博士論文(要約)

論文題目　Irregular Riemann–Hilbert correspondence and enhanced ind-sheaves
（不確定特異点型 Riemann–Hilbert 対応と拡大帰納層）

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1 Introduction

The aim of this thesis is to describe the essential image of the enhanced solution functor

$$\mathrm{Sol}_X^{\mathrm{E}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$$

of A. D'Agnolo and M. Kashiwara in [DK16, Thm. 9.5.3].

For this purpose, we define $\mathbb{C}$-constructible enhanced ind-sheaves (Definition 2.7) which are analogous to $\mathbb{C}$-constructible sheaves, and prove that they are nothing but its images of $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ via the enhanced solution functor $\mathrm{Sol}_X^{\mathrm{E}}$. Namely, we prove that the triangulated category of holonomic $\mathcal{D}_X$-modules is equivalent to the one of $\mathbb{C}$-constructible enhanced ind-sheaves (Theorem 2.8). Moreover we prove that the triangulated category of algebraic holonomic $\mathcal{D}$-modules is equivalent to the one of algebraic $\mathbb{C}$-constructible enhanced ind-sheaves when X is a smooth complex algebraic variety (Definition 2.14, Theorem 2.15).

1.1 Irregular Riemann–Hilbert Correspondence

First, we shall recall the irregular Riemann–Hilbert correspondence of D'Agnolo–Kashiwara [DK16, Thm. 9.5.3].

Let X be a complex analytic manifold. In 1984, the Riemann–Hilbert correspondence for regular holonomic $\mathcal{D}$-modules on X was proved by M. Kashiwara [Kas84, Main Theorem] (see also [Meb84, Thm. 2.1.1]). He established the equivalence of categories between the triangulated category $\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules and the one $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$ of $\mathbb{C}$-constructible sheaves on X.

Fact 1.1 ([Kas84, Main Theorem])**.** *There exists an equivalence of categories*

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{op} \underset{\mathrm{RH}_X}{\overset{\mathrm{Sol}_X}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X).$$

$$\mathcal{M} \longmapsto \mathrm{Sol}_X(\mathcal{M}) := \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

$$\mathrm{RH}_X(\mathcal{F}) := \mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{O}_X^{\mathrm{t}}) \longleftarrow\!\dashv \mathcal{F}$$

Here, $\mathcal{O}_X^{\mathrm{t}}$ is the ind-sheaf of tempered holomorphic functions (see [KS01, p.125] *for the definition). See also* [KS16b, § 4.3] *for the details of* RH_X*.*

The triangulated category $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$ has a t-structure $\left({}^{p}\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X), {}^{p}\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)\right)$ which is called the perverse t-structure. Let us denote by

$$\mathrm{Perv}(\mathbb{C}_X) := {}^{p}\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X) \cap {}^{p}\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$$

its heart and call an object of $\mathrm{Perv}(\mathbb{C}_X)$ a perverse sheaf. The above equivalence induces an equivalence of categories between the abelian category $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules and the one $\mathrm{Perv}(\mathbb{C}_X)$ of perverse sheaves.

The problem of extending the Riemann–Hilbert correspondence to cover the case of holonomic $\mathcal{D}$-modules with irregular singularities had been open for 30 years. After a groundbreaking development in the theory of irregular meromorphic connections by K. S. Kedlaya [Ked10, Ked11] and T. Mochizuki [Moc09, Moc11], A. D'Agnolo and M. Kashiwara established the Riemann–Hilbert correspondence for analytic irregular holonomic $\mathcal{D}$-modules in [DK16] as follows.

For this purpose, they introduced enhanced ind-sheaves extending the classical notion of ind-sheaves introduced by M. Kashiwara and P. Schapira in [KS01]. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ the triangulated category of holonomic $\mathcal{D}_X$-modules and by $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ the one of $\mathbb{R}$-constructible enhanced ind-sheaves on X. See [DK16, Def. 4.9.2] and also [DK19, Def. 3.3.1] for the details of $\mathbb{R}$-constructible enhanced ind-sheaves. We set

$$\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}) := \mathbf{R}\mathcal{I}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathrm{E}}_X).$$

Here $\mathcal{O}^{\mathrm{E}}_X$ is the enhanced ind-sheaf of tempered holomorphic functions, see [DK16, Def. 8.2.1] for the details.

Fact 1.2 ([DK16, Thm. 9.5.3])**.** *The enhanced solution functor* $\mathrm{Sol}^{\mathrm{E}}_X$ *induces a fully faithful embedding*

$$\mathrm{Sol}^{\mathrm{E}}_X \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{op} \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X).$$

Moreover, for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ *there exists an isomorphism in* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$

$$\mathcal{M} \xrightarrow{\sim} \mathrm{RH}^{\mathrm{E}}_X\big(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})\big),$$

where $\mathrm{RH}^{\mathrm{E}}_X(K) := \mathbf{R}\mathcal{H}om^{\mathrm{E}}(K, \mathcal{O}^{\mathrm{E}}_X)$*.*

Moreover, they gave a generalized t-structure $\left({}^{\frac{1}{2}}\mathbf{E}^{\leq c}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X), {}^{\frac{1}{2}}\mathbf{E}^{\geq c}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)\right)_{c\in\mathbb{R}}$ on $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ and proved that the enhanced solution functor $\mathrm{Sol}^{\mathrm{E}}_X$ induces a fully faithful embedding of the abelian category $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ of holonomic $\mathcal{D}_X$-modules into ${}^{\frac{1}{2}}\mathbf{E}^{\leq 0}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X) \cap {}^{\frac{1}{2}}\mathbf{E}^{\geq 0}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ in [DK19, Thm. 4.5.1].

Furthermore, T. Mochizuki proved that the image of $\mathrm{Sol}^{\mathrm{E}}_X$ can be characterized by the curve test [Moc16, Thm. 12.1]. On the other hand, in [Kas16b, Thm. 6.2], M. Kashiwara showed the similar result of Fact 1.2 by using enhanced subanalytic sheaves instead of enhanced ind-sheaves. In [Kuwa18, Thm. 8.5], T. Kuwagaki introduced another approach to the irregular Riemann–Hilbert correspondence via irregular constructible sheaves which are defined by $\mathbb{C}$-constructible sheaf with coefficients in a finite version of the Novikov ring and special gradings.

In this thesis, we define $\mathbb{C}$-constructible enhanced ind-sheaves which are analogous to $\mathbb{C}$-constructible sheaves and prove that the triangulated category of them is equivalent to the one of holonomic $\mathcal{D}$-modules via the enhanced solution functor $\mathrm{Sol}^{\mathrm{E}}$ of D'Agonolo and Kashiwara.

2 Main Results

2.1 $\mathbb{C}$-Constructible Enhanced Ind-Sheaves

The contents of this section are taken from a published paper [Ito20a].

First, we shall consider an analytic case. The main theorems of this section are Theorems 2.8 and 2.10.

Let X be a complex analytic manifold and D a normal crossing divisor of X. In this thesis, we regard the empty set as a normal crossing divisor for convenience. We denote by $\mathbf{E}^0(\mathrm{I}\mathbb{C}_X)$ the heart of $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ with respect to the t-structure which is induced by the one of the bounded derived category of ind-sheaves.

Definition 2.1. We say that $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_X)$ has a normal form along D if the following three conditions are satisfied:

(i) $\pi^{-1}\mathbb{C}_{X\setminus D} \otimes K \xrightarrow{\sim} K$,

(ii) for any $x \in X \setminus D$ there exist an open neighborhood $U_x \subset X \setminus D$ of x and a non-negative integer k such that $K|_{U_x} \simeq (\mathbb{C}^{\mathrm{E}}_{U_x})^{\oplus k}$,

(iii) for any $x \in D$ there exist an open neighborhood $U_x \subset X$ of x, a good set of irregular values $\{\varphi_i\}_i$ on $(U_x, D \cap U_x)$ and a finite sectorial open covering $\{U_{x,j}\}_j$ of $U_x \setminus D$ such that for any j

$$\pi^{-1}\mathbb{C}_{U_{x,j}} \otimes K|_{U_x} \simeq \pi^{-1}\mathbb{C}_{U_{x,j}} \otimes \Big(\bigoplus_i \mathbb{E}^{\mathrm{Re}\,\varphi_i}_{U_x\setminus D|U_x}\Big).$$

Here $\mathbb{E}^{\mathrm{Re}\,\varphi_i}_{U_x\setminus D|U_x}$ is an enhanced ind-sheaf which is induced by $\text{“}\varinjlim_{a\to\infty}\text{”}\ \mathbb{C}_{\{t+\mathrm{Re}\varphi_i \geq a\}}$. See [Moc11, Def. 2.1.2] for the definition of a good set of irregular values.

Note that any enhanced ind-sheaf which has a normal form along D is an $\mathbb{R}$-constructible enhanced ind-sheaf. Moreover we have an equivalence of categories as follows:

Proposition 2.2. *The enhanced solution functor* $\mathrm{Sol}_X^{\mathrm{E}}$ *induces an equivalence of categories between the full subcategory of* $\mathbf{E}^0(\mathrm{I}\mathbb{C}_X)$ *consisting of objects which have a normal form along* D *and the one of* $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ *consisting of objects which have a normal form along* D.

This proposition follows from the sectorial refinement irregular Riemann–Hilbert correspondence ([IT20a, Thms. 3.8 and 3.12]) and the curve test of Mochizuki [Moc16, Thm. 12.1].

Definition 2.3. We say that $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_X)$ has a quasi-normal form along D if it satisfies the conditions (i), (ii) in the Definition 2.1 and if for any $x \in D$ there exist an open neighborhood $U_x \subset X$ of x and a ramification $r_x\colon U_x^{\mathrm{rm}} \to U_x$ of U_x along $D_x := U_x \cap D$ such that $\mathbf{E}r_x^{-1}(K|_{U_x})$ has a normal form along $D_x^{\mathrm{rm}} := r_x^{-1}(D_x)$.

Note that any enhanced ind-sheaf which has a quasi-normal form along D is an $\mathbb{R}$-constructible enhanced ind-sheaf. Moreover we have an equivalence of categories as follows:

Proposition 2.4. *The enhanced solution functor* $\mathrm{Sol}_X^{\mathrm{E}}$ *induces an equivalence of categories between the full subcategory of* $\mathbf{E}^0(\mathrm{I}\mathbb{C}_X)$ *consisting of objects which have a quasi-normal form along* D *and the one of* $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ *consisting of objects which have a quasi-normal form along* D.

This proposition follows from Proposition 2.2.

Let H be an analytic hypersurface of X. In this thesis, we regard the empty set as an analytic hypersurface for convenience.

Definition 2.5. We say that $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_X)$ has a modified quasi-normal form along H if it satisfies the conditions (i), (ii) in the Definition 2.1 and if for any $x \in H$ there exist an open neighborhood $U_x \subset X$ of x and a modification $m_x\colon U_x^{\mathrm{md}} \to U_x$ of U_x along $H_x := U_x \cap H$ such that $\mathbf{E}m_x^{-1}(K|_{U_x})$ has a quasi-normal form along $D_x^{\mathrm{md}} := m_x^{-1}(H_x)$.

Note that any enhanced ind-sheaf which has a modified quasi-normal form along H is an $\mathbb{R}$-constructible enhanced ind-sheaf. Moreover the following result follows from Proposition 2.4 and results of K. S. Kedlaya and T. Mochizuki ([Ked10, Ked11, Moc09, Moc11]). We shall denote by $\mathrm{Conn}(X; H)$ the abelian category of meromorphic connections on X along H.

Proposition 2.6. *The enhanced solution functor* $\mathrm{Sol}_X^{\mathrm{E}}$ *induces an equivalence of categories between the full subcategory of* $\mathbf{E}^0(\mathrm{I}\mathbb{C}_X)$ *consisting of objects which have a modified quasi-normal form along* H *and the abelian category* $\mathrm{Conn}(X; H)$.

Let us define $\mathbb{C}$-constructible enhanced ind-sheaves.

Definition 2.7. Let X be a complex manifold. We say that $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_X)$ is (analytic) $\mathbb{C}$-constructible if there exists a complex stratification $\{X_\alpha\}_{\alpha\in A}$ of X such that $\pi^{-1}\mathbb{C}_{\overline{X}_\alpha^{\mathrm{bl}}\setminus D_\alpha} \otimes \mathbf{E}b_\alpha^{-1}K$ has a modified quasi-normal form along D_α for any $\alpha \in A$.

Here $b_\alpha \colon \overline{X}_\alpha^{\mathrm{bl}} \to X$ is a complex blow-up of $\overline{X_\alpha}$. along $\overline{X_\alpha}\setminus X_\alpha$, $D_\alpha := b_\alpha^{-1}\big(\overline{X_\alpha}\setminus X_\alpha\big)$ and $\pi \colon \overline{X}_\alpha^{\mathrm{bl}} \times \mathbb{R}_\infty \to \overline{X}_\alpha^{\mathrm{bl}}$ is the morphism of bordered spaces given by the projection $\overline{X}_\alpha^{\mathrm{bl}} \times \mathbb{R} \to \overline{X}_\alpha^{\mathrm{bl}}$.

Note that any $\mathbb{C}$-constructible enhanced ind-sheaf is $\mathbb{R}$-constructible. Let us denote by $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ the full subcategory of $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ consisting of objects whose cohomologies are $\mathbb{C}$-constructible enhanced ind-sheaves. The following theorem is the main theorem of this section.

Theorem 2.8. *For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, the enhanced solution complex $\mathrm{Sol}_X^{\mathrm{E}}(\mathcal{M})$ of $\mathcal{M}$ is an object of $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$. On the other hand, for any $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$, there exists an object $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ such that*

$$K \xrightarrow{\sim} \mathrm{Sol}_X^{\mathrm{E}}(\mathcal{M}).$$

Moreover, we obtain an equivalence of categories

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{op} \underset{\mathrm{RH}_X^{\mathrm{E}}}{\overset{\mathrm{Sol}_X^{\mathrm{E}}}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$$

where $\mathrm{RH}_X^{\mathrm{E}}(K) := \mathbf{R}\mathcal{H}om^{\mathrm{E}}(K, \mathcal{O}_X^{\mathrm{E}})$.

The second part of Theorem 2.8 follows from Proposition 2.6 and the first part follows from Lemma 2.9 below. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{mero}}(\mathcal{D}_{X(H)})$ the full subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ consisting of objects whose cohomologies are meromorphic connections on X along an analytic hypersurface H. Although Lemma 2.9 is well known by experts, we prove it for convenience.

Lemma 2.9. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then there exists a stratification $\{X_\alpha\}_{\alpha\in A}$ of X such that for any $\alpha \in A$ and any complex blow-up $b_\alpha \colon \overline{X}_\alpha^{\mathrm{bl}} \to X$ of $\overline{X_\alpha}$ along $\overline{X_\alpha}\setminus X_\alpha$, we have $(\mathbf{D}b_\alpha^*\mathcal{M})(*D_\alpha) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{mero}}(\mathcal{D}_{\overline{X}_\alpha^{\mathrm{bl}}(D_\alpha)})$, where $D_\alpha := b_\alpha^{-1}(\overline{X_\alpha}\setminus X_\alpha)$ is a normal crossing divisor of $\overline{X}_\alpha^{\mathrm{bl}}$.*

Moreover, we characterize a t-structure on the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ whose heart is equivalent to the abelian category $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ of holonomic $\mathcal{D}_X$-modules by using the sheafification functor

$$\mathrm{sh}_X := \alpha_X i_0^! \mathbf{R}^{\mathrm{E}} \colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$$

of A. D'Agnolo and M. Kashiwara as follows. We set

$$\begin{aligned} {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) &:= \{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) \mid \mathrm{sh}_X(K) \in {}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)\}, \\ {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) &:= \{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) \mid \mathrm{D}^{\mathrm{E}}_X(K) \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)\} \\ &= \{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) \mid \mathrm{sh}_X(K) \in {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)\}, \end{aligned}$$

where the pair $\big({}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X), {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)\big)$ is the perverse t-structure on $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$ and $\mathrm{D}^{\mathrm{E}}_X \colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ is the duality functor for enhanced ind-sheaves. See [DK16, Def. 4.8.1] and also [DK19, § 2.7] for the duality functor D^{E}.

Theorem 2.10. *The pair* $\big({}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X), {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)\big)$ *is a t-structure on* $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ *and its heart*

$$\mathrm{Perv}(\mathrm{I}\mathbb{C}_X) := {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) \cap {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$$

is equivalent to the abelian category $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ *of holonomic* $\mathcal{D}_X$*-modules.*

Moreover, the pair $\big({}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X), {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)\big)$ is related to the generalized t-structure $\big({}^{\frac{1}{2}}\mathbf{E}^{\leq c}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X), {}^{\frac{1}{2}}\mathbf{E}^{\geq c}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)\big)_{c\in\mathbb{R}}$ on $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ as follows. See [Kas16a, Def. 1.2] for the definition of generalized t-structures.

Corollary 2.11. *We have*

$$\begin{aligned} {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) &= {}^{\frac{1}{2}}\mathbf{E}^{\leq 0}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X), \\ {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) &= {}^{\frac{1}{2}}\mathbf{E}^{\geq 0}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X). \end{aligned}$$

2.2 Algebraic $\mathbb{C}$-Constructible Enhanced Ind-Sheaves

The contents of this section are taken from [Ito20b].

In Section 2.1, we define $\mathbb{C}$-constructible enhanced ind-sheaves on a complex manifold and proved that the triangulated category of them is equivalent to the one of analytic holonomic $\mathcal{D}$-modules via the enhanced solution functor of A. D'Agnolo and M. Kashiwara. In this section, we shall consider such an equivalence of categories on smooth complex algebraic varieties. The main theorems of this section are Theorems 2.15 and 2.16.

Recall that after the appearance of the regular Riemann–Hilbert correspondence of Kashiwara [Kas84, Main Theorem] (Fact 1.1), J. Bernstein proved an algebraic version of the Riemann–Hilbert correspondence for regular holonomic $\mathcal{D}$-modules stated as follows. Remark that M. Kashiwara also proved an algebraic version of the Riemann–Hilbert correspondence by using the analytic one.

Let X be a smooth complex algebraic variety. We denote by $X^{\rm an}$ the underling complex analytic manifold of X, by $\mathbf{D}^{\rm b}_{\rm rh}(\mathcal{D}_X)$ the triangulated category of regular holonomic $\mathcal{D}_X$-modules on X and by $\mathbf{D}^{\rm b}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$ the one of algebraic $\mathbb{C}$-constructible sheaves on $X^{\rm an}$. Then there exists an equivalence of triangulated categories

$$\mathrm{Sol}_X\colon \mathbf{D}^{\rm b}_{\rm rh}(\mathcal{D}_X)^{\rm op} \xrightarrow{\sim} \mathbf{D}^{\rm b}_{\mathbb{C}\text{-}c}(\mathbb{C}_X), \qquad \mathrm{Sol}_X(\mathcal{M}) := \mathrm{Sol}_{X^{\rm an}}(\mathcal{M}^{\rm an}).$$

See e.g., [Be, Main Theorem C (c)] or [HTT08, Thms. 4.7.7, 7.2.2] for the details. Thus, we try to consider an algebraic version of irregular Riemann–Hilbert correspondence in this thesis.

Let X be a smooth complex algebraic variety and denote by $\mathbf{D}^{\rm b}_{\rm hol}(\mathcal{D}_X)$ the triangulated category of (algebraic) holonomic $\mathcal{D}_X$-modules on X.

Definition 2.12. We say that an enhanced ind-sheaf $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_{X^{\rm an}})$ satisfies the condition ($\mathbf{AC}$) if there exists an algebraic stratification $\{X_\alpha\}_{\alpha\in A}$ of X such that $\pi^{-1}\mathbb{C}_{(\overline{X}^{\rm bl}_\alpha)^{\rm an}\setminus D^{\rm an}_\alpha} \otimes \mathbf{E}(b^{\rm an}_\alpha)^{-1}K$ has a modified quasi-normal form along $D^{\rm an}_\alpha$ for any $\alpha \in A$ (see Definition 2.5).

Here $b_\alpha\colon \overline{X}^{\rm bl}_\alpha \to X$ is a blow-up of $\overline{X_\alpha}$ along $\partial X_\alpha\colon = \overline{X_\alpha} \setminus X_\alpha$, $D^{\rm an}_\alpha := \big(\overline{X}^{\rm bl}_\alpha\big)^{\rm an} \setminus \big(\overline{X}^{\rm bl}_\alpha \setminus D_\alpha\big)^{\rm an}$ and $\pi\colon (\overline{X}^{\rm bl}_\alpha)^{\rm an} \times \mathbb{R}_\infty \to (\overline{X}^{\rm bl}_\alpha)^{\rm an}$ is the morphism of bordered spaces given by the projection $(\overline{X}^{\rm bl}_\alpha)^{\rm an} \times \mathbb{R} \to (\overline{X}^{\rm bl}_\alpha)^{\rm an}$.

Let us denote by $\mathbf{E}^{\rm b}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ the full triangulated subcategory of $\mathbf{E}^{\rm b}(\mathrm{I}\mathbb{C}_{X^{\rm an}})$ consisting of objects whose cohomologies satisfy the condition ($\mathbf{AC}$). Note that we have an essential surjective functor

$$\mathrm{Sol}^{\rm E}_X\colon \mathbf{D}^{\rm b}_{\rm hol}(\mathcal{D}_X)^{\rm op} \to \mathbf{E}^{\rm b}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X),\ \ \mathcal{M} \mapsto \mathrm{Sol}^{\rm E}_{X^{\rm an}}(\mathcal{M}^{\rm an}).$$

This is not fully faithful in general. However if X is complete, it is fully faithful.

Theorem 2.13. *Let X be a smooth complete algebraic variety over $\mathbb{C}$. There exists an equivalence of categories*

$$\mathrm{Sol}_X^{\mathrm{E}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{op} \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X).$$

Now, let us come back to the general case. Thanks to Hironaka's desingularization theorem [Hiro] (see also [Naga, Thm. 4.3]), for any smooth algebraic variety X over $\mathbb{C}$ we can take a smooth complete algebraic variety $\widetilde{X}$ such that $X \subset \widetilde{X}$ and $D := \widetilde{X} \setminus X$ is a normal crossing divisor of $\widetilde{X}$. Let us consider a bordered space $X^{\mathrm{an}}_\infty = (X^{\mathrm{an}}, \widetilde{X}^{\mathrm{an}})$. We denote by $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ the triangulated category of enhanced ind-sheaves on X^{an}_∞. Note that $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ does not depend on the choice of $\widetilde{X}$ and there exists an equivalence of categories

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty}) \underset{\mathbf{E}j^{-1}}{\overset{\mathbf{E}j_{!!}}{\underset{\sim}{\rightleftarrows}}} \{K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}}) \mid \pi^{-1}\mathbb{C}_{X^{\mathrm{an}}} \otimes K \xrightarrow{\sim} K\},$$

where $j\colon X^{\mathrm{an}}_\infty \to \widetilde{X}^{\mathrm{an}}$ is a morphism of bordered spaces given by the open embedding $X \hookrightarrow \widetilde{X}$. We shall denote the open embedding $X \hookrightarrow \widetilde{X}$ by the same symbol j and set

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) := \mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M}) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$$

for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$.

Definition 2.14. We say that an enhanced ind-sheaf $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ is algebraic $\mathbb{C}$-constructible if $\mathbf{E}j_{!!}K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}})$ is an object of $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$ (see Definition 2.12 for $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$).

Let us denote by $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty})$ the full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ consisting of algebraic $\mathbb{C}$-constructible enhanced ind-sheaves. The following result is the main theorem of this section.

Theorem 2.15. *Let X be a smooth complex algebraic variety. For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, the enhanced solution complex $\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M})$ of $\mathcal{M}$ is an algebraic $\mathbb{C}$-constructible enhanced ind-sheaf. On the other hand, for any algebraic $\mathbb{C}$-constructible enhanced ind-sheaf $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty})$, there exists an object $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ such that*

$$K \simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}).$$

Moreover, we obtain an equivalence of categories

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{op} \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}).$$

Furthermore, we characterize a t-structure on the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty})$ whose heart is equivalent to the abelian category $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ of holonomic $\mathcal{D}$-modules on X by using the sheafification functor

$$\mathrm{sh}_{X_\infty^{\mathrm{an}}} := \alpha_{X_\infty^{\mathrm{an}}} i_0^! \mathbf{R}^{\mathrm{E}} \colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$$

of A. D'Agnolo and M. Kashiwara as follows. Let us denote by $\mathrm{D}^{\mathrm{E}}_{X_\infty^{\mathrm{an}}} \colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}})^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}})$ the duality functor for enhanced ind-sheaves on X_∞^{an}. See [DK19, § 2.7] for the details. We set

$$\begin{aligned}
{}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}) &:= \{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}) \mid \mathrm{sh}_{X_\infty^{\mathrm{an}}}(K) \in {}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)\},\\
{}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}) &:= \{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}) \mid \mathrm{D}^{\mathrm{E}}_{X_\infty^{\mathrm{an}}}(K) \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty})\}\\
&= \{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}) \mid \mathrm{sh}_{X_\infty^{\mathrm{an}}}(K) \in {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)\},
\end{aligned}$$

where the pair $\left({}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X), {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)\right)$ is the perverse t-structure on the triangulated category $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$ of algebraic $\mathbb{C}$-constructible sheaves on X^{an}. Then we obtain the second main theorem of this section.

Theorem 2.16. *Let X be a smooth complex algebraic variety. The pair*

$$\left({}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}), {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty})\right)$$

is a t-structure on $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty})$ and its heart

$$\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}) := {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}) \cap {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty})$$

is equivalent to the abelian category $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ *of holonomic $\mathcal{D}_X$-modules.*

Furthermore the pair $\left({}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}), {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty})\right)$ is related to the generalized t-structure $\left({}^{\frac{1}{2}}\mathbf{E}^{\leq c}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}), {}^{\frac{1}{2}}\mathbf{E}^{\geq c}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}})\right)_{c\in\mathbb{R}}$ on $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}})$ as follows:

$$\begin{aligned}
{}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}) &= {}^{\frac{1}{2}}\mathbf{E}^{\leq 0}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}),\\
{}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}) &= {}^{\frac{1}{2}}\mathbf{E}^{\geq 0}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}).
\end{aligned}$$

Remark 2.17. T. Kuwagaki [Kuwa18, Thm. 10.8] proved an algebraic version of the irregular Riemann–Hilbert correspondence in the case of quasi-projective variety.

Thanks to the theory of minimal extensions of algebraic holonomic $\mathcal{D}$-modules, by using Theorem 2.16 we can describe simple objects of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ as follows.

Definition 2.18. Let X be a smooth complex algebraic variety. A non-zero object $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ is called simple if it contains no subobjects in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ other than K or 0.

Note that for any simple algebraic perverse sheaf $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}_X)$ the natural embedding $e_{X_\infty^{\mathrm{an}}}(\mathcal{F}) \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ is also simple.

In this thesis, we say that $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}})$ is an enhanced local system on X_∞ if for any $x \in X$ there exist an open neighborhood $U \subset X$ of x and a non-negative integer k such that $K|_{U_\infty^{\mathrm{an}}} \simeq (\mathbb{C}^{\mathrm{E}}_{U_\infty^{\mathrm{an}}})^{\oplus k}$. Note that for any enhanced local system K on X_∞, we have $K[d_X] \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$.

Proposition 2.19. *For any simple object K of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$, there exist a locally closed smooth connected subvariety Z of X whose natural embedding is affine and a simple enhanced local system L on Z_∞ such that*

$$K \simeq \mathrm{Im}\left(\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}L[d_Z] \to \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}L[d_Z]\right),$$

where a morphism $\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}L[d_Z] \to \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}L[d_Z]$ in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ is induced by a canonical morphism $\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!} \to \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}$ of functors $\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}, \mathbf{E}i_{Z_\infty^{\mathrm{an}}*} \colon \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}) \to \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$.

Moreover we shall consider the image of a canonical morphism

$${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}K \to {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}K$$

for $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$ and a locally closed smooth subvariety Z of X (not necessarily the natural embedding $i_Z \colon Z \hookrightarrow X$ is affine). Here we set

$$\begin{aligned}
{}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*} &:= {}^p\mathcal{H}^0 \circ \mathbf{E}i_{Z_\infty^{\mathrm{an}}*} \colon \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}) \to \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}),\\
{}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!} &:= {}^p\mathcal{H}^0 \circ \mathbf{E}i_{Z_\infty^{\mathrm{an}}!!} \colon \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}) \to \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}),
\end{aligned}$$

and ${}^p\mathcal{H}^0$ is the 0-th cohomology functor with respect to the perverse t-structures.

In this thesis, we shall define a minimal extension of $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$ along Z as follows.

Definition 2.20. In the situation as above, for any $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$, we call the image of the canonical morphism

$${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}K \to {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}K$$

the minimal extension of K along Z, and denote it by ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K$.

Note that if Z is closed then there exists an isomorphism ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K \simeq \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}K \simeq \mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}K$ in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ and there exists an equivalence of categories:

$$\mathrm{Perv}_Z(\mathrm{I}\mathbb{C}_{X_\infty}) \underset{\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}}{\overset{{}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}}^{-1}}{\rightleftarrows}} \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}),$$

where $\mathrm{Perv}_Z(\mathrm{I}\mathbb{C}_{X_\infty})$ is a full subcategory of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ consisting of objects whose support is contained in Z^{an}. This is nothing but a counter part of the Kashiwara's equivalence of categories for holonomic cases. Thus the minimal extension ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K$ of a simple object K of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$ along a closed smooth subvariety Z is also simple.

On the other hand, if U is open whose complement $X \setminus U$ is a smooth subvariety, then the minimal extension along U is characterized as follows:

Proposition 2.21. *In the situation as above, the minimal extension* ${}^p\mathbf{E}i_{UZ_\infty^{\mathrm{an}}!!*}K$ *of* $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$ *along* U *is characterized as the unique object* $L \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ *satisfying the conditions*

(1) $\mathbf{E}i_{U_\infty^{\mathrm{an}}}^{-1}L \simeq K$,
(2) $\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{-1}L \in \mathbf{E}_{\mathbb{C}\text{-}c}^{\leq -1}(\mathrm{I}\mathbb{C}_{W_\infty})$,
(3) $\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{!}L \in \mathbf{E}_{\mathbb{C}\text{-}c}^{\geq 1}(\mathrm{I}\mathbb{C}_{W_\infty})$.

Furthermore in the situation as above, the minimal extension ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K$ of a simple object K of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$ along U is also simple. Moreover, we obtain the following results.

Theorem 2.22. *Let* X *be a smooth complex algebraic variety and* Z *a locally closed smooth subvariety of* X *(not necessarily the natural embedding* $i_Z\colon Z \hookrightarrow X$ *is affine). We assume that* $Z = U \cap W$ *with an open subset* $U \subset X$ *whose complement* $X \setminus U$ *is smooth and a closed subvariety* $W \subset X$.

Then the minimal extension ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K$ *of a simple object* K *of* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$ *along* Z *is also simple.*

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References

[Be] Joseph Bernstein, Algebraic Theory of $\mathcal{D}$-Modules, unpublished notes.

[DK16] Andrea D'Agnolo and Masaki Kashiwara, Riemann–Hilbert correspondence for holonomic $\mathcal{D}$-modules, Publ. Math. Inst. Hautes Études Sci., **123(1)**, 2016, 69–197.

[DK19] Andrea D'Agnolo and Masaki Kashiwara, Enhanced perversities, J. Reine Angew. Math. (Crelle's Journal), **751**, 185–241, 2019.

[Hiro] Heisuke Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, Ann. Math., **79(1)**, I:109–203, 1964, II: 205–326, 1964.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, $\mathcal{D}$-modules, perverse sheaves, and representation theory, Progress in Mathematics, **236**, Birkhäuser, 2008.

[Ito20a] Yohei Ito, $\mathbb{C}$-Conctructible Enhanced Ind-Sheaves, Tsukuba journal of Mathematics, **44(1)**, 155–201, 2020.

[Ito20b] Yohei Ito, Note on Algebraic Irregular Riemann–Hilbert Correspondence, arXiv:2004.13518, 52 pages, preprint.

[IT20a] Yohei Ito and Kiyoshi Takeuchi, On irregularities of Fourier transforms of regular holonomic $\mathcal{D}$-Modules, Adv. in Math., **366**, 62 pages, 2020, doi:10.1016/j.aim.2020.107093 .

[Kas84] Masaki Kashiwara, The Riemann–Hilbert problem for holonomic systems, Publ. Res. Inst. Math. Sci., **20(2)**, 319–365, 1984.

[Kas16a] Masaki Kashiwara, Self-dual t-structure, Publ. Res. Inst. Math. Sci., **52(3)**, 271–295, 2016.

[Kas16b] Masaki Kashiwara, Riemann–Hilbert correspondence for irregular holonomic $\mathcal{D}$-modules, Japan J. Math., **11**, 13–149, 2016.

[KS01] Masaki Kashiwara and Pierre Schapira, Ind-sheaves, Astérisque, **271**, 2001.

[KS16b] Masaki Kashiwara and Pierre Schapira, Regular and irregular holonomic $\mathcal{D}$-modules, London Mathematical Society Lecture Note Series, **433**, Cambridge University Press, 2016.

[Ked10] Kiran S. Kedlaya, Good formal structures for flat meromorphic connections, I: surfaces, Duke Math. J., **154(2)**, 343–418, 2010.

[Ked11] Kiran S. Kedlaya, Good formal structures for flat meromorphic connections, II: excellent schemes, J. Amer. Math. Soc., **24(1)**, 183–229, 2011.

[Kuwa18] Tatsuki Kuwagaki, Irregular perverse sheaves, arXiv:1808.02760, preprint.

[Meb84] Zoghman Mebkhout, Une autre équivalence de catégories, Compositio Math. **51(1)**, 63–88, 1984.

[Moc09] Takuro Mochizuki, Good formal structure for meromorphic flat connections on smooth projective surfaces, In Algebraic analysis and around, Adv. Stud. Pure Math., **54**, 223–253, 2009.

[Moc11] Takuro Mochizuki, Wild harmonic bundles and wild pure twistor $\mathcal{D}$-modules, Astérisque, **340**, 2011.

[Moc16] Takuro Mochizuki, Curve test for enhanced ind-sheaves and holonomic $\mathcal{D}$-modules, arXiv:1610.08572, preprint.

[Naga] Masayoshi Nagata, Imbedding of an abstract variety in a complete variety, J. Math. Kyoto Univ. **2(1)**, 1–10, 1962.