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Nonconservative Reflectionless Inverse Scattering and Soliton Solutions of an Associated Nonlinear Evolution System

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Abstract. A nonconservative, reflectionless inverse scattering problem is discussed on an energy dependent Schrödinger equation. A scattering transform from the potential of the equation to the reflectionless scattering data is completely characterized by a function induced from a Gelfand-Levitan-Marchenko equation, with an expression of the inverse scattering transform in terms of the function. Based upon the inverse scattering theory, we establish an inverse scattering method by which N-soliton solutions of a nonlinear evolution system (Boussinesq system) are constructed.

1. Results

The objective of this paper is, firstly, to establish a reflectionless inverse scattering theory on an energy dependent Schrödinger equation, and secondly, to apply the theory to an isospectral flow associated with the equation for finding soliton solutions of the nonlinear evolution system. The energy Schrödinger equation we are concerned with is

$$f'' + [k^2 - (U(x) + 2k\,Q(x))]f = 0, \quad -\infty < x < \infty. \tag{1.1}$$

Here functions $U(x)$, $Q(x)$ on $\boldsymbol{R}$ are decreasing rapidly as $x = \pm\infty$, and we assume that $U(x)$ is real-valued and $Q(x)$ is purely imaginary-valued. Let $\psi_{\rightarrow}(x,k)$ and $\psi_{\leftarrow}(x,k)$ be the scattering solutions, namely, solutions of (1.1) having the asymptotics:

$$\begin{aligned} \psi_{\rightarrow}(x,k) &\sim \begin{cases} e^{ikx} + s_{12}(k)e^{-ikx}, & x \to -\infty, \\ s_{11}(k)e^{ikx}, & x \to +\infty; \end{cases} \\ \psi_{\leftarrow}(x,k) &\sim \begin{cases} e^{-ikx} + s_{21}(k)e^{ikx}, & x \to +\infty, \\ s_{22}(k)e^{-ikx}, & x \to -\infty. \end{cases} \end{aligned} \tag{1.2}$$

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The solution $\psi_{\rightarrow}(x,k)$ represents scattering of a free wave e^{ikx} coming from $-\infty$ scattered by the potential $(U(x),Q(x))$ with the transmitting wave $s_{11}(k)e^{ikx}$ and the reflecting wave $s_{12}(k)e^{-ikx}$. Similarly $\psi_{\leftarrow}(x,k)$ represents scattering of the wave e^{-ikx} from $+\infty$. The inverse scattering problem is to recover the potential $(U(x),Q(x))$ from the scattering matrix

$$S(k)=\begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k)\end{pmatrix}, \quad k\in\boldsymbol{R},$$

defined in terms of the transmission coefficients $s_{11}(k)$, $s_{22}(k)$, and the reflection coefficients $s_{12}(k)$, $s_{21}(k)$.

Unlike in the case $Q(x)$ is real-valued, $S(k)$ is not a unitary matrix in our case where $Q(x)$ is purely imaginary, in other words, the scattering for (1.1) is not conservative. However it admits a coupled unitarity: $S(k)S^{-}(k)^{*}=I$, where

$$S^{-}(k)=\begin{pmatrix} s^{-}_{11}(k) & s^{-}_{12}(k) \\ s^{-}_{21}(k) & s^{-}_{22}(k)\end{pmatrix}, \quad k\in\boldsymbol{R},$$

denotes the scattering matrix for the potential $(U,-Q)$. In addition, $S(k)$ has a conjugate symmetry: $S(-k)=\overline{S(k)}$.

Scattering is said to be reflectionless in the case

$$s_{21}(k)=s_{12}(k)=0 \quad \text{for any } k\in\boldsymbol{R}. \tag{1.3}$$

In the reflectionless scattering, the transmission coefficient $s_{11}(k)(=s_{22}(k))$ is a continuous function on $\boldsymbol{R}$ (see Lemma 2.2) and it can be analytically continued to a meromorphic function in the upper half plane $\boldsymbol{C}_{+}$ having at most finitely many poles k_n, $n=1,\cdots,N$. Let $f_{\pm}(x,k)$ be the Jost solutions of (1.1) with the asymptotic behaviors $f_{\pm}(x,k)\sim e^{\pm ikx}$ as $x\to\pm\infty$. Then, at the poles $k=k_n$ (bound states), there exist nonzero constants d_n^{+} (coupling constants) such that $f_{-}(x,k_n)=d_n^{+}f_{+}(x,k_n)$, $n=1,\cdots,N$. Furthermore the transmission coefficient $s^{-}_{11}(k)$ for $(U,-Q)$ coincides with $s_{11}(k)$ (see Proposition 2.3). Hence, in the reflectionless scattering, there exist also $d_n^{-}\neq 0$ such that $f^{-}_{-}(x,k_n)=d_n^{-}f^{-}_{+}(x,k_n)$, $n=1,\cdots,N$, associated with the Jost solution $f^{-}_{\pm}(x,k)$ for the potential $(U,-Q)$.

By means of coupling constant $d_n^{\pm}$, we define constant $c_n^{\pm}$ by

$$c_n^{\pm}=-i\,\mathrm{Res}_{k=k_n}s_{11}(k)\times d_n^{\pm}, \quad n=1,\cdots,N, \tag{1.4}$$

which is a generalized concept of the *norming constant* used in the inverse scattering theory for the standard Schrödinger case (Marchenko [25, 26], Faddeev [8], Deift and Trubowitz [6]), namely, the case $Q \equiv 0$ in (1.1). One can show that $c_n^{\pm}$ are nonzero (complex) numbers by the Poisson formula. In our reflectionless scattering the triplet $\{0, k_n, c_n^{\pm}\}$, $n = 1, \cdots, N$, are employed as *scattering data*, in which 0 indicates merely the reflectionless condition (1.3), $k_n \in \boldsymbol{C}_+$, $c_n^{\pm} \in \boldsymbol{C} \setminus \{0\}$. Given triplet $\{0, k_n, c_n^{\pm}\}$, we define $N \times N$ matrices $B^{\pm}$ and column vectors $\boldsymbol{v}^{\pm}$ by

$$B^{\pm} = \left(c_\ell^{\pm} \frac{e^{(ik_\ell + ik_j)x}}{ik_\ell + ik_j} \right), \quad \boldsymbol{v}^{\pm} := \left(c_\ell^{\pm} \frac{e^{ik_\ell x}}{ik_\ell} \right), \tag{1.5}$$

and set

$$\begin{aligned} \Delta^{\pm}(x) :=& \det(I - B^+ B^-) \\ &+ \begin{pmatrix} e^{ik_1 x} & \cdots & e^{ik_N x} \end{pmatrix} (I - B^{\mp} B^{\pm})\tilde{\ } (B^{\mp} \boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}), \end{aligned} \tag{1.6}$$

where I is the $N \times N$ identity matrix, $\begin{pmatrix} e^{ik_1 x} & \cdots & e^{ik_N x} \end{pmatrix}$ is a $1 \times N$ matrix and $(I - B^{\mp} B^{\pm})\tilde{\ }$ denote the cofactor matrices of $I - B^{\mp} B^{\pm}$. With the functions $\Delta^{\pm}(x)$ defined above, the main result on the reflectionless inverse scattering theory concerning (1.1) is stated as follows.

THEOREM 1.1. *A prescribed triplet $\{0, k_n, c_n^{\pm}\}$ is the scattering data for some $(U, Q) \in \mathcal{S} \times \mathcal{S}$ if and only if $\{0, k_n, c_n^{\pm}\}$ satisfies the following two conditions:*

(I) *there exists a permutation $\sigma \in \mathfrak{S}_N$ such that $k_{\sigma(n)} = -\overline{k_n}$, $c_{\sigma(n)}^{\pm} = \overline{c_n^{\pm}}$;*

(II) $\Delta^{\pm}(x) > 0$ *on $\boldsymbol{R}$.*

Under these conditions, (U, Q) is uniquely determined by

$$\begin{cases} Q(x) = -\frac{1}{2i} \frac{d}{dx} \left(\log \Delta^+(x) - \log \Delta^-(x) \right), \\ \\ U(x) + Q(x)^2 = -\frac{1}{2} \frac{d^2}{dx^2} \left(\log \Delta^+(x) + \log \Delta^-(x) \right). \end{cases} \tag{1.7}$$

Here $\mathcal{S}$ denotes the Schwartz class on $\boldsymbol{R}$.

Some remarks on Theorem 1.1 are helpful at this stage:

(1) A characteristic of (1.1) with purely imaginary $Q(x)$ is a symmetry $s_{11}(-\overline{k}) = \overline{s_{11}(k)}$ on $\overline{\boldsymbol{C}_+}$ with respect to the imaginary axis, which stems from $f_\pm(x, -\overline{k}) = \overline{f_\pm(x,k)}$ for each $k \in \overline{\boldsymbol{C}_+}$. It follows from this observation that the condition (I) in the theorem is necessary for $\{0, k_n, c_n^\pm\}$ to be the scattering data for some $(U, \pm Q)$. For this reason the symmetric group $\mathfrak{S}_N$ acts in the theory. The bound states either lie on or are symmetrically located with respect to the imaginary axis in $\boldsymbol{C}_+$. In particular, if N is odd then an odd number of bound states k_n lie on the imaginary axis in $\boldsymbol{C}_+$.

(2) Linear algebra tells us that $\det(I - AB) = \det(I - BA)$ and $(AB)\check{} = B\check{}A\check{}$. This enables us to see that, under the requirement (I), the functions $\Delta^\pm(x)$ defined in (1.6) are *real-valued* functions on $\boldsymbol{R}$. Hence $Q(x)$ defined firstly by *inversion formula* (1.7) is purely imaginary and $U(x)$ defined secondly by it is real.

(3) Conclusion in the theorem does not depend on a choice of function spaces, since, although we chose to work on a wider class, e.g., under condition (2.1), potentials U, Q are exponentially decaying as $|x| \to \infty$ and so, belong necessarily to the Schwartz class $\mathcal{S}$.

(4) Since the functions $\Delta^\pm(x)$ are real on $\boldsymbol{R}$ and tend to 1 as $x \to +\infty$ due to $\operatorname{Re} ik_n < 0$, the condition (II) is equivalent to that $\Delta^\pm(x)$ have no zeros on $\boldsymbol{R}$. By definition, $\Delta^\pm(x)$ are entire functions. Hence zeros of them are discrete. Whether $\Delta^\pm(x)$ have zeros on $\boldsymbol{R}$ or not is determined by a triplet $\{0, k_n, c_n^\pm\}$, which is said to be *regular* if neither $\Delta^+(x)$ nor $\Delta^-(x)$ has zeros on $\boldsymbol{R}$, and is said to be *singular* if either or both of these has zeros there. In the case $\{0, k_n, c_n^\pm\}$ is singular no pairs (U, Q) of continuous potentials on $\boldsymbol{R}$ can realize a prescribed scattering data $\{0, k_n, c_n^\pm\}$.

(5) The reflectionless scattering for (1.1) with purely imaginary Q is completely controlled by functions $\Delta^\pm(x)$; the range of the *scattering transform* $(U, Q) \mapsto \{0, k_n, c_n^\pm\}$ (ST) as well as the inverse of it (IST) is described by the functions.

(6) The potential (U, Q) determined from $\{0, k_n, c_n^\pm\}$ satisfies (see Propo-

sition 3.3)

$$\int_{-\infty}^{\infty} Q(x)dx = 0, \quad \int_{-\infty}^{\infty} [U(x) + Q(x)^2]dx = 4i\sum_{n=1}^{N} k_n.$$

In particular, these quantities are independent of the constants $c_n^{\pm}$.

(7) If $ik_n < 0$ and $c_n^+ = c_n^- > 0$ for each n in a triplet $\{0, k_n, c_n^{\pm}\}$ then $\Delta^+(x) = \Delta^-(x) = (\det(I - B^+))^2 > 0$ (see Corollary 3.8), and hence, by (1.7), $Q(x) \equiv 0$, $U(x) = -2\frac{d^2}{dx^2}\log\det(I - B^+)$, which is a case in the reflectionless inverse scattering theory on the standard Schrödinger equation (Kay and Moses [22], Gardner, Greene, Kruskal and Miura [9]); therefore Theorem 1.1 gives a generalization of the theory.

A mathematical research on inverse scattering theory for energy dependent Schrödinger equation (1.1) was begun by Jaulent [13, 14], Jaulent and Jean [15, 16]. They derived the following representation of the Jost solutions $f_+(x,k)$ via the *transformation kernel* $A(x,y)$ (see [15, Lemma 4.1]):

$$f_+(x,k) = e^{i\int_x^\infty Q(r)dr} e^{ikx} + \int_x^\infty A(x,y)e^{iky}dy, \quad k \in \overline{\boldsymbol{C}_+}. \tag{1.8}$$

This representation is valid even in the case where U, Q are complex-valued and $A(x,y)$ is uniquely determined from (U,Q) as a (unique) solution of the integral equation

$$\begin{aligned} A(x,y) = {} & \frac{1}{2}\int_{\frac{x+y}{2}}^{\infty} U(s)e^{i\int_s^\infty Q(r)dr}ds - \frac{i}{2}Q\left(\frac{x+y}{2}\right)e^{i\int_{\frac{x+y}{2}}^\infty Q(r)dr} \\ & + \frac{1}{2}\int_{\frac{x+y}{2}}^{\infty} U(s)ds\int_s^{y+s-x} A(s,u)du \\ & + \frac{1}{2}\int_x^{\frac{x+y}{2}} U(s)ds\int_{y+x-s}^{y+s-x} A(s,u)du \\ & + i\int_x^\infty Q(s)A(s,y+s-x)ds \\ & - i\int_x^{\frac{x+y}{2}} Q(s)A(s,y+x-s)ds, \quad x \le y. \end{aligned} \tag{1.9}$$

Transformation kernels are connected with scattering data via the so-called Gelfand-Levitan-Marchenko (GLM) equations.

Such an equation for (1.1) was found by [15], based on which the paper established a recovery procedure of a pair of real U, Q from the scattering matrix in the absence of bound states ($N = 0$). A complete solution in the case $N = 0$ corresponding to [6] was obtained by Kamimura [19]. Inverse scattering problem for (1.1) with bounds state is still open although the problem for a single bound state was studied by Sattinger and Szmigielski [31] as well as, recently, for the reflectionless case has been solved by [20]. The nonconservative case $Q(x)$ is purely imaginary was extensively studied by Aktosun, Klaus, and van der Mee [2, 3], where sufficient conditions for the unique solvability of the associated GLM equation were obtained, and by means of the knowledge, (U, Q) are recovered from scattering data including information on bound states.

The proof of Theorem 1.1 is given in Sections 2, 3. In course of the proof we find that the order of poles k_n is one, namely, every pole k_n, $n = 1, \cdots, N$, is simple.

As an application of Theorem 1.1 we have

Example 1.2. In the case $N = 1$, definition (1.6) becomes

$$\Delta^{\pm}(x) = 1 - c_1^{\mp}\frac{e^{2ik_1x}}{ik_1} + c_1^{+}c_1^{-}\frac{e^{4ik_1x}}{(2ik_1)^2},$$

where $-2ik_1 =: b > 0$ and $c_1^{\pm} \in \boldsymbol{R}$ due to condition (I). From asymptotic behaviors of this function as $x \to \pm\infty$, it turns out that $c_1^{+}c_1^{-} > 0$ is necessary for (II) to hold. By an elementary discussion on quadratic functions $\Delta^{\pm}$ of $\frac{e^{bx}}{b} > 0$ it follows that condition (II) is satisfied if and only if $c_1^{+} > 0$, $c_1^{-} > 0$. Then, by applying the inversion formula (1.7) to $\Delta^{\pm}$ and observing that $\Delta^{\pm}\Delta^{\pm\prime\prime} - \left(\Delta^{\pm\prime}\right)^2 = 2bc_1^{\mp}e^{-bx}\Delta^{\mp}$, potential (U, Q) corresponding to $\{0, k_1, c_1^{\pm}\}$ is determined as

$$(1.10)\quad iQ(x) = -\frac{e^{-bx}(c_1^{+} - c_1^{-})\left(1 - c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)}{\left(1 + \frac{2c_1^{-}e^{-bx}}{b} + c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)\left(1 + \frac{2c_1^{+}e^{-bx}}{b} + c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)},$$

(1.11)

$$U(x)+Q(x)^2$$
$$= -be^{-bx}\frac{c_1^+\left(1+\frac{2c_1^-e^{-bx}}{b}+c_1^+c_1^-\frac{e^{-2bx}}{b^2}\right)^3+c_1^-\left(1+\frac{2c_1^+e^{-bx}}{b}+c_1^+c_1^-\frac{e^{-2bx}}{b^2}\right)^3}{\left(1+\frac{2c_1^-e^{-bx}}{b}+c_1^+c_1^-\frac{e^{-2bx}}{b^2}\right)^2\left(1+\frac{2c_1^+e^{-bx}}{b}+c_1^+c_1^-\frac{e^{-2bx}}{b^2}\right)^2}.$$

If $c_1^+ = c_1^- > 0$ then $Q \equiv 0$ and U are the reflectionless potentials with a single bound state for the standard Schrödinger equation.

The latter half part of the present paper is devoted to a construction of soliton solutions of

$$\begin{cases} u_t + w_x + uu_x = 0, \\ w_t + u_{xxx} + (uw)_x = 0. \end{cases} \tag{1.12}$$

This evolution system, which is viewed as a Boussinesq system (see Broer [5], Kupershmidt [24], Ablowitz and Clarkson [1]), is a recast of an isospectral flow

$$\begin{cases} \frac{1}{i}Q_t - 6QQ_x - U_x = 0, \\ \\ \frac{1}{i}U_t - 4Q_xU - 2QU_x + Q_{xxx} = 0, \end{cases} \tag{1.13}$$

associated with energy dependent Schrödinger equation (1.1) via the transformation

$$iQ = -\frac{u}{4}, \quad U = -\frac{w}{4} + \frac{u^2}{16} \tag{1.14}$$
$$\left(\Longleftrightarrow u = -4iQ, \quad w = -4(U+Q^2)\right).$$

By this transformation, inversion formula (1.7) is rewritten as

$$\begin{cases} u(x,t) = 2\dfrac{\partial}{\partial x}\left(\log\Delta^+(x,t) - \log\Delta^-(x,t)\right), \\ \\ w(x,t) = 2\dfrac{\partial^2}{\partial x^2}\left(\log\Delta^+(x,t) + \log\Delta^-(x,t)\right), \end{cases} \tag{1.15}$$

where $\Delta^\pm(x,t)$ are functions defined by (1.6) with $c_n^\pm(t) = c_n^\pm(0)e^{\mp 4k_n^2 t}$. Notice that $(u(x,t), w(x,t))$ is real because $\Delta^\pm(x,t)$ are real-valued that stems from the purely imaginaryness of Q.

The following result (the proof will be given in Section 5) guarantees that $(u(x,t), w(x,t))$ defined by (1.15) are solutions of (1.12).

Theorem 1.3. *Let k_n, $c_n^{\pm}(0)$, $n = 1, \cdots, N$, satisfy conditions* (I), (II) *in Theorem* 1.1 *and set $c_n^{\pm}(t) = c_n^{\pm}(0)e^{\mp 4k_n^2 t}$. Then $(u(x,t), w(x,t))$ defined by* (1.15) *satisfies* (1.12) *as long as $\Delta^{\pm}(x,t) > 0$ on $\boldsymbol{R}$.*

The study on integrability of the Boussinesq system (1.12) was begun by Kaup [21], in which soliton solutions of it can be found. In Kaup's solutions, w tend to nonzero constants as $x \to \pm\infty$. On the other hand, our soliton solutions defined by (1.15) tend to zero then. Originally Kaup's solution was found from the inverse scattering approach for a Schrödinger equation with $k^2 + m^2$, $m \neq 0$, in place of k^2 in (1.1). Inverse scattering theory for the Schrödinger equation has been developed by Tsutsumi [33], Sattinger and Szmigielski [32], van der Mee and Pivovarchik [34]. Hirota [10] obtained soliton solutions of the Boussinesq system as a reduction of solutions of the first modified KP (Kadomtsev-Petviashvili) equation. Moreover Hirota [11] established a way by which exact solutions of the Boussinesq system can be obtained from solutions of the first modified KP equation in Wronskian forms. Sachs [29, 30] used a Painlevé analysis, an expression via tau functions (similar to formula (1.15)) to find rational solutions of the system. In addition Matveev and Yavor [27] found a family of almost periodic solutions. For related topics, refer to Alber, Luther, and Miller [4], El, Grimshaw, and Komchatnov [7]. Among of these papers, soliton solutions in [10, 11] are most closely related with solutions in the present paper. Although Hirota's solutions and our solutions defined by (1.15) are mutually different, we shall discuss the difference and a similarity in detail in Section 6 of the present paper.

Our strategy for constructing N-soliton solutions of (1.12) consists in an inverse scattering method in Figure 1 based upon the scattering transform (ST) and its inverse transform (IST) established in Theorem 1.1.

The inverse scattering method in Figure 1 will be established in Section 4 (see Proposition 4.1). The method means that if Cauchy problem (1.12) admits a rapidly decreasing solution then it is constructed by (1.15). On the other hand Theorem 1.3 asserts that the assumption can be dropped. The proof will be given in Section 5 by checking that the pair $(u(x,t), w(x,t))$ of functions constructed above is indeed a solution of the Cauchy problem,

$$\begin{tikzcd}
\boxed{(u(x,0),w(x,0))} \arrow[r, leftrightarrow, "(1.14)"] \arrow[d, "\text{Time evolution of } (u,w)"] & \boxed{(U(x,0),Q(x,0))} \arrow[r, "\text{ST}"] \arrow[d, "\text{Time evolution of } (U,Q)"] & \boxed{\text{Data}\,\{0,k_n,c_n^{\pm}(0)\}} \arrow[d, "\text{Time evolution of data}"] \\
\boxed{(u(x,t),w(x,t))} \arrow[r, leftrightarrow, "(1.14)"] & \boxed{(U(x,t),Q(x,t))} & \boxed{\text{Data}\ \{0,k_n,c_n^{\pm}(0)e^{\mp 4k_n^2 t}\}} \arrow[l, "\text{IST}"']
\end{tikzcd}$$

Fig. 1. Inverse scattering method.

based upon an observation (Proposition 4.2) of a character of (1.13). It follows from item (6) after Theorem 1.1 that our solution $(u(x,t),w(x,t))$ satisfies

$$\int_{-\infty}^{\infty} u(x,t)dx = 0, \quad \int_{-\infty}^{\infty} w(x,t)dx = -16i\sum_{n=1}^{N} k_n \ (>0).$$

Notice that the *mass* of wave $w(x,t)$ is conserved as a positive quantity.

The Boussinesq system (1.12) can be interpreted as a model of wave propagation of shallow water, where $u = u(x,t)$ represents the velocity at a horizontal displacement x and $w = w(x,t)$ represents the elevation of wave's surface above the bottom. Actually, the first equation in (1.12) is no other than the incompressible Euler equation $\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\frac{1}{\rho}\nabla p$ of one-dimensional $\boldsymbol{v} = (u,0)$ with $\nabla p = \rho g \frac{\partial w}{\partial x}$, where ρ is the density, and the second equation is understood as a dispersive version obtained by adding the dispersion u_{xxx} to the equation of the continuity $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0$, with $\rho \propto w$, provided that the gravity force dominates over capillary tension at the surface (see Kamchatnov, Kraenkel and Umarov [18, page 356], see also Korteweg and de Vries [23, page 2]).

The following corresponds to Example 1.2:

Example 1.4. The one-soliton solution of (1.12) is obtained by putting $c_1^{\pm} = c_1^{\pm}(0)e^{\pm b^2 t}$ ($b = -2ik_1 > 0$, $c_1^{\pm}(0) > 0$) and using transformation (1.14). Then rewriting the resultant formula in terms of new parameters δ, $\rho \in \boldsymbol{R}$, in place of $c_1^{\pm}(0)$, defined by

$$e^{b\delta} = \frac{b}{\sqrt{c_1^{+}(0)c_1^{-}(0)}}, \quad e^{b^2\rho} = \sqrt{\frac{c_1^{+}(0)}{c_1^{-}(0)}},$$

leads to the following representation:

$$(1.16)\quad u(x,t) = 2b\frac{\left(e^{b^2(t+\rho)} - e^{-b^2(t+\rho)}\right)\sinh b(x+\delta)}{\left(\cosh b(x+\delta) + e^{b^2(t+\rho)}\right)\left(\cosh b(x+\delta) + e^{-b^2(t+\rho)}\right)},$$

$$(1.17)\quad w(x,t) = 2b^2\left(\frac{1 + e^{b^2(t+\rho)}\cosh b(x+\delta)}{\left(\cosh b(x+\delta) + e^{b^2(t+\rho)}\right)^2} + \frac{1 + e^{-b^2(t+\rho)}\cosh b(x+\delta)}{\left(\cosh b(x+\delta) + e^{-b^2(t+\rho)}\right)^2}\right).$$

By an elementary calculus we see that (1) $0 < w(x,t) \leq 2b^2$, *where* $2b^2$ *is attained only at* $x = -\delta$ *for each* t, (2) *for* t *such that* $\cosh b^2(t+\rho) \leq \frac{5\sqrt{5}}{2}$, *the elevation* $w(x,t)$ *has only one maximum* $2b^2$, *while, for* t *such that* $\cosh b^2(t+\rho) > \frac{5\sqrt{5}}{2}$, *it has a local maximum (other than the maximum* $2b^2$*) located at two points* x *(symmetric with respect to* $x = -\delta$*) such that* $|b(x+\delta)| = \pm b^2(t+\rho) + \log 2 + o(1)$ *as* $t \to \pm\infty$, (3) *the local maximum is monotonically decreasing (from* $\frac{6}{11}b^2$*) to* $\frac{1}{2}b^2$ *as* t *tends to* $\pm\infty$. The wave splits into three peaks as time evolves, which behave almost independently (see Figure 2).

We find that the parts of peaks with small amplitude $\frac{1}{2}b^2$ behave asymp-

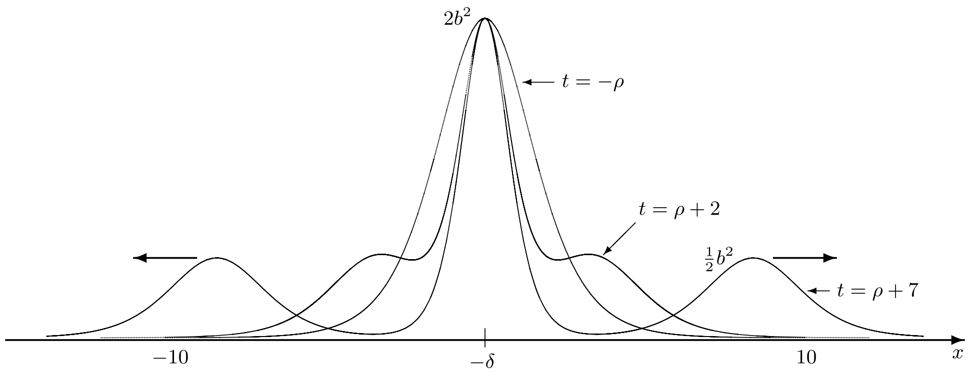

Fig. 2. Profile of $w(x,t)$ for $b = 1$, $\delta = 0$, $\rho = \log\sqrt{2}$ at $t = -\rho$, $t = \rho + 2$, $t = \rho + 7$. The part of the right peak with the amplitude $\approx \frac{1}{2}b^2$ is well approximated by the wave $\frac{1}{2}b^2\mathrm{sech}^2\frac{1}{2}(b(x+\delta) - b^2(t+\rho) - \log 2)$, going to $+\infty$ with the propagation velocity b.

totically (when $|t|$ is going to ∞) such as the solitary, travelling wave $\frac{1}{2}b^2\,\mathrm{sech}^2\frac{1}{2}\big(b|x+\delta| - b^2|t+\rho| - \log 2\big)$ with the propagation velocity b. For $t+\rho > 0$, the velocity $u(x,t)$ is positive (negative, resp.) if $x+\delta > 0$ (if $x+\delta < 0$, resp.). This makes the peaks of $w(x,t)$ with the height $\approx \frac{1}{2}b^2$ tend to $\pm\infty$ as time evolves.

Though $w(x,t)$ is a soliton corresponding to one pole k_1, it behaves such as a 3-soliton. This curious phenomenon can be explained by a typical behavior of the velocity $u(x,t)$ such as a boundary layer (see Figure 3). As is illustrated by the figure, for a fixed t, the velocity $u(x,t)$ disappears suddenly near a stationary point x_s at which the right soliton $w(x,t)$ has a peak with amplitude $\approx \frac{1}{2}b^2$; so the front of the peak has almost no motion, while the behind of it has a large motion. This makes a high elevation, the peak of $w(x,t)$. From a viewpoint of physical mechanism this phenomenon can be understood as a sort of *congestion* on water motion occurring based on an interaction between $u(x,t)$ and $w(x,t)$ in the system (1.12). From a viewpoint of mathematical structure it can be expected

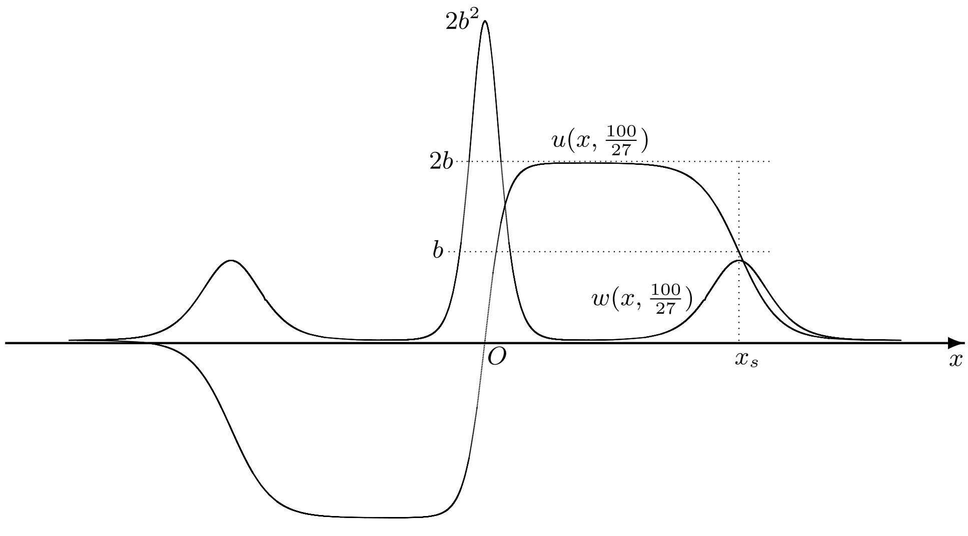

Fig. 3. Profile of $(u(x,\frac{100}{27}), w(x,\frac{100}{27}))$ at $t=\frac{100}{27}$ for $b=\frac{9}{5}$, $\delta=\rho=0$. The velocity $u(x,t)$ takes the value b approximately at the stationary point $x=x_s$ of the soliton $w(x,t)$, in general.

that $w(x,t)$ corresponding to N poles $k_1, \cdots, k_N$ splits into $(2N+1)$ peaks driven by boundary multi-layers of $u(x,t)$, as time evolves.

A hierarchy of the system (1.13) was found by Jaulent and Miodek [17]. For example, the second system of the hierarchy is given by

$$
\begin{cases} Q_t + Q_{xxx} - 6UQ_x - 6U_xQ - 30Q^2Q_x = 0, \\ U_t + U_{xxx} + 6QQ_{xxx} + 18Q_xQ_{xx} \\ \qquad -6UU_x - 24QQ_xU - 6Q^2U_x = 0, \end{cases} \tag{1.18}
$$

(see [17, equation (5.2)], where our Q becomes $\frac{1}{2}Q$). As was pointed out there, if $Q \equiv 0$ then the $2m$-th system of the hierarchy reduces to the m-th order KdV equation, such as (1.18) reduced to the KdV equation; notice that $\frac{1}{i}Q$ and U are real and so physical variables. Our inverse scattering method based upon Theorem 1.1 is applicable also to systems of the hierarchy: the method enables us to construct certain soliton solutions of the systems.

2. Forward Scattering Theory

We assume, in (1.1), that U is real and Q is purely imaginary and, in this section, we will work on (1.1) with the following conditions (see [19, 20]):

$$
\begin{aligned} &U(x) \in L^1_2(\boldsymbol{R}),\ Q(x) \in L^1_1(\boldsymbol{R}), \\ &Q(x) \text{ is absolutely continuous, and } Q'(x) \in L^1_2(\boldsymbol{R}). \end{aligned} \tag{2.1}
$$

Here $L^1_m(I)$ $(m = 0, 1, 2, \cdots)$ denotes the set of measurable functions $f(x)$ such that $\int_I (1+|x|^m)|f(x)|dx < \infty$. Let $f_\pm(x,k)$, $k \in \overline{\boldsymbol{C}_+}$, be the Jost solutions of (1.1), namely, solutions with the asymptotics

$$
\begin{aligned} &f_\pm(x,k) = e^{\pm ikx}[1+o(1)], \\ &f_\pm(x,k)' = \pm ike^{\pm ikx}[1+o(1)], \quad x \to \pm\infty. \end{aligned} \tag{2.2}
$$

These solutions are analytic with respect to k in $\boldsymbol{C}_+$. It follows from the uniqueness of the Jost solution that

$$
\overline{f_\pm(x,k)} = f_\pm(x,-\overline{k}), \quad k \in \overline{\boldsymbol{C}_+}. \tag{2.3}
$$

We denote the Jost solutions of (1.1) with $-Q(x)$ instead of $Q(x)$ by $f^-_\pm(x,k)$. Provided that $U(x)$ is real, $Q(x)$ is purely imaginary, we have four solutions of (1.1):

$$f_+(x,k), \quad f_-(x,k), \quad \overline{f^-_+(x,k)}, \quad \overline{f^-_-(x,k)},$$

by which, the scattering solutions $\psi_\rightarrow(x,k)$, $\psi_\leftarrow(x,k)$ in (1.2) are written as

$$\begin{aligned} \psi_\rightarrow(x,k) &= s_{11}(k)f_+(x,k) = \overline{f^-_-(x,k)} + s_{12}(k)f_-(x,k), \\ \psi_\leftarrow(x,k) &= s_{22}(k)f_-(x,k) = \overline{f^-_+(x,k)} + s_{21}(k)f_+(x,k). \end{aligned} \tag{2.4}$$

Since

$$W[f_\pm(x,k), \overline{f^-_\pm(x,k)}] = \mp 2ik, \quad k \in \boldsymbol{R} \setminus \{0\}, \tag{2.5}$$

where $W[f,g]$ denotes the Wronskian $fg' - f'g$, the coefficients $s_{\ell j}$ are determined as

$$\begin{aligned} &s_{11}(k) = s_{22}(k) = -\frac{2ik}{W[f_+(x,k), f_-(x,k)]}, \\ &s_{12}(k) = -\frac{W[f_+(x,k), \overline{f^-_-(x,k)}]}{W[f_+(x,k), f_-(x,k)]}, \quad s_{21}(k) = -\frac{W[\overline{f^-_+(x,k)}, f_-(x,k)]}{W[f_+(x,k), f_-(x,k)]}. \end{aligned} \tag{2.6}$$

We pick out basic properties (see [2]) of the scattering matrix $S(k)$:

Lemma 2.1. *Let $S^-(k)$ denote the scattering matrix for potential $(U, -Q)$. Then:*

(1) (*Symmetry*) $s_{11}(k) = s_{22}(k)$.

(2) (*Conjugate symmetry*) $S(-k) = \overline{S(k)}$, $k \in \boldsymbol{R}$, $\overline{s_{11}(-\overline{k})} = s_{11}(k)$, $k \in \overline{\boldsymbol{C}_+}$.

(3) (*Coupled unitarity*) $S(k)S^-(k)^* = I$.

A major difference between the case where both $U(x)$, $Q(x)$ are real (see [20]) and the case where $U(x)$ is real, $Q(x)$ is purely imaginary consists in the presence of singularities of the transmission coefficient $s_{11}(k)$ on $\boldsymbol{R}$. This difference stems from that the conservation $S(k)S(k)^* = I$ holds (conservative) in the former case and breaks down (nonconservative) in the latter case. For this reason we hereafter need to assume that

$$W[f_+(x,k), f_-(x,k)] \neq 0, \quad k \in \boldsymbol{R} \setminus \{0\}. \tag{2.7}$$

In the reflectionless case this condition is fulfilled since $s_{11}(k)\overline{s_{11}^-(k)} = 1$ on $\boldsymbol{R}$ by Lemma 2.1(3) and so, by (2.6) we have (2.7). From this observation, we draw the following

LEMMA 2.2. *We assume* (2.1). *Then, in the reflectionless scattering,*

(1) $s_{11}(k)$, $s_{11}^-(k)$ *are continuous functions on* $\boldsymbol{R}$.

(2) $s_{11}(k)$ *can be analytically continued to a meromorphic function in the upper half plane* $\boldsymbol{C}_+$ *having finitely many poles* k_n, $n = 1, \cdots, N$.

(3) $f_+(x,0)$ *and* $f_-(x,0)$ *are linearly dependent.*

PROOF. In the reflectionless scattering, $s_{11}(k)\overline{s_{11}^-(k)} = 1$ on $\boldsymbol{R}$. This is written as $s_{11}(k) = \overline{s_{11}^-(k)}^{-1}$. Since the right-hand side is continuous on $\boldsymbol{R} \setminus \{0\}$, $s_{11}(k)$ is also continuous there. But $s_{11}(k)$ is originally continuous at $k = 0$ (see [19, Lemma 2.2], which remains valid for purely imaginary-valued functions Q). Hence it is a continuous function on $\boldsymbol{R}$. Assertion (2) follows from the asymptotic behavior

$$s_{11}(k) = \gamma + O\left(\frac{1}{k}\right), \quad \gamma := e^{-i\int_{-\infty}^{\infty} Q(r)dr}, \quad |k| \to \infty \tag{2.8}$$

(see [19, Lemma 2.3(1)]). Since $s_{11}(0) \neq 0$, from (2.6), we have $W[f_+(x,0), f_-(x,0)] = 0$ (which is referred to as the exceptional case in [2]). This proves assertion (3). □

Our study on reflectionless scattering is based upon the following

PROPOSITION 2.3. *Let* U *be real,* Q *be purely imaginary and assume* (2.1). *Then, in the reflectionless scattering,*

(1) $Q(x)$ *satisfies*

$$\int_{-\infty}^{\infty} Q(r)dr = 0.$$

(2) $s_{11}^{-}(k) = s_{11}(k)$ *on* $\overline{\boldsymbol{C}_{+}}$, *which is written as*

$$(2.9)\qquad s_{11}(k) = \prod_{n=1}^{N}\left(\frac{k-\overline{k_n}}{k-k_n}\right)^{m_n}, \quad k \in \overline{\boldsymbol{C}_{+}}.$$

in terms of poles k_n *and their orders* m_n, $n = 1, \cdots, N$.

Proof. Let k_n, $n = 1, \cdots, N$, k_n^{-}, $n = 1, \cdots, N^{-}$ be poles of $s_{11}(k)$, $s_{11}^{-}(k)$ and let m_n, m_n^{-} be orders of the poles. Then, by the Poisson formula and (2.8), we get, for $k \in \boldsymbol{C}_{+}$,

$$s_{11}(k) = \gamma \prod_{n=1}^{N}\left(\frac{k-\overline{k_n}}{k-k_n}\right)^{m_n} \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\left|s_{11}(\zeta)\gamma^{-1}\right|^2}{\zeta-k}d\zeta\right\},$$

$$s_{11}^{-}(k) = \gamma^{-1} \prod_{n=1}^{N^{-}}\left(\frac{k-\overline{k_n^{-}}}{k-k_n^{-}}\right)^{m_n^{-}} \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log|s_{11}^{-}(\zeta)\gamma|^2}{\zeta-k}d\zeta\right\}.$$

Since $|s_{11}(\zeta)||s_{11}^{-}(\zeta)| = 1$ for $\zeta \in \boldsymbol{R}$, the product of $s_{11}(k)$ and $s_{11}^{-}(k)$ yields

$$s_{11}(k)s_{11}^{-}(k) = \prod_{n=1}^{N}\left(\frac{k-\overline{k_n}}{k-k_n}\right)^{m_n} \prod_{n=1}^{N^{-}}\left(\frac{k-\overline{k_n^{-}}}{k-k_n^{-}}\right)^{m_n^{-}}, \quad k \in \boldsymbol{C}_{+}.$$

Since, by Lemma 2.2(1), $s_{11}(k)$ and $s_{11}^{-}(k)$ are continuous on $\boldsymbol{R}$, this equality still holds on $\boldsymbol{R}$. This, together with $s_{11}^{-}(k)^{-1} = \overline{s_{11}(\overline{k})}$ for $k \in \boldsymbol{R}$, leads to the following equality:

$$(2.10)\qquad s_{11}(k)\prod_{n=1}^{N}\left(\frac{k-k_n}{k-\overline{k_n}}\right)^{m_n} = \overline{s_{11}(\overline{k})}\prod_{n=1}^{N^{-}}\left(\frac{k-\overline{k_n^{-}}}{k-k_n^{-}}\right)^{m_n^{-}}, \quad k \in \boldsymbol{R}.$$

The function in the left side of (2.10) is analytic and bounded on the upper half plane and that in its right side is analytic and bounded on the lower half plane. In view of Liouville's theorem in theory of analytic functions, this implies that

$$s_{11}(k)\prod_{n=1}^{N}\left(\frac{k-k_n}{k-\overline{k_n}}\right)^{m_n} = C, \quad k \in \overline{\boldsymbol{C}_{+}},$$

with some constant C. It follows from (2.8) that the constant must be $C = \gamma \in \boldsymbol{R}$. Hence, from (2.10), we have, for $k \in \overline{\boldsymbol{C}_+}$,

$$s_{11}(k) = \gamma \prod_{n=1}^{N} \left(\frac{k - \overline{k_n}}{k - k_n} \right)^{m_n} = \gamma \prod_{n=1}^{N^-} \left(\frac{k - \overline{k_n^-}}{k - k_n^-} \right)^{m_n^-} .$$

Due to the uniqueness of the factorization, this implies that $N^- = N$, and that, by renumbering $k_1^-, \cdots k_N^-$, we get $k_1^- = k_1, \cdots, k_N^- = k_N$, $m_1^- = m_1, \cdots, m_N^- = m_N$, namely, that there exists a permutation $\sigma \in \mathfrak{S}_N$ such that $k_n^- = k_{\sigma(n)}$, $m_n^- = m_{\sigma(n)}$. We thus have

$$(2.11) \qquad \begin{aligned} s_{11}(k) &= e^{-i \int_{-\infty}^{\infty} Q(r)dr} \prod_{n=1}^{N} \left(\frac{k - \overline{k_n}}{k - k_n} \right)^{m_n}, \\ s_{11}^-(k) &= e^{i \int_{-\infty}^{\infty} Q(r)dr} \prod_{n=1}^{N} \left(\frac{k - \overline{k_n}}{k - k_n} \right)^{m_n}, \end{aligned}$$

for $k \in \overline{\boldsymbol{C}_+}$.

On the other hand, from (2.4) and $\overline{f_\pm^-(x,0)} = f_\pm(x,0)$, we get $s_{11}(0) = \pm 1$. This, together with $s_{11}(0)\overline{s_{11}^-(0)} = 1$, leads to $s_{11}^-(0) = s_{11}(0) = \pm 1$. This is compatible with

$$s_{11}(0) = e^{-i \int_{-\infty}^{\infty} Q(r)dr} \prod_{n=1}^{N} \left(\frac{\overline{k_n}}{k_n} \right)^{m_n}, \quad s_{11}^-(0) = e^{i \int_{-\infty}^{\infty} Q(r)dr} \prod_{n=1}^{N} \left(\frac{\overline{k_n}}{k_n} \right)^{m_n}$$

only if $\int_{-\infty}^{\infty} Q(r)dr = 0$. This, combined with (2.11), proves the lemma. $\square$

The following properties of transformation kernels will be employed in our work.

Lemma 2.4. *Let U be real, Q be purely imaginary, assume* (2.1), *and let $A^\pm(x,y)$ be transformation kernels for $(U, \pm Q)$. Then:*

(1) *Functions $A^\pm(x,y)$, which belong to $L^1(x,\infty) \cap L^\infty(x,\infty)$ as functions of y for each x, are real.*

(2) *Potentials $(U, \pm Q)$ and $A^\pm(x,x)$ are connected by*

$$A^{\pm}(x,x) = \frac{1}{2} e^{\pm i \int_x^\infty Q(r)dr} \left(\int_x^\infty [U(r) + Q(r)^2] dr \mp iQ(x) \right). \tag{2.12}$$

Proof. By taking the complex conjugate of (1.8), using (2.3), and taking the uniqueness of transformation kernels into account, it follows that $\overline{A^{\pm}(x,y)} = A^{\pm}(x,y)$. This proves (1). By setting $x = y$ in (1.9), we get

$$\begin{aligned} & A^{\pm}(x,x) \mp i \int_x^\infty Q(s) A^{\pm}(s,s) ds \\ & = \frac{1}{2} \int_x^\infty U(s) e^{\pm i \int_s^\infty Q(r)dr} ds \mp \frac{i}{2} Q(x)\, e^{\pm i \int_x^\infty Q(r)dr}. \end{aligned}$$

This gives an integral equation for each $A^{\pm}(x,x)$, which is solved as (2.12). This proves (2). □

We consider the reflectionless scattering and set

$$F^{\pm}(y) = -\sum_{n=1}^{N} c_n^{\pm} e^{ik_n y}, \quad y \in \boldsymbol{R}. \tag{2.13}$$

in terms of the constants $c_n^{\pm}$ defined in (1.4). It follows from the conjugate symmetry in Lemma 2.1(2) that k_n is a pole of $s_{11}(k)$ if and only if $-\overline{k_n}$ is a pole of it, and hence, that there exists a permutation $\sigma \in \mathfrak{S}_N$ such that $k_{\sigma(n)} = -\overline{k_n}$. By definition, such a permutation is uniquely determined and, in view of $ik_{\sigma(n)} = \overline{ik_n}$, the permutation is involutive: $\sigma \circ \sigma = \mathrm{Id}$, where Id denotes the identity permutation. We mention that n is a fixed point of the permutation, namely $\sigma(n) = n$, if and only if $ik_n < 0$, namely k_n lies on the imaginary axis in $\boldsymbol{C}_+$.

Lemma 2.5. *The followings hold:*

(1) *The constants $c_n^{\pm}$ are nonzero numbers.*

(2) *The constants $c_n^{\pm}$ for $k_{\sigma(n)}$ and k_n are mutually conjugate : $c_{\sigma(n)}^{\pm} = \overline{c_n^{\pm}}$.*

(3) *The functions $F^{\pm}(y)$ are real.*

Proof. It follows from (2.9) that $\mathrm{Res}_{k=k_n} s_{11}(k) = 2i\, m_n \mathrm{Im}\, k_n$. This proves (1). By (2.3) we have $d^{\pm}_{\sigma(n)} = \overline{d^{\pm}_n}$. Hence, by remembering definition (1.4) and the conjugate symmetry $s_{11}(-\overline{k}) = \overline{s_{11}(k)}$, we can draw (2). Since σ is a permutation, from definition (2.13), we have

$$F^{\pm}(y) = -\sum_{n=1}^{N} c^{\pm}_{\sigma(n)}\, e^{ik_{\sigma(n)}y} = -\sum_{n=1}^{N} \overline{c^{\pm}_n}\, e^{\overline{ik_n y}} = \overline{F^{\pm}(y)}.$$

This proves (3). □

Generally transformation kernels are related with scattering data via GLM equations. Such integral equations can be deduced by a standard calculus of residues (see [8], [26, Chapter 3.5], [20, Proposition 2.2]). In our reflectionless scattering, the equation is given as follows.

Proposition 2.6. *Suppose that $U(x)$ is real, $Q(x)$ is purely imaginary, and assume* (2.1). *If $\{0, k_n, c^{\pm}_n\}$ is the scattering data for $(U, \pm Q)$ then $F^{\pm}(y)$ defined by* (2.13) *are related with $A^{\pm}(x,y)$ by the following, coupled integral equation*

$$(2.14)\qquad A^{\mp}(x,y) - \int_x^{\infty} A^{\pm}(x,r)F^{\pm}(r+y)dr - e^{\pm i\int_x^{\infty} Q(r)dr} F^{\pm}(x+y) = 0, \quad x \le y.$$

Proof. From (2.4), we get

$$(2.15)\qquad s_{11}(k)f_-(x,k) - e^{-i\int_x^{\infty} Q(r)dr} e^{-ikx} = \overline{f^-_+(x,k)} - e^{-i\int_x^{\infty} Q(r)dr} e^{-ikx}, \quad k \in \boldsymbol{R}.$$

By Lemma 2.4(1), $A^-(x,y)$ is real. So, from (1.8), we have

$$\overline{f^-_+(x,k)} = e^{-i\int_x^{\infty} Q(r)dr} e^{-ikx} + \int_x^{\infty} A^-(x,y)e^{-iky}dy, \quad k \in \boldsymbol{R}.$$

Hence (2.15) becomes

$$s_{11}(k)f_-(x,k) - e^{-i\int_x^{\infty} Q(r)dr} e^{-ikx} = \int_x^{\infty} A^-(x,y)e^{-iky}dy, \quad k \in \boldsymbol{R}.$$

This implies that the inverse Fourier transform of the left side is given by

$$A^-(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(s_{11}(k)f_-(x,k)e^{ikx} - e^{-i\int_x^\infty Q(r)dr}\right)e^{ik(y-x)}dk, \quad x < y.$$

It follows from Proposition 2.3, the transformation kernel representation of $f_-(x,k)$: $f_-(x,k) = e^{i\int_{-\infty}^x Q(r)dr}e^{-ikx} - \int_{-\infty}^x A_-(x,y)e^{-iky}dy$, and the Riemann-Lebesgue lemma that

$$s_{11}(k)f_-(x,k)e^{ikx} - e^{-i\int_x^\infty Q(r)dr} = o(1),$$

as $|k| \to \infty$ in $\overline{\boldsymbol{C}_+}$. By Lemma 2.2(1), this function is continuous on $\boldsymbol{R}$.

Let $y - x > 0$. Then, by means of the Jordan lemma in the calculus of residues, we get

$$\begin{aligned} A^-(x,y) &= i\sum_{n=1}^{N}\left(\operatorname{Res}_{k=k_n}s_{11}(k)\right)f_-(x,k_n)e^{ik_ny} \\ &= i\sum_{n=1}^{N}\left(\operatorname{Res}_{k=k_n}s_{11}(k)\right)d_n^+f_+(x,k_n)e^{ik_ny} \\ &= -\sum_{n=1}^{N}c_n^+\left(e^{i\int_x^\infty Q(r)dr}e^{ik_n(x+y)} + \int_x^\infty A^+(x,r)e^{ik_n(r+y)}dr\right). \end{aligned}$$

This gives an equation in (2.14) with F^+. Another one with F^- is deduced similarly. □

In order to derive a recovery algorithm for (U,Q) of equation (1.1) from the scattering data $\{0, k_n, c_n^\pm\}$, we introduce functions $\hat{J}^\pm(x,y)$ defined by

$$\hat{J}^\pm(x,y) := -\frac{1}{f_+(x,0)}\int_y^\infty A^\pm(x,s)ds, \quad x \le y, \tag{2.16}$$

for each $x \in \boldsymbol{R}$ such that $f_+(x,0) \neq 0$. Note that $f_+^-(x,0) = f_+(x,0)$ for $k = 0$.

Proposition 2.7. *Under the same assumption as in Proposition* 2.6, *for each $x \in \boldsymbol{R}$ such that $f_+(x,0) \neq 0$, functions $J^\pm(x,y) = \hat{J}^\pm(x,y)$ satisfy the followings.*

(1) *Relation*

$$f_+(x,0)(1+J^{\pm}(x,x)) = e^{\pm i\int_x^{\infty} Q(r)dr}. \tag{2.17}$$

(2) *Integral equation*

$$\begin{aligned} J^{\mp}(x,y) &+ \int_x^{\infty} J^{\pm}(x,r)F^{\pm}(r+y)dr \\ &+ \int_x^{\infty} F^{\pm}(r+y)dr = 0, \quad x \le y. \end{aligned} \tag{2.18}$$

(3) *Formula*

$$\frac{2J_y^{\pm}(x,x)}{1+J^{\pm}(x,x)} = \int_x^{\infty}[U(r)+Q(r)^2]dr \mp iQ(x). \tag{2.19}$$

PROOF. (1) is immediate by setting $k=0$ in (1.8) and using definition (2.16). We integrate (2.14) and rewrite the resultant in terms of $J^{\pm}(x.y)$. Then, by virtue of (2.17), we get

$$\begin{aligned} J^{\pm}(x,y) &+ \int_x^{\infty} \partial_r J^{\pm}(x,r)dr \int_y^{\infty} F^{\pm}(r+s)ds \\ &+ (1+J^{\pm}(x,x)) \int_y^{\infty} F^{\pm}(x+s)ds = 0. \end{aligned}$$

By performing an integration by parts, this is written in the form (2.18). Differentiating (2.16) with respect to y yields $A^{\pm}(x,y) = f_+(x,0)J_y^{\pm}(x,y)$. Hence, substituting this and (2.17) into (2.12) we obtain (2.19). Notice that relation (2.17) implies $1+J^{\pm}(x,x) \neq 0$ for x such that $f_+(x,0) \neq 0$. □

The function $F^{\pm}(y)$ defined in (2.13) is a finite dimensional function. Hence a GLM equation (2.18) gives a degenerate (finite dimensional) integral equation with unknowns $J^{\pm}(x,y)$ for each $x \in \boldsymbol{R}$. Its Fredholm determinant is given by

$$D(x) := \det(I - B^+B^-), \tag{2.20}$$

where $B^{\pm}$ are matrices defined in (1.5).

PROPOSITION 2.8. *Under condition* (I) *in Theorem* 1.1*, we have:*

(1) *$D(x)$ is real-valued functions on $\boldsymbol{R}$, whose zeros are discrete.*

(2) *The functions*

$$(e^{ik_1 y} \ \cdots \ e^{ik_N y})(I - B^{\mp}B^{\pm})\check{}\,(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp})$$

are real-valued functions on $\boldsymbol{R}^2$.

(3) *For each x such that $D(x) \neq 0$, integral equation* (2.18) *is uniquely solved as*

$$J^{\pm}(x,y) = (e^{ik_1 y} \ \cdots \ e^{ik_N y})(I - B^{\mp}B^{\pm})^{-1}(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}), \quad x \leq y. \tag{2.21}$$

Proof. Let P_σ be a fundamental matrix associated with σ. Since $\sigma \in \mathfrak{S}_N$ is involutive, $P_\sigma^{-1} = P_\sigma$ It follows from definition (1.5) and condition (I) that

$$\overline{B^{\pm}} = P_\sigma B^{\pm} P_\sigma, \quad \overline{\boldsymbol{v}^{\pm}} = \boldsymbol{v}^{\pm} P_\sigma. \tag{2.22}$$

Accordingly

$$\overline{D(x)} = \det(I - P_\sigma B^{+} P_\sigma P_\sigma B^{-} P_\sigma) = \det(P_\sigma(I - B^{+}B^{-})P_\sigma) = D(x),$$

due to $\det P_\sigma = \pm 1$. This shows that $D(x)$ is real. From this we obtain

$$\overline{D(x)} = \det(I - P_\sigma B^{+} P_\sigma P_\sigma B^{-} P_\sigma) = \det(P_\sigma(I - B^{+}B^{-})P_\sigma) = D(x),$$

since $\det P_\sigma = \pm 1$. It is clear from definition that $D(x)$ is an entire function of x. Hence zeros are discrete. This proves assertion (1).

By using the condition (I), (2.22), and $(AB)\check{} = B\check{}A\check{}$ the complex conjugate of the function in (2) is written as

$$\begin{aligned}
&(e^{ik_{\sigma(1)} y} \ \cdots \ e^{ik_{\sigma(N)} y})(I - \overline{B^{\mp}}\,\overline{B^{\pm}})\check{}\,(\overline{B^{\mp}}\,\overline{\boldsymbol{v}^{\pm}} - \overline{\boldsymbol{v}^{\mp}}) \\
&= (e^{ik_1 y} \ \cdots \ e^{ik_N y})P_\sigma(I - P_\sigma B^{\mp} P_\sigma P_\sigma B^{\pm} P_\sigma)\check{}\,(P_\sigma B^{\mp} P_\sigma\, P_\sigma \boldsymbol{v}^{\pm} - P_\sigma \boldsymbol{v}^{\mp}) \\
&= (e^{ik_1 y} \ \cdots \ e^{ik_N y})P_\sigma\,(P_\sigma(I - B^{\mp}B^{\pm})P_\sigma)\check{}\,P_\sigma(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}) \\
&= (e^{ik_1 y} \ \cdots \ e^{ik_N y})(I - B^{\mp}B^{\pm})\check{}\,(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}).
\end{aligned}$$

This proves assertion (2).

It follows from definition (2.13) that

$$
\begin{aligned}
&\int_x^\infty J^\pm(x,r)F^\pm(r+y)dr \\
&= -\int_x^\infty \sum_{n=1}^N c_n^\pm e^{ik_n(r+y)} \begin{pmatrix} e^{ik_1 r} & \cdots & e^{ik_N r} \end{pmatrix} dr\, (I - B^\mp B^\pm)^{-1} (B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp) \\
&= -\begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} \times \\
&\quad \int_x^\infty \begin{pmatrix} c_1^\pm e^{ik_1 r} \\ \vdots \\ c_N^\pm e^{ik_N x} \end{pmatrix} \begin{pmatrix} e^{ik_1 r} & \cdots & e^{ik_N r} \end{pmatrix} dr\, (I - B^\mp B^\pm)^{-1} (B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp) \\
&= -\begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} \int_x^\infty (B^\pm)' dr\, (I - B^\mp B^\pm)^{-1} (B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp) \\
&= \begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} B^\pm (I - B^\mp B^\pm)^{-1} (B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp).
\end{aligned}
$$

This, together with

$$
B^\pm (I - B^\mp B^\pm)^{-1} = (I - B^\pm B^\mp)^{-1} B^\pm, \tag{2.23}
$$

leads to

$$
\begin{aligned}
&\int_x^\infty J^\pm(x,r)F^\pm(r+y)dr + \int_x^\infty F^\pm(r+y)dr \\
&= \begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} \big((I - B^\pm B^\mp)^{-1} B^\pm (B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp) + \boldsymbol{v}^\pm\big) \\
&= \begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} (I - B^\pm B^\mp)^{-1} \big(B^\pm (B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp) + (I - B^\pm B^\mp)\boldsymbol{v}^\pm\big) \\
&= -\begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} (I - B^\pm B^\mp)^{-1} (B^\pm \boldsymbol{v}^\mp - \boldsymbol{v}^\pm) \\
&= -J^\mp(x,y).
\end{aligned}
$$

Therefore $J^\mp(x,y)$ satisfies (2.18). Since $D(x)$ is the Fredholm determinant of the equation, it follows from the Fredholm alternative that this solution is unique. □

If $\{0, k_n, c_n^\pm\}$ is the scattering data for some (U,Q) satisfying (2.1) then we have two definitions of solutions to the GLM equation (2.18); the first definition is (2.16), which is available for each $x \in \boldsymbol{R}$ such that $f_+(x,0) \neq 0$, and the second definition is (2.21) which is available for each $x \in \boldsymbol{R}$ such that $D(x) \neq 0$. Since $D(x)$ is the Fredholm determinant of the equation, it follows from the Fredholm alternative that they are both equivalent to each other, provided that $f_+(x,0) \neq 0$, $D(x) \neq 0$.

Concerning the solution $J^{\pm}(x,y)$ defined by (2.21), we have

LEMMA 2.9. *If $D(x) \neq 0$ then $1 + J^{\pm}(x,x) \neq 0$, and, for each x such that $D(x) \neq 0$,*

$$\frac{d}{dx} \log D(x) = -\frac{J_x^{\pm}(x,x) - J_y^{\pm}(x,x)}{1 + J^{\pm}(x,x)}. \tag{2.24}$$

PROOF. This lemma is obtained by a linear algebraic calculus similar to that in [20, Lemma 3.3] (just replacement of J, $\overline{J}$ there by $J^{\pm}$, $J^{\mp}$). $\square$

We now let $\Delta^{\pm}(x)$ be functions defined by (1.6). By Proposition 2.8(1), (2), the function $\Delta^{\pm}(x)$ are real. Moreover, due to $(\det A)A^{-1} = \tilde{A}$, for x such that $D(x) \neq 0$, definition (1.6) is rewritten as

$$\Delta^{\pm}(x) = D(x)(1 + J^{\pm}(x,x)), \tag{2.25}$$

in terms of $J^{\pm}(x,y)$ defined by (2.21). Notice that (2.25) is a straightforward definition of $\Delta^{\pm}(x)$, however, not valid unless $D(x) \neq 0$.

LEMMA 2.10. *Assume that (U,Q) satisfies (2.1) and let $\{0, k_n, c_n^{\pm}\}$ be its scattering data. Then, for any $x \in \boldsymbol{R}$,*

$$D(x) = f_+(x,0)\Delta^{\pm}(x)\, e^{\mp i \int_x^{\infty} Q(r)dr}, \tag{2.26}$$

$$\Delta^{+}(x)\, e^{-i \int_x^{\infty} Q(r)dr} = \Delta^{-}(x)\, e^{i \int_x^{\infty} Q(r)dr}. \tag{2.27}$$

PROOF. If $D(x) \neq 0$, $f_+(x,0) \neq 0$ then $J^{\pm}(x,y) = \hat{J}^{\pm}(x,y)$. Proposition 2.7(1) tells us that

$$f_+(x,0) \neq 0 \implies f_+(x,0)(1 + J^{\pm}(x,x)) = e^{\pm i \int_x^{\infty} Q(r)dr}.$$

On the other hand, by (2.25),

$$D(x) \neq 0 \implies D(x)(1 + J^{\pm}(x,x)) = \Delta^{\pm}(x).$$

Accordingly, if $D(x) \neq 0$, $f_+(x, 0) \neq 0$ then (2.26) holds. By Proposition 2.8(1) the set $\{x \in \boldsymbol{R} \,|\, D(x) = 0\}$ is discrete. Moreover, since $f_+(x, 0)$ is a nonzero solution of $f'' - U(x)f = 0$, the set $\{x \in \boldsymbol{R} \,|\, f_+(x, 0) = 0\}$ is also discrete. Hence (2.26) holds for almost every $x \in \boldsymbol{R}$. Since functions in both sides of it are continuous, (2.26) holds for *any* $x \in \boldsymbol{R}$. It follows from (2.26) that (2.27) holds for each x such that $f_+(x, 0) \neq 0$. But both sides of (2.27) are also continuous. Hence (2.27) holds for *any* $x \in \boldsymbol{R}$. □

We next deduce formula (1.7).

LEMMA 2.11. *Assume that* (U, Q) *satisfies* (2.1) *and let* $\{0, k_n, c_n^{\pm}\}$ *be its scattering data. Then* (1.7) *holds for each* $x \in \boldsymbol{R}$ *such that* $\Delta^{\pm}(x) \neq 0$.

PROOF. By (2.26), if $D(x) \neq 0$ then $f_+(x, 0) \neq 0$, and therefore, $J^{\pm}(x, y) = \hat{J}^{\pm}(x, y)$. Since, by Proposition 2.7, $\hat{J}^{\pm}(x, y)$ satisfies (2.19), the function $J^{\pm}(x, y)$ defined by (2.21) satisfies (2.19) for each x such that $D(x) \neq 0$.

If $D(x) \neq 0$ then, by (2.26), $\Delta^{\pm}(x) \neq 0$. By (2.25), (2.24) we obtain

$$\begin{aligned} \frac{d}{dx} \log \Delta^{\pm}(x) &= -\frac{J_x^{\pm}(x, x) - J_y^{\pm}(x, x)}{1 + J^{\pm}(x, x)} + \frac{J_x^{\pm}(x, x) + J_y^{\pm}(x, x)}{1 + J^{\pm}(x, x)} \\ &= \frac{2J_y^{\pm}(x, x)}{1 + J^{\pm}(x, x)}. \end{aligned}$$

This, combined with (2.19), shows that

$$\frac{d}{dx} \log \Delta^{\pm}(x) = \int_x^{\infty} [U(r) + Q(r)^2] dr \mp iQ(x) \tag{2.28}$$

holds for each $x \in \boldsymbol{R}$ such that $D(x) \neq 0$.

We now let $x \in \boldsymbol{R}$ be a point for which $\Delta^{\pm}(x) \neq 0$. Note that, by (2.27), $\Delta^{\pm}(x)$ have zeros in common. Since the set $\{x \in \boldsymbol{R} \,|\, D(x) = 0\}$ is discrete we may take a neighborhood of x where $D(x) \neq 0$ and so (2.28) holds. Hence, by taking the limit of (2.28) to x, we show that that (2.28) is valid for x, namely, that (2.28) holds for each x such that $\Delta^{\pm}(x) \neq 0$. Since (2.28) is equivalent to (1.7), this proves the lemma. □

So far we are concerned with the *forward* scattering problem under assumption (2.1). The conclusion on the forward problem is:

PROPOSITION 2.12. *Conditions* (I), (II) *in Theorem* 1.1 *are necessary for* $\{0, k_n, c_n^{\pm}\}$ *to be scattering data for some* (U, Q) *satisfying* (2.1).

PROOF. It suffices to show that (II) is necessary, because the necessity of (I) was already shown in an observation just after definition (2.13) and Lemma 2.5(2). It follows from (2.27) that

$$\{x \in \boldsymbol{R} \,|\, \Delta^+(x) = 0\} = \{x \in \boldsymbol{R} \,|\, \Delta^-(x) = 0\},$$

and the orders $m^{\pm}$ of a zero x_0 (if it exists) are independent of $\pm$. So, if there were a zero $x_0 \in \boldsymbol{R}$ of $\Delta^+(x)$ then $\Delta^{\pm}(x) = (x - x_0)^m \Delta_0^{\pm}(x)$ with analytic functions $\Delta_0^{\pm}(x)$ near x_0 satisfying $\Delta_0^{\pm}(x_0) \neq 0$. Hence, by logarithmic differentiation, we get

$$\frac{d}{dx} \log \Delta^{\pm}(x) = \frac{m}{x - x_0} + \frac{(\Delta_0^{\pm}(x))'}{\Delta_0^{\pm}(x)},$$

where $\frac{(\Delta_0^{\pm}(x))'}{\Delta_0^{\pm}(x)}$ are analytic in some neighborhood of x_0. Therefore from (1.7), which is valid for x $(\neq x_0)$ near x_0 by Lemma 2.11, it follows that, though $Q(x)$ is still continuous at x_0, $U(x)$ is not integrable at x_0. This contradicts the assumption $U(x) \in L^1(\boldsymbol{R})$ in (2.1). Consequently if $\{0, k_n, c_n^{\pm}\}$ is the scattering data for (U, Q) satisfying (2.1) then $\Delta^{\pm}(x) \neq 0$ everywhere. It is clear from definition (1.6) with $\operatorname{Re} ik_n < 0$ that $\Delta^{\pm}(x) \to 1$ as $x \to +\infty$, and hence $\Delta^{\pm}(x) > 0$ on $\boldsymbol{R}$. $\square$

REMARK 2.13. In Lemma 2.11 we have proved (1.7) under the assumption $\Delta^{\pm}(x) \neq 0$. This assumption however can be dropped because, as is shown in the proof of Proposition 2.12, if $\{0, k_n, c_n^{\pm}\}$ is the scattering data for (U, Q) satisfying (2.1) then $\Delta^{\pm}(x) \neq 0$ holds everywhere. Unlike this, the assumption $D(x) \neq 0$ employed in several propositions can *not* be dropped. For instance, $D(x)$ in Example 1.2 has a zero for each $\{0, k_1, c_1^{\pm}\}$ with $b, c_1^{\pm} > 0$.

3. Inverse Scattering Theory

We now embark on the *inverse* scattering problem. Since we have inversion formula (1.7), our task becomes to show that the scattering data for potential (U, Q) defined by the formula coincides with $\{0, k_n, c_n^{\pm}\}$. Throughout this section we assume that a prescribed triplet $\{0, k_n, c_n^{\pm}\}$ satisfies (I), (II) in Theorem 1.1, where $\Delta^{\pm}(x)$ are real functions on $\boldsymbol{R}$ defined by (1.6).

We begin with the following

LEMMA 3.1. *Let α_n, β_n, $n = 1, \cdots, N$, be complex numbers.*

(1) *If $\alpha_n + \beta_m \neq 0$ for any n, m, then*

$$\begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \cdots & \frac{1}{\alpha_1+\beta_j} & \cdots & \frac{1}{\alpha_1+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_N+\beta_1} & \cdots & \frac{1}{\alpha_N+\beta_j} & \cdots & \frac{1}{\alpha_N+\beta_N} \end{vmatrix} = \frac{\displaystyle\prod_{m<n}(\alpha_m - \alpha_n) \prod_{m<n}(\beta_m - \beta_n)}{\displaystyle\prod_{m,n=1}^{N}(\alpha_m + \beta_n)}.$$

(2) *If $\alpha_n + \beta_m \neq 0$ for any n, m and $\alpha_n \neq 0$ for any n, then, for each $z \in \boldsymbol{C}$ such that $z \neq \beta_n$ for any n,*

$$\sum_{j=1}^{N} \frac{\beta_j}{z - \beta_j} \begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \cdots & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_1+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_j+\beta_1} & \cdots & \frac{1}{\alpha_j} & \cdots & \frac{1}{\alpha_j+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_N+\beta_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\beta_N} \end{vmatrix}$$

$$= (-1)^N \prod_{n=1}^{N} \frac{\beta_n}{\alpha_n} \begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \cdots & \frac{1}{\alpha_1+\beta_j} & \cdots & \frac{1}{\alpha_1+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_j+\beta_1} & \cdots & \frac{1}{\alpha_j+\beta_j} & \cdots & \frac{1}{\alpha_j+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_N+\beta_1} & \cdots & \frac{1}{\alpha_N+\beta_j} & \cdots & \frac{1}{\alpha_N+\beta_N} \end{vmatrix} \times$$

$$\left(1 - \prod_{n=1}^{N} \frac{z + \alpha_n}{z - \beta_n}\right).$$

(3) *If* $\alpha_n + \beta_m \neq 0$ *for any* n, m *and* $\alpha_n \neq 0$ *for any* n, *then*

$$\sum_{j=1}^{N} \begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \cdots & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_1+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_j+\beta_1} & \cdots & \frac{1}{\alpha_j} & \cdots & \frac{1}{\alpha_j+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_N+\beta_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\beta_N} \end{vmatrix}$$

$$= \frac{\prod_{m<n}(\beta_m - \beta_n) \prod_{m<n}(\alpha_m - \alpha_n)}{\prod_{m,n=1}^{N}(\alpha_m + \beta_n)} \left(1 + (-1)^{N-1} \prod_{n=1}^{N} \frac{\beta_n}{\alpha_n}\right).$$

Proof. (1) is well-known as the Cauchy determinant. In what follows we denote it by C_N. To prove (2), let $\varphi(z)$, $\psi(z)$ be two polynomials with degree $N-1$ defined by

$$\varphi(z) := \sum_{j=1}^{N} \beta_j \prod_{n\neq j}(z - \beta_n) \begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \cdots & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_1+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_N+\beta_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\beta_N} \end{vmatrix},$$

$$\psi(z) := (-1)^N \prod_{n=1}^{N} \frac{\beta_n}{\alpha_n} C_N \left(\prod_{n=1}^{N}(z - \beta_n) - \prod_{n=1}^{N}(z + \alpha_n)\right).$$

For the proof of assertion (2) it suffices to show that $\varphi(z) \equiv \psi(z)$. It is easy to see that if $\beta_m = \beta_n$, $m \neq n$, then $\varphi(z) \equiv 0 \equiv \psi(z)$. Hence we may assume that β_n are mutually different.

By definition we get

$$\varphi(\beta_j) = \beta_j \prod_{n\neq j}(\beta_j - \beta_n) \begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \cdots & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_1+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_N+\beta_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\beta_N} \end{vmatrix}.$$

From (1) with $\beta_j = 0$, we have

$$\begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \cdots & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_1+\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_N+\beta_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\beta_N} \end{vmatrix} = (-1)^{N-1} \frac{\prod_{n=1}^{N}(\alpha_n+\beta_j)\prod_{n\neq j}\beta_n}{\prod_{n\neq j}(\beta_j-\beta_n)\prod_{n=1}^{N}\alpha_n} C_N.$$

Therefore

$$\varphi(\beta_j) = (-1)^{N-1}\prod_{n=1}^{N}(\alpha_n+\beta_j)\prod_{n=1}^{N}\frac{\beta_n}{\alpha_n}\,C_N.$$

If follows from the definition of $\psi(z)$ that the right-hand side of this equals $\psi(\beta_j)$. Hence $\varphi(\beta_j) = \psi(\beta_j)$, $j = 1, \cdots, N$. Since polynomials $\varphi(z)$ and $\psi(z)$ with degree $N-1$ coincides at mutually different N points, we conclude that $\varphi(z) \equiv \psi(z)$. This proves (2). Assertion (3) is immediate from (2) with $z = 0$ and (1). The proof is complete. □

By means of Lemma 3.1, we obtain asymptotic behaviors of $\Delta^{\pm}(x)$ as $x \to \pm\infty$:

LEMMA 3.2.

(1) *The determinant* $D(x) = \det(I - B^+B^-)$ *is written as*

$$D(x) = 1 + \cdots + (-1)^N \prod_{m<n}\left(\frac{ik_m - ik_n}{ik_m + ik_n}\right)^4 \prod_{n=1}^{N} c_n^+ c_n^- \left(\frac{e^{2ik_n x}}{2ik_n}\right)^2.$$

(2) *The functions*

$$\Gamma^{\pm}(x,k) := \left(\frac{ik_1 e^{ik_1 x}}{ik - ik_1} \cdots \frac{ik_N e^{ik_N x}}{ik - ik_N}\right)(I - B^{\mp}B^{\pm})^{\tilde{}}(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}), \quad k \in \boldsymbol{R},$$

are written as

$$\Gamma^{\pm}(x,k)$$
$$= \cdots + \prod_{m<n}\left(\frac{ik_m - ik_n}{ik_m + ik_n}\right)^4 \prod_{n=1}^{N} c_n^+ c_n^- \left(\frac{e^{2ik_n x}}{2ik_n}\right)^2 \left(-1 + \prod_{n=1}^{N}\frac{ik + ik_n}{ik - ik_n}\right).$$

(3) *The functions* $\Delta^{\pm}(x)$ *defined by* (1.6) *are written as*

$$\Delta^{\pm}(x) = 1 + \cdots + \prod_{m<n} \left(\frac{ik_m - ik_n}{ik_m + ik_n} \right)^4 \prod_{n=1}^{N} c_n^+ c_n^- \left(\frac{e^{2ik_n x}}{2ik_n} \right)^2 .$$

In these representations, dots represent terms of $o(1)$ *as* $x \to +\infty$ *and of*

$$o\left(\exp\left(4\sum_{n=1}^{N} ik_n\, x \right) \right)$$

as $x \to -\infty$*. Moreover, in the case* $N = 1$*, they have the convention*

$$\prod_{m<n} \left(\frac{ik_m - ik_n}{ik_m + ik_n} \right)^4 = 1.$$

Proof. (1) Set

$$G := \begin{pmatrix} \frac{1}{ik_1+ik_1} & \cdots & \frac{1}{ik_1+ik_N} \\ \vdots & \cdots & \vdots \\ \frac{1}{ik_N+ik_1} & \cdots & \frac{1}{ik_N+ik_N} \end{pmatrix}.$$

Then

$$\det(-B^+B^-) = (-1)^N \prod_{n=1}^{N} c_n^+ c_n^- (e^{2ik_n x})^2 (\det G)^2.$$

Using Lemma 3.1(1) with $\alpha_j = ik_j$, $\beta_j = ik_j$, we get

$$(\det G)^2 = \frac{1}{\prod_{n=1}^{N} (2ik_n)^2} \prod_{m<n} \left(\frac{ik_m - ik_n}{ik_m + ik_n} \right)^4 . \tag{3.1}$$

This proves (1).

(2) Set, for $\ell, j = 1, \cdots, N$,

$$\nu_{\ell j}^{\mp} = -\sum_{m=1}^{N} \frac{c_m^{\pm} e^{2ik_m x}}{(ik_\ell + ik_m)(ik_m + ik_j)}, \quad \rho_\ell^{\mp} := \sum_{m=1}^{N} \frac{c_m^{\pm} e^{2ik_m x}}{(ik_\ell + ik_m) ik_m} - \frac{1}{ik_\ell}.$$

Then the ℓj-component of $I-B^{\mp}B^{\pm}$ and the ℓth component of $B^{\mp}\boldsymbol{v}^{\pm}-\boldsymbol{v}^{\mp}$ are written as

$$(I-B^{\mp}B^{\pm})_{\ell j}=\delta_{\ell j}+c_{\ell}^{\mp}\,\nu_{\ell j}^{\mp}e^{ik_\ell x}e^{ik_j x},\quad (B^{\mp}\boldsymbol{v}^{\pm}-\boldsymbol{v}^{\mp})_{\ell}=c_{\ell}^{\mp}\,\rho_{\ell}^{\mp}e^{ik_\ell x}.$$

Hence, from a cofactor expansion

$$\begin{pmatrix} u_1 & \cdots & u_N \end{pmatrix}(a_{\ell j})^{\sim}\begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}=\sum_{j=1}^{N}u_j\begin{vmatrix} a_{11} & \cdots & w_1 & \cdots & a_{1N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & \cdots & w_j & \cdots & a_{jN} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N1} & \cdots & w_N & \cdots & a_{NN} \end{vmatrix},$$

we obtain

$$\begin{aligned}
&\Gamma^{\pm}(x,k)\\
&=\sum_{j=1}^{N}\frac{ik_j}{ik-ik_j}\times\\
&\begin{vmatrix} 1+c_1^{\mp}\nu_{11}^{\mp}e^{ik_1x}e^{ik_1x} & \cdots & c_1^{\mp}\,\rho_1^{\mp}e^{ik_1x}e^{ik_jx} & \cdots & c_1^{\mp}\,\nu_{1N}^{\mp}\,e^{ik_1x}e^{ik_Nx} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_j^{\mp}\,\nu_{j1}^{\mp}\,e^{ik_jx}e^{ik_1x} & \cdots & c_j^{\mp}\,\rho_j^{\mp}e^{ik_jx}e^{ik_jx} & \cdots & c_j^{\mp}\,\nu_{jN}^{\mp}\,e^{ik_jx}e^{ik_Nx} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_N^{\mp}\nu_{N1}^{\mp}\,e^{ik_Nx}e^{ik_1x} & \cdots & c_N^{\mp}\,\rho_N^{\mp}e^{ik_Nx}e^{ik_jx} & \cdots & 1+c_N^{\mp}\,\nu_{NN}^{\mp}\,e^{ik_Nx}\,e^{ik_Nx} \end{vmatrix}\\
&=\prod_{n=1}^{N}c_n^{\mp}\,e^{2ik_nx}\,\sum_{j=1}^{N}\frac{ik_j}{ik-ik_j}\times\\
&\begin{vmatrix} (c_1^{\mp})^{-1}e^{-2ik_1x}+\nu_{11}^{\mp} & \cdots & \rho_1^{\mp} & \cdots & \nu_{1N}^{\mp} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \nu_{j1}^{\mp} & \cdots & \rho_j^{\mp} & \cdots & \nu_{jN}^{\mp} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \nu_{N1}^{\mp} & \cdots & \rho_N^{\mp} & \cdots & (c_N^{\mp})^{-1}e^{-2ik_Nx}+\nu_{NN}^{\mp} \end{vmatrix},
\end{aligned}$$

from which we compute

$$\begin{aligned}
&\text{the primary terms of }\Gamma^{\pm}(x,k)\\
&=(-1)^{N-1}\prod_{n=1}^{N}c_n^{\mp}\,e^{2ik_nx}\,\times\sum_{j=1}^{N}\frac{ik_j}{ik-ik_j}\times
\end{aligned}$$

$$\left|\begin{pmatrix} \frac{c_1^{\pm}e^{2ik_1x}}{ik_1+ik_1} & \cdots & \frac{c_N^{\pm}e^{2ik_Nx}}{ik_1+ik_N} \\ \vdots & \vdots & \vdots \\ \frac{c_1^{\pm}e^{2ik_1x}}{ik_N+ik_1} & \cdots & \frac{c_N^{\pm}e^{2ik_Nx}}{ik_N+ik_N} \end{pmatrix} \begin{pmatrix} \frac{1}{ik_1+ik_1} & \cdot\cdot & \frac{1}{ik_1} & \cdot\cdot & \frac{1}{ik_N+ik_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{ik_1+ik_N} & \cdot\cdot & \frac{1}{ik_N} & \cdot\cdot & \frac{1}{ik_N+ik_N} \end{pmatrix}\right|$$

$$= (-1)^{N-1} \det G \prod_{n=1}^{N} c_n^+ c_n^- \left(e^{2ik_n x}\right)^2 \times$$

$$\sum_{j=1}^{N} \frac{ik_j}{ik - ik_j} \begin{vmatrix} \frac{1}{ik_1+ik_1} & \cdot\cdot & \frac{1}{ik_1} & \cdot\cdot & \frac{1}{ik_1+ik_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{ik_N+ik_1} & \cdot\cdot & \frac{1}{ik_N} & \cdot\cdot & \frac{1}{ik_N+ik_N} \end{vmatrix}.$$

This, together with (3.1), yields (2), since, by Lemma 3.1(2) with $\alpha_n = \beta_n = ik_n$, $z = ik$,

$$\sum_{j=1}^{N} \frac{ik_j}{ik - ik_j} \begin{vmatrix} \frac{1}{ik_1+ik_1} & \cdot\cdot & \frac{1}{ik_1} & \cdot\cdot & \frac{1}{ik_1+ik_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{ik_N+ik_1} & \cdot\cdot & \frac{1}{ik_N} & \cdot\cdot & \frac{1}{ik_N+ik_N} \end{vmatrix}$$
$$= (-1)^N \det G \left(1 - \prod_{n=1}^{N} \frac{ik + ik_n}{ik - ik_n}\right).$$

(3) By definition we have

$$\begin{aligned} \Delta^{\pm}(x) &= D(x) + \begin{pmatrix} e^{ik_1x} & \cdots & e^{ik_Nx} \end{pmatrix} (I - B^{\mp}B^{\pm})^{\tilde{}} (B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}) \\ &= D(x) - \Gamma^{\pm}(x, 0). \end{aligned}$$

Hence subtracting (2) with $k = 0$ from (1) completes the proof. □

PROPOSITION 3.3. *Let* $\{0, k_n, c_n^{\pm}\}$ *be a prescribed triplet satisfying* (I), (II) *in Theorem* 1.1, *and let* U, Q *are functions defined by* (1.7). *Then*

(1) *U, Q belong to $C^{\infty}(\boldsymbol{R})$, and they are exponentially decaying as $x \to \pm\infty$ with their derivatives. In particular, (U, Q) belongs to $\mathcal{S} \times \mathcal{S}$.*

(2) *Q satisfies $\int_{-\infty}^{\infty} Q(x)dx = 0$. In addition, $\int_{-\infty}^{\infty}[U(x) + Q(x)^2]dx = 4i\sum_{n=1}^{N} k_n$.*

Proof. (1) As in Proposition 2.8(1), functions $\Delta^{\pm}(x)$ defined in (1.6) are real-valued on $\boldsymbol{R}$. We define a constant κ by

$$\kappa := \left(\prod_{m<n} \frac{ik_m - ik_n}{ik_m + ik_n}\right)^4 \left(\prod_{n=1}^{N} \frac{1}{2ik_n}\right)^2 \prod_{n=1}^{N} c_n^+ c_n^- = (\det G)^2 \prod_{n=1}^{N} c_n^+ c_n^- .$$

Then, from Lemma 3.2(3), we get expressions

$$\Delta^{\pm}(x) = 1 + \cdots + \kappa \prod_{n=1}^{N} e^{4ik_n x}. \tag{3.2}$$

It follows from (I) that $(\det G)^2 > 0$, $\prod_{n=1}^{N} c_n^+ c_n^-$ are nonzero, real numbers, and so that κ is a nonzero real number. The number κ is positive if and only if $\prod_{n=1}^{N} c_n^+ c_n^-$ is positive. In view of (3.2), real-valued functions $\Delta^{\pm}(x)$ tend to 1 as $x \to +\infty$, and moreover, tend to $\pm\infty$ as $x \to -\infty$ according to $\pm\kappa > 0$. Consequently, if $\prod_{n=1}^{N} c_n^+ c_n^- < 0$ then $\Delta^{\pm}(x)$ must have zeros on $\boldsymbol{R}$, which contradicts the assumption (II). Thus

$$\prod_{n=1}^{N} c_n^+ c_n^- > 0,$$

under the assumption (II), and so, $\kappa > 0$. Since, by (3.2),

$$\log \Delta^{\pm}(x) = 4\left(\sum_{n=1}^{N} ik_n\right) x + \log\left(\kappa + O(e^{\mu x})\right), \quad x \to -\infty,$$

with some $\mu > 0$, Noting that $4\left(\sum_{n=1}^{N} ik_n\right) < 0$, we prove (1).

(2) By (1.7) we get

$$e^{\pm i \int_x^{\infty} Q(r)dr} = \sqrt{\frac{\Delta^{\pm}(x)}{\Delta^{\mp}(x)}}. \tag{3.3}$$

By letting $x \to -\infty$ in (3.3) and using (3.2), we get the first equality in (2). Similarly, by letting $x \to -\infty$ in (2.28), we get the second equality there. □

Let $J^{\pm}(x,y)$ be functions in the right side of (2.21), namely,

$$J^{\pm}(x,y) := \left(e^{ik_1 y} \ \cdots \ e^{ik_N y} \right)(I - B^{\mp}B^{\pm})^{-1} (B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}), \quad x \le y.$$

Then, by (2.25) and (2.24), for each x such that $D(x) \neq 0$,

$$\begin{aligned}\frac{d}{dx}\log\Delta^{\pm}(x) &= \frac{\frac{d}{dx}J^{\pm}(x,x)}{1+J^{\pm}(x,x)} + \frac{d}{dx}\log D(x)\\ &= \frac{\frac{d}{dx}J^{\pm}(x,x)}{1+J^{\pm}(x,x)} - \frac{J^{\pm}_x(x,x)-J^{\pm}_y(x,x)}{1+J^{\pm}(x,x)}.\end{aligned}$$

Therefore, for each x such that $D(x) \neq 0$, functions $Q(x)$, $U(x)$ defined by (1.7) are expressed as

$$Q(x) = -\frac{1}{2i}\left(\frac{\frac{d}{dx}J^{+}(x,x)}{1+J^{+}(x,x)} - \frac{\frac{d}{dx}J^{-}(x,x)}{1+J^{-}(x,x)}\right), \tag{3.4}$$

$$\begin{aligned}U(x) &= -\frac{1}{2}\frac{d}{dx}\left(\frac{\frac{d}{dx}J^{+}(x,x)}{1+J^{+}(x,x)} + \frac{\frac{d}{dx}J^{-}(x,x)}{1+J^{-}(x,x)}\right)\\ &\quad + \frac{d}{dx}\frac{J^{\pm}_x(x,x)-J^{\pm}_y(x,x)}{1+J^{\pm}(x,x)} + (iQ(x))^2,\end{aligned} \tag{3.5}$$

in terms of $J^{\pm}(x,y)$.

In order to rewrite (3.5) in a simpler form, we require:

Lemma 3.4. *For each $x \in \boldsymbol{R}$ such that $D(x) \neq 0$ and each $z \geq 0$,*

$$\frac{d}{dx}\frac{J^{\pm}_x(x,x+z)-J^{\pm}_y(x,x+z)}{1+J^{\pm}(x,x)} = \frac{\frac{d}{dx}J^{\pm}(x,x+z)}{1+J^{\pm}(x,x)}\frac{\frac{d}{dx}J^{\mp}(x,x)}{1+J^{\mp}(x,x)}.$$

Proof. By differentiating (2.18) with respect to x, we get

$$J^{\mp}_x(x,y) + \int_x^{\infty} J^{\pm}_x(x,r)F^{\pm}(r+y)dr = (1+J^{\pm}(x,x))F^{\pm}(x+y), \quad x<y.$$

Provided that $D(x) \neq 0$, this is solved as

$$\begin{aligned}J^{\pm}_x(x,y) = (\, e^{ik_1 y} \ \cdots \ e^{ik_N y}\,)(I-B^{\mp}B^{\pm})^{-1} \times\\ \left((1+J^{\pm}(x,x))B^{\mp}\boldsymbol{b}^{\pm} - (1+J^{\mp}(x,x))\boldsymbol{b}^{\mp}\right),\end{aligned}$$

where we put

$$\boldsymbol{b}^{\pm} := \begin{pmatrix} c_1^{\pm}e^{ik_1x} \\ \vdots \\ c_N^{\pm}e^{ik_Nx}\end{pmatrix}.$$

Moreover, by differentiating (2.18) with respect to y, and performing an integration by parts, we get

$$(3.6)\qquad \begin{aligned} &J_y^{\mp}(x,y) - \int_x^{\infty} J_y^{\pm}(x,r)F^{\pm}(r+y)dr \\ &= (1+J^{\pm}(x,x))F^{\pm}(x+y), \quad x<y, \end{aligned}$$

which is solved as

$$(3.7)\qquad \begin{aligned} J_y^{\pm}(x,y) = (\, e^{ik_1 y} \ \cdots \ e^{ik_N y}\,)(I-B^{\mp}B^{\pm})^{-1} \times \\ \left(-(1+J^{\pm}(x,x))B^{\mp}\boldsymbol{b}^{\pm} - (1+J^{\mp}(x,x))\boldsymbol{b}^{\mp}\right). \end{aligned}$$

Hence, for $z \geq 0$,

$$(3.8)\qquad \begin{aligned} &\frac{J_x^{\pm}(x,x+z) - J_y^{\pm}(x,x+z)}{1+J^{\pm}(x,x)} \\ &\quad = 2(\, e^{ik_1(x+z)} \ \cdots \ e^{ik_N(x+z)}\,)(I-B^{\mp}B^{\pm})^{-1}B^{\mp}\boldsymbol{b}^{\pm}, \\ &\frac{\frac{d}{dx}J^{\pm}(x,x+z)}{1+J^{\mp}(x,x)} = -2(\, e^{ik_1(x+z)} \ \cdots \ e^{ik_N(x+z)}\,)(I-B^{\mp}B^{\pm})^{-1}\boldsymbol{b}^{\mp}. \end{aligned}$$

By a straightforward, linear algebraic computation (refer to [20, Proof of Lemma 5.1]) for the right-hand sides of (3.8) with the aid of (2.23) we can prove the lemma. □

From Lemma 3.4 with $z=0$ and (3.4), expression (3.5) is recast as

$$(3.9)\qquad \begin{aligned} U(x) = \frac{1}{4}\left(\frac{\frac{d}{dx}J^{+}(x,x)}{1+J^{+}(x,x)} + \frac{\frac{d}{dx}J^{-}(x,x)}{1+J^{-}(x,x)}\right)^2 \\ - \frac{1}{2}\frac{d}{dx}\left(\frac{\frac{d}{dx}J^{+}(x,x)}{1+J^{+}(x,x)} + \frac{\frac{d}{dx}J^{-}(x,x)}{1+J^{-}(x,x)}\right) \end{aligned}$$

for each $x \in \boldsymbol{R}$ such that $D(x) \neq 0$.

We now define a function $f_0(x)$ that will play a role of the Jost function $f_+(x,0)$ by

$$(3.10)\qquad f_0(x) = \frac{D(x)}{\Delta^{\pm}(x)} e^{\pm i\int_x^{\infty} Q(r)dr} = \frac{D(x)}{\sqrt{\Delta^{+}(x)\Delta^{-}(x)}},$$

in light of (2.26), (3.3). It is clear from this definition that $f_0(x) = 0$ if and only if $D(x) = 0$.

LEMMA 3.5. *The function* $f_0(x)$ *defined by* (3.10) *satisfies* $f_0''(x) = U(x)f_0(x)$ *on* $\boldsymbol{R}$.

PROOF. In view of (2.25), $f_0(x)$ is written as

$$f_0(x) = \frac{1}{1 + J^\pm(x,x)} e^{\pm i \int_x^\infty Q(r)dr} \tag{3.11}$$

for each $x \in \boldsymbol{R}$ such that $D(x) \neq 0$. Hence, by using (3.4), the derivative $f_0'(x)$ can be computed as

$$f_0'(x) = -\frac{1}{2}\left(\frac{\frac{d}{dx}J^+(x,x)}{1 + J^+(x,x)} + \frac{\frac{d}{dx}J^-(x,x)}{1 + J^-(x,x)}\right) f_0(x). \tag{3.12}$$

Differentiating this and taking (3.9) into account, we get $f_0''(x) = U(x)f_0(x)$. Since, under the assumption (II), functions $f_0(x)$, $U(x)$ are in the class $C^\infty(\boldsymbol{R})$, this equality holds for any $x \in \boldsymbol{R}$. $\square$

We next define a function $\tilde{A}^\pm(x,y)$ for $x \le y < \infty$ by

$$\begin{aligned} \tilde{A}^\pm(x,y) = {} & \frac{1}{\sqrt{\Delta^+(x)\Delta^-(x)}} \partial_y \times \\ & \Big(\begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} (I - B^\mp B^\pm)\check{}\,(B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp) \Big), \end{aligned} \tag{3.13}$$

where ∂_y denotes the differentiation with respect to y. It follows from Proposition 2.8 that $\tilde{A}^\pm(x,y)$ are real-valued functions, under the assumption (I). By definition, $\tilde{A}^\pm(x,y)$ are continuous functions on a closed region $\{(x,y) \in \boldsymbol{R}^2 | x \le y\}$ with $\tilde{A}^\pm(x,\cdot) \in C^\infty[x,\infty) \cap L^1(x,\infty)$ for each $x \in \boldsymbol{R}$, under the assumption (II).

Assertion (2) of the following proposition, in which $f_+^\pm(x,k)$ play the roles of $f_+(x,k)$ and $f_+^-(x,k)$ in the previous section, states that the functions $\tilde{A}^\pm(x,y)$ defined by (3.13) are the transformation kernels for $(U, \pm Q)$ defined by (1.7).

PROPOSITION 3.6. *Let* $\{0, k_n, c_n^\pm\}$ *be a prescribed triplet satisfying* (I), (II) *in Theorem* 1.1 *and let* U, Q *be functions defined by* (1.7). *Then:*

(1) *Functions* $\tilde{A}^{\pm}(x,y)$ *defined by* (3.13) *satisfy*

$$\tilde{A}^{\mp}(x,y)-\int_x^\infty \tilde{A}^{\pm}(x,r)F^{\pm}(r+y)dr-e^{\pm i\int_x^\infty Q(r)dr}F^{\pm}(x+y)=0,\quad x\le y,$$

where

$$F^{\pm}(y)=-\sum_{n=1}^{N}c_n^{\pm}e^{ik_ny},\quad y\in\boldsymbol{R}.$$

(2) *Functions* $f_+^{\pm}(x,k)$ *defined by*

$$f_+^{\pm}(x,k)=e^{\pm i\int_x^\infty Q(r)dr}e^{ikx}+\int_x^\infty \tilde{A}^{\pm}(x,y)e^{iky}dy \tag{3.14}$$

become the Jost solutions of (1.1) *with the potentials* $(U,\pm Q)$.

PROOF. By definition of $J^{\pm}$ and $(I-B^{\mp}B^{\pm})\check{}=D(x)(I-B^{\mp}B^{\pm})^{-1}$ we get

$$(\,e^{ik_1y}\ \cdots\ e^{ik_Ny}\,)(I-B^{\mp}B^{\pm})\check{}\,(B^{\mp}\boldsymbol{v}^{\pm}-\boldsymbol{v}^{\mp})=D(x)J^{\pm}(x,y).$$

Hence it follows from (3.3), (2.25), (3.11) that

$$\tilde{A}^{\pm}(x,y)=e^{\pm i\int_x^\infty Q(r)dr}\frac{J_y^{\pm}(x,y)}{1+J^{\pm}(x,x)}=f_0(x)J_y^{\pm}(x,y), \tag{3.15}$$

for each $x\in\boldsymbol{R}$ such that $D(x)\neq 0$. It suffices to show assertions for each $x\in\boldsymbol{R}$ such that $D(x)\neq 0$ because $\tilde{A}^{\pm}(x,y)$ are continuous extensions of the functions in right sides in (3.15).

(1) Multiplying (3.6) by the function $f_0(x)$, we get

$$\begin{aligned}&f_0(x)J_y^{\mp}(x,y)\\&\quad-\int_x^\infty f_0(x)J_y^{\pm}(x,r)F^{\pm}(r+y)dr-f_0(x)(1+J^{\pm}(x,x))F^{\pm}(x+y)=0.\end{aligned}$$

In view of (3.15), (3.11), this can be written in the form (2.14).

(2) By using (3.15), an integration by parts, and (3.11), expressions (3.14) can be recast as

$$f_+^{\pm}(x,k)=f_0(x)\left(e^{ikx}-ik\int_x^\infty J^{\pm}(x,y)e^{iky}dy\right). \tag{3.16}$$

Hence, with the aid of Lemma 3.5, we obtain

$$
\begin{aligned}
(3.17)\qquad &(f_+^{\pm}(x,k))'' - U(x)f_+^{\pm}(x,k) \\
&= 2ikf_0'(x)e^{ikx} - 2ikf_0'(x)\left(\int_x^\infty J^{\pm}(x,y)e^{iky}dy\right)' \\
&\quad - k^2 f_0(x)e^{ikx} - ikf_0(x)\left(\int_x^\infty J^{\pm}(x,y)e^{iky}dy\right)''.
\end{aligned}
$$

The first derivative of $\int_x^\infty J^{\pm}(x,y)e^{iky}dy$ can be computed as

$$
\begin{aligned}
(3.18)\qquad &\left(\int_x^\infty J^{\pm}(x,y)e^{iky}dy\right)' \\
&= -2J^{\pm}(x,x)e^{ikx} - ik\int_x^\infty J^{\pm}(x,y)e^{iky}dy \\
&\quad + \int_0^\infty (J_x^{\pm}(x,x+z) - J_y^{\pm}(x,x+z))e^{ikz}dz\, e^{ikx}.
\end{aligned}
$$

Differentiating this we have

$$
\begin{aligned}
&\left(\int_x^\infty J^{\pm}(x,y)e^{iky}dy\right)'' \\
&= -2\left(\frac{d}{dx}J^{\pm}(x,x)\right)e^{ikx} - k^2\int_x^\infty J^{\pm}(x,y)e^{iky}dy \\
&\quad + \int_0^\infty \frac{d}{dx}(J_x^{\pm}(x,x+z) - J_y^{\pm}(x,x+z))e^{ikz}dz\, e^{ikx}.
\end{aligned}
$$

But, by Lemma 3.4, (3.12) and (3.4), we obtain

$$
\begin{aligned}
&\frac{d}{dx}(J_x^{\pm}(x,x+z) - J_y^{\pm}(x,x+z)) \\
&= \frac{d}{dx}\left(\frac{J_x^{\pm}(x,x+z) - J_y^{\pm}(x,x+z)}{1+J^{\pm}(x,x)}(1+J^{\pm}(x,x))\right) \\
&= \left(\frac{\frac{d}{dx}J^{+}(x,x)}{1+J^{+}(x,x)} + \frac{\frac{d}{dx}J^{-}(x,x)}{1+J^{-}(x,x)}\right)J_x^{\pm}(x,x+z) \\
&\quad \pm \left(\frac{\frac{d}{dx}J^{-}(x,x)}{1+J^{-}(x,x)} - \frac{\frac{d}{dx}J^{+}(x,x)}{1+J^{+}(x,x)}\right)J_y^{\pm}(x,x+z) \\
&= \pm 2iQ(x)J_y^{\pm}(x,x+z) - 2\frac{f_0'(x)}{f_0(x)}J_x^{\pm}(x,x+z).
\end{aligned}
$$

Consequently

$$\begin{aligned}&\left(\int_x^\infty J^\pm(x,y)e^{iky}dy\right)'' \\ &= -2\left(\frac{d}{dx}J^\pm(x,x)\right)e^{ikx} - k^2\int_x^\infty J^\pm(x,y)e^{iky}dy \\ &\quad \pm 2iQ(x)\int_x^\infty J_y^\pm(x,y)e^{iky}dy - 2\frac{f_0'(x)}{f_0(x)}\int_x^\infty J_x^\pm(x,y)e^{iky}dy.\end{aligned}$$

By substituting this and (3.18) to (3.17), it follows from (3.16), (3.4), and (3.12) that $f_+^\pm(x,k)$ satisfy, for each $x \in \boldsymbol{R}$ such that $D(x) \neq 0$,

$$(f_+^\pm(x,k))'' + [k^2 - (U(x) \pm 2kQ(x))]f_+^\pm(x,k) = 0.$$

The asymptotics $f_+^\pm(x,k)$ are immediate from $\tilde{A}^\pm(x,\cdot) \in L^1(x,\infty)$ and $\tilde{A}^\pm(x,x) \to 0$ as $x \to +\infty$. $\square$

We are now in a position to establish Theorem 1.1.

Proof of Theorem 1.1. The "only if" part was already proved in Proposition 2.12. We will prove that the scattering data for $(U, \pm Q)$ defined by (1.7) are equal to the prescribed data $\{0, k_n, c_n^\pm\}$.

Let $S(k)$ and $S^-(k)$ be the scattering matrices for (U,Q) and $(U,-Q)$, respectively. In view of Proposition 3.6(2), the Jost solutions $f_+^\pm(x,k)$ for $(U,\pm Q)$ are given by (3.14) via $\tilde{A}^\pm(x,y)$ defined by (3.13). Since $\pm iQ(x)$, $\tilde{A}^\pm(x,y)$ are real-valued, we have

$$\overline{f_+^\pm(x,k)} = e^{\pm i\int_x^\infty Q(r)dr}e^{-ikx} + \int_x^\infty \tilde{A}^\pm(x,y)e^{-iky}dy, \quad k \in \boldsymbol{R}. \tag{3.19}$$

From (3.13), (3.3) we get

$$\begin{aligned}&\int_x^\infty \tilde{A}^\pm(x,y)e^{-iky}dy \\ &= e^{\pm i\int_x^\infty Q(r)dr}\frac{1}{\Delta^\pm(x)}\left(\frac{ik_1e^{ik_1x}}{ik-ik_1} \ \cdots \ \frac{ik_Ne^{ik_Nx}}{ik-ik_N}\right)\times \\ &\quad (I - B^\mp B^\pm)\check{}\,(B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp)e^{-ikx}.\end{aligned}$$

Hence

$$\overline{f_+^\pm(x,k)}e^{\mp i\int_x^\infty Q(r)dr}e^{ikx}-1 \tag{3.20}$$
$$=e^{\mp i\int_x^\infty Q(r)dr}e^{ikx}\int_x^\infty \tilde{A}^\pm(x,y)e^{-iky}dy=\frac{\Gamma^\pm(x,k)}{\Delta^\pm(x)}.$$

By Lemma 3.2(2) we have $\Gamma^\pm(x,k)=\cdots+\omega\prod_{n=1}^N e^{4ik_nx}$, where we set

$$\omega:=(\det G)^2\prod_{n=1}^N c_n^+c_n^-\left(-1+\prod_{n=1}^N\frac{ik+ik_n}{ik-ik_n}\right).$$

This, together with (3.2), leads to

$$\frac{\Gamma^\pm(x,k)}{\Delta^\pm(x)}=\frac{\cdots+\omega\prod_{n=1}^N e^{4ik_nx}}{1+\cdots+\kappa\prod_{n=1}^N e^{4ik_nx}}.$$

It follows from this formula that

$$\frac{\Gamma^\pm(x,k)}{\Delta^\pm(x)}=\frac{\omega}{\kappa}+o(1),\quad \frac{d}{dx}\frac{\Gamma^\pm(x,k)}{\Delta^\pm(x)}=o(1),\quad x\to-\infty,$$

with

$$\frac{\omega}{\kappa}=-1+\prod_{n=1}^N\frac{ik+ik_n}{ik-ik_n}.$$

Therefore, by (3.20) and (3.19),

$$\overline{f_+^\pm(x,k)}e^{-i\int_x^\infty Q(r)dr}e^{ikx}=\prod_{n=1}^N\frac{k+k_n}{k-k_n}[1+o(1)],$$
$$\frac{d}{dx}\left(\overline{f_+^\pm(x,k)}e^{-i\int_x^\infty Q(r)dr}e^{ikx}\right)=o(1).$$

as $x\to-\infty$ for each $k\in\boldsymbol{R}$. But, by Proposition 3.3(2), $i\int_{-\infty}^\infty Q(r)dr=0$. Hence this implies that

$$\overline{f_+^\pm(x,k)}=\prod_{n=1}^N\frac{k+k_n}{k-k_n}e^{-ikx}[1+o(1)],$$
$$\overline{f_+^\pm(x,k)}\,'=-ik\prod_{n=1}^N\frac{k+k_n}{k-k_n}e^{-ikx}[1+o(1)]$$

as $x \to -\infty$ for each $k \in \boldsymbol{R}$. Since $\overline{f_+^{\pm}(x,k)}$ satisfies the same equation as for $f_-^{\mp}(x,k)$, it follows from the uniqueness of the Jost solution that

$$\overline{f_+^{\pm}(x,k)} = \prod_{n=1}^{N} \frac{k+k_n}{k-k_n} f_-^{\mp}(x,k), \quad k \in \boldsymbol{R}.$$

It is equivalent to

$$\left(\prod_{n=1}^{N} \frac{k+k_n}{k-k_n}\right) f_+^{\pm}(x,k) = \overline{f_-^{\mp}(x,k)}, \quad k \in \boldsymbol{R},$$

because, by (I),

$$\overline{\left(\prod_{n=1}^{N} \frac{k+k_n}{k-k_n}\right)} = \prod_{n=1}^{N} \frac{k+\overline{k_n}}{k-\overline{k_n}} = \prod_{n=1}^{N} \frac{k-k_{\sigma(n)}}{k+k_{\sigma(n)}} = \left(\prod_{n=1}^{N} \frac{k+k_n}{k-k_n}\right)^{-1}, \quad k \in \boldsymbol{R}.$$

In view of (2.4), this implies that

$$s_{11}(k) = s_{11}^-(k) = \prod_{n=1}^{N} \frac{k+k_n}{k-k_n}, \quad s_{12}(k) = s_{12}^-(k) \equiv 0. \tag{3.21}$$

By virtue of the coupled unitarity in Lemma 2.1, this yields

$$s_{21}(k) = s_{21}^-(k) \equiv 0, \quad k \in \boldsymbol{R}.$$

In this way we have proved that scatterings for $(U, \pm Q)$ defined by (1.7) are reflectionless.

We now let $\tilde{c}_n^{\pm}$ be constants defined by

$$\tilde{c}_n^{\pm} = -i \operatorname{Res}_{k=k_n} s_{11}(k) \times \tilde{d}_n^{\pm},$$

where $\tilde{d}_n^{\pm}$ are coupling constants such that $f_-^{\pm}(x,k_n) = \tilde{d}_n^{\pm} f_+^{\pm}(x,k_n)$ (cf: (1.4)), and set

$$\tilde{F}^{\pm}(y) := \sum_{n=1}^{N} \tilde{c}_n^{\pm} e^{ik_n y}, \quad y \in \boldsymbol{R}.$$

By Proposition 3.6(2), functions $\tilde{A}^{\pm}(x,y)$ defined by (3.13) are transformation kernels for $(U, \pm Q)$ defined by (1.7). Hence, by applying Proposition

2.6 to $\tilde{A}^{\pm}(x,y)$, $\tilde{F}^{\pm}(y)$, it follows that

$$\begin{aligned}(3.22)\qquad &\tilde{A}^{\mp}(x,y) - \int_x^\infty \tilde{A}^{\pm}(x,r)\tilde{F}^{\pm}(r+y)dr - e^{\pm i\int_x^\infty Q(r)dr}\tilde{F}^{\pm}(x+y)\\ &= 0, \quad x \le y.\end{aligned}$$

On the other hand, by Proposition 3.6(1), the functions

$$F^{\pm}(y) := \sum_{n=1}^{N} c_n^{\pm} e^{ik_n y}, \quad y \in \boldsymbol{R},$$

satisfy the same equation

$$\tilde{A}^{\mp}(x,y) - \int_x^\infty \tilde{A}^{\pm}(x,r)F^{\pm}(r+y)dr - e^{\pm i\int_x^\infty Q(r)dr}F^{\pm}(x+y) = 0, \quad x \le y.$$

Subtracting (3.22) from this equalities and then putting $y = x$, we have

$$\begin{aligned}&\tilde{F}^{\pm}(2x) - F^{\pm}(2x)\\ &= -e^{\mp i\int_x^\infty Q(r)dr}\int_x^\infty \tilde{A}^{\pm}(x,r)\left(\tilde{F}^{\pm}(r+x) - F^{\pm}(r+x)\right)dr.\end{aligned}$$

It follows from definition of $\tilde{A}^{\pm}(x,y)$, assumption (II), and $\Delta^{\pm}(x) \to 1$ as $x \to +\infty$ that, for each fixed $a \in \boldsymbol{R}$, there exists M, $\lambda > 0$ such that

$$|e^{\mp i\int_x^\infty Q(r)dr}\tilde{A}^{\pm}(x,r)| \le Me^{-\lambda(x+r)}, \quad x \ge \frac{a}{2}.$$

Set $t = 2x$, $\varphi(t) := |\tilde{F}^{\pm}(t) - F^{\pm}(t)|$. Then, by the substitution $s = r + \frac{t}{2}$, we get

$$\varphi(t) \le \int_t^\infty \left| e^{\mp i\int_{\frac{t}{2}}^\infty Q(r)dr}\tilde{A}^{\pm}\left(\frac{t}{2}, s - \frac{t}{2}\right)\right| \varphi(s)ds$$

Hence, by

$$\left| e^{\mp i\int_{\frac{t}{2}}^\infty Q(r)dr}\tilde{A}^{\pm}\left(\frac{t}{2}, s - \frac{t}{2}\right)\right| \le Me^{-\lambda s}, \quad t \ge a,$$

we obtain

$$0 \le \varphi(t) \le M\int_t^\infty e^{-\lambda s}\varphi(s)ds, \quad a \le t < \infty.$$

It is easy to see from this that $\varphi(t) \equiv 0$, $t \geq a$. Since a is arbitrary, this implies that

$$\sum_{n=1}^{N} \tilde{c}_n^{\pm} e^{ik_n y} = \sum_{n=1}^{N} c_n^{\pm} e^{ik_n y}, \quad y \in \boldsymbol{R}.$$

Since ik_n are mutually different, we conclude that $\tilde{c}_n^{\pm} = c_n^{\pm}$, $n = 1, \cdots, N$. □

In the proof above we have deduced (3.21). Noting that $\prod_{n=1}^{N}(k+k_n) = \prod_{n=1}^{N}(k - \overline{k_n})$ for $k \in \boldsymbol{C}$ by (I) and comparing (3.21) with (2.9), we have proved the following:

Corollary 3.7. *In reflectionless scattering for* (1.1), *every pole* k_n, $n = 1, \cdots, N$, *is simple. Consequently, in the reflectionless scattering,*

$$S(k) = \begin{pmatrix} \prod_{n=1}^{N} \frac{k+k_n}{k-k_n} & 0 \\ 0 & \prod_{n=1}^{N} \frac{k+k_n}{k-k_n} \end{pmatrix}.$$

Our scattering theory involves the reflectionless scattering on the standard Schrödinger equation as a special case:

Corollary 3.8. *Assume that* $ik_n < 0$, $c_n^+ = c_n^- =: c_n > 0$, $n = 1, \cdots, N$. *Then* $\{0, k_n, c_n\}$ *is the scattering data for*

$$U(x) = -2\frac{d^2}{dx^2} \log \det(I - B), \quad Q(x) \equiv 0,$$

where $B := B^{\pm}$.

Proof. By assumption, $B^+ = B^-$, $\boldsymbol{v}^+ = \boldsymbol{v}^-$, and so $\Delta^+(x) = \Delta^-(x)$, which we denote simply by B, $\boldsymbol{v}$, $\Delta(x)$. The function $\Delta(x)$ is computed (see [20, equation (7.4)]) as

$$\begin{aligned} \Delta(x) &= \det(I - B^2) + \begin{pmatrix} e^{ik_1 x} & \cdots & e^{ik_N x} \end{pmatrix} (I - B^2)^{\tilde{}} (B\boldsymbol{v} - \boldsymbol{v}) \\ &= \det(I - B) \Big(\det(I + B) - \begin{pmatrix} e^{ik_1 x} \cdots & e^{ik_N x} \end{pmatrix} (I + B)^{\tilde{}} \boldsymbol{v} \Big) \\ &= (\det(I - B))^2. \end{aligned}$$

Hence, by (1.7), we can prove the corollary. □

4. Isospectral Flow and Inverse Scattering Method

An isospectral flow (1.13) associated with the energy dependent Schrödinger operator

$$L := D^2 - (U + 2kQ), \ \text{ where } \ D := \frac{d}{dx}, \quad Q : \text{purely imaginary},$$

can be derived by a manipulation (see [32, section 5]) of the Lax pair

$$\frac{1}{i}\frac{d}{dt}L = [A, L]$$

with the differential operator

$$A = 2(Q + k)D - Q_x. \tag{4.1}$$

Our inverse scattering method (see Figure 1) for (1.12), (1.13) is based upon the following

PROPOSITION 4.1. *Suppose that $(U, Q) = (U(x,t), Q(x,t))$ with real U and purely imaginary Q decreasing rapidly as $x = \pm\infty$, satisfies (1.13). Then*

(1) *Time evolutions of the scattering matrices*

$$S(k,t) = \begin{pmatrix} s_{11}(k,t) & s_{12}(k,t) \\ s_{21}(k,t) & s_{22}(k,t) \end{pmatrix}, \quad S^-(k,t) = \begin{pmatrix} s_{11}^-(k,t) & s_{12}^-(k,t) \\ s_{21}^-(k,t) & s_{22}^-(k,t) \end{pmatrix},$$

of (1.1) with $(U, \pm Q) = (U(x,t), \pm Q(x,t))$ are given by

$$s_{11}(k,t) = s_{11}(k,0), \quad s_{11}^-(k,t) = s_{11}^-(k,0), \qquad k \in \overline{\boldsymbol{C}_+}, \tag{4.2}$$

$$\begin{aligned} &s_{12}(k,t) = s_{12}(k,0)e^{4k^2t}, \quad s_{12}^-(k,t) = s_{12}^-(k,0)e^{-4k^2t}, \\ &s_{21}(k,t) = s_{21}(k,0)e^{-4k^2t}, \quad s_{21}^-(k,t) = s_{21}^-(k,0)e^{4k^2t}, \quad k \in \boldsymbol{R}. \end{aligned} \tag{4.3}$$

In particular "reflectionless" is preserved in time evolution.

(2) *In the reflectionless scattering, time evolutions of constants $c_n^+(t)$, $c_n^-(t)$ in (1.4) are given by*

$$c_n^+(t) = c_n^+(0)e^{-4k_n{}^2t}, \quad c_n^-(t) = c_n^-(0)e^{4k_n{}^2t},$$

where k_n, $n = 1, \cdots, N$, are poles in $\boldsymbol{C}_+$ of $s_{11}(k,t)$ which are t-invariant by (4.2).

Proof. In the proof we use the notation $a(x) \sim b(x)$, which implies $a(x) = b(x)[1 + o(1)]$.

(1) Let $f_\pm = f_\pm(x,k,t)$ be the Jost solutions of (1.1) with $(U,Q) = (U(x,t), Q(x,t))$, and define functions $g_\pm = g_\pm(x,k,t)$ in $x, t \in \boldsymbol{R}$, $k \in \overline{\boldsymbol{C}_+}$ by

$$g_\pm := \frac{1}{i}\dot{f}_\pm - (Af_\pm \mp 2ik^2 f_\pm), \tag{4.4}$$

where A is the operator defined in (4.1). We first show that

$$g_\pm = 0, \quad k \in \overline{\boldsymbol{C}_+}, \tag{4.5}$$

provided that (U,Q) satisfies (1.13). By applying the variation of constants method to the equation

$$\begin{aligned}&\dot{f}_+'' + [k^2 - (U(x,t) + 2k\,Q(x,t))]\dot{f}_+\\ &= (U_t(x,t) + 2k\,Q_t(x,t))f_+, \quad -\infty < x < \infty,\end{aligned}$$

it follows from (2.5) that $\dot{f}_+ = \dot{f}_+(x,k,t)$ is expressed as

$$\begin{aligned}\frac{1}{i}\dot{f}_+ = -\frac{1}{2ik}\int_x^\infty \left(\frac{1}{i}U_t(y,t) + 2k\frac{1}{i}Q_t(y,t)\right) \times&\\ f_+(y,k,t)G(x,y,k,t)dy, \quad k \in \boldsymbol{R},&\end{aligned} \tag{4.6}$$

where

$$G(x,y,k,t) := f_+(x,k,t)\overline{f_+^-(y,k,t)} - f_+(y,k,t)\overline{f_+^-(x,k,t)},$$

having properties

$$G_y(x,x,k,t) = -2ik, \quad G(x,x,k,t) = 0.$$

We rewrite (4.6) as

$$\begin{aligned}\frac{1}{i}\dot{f}_+ = -\frac{1}{2ik}\int_x^\infty &\left(\frac{1}{i}U_t - 4Q_yU - 2QU_y + Q_{yyy}\right)(y,t) \times\\ &f_+(y,k,t)G(x,y,k,t)dy\\ &-\frac{1}{2ik}\int_x^\infty 2k\left(\frac{1}{i}Q_t - 6QQ_y - U_y\right)(y,t)f_+(y,k,t)G(x,y,k,t)dy\\ &-\frac{1}{2ik}\int_x^\infty (4Q_yU + 2QU_y - Q_{yyy})\, f_+(y,k,t)G(x,y,k,t)dy\\ &-\frac{1}{2ik}\int_x^\infty 2k\,(6QQ_y + U_y)\, f_+(y,k,t)G(x,y,k,t)dy.\end{aligned}$$

The third and fourth terms in the right-hand side is written as $Af_+ - 2ik^2 f_+$ by exactly the same computation (with $\overline{f_+^-(z,k,t)}$ instead of $\overline{f_+(z,k,t)}$) as in [20, section 8]. Therefore, for $k \in \boldsymbol{R}$,

$$
\begin{aligned}
(4.7) \qquad g_+(x,k,t) = & -\frac{1}{2ik}\int_x^\infty \left(\frac{1}{i}U_t - 4Q_yU - 2QU_y + Q_{yyy}\right)(y,t) \times \\
& \qquad f_+(y,k,t)G(x,y,k,t)dy \\
& -\frac{1}{2ik}\int_x^\infty 2k\left(\frac{1}{i}Q_t - 6QQ_y - U_y\right)(y,t) \times \\
& \qquad f_+(y,k,t)G(x,y,k,t)dy.
\end{aligned}
$$

This shows that if $(U,Q) = (U(x,t), Q(x,t))$ satisfies (1.13) then g_+ satisfies (4.5), because an analytic continuation to the upper half-plane retains this functional relation. By observing that system (1.13) is invariant in the transformation $(x,t) \to (-x,-t)$, it turns out that g_- satisfies (4.5).

We next deduce a necessary and sufficient condition of $g_\pm$ for $s_{11}(k,t)$ to be time-invariant. By $(Af_\pm)' = 2(Q+k)f_\pm'' - Q_{xx}f_\pm + Q_xf_\pm'$, we easily verify that

$$
\det\begin{pmatrix} Af_+ & Af_- \\ f_+' & f_-' \end{pmatrix} + \det\begin{pmatrix} f_+ & f_- \\ (Af_+)' & (Af_-)' \end{pmatrix} = 0,
$$

from which it follows that

$$
\begin{aligned}
& \det\begin{pmatrix} g_+ & g_- \\ f_+' & f_-' \end{pmatrix} + \det\begin{pmatrix} f_+ & f_- \\ g_+' & g_-' \end{pmatrix} \\
& = \det\begin{pmatrix} \frac{1}{i}\dot{f}_+ & \frac{1}{i}\dot{f}_- \\ f_+' & f_-' \end{pmatrix} + \det\begin{pmatrix} f_+ & f_- \\ \frac{1}{i}\dot{f}_+' & \frac{1}{i}\dot{f}_-' \end{pmatrix} \\
& \quad - \left(\det\begin{pmatrix} Af_+ & Af_- \\ f_+' & f_-' \end{pmatrix} + \det\begin{pmatrix} f_+ & f_- \\ (Af_+)' & (Af_-)' \end{pmatrix}\right) \\
& = \frac{1}{i}\frac{d}{dt}W[f_+, f_-].
\end{aligned}
$$

We have thus proved:

$$
\begin{aligned}
(4.8) \qquad & \frac{d}{dt}s_{11}(k,t) = 0 \\
& \Longleftrightarrow \det\begin{pmatrix} g_+ & g_- \\ f_+' & f_-' \end{pmatrix} + \det\begin{pmatrix} f_+ & f_- \\ g_+' & g_-' \end{pmatrix} = 0, \quad k \in \overline{\boldsymbol{C}_+}.
\end{aligned}
$$

If (U, Q) satisfies (1.13) then $g_{\pm} = 0$, and therefore, by (4.8), $s_{11}(k,t)$ is time-invariant. Observing that system (1.13) is invariant in the transformation

$$(U(x,t), Q(x,t)) \to (U(x,-t), -Q(x,-t)), \tag{4.9}$$

we can see that $s_{11}^{-}(k,t)$ is also time-invariant.

By (4.5) we get

$$\frac{1}{i}\dot{f_+} = 2(Q+k)(f_+)' - (Q_x + 2ik^2)f_+, \quad k \in \overline{\boldsymbol{C}_+}.$$

Also, by using (4.9), we get

$$\frac{1}{i}\dot{f_-^-} = 2(Q-k)(f_-^-)' - (Q_x + 2ik^2)f_-^-, \quad k \in \overline{\boldsymbol{C}_+}. \tag{4.10}$$

This enables us to compute, for $k \in \boldsymbol{R}$,

$$\begin{aligned}
\frac{1}{i}\frac{d}{dt}W[f_+, \overline{f_-^-}] &= \det\begin{pmatrix} \frac{1}{i}\dot{f_+} & \frac{1}{i}\dot{(\overline{f_-^-})} \\ (f_+)' & (\overline{f_-^-})' \end{pmatrix} + \det\begin{pmatrix} f_+ & \overline{f_-^-} \\ \frac{1}{i}(\dot{f_+})' & \frac{1}{i}(\dot{\overline{f_-^-}})' \end{pmatrix} \\
&= -(Q_x + 2ik^2)W[f_+, \overline{f_-^-}] + (Q_x - 2ik^2)W[f_+, \overline{f_-^-}] \\
&= -4ik^2 W[f_+, \overline{f_-^-}].
\end{aligned}$$

Hence, remembering (2.6) and noting $W[f_+, f_-]$ is invariant in t, we get $\dot{s_{12}} = 4k^2 s_{12}$. Similarly we get $\dot{s_{12}^-} = -4k^2 s_{12}^-$. We thus proved (4.3).

(2) We recall that, in the reflectionless scattering, $s_{11}(k,t) = s_{11}^-(k,t)$ by Proposition 2.3(2). It follows from $f_-^-(x,k_n,t) = d_n^-(t) f_+^-(x,k_n,t)$ that

$$\dot{f_-^-}(x,k_n,t) = \dot{d_n^-}(t) f_+^-(x,k_n,t) + d_n^-(t)\dot{f_+^-}(x,k_n,t) \sim \dot{d_n^-}(t)e^{ik_n x}, \quad x \to +\infty.$$

By (4.10),

$$\begin{aligned}
\frac{1}{i}\dot{f_-^-}(x,k_n,t) &= 2(Q-k_n)(f_-^-)'(x,k_n,t) - Q_x f_-^-(x,k_n,t) - 2ik_n^2 f_-^-(x,k_n,t) \\
&= 2(Q-k_n)d_n^-(t)(f_+^-)'(x,k_n,t) - Q_x f_-^-(x,k_n,t) \\
&\quad - 2ik_n^2 d_n^-(t) f_+^-(x,k_n,t) \\
&\sim -4ik_n^2 d_n^-(t)e^{ik_n x}, \quad x \to +\infty,
\end{aligned}$$

from which we get $\dot{f_-^-}(x,k_n,t) \sim 4k_n^2 d_n^-(t) e^{ik_n x}$ as $x \to +\infty$. This, combined with $\dot{f_-^-}(x,k_n,t) \sim \dot{d_n^-}(t) f_+^-(x,k_n,t)$ as $x \to +\infty$, shows that $\dot{d_n^-}(t) =$

$4k_n^2 d_n^-(t)$. This leads to $c_n^-(t) = c_n^-(0)e^{4k_n{}^2 t}$. Similarly we can show $c_n^+(t) = c_n^+(0)e^{-4k_n{}^2 t}$. This proves (2). □

In the reflectionless scattering, we have a kind of converse for Proposition 4.1:

PROPOSITION 4.2. *Suppose that* $(U(x,t), Q(x,t))$ *is a potential whose scattering data is* $\{0, k_n, c_n^\pm(0)e^{\mp 4k_n^2 t}\}$ *for each* t. *If* $(U(x,t), Q(x,t))$ *satisfies the first equation of system* (1.13) *then* $(U(x,t), Q(x,t))$ *is a solution of the system.*

PROOF. As in the proof of Proposition 4.1, we use functions $g_\pm$ defined in (4.4). By differentiating in t the transformation kernel representation

$$f_\pm(x,k,t) = e^{\pm i\int_x^{\pm\infty} Q(r,t)dr} e^{\pm ikx} + \int_x^{\pm\infty} A_\pm(x,y,t)e^{\pm iky}dy,$$

substituting the resultant to definition of $g_\pm(x,y,t)$, and performing integrations by parts, we have

$$\begin{aligned}
g_\pm(x,k,t) = &\pm \int_x^{\pm\infty} Q_t(r,t)dr\, e^{\pm i\int_x^{\pm\infty} Q(r,t)dr} e^{\pm ikx} \\
&\pm 2iQ^2 e^{\pm i\int_x^{\pm\infty} Q(r,t)dr} e^{\pm ikx} \\
&+ \left(\mp 2i\frac{\partial}{\partial x}A_\pm(x,x,t) + 2QA_\pm(x,x,t) + Q_x e^{\pm i\int_x^{\pm\infty} Q(r,t)dr}\right) e^{\pm ikx} \\
&+ \int_x^{\pm\infty} \frac{1}{i}\frac{\partial A_\pm}{\partial t}(x,y,t)e^{\pm iky}dy + \int_x^{\pm\infty} Q_x(x,t)A_\pm(x,y,t)e^{\pm iky}dy \\
&- 2\int_x^{\pm\infty} Q(x,t)\frac{\partial A_\pm}{\partial x}(x,y,t)e^{\pm iky}dy \\
&\mp 2i\int_x^{\pm\infty} \left(\frac{\partial^2 A_\pm}{\partial y^2} + \frac{\partial^2 A_\pm}{\partial x \partial y}\right) e^{\pm iky}dy.
\end{aligned}$$

But, by differentiating

$$A_\pm(x,x,t) = e^{\pm i\int_x^{\pm\infty} Q(r,t)dr}\left(\int_x^{\pm\infty} [U(r,t) + Q(r,t)^2]dr \mp iQ(x,t)\right)$$

(see formula (2.12)) in x, it follows that

$$\begin{aligned}&\mp 2i\frac{\partial}{\partial x}A_{\pm}(x,x,t)+2QA_{\pm}(x,x,t)+Q_x e^{\pm i\int_x^{\pm\infty}Q(r,t)dr}\\&=\pm i(U+Q^2)e^{\pm i\int_x^{\pm\infty}Q(r,t)dr}.\end{aligned}$$

We have thus found the following, transformation kernel representation (in the Faddeev form) of $g_{\pm}(x,k,t)$:

Lemma 4.3. *For* $k\in\overline{\boldsymbol{C}_+}$,

$$\begin{aligned}&g_{\pm}(x,k,t)\\&=\left(m_{\pm}(x,t)e^{\pm i\int_x^{\pm\infty}Q(r,t)dr}+\int_0^{\pm\infty}M_{\pm}(x,x+z,t)e^{\pm ikz}dz\right)e^{\pm ikx},\end{aligned}$$

where

$$\begin{aligned}m_{\pm}(x,t)&=\pm i\int_x^{\pm\infty}\left(\frac{1}{i}Q_t-6QQ_x-U_x\right)dx,\\M_{\pm}(x,y,t)&=\frac{1}{i}\frac{\partial A_{\pm}}{\partial t}-2Q(x,t)\frac{\partial A_{\pm}}{\partial x}(x,y,t)\\&\qquad+Q_x(x,t)A_{\pm}(x,y,t)\mp 2i\left(\frac{\partial^2 A_{\pm}}{\partial y^2}+\frac{\partial^2 A_{\pm}}{\partial x\partial y}\right)(x,y,t).\end{aligned}$$

Observing that $m_{\pm}(x,t)$ vanish, provided that (U,Q) satisfies the first equation of system (1.13), we have the representation

$$\begin{aligned}&f_+(x,k,t)g_-(x,k,t)-f_-(x,k,t)g_+(x,k,t)\\&=\int_0^{\infty}{}^{\exists}\Xi(x,z,t)e^{ikz}dz,\quad k\in\overline{\boldsymbol{C}_+},\end{aligned}$$

which is an analytic and bounded function on $\overline{\boldsymbol{C}_+}$. In view of the Riemann-Lebesgue lemma, it tends to zero as $|k|\to\infty$ in $\in\overline{\boldsymbol{C}_+}$. On the other hand, it follows from definition (4.4), $f_-(x,k_n,t)=d_n(t)f_+(x,k_n,t)$, and $\dot{d}_n(t)=-4k_n^2d_n(t)$ that, for $k=k_n$,

$$g_-(x,k_n,t)=d_n(t)g_+(x,k_n,t).$$

This implies that the analytic function has k_n as zeros. Remembering that k_n are simple poles of $s_{11}(k)$ (see Corollary 3.7) leads to:

LEMMA 4.4. *For each fixed $(x,t) \in \boldsymbol{R}^2$, the function*

$$\phi(k) := is_{11}(k)\left(f_+(x,k,t)g_-(x,k,t) - f_-(x,k,t)g_+(x,k,t)\right)$$

is analytic and bounded on $\overline{\boldsymbol{C}_+}$, which tends to zero as $|k| \to \infty$.

In addition, we define functions $g_\pm^- = g_\pm^-(x,k,t)$ on $\overline{\boldsymbol{C}_+}$ by

$$g_\pm^- := -\frac{1}{i}\dot{f}_\pm^- - (\tilde{A}^- f_\pm^- \mp 2ik^2 f_\pm^-), \quad \tilde{A}^- := 2(-Q+k)D + Q_x.$$

Then, in a similar way to that for Lemma 4.4, we have:

LEMMA 4.5. *For each fixed $(x,t) \in \boldsymbol{R}^2$, the function*

$$\phi^-(k) := is_{11}(k)\left(f_+^-(x,k,t)g_-^-(x,k,t) - f_-^-(x,k,t)g_+^-(x,k,t)\right)$$

is analytic and bounded on $\overline{\boldsymbol{C}_+}$ tending to zero as $|k| \to \infty$, provided that (U,Q) satisfies the first equation of (1.13).

Two functions $\phi(k)$, $\phi^-(k)$ are connected on the real line via a conjugate relation. To see the relation, we note that, in the reflectionless scattering, (2.4) is written as

$$\overline{s_{11}(k)f_\pm(x,k,t)} = f_\mp^-(x,k,t), \quad k \in \boldsymbol{R}.$$

This property is passed on to g as:

$$\overline{s_{11}(k)g_\pm(x,k,t)} = g_\mp^-(x,k,t), \quad k \in \boldsymbol{R}.$$

This, together with $s_{11}(k)\overline{s_{11}(k)} = 1$, leads to:

LEMMA 4.6. *For $k \in \boldsymbol{R}$, $\overline{\phi(k)} = \phi^-(k)$.*

This lemma, together with Lemmas 4.4 and 4.5, shows that $\phi(k)$ is the restriction to $\overline{\boldsymbol{C}_+}$ of an analytic function on the whole plane $\boldsymbol{C}$ with the condition $\phi(k) = \overline{\phi^-(\overline{k})}$. But $\phi(k)$ is bounded on $\boldsymbol{C}$, and therefore, by Liouville's Theorem, $\phi(k)$ is a constant in k. By letting $|k| \to \infty$ it turns out the constant must be zero. We thus conclude that $\phi(k) \equiv 0$ on $\overline{\boldsymbol{C}_+}$. In other words,

$$\det \begin{pmatrix} f_+ & f_- \\ g_+ & g_- \end{pmatrix} = 0, \quad k \in \overline{\boldsymbol{C}_+}. \tag{4.11}$$

Differentiating this in x to get

$$\det \begin{pmatrix} f_+' & f_-' \\ g_+ & g_- \end{pmatrix} + \det \begin{pmatrix} f_+ & f_- \\ g_+' & g_-' \end{pmatrix} = 0.$$

By using (4.8) this can be rewritten as

$$\det \begin{pmatrix} f_+' & f_-' \\ g_+ & g_- \end{pmatrix} = 0.$$

This, combined with (4.11), yields

$$\begin{pmatrix} f_+ & f_- \\ f_+' & f_-' \end{pmatrix} \begin{pmatrix} g_- \\ -g_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $W[f_+, f_-] \neq 0$ for $k \in \boldsymbol{R} \setminus \{0\}$, this means that $g_+(x,k,t) = 0$ for $k \in \boldsymbol{R}$. From (4.7) we conclude that $\frac{1}{i} U_t - 4Q_x U - 2QU_x + Q_{xxx} = 0$. We complete the proof of Proposition 4.1. □

Remark 4.7. In the case where $Q(x,t)$ is real-valued, we can prove the following: *Let $(U(x,t), Q(x,t))$ be the potential determined from the scattering data $\{0, k_n, c_n(0)e^{4ik_n^2 t}\}$. If $(U(x,t), Q(x,t))$ satisfies $Q_t - 6QQ_x - U_x = 0$ then it satisfies*

$$\begin{cases} Q_t - 6QQ_x - U_x = 0, \\ U_t - 4Q_x U - 2QU_x + Q_{xxx} = 0. \end{cases}$$

The proof is obtained in a similar manner to that in the proof of Proposition 4.2. It is just enough to replace $\frac{1}{i}\frac{d}{dt}$ by $\frac{d}{dt}$, for instance, use Q_t instead of $\frac{1}{i}Q_t$. We need only to see $\phi(k)$ defined by exactly the same form as in Lemma 4.4 can be continued analytically to the whole plane $\boldsymbol{C}$ by the symmetry condition $\phi(k) = \overline{\phi(\overline{k})}$ by showing that $\phi(k)$ is real-valued on $\boldsymbol{R}$. This makes the proof easier than that for our case where $Q(x,t)$ is purely imaginary.

5. Soliton Solutions

In this section we will present the proof of Theorem 1.3. We require the following

LEMMA 5.1. *Let* $c_n^{\pm} = c_n^{\pm}(t)$ *satisfy* $\frac{d}{dt}c_n^{\pm}(t) = \mp 4{k_n}^2 c_n^{\pm}(t)$, $n = 1, \cdots, N$, *and let* $J^{\pm}(x, y, t)$ *be a function defined by* (2.21) *with* $c_n^{\pm} = c_n^{\pm}(t)$. *Then, for each* x *such that* $D(x, t) \neq 0$,

$$\frac{\left(\frac{\partial^2}{\partial x^2} \pm \frac{\partial}{\partial t}\right) J^{\pm}(x, x, t)}{1 + J^{\pm}(x, x, t)} = 2\frac{\frac{\partial}{\partial x}J^{+}(x, x, t)}{1 + J^{+}(x, x, t)}\,\frac{\frac{\partial}{\partial x}J^{-}(x, x, t)}{1 + J^{-}(x, x, t)}.$$

PROOF. We employ the following notation:

$$\boldsymbol{e} = \begin{pmatrix} e^{ik_1 x} \\ \vdots \\ e^{ik_N x} \end{pmatrix}, \quad {}^t\boldsymbol{e} = \begin{pmatrix} e^{ik_1 x} & \cdots & e^{ik_N x} \end{pmatrix}, \quad \boldsymbol{b}^{\pm} := \begin{pmatrix} c_1^{\pm} e^{ik_1 x} \\ \vdots \\ c_N^{\pm} e^{ik_N x} \end{pmatrix}.$$

By setting $z = 0$ in (3.8) we have

$$\frac{\frac{\partial}{\partial x}J^{\pm}(x, x, t)}{1 + J^{\mp}(x, x, t)} = -2\,{}^t\boldsymbol{e}\,(I - B^{\mp}B^{\pm})^{-1}\boldsymbol{b}^{\mp}, \tag{5.1}$$

which leads to

$$\begin{aligned}&\frac{\frac{\partial}{\partial x}J^{+}(x, x, t)}{1 + J^{-}(x, x, t)}\,\frac{\frac{\partial}{\partial x}J^{-}(x, x, t)}{1 + J^{+}(x, x, t)} \\ &= 4\,{}^t\boldsymbol{e}\,(I - B^{\mp}B^{\pm})^{-1}(B^{\mp})'\,(I - B^{\pm}B^{\mp})^{-1}\boldsymbol{b}^{\pm}.\end{aligned} \tag{5.2}$$

Let K be a diagonal matrix whose $\ell\ell$-component is given by ik_ℓ. By differentiating (5.1) in x and noting $(B^{\pm})' = B^{\pm}K + K\,B^{\pm}$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial x}\frac{\frac{\partial}{\partial x}J^{\pm}(x, x, t)}{1 + J^{\mp}(x, x, t)} \\ &= -4\,{}^t\boldsymbol{e}\,(I - B^{\mp}B^{\pm})^{-1}(K + B^{\mp}K\,B^{\pm})(I - B^{\mp}B^{\pm})^{-1}\,\boldsymbol{b}^{\mp}.\end{aligned} \tag{5.3}$$

Since

$$\frac{\partial^2}{\partial x^2}J^{\pm}(x,x,t) = \left(\frac{\frac{\partial}{\partial x}J^{+}(x,x,t)}{1+J^{+}(x,x,t)}\frac{\frac{\partial}{\partial x}J^{-}(x,x,t)}{1+J^{-}(x,x,t)}\right)(1+J^{\pm}(x,x,t)) + \left(\frac{\partial}{\partial x}\frac{\frac{\partial}{\partial x}J^{\pm}(x,x,t)}{1+J^{\mp}(x,x,t)}\right)\left(1+J^{\mp}(x,x,t)\right), \tag{5.4}$$

from (5.2) and (5.3), we have

$$\begin{aligned}\frac{\partial^2}{\partial x^2}J^{\pm}(x,x,t) = 4\,{}^t\boldsymbol{e}\,(I-B^{\mp}B^{\pm})^{-1} \times \\ \big((B^{\mp})'(I-B^{\pm}B^{\mp})^{-1}(1+J^{\pm}(x,x,t))\,\boldsymbol{b}^{\pm} \\ -(K+B^{\mp}K\,B^{\pm})(I-B^{\mp}B^{\pm})^{-1}\left(1+J^{\mp}(x,x,t)\right)\boldsymbol{b}^{\mp}\big).\end{aligned} \tag{5.5}$$

In the case where $c_n^{\pm}$ depend on t, (2.18) becomes a GLM equation

$$J^{\mp}(x,y,t)+\int_x^{\infty} J^{\pm}(x,r,t)F^{\pm}(r+y,t)dr+\int_x^{\infty} F^{\pm}(r+y,t)dr = 0, \quad x<y,$$

where

$$F^{\pm}(y,t) := -\sum_{n=1}^{N} c_n^{\pm}(t)e^{ik_ny}.$$

By $\frac{d}{dt}c_n^{\pm}(t) = \mp 4{k_n}^2 c_n^{\pm}(t)$, we have

$$\frac{\partial F^{\pm}}{\partial t} = \pm 4\frac{\partial^2 F^{\pm}}{\partial y^2}.$$

Hence, differentiating the GLM equation once in t and twice in y, we find an equation

$$J_t^{\mp}(x,y,t)+\int_x^{\infty} J_t^{\pm}(x,r,t)F^{\pm}(r+y,t)dr \mp 4J_{yy}^{\mp}(x,y,t) = 0, \tag{5.6}$$

for unknown $J_t^{\pm}(x,y,t)$. In view of (2.21),

$$\begin{aligned}\mp 4J_{yy}^{\mp}(x,y,t) = \mp(\,e^{ik_1y}\ \cdots\ e^{ik_Ny}\,)\,4K^2\boldsymbol{p}^{\pm}, \\ \boldsymbol{p}^{\pm} := (I-B^{\pm}B^{\mp})^{-1}\,(B^{\pm}\boldsymbol{v}^{\mp}-\boldsymbol{v}^{\pm}).\end{aligned} \tag{5.7}$$

This implies that (5.6) is an equation with $\boldsymbol{w}^{\pm} := \mp 4K^2\boldsymbol{p}^{\pm}$ instead of the last term $\int_x^\infty F^{\pm}(r+y)dr = \left(e^{ik_1 y} \ \cdots \ e^{ik_N y} \right) \boldsymbol{v}^{\pm}$ in (2.18). Therefore, equation (5.6) can be solved in the form (2.21) by replacing $\boldsymbol{v}^{\pm}$ there by $\boldsymbol{w}^{\pm}$. This discussion yields

$$\begin{aligned} \pm J_t^{\pm}(x,x,t) = 4\, {}^t\boldsymbol{e}\,(I - B^{\mp}B^{\pm})^{-1} \times \\ \big(-B^{\mp}K^2\,(I - B^{\pm}B^{\mp})^{-1}\,(B^{\pm}\boldsymbol{v}^{\mp} - \boldsymbol{v}^{\pm}) \\ -K^2\,(I - B^{\mp}B^{\pm})^{-1}\,(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp})\big)\,. \end{aligned}$$

Adding (5.5) to this, we have

$$\begin{aligned} &\frac{\partial^2}{\partial x^2} J^{\pm}(x,x,t) \pm \frac{\partial}{\partial t} J^{\pm}(x,x,t) = 4\, {}^t\boldsymbol{e}\,(I - B^{\mp}B^{\pm})^{-1} \times \\ &\big((B^{\mp})'(I - B^{\pm}B^{\mp})^{-1}(1 + J^{\pm}(x,x,t))\,\boldsymbol{b}^{\pm} \\ &\quad -(K + B^{\mp}K\,B^{\pm})(I - B^{\mp}B^{\pm})^{-1}\left(1 + J^{\mp}(x,x,t)\right)\boldsymbol{b}^{\mp} \\ &\quad - B^{\mp}K^2\,(I - B^{\pm}B^{\mp})^{-1}\,(B^{\pm}\boldsymbol{v}^{\mp} - \boldsymbol{v}^{\pm}) \\ &\quad -K^2\,(I - B^{\mp}B^{\pm})^{-1}\,(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp})\big)\,. \end{aligned} \tag{5.8}$$

Differentiating (2.21) in y and comparing the resultant with (3.7), we get

$$\begin{aligned} &- K(I - B^{\mp}B^{\pm})^{-1}\,(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}) \\ &= (I - B^{\mp}B^{\pm})^{-1}\left((1 + J^{\pm}(x,x,t))B^{\mp}\boldsymbol{b}^{\pm} + (1 + J^{\mp}(x,x,t))\boldsymbol{b}^{\mp}\right). \end{aligned} \tag{5.9}$$

It follows from (5.9), (2.23), and $B^{\mp}K + K\,B^{\mp} = (B^{\mp})'$ that

$$\begin{aligned} &- (K + B^{\mp}K\,B^{\pm})(I - B^{\mp}B^{\pm})^{-1}\left(1 + J^{\mp}(x,x,t)\right)\boldsymbol{b}^{\mp} \\ &- B^{\mp}K^2\,(I - B^{\pm}B^{\mp})^{-1}\,(B^{\pm}\boldsymbol{v}^{\mp} - \boldsymbol{v}^{\pm}) - K^2\,(I - B^{\mp}B^{\pm})^{-1}\,(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}) \\ &= (B^{\mp})'(I - B^{\pm}B^{\mp})^{-1}(1 + J^{\pm}(x,x,t))\boldsymbol{b}^{\pm}. \end{aligned}$$

Hence, by (5.8), we arrive at

$$\frac{\left(\frac{\partial^2}{\partial x^2} \pm \frac{\partial}{\partial t}\right) J^{\pm}(x,x,t)}{1 + J^{\pm}(x,x,t)} = 8\, {}^t\boldsymbol{e}\,(I - B^{\mp}B^{\pm})^{-1}(B^{\mp})'(I - B^{\pm}\,B^{\mp})^{-1}\,\boldsymbol{b}^{\pm}.$$

This, combined with (5.2), shows that

$$\frac{\left(\frac{\partial^2}{\partial x^2} \pm \frac{\partial}{\partial t}\right) J^{\pm}(x,x,t)}{1 + J^{\pm}(x,x,t)} = 2\frac{\frac{\partial}{\partial x}J^{+}(x,x,t)}{1 + J^{-}(x,x,t)}\,\frac{\frac{\partial}{\partial x}J^{-}(x,x,t)}{1 + J^{+}(x,x,t)}.$$

The proof is complete. $\square$

PROOF OF THEOREM 1.3. Due to Proposition 4.2 it is enough to show that $(U(x,t),Q(x,t))$ transformed from $(u(x,t),w(x,t))$ via (1.14) satisfies the first equation of (1.13). The first equation of (1.13) is however equivalent to the first equation of (1.12). Hence it suffices to prove that $(u(x,t),w(x,t))$ defined by (1.15) satisfies $u_t+w_x+uu_x=0$.

Due to (2.25), $u(x,t)$ is written as

$$u(x,t)=2\frac{\partial}{\partial x}\big(\log\big(1+J^{+}(x,x,t)\big)-\log\big(1+J^{-}(x,x,t)\big)\big)$$

for each x such that $D(x,t)\neq 0$. From this we get

$$\frac{1}{8}u(x,t)^2=\frac{1}{2}\left(\frac{\frac{\partial}{\partial x}J^{+}(x,x,t)}{1+J^{+}(x,x,t)}-\frac{\frac{\partial}{\partial x}J^{-}(x,x,t)}{1+J^{-}(x,x,t)}\right)^2.$$

Moreover, by (1.14), (3.4), (3.5) and Lemma 3.4, we have

$$\begin{aligned}
&\frac{1}{4}w(x,t)\\
&=\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\frac{\partial}{\partial x}J^{+}(x,x,t)}{1+J^{+}(x,x,t)}+\frac{\frac{\partial}{\partial x}J^{-}(x,x,t)}{1+J^{-}(x,x,t)}\right)-\frac{\partial}{\partial x}\frac{J_x^{\pm}(x,x,t)-J_y^{\pm}(x,x,t)}{1+J^{\pm}(x,x,t)}\\
&=\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\frac{\partial}{\partial x}J^{+}(x,x,t)}{1+J^{+}(x,x,t)}+\frac{\frac{\partial}{\partial x}J^{-}(x,x,t)}{1+J^{-}(x,x,t)}\right)\\
&\quad-\frac{\frac{\partial}{\partial x}J^{+}(x,x,t)}{1+J^{+}(x,x,t)}\frac{\frac{\partial}{\partial x}J^{-}(x,x,t)}{1+J^{-}(x,x,t)}\\
&=\frac{1}{2}\left(\frac{\frac{\partial^2}{\partial x^2}J^{+}(x,x,t)}{1+J^{+}(x,x,t)}+\frac{\frac{\partial^2}{\partial x^2}J^{-}(x,x,t)}{1+J^{-}(x,x,t)}\right)\\
&\quad-\frac{1}{2}\left(\left(\frac{\frac{\partial}{\partial x}J^{+}(x,x,t)}{1+J^{+}(x,x,t)}\right)^2+\left(\frac{\frac{\partial}{\partial x}J^{-}(x,x,t)}{1+J^{-}(x,x,t)}\right)^2\right)\\
&\quad-\frac{\frac{\partial}{\partial x}J^{+}(x,x,t)}{1+J^{+}(x,x,t)}\frac{\frac{\partial}{\partial x}J^{-}(x,x,t)}{1+J^{-}(x,x,t)}.
\end{aligned}$$

Adding these two equalities and using Lemma 5.1, we find that

$$
\begin{aligned}
&\frac{1}{4}w(x,t)+\frac{1}{8}u(x,t)^2\\
&\quad=\frac{1}{2}\left(\frac{\frac{\partial^2}{\partial x^2}J^+(x,x,t)}{1+J^+(x,x,t)}+\frac{\frac{\partial^2}{\partial x^2}J^-(x,x,t)}{1+J^-(x,x,t)}\right)\\
&\quad-2\frac{\frac{\partial}{\partial x}J^+(x,x,t)}{1+J^+(x,x,t)}\frac{\frac{\partial}{\partial x}J^-(x,x,t)}{1+J^-(x,x,t)}\\
&\quad=-\frac{1}{2}\left(\frac{\frac{\partial}{\partial t}J^+(x,x,t)}{1+J^+(x,x,t)}-\frac{\frac{\partial}{\partial t}J^-(x,x,t)}{1+J^-(x,x,t)}\right)=-\frac{1}{2}\frac{\partial}{\partial t}\log\frac{1+J^+(x,x,t)}{1+J^-(x,x,t)}\\
&\quad=-\frac{1}{2}\frac{\partial}{\partial t}\log\frac{\Delta^+(x,x,t)}{\Delta^-(x,x,t)},
\end{aligned}
$$

for each x such that $D(x,t)\neq 0$. Differentiating this in x and taking (1.15) into account, we arrive at $u_t+w_x+uu_x=0$ for each x such that $D(x,t)\neq 0$. Since the left-hand side is continuous, it holds for each (x,t) as long as $\Delta^{\pm}(x,t)>0$. The proof is complete. □

6. Concluding Remarks

Sections 2, 3 of the present paper have been devoted to an establishment of the scattering theory for (1.1). Our success in proving Theorem 1.1 depends on our ability to overcome the obstacle that a GLM equation (2.18) does not admit the solvability (see Remark 2.13). This means that, in the reflectionless inverse scattering, the obstacle is not necessarily crucial. From this point of view the present paper is quite different from [17], whose inverse scattering methodis based on the solvability (see assumption [17, D4]) of a GLM equation. This solvability is too strong for inverse scattering method to be applicable for a wider class of integrable systems. The first observation that the obstacle is not essential is found in [31], though the paper treated the case with real Q.

A characteristic of the solution formula (1.15) for the Boussinesq system (1.12) consists in the form

$$\Delta^{\pm}(x,t)=1+\cdots+(\det G)^2\prod_{n=1}^{N}e^{4ik_nx}, \tag{6.1}$$

where $(\det G)^2$ is defined in (3.1) that is independent of t, as well as the signature $\pm$. This stems from the invariance $c_n^+(t)c_n^-(t) = c_n^+(0)c_n^-(0)$ of the Boussinesq system: dependence on t and the signature appears only in the middle terms in (6.1). In particular, $\Delta^+(x,t) \not\equiv 1$, $\Delta^-(x,t) \not\equiv 1$ for each t.

It is of significance to ask a relation between soliton solutions of the Boussinesq system (1.12) constructed by Hirota [10, 11] and them of the present paper, because if we find some relation then the soliton solution can be obtained also by the inverse scattering method from a pair (U, Q) in (1.1) and so the solution admits two different approaches.

A candidate is found in Hirota [10, §2]. In the paper a way called "$pq = c$" reduction was developed, by which soliton solutions of the Boussinesq system (1.12) can be obtained from an N-soliton solution of the first modified KP equation. Then (u, w) in the Boussinesq system (1.12) is written as

$$u(x,t) = 2[\log(f/g)]_x, \quad w(x,t) = 4c + 2[\log(fg)]_{xx} \tag{6.2}$$

with a constant c, under a replacement $t \to -t$. Notice that (6.2) is a recast of [10, equation (1.13) with (2.9b)] by our notation. Here f, g are defined by [10, equation (2.3)], namely,

$$\begin{aligned} f &:= 1 + \sum_{i=1}^{\mathrm{N}} p_i e^{\eta_i} \\ &\quad + \sum_{r=2}^{\mathrm{N}} \sum_{i_1<\cdots<i_r} \prod_{\mu<\nu} p_{i_1}\cdots p_{i_r} \frac{(p_{i_\mu} - p_{i_\nu})(q_{i_\mu} - q_{i_\nu})}{(p_{i_\mu} - q_{i_\nu})(q_{i_\mu} - p_{i_\nu})} e^{\eta_{i_1}+\cdots+\eta_{i_r}}, \\ g &:= 1 + \sum_{i=1}^{\mathrm{N}} q_i e^{\eta_i} \\ &\quad + \sum_{r=2}^{\mathrm{N}} \sum_{i_1<\cdots<i_r} \prod_{\mu<\nu} q_{i_1}\cdots q_{i_r} \frac{(p_{i_\mu} - p_{i_\nu})(q_{i_\mu} - q_{i_\nu})}{(p_{i_\mu} - q_{i_\nu})(q_{i_\mu} - p_{i_\nu})} e^{\eta_{i_1}+\cdots+\eta_{i_r}}, \end{aligned}$$

with $\eta_i = (p_i - q_i)x_1 + (p_i^2 - q_i^2)x_2 + (p_i^3 - q_i^3)x_3$, $i = 1, \cdots, \mathrm{N}$. Hirota's "$pq = c$" reduction guarantees that if $p_i q_i = c$, $i = 1, \cdots, \mathrm{N}$, and f, g satisfy the bilinear equation

$$(D_1^3 + 3cD_2 - D_3) g \cdot f = 0$$

in the Hirota form (see Hirota [12]) then the pair (u, w) defined in (6.2) with $x_2 = -t$ satisfies the Boussinesq system (1.12). In the case $c \neq 0$, the element w of this solution tends to nonzero constant as $x \to \pm\infty$; this solution is different from our solutions defined in (1.15) in the present paper because our solutions tend to zero as $x \to \pm\infty$. On the other hand, the case $c = 0$ is somewhat delicate, because in order for g, f to satisfy

$$(D_1^3 - D_3)g \cdot f = 0, \tag{6.3}$$

additional conditions besides $p_i q_i = 0$, $i = 1, \cdots, \mathrm{N}$, are imposed on $\boldsymbol{p} = (p_1, \cdots, p_\mathrm{N})$ and $\boldsymbol{q} = (q_1, \cdots, q_\mathrm{N})$, as is shown in what follows.

In the case $c = 0$, the terms of f with $r \geq 2$ disappear since $p_{i_i} \cdots p_{i_r} \neq 0$ means $q_{i_\mu} - q_{i_\nu} = 0$ by assumption $p_i q_i = 0$. Also the terms of g with $r \geq 2$ disappear. Furthermore, by the assumption, $(D_1^3 - D_3)e^{\eta_i} \cdot 1 = (D_1^3 - D_3)1 \cdot e^{\eta_j} = 0$. Hence we get

$$\begin{aligned}(D_1^3 - D_3)g \cdot f &= (D_1^3 - D_3)\left(1 + \sum_{i=1}^{\mathrm{N}} q_i e^{\eta_i}\right) \cdot \left(1 + \sum_{j=1}^{\mathrm{N}} p_j e^{\eta_j}\right) \\ &= \sum_{i,j=1,\cdots,\mathrm{N}} q_i p_j (D_1^3 - D_3) e^{\eta_i} \cdot e^{\eta_j}.\end{aligned}$$

By renumbering, we may assume that

$$\boldsymbol{p} = (0, \cdots, 0, p_{\mathrm{M}+1}, \cdots, p_\mathrm{N}), \quad \boldsymbol{q} = (q_1, \cdots, q_\mathrm{M}, 0, \cdots, 0).$$

Then the equality above is written as

$$(D_1^3 - D_3)g \cdot f = \sum_{i \leq \mathrm{M} < j} q_i p_j (D_1^3 - D_3) e^{\eta_i} \cdot e^{\eta_j} = -3 \sum_{i \leq \mathrm{M} < j} q_i^2 p_j^2 \left(q_i + p_j\right) e^{\eta_i + \eta_j},$$

since $(D_1^3 - D_3)e^{\eta_i} \cdot e^{\eta_j} = -3 q_i p_j \left(q_i + p_j\right) e^{\eta_i + \eta_j}$ for $i \leq \mathrm{M} < j$. This implies that, for (6.3), it is necessary that $q_i + p_j = 0$ for each $i = 1, \cdots, \mathrm{M}$, $j = \mathrm{M} + 1, \cdots, \mathrm{N}$. Accordingly $\boldsymbol{p}$, $\boldsymbol{q}$ must be the forms

$$\boldsymbol{p} = (0, \cdots, 0, -q, \cdots, -q), \quad \boldsymbol{q} = (q, \cdots q, 0, \cdots, 0),$$

which lead to

$$\begin{aligned}\eta_i &= -q x_1 - q^2 x_2 - q^3 x_3, \quad i = 1, \cdots, \mathrm{M}; \\ \eta_i &= -q x_1 + q^2 x_2 - q^3 x_3, \quad i = \mathrm{M} + 1, \cdots, \mathrm{N}.\end{aligned}$$

This shows that, in "$pq = 0$" reduction, if (6.3) holds then

$$f = 1 - (\mathrm{N} - \mathrm{M})qe^{-qx_1+q^2x_2-q^3x_3}, \quad g = 1 + \mathrm{M}qe^{-qx_1-q^2x_2-q^3x_3},$$

and therefore fg has zeros necessarily as x_1 moves from $-\infty$ to $+\infty$ for each $x_2 = -t$, except for $\mathrm{M} = 0$ or $\mathrm{M} = \mathrm{N}$. This shows that the solution $(u(x,t), w(x,t))$ defined on the whole real x-line $\boldsymbol{R}$ of the Boussinesq system (1.12) can be obtained by "$pq = 0$" reduction only in the case $f \equiv 1$ or $g \equiv 1$, which is not the case with (6.1). We have thus proved that solutions of the Boussinesq system (1.12) constructed from "$pq = c$" reduction and our solutions by (1.15) are mutually different.

Another candidate is found in Hirota [11]. In the paper, Hirota treated the Wronskians

$$\tau := W[r_1, \cdots, r_n] = \begin{vmatrix} r_1 & \frac{\partial}{\partial x_1} r_1 & \cdots & \frac{\partial^{n-1}}{\partial x_1^{n-1}} r_1 \\ \vdots & \vdots & \vdots & \vdots \\ r_n & \frac{\partial}{\partial x_1} r_n & \cdots & \frac{\partial^{n-1}}{\partial x_n^{n-1}} r_n \end{vmatrix},$$
$$\tau' := W[r_1, \cdots, r_n, r_{n+1}],$$

with functions $r_i(x_1, x_2, x_3)$ satisfying

$$\frac{\partial}{\partial x_2} r_i = \frac{\partial^2}{\partial x_1^2} r_i, \quad \frac{\partial}{\partial x_3} r_i = \frac{\partial^3}{\partial x_1^3} r_i, \quad i = 1, \cdots, n+1,$$

to establish, in [11, §3], the theorem: *if the Wronskians are symmetric, namely,*

$$\frac{\partial}{\partial x_1} r_1 = r_2, \cdots, \frac{\partial^n}{\partial x_1^n} r_1 = r_{n+1}, \tag{6.4}$$

then

$$u(x,t) = 2[\log(\tau'/\tau)]_x, \quad w(x,t) = 2[\log(\tau'\tau)]_{xx}, \quad x_1 = x,\ x_2 = -t, \tag{6.5}$$

satisfy the Boussinesq system (1.12). This result can be regarded as a "$pq = 0$" reduction for Wronskian forms, although no parameter p, q appears explicitly. By applying this theorem to Wronskians of Hermite polynomials, it was shown in [11, §2] that rational functions expected to be solutions of the

Boussinesq system in Nakamura and Hirota [28] are actually the solutions of it.

Apart from the Wronskians of Hermite polynomials, Hirota [11, Appendix B] treated the case

$$\begin{aligned} r_i &= e^{\xi_i} + e^{\hat{\xi}_i}, \quad \xi_i = x_0 + x_1 p_i + x_2 p_i^2 + x_3 p_i^3, \\ \hat{\xi}_i &= x_0 + x_1 q_i + x_2 q_i^2 + x_3 q_i^3 \end{aligned} \tag{6.6}$$

to show that, in this case, τ gives the soliton solutions of the KP equation in the usual form

$$\eta_i = \varphi_i + (p_i - q_i)x_1 + (p_i^2 - q_i^2)x_2 + (p_i^3 - q_i^3)x_3,$$

where φ_i are constants. Therefore it can be expected that by applying Hirota's theorem to r_i in (6.6) we can obtain n-soliton solutions of the Boussinesq system (1.12). In the case (6.6), symmetry condition (6.4) is somewhat strong because the condition leads to

$$\begin{aligned} & e^{x_0 + x_1 p_i + x_2 p_i^2 + x_3 p_i^3} + e^{x_0 + x_1 q_i + x_2 q_i^2 + x_3 q_i^3} \\ & = p_1^{i-1} e^{x_0 + x_1 p_1 + x_2 p_1^2 + x_3 p_1^3} + q_1^{i-1} e^{x_0 + x_1 q_1 + x_2 q_1^2 + x_3 q_1^3} \end{aligned}$$

for each $(x_1, x_2) \in \boldsymbol{R}^2$. For $i = 2$, by setting $x_2 = 0$, we have

$$e^{x_3 p_1^3} e^{p_2 x_1} + e^{x_3 p_1^3} e^{q_2 x_1} - p_1 e^{x_3 p_1^3} e^{p_1 x_1} - q_1 e^{x_3 q_1^3} e^{q_1 x_1} = 0.$$

This is possible only if

$$p_2 = p_1 = q_2 = q_1 = 1$$

or

$$p_1 = 0,\ q_1 = q_2 = p_2,\ q_1 e^{x_3 q_1^3} = 2 \quad (q_1 = 0,\ p_1 = p_2 = q_2,\ p_1 e^{x_3 p_1^3} = 2).$$

The former case is meaningless because $r_1 = r_2 = \cdots = r_{n+1}$. In the latter case, by (6.4), we obtain

$$\begin{aligned} r_1 &= e^{\xi_1} + e^{\hat{\xi}_1} = e^{x_0}(1 + 2q_1^{-1} e^{x_1 q_1 + x_2 q_1^2}), \\ r_i &= 2q_1^{i-2} e^{x_0 + x_1 q_1 + x_2 q_1^2}, \quad i = 2, 3, \cdots. \end{aligned}$$

This implies that r_2, r_3,$\cdots$ are linearly dependent and so $\tau' = 0$ for $n \geq 2$. Therefore we have only one-soliton solution defined by (6.5). The solution is easily calculated from $\tau = W[r_1] = e^{x_0}(1 + 2q_1^{-1}e^{q_1x - q_1^2t})$, $\tau' = W[r_1, r_2] = 2e^{2x_0}q_1e^{q_1x-q_1^2t}$ as

$$(6.7) \qquad u(x,t) = \frac{2q^2e^{q^2t}e^{-qx}}{2 + qe^{q^2t}e^{-qx}}, \quad w(x,t) = \frac{4q^3e^{q^2t}e^{-qx}}{(2 + qe^{q^2t}e^{-qx})^2},$$

where we set $q := q_1$. Note that $q > 0$. The function $u(x, t)$ tends to $2q > 0$ as $x \to -\infty$. Accordingly the solution (6.7) is different from our solution that tends to 0 as $x \to \pm\infty$.

It is interesting that the solution (6.7) has a similar character to 1-soliton solution in Example 1.4: $w(x, t)$ is such as a water congestion driven by the velocity $u(x, t)$ like a boundary layer. As a matter of fact, the solution (6.7) can be obtained by setting $c_1^- = 0$, $c_1^+ = \frac{1}{4}q^2e^{q^2t}$ with $b = q$ in (1.10), (1.11) via (1.14). It is however a formal manipulation since the form (6.1) is robust. Therefore an inverse scattering theoretical approach to the Hirota's soliton solution is left to the future as a bidirectional subject. It should be also noticed that the situation is different for other systems such as (1.18) where the time invariance $c_n^+(t)c_n^-(t) = c_n^+(0)c_n^-(0)$ becomes relaxed.

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