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On the welfare role of redundant assets with heterogeneous forecasts

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Abstract

We study a multiperiod model with a nominal bond that matures in one period and identify the set of efficient allocations that can be sustained as Walrasian equilibria with heterogeneous forecasts. We next add a long maturity bond, which under perfect foresight would be a redundant asset, and show that it fundamentally expands the set of efficient allocations that can be sustained as Walrasian equilibria. Indeed all wealth transfers compatible with efficiency can arise endogenously. The key feature driving this conclusion are forecasting errors, which lead to ex post arbitrage opportunities that induce these income transfers. (JEL classification numbers: D51, D53, D61)

1 Introduction

No arbitrage conditions play a fundamental role in the way assets are priced and therefore are instrumental in deciding the set of allocations that can be generated by Walrasian markets. The axiom of perfect foresight is built into the methodology most frequently used to price assets.[1] This paper investigates the allocational implications of relaxing

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[1] However, this axiom does not sit well with the Walrasian paradigm since decentralized households cannot be expected to coordinate on prices that are not commonly observed (Radner [1982], Grandmont

perfect foresight in a model where a short term bond coexists with a longer maturity bond, where the latter under perfect foresight would be a redundant asset. Forecasts are required to satisfy no arbitrage conditions so that market equilibrium is well defined in each period. However, due to errors in forecasting, there may exist arbitrage possibilities in an ex post sense, which allows the presence of the long term bond to expand significantly the set of intertemporally efficient allocations that can be sustained as Walrasian equilibria.

In earlier work (Chatterji and Kajii [2023]) we have studied the structure of intertemporally efficient allocations which can arise as a sequence of temporary equilibria (ETE) in a two period model with multiple non-storable goods with one nominal asset (bond) and have shown that there is a one dimensional set of ETE allocations around each perfect foresight equilibrium (PFE) allocation, generically in endowments. To address the aforementioned issue about bonds with different maturity, we analyze the structure of the set of ETE allocations for a multiperiod model, where in each period consumption and saving take place competitively.

To understand the structure of ETE with multiple periods, we chooses to study a model with a single non-storable good, with utility functions and endowments which ensure the equivalence of intertemporal efficiency and perfect smoothing of consumption. There are H households, $H > 1$, and there are $T + 1$ periods, $T \geq 1$, and so the set of efficient allocations is an $H - 1$ dimensional set parameterized by the time invariant consumption level of H households. While restrictive, the simple structure of efficient allocations yields a transparent characterization of ETE in this model. It will be seen that the essence of our analysis does not rely on the restrictive set up.

First, we consider the benchmark case of a single asset, a discount bond which matures in one period. As is well known, the set of perfect foresight equilibria (PFE) coincides with that of Arrow Debreu equilibria. We show that the dimension of ETE allocation is at most T, one less than the number of trading periods. Note that $T = 1$ corresponds to the previous finding, and so this result can be seen as a natural extension. The indeterminacy grows with the time periods, and if $H < T$, any efficient allocation

[1988]). Various kinds of evidence against alignment of forecasts can be supplied even in more restrictive contexts of asset pricing (see e.g., Bossaerts [2002]).

can arise as a sequence of TE.

While the case $H < T$ appears to be of interest in modelling scenarios where trading opportunities arise frequently, it is at odds with the spirit of perfect competition since it means in effect that there are more markets than traders. Our emphasis will therefore be on the case $H > T$.

Secondly, we consider a model of two assets to address the issue concerning no arbitrage: in addition to the discount bond, there is another bond which matures in the last period and that is competitively traded in every period. Given our basic set up, one might however wonder if this additional asset is redundant in our model without uncertainty, since the long maturity bond could be replicated by a series of trading bonds of one period maturity, and thus their prices must be computable theoretically using no arbitrage conditions: the one period return of the long maturity bond should be the same as the return of one period maturity bond in the respective period. If it is not redundant, i.e., the theoretical price of the long term maturity bond does not match its price observed in the market, traders will find some arbitrage opportunities. Even when perfect foresight is not assumed, if a trader finds such a free lunch under its price forecasts, the markets cannot be in a temporary equilibrium. Although we do not assume perfect foresight up front, we do consider a temporary equilibrium, and hence at every trading opportunity, every household must have price forecasts which do not admit any arbitrage opportunity in the markets to take advantage of.

A plausible conjecture might be that this additional requirement of no arbitrage implies alignment of forecast prices, since it effectively requires the same theory about prices must be adopted by the households. Thus all temporary equilibria but the ones supported by common and perfect forecasts would be ruled out. Our result should then come as a surprise: in fact, the set of ETE allocations is of dimension $H - 1$, i.e., all intertemporally efficient allocations (in the vicinity of the Arrow Debreu allocation) are supportable by Walrasian markets since any direction of income transfers can arise in a temporary equilibrium for any T. When $H > T$ in particular, the set of ETE expands rather dramatically with the additional bond since without it the set of ETE is of dimension T.

Notice that an ETE by definition yields an efficient outcome and in this sense the markets function ideally, fulfilling their mandates. Every household' mathematical model explains the bond prices perfectly at any time under its forecast prices. In our simple world, each household is no worse than a financial firm which uses very sophisticated model to find out the correct valuations of assets in every trading opportunity. In spite of these aspects, our analysis concludes that any kind of income transfer can arise implicitly in competitive markets, i.e., there will be winners and losers.

The point will be seen clearly if the households are identical, and they might trade only when their forecasts differ from each other. One can view this special case as an adequate setup to explore a version of purely speculative trade. We show that the no-trade outcome is a unique ETE when the one period bond is traded, but any efficient allocation with some lower bound can arise as an ETE when both the one period maturity bond and the long maturity bond are traded in every period.

Our analysis will reveal that it is an *ex post* arbitrage opportunity which drives the income transfers. The discrepancy between the realized return of one period bond and the one period return of the long maturity bond may be arbitrarily small. Since households forecast future prices imperfectly, notwithstanding that their model is very sophisticated in the sense that they utilize all available information in the model in a logically consistent manner, they are not necessarily capable of identifying the opportunity *ex ante*; of course if they did, they would take advantage of it until the opportunity vanishes.

As far as we are aware of, (publicly known) asset pricing models about derivative securities are built upon an arguably stringent assumption that the assumed price processes of underlying assets are correct. In our model, the long maturity bond is a derivative security *if perfect foresight is assumed.* A liberal interpretation of our results is that unless the assumed process that is commonly known to be correct actually turns out to be correct, any income transfer that is consistent with efficiency can occur owing to ex post forecasting errors that lead to ex post arbitrage opportunities, in perfectly competitive markets.

The remainder of the paper is organized as follows. Section 2 introduces the benchmark model while Sections 3 and 4 study the temporary equilibria of model using the key intermediary concept of a quasi-temporary equilibrium. Section 5 studies the case where the additional longer maturity bond is added to the economy. Section 6 explores the possibility of speculative trade in this set up while Section 7 concludes.

2 Multiperiod Model: Benchmark

2.1 Set up

We consider a very simple market economy whose properties are well-known, but we nonetheless summarize its properties for completeness. Let there be $T+1$ periods starting with period 0, where $T > 0$. A single non-storable good is available in every period. There is no uncertainty in the economy.

There are H households, labeled by $h = 1, 2, ..., H$. To avoid triviality, we assume $H > 1$. Household h is endowed with e_h^t units of the good in period t, $t = 0, 1, ..., T$. There is no uncertainty and household h knows e_h^t, $t = 0, 1, ..., T$. We shall write $x_h^t \geq 0$ for the consumption of household h in period t. An allocation of the goods, $x = (\cdots, x_h, \cdots) \in \left(\mathbb{R}_+^{T+1}\right)^H$, is feasible if $\sum_{h=1}^H \left(x_h^t - e_h^t\right) = 0$ for $t = 0, 1, ..., T$.

For ease of illustration of main issues, we assume throughout the paper the following conditions to keep the structure of intertemporally efficient allocations simple:

1. The total endowment is one in every period, i.e., $\sum_{h=1}^H e_h^t = 1$ for $t = 0, 1, ..., T$.
2. Each household h has an additively time separable utility function $u_h\left(x_h^0\right) + u_h\left(x_h^1\right) + \cdots + u_h\left(x_h^T\right)$, with $u_h' > 0$ and $u_h'' < 0$. Also, $u_h'(0) = +\infty$ is assumed, in order to avoid boundary consumption.

Consequently, a feasible allocation of goods is efficient intertemporally if and only if assigns a time invariant consumption to every household.[2] An efficient allocation can therefore be parameterized by a tuple of positive numbers $\xi_1, \xi_2, \cdots, \xi_H$ with $\sum_{h=1}^H \xi_h = 1$, where ξ_h is the time invariant consumption level of household h. Thus from now on, we shall identify an efficient allocation with a tuple $(\cdots, \xi_h, \cdots)$ of H positive numbers summing up to one.

[2] Indeed, it can be readily confirmed that a time invariant feasible allocation is efficient. Conversely, for any consumption path $\left(x^0, x^1, ..., x^T\right)$, every household prefers the path which provides its average consumption $\frac{1}{T+1} \sum_{t=0}^T x^t$ every period, so an efficient allocation must be time invariant.

Consider an Arrow Debreu equilibrium $(p, x) \in \mathbb{R}^{T+1} \times \left(\mathbb{R}^{T+1}\right)^H$ of this economy (AD equilibrium, hereafter), where $p = \left(\cdots, p^t, \cdots\right)$ are positive prices of the goods and $x = (\cdots, x_h, \cdots)$ is the associated allocation of the goods; that is, each household is utility maximizing at x_h given prices p and income $p \cdot e_h$, and x is feasible. The allocation x is of course efficient by the first fundamental theorem of welfare economics. From utility maximization and the additive time separability of the utility function, prices p must be proportional to the gradient vector $u_h'(x_h) = \left(u_h'\left(x_h^0\right), u_h'\left(x_h^1\right), \cdots, u_h'\left(x_h^T\right)\right)$ for every household h. The observation about the efficient allocations above implies that $\left(u_h'\left(x_h^0\right), u_h'\left(x_h^1\right), \cdots, u_h'\left(x_h^T\right)\right) = \left(\cdots, u_h'(\xi_h), \cdots\right)$, which is time invariant. Thus the AD equilibrium price system must also be time invariant.

By the homogeneity of equilibrium prices, the AD equilibrium price of the good can be normalized to be $\frac{1}{T+1}$ in each period, in particular, the AD equilibrium allocation is unique. With the normalized AD equilibrium prices, the value of the total endowments (one in every period) is one, and the market value of household h's endowments is $\frac{1}{T+1}\sum_{t=0}^{T} e_h^t$, and the market value of the consumption consuming ξ_h in every period is ξ_h. So from the budget constraint, we conclude that in the unique Arrow Debreu equilibrium, household h consumes $\xi_h = \frac{\sum_{t=0}^{T} e_h^t}{T+1}$ in every period.

Example 1 *$H = 4$ and $T = 2$. The endowments are given as in the following table:*

$h \backslash t$	$t = 0$	$t = 1$	$t = 2$
$h = 1$	$\frac{1}{4} + 2\eta$	$\frac{1}{4} - \eta$	$\frac{1}{4} - \eta$
$h = 2$	$\frac{1}{4} - \eta$	$\frac{1}{4} + 2\eta$	$\frac{1}{4} - \eta$
$h = 3$	$\frac{1}{4} - \eta$	$\frac{1}{4} - \eta$	$\frac{1}{4} + 2\eta$
$h = 4$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

where $0 \leq \eta < \frac{1}{4}$. That is, household h, $h = 1, 2, 3$, has a high endowment in period $t = h - 1$, and a low endowment in the other periods, whereas household $h = 4$ has a constant endowment $\frac{1}{4}$ in every period. Notice that $e_h^0 + e_h^1 + e_h^2 = \frac{3}{4}$ for all households. Thus in a unique PFE, every household consumes $\frac{1}{4}$ in every period.

3 Temporary Equilibrium

There is a discount bond traded in periods $t = 0, 1, ..., T - 1$, which matures in one period. That is, the bond traded in period t pays out \$1 in period $t + 1$. The net supply of any of these bonds is zero. By assumption no default occurs. Later, we will consider bonds with longer maturity. Note that the payout is fixed in units of account, not in units of good. As is known, when markets are complete and perfect foresight *is assumed*, the distinction is not essential since the real return of the asset is commonly known. However, when heterogeneous forecasts are permitted, the expected real returns of a nominal bond might be different across households.

In period t, the good and the bond are traded competitively. We write p^t for the price of the good in units of account in period t. Write q^t for the price of the bond traded in period t in units of account. Write b_h^t for the amount of the bond household h holds at the end of period t. Unlimited short sales is allowed, so b_h^t is possibly any negative number. However, default is not allowed.

If perfect foresight (rational expectation) is assumed, as is well known, with the bond traded in every period, the markets are complete and a perfect foresight dynamic equilibrium allocation of this economy must be an AD equilibrium allocation, and vice versa. Hence there is a unique competitive equilibrium where household h consumes $\xi_h = \frac{\sum_{t=0}^{T} e_h^t}{T+1}$ in every period.

However, we do not assume perfect foresight a priori. In every period t, household h will trade with some forecast prices in mind, which are not necessarily correct ex post, given market prices available that period, i.e., p^t and q^t. It means in particular that households might anticipate different rate of real returns of the bond. Write $\hat{p}_{h|t} = \left(\hat{p}_h^{t+1}, ..., \hat{p}_h^{T+1}\right)$ and $\hat{q}_{h|t} = \left(\hat{q}_h^{t+1}, ..., \hat{q}_h^{T-1}\right)$ for the forecast prices of the good and the bond of household h. Then household h optimizes given prices p^t and q^t as well as the outstanding bond holding b_h^{t-1} (where $b_h^{-1} = 0$ by convention) under the following constraints,

$$\begin{aligned} p^t x^t + q^t b^t &\leq p^t e_h^t + b_h^{t-1}, \\ \hat{p}_h^{t+1} \hat{x}^{t+1} + \hat{q}_h^{t+1} \hat{b}^{t+1} &\leq \hat{p}_h^{t+1} e_h^{t+1} + b^t, \\ &\vdots \\ \hat{p}_h^{t+1} \hat{x}^{T-1} + \hat{q}_h^{T-1} \hat{b}^{T-1} &\leq \hat{p}^{T-1} e_h^{T-1} + \hat{b}^{T-2}, \\ \hat{p}_h^{t+1} \hat{x}^T &\leq \hat{p}^T e_h^T + \hat{b}^{T-1}, \end{aligned} \tag{1}$$

with variables x^t, b^t, $\hat{x}^{t+1}, ..., \hat{x}^T$, and $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$. Note that the choice variables with hats are also forecasts and yet to be realized at the time household h trades x^t and b^t. Notice that one can normalize $p^0 = 1$ but not necessarily the other variables; the real market value of the outstanding bond holding depends on various forecast prices.

For future reference, say that $\left(x^t, b^t\right)$ is *justifiable* at prices $\left(p^t, q^t\right)$ (and b_h^{t-1}) for household h if there exist forecasts $\hat{p}_{h|t}$ and $\hat{q}_{h|t}$ about future prices such that household h's utility is maximized at x^t, b^t, $\hat{x}^{t+1}, ..., \hat{x}^T$, and $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$ for some $\hat{x}^{t+1}, ..., \hat{x}^T$, and $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$. That is, $\left(x^t, b^t\right)$ is justifiable if they constitute household h's demand in period t for some forecasts. Note that the justifiability is purely individualistic, specific to each household, if there are no constraints on forecasts.

The classical (sequential) temporary equilibrium is defined as follows: a sequence of prices and consumption allocation $((p, q), x)$ constitutes a temporal equilibrium (TE) if x is feasible and for every household h, $\left(x^t, b^t\right)$ is justifiable at prices $\left(p^t, q^t\right)$ at $t = 0, 1, ..., T$. It is tempting to set $p^t = 1$ for $t = 0, 1, ..., T$ for normalization, but for the reason mentioned above, normalization of prices is delicate under heterogeneous forecasts and nominal bonds. So we choose to proceed without normalization for now.

Under our maintained assumptions on the utility functions, the budget constraint holds with equality for any household in any TE. Notice that even when forecast prices are incorrect, the ex post budget constraint of a household h must hold as an equality nonetheless. So write x_h^t for the amount of the good consumed in period t by household

h, the following constraints must hold:

$$
\begin{aligned}
p^0 x_h^0 + q^0 b_h^0 &= p^0 e_h^0 \\
p^1 x_h^1 + q^1 b_h^1 &= p^1 e_h^0 + b_h^0 \\
&\vdots \\
p^t x_h^t + q^t b_h^t &= p^t e_h^t + b_h^{t-1} \\
&\vdots \\
p^{T-1} x_h^{T-1} + q^{T-1} b_h^{T-1} &= p^{T-1} e_h^{T-1} + b_h^{T-2} \\
p^T x_h^T &= p^T e_h^T + b_h^{T-1}
\end{aligned}
\tag{2}
$$

Notice that prices p^t and q^t appearing the equations above are realized prices in the respective competitive markets, thus the equations hold irrespective of price forecasts household h may have at any time.

Multiplying period t constraint above by $q^0 q^1 \cdots q^{t-1}$ and summing them up, we obtain an ex post budget constraint:

$$\sum_{t=0}^{T} \tilde{p}^t \left(x_h^t - e_h^t \right) = 0 \tag{3}$$

where $\tilde{p}^t$ is the discounted period t price, i.e., $\tilde{p}^t = q^0 q^1 \cdots q^{t-1} p^t$. If perfect forecast is assumed, constraint (3) holds ex ante with forecast prices, i.e., household h plans to choose a utility maximizing x_h given constraint (3), and trades the bond to finance, i.e., to satisfy (2).

If there is no requirement about forecasts, justifiability is a mild condition and so the set of TE allocations tend to be a large set; as long as $\left(x_h^t, b_h^t \right)$ is in the range of household h's demand, the ex post budget constraint (3) is the only binding requirement. TE for itself has insufficient predictive power, and there ought to be more structure. But perfect foresight seems too extreme.

We advocate an allocation based requirement, and as its extreme form we consider efficient allocations which appear as TE. Since efficiency implies considerable alignment of marginal rates of substitution, one might expect that an efficient TE might necessarily entail perfect foresight. For the case of $T = 1$, Chatterji and Kajii [2023] have shown that there is one dimensional set of TE allocations around an AD equilibrium allocation, generically in endowments, in a much more general setup: the result holds for the case of multiple goods under mild assumption on utility functions. But the case of $T > 1$ has been unknown.

4 Structure of Quasi-ETE

We begin with the intermediate notion of a Quasi- ETE (Q-ETE), which obtains when an efficient allocation satisfies the ex post budget constraint (3) for all households with common (discounted) prices which clear spot markets. There is one good in each period in this model, and hence the spot market clearing condition is a trivial implication of the feasibility of consumption. The ex post budget constraint is an inevitable consequence of price taking trade as we have discussed. Thus any efficient allocation which might arise in perfectly competitive markets must be a Q-ETE.

By construction, a Q-ETE does not contain any information about households' forecasts, and so in particular it might not arise in the competitive markets as a result of voluntary utility maximization. Indeed, if in addition the consumption is justifiable in every period for all households, then we have a TE, and thus an efficient TE (ETE).

We shall first characterize Q-ETE in this benchmark model of a single type of bond. Then we will ask if they are justifiable for each household.

Recall that the efficiency of an allocation is equivalent to the time invariance and the feasibility of the allocation in this model. Thus, a sequence of positive (discounted) ex post prices $p = \left(\cdots, p^t, \cdots\right)$ and a time invariant consumption $x_h > 0$ allocated to households $h = 1, 2, ..., H$ constitute a Quasi-ETE if and only if the following $H + 1$ equations hold:

$$\begin{array}{rcl} \sum_{t=0}^{T} p^t \left(x_h - e_h^t\right) & = & 0 \quad \text{for } h = 1, 2, .., H \\ \left(\sum_{h=1}^{H} x_h\right) - 1 & = & 0 \end{array} \tag{4}$$

The first set of H equations implies that the ex post budget balance for all households and the second of course implies feasibility.

The set of Q-ETE can be succinctly parametrized by prices. Notice that the last feasibility equation is redundant in (4). Indeed, if $\sum_{t=0}^{T} p^t \left(x_h - e_h^t\right) = \left(\sum_{t=0}^{T} p^t\right) x_h -$

$\sum_{t=0}^{T} p^t e_h^t = 0$ holds for each h, i.e., when the first H equations hold, then

$$\begin{aligned}
\sum_{h=1}^{H} x_h &= \sum_{h=1}^{H} \frac{\sum_{t=0}^{T} p^t e_h^t}{\left(\sum_{t=0}^{T} p^t\right)} \\
&= \frac{1}{\sum_{t=0}^{T} p^t} \sum_{t=0}^{T} p^t \left(\sum_{h=1}^{H} e_h^t\right) \\
&= \frac{1}{\sum_{t=0}^{T} p^t} \sum_{t=0}^{T} p^t \\
&= 1,
\end{aligned}$$

hence the last equation holds automatically.

Therefore, $(x_h)_{h=1}^{H}$ is a Q-ETE allocation if and only if there are prices $p = \left(p^0, p^1, \cdots, p^T\right)$ for which the budget constraint (3) is satisfied for every household. Since they are homogeneous equations in p, we might as well normalize $\sum_{t=0}^{T} p^t = 1$. Then the set of equations (4) is equivalent to

$$\sum_{t=0}^{T} p^t e_h^t \quad = \quad x_h \quad \text{for } h = 1, 2, .., H \quad . \tag{5}$$

Notice that if (5) is satisfied, since $\sum_{h=1}^{H} x_h = 1$, $1 = \sum_{h=1}^{H} \left(\sum_{t=0}^{T} p^t e_h^t\right) = \sum_{t=0}^{T} p^t \left(\sum_{h=1}^{H} e_h^t\right) = \sum_{t=0}^{T} p^t$, so the prices are normalized as required. This shows that for any normalized positive prices, there is a Q-ETE allocation.

Or equivalently, (5) can be expressed with matrices: let E be a $(T+1) \times H$ matrix given by the rule

$$E = [e_1, e_2, ..., e_H]$$

where e_h is the column vector of endowments for household h. Then (5) can be written as

$$pE = x, \tag{6}$$

where $p = \left(p^0, p^1, \cdots, p^T\right)$ and $x = (x_1, x_2, \cdots, x_H)$ are row vectors.

To sum up, a row vector of an efficient time invariant consumption allocation, $x = (\cdots, x_h, \cdots) \in \mathbb{R}^H$, is a Q-ETE allocation if and only if the simultaneous equations (5), or equivalently (6), have a positive solution $p = \left(p^0, p^1, ..., p^T\right) \in \mathbb{R}^T$.

It means that the set of Q-ETE allocations is the image of the set of normalized positive prices for the linear function $p \mapsto pE$. The dimension of this set is the rank of matrix E minus one. More specifically, notice that pE can be expressed as a convex combination of row vectors of initially endowed goods among households in period t:

$$pE = \sum_{t=0}^{T} p^t e^t$$

where $e^t = \left(\cdots, e_h^t, \cdots\right)$ is the row vector of initially endowed goods among households in period t. Recall that a unique AD equilibrium normalized prices are $p^t = \frac{1}{T+1}$ for $t = 0, 1, ..., T$, which certainly satisfies the equation above**.** Thus we obtain the following result:

Proposition 2 *The set of Q-ETE allocation is* $\left\{\sum_{t=0}^{T} p^t e^t : \sum_{t=0}^{T} p^t = 1, p^t > 0, \text{ for } t = 0, 1, ..., T\right\}$. *That is, the set of Q-ETE allocations is the relative interior of the convex hull of* $T+1$ *vectors* $e^0, e^1, ..., e^T$ *in* $\mathbb{R}^H$.

When $T = 1$, i.e., there are two periods, the relative interior of the convex hull of $T + 1 = 2$ vectors $e^0, e^1 \in \mathbb{R}_{++}^H$ has dimension 1 unless e^0 and e^1 are collinear, since $H \geq 1$. When they are collinear, they must be the same vector since the total resource is constant, which means that the set of Q-ETE is a singleton (in which case we regard the relative interior is the point itself by convention) So, in particular, generically in endowments, we conclude that the dimension of Q-ETE allocation is one.

Since the total resource is one for any t, the dimension of the convex set in question is at most $\min(T, H-1)$. If any choice of $\min(T+1, H)$ vectors among $e^0, e^1, e^2, \cdots, e^T$ are affine independent, then the dimension is exactly $\min(T, H-1)$. Since such affine independence is a generic property, we conclude that generically in endowments (with total resource equal to one in every period), the dimension of Q-ETE allocation is $\min(T, H-1)$. In particular, when $T \geq H$, any efficient allocation is a Q-ETE allocation, generically in endowments.

The result above is a consequence of the dimensionality of the set of efficient allocations and the ex post budget sets. It is transparent owing to the simple linear structure of the set of efficient allocations, but even for general utility functions and endowments, we conjecture that the set of efficient allocations is a manifold of the same dimension as in this setup, generically.[3] Thus in our view, the implication about the dimension of the

[3] We have proved this for the special case $T = 1$ with arbitrarily many goods in each of the two periods

set of Q-ETE allocations does not appear to hinge upon the special structure employed in this note.

Example 3 *Consider the economy in Example 1. The set of Q-ETE is the convex combination of three column vectors*

$$\begin{pmatrix} \frac{1}{4}+2\eta \\ \frac{1}{4}-\eta \\ \frac{1}{4}-\eta \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{4}-\eta \\ \frac{1}{4}+2\eta \\ \frac{1}{4}-\eta \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{4}-\eta \\ \frac{1}{4}-\eta \\ \frac{1}{4}+2\eta \\ \frac{1}{4} \end{pmatrix},$$

which is a two dimensional subset of $\mathbb{R}^4$*, as long as* $\eta > 0$*. In any Q-ETE, household 4 consumes* $\frac{1}{4}$ *in every period.*

Given a Q-ETE allocation, we ask if every household chooses to consume the allocated bundle of goods, i.e., consumption is justifiable. Say, household h is to consume $\bar{x}_h$ in every period. Fix period $t < T$, and ask if and how consumption of $\bar{x}_h$ in period t is justified.

We will not formally demonstrate the justifiability of Q-ETE allocations in the section as it is not the central finding of this paper. We provide below an informal discussion of how justifiability is obtained in this framework. [4]

Since transactions before t are already completed and they meet the budget equation (2), b_h^{t-1} is equal to the total net saving accumulated before t, which is a exogenously fixed parameter[5]. In period t, household h maximizes (continuation) utility subject to the (continuation) budget (1). That is, $\left(x^t, b^t\right)$ of household h is justified if and only if x^t is the quantity demanded for some forecasts, as the associated b^t is found automatically from the budget.

in Chatterji and Kajii [2023]. The one dimensionality of the Q-ETE for $T = 1$ we obtained above is a variant of the local indeterminacy result we have established in the aforementioned earlier paper. We believe that the proof of that paper can be generalized to the multiple good version of the multi-period model considered here.

[4] We refer to Chatterji-Kajii [2023] for general techniques and issues therein.

[5] Since the saving decision in period $t-1$ might be done with very incorrect forecasts, it is possible that the household is practically bankrupt. However, as long as endowments are positive, there always exist (very optimisitic) forecasts with which household can repay the debt in future.

Suppose that the demand function is responsive to forecasts in the sense that as a function of price forecasts, the demand changes in any direction. This property will be generically true under some mild and plausible conditions on utility functions and endowments, at least if x^t is the AD (thus PFE) consumption in period t. We therefore contend that any Q-ETE close enough to the AD equilibrium is an ETE.

For a specific case of additive log function, i.e., $u_h(z) = \ln(z)$, the problem of justifiability is reduced to find a forecast future income $\hat{m}$ (which depends on price forecasts, but not time t market variables) such that x_h^t is the quantity demanded in period t, which can be explicitly written as $\gamma \frac{p^t e^t + b^{t-1} + \hat{m}}{p^t}$ where γ, $1 > \gamma > 0$ is a parameter which depends on the length of the remaining periods. It is clear that any positive $\hat{m}$ can be found by choosing suitable forecasts exists as long as endowments are strictly positive. Then to verify the existence of a suitable $\hat{m}$ to justify consumption, it suffices to show $x_h^t > \gamma \frac{p^t e^t + b^{t-1}}{p^t}$, which follows if the stream of consumption in question satisfies the ex post budget constraints since $1 > \gamma$. Note that this observation is not restricted to being in a neighborhood of the AD equilibrium consumption. Thus in fact a wide range of Q-ETE allocations are ETE allocations. We shall re-examine this property in the next section so we summarize it as follows:

Lemma 4 *When $u_h(z) = \ln(z)$ and $e_h^t > 0$ for every t, any stream of positive consumption which meets the ex post budget constraints at given positive prices can be justified in $t = 0, 1, \cdots, T-1$.*

Example 5 *In the economy of Example 1, let $u_h(z) = \ln(z)$ for every h. Then the set of ETE coincides with the set of Q-ETE by Lemma 4. Since household 4 consumes $\frac{1}{4}$ in any ETE, household 4 must think its income is invariant over time, hence the discounted prices are constant over time, just as in the unique PFE. In this sense, household 4' forecasts are always on the right track. But it does not imply household 4's forecasts are correct ex post. Indeed, there are ETE where discounted ex post prices $\tilde{p}^1$ and $\tilde{p}^2$ are different from p^0, and then household 4's forecasts cannot be correct ex post.*

5 Model with a Longer Maturity Bond

5.1 Set up

Here we want to consider additional assets in the model. Since the markets are already complete with bonds with one period maturity, one may wonder why such an exercise is of interest. Indeed under perfect foresight, the payoffs of an additional asset can be replicated by a plan of dynamic transactions of the bonds. Thus any additional asset is redundant in this sense, and its market price must be the same as the cost of the dynamic transactions, or else the markets would admit an arbitrage opportunity. In particular, *under the assumption of perfect foresight*, an additional asset does not create new trading opportunities of the underlying consumption goods. However, in the absence of perfect foresight, we show that the set of Q-ETE might be different when there are multiple assets. Furthermore, as we later demonstrate, the structure of bond markets might influence the nature of the justifiability issue, since it might affect the property of price forecasts of households. Since additional assets are all redundant in the perfect foresight model, our formulation thus marks a stark contrast to the standard rational expectations paradigm.

We consider the simplest case of one additional asset to address the issue. There is a discount bond with a longer maturity of zero net supply, which can be traded in every period. The long term bond pays \$1 in period T, nothing in other periods. Write q_L^t for its (realized) market price in period t, $t = 0, 1, ..., T-1$. If perfect foresight is assumed, the long term bond can be replicated by a familiar synthetic trading plan of the (short term) bonds: to assure \$1 in period T, it plan to hold a unit of the bond in period $T-1$, which costs q^{T-1} in period $T-1$. Rolling the procedure backward, it plans to hold $q^{t+1} \cdots q^{T-1}$ units of the bond in period t, which costs $q^t q^{t+1} \cdots q^{T-1}$.

Since the synthetic trading plan has exactly the same yield as the long term bond, by the no arbitrage principle, we find that the market price long term bond in period t, q_L^t, must be the same as the cost of the plan, hence $q_L^t = q^t q^{t+1} \cdots q^{T-1}$ holds for $t = 0, 1, ..., T-1$, in any PFE. Moreover, the set of PFE allocations and their associated (discount) prices is the same as in the benchmark model. In this sense introduction of the long term maturity bond does not affect the market outcomes.

5.2 Structure of Q-ETE

Since Q-ETE is defined by the ex post budget (3) and efficiency, as long as the same the ex post budget (3) is required, the set of Q-ETE allocation remains the same if the asset (bond) structure is modified. Therefore the key question here is whether or not the ex post budget (3) suffices to describe the ex post budget constraint in this modified asset structure.

Write q_L^t for its (ex post) market price in period t. Write l_h^t for the amount of the long term bond held by household h at the end of period t. Thus $l_h^t - l_h^{t-1}$ is the amount traded in period t, which costs $q_L^t \left(l_h^t - l_h^{t-1}\right)$. Then the following budget equations must hold, irrespective of forecasts:

$$\begin{aligned} p^0 x_h^0 + q^0 b_h^0 + q_L^0 l_h^0 &= p^0 e_h^0 \\ p^1 x_h^1 + q^1 b_h^1 + q_L^1 \left(l_h^1 - l_h^0\right) &= p^1 e_h^0 + b_h^0 \\ &\vdots \\ p^t x_h^t + q^t b_h^t + q_L^t \left(l_h^t - l_h^{t-1}\right) &= p^t e_h^t + b_h^{t-1} \\ &\vdots \\ p^{T-1} x_h^{T-1} + q^{T-1} b_h^{T-1} + q_L^{T-1} \left(l_h^{T-1} - l_h^{T-2}\right) &= p^{T-1} e_h^{T-1} + b_h^{T-2} \\ p^T x_h^T &= p^T e_h^T + b_h^{T-1} + l_h^{T-1} \end{aligned} \tag{7}$$

Suppose that among the ex post market prices, the relation $q_L^t = q^t q^{t+1} \cdots q^{T-1}$ holds for $t = 0, 1, ..., T-1$, just like in PFE. Then multiplying the period t equation by $q^0 q^1 \cdots q^{t-1}$ and summing up, we see the ex post budget equation (3) holds. Starting with a Q-ETE without the long term bond, if the long term bond price is set by the formula $q_L^t = q^t q^{t+1} \cdots q^{T-1}$ to hold ex post, then the same Q-ETE allocation is sustained. Therefore, we obtain a corollary to Proposition 2:

Corollary 6 *The set of Q-ETE allocation in the model with the long maturity bond contains* $\left\{\sum_{t=0}^T p^t e^t : \sum_{t=0}^T p^t = 1, p^t > 0, \text{ for } t = 0, 1, ..., T\right\}$.

We emphasize however that the relation $q_L^t = q^t q^{t+1} \cdots q^{T-1}$ of ex post prices *is not warranted* in our set up, since in period t, the bond prices $q^{t+1}, \cdots, q^{T-1}$ are not observed. Hence to find the aforementioned synthetic trading, household h must rely on forecasts about these short term bond prices. On the other hand, even with heterogeneous forecasts, the bond prices must be subject to some constraints or else they might admit what may seem like arbitrage opportunities. Forecasts which predict arbitrage opportunities are not consistent with TE; since there is no limit on the volume of trade, if a household finds an arbitrage opportunity, it will engage in an indefinitely large amount of trade. Thus it might appear that the law of one price of this sort implies that the long term bond does not add any distributional effect, just as for PFE.

Observe that households must agree that in period $T-1$, the (short term) bond and the long term bond are identical objects, and thus unless $q^{T-1} = q_L^{T-1}$, the asset markets in period $T-1$ will not clear, *regardless of* forecasts about prices. Thus we conclude that $q^{T-1} = q_L^{T-1}$ holds ex post. Each household should take this conclusion into account, or else it allows itself an arbitrage opportunity.

In period $T-2$, as is mentioned above, each household's forecast future prices should not allow any arbitrage opportunity. Hence if $\hat{q}^{T-1}$ and $\hat{q}_L^{T-1}$ are forecast bond prices of household h, $\hat{q}^{T-1} = \hat{q}_L^{T-1}$ must hold or else household h allows itself an arbitrage opportunity in period $T-1$. Moreover, since the cost of the synthetic trading for the long term bonds is forecast as $q^{T-2}\hat{q}^{T-1}$, $q_L^{T-2} = q^{T-2}\hat{q}^{T-1}$ must hold or else household h finds an arbitrage opportunity in period $T-2$ at the prevailing bond prices. Hence $\hat{q}_L^{T-1} = \hat{q}^{T-1} = \frac{q_L^{T-2}}{q^{T-2}}$ must hold. Notice that the forecast values $\hat{q}_L^{T-1}$ and $\hat{q}^{T-1}$ are tied to the market prices observed in $T-2$, which means that all households must *have common forecasts* about *the relative* bond prices, although it might first appear that the households could disagree.

The argument iterates similarly: in period $T-3$, let $\hat{q}^{T-1}$, $\hat{q}_L^{T-1}$ and $\hat{q}^{T-2}$ and $\hat{q}_L^{T-2}$ be forecast bond prices which household h has when trading in period $T-3$. Then $\hat{q}^{T-1} = \hat{q}_L^{T-1} = \frac{\hat{q}_L^{T-2}}{\hat{q}^{T-2}}$, and hence $\hat{q}^{T-2}\hat{q}^{T-1} = \hat{q}_L^{T-2}$ must follow or else the household allows itself an arbitrage opportunity in period $T-1$ or $T-2$. Moreover, since the cost of the synthetic trading for the long term bonds is forecast as $q^{T-3}\hat{q}^{T-2}\hat{q}^{T-1}$, $q_L^{T-3} = q^{T-3}\hat{q}^{T-2}\hat{q}^{T-1}$ must hold or else household h finds an arbitrage opportunity in period $T-3$ at the prevailing bond prices. Combining this with $\hat{q}^{T-2}\hat{q}^{T-1} = \hat{q}_L^{T-2}$, we have $\frac{q_L^{T-3}}{q^{T-3}} = \hat{q}^{T-2}\hat{q}^{T-1} = \hat{q}_L^{T-2}$. Proceeding analogously we obtain:

Lemma 7 *The bond price forecasts satisfy* $\frac{q_L^{T-j-1}}{q^{T-j-1}} = \hat{q}_L^{T-j}$ *for* $j = 1, ..., T-1$. *Thus the* *forecasts of the long term bond prices are commonly held across the household in any period.*

It is worth remarking that the presence of the long bond serves as a coordination device, via a no-arbitrage condition, that leads households to a common bond price forecast for the next period. Despite these common forecasts, households might still disagree on real rates of interest as commodity price forecasts are not coordinated, and the payouts of the bonds are fixed nominally. Moreover, these constraints specified in Lemma 7 do not imply that the ex post price relation $q_L^t = q^t q^{t+1} \cdots q^{T-1}$, and these constraints exhaust the implications of no arbitrage requirement. To confirm these points, we shall focus on the minimal set up of $T = 2$ from now on. It can be readily seen that the argument can be extended for $T > 2$.

When $T = 2$, the ex post budget equations are:

$$\begin{aligned} p^0 x_h^0 + q^0 b_h^0 + q_L^0 l_h^0 &= p^0 e_h^0 \\ p^1 x_h^1 + q^1 b_h^1 + q_L^1 \left(l_h^1 - l_h^0\right) &= p^1 e_h^1 + b_h^0 \\ p^2 x_h^2 &= p^2 e_h^2 + b_h^1 + l_h^1 \end{aligned} \tag{8}$$

The implications of no-arbitrage requirement explored above are:

1. $q^1 = q_L^1$ must hold, since in period 1, these two bonds are equivalent. This condition holds regardless of forecasts.

2. in period 0, for any household h, $\hat{q}^1 = \hat{q}_L^1 = \frac{q_L^0}{q^0}$ must hold since household must not forecast prices which admits arbitrage.

Note that the condition $\hat{q}^1 = \hat{q}_L^1 = \frac{q_L^0}{q^0}$ implies the households' forecasts must coincide, but it does not add any restriction on the relative price of the two assets in period 0. It is readily confirmed that the no arbitrage condition imposes no other conditions since the two conditions mean that the bond prices are exactly the prices which are found from the equivalence of the synthetic trade and the long term bond, for every household. [6]

[6] As is well known (e.g., Lemma 19.E.1 of Mas-Collel et al [1995]), in the rational expectations model, no free lunch is equivalent to the existence of state prices, and asset prices are derived using the state prices. In our context, bond prices are discounted values of forecast future bond prices, and therefore the relations derived above correspond to no free lunch.

In order to identify the set of Q-ETE, first use condition 1 above and set $q^1 = q_L^1$. Multiply period 1 budget equation by q^0 and period 2 budget equation by $q^0 q^1$, and then add them up we have

$$p^0 \left(x_h^0 - e_h^0\right) + \tilde{p}^1 \left(x_h^1 - e_h^1\right) + \tilde{p}^2 \left(x_h^2 - e_h^2\right) = \left(q^0 q^1 - q_L^0\right) l_h^0 \tag{9}$$

where $\tilde{p}^1 = q^0 p^1$ and $\tilde{p}^2 = q^0 q^1 p^2$ are the discounted prices. With perfect foresight, the bond price relation $q^0 q^1 - q_L^0 = 0$ holds by no arbitrage, and so (9) is reduced to (3), confirming that the set of Q-ETE with the long term bond includes the set of Q-ETE without it.

The set of Q-ETE with the long term bond turns out to be strictly larger than that without it. In fact, we shall argue that any efficient allocation is a Q-ETE (even when $H > T$), and moreover the nominal prices of the good and the short term bond are completely indeterminate. Fix an efficient allocation, and fix positive p^0, p^1, p^2 and q^0, q^1 arbitrarily, hence discounted prices $\tilde{p}^1$ and $\tilde{p}^2$ are also fixed. Set q_L^0 to be (slightly) different from $q^0 q^1$. Set $q_L^1 = q^1$, or else there will be some arbitrage opportunity. Write $\bar{x}_h$ for the constant consumption of household h in the efficient allocation. Then (9) is reduced to

$$\left(p^0 + \tilde{p}^1 + \tilde{p}^2\right) \bar{x}_h - \left(p^0 e_h^0 + \tilde{p}^1 e_h^1 + \tilde{p}^2 e_h^2\right) = \left(q^0 q^1 - q_L^0\right) l_h^0.$$

Set

$$l_h^0 = \frac{\left(p^0 + \tilde{p}^1 + \tilde{p}^2\right) \bar{x}_h - \left(p^0 e_h^0 + \tilde{p}^1 e_h^1 + \tilde{p}^2 e_h^2\right)}{\left(q^0 q^1 - q_L^0\right)}, \tag{10}$$

and so (9) holds all households. By construction, $\sum_{h=1}^H l_h^0 = 0$, i.e., market clearing for for the long term bond in period 0 is guaranteed. Set $l_h^1 = l_h^0$ (i.e., no re-trade of the long term bond) and for the shot term bond, just find the unique b_h^0 and b_h^1 so that (8) is satisfied: that is,

$$\begin{aligned} b_h^0 &= \frac{1}{q^0} \left(p^0 e_h^0 - p^0 x_h^0 - q_L^0 l_h^0\right) \\ b_h^1 &= \frac{1}{q^1} \left(p^1 e_h^1 + b_h^0 - p^1 x_h^1\right) \end{aligned} \tag{11}$$

By construction, $\sum_{h=1}^H b_h^0 = \sum_{h=1}^H b_h^1 = 0$. Therefore, we have shown that the efficient allocation satisfies the ex post budget constraint given some prices for every households, hence it is a Q-ETE.

To sum up, we have shown:

Proposition 8 *Under the market structure where the short term bond coexists with the long term bond, every efficient allocation constitutes a Q-ETE allocation. Moreover, ex post prices* p^0, p^1, p^2 *and* q^0, q^1 *are indeterminate.*

Recall that in the benchmark model the set of Q-ETE is expressed as the convex hull of the endowment vectors, and so in particular, an efficient allocation which assigns to household h an amount $\bar{x}_h$ smaller than the minimum of $e_h^0, e_h^1, ..., e_h^T$ is not a Q-ETE. On the other hand, Proposition 8 shows that any efficient allocation can arise.

One might wonder why there is no constraint on prices p^0, p^1, p^2 and q^0, q^1. Indeed, in the bench mark model with the short term bond, to sustain a particular Q-ETE, these prices must be configured in a certain way depending on the allocation. In the construction above, required income transfers effectively occur through trade of the short term bond and the long term bond in period 0. The key is $q^0 q^1 \neq q_L^0$, i.e., the law of one price is broken ex post. If the same object has two prices, any kind of transfers can be established.

This point can also be seen from comparison of (10) and (11). Fix $q^0 q^1 > q_L^0$, so ex post, the long term bond was less expensive the short term bond in period 0. From (10) and (11), the consumption level is larger if household h sells an extra amount of the expensive short term bond to buy the inexpensive long term bond. Note that the second implication of no arbitrage, condition 2 about homogeneous forecasts, is irrelevant so far. The reason is that the forecasts matter only for justifiability of the intended consumption, while Q-ETE by definition does not address justifiability. Justifiability is a more stringent requirement than in the benchmark case, since forecasts must be homogeneous as condition 2 requires. In particular, households must hold some arbitrage free forecasts consistent with the ex post arbitrage opportunity of $q^0 q^1 \neq q_L^0$.

5.3 Justifiability and Structure of ETE

Recall that we allow for heterogeneous forecasts of commodity prices: in principle, households having different forecasts might expect different real payouts, and hence they can disagree on the real payouts of a plan of dynamic transactions of the bonds. However, as we have seen above, no arbitrage for individual forecasts of the long term bond forces the forecasts of the bonds to agree. The presence of additional assets do therefore induce further alignment of forecasts to an extent. Therefore the justifiability issue is more nuanced in this set up than in the benchmark case. To simplify matters, we focus here on the case where every household has the log utility, i.e., $u_h(z) = \ln(z)$. For non log utilities, we believe that a generic local argument around PFE, similar to the one in the benchmark case, can be done with some effort.

So now we fix a Q-ETE allocation and ask whether it can be justified. In view of Proposition 8, we might as well set ex post prices p^0, p^1, p^2 and q^0, q^1 equal to one. Write q_L^0 for the price of the long term bond in period 0. Let $\bar{x}_h$ be the amount household h consumes in every period in the Q-ETE. Recall that $q_L^1 = 1$ must hold by no arbitrage, and the short term bond and the long term bond is equivalent in period 1. So we might as well make the household trade the short term bond only, i.e., $l_h^1 = l_h^0$, in the following discussion. Our goal is to find forecasts which induce household h to consume $\bar{x}_h$ under the prices specified above.

Let b_h^0, b_h^1, and l_h^0 be the amount of bond holdings which sustain the consumption ex post, as found in (11) and (10). With the price normalized to be one as above, they satisfy by construction, for all h,

$$\begin{aligned} \bar{x}_h + b_h^0 + q_L^0 l_h^0 &= e_h^0 \\ \bar{x}_h + b_h^1 &= e_h^1 + b_h^0 \\ \bar{x}_h &= e_h^2 + b_h^1 + l_h^0 \end{aligned} \tag{12}$$

We shall show that in each period, household h will demand $\bar{x}_h$ units of the good, with some price forecasts.

Period 2: This holds automatically from the period 2 budget, since the budget constraints of the earlier periods are satisfied, and by the definition of Q-ETE, the ex post budget constraint is satisfied.

Period 1: The argument is analogous to that for Lemma 4 where only the short term bond is available. With the log utility, $\bar{x}_h$ should be a half of the forecast income given $p^1 = q^1 = 1$ as well as b_h^0 and l_h^0, so

$$\bar{x}_h = \frac{e_h^1 + b_h^0 + \hat{p}^2 e_h^2 + l_h^0}{2}.$$

Substituting (11) and (10), we can explicitly solve the equation above to the forecast:

$$\begin{aligned}\hat{p}^2 &= \frac{2\bar{x}_h - (e_h^1 + b_h^0 + l_h^0))}{e_h^2} \\ &= \frac{3\bar{x}_h - (e_h^1 + e_h^0 + l_h^0(1 - q_L^0))}{e_h^2} \\ &= 1.\end{aligned}$$

Notice that the required forecast $\hat{p}^2 = 1$ $(= p^2)$ is independent of h, i.e., they have a common and correct forecast.

Period 0: we note that $\bar{x}_h$ should be a third of the forecast income, and hence

$$\bar{x}_h = \frac{e_h^0 + \hat{p}^1 e_h^1 + \hat{q}^1 \hat{p}^2 e_h^2}{3}.$$

Although $\hat{p}^2$ can be set arbitrarily and independently of the forecast to be held in period 1, we might as well set it as $\hat{p}^2 = 1$; that is, every household correctly forecasts $p^2 = 1$. Set $\hat{q}^1 = q^0/q_L^0 = /q_L^0$ so that the second no arbitrage requirement is met. Then the condition above yields

$$\hat{p}^1 = \frac{3\bar{x}_h - e_h^0 - q_L^0 e_h^2}{e_h^1} \tag{13}$$

for household h. Note that the forecasts are heterogeneous in general. The positivity restriction on $\hat{p}^1$ requires

$$\bar{x}_h > \frac{e_h^0 + e_h^2 q_L^0}{3}, \tag{14}$$

which is certainly satisfied if q_L^0 is close enough to 1 (which would be the case at the unique PFE) and $\bar{x}_h$ is not too far from the PFE consumption level $\frac{1}{3}\left(e_h^0 + e_h^1 + e_h^2\right)$. In the extreme case of $q_L^0 = 0$, (14) yields $\bar{x}_h > \frac{1}{3}e_h^0$.

To sum up our argument above:

Proposition 9 *The set of ETE allocations is a $H - 1$ dimensional set containing the PFE allocation when every household's utility function is* ln. *The set contains all efficient allocations such that $\bar{x}_h > \frac{1}{3}e_h^0$ is satisfied for every h.*

The inequality constraint above is effectively the same as the ones show up in the characterization of Q-ETE in the benchmark model. But they arise for different reasons: in the benchmark model, it is an implication of ex post budget constraint with positive prices, whereas in the result above it is an implication of utility maximization under positive price forecasts.

Example 10 *In the economy of Example 1 with the log utility functions, the set of ETE contains all positive $\bar{x}_1, \bar{x}_2, \bar{x}_3$ and $\bar{x}_4$ with $\sum_h \bar{x}_h = 1$ such that $3\bar{x}_1 > \frac{1}{4} + 2\eta$, $3\bar{x}_2 > \frac{1}{4} - \eta$, $3\bar{x}_3 > \frac{1}{4} - \eta$, and $3\bar{x}_4 > \frac{1}{4}$. In particular, household 4's consumption may be different from $\frac{1}{4}$ whereas it must consume $\bar{x}_3 = \frac{1}{4}$ in any of ETE in Example 5.*

Finally, we comment on the accuracy of forecasts and welfare. Since the construction of a Q-ETE and its justifiability issue can be established separately, it can be readily inferred that those who are benefitted from the implicit transfers do not necessarily have forecasts which are accurate ex post; the former is determined by the construction of an Q-ETE, whereas the latter is related to the issue of justifiability. This observation is indeed valid, even in the common log utility model where the quality of a forecast might appear to be the only source for an advantageous trade. Here we explicitly compute forecasts which justify the ETE found in Example 10:

h	$\hat{p}^1$
1	$\frac{1}{1-4\eta}\left(-q_L + 12\bar{x}_1 - 8\eta + 4q_L\eta - 1\right)$
2	$\frac{1}{1+8\eta}\left(-q_L + 12\bar{x}_2 + 4\eta + 4q_L\eta - 1\right)$
3	$\frac{1}{1-4\eta}\left(-q_L + 12\bar{x}_3 + 4\eta - 8q_L\eta - 1\right)$
4	$2 - q_L$

(15)

Recall that we set the ex post price $p^1 = 1$ so a perfectly accurate forecast is $\hat{p}^1 = 1$. Household 4, who must consume $\frac{1}{4}$ in every period, has a correct forecast 1 only when $q_L = 1$, i.e., there is no ex post arbitrage opportunity. Observe that forecast $\hat{p}^1$ is increasing in $\bar{x}_i$ for $i = 1, 2, 3$, and thus a household who is benefited from the implicit income transfer tends to forecast a higher price, not an accurate forecast 1.

6 On Speculative Trade

The so called no-trade theorem asks if a purely speculative trade based on private information is possible in a rational expectation equilibrium. Since it is hard to distinguish speculative motive from other genuine motives based on gains from trade, work in this literature typically start with an ex ante efficient allocation and ask if there is an equilibrium where trade takes place. If there is one, it can be regarded as a result of pure speculation. A general conclusion in this literature is that there tends not to be any purely speculative trade, which is referred to as the no (speculative) trade theorem.[7]

We can carry out the following exercise with a similar motivation in spirit in our framework. Suppose that there are many, identical households. The initial allocation is efficient, and a unique perfect foresight equilibrium occurs with no trade. The question is whether not if there is a non-trivial ETE where households trade in this economy. If there is, one might interpret that the trade is driven by heterogeneous (and incorrect) forecasts, i.e., lack of rational expectation.

Of course, since there are many inefficient TE, i.e., households might choose trades that distort intertemporal efficiency. One might think that there might also be trades based on heterogeneity of forecasts that preserve efficiency: households whose price forecasts disagree seem to find (incorrectly) that they have mutually beneficial trading opportunities. Even if the initial endowments are efficient, a household which thinks the price will be very low is willing to sell the good today to another household which thinks the price will be very high. This process might induce effective income transfers among households from the ones with good forecasts to the ones with bad forecasts, without distorting efficiency.

It turns out that there is no ETE other than PFE for the case where only the short term bond is traded in every period. To see this recall the characterization result Proposition 2: The set of Q-ETE allocation is $\left\{\sum_{t=0}^{T} p^t e^t : \sum_{t=0}^{T} p^t = 1, p^t > 0, \text{ for } t = 0, 1, ..., T\right\}$, where $e^t = \left(\cdots, e^t, \cdots\right)$. Since e^t is equal to a constant vector $\bar{e}$ for $t = 0, 1, ..., T$, and since the total resource is time invariant, it can be readily seen that $\sum_{t=0}^{T} p^t e^t = \left(\sum_{t=0}^{T} p^t\right)\bar{e} = \bar{e}$ for any element of this set; that is, it is a singleton set consisting of the initial, no trade allocation, and it is exactly the set of ETE allocations consisting of the PFE allocation. In conclusion, lack of rational expectation does not necessarily invoke trade that leads to efficient allocations.

The conclusion, however, is sensitive to the structure of the market: to be more specific, let $T = 2$ and consider the model where the short term bond and the long term bond coexist. There are H identical households, and each gets $\frac{1}{H}$ in every period.

[7] The literature was initiated by Milgrom and Stokey [1982], and a clean, efficiency based formulation was given in Morris [1995].

Assume that the common utility function is $u_h(x) = \ln(x)$. The analysis of the previous section has shown that any efficient allocation where each household's consumption is greater than $\frac{1}{3H}$ can arise as an ETE, as reported in Proposition 9.

The analysis of this section suggests first that heterogeneity of forecasts alone might not constitute motives for market trade without distortion of efficiency. Secondly, it suggests that heterogeneity of forecasts combined with ex post arbitrage opportunity might generate market trades which preserve efficiency. However, households are engaged in trade not because they see arbitrage opportunities for easy profits. Recall that by the construction of ETE each household holds forecasts which do not permit itself any arbitrage opportunity. Each household thinks, at any time, there is no such thing as free lunch in the markets, but nevertheless their trade creates income transfers from the winners to the losers.

Example 11 *In the economy of Example 1, set $\eta = 0$, and assume the log utility function for all households. The set of ETE coincides with PFE when there is only short term bond. But with both the short term bond and the long term bond, any efficient allocation which gives more than $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ to every household can arise as an ETE. When $q_L^0 < 1$, i.e., where the long term bond is inexpensive according to the ex post prices, those who enjoy consumption higher than $\frac{1}{4}$ every period, i.e., the winners, sell the long term bond in period 0 as in (10), whereas those who buy the long term bond consume less than $\frac{1}{4}$ to be the losers.*

As in the previous section, we can compute forecasts explicitly: it is $\hat{p}^1 = -q_L + 12\bar{x}_i - 1$ for household i. Since $\hat{p}^1$ is increasing in $\bar{x}_i$, for these two households,

1. if $q_L < 1$ (thus $2 - q_L > 1$), a household with $\bar{x}_i > \frac{1}{4}$ must have $\hat{p}^1 > 2 - q_L > 1$, and a household with $\bar{x}_i < \frac{1}{4}$ must have $\hat{p}^1 < 2 - q$. Thus a household who is worse off than in the PFE may be ex post correct, but a household who is better off than in the PFE never has a correct forecast.

2. if $q_L > 1$ (thus $2 - q_L < 1$), a household with $\bar{x}_i > \frac{1}{4}$ must have $\hat{p}^1 > 2 - q_L$, and a household with $\bar{x}_i < \frac{1}{4}$ must have $\hat{p}^1 < 2 - q_L < 1$. Thus a household who is better off than in the PFE may also be ex post correct, but a household who is worse off than in the PFE never has a correct forecast.

7 Remarks

We first comment on the generality of our findings. For the benchmark case, one can accommodate more than one good in each period. Since efficiency requires that the marginal rates of substitution are equated within each period, the analysis of Q-ETE can be readily generalized, although the expression of the set of Q-ETE will be more convoluted. The (local) justifiability can also be readily generalized, as the logic there does not really depend on the number of the available goods in temporary markets. On the other hand, allowing more general form of utility functions create some non-trivial problems: it changes the shape of efficient allocations, and also the maximization problem in each period may be delicately related to other periods, which might generate both technical and conceptual problems. We believe however that a general treatment we developed in Chatterji- Kajii [2023] for a general two period economy can be extended to cover this case.

For the model with the short term and the long term bond, the key observation about the distributional role of redundant assets is very general, as it does not rely on anything but budget equations: any direction of income transfer can be accommodated without distorting efficiency if the theoretical price of the long term bond turns out to be different from the observed price ex post. It holds in a model with more goods, a longer period, or with general utility functions. The justifiability problem can be more complex, because the no arbitrage condition does require some alignment of forecasts across the households, and hence we cannot employ the individualistic logic applicable to the benchmark case. In our analysis with the common log utility, we are able to bypass the issue, since for a homogeneous log utility economy, the market clearing price does not depend on the distribution of wealth, in particular, the kind of income transfers generated through bond trading. Hence we could set the forecast equal to this invariant market clearing price for every household, independently of the intended transfer in our construction, making it very simple and transparent. In general, the construction will be more complicated, but we foresee no serious obstacles other than complications.

A natural and important extension is to accommodate uncertainty in the model. It is especially important in the context of our interest in studying the role of redundant assets. In particular, derivative securities, which constitute a rich class of financial assets, can only be studied in models that explicitly incorporate uncertainty.[8] It would be interesting to investigate whether our observation regarding the indeterminacy of wealth transfers under efficiency can be made by studying a two period model with two or more states in the second period, and with a wider variety of financial assets. We expect that under our formulation, the presence of assets which are equivalent under rational expectations would provide some channels of income transfers, and thus would expand the set of attainable (ex ante) efficient allocations beyond the set of rational expectations allocations.

Recall that in the construction the households choose to hold a mixed portfolio of bonds of both kinds, and they are indifferent to various portfolios since according to their forecasts, both bonds are equivalent and the long term bond is priced at its theoretical value. Though such a portfolio is rational from an individual household's perspective, we offer no explanation here of how a household chooses the particular combination when there are indifferent choices.

Finally, we make some observations on the quality of forecasts in our framework.[9] For instance, in the benchmark model, an ex post correct forecast might not be a good forecasts from the view point of individual utility level. In the model where the short term bond and the long term bond coexist, with log utilities, households forecasts about period 2 are ex post correct. In fact, one can write an example with longer periods, where households' forecasts are ex post correct from period 2 and after (so they learn to be perfect forecasters, but welfare shifts takes place before they become perfect). While this is an artifact of the log form of utilities, it is nonetheless interesting that efficiency preserving transfers can arise due to short term forecasting problems which

[8] The classical Black-Scholes option pricing formula finds the theoretical price of an option contract as a derivative asset assuming the relevant price processes are rationally expected. It has also been argued (first by Ross [1976] and subsequently elaborated by Polemarchakis and Ku [1990], Krasa and Werner [1991], Kajii [1997], among others) that under rational expectations, the presence of options might complete the markets, and consequently a rational expectations equilibrium with options is efficient and determinate.

[9] We have observed in earlier work (Chatterji *et al* [2018]) that households whose forecasts turn out more accurate need not be the beneficiaries in the ETE induced by heterogeneous forecasts. The same remains true here.

resolve themselves over the longer term. It suggests that the phenomenon we have identified need not be incompatible with some sort of learning behavior that leads to improvements in forecasts over time.

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