論文の内容の要旨

論文題目　Twisted arrow categories of operads and Segal conditions
(オペラッドの捻れ射圏とシーガル条件)

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Operads are generalizations of categories, and there are many similarities between category and operad theory. Categories are multi-object algebras over the operad uAs of monoids, and coloured operads are multi-object algebras over the operad sOp of symmetric operads. One of the main purposes of operads is also similar to that of categories: operads are used to better understand and to unify different instances of fundamentally the same general facts and methods in homotopy theory. This point of view applies to the main subject of the present work: twisted arrow categories. Twisted arrow categories of operads generalize twisted arrow categories of categories both formally and in their place and function in homotopy theory.

Twisted arrow categories of categories belong to the following sequence of categories:

$$\Delta/C \to Tw(C) \to \mathcal{U}(C) \to C \to I,$$

where Δ/C is the category of elements of the nerve of a category C, the category $Tw(C)$ is the twisted arrow category of the category C, and the category $\mathcal{U}(C)$ is universal enveloping category of the category C. The category $\mathcal{U}(C)$ is equivalent to the category $C^{op} \times C$.

We show that twisted arrow categories of operads belong to similar sequence of categories:

$$\Omega/P \to Tw_{sOp}(P) \to \mathcal{U}_{sOp}(P) \to (\Gamma_P)^{op} \to I,$$

where Ω/P is the category of elements of the operadic nerve of a coloured symmetric operad P, the category $Tw_{sOp}(P)$ is the twisted arrow category of the operad P, the category $\mathcal{U}_{sOp}(P)$ is the universal enveloping category of the operad P, and Γ_P is the PROP corresponding to the operad P. The category $\mathcal{U}_{sOp}$ coincides with the category $(\Gamma^+_P)^{op}$ that was explicitly constructed in [1].

For categories functors from the categories in the first sequence above are coefficient systems for cohomology theories. This remains true for $(\infty, 1)$-categories

([2]). This also generalizes correctly to operads: this is shown in recent work [3], in which twisted arrow categories of ∞-operads are introduced. The latter paper has little intersection with the present thesis, since our point of view is different. We use twisted arrow categories to develop theory of Segal presheaves and multi-object algebras.

This is motivated by the following computations: twisted arrow category of the operad uAs of monoids is equivalent to the simplex category Δ, twisted arrow category of the operad $uCom$ of commutative monoids is equivalent to Segal's category Γ, and twisted arrow category of the operad sOp of single-colour operads is equivalent to Moerdijk–Weiss category Ω. This suggests that Segal presheaves over twisted arrow categories of operads should be seen as multi-object algebras over these operads.

There is further connection between twisted arrow categories and Segal presheaves. Twisted arrow categories can be defined not only for operads, but for algebras over operads. Theorem 2.2.3 and Theorem 2.2.4 show that for any algebra A over an operad P there is a Segal presheaf A and a canonical map $Tw_{sOp}(P)/A \to Tw_P(A)$. In particular, the map $\Omega/P \to Tw_{sOp}(P)$ is the map $Tw_{sOp}(sOp)/P \to Tw_{sOp}(P)$, and the map $\Delta/C \to Tw(C)$ is the map $Tw_{sOp}(uAs)/C \to Tw_{uAs}(C)$.

There are two versions of Segal conditions: for operads, and for filtered operads. For all operads in the present work these two versions define the same presheaves. They differ in their interpretation of what is to be considered as the objects of generalized morphisms. And while they define the same presheaves, they define different single-object Segal presheaves. For the non-filtered version the following theorem holds.

Theorem (2.2.3)**.** *For any operad P the category of single-object Segal P-presheaves is equivalent to the category of P-algebras.*

In general this theorem does not hold for filtered version of Segal condition. The difference can be seen in the case of operads that are related to works [4, 5] and to the thesis [6]. These works attempt to construct category $\widetilde{U}$ such that Segal presheaves over $\widetilde{U}$ are multi-object modular operads or compact symmetric multi-categories. We are not sure whether the correct construction exists at all: we show that there is no operad P such that $Tw_{sOp}(P)$ is $\widetilde{U}$. Yet there are several related operads. The twisted arrow category of the operad mOp is very similar to the category $\widetilde{U}$, and the operad mOp itself is connected with the category of cobordisms. There are also four operads of unital associative algebras endowed with a trace map and possibly with anti-involution. These operads explain the inherent meaning of the subcategory of $\widetilde{U}$ which has caused some confusion in the works above. This confusion was justified, since these are the only operads in the present work that provide counter-examples to Theorem 2.2.3 and to several other results. At the same time these operads are reasonable: Segal condition for one of them gives cyclic nerves of categories, and the rest should give cyclic nerves of categories with anti-involution.

There is a class of (filtered and non-filtered) operads called palatable, which enjoy the following properties:

- For any operad there is a canonical map from any presheaf X over $Tw_{sOp}(P)$ to the presheaf $Pl(X)$. For palatable operads the presheaf $Pl(X)$ corresponds to the discrete fibration of categories of the form $Tw_{sOp}(P_{Pl(x)}) \to Tw_{sOp}(P)$, and in particular gives an operad $P_{Pl(X)}$. This is Theorem 2.2.23. Additionally Proposition 2.2.20 shows that for the operad mOp and for the four operads of algebras with trace map mentioned above the corresponding operads $P_{Pl(X)}$ in general do not exist, unless these operads are considered to be filtered. It this case the operad mOp still does not produce the corresponding operads $P_{Pl(X)}$, but for three out of the four operads of algebras with trace map the corresponding operads $P_{Pl(X)}$ always exists.

- For a palatable operad P the canonical map $X \to Pl(X)$ from a Segal presheaf X corresponds to a single-object Segal presheaf X' over $Tw_{sOp}(P_{Pl(X)})$.

- Twisted arrow categories of Segal presheaves are well-defined and coincide with twisted arrow categories of single-object Segal presheaves X'.

Unlike in the single-object case, the full form of Segal condition behaves better for filtered operads than for operads.

Finally, we show how to construct a simplicial set from a dendroidal set so that nerves of operads are sent to nerves of categories. We expect that this construction is the dendroidal version of the construction of twisted arrow quasi-categories from ∞-operads that appeared in [3].

Many of the known twisted arrow categories of operads are generalized Reedy categories. We study properties that an operad should satisfy so that its twisted arrow category is generalized Reedy. With this goal we introduce the notion of unique factorization system. This is a structure on a category which can be seen as a stronger version of orthogonal factorization system. Any unique factorization system generates an orthogonal factorization system. Unique factorization systems are not preserved under equivalences of categories, and yet twisted arrow categories of operads have two unique factorization systems, one of which is related to $(Inert, Active)$ orthogonal factorization, and another is often a part of Reedy structure. In Theorem 2.4.2 we prove that twisted arrow category of an operad is generalized Reedy if and only if the underlying category of the operad is a groupoid.

References

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