J. Math. Sci. Univ. Tokyo
28 (2021), 557–582.

Dirichlet Problem for Critical 2D Quasi-Geostrophic Equation with Large Data

By Tomasz DLOTKO, Tongtong LIANG and Yejuan WANG

Abstract. The 2D Quasi-geostrophic equation attracts attention of mathematicians through recent years; see for example [2, 7, 8, 10, 12, 13, 16, 17, 20, 21, 39]. While the sub-critical problems are rather standard (but not classical), the critical equation contains nonlinearity of the same order as the main dissipative half negative Laplace operator. Therefore we face a balance of the two terms in that case, which makes the problem interesting. We construct a weak solution of the critical problem, and associate it with a multivalued semiflow, since the solution may not be unique. A compact global attractor is shown to exist for that multivalued semiflow.

1. Introduction

In this paper, we consider Dirichlet problem for the critical Quasi-geostrophic equation, having the form

$$
\begin{aligned}
&\theta_t + u\cdot\nabla\theta + \kappa(-\Delta)^{\frac{1}{2}}\theta = f, \quad x\in\Omega,\ t>0,\\
&\theta = 0, \quad x\in\partial\Omega,\\
&\theta(0,x) = \theta_0(x),
\end{aligned}
\tag{1}
$$

where θ represents the potential temperature, $\kappa > 0$ is a diffusivity coefficient, f is a free force independent on time, and $u = (u_1, u_2)$ is the *velocity field* determined by θ through the relation:

$$
u = \big(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\big), \quad \text{where} \quad (-\Delta)^{\frac{1}{2}}\psi = -\theta,
\tag{2}
$$

or, in a more explicit way:

$$
u = \big(-\mathcal{R}_2\theta, \mathcal{R}_1\theta\big),
\tag{3}
$$

2010 *Mathematics Subject Classification.* 35S15, 35Q35, 35B41.
Key words: Quasi-geostrophic equation, fractional approximations, global attractor.

where $\mathcal{R}_i, i = 1, 2$, are the *Riesz transforms* in bounded domain Ω (see (4) for the definition).

Working in a bounded domain $\Omega \subset \mathbb{R}^N$ with C^2 boundary we are using the Balakrishnan/Komatsu definition of fractional powers of negative Laplace operator with Dirichlet boundary condition (see e.g. the Appendix). Through an analogy to the whole of $\mathbb{R}^N$ (e.g. [28, p. 299 (12.22)]), the Riesz transform in a bounded regular domain $\Omega \subset \mathbb{R}^2$ will be defined for $v \in D((-\Delta)^\beta))$ (domain of the β-power of negative Dirichlet Laplacian; see (7)), through the formula

$$(4)\quad \mathcal{R} = -\nabla(-\Delta)^{-\frac{1}{2}}, \quad \text{with components } \mathcal{R}_j = -\frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}, j = 1, 2.$$

A comment. Note that whenever $v \in D((-\Delta)^\beta), \beta \geq 0$, then $(-\Delta)^{-\frac{1}{2}} v \in D((-\Delta)^{\beta+\frac{1}{2}})$ and further, since any partial derivative will be extended through interpolation argument to a bounded linear operator from $H^{s+1}(\Omega)$ to $H^s(\Omega), s \in [0, \infty)$ (e.g. [27, Theorem 2.6]), we notify that $-\frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}} v \in H^{2\beta}(\Omega)$. Consequently, we observe that

$$(5)\qquad \mathcal{R}_j : D((-\Delta)^\beta) \to H^{2\beta}(\Omega), \quad j = 1, 2, \text{ whenever } \beta \geq 0.$$

The definitions and technique we are using here have a long history. In Remark 5.6 we will present shortly basic steps in the studies of fractional powers of sectorial positive operators and their application in the theory of abstract parabolic problems.

In our approach, we study first a family of *sub-critical problems* (6) with $\alpha \in (\frac{1}{2}, 1]$. Their solutions are obtained inside the semigroup theory, as in Dan Henry's monograph [19]. A weak solution of the *critical problem* (1) is constructed next as a limit of a sequence, when $\alpha_n \to \frac{1}{2}^+$, of the regular solutions to sub-critical problems. Since weak solutions of the critical problem (1) are eventually not unique, we are treating them as a *multivalued mapping*. Finally, following the result of [29], we construct a compact global attractor suitable for such a multivalued mapping.

The Cauchy problem for 2D Quasi-geostrophic equation was very popular through the last 20 years [2, 7, 12, 13, 16, 17, 20, 21, 39], while not many publications were devoted to the Dirichlet boundary value problem. We should mention here our earlier publications [15, 16] and a series of publications by P. Constantin and collaborators [8, 10, 30], all devoted mostly to existence considerations for sub-critical and critical problem (1). The latter three references contain interesting interior regularity considerations; the gradient $|\nabla_x \theta(t,x)|$ is bounded in $[0,T] \times \Omega'$, where Ω' is a strict subdomain of Ω, in terms of $|\nabla_x \theta_0|$ in Ω' and $\|\theta_0\|_{L^\infty(\Omega)}$. The latter papers are devoted to homogeneous problem (1), with $f = 0$. In such case (see e.g. [16, Lemma 5.7]) the $L^p(\Omega), 2 \leq p \leq \infty$ norms of solutions decay to zero as $t \to \infty$. We are considering further dynamics of the non-homogeneous problem (1), with $f \neq 0$.

Our approach here is a continuation of those in [6, 14], papers devoted to fractional generalization of the celebrated Navier-Stokes equations. The same technique based on approximation of the critical problem through its sub-critical approximations is used now to study (1) as a limit of equations (6) when $\alpha \to \frac{1}{2}$. But our present task is the asymptotic behavior of solutions to the critical problem (1), formulated in the language of *multivalued mappings* and the *global attractors*.

For $r \in \mathbb{R}$, symbol r^- denotes a number strictly less than r, but close to it; similarly, $r^+ > r$ and r^+ close to r. The constant c may vary from line to line.

2. Preliminaries

In this paper we are dealing with weak solutions of the critical Quasi-geostrophic equation, which are defined for arbitrary large data. Such approach allows to avoid involved study of the global regularity, as in the very interesting paper [2]. The eventual non-uniqueness of solutions is resolved by considering them as a *multivalued semiflow.* We recall some standard definitions for multivalued semiflow and some results ensuring the existence of a global attractor for these kinds of systems. We continue here the studies reported in [14, 15, 16, 17, 38] clarifying all the details in case of the Dirichlet problem (1). In particular we are considering less regular local solutions than in [16], as specified in Theorem 2.5, since more regular solutions are not available in our approach (see Observation 2.6).

Let X be a Banach space. By $\mathcal{P}(X)$ we denote the family of nonempty subsets of X and $\mathcal{B}(X)$ the family of nonempty and bounded subsets of X.

We also denote by $dist(A, B)$ the Hausdorff semi-distance, i.e., for given subsets A and B, we set

$$dist_X(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X.$$

DEFINITION 2.1. A family of multivalued mappings $S(t) : X \to \mathcal{P}(X)$, $t \in \mathbb{R}^+$, is called a *multivalued semiflow* (m-semiflow for short) if

(i) $S(0)x = \{x\}$, $\forall x \in X$;

(ii) $S(t + s)x \subset S(t)S(s)x$, $\forall t, s \in \mathbb{R}^+$, $\forall x \in X$.

If in (ii) we have the equality $S(t+s)x = S(t)S(s)x$ in place of the inclusion, then the m-semiflow is said to be *strict*.

DEFINITION 2.2. Let $\{S(t)\}_{t\geq 0}$ be an m-semiflow on X.

(1) A bounded set $B_0 \in \mathcal{B}(X)$ is said to be *an absorbing set for* $\{S(t)\}_{t\geq 0}$ if for every bounded set $B \subset X$, there exists a large time $T_0 = T_0(B) \in \mathbb{R}^+$ such that

$$\bigcup_{t \geq T_0} S(t)B \subset B_0.$$

(2) The m-semiflow $\{S(t)\}_{t\geq 0}$ is called *asymptotically upper semicompact* if for every bounded set $B \subset X$, each sequence $\xi_n \in S(t_n)B$ with $t_n \to \infty$ $(n \to \infty)$ is precompact in X.

DEFINITION 2.3. A nonempty compact subset $\mathcal{A}$ of X is called *a global attractor for the m-semiflow* $\{S(t)\}_{t\geq 0}$, if it satisfies

(i) $\mathcal{A}$ is negatively semiinvariant, i.e., $\mathcal{A} \subset S(t)\mathcal{A}$ for all $t \in \mathbb{R}^+$;

(ii) $\mathcal{A}$ uniformly attracts all bounded subsets B of X in X, i.e.,

$$\lim_{t \to \infty} dist_X(S(t)B, \mathcal{A}) = 0.$$

For further needs, we recall a general result on the existence and uniqueness of global attractors associated to m-semiflow, which was proved in [29].

THEOREM 2.4. *Let $\{S(t)\}_{t\geq 0}$ be an m-semiflow on X. Suppose that $S(t) : X \to \mathcal{P}(X)$ is upper semicontinuous for any $t \in \mathbb{R}^+$, that means; if $x_n \to x$ in X and $y_n \in S(t)x_n$, there exists $y \in S(t)x$ such that $y_n \to y$ as $n \to \infty$. Then $\{S(t)\}_{t\geq 0}$ has a unique global attractor $\mathcal{A}$ given by*

$$\mathcal{A} = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B_0}$$

if and only if $\{S(t)\}_{t\geq 0}$ is asymptotically upper semicompact and $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set $B_0 \subset X$.

2.1. Local in time solvability when $\alpha \in (\frac{1}{2}, 1]$

We start with the local existence result for sub-critical equations with $\alpha \in (\frac{1}{2}, 1]$. The Dirichlet problem for the Quasi-geostrophic equation is considered

$$\begin{aligned} &\theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha \theta = f, \quad x \in \Omega,\, t > 0,\\ &\theta = 0, \quad x \in \partial\Omega, \\ &\theta(0, x) = \theta_0(x), \end{aligned} \tag{6}$$

where $\kappa > 0$, $\alpha \in (\frac{1}{2}, 1]$, $u = \mathcal{R}^\perp\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta)$, and Ω is a bounded domain in $\mathbb{R}^2$ with C^2 boundary. Since we will be changing the value of parameter $\alpha \in (\frac{1}{2}, 1]$, to avoid confusion, the solutions of the problem (6) will be denoted by θ^α with the corresponding function u^α, while the notation θ and u will be reserved for solutions of the critical problem (6) with $\alpha = \frac{1}{2}$.

In the approach of solutions [3, 19], (6) can be written in an abstract form

$$\theta_t + \kappa(-\Delta)^\alpha\theta = F(\theta) := -u \cdot \nabla\theta + f,$$

which is located in the scale of Banach spaces; domains of fractional powers of the negative Dirichlet Laplacian $(-\Delta)^\beta, \beta \in \mathbb{R}$. The linear operator $(-\Delta)^\alpha$ acts between elements of that scale according to the formula

$$(-\Delta)^\alpha : D((-\Delta)^\beta) \to D((-\Delta)^{\beta-\alpha})), \quad \beta \in \mathbb{R}, \tag{7}$$

as an isometry (see e.g. [25, (3.9)], [37, Section 1.15.2]). We will also use the property ([25, p. 294], [27, p. 90]), that for $\gamma > \delta$ and $\delta < 0$, we will extend an operator $(-\Delta)^\gamma$ to be defined on $D((-\Delta)^\delta)$ (where $D((-\Delta)^\delta) \equiv \big(D((-\Delta)^{-\delta})\big)'$; the dual space), as

$$(-\Delta)^\gamma v := (-\Delta)^{\gamma-\delta}(-\Delta)^\delta v, \quad v \in D((-\Delta)^\delta).$$

We mention also, for further use, an estimate of the Riesz operator in a bounded domain:

$$\text{(8)} \quad \forall_{v \in D((-\Delta)^{\frac{j}{2}})} \quad \|D^j \mathcal{R}_i v\|_{L^q(\Omega)} \le c\|(-\Delta)^{\frac{j}{2}} v\|_{L^q(\Omega)},\ i=1,2,\ q\in(1,\infty),$$

with a constant $c = c(j) > 0$, where the symbol D^j represents any partial derivative of order $j \in \mathbb{N}$. The proof is given in Observation 5.4.

Recall further (e.g. [4, 25, 40]) characterization of the domains of fractional powers of $(-\Delta)$ with zero Dirichlet boundary condition, build on $L^2(\Omega)$:

$$\text{(9)} \qquad D((-\Delta)^{\frac{s}{2}}) = \begin{cases} H^s(\Omega) & \text{for} \quad 0 \le s < \frac{1}{2}, \\ H^s_{\{Id\}}(\Omega) & \text{if} \quad \frac{1}{2} < s < 1, \\ H^s(\Omega) \cap H^1_{\{Id\}}(\Omega) & \text{if} \quad 1 \le s < \frac{5}{2}, s \ne \frac{3}{2} \\ H^s(\Omega) \cap H^2_{\{Id,\Delta\}}(\Omega) & \text{if} \quad \frac{5}{2} < s < \frac{9}{2}, s \ne \frac{7}{2}, \end{cases}$$

where $H^k(\Omega)$ denotes the standard Sobolev spaces (see, e.g. [37, 40]), $H^1_{\{Id\}}(\Omega)$ stands for the subspace of $H^1(\Omega)$ consisting of functions with zero value (trace) on $\partial\Omega$, and $H^2_{\{Id,\Delta\}}(\Omega)$ stands for the subspace of $H^2(\Omega)$ of elements v fulfilling: $v = \Delta v = 0$ on $\partial\Omega$. For more complete descrption of the positive part of the scale corresponding to Dirichlet Laplacian, see e.g. [37, Section 4.3.3], [4, 31]; in particular Remark 5 in [4] (considering higher powers of the Laplacian one need to assume simultaneously higher regularity of $\partial\Omega$, usually the authors just set $\partial\Omega \in C^\infty$). Description (9) contains in particular boundary conditions required for elements on various levels of that scale.

We choose $L^2(\Omega)$ as a base space in which the equation (6) will be considered. For fixed $\alpha > \frac{1}{2}$ the space $D((-\Delta)^{\frac{s}{2}}) \subset H^s(\Omega), s \in (1, 2\alpha)$, will be the phase-space (in which the solution varies). Since the difference of the exponents fulfills $\frac{s}{2} - 0 < \alpha \in (\frac{1}{2}, 1]$, we see that: *"the nonlinear term*

F acts between elements of the scale of fractional power spaces associated with the main part linear operator $(-\Delta)^\alpha$ whose difference (of exponents) is strictly less than α". This is the necessary condition for local solvability in [3, 19].

The solutions, for arbitrary $\alpha \in (\frac{1}{2}, 1]$, will vary in the phase space $D((-\Delta)^{\frac{s}{2}})$, $s \in (1, 2\alpha)$. To justify local in time solvability of (6), we need to check (e.g. [3, 19]) that for $f \in L^2(\Omega)$ the nonlinearity $F(\theta^\alpha) = -u^\alpha \cdot \nabla\theta^\alpha + f$ is Lipschitz on bounded sets as a map from $D((-\Delta)^{\frac{s}{2}})$ into $D((-\Delta)^0) = L^2(\Omega)$ (the norms of $D((-\Delta)^{\frac{r}{2}})$ and $H^r(\Omega)$ are equivalent on $D((-\Delta)^{\frac{r}{2}})$). Indeed, for θ_1, θ_2 in a bounded set K in $D((-\Delta)^{\frac{s}{2}}), s \in (1, 2\alpha)$, we have

$$\begin{aligned} \|F(\theta_1) - F(\theta_2)\|_{L^2(\Omega)} &\le \|\mathcal{R}_2(\theta_1 - \theta_2)\frac{\partial\theta_1}{\partial x_1} + \mathcal{R}_2\theta_2\frac{\partial(\theta_1 - \theta_2)}{\partial x_1}\|_{L^2(\Omega)} \\ &\quad + \|\mathcal{R}_1(\theta_1 - \theta_2)\frac{\partial\theta_1}{\partial x_2} + \mathcal{R}_1\theta_2\frac{\partial(\theta_1 - \theta_2)}{\partial x_2}\|_{L^2(\Omega)}. \end{aligned} \tag{10}$$

First we estimate the first term on the right-hand side of (10). Since $s > 1$, using equivalent norms on the domains of fractional powers ($\|v\|_{H^r(\Omega)} \sim \|(-\Delta)^{\frac{r}{2}}v\|_{L^2(\Omega)}$ for $v \in D(-\Delta)^{\frac{r}{2}})$), we obtain

$$\begin{aligned} \|\mathcal{R}_2(\theta_1 - \theta_2)\frac{\partial\theta_1}{\partial x_1}\|_{L^2(\Omega)} &\le \|\mathcal{R}_2(\theta_1 - \theta_2)\|_{L^\infty(\Omega)}\|\frac{\partial\theta_1}{\partial x_1}\|_{L^2(\Omega)} \\ &\le c\|\mathcal{R}_2(\theta_1 - \theta_2)\|_{H^s(\Omega)}\|\frac{\partial\theta_1}{\partial x_1}\|_{L^2(\Omega)} \\ &\le c\|(-\Delta)^{-\frac{1}{2}}(\theta_1 - \theta_2)\|_{H^{s+1}(\Omega)}\|\theta_1\|_{H^1(\Omega)} \\ &\le c\|\theta_1 - \theta_2\|_{D((-\Delta)^{\frac{s}{2}})}\|\theta_1\|_{D((-\Delta)^{\frac{s}{2}})}, \end{aligned} \tag{11}$$

where u_i, $i = 1, 2$, correspond to θ_i through relation $u = \mathcal{R}^\perp\theta$.

For the second component on the right-hand side of (10) we have a similar estimate

$$\begin{aligned} \|\mathcal{R}_2\theta_2\frac{\partial(\theta_1 - \theta_2)}{\partial x_1}\|_{L^2(\Omega)} &\le c\|\mathcal{R}_2\theta_2\|_{L^\infty(\Omega)}\|\frac{\partial(\theta_1 - \theta_2)}{\partial x_1}\|_{L^2(\Omega)} \\ &\le c\|\mathcal{R}_2\theta_2\|_{H^s(\Omega)}\|\theta_1 - \theta_2\|_{H^1(\Omega)} \\ &\le c\|(-\Delta)^{-\frac{1}{2}}\theta_2\|_{H^{s+1}(\Omega)}\|\theta_1 - \theta_2\|_{H^1(\Omega)} \\ &\le c\|\theta_2\|_{D((-\Delta)^{\frac{s}{2}})}\|\theta_1 - \theta_2\|_{D((-\Delta)^{\frac{s}{2}})}. \end{aligned}$$

The other components in (10) are treated analogously. Consequently we obtain that for each $s \in (1, 2\alpha)$ (with a non-decreasing function c')

$$\|F(\theta_1) - F(\theta_2)\|_{L^2(\Omega)} \leq c'\big(\|\theta_1\|_{D((-\Delta)^{\frac{s}{2}})}, \|\theta_2\|_{D((-\Delta)^{\frac{s}{2}})}\big)\|\theta_1 - \theta_2\|_{D((-\Delta)^{\frac{s}{2}})}.$$

We have thus shown that $F : D((-\Delta)^{\frac{s}{2}}) \to L^2(\Omega)$ is Lipschitz on bounded sets, which proves local solvability of (6) in the phase space $D((-\Delta)^{\frac{s}{2}})$. More precisely, following [3, 19], we formulate:

Theorem 2.5. *Let $s \in (1, 2\alpha)$ be fixed. Then for $f \in L^2(\Omega)$ and any $\theta_0 \in D((-\Delta)^{\frac{s}{2}}) \subset H^s(\Omega)$, there exists a unique local in time mild solution θ^α to the problem* (6) *in the phase space $D((-\Delta)^{\frac{s}{2}})$. Moreover,*

$$\theta^\alpha \in C([0, \tau); D((-\Delta)^{\frac{s}{2}})) \cap C((0, \tau); D((-\Delta)^{\alpha})),$$
$$\theta^\alpha_t \in C((0, \tau); D((-\Delta)^{\gamma})),$$

with arbitrary $\gamma < \alpha$. Here $\tau > 0$ is the 'life time' of that local in time solution. Moreover, the Duhamel formula *is satisfied:*

$$\theta^\alpha(t) = e^{-A_\alpha t}\theta_0 + \int_0^t e^{-A_\alpha(t-s)}F(\theta^\alpha(s))ds,\ t \in [0, \tau),$$

where $e^{-A_\alpha t}$ denotes the linear semigroup corresponding to the operator $A_\alpha := (-\Delta)^\alpha$ on $D((-\Delta)^{\frac{s}{2}})$, and $F(\theta^\alpha) = -u^\alpha \cdot \nabla\theta^\alpha + f$.

Observation 2.6. As stated in (5), for $\beta \geq 0$ the Riesz operator 'switch' from the scale $D((-\Delta)^\beta)$ into the scale $H^{2\beta}(\Omega)$ ('erasing' the boundary condition). As a consequence of that observation, there exists a maximal regularity of the considered here local solutions $\theta^\alpha, \alpha \in (\frac{1}{2}, 1]$. We are allowed to choose, as a *base space*, a space $D((-\Delta)^\beta)$ as far as it coincides with $H^{2\beta}(\Omega)$; that is whenever $\beta < \frac{1}{4}$ (see (9)).

Recall that, because of the form of u^α, the following equality holds

$$u^\alpha \cdot \nabla\theta^\alpha = (-\mathcal{R}_2\theta^\alpha, \mathcal{R}_1\theta^\alpha) \cdot (\frac{\partial\theta^\alpha}{\partial x_1}, \frac{\partial\theta^\alpha}{\partial x_2}) = \nabla \cdot (u^\alpha\theta^\alpha). \tag{12}$$

A procedure, similar as in the proof of Theorem 2.5, allows us to build a local in time solution to (6) varying in the *phase space* $D((-\Delta)^{\frac{3}{4}^-})$. The key point is the Lipschitz continuity of the nonlinearity, on bounded sets

$B \subset D((-\Delta)^{\frac{3}{4}-\frac{\epsilon}{2}})$, $0<\epsilon<\min\{2\alpha-1,\frac{1}{2}\}$, into $H^{\frac{1}{2}-\epsilon}(\Omega) \equiv D((-\Delta)^{\frac{1}{4}-\frac{\epsilon}{2}})$. Indeed, if $\theta_1,\theta_2 \in B$ then

$$\begin{aligned}
&\|\nabla\cdot(u_1\theta_1)-\nabla\cdot(u_2\theta_2)\|_{H^{\frac{1}{2}-\epsilon}(\Omega)}\\
&\le \|\nabla\cdot(u_1\theta_1)-\nabla\cdot(u_1\theta_2)+\nabla\cdot(u_1\theta_2)-\nabla\cdot(u_2\theta_2)\|_{H^{\frac{1}{2}-\epsilon}(\Omega)}\\
&\le \|\nabla\cdot(u_1(\theta_1-\theta_2))\|_{H^{\frac{1}{2}-\epsilon}(\Omega)}+\|\nabla\cdot((u_1-u_2)\theta_2)\|_{H^{\frac{1}{2}-\epsilon}(\Omega)}\\
&\le c\left(\|u_1(\theta_1-\theta_2)\|_{H^{\frac{3}{2}-\epsilon}(\Omega)}+\|(u_1-u_2)\theta_2\|_{H^{\frac{3}{2}-\epsilon}(\Omega)}\right)\\
&\le c'\Big(\|u_1\|_{D((-\Delta)^{\frac{3}{4}-\frac{\epsilon}{2}})}\|\theta_1-\theta_2\|_{D((-\Delta)^{\frac{3}{4}-\frac{\epsilon}{2}})}\\
&\qquad+\|u_1-u_2\|_{D((-\Delta)^{\frac{3}{4}-\frac{\epsilon}{2}})}\|\theta_2\|_{D((-\Delta)^{\frac{3}{4}-\frac{\epsilon}{2}})}\Big),
\end{aligned}$$

where the property that ∇ is a bounded linear operator from $H^{\frac{3}{2}-\epsilon}(\Omega)$ into $H^{\frac{1}{2}-\epsilon}(\Omega)$ was used together with the fact that $H^{\frac{3}{2}-\epsilon}(\Omega)$ is a Banach algebra (since $\frac{3}{2}-\epsilon>1$).

Choosing $H^{\frac{1}{2}-\epsilon}(\Omega)$ as a *base space*, we thus obtain a local in time solution θ^α of (6) enjoing the following regularity properties

$$\begin{aligned}
&\theta^\alpha \in C([0,\tau);D((-\Delta)^{\frac{3}{4}-\frac{\epsilon}{2}}))\cap C((0,\tau);D((-\Delta)^{\alpha+\frac{1}{4}-\frac{\epsilon}{2}})),\\
&\theta^\alpha_t \in C((0,\tau);D((-\Delta)^{\gamma})),
\end{aligned}$$

where $\gamma<\alpha+\frac{1}{4}-\frac{\epsilon}{2}$. Note that the value $\alpha+\frac{1}{4}-\frac{\epsilon}{2}>\frac{3}{4}$ is allowed for ϵ near 0; consequently the constructed above local solutions starting at $\theta^\alpha_0 \in D((-\Delta)^{\frac{3}{4}-\frac{\epsilon}{2}})$ will be regularized into $H^{\frac{3}{2}}(\Omega)$ for $t>0$.

2.2. Natural a priori estimate of the Quasi-geostrophic equation

Note that, for solutions as in Theorem 2.5, after multiplying the nonlinear term $u^\alpha\cdot\nabla\theta^\alpha$ of (6) by θ^α and integrating over Ω, the resulting term will vanish:

$$\begin{aligned}
\int_\Omega u^\alpha\cdot\nabla\theta^\alpha\theta^\alpha dx &= \int_\Omega\left(\frac{\partial}{\partial x_2}(-\Delta)^{-\frac{1}{2}}\theta^\alpha\frac{\partial\theta^\alpha}{\partial x_1}-\frac{\partial}{\partial x_1}(-\Delta)^{-\frac{1}{2}}\theta^\alpha\frac{\partial\theta^\alpha}{\partial x_2}\right)\theta^\alpha dx\\
&= -\frac{1}{2}\int_\Omega\left(\frac{\partial(\theta^\alpha)^2}{\partial x_1}\frac{\partial}{\partial x_2}(-\Delta)^{-\frac{1}{2}}\theta^\alpha-\frac{\partial(\theta^\alpha)^2}{\partial x_2}\frac{\partial}{\partial x_1}(-\Delta)^{-\frac{1}{2}}\theta^\alpha\right)dx=0,
\end{aligned}$$

thanks to integration by parts. When $\theta^\alpha \in D((-\Delta)^{\frac{s}{2}}) \subset H^1_0(\Omega)$, $s\in(1,2\alpha)$, then $(-\Delta)^{-\frac{1}{2}}\theta^\alpha \in D((-\Delta)^{\frac{s+1}{2}}) \subset H^2(\Omega)$. Consequently, multiplying (6) by

θ^α, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta^\alpha\|^2_{L^2(\Omega)} + \kappa\|(-\Delta)^{\frac{\alpha}{2}}\theta^\alpha\|^2_{L^2(\Omega)} = \int_\Omega f\theta^\alpha dx. \tag{13}$$

Recall next the *generalized Poincaré inequality* (following e.g. [34, Chapter II, Section 3.2]):

$$\begin{aligned} &\lambda_1^\alpha \int_\Omega \phi^2 dx \le \int_\Omega \big[(-\Delta)^{\frac{\alpha}{2}}\phi\big]^2 dx, \quad \text{or more general} \\ &\lambda_1^{\beta-\gamma} \int_\Omega \big[(-\Delta)^{\frac{\gamma}{2}}\phi\big]^2 dx \le \int_\Omega \big[(-\Delta)^{\frac{\beta}{2}}\phi\big]^2 dx, \quad \forall \beta > \gamma, \end{aligned} \tag{14}$$

where λ_1 denotes the first eigenvalue of negative Laplacian in Ω. Equality (13) will be thus extended to a uniform in $\alpha \in [\frac{1}{2}, 1]$ estimate

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta^\alpha\|^2_{L^2(\Omega)} + \kappa\lambda_1^{\alpha-\frac{1}{2}}\|(-\Delta)^{\frac{1}{4}}\theta^\alpha\|^2_{L^2(\Omega)} &\le \int_\Omega f\theta^\alpha dx \\ &\le \|f\|_{H^{-\frac{1}{2}}(\Omega)}\|\theta^\alpha\|_{H^{\frac{1}{2}}(\Omega)}. \end{aligned} \tag{15}$$

Using the first inequality in (14) and the Young inequality to (13), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta^\alpha\|^2_{L^2(\Omega)} &\le -\frac{\kappa}{2}\|(-\Delta)^{\frac{\alpha}{2}}\theta^\alpha\|^2_{L^2(\Omega)} + \frac{1}{2\kappa\lambda_1^\alpha}\|f\|^2_{L^2(\Omega)} \\ &\le -\frac{\kappa\lambda_1^\alpha}{2}\|\theta^\alpha\|^2_{L^2(\Omega)} + \frac{1}{2\kappa\lambda_1^\alpha}\|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{16}$$

Applying the Gronwall lemma, we obtain

$$\begin{aligned} \|\theta^\alpha(t)\|^2_{L^2(\Omega)} &\le e^{-\kappa\lambda_1^\alpha t}\|\theta_0\|^2_{L^2(\Omega)} \\ &\quad + \frac{1}{\kappa^2\lambda_1^{2\alpha}}\|f\|^2_{L^2(\Omega)}(1 - e^{-\kappa\lambda_1^\alpha t}), \quad \text{for all } t \ge 0, \end{aligned} \tag{17}$$

and integrating both sides of the first inequality in (16) from 0 to T, we find that

$$\begin{aligned} &\|\theta^\alpha(T)\|^2_{L^2(\Omega)} + \kappa\int_0^T \|(-\Delta)^{\frac{\alpha}{2}}\theta^\alpha(t)\|^2_{L^2(\Omega)} dt \\ &\le \|\theta_0^\alpha\|^2_{L^2(\Omega)} + \frac{T}{\kappa\lambda_1^\alpha}\|f\|^2_{L^2(\Omega)}, \quad \forall T > 0. \end{aligned} \tag{18}$$

The latter gives us immediately the *uniform in* $\alpha \in [\frac{1}{2}, 1]$ estimate:

$$\theta^\alpha \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^\alpha(\Omega)). \tag{19}$$

2.3. Estimates in $L^q(\Omega)$, $2 \le q \le \infty$

Similarly as above, multiplying the nonlinear term $u^\alpha \cdot \nabla\theta^\alpha$ of (6) by $|\theta^\alpha|^{q-1} sgn(\theta^\alpha)$ with $q \ge 2$ and integrating the result over Ω, the resulting term will vanish:

$$\int_\Omega \left(\frac{\partial\theta^\alpha}{\partial x_1}\frac{\partial}{\partial x_2}[(-\Delta)^{-\frac{1}{2}}\theta^\alpha] - \frac{\partial\theta^\alpha}{\partial x_2}\frac{\partial}{\partial x_1}[(-\Delta)^{-\frac{1}{2}}\theta^\alpha] \right) |\theta^\alpha|^{q-1} sgn(\theta^\alpha)dx$$
$$= \frac{1}{q}\int_\Omega \left(\frac{\partial(|\theta^\alpha|^q)}{\partial x_1}\frac{\partial}{\partial x_2}[(-\Delta)^{-\frac{1}{2}}\theta^\alpha] - \frac{\partial(|\theta^\alpha|^q)}{\partial x_2}\frac{\partial}{\partial x_1}[(-\Delta)^{-\frac{1}{2}}\theta^\alpha] \right) dx = 0,$$

thanks to integration by parts. Consequently, multiplying (6) by $|\theta^\alpha|^{q-1} sgn(\theta^\alpha)$, we obtain

$$\begin{aligned} &\int_\Omega \theta^\alpha_t |\theta^\alpha|^{q-1} sgn(\theta^\alpha)dx + \kappa \int_\Omega (-\Delta)^\alpha\theta^\alpha |\theta^\alpha|^{q-1} sgn(\theta^\alpha)dx \\ &\quad = \int_\Omega f|\theta^\alpha|^{q-1} sgn(\theta^\alpha)dx. \end{aligned} \tag{20}$$

Applying the Kato-Beurling-Deny inequality (see Corollary 5.5) to (20), then using Hölder's and Young's inequalities, we obtain

$$\begin{aligned} &\frac{1}{q}\frac{d}{dt}\int_\Omega |\theta^\alpha|^q dx + \kappa\frac{4(q-1)}{q^2}\int_\Omega \left[(-\Delta)^{\frac{\alpha}{2}}(|\theta^\alpha|^{\frac{q}{2}})\right]^2 dx \\ &\quad \le \int_\Omega f|\theta^\alpha|^{q-1} sgn(\theta^\alpha)dx \le \frac{C_\epsilon}{q}\|f\|^q_{L^q(\Omega)} + \frac{\epsilon(q-1)}{q}\|\theta^\alpha\|^q_{L^q(\Omega)}, \end{aligned} \tag{21}$$

with arbitrary $\epsilon > 0$. By the first estimate of (14) we have

$$\kappa\frac{4(q-1)}{q^2}\int_\Omega [(-\Delta)^{\frac{\alpha}{2}}(|\theta^\alpha|^{\frac{q}{2}})]^2 dx \ge \lambda_1^\alpha\kappa\frac{4(q-1)}{q^2}\int_\Omega |\theta^\alpha|^q dx,$$

which, for a sufficiently small $\epsilon > 0$, extends (21) to

$$\frac{d}{dt}\int_\Omega |\theta^\alpha|^q dx + \lambda_1^\alpha\kappa\frac{2(q-1)}{q}\int_\Omega |\theta^\alpha|^q dx \le C_\epsilon\|f\|^q_{L^q(\Omega)}. \tag{22}$$

Solving the above differential inequality we get

$$\begin{aligned} &\|\theta^\alpha(t)\|^q_{L^q(\Omega)} \\ &\quad \le \left(\|\theta_0\|^q_{L^q(\Omega)} + C_\epsilon\|f\|^q_{L^q(\Omega)}\frac{e^{\lambda_1^\alpha\kappa\frac{2(q-1)}{q}t} - 1}{\lambda_1^\alpha\kappa\frac{2(q-1)}{q}} \right) e^{-\lambda_1^\alpha\kappa\frac{2(q-1)}{q}t}. \end{aligned} \tag{23}$$

Finally, taking first q-th roots, we will let $q \to \infty$ to get a uniform estimate for $t \in [0, T]$

$$\|\theta^\alpha(t)\|_{L^\infty(\Omega)} \leq \|\theta_0\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}. \tag{24}$$

2.4. Passing to the limit in weakly formulated approximating problems

The limit of functions θ^α will be denoted by θ, with the corresponding function $u = \mathcal{R}^\perp\theta$. We introduce next a *weak solution* to (6).

DEFINITION 2.7. A *weak solution* to (6) ($\alpha \in (\frac{1}{2}, 1]$) or (1) ($\alpha = \frac{1}{2}$), is a function

$$\theta \in L^2_{loc}(0, \infty; H^\alpha(\Omega)) \cap C_w([0, \infty); L^2(\Omega)), \tag{25}$$

such that for every $T > 0$ the following equality holds

$$\begin{aligned} &- \int_0^T \langle \theta(t), \phi_t(t) \rangle dt + \langle \theta(t), \phi(t) \rangle |_0^T \\ &= \int_0^T \langle \theta(t), u(t) \cdot \nabla\phi(t) \rangle dt - \int_0^T \langle (-\Delta)^{\frac{\alpha}{2}}\theta(t), (-\Delta)^{\frac{\alpha}{2}}\phi(t) \rangle dt \\ &\quad + \int_0^T \langle f, \phi(t) \rangle dt, \end{aligned} \tag{26}$$

for arbitrary *test function*

$$\phi \in C_0^\infty([0, T]; C_0^\infty(\Omega)).$$

Note that the regular solutions reported in Theorem 2.5 are evidently weak solutions in the specified above sense, so that for arbitrary $\alpha \in (\frac{1}{2}, 1]$ the problem (6) has a weak solution.

Through the described above natural a priori estimate (19) we have the, uniform in $\alpha \in (\frac{1}{2}, 1]$, boundedness of the approximating solutions θ^α:

$$\|\theta^\alpha\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta^\alpha\|_{L^2(0,T;H^{\frac{1}{2}}(\Omega))} \leq c, \tag{27}$$

and also, through (19),

$$\|\theta^\alpha\|_{L^2(0,T;H^\alpha(\Omega))} \leq c. \tag{28}$$

The last information allows us to let $\alpha \to \frac{1}{2}^+$ (over a sequence $\alpha_n \to \frac{1}{2}^+$, passing several times to a subsquences, $\{\theta^{\alpha_{n_k}}\}$) in a weakly formulated approximating equation (6).

Note first, that the uniform boundedness of θ^α in $L^\infty(0,T;L^2(\Omega))$ allows to pass to the limit in linear components in (26). Indeed, thanks to the weak* $L^\infty(0,T;L^2(\Omega))$ compactness

$$-\int_0^T \langle \theta^\alpha(t), \phi_t(t)\rangle dt \to -\int_0^T \langle \theta(t), \phi_t(t)\rangle dt, \quad \text{as } \alpha \to \frac{1}{2}^+.$$

For the nonlinear term, using Hölder's inequality, bilinearity of the nonlinear term and (8), we have

$$\begin{aligned}
&|\int_0^T \langle \theta^\alpha(t), u^\alpha(t)\cdot\nabla\phi(t)\rangle dt - \int_0^T \langle \theta(t), u(t)\cdot\nabla\phi(t)\rangle dt| \\
&\quad \le |\int_0^T \langle (\theta^\alpha(t)-\theta(t)), u^\alpha(t)\cdot\nabla\phi(t)\rangle dt| \\
&\qquad + |\int_0^T \langle \theta(t), (u(t)-u^\alpha(t))\cdot\nabla\phi(t)\rangle dt| \\
&\quad \le \|\theta^\alpha-\theta\|_{L^2(0,T;L^3(\Omega))}\|u^\alpha\|_{L^\infty(0,T;L^2(\Omega))}\|\nabla\phi\|_{L^2(0,T;L^6(\Omega))} \\
&\qquad + \|u^\alpha-u\|_{L^2(0,T;L^3(\Omega))}\|\theta\|_{L^\infty(0,T;L^2(\Omega))}\|\nabla\phi\|_{L^2(0,T;L^6(\Omega))} \\
&\quad \le c(\|\theta^\alpha\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{L^\infty(0,T;L^2(\Omega))}) \\
&\qquad \cdot \|\phi\|_{L^2(0,T;W^{1,6}(\Omega))}\|\theta^\alpha-\theta\|_{L^2(0,T;H^{\frac{1}{3}}(\Omega))}.
\end{aligned} \tag{29}$$

Since $u^\alpha = (-\mathcal{R}_2\theta^\alpha, \mathcal{R}_1\theta^\alpha)$, then $div\ u^\alpha = 0$ and the nonlinear term in (6) will be rewritten as

$$u^\alpha\cdot\nabla\theta^\alpha = \nabla\cdot(u^\alpha\theta^\alpha) - \theta^\alpha\nabla\cdot\ u^\alpha = \nabla\cdot(u^\alpha\theta^\alpha).$$

Further, thanks to the uniform in $\alpha \in (\frac{1}{2}, 1]$ estimates of θ^α ((27) and (28)), for nonlinearity given in that form, we have from (8) and Proposition 5.3 that

$$\begin{aligned}
\|\nabla\cdot(u^\alpha\theta^\alpha)\|_{H^{-1}(\Omega)} &= \|(-\Delta)^{-\frac{1}{2}}\nabla\cdot(u^\alpha\theta^\alpha)\|_{L^2(\Omega)} \le c\||u^\alpha\theta^\alpha|\|_{L^2(\Omega)} \\
&\le c\|u^\alpha\|_{L^4(\Omega)}\|\theta^\alpha\|_{L^4(\Omega)} \le c\|\theta^\alpha\|_{L^4(\Omega)}\|\theta^\alpha\|_{H^{\frac{1}{2}}(\Omega)},
\end{aligned} \tag{30}$$

where we have used the fact that $D((-\Delta)^{-\frac{1}{2}}) = H^{-1}(\Omega)$. Thus, it will be seen from equation (6) that θ^α_t are bounded in $L^1(0,T;H^{-1}(\Omega))$ uniformly in $\alpha \in (\frac{1}{2}, 1]$. Therefore, due to (19) and the Lions-Aubin compactness lemma [33, Corollary 4], the following strong convergence holds

$$\|\theta^\alpha - \theta\|_{L^2(0,T;H^{\frac{1}{3}}(\Omega))} \to 0,$$

as $\alpha \to \frac{1}{2}^+$, consequently it follows from (29) that

$$\int_0^T \langle \theta^\alpha(t), u^\alpha(t) \cdot \nabla\phi(t)\rangle dt \to \int_0^T \langle \theta(t), u(t) \cdot \nabla\phi(t)\rangle dt, \quad \text{as } \alpha \to \frac{1}{2}^+.$$

Weak continuity, $\theta \in C_w([0,T]; L^2(\Omega))$, follows from [36, Corollary 2.1] since $\theta \in L^\infty(0,T; L^2(\Omega))$ with $\theta_t \in L^1(0,T; H^{-1}(\Omega))$. Consequently the limit function $\theta \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^{\frac{1}{2}}(\Omega))$ fulfills the Definition 2.7 with $\alpha = \frac{1}{2}$, therefore it is a weak solution of the critical equation.

We are thus able to conclude.

Theorem 2.8. *When $\theta_0 \in D((-\Delta)^{\frac{s}{2}}) \subset H^s(\Omega)$, with $s \in (1, 2\alpha)$, and $f \in L^2(\Omega)$, then there exists a (not necessary unique which is connected with passing to the limits over different sequences) weak solution θ of the critical Quasi-geostrophic equation.*

Observation 2.9. The latter theorem will be generalized to cover a larger class of initial data. Inspecting the proof, it is evident that in the process of approximating weak solution of the critical problem (1), $\alpha = \frac{1}{2}$, we can choose a sequence of approximating solutions $\{\theta^{\alpha_n}\}$, $\alpha_n \to \frac{1}{2}$, in such a way that they correspond to initial data $\theta_0^{\alpha_n}$ satisfying (here $s_n \in (1, 2\alpha_n)$)

$$\theta_0^{\alpha_n} \in D((-\Delta)^{\frac{s_n}{2}}), \quad \theta_0^{\alpha_n} \to \theta_0 \text{ in } L^2(\Omega) \text{ as } \alpha_n \to \frac{1}{2}^+.$$

This is possible thanks to density of $D((-\Delta)^{\frac{s}{2}})$ in $L^2(\Omega)$ (e.g. [19, p.29]). For such solutions θ^{α_n} of (6), $\alpha = \alpha_n$, the uniform estimates (27) and (28) are still valid. While the whole range of initial data $\theta_0 \in L^2(\Omega)$ will be reached in such a construction (see similar consideration in [16, p.54]).

In the light of the latter observation we will sharpen Theorem 2.8 to the form which will be used below.

COROLLARY 2.10. *When $\theta_0 \in L^2(\Omega)$ and $f \in L^2(\Omega)$, there exists a (not necessary unique) global in time weak solution θ of the critical Quasi-geostrophic equation.*

REMARK 2.11. Existence of weak solutions of the Cauchy problem in $\mathbb{R}^2$ is a long established fact (see Resnick's thesis [32], or [11, Proposition 1.1]). In [32] the Fourier methods were used, so that the result applies to the case of the whole $\mathbb{R}^2$, eventually to the problem on a torus $\mathbb{T}^2$.

The Dirichlet problem in a bounded domain (zero boundary condition) was studied only recently in [8, 9] in case of the *homogeneous* equation ($f \equiv 0$). More precisely, in [9, Theorem 5], a weak solution; $\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ was obtained through the Galerkin approximation technique mixed with the commutator estimates.

Existence of a non-trivial (not reduced to the single function 0) global attractor requires a non-trivial free force ($f \neq 0$ on a set of positive measure). That case is treated in the present manuscript (see also [15, 16]) using the regular sub-critical approximations (6) together with their weak limits as $\frac{1}{2} < \alpha \to \frac{1}{2}$. We recommend the considerations in [9] showing the difficulties appearing in case of the Dirichlet boundary condition when estimating solutions near the boundary $\partial\Omega$.

3. Uniform Global in Time Estimates of Solutions

In this section, we present uniform estimates of solutions of problem (1) in $L^2(\Omega)$ which are needed for proving the existence of absorbing sets.

LEMMA 3.1. *For any $\theta_0 \in L^2(\Omega)$ and $f \in L^2(\Omega)$, the weak solution of (1) satisfies*

$$\begin{aligned}\|\theta(t)\|^2_{L^2(\Omega)} \le e^{-\kappa\lambda_1^{\frac{1}{2}}t}\|\theta_0\|^2_{L^2(\Omega)} \\ + \frac{1}{\kappa^2\lambda_1}\|f\|^2_{L^2(\Omega)}(1-e^{-\kappa\lambda_1^{\frac{1}{2}}t}), \quad \text{for all } t \ge 0,\end{aligned} \tag{31}$$

and

$$\begin{aligned}\kappa\int_0^T \|(-\Delta)^{\frac{1}{4}}\theta(t)\|^2_{L^2(\Omega)}dt \le \|\theta_0\|^2_{L^2(\Omega)} \\ + \frac{T}{\kappa\lambda_1^{\frac{1}{2}}}\|f\|^2_{L^2(\Omega)}, \quad \text{for any } T > 0.\end{aligned} \tag{32}$$

The proof was given in Subsection 2.2. Using the reported above estimates we will find an *attracting set* in $L^2(\Omega)$.

Let $B_0 = B(0, r)$ denote a closed ball in $L^2(\Omega)$ centered at zero with radius $r := \frac{1}{\kappa^2 \lambda_1} \|f\|^2_{L^2(\Omega)} + 1$. Then we find that B_0 is a bounded absorbing set of the m-semiflow $\{S(t)\}_{t \geq 0}$.

4. The m-Semiflow Generated by Weak Solutions to the Critical q-g Equation

It follows from Theorem 2.8 that problem (1) has at least one weak solution, thus we will define a family of multivalued mappings $S(t) : L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ by

$$S(t)\theta_0 = \{\theta(t) : \theta \text{ is a weak solution given by } \\ \text{Definition 2.7 with } \theta(0) = \theta_0\}.$$

In a standard way, as in [13, Lemma 2.4 and Lemma 2.6], we see that $\{S(t)\}_{t \geq 0}$ is a strict m-semiflow. First, we will establish that $\{S(t)\}_{t \geq 0}$ is weakly closed for each $t \geq 0$. This result will be necessary for the proof of the asymptotical upper semicompactness.

LEMMA 4.1. *The m-semiflow $\{S(t)\}_{t \geq 0}$ is weakly closed for any $t \geq 0$, i.e., $\theta_0^n \to \theta_0$ weakly in $L^2(\Omega)$, $\xi_n \in S(t)\theta_0^n$ and $\xi_n \to \xi$ weakly in $L^2(\Omega)$, then $\xi \in S(t)\theta_0$.*

PROOF. From $\xi_n \in S(t)\theta_0^n$ and Observation 2.9 we have that there exists a sequence θ^n of weak solutions verifying $\theta^n(0) = \theta_0^n$ and $\theta^n(t) = \xi_n$. Let $T \geq t$. Since $\theta^n(0) = \theta_0^n$ is bounded in $L^2(\Omega)$, it follows from Lemma 3.1 that θ^n is bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^{\frac{1}{2}}(\Omega))$. Thus, for $\theta = \theta^n$ and $\phi \in C_0^\infty([0, T]; C_0^\infty(\Omega))$ in (26) we get

$$\begin{aligned}
&- \int_0^T \langle \theta^n(t), \phi_t(t) \rangle dt + \langle \theta^n, \phi \rangle |_0^T \\
&= \int_0^T \langle \theta^n(t), u^n(t) \cdot \nabla \phi(t) \rangle dt - \int_0^T \langle (-\Delta)^{\frac{1}{4}} \theta^n(t), (-\Delta)^{\frac{1}{4}} \phi(t) \rangle dt \\
&\quad + \int_0^T \langle f, \phi(t) \rangle dt.
\end{aligned}$$

In the following, we show that each term at the right hand side of the above expression is uniformly bounded. First, by Sobolev type embeddings, we get

$$
\begin{aligned}
&|\int_0^T \langle \theta^n(t), u^n(t)\cdot\nabla\phi(t)\rangle dt| \\
&\quad\leq \int_0^T \|\theta^n(t)\|_{L^4(\Omega)}\|u^n(t)\|_{L^4(\Omega)}\|\nabla\phi(t)\|_{L^2(\Omega)}dt \\
&\quad\leq c\|\theta^n\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Omega))}\|\phi\|_{L^\infty(0,T;H^1(\Omega))}.
\end{aligned}
\tag{33}
$$

For the last two terms, it is clear that

$$
\begin{aligned}
&|\int_0^T \langle(-\Delta)^{\frac{1}{4}}\theta^n(t),(-\Delta)^{\frac{1}{4}}\phi(t)\rangle dt| + |\int_0^T\langle f,\phi(t)\rangle dt| \\
&\leq \int_0^T \|\theta^n(t)\|_{H^{\frac{1}{2}}(\Omega)}\|\phi(t)\|_{H^{\frac{1}{2}}(\Omega)}dt + \int_0^T \|f\|_{L^2(\Omega)}\|\phi(t)\|_{L^2(\Omega)}dt \\
&\leq \|\theta^n\|_{L^2(0,T;H^{\frac{1}{2}}(\Omega))}\|\phi\|_{L^2(0,T;H^{\frac{1}{2}}(\Omega))} + \sqrt{T}\|f\|_{L^2(\Omega)}\|\phi\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
\tag{34}
$$

It follows from (33) and (34) that the distributional derivative θ_t^n is bounded in $L^1(0,T;H^{-1}(\Omega))$. Hence, for a subsequence, not renumerating, we have that $\theta^n \to \theta$ weakly in $L^2(0,T;H^{\frac{1}{2}}(\Omega))$ and $\theta_t^n \to \theta_t$ weakly in $L^1(0,T;H^{-1}(\Omega))$. We can pass to the limit in all the terms in (26) written for θ^n, then we obtain that θ satisfies (26), which implies that θ is a weak solution of (1). Moreover, $\theta^n(T)\to\theta(T)$ weakly in $L^2(\Omega)$, so we have $\xi=\theta(T)\in S(T)\theta_0$. $\square$

Lemma 4.2. *The m-semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically upper semicompact for any $t\geq 0$.*

Proof. Suppose that $\theta_0^n\in B$ and $\xi_n\in S(t_n)\theta_0^n$, where B is a bounded set in $L^2(\Omega)$. Then, from the uniform estimates (31), we find that there exists a sequence (again denoted by ξ_n) such that $\xi_n\to\xi$ weakly in $L^2(\Omega)$. Thus, we easily obtain

$$
\|\xi\|_{L^2(\Omega)} \leq \lim\inf_{n\to\infty}\|\xi_n\|_{L^2(\Omega)}. \tag{35}
$$

If we prove that $\|\xi\|_{L^2(\Omega)}\geq \lim\sup_{n\to\infty}\|\xi_n\|_{L^2(\Omega)}$, then $\xi_n\to\xi$ strongly in $L^2(\Omega)$, as we need.

Clearly, any weak solution to (1) satisfies

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\Omega)}+\frac{1}{2}\|\theta\|^2_{L^2(\Omega)}+\kappa\|(-\Delta)^{\frac{1}{4}}\theta\|^2_{L^2(\Omega)}=\int_\Omega f\theta dx+\frac{1}{2}\|\theta\|^2_{L^2(\Omega)}.$$

Let $T>0$ be an arbitrary number. Furthermore, by integrating the above equality over $[0,T]$, we have

$$\begin{aligned}\|\theta(T)\|^2_{L^2(\Omega)}=e^{-T}\|\theta(0)\|^2_{L^2(\Omega)}&+2\int_0^T\int_\Omega e^{-(T-s)}f\theta(s)dxds\\&+\int_0^T e^{-(T-s)}\|\theta(s)\|^2_{L^2(\Omega)}ds\\&-2\kappa\int_0^T e^{-(T-s)}\|(-\Delta)^{\frac{1}{4}}\theta(s)\|^2_{L^2(\Omega)}ds.\end{aligned}\tag{36}$$

Since the m-semiflow $\{S(t)\}_{t\geq 0}$ is a strict m-semiflow, we have that $\xi_n\in S(t_n)\theta_0^n=S(T)S(t_n-T)\theta_0^n$, and then there must be $\eta_n\in S(t_n-T)\theta_0^n$ satisfying $\xi_n\in S(T)\eta_n$. Let θ^n be a sequence of weak solutions verifying $\theta^n(T)=\xi_n$ and $\theta^n(0)=\eta_n$. Then θ^n satisfies (36), i.e.,

$$\begin{aligned}\|\xi_n\|^2_{L^2(\Omega)}=e^{-T}\|\eta_n\|^2_{L^2(\Omega)}&+2\int_0^T\int_\Omega e^{-(T-s)}f\theta^n(s)dxds\\&+\int_0^T e^{-(T-s)}\|\theta^n(s)\|^2_{L^2(\Omega)}ds\\&-2\kappa\int_0^T e^{-(T-s)}\|(-\Delta)^{\frac{1}{4}}\theta^n(s)\|^2_{L^2(\Omega)}ds.\end{aligned}\tag{37}$$

Choosing $N=N(T)$ such that for all $n\geq N$, $\eta_n\in S(t_n-T)\theta_0^n\subset B_0$, we get $\eta_n\to\xi_T$ weakly in $L^2(\Omega)$. We know from Lemma 4.1 that there exists a weak solution θ such that

$$\theta^n\to\theta \text{ weakly in } L^2(\Omega) \text{ and } \theta(T)=\xi\in S(T)\xi_T,\quad \theta(0)=\xi_T.$$

In the following, we handle each term in (37) separately for the sequence θ^n. First, since η_n is bounded, there exists some constant $M>0$ such that

$$e^{-T}\|\eta_n\|^2_{L^2(\Omega)}\leq e^{-T}M,\qquad \text{for all } n.\tag{38}$$

Using an argument similar to Lemma 4.1, we have that $\theta^n \in L^2(0,T;H^{\frac{1}{2}}(\Omega))$ and $\theta^n_t \in L^1(0,T;H^{-1}(\Omega))$, then we apply the Lions-Aubin compactness lemma to get a subsequence θ^n (after relabeling) such that $\theta^n \to \theta$ strongly in $L^2(0,T;L^2(\Omega))$, so that

$$\limsup_{n\to\infty} 2\int_0^T\int_\Omega e^{-(T-s)} f\theta^n(s)dxds \le 2\int_0^T\int_\Omega e^{-(T-s)} f\theta(s)dxds, \tag{39}$$

and

$$\limsup_{n\to\infty}\int_0^T e^{-(T-s)}\|\theta^n(s)\|^2_{L^2(\Omega)}ds \le \int_0^T e^{-(T-s)}\|\theta(s)\|^2_{L^2(\Omega)}ds. \tag{40}$$

Furthermore, since $\theta^n \to \theta$ weakly in $L^2(0,T;H^{\frac{1}{2}}(\Omega))$, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left(-2\kappa\int_0^T e^{-(T-s)}\|(-\Delta)^{\frac{1}{4}}\theta^n(s)\|^2_{L^2(\Omega)}ds\right)\\
&= -2\kappa\liminf_{n\to\infty}\int_0^T e^{-(T-s)}\|(-\Delta)^{\frac{1}{4}}\theta^n(s)\|^2_{L^2(\Omega)}ds\\
&\le -2\kappa\int_0^T e^{-(T-s)}\|(-\Delta)^{\frac{1}{4}}\theta(s)\|^2_{L^2(\Omega)}ds.
\end{aligned} \tag{41}$$

Combining (38)-(41) and applying the identity (36) to θ, it follows that

$$\begin{aligned}
\limsup_{n\to\infty}\|\xi_n\|^2_{L^2(\Omega)} &\le e^{-T}M + 2\int_0^T\int_\Omega e^{-(T-s)}f\theta(s)dxds\\
&\quad + \int_0^T e^{-(T-s)}\|\theta(s)\|^2_{L^2(\Omega)}ds\\
&\quad - 2\kappa\int_0^T e^{-(T-s)}\|(-\Delta)^{\frac{1}{4}}\theta(s)\|^2_{L^2(\Omega)}ds\\
&= \|\xi\|^2_{L^2(\Omega)} + e^{-T}M - e^{-T}\|\xi_T\|^2_{L^2(\Omega)}.
\end{aligned}$$

Taking the limit as $T\to\infty$, we get the inequality

$$\limsup_{n\to\infty}\|\xi_n\|^2_{L^2(\Omega)} \le \|\xi\|^2_{L^2(\Omega)},$$

and thus the proof is complete. $\square$

We next show continuity of the m-semiflow $\{S(t)\}_{t\ge 0}$.

Lemma 4.3. *The m-semiflow $\{S(t)\}_{t\geq 0}$ is upper semicontinuous and has compact values for any $t \geq 0$.*

Proof. Let $\xi_n \in S(t)\theta_0^n$ and $\theta_0^n \to \theta_0$. We will show that ξ_n is precompact in $L^2(\Omega)$. It follows from Lemma 3.1 that the sequence ξ_n is bounded, so by extracting a subsequence (again denoted by ξ_n) we will ensure that ξ_n is weakly convergent to some ξ. Arguing in a similar way as in the proof of Lemma 4.1, there exist weak solutions θ^n and θ such that $\theta^n(t) = \xi_n$, $\theta^n(0) = \theta_0^n$, $\theta(t) = \xi$, $\theta(0) = \theta_0$ and $\theta^n \to \theta$ weakly in $L^2(\Omega)$. Moreover, they satisfy the equality

$$\|\theta(t)\|_{L^2(\Omega)} = \|\theta(0)\|_{L^2(\Omega)} - 2\kappa \int_0^t \|(-\Delta)^{\frac{1}{4}}\theta(s)\|^2_{L^2(\Omega)}ds + 2\int_0^t\int_\Omega f\theta(s)dxds.$$

Repeating the arguments of Lemma 4.2, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \|\xi_n\|^2_{L^2(\Omega)} \\
&\quad\leq \lim_{n\to\infty} \|\theta_0^n\|^2_{L^2(\Omega)} - 2\kappa\int_0^T \|(-\Delta)^{\frac{1}{4}}\theta(s)\|^2_{L^2(\Omega)}ds + 2\int_0^T\int_\Omega f\theta(s)dxds \\
&\quad= \|\theta_0\|^2_{L^2(\Omega)} - 2\kappa\int_0^T \|(-\Delta)^{\frac{1}{4}}\theta(s)\|^2_{L^2(\Omega)}ds + 2\int_0^T\int_\Omega f\theta(s)dxds \\
&\quad= \|\xi\|_{L^2(\Omega)}.
\end{aligned}$$

Hence, $\xi_n \to \xi$ strongly in $L^2(\Omega)$, which implies that $S(t)\theta_0$ is precompact in $L^2(\Omega)$ for any θ_0. We know from Lemma 4.1 that $S(t)\theta_0$ is weakly closed, hence closed, so that $S(t)\theta_0$ is a compact set.

Now, if $\{S(t)\}_{t\geq 0}$ is not upper semicontinuous. Then there exist a point θ_0, a neighborhood $\mathcal{O}$ of $S(t)\theta_0$ and a sequence $\theta_0^n \to \theta_0$ in $L^2(\Omega)$, $\xi_n \in S(t)\theta_0^n$ such that $\xi_n \notin \mathcal{O}$. Since $S(t)\theta_0^n$ is compact, there exists a sequence ξ_{n_k} such that $\xi_{n_k} \to \xi$ in $L^2(\Omega)$. Note that $\theta_0^n \to \theta_0$ in $L^2(\Omega)$, it follows from Lemma 4.1 that $\xi \in S(t)\theta_0$, which is a contradiction. □

Concluding the above results, by Theorem 2.4 we formulate the main result concerning the existence of a compact global attractor for the m-semiflow $\{S(t)\}_{t\geq 0}$.

Theorem 4.4. *When $\theta_0 \in L^2(\Omega)$ and $f \in L^2(\Omega)$, then the m-semiflow $\{S(t)\}_{t\geq 0}$ associated with problem* (1) *has a compact global attractor $\mathcal{A}$.*

5. Appendix

Some technicalities. This short section contains various technical tools needed in the paper. Below $(-\Delta_p)$ denotes negative Dirichlet Laplacian in a bounded smooth domain Ω, densely defined in $L^p(\Omega)$.

Following the famous considerations of H. Komatsu (e.g. [22, 23, 28]), fractional powers will be defined for any *non-negative operator*, that is a closed linear densely defined operator A in a Banach space X, such that the resolvent set of A contains $(-\infty, 0)$ and the resolvent satisfies

$$\|\lambda(\lambda + A)^{-1}\| \leq M, \ \lambda > 0,$$

(where M is a constant independent of λ), through the formula (written here in case $0 < \alpha < 1$)

$$A^\alpha \phi = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{\alpha-1} A(\lambda + A)^{-1} \phi \, d\lambda. \tag{42}$$

We will recall next that *the proper fractional powers of* $(-\Delta_p)$ *are sectorial operators*. To verify the last claim it is convenient to use a notion, due to H. Komatsu [22, p. 288], of an *operator of the type* $(\omega, M(\theta))$ *in a Banach space* X.

Definition 5.1. We say that A is of type $(\omega, M(\theta))$, $0 \leq \omega < \pi$, if the domain $D(A)$ is dense in X, the resolvent set of $-A$ contains the sector $|arg\,\lambda| < \pi - \omega$ and the condition $\|\lambda(\lambda + A)^{-1}\| \leq M(\theta)$ holds on each ray $\lambda = re^{i\theta}$, $r \in (0, +\infty)$, $|\theta| < \pi - \omega$.

One may easily verify that A is of the type $(\omega, M(\theta))$ with $\omega < \frac{\pi}{2}$ *if and only if* A is a sectorial operator in the sense of [19]. A theorem by T. Kato (see [22, p. 320], also [40, p. 97]) ensures that:

Proposition 5.2. *If* A *is of type* $(\omega, M(\theta))$ *and if* $0 < \alpha < \frac{\pi}{\omega}$, *then* A^α *is of type* $(\alpha\omega, M_\alpha(\theta))$ *with certain positive constant* $M_\alpha(\theta)$. *Furthermore, the resolvent* $(\lambda + A^\alpha)^{-1}$ *is analytic in* α *and* λ *in the domain* $0 < \alpha < \frac{\pi}{\omega}$, $|arg\lambda| < \pi - \alpha\omega$.

We recall next a property similar to the one obtained for the Stokes operator in [18] (see [16, p.57] for the proof).

PROPOSITION 5.3. *For each $j = 1, \ldots, N$ and $1 < p < +\infty$ the operator $(-\Delta_p)^{-\frac{1}{2}} \frac{\partial}{\partial x_j}$ extends uniquely to a bounded linear operator from $L^p(\Omega)$ into itself. In particular, for the nonlinearity in the N-D Quasi-geostrophic equation, an estimate holds:*

$$\forall_{p\in(1,+\infty)} \exists_{M_p>0} \ \|(-\Delta_p)^{-\frac{1}{2}} \nabla \cdot (u\theta)\|_{L^p(\Omega)} \leq M_p \| |u\theta| \|_{L^p(\Omega)}. \tag{43}$$

It will be seen also, that the operators $\frac{\partial}{\partial x_j}(-\Delta_p)^{-\frac{1}{2}} : L^p(\Omega) \to L^p(\Omega)$, $j = 1, \ldots, N$, $1 < p < \infty$, are bounded linear (see [19, p.18] when $N = 1$, [18, p.270] in case of the Stokes operator, and [16, Proposition A.1]).

OBSERVATION 5.4. The Riesz transforms $\mathcal{R}_j$, $j = 1, 2$, in a bounded domain are bounded operators from $D((-\Delta)^{\frac{s}{2}})$ into $H^s(\Omega)$ for any $s \geq 0$. Here, for given s, we need to assume that $\partial\Omega \in C^k$, where k is the smallest even number dominating s.

We will only prove that for any $s \geq 0$,

$$\|\mathcal{R}_2 \theta\|_{H^s(\Omega)} \leq c \|\theta\|_{H^s(\Omega)},$$

since the second estimate is analogous. Let $\theta \in D((-\Delta)^{\frac{s}{2}})$ with $s \geq 0$, then also $(-\Delta)^{-\frac{1}{2}}\theta \in D((-\Delta)^{\frac{s+1}{2}})$ and

$$\begin{aligned} \|\mathcal{R}_2 \theta\|_{H^s(\Omega)} = \|\frac{\partial}{\partial x_2}(-\Delta)^{-\frac{1}{2}}\theta\|_{H^s(\Omega)} &\leq c\|(-\Delta)^{-\frac{1}{2}}\theta\|_{H^{s+1}(\Omega)} \\ &\leq \overline{c}\|(-\Delta)^{\frac{s+1}{2}}(-\Delta)^{-\frac{1}{2}}\theta\|_{L^2(\Omega)} \\ &= \overline{c}\|(-\Delta)^{\frac{s}{2}}\theta\|_{L^2(\Omega)} = \overline{c}\|\theta\|_{H^s(\Omega)}, \end{aligned}$$

where an equivalent norm of $\theta \in D((-\Delta)^{\frac{s}{2}})$ in $H^s(\Omega)$, given by $\|(-\Delta)^{\frac{s}{2}}\theta\|_{L^2(\Omega)}$, has been used.

Analogous property holds, with similar proof, for the Riesz transforms considered on the scale build on $L^p(\Omega)$ with $p \in (1, \infty)$.

The theory of the Riesz operator in the whole of $\mathbb{R}^N$ was studied in the famous monograph [35].

Further, we recall a version (see [5] for the proof) of the famous Kato-Beurling-Deny inequality.

Corollary 5.5. *For $\alpha \in [0,1]$, $q \in [2,+\infty)$, $\phi \in D((-\Delta_2)^\alpha)$ with $sgn\phi|\phi|^{q-1} \in D((-\Delta_2)^{\frac{\alpha}{2}})$, the following estimate holds:*

$$\int_\Omega (-\Delta_2)^\alpha \phi \ \mathrm{sgn}\phi \ |\phi|^{q-1} dx \geqslant \frac{4(q-1)}{q^2} \int_\Omega \left[(-\Delta_2)^{\frac{\alpha}{2}}(|\phi|^{\frac{q}{2}})\right]^2 dx. \tag{44}$$

Note that taking $q = 4p, p \in \mathbb{N}$, in the latter estimate, the absolute value will not be needed.

Remark 5.6. The definition of fractional powers (42) was proposed by A.V. Balakrishnan in 1960 and studied in a series of 6 papers by H. Komatsu through the years 1966-1972 ([22] is one of them). Application of that notion to abstract parabolic equation in Hilbert or Banach spaces was given by T. Kato and P.E. Sobolevskii around 1960; see the Comments on that results in Chapter XIV of K. Yosida's famous "Functional Analysis". An early reference is also the monograph [23], with Russian edition from 1967. Formulation of the Navier-Stokes equation as an integral equation in a Banach space connected with the fractional powers of the Stokes operator was used in particular in a semigroup studies of the Navier-Stokes equation by Japan mathematicians; among them T. Kato and H. Fujita (1962, 1964), Y. Giga and T. Miyakawa (starting from 1980); and by P.E. Sobolevskii (1959). The 1981 monograph [19] by D. Henry provides an important abstract approach, using semigroup technique, to general parabolic equations and systems. Later on that approach extends (see in particular [3]), so that we will call here only the recent monograph by A. Yagi [40], providing in Chapter 16 an explicit description of the domains of fractional powers of sectorial positive/non-negative operators, also nice monograph [28] in which the properties of fractional powers (in the Balakrishnan/Komatsu approach) are described in details.

All these references form an approach inside of which fractional powers can be used to study parabolic problems both in bounded domains (as in the present paper), or in $\mathbb{R}^N$. In a particular case of the $-\Delta$ operator in $\mathbb{R}^N$ (considered in $L^p(\mathbb{R}^N), 1 < p < \infty$, spaces) various definitions of fractional powers were present in the literature; they are known to be equivalent in general (however, we must specify on which sets); a nice comparison was given recently in [24]. While, in case of the Dirichlet or Neumann minus Laplacian in a bounded regular (C^2, say) domain $\Omega \subset \mathbb{R}^N$ equivalence of definitions is restricted only to the Balakrishnan/Komatsu and the Bochner definitions (e.g. [28, Section 6.1.1]). In a particular case of the Hilbert spaces (e.g. in $L^2(\Omega)$), also the definition through spectral resolution of selfadjoint positive definite operators (e.g. [34, p.95]) will be used equivalently with the two just called. Thus, we need to make a warning, that the symbol $(-\Delta)^\alpha$ used by different authors need not have the same meaning.

Acknowledgements. T. Liang and Y. Wang were supported by NSF of China (Grants No. 41875084, 11571153), the Fundamental Research Funds for the Central Universities under Grant Nos. lzujbky-2018-ot03 and lzujbky-2018-it58.

The authors are very grateful to the Referee for plenty of valuable comments significantly improving the original version of the manuscript.

References

[1] Adams, R. A., *Sobolev Spaces*, New York, Academic Press (1975).

[2] Caffarelli, L. A. and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. Math. **171** (2010), 1903–1930.

[3] Cholewa, J. W. and T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, Cambridge (2000).

[4] Cholewa, J. W. and T. Dlotko, Remarks on the powers of ellliptic operators, Rev. Math. Complut. **13** (2000), 325–337.

[5] Cholewa, J. W., Dlotko, T. and A. W. Turski, Asymptotics of pseudodifferential parabolic equations, Demonstratio Math. **35** (2002), 75–91.

[6] Cholewa, J. W. and T. Dlotko, Fractional Navier-Stokes equations, Discrete Contin. Dyn. Syst. Ser. B **23** (2018), 2967–2988.

[7] Constantin, P., Tarfulea, A. and V. Vicol, Long time dynamics of forced critical SQG, Comm. Math. Phys. **335** (2015), 93–141.

[8] Constantin, P. and M. Ignatova, Critical SQG in bounded domains, Annals of PDE **2** (2016), 2–42.

[9] Constantin, P. and M. Ignatova, Remarks on the fractional Laplacian with Dirichlet boundary conditions and applications, Int. Math. Res. Not. Vol. **2017** (2017), 1653–1672.

[10] Constantin, P. and H. Q. Nguyen, Local and global strong solutions for SQG in bounded domains, Physica D: Nonlinear Phenomena **376** (2018), 195–203.

[11] Constantin, P. and J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, SIAM J. Math. Anal. **30** (1999), 937–948.

[12] Cordoba, A. and D. Cordoba, A Maximum Principle applied to quasi-geostrophic equations, Commun. Math. Phys. **249** (2004), 511–528.

[13] Coti Zelati, M. and P. Kalita, Smooth attractors for weak solutions of the SQG equation with critical dissipation, Discrete Contin. Dyn. Syst. Series B **22** (2017), 1857–1873.

[14] Dlotko, T., Navier-Stokes equation and its fractional approximations, Appl. Math. Optim. **77** (2018), 99–128.

[15] Dlotko, T., Kania, M. B. and C. Sun, Quasi-geostrophic equation in $\mathbb{R}^2$, J. Differential Equations **259** (2015), 531–561.

[16] Dlotko, T. and C. Sun, 2-D Quasi-geostrophic equation; sub-critical and critical cases, Nonlinear Anal. **150** (2017), 38–60.

[17] Dlotko, T., Liang, T. and Y. Wang, Critical and super-critical abstract parabolic equations, Discrete Contin. Dyn. Syst. Series B **25** (2020), 1517–1541.

[18] Giga, Y. and T. Miyakawa, Solutions in L_r of the Navier-Stokes initial value problem, Arch. Rational Mech. Anal. **89** (1985), 267–281.

[19] Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Berlin, Springer-Verlag (1981).

[20] Ju, N., Global solutions to the two dimensional quasi-geostrophic equation with critical or super-critical dissipation, Math. Ann. **334** (2006), 627–642.

[21] Kiselev, A., Nazarov, F. and A. Volberg, Global well-posedness for the critical $2D$ dissipative quasi-geostrophic equation, Invent. Math. **167** (2007), 445–453.

[22] Komatsu, H., Fractional powers of operators, Pacific J. Math. **19** (1966), 285–346.

[23] Krein, S. G., *Linear Equations in Banach Spaces, Birkhäuser*, Mass., Boston (1982).

[24] Kwasnicki, M., Ten equivalent definitions of the fractional Laplace operator, Frac. Calc. Appl. Anal. **20** (2017), 7–51.

[25] Lasiecka, I., Unified theory for abstract parabolic boundary problems-A semi-group approach, Appl. Math. Optim. **6** (1980), 287–333.

[26] Lions, J.-L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Paris, Dunod Gauthier-Villars (1969).

[27] Lunardi, A., *Interpolation Theory*, Edizioni della Normale, Pisa (2018).

[28] Martínez Carracedo, C. and M. Sanz Alix, *The Theory of Fractional Powers of Operators*, Amsterdam, Elsevier (2001).

[29] Mel'nik, V. S. and J. Valero, On attractors of multivalued semi-flows and differential inclusions, Set-Values Anal. **6** (1998), 83–111.

[30] Nguyen, H. Q., Global weak solutions for generalized SQG in bounded domains, Analysis & PDE **11** (2018), 1029–1047.

[31] Prüss, J. and H. Sohr, Imaginary powers of elliptic second order differential operators in L^p-spaces, Hiroshima Math. J. **23** (1993), 161–192.

[32] Resnick, S., *Dynamical Problems in Non-linear Advective Partial Differential Equations*, Ph.D. thesis, Chicago, IL, University of Chicago (1995).

[33] Simon, J., Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. **146** (1987), 65–96.

[34] Sohr, H., *The Navier-Stokes Equations, An Elementary Functional Analytic Approach*, Basel, Birkhaüser Verlag (2001).

[35] Stein, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton, Princeton University Press (1970).

[36] Strauss, W. A., On continuity of functions with values in various Banach spaces, Pacific J. Math. **19** (1966), 543–551.

[37] Triebel, H., *Interpolation Theory, Function Spaces, Differential Operators*, Berlin, Veb Deutscher Verlag (1978).

[38] Wang, Y. and T. Liang, Mild solutions to the time fractional Navier-Stokes delay differential inclusions, Discrete Contin. Dyn. Syst. Series B **24** (2019), 3713–3740.

[39] Wu, J., Dissipative quasi-geostrophic equations with L^p data, Electron. J. Differential Equations **56** (2001), 1–13.

[40] Yagi, A., *Abstract Parabolic Evolution Equations and their Applications*, Heidelberg, Springer-Verlag (2010).

(Received March 30, 2020)
(Revised April 28, 2021)

Tomasz Dlotko
Institute of Mathematics
University of Silesia
Katowice, Poland
E-mail: tomasz.dlotko@us.edu.pl

Tongtong Liang and Yejuan Wang
School of Mathematics and Statistics
Gansu Key Laboratory of Applied
Mathematics and Complex Systems
Lanzhou University
Lanzhou 730000, China
E-mail: liangtt18@lzu.edu.cn
wangyj@lzu.edu.cn