\psi) = \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{a+b} \sum_{\substack{\mathbf{X}\in I^a \\ \mathbf{Y}\in I^b}} \widehat{E}_{a,b}^{(n)}(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}.$$

Moreover, the following inequalities hold for any $a, b \in \{2, 4, \cdots, N\}$, $n \in \mathbb{N}_{\ge 2}$ and anti-symmetric function $g : I^2 \to \mathbb{C}$.

(3.56)
$$\|E_{a,b}^{(1)}\|_{1,\infty} \le \sum_{p=a}^{N}\sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} \begin{pmatrix} p \\ a \end{pmatrix} \begin{pmatrix} q \\ b \end{pmatrix} D^{\frac{1}{2}(p+q-a-b)} \|G_{p,q}\|_{1,\infty}.$$

(3.57)

$$[E^{(1)}_{a,b}, g]_{1,\infty} \le \sum_{p=a}^{N}\sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} \begin{pmatrix} p \\ a \end{pmatrix}\begin{pmatrix} q \\ b \end{pmatrix} D^{\frac{1}{2}(p+q-a-b)}[G_{p,q}, g]_{1,\infty}.$$

(3.58)

$$\begin{aligned}
&\|E^{(n)}_{a,b}\|_{1,\infty}\\
&\le (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\quad\cdot 2^{-2a-2b}D^{-n+1-\frac{1}{2}(a+b)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}}2^{3p_1+3q_1}D^{\frac{p_1+q_1}{2}}\|G_{p_1,q_1}\|_{1,\infty}\\
&\quad\cdot\prod_{j=2}^{m+1}\left(\sum_{p_j=4}^{N}2^{3p_j}D^{\frac{p_j}{2}}\|G^{s_j}_{p_j}\|_{1,\infty}\right)\prod_{k=2}^{n-m}\left(\sum_{q_k=4}^{N}2^{3q_k}D^{\frac{q_k}{2}}\|G^{t_k}_{q_k}\|_{1,\infty}\right)\\
&\quad\cdot 1_{\sum_{j=1}^{m+1}p_j-2m\ge a\ge 2m+2}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\ge b\ge 2(n-m)}.
\end{aligned}$$

(3.59)

$$\begin{aligned}
&[E^{(n)}_{a,b}, g]_{1,\infty}\\
&\le (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\quad\cdot 2^{-2a-2b}D^{-n+1-\frac{1}{2}(a+b)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-2}\\
&\quad\cdot\sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}}2^{3p_1+3q_1}D^{\frac{p_1+q_1}{2}}\\
&\quad\cdot([G_{p_1,q_1}, g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty} + [G_{p_1,q_1}, \tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty})\\
&\quad\cdot\prod_{j=2}^{m+1}\left(\sum_{p_j=4}^{N}2^{3p_j}D^{\frac{p_j}{2}}\|G^{s_j}_{p_j}\|_{1,\infty}\right)\prod_{k=2}^{n-m}\left(\sum_{q_k=4}^{N}2^{3q_k}D^{\frac{q_k}{2}}\|G^{t_k}_{q_k}\|_{1,\infty}\right)\\
&\quad\cdot 1_{\sum_{j=1}^{m+1}p_j-2m\ge a\ge 2m+2}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\ge b\ge 2(n-m)}.
\end{aligned}$$

(3.60)

$$\begin{aligned}
&\|\widehat{E}^{(n)}_{a,b}\|_{1}\\
&\le (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\quad\cdot 2^{-2a-2b}D^{-n+1-\frac{1}{2}(a+b)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}}2^{3p_1+3q_1}D^{\frac{p_1+q_1}{2}}\|G_{p_1,q_1}\|_{1,\infty}
\end{aligned}$$

$$\cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N} 2^{3p_j} D^{\frac{p_j}{2}} (1_{s_j\neq n}\|G_{p_j}^{s_j}\|_{1,\infty} + 1_{s_j=n}\|F_{p_j}\|_1)\right)$$
$$\cdot \prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N} 2^{3q_k} D^{\frac{q_k}{2}} (1_{t_k\neq n}\|G_{q_k}^{t_k}\|_{1,\infty} + 1_{t_k=n}\|F_{q_k}\|_1)\right)$$
$$\cdot 1_{\sum_{j=1}^{m+1} p_j-2m\geq a\geq 2m+2-2} 1_{n\in\{s_j\}_{j=2}^{m+1}}$$
$$\cdot 1_{\sum_{k=1}^{n-m} q_k-2(n-m-1)\geq b\geq 2(n-m)-2} 1_{n\in\{t_k\}_{k=2}^{n-m}}.$$

(3.61)

$$\begin{aligned}
&[\widehat{E}_{a,b}^{(n)}, g]_1\\
&\leq (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\quad \cdot 2^{-2a-2b} D^{-n+1-\frac{1}{2}(a+b)} \|\tilde{\mathcal{C}}\|_{1,\infty}^{n-2}\\
&\quad \cdot \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}} 2^{3p_1+3q_1} D^{\frac{p_1+q_1}{2}}\\
&\quad \cdot ([G_{p_1,q_1}, g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty} + [G_{p_1,q_1}, \tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty})\\
&\quad \cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N} 2^{3p_j} D^{\frac{p_j}{2}} (1_{s_j\neq n}\|G_{p_j}^{s_j}\|_{1,\infty} + 1_{s_j=n}\|F_{p_j}\|_1)\right)\\
&\quad \cdot \prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N} 2^{3q_k} D^{\frac{q_k}{2}} (1_{t_k\neq n}\|G_{q_k}^{t_k}\|_{1,\infty} + 1_{t_k=n}\|F_{q_k}\|_1)\right)\\
&\quad \cdot 1_{\sum_{j=1}^{m+1} p_j-2m\geq a\geq 2m+2-2} 1_{n\in\{s_j\}_{j=2}^{m+1}}\\
&\quad \cdot 1_{\sum_{k=1}^{n-m} q_k-2(n-m-1)\geq b\geq 2(n-m)-2} 1_{n\in\{t_k\}_{k=2}^{n-m}}.
\end{aligned}$$

Remark 3.8. In fact the inequalities (3.56), (3.57) are same as "(4.16)", "(4.18)" of [14, Lemma 4.4] respectively. However, we present them for convenience in the subsequent application.

Proof. The existence of the bi-anti-symmetric kernels satisfying the claimed properties is essentially implied by [13, Lemma 3.3]. In fact the kernels are explicitly given in [14, (4.15)] in a more general setting. To make clear, let us present the kernel $\widehat{E}^{(n)}_{a,b}: I^a \times I^b \to \mathbb{C}$ for $n \in \mathbb{N}$, $a, b \in \{2, 4, \cdots, N\}$. For $\mathbf{X} = (X_1, \cdots, X_a) \in I^a$, $\mathbf{Y} = (Y_1, \cdots, Y_b) \in I^b$

$$\begin{aligned}
&\widehat{E}^{(n)}_{a,b}(\mathbf{X}, \mathbf{Y}) \\
&= \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1 \in 2\mathbb{N}} \sum_{u_1=0}^{p_1} (1_{m=0} + 1_{m\neq 0} 1_{u_1 \le p_1 - 1}) \left(\begin{array}{c} p_1 \\ u_1 \end{array}\right) \\
&\quad \cdot \sum_{v_1=0}^{q_1} (1_{m=n-1} + 1_{m\neq n-1} 1_{v_1 \le q_1 - 1}) \left(\begin{array}{c} q_1 \\ v_1 \end{array}\right) \\
&\quad \cdot \left(\frac{1}{h}\right)^{p_1+q_1-u_1-v_1} \sum_{\mathbf{W}_1 \in I^{p_1-u_1}} \sum_{\mathbf{Z}_1 \in I^{q_1-v_1}} G_{p_1,q_1}((\mathbf{W}_1, \mathbf{X}'_1), (\mathbf{Z}_1, \mathbf{Y}'_1)) \\
&\quad \cdot \prod_{j=2}^{m+1} \Bigg(1_{s_j \neq n} \sum_{p_j=4}^{N} \sum_{u_j=0}^{p_j-1} \left(\begin{array}{c} p_j \\ u_j \end{array}\right) \left(\frac{1}{h}\right)^{p_j-u_j} \sum_{\mathbf{W}_j \in I^{p_j-u_j}} G^{s_j}_{p_j}(\mathbf{W}_j, \mathbf{X}'_j) \\
&\qquad + 1_{s_j = n} \sum_{p_j=2}^{N} \sum_{u_j=0}^{p_j-1} \left(\begin{array}{c} p_j \\ u_j \end{array}\right) \left(\frac{1}{h}\right)^{p_j-u_j} \sum_{\mathbf{W}_j \in I^{p_j-u_j}} F_{p_j}(\mathbf{W}_j, \mathbf{X}'_j) \Bigg) \\
&\quad \cdot \prod_{k=2}^{n-m} \Bigg(1_{t_k \neq n} \sum_{q_k=4}^{N} \sum_{v_k=0}^{q_k-1} \left(\begin{array}{c} q_k \\ v_k \end{array}\right) \left(\frac{1}{h}\right)^{q_k-v_k} \sum_{\mathbf{Z}_k \in I^{q_k-v_k}} G^{t_k}_{q_k}(\mathbf{Z}_k, \mathbf{Y}'_k) \\
&\qquad + 1_{t_k = n} \sum_{q_k=2}^{N} \sum_{v_k=0}^{q_k-1} \left(\begin{array}{c} q_k \\ v_k \end{array}\right) \left(\frac{1}{h}\right)^{q_k-v_k} \sum_{\mathbf{Z}_k \in I^{q_k-v_k}} F_{q_k}(\mathbf{Z}_k, \mathbf{Y}'_k) \Bigg) \\
&\quad \cdot Tree(\{s_j\}_{j=1}^{m+1}, \mathcal{C}) \prod_{j=1}^{m+1} \psi^{s_j}_{\mathbf{W}_j} \Bigg|_{\substack{\psi^{s_j}=0 \\ (\forall j \in \{1,\cdots,m+1\})}} \\
&\quad \cdot Tree(\{t_k\}_{k=1}^{n-m}, \mathcal{C}) \prod_{k=1}^{n-m} \psi^{t_k}_{\mathbf{Z}_k} \Bigg|_{\substack{\psi^{t_k}=0 \\ (\forall k \in \{1,\cdots,n-m\})}} \\
&\quad \cdot (-1)^{\sum_{j=1}^{m} u_j \sum_{i=j+1}^{m+1}(p_i-u_i) + \sum_{k=1}^{n-m-1} v_k \sum_{i=k+1}^{n-m}(q_i - v_i)} 1_{\sum_{j=1}^{m+1} u_j = a} 1_{\sum_{k=1}^{n-m} v_k = b} \\
&\quad \cdot \frac{1}{a!b!} \sum_{\substack{\sigma \in \mathbb{S}_a \\ \tau \in \mathbb{S}_b}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) 1_{(\mathbf{X}'_1,\cdots,\mathbf{X}'_{m+1}) = (X_{\sigma(1)},\cdots,X_{\sigma(a)})} \\
&\quad \cdot 1_{(\mathbf{Y}'_1,\cdots,\mathbf{Y}'_{n-m}) = (Y_{\tau(1)},\cdots,Y_{\tau(b)})}.
\end{aligned}$$

By considering (3.48), (3.50) we can substitute the constraints

$$u_1 \ge 1, \quad u_j \ge 2 1_{s_j \neq n}, \quad (\forall j \in \{2, \cdots, m+1\}),$$
$$v_1 \ge 1, \quad v_k \ge 2 1_{t_k \neq n}, \quad (\forall k \in \{2, \cdots, n-m\}).$$

Moreover, by using the fact that a, b must be even we have that

$$\widehat{E}^{(n)}_{a,b}(\mathbf{X}, \mathbf{Y}) = 1_{a \ge 2m+2-2 1_{n \in \{s_j\}_{j=2}^{m+1}}} 1_{b \ge 2(n-m)-2 1_{n \in \{t_k\}_{k=2}^{n-m}}} \widehat{E}^{(n)}_{a,b}(\mathbf{X}, \mathbf{Y}),$$

and thus

$$\|\widehat{E}^{(n)}_{a,b}\|_1 = 1_{a \ge 2m+2-2 1_{n \in \{s_j\}_{j=2}^{m+1}}} 1_{b \ge 2(n-m)-2 1_{n \in \{t_k\}_{k=2}^{n-m}}} \|\widehat{E}^{(n)}_{a,b}\|_1,$$
$$[\widehat{E}^{(n)}_{a,b}, g]_1 = 1_{a \ge 2m+2-2 1_{n \in \{s_j\}_{j=2}^{m+1}}} 1_{b \ge 2(n-m)-2 1_{n \in \{t_k\}_{k=2}^{n-m}}} [\widehat{E}^{(n)}_{a,b}, g]_1$$

for any anti-symmetric function $g : I^2 \to \mathbb{C}$. Then we can apply "(3.37)" of [13, Lemma 3.3] (or "(4.21)" of [14, Lemma 4.4]), "(4.23)" of [14, Lemma 4.4] to obtain (3.60), (3.61) respectively. By the same consideration based on (3.48), (3.50) and the parity of a, b we see that

$$\|E^{(n)}_{a,b}\|_{1,\infty} = 1_{a \ge 2m+2} 1_{b \ge 2(n-m)} \|E^{(n)}_{a,b}\|_{1,\infty},$$
$$[E^{(n)}_{a,b}, g]_{1,\infty} = 1_{a \ge 2m+2} 1_{b \ge 2(n-m)} [E^{(n)}_{a,b}, g]_{1,\infty}$$

for any anti-symmetric function $g : I^2 \to \mathbb{C}$. Then combination with "(3.36)" of [13, Lemma 3.3] (or "(4.20)" of [14, Lemma 4.4]), "(4.22)" of [14, Lemma 4.4] leads to (3.58), (3.59) respectively. □

3.3. Double-scale integration

In this subsection we construct a double-scale integration scheme based on some general properties of a couple of covariances. With $c_0 \in \mathbb{R}_{\ge 1}$, A, $B \in \mathbb{R}_{>0}$ the covariances $\mathcal{C}_0, \mathcal{C}_1 : I_0^2 \to \mathbb{C}$ are assumed to satisfy the following conditions.

•

$$\mathcal{C}_0(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t) = \mathcal{C}_0(\overline{\rho}\rho\mathbf{x}0, \overline{\eta}\eta\mathbf{y}0), \quad (\forall (\overline{\rho}, \rho, \mathbf{x}, s),\ (\overline{\eta}, \eta, \mathbf{y}, t) \in I_0). \tag{3.62}$$

•

$$\mathcal{C}_1(\mathcal{R}_\beta(\mathbf{X} + s)) = \mathcal{C}_1(\mathbf{X}), \quad \left(\forall \mathbf{X} \in I_0^2,\ s \in \frac{1}{h}\mathbb{Z}\right). \tag{3.63}$$

•

$$|\det(\langle \mathbf{u}_i, \mathbf{w}_j\rangle_{\mathbb{C}^m}\mathcal{C}_a(X_i,Y_j))_{1\le i,j\le n}| \le (c_0(1_{a=0}A + 1_{a=1}))^n, \tag{3.64}$$
$$(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{w}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m}, \|\mathbf{w}_i\|_{\mathbb{C}^m}\le 1,\ X_i,Y_i\in I_0\ (i=1,\cdots,n),\ a\in\{0,1\}).$$

•

$$\|\tilde{\mathcal{C}}_a\|_{1,\infty} \le c_0 B, \quad (\forall a\in\{0,1\}). \tag{3.65}$$

•

$$\|\tilde{\mathcal{C}}_a\|'_{1,\infty} \le c_0 A, \quad (\forall a\in\{0,1\}). \tag{3.66}$$

We should think of them as generalizations of the covariances C_0, C_1 introduced in Subsection 3.1. It is efficient to define the covariances by abstracting the dependency on the physical parameters at this stage. On the contrary, we explicitly define the input Grassmann polynomials to the double-scale integration process as follows.

$$V^{0-1,0}(u)(\psi) := \left(\frac{1}{h}\right)^2 \sum_{\mathbf{X}\in I^2} V_2^{0-1,0}(u)(\mathbf{X})\psi_{\mathbf{X}},$$
$$V^{0-2,0}(u)(\psi) := \left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}^{0-2,0}(u)(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}},$$

where the anti-symmetric kernel $V_2^{0-1,0}(u) : I^2 \to \mathbb{C}$ and the bi-anti-symmetric kernel $V_{2,2}^{0-2,0}(u) : I^2\times I^2 \to \mathbb{C}$ are defined by

$$\begin{aligned}
&V_2^{0-1,0}(u)(\overline{\rho}_1\rho_1\mathbf{x}_1 s_1\xi_1, \overline{\rho}_2\rho_2\mathbf{x}_2 s_2\xi_2)\\
&:= -\frac{1}{2}uL^{-d}h 1_{(\overline{\rho}_1,\rho_1,\mathbf{x}_1,s_1)=(\overline{\rho}_2,\rho_2,\mathbf{x}_2,s_2)} 1_{\overline{\rho}_1=1}\big(1_{(\xi_1,\xi_2)=(1,-1)} - 1_{(\xi_1,\xi_2)=(-1,1)}\big),\\
&V_{2,2}^{0-2,0}(u)(\overline{\rho}_1\rho_1\mathbf{x}_1 s_1\xi_1, \overline{\rho}_2\rho_2\mathbf{x}_2 s_2\xi_2, \overline{\eta}_1\eta_1\mathbf{y}_1 t_1\zeta_1, \overline{\eta}_2\eta_2\mathbf{y}_2 t_2\zeta_2)\\
&:= -\frac{1}{4}uL^{-d}h^2(h1_{s_1=t_1} - \beta^{-1})1_{(\rho_1,\mathbf{x}_1,s_1,\eta_1,\mathbf{y}_1,t_1)=(\rho_2,\mathbf{x}_2,s_2,\eta_2,\mathbf{y}_2,t_2)}\\
&\quad\cdot \sum_{\sigma,\tau\in\mathbb{S}_2} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) 1_{(\overline{\rho}_{\sigma(1)},\overline{\rho}_{\sigma(2)},\overline{\eta}_{\tau(1)},\overline{\eta}_{\tau(2)})=(1,2,2,1)}\\
&\quad\cdot 1_{(\xi_{\sigma(1)},\xi_{\sigma(2)},\zeta_{\tau(1)},\zeta_{\tau(2)})=(1,-1,1,-1)}.
\end{aligned}$$

Here u is a complex parameter and should be considered as an extension of the coupling constant U. Though the definitions seem complicated, they can be simply rewritten as follows.

$$V^{0-1,0}(u)(\psi) = -\frac{u}{L^d h}\sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\rho\mathbf{x}s}\psi_{1\rho\mathbf{x}s},$$

(3.67)

$$\begin{aligned}V^{0-2,0}(u)(\psi) = &-\frac{u}{L^d h}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\rho\mathbf{x}s}\psi_{2\rho\mathbf{x}s}\overline{\psi}_{2\eta\mathbf{y}s}\psi_{1\eta\mathbf{y}s}\\ &+\frac{u}{\beta L^d h^2}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\sum_{s,t\in[0,\beta)_h}\overline{\psi}_{1\rho\mathbf{x}s}\psi_{2\rho\mathbf{x}s}\overline{\psi}_{2\eta\mathbf{y}t}\psi_{1\eta\mathbf{y}t}.\end{aligned}$$

We adopt [14, Lemma 3.6] as the formulation of our system. We can see from (3.67) and [14, Lemma 3.6] that the Grassmann polynomial $V^{0-1,0}(U)(\psi)+V^{0-2,0}(U)(\psi)$ appears in the Grassmann integral formulation as the effective interaction. Our first goal in this subsection is to construct an analytic continuation of the $\bigwedge_{even}\mathcal{V}$-valued function

$$u \mapsto \log\left(\int e^{V^{0-1,0}(u)(\psi^0+\psi)+V^{0-2,0}(u)(\psi^0+\psi)}d\mu_{\mathcal{C}_0}(\psi^0)\right)$$

in a neighborhood of the origin. Let us remark that we integrate with the time-independent covariance $\mathcal{C}_0$ as the first step, while the integration with the time-independent covariance was performed in the last step of the multi-scale integrations in [13], [14]. The determinant bound on $\mathcal{C}_0$ is the main problematic contribution from the sliced covariances, while the $\|\cdot\|_{1,\infty}$-norm bound on the time-independent covariance was so in [13], [14]. We integrate with the covariance $\mathcal{C}_0$ first in order to remove the main burden on the possible magnitude of u. The output of the integration with $\mathcal{C}_0$ will be integrated with $\mathcal{C}_1$ in the second step.

It will help us to organize our analysis if we prepare some sets of $\bigwedge_{even}\mathcal{V}$-valued functions in advance. For $r\in\mathbb{R}_{>0}$, set $D(r) := \{z\in\mathbb{C} \mid |z| < r\}$. In the following α denotes a parameter belonging to $\mathbb{R}_{\ge 1}$. Admitting the convention concerning choice of a norm of $\bigwedge_{even}\mathcal{V}$ explained in the beginning of [14, Subsection 4.4], for any domain D of $\mathbb{C}^n$ we let $C(\overline{D},\bigwedge_{even}\mathcal{V})$, $C^{\omega}(D,\bigwedge_{even}\mathcal{V})$ denote the set of continuous maps from $\overline{D}$ to $\bigwedge_{even}\mathcal{V}$, the set of analytic maps from D to $\bigwedge_{even}\mathcal{V}$ respectively. Let us also refer to the beginning of [14, Subsection 4.4] for the definitions of the norm $\|\cdot\|_{1,\infty,r}$ of $C(\overline{D(r)},\mathbb{C})$ and $C(\overline{D(r)},\mathrm{Map}(I^m,\mathbb{C}))$ and the measurement $[\cdot,\cdot]_{1,\infty,r}$ for a coupling between a function belonging to $C(\overline{D(r)},\mathrm{Map}(I^m,\mathbb{C}))$ and an anti-symmetric function on I^2. For $r\in\mathbb{R}_{>0}$ we define the subsets $\mathcal{Q}(r)$, $\mathcal{R}(r)$ of $\mathrm{Map}(\overline{D(r)},\bigwedge_{even}\mathcal{V})$ as follows.

$f\in\mathcal{Q}(r)$ if and only if

- $$f\in C\left(\overline{D(r)},\bigwedge_{even}\mathcal{V}\right)\cap C^{\omega}\left(D(r),\bigwedge_{even}\mathcal{V}\right).$$
- For any $u\in\overline{D(r)}$ the anti-symmetric kernels $f(u)_m:I^m\to\mathbb{C}$ ($m=2,4,\cdots,N$) satisfy (3.49) and

$$\frac{h}{N}\|f_0\|_{1,\infty,r}\le\alpha^{-1}AB^{-1}L^{-d},\tag{3.68}$$
$$\sum_{m=2}^{N}c_0^{\frac{m}{2}}\alpha^m\|f_m\|_{1,\infty,r}\le(A+1)B^{-1}L^{-d}.$$

$f\in\mathcal{R}(r)$ if and only if

- $$f\in C\left(\overline{D(r)},\bigwedge_{even}\mathcal{V}\right)\cap C^{\omega}\left(D(r),\bigwedge_{even}\mathcal{V}\right).$$
- There exist $f_{p,q}\in C(\overline{D(r)},\mathrm{Map}(I^p\times I^q,\mathbb{C}))$ ($p,q\in\{2,4,\cdots,N\}$) such that for any $u\in\overline{D(r)}$, $p,q\in\{2,4,\cdots,N\}$ $f_{p,q}(u):I^p\times I^q\to\mathbb{C}$ is bi-anti-symmetric, satisfies (3.49), (3.50),

$$f(u)(\psi)=\sum_{p,q=2}^{N}1_{p,q\in2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}f_{p,q}(u)(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}$$

and

$$\sum_{p,q=2}^{N}c_0^{\frac{p+q}{2}}\alpha^{p+q}\|f_{p,q}\|_{1,\infty,r}\le B^{-1},\tag{3.69}$$

$$\sum_{p,q=2}^{N}c_0^{\frac{p+q}{2}}\alpha^{p+q}[f_{p,q},g]_{1,\infty,r}\le B^{-1}(\|g\|_{1,\infty}'+AB^{-1}\|g\|_{1,\infty})L^{-d}\tag{3.70}$$

for any anti-symmetric function $g : I^2 \to \mathbb{C}$.

Next we arrange the Grassmann polynomials

$$\frac{1}{n!}\left(\frac{d}{dz}\right)^n \log\left(\int e^{zV^{0-1,0}(u)(\psi^0+\psi)+zV^{0-2,0}(u)(\psi^0+\psi)}d\mu_{\mathcal{C}_0}(\psi^0)\right)\Bigg|_{z=0} \tag{3.71}$$

($n \in \mathbb{N}$) in the same way as in [13, Subsection 3.4]. One apparent difference is that here we have the covariance $\mathcal{C}_0$ rather than $\mathcal{C}_1$. The difference in the index of the covariances results in the difference in the second superscript of the Grassmann polynomials. Let us remark that here the input polynomials have 0 and the output polynomials have 1 in the second superscript. In [13, Subsection 3.4] the Grassmann polynomials had the opposite numbers in the second superscript. For $n \in \mathbb{N}$ we define $V^{0-1-1,1,(n)}$, $V^{0-1-2,1,(n)}$, $V^{0-2,1,(n)} \in \mathrm{Map}(\mathbb{C}, \bigwedge_{even} \mathcal{V})$ as follows.

$$\begin{aligned}
&V^{0-1-1,1,(n)}(u)(\psi)\\
&:= \frac{1}{n!}Tree(\{1,\cdots,n\},\mathcal{C}_0)\prod_{j=1}^{n}\left(\sum_{b_j\in\{1,2\}}V^{0-b_j,0}(u)(\psi^j+\psi)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,\cdots,n\})}}\\
&\quad\cdot 1_{\exists j(b_j=1)},\\
&V^{0-1-2,1,(n)}(u)(\psi)\\
&:= \left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V^{0-2,0}_{2,2}(u)(\mathbf{X},\mathbf{Y})\frac{1}{n!}Tree(\{1,\cdots,n+1\},\mathcal{C}_0)\\
&\quad\cdot(\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}\prod_{j=3}^{n+1}V^{0-2,0}(u)(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,\cdots,n+1\})}},\\
&V^{0-2,1,(n)}(u)(\psi)\\
&:= \frac{1}{n!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}\left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V^{0-2,0}_{2,2}(u)(\mathbf{X},\mathbf{Y})\\
&\quad\cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C}_0)(\psi^{s_1}+\psi)_{\mathbf{X}}\prod_{j=2}^{m+1}V^{0-2,0}(u)(\psi^{s_j}+\psi)\Bigg|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,\cdots,m+1\})}}\\
&\quad\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C}_0)(\psi^{t_1}+\psi)_{\mathbf{Y}}\prod_{k=2}^{n-m}V^{0-2,0}(u)(\psi^{t_k}+\psi)\Bigg|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,\cdots,n-m\})}},
\end{aligned}$$

where

$$S(n,m) := \left\{ (\{s_j\}_{j=1}^{m+1}, \{t_k\}_{k=1}^{n-m}) \;\middle|\; \begin{array}{l} 1 = s_1 < s_2 < \cdots < s_{m+1} \le n, \\ 1 = t_1 < t_2 < \cdots < t_{n-m} \le n, \\ \{s_j\}_{j=2}^{m+1} \cup \{t_k\}_{k=2}^{n-m} = \{2,3,\cdots,n\}, \\ \{s_j\}_{j=2}^{m+1} \cap \{t_k\}_{k=2}^{n-m} = \emptyset. \end{array} \right\}.$$

The following equality is structurally same as [13, (3.56)], [14, (4.41)] and originates from [17, (3.38)], [16, (IV.15)].

$$\begin{aligned} &\text{(The Grassmann polynomial (3.71))} \\ &= V^{0-1-1,1,(n)}(u)(\psi) + V^{0-1-2,1,(n)}(u)(\psi) + V^{0-2,1,(n)}(u)(\psi). \end{aligned} \tag{3.72}$$

Moreover, we set

$$\begin{aligned} V^{0-1-j,1}(u)(\psi) &:= \sum_{n=1}^{\infty} V^{0-1-j,1,(n)}(u)(\psi), \quad (j=1,2), \\ V^{0-1,1}(u)(\psi) &:= \sum_{j=1}^{2} V^{0-1-j,1}(u)(\psi), \\ V^{0-2,1}(u)(\psi) &:= \sum_{n=1}^{\infty} V^{0-2,1,(n)}(u)(\psi), \end{aligned}$$

if they converge in $\bigwedge_{even} \mathcal{V}$. Bearing in mind that the constant A will be β-dependent in practice, we want to prove the analyticity of $u \mapsto V^{0-1,1}(u)$, $u \mapsto V^{0-2,1}(u)$ in an A-independent neighborhood of the origin. The machinery which essentially enables us to achieve this goal is the general estimations summarized in Subsection 3.2. They are applicable in the proof below, mainly because $V_2^{0-1,0}(u) : I^2 \to \mathbb{C}$, $V_{2,2}^{0-2,0}(u) : I^2 \times I^2 \to \mathbb{C}$ satisfy (3.49), $V_{2,2}^{0-2,0}(u)(\cdot)$ satisfies (3.50) and the covariance $\mathcal{C}_0$ satisfies (3.62).

Lemma 3.9. *There exists $c \in \mathbb{R}_{>0}$ independent of any parameter such that if $\alpha \ge c$,*

$$V^{0-1,1} \in \mathcal{Q}(c_0^{-2}\alpha^{-5}b^{-1}B^{-1}), \quad V^{0-2,1} \in \mathcal{R}(c_0^{-2}\alpha^{-5}b^{-1}B^{-1}).$$

Proof. We set $r := c_0^{-2}\alpha^{-5}b^{-1}B^{-1}$. Let us begin by listing necessary bounds on the input. It follows from the definitions that

$$\|V_2^{0-1,0}\|_{1,\infty,r} \le rL^{-d}, \tag{3.73}$$

$$\|V_{2,2}^{0-2,0}\|_{1,\infty,r} \le br, \tag{3.74}$$

$$\|V_4^{0-2,0}\|_{1,\infty,r} \le \|V_{2,2}^{0-2,0}\|_{1,\infty,r} \le br, \tag{3.75}$$

$$[V_{2,2}^{0-2,0}, g]_{1,\infty,r} \le rL^{-d}(\|g\|'_{1,\infty} + \beta^{-1}\|g\|_{1,\infty}) \le 2rL^{-d}\|g\|'_{1,\infty}. \tag{3.76}$$

First let us consider $V^{0-1-1,1,(n)}$. By "(3.14)" of [13, Lemma 3.1] or "(4.6)" of [14, Lemma 4.1], (3.64) and (3.73), for $m \in \{0, 2, \cdots, N\}$

$$\begin{aligned}\|V_m^{0-1-1,1,(1)}\|_{1,\infty,r} &\le \left(\frac{N}{h}\right)^{1_{m=0}} (c_0A)^{\frac{2-m}{2}} rL^{-d} 1_{m\le 2}\\ &\le \left(\frac{N}{h}\right)^{1_{m=0}} c_0^{-\frac{m}{2}} A^{1-\frac{m}{2}} \alpha^{-5} B^{-1} L^{-d} 1_{m\le 2},\end{aligned}$$

where we also used that $c_0^{-1} \le 1$. Moreover, by (3.51), (3.64), (3.65), (3.73) and (3.75) for any $n \in \mathbb{N}_{\ge 2}$, $m \in \{0, 2, \cdots, N\}$

$$\begin{aligned}\|V_m^{0-1-1,1,(n)}\|_{1,\infty,r} \le& \left(\frac{N}{h}\right)^{1_{m=0}} \sum_{l=1}^{n} \left(\begin{array}{c} n \\ l \end{array}\right) (c_0A)^{-n+1-\frac{m}{2}} 2^{-2m} (c_0B)^{n-1}\\ &\cdot (2^6 c_0 ArL^{-d})^l (2^{12} c_0^2 A^2 br)^{n-l} 1_{2(n-l)+2\ge m \ge 2(n-l)}.\end{aligned}$$

Here we remark that when $m = 0$, only the term with $l = n$ remains in the right-hand side of the above inequality. It follows that

$$\|V_0^{0-1-1,1,(1)}\|_{1,\infty,r} \le \frac{N}{h} AB^{-1}L^{-d}\alpha^{-5}, \tag{3.77}$$

$$\begin{aligned}\|V_0^{0-1-1,1,(n)}\|_{1,\infty,r} &\le \frac{N}{h} A^{-n+1}B^{n-1}(2^6 c_0 ArL^{-d})^n\\ &\le \frac{N}{h} AB^{-1}L^{-d}(2^6\alpha^{-5})^n,\end{aligned} \tag{3.78}$$

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{0-1-1,1,(1)}\|_{1,\infty,r} \le c_0\alpha^2 rL^{-d} \le \alpha^{-3}B^{-1}L^{-d}, \tag{3.79}$$

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{0-1-1,1,(n)}\|_{1,\infty,r} \tag{3.80}$$

$$\leq c\sum_{l=1}^{n}\begin{pmatrix} n \\ l \end{pmatrix} A^{-n+1}B^{n-1}(2^6c_0ArL^{-d})^l(2^{12}c_0^2A^2br)^{n-l}$$
$$\cdot 2^{-4(n-l)}A^{-(n-l)}\alpha^{2(n-l)}(1+A^{-1}\alpha^2)$$
$$\leq cAB^{-1}(1+A^{-1}\alpha^2)\sum_{l=1}^{n}\begin{pmatrix} n \\ l \end{pmatrix}(2^6c_0BrL^{-d})^l(2^8c_0^2B\alpha^2br)^{n-l}$$
$$\leq cAB^{-1}(1+A^{-1}\alpha^2)\sum_{l=1}^{n}\begin{pmatrix} n \\ l \end{pmatrix}(2^6\alpha^{-5}L^{-d})^l(2^8\alpha^{-3})^{n-l}$$
$$\leq cB^{-1}(A+\alpha^2)L^{-d}(2^9\alpha^{-3})^n.$$

Next let us study $V^{0-1-2,1,(n)}$. We can apply (3.53), (3.64), (3.66), (3.76) to derive that for $m \in \{0,2,\cdots,N\}$

$$\|V_m^{0-1-2,1,(1)}\|_{1,\infty,r} \leq c(c_0A)^{-2}(c_0A)^2rL^{-d}c_0A1_{m=2}$$
$$\leq cc_0^{-1}\alpha^{-5}AB^{-1}L^{-d}1_{m=2}.$$

For $n \in \mathbb{N}_{\geq 2}$ we use (3.54) instead of (3.53) and (3.75) together with (3.76) to derive that

$$\|V_m^{0-1-2,1,(n)}\|_{1,\infty,r}$$
$$\leq c(c_0A)^{-n-\frac{m}{2}}2^{-2m}(c_0B)^{n-1}(c_0A)^2rL^{-d}c_0A(2^{12}(c_0A)^2br)^{n-1}1_{m=2n}$$
$$\leq cc_0^{-\frac{m}{2}}AB^{-1}L^{-d}(2^8\alpha^{-5})^n1_{m=2n}.$$

Thus

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m\|V_m^{0-1-2,1,(1)}\|_{1,\infty,r} \leq c\alpha^{-3}AB^{-1}L^{-d}, \tag{3.81}$$
$$\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m\|V_m^{0-1-2,1,(n)}\|_{1,\infty,r} \leq cAB^{-1}L^{-d}(2^8\alpha^{-3})^n. \tag{3.82}$$

Assume that $\alpha^3 \geq 2^{10}$. Then by (3.77), (3.78), (3.79), (3.80), (3.81) and (3.82)

$$\frac{h}{N}\sum_{n=1}^{\infty}\|V_0^{0-1-1,1,(n)}\|_{1,\infty,r} \leq c\alpha^{-5}AB^{-1}L^{-d},$$

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \sum_{n=1}^{\infty} (\|V_m^{0-1-1,1,(n)}\|_{1,\infty,r} + \|V_m^{0-1-2,1,(n)}\|_{1,\infty,r})$$
$$\le c\alpha^{-3}(A+1)B^{-1}L^{-d}.$$

These uniform convergent properties imply the well-definedness of $V^{0-1,1}$ and the claimed regularity with u. It follows from the statements of [13, Lemma 3.1] (or [14, Lemma 4.1]), Lemma 3.5, Lemma 3.6 that the kernels of $V^{0-1,1}$ satisfy (3.49). Moreover, the above inequalities ensure that if $\alpha \ge c$, $V^{0-1,1}$ satisfies (3.68). Therefore, $V^{0-1,1} \in \mathcal{Q}(r)$ on the assumption $\alpha \ge c$.

Let us treat $V^{0-2,1,(n)}$. By Lemma 3.7 (or more originally by [13, Lemma 3.3], [14, Lemma 4.4]) there are bi-anti-symmetric functions $V_{a,b}^{0-2,1,(n)}(u) : I^a \times I^b \to \mathbb{C}$ ($n \in \mathbb{N}$, $a, b \in \{2, 4, \cdots, N\}$, $u \in \mathbb{C}$) satisfying (3.49), (3.50) such that

$$V^{0-2,1,(n)}(u)(\psi) = \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{a+b} \sum_{\substack{\mathbf{X}\in I^a \\ \mathbf{Y}\in I^b}} V_{a,b}^{0-2,1,(n)}(u)(\mathbf{X}, \mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}},$$

$(\forall n \in \mathbb{N},\ u \in \mathbb{C})$.

By (3.56) and (3.74), for $a, b \in \{2, 4, \cdots, N\}$

$$\|V_{a,b}^{0-2,1,(1)}\|_{1,\infty,r} \le \|V_{a,b}^{0-2,0}\|_{1,\infty,r} 1_{a=b=2} \le c_0^{-2}\alpha^{-5}B^{-1}1_{a=b=2},$$

and thus

$$\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} c_0^{\frac{a+b}{2}} \alpha^{a+b} \|V_{a,b}^{0-2,1,(1)}\|_{1,\infty,r} \le \alpha^{-1}B^{-1}. \tag{3.83}$$

For $n \in \mathbb{N}_{\ge 2}$, $a, b \in \{2, 4, \cdots, N\}$ the inequalities (3.58), (3.64), (3.65), (3.74) and (3.75) yield that

$$\|V_{a,b}^{0-2,1,(n)}\|_{1,\infty,r}$$
$$\le \frac{1}{n!}\sum_{m=0}^{n-1} \sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}$$
$$\cdot (1_{m\ne 0}(m-1)! + 1_{m=0})(1_{m\ne n-1}(n-m-2)! + 1_{m=n-1})$$

$$\cdot 2^{-2a-2b}(c_0A)^{-n+1-\frac{1}{2}(a+b)}(c_0B)^{n-1}(2^{12}c_0^2A^2br)^n 1_{a=2m+2}1_{b=2(n-m)}$$
$$= \frac{1}{n!}\sum_{m=0}^{n-1}\left(\begin{array}{c} n-1 \\ m \end{array}\right)(1_{m\neq 0}(m-1)!+1_{m=0})$$
$$\cdot (1_{m\neq n-1}(n-m-2)!+1_{m=n-1})$$
$$\cdot 2^{8n-4}c_0^{-\frac{a+b}{2}}B^{-1}\alpha^{-5n}1_{a=2m+2}1_{b=2(n-m)}.$$

Therefore,

$$\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} c_0^{\frac{a+b}{2}}\alpha^{a+b}\|V_{a,b}^{0-2,1,(n)}\|_{1,\infty,r} \le c\alpha^2 B^{-1}(2^8\alpha^{-3})^n. \tag{3.84}$$

On the other hand, let us take an anti-symmetric function $g : I^2 \to \mathbb{C}$. By (3.57) and (3.76), for any $a, b \in \{2, 4, \cdots, N\}$

$$\begin{aligned}[V_{a,b}^{0-2,1,(1)}, g]_{1,\infty,r} &\le [V_{2,2}^{0-2,0}, g]_{1,\infty,r}1_{a=b=2} \\ &\le 2c_0^{-2}\alpha^{-5}B^{-1}L^{-d}\|g\|'_{1,\infty}1_{a=b=2}.\end{aligned}$$

Thus

$$\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} c_0^{\frac{a+b}{2}}\alpha^{a+b}[V_{a,b}^{0-2,1,(1)}, g]_{1,\infty,r} \le 2\alpha^{-1}B^{-1}L^{-d}\|g\|'_{1,\infty}. \tag{3.85}$$

For $n \in \mathbb{N}_{\ge 2}$, $a, b \in \{2, 4, \cdots, N\}$ we can apply (3.59), (3.64), (3.65), (3.66), (3.75) and (3.76) to deduce that

$$[V_{a,b}^{0-2,1,(n)}, g]_{1,\infty,r}$$
$$\le \frac{c}{n!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}$$
$$\cdot (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})$$
$$\cdot 2^{-2a-2b}(c_0A)^{-n+1-\frac{1}{2}(a+b)}(c_0B)^{n-2}c_0^2A^2$$
$$\cdot (rL^{-d}\|g\|'_{1,\infty}c_0B + rL^{-d}c_0A\|g\|_{1,\infty})$$
$$\cdot (2^{12}c_0^2A^2br)^{n-1}1_{a=2m+2}1_{b=2(n-m)}$$

$$\leq \frac{c}{n!}\sum_{m=0}^{n-1}\left(\begin{array}{c} n-1 \\ m \end{array}\right)(1_{m\neq 0}(m-1)!+1_{m=0})$$
$$\cdot(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})$$
$$\cdot 2^{8n}c_0^{-\frac{a+b}{2}}B^{-1}\alpha^{-5n}L^{-d}(\|g\|'_{1,\infty}+AB^{-1}\|g\|_{1,\infty})1_{a=2m+2}1_{b=2(n-m)},$$

and thus

$$\begin{aligned}(3.86)\qquad &\sum_{a,b=2}^{N}1_{a,b\in 2\mathbb{N}}c_0^{\frac{a+b}{2}}\alpha^{a+b}[V_{a,b}^{0-2,1,(n)},g]_{1,\infty,r}\\ &\leq c\alpha^2B^{-1}L^{-d}(\|g\|'_{1,\infty}+AB^{-1}\|g\|_{1,\infty})(2^8\alpha^{-3})^n.\end{aligned}$$

Assume that $\alpha^3\geq 2^9$. By summing up (3.83), (3.84), (3.85), (3.86) we observe that

$$\begin{aligned}(3.87)\qquad &\sum_{a,b=2}^{N}1_{a,b\in 2\mathbb{N}}c_0^{\frac{a+b}{2}}\alpha^{a+b}\sum_{n=1}^{\infty}\|V_{a,b}^{0-2,1,(n)}\|_{1,\infty,r}\leq(\alpha^{-1}+c\alpha^{-4})B^{-1},\\ &\sum_{a,b=2}^{N}1_{a,b\in 2\mathbb{N}}c_0^{\frac{a+b}{2}}\alpha^{a+b}\sum_{n=1}^{\infty}[V_{a,b}^{0-2,1,(n)},g]_{1,\infty,r}\\ &\leq(2\alpha^{-1}+c\alpha^{-4})B^{-1}(\|g\|'_{1,\infty}+AB^{-1}\|g\|_{1,\infty})L^{-d}.\end{aligned}$$

The uniform convergence property (3.87) ensures the well-definedness of $V^{0-2,1}$ and the claimed regularity with u. On the assumption $\alpha\geq c$ we can conclude from the above inequalities that $V^{0-2,1}\in\mathcal{R}(r)$. □

Lemma 3.9 will support us in the derivation of the free energy density. In order to derive the thermal expectations, on the other hand, we need to add an artificial term to the input Grassmann polynomials and construct the double-scale integration process by clarifying how the artificial term affects the output. Let us fix $(\hat{\rho},\hat{\mathbf{x}})$, $(\hat{\eta},\hat{\mathbf{y}})\in\mathcal{B}\times\Gamma_\infty$, which are to represent the sites where the Cooper pair density is measured. The artificial Grassmann polynomial $V^{1,0}(\boldsymbol{\lambda})(\psi)\in\bigwedge_{even}\mathcal{V}$ parameterized by the artificial parameter $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)\in\mathbb{C}^2$ is defined as follows.

$$V^{1,0}(\boldsymbol{\lambda})(\psi):=\sum_{m\in\{2,4\}}\left(\frac{1}{h}\right)^m\sum_{\mathbf{X}\in I^m}V_m^{1,0}(\boldsymbol{\lambda})(\mathbf{X})\psi_{\mathbf{X}}$$

with the anti-symmetric kernels $V_m^{1,0}(\boldsymbol{\lambda}) : I^m \to \mathbb{C}$ $(m = 2, 4)$ defined by

$$\begin{aligned}
&V_2^{1,0}(\boldsymbol{\lambda})(\overline{\rho}_1\rho_1\mathbf{x}_1s_1\xi_1, \overline{\rho}_2\rho_2\mathbf{x}_2s_2\xi_2)\\
&:= -\frac{h}{2}1_{s_1=s_2}\sum_{\sigma\in\mathbb{S}_2}\mathrm{sgn}(\sigma)\Big(\lambda_1 1_{\substack{((\overline{\rho}_{\sigma(1)},\rho_{\sigma(1)},\mathbf{x}_{\sigma(1)},\xi_{\sigma(1)}),(\overline{\rho}_{\sigma(2)},\rho_{\sigma(2)},\mathbf{x}_{\sigma(2)},\xi_{\sigma(2)}))\\ =((1,\hat{\rho},r_L(\hat{\mathbf{x}}),1),(2,\hat{\rho},r_L(\hat{\mathbf{x}}),-1))}}\\
&\qquad\qquad + \lambda_2 1_{\substack{((\overline{\rho}_{\sigma(1)},\rho_{\sigma(1)},\mathbf{x}_{\sigma(1)},\xi_{\sigma(1)}),(\overline{\rho}_{\sigma(2)},\rho_{\sigma(2)},\mathbf{x}_{\sigma(2)},\xi_{\sigma(2)}))\\ =((1,\hat{\rho},r_L(\hat{\mathbf{x}}),1),(1,\hat{\rho},r_L(\hat{\mathbf{x}}),-1))}} 1_{(\hat{\rho},r_L(\hat{\mathbf{x}}))=(\hat{\eta},r_L(\hat{\mathbf{y}}))}\Big),\\
&V_4^{1,0}(\boldsymbol{\lambda})(\overline{\rho}_1\rho_1\mathbf{x}_1s_1\xi_1, \overline{\rho}_2\rho_2\mathbf{x}_2s_2\xi_2, \overline{\rho}_3\rho_3\mathbf{x}_3s_3\xi_3, \overline{\rho}_4\rho_4\mathbf{x}_4s_4\xi_4)\\
&:= -\frac{h^3}{4!}\lambda_2 1_{s_1=s_2=s_3=s_4}\sum_{\sigma\in\mathbb{S}_4}\mathrm{sgn}(\sigma)\\
&\cdot 1_{\substack{((\overline{\rho}_{\sigma(1)},\rho_{\sigma(1)},\mathbf{x}_{\sigma(1)},\xi_{\sigma(1)}),(\overline{\rho}_{\sigma(2)},\rho_{\sigma(2)},\mathbf{x}_{\sigma(2)},\xi_{\sigma(2)}),(\overline{\rho}_{\sigma(3)},\rho_{\sigma(3)},\mathbf{x}_{\sigma(3)},\xi_{\sigma(3)}),(\overline{\rho}_{\sigma(4)},\rho_{\sigma(4)},\mathbf{x}_{\sigma(4)},\xi_{\sigma(4)}))\\ =((1,\hat{\rho},r_L(\hat{\mathbf{x}}),1),(2,\hat{\rho},r_L(\hat{\mathbf{x}}),-1),(2,\hat{\eta},r_L(\hat{\mathbf{y}}),1),(1,\hat{\eta},r_L(\hat{\mathbf{y}}),-1))}}.
\end{aligned}$$

Remind us that the map $r_L : \Gamma_\infty \to \Gamma$ was defined just before the statement of Theorem 1.3 in Subsection 1.2. We can confirm that

$$\begin{aligned}
&V^{1,0}(\boldsymbol{\lambda})(\psi)\\
&= -\frac{\lambda_1}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\hat{\rho}r_L(\hat{\mathbf{x}})s}\psi_{2\hat{\rho}r_L(\hat{\mathbf{x}})s}\\
&\quad - 1_{(\hat{\rho},r_L(\hat{\mathbf{x}}))=(\hat{\eta},r_L(\hat{\mathbf{y}}))}\frac{\lambda_2}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\hat{\rho}r_L(\hat{\mathbf{x}})s}\psi_{1\hat{\rho}r_L(\hat{\mathbf{x}})s}\\
&\quad - \frac{\lambda_2}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\hat{\rho}r_L(\hat{\mathbf{x}})s}\psi_{2\hat{\rho}r_L(\hat{\mathbf{x}})s}\overline{\psi}_{2\hat{\eta}r_L(\hat{\mathbf{y}})s}\psi_{1\hat{\eta}r_L(\hat{\mathbf{y}})s}.
\end{aligned} \tag{3.88}$$

As the second goal of this subsection we construct an analytic continuation of the $\bigwedge_{even}\mathcal{V}$-valued function

$$(u,\boldsymbol{\lambda}) \mapsto \log\left(\int e^{V^{0-1,0}(u)(\psi^0+\psi)+V^{0-2,0}(u)(\psi^0+\psi)+V^{1,0}(\boldsymbol{\lambda})(\psi^0+\psi)}d\mu_{\mathcal{C}_0}(\psi^0)\right)$$

in a neighborhood of the origin. The mission is seemingly close to that in [13, Subsection 3.5]. However, the fact that the covariance is independent of the time variables makes non-trivial differences in analysis. Let us introduce sets of $\bigwedge_{even}\mathcal{V}$-valued functions in order to concisely describe properties of the output of this single-scale integration. Let $r, r' \in \mathbb{R}_{>0}$. We use the norm $\|\cdot\|_{1,r,r'}$ on $C(\overline{D(r)}\times\overline{D(r')}^2,\mathbb{C})$ and $C(\overline{D(r)}\times\overline{D(r')}^2,\mathrm{Map}(I^m,\mathbb{C}))$ and the measurement $[\cdot,\cdot]_{1,r,r'}$ for a coupling between a function belonging to $C(\overline{D(r)}\times\overline{D(r')}^2,\mathrm{Map}(I^m,\mathbb{C}))$ and an anti-symmetric function on I^2. The definition of these notions is found in [14, Subsection 4.5]. We define the subset $\mathcal{Q}'(r,r')$ of $\mathrm{Map}(\overline{D(r)}\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V})$ as follows.

$f\in\mathcal{Q}'(r,r')$ if and only if

- $$f\in C\left(\overline{D(r)}\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right)\cap C^{\omega}\left(D(r)\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right).$$
- For any $u\in\overline{D(r)}$, $\boldsymbol{\lambda}\mapsto f(u,\boldsymbol{\lambda})(\psi):\mathbb{C}^2\to\bigwedge_{even}\mathcal{V}$ is linear.
- For any $(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\mathbb{C}^2$ the anti-symmetric kernels $f(u,\boldsymbol{\lambda})_m:I^m\to\mathbb{C}$ $(m=2,4,\cdots,N)$ satisfy (3.49) and

$$\|f_0\|_{1,r,r'}\le\alpha^{-1}L^{-d},\quad\sum_{m=2}^{N}c_0^{\frac{m}{2}}\alpha^m\|f_m\|_{1,r,r'}\le L^{-d}.\tag{3.89}$$

We also need a set of $\bigwedge_{even}\mathcal{V}$-valued functions with bi-anti-symmetric kernels. Let us define the set $\mathcal{R}'(r,r')$ as follows.

$f\in\mathcal{R}'(r,r')$ if and only if

- $$f\in C\left(\overline{D(r)}\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right)\cap C^{\omega}\left(D(r)\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right).$$
- For any $u\in\overline{D(r)}$, $\boldsymbol{\lambda}\mapsto f(u,\boldsymbol{\lambda})(\psi):\mathbb{C}^2\to\bigwedge_{even}\mathcal{V}$ is linear.
- There exist $f_{p,q}\in C(\overline{D(r)}\times\mathbb{C}^2,\mathrm{Map}(I^p\times I^q,\mathbb{C}))$ $(p,q=2,4,\cdots,N)$ such that for any $(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\mathbb{C}^2$, $p,q\in\{2,4,\cdots,N\}$ $f_{p,q}(u,\boldsymbol{\lambda}):I^p\times I^q\to\mathbb{C}$ is bi-anti-symmetric, satisfies (3.49), (3.50),

$$f(u,\boldsymbol{\lambda})(\psi)=\sum_{p,q=2}^{N}1_{p,q\in2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}f_{p,q}(u,\boldsymbol{\lambda})(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}$$

and

$$\sum_{p,q=2}^{N} c_0^{\frac{p+q}{2}} \alpha^{p+q} \|f_{p,q}\|_{1,r,r'} \le 1, \tag{3.90}$$

$$\sum_{p,q=2}^{N} c_0^{\frac{p+q}{2}} \alpha^{p+q} [f_{p,q}, g]_{1,r,r'} \le (\|g\|'_{1,\infty} + AB^{-1}\|g\|_{1,\infty}) L^{-d} \tag{3.91}$$

for any anti-symmetric function $g : I^2 \to \mathbb{C}$.

We must prepare a set which can contain the direct descent from $V^{1,0}$. The definition is as below.

$f \in \mathcal{S}(r, r')$ if and only if

- $$f \in C\left(\overline{D(r)} \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V}\right) \cap C^{\omega}\left(D(r) \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V}\right).$$
- For any $u \in \overline{D(r)}$, $\boldsymbol{\lambda} \mapsto f(u, \boldsymbol{\lambda})(\psi) : \mathbb{C}^2 \to \bigwedge_{even} \mathcal{V}$ is linear.
- For any $(u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \mathbb{C}^2$ the anti-symmetric kernels $f(u, \boldsymbol{\lambda})_m : I^m \to \mathbb{C}$ $(m = 2, 4, \cdots, N)$ satisfy (3.49) and

$$\|f_0\|_{1,r,r'} \le \alpha^{-1}, \quad \sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|f_m\|_{1,r,r'} \le 1. \tag{3.92}$$

Finally we define a set of $\bigwedge_{even} \mathcal{V}$-valued functions depending on $\boldsymbol{\lambda}$ at least quadratically.

$f \in \mathcal{W}(r, r')$ if and only if

- $$f \in C\left(\overline{D(r)} \times \overline{D(r')}^2, \bigwedge_{even} \mathcal{V}\right) \cap C^{\omega}\left(D(r) \times D(r')^2, \bigwedge_{even} \mathcal{V}\right).$$
- For any $u \in D(r)$, $j \in \{1, 2\}$ $f(u, \mathbf{0})(\psi) = \frac{\partial}{\partial \lambda_j} f(u, \mathbf{0})(\psi) = 0$.
- For any $(u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \overline{D(r')}^2$ the anti-symmetric kernels $f(u, \boldsymbol{\lambda})_m$ $: I^m \to \mathbb{C}$ $(m = 2, 4, \cdots, N)$ satisfy (3.49) and

$$\|f_0\|_{1,r,r'} \le \alpha^{-1}, \quad \sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|f_m\|_{1,r,r'} \le 1. \tag{3.93}$$

Let us organize the Grassmann polynomials

$$\begin{aligned}&\frac{1}{n!}\left(\frac{d}{dz}\right)^n \\ &\quad\cdot \log\left(\int e^{zV^{0-1,0}(u)(\psi^0+\psi)+zV^{0-2,0}(u)(\psi^0+\psi)+zV^{1,0}(\boldsymbol{\lambda})(\psi^0+\psi)} d\mu_{\mathcal{C}_0}(\psi^0)\right)\Bigg|_{z=0}\end{aligned} \tag{3.94}$$

in the same way as in [13, Subsection 3.5]. The only difference from the previous work is that here the second superscript of the input polynomials is 0 and that of the output polynomials is 1. This is in accordance with the index of the covariances. Define $V^{0,0}$, $V^{0,1,(n)} \in \mathrm{Map}(\mathbb{C}, \bigwedge_{even} \mathcal{V})$ $(n \in \mathbb{N})$, $V^{1-3,1} \in \mathrm{Map}(\mathbb{C} \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V})$ by

$$\begin{aligned}
&V^{0,0}(u)(\psi) := V^{0-1,0}(u)(\psi) + V^{0-2,0}(u)(\psi),\\
&V^{0,1,(n)}(u)(\psi) := \frac{1}{n!} Tree(\{1, \cdots, n\}, \mathcal{C}_0) \prod_{j=1}^{n} V^{0,0}(u)(\psi^j + \psi)\Bigg|_{\substack{\psi^j=0\\(\forall j \in \{1,\cdots,n\})}},\\
&V^{1-3,1}(u, \boldsymbol{\lambda})(\psi) := Tree(\{1\}, \mathcal{C}_0) V^{1,0}(\boldsymbol{\lambda})(\psi^1 + \psi)\Big|_{\psi^1=0}.
\end{aligned}$$

Apparently $V^{1-3,1}$ is independent of u. However, by defining as if it depends on $(u, \boldsymbol{\lambda})$ we can estimate $V^{1-3,1}$ with the norm $\| \cdot \|_{1,r,r'}$. This saves us introducing another norm. For $n \in \mathbb{N}_{\ge 2}$ we define $V^{1-1-1,1,(n)}$, $V^{1-1-2,1,(n)}$, $V^{1-2,1,(n)}$, $V^{2,1,(n)} \in \mathrm{Map}(\mathbb{C} \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V})$ as follows.

$$\begin{aligned}
&V^{1-1-1,1,(n)}(u, \boldsymbol{\lambda})(\psi)\\
&:= \frac{1}{(n-1)!} Tree(\{1, \cdots, n\}, \mathcal{C}_0)\\
&\quad\cdot \prod_{j=1}^{n-1}\left(\sum_{b_j \in \{1,2\}} V^{0-b_j,0}(u)(\psi^j + \psi)\right) V^{1,0}(\boldsymbol{\lambda})(\psi^n + \psi)\Bigg|_{\substack{\psi^j=0\\(\forall j \in \{1,\cdots,n\})}}\\
&\quad\cdot 1_{\exists j (b_j = 1)},\\
&V^{1-1-2,1,(n)}(u, \boldsymbol{\lambda})(\psi)
\end{aligned}$$

$$:= \left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}^{0-2,0}(u)(\mathbf{X},\mathbf{Y}) \frac{1}{(n-1)!} Tree(\{1,\cdots,n+1\},\mathcal{C}_0)$$
$$\cdot (\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}$$
$$\cdot \prod_{j=3}^{n} V^{0-2,0}(u)(\psi^j+\psi)V^{1,0}(\boldsymbol{\lambda})(\psi^{n+1}+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,\cdots,n+1\})}},$$

$$V^{1-2,1,(n)}(u,\boldsymbol{\lambda})(\psi)$$
$$:= \frac{1}{(n-1)!}\sum_{m=0}^{n-1} \sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)} \left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}^{0-2,0}(u)(\mathbf{X},\mathbf{Y})$$
$$\cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C}_0)(\psi^{s_1}+\psi)_{\mathbf{X}}$$
$$\cdot \prod_{j=2}^{m+1}(1_{s_j\neq n}V^{0-2,0}(u)(\psi^{s_j}+\psi)+1_{s_j=n}V^{1,0}(\boldsymbol{\lambda})(\psi^{s_j}+\psi))\Bigg|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,\cdots,m+1\})}}$$
$$\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C}_0)(\psi^{t_1}+\psi)_{\mathbf{Y}}$$
$$\cdot \prod_{k=2}^{n-m}(1_{t_k\neq n}V^{0-2,0}(u)(\psi^{t_k}+\psi)+1_{t_k=n}V^{1,0}(\boldsymbol{\lambda})(\psi^{t_k}+\psi))\Bigg|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,\cdots,n-m\})}},$$

$$V^{2,1,(n)}(u,\boldsymbol{\lambda})(\psi)$$
$$:= \frac{1}{n!}Tree(\{1,\cdots,n\},\mathcal{C}_0)\prod_{j=1}^{n}\left(\sum_{b_j\in\{0,1\}} V^{b_j,0}(\psi^j+\psi)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,\cdots,n\})}}$$
$$\cdot 1_{\sum_{j=1}^n b_j\geq 2}.$$

Then, the following equality holds.

$$\begin{aligned}&(\text{The Grassmann polynomial (3.94)})\\ &= V^{0,1,(n)}(u)(\psi)+1_{n=1}V^{1-3,1}(u,\boldsymbol{\lambda})(\psi)\\ &\quad +1_{n\geq 2}(V^{1-1-1,1,(n)}(u,\boldsymbol{\lambda})(\psi)+V^{1-1-2,1,(n)}(u,\boldsymbol{\lambda})(\psi)\\ &\qquad +V^{1-2,1,(n)}(u,\boldsymbol{\lambda})(\psi)+V^{2,1,(n)}(u,\boldsymbol{\lambda})(\psi)).\end{aligned}$$

We should remark that the above decomposition is essentially same as that presented in [13, Subsection 3.5]. Assuming their convergence, we set

$$V^{0,1}(u)(\psi) := \sum_{n=1}^{\infty} V^{0,1,(n)}(u)(\psi),$$

$$V^{1-1-j,1}(u,\boldsymbol{\lambda})(\psi) := \sum_{n=2}^{\infty} V^{1-1-j,1,(n)}(u,\boldsymbol{\lambda})(\psi), \quad (\forall j \in \{1,2\}),$$

$$V^{1-1,1}(u,\boldsymbol{\lambda})(\psi) := \sum_{j=1}^{2} V^{1-1-j,1}(u,\boldsymbol{\lambda})(\psi),$$

$$V^{1-2,1}(u,\boldsymbol{\lambda})(\psi) := \sum_{n=2}^{\infty} V^{1-2,1,(n)}(u,\boldsymbol{\lambda})(\psi),$$

$$V^{2,1}(u,\boldsymbol{\lambda})(\psi) := \sum_{n=2}^{\infty} V^{2,1,(n)}(u,\boldsymbol{\lambda})(\psi).$$

We want to prove that these $\bigwedge_{even}\mathcal{V}$-valued functions are analytic with $(u,\boldsymbol{\lambda})$ in a neighborhood of the origin. In particular the analyticity with u must be ensured independently of A. We have developed the general estimates (3.52), (3.55), (3.60), (3.61) for this particular purpose.

Lemma 3.10. *There exists $c \in \mathbb{R}_{>0}$ independent of any parameter such that if $\alpha \geq c$,*

$$V^{1-1,1} \in \mathcal{Q}'(r,r'),\ V^{1-2,1} \in \mathcal{R}'(r,r'),\ V^{1-3,1} \in \mathcal{S}(r,r'),\ V^{2,1} \in \mathcal{W}(r,r')$$

with $r := c_0^{-2}\alpha^{-5}b^{-1}B^{-1}$, $r' := (A+1)^{-2}(B+1)^{-1}(\beta+1)^{-1}c_0^{-2}\alpha^{-5}$.

Proof. We will repeatedly use the following inequalities, which can be directly derived from the definitions.

$$\|V_2^{1,0}\|_{1,r,r'} \leq 2\beta r', \tag{3.95}$$

$$\sup_{\boldsymbol{\lambda}\in\overline{D(r')}^2} \|V_2^{1,0}(\boldsymbol{\lambda})\|_{1,\infty} \leq 2r', \tag{3.96}$$

$$\|V_4^{1,0}\|_{1,r,r'} \leq \beta r', \tag{3.97}$$

$$\sup_{\boldsymbol{\lambda}\in\overline{D(r')}^2} \|V_4^{1,0}(\boldsymbol{\lambda})\|_{1,\infty} \leq r'. \tag{3.98}$$

First let us summarize properties of $V^{1-3,1}$. By "(4.7)" of [14, Lemma 4.1] (or "(3.15)" of [13, Lemma 3.1]), (3.64), (3.95) and (3.97)

$$\|V_0^{1-3,1}\|_{1,r,r'} \leq c_0 A\|V_2^{1,0}\|_{1,r,r'} + (c_0A)^2\|V_4^{1,0}\|_{1,r,r'} \leq c\alpha^{-5}, \tag{3.99}$$

$$\|V_2^{1-3,1}\|_{1,r,r'} \le \|V_2^{1,0}\|_{1,r,r'} + cc_0A\|V_4^{1,0}\|_{1,r,r'} \le c(1+c_0A)\beta r'. \tag{3.100}$$

Since $V_4^{1-3,1} = V_4^{1,0}$, we can derive from (3.97), (3.100) that

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{1-3,1}\|_{1,r,r'} \le cc_0\alpha^2(1+c_0A)\beta r' + c_0^2\alpha^4\beta r' \le c\alpha^{-1}. \tag{3.101}$$

One part of the claims of [14, Lemma 4.1] or [13, Lemma 3.1] implies that the kernels of $V^{1-3,1}$ satisfy (3.49). The linearity with $\boldsymbol{\lambda} \in \mathbb{C}^2$ is clear from the definition. Therefore we can conclude from (3.99), (3.101) that if $\alpha \ge c$, $V^{1-3,1} \in \mathcal{S}(r,r')$.

Next let us consider $V^{1-1-1,1,(n)}$ ($n \in \mathbb{N}_{\ge 2}$). One can rewrite the defining equality as follows.

$$\begin{aligned}
&V^{1-1-1,1,(n)}(u,\boldsymbol{\lambda})(\psi)\\
&= \frac{1}{(n-1)!}\sum_{l=2}^{n}\begin{pmatrix} n-1\\ l-1\end{pmatrix} Tree(\{1,\cdots,n\},\mathcal{C}_0)V^{1,0}(\boldsymbol{\lambda})(\psi^1+\psi)\\
&\quad\cdot\prod_{j=2}^{l}V^{0-1,0}(u)(\psi^j+\psi)\prod_{k=l+1}^{n}V^{0-2,0}(u)(\psi^k+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,\cdots,n\})}}.
\end{aligned}$$

Then we can apply (3.52), (3.64), (3.65), (3.73), (3.75), (3.95) and (3.97) to derive that for $m \in \{0,2,4,\cdots,N\}$

$$\begin{aligned}
&\|V_m^{1-1-1,1,(n)}\|_{1,r,r'}\\
&\le c\sum_{l=2}^{n}\begin{pmatrix} n-1\\ l-1\end{pmatrix}(c_0A)^{-n+1-\frac{m}{2}}2^{-2m}(c_0B)^{n-1}\sum_{p\in\{2,4\}}2^{3p}(c_0A)^{\frac{p}{2}}\beta r'\\
&\quad\cdot(2^6c_0ArL^{-d})^{l-1}(2^{12}c_0^2A^2br)^{n-l}1_{p+2(n-l)\ge m\ge 2(n-l)}\\
&\le c\sum_{l=2}^{n}\begin{pmatrix} n-1\\ l-1\end{pmatrix}c_0^{-\frac{m}{2}}A^{-\frac{m}{2}}2^{-2m}\sum_{p\in\{2,4\}}2^{3p}(c_0A)^{\frac{p}{2}}\beta r'\\
&\quad\cdot(2^6c_0BrL^{-d})^{l-1}(2^{12}c_0^2ABbr)^{n-l}1_{p+2(n-l)\ge m\ge 2(n-l)}.
\end{aligned}$$

Therefore,

$$\|V_0^{1-1-1,1,(n)}\|_{1,r,r'} \le c\sum_{p\in\{2,4\}}2^{3p}(c_0A)^{\frac{p}{2}}\beta r'(2^6c_0BrL^{-d})^{n-1} \tag{3.102}$$

$$\leq cL^{-d}(2^6\alpha^{-5})^n,$$

(3.103)
$$\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m \|V_m^{1-1-1,1,(n)}\|_{1,r,r'}$$
$$\leq c\sum_{l=2}^{n}\left(\begin{array}{c} n-1 \\ l-1 \end{array}\right)\sum_{p\in\{2,4\}} 2^{3p}(c_0A)^{\frac{p}{2}}\beta r'$$
$$\cdot (2^6c_0BrL^{-d})^{l-1}(2^8\alpha^2c_0^2Bbr)^{n-l}(1+A^{-1}\alpha^2+1_{p=4}A^{-2}\alpha^4)$$
$$\leq cL^{-d}c_0^2\alpha^4(A+1)^2\beta r'(2^6c_0Br+2^8\alpha^2c_0^2Bbr)^{n-1}$$
$$\leq c\alpha^{-1}L^{-d}(2^9\alpha^{-3})^{n-1}.$$

Next let us study $V^{1-1-2,l,(n)}$ ($n \in \mathbb{N}_{\geq 2}$). In this case the main tool is the inequality (3.55). By combining (3.55) with (3.64), (3.65), (3.66), (3.75), (3.76), (3.95), (3.97) we observe that for any $m \in \{0, 2, 4, \cdots, N\}$

$$\|V_m^{1-1-2,1,(n)}\|_{1,r,r'}$$
$$\leq c(c_0A)^{-n-\frac{m}{2}}2^{-2m}(c_0B)^{n-1}(c_0A)^2rL^{-d}c_0A(2^{12}c_0^2A^2br)^{n-2}$$
$$\cdot \sum_{p\in\{2,4\}} 2^{3p}(c_0A)^{\frac{p}{2}}\beta r' 1_{p+2n-4\geq m\geq 2n-2}$$
$$\leq c(c_0A)^{-\frac{m}{2}}2^{-2m}c_0^2ABrL^{-d}(2^{12}c_0^2ABbr)^{n-2}$$
$$\cdot \sum_{p\in\{2,4\}} 2^{3p}(c_0A)^{\frac{p}{2}}\beta r' 1_{p+2n-4\geq m\geq 2n-2}.$$

Since $n \geq 2$, this implies that

(3.104)
$$\|V_0^{1-1-2,1,(n)}\|_{1,r,r'} = 0.$$

Moreover,

(3.105)
$$\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m \|V_m^{1-1-2,1,(n)}\|_{1,r,r'}$$
$$\leq cc_0^2\alpha^2BrL^{-d}(2^8c_0^2\alpha^2Bbr)^{n-2}\sum_{p\in\{2,4\}} 2^{3p}(c_0A)^{\frac{p}{2}}\beta r'(1+1_{p=4}A^{-1}\alpha^2)$$
$$\leq cL^{-d}(2^8\alpha^{-3})^n.$$

By summing up (3.102), (3.103), (3.104), (3.105) and assuming $\alpha^3 \ge 2^{10}$ we obtain that

$$\sum_{n=2}^{\infty}(\|V_0^{1-1-1,1,(n)}\|_{1,r,r'} + \|V_0^{1-1-2,1,(n)}\|_{1,r,r'}) \le c\alpha^{-10}L^{-d},$$
$$\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m \sum_{n=2}^{\infty}(\|V_m^{1-1-1,1,(n)}\|_{1,r,r'} + \|V_m^{1-1-2,1,(n)}\|_{1,r,r'}) \le c\alpha^{-4}L^{-d}.$$

These inequalities imply that $V^{1-1,1}$ is well-defined and

$$V^{1-1,1} \in C\left(\overline{D(r)} \times \overline{D(r')}^2, \bigwedge_{even} \mathcal{V}\right) \cap C^{\omega}\left(D(r) \times D(r')^2, \bigwedge_{even} \mathcal{V}\right).$$

By the definition $V^{1-1,1}$ is linear with $\boldsymbol{\lambda} \in \mathbb{C}^2$. It is implied by Lemma 3.5 and Lemma 3.6 that the kernels of $V^{1-1,1}$ satisfy (3.49). Thus by assuming $\alpha \ge c$ we can conclude from the above inequalities that $V^{1-1,1} \in \mathcal{Q}'(r, r')$.

Next let us analyze $V^{1-2,1,(n)}$ ($n \in \mathbb{N}_{\ge 2}$). Lemma 3.7 ensures the existence of bi-anti-symmetric functions $V_{a,b}^{1-2,1,(n)}(u, \boldsymbol{\lambda}) : I^a \times I^b \to \mathbb{C}$ ($n \in \mathbb{N}_{\ge 2}$, $a, b \in \{2, 4, \cdots, N\}$, $(u, \boldsymbol{\lambda}) \in \mathbb{C} \times \mathbb{C}^2$) such that they satisfy (3.49), (3.50) and

$$V^{1-2,1,(n)}(u, \boldsymbol{\lambda})(\psi)$$
$$= \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{a+b} \sum_{\substack{\mathbf{X}\in I^a \\ \mathbf{Y}\in I^b}} V_{a,b}^{1-2,1,(n)}(u, \boldsymbol{\lambda})(\mathbf{X}, \mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}.$$

It is clear from the definition that $\boldsymbol{\lambda} \mapsto V^{1-2,1,(n)}(u, \boldsymbol{\lambda})(\psi) : \mathbb{C}^2 \to \bigwedge_{even} \mathcal{V}$ is linear for any $n \in \mathbb{N}_{\ge 2}$, $u \in \mathbb{C}$. Once the uniform convergence of $\sum_{n=2}^{\infty} V^{1-2,1,(n)}(u, \boldsymbol{\lambda})(\psi)$ with $(u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \overline{D(r')}^2$ is proved, the properties (3.49), (3.50), the linearity with $\boldsymbol{\lambda}$ and the claimed regularity with $(u, \boldsymbol{\lambda})$ are automatically satisfied by $V^{1-2,1}$. Let us establish desirable norm bounds. The inequalities (3.60), (3.64), (3.65), (3.74), (3.75), (3.95), (3.97) lead to that for any $n \in \mathbb{N}_{\ge 2}$, $a, b \in \{2, 4, \cdots, N\}$

(3.106) $$\|V_{a,b}^{1-2,1,(n)}\|_{1,r,r'}$$
$$\le \frac{c}{(n-1)!} \sum_{m=0}^{n-1} \sum_{(\{s_j\}_{j=1}^{m+1}, \{t_k\}_{k=1}^{n-m}) \in S(n,m)}$$

$$
\begin{aligned}
&\cdot (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\
&\cdot 2^{-2a-2b}(c_0A)^{-n+1-\frac{1}{2}(a+b)}(c_0B)^{n-1}c_0^2A^2br\\
&\cdot \sum_{p\in\{2,4\}} 2^{3p}(c_0A)^{\frac{p}{2}}\beta r'(2^{12}c_0^2A^2br)^{n-2}\\
&\cdot \big(1_{n\in\{s_j\}_{j=2}^{m+1}}1_{2m-2+p\ge a\ge 2m}1_{b=2(n-m)}\\
&\quad + 1_{n\in\{t_k\}_{k=2}^{n-m}}1_{a=2m+2}1_{2(n-m)-4+p\ge b\ge 2(n-m)-2}\big)\\
\le{}& \frac{c}{(n-1)!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}\\
&\cdot (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\
&\cdot 2^{-2a-2b}(c_0A)^{-\frac{1}{2}(a+b)}\sum_{p\in\{2,4\}} 2^{3p}(c_0A)^{\frac{p}{2}}\beta r'(2^{12}c_0^2ABbr)^{n-1}\\
&\cdot \big(1_{n\in\{s_j\}_{j=2}^{m+1}}1_{2m-2+p\ge a\ge 2m}1_{b=2(n-m)}\\
&\quad + 1_{n\in\{t_k\}_{k=2}^{n-m}}1_{a=2m+2}1_{2(n-m)-4+p\ge b\ge 2(n-m)-2}\big).
\end{aligned}
$$

Thus

(3.107)

$$
\begin{aligned}
&\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}}c_0^{\frac{a+b}{2}}\alpha^{a+b}\|V_{a,b}^{1-2,1,(n)}\|_{1,r,r'}\\
&\le c\sum_{p\in\{2,4\}} 2^{3p}(c_0A)^{\frac{p}{2}}\beta r'(2^{12}c_0^2ABbr)^{n-1}2^{-4n}\alpha^{2n}A^{-n}(1+1_{p=4}\alpha^2A^{-1})\\
&\le cc_0^2\alpha^4(1+A)\beta r'(2^8c_0^2\alpha^2Bbr)^{n-1}\\
&\le c\alpha^{-1}(2^8\alpha^{-3})^{n-1}.
\end{aligned}
$$

On the other hand, by applying (3.61) instead of (3.60) and (3.66), (3.76) in addition we observe that for any $n\in\mathbb{N}_{\ge 2}$, $a,b\in\{2,4,\cdots,N\}$ and anti-symmetric function $g: I^2\to\mathbb{C}$,

$$
\begin{aligned}
&[V_{a,b}^{1-2,1,(n)},g]_{1,r,r'}\\
&\le \frac{c}{(n-1)!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\cdot 2^{-2a-2b}(c_0A)^{-n+1-\frac{1}{2}(a+b)}(c_0B)^{n-2}c_0^2A^2\\
&\cdot (rL^{-d}\|g\|'_{1,\infty}c_0B + rL^{-d}c_0A\|g\|_{1,\infty})\\
&\cdot \sum_{p\in\{2,4\}} 2^{3p}(c_0A)^{\frac{p}{2}}\beta r'(2^{12}c_0^2A^2br)^{n-2}\\
&\cdot \big(1_{n\in\{s_j\}_{j=2}^{m+1}}1_{2m-2+p\ge a\ge 2m}1_{b=2(n-m)}\\
&\quad + 1_{n\in\{t_k\}_{k=2}^{n-m}}1_{a=2m+2}1_{2(n-m)-4+p\ge b\ge 2(n-m)-2}\big)\\
\le\ & cL^{-d}(\|g\|'_{1,\infty} + AB^{-1}\|g\|_{1,\infty})\cdot(\text{R.H.S of (3.106)}).
\end{aligned}
$$

Therefore, by the same calculation as in (3.107) we reach that

$$
\begin{aligned}
&\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}}c_0^{\frac{a+b}{2}}\alpha^{a+b}[V_{a,b}^{1-2,1,(n)}, g]_{1,r,r'}\\
&\le cL^{-d}(\|g\|'_{1,\infty} + AB^{-1}\|g\|_{1,\infty})\alpha^{-1}(2^8\alpha^{-3})^{n-1}.
\end{aligned}
\tag{3.108}
$$

Assuming $\alpha^3 \ge 2^9$, we deduce from (3.107), (3.108) that

$$
\begin{aligned}
&\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}}c_0^{\frac{a+b}{2}}\alpha^{a+b}\sum_{n=2}^{\infty}\|V_{a,b}^{1-2,1,(n)}\|_{1,r,r'} \le c\alpha^{-4},\\
&\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}}c_0^{\frac{a+b}{2}}\alpha^{a+b}\sum_{n=2}^{\infty}[V_{a,b}^{1-2,1,(n)}, g]_{1,r,r'}\\
&\le cL^{-d}(\|g\|'_{1,\infty} + AB^{-1}\|g\|_{1,\infty})\alpha^{-4}.
\end{aligned}
$$

These inequalities enable us to conclude that if $\alpha \ge c$, $V^{1-2,1} \in \mathcal{R}'(r,r')$.

Finally let us treat $V^{2,1,(n)}$ ($n \in \mathbb{N}_{\ge 2}$). Observe that for any $n \in \mathbb{N}_{\ge 2}$, $(u,\boldsymbol{\lambda}) \in \mathbb{C}\times\mathbb{C}^2$

$$
\begin{aligned}
&V^{2,1,(n)}(u,\boldsymbol{\lambda})(\psi)\\
&= \frac{1}{n!}\sum_{l=2}^{n}\begin{pmatrix} n \\ l \end{pmatrix}\sum_{p=0}^{n-l}\begin{pmatrix} n-l \\ p \end{pmatrix} Tree(\{1,\cdots,n\},\mathcal{C}_0)\prod_{j=1}^{l}V^{1,0}(\boldsymbol{\lambda})(\psi^j+\psi)\\
&\quad\cdot\prod_{k=l+1}^{l+p}V^{0-1,0}(u)(\psi^k+\psi)\prod_{i=l+p+1}^{n}V^{0-2,0}(u)(\psi^i+\psi)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,\cdots,n\})}}.
\end{aligned}
$$

We can see from this equality that

$$V^{2,1,(n)} \in C^{\omega}\left(\mathbb{C}\times\mathbb{C}^2, \bigwedge_{even}\mathcal{V}\right),$$
$$V^{2,1,(n)}(u,\mathbf{0})(\psi) = \frac{\partial}{\partial\lambda_j}V^{2,1,(n)}(u,\mathbf{0})(\psi) = 0, \quad (\forall j\in\{1,2\},\ u\in\mathbb{C}).$$

Moreover, Lemma 3.5 guarantees that for any $(u,\boldsymbol{\lambda})\in\mathbb{C}\times\mathbb{C}^2$ the kernels of $V^{2,1,(n)}(u,\boldsymbol{\lambda})(\psi)$ satisfy (3.49). If a uniform convergence of $\sum_{n=2}^{\infty}V^{2,1,(n)}(u,\boldsymbol{\lambda})(\psi)$ with $(u,\boldsymbol{\lambda})$ in a neighborhood of the origin is established, then $V^{2,1}(u,\boldsymbol{\lambda})(\psi)$ will have the regularity with $(u,\boldsymbol{\lambda})$ and the other properties described above in the domain. Thus it suffices to prove suitable norm bounds which imply the desired convergence of $\sum_{n=2}^{\infty}V^{2,1,(n)}$ together with the claimed inequalities. We can combine (3.52) with (3.64), (3.65), (3.73), (3.75), (3.95), (3.96), (3.97), (3.98) to derive that for any $n\in\mathbb{N}_{\ge 2}$, $m\in\{0,2,\cdots,N\}$

$$\begin{aligned}
&\|V_m^{2,1,(n)}\|_{1,r,r'}\\
&\le c\sum_{l=2}^{n}\begin{pmatrix} n \\ l\end{pmatrix}\sum_{p=0}^{n-l}\begin{pmatrix} n-l \\ p\end{pmatrix}(c_0A)^{-n+1-\frac{m}{2}}2^{-2m}(c_0B)^{n-1}\\
&\quad\cdot\sum_{p_1\in\{2,4\}}2^{3p_1}(c_0A)^{\frac{p_1}{2}}\beta r'\\
&\quad\cdot\prod_{j=2}^{l}\left(\sum_{p_j\in\{2,4\}}2^{3p_j+1}(c_0A)^{\frac{p_j}{2}}r'\right)\prod_{k=l+1}^{l+p}(2^6c_0ArL^{-d})\prod_{i=l+p+1}^{n}(2^{12}c_0^2A^2br)\\
&\quad\cdot 1_{\sum_{j=1}^{l}p_j+2n-4l-2p+2\ge m\ge 2(n-l-p)}\\
&\le c2^{-2m}c_0^{-\frac{m}{2}}A^{-\frac{m}{2}}\sum_{l=2}^{n}\begin{pmatrix} n \\ l\end{pmatrix}\sum_{p=0}^{n-l}\begin{pmatrix} n-l \\ p\end{pmatrix}\sum_{p_1\in\{2,4\}}2^{3p_1}(c_0A)^{\frac{p_1}{2}}\beta r'\\
&\quad\cdot\prod_{j=2}^{l}\left(\sum_{p_j\in\{2,4\}}2^{3p_j+1}c_0^{\frac{p_j}{2}}A^{\frac{p_j}{2}-1}Br'\right)(2^6c_0BrL^{-d})^p(2^{12}c_0^2ABbr)^{n-l-p}\\
&\quad\cdot 1_{\sum_{j=1}^{l}p_j+2n-4l-2p+2\ge m\ge 2(n-l-p)}.
\end{aligned}$$

It follows that

(3.109)

$$
\begin{aligned}
&\|V_0^{2,1,(n)}\|_{1,r,r'}\\
&\le c\sum_{l=2}^{n}\left(\begin{array}{c} n \\ l \end{array}\right) c_0^2(A+1)^2\beta r'(2^{13}c_0^2(A+1)Br')^{l-1}(2^6c_0BrL^{-d})^{n-l}\\
&\le c\sum_{l=2}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)(2^{13}\alpha^{-5})^l(2^6\alpha^{-5})^{n-l}\\
&\le c(2^{14}\alpha^{-5})^n,
\end{aligned}
$$

(3.110)

$$
\begin{aligned}
&\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m\|V_m^{2,1,(n)}\|_{1,r,r'}\\
&\le c\sum_{l=2}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)\sum_{p=0}^{n-l}\left(\begin{array}{c} n-l \\ p \end{array}\right)\sum_{p_1\in\{2,4\}} 2^{3p_1}(c_0A)^{\frac{p_1}{2}}\beta r'\\
&\quad\cdot\prod_{j=2}^{l}\left(\sum_{p_j\in\{2,4\}} 2^{3p_j+1}c_0^{\frac{p_j}{2}}A^{\frac{p_j}{2}-1}Br'\right)(2^6c_0BrL^{-d})^p(2^8c_0^2\alpha^2Bbr)^{n-l-p}\\
&\quad\cdot(1+\alpha A^{-\frac{1}{2}})^{\sum_{j=1}^{l}p_j-2l+2}\\
&\le c\sum_{l=2}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)\sum_{p=0}^{n-l}\left(\begin{array}{c} n-l \\ p \end{array}\right)\sum_{p_1\in\{2,4\}} 2^{3p_1}(c_0A)^{\frac{p_1}{2}}\beta r'(1+\alpha A^{-\frac{1}{2}})^{p_1}\\
&\quad\cdot\left(\sum_{m\in\{2,4\}} 2^{3m+1}c_0^{\frac{m}{2}}A^{\frac{m}{2}-1}Br'(1+\alpha A^{-\frac{1}{2}})^{m-2}\right)^{l-1}(2^6c_0BrL^{-d})^p\\
&\quad\cdot(2^8c_0^2\alpha^2Bbr)^{n-l-p}\\
&\le c\sum_{l=2}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)c_0^2\alpha^4(A+1)^2\beta r'(2^{15}c_0^2\alpha^2(A+1)Br')^{l-1}\\
&\quad\cdot(2^6c_0BrL^{-d}+2^8c_0^2\alpha^2Bbr)^{n-l}\\
&\le c\alpha^2(2^{15}\alpha^{-3}+2^6\alpha^{-5}+2^8\alpha^{-3})^n\\
&\le c\alpha^2(2^{16}\alpha^{-3})^n.
\end{aligned}
$$

On the assumption $\alpha^3 \ge 2^{17}$, the inequalities (3.109), (3.110) yield that

$$\sum_{n=2}^{\infty} \|V_0^{2,1,(n)}\|_{1,r,r'} \le c\alpha^{-10}, \quad \sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \sum_{n=2}^{\infty} \|V_m^{2,1,(n)}\|_{1,r,r'} \le c\alpha^{-4}.$$

Assuming additionally that $\alpha \ge c$, we can conclude that $V^{2,1} \in \mathcal{W}(r, r')$. □

Using the results obtained in Lemma 3.9 and Lemma 3.10, we can construct an analytic continuation of the function

$$(3.111) \quad (u, \boldsymbol{\lambda}) \mapsto \log\left(\int e^{V^{0-1,0}(u)(\psi)+V^{0-2,0}(u)(\psi)+V^{1,0}(\boldsymbol{\lambda})(\psi)} d\mu_{\mathcal{C}_0+\mathcal{C}_1}(\psi)\right)$$

in a neighborhood of the origin. This can be achieved by integrating the output of the first integration with the covariance $\mathcal{C}_1$. We want to keep the analyticity with the variable u in the same domain as in Lemma 3.9, Lemma 3.10, while the domain of the artificial variable $\boldsymbol{\lambda}$ can be taken smaller. We only need estimates previously proved in [13, Subsection 3.2], [14, Subsection 4.2] for this purpose. We will not use the estimates presented in Subsection 3.2 in the rest of this paper. However, we need to argue differently from the previous final integration steps [13, Lemma 3.8], [14, Lemma 4.10], since here the final covariance $\mathcal{C}_1$ depends on time variables. Let

$$r = c_0^{-2}\alpha^{-5}b^{-1}B^{-1}, \quad r' = (A+1)^{-2}(B+1)^{-1}(\beta+1)^{-1}c_0^{-2}\alpha^{-5}$$

as we set in Lemma 3.10. Then let us define the functions $V^{end,(n)}$, $V^{1-3,end}$: $\overline{D(r)} \times \overline{D(r')}^2 \to \mathbb{C}$ $(n \in \mathbb{N})$ by

$$\begin{aligned} &V^{end,(n)}(u, \boldsymbol{\lambda}) \\ &:= \frac{1}{n!} Tree(\{1, \cdots, n\}, \mathcal{C}_1) \\ &\quad \cdot \prod_{j=1}^{n} \left(\sum_{m=1}^{2} V^{0-m,1}(u)(\psi^j) + \sum_{k=1}^{3} V^{1-k,1}(u, \boldsymbol{\lambda})(\psi^j) + V^{2,1}(u, \boldsymbol{\lambda})(\psi^j) \right) \\ &\quad \cdot \Big|_{\substack{\psi^j = 0 \\ (\forall j \in \{1, \cdots, n\})}}, \end{aligned}$$

$$V^{1-3,end}(u,\boldsymbol{\lambda}) := Tree(\{1\},\mathcal{C}_1)V^{1-3,1}(u,\boldsymbol{\lambda})(\psi^1).$$

Moreover, we set

$$V^{end}(u,\boldsymbol{\lambda}) := \sum_{n=1}^{\infty} V^{end,(n)}(u,\boldsymbol{\lambda})$$

if it converges. By the definition and the division formula of Grassmann Gaussian integral (see e.g. [5, Proposition I.21]) one can check that V^{end} is an analytic continuation of the function (3.111) if it is proved to be analytic in a neighborhood of the origin. It is obvious that $V^{end,1-3}$ is actually independent of the variable u and linear with $\boldsymbol{\lambda} \in \mathbb{C}^2$. We write as if it depends on u only for notational consistency. The result is claimed as follows.

LEMMA 3.11. *There exists $c \in \mathbb{R}_{>0}$ independent of any parameter such that if $\alpha \geq c$, $L^d \geq A+1$, the following statements hold.*

- $$V^{end} \in C\left(\overline{D(r)} \times \overline{D(\hat{r})}^2\right) \cap C^{\omega}\left(D(r) \times D(\hat{r})^2\right). \tag{3.112}$$

- $$\frac{h}{N} \sup_{u\in\overline{D(r)}} |V^{end}(u,\mathbf{0})| \leq (A+1)B^{-1}L^{-d}. \tag{3.113}$$

- $$\left|\frac{\partial}{\partial\lambda_j}V^{end}(u,\mathbf{0}) - \frac{\partial}{\partial\lambda_j}V^{1-3,end}(u,\mathbf{0})\right| \leq (A+1)^3(B+1)(\beta+1)c_0^2\alpha^5L^{-d}, \quad (\forall j\in\{1,2\},\ u\in D(r)). \tag{3.114}$$

Here

$$r = c_0^{-2}\alpha^{-5}b^{-1}B^{-1},$$
$$\hat{r} := 2^{-1}(h+1)^{-1}(A+1)^{-2}(B+1)^{-2}(\beta+1)^{-1}c_0^{-2}\alpha^{-5}.$$

Proof. The following inequalities will be often used. For $m \in \{0,2,\cdots,N\}$

$$\|V_m^{0-2,1}\|_{1,\infty,r} \le \sum_{p,q\in 2\mathbb{N}} 1_{p+q=m}\|V_{p,q}^{0-2,1}\|_{1,\infty,r}, \tag{3.115}$$

$$\|V_m^{1-2,1}\|_{1,r,r'} \le \sum_{p,q\in 2\mathbb{N}} 1_{p+q=m}\|V_{p,q}^{1-2,1}\|_{1,r,r'}. \tag{3.116}$$

The following inequality is essentially same as [13, (3.92)], [14, Lemma 4.9].

$$\begin{aligned}&\|V_m^{a,1}(u,\varepsilon\boldsymbol{\lambda})\|_{1,\infty} \le h\varepsilon\|V_m^{a,1}\|_{1,r,r'},\\ &(\forall u\in\overline{D(r)},\ \boldsymbol{\lambda}\in\overline{D(r')}^2,\ \varepsilon\in[0,1/2],\ a\in\{1-1,1-2,1-3,2\},\\ &\quad m\in\{0,2,\cdots,N\}).\end{aligned} \tag{3.117}$$

Cauchy's integral formula can be used to prove it in the case $a=2$. Set $\varepsilon := 2^{-1}(h+1)^{-1}(B+1)^{-1}$ so that $\varepsilon\in(0,1/2]$. We can deduce from "(3.16)" of [13, Lemma 3.1] (or "(4.8)" of [14, Lemma 4.1]), (3.64), (3.65), (3.68), (3.69), (3.89), (3.90), (3.92), (3.93), (3.115), (3.116), (3.117) and the assumption $\alpha\ge 2^3$ that for $n\in\mathbb{N}_{\ge 2}$, $(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\overline{D(r')}^2$

$$\begin{aligned}&|V^{end,(n)}(u,\varepsilon\boldsymbol{\lambda})|\\ &\le \frac{N}{h}c_0^{-n+1}(c_0B)^{n-1}\\ &\quad\cdot\Bigg(\sum_{p=2}^{N}2^{3p}c_0^{\frac{p}{2}}\Bigg(\|V_p^{0-1,1}\|_{1,\infty,r}+1_{p\ge4}\|V_p^{0-2,1}\|_{1,\infty,r}\\ &\qquad+\sum_{a\in\{1-1,1-3,2\}}\|V_p^{a,1}(u,\varepsilon\boldsymbol{\lambda})\|_{1,\infty}+1_{p\ge4}\|V_p^{1-2,1}(u,\varepsilon\boldsymbol{\lambda})\|_{1,\infty}\Bigg)\Bigg)^n\\ &\le\frac{N}{h}B^{n-1}\big(2^6\alpha^{-2}(A+1)B^{-1}L^{-d}+2^{12}\alpha^{-4}B^{-1}\\ &\qquad\qquad+3\cdot2^6\alpha^{-2}h\varepsilon+2^{12}\alpha^{-4}h\varepsilon\big)^n\\ &\le\frac{N}{h}B^{-1}(2^{14}\alpha^{-2})^n.\end{aligned}$$

In the last inequality we also used that $L^d\ge A+1$, $h\varepsilon\le B^{-1}$. Thus, if $\alpha^2\ge 2^{15}$,

$$\sum_{n=2}^{\infty}\sup_{(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\overline{D(\hat r)}^2}|V^{end,(n)}(u,\boldsymbol{\lambda})|<\infty,$$

which implies (3.112).

To derive (3.113), let us observe that for $n \in \mathbb{N}_{\ge 1}$, $u \in \overline{D(r)}$

(3.118)

$$\begin{aligned}
&V^{end,(n)}(u,\mathbf{0})\\
&= \frac{1}{n!} Tree(\{1,\cdots,n\},\mathcal{C}_1)\prod_{j=1}^{n}\left(\sum_{m=1}^{2} V^{0-m,1}(u)(\psi^j)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,\cdots,n\})}}\\
&= \sum_{l=1}^{n}\begin{pmatrix} n\\ l\end{pmatrix}\frac{1}{n!}Tree(\{1,\cdots,n\},\mathcal{C}_1)\\
&\quad \cdot \prod_{j=1}^{l} V^{0-1,1}(u)(\psi^j)\prod_{k=l+1}^{n} V^{0-2,1}(u)(\psi^k)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,\cdots,n\})}}\\
&\quad + \frac{1}{n!}Tree(\{1,\cdots,n+1\},\mathcal{C}_1)\sum_{p,q=2}^{N}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}} V_{p,q}^{0-2,1}(u)(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}^1\psi_{\mathbf{Y}}^2\\
&\qquad \cdot \prod_{j=3}^{n+1} V^{0-2,1}(u)(\psi^j)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,\cdots,n+1\})}}\\
&\quad + \frac{1}{n!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}\sum_{p,q=2}^{N}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}} V_{p,q}^{0-2,1}(u)(\mathbf{X},\mathbf{Y})\\
&\qquad \cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C}_1)\psi_{\mathbf{X}}^{s_1}\prod_{j=2}^{m+1} V^{0-2,1}(u)(\psi^{s_j})\Bigg|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,\cdots,m+1\})}}\\
&\qquad \cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C}_1)\psi_{\mathbf{Y}}^{t_1}\prod_{k=2}^{n-m} V^{0-2,1}(u)(\psi^{t_k})\Bigg|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,\cdots,n-m\})}}.
\end{aligned}$$

The above transformation is based on the same idea as that behind (3.72). By the properties (3.49), (3.50) of the kernels of $V^{0-2,1}$ and (3.63) the third term in the right-hand side of (3.118) vanishes. Then, combination of "(3.14)" of [13, Lemma 3.1], "(3.24)" of [13, Lemma 3.2] (or "(4.6)" of [14, Lemma 4.1], "(4.11)" of [14, Lemma 4.2]), (3.64), (3.65), (3.66), (3.68), (3.70) and the assumption $\alpha \ge 2^2$ yields that

$$\begin{aligned}
&\sup_{u\in\overline{D(r)}} |V^{end,(1)}(u,\mathbf{0})| \\
&\le \|V_0^{0-1,1}\|_{1,\infty,r} + \frac{N}{h}\sum_{m=2}^{N} c_0^{\frac{m}{2}} \|V_m^{0-1,1}\|_{1,\infty,r} \\
&\quad + \frac{N}{h}c_0^{-1}\sum_{p,q=2}^{N} 2^{2p+2q} c_0^{\frac{p+q}{2}} [V_{p,q}^{0-2,1}, \tilde{\mathcal{C}}_1]_{1,\infty,r} \\
&\le \frac{N}{h}\alpha^{-1}AB^{-1}L^{-d} + \frac{N}{h}\alpha^{-2}(A+1)B^{-1}L^{-d} + c\frac{N}{h}\alpha^{-4}AB^{-1}L^{-d} \\
&\le c\frac{N}{h}\alpha^{-1}(A+1)B^{-1}L^{-d}.
\end{aligned}$$

On the other hand, for $n \in \mathbb{N}_{\ge 2}$ we can use "(3.16)" of [13, Lemma 3.1], "(3.26)" of [13, Lemma 3.2] (or "(4.8)" of [14, Lemma 4.1], "(4.13)" of [14, Lemma 4.2]), (3.64), (3.65), (3.66), (3.68), (3.69), (3.70), (3.115) and the assumptions $\alpha \ge 2^3$, $L^d \ge A+1$ to derive that

$$\begin{aligned}
&\sup_{u\in\overline{D(r)}} |V^{end,(n)}(u,\mathbf{0})| \\
&\le \frac{N}{h}\sum_{l=1}^{n} \begin{pmatrix} n \\ l \end{pmatrix} c_0^{-n+1}(c_0B)^{n-1} \\
&\quad \cdot \left(\sum_{m=2}^{N} 2^{3m} c_0^{\frac{m}{2}} \|V_m^{0-1,1}\|_{1,\infty,r}\right)^{l} \left(\sum_{p=4}^{N} 2^{3p} c_0^{\frac{p}{2}} \|V_p^{0-2,1}\|_{1,\infty,r}\right)^{n-l} \\
&\quad + \frac{N}{h}c_0^{-n}(c_0B)^{n-1}\sum_{p,q=2}^{N} 2^{3p+3q} c_0^{\frac{p+q}{2}} [V_{p,q}^{0-2,1}, \tilde{\mathcal{C}}_1]_{1,\infty,r} \\
&\qquad \cdot \left(\sum_{m=4}^{N} 2^{3m} c_0^{\frac{m}{2}} \|V_m^{0-2,1}\|_{1,\infty,r}\right)^{n-1} \\
&\le \frac{N}{h}\sum_{l=1}^{n} \begin{pmatrix} n \\ l \end{pmatrix} B^{n-1}(2^6\alpha^{-2}(A+1)B^{-1}L^{-d})^l (2^{12}\alpha^{-4}B^{-1})^{n-l} \\
&\quad + c\frac{N}{h}\alpha^{-4}AB^{-1}L^{-d}(2^{12}\alpha^{-4})^{n-1} \\
&\le c\frac{N}{h}(A+1)B^{-1}L^{-d}(2^{13}\alpha^{-2})^n.
\end{aligned}$$

Therefore, on the assumption $\alpha^2 \ge 2^{14}$

$$\frac{h}{N}\sum_{n=1}^{\infty}\sup_{u\in\overline{D(r)}}|V^{end,(n)}(u,\mathbf{0})| \le c\alpha^{-1}(A+1)B^{-1}L^{-d},$$

which coupled with the further assumption $\alpha \ge c$ gives (3.113).

Finally let us prove (3.114). For any $u \in D(r)$, $j \in \{1,2\}$

$$\begin{aligned}
&\frac{\partial}{\partial\lambda_j}V^{end,(1)}(u,\mathbf{0}) - \frac{\partial}{\partial\lambda_j}V^{1-3,end}(u,\mathbf{0})\\
&= \frac{1}{r'}Tree(\{1\},\mathcal{C}_1)(V^{1-1,1}(u,r'\mathbf{e}_j)(\psi^1) + V^{1-2,1}(u,r'\mathbf{e}_j)(\psi^1))\\
&= \frac{1}{r'}Tree(\{1\},\mathcal{C}_1)V^{1-1,1}(u,r'\mathbf{e}_j)(\psi^1)\\
&\quad + \frac{1}{r'}Tree(\{1,2\},\mathcal{C}_1)\\
&\quad\quad \cdot \sum_{p,q=2}^{N}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}} V_{p,q}^{1-2,1}(u,r'\mathbf{e}_j)(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}^1\psi_{\mathbf{Y}}^2\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2\})}},
\end{aligned}$$

where $\mathbf{e}_1 := (1,0)$, $\mathbf{e}_2 := (0,1) \in \mathbb{R}^2$. To derive the last equality, we transformed the integral of $V^{1-2,1}$ in the same manner as in (3.118) and erased one part by taking into account the property (3.50) of the kernels of $V^{1-2,1}$ and (3.63). Moreover, by "(3.15)" of [13, Lemma 3.1], "(3.25)" of [13, Lemma 3.2] (or "(4.7)" of [14, Lemma 4.1], "(4.12)" of [14, Lemma 4.2]), (3.64), (3.65), (3.66), (3.89), (3.91) and the assumption $\alpha \ge 2^2$

$$\begin{aligned}
&\left|\frac{\partial}{\partial\lambda_j}V^{end,(1)}(u,\mathbf{0}) - \frac{\partial}{\partial\lambda_j}V^{1-3,end}(u,\mathbf{0})\right| \\
&\le \frac{1}{r'}\|V_0^{1-1,1}\|_{1,r,r'} + \frac{1}{r'}\sum_{m=2}^{N}c_0^{\frac{m}{2}}\|V_m^{1-1,1}\|_{1,r,r'}\\
&\quad + \frac{1}{r'}c_0^{-1}\sum_{p,q=2}^{N}2^{2p+2q}c_0^{\frac{p+q}{2}}[V_{p,q}^{1-2,1},\tilde{\mathcal{C}}_1]_{1,r,r'}\\
&\le \frac{c}{r'}\alpha^{-1}(A+1)L^{-d}.
\end{aligned} \tag{3.119}$$

Let $n \in \mathbb{N}_{\ge 2}$. Based on the properties (3.49), (3.50) of the kernels of $V^{0-2,1}$, the property (3.49) of the kernels of $V^{1-a,1}$ $(a = 1,2,3)$ and (3.63), we can transform the defining equality in the same way as above and obtain that for $u\in D(r)$, $j\in\{1,2\}$

$$\begin{aligned}
&\frac{\partial}{\partial\lambda_j}V^{end,(n)}(u,\mathbf{0})\\
&=\frac{1}{(n-1)!r'}Tree(\{1,\cdots,n\},\mathcal{C}_1)\\
&\quad\cdot\sum_{a=1}^{3}V^{1-a,1}(u,r'\mathbf{e}_j)(\psi^1)\prod_{k=2}^{n}\left(\sum_{p=1}^{2}V^{0-p,1}(u)(\psi^k)\right)\Bigg|_{\substack{\psi^i=0\\(\forall i\in\{1,\cdots,n\})}}\\
&=\frac{1}{(n-1)!r'}\sum_{l=2}^{n}\left(\begin{array}{c}n-1\\l-1\end{array}\right)Tree(\{1,\cdots,n\},\mathcal{C}_1)\\
&\quad\cdot\sum_{a=1}^{3}V^{1-a,1}(u,r'\mathbf{e}_j)(\psi^1)\prod_{k=2}^{l}V^{0-1,1}(u)(\psi^k)\prod_{s=l+1}^{n}V^{0-2,1}(u)(\psi^s)\Bigg|_{\substack{\psi^i=0\\(\forall i\in\{1,\cdots,n\})}}\\
&\quad+\frac{1}{(n-1)!r'}Tree(\{1,\cdots,n+1\},\mathcal{C}_1)\\
&\qquad\cdot\sum_{p,q=2}^{N}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\\mathbf{Y}\in I^q}}V^{0-2,1}_{p,q}(u)(\mathbf{X},\mathbf{Y})\psi^1_{\mathbf{X}}\psi^2_{\mathbf{Y}}\\
&\qquad\cdot\prod_{k=3}^{n}V^{0-2,1}(u)(\psi^k)\sum_{a=1}^{3}V^{1-a,1}(u,r'\mathbf{e}_j)(\psi^{n+1})\Bigg|_{\substack{\psi^i=0\\(\forall i\in\{1,\cdots,n+1\})}}.
\end{aligned}$$

In this situation we can apply "(3.17)" of [13, Lemma 3.1], "(3.27)" of [13, Lemma 3.2] (or "(4.9)" of [14, Lemma 4.1], "(4.14)" of [14, Lemma 4.2]), (3.64), (3.65), (3.66), (3.68), (3.69), (3.70), (3.89), (3.90), (3.92), (3.115), (3.116) and the inequalities $\alpha\ge 2^3$, $L^d\ge A+1$ to deduce that

$$\begin{aligned}
&\left|\frac{\partial}{\partial\lambda_j}V^{end,(n)}(u,\mathbf{0})\right|\\
&\le\frac{1}{r'}\sum_{l=2}^{n}\left(\begin{array}{c}n-1\\l-1\end{array}\right)c_0^{-n+1}(c_0B)^{n-1}\\
&\quad\cdot\sum_{p=2}^{N}2^{3p}c_0^{\frac{p}{2}}(\|V_p^{1-1,1}\|_{1,r,r'}+1_{p\ge 4}\|V_p^{1-2,1}\|_{1,r,r'}+\|V_p^{1-3,1}\|_{1,r,r'})
\end{aligned}$$

$$
\begin{aligned}
&\cdot \left(\sum_{q=2}^{N} 2^{3q} c_0^{\frac{q}{2}} \|V_q^{0-1,1}\|_{1,\infty,r} \right)^{l-1} \left(\sum_{m=4}^{N} 2^{3m} c_0^{\frac{m}{2}} \|V_m^{0-2,1}\|_{1,\infty,r} \right)^{n-l} \\
&+ \frac{1}{r'} c_0^{-n} (c_0 B)^{n-1} \sum_{p,q=2}^{N} 2^{3p+3q} c_0^{\frac{p+q}{2}} [V_{p,q}^{0-2,1}, \tilde{\mathcal{C}}_1]_{1,\infty,r} \\
&\quad \cdot \left(\sum_{m=4}^{N} 2^{3m} c_0^{\frac{m}{2}} \|V_m^{0-2,1}\|_{1,\infty,r} \right)^{n-2} \\
&\quad \cdot \left(\sum_{s=2}^{N} 2^{3s} c_0^{\frac{s}{2}} (\|V_s^{1-1,1}\|_{1,r,r'} + 1_{s\ge 4} \|V_s^{1-2,1}\|_{1,r,r'} + \|V_s^{1-3,1}\|_{1,r,r'}) \right) \\
&\le \frac{c}{r'} B^{n-1} \alpha^{-2} \sum_{l=2}^{n} \begin{pmatrix} n-1 \\ l-1 \end{pmatrix} (2^6 \alpha^{-2} (A+1) B^{-1} L^{-d})^{l-1} (2^{12} \alpha^{-4} B^{-1})^{n-l} \\
&\quad + \frac{c}{r'} B^{n-2} \alpha^{-6} A L^{-d} (2^{12} \alpha^{-4} B^{-1})^{n-2} \\
&\le \frac{c}{r'} (A+1) L^{-d} (2^{13} \alpha^{-2})^n.
\end{aligned}
$$

Thus by assuming that $\alpha^2 \ge 2^{14}$ we have that

$$
\sum_{n=2}^{\infty} \left| \frac{\partial}{\partial \lambda_j} V^{end,(n)}(u, \mathbf{0}) \right| \le \frac{c}{r'} \alpha^{-4} (A+1) L^{-d}. \tag{3.120}
$$

By coupling (3.119) with (3.120) and assuming that $\alpha \ge c$ once more we reach (3.114). □

3.4. The infinite-volume limit

Among all the lemmas prepared in this section so far, Lemma 3.1, Lemma 3.4, Lemma 3.11 are the main necessary tools to prove Theorem 1.3. With these lemmas we can straightforwardly follow the arguments of [14, Subsection 5.2] to complete the proof of Theorem 1.3. Though we should not lengthen the paper by repeating the same statements as before, let us state a few pivotal lemmas for the sake of readability. These are close to lemmas proved in [13], [14] but are adjusted to the present situation. Let us recall the definitions of $V(u)(\psi)$, $W(u)(\psi)$ given in the beginning of [14, Subsection 4.4] and $A^1(\psi)$, $A^2(\psi)$, $A(\psi)$ given in [14, Section 3]. It is apparent from (3.67), (3.88) that

$$
V^{0-1,0}(u)(\psi) + V^{0-2,0}(u)(\psi) = -V(u)(\psi) + W(u)(\psi),
$$

$$V^{1,0}(\boldsymbol{\lambda})(\psi) = -A(\psi).$$

A practical application of Lemma 3.1, Lemma 3.4, Lemma 3.11 results in the following lemma.

LEMMA 3.12. *Set*

$$\hat{A} := (e_{min} + \beta^{-1} + \beta^{-1}e_{min}^{-1} + 1)\max\{e_{min}^{-1}, e_{min}^{-d-1}\},$$
$$\hat{B} := \max\{e_{min}^{-1}, e_{min}^{-d-1}\}.$$

Then there exist $c \in \mathbb{R}_{>0}$ independent of any parameter and $\hat{c}_0 \in \mathbb{R}_{\ge 1}$ depending only on d, b, $(\hat{\mathbf{v}}_j)_{j=1}^d$, c_E such that the following statements hold for any $\alpha \in \mathbb{R}_{\ge 1}$, $h \in \frac{2}{\beta}\mathbb{N}$, $L \in \mathbb{N}$, $\phi \in \mathbb{C}$ satisfying that

$$\alpha \ge c, \quad L^d \ge \hat{A} + 1,$$
$$h \ge \max\{\sqrt{e_{max}^2 + |\phi|^2}, 1\} + \frac{1}{\beta}(3\pi + 2). \tag{3.121}$$

(i)

$$e^{-4b\beta(\hat{A}+1)\hat{B}^{-1}} \le \left|\int e^{-V(u)(\psi)+W(u)(\psi)} d\mu_{C(\phi)}(\psi)\right| \le e^{4b\beta(\hat{A}+1)\hat{B}^{-1}},$$
$$\left(\forall u \in \overline{D(\hat{c}_0^{-2}\alpha^{-5}b^{-1}\hat{B}^{-1})}\right).$$

(ii)

$$\left|\frac{\int e^{-V(u)(\psi)+W(u)(\psi)} A^j(\psi) d\mu_{C(\phi)}(\psi)}{\int e^{-V(u)(\psi)+W(u)(\psi)} d\mu_{C(\phi)}(\psi)} - \int A^j(\psi) d\mu_{C(\phi)}(\psi)\right|$$
$$\le (\hat{A}+1)^3(\hat{B}+1)(\beta+1)\hat{c}_0^2\alpha^5 L^{-d},$$
$$\left(\forall j \in \{1,2\},\ u \in \overline{D(\hat{c}_0^{-2}\alpha^{-5}b^{-1}\hat{B}^{-1})}\right).$$

PROOF. We take the generalized covariances $\mathcal{C}_0$, $\mathcal{C}_1$ to be C_0, C_1, which were analyzed in Subsection 3.1, respectively. We can see from Lemma 3.1, Lemma 3.4 that on the assumption (3.121) c_0, A, B can be taken to be $\hat{c}_0$, $\hat{A}$, $\hat{B}$ respectively. Accordingly the claims of Lemma 3.11 hold with $\hat{c}_0$, $\hat{A}$,

$\hat{B}$ in place of c_0, A, B. By using the relation (3.1) and the gauge transform $\psi_{\overline{\rho}\rho\mathbf{x}s\xi} \mapsto e^{-i\xi\frac{\pi}{\beta}s}\psi_{\overline{\rho}\rho\mathbf{x}s\xi}$ we can prove that if $|u|$, $\|\boldsymbol{\lambda}\|_{\mathbb{C}^2}$ are sufficiently small,

$$\begin{aligned}&\mathrm{Re}\int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi) > 0,\\ &V^{end}(u,\boldsymbol{\lambda}) = \log\left(\int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi)\right).\end{aligned}$$

For the proof of the above properties let us refer to the proof of [13, Lemma 4.13] or [14, Proposition 5.9] where a similar claim was proved. Then it follows from (3.112), the identity theorem and continuity that on the assumptions of this lemma

(3.122)

$$\begin{aligned}&e^{V^{end}(u,\boldsymbol{\lambda})} = \int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi),\\ &\Big(\forall(u,\boldsymbol{\lambda}) \in \overline{D(\hat{c}_0^{-2}\alpha^{-5}b^{-1}\hat{B}^{-1})}\\ &\qquad\times \overline{D(2^{-1}(h+1)^{-1}(\hat{A}+1)^{-2}(\hat{B}+1)^{-2}(\beta+1)^{-1}\hat{c}_0^{-2}\alpha^{-5})}^2\Big).\end{aligned}$$

On the other hand, by the definition and the same gauge transform as above

$$V^{1-3,end} = -\int A(\psi)d\mu_{C(\phi)}(\psi). \tag{3.123}$$

By combining (3.122), (3.123) with (3.113), (3.114) we can derive the claimed inequalities. □

The next lemma is essentially based on [13, Proposition 4.16]. The proof of [14, Proposition 5.10] can be read as a guide to deduce the lemma from [13, Proposition 4.16].

LEMMA 3.13. *Let $\hat{A}$, $\hat{B}$, c, $\hat{c}_0$ be those introduced in Lemma 3.12. Assume that $L^d \ge \hat{A}+1$ and $\alpha \ge c$. Then for any non-empty compact set Q of $\mathbb{C}$*

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi),$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)$$

converge in $C(Q\times\overline{D(2^{-1}\hat{c}_0^{-2}\alpha^{-5}b^{-1}\hat{B}^{-1})})$ *as sequences of functions of the variable* (ϕ,u). *Here we consider* $C(Q\times\overline{D(2^{-1}\hat{c}_0^{-2}\alpha^{-5}b^{-1}\hat{B}^{-1})})$ *as the Banach space with the uniform norm.*

Now we can describe how to derive the claims of Theorem 1.3 by following the final part of the proof of [14, Theorem 1.3] presented in [14, Subsection 5.2].

Proof of Theorem 1.3. The proof of the claims "(i), (ii), (iii), (iv), (v)" of [14, Theorem 1.3] straightforwardly applies to prove (i), (ii), (iii), (iv), (v) of Theorem 1.3 respectively. In the proof of [14, Theorem 1.3] the basic lemmas "Lemma 3.1", "Lemma 3.2", "Lemma 3.6", "Lemma 5.11" of [14] were frequently used. We should remark that here the same statements as these lemmas hold for any $\beta\in\mathbb{R}_{>0}$, $\theta\in\mathbb{R}$ including the case $\beta\theta/2\in\pi(2\mathbb{Z}+1)$. This is because in this paper the free partition function does not vanish for any $\theta\in\mathbb{R}$ thanks to the assumption (1.6). Let us fix $\alpha\in\mathbb{R}_{\ge 1}$ satisfying the condition $\alpha\ge c$ required in Lemma 3.12 and Lemma 3.13. Set $c':=4^{-1}\hat{c}_0^{-2}\alpha^{-5}$. We see that $c'\in(0,1]$, it depends only on d, b, $(\hat{\mathbf{v}}_j)_{j=1}^d$, c_E and

$$\left(-\frac{2c'}{b}\min\{e_{min},e_{min}^{d+1}\},0\right)\subset\overline{D(2^{-1}\hat{c}_0^{-2}\alpha^{-5}b^{-1}\hat{B}^{-1})}.$$

This means that the inequalities and the convergence properties stated in Lemma 3.12, Lemma 3.13 are applicable to the Grassmann integral formulation with the coupling constant

$$U\in\left(-\frac{2c'}{b}\min\{e_{min},e_{min}^{d+1}\},0\right).$$

Subsequently, for U belonging to this open interval the claims of Theorem 1.3 can be proved.

Here we only summarize which lemmas are necessary to conclude the claims of Theorem 1.3 if we straightforwardly follow the proof of [14, Theorem 1.3]. We avoid fully repeating the same arguments as before. The key point of translating the proof of [14, Theorem 1.3] into the proof of Theorem 1.3 is to replace "Proposition 5.9 (i),(ii)", "Proposition 5.10" of [14] by Lemma 3.12 (i),(ii), Lemma 3.13 respectively. We can prove (i), (iii), (iv), (v), (ii) in this order as in the proof of [14, Theorem 1.3].

(i): "Lemma 3.1", "Lemma 3.2", "Lemma 3.6 (i),(iii),(iv)" of [14], Lemma 3.12 (i) and Lemma 3.13 of this paper.

(iii): "Lemma 3.1", "Lemma 3.6 (i),(iii)", "Lemma 5.11", "Lemma A.1" of [14], Lemma 3.1 (i), Lemma 3.12 (i),(ii) and Lemma 3.13 of this paper.

(iv), (v): "Lemma 3.1", "Lemma 3.6 (i),(iii)", "Lemma 5.11", "Lemma A.2", "Lemma A.3" of [14], Lemma 3.1 (i), Lemma 3.12 (i),(ii) and Lemma 3.13 of this paper.

(ii): "Lemma 3.1", "Lemma 3.2", "Lemma 3.6 (iii)", "Lemma A.4" of [14]. □

Appendix A. A Special Matrix-Valued Function

Here we construct a matrix-valued function, which is used to prove that the function $\tau(\cdot)$ can have more than one local minimum points in Subsection 2.2.

Lemma A.1. *For any d, $b \in \mathbb{N}$, basis $(\hat{\mathbf{v}}_j)_{j=1}^d$ of $\mathbb{R}^d$, $s, t \in \mathbb{R}_{>0}$ satisfying $0 < s < t < 1$, e_{max}, $e_{min} \in \mathbb{R}_{>0}$ satisfying $0 < e_{min} < e_{max}$ there exists $E \in \mathcal{E}(e_{min}, e_{max})$ such that*

$$\begin{aligned} &D_d|\{\mathbf{k} \in \Gamma_\infty^* \mid \operatorname{Tr}|E(\mathbf{k})| = be_{max}\}| = s, \\ &D_d|\{\mathbf{k} \in \Gamma_\infty^* \mid \operatorname{Tr}|E(\mathbf{k})| = be_{min}\}| = 1 - t, \end{aligned}$$

where $|S|$ denotes the Lebesgue measure of a measurable set S $(\subset \mathbb{R}^d)$.

Proof. By a standard procedure one can construct a function ϕ $(\in C^\infty(\mathbb{R}))$ satisfying that

$$\begin{aligned} &\phi(x) = (e_{max} - e_{min})^{\frac{1}{d}} \text{ if } |x - \pi| \le \pi s^{\frac{1}{d}}, \\ &\phi(x) = 0 \text{ if } |x - \pi| \ge \pi t^{\frac{1}{d}}, \\ &\phi(x) \in (0, (e_{max} - e_{min})^{\frac{1}{d}}) \text{ if } \pi s^{\frac{1}{d}} < |x - \pi| < \pi t^{\frac{1}{d}}, \\ &\phi(\pi + x) = \phi(\pi - x), \quad (\forall x \in \mathbb{R}). \end{aligned} \tag{A.1}$$

Let us define the function Φ ($\in C^\infty(\mathbb{R}^d)$) by $\Phi(x_1,\cdots,x_d) := \prod_{j=1}^d \phi(x_j) + e_{min}$. Observe that

$$\Phi(x_1,\cdots,x_d) = e_{max} \text{ if } |x_j-\pi| \le \pi s^{\frac{1}{d}} \ (\forall j\in\{1,\cdots,d\}),$$
$$\Phi(x_1,\cdots,x_d) = e_{min} \text{ if } \exists j\in\{1,\cdots,d\} \text{ s.t. } |x_j-\pi| \ge \pi t^{\frac{1}{d}},$$
$$\Phi(x_1,\cdots,x_d) \in (e_{min}, e_{max}) \text{ otherwise.}$$

Then let us define the matrix-valued function $\hat{E} : \Gamma_\infty^* \to \mathrm{Mat}(b,\mathbb{C})$ by $\hat{E}(\mathbf{k}) := \Phi((\hat{\mathbf{v}}_1,\cdots,\hat{\mathbf{v}}_d)^{-1}\mathbf{k})I_b$ ($\mathbf{k}\in\Gamma_\infty^*$). We can periodically extend $\hat{E}$ to be a map from $\mathbb{R}^d$ to $\mathrm{Mat}(b,\mathbb{C})$. If E denotes the extension, it follows that $E\in\mathcal{E}(e_{min},e_{max})$. Let us confirm the property (1.5). The other properties are obvious. Take $\mathbf{k}\in\mathbb{R}^d$. There exist $\hat{k}_j\in[0,2\pi)$, $m_j\in\mathbb{Z}$ ($j=1,\cdots,d$) such that $\mathbf{k}=\sum_{j=1}^d(\hat{k}_j+2\pi m_j)\hat{\mathbf{v}}_j$. By the periodicity and (A.1),

$$\begin{aligned}\overline{E(-\mathbf{k})} &= E\left(\sum_{j=1}^d(2\pi-\hat{k}_j)\hat{\mathbf{v}}_j\right) = \Phi(2\pi-\hat{k}_1,\cdots,2\pi-\hat{k}_d)I_b\\ &= \Phi(\hat{k}_1,\cdots,\hat{k}_d)I_b = E(\mathbf{k}).\end{aligned}$$

Moreover, we can verify that

$$\begin{aligned}&D_d|\{\mathbf{k}\in\Gamma_\infty^* \mid \operatorname{Tr}|E(\mathbf{k})| = be_{max}\}|\\ &= D_d|\{\mathbf{k}\in\Gamma_\infty^* \mid (\hat{\mathbf{v}}_1,\cdots,\hat{\mathbf{v}}_d)^{-1}\mathbf{k}\in[\pi-\pi s^{\frac{1}{d}},\pi+\pi s^{\frac{1}{d}}]^d\}| = s,\\ &D_d|\{\mathbf{k}\in\Gamma_\infty^* \mid \operatorname{Tr}|E(\mathbf{k})| = be_{min}\}|\\ &= D_d|\{\mathbf{k}\in\Gamma_\infty^* \mid (\hat{\mathbf{v}}_1,\cdots,\hat{\mathbf{v}}_d)^{-1}\mathbf{k}\in[0,2\pi]^d\backslash(\pi-\pi t^{\frac{1}{d}},\pi+\pi t^{\frac{1}{d}})^d\}|\\ &= 1-t. \ \square\end{aligned}$$

Appendix B. A Definite Integral Formula

Here we derive an explicit formula of a definite integral, which is used in the proof of Proposition 2.26.

LEMMA B.1. *For* x, $t\in\mathbb{R}_{\ge 0}$

$$\frac{1}{2\pi}\int_0^{2\pi} dk \frac{1}{1+x(t(\cos k+1)+1)^2} \tag{B.1}$$

$$
\begin{aligned}
&= \left(((2t+1)^2x+1)^{\frac{1}{2}} + (x+1)^{\frac{1}{2}}\right) \\
&\quad \cdot \Big/ \Big(\sqrt{2}(x+1)^{\frac{1}{2}}((2t+1)^2x+1)^{\frac{1}{2}} \\
&\qquad \cdot \left((x+1)^{\frac{1}{2}}((2t+1)^2x+1)^{\frac{1}{2}} + (2t+1)x+1\right)^{\frac{1}{2}}\Big).
\end{aligned}
$$

PROOF. When $x=0$ or $t=0$, the equality obviously holds. Let us assume that $x>0$, $t>0$. One can prove by applying the residue theorem that

$$
\int_0^\infty ds \frac{1}{s^2 + re^{i\theta}} = \frac{\pi}{2} r^{-\frac{1}{2}} e^{-i\frac{\theta}{2}}, \quad (\forall r \in \mathbb{R}_{>0},\ \theta \in (0,\pi)). \tag{B.2}
$$

By introducing a new variable s by $s = \tan(k/2)$ we observe that

$$
\begin{aligned}
&(\text{L.H.S of (B.1)}) \\
&= \frac{2}{\pi x t^2} \int_0^\infty ds (1+s^2)^{-1} \left(\left(\frac{2}{1+s^2} + \frac{1}{t} \right)^2 + \frac{1}{xt^2} \right)^{-1} \\
&= \frac{1}{i\pi} \left((x^{\frac{1}{2}} - i)^{-1} \int_0^\infty ds \left(s^2 + \frac{(2t+1)x+1}{x+1} + \frac{2x^{\frac{1}{2}}t}{x+1} i \right)^{-1} \right. \\
&\qquad \left. - (x^{\frac{1}{2}} + i)^{-1} \int_0^\infty ds \left(s^2 + \frac{(2t+1)x+1}{x+1} - \frac{2x^{\frac{1}{2}}t}{x+1} i \right)^{-1} \right).
\end{aligned}
$$

Here we can apply (B.2) with

$$
r = \left(\frac{(2t+1)^2x+1}{x+1} \right)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left(\frac{2tx^{\frac{1}{2}}}{(2t+1)x+1} \right) \ \left(\in \left(0, \frac{\pi}{2}\right) \right)
$$

to derive that

$$
\begin{aligned}
(\text{L.H.S of (B.1)}) &= \frac{1}{i\pi} \left((x^{\frac{1}{2}} - i)^{-1} \frac{\pi}{2} r^{-\frac{1}{2}} e^{-i\frac{\theta}{2}} - (x^{\frac{1}{2}} + i)^{-1} \frac{\pi}{2} r^{-\frac{1}{2}} e^{i\frac{\theta}{2}} \right) \\
&= \frac{\cos\left(\frac{\theta}{2}\right)\left(1 - x^{\frac{1}{2}} \tan\left(\frac{\theta}{2}\right)\right)}{((2t+1)^2x+1)^{\frac{1}{4}} (x+1)^{\frac{3}{4}}}.
\end{aligned}
$$

Substitution of the equalities

$$\tan\left(\frac{\theta}{2}\right)=\frac{1}{2tx^{\frac{1}{2}}}\Big((x+1)^{\frac{1}{2}}((2t+1)^2x+1)^{\frac{1}{2}}-((2t+1)x+1)\Big),$$
$$\cos\left(\frac{\theta}{2}\right)=\frac{\Big((x+1)^{\frac{1}{2}}((2t+1)^2x+1)^{\frac{1}{2}}+(2t+1)x+1\Big)^{\frac{1}{2}}}{\sqrt{2}(x+1)^{\frac{1}{4}}((2t+1)^2x+1)^{\frac{1}{4}}}$$

leads to the right-hand side of (B.1). □

Supplementary List of Notations

Notation	Description	Reference
e_{min}	minimum of magnitude of free dispersion relation	Subsection 1.2
e_{max}	maximum of magnitude of free dispersion relation	Subsection 1.2
$\mathcal{E}(e_{min}, e_{max})$	set of matrix-valued functions	Subsection 1.2
$g_E(\cdot)$	real-valued function on $\mathbb{R}_{>0}\times\mathbb{R}\times\mathbb{R}$	Subsection 1.2
β_c	critical inverse temperature	Lemma 1.2
c_E	positive constant depending only on $E(\cdot)$	(1.8)

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