

# 博士論文

論文題目 Studies on singular Hermitian metrics on holomorphic  
vector bundles via  $L^2$  estimates and  $L^2$  extension theorems

( $L^2$  評価及び  $L^2$  拡張定理による正則ベクトル束の特異エルミート計量の研究)

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# STUDIES ON SINGULAR HERMITIAN METRICS ON HOLOMORPHIC VECTOR BUNDLES VIA $L^2$ ESTIMATES AND $L^2$ EXTENSION THEOREMS

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## Preface

Singular Hermitian metrics on holomorphic line bundles have an important role in algebraic and complex geometry. They make it possible that we apply complex analytic methods to the study of complex algebraic geometry. Singular Hermitian metrics on holomorphic vector bundles were also introduced and investigated by many people. However, it turns out that we cannot always define the Chern curvature currents of Griffiths semi-negative singular Hermitian metrics with measure coefficients. Hence, we need to define and study positivity notions without using the Chern currents.

In this thesis, we study singular Hermitian metrics on holomorphic vector bundles via  $L^2$  estimates and  $L^2$  extensions. Specifically, we characterize positivity notions by using  $L^2$  estimates and  $L^2$  extension theorems and investigate  $L^2$  estimates and vanishing theorems for holomorphic vector bundles with positive singular Hermitian metrics.

The content in Part 1 is a joint research with Dr. Genki Hosono [HI]. Deng, Wang, Zhang and Zhou introduce the multiple  $L^2$  extension property in their paper [DWZZ18], which is defined as follows:

Let  $(E, h) \rightarrow D$  be a holomorphic vector bundle and a singular Hermitian metric over a domain  $D \subset \mathbb{C}^n$ . Suppose that for any point  $x \in D$  and any non-zero element  $a \in E_x$  with finite norm  $|a|_h < +\infty$ , and any  $m \in \mathbb{N}$ , there exists a holomorphic section  $f_m \in H^0(D, E^{\otimes m})$  such that  $f_m(x) = a^{\otimes m}$  and

$$\int_D |f_m|^2 \leq C_m |a^{\otimes m}|_{h^{\otimes m}}^2 = C_m |a|_h^{2m},$$

where  $C_m$  are constants satisfy  $\log C_m/m \rightarrow +\infty$ . Then  $(E, h)$  is said to have multiple  $L^2$ -extension property.

The authors proved that if  $h^*$  is upper semi-continuous and  $(E, h)$  satisfies the multiple  $L^2$ -extension property,  $(E, h)$  is Griffiths semi-positive [DWZZ18]. They also proposed a conjecture that the multiple  $L^2$ -extension property is equivalent to Nakano semi-positivity. We give a counterexample to this conjecture.

**THEOREM 0.1.** [HI] *Let  $Q$  be the vector bundle of rank  $n$  over  $\mathbb{P}^n$  defined by*

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \underline{\mathbb{C}}^{n+1} \rightarrow Q \rightarrow 0,$$

*where  $\underline{\mathbb{C}}^{n+1}$  is the trivial vector bundle and  $\mathcal{O}(-1)$  be the tautological line bundle. Let  $h_Q$  be the metric on  $Q$  induced by the standard metric on  $\underline{\mathbb{C}}^{n+1}$ . Then  $(Q, h_Q)$  satisfies the multiple  $L^2$ -extension property. However,  $(Q, h_Q)$  is not Nakano semi-positive.*

We also introduce the twisted Hörmander condition and show that this condition implies Griffiths semi-positivity under some regularity assumptions.

In Part 2, we study pseudonorms on direct images of pluricanonical bundles [Ina3]. The space of holomorphic functions with finite  $L^p$ -norm  $A^p(\Omega)$  ( $0 < p < 2$ ) on a bounded domain  $\Omega$ , or the  $m$ -pluricanonical space  $H^0(X, mK_X)$  ( $m \geq 2$ ) for a complex manifold  $X$  has a natural pseudonorm. If there exists a biholomorphic map between two bounded

simply connected hyperconvex domains  $\Omega_1, \Omega_2$ , we get a natural linear isometry between  $A^p(\Omega_1)$  and  $A^p(\Omega_2)$ . On the other hand, it is known that if there is a linear isometry between  $A^p(\Omega_1)$  and  $A^p(\Omega_2)$ , we obtain a biholomorphic map between  $\Omega_1$  and  $\Omega_2$  [DWZZ20]. In algebraic geometry, there are other generalized results for projective manifolds of general type. These kinds of programs are called Yau's pseudonorm projects [CY08]. We show that the pseudonorms also determine holomorphic structures of Stein morphisms. One important technique is an  $L^2$ -variant of the Ohsawa-Takegoshi extension theorem.

**THEOREM 0.2.** [Ina3] *Let  $\tilde{f} : \tilde{X} \rightarrow B$  and  $\tilde{g} : \tilde{Y} \rightarrow B$  be smooth Stein morphisms over the open unit ball  $B \subset \mathbb{C}^r$  with  $\dim \tilde{X} = r + n$  and  $\dim \tilde{Y} = r + \ell$ . Suppose that  $f = (f_1, \dots, f_r) : X \rightarrow B$  and  $g = (g_1, \dots, g_r) : Y \rightarrow B$  are relatively compact Stein morphisms of  $\tilde{f} : \tilde{X} \rightarrow B$  and  $\tilde{g} : \tilde{Y} \rightarrow B$ , respectively. Assume that there exists  $m \geq 2$  such that*

- (i)  $X_t := f^{-1}(t)$  and  $Y_t := g^{-1}(t)$  are hyperconvex, and
- (ii) there exists a linear isomorphism

$$T : A(X, mK_X) \longrightarrow A(Y, mK_Y)$$

such that

$$(0.1) \quad \int_{X_t} |U_t \wedge \overline{U}_t|^{1/m} = \int_{Y_t} |(TU)_t \wedge \overline{(TU)}_t|^{1/m}$$

for any  $U \in A(X, mK_X)$  and  $t \in B$ , where  $A(X, mK_X) = \{U \in H^0(X, mK_X) \mid \int_X |U \wedge \overline{U}|^{1/m} < +\infty\}$  and  $U_t \in H^0(X_t, mK_{X_t})$  and  $(TU)_t \in H^0(Y_t, mK_{Y_t})$  are uniquely determined  $m$ -canonical forms such that

$$U|_{X_t} = U_t \wedge (df_1 \wedge \dots \wedge df_r)^{\otimes m}, \quad TU|_{Y_t} = (TU)_t \wedge (dg_1 \wedge \dots \wedge dg_r)^{\otimes m}.$$

Then we have that  $T$  induces linear isometries  $T_t : A(X_t, mK_{X_t}) \rightarrow A(Y_t, mK_{Y_t})$ ,  $n = \ell$ , and there exists a unique biholomorphic map  $F : X \rightarrow Y$  such that  $f = g \circ F$ .

In Part 3, we investigate singular Hermitian metrics on holomorphic vector bundles [Ina1]. Although we can define Griffiths semi-positivity of singular Hermitian metrics, the definition of Nakano semi-positivity of singular Hermitian metrics was not known. By using Demailly and Skoda's theorem, we establish  $L^2$ -estimates with coefficients in singular Hermitian metrics. The Demailly-Skoda theorem states that if  $(E, h)$  is Griffiths positive, then  $(E \otimes \det E, h \otimes \det h)$  is Nakano positive. By introducing a notion of strict  $\delta_\omega$ -positivity, we prove the following theorem. Applying this  $L^2$ -estimate, we also show vanishing theorems.

**THEOREM 0.3.** [Ina1] *Let  $(X, \omega)$  be a projective manifold and a Kähler form and  $E \rightarrow X$  be a holomorphic vector bundle over  $X$ . We also let  $h$  be a strictly Griffiths  $\delta_\omega$ -positive singular Hermitian metric on  $E$ . If  $f$  is a  $\bar{\partial}$ -closed  $E \otimes \det E$ -valued  $(n, q)$ -form with finite norm, then there exists an  $E \otimes \det E$ -valued  $(n, q-1)$ -form  $g$  satisfying  $\bar{\partial}g = f$  and*

$$\int_X |g|_{\omega, h \otimes \det h}^2 dV_\omega \leq \frac{1}{\delta_{qr}} \int_X |f|_{(\omega, h \otimes \det h)}^2 dV_\omega.$$

THEOREM 0.4. [Ina1] *Let the notation be the same as in Theorem 0.3. Suppose that the Lelong number  $\nu(-\log \det h, x) < 1$  for every point  $x \in X$ . Then we have*

$$H^q(X, K_X \otimes E \otimes \det E) = 0$$

for  $q \geq 1$ .

Theorem 0.4 can be regarded as a generalization of the Griffiths vanishing theorem.

In part4, we propose a definition of Nakano semi-positivity of singular Hermitian metrics [Ina4]. Deng, Ning, Wang and Zhou [DNWZ20] succeeded in characterizing smooth Nakano positivity via  $L^2$ -estimates by using the twisted Hörmander condition introduced in Part 1. Applying their result, we obtain the following definition.

DEFINITION 0.5. [Ina4] Suppose that  $h$  is a Griffiths semi-positive singular Hermitian metric. We say that  $h$  is *Nakano semi-positive* if for any Stein coordinate  $(\Omega, \iota)$  around any point  $x \in X$  such that  $E|_{\iota(\Omega)}$  is trivial on  $\iota(\Omega)$ , for any Kähler form  $\omega_\Omega$  on  $\Omega$ , for any smooth strictly plurisubharmonic function  $\psi$  on  $\Omega$ , for any positive integer  $q$  such that  $1 \leq q \leq n$ , and for any  $\bar{\partial}$ -closed  $f \in L^2_{(n,q)}(\Omega, \iota^*E; \omega_\Omega, (\iota^*h)e^{-\psi})$ , there exists  $u \in L^2_{(n,q-1)}(\Omega, \iota^*E; \omega_\Omega, (\iota^*h)e^{-\psi})$  satisfying  $\bar{\partial}u = f$  and

$$\int_{\Omega} |u|_{(\omega_\Omega, \iota^*h)}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_{\Omega} \langle [\sqrt{-1} \partial \bar{\partial} \psi \otimes Id_E, \Lambda_{\omega_\Omega}]^{-1} f, f \rangle_{(\omega_\Omega, \iota^*h)} e^{-\psi} dV_{\omega_\Omega}.$$

Here we suppose that the right-hand side is finite.

Similarly, we define strict Nakano  $\delta_\omega$ -positivity and establish the following  $L^2$ -estimate and vanishing theorem.

THEOREM 0.6. [Ina4] *Let  $(X, \omega)$  be a projective manifold and a Kähler form and  $E \rightarrow X$  be a holomorphic vector bundle over  $X$ . We also let  $h$  be a strictly Nakano  $\delta_\omega$ -positive singular Hermitian metric on  $E$ . If  $f$  is a  $\bar{\partial}$ -closed  $E$ -valued  $(n, q)$ -form with finite norm, then there exists an  $E$ -valued  $(n, q-1)$ -form  $u$  satisfying  $\bar{\partial}u = f$  and*

$$\int_X |u|_{\omega, h}^2 dV_\omega \leq \frac{1}{\delta q} \int_X |f|_{(\omega, h)}^2 dV_\omega.$$

THEOREM 0.7. [Ina4] *Let the notation be the same as in 0.6. Then we have*

$$H^q(X, K_X \otimes \mathcal{E}(h)) = 0$$

for  $q \geq 1$ , where  $\mathcal{E}(h)$  is the sheaf of germs of locally square integrable holomorphic sections of  $E$  with respect to  $h$ .

We prove the coherence of  $\mathcal{E}(h)$  in Part 1. Theorem 0.6 is a generalization of Theorem 0.3 and Theorem 0.7 is a generalization of Theorem 0.4. Theorem 0.7 also generalizes Nakano's vanishing theorem.

In Part 5, we study the Chern current of Griffiths semi-negative or Griffiths semi-positive singular Hermitian metrics on holomorphic vector bundles [Ina2]. For a holomorphic vector bundle  $E$  of rank  $E \geq 2$ , it is not possible to define the Chern curvature current with measure coefficients in general. This phenomenon was observed by Raufi. Then

Raufi proved that for a Griffiths semi-negative singular Hermitian metric  $h$ , if  $\det h > \epsilon$  for some positive constant  $\epsilon > 0$ , the Chern curvature current  $\Theta_h$  can be defined with measure coefficients. We find some condition that the curvature current can be defined with measure coefficients over a set including the degeneracy set  $\{\det h = 0\}$ .

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## Part 1. A converse of Hörmander's $L^2$ -estimate and new positivity notions for vector bundles

ABSTRACT. We study conditions of Hörmander's  $L^2$ -estimate and the Ohsawa-Takegoshi extension theorem. Introducing a twisted version of Hörmander-type condition, we show a converse of Hörmander  $L^2$ -estimate under some regularity assumptions on an  $n$ -dimensional domain. This result is a partial generalization of the 1-dimensional result obtained by Berndtsson. We also define new positivity notions for vector bundles with singular Hermitian metrics by using these conditions. We investigate these positivity notions and compare them with classical positivity notions.

### 1. INTRODUCTION

The content in Part 1 is based on the paper [HI]. There are many curvature-positivity notions for Hermitian holomorphic vector bundles on complex manifolds. The situation is simple for line bundles: a Hermitian metric  $h = e^{-\phi}$  on a line bundle has positive curvature if and only if the corresponding local weight  $\phi$  is plurisubharmonic. When we consider a singular Hermitian metric, its curvature, defined as a current, is positive.

For general vector bundles, the situation is much complicated. There are some positivity notions which are not equivalent to each other. When we consider a singular metric on vector bundles, we cannot even define its curvature ([Rau15, Theorem 1.5]). Therefore, to generalize curvature-positivity concepts of vector bundles for singular metrics, we have to seek their characterizations without using curvature tensors.

For the Griffiths semi-positivity, such a characterization has been obtained (cf. [DWZZ18], [DWZZ20], [HPS18], [PT18], [Rau15]). On the other hand, we do not know such a characterization for the Nakano semi-positivity, which is stronger than the Griffiths positivity and a natural setting for using Hörmander's  $L^2$ -methods. This is one of the difficulties in the study of singular Hermitian metrics on vector bundles.

In [DWZZ18], a new characterization of a plurisubharmonicity was obtained. Namely, the possibility of the  $L^2$ -extension with a certain condition on its estimate is equivalent to the plurisubharmonicity. Here we state the precise condition in the trivial line bundle case (for more general settings, see Definition 3.2):

DEFINITION 1.1 ([DWZZ18]). Let  $\Omega \subset \mathbb{C}^n$  be a domain and  $\phi$  be an upper semi-continuous function. We say that  $(\Omega, \phi)$  satisfies *the multiple  $L^2$ -extension property* if there exists a number  $C_m > 0$  for each  $m$  such that

- for every  $p \in \Omega$  with  $\phi(p) \neq -\infty$ , there exists a holomorphic function  $f$  satisfying  $f(p) = 1$  and

$$\int_{\Omega} |f|^2 e^{-m\phi} \leq C_m e^{-m\phi(p)},$$

and

- (Growth condition)  $C_m$  satisfies the condition  $\lim_{m \rightarrow +\infty} \frac{\log C_m}{m} = 0$ .

For vector bundles, it is proved in [DWZZ18] that the multiple  $L^2$ -extension property implies the Griffiths positivity.

In [Ber98], it is proved that, for a continuous function on a one-dimensional domain, the availability of the Hörmander estimate implies the subharmonicity. This kind of study has been applied to various fields (cf. [DZZ14]). In [DWZZ18], Deng, Wang, Zhang, and Zhou asked a problem if one can extend this result to higher dimensional cases. In Section 2, we show a partial converse of Hörmander's  $L^2$ -estimate for line bundles on an  $n$ -dimensional domain. We define the twisted Hörmander condition as follows:

**DEFINITION 1.2.** Let  $\Omega \subset \mathbb{C}^n$  be a domain and  $\phi$  be an upper semi-continuous function. We say that  $(\Omega, \phi)$  satisfies *the twisted Hörmander condition* if, for every positive integer  $m$ , smooth strictly plurisubharmonic function  $\psi$  on  $\Omega$ , and smooth  $\bar{\partial}$ -closed  $(0, 1)$ -form  $\alpha$  with compact support and finite norm  $\int_{\Omega} |\alpha|^2_{\sqrt{-1}\partial\bar{\partial}\psi} e^{-(m\phi+\psi)} < +\infty$ , there exists a smooth function  $u$  such that

- $\bar{\partial}u = \alpha$ , and
- $\int_{\Omega} |u|^2 e^{-(m\phi+\psi)} \leq \int_{\Omega} |\alpha|^2_{\sqrt{-1}\partial\bar{\partial}\psi} e^{-(m\phi+\psi)}.$

Using this, we state a converse of Hörmander's  $L^2$ -estimate as follows:

**THEOREM 1.3.** *Let  $\Omega \subset \mathbb{C}^n$  be a domain. Assume that  $\text{Pole}(\phi)$  is closed and  $\phi$  is a locally Hölder continuous function on  $\Omega \setminus \text{Pole}(\phi)$ . If the twisted Hörmander condition is satisfied for  $(\Omega, \phi)$ ,  $\phi$  is plurisubharmonic.*

Here we consider a *twisted* version of Hörmander conditions. Our proof is an application of the theorem in [DWZZ18]. Twisting with an additional weight  $\psi$  enables us to prove the multiple  $L^2$ -extension property and thus the plurisubharmonicity of  $\phi$ .

We shall also prove a partial converse of Hörmander's  $L^2$ -estimate for vector bundles (see Theorem 3.5). We also introduce several positivity notions for vector bundles. To be precise, we study a singular Hermitian metric which is positive in the sense of twisted Hörmander (see Definition 3.7) and a singular Hermitian metric which is positive in the sense of multiple  $L^2$ -extension (see Definition 3.8).

It is important to consider the sheaf of square integrable holomorphic sections of vector bundles with respect to positively curved singular Hermitian metrics. For line bundles, such a sheaf is called *the multiplier ideal sheaf*. Multiplier ideal sheaves are proved to be coherent by  $L^2$ -estimates. For general vector bundles, little is known about the coherence of such sheaves, because of the lack of  $L^2$ -estimates for general singular Hermitian metrics. The first author has proved the coherence of such sheaves for singular Hermitian metrics induced by holomorphic sections [Hos17, Theorem 1.1]. We prove that the sheaf of locally square integrable holomorphic sections with respect a metric which is positive in the sense of twisted Hörmander is coherent.

**THEOREM 1.4.** *Let  $(E, h)$  be positively curved in the sense of twisted Hörmander. Assume that  $|u|_{h^*}$  is upper semi-continuous for any local holomorphic section  $u \in \mathcal{O}(E^*)$ .*

Then  $\mathcal{E}(h)$  is a coherent subsheaf of  $\mathcal{O}(E)$ , where  $\mathcal{E}(h)$  is the sheaf of germs of locally square integrable holomorphic sections of  $E$  with respect to  $h$ .

We also study metrics which is positive in the sense of multiple  $L^2$ -extension. We show that the positivity in this sense is strictly weaker than the Nakano semi-positivity.

**THEOREM 1.5.** *There exists a positively curved vector bundle  $(E, h)$  in the sense of multiple  $L^2$ -extension such that  $(E, h)$  is not Nakano semi-positive.*

This is a partial answer to the question proposed by Deng, Wang, Zhang, and Zhou in [DWZZ18] (see Question 3.10 below). We also propose some questions about the above new positivity notions.

## 2. A CONVERSE OF HÖRMANDER'S $L^2$ ESTIMATE FOR LINE BUNDLES

In this section, we formulate a Hörmander-type condition and prove the equivalence to plurisubharmonicity under some regularity assumption.

**DEFINITION 2.1.** Let  $\Omega \subset \mathbb{C}^n$  be a domain and  $\phi : \Omega \rightarrow [-\infty, +\infty)$  be an upper semi-continuous function. We say that  $(\Omega, \phi)$  satisfies *the twisted Hörmander condition* if, for every positive integer  $m$ , smooth strictly plurisubharmonic function  $\psi$  on  $\Omega$ , and smooth  $\bar{\partial}$ -closed  $(0, 1)$ -form  $\alpha$  with compact support and finite norm  $\int_{\Omega} |\alpha|_{\sqrt{-1}\partial\bar{\partial}\psi}^2 e^{-(m\phi+\psi)} < +\infty$ , there exists a smooth function  $u$  such that

- $\bar{\partial}u = \alpha$ , and
- $\int_{\Omega} |u|^2 e^{-(m\phi+\psi)} \leq \int_{\Omega} |\alpha|_{\sqrt{-1}\partial\bar{\partial}\psi}^2 e^{-(m\phi+\psi)}.$

**REMARK 2.2.** (1) To formulate the condition, we used an additional weight function  $\psi$ . This enables us to prove the multiple  $L^2$ -extension property (under some regularity assumption) and therefore the plurisubharmonicity of  $\phi$ .

(2) We clearly see that, on a domain in  $\mathbb{C}^n$ , an upper semi-continuous function  $\phi$  is plurisubharmonic if and only if for every smooth strictly plurisubharmonic function  $\psi$ ,  $\phi + \psi$  is plurisubharmonic. Hence, it is worth considering a twisted version of Hörmander's  $L^2$  estimate.

(3) If  $\phi$  is a plurisubharmonic and  $\Omega$  is pseudoconvex, the twisted Hörmander condition is automatically satisfied. Indeed, the standard estimate gives the following:

$$\int_{\Omega} |u|^2 e^{-(m\phi+\psi)} \leq \int_{\Omega} |\alpha|_{\sqrt{-1}\partial\bar{\partial}(m\phi+\psi)}^2 e^{-(m\phi+\psi)}.$$

Since  $\phi$  is plurisubharmonic, we have that  $\sqrt{-1}\partial\bar{\partial}(m\phi + \psi) \geq \sqrt{-1}\partial\bar{\partial}\psi$  (in the sense of currents). Then we have that  $|\cdot|_{\sqrt{-1}\partial\bar{\partial}(m\phi+\psi)}^2 \leq |\cdot|_{\sqrt{-1}\partial\bar{\partial}\psi}^2$  and thus

$$\int_{\Omega} |u|^2 e^{-(m\phi+\psi)} \leq \int_{\Omega} |\alpha|_{\sqrt{-1}\partial\bar{\partial}(m\phi+\psi)}^2 e^{-(m\phi+\psi)} \leq \int_{\Omega} |\alpha|_{\sqrt{-1}\partial\bar{\partial}\psi}^2 e^{-(m\phi+\psi)}.$$

(4) One may formulate, as in Definition 1.1, the twisted Hörmander condition with constants  $C_m$  and the same growth condition  $\lim_{m \rightarrow \infty} \frac{\log C_m}{m} = 0$ .

We will show that, under some continuity assumption, the Hörmander condition implies the multiple  $L^2$ -extension property. We set  $\text{Pole}(\phi) := \{\phi^{-1}(-\infty)\}$ , the set of poles of  $\phi$ .

**THEOREM 2.3.** (= Theorem 1.3) *Let  $\Omega \subset \mathbb{C}^n$  be a domain. Assume that  $\text{Pole}(\phi)$  is closed and  $\phi$  is a locally Hölder continuous function on  $\Omega \setminus \text{Pole}(\phi)$ , i.e. for every  $\Omega' \Subset \Omega \setminus \text{Pole}(\phi)$ , there exist constants  $\alpha = \alpha_{\Omega'} \in (0, 1]$  and  $C = C_{\Omega'} > 0$  such that  $|\phi(z) - \phi(w)| \leq C|z - w|^\alpha$  for every  $z, w \in \Omega'$ . If the twisted Hörmander condition (Definition 2.1) is satisfied for  $(\Omega, \phi)$ ,  $\phi$  is plurisubharmonic.*

**PROOF.** Fix a domain  $\Omega' \Subset \Omega \setminus \text{Pole}(\phi)$ . We will show that  $(\Omega', \phi)$  satisfies the multiple  $L^2$ -extension property (Definition 1.1).

Fix a point  $w \in \Omega'$  with  $\phi(w) > -\infty$  and an integer  $m > 0$ . We will construct a holomorphic function  $f \in \mathcal{O}(\Omega')$  such that  $f(w) = 1$  and

$$\int_{\Omega'} |f|^2 e^{-m\phi} \leq C_m e^{-m\phi(w)},$$

where  $C_m$  is a constant independent of the choice of  $w \in \Omega'$ .

Take  $\chi = \chi(t)$  be a smooth function on  $\mathbb{R}$ , such that

- $\chi(t) = 1$  for  $t \leq 1/2$ ,
- $\chi(t) = 0$  for  $t \geq 1$ , and
- $|\chi'(t)| \leq 3$  on  $\mathbb{R}$ .

Define a  $(0,1)$ -form  $\alpha$  by

$$\alpha := \bar{\partial}\chi \left( \frac{|z - w|^2}{\epsilon^2} \right) = \chi' \left( \frac{|z - w|^2}{\epsilon^2} \right) \sum_j \frac{z_j - w_j}{\epsilon^2} d\bar{z}_j.$$

We apply the twisted Hörmander condition for the weight

$$\psi_{\epsilon, \delta} := \log(|z - w|^2 + \epsilon^2) + n \log(|z - w|^2 + \delta^2),$$

where  $\epsilon$  and  $\delta$  are positive parameters.

Then we obtain a smooth function  $u_{\epsilon, \delta}$  on  $\Omega'$  such that

- $\bar{\partial}u_{\epsilon, \delta} = \alpha$  and
- 

$$(2.1) \quad \int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-(m\phi + \psi_{\epsilon, \delta})} \leq \int_{\Omega'} |\alpha|_{\sqrt{-1}\partial\bar{\partial}\psi_{\epsilon, \delta}}^2 e^{-(m\phi + \psi_{\epsilon, \delta})}.$$

Since

$$|\alpha|_{\sqrt{-1}\partial\bar{\partial}\psi_{\epsilon, \delta}}^2 = \left| \chi' \left( \frac{|z - w|^2}{\epsilon^2} \right) \right|^2 \cdot \frac{1}{\epsilon^4} \cdot \left| \sum_j (z_j - w_j) d\bar{z}_j \right|_{\sqrt{-1}\partial\bar{\partial}\psi_{\epsilon, \delta}}^2$$

and the support of  $\chi' \left( \frac{|z-w|^2}{\epsilon^2} \right)$  is in  $\{1/2 \leq |z-w|^2/\epsilon^2 \leq 1\}$ , we have that

(2.2)

$$\begin{aligned} (\text{RHS of (2.1)}) &= \int_{\{1/2 \leq |z-w|^2/\epsilon^2 \leq 1\}} \left| \chi' \left( \frac{|z-w|^2}{\epsilon^2} \right) \right|^2 \cdot \frac{1}{\epsilon^4} \cdot \left| \sum_j (z_j - w_j) d\bar{z}_j \right|_{\sqrt{-1}\partial\bar{\partial}\psi_{\epsilon,\delta}}^2 e^{-(m\phi+\psi_{\epsilon,\delta})} \\ (2.3) \quad &\leq \frac{9}{\epsilon^4} \int_{\{\epsilon^2/2 \leq |z-w|^2 \leq \epsilon^2\}} \left| \sum_j (z_j - w_j) d\bar{z}_j \right|_{\sqrt{-1}\partial\bar{\partial}\psi_{\epsilon,\delta}}^2 e^{-(m\phi+\psi_{\epsilon,\delta})}. \end{aligned}$$

Letting  $\omega := i\partial\bar{\partial}|z|^2$ , we have that

$$\sqrt{-1}\partial\bar{\partial}\psi_{\epsilon,\delta} \geq \sqrt{-1}\partial\bar{\partial} \log(|z-w|^2 + \epsilon^2) \geq \frac{\epsilon^2}{(|z-w|^2 + \epsilon^2)^2} \omega,$$

and thus

$$(2.4) \quad |\cdot|_{\sqrt{-1}\partial\bar{\partial}\psi_{\epsilon,\delta}}^2 \leq |\cdot|_{(\epsilon^2/(|z-w|^2 + \epsilon^2)^2)\omega}^2.$$

Combining (2.3) and (2.4), we have the following:

$$\begin{aligned} (2.3) &\leq \frac{9}{\epsilon^4} \int_{\{\epsilon^2/2 \leq |z-w|^2 \leq \epsilon^2\}} |z-w|^2 \frac{(|z-w|^2 + \epsilon^2)^2}{\epsilon^2} e^{-(m\phi+\psi_{\epsilon,\delta})} \\ &= \frac{9}{\epsilon^4} \int_{\{\epsilon^2/2 \leq |z-w|^2 \leq \epsilon^2\}} |z-w|^2 \frac{(|z-w|^2 + \epsilon^2)^2}{\epsilon^2} \frac{1}{|z-w|^2 + \epsilon^2} \frac{1}{(|z-w|^2 + \delta^2)^n} e^{-m\phi} \\ &= \frac{9}{\epsilon^4} \int_{\{\epsilon^2/2 \leq |z-w|^2 \leq \epsilon^2\}} |z-w|^2 \frac{|z-w|^2 + \epsilon^2}{\epsilon^2} \frac{1}{(|z-w|^2 + \delta^2)^n} e^{-m\phi} \\ &\leq \frac{9}{\epsilon^4} \text{Vol}(\{\epsilon^2/2 \leq |z-w|^2 \leq \epsilon^2\}) \frac{\epsilon^2(\epsilon^2 + \epsilon^2)}{\epsilon^2} \frac{1}{(\epsilon^2/2 + \delta^2)^n} e^{-m \inf_{B(w,\epsilon)} \phi} \\ &= \frac{9}{\epsilon^4} C_n \epsilon^{2n} \frac{\epsilon^2(\epsilon^2 + \epsilon^2)}{\epsilon^2} \frac{1}{(\epsilon^2/2 + \delta^2)^n} e^{-m \inf_{B(w,\epsilon)} \phi} \\ &= 9C_n \epsilon^{2n-2} \frac{1}{(\epsilon^2/2 + \delta^2)^n} e^{-m \inf_{B(w,\epsilon)} \phi}, \end{aligned}$$

where  $C_n$  is a positive constant depending only on  $n$ .

To summarize, we have obtained a smooth function  $u_{\epsilon,\delta}$  on  $\Omega'$  such that

- $\bar{\partial}u_{\epsilon,\delta} = \alpha$ , and
- the following estimate holds:

$$(2.5) \quad \int_{\Omega'} |u_{\epsilon,\delta}|^2 e^{-(m\phi+\psi_{\epsilon,\delta})} \leq 9C_n \epsilon^{2n-2} \frac{1}{(\epsilon^2/2 + \delta^2)^n} e^{-m \inf_{B(w,\epsilon)} \phi}.$$

Let  $\delta \rightarrow 0$ . The right-hand side of (2.5) is increasing to

$$9C_n \epsilon^{2n-2} \frac{2^n}{\epsilon^{2n}} e^{-m \inf_{B(w,\epsilon)} \phi} = 9C_n \frac{2^n}{\epsilon^2} e^{-m \inf_{B(w,\epsilon)} \phi}.$$

Let us show the convergence of  $u_{\epsilon,\delta}$ . We have that

$$\int_{\Omega'} |u_{\epsilon,\delta}|^2 e^{-(m\phi+\psi_{\epsilon,\delta})} \leq 9C_n \frac{2^n}{\epsilon^2} e^{-m \inf \phi}.$$

Note that the weight function  $\psi_{\epsilon,\delta}$  is decreasing when  $\delta \rightarrow 0$ . Therefore  $e^{-\psi_{\epsilon,\delta}}$  is increasing when  $\delta \rightarrow 0$ .

Fix  $\delta_0 > 0$ . Then, for  $\delta < \delta_0$ , (since  $e^{-\psi_{\epsilon,\delta}} > e^{-\psi_{\epsilon,\delta_0}}$ ) we have that

$$\int_{\Omega'} |u_{\epsilon,\delta}|^2 e^{-(m\phi+\psi_{\epsilon,\delta_0})} \leq \int_{\Omega'} |u_{\epsilon,\delta}|^2 e^{-(m\phi+\psi_{\epsilon,\delta})} \leq 9C_n \frac{2^n}{\epsilon^2} e^{-m \inf \phi}.$$

Thus  $\{u_{\epsilon,\delta}\}_{\delta < \delta_0}$  forms a bounded sequence in  $L^2(\Omega', e^{-(m\phi+\psi_{\epsilon,\delta_0})})$ . We can choose a weakly convergent sequence  $\{u_{\epsilon,\delta(k)}\}_k$  in  $L^2(\Omega', e^{-(m\phi+\psi_{\epsilon,\delta_0})})$ . Since the  $L^2$ -norm is lower semi-continuous under weak limits, the (weak) limit function  $u_\epsilon$  satisfies

$$\int_{\Omega'} |u_\epsilon|^2 e^{-(m\phi+\psi_{\epsilon,\delta_0})} \leq 9C_n \frac{2^n}{\epsilon^2} e^{-m \inf \phi}.$$

Next, fix  $\delta_1 < \delta_0$ . By the same argument, we can choose a subsequence of  $\{u_{\epsilon,\delta(k)}\}_k$  weakly convergent in  $L^2(\Omega', e^{-(m\phi+\psi_{\epsilon,\delta_1})})$ . Repeating this argument for a sequence  $\{\delta_j\}$  decreasing to 0 and taking a diagonal sequence, finally we can obtain a sequence  $\{u_{\epsilon,\delta(k)}\}_k$  weakly convergent in  $L^2(\Omega', e^{-(m\phi+\psi_{\epsilon,\delta_j})})$  for every  $j$ . Then we have

$$\int_{\Omega'} |u_\epsilon|^2 e^{-(m\phi+\psi_{\epsilon,\delta_j})} \leq 9C_n \frac{2^n}{\epsilon^2} e^{-m \inf \phi}$$

for every  $j$  and, by the monotone convergence theorem,

$$\int_{\Omega'} |u_\epsilon|^2 e^{-(m\phi+\psi_\epsilon)} \leq 9C_n \frac{2^n}{\epsilon^2} e^{-m \inf \phi},$$

where  $\psi_\epsilon := \psi_{\epsilon,0}$ .

Since differential operators are continuous under weak limits, we have  $\bar{\partial}u_\epsilon = \alpha$ .

The integral

$$\int_{\Omega'} |u_\epsilon|^2 e^{-(m\phi+\psi_\epsilon)} = \int_{\Omega'} |u_\epsilon|^2 e^{-m\phi} \frac{1}{(|z-w|^2 + \epsilon^2)} \frac{1}{|z-w|^{2n}}$$

is finite while the weight  $\frac{1}{|z-w|^{2n}}$  is not integrable near  $w$ , thus  $u_\epsilon(w)$  must be 0.

Let  $f_\epsilon := \chi(|z-w|^2/\epsilon^2) - u_\epsilon$ . Then  $f_\epsilon(0) = 1$  and

$$(2.6) \quad \left( \int_{\Omega'} |f_\epsilon|^2 e^{-m\phi} \right)^{1/2} \leq \left( \int_{\Omega'} \left| \chi \left( \frac{|z-w|^2}{\epsilon^2} \right) \right|^2 e^{-m\phi} \right)^{1/2} + \left( \int_{\Omega'} |u_\epsilon|^2 e^{-m\phi} \right)^{1/2}.$$

We will estimate each term of the right-hand side of (2.6). Since  $\chi \leq 1$  and the support of  $\chi(|z-w|^2/\epsilon^2)$  is contained in  $\{|z-w|^2 \leq \epsilon^2\}$ , the first term can be bounded by

$$(C_n \epsilon^{2n} e^{-m \inf_{B(w,\epsilon)} \phi})^{\frac{1}{2}}.$$

Next we consider the second term. We have that

$$\begin{aligned} \int_{\Omega'} |u|^2 e^{-m\phi} &\leq \left[ \sup_{z \in \Omega'} |z - w|^{2n} (|z - w|^2 + \epsilon) \right] \cdot \int_{\Omega'} |u|^2 e^{-m\phi} \frac{1}{(|z - w|^2 + \epsilon^2)} \frac{1}{|z - w|^{2n}} \\ &\leq \left[ \sup_{z \in \Omega'} |z - w|^{2n} (|z - w|^2 + \epsilon) \right] \cdot 9C_n \frac{2^n}{\epsilon^2} e^{-m \inf_{B(w, \epsilon)} \phi}. \end{aligned}$$

Assuming  $\epsilon < 1$ , we can bound the sup by  $(R + 1)^{2n+2}$ , where  $R$  is the radius of  $\Omega'$ . Therefore the second term is bounded by  $C \frac{1}{\epsilon^2} e^{-m \inf_{B(w, \epsilon)} \phi}$ .

Therefore, we have that

$$\int_{\Omega'} |f|^2 e^{-m\phi} \leq C' \left( \epsilon^n + \frac{1}{\epsilon} \right)^2 e^{-m \inf_{B(w, \epsilon)} \phi},$$

where  $C'$  is a constant independent of  $m$ .

By the assumption that  $\phi$  is locally Hölder continuous on  $\Omega \setminus \text{Pole}(\phi)$ , we have that  $|\phi(z) - \phi(w)| \leq C_{\Omega'} |z - w|^\alpha$  for every  $z \in \Omega'$ . Let  $\epsilon := 1/(mC_{\Omega'})^{1/\alpha}$ . Then we have that  $|m\phi(z) - m\phi(w)| \leq 1$  for  $|z - w| \leq \epsilon$ . Thus,

$$\begin{aligned} \int_{\Omega'} |f|^2 e^{-m\phi} &\leq C' \left( \frac{1}{m^{\frac{n}{\alpha}} C_{\Omega'}^{\frac{n}{\alpha}}} + m^{\frac{1}{\alpha}} C_{\Omega'}^{\frac{1}{\alpha}} \right)^2 e^{-m\phi(w)+1} \\ (2.7) \quad &= C' e \cdot \left( \frac{1}{m^{\frac{n}{\alpha}} C_{\Omega'}^{\frac{n}{\alpha}}} + m^{\frac{1}{\alpha}} C_{\Omega'}^{\frac{1}{\alpha}} \right)^2 e^{-m\phi(w)}. \end{aligned}$$

The coefficient of (2.7) satisfies the growth condition  $\frac{\log C_m}{m} \rightarrow 0$ , thus we have verified the multiple  $L^2$ -extension property for  $(\Omega', \phi)$ .

Then, by [DWZZ18], it follows that  $\phi$  is plurisubhammonic on  $\Omega \setminus \text{Pole}(\phi)$ . Since  $\phi$  takes the value  $-\infty$  on  $\text{Pole}(\phi)$ ,  $\phi$  is also plurisubharmonic on  $\Omega$ .  $\square$

**REMARK 2.4.** In the last part of the proof, we used the assumption that  $\phi$  is locally Hölder continuous. After the first version of the manuscript was completed, a proof only assuming continuity of  $\phi$  was obtained in [DNW19]. They just used the uniform convergence  $\inf_{B(w, \epsilon)} \phi \rightarrow \phi$  on compact subsets. This was also pointed out by a referee.

### 3. A CONVERSE OF HÖRMANDER'S $L^2$ ESTIMATE FOR VECTOR BUNDLES AND RELATIONS TO VARIOUS POSITIVITY NOTIONS

**3.1. A converse of Hörmander's  $L^2$  estimate for vector bundles.** In this section, we prove a version of Theorem 2.3 for vector bundles. First of all, we introduce the definition of singular Hermitian metrics on vector bundles. Throughout this section,  $E \rightarrow \Omega$  denotes a holomorphic vector bundle over a domain  $\Omega \subset \mathbb{C}^n$ ,  $\omega$  denotes the standard Kähler metric on  $\Omega$ ,  $dV_\omega$  be the volume form determined by  $\omega$ ,  $h$  denotes a singular Hermitian metric on  $E$ , and  $h^*$  denotes the dual metric on the dual vector bundle  $E^*$ .

DEFINITION 3.1. (cf. [HPS18, Definition 17.1], [Rau15, Definition 1]) A *singular Hermitian metric*  $h$  on  $E$  is a measurable map from  $\Omega$  to the space of non-negative Hermitian forms on the fibers, i.e.  $h$  satisfies the following conditions:

- (1)  $h$  is finite and positive definite almost everywhere on each fiber.
- (2) the function  $|s|_h : U \rightarrow [0, +\infty]$  is measurable whenever  $U \subset \Omega$  is an open set and  $s \in H^0(U, E)$ .

The Ohsawa-Takegoshi type condition, which is called the multiple  $L^p$ -extension property, for vector bundles is introduced in [DWZZ18]. The precise definition is as follows.

DEFINITION 3.2 ([DWZZ18]). (Multiple  $L^p$ -extension property) Let  $p > 0$  be a fixed constant. Assume that for any  $z \in \Omega$ , any nonzero element  $a \in E_z$  with finite norm  $|a|_h < +\infty$ , and any  $m \geq 1$ , there is a holomorphic section  $f_m \in H^0(\Omega, E^{\otimes m})$  such that  $f_m(z) = a^{\otimes m}$  and satisfies the following condition:

$$(3.1) \quad \int_{\Omega} |f_m|_{h^{\otimes m}}^p dV_{\omega} \leq C_m |a^{\otimes m}|_{h^{\otimes m}}^p = C_m |a|_h^{mp},$$

where  $C_m$  are constants independent of  $z \in \Omega$  and satisfy the growth condition  $\lim_{m \rightarrow \infty} \frac{1}{m} \log C_m = 0$ . Then  $(E, h)$  is said to have *the multiple  $L^p$ -extension property*.

In this paper, we only consider the multiple  $L^2$ -extension property. We also define the Hörmander-type condition for vector bundles.

DEFINITION 3.3. We say that  $(E, h)$  satisfies *the twisted Hörmander condition* on  $\Omega$  if, for every positive integer  $m$ , smooth strictly plurisubharmonic function  $\psi$  on  $\Omega$  and smooth  $\bar{\partial}$ -closed  $E^{\otimes m}$ -valued  $(n, 1)$ -form  $\alpha = \sum_j \alpha_j dz \wedge d\bar{z}_j$  ( $\alpha_j$  is a smooth section of  $E^{\otimes m}$ ) with compact support and finite norm  $\int_{\Omega} \sum_{1 \leq i, j \leq n} (\psi^{(i\bar{j})} \alpha_i, \alpha_j)_{h^{\otimes m}} e^{-\psi} dV_{\omega} < +\infty$ , there exists a smooth  $E^{\otimes m}$ -valued  $(n, 0)$  form  $u$  such that

- $\bar{\partial}u = \alpha$ , and
- $\int_{\Omega} |u|_{(h^{\otimes m}, \omega)}^2 e^{-\psi} dV_{\omega} \leq \int_{\Omega} \sum_{1 \leq i, j \leq n} (\psi^{(i\bar{j})} \alpha_i, \alpha_j)_{h^{\otimes m}} e^{-\psi} dV_{\omega},$

where  $dz := dz_1 \wedge \dots \wedge dz_n$  and  $(\psi^{(i\bar{j})})_{1 \leq i, j \leq n}$  is the inverse matrix of  $(\frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j})_{1 \leq i, j \leq n}$ .

REMARK 3.4. (1) The above  $(\psi^{(i\bar{j})})_{1 \leq i, j \leq n}$  corresponds to the inverse operator of  $[\sqrt{-1} \partial \bar{\partial} \psi \otimes Id_{E^{\otimes m}}, \Lambda_{\omega}]$ . Here  $[\cdot, \cdot]$  denotes a graded Lie bracket, and  $\Lambda_{\omega}$  denotes an adjoint of the operator  $L := \omega \wedge \cdot$ .

(2) When  $h$  is smooth and Nakano positive and  $\Omega$  is pseudoconvex, we can show that  $h$  satisfies the twisted Hörmander condition.

We will show that, under some regularity condition, the twisted Hörmander condition implies the multiple  $L^2$ -extension property for vector bundles.

THEOREM 3.5. We fix a bounded domain  $\Omega' \Subset \Omega$ . Assume that  $|u|_{h^*}$  is upper semi-continuous for any local holomorphic section  $u \in \mathcal{O}(E^*)$ . Moreover we assume that  $\log |u|_h$



is locally Hölder continuous on  $\Omega$  for any non-zero local holomorphic section  $u$ , i.e. for every  $\Omega'' \Subset \Omega$ ,  $|\log |u(z)|_{h(z)} - \log |u(w)|_{h(w)}| \leq C_{\Omega''} |z-w|^\alpha$  for positive constants  $\alpha \in (0, 1]$  and  $C_{\Omega''} > 0$  independent of  $u \in H^0(\Omega'', E)$ . If  $(E, h)$  satisfies the twisted Hörmander condition (Definition 3.3) on  $\Omega$ ,  $(E, h)$  has the multiple  $L^2$ -extension property on  $\Omega'$ . Then,  $(E, h)$  is also Griffiths semi-positive.

PROOF. For any  $w \in \Omega'$ , we take any non-zero element  $a \in E_w$ . Taking the smooth function  $\chi$  in the proof of Theorem 2.3, we define a  $\bar{\partial}$ -closed  $E^{\otimes m}$ -valued  $(n, 1)$ -form  $\alpha$  by

$$\alpha := \bar{\partial} \left( a^{\otimes m} \chi \left( \frac{|z-w|^2}{\epsilon^2} \right) dz \right) = a^{\otimes m} \chi' \left( \frac{|z-w|^2}{\epsilon^2} \right) \sum_j \frac{(z_j - w_j)}{\epsilon^2} dz \wedge d\bar{z}_j.$$

We also take the weight function

$$\psi_{\epsilon, \delta} := \log(|z-w|^2 + \epsilon^2) + n \log(|z-w|^2 + \delta^2)$$

for  $\epsilon, \delta > 0$ . Since

$$\begin{aligned} \sum_{1 \leq i, j \leq n} \left( \psi_{\epsilon, \delta}^{(i\bar{j})} \frac{(z_i - w_i) \chi'}{\epsilon^2} a^{\otimes m}, \frac{(z_j - w_j) \chi'}{\epsilon^2} a^{\otimes m} \right)_{h^{\otimes m}} &= \frac{|\chi'|^2}{\epsilon^4} |a^{\otimes m}|_{h^{\otimes m}}^2 \sum_{1 \leq i, j \leq n} \psi_{\epsilon, \delta}^{(i\bar{j})} (z_i - w_i)(\bar{z}_j - \bar{w}_j) \\ &\leq \frac{|\chi'|^2}{\epsilon^4} |a^{\otimes m}|_{h^{\otimes m}}^2 \frac{|z-w|^2 (|z-w|^2 + \epsilon^2)^2}{\epsilon^2}, \end{aligned}$$

we have

$$\begin{aligned} &\int_{\Omega} \sum_{1 \leq i, j \leq n} \left( \psi_{\epsilon, \delta}^{(i\bar{j})} \frac{(z_i - w_i) \chi'}{\epsilon^2} a^{\otimes m}, \frac{(z_j - w_j) \chi'}{\epsilon^2} a^{\otimes m} \right)_{h^{\otimes m}} e^{-\psi_{\epsilon, \delta}} dV_{\omega} \\ &\leq \frac{1}{\epsilon^6} \int_{\Omega} |\chi'|^2 |a^{\otimes m}|_{h^{\otimes m}}^2 |z-w|^2 (|z-w|^2 + \epsilon^2)^2 e^{-\psi_{\epsilon, \delta}} dV_{\omega} \\ &\leq \frac{9}{\epsilon^6} \int_{\{\epsilon^2/2 \leq |z-w|^2 \leq \epsilon^2\}} \frac{|z-w|^2 (|z-w|^2 + \epsilon^2)}{(|z-w|^2 + \delta^2)^n} |a|_{h(z)}^{2m} dV_{\omega} \\ &\leq 9C_n \epsilon^{2n-2} \frac{1}{(\epsilon^2/2 + \delta^2)^n} \sup_{|z-w| \leq \epsilon} |a|_{h(z)}^{2m} \end{aligned}$$

for some positive constant  $C_n$  depending only on  $n$ . Repeating the same argument of the proof of Theorem 2.3, we find a smooth  $E^{\otimes m}$ -valued  $(n, 0)$  form  $u_{\epsilon}$  such that

- $\bar{\partial} u_{\epsilon} = \alpha$ , and
- $\int_{\Omega} |u_{\epsilon}|_{(h^{\otimes m}, \omega)}^2 e^{-\psi_{\epsilon}} dV_{\omega} \leq \frac{9C_n 2^n}{\epsilon^2} \sup_{|z-w| \leq \epsilon} |a|_{h(z)}^{2m}.$

Since  $\log |a|_{h(z)}$  is Hölder continuous,

$$\sup_{|z-w| \leq \epsilon} |a|_{h(z)}^{2m} \leq e^{2mC_{\Omega'} \epsilon^{\alpha}} |a|_{h(w)}^{2m}$$

for constants  $\alpha \in (0, 1]$  and  $C_{\Omega'} > 0$ .

Let  $f_\epsilon := a^{\otimes m} \chi dz - u_\epsilon$ . Then  $f_\epsilon$  is a holomorphic  $(n, 0)$ -form,  $f_\epsilon(w) = a^{\otimes m}$ , and satisfies the following inequality

$$\int_{\Omega} |f_\epsilon|_{(h^{\otimes m}, \omega)}^2 dV_\omega \leq C \left( \epsilon^{2n} + \frac{1}{\epsilon^2} \right) e^{2mC_{\Omega'} \epsilon^\alpha} |a|_{h(w)}^{2m},$$

where  $C > 0$  is a constant depending on  $\Omega'$ . Taking  $\epsilon = 1/(mC_{\Omega'})^{1/\alpha}$ , we have

$$\int_{\Omega} |f_\epsilon|_{(h^{\otimes m}, \omega)}^2 dV_\omega \leq e^2 C \left( \frac{1}{(mC_{\Omega'})^{\frac{2n}{\alpha}}} + (mC_{\Omega'})^{\frac{2}{\alpha}} \right) |a|_{h(w)}^{2m}.$$

Since the above coefficient satisfies the growth condition  $\frac{\log C_m}{m} \rightarrow 0$ , we can conclude that  $(E, h)$  has the multiple  $L^2$ -extension property on  $\Omega'$ . We also see that  $(E, h)$  is Griffiths semi-positive from the results of [DWZZ18, Theorem 6.4].  $\square$

**REMARK 3.6.** Continuity of  $\log |u|_h$  for any non-zero local holomorphic section  $u$  implies that  $h$  is positive definite and finite on  $\Omega$ . In Theorem 2.3,  $\phi$  has a plurisubharmonic extension from  $\Omega \setminus \text{Pole}(\phi)$  to  $\Omega$ . However, “the singular set” of singular Hermitian metrics on vector bundles is much more complicated. In the case that  $h$  has general singularity, we are not sure that Theorem 3.5 holds.

**3.2. New positivity notions for vector bundles.** In this subsection, we introduce various positivity notions for vector bundles and compare them. Throughout this section, we let  $(X, \omega)$  be a smooth Hermitian manifold,  $E \rightarrow X$  be a holomorphic vector bundle of rank  $r$  over  $X$ , and  $h$  be a singular Hermitian metric on  $E$ .

Firstly, we define a positively curved singular Hermitian metric in the sense of twisted Hörmander.

**DEFINITION 3.7.** We say that  $(E, h)$  is *positively curved in the sense of twisted Hörmander* if for any point  $x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $(E, h)$  satisfies the *twisted Hörmander condition* (cf. Definition 3.3) on  $U$ .

We also define a positively curved metric in the sense of multiple  $L^2$ -extension.

**DEFINITION 3.8.** We say that  $(E, h)$  is *positively curved in the sense of multiple  $L^2$ -extension* if for any point  $x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $(E, h)$  has the *multiple  $L^2$ -extension property* (cf. Definition 3.2) on  $U$ .

Theorem 3.5 implies the following theorem.

**THEOREM 3.9.** *Let  $(E, h)$  be positively curved in the sense of twisted Hörmander. Assume that  $|u|_{h^*}$  is upper semi-continuous for any local holomorphic section  $u \in \mathcal{O}(E^*)$ . If  $\log |u|_h$  is locally Hölder continuous for any non-zero local holomorphic section  $u$ ,  $(E, h)$  is positively curved in the sense of multiple  $L^2$ -extension.*

In [DWZZ18], the authors proved that a vector bundle which has the multiple  $L^2$ -extension property is positively curved in the sense of Griffiths. Moreover, they proposed the next question.

QUESTION 3.10. Is the multiple  $L^2$ -extension property stronger than Griffiths positivity? Is it more or less equivalent to Nakano positivity?

In this subsection, we give a partial answer to Question 3.10. To be precise, we show the following theorem.

THEOREM 3.11. (= Theorem 1.5) *There is a positively curved vector bundle  $(E, h)$  in the sense of multiple  $L^2$ -extension such that  $(E, h)$  is not Nakano semi-positive.*

To prove this theorem, we prepare the following essential lemma.

LEMMA 3.12. *Let  $(E, h)$  be positively curved in the sense of multiple  $L^2$ -extension, and  $(Q, h_Q)$  be the quotient bundle and quotient metric of  $(E, h)$ . Then,  $(Q, h_Q)$  is also positively curved in the sense of multiple  $L^2$ -extension.*

PROOF. Let  $\beta : E \rightarrow Q$  be the quotient map. For any point  $x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $(E, h)$  has the multiple  $L^2$ -extension property on  $U$ . For any  $z \in U$  and a nonzero element  $a \in Q_z$  with finite norm  $|a|_{h_Q} < +\infty$ , there is a nonzero element  $b \in E_z$  such that  $\beta(b) = a$  and  $|a|_{h_Q} = |b|_h$ . Since  $(E, h)$  has the multiple  $L^2$ -extension property on  $U$ , for any  $m \in \mathbb{N}$ , there is a holomorphic section  $f_m \in H^0(U, E^{\otimes m})$  such that  $f_m(z) = b^{\otimes m}$  and

$$\int_U |f_m|_{h^{\otimes m}}^2 dV_\omega \leq C_m |b^{\otimes m}|_{h^{\otimes m}}^2,$$

where  $C_m$  are constants satisfying the growth condition  $\lim_{m \rightarrow \infty} \frac{1}{m} \log C_m = 0$ . Therefore we have

$$\begin{aligned} \int_U |\beta^{\otimes m} \circ f_m|_{h_Q^{\otimes m}}^2 dV_\omega &\leq \int_U |f_m|_{h^{\otimes m}}^2 dV_\omega \\ &\leq C_m |b|_h^{2m} \\ &= C_m |a|_{h_Q}^{2m}. \end{aligned}$$

Consequently, we can conclude that  $(Q, h_Q)$  has the multiple  $L^2$ -extension property on  $U$ . Hence,  $(Q, h_Q)$  is positively curved in the sense of multiple  $L^2$ -extension.  $\square$

Here we give the following example.

EXAMPLE 3.13. Let  $Q$  be the vector bundle of rank  $n$  over  $\mathbb{P}^n$  defined by

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \underline{\mathbb{C}}^{n+1} \rightarrow Q \rightarrow 0,$$

where  $\underline{\mathbb{C}}^{n+1}$  is the trivial vector bundle of rank  $n+1$  over  $\mathbb{P}^n$  and  $\mathcal{O}(-1)$  be the tautological line bundle. It is known that  $Q$  is not Nakano semi-positive (cf. [Dem-book, Chapter VII, Example 6.8]).

Lemma 3.12 and Example 3.13 imply Theorem 3.11.

PROOF OF THEOREM 3.11. We consider the vector bundles  $\underline{\mathbb{C}}^{n+1}$  and  $Q$  over  $\mathbb{P}^n$  in Example 3.13. Let  $h_0$  be the standard Euclidean metric on  $\underline{\mathbb{C}}^{n+1}$  and  $h_q$  be the quotient metric of  $h_0$  on  $Q$ . Since  $(\underline{\mathbb{C}}^{n+1}, h_0)$  is positively curved in the sense of multiple  $L^2$ -extension,  $(Q, h_q)$  is also positively curved in the sense of multiple  $L^2$ -extension. However,  $(Q, h_q)$  is not Nakano semi-positive. Hence,  $(Q, h_q)$  satisfies the conclusion of Theorem 3.11.  $\square$

Now we consider the coherence of higher rank analogue of multiplier ideal sheaves. For a line bundle with a singular Hermitian metric, the sheaf of locally square integrable holomorphic sections is called the multiplier ideal sheaf. It is known that multiplier ideal sheaves are coherent for positively curved singular Hermitian metrics. For vector bundles, the coherence of these sheaves is not known in general due to the lack of results like  $L^2$ -estimates. Here we prove that the twisted Hörmander condition implies the coherence of the sheaf of square integrable holomorphic sections.

THEOREM 3.14. (= Theorem 1.4) *Let  $(E, h)$  be positively curved in the sense of twisted Hörmander. Assume that  $|u|_{h^*}$  is upper semi-continuous for any local holomorphic section  $u \in \mathcal{O}(E^*)$ . Then  $\mathcal{E}(h)$  is a coherent subsheaf of  $\mathcal{O}(E)$ , where  $\mathcal{E}(h)$  is the sheaf of germs of locally square integrable holomorphic sections of  $E$  with respect to  $h$ .*

PROOF. The following proof is based on the proof of [Dem, Proposition 5.7].

For any point  $x \in X$ , there exists an open neighborhood  $\Omega$  of  $x$  such that  $(E, h)$  satisfies the twisted Hörmander condition on  $\Omega$ . Since coherence is a local property, we can assume that  $\Omega$  is a bounded domain in  $\mathbb{C}^n$ ,  $E = \Omega \times \underline{\mathbb{C}}^r$  is the trivial bundle over  $\Omega$ , and each element of  $h^*$  is bounded on  $\Omega$ . Let  $H_{(2,h)}^0(\Omega, \underline{\mathbb{C}}^r)$  be the square integrable  $\mathbb{C}^r$ -valued holomorphic functions with respect to  $h$  on  $\Omega$ . By the strong Noetherian property of coherent sheaves,  $H_{(2,h)}^0(\Omega, \underline{\mathbb{C}}^r)$  generates a coherent ideal sheaf  $\mathcal{F} \subset \mathcal{O}(E) = \mathcal{O}(\underline{\mathbb{C}}^r)$ . First of all, we will show that

$$\mathcal{F}_x + \mathcal{E}(h)_x \cap \mathfrak{m}_x^{k+1} \cdot \mathcal{O}(\underline{\mathbb{C}}^r)_x = \mathcal{E}(h)_x,$$

where  $\mathfrak{m}_x$  is a maximal ideal of  $\mathcal{O}_{\Omega,x}$  and  $k$  is any positive integer. It is enough to show that

$$(3.2) \quad \mathcal{E}(h)_x \subset \mathcal{F}_x + \mathcal{E}(h)_x \cap \mathfrak{m}_x^{k+1} \cdot \mathcal{O}(\underline{\mathbb{C}}^r)_x.$$

We take any element  $f = {}^t(f_{1,x}, \dots, f_{r,x}) \in \mathcal{E}(h)_x$ , where  $f_i$  is a holomorphic function defined in a neighborhood  $U$  of  $x$ . Let  $\theta$  be a cut-off function with support in  $U$  such that  $\theta = 1$  in a neighborhood of  $x$ . We define a  $\bar{\partial}$ -closed  $\mathbb{C}^r$ -valued  $(n, 1)$ -form  $\alpha$  as

$$\alpha = \bar{\partial}(\theta f dz).$$

We also take a weight function

$$\psi_\delta(z) = (n+k) \log(|z-x|^2 + \delta^2) + |z|^2.$$

Solving a  $\bar{\partial}$ -equation, we get a smooth  $\mathbb{C}^r$ -valued  $(n, 0)$ -form  $u_\delta$  such that

- $\bar{\partial}u_\delta = \alpha$ , and
- $\int_\Omega |u_\delta|_{(h,\omega)}^2 e^{-\psi_\delta} dV_\omega \leq \int_\Omega \sum_{1 \leq i,j \leq n} (\psi_\delta^{(i\bar{j})} \alpha_i, \alpha_j)_h e^{-\psi_\delta} dV_\omega$ .

The right-hand side of the above inequality has an upper bound independent of  $\delta$ . Taking limits  $\delta \rightarrow 0$  and repeating the argument of the proof of Theorem 3.5, we obtain a smooth  $\mathbb{C}^r$ -valued  $(n, 0)$ -form  $udz$  such that

- $\bar{\partial}(udz) = \alpha$ , and
- $\int_\Omega \frac{|u|_h^2}{|z-x|^{2(n+k)}} dV_\omega < +\infty$ .

Since each element of  $h^*$  is bounded, there exists a positive constant  $C$  such that

$$|g|_h^2 \geq C|g|^2 = C(|g_1|^2 + \cdots + |g_r|^2)$$

for any  $\mathbb{C}^r$ -valued smooth function  $g$ . Hence we get

$$\int_\Omega \frac{|u_i|^2}{|z-x|^{2(n+k)}} dV_\omega < +\infty.$$

Letting  $F := \theta f - u$ , we obtain  $F \in H_{(2,h)}^0(\Omega, \mathbb{C}^r)$  and  $f_x - F_x = u_x \in \mathcal{E}(h)_x \cap \mathfrak{m}^{k+1} \cdot \mathcal{O}(\mathbb{C}^r)$ . This proves (3.2).

Finally, we will prove  $\mathcal{F}_x = \mathcal{E}(h)_x$ . The Artin-Rees lemma implies that there exists an integer  $l \geq 1$  such that

$$\mathfrak{m}_x^{k+1} \cdot \mathcal{O}(\mathbb{C}^r)_x \cap \mathcal{E}(h)_x = \mathfrak{m}_x^{k-l+1} (\mathfrak{m}_x^l \cdot \mathcal{O}(\mathbb{C}^r)_x \cap \mathcal{E}(h)_x)$$

holds for any  $k \geq l-1$ . Therefore, for  $k > l-1$ , we have

$$\begin{aligned} \mathcal{E}(h)_x &= \mathcal{F}_x + \mathcal{E}(h)_x \cap \mathfrak{m}_x^{k+1} \cdot \mathcal{O}(\mathbb{C}^r)_x \\ &= \mathcal{F}_x + \mathfrak{m}_x^{k-l+1} (\mathfrak{m}_x^l \cdot \mathcal{O}(\mathbb{C}^r)_x \cap \mathcal{E}(h)_x) \\ &\subset \mathcal{F}_x + \mathfrak{m}_x \cdot \mathcal{E}(h)_x \\ &\subset \mathcal{E}(h)_x. \end{aligned}$$

By Nakayama's lemma, we obtain  $\mathcal{F}_x = \mathcal{E}(h)_x$ . Thus we can conclude that  $\mathcal{E}(h)$  is coherent.  $\square$

Finally, we give a simple example of a singular Hermitian metric satisfying the twisted Hörmander condition.

**THEOREM 3.15.** *Let  $h$  be a Griffiths semi-positive singular Hermitian metric. Then,  $(E \otimes \det E, h \otimes \det h)$  is positively curved in the sense of twisted Hörmander.*

**PROOF.** We set  $G := E \otimes \det E$  and  $g := h \otimes \det h$  for simplicity. For any point  $x \in X$ , we take an open Stein neighborhood  $U$  of  $x$  such that  $E$  is trivial over  $U$ . It is enough to show that  $(G, g)$  satisfies the twisted Hörmander condition on  $U$ . Let  $m$  be a positive integer,  $\psi$  be a smooth strictly plurisubharmonic function on  $U$ , and  $\alpha = \sum_j \alpha_j dz \wedge d\bar{z}_j$  be a smooth  $\bar{\partial}$ -closed  $G^{\otimes m}$ -valued  $(n, 1)$ -form with compact support and finite norm  $\int_U \sum_{1 \leq i,j \leq n} (\psi^{(i\bar{j})} \alpha_i, \alpha_j)_{g^{\otimes m}} e^{-\psi} dV_\omega < +\infty$ . We take a Stein exhaustion

$\{U_\lambda\}_\lambda$  of  $U$  such that  $U_\lambda \subseteq U_{\lambda+1} \subseteq \dots \subseteq U$  and each  $U_\lambda$  is a Stein domain. It is known that there exists a sequence of smooth Hermitian metrics  $\{h_\nu\}_{\nu=1}^\infty$  with positive Griffiths curvature, increasing pointwise to  $h$  on each  $U_\lambda$  ([BP08, Proposition 3.1]). From the results of Demailly-Skoda ([DS]), we see that  $g_\nu$  is Nakano semi-positive for each  $\nu$ , where  $g_\nu := h_\nu \otimes \det h_\nu$ . Since  $g_\nu^{\otimes m} e^{-\psi}$  is Nakano positive, we obtain a smooth  $G^{\otimes m}$ -valued  $(n, 0)$ -form  $u_{\lambda, \nu}$  such that  $\bar{\partial} u_{\lambda, \nu} = \alpha$  and

$$\begin{aligned} \int_{U_\lambda} |u_{\lambda, \nu}|_{(g_\nu^{\otimes m}, \omega)}^2 e^{-\psi} dV_\omega &\leq \int_{U_\lambda} ([\sqrt{-1} \Theta_{g_\nu^{\otimes m} e^{-\psi}}, \Lambda_\omega]^{-1} \alpha, \alpha)_{(g_\nu^{\otimes m}, \omega)} e^{-\psi} dV_\omega \\ &\leq \int_{U_\lambda} \sum_{1 \leq i, j \leq n} (\psi^{(i\bar{j})} \alpha_i, \alpha_j)_{g_\nu^{\otimes m}} e^{-\psi} dV_\omega \\ &\leq \int_U \sum_{1 \leq i, j \leq n} (\psi^{(i\bar{j})} \alpha_i, \alpha_j)_{g^{\otimes m}} e^{-\psi} dV_\omega \end{aligned}$$

by using Hörmander's  $L^2$ -estimate. Using a diagonal argument and taking weak limits  $\lambda \rightarrow \infty, \nu \rightarrow \infty$ , we get a smooth solution  $u$  such that

- $\bar{\partial} u = \alpha$ , and
- $\int_U |u|_{(g^{\otimes m}, \omega)}^2 e^{-\psi} dV_\omega \leq \int_U \sum_{1 \leq i, j \leq n} (\psi^{(i\bar{j})} \alpha_i, \alpha_j)_{g^{\otimes m}} e^{-\psi} dV_\omega$ .

Therefore, we can conclude that  $(G, g)$  satisfies the twisted Hörmander condition on  $U$ .  $\square$

Inspired by Theorem 3.14 and Theorem 3.15, we propose the following question in relation to Question 3.10.

**QUESTION 3.16.** Is positivity in the sense of twisted Hörmander is more or less equivalent to Nakano positivity?

## Part 2. Pseudonorms on direct images of pluricanonical bundles

**ABSTRACT.** We study pseudonorms on pluricanonical bundles over Stein manifolds. We prove that the pseudonorms determine holomorphic structures of Stein manifolds under certain assumptions. This theorem is a generalization of the result obtained by Deng, Wang, Zhang, and Zhou for bounded domains in  $\mathbb{C}^n$ . We also investigate Stein morphisms and the pseudonorms on direct images of pluricanonical bundles. Our main goal in this paper is to show that the pseudonorms also determine holomorphic structures of Stein morphisms. One important technique is an  $L^{2/m}$ -variant of the Ohsawa-Takegoshi extension theorem.

### 4. INTRODUCTION

Part 2 corresponds to the paper [Ina3]. The space of holomorphic functions with finite  $L^p$ -norm  $A^p(\Omega)$  on a bounded domain  $\Omega$ , or the  $m$ -pluricanonical space  $H^0(X, mK_X)$  for a complex manifold  $X$  plays an important role in understanding its geometric property. Initially, Royden proved that if  $H^0(C, 2K_C)$  is isomorphic to  $H^0(C', 2K_{C'})$  with respect to the canonical norm for compact Riemann surfaces  $C, C'$  of genus  $g \geq 2$ , then  $C$  is isomorphic to  $C'$  [Roy71]. There are many other generalized results obtained by Markovic [Mar03] for more general classes of Riemann surfaces, by Chi and Yau for projective manifolds of general type [Chi16], [CY08], and by Deng, Wang, Zhang, and Zhou for bounded hyperconvex domains in  $\mathbb{C}^n$  [DWZZ18]. In any case, the pseudonorm, called  $L^p$ -norm or  $L^{2/m}$ -norm, plays an essential role. These kinds of programs are often called Yau's pseudonorm projects (cf. [CY08]).

In this paper, we study a relatively compact hyperconvex domain in a Stein manifold  $X \Subset \tilde{X}$ . We prove that the space of pluricanonical forms with pseudonorms determines a holomorphic structure of the base space. To be precise, we obtain the following theorem.

**THEOREM 4.1.** *Let  $\tilde{X}$  be an  $n$ -dimensional Stein manifold, and  $\tilde{Y}$  be an  $l$ -dimensional Stein manifold. We also let  $X \Subset \tilde{X}$  and  $Y \Subset \tilde{Y}$  be relatively compact hyperconvex domains. Assume that there exist  $m \geq 2$  and a linear isometry*

$$T : A(X, mK_X) \longrightarrow A(Y, mK_Y)$$

*such that*

$$\int_X |u \wedge \bar{u}|^{1/m} = \int_Y |Tu \wedge \overline{Tu}|^{1/m}$$

*for all  $u \in A(X, mK_X)$ . Here we define the space  $A(X, mK_X)$  as*

$$A(X, mK_X) := \{u \in H^0(X, mK_X) \mid \int_X |u \wedge \bar{u}|^{1/m} < +\infty\},$$

*and*

$$|u \wedge \bar{u}|^{1/m} := (\sqrt{-1}^{mn^2} u \wedge \bar{u})^{1/m}$$

*for any  $u \in H^0(X, mK_X)$  (see Definition 5.1).*

Then we have that  $n = l$ , and there exists a unique biholomorphic map

$$F : X \longrightarrow Y$$

satisfying the following equation

$$|u(z) \wedge \overline{u(z)}|^{1/m} = |F^*(Tu)(z) \wedge \overline{F^*(Tu)(z)}|^{1/m}$$

for  $z \in X$  and  $u \in A(X, mK_X)$ .

We call a relatively compact domain  $D$  in a Stein manifold  $\tilde{X}$  hyperconvex if there exists a negative plurisubharmonic function  $\varphi$  on  $D$  such that the set  $\{\varphi < c\}$  is relatively compact in  $D$  for every  $c < 0$ . We can easily see that a hyperconvex domain in  $\mathbb{C}^n$  is pseudoconvex. We do not have a complete converse. However, many pseudoconvex domains become hyperconvex. For instance, it was proved that a pseudoconvex domain with Lipschitz boundary is hyperconvex [Dem87].

Theorem 4.1 is a natural generalization of the theorem obtained by Deng, Wang, Zhang, and Zhou [DWZZ18] for bounded domains in  $\mathbb{C}^n$ . For any point in Stein manifold, there exist global holomorphic functions that define a coordinate around this point. By using this property, we can apply local results obtained by them to Stein cases.

We also investigate Stein manifolds fibered over a complex manifold  $T$  with  $\dim T = r$ . Here we recall a notion of a Stein morphism. Let  $\tilde{X}$  be an  $r + n$ -dimensional complex manifold and  $f : \tilde{X} \rightarrow T$  be a holomorphic map. In this paper, we say that  $f$  is a *Stein morphism* if  $f$  is a surjective submersive map with connected fibers, and for any point  $t \in T$ , there exists an open neighborhood  $U \subset T$  of  $t$  such that  $f^{-1}(U)$  is a Stein manifold. Then we introduce a notion of a relatively compact Stein morphism.

**DEFINITION 4.2.** Let  $\tilde{f} : \tilde{X} \rightarrow T$  be a Stein morphism over  $T$ , and  $X \subset \tilde{X}$  be an open submanifold in  $\tilde{X}$ . We call  $f := \tilde{f}|_X : X \rightarrow T$  a *relatively compact Stein morphism* of  $\tilde{X}$  if the following conditions are satisfied:

- (i) The map  $f$  is a Stein morphism in the above sense.
- (ii) For each  $t \in T$ ,  $X_t := f^{-1}(t)$  is a relatively compact domain in  $\tilde{X}_t := \tilde{f}^{-1}(t)$ .

In this setting, we prove the following theorem.

**THEOREM 4.3.** Let  $\tilde{f} : \tilde{X} \rightarrow B$  and  $\tilde{g} : \tilde{Y} \rightarrow B$  be Stein morphisms over the open unit ball  $B \subset \mathbb{C}^r$  with  $\dim \tilde{X} = r + n$  and  $\dim \tilde{Y} = r + l$ . Suppose that  $f = (f_1, \dots, f_r) : X \rightarrow B$  and  $g = (g_1, \dots, g_r) : Y \rightarrow B$  are relatively compact Stein morphisms of  $\tilde{f} : \tilde{X} \rightarrow B$  and  $\tilde{g} : \tilde{Y} \rightarrow B$ , respectively. In this local setting, we also let  $\tilde{X}, X, \tilde{Y}$ , and  $Y$  be Stein. Assume that there exists  $m \geq 2$  such that

- (i)  $X_t := f^{-1}(t)$  and  $Y_t := g^{-1}(t)$  are hyperconvex, and
- (ii) there exists a linear isomorphism

$$T : A(X, mK_X) \longrightarrow A(Y, mK_Y)$$



such that

$$(4.1) \quad \int_{X_t} |U_t \wedge \overline{U}_t|^{1/m} = \int_{Y_t} |(TU)_t \wedge \overline{(TU)_t}|^{1/m}$$

for any  $U \in A(X, mK_X)$  and  $t \in B$ , where  $U_t \in H^0(X_t, mK_{X_t})$  and  $(TU)_t \in H^0(Y_t, mK_{Y_t})$  are uniquely determined  $m$ -canonical forms such that

$$U|_{X_t} = U_t \wedge (df_1 \wedge \cdots \wedge df_r)^{\otimes m}, \quad TU|_{Y_t} = (TU)_t \wedge (dg_1 \wedge \cdots \wedge dg_r)^{\otimes m}.$$

The equation (4.1) includes the case  $+\infty = +\infty$ .

Then we have that  $T$  induces linear isometries  $T_t : A(X_t, mK_{X_t}) \rightarrow A(Y_t, mK_{Y_t})$ ,  $n = l$ , and there exists a unique biholomorphic map  $F : X \rightarrow Y$  such that  $f = g \circ F$  and the following equation is satisfied

$$|u(z) \wedge \overline{u(z)}|^{1/m} = |F_t^*(T_t u)(z) \wedge \overline{F_t^*(T_t u)(z)}|^{1/m}$$

for  $t \in B$ ,  $z \in X_t$ , and  $u \in A(X_t, mK_{X_t})$ , where  $F_t := F|_{X_t} : X_t \rightarrow Y_t$  is a well-defined biholomorphic map.

The fiberwise uniqueness of  $F$  implies the following theorem in the global setting.

**THEOREM 4.4.** *Let  $\tilde{X}$  and  $\tilde{Y}$  be complex manifolds with  $\dim \tilde{X} = r+n$  and  $\dim \tilde{Y} = r+l$ , and  $\tilde{f} : \tilde{X} \rightarrow T$  and  $\tilde{g} : \tilde{Y} \rightarrow T$  be Stein morphisms over an  $r$ -dimensional complex manifold  $T$ . Suppose that  $f : X \rightarrow T$  and  $g : Y \rightarrow T$  are relatively compact Stein morphisms of  $\tilde{f}$  and  $\tilde{g}$ , respectively. Assume that there exists  $m \geq 2$  such that*

- (i)  $X_t$  and  $Y_t$  are hyperconvex, and
- (ii) there exists an isomorphism of sheaves

$$T : f_*(mK_X) \rightarrow g_*(mK_Y)$$

which satisfies the following conditions:

For any open set  $D \subset T$ , there is a linear isomorphism

$$T_D : H^0(f^{-1}(D), mK_X) \rightarrow H^0(g^{-1}(D), mK_Y)$$

such that

$$(4.2) \quad \int_{X_t} |U_t \wedge \overline{U}_t|^{1/m} = \int_{Y_t} |(T_D U)_t \wedge \overline{(T_D U)_t}|^{1/m}$$

for any  $U \in H^0(f^{-1}(D), mK_X)$  and  $t \in D$ , where  $U|_{X_t} = U_t \wedge (df_1 \wedge \cdots \wedge df_r)^{\otimes m}$  and  $(T_D U)|_{Y_t} = (T_D U)_t \wedge (dg_1 \wedge \cdots \wedge dg_r)^{\otimes m}$  for some fixed coordinate  $(t_1, \dots, t_r) \subset D$  around  $t$  and  $f_i = t_i \circ f, g_i = t_i \circ g$ . The equation (4.2) includes that case  $+\infty = +\infty$ .

Then we have that  $T$  induces linear isometries  $T_t : A(X_t, mK_{X_t}) \rightarrow A(Y_t, mK_{Y_t})$ ,  $n = l$ , and there exists a unique biholomorphic map  $F : X \rightarrow Y$  such that  $f = g \circ F$  and the following equation is satisfied

$$|u(z) \wedge \overline{u(z)}|^{1/m} = |F_t^*(T_t u)(z) \wedge \overline{F_t^*(T_t u)(z)}|^{1/m}$$

for  $t \in D$ ,  $z \in X_t$ , and  $u \in A(X_t, mK_{X_t})$ , where  $F_t := F|_{X_t} : X_t \rightarrow Y_t$  is a well-defined biholomorphic map.

Theorem 4.3 and 4.4 say that the  $L^{2/m}$ -norms on direct images of pluricanonical bundles determine holomorphic structures of fibrations. The so-called  $m$ -th Narasimhan-Simha Hermitian metric on direct images of relative pluricanonical bundles has been studied by several people (cf. [BP08], [HPS18], [PT18]). The above theorems also demonstrate the importance of the  $m$ -th pseudonorms on them.

For bounded domains in  $\mathbb{C}^n$ , we obtain the following corollary.

**COROLLARY 4.5.** *Let  $X \Subset \mathbb{C}_t^r \times \mathbb{C}_z^n$  and  $Y \Subset \mathbb{C}_t^r \times \mathbb{C}_w^l$  be bounded pseudoconvex domains fibered over the open unit ball  $B \subset \mathbb{C}^r$ . We also let  $f : X \rightarrow \mathbb{C}^r$  and  $g : Y \rightarrow \mathbb{C}^r$  be natural projections such that  $f(t, z) = t$  and  $g(s, w) = s$  with  $f(X) = g(Y) = B$ . Assume that there exists  $0 < p < 2$  such that*

- (i)  $X_t := f^{-1}(t)$  and  $Y_t := g^{-1}(t)$  are hyperconvex domains for each  $t \in B$ ,
- (ii) *there exists a linear isomorphism*

$$T : A^p(X) \longrightarrow A^p(Y)$$

with

$$\int_{X_t} |\Phi|_{X_t}|^p d\mu_n = \int_{Y_t} |(T\Phi)|_{Y_t}|^p d\mu_l$$

for any  $\Phi \in A^p(X)$  and  $t \in B$ , where  $d\mu_n$  and  $d\mu_l$  are the standard Lebesgue measures on  $\mathbb{C}^n$  and  $\mathbb{C}^l$ , respectively. Then we have that  $T$  induces linear isometries  $T_t : A^p(X_t) \rightarrow A^p(Y_t)$ ,  $n = l$ , and there exists a unique biholomorphic map  $F : X \rightarrow Y$  such that  $f = g \circ F$  and the following equation is satisfied

$$|\phi(z)| = |T_t \phi(F_t(z))| |J_{F_t}(z)|^{2/p}$$

for  $t \in B$ ,  $z \in X_t$ , and  $\phi \in A^p(X_t)$ , where  $F_t := F|_{X_t} : X_t \rightarrow Y_t$  is a well-defined biholomorphic map and  $J_{F_t}$  is the holomorphic Jacobian of  $F_t$ .

Theorem 4.3 holds only for  $p = 2/m$ , whereas we can prove Corollary 4.5 for all  $0 < p < 2$  by using the same argument of the proof of Theorem 4.3. Corollary 4.5 is a relative version of Theorem 1.2 in [DWZZ18]. In Theorem 4.3, 4.4, and Corollary 4.5, we do not assume the hyperconvexity of  $X$ , whereas we can construct the biholomorphic map between total spaces. This is an important and new point. A key proposition to prove the main theorems is an  $L^{2/m}$ -variant of the Ohsawa-Takegoshi extension theorem.

On the other hand, we can prove Corollary 4.5 without using the  $L^{2/m}$ -variant of the Ohsawa-Takegoshi extension theorem. We will show the proof in Appendix 8.

The organization of this paper is as follows. In Section 5, we introduce some definitions and properties of  $L^{2/m}$ -norms and  $m$ -th Bergman kernels. In Section 6, we give a proof of Theorem 4.1. In Section 7, we prove Theorem 4.3, 4.4, and Corollary 4.5. At last, in Appendix 8, we show a simple proof of Corollary 4.5.

## 5. PRELIMINARIES

**5.1.  $L^{2/m}$ -norm.** We prepare some basic definitions and properties to show the main theorem. Throughout this section, we denote by  $X$  an  $n$ -dimensional complex manifold, and by  $K_X$  the canonical line bundle over  $X$ .

Firstly, we confirm the following notation.

**DEFINITION 5.1** ( $L^{2/m}$ -norm). We take a local coordinate  $\{U, (z_1, \dots, z_n)\}$  on  $X$ . For a holomorphic section  $u \in H^0(X, mK_X)$ , we locally define  $(\sqrt{-1}^{mn^2} u \wedge \bar{u})^{1/m}$  as

$$(\sqrt{-1}^{mn^2} u \wedge \bar{u})^{1/m} = |f_U|^{2/m} \sqrt{-1}^{n^2} (dz_1 \wedge \dots \wedge dz_n) \wedge (d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n),$$

where  $u = f_U(z_1, \dots, z_n)(dz_1 \wedge \dots \wedge dz_n)^{\otimes m}$  on  $U$ . We can verify that  $(\sqrt{-1}^{mn^2} u \wedge \bar{u})^{1/m}$  gives a globally defined real non-negative  $(n, n)$ -form. For simplicity, we define  $|u \wedge \bar{u}|^{1/m} := (\sqrt{-1}^{mn^2} u \wedge \bar{u})^{1/m}$ . Then, we can define the  $L^{2/m}$ -norm  $\|\cdot\|_{X,m}$  of  $u$  as

$$\|u\|_{X,m} = \int_X |u \wedge \bar{u}|^{1/m}.$$

For  $m > 2$ ,  $\|\cdot\|_{X,m}$  become only pseudonorms, i.e. they satisfy the norm axioms except the homogeneity. However, we call them  $L^{2/m}$ -norms for all  $m \in \mathbb{N}$ .

We remark that the space  $A(X, mK_X)$  might be  $\{0\}$  even though  $X$  is a Stein manifold. Hence, in this paper, we mainly consider the space  $A(X, mK_X)$  over a relatively compact Stein domain in Stein manifold. If  $X$  is a relatively compact Stein domain in some Stein manifold  $\tilde{X}$ ,  $A(X, mK_X)$  is an infinite-dimensional vector space and has the separation property since  $H(\tilde{X}, mK_{\tilde{X}})|_X \subset A(X, mK_X)$  and  $\tilde{X}$  is Stein. Here the separation property means that for any points  $x \neq y \in X$ , there exist sections  $\sigma_1, \sigma_2 \in A(X, mK_X)$  such that  $\sigma_1(x) = 0, \sigma_1(y) \neq 0, \sigma_2(x) \neq 0, \sigma_2(y) = 0$ . We also know that  $A(X, mK_X)$  are complete separable metric space with respect to the metric  $d(u_1, u_2) := \|u_1 - u_2\|_{X,m}$  for  $m \geq 2$ .

A fundamental lemma to prove the main theorems is the following result about isometries between  $L^p$ -spaces.

**THEOREM 5.2.** ([Rud76]) *Let  $\mu$  and  $\nu$  be finite positive measures on two sets  $U$  and  $V$ . We also let  $p \in \mathbb{R}_{>0}$  be not even, and  $N$  be a positive integer. If  $\{f_i\}_{1 \leq i \leq N} \subset L^p(U, \mu), \{g_i\}_{1 \leq i \leq N} \in L^p(V, \nu)$  satisfy*

$$\int_U |1 + \sum_{1 \leq i \leq N} \alpha_i f_i|^p d\mu = \int_V |1 + \sum_{1 \leq i \leq N} \alpha_i g_i|^p d\nu$$

*for all  $(\alpha_1, \dots, \alpha_N) \in \mathbb{C}^N$ , then  $(f_1, \dots, f_N)$  and  $(g_1, \dots, g_N)$  are called equimeasurable, i.e. for every real-valued non-negative Borel function  $u : \mathbb{C}^N \rightarrow \mathbb{R}_{\geq 0}$ , we get*

$$\int_U u(f_1, \dots, f_N) d\mu = \int_V u(g_1, \dots, g_N) d\nu.$$

*Moreover, let  $I : X \rightarrow \mathbb{C}^N$  and  $J : Y \rightarrow \mathbb{C}^N$  be the maps  $I = (f_1, \dots, f_N)$  and  $J = (g_1, \dots, g_N)$ . Then we obtain*

$$\mu(I^{-1}(E)) = \nu(J^{-1}(E))$$

for every Borel set  $E \subset \mathbb{C}^N$ .

This theorem implies the next lemma, which is necessary for us to prove the main theorems. This is a version of Lemma 2.1 in [Mar03], and of Lemma 2.2 in [DWZZ18].

LEMMA 5.3. *Let  $X$  and  $Y$  be two relatively compact domains in an  $n$ -dimensional Stein manifold  $\tilde{X}$  and an  $l$ -dimensional Stein manifold  $\tilde{Y}$ , respectively. Suppose that  $\{u_k\}_{k=0}^N \subset A(X, mK_X)$ ,  $\{v_k\}_{k=0}^N \subset A(Y, mK_Y)$  are pluricanonical sections such that for every  $N$ -tuple of complex numbers  $\{\alpha_k\}_{k=1}^N$ , we have*

$$\begin{aligned} & \int_X |(u_0 + \sum_{1 \leq k \leq N} \alpha_k u_k) \wedge \overline{(u_0 + \sum_{1 \leq k \leq N} \alpha_k u_k)}|^{1/m} \\ &= \int_Y |(v_0 + \sum_{1 \leq k \leq N} \alpha_k v_k) \wedge \overline{(v_0 + \sum_{1 \leq k \leq N} \alpha_k v_k)}|^{1/m}. \end{aligned}$$

If neither  $u_0$  nor  $v_0$  is constantly zero, then for every real-valued non-negative Borel function  $f : \mathbb{C}^N \rightarrow \mathbb{R}_{\geq 0}$ , we have

$$\int_X f\left(\frac{u_1}{u_0}, \dots, \frac{u_N}{u_0}\right) |u_0 \wedge \bar{u}_0|^{1/m} = \int_Y f\left(\frac{v_1}{v_0}, \dots, \frac{v_N}{v_0}\right) |v_0 \wedge \bar{v}_0|^{1/m}.$$

Here we regard  $u_k/u_0$  and  $v_k/v_0$  as a function on  $X$  and  $Y$ , respectively.

PROOF. Set  $d\mu = |u_0 \wedge \bar{u}_0|^{1/m}$  and  $d\nu = |v_0 \wedge \bar{v}_0|^{1/m}$ . The measures  $\mu$  and  $\nu$  are well-defined on  $X$  and  $Y$ , respectively. Then we have that  $u_k/u_0 \in L^{2/m}(X, d\mu)$  and  $v_k/v_0 \in L^{2/m}(Y, d\nu)$ , respectively. From the assumption, we have that

$$\int_X |1 + \sum_{1 \leq k \leq N} \alpha_k \frac{u_k}{u_0}|^{2/m} d\mu = \int_Y |1 + \sum_{1 \leq k \leq N} \alpha_k \frac{v_k}{v_0}|^{2/m} d\nu$$

for every  $N$ -tuple of complex numbers  $\{\alpha_k\}_{k=1}^N$ . Therefore Lemma 5.3 follows from Theorem 5.2.  $\square$

We introduce the following result obtained by Deng, Wang, Zhang, and Zhou. This is a fundamental result and the absolute version of Corollary 4.5.

THEOREM 5.4. ([DWZZ18, Theorem 1.2]) *Let  $\Omega_1 \subset \mathbb{C}^n$  and  $\Omega_2 \subset \mathbb{C}^m$  be bounded hyperconvex domains. If there is a linear isometry  $T : A^p(\Omega_1) \rightarrow A^p(\Omega_2)$  for some  $0 < p < 2$ , then  $n = m$  and there exists a unique biholomorphic map  $F : \Omega_1 \rightarrow \Omega_2$  such that*

$$|T\phi \circ F| |J_F|^{2/p} = |\phi|$$

for all  $\phi \in A^p(\Omega_1)$ , where  $J_F$  is the holomorphic Jacobian of  $F$ .

**5.2.  $m$ -th Bergman kernel.** In this subsection, we let  $\tilde{X}$  denote an  $n$ -dimensional Stein manifold, and  $X$  denote a relatively compact Stein domain in  $\tilde{X}$ . Firstly, we introduce the definition of the  $m$ -th Bergman kernel  $K_{X,m}$  and the exhaustivity of it.

**DEFINITION 5.5** ( $m$ -th Bergman kernel). We set

$$K_{X,m}(z) := \sup\{|u \wedge \bar{u}|^{1/m}(z) \mid u \in A(X, mK_X), \int_X |u \wedge \bar{u}|^{1/m} \leq 1\},$$

for each point  $z \in X$ .

We also say that  $K_{X,m}$  is *exhaustive* if the following function

$$K'_{X,m}(z) := \sup\{|u \wedge \bar{u}|^{1/m}/dV_{\tilde{X}}(z) \mid u \in A(X, mK_X), \int_X |u \wedge \bar{u}|^{1/m} \leq 1\}$$

is exhaustive on  $X$  for a volume form  $dV_{\tilde{X}}$  on  $\tilde{X}$ .

**REMARK 5.6.** The function  $K'_{X,m}$  depends on the choice of volume forms on  $\tilde{X}$ . However, the exhaustivity of  $K_{X,m}$  is independent of them.

Since  $\tilde{X}$  is Stein, we can take a volume form  $dV_{\tilde{X}}$  (= smooth Hermitian metric on  $-K_{\tilde{X}}$ ) with curvature positive on  $X$ . Taking this metric, we have that  $K'_{X,m}$  is a continuous plurisubharmonic function on  $X$  (cf. [DWZZ18, Lemma 6.2 and 6.3], [HPS18, Proposition 28.3], [PT18]). Hence, the exhaustivity of  $K_{X,m}$  implies the pseudoconvexity of  $X$ .

On the other hand, if  $\tilde{X} = \mathbb{C}^n$  and  $X$  is a bounded domain in  $\mathbb{C}^n$ , it is known that the pseudoconvexity of  $\Omega$  implies the exhaustivity of  $K_{\Omega,m}$  for  $m \geq 2$ .

**THEOREM 5.7.** ([NZZ16, Theorem 2.7]) *Let  $\Omega \Subset \mathbb{C}^n$  be any bounded domain. Then  $\Omega$  is pseudoconvex if and only if  $K_{\Omega,p}$  is an exhaustion function for  $p \in (0, 2)$ .*

We also prove that  $K_{X,m}$  is exhaustive when  $\tilde{X}$  is a Stein manifold and  $X \Subset \tilde{X}$  is a relatively compact hyperconvex domain. To prove this theorem, we prepare the following results. The first one is a localization principle. This is obtained by Ohsawa for bounded pseudoconvex domains in  $\mathbb{C}^n$  [Ohs84]. A more general result appears in [Ohs15]. We can prove this principle by using Hörmander's  $L^2$ -estimate.

**LEMMA 5.8.** *In the above setting, we let  $a \in \partial X$  be a boundary point. Then there exists an open neighborhood  $U_0$  of  $a$  such that for any two open neighborhoods  $V \Subset U \subset U_0$  of  $a$ , there is a positive constant  $C$  such that*

$$K_{U \cap X}(x) \leq CK_X(x)$$

for any  $x \in V \cap X$ . Here  $K_{U \cap X}$  and  $K_X$  are Bergman kernels of  $U \cap X$  and  $X$ , respectively, and  $C$  is independent of  $x$ .

**PROOF OF LEMMA 5.8.** We fix a Kähler form  $\tilde{\omega}$  on  $\tilde{X}$ , and set  $\omega := \tilde{\omega}|_X$ . Since  $\tilde{X}$  is a Stein manifold, we can take global holomorphic functions  $(g_1, \dots, g_n) \in \mathcal{O}(\tilde{X})^n$  and an

open neighborhood  $U_0$  of  $a$  such that  $(g_1, \dots, g_n)$  defines a biholomorphic coordinate map on  $U_0$ . We will modify the norms of  $g_i$  such that

$$\sup_X |g_i| \leq \frac{1}{2\sqrt{n}}$$

for  $1 \leq i \leq n$ . We also take a smooth strictly plurisubharmonic function  $\psi$  on  $\tilde{X}$  such that  $\psi < 0$  and

$$\sqrt{-1}\partial\bar{\partial}\psi \geq \omega$$

on  $X$ . Then  $\phi(z) := (n+1)\log(|g_1(z) - g_1(x)|^2 + \dots + |g_n(z) - g_n(x)|^2) + \psi(z)$  satisfies  $\phi < 0$  and

$$\sqrt{-1}\partial\bar{\partial}\phi \geq \omega$$

on  $X$ .

Suppose that  $V$  and  $U$  are open neighborhoods of  $a$  with  $V \Subset U \subset U_0$ . Let  $x \in V \cap X$  be any point. We take a cut-off function  $\chi \in C_c^\infty(U)$  which satisfies  $0 \leq \chi \leq 1$  and  $\chi = 1$  on a neighborhood of  $V$ . We have a holomorphic  $(n, 0)$ -form  $f$  on  $U \cap X$  such that  $|f \wedge \bar{f}|(x) = K_{U \cap X}(x)$  and

$$\int_{U \cap X} |f \wedge \bar{f}| = 1.$$

We define an  $(n, 1)$ -form  $\alpha := \bar{\partial}(\chi f)$  on  $X$ . We get

$$\int_X |\alpha|_\omega^2 e^{-\phi} dV_\omega < +\infty.$$

Thanks to Hörmander's  $L^2$ -estimate (cf. [Dem-book]), we can obtain a solution  $u$  satisfying  $\bar{\partial}u = \alpha$  and

$$\int_X |u \wedge \bar{u}| e^{-\phi} \leq \int_X |\alpha|_\omega^2 e^{-\phi} dV_\omega.$$

Let  $\beta := \chi f - u$ . Then  $\beta$  is a holomorphic  $(n, 0)$ -form on  $X$ ,  $|\beta \wedge \bar{\beta}|(x) = |f \wedge \bar{f}|(x)$ , and

$$\begin{aligned} \left( \int_X |\beta \wedge \bar{\beta}| \right)^{1/2} &\leq 1 + \left( \int_X |u \wedge \bar{u}|^2 e^{-\phi} \right)^{1/2} \\ &\leq 1 + \left( \int_{\text{supp}|\bar{\partial}\chi|} |\alpha|_\omega^2 e^{-\phi} dV_\omega \right)^{1/2} \\ &< C. \end{aligned}$$

For the reason that

$$\inf_{z \in \text{supp}|\bar{\partial}\chi|} \inf_{x \in V \cap X} (|g_1(z) - g_1(x)|^2 + \dots + |g_n(z) - g_n(x)|^2) > 0,$$

the above constant  $C$  is independent of  $x$ . Therefore, we have

$$K_{U \cap X}(x) \leq \frac{C^2 |\beta \wedge \bar{\beta}|(x)}{\int_X |\beta \wedge \bar{\beta}|},$$

which completes the proof.  $\square$

The second one is the exhaustivity of the Bergman kernel of bounded hyperconvex domains. This result was obtained by Ohsawa.

**THEOREM 5.9.** ([Ohs93]) *Let  $D$  be a bounded hyperconvex domain in  $\mathbb{C}^n$ . Then  $\lim_{z \rightarrow \partial D} K_D(z) = +\infty$ .*

Combining the above results, we can prove the following theorem.

**THEOREM 5.10.** *In the above setting, we have that  $\lim_{z \rightarrow \partial X} K_{X,m}(z) = +\infty$ , that is, the  $m$ -th Bergman kernel is exhaustive.*

**PROOF OF THEOREM 5.10.** For any point  $a \in \partial X$ , we take an open neighborhood  $U_0$  and two open hyperconvex neighborhoods  $V \Subset U \subset U_0$  of  $a$  which satisfy the condition of Lemma 5.8. We have  $K_{U \cap X}(z) \leq CK_X(z)$  for  $z \in V \cap X$  and some positive constant  $C > 0$ . Then Theorem 5.9 implies that  $\lim_{z \rightarrow a} K_X(z) = +\infty$ . We also obtain  $K_X(z) \leq K_{X,m}(z)$  since for any holomorphic  $(n, 0)$ -form  $u \in A(X, K_X)$ ,  $u^{\otimes m} \in A(X, mK_X)$  and

$$|u^{\otimes m} \wedge \bar{u}^{\otimes m}|^{1/m} = |u \wedge \bar{u}|.$$

Hence, we obtain the conclusion. □

## 6. PSEUDONORMS ON PLURICANONICAL BUNDLES

In this section, we prove Theorem 4.1. The main argument of the proof of this theorem is almost the same as the proof of Theorem 1.2 in [DWZZ18]. Before proving Theorem 4.1, we provide basic settings and some lemmas.

We set  $H_{X,z} := \{u \in A(X, mK_X) \mid u(z) = 0\}$  be a hyperplane in  $A(X, mK_X)$ . First, we define a subset  $X'$  of  $X$  as follows. We say that  $z \in X'$  if and only if there exists  $w \in Y$  such that  $T(H_{X,z}) = H_{Y,w}$ . The separation property of  $A(X, mK_X)$  implies that there exists a unique  $w \in Y$  such that the equation  $T(H_{X,z}) = H_{Y,w}$  holds. Therefore, we can define a map  $F : X' \rightarrow Y$  by setting  $F(z) = w$  if  $T(H_{X,z}) = H_{Y,w}$ . Set  $Y' := F(X')$ . Then  $F$  is a bijection from  $X'$  to  $Y'$ . Here we take a countable dense subset  $\{u_0, u_1, \dots\}$  of  $A(X, mK_X)$  with  $u_0 \neq 0$ . Then  $\{v_0 := Tu_0, v_1 := Tv_1, \dots\}$  is a countable dense subset of  $A(Y, mK_Y)$ . We define maps  $I_N, J_N, I_\infty$ , and  $J_\infty$  as follows.

$$\begin{array}{ccc} I_N : & X & \longrightarrow \mathbb{C}^N \\ & \Psi & \Psi \\ & z & \longmapsto \left( \frac{u_1(z)}{u_0(z)}, \dots, \frac{u_N(z)}{u_0(z)} \right) \end{array}$$

$$\begin{array}{ccc} J_N : & Y & \longrightarrow \mathbb{C}^N \\ & \Psi & \Psi \\ & w & \longmapsto \left( \frac{v_1(w)}{v_0(w)}, \dots, \frac{v_N(w)}{v_0(w)} \right) \end{array}$$

$$\begin{aligned}
I_\infty : X &\longrightarrow \mathbb{C}^\infty \\
\Downarrow &\qquad \qquad \Downarrow \\
z &\longmapsto \left( \frac{u_1(z)}{u_0(z)}, \frac{u_2(z)}{u_0(z)}, \dots \right) \\
\\
J_\infty : Y &\longrightarrow \mathbb{C}^\infty \\
\Downarrow &\qquad \qquad \Downarrow \\
w &\longmapsto \left( \frac{v_1(w)}{v_0(w)}, \frac{v_2(w)}{v_0(w)}, \dots \right).
\end{aligned}$$

The maps are well-defined on  $X \setminus u_0^{-1}(0)$  and  $Y \setminus v_0^{-1}(0)$ , respectively. We also obtain that  $X' \setminus u_0^{-1}(0) = \cap_N I_N^{-1} J_N(Y \setminus v_0^{-1}(0)) = I_\infty^{-1} J_\infty(Y \setminus v_0^{-1}(0))$  and  $Y' \setminus v_0^{-1}(0) = \cap_N J_N^{-1} I_N(X \setminus u_0^{-1}(0)) = J_\infty^{-1} I_\infty(X \setminus u_0^{-1}(0))$ . The separation property of  $A(X, mK_X)$  and  $A(Y, mK_Y)$  imply that  $I_\infty$  and  $J_\infty$  are injective. For  $z \in I_\infty^{-1} J_\infty(Y \setminus v_0^{-1}(0))$ , we get  $F(z) = J_\infty^{-1}(I_\infty(z))$ .

Lemma 5.3 implies the following lemma.

LEMMA 6.1. (*cf.* [DWZZ18, Lemma 2.4]) *A measure of  $X \setminus X'$  and  $Y \setminus Y'$  are zero with respect to any smooth positive volume form on  $X$  and  $Y$ , respectively.*

We fix smooth positive volume forms on  $\tilde{X}$  and  $\tilde{Y}$ . Let  $V \subset Y \setminus v_0^{-1}(0)$  be a local open coordinate such that  $V \cap (Y' \setminus v_0^{-1}(0))$  has positive measure in  $Y$ . Since  $\tilde{Y}$  is a Stein manifold, there exist global holomorphic functions  $\{g_j\}_{j=1}^l$  on  $Y$  such that  $(w_1 := g_1|_V, \dots, w_l := g_l|_V)$  defines a coordinate function. We choose  $u_0, u_1, \dots \in A(X, mK_X)$  such that  $Tu_0 = v_0, Tu_1 = g_1 v_0, \dots, Tu_l = g_l v_l$ . Then  $v_j/v_0 = g_j$  for  $1 \leq j \leq l$ . Set  $V' := V \cap (Y' \setminus v_0^{-1}(0))$ ,  $U' := F^{-1}(V') \subset X' \setminus u_0^{-1}(0)$ . Since  $V' = J_\infty^{-1}(I_\infty(U')) \subset J_l^{-1}(I_l(U'))$ ,  $J_l^{-1}(I_l(U'))$  also has positive measure in  $Y$ . Then  $I_l(U')$  has positive measure in  $\mathbb{C}^l$  for the reason that  $J_l|_{V'}$  is a biholomorphic coordinate function on  $V'$ . Note that  $I_l$  is a non-constant holomorphic map on  $X \setminus u_0^{-1}(0)$ , and  $I_l(X \setminus v_0^{-1}(0))$  also has positive measure in  $\mathbb{C}^l$ . Then we have  $n \geq l$ . The same argument implies that  $l \geq n$ . Hence, we get the following lemma.

LEMMA 6.2. *The dimensions of  $X$  and  $Y$  are equal, i.e.  $n = l$ .*

We can also prove the following lemma.

LEMMA 6.3. *The sets  $X'$  and  $Y'$  are open in  $X$  and  $Y$ , respectively.*

PROOF OF LEMMA 6.3. We take points  $z_0 \in X'$  and  $F(z_0) =: w_0 \in Y'$ . We take a local open coordinate  $V \subset Y$  around  $w_0$ , global holomorphic functions  $(g_1, \dots, g_n)$  on  $Y$  which define a local coordinate on  $V$ , and  $v_0 \in A(Y, mK_Y)$  such that  $v_0 \neq 0$  on  $V$ . We choose a countable dense set  $u_0, u_1, \dots \in A(X, mK_X)$  such that  $Tu_0 = v_0, Tu_1 = g_1 v_0, \dots, Tu_n = g_n v_n$ . Since  $I_n : X \setminus u_0^{-1}(0) \rightarrow \mathbb{C}^n$  is holomorphic,  $I_n^{-1}(J_n(V)) =: U$  is an open set in  $X \setminus u_0^{-1}(0)$  around  $z_0$ .

Set  $U' := U \cap X'$ . It follows that

$$F = J_\infty^{-1} \circ I_\infty = J_n^{-1} \circ I_n$$



on  $U'$ . Therefore, we have

$$(6.1) \quad \frac{u(z)}{u_0(z)} = \frac{Tu(J_n^{-1} \circ I_n(z))}{Tu_0(J_n^{-1} \circ I_n(z))}$$

for all  $u \in A(X, mK_X)$  and  $z \in U'$ . Lemma 6.1 implies that  $U'$  is dense in  $U$ . By continuity of  $J_n^{-1} \circ I_n$ , the equation (6.1) holds on  $U$ . Then we get  $U \subset X'$ . Since  $U$  is also open in  $X$  and  $z_0 \in U$ , we have  $X'$  is open in  $X$ . The same argument implies that  $Y'$  is open in  $Y$ .  $\square$

The above argument also implies that  $F$  is holomorphic on  $X'$ , i.e.  $F$  is a biholomorphic map from  $X'$  to  $Y'$ .

Theorem 5.2 and Lemma 5.3 imply the following result.

LEMMA 6.4. (cf. [DWZZ18, Lemma 2.7]) *For any  $z \in X'$  and  $u \in A(X, mK_X)$ , we have*

$$|u(z) \wedge \overline{u(z)}|^{1/m} = |F^*(Tu)(z) \wedge \overline{F^*(Tu)(z)}|^{1/m}$$

We will show that the biholomorphic map  $F : X' \rightarrow Y'$  can be extended to a biholomorphic map from  $X$  to  $Y$ . Before proving it, we give the following result.

LEMMA 6.5. *Let  $S := X \setminus X'$ . Then  $S$  is a closed pluripolar set, i.e. for any point  $a \in S$ , there exist an open neighborhood  $U$  of  $a$  and a plurisubharmonic function  $\rho$  on  $U$  such that  $S \cap U \subset \rho^{-1}(-\infty)$ .*

PROOF OF LEMMA 6.5. We take any convergent sequence  $\{a_j\} \subset X \setminus S \rightarrow a \in S$ . Since  $Y$  is relatively compact, by passing to a subsequence, we can assume that there is  $b \in \overline{Y}$  such that  $F(a_j) \rightarrow b$ . If  $b \in Y$ , we have  $a \in X'$  from equation 6.1 or the definition of  $F$ . Therefore  $b \in \partial Y$ .

We take open coordinates  $U \Subset \tilde{U} = (z_1, \dots, z_n) \subset \tilde{X}$  around  $a$  and  $V \Subset \tilde{V} = (w_1, \dots, w_n) \subset \tilde{Y}$  around  $b$ . In this local setting, the equation of Lemma 6.4 gives us the following expression

$$|g_u(z)| = |h_{Tu}(F(z))| \left| \frac{\partial(F_1, \dots, F_n)}{\partial(z_1, \dots, z_n)}(z) \right|^m$$

for  $z \in X'$  and any  $u \in A(X, mK_X)$ . Here  $F_j = w_j \circ F$ ,  $u = g_u(dz_1 \wedge \dots \wedge dz_n)^{\otimes m}$ , and  $Tu = h_{Tu}(dw_1 \wedge \dots \wedge dw_n)^{\otimes m}$ . The exhaustivity of  $K_{Y,m}$  implies that  $K_{Y,m}(F(a_j)) \rightarrow +\infty$ . The choices of coordinates imply that boundary behaviors of  $K_{X,m}/(\sqrt{-1}dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge \sqrt{-1}dz_n \wedge d\bar{z}_n)$  and  $K_{Y,m}/(\sqrt{-1}dw_1 \wedge d\bar{w}_1 \wedge \dots \wedge \sqrt{-1}dw_n \wedge d\bar{w}_n)$  coincide with  $K'_{X,m}$  and  $K'_{Y,m}$  on  $U \cap X$  and  $V \cap Y$ , respectively. Then, for each  $j$ , we have  $v_j \in A(Y, mK_Y)$  such that  $\|v_j\|_{Y,m} = 1$  and  $|h_{v_j}(F(a_j))| \rightarrow +\infty$  as  $j \rightarrow +\infty$ , where  $v_j = h_{v_j}(dw_1 \wedge \dots \wedge dw_n)^{\otimes m}$ . Set  $u_j := T^{-1}(v_j)$ . Then we have

$$|g_{u_j}(a_j)| = |h_{v_j}(F(a_j))| \left| \frac{\partial(F_1, \dots, F_n)}{\partial(z_1, \dots, z_n)}(a_j) \right|^m,$$

and  $\|u_j\|_{X,m} = 1$ . Since  $K_{X,m}$  is locally bounded on  $X$ , the left-hand side has an upper bound. Then it follows that

$$\left| \frac{\partial(F_1, \dots, F_n)}{\partial(z_1, \dots, z_n)}(a_j) \right| \rightarrow 0.$$

We define a function  $\rho : U \rightarrow \mathbb{R} \cup \{-\infty\}$  as

$$\rho(z) = \begin{cases} \log \left| \frac{\partial(F_1, \dots, F_n)}{\partial(z_1, \dots, z_n)}(z) \right| & (z \in U \setminus S) \\ -\infty & (z \in U \cap S). \end{cases}$$

This function satisfies the mean-value inequality. The above argument ensures that  $\rho$  is upper semi-continuous on  $U$  for the following reason. If  $\limsup_{z \rightarrow a} |\rho(z)| > -\infty$  for some point  $a \in S$ , passing to a subsequence, we have that  $K_{X,m}(z) \rightarrow +\infty$  as  $z \rightarrow a$ , which is a contradiction. Hence,  $\rho$  is a plurisubharmonic function on  $U$ . By definition, we see that  $S \cap U \subset \rho^{-1}(-\infty)$ . □

By using the following fact, we can prove Theorem 4.1.

**THEOREM 6.6.** (*cf.* [Dem-book, Theorem 5.24, Corollary 5.25]) *Let  $A \subset X$  be a closed pluripolar set in a complex analytic manifold  $X$ . Then*

- (i) *every plurisubharmonic function  $v$  on  $X \setminus A$  that is locally bounded above near  $A$  extends uniquely into a function  $\tilde{v}$  on  $X$ , and*
- (ii) *every holomorphic function  $f$  on  $X \setminus A$  that is locally bounded near  $A$  extends to a holomorphic function on  $X$ .*

**PROOF OF THEOREM 4.1.** Taking a local coordinate or embedding  $Y$  into the complex Euclidean space, we can regard  $F$  as a bounded function. Since  $S$  is a closed pluripolar set, there exists a holomorphic function  $\tilde{F}$  on  $X$  such that  $\tilde{F}|_{X \setminus S} = F$ . We also denote by  $F$  this extension. The hyperconvexity of  $Y$  implies the existence of a negative plurisubharmonic function  $\varphi$  on  $Y$  such that  $\{w \in Y \mid \varphi(w) < c\}$  is relatively compact for any  $c < 0$ . Then  $\tilde{\varphi} := \varphi \circ F$  is also negative plurisubharmonic function on  $X \setminus S$ , and can be extended to a plurisubharmonic function on  $X$  by Theorem 6.6. Hence,  $\tilde{\varphi}$  attains its maximum on  $S$  for the reason that  $F(S) \subset \bar{Y} \setminus Y$ . By the maximum principle,  $\varphi$  must be a constant function, which is a contradiction. Then  $S = \emptyset$ .

By applying the same method to  $Y$ , we obtain  $Y \setminus Y' = \emptyset$ . Consequently,  $F$  is globally defined on  $X$  and a biholomorphic map from  $X$  to  $Y$ . □

We can also prove the uniqueness of  $F$ . The equation

$$|u(z) \wedge \overline{u(z)}|^{1/m} = |F^*(Tu)(z) \wedge \overline{F^*(Tu)(z)}|^{1/m}$$

for any  $u \in A(X, mK_X)$  implies that the condition  $u(z) = 0$  is equivalent to  $Tu(F(z)) = 0$ . Namely,  $F$  is uniquely determined by  $T$  as  $T(H_{X,z}) = H_{Y,F(z)}$ .

## 7. PSEUDONORMS ON DIRECT IMAGES

In this section, we prove Theorem 4.3 and 4.4. A key ingredient to prove them is the following  $L^{2/m}$ -variant of the Ohsawa-Takegoshi extension theorem.

**THEOREM 7.1.** (*cf.* [BP10], [GZ15], [HPS18], [PT18]) *Let  $X$ ,  $B$ ,  $f$ , and the notation be the same as in Theorem 4.3. Then for any  $t \in B$  and  $u \in A(X_t, mK_{X_t})$ , there exists an extension  $U \in A(X, mK_X)$  such that  $U|_{X_t} = u \wedge (df_1 \wedge \cdots \wedge df_r)^{\otimes m}$  and*

$$\int_X |U \wedge \overline{U}|^{1/m} \leq C \int_{X_t} |u \wedge \overline{u}|^{1/m},$$

for positive constant  $C > 0$  which is independent of  $t$ ,  $u$ , and  $U$ .

**LEMMA 7.2.** *The linear isomorphism  $T$  in Theorem 4.3 induces fiberwise linear isometries  $\{T_t\}_{t \in B}$ , i.e.*

$$T_t : A(X_t, mK_{X_t}) \longrightarrow A(Y_t, mK_{Y_t})$$

is a linear isomorphism and

$$\int_{X_t} |u \wedge \overline{u}|^{1/m} = \int_{Y_t} |T_t u \wedge \overline{T_t u}|^{1/m}$$

for all  $t \in B$  and  $u \in A(X_t, mK_{X_t})$ .

**PROOF OF LEMMA 7.2.** We define the well-defined maps  $\{T_t\}_{t \in B}$ . Let  $u \in A(X_t, mK_{X_t})$ . We can take an extension  $U \in A(X, mK_X)$  such that  $U|_{X_t} = u \otimes (df_1 \wedge \cdots \wedge df_r)^{\otimes m}$  and

$$\int_X |U \wedge \overline{U}|^{1/m} \leq C \int_{X_t} |u \wedge \overline{u}|^{1/m}$$

for some positive constant  $C > 0$  from Theorem 7.1. Then we define  $T_t(u) := (TU)_t$ , where  $(TU)_t \in A(Y_t, mK_{Y_t})$  and  $(TU)|_{Y_t} = (TU)_t \otimes (df_1 \wedge \cdots \wedge df_r)^{\otimes m}$ . We have to show that  $T_t$  are well-defined. If  $U_1, U_2 \in A(X, mK_X)$  are both extensions of  $u$  satisfying the above properties,  $(U_1 - U_2)_t = 0$  and

$$\begin{aligned} & \int_{X_t} |(U_1 - U_2)_t \wedge \overline{(U_1 - U_2)_t}|^{1/m} \\ &= \int_{Y_t} |(TU_1 - TU_2)_t \wedge \overline{(TU_1 - TU_2)_t}|^{1/m} \\ &= 0. \end{aligned}$$

Therefore we get  $(TU_1)_t = (TU_2)_t$ .

We also have

$$\begin{aligned} \int_{X_t} |u \wedge \overline{u}|^{1/m} &= \int_{X_t} |U_t \wedge \overline{U_t}|^{1/m} \\ &= \int_{Y_t} |(TU)_t \wedge \overline{(TU)_t}|^{1/m} \\ &= \int_{Y_t} |T_t(u) \wedge \overline{T_t(u)}|^{1/m}. \end{aligned}$$

Similarly, we can prove that  $T_t$  is surjective. Hence,  $T_t : A(X_t, mK_{X_t}) \rightarrow A(Y_t, mK_{Y_t})$  is a linear isometry.  $\square$

**PROOF OF THEOREM 4.3.** We can construct biholomorphic maps  $\{F_t : X_t \rightarrow Y_t\}_{t \in B}$  induced by linear isometries  $\{T_t\}_{t \in B}$  by Theorem 4.1. Hence, we get  $n = l$ . Then we will make a global holomorphic map  $F$  from  $X$  to  $Y$ . We define a map  $F$  as follows:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \Psi & & \Psi \\ z & \longmapsto & F_{f(z)}(z). \end{array}$$

We know that  $F_t(z)$  is holomorphic in the fiber directions for each fixed  $t$ . It is sufficient to show that the map  $F$  is holomorphic in all the directions.

We take a countable dense subset  $\{U_0, U_1, \dots\} \subset A(X, mK_X)$ . Then  $\{V_0 := TU_0, V_1 := TU_1, \dots\}$  is a countable dense subset of  $A(Y, mK_Y)$ . We remark that Theorem 7.1 ensures that  $A(X, mK_X)$  is infinite-dimensional. We define maps  $I_\infty, J_\infty$  as follows:

$$\begin{array}{ccc} I_\infty : X \setminus U_0^{-1}(0) & \longrightarrow & \mathbb{C}^\infty \\ \Psi & & \Psi \\ z & \longmapsto & (f(z), (\frac{U_1(z)}{U_0(z)}, \frac{U_2(z)}{U_0(z)}, \dots)) \end{array}$$

$$\begin{array}{ccc} J_\infty : Y \setminus V_0^{-1}(0) & \longrightarrow & \mathbb{C}^\infty \\ \Psi & & \Psi \\ w & \longmapsto & (g(w), (\frac{V_1(w)}{V_0(w)}, \frac{V_2(w)}{V_0(w)}, \dots)). \end{array}$$

Next, we show that  $F = J_\infty^{-1} \circ I_\infty$  on  $I_\infty^{-1} \circ J_\infty(Y \setminus V_0^{-1}(0))$ . A separation property of  $A(X_t, mK_{X_t})$  and  $A(Y_t, mK_{Y_t})$  implies that  $I_\infty$  and  $J_\infty$  are injective maps. For any  $u \in A(X_t, mK_{X_t})$ , we get an extension  $U \in A(X, mK_X)$  of  $u$  such that  $U_t = u$ . Since  $\{U_j\}$  (resp.  $\{TU_j\}$ ) is a dense subset of  $A(X, mK_X)$  (resp.  $A(Y, mK_Y)$ ), we can take a sequence  $\{U_{j_k}\} \subset \{U_j\}$  such that  $\|U_{j_k} - U\|_{X,m} \rightarrow 0$  and  $\|TU_{j_k} - TU\|_{Y,m} \rightarrow 0$  as  $k \rightarrow +\infty$ . Therefore, we have that  $U_{j_k}$  (resp.  $TU_{j_k}$ ) converges compactly to  $U$  (resp.  $TU$ ) on  $X$  (resp.  $Y$ ). Since all compact sets of  $X_t$  (resp.  $Y_t$ ) are compact in  $X$  (resp.  $Y$ ),  $(U_{j_k})_t$  (resp.  $(TU_{j_k})_t$ ) also converges compactly to  $u$  (resp.  $T_t u$ ) on  $X_t$  (resp.  $Y_t$ ). Here, we remark that we do not know whether  $(U_{j_k})_t \in A(X_t, mK_{X_t})$  (resp.  $(TU_{j_k})_t \in A(Y_t, mK_{Y_t})$ ).

For any point  $z \in I_\infty^{-1} \circ J_\infty(Y \setminus V_0^{-1}(0))$ , there exists a unique point  $w \in Y \setminus V_0^{-1}(0)$  such that

$$(f(z), (\frac{U_1(z)}{U_0(z)}, \frac{U_2(z)}{U_0(z)}, \dots)) = (g(w), (\frac{V_1(w)}{V_0(w)}, \frac{V_2(w)}{V_0(w)}, \dots)).$$

Therefore,  $f(z) = g(w)$  and

$$\frac{U_j(z)}{U_0(z)} = \frac{V_j(w)}{V_0(w)}$$

for all  $j \in \mathbb{N}$ . Set  $t := f(z) = g(w)$ . The above argument implies that for any  $u \in A(X_t, mK_{X_t})$ , we have

$$\frac{u(z)}{(U_0)_t(z)} = \frac{T_t u(w)}{(V_0)_t(w)}.$$

We have that  $u(z) = 0$  if and only if  $T_t u(w) = 0$ . Hence, this  $w$  satisfies the following equations

$$H_{Y_t, w} = T_t(H_{X_t, z})$$

and  $F_t(z) = w$ . Consequently, we obtain that

$$J_\infty^{-1}(I_\infty(z)) = F_{f(z)}(z) = F(z)$$

for any  $z \in I_\infty^{-1} \circ J_\infty(Y \setminus V_0^{-1}(0))$ .

Next, we will show that  $X \setminus U_0^{-1}(0) = I_\infty^{-1} \circ J_\infty(Y \setminus V_0^{-1}(0))$ . Since  $J_\infty^{-1} \circ I_\infty|_{X_t} = F_t$ ,  $I_\infty^{-1} \circ J_\infty(Y \setminus V_0^{-1}(0)) \cap X_t = X'_t \setminus (U_0)_t^{-1}(0)$ . From the arguments in Section 6,  $X'_t$  is actually  $X_t$ . Therefore, it follows that

$$I_\infty^{-1} \circ J_\infty(Y \setminus V_0^{-1}(0)) = X \setminus U_0^{-1}(0).$$

Finally, we also show that the map  $F = J_\infty^{-1} \circ I_\infty$  is holomorphic on  $X \setminus U_0^{-1}(0)$ . It is enough to show that  $F$  is holomorphic around any point  $z \in X \setminus U_0^{-1}(0)$  locally. We let  $w := F(z) = J_\infty^{-1} \circ I_\infty(z)$ . Since  $g = (g_1, \dots, g_r)$  defines a submersive map, we can take global holomorphic functions  $(g_1, \dots, g_r, \eta_{r+1}, \dots, \eta_{r+n}) \in \mathcal{O}(Y)^{n+r}$  which define a coordinate around  $w$ . By taking  $V_j = \eta_{j+r} V_0$  and  $U_j = T^{-1}(V_j)$  for  $j \in \{1, 2, \dots, n\}$ , we obtain that

$$F(z) = (f_1(z), \dots, f_r(z), (\frac{U_1(z)}{U_0(z)}, \dots, \frac{U_n(z)}{U_0(z)}))$$

locally. Choosing  $U_0$  arbitrarily, we conclude that  $F$  is holomorphic on  $X$ . Then  $F : X \rightarrow Y$  is a biholomorphic map and satisfies  $f = g \circ F$ .

By the construction of  $F$ , we also have that  $F|_{X_t} = F_t$ . Hence, we get

$$|u(z) \wedge \overline{u(z)}|^{1/m} = |F_t^*(T_t u)(z) \wedge \overline{F_t^*(T_t u)(z)}|^{1/m}$$

for  $t \in B$ ,  $z \in X_t$ , and  $u \in A(X, mK_X)$ . □

Gluing the above local result, we can prove Theorem 4.4.

**PROOF OF THEOREM 4.4.** Since  $f$  and  $g$  are relatively compact Stein morphisms, for each point  $t \in T$ , we can take an open neighborhood  $D$  of  $t$  such that  $f^{-1}(D)$  and  $g^{-1}(D)$  are Stein. Then there exists a linear isomorphism  $T_D : H^0(f^{-1}(D), mK_X) \rightarrow H^0(g^{-1}(D), mK_Y)$  which preserves the  $L^{2/m}$ -norm on each fiber. By Fubini's theorem, we see that  $T_D$  induces a linear isomorphism

$$T_D : A(f^{-1}(D), mK_X) \rightarrow A(g^{-1}(D), mK_Y).$$

Then  $f^{-1}(D)$  and  $g^{-1}(D)$  satisfy the assumptions of Theorem 4.3. Hence,  $n = l$ . We can also construct the unique biholomorphism  $F_D : f^{-1}(D) \rightarrow g^{-1}(D)$  satisfying the conditions of Theorem 4.3.

Then we define a global biholomorphic map  $F : X \rightarrow Y$  as

$$F(z) = F_D(z)$$

for some open neighborhood  $D$  around  $f(z) =: t$ . The map  $F$  is independent of the choice of  $D$  for the following reason. Let  $D_1$  and  $D_2$  be two open neighborhoods such that  $t \in D_1 \cap D_2$ . Repeating the proof of Lemma 7.2, we have that  $T_{(D_1)t} = T_{(D_2)t}$  since  $T_{D_1}$  and  $T_{D_2}$  commute with the restriction maps of sheaves. Therefore, fiberwise linear isometries  $T_t : A(X_t, mK_{X_t}) \rightarrow A(Y_t, mK_{Y_t})$  are uniquely induced by  $T$ . Then  $F_{D_1}|_{X_t}$  and  $F_{D_2}|_{X_t}$  are uniquely determined by  $T_t$ . In other words, from the results of Theorem 4.3,  $F_{D_1}$  and  $F_{D_2}$  satisfy the following equations

$$|u(z) \wedge \overline{u(z)}|^{1/m} = |F_{(D_1)t}^*(T_t u)(z) \wedge \overline{F_{(D_1)t}^*(T_t u)(z)}|^{1/m}$$

$$|u(z) \wedge \overline{u(z)}|^{1/m} = |F_{(D_2)t}^*(T_t u)(z) \wedge \overline{F_{(D_2)t}^*(T_t u)(z)}|^{1/m}$$

for  $z \in X_t$  and  $u \in A(X_t, mK_{X_t})$ . Hence,

$$F_{D_1}(z) = F_{(D_1)t}(z) = F_{(D_2)t}(z) = F_{D_2}(z).$$

Then  $F$  gives the biholomorphism  $F : X \rightarrow Y$  which satisfies the condition of Theorem 4.4.  $\square$

## 8. APPENDIX

We introduce the direct proof of Corollary 4.5. We do not need explicitly the  $L^{2/m}$ -variant of the Ohsawa-Takegoshi extension theorem.

PROOF OF COROLLARY 4.5. We obtain fiberwise linear isometries  $T_t : A^p(X_t) \rightarrow A^p(Y_t)$  and biholomorphic maps  $F_t : X_t \rightarrow Y_t$  for each  $t \in B$  (cf. Lemma 7.2). Hence, we get  $n = l$ . Since  $Y$  is a domain in  $\mathbb{C}^{r+n}$ , we take a global coordinate  $(t_1, \dots, t_r, w_1, \dots, w_n)$  on  $Y$ . We can assume that  $g : Y \rightarrow B$  is defined as

$$g(t_1, \dots, t_r, w_1, \dots, w_n) = (t_1, \dots, t_r)$$

without any loss of generality. Then  $(w_1, \dots, w_n)$  gives a global coordinate function on each  $Y_t$ . Let  $\phi_0 = T_t^{-1}(1), \phi_1 = T_t^{-1}(w_1), \dots, \phi_n = T_t^{-1}(w_n)$  in  $A^p(X_t)$ . The results of [DWZZ18] (cf. Theorem 5.4) imply that the expression of  $F_t$  is given by

$$F_t = J_\infty^{-1} \circ I_\infty = J_n^{-1} \circ I_n = \left( \frac{\phi_1}{\phi_0}, \dots, \frac{\phi_n}{\phi_0} \right)$$

globally on  $X_t$ .

In this setting, we let  $\Psi_0 = 1, \Psi_1 = w_1, \dots, \Psi_n = w_n$  in  $A^p(Y)$ , and define  $\Phi_0 = T^{-1}(\Psi_0), \Phi_1 = T^{-1}(\Psi_1), \dots, \Phi_n = T^{-1}(\Psi_n)$  in  $A^p(X)$ . By the definition of  $T_t$ , we have

$$\Phi_0|_{X_t} = T_t^{-1}((T\Phi_0)|_{Y_t}) = T_t^{-1}(1), \quad \Phi_j|_{X_t} = T_t^{-1}((T\Phi_j)|_{Y_t}) = T_t^{-1}(w_j).$$

Letting

$$F := \left( \frac{\Phi_1}{\Phi_0}, \dots, \frac{\Phi_n}{\Phi_0} \right),$$

we see that  $F$  is a well-defined holomorphic map and  $F|_{X_t} = F_t$ . Hence,  $F$  is a biholomorphic map from  $X$  to  $Y$  and gives the condition of Corollary 4.5.  $\square$

### Part 3. $L^2$ estimates and vanishing theorems for holomorphic vector bundles equipped with singular Hermitian metrics

ABSTRACT. We investigate singular Hermitian metrics on vector bundles, especially strictly Griffiths positive ones.  $L^2$  estimates and vanishing theorems usually require an assumption that vector bundles are Nakano positive. However, there is no general definition of the Nakano positivity in singular settings. In this paper, we show various  $L^2$  estimates and vanishing theorems by assuming that the vector bundle is strictly Griffiths positive and the base manifold is projective.

#### 9. INTRODUCTION

The research in Part 3 is based on the paper [Ina1]. We investigate singular Hermitian metrics on vector bundles. Singular Hermitian metrics on line bundles have a key role in complex geometry. They make it possible that we apply complex analytic methods to complex algebraic geometry (cf. [Dem]). Singular Hermitian metrics on vector bundles were also introduced and investigated in many papers (for examples, [BP08], [deC98], [HPS18], [PT18], [Rau15], etc.). However, it is known that curvature currents of singular Hermitian metrics on vector bundles are not always defined with measure coefficients [Rau15, Theorem 1.5]. As a result, a positivity of singular Hermitian metrics on vector bundles generally cannot be dealt with directly by using the curvature currents. Griffiths semi-positivity or semi-negativity of singular Hermitian metrics is defined without using the curvature currents ([BP08], [PT18], [Rau15], see Definition 10.4). Nevertheless, a general definition of Nakano positivity has not been formulated even though  $L^2$  estimates and vanishing theorems usually require an assumption that vector bundles are Nakano positive.

In this paper, we show various  $L^2$  estimates and vanishing theorems. To be precise, we have the following result. We let  $X$  be an  $n$ -dimensional complex projective manifold, let  $\omega$  be a Kähler form on  $X$ , let  $dV_\omega = \frac{\omega^n}{n!}$  be the volume form determined by  $\omega$ , and let  $E \rightarrow X$  be a holomorphic vector bundle of rank  $r$  over  $X$ .

**THEOREM 9.1.** *Let  $h$  be a Griffiths semi-positive singular Hermitian metric on  $E$ . We assume that there exists a proper analytic subset  $S$  such that  $h$  is strictly Griffiths  $\delta_\omega$ -positive on  $X \setminus S$  in the sense of Definition 10.6. Suppose that  $f$  is an  $E$ -valued  $(n, n)$ -form with finite  $L^2$ -norm. Then there is an  $E$ -valued  $(n, n-1)$ -form  $g$  such that*

$$\bar{\partial}g = f, \quad \int_X |g|_{h,\omega}^2 dV_\omega \leq \frac{1}{\delta n} \int_X |f|_{h,\omega}^2 dV_\omega.$$

Roughly speaking, the singular Hermitian metric  $h$  is strictly Griffiths  $\delta$ -positive (or simply strictly Griffiths positive) if the curvature current  $\Theta_{h^*}$  of the dual Hermitian vector bundle  $(E^*, h^*)$  satisfies the following inequality

$$\sum_{j,k=1}^n (\Theta_{j\bar{k}}^{h^*} s, s)_h \xi_j \bar{\xi}_k \leq -\delta \|s\|_{h^*}^2 |\xi|^2$$



in the sense of distributions for any section  $s$  of  $E^*$  and any vector field  $\xi = \sum \xi_j \frac{\partial}{\partial z_j}$ , where  $\sqrt{-1}\Theta_h^* = \sum_{j,k=1}^n \Theta_{j\bar{k}}^* dz_j \wedge d\bar{z}_k$  (see Definition 10.5). This type of theorem was announced by H. Raufi [Rau12] in the case  $n = 1$ . In [Rau12],  $L^2$  estimates are established on Riemann surfaces for the reason that Griffiths positivity of vector bundles coincides with Nakano positivity of those on Riemann surfaces. As we have said above, we allow the dimension of the base manifold  $X$  to be larger than 1. As an application of Theorem 9.1, we have the next statement.

**COROLLARY 9.2.** *Let  $h$  be a Griffiths semi-positive singular Hermitian metric such that  $h$  is strictly Griffiths  $\delta_\omega$ -positive on  $X \setminus S$ , and  $K_X$  be the canonical bundle of  $X$ . If the Lelong number  $\nu(-\log \det h, x) < 2$  for all points  $x \in X$ , the  $n$ -th cohomology group of  $X$  with coefficients in the sheaf of holomorphic sections of  $K_X \otimes E$  vanishes :*

$$H^n(X, K_X \otimes E) = 0.$$

Here  $-\log \det h$  is locally plurisubharmonic. Then the Lelong number  $\nu(-\log \det h, x)$  is naturally defined (see Definition 12.5).

We show another  $L^2$  estimate. Let  $h$  be a smooth Hermitian metric on  $E$ . If  $(E, h)$  is positive in the sense of Griffiths,  $(E \otimes \det E, h \otimes \det h)$  is positive in the sense of Nakano. This is a well-known result of [DS]. As described above, Nakano positivity of a singular Hermitian metric has not been defined. However, we show some  $L^2$  estimate by applying this result to a vector bundle equipped with a singular Hermitian metric.

**THEOREM 9.3.** *Let  $h$  be a Griffiths semi-positive singular Hermitian metric on  $E$  such that  $h$  is strictly Griffiths  $\delta_\omega$ -positive on  $X \setminus S$ . Suppose that  $q$  is a positive integer,  $f$  is an  $E \otimes \det E$ -valued  $\bar{\partial}$ -closed  $(n, q)$ -form with finite  $L^2$  norm. Then there is an  $E \otimes \det E$ -valued  $(n, q-1)$ -form  $g$  such that*

$$\bar{\partial}g = f, \quad \int_X |g|_{h \otimes \det h, \omega}^2 dV_\omega \leq \frac{1}{\delta q r} \int_X |f|_{h \otimes \det h, \omega}^2 dV_\omega.$$

As is the case in Corollary 9.2, we have the following statement.

**COROLLARY 9.4.** *Let  $h$  be a Griffiths semi-positive singular Hermitian metric on  $E$  such that  $h$  is strictly Griffiths  $\delta_\omega$ -positive on  $X \setminus S$ . If the Lelong number  $\nu(-\log \det h, x) < 1$  for all points  $x \in X$ , then*

$$H^q(X, K_X \otimes E \otimes \det E) = 0.$$

Corollary 9.4 is a generalization of the so-called Griffiths vanishing theorem in the smooth setting (cf. [Dem-book, Chapter VII, Corollary 9.4]). For a (strictly) Griffiths positive singular Hermitian metric  $h$ , we cannot conclude that  $h \otimes \det h$  is (strictly) Nakano positive for the reason that the definition of Nakano positive singular Hermitian metrics has not been formulated. However,  $h \otimes \det h$  behaves like a Nakano positive metric as in Theorem 9.3 and Corollary 9.4. Corollary 9.4 can be used to discriminate the existence

of a Griffiths positive singular Hermitian metric on a certain vector bundle. We have the following example.

**EXAMPLE 9.5.** Let  $V$  be a complex vector space of dimension  $n + 1$ , and  $X = P(V) := (V \setminus \{0\})/\mathbb{C}^*$  be the projective space of  $V$ . We also let  $\mathcal{O}(-1)$  denote the tautological line bundle, and  $Q$  denote the quotient bundle  $V/\mathcal{O}(-1)$ . Then there do not exist any Griffiths semi-positive singular Hermitian metrics satisfying the strict Griffiths  $\delta_\omega$ -positivity outside some proper analytic subset  $S$  and  $\nu(-\log \det h, x) < 1$  for all  $x \in X$  on  $Q$ .

It is known that the above vector bundle  $Q$  has a Griffiths semipositive smooth Hermitian metric, but does not have any strictly Griffiths positive ones (cf. [Dem-book, Chapter VII, Example 8.4]). Roughly speaking, Example 9.5 asserts that there cannot exist any strictly Griffiths positive Hermitian metrics even if they are (mildly) singular. The criterion which determines whether a Griffiths positive singular Hermitian metric of a vector bundle exist is one of important applications of Corollary 9.4.

The organization of this paper is as follows. In Section 10, we explain basic definitions and properties of singular Hermitian metrics on vector bundles. In Section 11, we prepare some lemmas and properties about Griffiths positive or negative singular Hermitian metrics on vector bundles. In Section 12 and 13, we prove the main theorems and corollaries.

## 10. SINGULAR HERMITIAN METRICS ON VECTOR BUNDLES

In this section, we introduce a definition and property of a singular Hermitian metric on a vector bundle. To start with, we refer to basic notions of smooth Hermitian metrics on vector bundles. Throughout this section,  $X$  denotes a complex manifold with a positive Hermitian form  $\omega$ , and  $E$  denotes a holomorphic vector bundle over  $X$ .

**10.1. Positivity concepts of smooth Hermitian metrics on vector bundles.** Let  $h$  be a smooth Hermitian metric on  $E$  and the Chern curvature of  $(E, h)$  be  $\Theta_{E,h}$  or simply  $\Theta$ . The curvature  $\Theta_{E,h}$  is a  $\text{Hom}(E, E)$  valued  $(1, 1)$  form, thus let  $(z_1, \dots, z_n)$  be holomorphic coordinates on  $X$  around  $p$  and let  $(e_1, \dots, e_r)$  be an orthonormal frame of  $E$ . By referring to [Dem, (3.8)], writing

$$\sqrt{-1}\Theta_{E,h} = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j\bar{k}\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu,$$

we can identify the curvature tensor to a Hermitian form

$$\Theta_{E,h}(\xi \otimes v, \xi \otimes v) = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j\bar{k}\lambda\mu} \xi_j \bar{\xi}_k v_\lambda \bar{v}_\mu \quad (\xi = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j} \in T_X, v = \sum_{\lambda=1}^r v_\lambda e_\lambda \in E)$$

on  $T_X \otimes E$ . This leads to positivity concepts.

**DEFINITION 10.1.** The smooth Hermitian vector bundle  $(E, h)$  is

(1) *Griffiths semi-positive* (resp. *Griffiths positive*) if  $\Theta_{E,h}(\xi \otimes v, \xi \otimes v) \geq 0$  (resp.  $> 0$ ) for every local non-zero section  $\xi \in T_X, v \in E$ . We denote it by  $\Theta_{E,h} \geq_{\text{Grif}} 0$  (resp.

$\Theta_{E,h} >_{\text{Grif}} 0$ ).

(2) *Nakano semi-positive* (resp. *Nakano positive*) if  $\Theta_{E,h}(\tau, \tau) \geq 0$  (resp.  $> 0$ ) for all non-zero tensors  $\tau = \sum \tau_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_X \otimes E$ . We denote it by  $\Theta_{E,h} \geq_{\text{Nak}} 0$  (resp.  $\Theta_{E,h} >_{\text{Nak}} 0$ ).

If we consider reverse inequalities in the above definition, we can get a definition of Griffiths and Nakano negativity of the smooth Hermitian vector bundle.

It is known that Griffiths positivity and negativity can be defined without using the curvature current. This characterization and the following property will be used in the singular setting.

**PROPOSITION 10.2.** ([Rau15, Section 2]). *Let  $h$  be a smooth Hermitian metric on  $E$ . Then the followings are equivalent.*

- (1)  *$h$  is Griffiths semi-negative.*
- (2)  *$|u|_h^2$  is plurisubharmonic for every local holomorphic section of  $E$ .*
- (3)  *$\log |u|_h^2$  is plurisubharmonic for every local holomorphic section of  $E$ .*
- (4)  *$h^*$  is Griffiths semi-positive ( $h^*$  is the dual metric on the dual bundle  $E^*$ ).*

**10.2. Singular Hermitian metrics on vector bundles.** We adopt the definition of singular Hermitian metrics introduced in [HPS18], [PT18], and [Rau15].

**DEFINITION 10.3.** ([BP08, Section 3], [HPS18, Definition 17.1], [PT18, Definition 2.2.1] and [Rau15, Definition 1.1]). A *singular Hermitian metric  $h$*  on  $E$  is a measurable map from the base manifold  $X$  to the space of non-negative Hermitian forms on the fibers satisfying  $0 < \det h < +\infty$  almost everywhere.

The (Chern) curvature current  $\Theta_h$  of a singular Hermitian metric  $h$  is locally defined as  $\Theta_h = \bar{\partial}(h^{-1}\partial h)$ . However, its coefficients are not always measure. This example is found by Raufi in [Rau15, Theorem 1.5]. For this reason, we cannot define the positivity of a singular Hermitian metric by using its curvature current. The definition of Griffiths positivity and negativity that we will adopt is the following.

**DEFINITION 10.4.** ([BP08, Definition 3.1], [PT18, Definition 2.2.2] and [Rau15, Definition 1.2]) A singular Hermitian metric  $h$  is

- (1) *Griffiths semi-negative*, if for any open subset  $U \subset X$  and any  $u \in H^0(U, E)$ ,  $\log |u|_h^2$  is plurisubharmonic on  $U$ .
- (1') *Griffiths semi-negative*, if for any open subset  $U \subset X$  and any  $u \in H^0(U, E)$ ,  $|u|_h^2$  is plurisubharmonic on  $U$ .
- (2) *Griffiths semi-positive*, if the dual metric  $h^*$  is Griffiths semi-negative, or equivalently if for any open subset  $U \subset X$  and any  $u \in H^0(U, E^*)$ ,  $\log |u|_{h^*}^2$  or  $|u|_{h^*}^2$  is plurisubharmonic on  $U$ .

The above definition (1) is equivalent to (1') (cf. [Rau15, Section 2]). Nakano semi-negativity of a singular Hermitian metric is also defined without using the curvature

current (cf. [Rau15, Definition 1.8]). However, the dual bundle of a Nakano negative vector bundle is not necessarily Nakano positive. Hence, the Nakano (semi-)positive singular Hermitian metric has not been defined. Furthermore, strictly positivity or negativity of a singular Hermitian metric is not generally formulated. The following result is only known as a definition of strict Griffiths negativity of a singular Hermitian metric.

DEFINITION 10.5. ([Rau15, Definition 6.1]) A singular Hermitian metric  $h$  on  $E$  is strictly *Griffiths negative* or  $\delta$ -*negative* if:

- (1)  $h$  is Griffiths semi-negative in the sense of Definition 10.4.
- (2)  $F = \{z \in X : \det h(z) = 0\}$  is a closed set, and there exists an exhaustion of open sets  $\{U_j\}_{j=1}^\infty$  such that  $\det h > \frac{1}{j}$  on  $U_j$ , and  $\bigcup_{j=1}^\infty U_j = X \setminus F$ .
- (3) There exists some  $\delta > 0$  such that on  $X \setminus F$

$$\sum_{j,k=1}^n (\Theta_{j\bar{k}}^h s, s)_h \xi_j \bar{\xi}_k \leq -\delta \|s\|_h^2 \omega(\xi, \xi)$$

in the sense of distributions, for any section  $s$  and any vector field  $\xi = \sum \xi_j \frac{\partial}{\partial z_j}$ . Here we let

$$\sqrt{-1}\Theta_h = \sum_{j,k=1}^n \Theta_{j\bar{k}}^h dz_j \wedge d\bar{z}_k, \quad \Theta_{j\bar{k}}^h \in \mathcal{O}(\text{Hom}(E, E)).$$

We say that a singular Hermitian metric  $h$  is *strictly*  $(\delta)$ -*Griffiths positive* if the dual metric is strictly Griffiths negative. The above condition (2) certifies that the curvature current  $\Theta_h$  exists as a current with measurable coefficients on  $X \setminus F$  [Rau15, Corollary 1.7].

If  $\omega$  is a Kähler form, we can define the strict Griffiths negativity and positivity in a more general setting.

DEFINITION 10.6. Let  $\omega$  be a Kähler form. We say that a singular Hermitian metric  $h$  is *strictly Griffiths*  $\delta_\omega$ -*negative* around  $x \in X$  if for any open neighborhood  $U$  of  $x$  and any Kähler potential  $\varphi$  of  $\omega$  on  $U$ , i.e.  $\sqrt{-1}\partial\bar{\partial}\varphi = \omega$  on  $U$ ,  $h \cdot e^{-\delta\varphi}$  is Griffiths semi-negative in the sense of Definition 10.4. We also say that  $h$  is *strictly Griffiths*  $\delta_\omega$ -*negative* on  $X$  if for any point  $x \in X$ ,  $h$  is *strictly Griffiths*  $\delta_\omega$ -*negative* around  $x$ .

Similarly, we say that  $h$  is *strictly Griffiths*  $\delta_\omega$ -*positive* if the dual metric  $h^*$  is strictly Griffiths  $\delta_\omega$ -negative, i.e.  $h \cdot e^{\delta\varphi}$  is Griffiths semi-positive in the sense of Definition 10.4.

In the above definition, we define the strict negativity without assuming the condition (2) in Definition 10.5. We remark that we can find this  $\varphi$  locally by  $\partial\bar{\partial}$ -lemma and the above definition is independent of the choice of local potentials.

## 11. SOME PROPERTIES OF GRIFFITHS POSITIVE SINGULAR HERMITIAN METRICS

In this section, we prepare some lemmas and propositions about Griffiths negative or positive singular Hermitian metrics. Some of them are not directly related to the proof of the main theorems. However, we summarize these properties in order to improve the

outlook. Throughout this section, let  $E$  be a holomorphic vector bundle on the base manifold  $X$ , and  $h$  be a singular Hermitian metric on  $E$ .

In order to study analytic properties about the curvature current of  $h$ , we will take an approximating sequence  $\{h_\nu\}_{\nu=1}^\infty$  of  $h$ . Locally, we can take such a sequence, through convolution with an approximate identity, from the following lemma.

**LEMMA 11.1.** [BP08, Proposition 3.1] *Let  $h$  be a Griffiths semi-negative singular Hermitian metric. If  $E$  is a trivial vector bundle over a polydisk, then there exists a sequence of smooth Hermitian metrics  $\{h_\nu\}_{\nu=1}^\infty$ , with negative Griffiths curvature, decreasing point-wise to  $h$  on any smaller polydisk.*

If  $h$  is Griffiths semi-positive, we can also get an increasing approximating sequence by taking the dual metric of above Lemma 11.1. Let  $\{h_\nu\}_{\nu=1}^\infty$  be an approximating sequence of  $h$ . We investigate analytic properties about  $\theta_{h_\nu}$ ,  $\theta_h$ ,  $\Theta_{h_\nu}$ , and  $\Theta_h$ , where  $\theta_{h_\nu}$  (resp.  $\theta_h$ ) is the connection form associated with  $h_\nu$  (resp.  $h$ ). For an arbitrary smooth vector field  $\xi$ ,  $\tilde{\theta}_h$  denotes  $\theta_h(\xi)$ , and  $\tilde{\Theta}_h$  denotes  $\Theta_h(\xi, \xi)$ .

**THEOREM 11.2.** [Rau15, Theorem 1.6] *Let  $X$  be a complex manifold with a positive Hermitian form  $\omega$ , and let  $h$  be a singular Hermitian metric that is Griffiths semi-negative. Moreover let  $\{h_\nu\}_{\nu=1}^\infty$  be any approximating sequence of smooth Hermitian metrics with Griffiths semi-negative curvature, decreasing to  $h$ .*

*If there exists  $\epsilon > 0$  such that  $\det h > \epsilon$ , then*

- (1)  $\tilde{\theta}_{h_\nu} \in L_{loc}^2(X)$  uniformly in  $\nu$ , and  $\tilde{\theta}_h \in L_{loc}^2(X)$ ,
- (2)  $\tilde{\theta}_{h_\nu} \in L_{loc}^2(X)$  weakly converges to  $\tilde{\theta}_h$  in  $L_{loc}^2(X)$ ,
- (3)  $\tilde{\Theta}_{h_\nu} \in L_{loc}^1(X)$  uniformly in  $\nu$ ,  $\tilde{\Theta}_h$  has measure coefficients, and  $\tilde{\Theta}_{h_\nu}$  weakly converges to  $\tilde{\Theta}_h$  in the sense of measures.

The above notations mean that after choosing a basis for  $E$  and representing  $\tilde{\theta}_{h_\nu}$ ,  $\tilde{\theta}_h$ ,  $\tilde{\Theta}_{h_\nu}$ , and  $\tilde{\Theta}_h$  as a matrix, each element of the matrix has measure coefficients, weakly converges, and so on. If the  $L^2$  norm, on a fixed compact subset of  $X$ , of each element of  $\tilde{\theta}_{h_\nu}$  has an upper bound which is independent of  $\nu$ , we say  $\tilde{\theta}_{h_\nu} \in L_{loc}^2(X)$  uniformly in  $\nu$ .

This type of lemma is only known in the case that the singular Hermitian metric  $h$  is Griffiths semi-negative. We show it in the situation that  $h$  is Griffiths semi-positive. We start by preparing some lemmas for it.

**LEMMA 11.3.** *Let  $h$  be a Griffiths semi-negative singular Hermitian metric, and  $h_{i\bar{j}}$  be the  $(i, j)$  element of  $h$ . Then it follows that*

$$\partial(h_{i\bar{j}}h_{k\bar{l}}) = (\partial h_{i\bar{j}})h_{k\bar{l}} + h_{i\bar{j}}(\partial h_{k\bar{l}})$$

*in the sense of distributions. Moreover, we have*

- (1)  $\partial(\det h) \in L_{loc}^2(X)$ ,
- (2)  $\partial(\det h_\nu) \in L_{loc}^2(X)$  uniformly in  $\nu$ ,
- (3)  $\partial(\det h_\nu) \in L_{loc}^2(X)$  weakly converges to  $\partial(\det h) \in L_{loc}^2(X)$  in  $L_{loc}^2(X)$ .

PROOF. Since the setting is local, we can assume that  $X$  is a polydisk in  $\mathbb{C}^n$ ,  $E$  is trivial over  $X$ , and  $h$  is represented as a matrix. Without any loss of generality, we also can take an approximating sequence  $\{h_\nu\}_{\nu=1}^\infty$  of smooth Hermitian metrics with Griffiths semi-negative curvature, decreasing to  $h$  on  $X$ . We begin to show that

$$\int_X \partial(h_{i\bar{j}}h_{k\bar{l}})\chi = \int_X \{(\partial h_{i\bar{j}})h_{k\bar{l}} + h_{i\bar{j}}(\partial h_{k\bar{l}})\}\chi$$

for any test form  $\chi \in C_c^{\infty(n-1,n)}(X)$ . It is known that

$$\int_X \partial(h_{\nu_{i\bar{j}}}h_{\nu_{k\bar{l}}})\chi = \int_X \{(\partial h_{\nu_{i\bar{j}}})h_{\nu_{k\bar{l}}} + h_{\nu_{i\bar{j}}}(\partial h_{\nu_{k\bar{l}}})\}\chi \quad \cdots (\diamond)$$

for each  $\nu$ . Firstly, we have

$$\left| \int_X \{\partial(h_{\nu_{i\bar{j}}}h_{\nu_{k\bar{l}}}) - \partial(h_{i\bar{j}}h_{k\bar{l}})\}\chi \right| = \left| \int_X (h_{\nu_{i\bar{j}}}h_{\nu_{k\bar{l}}} - h_{i\bar{j}}h_{k\bar{l}})\partial\chi \right|.$$

Each element of  $h$  and  $h_\nu$  is locally bounded uniformly in  $\nu$  [PT18, Remark 2.2.3]. Hence this integral value goes to zero by the Lebesgue convergence theorem.

Secondly, we get

$$\begin{aligned} & \left| \int_X \{(\partial h_{\nu_{i\bar{j}}})h_{\nu_{k\bar{l}}} - (\partial h_{i\bar{j}})h_{k\bar{l}}\}\chi \right| \\ & \leq \left| \int_X \partial h_{\nu_{i\bar{j}}}(h_{\nu_{k\bar{l}}} - h_{k\bar{l}})\chi \right| + \left| \int_X (\partial h_{\nu_{i\bar{j}}} - \partial h_{i\bar{j}})h_{k\bar{l}}\chi \right| \\ & \leq \|\partial h_{\nu_{i\bar{j}}}\|_{L^2(K)} \|h_{\nu_{k\bar{l}}} - h_{k\bar{l}}\|_{L^2(K)} + \left| \int_X (\partial h_{\nu_{i\bar{j}}} - \partial h_{i\bar{j}})h_{k\bar{l}}\chi \right|, \end{aligned}$$

where  $K$  denotes the support of  $\chi$ . Locally boundness of  $h$  leads to that  $\|h_{\nu_{k\bar{l}}} - h_{k\bar{l}}\|_{L^2(K)} \rightarrow 0$  as  $\nu \rightarrow \infty$ . Then  $\|\partial h_{\nu_{i\bar{j}}}\|_{L^2(K)}$  is uniformly bounded in  $\nu$  [Rau15, Proposition 1.4]. Therefore, the first term goes to zero. The second term also goes to zero for the reason that  $\partial h_{\nu_{i\bar{j}}}$  weakly converges to  $\partial h_{i\bar{j}}$  in  $L^2(K)$  and  $h\chi \in L^2(K)$ . Then we can conclude that  $(\partial h_{\nu_{i\bar{j}}})h_{\nu_{k\bar{l}}}$  weakly converges to  $(\partial h_{i\bar{j}})h_{k\bar{l}}$ , and  $h_{\nu_{i\bar{j}}}(\partial h_{\nu_{k\bar{l}}})$  also weakly converges to  $h_{i\bar{j}}(\partial h_{k\bar{l}})$ .

Finally, taking weak limits of  $(\diamond)$ , we obtain

$$\partial(h_{i\bar{j}}h_{k\bar{l}}) = (\partial h_{i\bar{j}})h_{k\bar{l}} + h_{i\bar{j}}(\partial h_{k\bar{l}}).$$

Repeating this argument, we consequently prove (1), (2), and (3) for the reason that  $\partial h \in L_{loc}^2(X)$ ,  $\partial h_\nu \in L_{loc}^2(X)$  uniformly in  $\nu$ , and  $\partial h_\nu$  weakly converges to  $\partial h$  in  $L_{loc}^2(X)$  [Rau15, Proposition 1.4, Lemma 5.1].  $\square$

Subsequently, we prepare lemmas in the local setting.

LEMMA 11.4. *Let  $h$  be a Griffiths semi-negative singular Hermitian metric. We assume that  $\det h > \epsilon$  for some positive constant  $\epsilon > 0$ , then we have*

$$\partial \left( \frac{1}{\det h} \right) = - \frac{\partial \det h}{\det^2 h}$$

in the sense of distributions.

PROOF. It is sufficient to show that

$$\int_X \partial \left( \frac{1}{\det h} \right) \chi = \int_X -\frac{\partial \det h}{\det^2 h} \chi$$

for any test form  $\chi \in C_c^{\infty(n-1,n)}(X)$ . It is known that

$$\int_X \partial \left( \frac{1}{\det h_\nu} \right) \chi = \int_X -\frac{\partial \det h_\nu}{\det^2 h_\nu} \chi \quad \dots (\diamond)$$

for each  $\nu$ .

Firstly, it follows that  $\partial \left( \frac{1}{\det h_\nu} \right)$  weakly converges to  $\partial \left( \frac{1}{\det h} \right)$  for the reason that  $\frac{1}{\det h_\nu}$  is increasing to  $\frac{1}{\det h}$ .

Secondly, we begin to show that  $\frac{\partial \det h_\nu}{\det^2 h_\nu}$  weakly converges to  $\frac{\partial \det h}{\det^2 h}$ . We have

$$\begin{aligned} & \left| \int_X \frac{\partial \det h_\nu}{\det^2 h_\nu} \chi - \int_X \frac{\partial \det h}{\det^2 h} \chi \right| \\ & \leq \left| \int_X \left( \frac{1}{\det^2 h_\nu} - \frac{1}{\det^2 h} \right) \partial \det h_\nu \chi \right| + \left| \int_X \frac{1}{\det^2 h} (\partial \det h_\nu - \partial \det h) \chi \right| \\ & \leq C \left\| \frac{1}{\det^2 h_\nu} - \frac{1}{\det^2 h} \right\|_{L^2(K)} \|\partial \det h_\nu\|_{L^2(K)} + \left| \int_X \frac{1}{\det^2 h} (\partial \det h_\nu - \partial \det h) \chi \right|, \end{aligned}$$

where  $C$  denotes the supremum of  $\chi$  on  $X$ , and  $K$  denotes a support of  $\chi$ . The first term goes to zero as  $\nu \rightarrow \infty$  for the reason that  $\|\partial \det h_\nu\|_{L^2(K)}$  uniformly in  $\nu$  and  $\frac{1}{\det^2 h_\nu}$  is increasing to  $\frac{1}{\det^2 h} < \frac{1}{\epsilon^2}$ . For the second term, we know that  $\partial \det h_\nu$  weakly converges to  $\partial \det h$  in  $L^2(K)$  by the Lemma 11.3 and  $\frac{1}{\det^2 h} < \frac{1}{\epsilon^2}$  is of course  $L^2(K)$  function. Hence it goes to zero as  $\nu \rightarrow \infty$ .

Finally, we can conclude that  $\frac{\partial \det h_\nu}{\det^2 h_\nu}$  weakly converges to  $\frac{\partial \det h}{\det^2 h}$ . Taking weak limits of  $(\diamond)$ , we obtain

$$\partial \left( \frac{1}{\det h} \right) = -\frac{\partial \det h}{\det^2 h}$$

in the sense of the distributions.  $\square$

LEMMA 11.5. *Let  $h$  be a Griffiths semi-negative singular Hermitian metric, and  $\hat{h}$  be the adjugate matrix of  $h$ . We assume that  $\det h > \epsilon$  for some positive constant  $\epsilon > 0$ , then we have*

$$\partial \left( \frac{1}{\det h} h \right) = \partial \left( \frac{1}{\det h} \right) h + \frac{1}{\det h} \partial h, \quad \partial \left( \frac{1}{\det h} \hat{h} \right) = \partial \left( \frac{1}{\det h} \right) \hat{h} + \frac{1}{\det h} \partial \hat{h}$$

in the sense of distributions.

PROOF. The proof of the first part is almost the same as the second part. It is enough to show that

$$\int_X \partial \left( \frac{1}{\det h} \hat{h} \right) \chi = \int_X \left\{ \partial \left( \frac{1}{\det h} \right) \hat{h} + \frac{1}{\det h} \partial \hat{h} \right\} \chi$$

for any test form  $\chi \in C_c^{\infty(n-1,n)}(X)$ . It is known that

$$\int_X \partial \left( \frac{1}{\det h_\nu} \widehat{h}_\nu \right) \chi = \int_X \left\{ \partial \left( \frac{1}{\det h_\nu} \right) \widehat{h}_\nu + \frac{1}{\det h_\nu} \partial \widehat{h}_\nu \right\} \chi \quad \cdots (\diamond)$$

for each  $\nu$ . For the left-hand side of the above equation, we will show that  $\partial \left( \frac{1}{\det h_\nu} \widehat{h}_\nu \right)$  weakly converges to  $\partial \left( \frac{1}{\det h} \widehat{h} \right)$ . We have

$$\left| \int_X \left\{ \partial \left( \frac{\widehat{h}_\nu}{\det h_\nu} \right) - \partial \left( \frac{\widehat{h}}{\det h} \right) \right\} \chi \right| = \left| \int_K \left( \frac{\widehat{h}_\nu}{\det h_\nu} - \frac{\widehat{h}}{\det h} \right) \partial \chi \right|.$$

Here

$$\left| \left( \frac{\widehat{h}_\nu}{\det h_\nu} - \frac{\widehat{h}}{\det h} \right) \partial \chi \right| \leq C' \left( \frac{|\widehat{h}_\nu|}{\det h_\nu} + \frac{|\widehat{h}|}{\det h} \right) \leq \frac{2(r-1)!C^{r-1}C'}{\epsilon} \in L^1(K),$$

where  $K$  denotes a support of  $\chi$ ,  $C$  denotes the supremum of  $h_0$  on  $K$ , and  $C'$  denotes the supremum of  $\partial \chi$  on  $K$ . The constant  $C$  satisfies the following inequalities that  $|h_\nu| \leq C$  and  $|h| \leq C$  on  $K$  [PT18, Remark 2.2.3]. Then we conclude that  $\partial \left( \frac{1}{\det h_\nu} \widehat{h}_\nu \right)$  weakly converges to  $\partial \left( \frac{1}{\det h} \widehat{h} \right)$  by the Lebesgue convergence theorem.

For the right-hand side of the above equation, we will prove that  $\partial \left( \frac{1}{\det h_\nu} \right) \widehat{h}_\nu$  weakly converges to  $\partial \left( \frac{1}{\det h} \right) \widehat{h}$  and  $\frac{1}{\det h_\nu} \partial \widehat{h}_\nu$  weakly converges to  $\frac{1}{\det h} \partial \widehat{h}$ .

Firstly, we have

$$\begin{aligned} & \left| \int_X \left\{ \partial \left( \frac{1}{\det h_\nu} \right) \widehat{h}_\nu - \partial \left( \frac{1}{\det h} \right) \widehat{h} \right\} \chi \right| \\ &= \left| \int_X \left( \frac{\partial \det h}{\det^2 h} \widehat{h} - \frac{\partial \det h_\nu}{\det^2 h_\nu} \widehat{h}_\nu \right) \chi \right| \\ &\leq \left| \int_X (\partial \det h - \partial \det h_\nu) \frac{\widehat{h}}{\det^2 h} \chi \right| + \left| \int_X \partial \det h_\nu \left( \frac{\widehat{h}}{\det^2 h} - \frac{\widehat{h}_\nu}{\det^2 h_\nu} \right) \chi \right| \\ &\leq \left| \int_X (\partial \det h - \partial \det h_\nu) \frac{\widehat{h}}{\det^2 h} \chi \right| + C''' \|\partial \det h_\nu\|_{L^2(K)} \left\| \frac{\widehat{h}}{\det^2 h} - \frac{\widehat{h}_\nu}{\det^2 h_\nu} \right\|_{L^2(K)} \end{aligned}$$

by Lemma 11.4, where  $K$  deotes a support of  $\chi$ , and  $C'''$  denotes the supremum of  $\chi$  on  $K$ . The first term goes to zero for the reason why  $\partial \det h_\nu$  weakly converges to  $\partial \det h$  in  $L^2(K)$  from the results of Lemma 11.3 and  $\frac{\widehat{h}}{\det^2 h} \in L^2(K)$ . For the second term, we have  $\|\partial \det h_\nu\|_{L^2(K)}$  uniformly in  $\nu$ , and

$$\left| \frac{\widehat{h}}{\det^2 h} - \frac{\widehat{h}_\nu}{\det^2 h_\nu} \right|^2 \leq \frac{4C^2}{\epsilon^4} \in L^1(K).$$

Therefore, it goes to zero by the Lebesgue convergence theorem.



Finally, taking weak limits of  $(\diamond)$ , we have

$$\partial \left( \frac{1}{\det h} \widehat{h} \right) = \partial \left( \frac{1}{\det h} \right) \widehat{h} + \frac{1}{\det h} \partial \widehat{h}$$

in the sense of distributions.  $\square$

Using the above lemmas, we can get the positive version of Theorem 11.2. More precisely, we have the following theorem.

**THEOREM 11.6.** *Let  $X$  be a complex manifold with a positive Hermitian form  $\omega$ , and let  $h$  be a singular Hermitian metric that is Griffiths semi-positive. Moreover let  $\{h_\nu\}_{\nu=1}^\infty$  be any approximating sequence of smooth Hermitian metrics with Griffiths semi-positive curvature, increasing to  $h$ .*

*If there exists  $C > 0$  such that  $\det h < C$ , then*

- (1)  $\widetilde{\theta}_{h_\nu} \in L_{loc}^2(X)$  uniformly in  $\nu$ , and  $\widetilde{\theta}_h \in L_{loc}^2(X)$ ,
- (2)  $\widetilde{\theta}_{h_\nu}$  weakly converges to  $\widetilde{\theta}_h$  in  $L_{loc}^2(X)$ , and
- (3)  $\widetilde{\Theta}_{h_\nu} \in L_{loc}^1(X)$  uniformly in  $\nu$ ,  $\widetilde{\Theta}_h$  has measure coefficients, and  $\widetilde{\Theta}_{h_\nu}$  weakly converges to  $\widetilde{\Theta}_h$  in the sense of measures.

**PROOF.** First of all, the dual metric  $h^*$  satisfies the assumption of the above lemmas. Repeating the proof of Lemma 11.5, we get

$$\partial \left( \frac{1}{\det h^*} \widehat{h^*} h^* \right) = \left( \partial \left( \frac{1}{\det h^*} \right) \right) \widehat{h^*} h^* + \frac{1}{\det h^*} \partial (\widehat{h^*} h^*).$$

Using Lemma 11.3, we obtain

$$\begin{aligned} 0 &= \partial(h^{*-1} h^*) \\ &= \partial \left( \frac{1}{\det h^*} \widehat{h^*} h^* \right) \\ &= \left( \partial \left( \frac{1}{\det h^*} \right) \right) \widehat{h^*} h^* + \frac{1}{\det h^*} (\partial \widehat{h^*}) h^* + \frac{1}{\det h^*} \widehat{h^*} (\partial h^*) \\ &= (\partial h^{*-1}) h^* + h^{*-1} (\partial h^*) \end{aligned}$$

in the sense of distributions, hence we have  $\theta_h = -{}^t \theta_{h^*}$ . For each  $\nu$ , we also have  $\theta_{h_\nu} = -{}^t \theta_{h_\nu^*}$ . Using Theorem 11.2, we can prove the part (1) and (2). Part (3) also follows since

$$\Theta_h = \bar{\partial} \theta_h = -{}^t \bar{\partial} \theta_{h^*} = -{}^t \Theta_{h^*}.$$

$\square$

**REMARK 11.7.** If  $h$  is smooth, the above equation  $\Theta_h = -{}^t \Theta_{h^*}$  is a well-known fact. However, if  $h$  is singular, differential is in the sense of distributions. Hence  $0 = \partial h^{-1} h + h^{-1} \partial h$  makes no sense. For this reason, we do not know whether the equation  $\Theta_h = -{}^t \Theta_{h^*}$  holds when  $h$  is singular.

12.  $L^2$  ESTIMATES FOR TOP DEGREE FORMS

In this section, we will prove Theorem 9.1. First of all, we need some lemmas. Throughout section 12 and 13,  $X$  denotes an  $n$ -dimensional projective manifold,  $\omega$  denotes a Kähler form on  $X$ ,  $E$  denotes a holomorphic vector bundle over of rank  $r$ , and  $h$  denotes a Hermitian metric on  $E$ . Let  $L^2_{(p,q)}(X, E, h, \omega)$  (resp.  $L^2_{loc(p,q)}(X, E, h, \omega)$ ) be the space of square integrable (resp. locally square integrable)  $E$ -valued  $(p, q)$ -forms on  $X$ .

LEMMA 12.1. *Let  $h'$  be a smooth and strictly Griffiths  $\delta_\omega$ -positive metric on  $E$ . For any  $u \in L^2_{(n,n)}(X, E, h')$ , there exists  $g \in L^2_{(n,n-1)}(X, E, h')$  such that*

$$\bar{\partial}g = u, \quad \|g\|_{L^2}^2 \leq \frac{1}{\delta n} \|u\|_{L^2}^2.$$

PROOF. We will compute the Hermitian operator  $[\sqrt{-1}\Theta_{h'}, \Lambda]$ , where  $\Lambda$  is the adjoint operators of  $L$  which is defined by  $Lu = \omega \wedge u$ , and  $[\ , \ ]$  is graded Lie bracket. Writing

$$\begin{aligned} \sqrt{-1}\Theta_{h'} &= \sqrt{-1} \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j\bar{k}\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu, \\ \omega &= \sqrt{-1} \sum_{1 \leq j \leq n} dz_j \wedge d\bar{z}_j \end{aligned}$$

at a fixed point  $p \in X$  as in Section 10, we compute the operator  $[\sqrt{-1}\Theta_{h'}, \Lambda]$ . For any  $E$ -valued  $(n, n)$ - $L^2$  form  $u = \sum_{\lambda=1}^r u_\lambda dz_1 \wedge \dots \wedge d\bar{z}_n \otimes e_\lambda$ , we get

$$[\sqrt{-1}\Theta_{h'}, \Lambda] u = \sqrt{-1}\Theta_{h'} \Lambda u = \sum_{1 \leq j \leq n, 1 \leq \lambda, \mu \leq r} c_{j\bar{j}\lambda\mu} u_\lambda dz_1 \wedge \dots \wedge d\bar{z}_n \otimes e_\mu.$$

Therefore, the following equations hold

$$\begin{aligned} &([\sqrt{-1}\Theta_{h'}, \Lambda] u, u)_{h'} \\ &= \left( \sum_{1 \leq j \leq n, 1 \leq \lambda, \mu \leq r} c_{j\bar{j}\lambda\mu} u_\lambda dz_1 \wedge \dots \wedge d\bar{z}_n \otimes e_\mu, \sum_{1 \leq \nu \leq r} u_\nu dz_1 \wedge \dots \wedge d\bar{z}_n \otimes e_\nu \right)_{h'} \\ &= \sum_{1 \leq j \leq n, 1 \leq \lambda, \mu \leq r} c_{j\bar{j}\lambda\mu} u_\lambda \bar{u}_\mu \\ &= \sum_{1 \leq j \leq n} \left( \Theta_{j\bar{j}}^{h'} \left( \sum_{\lambda=1}^r u_\lambda \right), \left( \sum_{\lambda=1}^r u_\lambda \right) \right)_{h'}. \end{aligned}$$

For each  $j$ , by taking a vector field  $\xi = \frac{\partial}{\partial z_j}$  we show that

$$\left( \Theta_{j\bar{j}}^{h'} \left( \sum_{\lambda=1}^r u_\lambda \right), \left( \sum_{\lambda=1}^r u_\lambda \right) \right)_{h'} \geq \delta |u|_{h'}^2$$

from the definition of a strictly Griffiths  $\delta$ -positive metric. Then we get

$$\sum_{1 \leq j \leq n} \left( \Theta_{j\bar{j}}^{h'} \left( \sum_{\lambda=1}^r u_\lambda \right), \left( \sum_{\lambda=1}^r u_\lambda \right) \right)_{h'} \geq \delta n |u|_{h'}^2,$$

and we see that the operator  $[\sqrt{-1}\Theta_{h'}, \Lambda]$  is positive definite. Hence it follows that

$$([\sqrt{-1}\Theta_{h'}, \Lambda]^{-1} u, u)_{h'} \leq \frac{1}{\delta n} (u, u)_{h'},$$

$$\int_X ([\sqrt{-1}\Theta_{h'}, \Lambda]^{-1} u, u)_{h'} dV_\omega \leq \frac{1}{\delta n} \|u\|_{L^2}^2 < +\infty.$$

We then can conclude that there exists  $g \in L^2_{(n, n-1)}(X, E, h')$  such that  $\bar{\partial}g = u$  and  $\|g\|_{L^2}^2 \leq \frac{1}{\delta n} \|u\|_{L^2}^2$ , thanks to Hörmander's  $L^2$  estimate (cf. [Dem, Theorem 5.1]).  $\square$

We prove one more lemma. It is a generalization of the argument of [Rau12, Section 4].

**LEMMA 12.2.** *Let  $Z$  be a  $n$ -dimensional submanifold of  $\mathbb{C}^N$ ,  $U$  be an open neighborhood of  $Z$  in  $\mathbb{C}^N$ , and  $p: U \rightarrow Z$  be a holomorphic retraction map such that  $p \circ i = \text{id}_Z$ , where  $i$  is an inclusion map  $i: Z \rightarrow U$ . We also fix a Kähler metric  $\omega$  such that  $\omega$  is  $\partial\bar{\partial}$ -exact on  $Z$ . Assume that there is a trivial vector bundle  $E$  over  $Z$  equipped with a singular Hermitian metric  $h$  which is strictly Griffiths  $\delta_\omega$ -positive in the sense of Definition 10.6.*

*Then we can take a sequence of smooth Hermitian metrics  $\{h_\nu\}_{\nu=1}^\infty$  approximating  $h$  on any relatively compact subset of  $Z$  such that  $h_\nu$  is strictly Griffiths  $\delta_\omega$ -positive for each  $\nu$ .*

**PROOF.** The assumption implies that there exists a smooth Kähler potential  $\varphi$  of  $\omega$  on  $Z$  such that  $\sqrt{-1}\partial\bar{\partial}\varphi = \omega$  and  $h \cdot e^{\delta\varphi}$  is Griffiths semi-positive. We see that  $p^*(h \cdot e^{\delta\varphi})$  is also Griffiths semi-positive from [PT18, Lemma 2.3.2].

Then we take an approximate identity, i.e.  $\chi \in C_c^\infty(U)$ ,  $\chi \geq 0$ ,  $\chi(z) = \chi(|z|)$ ,  $\int_{\mathbb{C}^n} \chi dV = 1$ , and  $\chi_\nu(z) = \nu^n \chi(\nu z)$ . Since  $p^*E$  is also trivial over  $U$ , we consider smooth Griffiths semi-positive Hermitian metrics  $((p^*(h \cdot e^{\delta\varphi}))^* \chi_\nu)^*$  on  $p^*E$  increasing pointwise to  $p^*(h \cdot e^{\delta\varphi})$  on any relatively compact subset of  $U$ . Set  $g_\nu := i^*((p^*(h \cdot e^{\delta\varphi}))^* \chi_\nu)^*$  and  $h_\nu := g_\nu e^{-\delta\varphi}$ . Since  $i$  is an inclusion map, we see that  $\{g_\nu\}$  is a sequence of smooth Hermitian metrics on  $E$  with semi-positive Griffiths curvature, increasing pointwise to  $h \cdot e^{\delta\varphi}$  on any relatively compact subset of  $Z$ . Hence,  $h_\nu$  is strictly Griffiths  $\delta_\omega$ -positive and increasing to  $h$  on any relatively compact subset of  $Z$ .  $\square$

Then we will prove Theorem 9.1.

**PROOF OF THEOREM 9.1.** By Serre's GAGA, there exists a Zariski open subset  $Z \neq \emptyset$  such that  $Z \subset X \setminus S$ ,  $E|_Z$  is a trivial over  $Z$ , and  $\omega$  is  $\partial\bar{\partial}$ -exact on  $Z$  for the reason that  $X$  is a projective manifold. We can take  $Z$  as a Stein open subset. Then  $Z$  can be properly imbedded in  $\mathbb{C}^N$  for some large  $N$ . We regard  $Z$  as a submanifold of  $\mathbb{C}^N$ . From Siu's result in [Siu76], there exists an open neighborhood  $U$  of  $Z$  in  $\mathbb{C}^N$  which is a holomorphic retract of  $Z$ . Let  $p: U \rightarrow Z$  be a holomorphic retraction map such that  $p \circ i = \text{id}_Z$ , where  $i$  is an inclusion map  $i: Z \rightarrow U$ . Since  $E|_Z$  is a trivial vector bundle,  $p^*E$  is also trivial on  $U$ . We can take an exhaustion  $\{Z_j\}_{j=1}^\infty$  of  $Z$ , where each  $Z_j$  is a relatively compact Stein subdomain. From the results of Lemma 12.2, we can take a sequence of smooth Hermitian

metrics  $\{h_\nu\}_{\nu=1}^\infty$  with strictly Griffiths  $\delta_\omega$ -positive curvature, increasing pointwise to  $h$  on  $Z_j$  for all  $j$ .

For fixed  $j$ , we get the following inequality

$$\int_{Z_j} |f|_{h_\nu, \omega}^2 dV_\omega \leq \int_{Z_j} |f|_{h, \omega}^2 dV_\omega \leq \int_X |f|_{h, \omega}^2 dV_\omega < +\infty$$

for the reason that  $\{h_\nu\}_{\nu=1}^\infty$  is increasing to  $h$  and  $f \in L^2_{(n,n)}(X, E, h, \omega)$ . Applying Lemma 12.1 and Lemma 12.2 on  $Z_j$ , we get  $E$ -valued  $(n, n-1)$ - $L^2$  form  $g_\nu$  on  $Z_j$  such that  $\bar{\partial}g_\nu = f$ , and

$$\int_{Z_j} |g_\nu|_{h_\nu, \omega}^2 dV_\omega \leq \frac{1}{\delta n} \int_{Z_j} |f|_{h_\nu, \omega}^2 dV_\omega \leq \frac{1}{\delta n} \int_{Z_j} |f|_{h, \omega}^2 dV_\omega < +\infty,$$

since Lemma 12.1 also holds for Stein manifolds. Moreover, the right-hand side of above inequalities has upper bound independent of  $\nu$ . Therefore, we can find a weakly convergent subsequence  $\{g_{\nu_k}\}_{k=1}^\infty$  by using a diagonal argument and monotonicity of  $\{h_\nu\}_{\nu=1}^\infty$ . It follows that  $\{g_{\nu_k}\}_{k=1}^\infty$  weakly converges in  $L^2_{(n,n)}(Z_j, E, h_\nu, \omega)$  and the weak limit  $g_j$  is in  $L^2_{(n,n)}(Z_j, E, h, \omega)$ , i.e.

$$\int_{Z_j} |g_j|_{h, \omega}^2 dV_\omega \leq \frac{1}{\delta n} \int_{Z_j} |f|_{h, \omega}^2 dV_\omega \leq \frac{1}{\delta n} \int_Z |f|_{h, \omega}^2 dV_\omega,$$

equivalently,

$$\delta n \int_{Z_j} |g_j|_{h, \omega}^2 dV_\omega \leq \int_Z |f|_{h, \omega}^2 dV_\omega.$$

The right-hand side of the above inequality is independent of  $j$ , repeating a diagonal argument and taking weak limits, then we obtain  $E$ -valued  $(n, n-1)$ - $L^2$  form  $g$  on  $Z$  such that

$$\int_Z |g|_{h, \omega}^2 dV_\omega \leq \frac{1}{\delta n} \int_Z |f|_{h, \omega}^2 dV_\omega.$$

Letting  $g$  be 0 on  $X \setminus Z$ ,  $g$  is in  $L^2_{(n,n-1)}(X, E, h, \omega)$ , and we get

$$\bar{\partial}g = f, \int_X |g|_{h, \omega}^2 dV_\omega \leq \frac{1}{\delta n} \int_X |f|_{h, \omega}^2 dV_\omega$$

on  $X$  from the following lemma. □

**LEMMA 12.3.** ([B-book, Lecture 5]) *Let  $X$  be a complex manifold and let  $S$  be a complex hypersurface in  $X$ . Let  $f$  and  $g$  be (possibly bundle valued) forms with  $L^2_{loc}$  coefficients on  $X$  satisfying  $\bar{\partial}g = f$  on  $X \setminus S$ . Then the equation  $\bar{\partial}g = f$  also holds on  $X$  in the sense of distributions.*

**REMARK 12.4.** We only have to show  $\bar{\partial}g = f$  on a local open set  $U \subset X$ , where  $U$  contains points of  $X \setminus Z$  and  $E|_U$  is trivial on  $U$ . Since  $h$  is Griffiths semi-positive, there exists a sequence of smooth Hermitian metrics  $\{h_\nu\}_{\nu=1}^\infty$  increasing to  $h$  in this local setting. For this reason, if  $f$  and  $g$  are  $L^2_{loc}$  with respect to the singular Hermitian metric  $h$ , then  $f$  and  $g$  are  $L^2_{loc}$  with respect to the smooth Hermitian metric  $h_0$ . Therefore we can apply Lemma 12.3 to Theorem 9.1.

Before we prove Corollary 9.2, we introduce the definition of the Lelong number of a singular Hermitian metric on a line bundle.

DEFINITION 12.5. Let  $L \rightarrow X$  be a line bundle over  $X$ , and  $g$  be a singular Hermitian metric on  $L$ .

If  $g$  is semi-negative, the Lelong number of  $g$  at  $x$  is defined as

$$\nu(\log g, x) = \liminf_{z \rightarrow x} \frac{\log g(z)}{\log |z - x|}.$$

If  $g$  is semi-positive, the Lelong number of  $g$  at  $x$  is defined as

$$\nu(-\log g, x) = \liminf_{z \rightarrow x} \frac{-\log g(z)}{\log |z - x|}.$$

Here  $x$  is a point of  $X$ ,  $z$  is a coordinate around  $x$ . Taking a basis for  $L$ , we represent  $g$  as a positive function.

The above definition is independent of the choice of local coordinates. If  $g$  is semi-negative (resp. positive),  $\log g$  (resp.  $-\log g$ ) is locally plurisubharmonic. Therefore Definition 12.5 coincide with the usual definition of the Lelong number of a closed positive current (cf. [Dem, Theorem 2.8]).

Using these notions, we will prove Corollary 9.2. We recall that  $\det h$  is a semi-positive (resp. negative) if  $h$  is a Griffiths semi-positive (resp. negative) singular Hermitian metric (cf. [HPS18, Proposition 25.1], [Rau15, Proposition 1.3]).

PROOF OF COROLLARY 9.2. Let  $C_{(n,n)}^\infty(X, E)$  be the space of smooth  $E$ -valued  $(n, n)$  forms on  $X$  and  $\mathcal{U} = \{U_i\}_{i \in I}$  be a locally finite open cover of  $X$  such that  $U_i$  are biholomorphic to a polydisc. For the reason that the Lelong number of  $\frac{1}{2} \log \det h^*$  is less than 1 for a point  $x \in X$ , we have

$$e^{-\log \det h^*} = \frac{1}{\det h^*} \in L_{loc}^1(X)$$

from the results of Skoda [Sko72]. Then  $|s|_h^2$  is an  $L_{loc}^1$  form for any  $s \in C_{(n,n)}^\infty(X, E)$  since  $h = \frac{1}{\det h^*} \hat{h}^*$  and each element of  $\hat{h}^*$  is locally bounded [PT18, Remark 2.2.3]. Here  $\hat{h}^*$  is the adjugate matrix of  $h^*$ . Hence, there is an inclusion map

$$C_{(n,n)}^\infty(X, E) \hookrightarrow L_{loc(n,n)}^2(X, E, h, \omega).$$

We know that  $U_{i_0} \cap \cdots \cap U_{i_l}$  is a pseudoconvex domain for all  $\{i_0, \dots, i_l\} \subset I$ . Taking a sequence of smooth Hermitian metrics  $\{h_\nu\}_{\nu=1}^\infty$  approximating  $h$  and repeating the argument of the proof of Theorem 9.1, we can solve the  $\bar{\partial}$ -equation on  $U_{i_0} \cap \cdots \cap U_{i_l}$  with respect to  $h$ . Hence, we have the isomorphism

$$\begin{aligned} & H^n(X, K_X \otimes E) \\ & \cong \frac{\{f \in L_{loc(n,n)}^2(X, E, h, \omega)\}}{\{h \in L_{loc(n,n)}^2(X, E, h, \omega) : \bar{\partial}g = h, g \in L_{loc(n,n-1)}^2(X, E, h, \omega)\}} \end{aligned}$$

from the results of sheaf cohomology. This is a singular version of isomorphism theorems (cf. [Ohs82, Proposition 3.1]). For any  $f \in L_{loc(n,q)}^2(X, E, h, \omega)$ , we can obtain an  $E$ -valued  $(n, n-1)$ - $L^2$  form  $g$  on  $X$  such that  $\bar{\partial}g = f$  and  $g \in L_{(n,n-1)}^2(X, E, h, \omega)$  by using Theorem 9.1. We can conclude that  $H^n(X, K_X \otimes E) = 0$ .  $\square$

### 13. DEMAILLY AND SKODA'S THEOREM AND $L^2$ ESTIMATES

We will prove Theorem 9.3. The proof of Theorem 9.3 is same as that of Theorem 9.1. To begin with, we prepare a lemma with respect to a smooth Hermitian metric.

LEMMA 13.1. *Let  $h'$  be a smooth Hermitian metric that is strictly  $\delta$ -Griffiths positive. Then the metric  $h' \otimes \det h'$  is strictly  $\delta r$ -Nakano positive, i.e. the inequality*

$$\Theta_{h' \otimes \det h'}(\tau, \tau) \geq \delta r |\tau|_{h' \otimes \det h', \omega}^2$$

*holds for all non-zero tensors  $\tau \in T_X \otimes E \otimes \det E$ .*

PROOF. We write

$$\sqrt{-1}\Theta_{h'} = \sqrt{-1} \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j\bar{k}\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

as in Section 10, where  $(e_1, \dots, e_r)$  is orthonormal with respect to  $h'$  at a fixed point. It follows that

$$\Theta_{E \otimes \det E} = \Theta_E + \text{Tr } \Theta_E \otimes h'.$$

Thus we should prove the following inequality

$$(\Theta_E + \text{Tr } \Theta_E \otimes h')(u, u) \geq \delta r |u|_{h', \omega}^2$$

for any section  $u = \sum u_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_X \otimes E$ . We have the inequality

$$(\Theta_E + \text{Tr } \Theta_E \otimes h')(u, u) \geq \sum_{1 \leq j, k \leq n, 1 \leq \lambda \leq r} c_{j\bar{k}\lambda\lambda} u_{j\lambda} \bar{u}_{k\lambda}$$

from the result of [Dem, Proposition 10.14]. Computing the right-hand side of the above inequality, we obtain

$$\sum_{1 \leq j, k \leq n, 1 \leq \lambda \leq r} c_{j\bar{k}\lambda\lambda} u_{j\lambda} \bar{u}_{k\lambda} = \sum_{1 \leq j, k \leq n, 1 \leq \lambda \leq r} (\Theta_{j\bar{k}}^{h'} e_\lambda, e_\lambda)_{h'} u_{j\lambda} \bar{u}_{k\lambda} \geq \delta r |u|_{h', \omega}^2$$

for the reason that  $h'$  is  $\delta$ -Griffiths positive.  $\square$

Then we will prove Theorem 9.3.

PROOF OF THEOREM 9.3. We take  $Z, \{Z_j\}_{j=1}^\infty, U$ , and  $p$  as in the proof of Theorem 9.1. Thanks to Lemma 12.2, we can take a sequence of smooth Hermitian metrics  $\{h_\nu\}_{\nu=1}^\infty$  with strictly Griffiths  $\delta_\omega$ -positive curvature, approximating  $h$  on any relatively compact subset of  $Z$ . From the above Lemma 13.1, the Hermitian metric  $h_\nu \otimes \det h_\nu$  is  $\delta r$  Nakano

positive. Then for each  $\nu$ , there exists an  $E \otimes \det E$ -valued  $(n, q-1)$ - $L^2$  form  $g_\nu$  on fixed  $Z_j$  such that  $\bar{\partial}g_\nu = f$ , and

$$\begin{aligned} \int_{Z_j} |g_\nu|_{h_\nu \otimes \det h_\nu, \omega}^2 dV_\omega &\leq \frac{1}{\delta q r} \int_{Z_j} |f|_{h_\nu \otimes \det h_\nu, \omega}^2 dV_\omega \\ &\leq \frac{1}{\delta q r} \int_{Z_j} |f|_{h \otimes \det h, \omega}^2 dV_\omega < +\infty \end{aligned}$$

for the reason that  $\{h_\nu \otimes \det h_\nu\}_{\nu=1}^\infty$  is also increasing pointwise to  $h \otimes \det h$  on any relatively compact subset of  $Z$  and we can apply Hörmander's  $L^2$  estimates to it (cf. [Dem-book, Chapter VIII, Theorem 6.1], [Rau12]). Taking weak limits  $\nu \rightarrow \infty$  as in the proof of Theorem 9.1, we can take an  $E \otimes \det E$ -valued  $(n, q-1)$ - $L^2$  form  $g_j$  on fixed  $Z_j$  such that

$$\int_{Z_j} |g_j|_{h \otimes \det h, \omega}^2 dV_\omega \leq \frac{1}{\delta q r} \int_{Z_j} |f|_{h \otimes \det h, \omega}^2 dV_\omega \leq \frac{1}{\delta q r} \int_Z |f|_{h \otimes \det h, \omega}^2 dV_\omega.$$

Moreover taking limits  $j \rightarrow \infty$  as in the proof of Theorem 9.1, we obtain an  $E \otimes \det E$ -valued  $(n, q-1)$ - $L^2$  form  $g$  on  $Z$  such that

$$\int_Z |g|_{h \otimes \det h, \omega}^2 dV_\omega \leq \frac{1}{\delta q r} \int_Z |f|_{h \otimes \det h, \omega}^2 dV_\omega.$$

Letting  $g$  be 0 on  $X \setminus Z$ , we see that  $g$  is in  $L^2_{(n, q-1)}(X, E \otimes \det E, h \otimes \det h, \omega)$ . Then we get  $\bar{\partial}g = f$  on  $X$ , and

$$\int_X |g|_{h \otimes \det h, \omega}^2 dV_\omega \leq \frac{1}{\delta q r} \int_X |f|_{h \otimes \det h, \omega}^2 dV_\omega.$$

□

Theorem 9.3 lead to Corollary 9.4. The proof of it is the same as the one of Corollary 9.2.

PROOF OF COROLLARY 9.4. Locally, we have

$$h \otimes \det h = \frac{1}{\det^2 h^\star} \widehat{h}^\star,$$

where  $\widehat{h}^\star$  is the adjugate matrix of  $h^\star$ . From the results of Skoda [Sko72], we have

$$\frac{1}{\det^2 h^\star} \in L^1_{loc}(X).$$

Repeating the argument of the proof of Corollary 9.2, we can conclude that

$$H^q(X, K_X \otimes E \otimes \det E) = 0$$

for  $q > 0$ .

□

We have an application of Corollary 9.4. We show the following example.

EXAMPLE 13.2. Let  $V$  be a complex vector space of dimension  $n+1$ , and  $X = P(V) := (V \setminus \{0\})/\mathbb{C}^* = \mathbb{P}^n$  be the projective space of  $V$ . We also let  $\mathcal{O}(-1)$  denote the tautological line bundle, and  $Q$  denote the quotient bundle  $V/\mathcal{O}(-1)$ . Then there do not exist any Griffiths semi-positive singular Hermitian metrics satisfying strict Griffiths  $\delta_\omega$ -positivity outside some proper analytic subset  $S$  and  $\nu(-\log \det h, x) < 1$  for all  $x \in X$  on  $Q$ .

PROOF. There exists an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow V \longrightarrow Q \longrightarrow 0$$

over  $X = \mathbb{P}^n$ . We have the isomorphisms

$$\det Q \cong \mathcal{O}(1),$$

$$T\mathbb{P}^n = Q \otimes \mathcal{O}(1) \cong Q \otimes \det Q$$

from [Dem-book, Chapter VII, Example 8.4]. If  $Q$  has a Griffiths semi-positive singular Hermitian metric satisfying strict Griffiths  $\delta_\omega$ -positivity outside some proper analytic subset  $S$  and  $\nu(-\log \det h, x) < 1$ , the cohomology group  $H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes Q \otimes \det Q) = H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T\mathbb{P}^n)$  vanishes for any positive integer  $q > 0$  from Corollary 9.4. However, the Serre duality theorem implies that

$$H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T\mathbb{P}^n)^* \cong H^{n-q}(\mathbb{P}^n, T^*\mathbb{P}^n) \cong H^{(1, n-q)}(\mathbb{P}^n, \mathbb{C}) \cong \begin{cases} \mathbb{C} & (q = n-1) \\ 0 & (q \neq n-1). \end{cases}$$

This contradicts to Corollary 9.4. □



## Part 4. Nakano positivity of singular Hermitian metrics and vanishing theorems of Demailly-Nadel-Nakano type

ABSTRACT. In this article, we propose a general definition of Nakano semi-positivity of singular Hermitian metrics on holomorphic vector bundles. By using this positivity notion, we establish  $L^2$ -estimates for holomorphic vector bundles with Nakano positive singular Hermitian metrics. We also show vanishing theorems, which generalize both Nakano type and Demailly-Nadel type vanishing theorems.

### 14. INTRODUCTION

In Part 4, we discuss the content in [Ina4]. We study positivity notions of singular Hermitian metrics on holomorphic vector bundles. On holomorphic line bundles, positivity of singular Hermitian metrics has been widely studied in various ways. In this situation, a singular Hermitian metric is semi-positive if and only if the corresponding local weight is plurisubharmonic. Hence, we can use complex analytic methods to study properties of positive singular Hermitian metrics.

For holomorphic vector bundles, notions of singular Hermitian metrics were initially observed in [deC98]. Then Berndtsson and Paun investigated general notions of singular Hermitian metrics in [BP08]. Properties and positivity notions of singular Hermitian metrics have been investigated by many people.

However, it turns out that we cannot always define the curvature currents with measure coefficients. This example was found by Raufi in [Rau15]. Hence, we need to define positivity notions without using curvature currents. We have such a characterization for Griffiths semi-positivity or semi-negativity (see Proposition 15.4). On the other hand, it was not known the way to define Nakano positivity of singular Hermitian metrics without using the expression of the curvature currents.

Recently, new positivity notions of singular Hermitian metrics have been introduced and studied by several people (cf. [DNW19], [DNWZ20], [DWZZ18], [DWZZ20], Part 1). These properties are defined via Hörmander type  $L^p$ -estimates or Ohsawa-Takegoshi type  $L^p$ -extension theorems for  $p > 0$ . In [DNWZ20], Deng, Ning, Wang, and Zhou introduced *the optimal  $L^p$ -estimate condition* and obtained a new characterization of Nakano positive smooth Hermitian metrics by using this condition (see Theorem 15.7).

Applying and modifying the above result, we get the following definition. Before describing the definition of Nakano semi-positivity, we introduce the notion of Stein coordinates. Throughout this paper, we let  $X$  be an  $n$ -dimensional complex manifold, let  $E \rightarrow X$  be a holomorphic vector bundle of finite rank  $r > 0$ , and let  $h$  be a singular Hermitian metric on  $E$  (see Definition 15.10).

DEFINITION 14.1. Let  $\Omega$  be an  $n$ -dimensional Stein manifold and  $\iota : \Omega \rightarrow X$  be a holomorphic map from  $\Omega$  to  $X$ . We say that  $(\Omega, \iota)$  is a *Stein coordinate* around  $x_0 \in X$  if and only if the following conditions are satisfied:

- (1)  $\iota : \Omega \rightarrow X$  is an injective holomorphic map, i.e.  $\Omega \rightarrow \iota(\Omega)$  defines a biholomorphic map.
- (2)  $\iota(\Omega)$  is an open subset of  $X$  such that  $x_0 \in \iota(\Omega)$ .

By definition, every complex manifold admits a Stein coordinate around any point. Using this notion, we define the following positivity.

DEFINITION 14.2. Suppose that  $h$  is a Griffiths semi-positive singular Hermitian metric. We say that  $h$  is *Nakano semi-positive in the sense of singular Hermitian metrics* or simply *Nakano semi-positive* if for any Stein coordinate  $(\Omega, \iota)$  around any point  $x \in X$  such that  $E|_{\iota(\Omega)}$  is trivial on  $\iota(\Omega)$ , for any Kähler form  $\omega_\Omega$  on  $\Omega$ , for any smooth strictly plurisubharmonic function  $\psi$  on  $\Omega$ , for any positive integer  $q$  such that  $1 \leq q \leq n$ , and for any  $\bar{\partial}$ -closed  $f \in L^2_{(n,q)}(\Omega, \iota^*E; \omega_\Omega, (\iota^*h)e^{-\psi})$ , there exists  $u \in L^2_{(n,q-1)}(\Omega, \iota^*E; \omega_\Omega, (\iota^*h)e^{-\psi})$  satisfying  $\bar{\partial}u = f$  and

$$\int_{\Omega} |u|_{(\omega_\Omega, \iota^*h)}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_{\Omega} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, \iota^*h)} e^{-\psi} dV_{\omega_\Omega},$$

where  $B_{\omega_\Omega, \psi} = [\sqrt{-1} \partial \bar{\partial} \psi \otimes Id_E, \Lambda_{\omega_\Omega}]$ . Here we suppose that the right-hand side is finite (for detailed notation, see Notation in Section 15).

REMARK 14.3. In this paper, we always assume the Griffiths semi-positivity of  $h$  when we say that  $h$  is Nakano semi-positive in the sense of singular Hermitian metrics. We do not know whether the assumption that  $h$  is Griffiths semi-positive is necessary or not (see Question 19.3 in Section 19).

We explain the reason that we use the above condition to define Nakano positivity in Section 15. Here we only assume that  $X$  is a complex manifold, not Hermitian or Kähler. Hence, we can define Nakano semi-positivity in a general setting. That is one of the advantages of Definition 14.2.

In this setting, we can also show the following result, which is a generalization of Demailly and Skoda's theorem in the singular setting.

THEOREM 14.4. *Let  $h$  be a Griffiths semi-positive singular Hermitian metric on  $E$ . Then  $h \otimes \det h$  is a Nakano semi-positive singular Hermitian metric on  $E \otimes \det E$ .*

Next, we consider the case that  $X$  admits a Kähler metric  $\omega_X$ . In this situation, we can define strict Nakano positivity for singular Hermitian metrics in a simple way (see Definition 15.16). By using this notion, we prove the following  $L^2$ -estimate.

THEOREM 14.5. *Let  $(X, \omega_X)$  be a projective manifold and a Kähler metric on  $X$ , and  $q$  be a positive integer. We assume that  $(E, h)$  is strictly Nakano  $\delta_{\omega_X}$ -positive in the sense of Definition 15.16 on  $X$ . Then for any  $\bar{\partial}$ -closed  $f \in L^2_{(n,q)}(X, E; \omega_X, h)$ , there exists  $u \in L^2_{(n,q-1)}(X, E; \omega_X, h)$  satisfying  $\bar{\partial}u = f$  and*

$$\int_X |u|_{(\omega_X, h)}^2 dV_{\omega_X} \leq \frac{1}{\delta q} \int_X |f|_{(\omega_X, h)}^2 dV_{\omega_X}$$

This estimate generalizes usual  $L^2$ -estimates (cf. Theorem 15.9). Applying Theorem 14.5, we get the following vanishing theorem. This is a generalization of both the Nakano vanishing theorem and the Demailly-Nadel vanishing theorem.

**THEOREM 14.6.** *Let  $(X, \omega_X)$  be a projective manifold and a Kähler metric on  $X$ . We assume that  $(E, h)$  is strictly Nakano  $\delta_{\omega_X}$ -positive in the sense of Definition 15.16 on  $X$ . Then the  $q$ -th cohomology group of  $X$  with coefficients in the sheaf of germs of holomorphic sections of  $K_X \otimes \mathcal{E}(h)$  vanishes for  $q > 0$ :*

$$H^q(X, K_X \otimes \mathcal{E}(h)) = 0,$$

where  $\mathcal{E}(h)$  is the sheaf of germs of locally square integrable holomorphic sections of  $E$  with respect to  $h$ .

Here we can prove that the sheaf  $\mathcal{E}(h)$  is coherent when  $h$  is a Nakano (semi-)positive singular Hermitian metric (see Proposition 17.4). As an application of Theorem 14.4 and Theorem 14.6, we get the following result.

**THEOREM 14.7.** *Let  $(X, \omega_X)$  be a projective manifold and a Kähler metric on  $X$ . We assume that  $h$  is strictly Griffiths  $\delta_{\omega_X}$ -positive on  $X$  (see Definition 15.15). Then the  $q$ -th cohomology group of  $X$  with coefficients in the sheaf of germs of holomorphic sections of  $K_X \otimes \mathcal{E}(h \otimes \det h)$  vanishes for  $q > 0$ :*

$$H^q(X, K_X \otimes \mathcal{E}(h \otimes \det h)) = 0.$$

Theorem 14.7 can be regarded as a generalization of the Griffiths vanishing theorem (cf. [Dem-book, Chapter VII, Corollary 9.4]) If the Lelong number  $\nu(\det h, x) < 1$  for all points  $x \in X$ , this kind of result was obtained by us (cf. Corollary 9.4).

The organization of this paper is as follows. We start with Section 15 a general discussion of smooth and singular Hermitian metrics on holomorphic vector bundles. Here we introduce several Hörmander type conditions. In Section 16, we explain the result of Demailly and Skoda. Here we also generalize the result in the singular setting. In Section 17, we establish  $L^2$ -estimates and vanishing theorems for holomorphic vector bundles with Nakano positive singular Hermitian metrics. In Section 18, we verify that our definition of Nakano semi-positivity is an appropriate positivity notion when we compare it with the definition of Griffiths semi-positivity. Finally, in Section 19, we propose some questions which might be worth thinking about.

## 15. PRELIMINARIES

Throughout this paper, we use the following notations.

**NOTATION 15.1.** •  $K_X$  : the canonical line bundle of  $X$ .

- $dV_\omega := \frac{\omega^n}{n!}$  : the volume form determined by  $\omega$ .
- $E^*$  : the dual bundle of  $E$ .

- $h^*$  : the dual metric of  $h$  on  $E^*$ .
- $\mathcal{O}(E)$  : the sheaf of germs of local holomorphic sections of  $E$ .
- $C_{(p,q)}^k(X, E) := C^k(X, \wedge^{(p,q)} T_X^* \otimes E)$  for  $0 \leq k \leq +\infty$ .
- $\mathcal{D}_{(p,q)}(X, E)$  : the space of smooth sections of  $\wedge^{(p,q)} T_X^* \otimes E$  with compact support.
- $L_{(p,q)}^p(X, E; \omega, h)$  : the space of  $L^p$  sections of  $\wedge^{(p,q)} T_X^* \otimes E$  with respect to  $\omega$  and  $h$ .
- $\langle\langle \alpha, \beta \rangle\rangle_{(\omega, h)} := \int_X \langle \alpha, \beta \rangle_{(\omega, h)} dV_\omega$ .
- $\|\alpha\|_{(\omega, h)}^2 := \langle\langle \alpha, \alpha \rangle\rangle_{(\omega, h)}$ .
- $D'_\psi$  : the adjoint operator of  $D'_\psi$  with respect to  $\langle\langle \cdot, \cdot \rangle\rangle_{(\omega, h e^{-\psi})}$ .
- $\bar{D}'_\psi$  : the adjoint operator of  $\bar{D}$  with respect to  $\langle\langle \cdot, \cdot \rangle\rangle_{(\omega, h e^{-\psi})}$ .
- $\Delta'_\psi := D'_\psi D'^*_\psi + D'^*_\psi D'_\psi$ ,  $\Delta''_\psi = \bar{D} \bar{D}'_\psi + \bar{D}'_\psi \bar{D}$  with respect to  $\langle\langle \cdot, \cdot \rangle\rangle_{(\omega, h e^{-\psi})}$ .
- $L_\omega : C_{(p,q)}^\infty(X, E) \rightarrow C_{(p+1, q+1)}^\infty(X, E)$  : the operator defined by  $\omega \wedge \cdot$ .
- $\Lambda_\omega$  : the adjoint operator of  $L_\omega$ .
- $[\cdot, \cdot]$  : the graded Lie bracket.
- $\Delta^n(p; r) := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i - p_i| < r\}$  for  $p = (p_1, \dots, p_n) \in \mathbb{C}^n$ .
- $\Delta_r^n := \Delta^n(0; r)$ .

**15.1. Smooth Hermitian metrics.** We explain some definitions and properties of smooth Hermitian metrics. In this subsection, we always assume that a Hermitian metric  $h$  is smooth.

Let  $\Theta_{(E, h)}$  be the Chern curvature tensor of  $(E, h)$ . Taking a local coordinate  $(z_1, \dots, z_n)$  of  $X$  and an orthonormal frame  $(e_1, \dots, e_r)$  of  $E$ , we can write

$$\sqrt{-1}\Theta_{(E, h)} = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j\bar{k}\lambda\bar{\mu}} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu.$$

We identify the curvature tensor with a Hermitian form

$$\tilde{\Theta}_{(E, h)}(\tau, \tau) = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j\bar{k}\lambda\bar{\mu}} \tau_{j\lambda} \bar{\tau}_{k\mu}$$

for  $\tau = \sum_{j, \lambda} \tau_{j\lambda} \frac{\partial}{\partial z_i} \otimes e_\lambda \in T_X \otimes E$  on  $T_X \otimes E$ . By using this Hermitian form, we define the following positivity notions.

**DEFINITION 15.2.** The Hermitian vector bundle  $(E, h)$  is said to be :

- (1) *Griffiths positive* (resp. *Griffiths negative*) if we have  $\tilde{\Theta}_{(E, h)}(\xi \otimes s, \xi \otimes s) > 0$  (resp.  $\tilde{\Theta}_{(E, h)}(\xi \otimes s, \xi \otimes s) < 0$ ) for all non-zero elements  $\xi \in T_X, s \in E$ . We denote it by  $\Theta_{(E, h)} >_{\text{Grif.}} 0$  (resp.  $\Theta_{(E, h)} <_{\text{Grif.}} 0$ ).
- (2) *Nakano positive* (resp. *Nakano negative*) if we have  $\tilde{\Theta}_{(E, h)}(\tau, \tau) > 0$  (resp.  $\tilde{\Theta}_{(E, h)}(\tau, \tau) < 0$ ) for all non-zero elements  $\tau \in T_X \otimes E$ . We denote it by  $\Theta_{(E, h)} >_{\text{Nak.}} 0$  (resp.  $\Theta_{(E, h)} <_{\text{Nak.}} 0$ ).

Corresponding semi-positivity and semi-negativity are defined by relaxing the strict inequalities.

We can associate Nakano positivity with the positivity of the operator  $[\sqrt{-1}\Theta_{(E,h)}, \Lambda_\omega]$  from the following lemma.

LEMMA 15.3. (*cf.* [Dem-book, Chapter VII, Lemma 7.2], [DNWZ20, Lemma 2.5]) *Let  $(X, \omega)$  be a Kähler manifold. We have that  $(E, h) >_{\text{Nak.}} 0$  (resp.  $(E, h) \geq_{\text{Nak.}} 0$ ) if and only if the Hermitian operator  $[\sqrt{-1}\Theta_{(E,h)}, \Lambda_\omega]$  is positive definite (resp. semi-positive definite) on  $\wedge^{(n,1)}T_X^* \otimes E$ .*

We can define Griffiths positivity and negativity without using the curvature tensor. We have the following result.

PROPOSITION 15.4. (*cf.* [Rau15, Section 2]) *The following properties are equivalent:*

- (1)  *$h$  is Griffiths semi-negative.*
- (2)  *$|u|_h^2$  is plurisubharmonic for any local holomorphic section  $u$  of  $E$ .*
- (3)  *$\log |u|_h^2$  is plurisubharmonic for any local holomorphic section  $u$  of  $E$ .*
- (4) *the dual metric  $h^*$  on  $E^*$  is Griffiths semi-positive.*

We can treat the above conditions (2) and (3) without using the curvature tensor. Hence, we use these conditions to define Griffiths semi-positivity and semi-negativity of singular Hermitian metrics (see Definition 15.13). On the other hand, we did not know such a characterization of Nakano positivity.

Recently, new positivity notions defined via the Hörmander  $L^p$ -estimate were widely investigated. These studies can be regarded as a converse of Hörmander's estimate which is essentially due to Andreotti and Vesentini [AV65], and Hörmander [Hör65] (see also Theorem 15.9). Initially, Berndtsson established a converse of Hörmander's  $L^2$ -estimate for a continuous function on a 1-dimensional domain, and use this result to prove the complex Prékopa theorem in [Ber98]. In Part 1, we introduced the following condition which is named as the twisted Hörmander condition for holomorphic vector bundles on an  $n$ -dimensional domain.

DEFINITION 15.5. (= Definition 3.3) Let  $h$  be a singular Hermitian metric on  $E \rightarrow \Omega$  over a domain  $\Omega \subset \mathbb{C}^n$ . We say that  $(E, h)$  satisfies *the twisted Hörmander condition* if for any positive integer  $m$ , for any smooth strictly plurisubharmonic function  $\psi$  on  $\Omega$ , and for any  $\bar{\partial}$ -closed  $f = \sum_j f_j dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_j \in \mathcal{D}_{(n,1)}(\Omega, E^{\otimes m})$ , there exists  $u \in C_{(n,0)}^\infty(\Omega, E^{\otimes m})$  satisfying  $\bar{\partial}u = f$  and

$$\int_\Omega |u|_{(\omega_\Omega, h^{\otimes m})}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_\Omega \sum_{1 \leq i, j \leq n} \langle \psi^{i\bar{j}} f_i, f_j \rangle_{(\omega_\Omega, h^{\otimes m})} e^{-\psi} dV_{\omega_\Omega},$$

where we assume that the right-hand side is finite. Here  $(\psi^{i\bar{j}})_{1 \leq i, j \leq n}$  denotes the inverse matrix of  $(\frac{\partial^2}{\partial z_i \partial \bar{z}_j})_{1 \leq i, j \leq n}$ .

We remark that the matrix  $(\psi^{i\bar{j}})_{1 \leq i, j \leq n}$  corresponds to the inverse operator of  $B_{\omega_\Omega, \psi} = [\sqrt{-1}\partial\bar{\partial}\psi \otimes Id_{E^{\otimes m}}, \Lambda_{\omega_\Omega}]$ . It is known that this twisted Hörmander condition implies

Griffiths semi-positivity under some regularity assumptions (cf. [DNWZ20, Theorem 1.2], Theorem 3.5).

Then Deng, Ning, Wang, and Zhou introduced and improved various Hörmander type positivity notions for holomorphic vector bundles, which were named as the multiple coarse  $L^p$ -estimate condition and the optimal  $L^p$ -estimate condition in [DNWZ20]. We mention that the twisted Hörmander condition above is something like a multiple optimal  $L^2$ -estimate type condition. In this paper, we focus on the optimal  $L^p$ -estimate condition.

**DEFINITION 15.6.** ([DNWZ20, Definition 1.1]) Assume that a Kähler manifold  $(X, \omega)$  admits a positive holomorphic line bundle,  $(E, h)$  is a (singular) Hermitian vector bundle (maybe of infinite rank) over  $X$ , and  $p > 0$ . Then we say that  $(E, h)$  satisfies *the optimal  $L^p$ -estimate condition* if for any positive holomorphic line bundle  $(A, h_A)$  on  $X$ , for any  $\bar{\partial}$ -closed  $f \in \mathcal{D}_{(n,1)}(X, E \otimes A)$ , there exists  $u \in L^p_{(n,0)}(X, E \otimes A)$  satisfying  $\bar{\partial}u = f$  and

$$\int_X |u|_{(\omega, h \otimes h_A)}^p dV_\omega \leq \int_X \langle B_{h_A}^{-1} f, f \rangle_{(\omega, h \otimes h_A)}^{\frac{p}{2}} dV_\omega,$$

where  $B_{h_A} = [\sqrt{-1}\Theta_{(A, h_A)} \otimes Id_E, \Lambda_\omega]$  and we assume that the right-hand side is finite.

Furthermore, they succeeded in characterizing Nakano semi-positivity by using the above condition. To be precise, they proved the following theorem.

**THEOREM 15.7.** ([DNWZ20, Theorem 1.1]) Suppose that a Kähler manifold  $(X, \omega)$  admits a positive holomorphic line bundle,  $(E, h)$  is a smooth Hermitian vector bundle over  $X$ , and  $\theta \in C^0_{(1,1)}(X, \text{End}(E))$  with  $\theta^* = \theta$ . We assume that for any  $\bar{\partial}$ -closed  $f \in \mathcal{D}_{(n,1)}(X, E \otimes A)$ , and for any positive holomorphic line bundle  $(A, h_A)$  such that  $\sqrt{-1}\Theta_{(A, h_A)} \otimes Id_E + \theta >_{\text{Nak.}} 0$  on  $\text{supp} f$ , there exists  $u \in L^2_{(n,0)}(X, E \otimes A)$  satisfying  $\bar{\partial}u = f$  and

$$\int_X |u|_{(\omega, h \otimes h_A)}^2 dV_\omega \leq \int_X \langle B_{h_A, \theta}^{-1} f, f \rangle_{(\omega, h \otimes h_A)} dV_\omega,$$

where  $B_{h_A, \theta} = [\sqrt{-1}\Theta_{(h_A, \theta)} \otimes Id_E + \theta, \Lambda_\omega]$  and we assume that the right-hand side is finite. Then  $\sqrt{-1}\Theta_{(E, h)} \geq_{\text{Nak.}} \theta$ .

Here we consider the case that  $\theta = 0$ . In this situation, the condition in Theorem 15.7 is just the optimal  $L^2$ -estimate condition introduced in Definition 15.6. By applying and modifying this theorem, we get the following proposition.

**PROPOSITION 15.8.** Let  $h$  be a smooth Hermitian metric on  $E$ . We consider the following conditions:

- (1)  $h$  is Nakano semi-positive.
- (2) For any Stein coordinate  $(\Omega, \iota)$  such that  $E|_{\iota(\Omega)}$  is trivial on  $\iota(\Omega)$ , for any Kähler form  $\omega_\Omega$  on  $\Omega$ , for any smooth strictly plurisubharmonic function  $\psi$  on  $\Omega$ , for any positive integer  $q$  such that  $1 \leq q \leq n$ , and for any  $\bar{\partial}$ -closed  $f \in L^2_{(n,q)}(\Omega, \iota^* E; \omega_\Omega, (\iota^* h)e^{-\psi})$ ,

there exists  $u \in L^2_{(n,q-1)}(\Omega, \iota^*E; \omega_\Omega, (\iota^*h)e^{-\psi})$  satisfying  $\bar{\partial}u = f$  and

$$\int_{\Omega} |u|_{(\omega_\Omega, \iota^*h)}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_{\Omega} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, \iota^*h)} e^{-\psi} dV_{\omega_\Omega},$$

provided the right-hand side is finite.

(3)  $(E, h)$  satisfies the optimal  $L^2$ -estimate condition.

Then the condition (1) is equivalent to the condition (2). If  $X$  admits a Kähler metric  $\omega$  and a positive holomorphic line bundle on  $X$ , the above three conditions are equivalent.

Obviously, the above condition (2) corresponds to the condition in Definition 14.2. Theorem 15.7 and the following Theorem 15.9 imply that the condition (1) is equivalent to the condition (3). The way to prove that the condition (1) is equivalent to the condition (2) is essentially contained in the proof of Theorem 15.7 in [DNWZ20]. However, our situation is slightly different. Hence, for the sake of completeness, we show the equivalence of (1) and (2) here. In our situation, the proof is a little bit simpler. Before proving that, we prepare the following  $L^2$ -estimate theorem.

**THEOREM 15.9.** (cf. [Dem82], [Dem-book, Chapter VIII, Theorem 6.1]) *Let  $(X, \hat{\omega})$  be a complete Kähler manifold,  $\omega$  be another Kähler metric which is not necessarily complete, and  $(E, h) \rightarrow X$  be Nakano semi-positive vector bundle. We also let  $A_{q, \omega, h} = [\sqrt{-1}\Theta_{(E, h)}, \Lambda_\omega]$  be the operator in bidegree  $(n, q)$  for  $q \geq 1$ . Then for any  $\bar{\partial}$ -closed  $f \in L^2_{(n, q)}(X, E; \omega, h)$ , there exists  $u \in L^2_{(n, q-1)}(X, E; \omega, h)$  satisfying  $\bar{\partial}u = f$  and*

$$\int_X |u|_{(\omega, h)}^2 dV_\omega \leq \int_X \langle A_{q, \omega, h}^{-1} f, f \rangle_{(\omega, h)} dV_\omega,$$

where we assume that the right-hand side is finite.

**PROOF OF PROPOSITION 15.8.** First, we assume that  $h$  is Nakano semi-positive. We take an arbitrary Stein coordinate  $(\Omega, \iota)$  such that  $E|_{\iota(\Omega)}$  is trivial on  $\iota(\Omega)$ , an arbitrary Kähler metric  $\omega_\Omega$  on  $\Omega$ , and an arbitrary smooth strictly plurisubharmonic function  $\psi$  on  $\Omega$ . Considering the twisted weight  $(\iota^*h)e^{-\psi}$ , we have that  $\sqrt{-1}\Theta_{(\iota^*E, (\iota^*h)e^{-\psi})} = \sqrt{-1}\Theta_{(\iota^*E, (\iota^*h))} + \sqrt{-1}\partial\bar{\partial}\psi \otimes Id_{\iota^*E}$  and

$$\begin{aligned} A_{q, \omega_\Omega, (\iota^*h)e^{-\psi}} &= [\sqrt{-1}\Theta_{(\iota^*E, \iota^*h)}, \Lambda_{\omega_\Omega}] + [\sqrt{-1}\partial\bar{\partial}\psi \otimes Id_{\iota^*E}, \Lambda_{\omega_\Omega}] \\ &= A_{q, \omega_\Omega, \iota^*h} + B_{\omega_\Omega, \psi}. \end{aligned}$$

We have  $(\iota^*h)e^{-\psi}$  is Nakano positive on  $\iota^*E$ . Then Theorem 15.9 implies that for any  $q \geq 1$  and for any  $\bar{\partial}$ -closed  $f \in L^2_{(n, q)}(\Omega, \iota^*E; \omega_\Omega, (\iota^*h)e^{-\psi})$ , we have  $u \in L^2_{(n, q-1)}(\Omega, \iota^*E; \omega_\Omega, (\iota^*h)e^{-\psi})$  satisfying  $\bar{\partial}u = f$  and

$$\int_{\Omega} |u|_{(\omega_\Omega, \iota^*h)}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_{\Omega} \langle A_{q, \omega_\Omega, (\iota^*h)e^{-\psi}}^{-1} f, f \rangle_{(\omega_\Omega, \iota^*h)} e^{-\psi} dV_{\omega_\Omega}.$$

Since  $\iota^*h$  is also Nakano semi-positive, we have the inequality

$$\langle A_{q, \omega_\Omega, (\iota^*h)e^{-\psi}}^{-1} f, f \rangle_{(\omega_\Omega, \iota^*h)} \leq \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, \iota^*h)}.$$

Therefore, we also have the estimate

$$\int_{\Omega} |u|_{(\omega_{\Omega}, \iota^* h)}^2 e^{-\psi} dV_{\omega_{\Omega}} \leq \int_{\Omega} \langle B_{\omega_{\Omega}, \psi}^{-1} f, f \rangle_{(\omega_{\Omega}, \iota^* h)} e^{-\psi} dV_{\omega_{\Omega}}.$$

Next, we assume that the condition (2). Suppose that  $h$  is not Nakano semi-positive. Then, there exist  $x_0 \in X$  and  $f_0 \in \wedge^{(n,1)} T_{X, x_0}^* \otimes E_{x_0}$  such that

$$\tilde{\Theta}_{(E, h)}(f_0, f_0) < 0.$$

We take a Stein coordinate  $(\Delta_r^n, \iota)$  such that  $\iota(0) = x_0$  and  $E|_{\iota(\Delta_r^n)}$  is trivial for some  $r > 0$ , take the standard Kähler metric  $\omega_0 = \sqrt{-1} \partial \bar{\partial} |z|^2$  on  $\Delta_r^n$ , and take a frame  $(e_1, \dots, e_r)$  of  $\iota^* E$  on  $\Delta_r^n$  such that  $(e_1, \dots, e_r)$  is orthonormal at  $0 \in \Delta_r^n$ . Then  $(\iota^* E, \iota^* h)$  is not Nakano semi-positive at  $0 \in \Delta_r^n$ . For the sake of simplicity, we also write  $(E, h) (= (\iota^* E, \iota^* h))$  on  $\Delta_r^n$ .

We fix a smooth strictly plurisubharmonic function  $\psi$  on  $\Delta_r^n$ . Then for any  $\bar{\partial}$ -closed  $f \in \mathcal{D}_{(n,1)}(\Delta_r^n, E) \subset L_{(n,1)}^2(\Delta_r^n, E; \omega_0, h e^{-\psi})$ , there exists  $u \in C_{(n,0)}^\infty(\Delta_r^n, E)$  satisfying  $\bar{\partial} u = f$  and

$$\int_{\Delta_r^n} |u|_{(\omega_0, h)}^2 e^{-\psi} dV_{\omega_0} \leq \int_{\Delta_r^n} \langle B_{\omega_0, \psi}^{-1} f, f \rangle_{(\omega_0, h)} e^{-\psi} dV_{\omega_0}.$$

Therefore, we have

$$\begin{aligned} |\langle \langle B_{\omega_0, \psi}^{-1} f, f \rangle \rangle_{(\omega_0, h e^{-\psi})}|^2 &= |\langle \langle B_{\omega_0, \psi}^{-1} f, \bar{\partial} u \rangle \rangle_{(\omega_0, h e^{-\psi})}|^2 \\ &= |\langle \langle B_{\omega_0, \psi}^{-1} f, \bar{\partial} u \rangle \rangle_{(\omega_0, h e^{-\psi})}|^2 \\ &= |\langle \langle \bar{\partial}_\psi^* (B_{\omega_0, \psi}^{-1} f), u \rangle \rangle_{(\omega_0, h e^{-\psi})}|^2 \\ &= \|\bar{\partial}_\psi^* (B_{\omega_0, \psi}^{-1} f)\|_{(\omega_0, h e^{-\psi})}^2 \|u\|_{(\omega_0, h e^{-\psi})}^2 \\ &= \|\bar{\partial}_\psi^* (B_{\omega_0, \psi}^{-1} f)\|_{(\omega_0, h e^{-\psi})}^2 |\langle \langle B_{\omega_0, \psi}^{-1} f, f \rangle \rangle_{(\omega_0, h e^{-\psi})}|. \end{aligned}$$

In short, we have  $|\langle \langle B_{\omega_0, \psi}^{-1} f, f \rangle \rangle_{(\omega_0, h e^{-\psi})}| \leq \|\bar{\partial}_\psi^* (B_{\omega_0, \psi}^{-1} f)\|_{(\omega_0, h e^{-\psi})}^2$  for any  $\bar{\partial}$ -closed  $f$ . By using the Bochner-Kodaira-Nakano identity  $\Delta_\psi' = \Delta_\psi'' + [\sqrt{-1} \Theta_{(E, h e^{-\psi})}, \Lambda_{\omega_0}] = \Delta_\psi' + A_{1, \omega_0, h} + B_{\omega_0, \psi}$  (cf. [Dem, (4.6)]), we get

$$\begin{aligned} &\|\bar{\partial}_\psi^* (B_{\omega_0, \psi}^{-1} f)\|_{(\omega_0, h e^{-\psi})}^2 \\ &= \langle \langle \Delta_\psi'' (B_{\omega_0, \psi}^{-1} f), B_{\omega_0, \psi}^{-1} f \rangle \rangle_{(\omega_0, h e^{-\psi})} - \|\bar{\partial} (B_{\omega_0, \psi}^{-1} f)\|_{(\omega_0, h e^{-\psi})}^2 \\ &\leq \langle \langle \Delta_\psi' (B_{\omega_0, \psi}^{-1} f), B_{\omega_0, \psi}^{-1} f \rangle \rangle_{(\omega_0, h e^{-\psi})} + \langle \langle A_{1, \omega_0, h} (B_{\omega_0, \psi}^{-1} f), B_{\omega_0, \psi}^{-1} f \rangle \rangle_{(\omega_0, h e^{-\psi})} + \langle \langle f, B_{\omega_0, \psi}^{-1} f \rangle \rangle_{(\omega_0, h e^{-\psi})}. \end{aligned}$$

Then we obtain

$$\langle \langle A_{1, \omega_0, h} (B_{\omega_0, \psi}^{-1} f), B_{\omega_0, \psi}^{-1} f \rangle \rangle_{(\omega_0, h e^{-\psi})} + \|D_\psi' (B_{\omega_0, \psi}^{-1} f)\|_{(\omega_0, h e^{-\psi})}^2 \geq 0.$$

We let  $f = \sum_{j, \lambda} f_{j, \lambda} dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_j \otimes e_\lambda \in C_{(n,1)}^\infty(\Delta_r^n, E)$  be a  $\bar{\partial}$ -closed  $(n, 1)$ -form with constant coefficients such that  $f(0) = f_0$ . By Lemma 15.3, We can take a positive constant  $R \in (0, r)$  such that

$$\langle [\sqrt{-1} \Theta_{(E, h)}, \Lambda_{\omega_0}] f, f \rangle_{(\omega_0, h)} = \langle A_{1, \omega_0, h} f, f \rangle_{(\omega_0, h)} < -c$$

on  $\Delta_R^n$  for some positive constant  $c > 0$ .



Choose a cut-off function  $\chi \in \mathcal{D}_{(0,0)}(\Delta_R^n, \mathbb{R})$  such that  $0 \leq \chi \leq 1$  and  $\chi|_{\Delta_{\frac{R}{2}}^n} \equiv 1$ . We define  $v \in \mathcal{D}_{(n,0)}(\Delta_r^n, E)$  by

$$v = (-1)^n \sum_{j,\lambda} f_{j\lambda} \bar{z}_j \chi dz_1 \wedge \cdots \wedge dz_n \otimes e_\lambda,$$

and define  $g$  by  $\bar{\partial}v = g$ . Then  $g \in \mathcal{D}_{(n,1)}(\Delta_r^n, E)$  and  $g = f$  on  $\Delta_{\frac{R}{2}}^n$ . Set  $\phi(z) = |z|^2 - \frac{R^2}{4}$ . Then we have  $B_{(\omega_0, m\phi)} = m \cdot$ . We define  $\alpha_m := B_{(\omega_0, m\phi)}^{-1} g = \frac{1}{m} g$ . Considering the commutation relation  $\sqrt{-1}[\Lambda_{\omega_0}, \bar{\partial}] = D_{m\phi}'^*$  (cf. [Dem, (4.5)]), we obtain  $D_{m\phi}'^* \alpha_m = 0$  on  $\Delta_{\frac{R}{2}}^n$  and  $|D_{m\phi}'^* \alpha_m|_{(\omega_0, h)} \leq \frac{C}{m}$  for some positive constant  $C > 0$  on  $\Delta_R^n \setminus \bar{\Delta}_{\frac{R}{2}}^n$ . We also have  $\langle A_{1, \omega_0, h} \alpha_m, \alpha_m \rangle_{(\omega_0, h)} < -\frac{c}{m^2}$  on  $\Delta_{\frac{R}{2}}^n$  and  $\langle A_{1, \omega_0, h} \alpha_m, \alpha_m \rangle_{(\omega_0, h)} \leq \frac{C'}{m^2}$  for some  $C' > 0$  on  $\Delta_R^n \setminus \bar{\Delta}_{\frac{R}{2}}^n$  since  $g$  has compact support in  $\Delta_R^n$ . Set  $C'' := C^2 + C'$ . To summarize, we obtain

$$\begin{aligned} 0 &\leq \langle A_{1, \omega_0, h} (B_{(\omega_0, m\phi)}^{-1} g), B_{(\omega_0, m\phi)}^{-1} g \rangle_{(\omega_0, h e^{-m\phi})} + \|D_{m\phi}'^* (B_{(\omega_0, m\phi)}^{-1} g)\|_{(\omega_0, h e^{-m\phi})}^2 \\ &= \langle A_{1, \omega_0, h} \alpha_m, \alpha_m \rangle_{(\omega_0, h e^{-m\phi})} + \|D_{m\phi}'^* \alpha_m\|_{(\omega_0, h e^{-m\phi})}^2 \\ &= \int_{\Delta_{\frac{R}{2}}^n} \langle A_{1, \omega_0, h} \alpha_m, \alpha_m \rangle_{(\omega_0, h)} e^{-m\phi} dV_{\omega_0} + \int_{\Delta_R^n \setminus \bar{\Delta}_{\frac{R}{2}}^n} \langle A_{1, \omega_0, h} \alpha_m, \alpha_m \rangle_{(\omega_0, h)} e^{-m\phi} dV_{\omega_0} \\ &\quad + \int_{\Delta_R^n \setminus \bar{\Delta}_{\frac{R}{2}}^n} |D_{m\phi}'^* \alpha_m|_{(\omega_0, h)}^2 e^{-m\phi} dV_{\omega_0} \\ &\leq -\frac{c}{m^2} \int_{\Delta_{\frac{R}{2}}^n} e^{-m\phi} dV_{\omega_0} + \frac{C''}{m^2} \int_{\Delta_R^n \setminus \bar{\Delta}_{\frac{R}{2}}^n} e^{-m\phi} dV_{\omega_0} \end{aligned}$$

for any  $m \in \mathbb{N}$ . Hence, we get

$$-c \int_{\Delta_{\frac{R}{2}}^n} e^{-m\phi} dV_{\omega_0} + C'' \int_{\Delta_R^n \setminus \bar{\Delta}_{\frac{R}{2}}^n} e^{-m\phi} dV_{\omega_0} \geq 0.$$

Since  $\phi < 0$  on  $\Delta_{\frac{R}{2}}^n$  and  $\phi > 0$  on  $\Delta_R^n \setminus \bar{\Delta}_{\frac{R}{2}}^n$ , the first term has a negative upper bound which is independent of  $m$

$$-c \int_{\Delta_{\frac{R}{2}}^n} e^{-m\phi} dV_{\omega_0} < -c |\Delta_{\frac{R}{2}}^n|.$$

The second term goes to zero as  $m \rightarrow +\infty$  by Lebesgue's dominated convergence theorem. Then for sufficiently large  $m \gg 1$ , we have

$$-c \int_{\Delta_{\frac{R}{2}}^n} e^{-m\phi} dV_{\omega_0} + C'' \int_{\Delta_R^n \setminus \bar{\Delta}_{\frac{R}{2}}^n} e^{-m\phi} dV_{\omega_0} < 0,$$

which is a contradiction. Consequently, we can conclude that  $h$  is Nakano semi-positive on  $\Delta_r^n$ .  $\square$

**15.2. Singular Hermitian metrics.** In this subsection, we consider the case that a Hermitian metric has singularities. First, we introduce the definition of singular Hermitian metrics on vector bundles.

**DEFINITION 15.10.** ([BP08, Section 3], [HPS18, Definition 17.1], [PT18, Definition 2.2.1] and [Rau15, Definition 1.1]) We say that  $h$  is a *singular Hermitian metric* on  $E$  if  $h$  is a measurable map from the base manifold  $X$  to the space of non-negative Hermitian forms on the fibers satisfying  $0 < \det h < +\infty$  almost everywhere.

Related to the notion of singular Hermitian metrics, we introduce the ideal sheaves.

**DEFINITION 15.11.** ([Nad90]) Let  $h$  be a singular Hermitian metric on a holomorphic line bundle  $L \rightarrow X$ , and  $\varphi$  be the local weight of  $h$ , i.e.  $h = e^{-\varphi}$  locally. Then we define the ideal subsheaf  $\mathcal{I}(h) \subset \mathcal{O}_X$  of germs of holomorphic functions as follows:

$$\mathcal{I}(h)_x := \{f_x \in \mathcal{O}_{X,x} \mid |f_x|^2 e^{-\varphi} \text{ is locally integrable around } x\}.$$

We can easily verify that the above definition is independent of the choice of local weights. In [Nad90], Nadel proved that  $\mathcal{I}(h)$  is coherent by using the Hörmander  $L^2$ -estimate. We can also define a higher-rank analogue of the multiplier ideal sheaf  $\mathcal{I}(h)$ .

**DEFINITION 15.12.** (cf. [deC98]) Let  $h$  be a singular Hermitian metric on a holomorphic vector bundle  $E \rightarrow X$ . We define the ideal subsheaf  $\mathcal{E}(h)$  of germs of local holomorphic sections of  $E$  as follows:

$$\mathcal{E}(h)_x := \{s_x \in \mathcal{O}(E)_x \mid |s_x|_h^2 \text{ is locally integrable around } x\}.$$

In Part 1, we prove that  $\mathcal{E}(h)$  is coherent if  $h$  satisfies the twisted Hörmander condition above. We can also show that  $\mathcal{E}(h)$  is coherent when  $h$  is a Nakano semi-positive singular Hermitian metric (cf. Proposition 17.4).

The Chern curvature tensor  $\Theta_{(E,h)}$  of a smooth Hermitian metric  $h$  can be locally defined by  $\bar{\partial}(h^{-1}\partial h)$ . On a holomorphic line bundle, the Chern curvature of a positive or negative singular Hermitian metric can be also defined in the sense of currents. However, for a holomorphic vector bundle  $E$  of rank  $E \geq 2$ , it is not possible to define the Chern curvature currents with measure coefficients in general. This phenomenon was observed by Raufi in [Rau15]. Before showing the example, we introduce the definitions of Griffiths semi-negativity and Griffiths semi-positivity.

**DEFINITION 15.13.** ([BP08, Definition 3.1], [PT18, Definition 2.2.2] and [Rau15, Definition 1.2]) We say that a singular Hermitian metric  $h$  is:

- (1) *Griffiths semi-negative* if  $|u|_h$  is plurisubharmonic for any local holomorphic section  $u \in \mathcal{O}(E)$  of  $E$ .
- (2) *Griffiths semi-positive* if the dual metric  $h^*$  on  $E^*$  is Griffiths semi-positive.

This definition arises from a characterization of Griffiths semi-positivity (see Proposition 15.4). Then Raufi found the following example.

THEOREM 15.14. ([Rau15, Theorem 1.5]) *Let  $E$  be the trivial vector bundle  $\Delta \times \mathbb{C}^2$  over  $\Delta := \Delta_1^1 \subset \mathbb{C}$ . Let  $h$  be the singular Hermitian metric*

$$h = \begin{pmatrix} 1 + |z|^2 & z \\ \bar{z} & |z|^2 \end{pmatrix}.$$

*Then,  $h$  is Griffiths semi-negative, and  $\Theta_{(E,h)}$  is not a current with measure coefficients.*

This result implies that we cannot define the positivity or negativity by using the Chern curvature currents. Furthermore, the strict positivity or negativity is not generally formulated. If there is a Kähler metric on  $X$ , we can define the strict Griffiths positivity as follows.

DEFINITION 15.15. (= Definition 10.6) Let  $\omega_X$  be a Kähler metric on  $X$ . We say that a singular Hermitian metric  $h$  is *strictly Griffiths  $\delta_{\omega_X}$ -positive* if for any open subset  $U$  and for any Kähler potential  $\varphi$  of  $\omega_X$  on  $U$ , i.e.  $\sqrt{-1}\partial\bar{\partial}\varphi = \omega_X$  on  $U$ ,  $he^{\delta\varphi}$  is Griffiths semi-positive on  $U$ .

For Nakano semi-positivity of singular Hermitian metrics, we can characterize it by using Proposition 15.8 (see Definition 14.2). We can also define the strict Nakano  $\delta_{\omega_X}$ -positivity of singular Hermitian metrics as follows.

DEFINITION 15.16. Let  $(X, \omega_X)$  be a Kähler manifold. We say that  $h$  is *strictly Nakano  $\delta_{\omega_X}$ -positive* if for any open subset  $U$  and for any Kähler potential  $\varphi$  of  $\omega_X$ , i.e.  $\sqrt{-1}\partial\bar{\partial}\varphi = \omega_X$  on  $U$ ,  $he^{\delta\varphi}$  is Nakano semi-positive on  $U$  in the sense of Definition 14.2.

REMARK 15.17. We consider the following condition related to the condition (2) in Proposition 15.8 for  $k \geq 1$ .

(2-k): For any Stein coordinate  $(\Omega, \iota)$  such that  $E|_{\iota(\Omega)}$  is trivial on  $\iota(\Omega)$ , for any Kähler form  $\omega_\Omega$  on  $\Omega$ , for any smooth strictly plurisubharmonic function  $\psi$  on  $\Omega$ , for any positive integer  $q$  such that  $1 \leq q \leq k$ , and for any  $\bar{\partial}$ -closed  $f \in L^2_{(n,q)}(\Omega, \iota^*E; \omega_\Omega, (\iota^*h)e^{-\psi})$ , there exists  $u \in L^2_{(n,q-1)}(\Omega, \iota^*E; \omega_\Omega, (\iota^*h)e^{-\psi})$  satisfying  $\bar{\partial}u = f$  and

$$\int_{\Omega} |u|_{(\omega_\Omega, \iota^*h)}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_{\Omega} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, \iota^*h)} e^{-\psi} dV_{\omega_\Omega},$$

provided the right-hand side is finite.

The proof of Proposition 15.8 suggests that we only have to consider all  $(n, 1)$ -forms  $f$ , not all  $(n, q)$ -forms for  $1 \leq q \leq n$ . However, the conditions (2-1),  $\dots$ , (2-n) are equivalent to each other under the assumption that  $h$  is smooth. Hence, in this paper, we adopt the seemingly stronger condition (2-n) (=Definition 14.2) to define Nakano semi-positivity of singular Hermitian metrics. Related to this remark, we propose Question 19.4 in Section 19.

## 16. DEMAILLY AND SKODA'S THEOREM IN THE SINGULAR SETTING

In this section, we prove Theorem 14.4, which is a generalization of Demailly and Skoda's result. Before proving that, we explain Demailly and Skoda's theorem.

**THEOREM 16.1.** ([DS]) *Let  $h$  be a smooth Hermitian metric on  $E$ . If  $(E, h)$  is Griffiths semi-positive, then  $(E \otimes \det E, h \otimes \det h)$  is Nakano semi-positive.*

Taking a smooth approximating sequence  $\{h_\nu\}_{\nu=1}^\infty$  of  $h$ , we give a proof of Theorem 14.4. Our main approximation technique is based on the following proposition obtained by Berndtsson and Paun.

**PROPOSITION 16.2.** (cf. [BP08, Proposition 3.1], [Rau15]) *Let  $E$  be a trivial vector bundle over a polydisc  $U$  and  $h$  be a Griffiths semi-positive singular Hermitian metric on  $E$ . Then there exists a sequence of smooth Hermitian metrics  $\{h_\nu\}_{\nu=1}^\infty$ , with positive Griffiths curvature, increasing to  $h$  on smaller polydiscs.*

We remark that the above proposition is valid if  $U$  is not a polydisc but a domain. A sequence of smooth Hermitian metrics approximating  $h$  is obtained through convolution of  $h$  with an approximate identity. In this way, we can only get an approximating sequence when  $E$  is a trivial vector bundle over a domain in  $\mathbb{C}^n$ .

To prove Theorem 14.4, we also need the following theorem.

**THEOREM 16.3.** ([Siu76, Corollary 1]) *Let  $X$  be a Stein submanifold of  $\mathbb{C}^N$  for some  $N > n = \dim X$ . Let  $i : X \rightarrow \mathbb{C}^N$  be an inclusion map. Then there exists an open neighborhood  $U$  of  $X$  in  $\mathbb{C}^N$  such that  $U$  is a holomorphic retraction of  $X$ , i.e. there exists a holomorphic map  $p : U \rightarrow X$  such that  $p \circ i = \text{id}_X$ .*

Then we give a proof of the following result.

**THEOREM 16.4.** (= Theorem 14.4) *Let  $h$  be a singular Hermitian metric on  $E$ . If  $(E, h)$  is Griffiths semi-positive, then  $(E \otimes \det E, h \otimes \det h)$  is Nakano semi-positive in the sense of singular Hermitian metrics.*

**PROOF.** It is clear that Griffiths semi-positivity of  $h$  yields the Griffiths semi-positivity of  $h \otimes \det h$  (cf. [Rau15, Proposition 1.3]). Then it is enough to show that  $(E \otimes \det E, h \otimes \det h)$  satisfies the condition in Definition 14.2.

Let  $(\Omega, \iota)$  be an arbitrary Stein coordinate of  $X$  such that  $(E \otimes \det E)|_{\iota(\Omega)}$  is trivial on  $\iota(\Omega)$ . Since  $\Omega$  can be properly embedded into  $\mathbb{C}^N$  for some large  $N$ , we can regard  $\Omega$  as a submanifold of  $\mathbb{C}^N$  without any loss of generality. From Theorem 16.3, we take an open neighborhood  $U$  of  $\Omega$  in  $\mathbb{C}^N$  and a holomorphic map  $p : U \rightarrow \Omega$  which defines a holomorphic retraction of  $\Omega$ , i.e.  $p \circ i = \text{id}_\Omega$ , where  $i : \Omega \rightarrow \mathbb{C}^N$  is an inclusion map. Since  $(E \otimes \det E)|_{\iota(\Omega)}$  is a trivial bundle,  $\iota^*(E \otimes \det E)$  and  $p^*\iota^*(E \otimes \det E)$  are also trivial on  $\Omega$  and  $U$ . Thanks to [PT18, Lemma 2.3.2],  $\iota^*h$  and  $p^*\iota^*h$  are also Griffiths semi-positive. For the sake of clarity, we omit the map  $\iota$  and simply write  $(E, h) = (\iota^*E, \iota^*h)$  on  $\Omega$ .

Since  $E \otimes \det E$  is trivial on  $\Omega$ , we fix a holomorphic global frame  $(e_1, \dots, e_r)$  of  $E \otimes \det E$  on  $\Omega$ . Then  $(\det(E \otimes \det E), \det(h \otimes \det h)) \cong ((\det E)^{\otimes r+1}, (\det h)^{\otimes r+1})$  is also trivial on  $\Omega$  with respect to the frame  $e_1 \wedge \dots \wedge e_r$ . We define the function  $\Psi$  by

$$|e_1 \wedge \dots \wedge e_r|_{(\det h)^{\otimes r+1}} = e^{-\Psi}.$$

Since  $(\det h)^{\otimes r+1}$  is Griffiths semi-positive (cf. [Rau15, Proposition 1.3]),  $\Psi$  is a plurisubharmonic function on  $\Omega$ . We construct the metric  $h \otimes \det h e^{\frac{\Psi}{r+1}}$  on  $E \otimes \det E$ . We can easily see that  $h \otimes \det h e^{\frac{\Psi}{r+1}}$  is Griffiths semi-positive (for the detailed proof, see Proposition 16.5 below). From Proposition 16.2, we get a sequence of smooth Hermitian metrics  $\{h_\nu\}_{\nu=1}^\infty$ , with positive Griffiths curvature, increasing to  $p^*(h \otimes \det h e^{\frac{\Psi}{r+1}})$  on  $p^*(E \otimes \det E)$  over any relatively compact subdomain of  $U$ . Set  $g_\nu := i^* h_\nu$ . Since  $p \circ i = id_\Omega$ ,  $\{g_\nu\}_{\nu=1}^\infty$  is also a sequence of smooth Hermitian metrics, with positive Griffiths curvature, increasing to  $h \otimes \det h e^{\frac{\Psi}{r+1}}$  on  $E \otimes \det E$  over any relatively compact subset of  $\Omega$ . We also have that  $\{\det g_\nu\}_{\nu=1}^\infty$  becomes a sequence of smooth Hermitian metrics, with positive curvature, increasing to

$$\begin{aligned} (\det(E \otimes \det E), \det(h \otimes \det h e^{\frac{\Psi}{r+1}})) &= ((\det E)^{\otimes r+1}, (\det h)^{\otimes r+1} e^{\frac{r\Psi}{r+1}}) \\ &\cong (\mathbb{C}, e^{-\frac{\Psi}{r+1}}) \end{aligned}$$

(cf. [Rau15, the proof of Proposition 1.3]). Then, from the result of Demailly-Skoda (Theorem 16.1),  $\{g_\nu \otimes \det g_\nu\}_{\nu=1}^\infty$  gives a sequence of smooth Hermitian metrics, with positive Nakano curvature, increasing to  $h \otimes \det h$  on  $E \otimes \det E$  over any relatively compact subset of  $\Omega$ . Here we regard  $g_\nu \otimes \det g_\nu$  as the metric on  $E \otimes \det E$  via the trivialization of  $(\det E)^{\otimes r+1}$  for every  $\nu \in \mathbb{N}$ .

Then we take an arbitrary Kähler metric  $\omega_\Omega$ , an arbitrary smooth strictly plurisubharmonic function  $\psi$ , and an arbitrary  $\bar{\partial}$ -closed  $f \in L^2_{(n,q)}(\Omega, E \otimes \det E; \omega_\Omega, h \otimes \det h e^{-\psi})$  for any  $q > 0$  on  $\Omega$ . We also take a Stein exhaustion  $\{\Omega_j\}_{j=1}^\infty$  of  $\Omega$ , where  $\Omega_j$  is a relatively compact Stein subdomain. We assume that

$$\int_\Omega \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, h \otimes \det h)} e^{-\psi} dV_{\omega_\Omega} < +\infty.$$

Since  $\{g_\nu \otimes \det g_\nu\}_{\nu=1}^\infty$  is an increasing sequence on any relatively compact subset, we have

$$\int_{\Omega_j} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, g_\nu \otimes \det g_\nu)} e^{-\psi} dV_{\omega_\Omega} < +\infty$$

for fixed  $j \in \mathbb{N}$ . Thanks to Hörmander's  $L^2$ -estimate for smooth Hermitian metrics (cf. Theorem 15.9) and the proof of Proposition 15.8, we get a solution  $u_\nu \in L^2_{(n,q-1)}(\Omega_j, E \otimes$

$\det E; \omega_\Omega, g_\nu \otimes \det g_\nu e^{-\psi})$  of  $\bar{\partial}u_\nu = g$  such that

$$\begin{aligned} \int_{\Omega_j} |u_\nu|_{(\omega_\Omega, g_\nu \otimes \det g_\nu)}^2 e^{-\psi} dV_{\omega_\Omega} &\leq \int_{\Omega_j} \langle A_{q, \omega_\Omega, g_\nu \otimes \det g_\nu}^{-1} f, f \rangle_{(\omega_\Omega, g_\nu \otimes \det g_\nu)} e^{-\psi} dV_{\omega_\Omega} \\ &\leq \int_{\Omega_j} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, g_\nu \otimes \det g_\nu)} e^{-\psi} dV_{\omega_\Omega} \\ &\leq \int_{\Omega} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, h \otimes \det h)} e^{-\psi} dV_{\omega_\Omega} < +\infty \end{aligned}$$

since  $g_\nu \otimes \det g_\nu$  is Nakano semi-positive. For fixed  $\nu_0$ ,  $\{u_\nu\}_{\nu \geq \nu_0}$  forms a bounded sequence in  $L_{(n, q-1)}^2(\Omega_j, E \otimes \det E; \omega_\Omega, g_{\nu_0} \otimes \det g_{\nu_0} e^{-\psi})$  due to the monotonicity of  $\{g_\nu \otimes \det g_\nu\}_{\nu=1}^\infty$ . Hence, we can obtain a weakly convergent subsequence in  $L_{(n, q-1)}^2(\Omega_j, E \otimes \det E; \omega_\Omega, g_{\nu_0} \otimes \det g_{\nu_0} e^{-\psi})$ . By using a diagonal argument, we get a subsequence  $\{u_{\nu_k}\}_{k=1}^\infty$  of  $\{u_\nu\}_{\nu=1}^\infty$  converging weakly in  $L_{(n, q-1)}^2(\Omega_j, E \otimes \det E; \omega_\Omega, g_{\nu_0} \otimes \det g_{\nu_0} e^{-\psi})$  for any  $\nu_0$ . We denote by  $u_j$  the weak limit of  $\{u_{\nu_k}\}_{k=1}^\infty$ . Then  $u_j$  satisfies  $\bar{\partial}u_j = f$  on  $\Omega_j$  and

$$\int_{\Omega_j} |u_j|_{(\omega_\Omega, g_{\nu_0} \otimes \det g_{\nu_0})}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_{\Omega} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, h \otimes \det h)} e^{-\psi} dV_{\omega_\Omega}$$

for each  $\nu_0$ . Taking weak limits  $\nu_0 \rightarrow +\infty$  and using the monotone convergence theorem, we have the following estimate

$$\int_{\Omega_j} |u_j|_{(\omega_\Omega, h \otimes \det h)}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_{\Omega} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, h \otimes \det h)} e^{-\psi} dV_{\omega_\Omega}.$$

Repeating the above argument and taking the weak limit  $j \rightarrow \infty$ , we get a solution  $u \in L_{(n, q-1)}^2(\Omega, E \otimes \det E; \omega_\Omega, h \otimes \det h e^{-\psi})$  of  $\bar{\partial}u = f$  such that

$$\int_{\Omega} |u|_{(\omega_\Omega, h \otimes \det h)}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_{\Omega} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, h \otimes \det h)} e^{-\psi} dV_{\omega_\Omega}$$

on  $\Omega$ . Consequently, we can conclude that  $h \otimes \det h$  is Nakano semi-positive in the sense of singular Hermitian metrics.  $\square$

**PROPOSITION 16.5.** *Let notation be the same as one in the proof of Theorem 16.4. Then the metric  $h \otimes \det h e^{\frac{\Psi}{r+1}}$  is Griffiths semi-positive on  $E \otimes \det E$ .*

**PROOF.** We have to show that  $\log |u|_{h^* \otimes \det h^* e^{-\frac{\Psi}{r+1}}}$  is plurisubharmonic for any local holomorphic section  $u \in \mathcal{O}(E^* \otimes \det E^*)$  of  $E^* \otimes \det E^*$ . Let  $(e_1^*, \dots, e_r^*)$  be the global dual frame of  $(e_1, \dots, e_r)$ . We also take a local frame of  $(\epsilon_1, \dots, \epsilon_r)$  of  $E$  and let  $(\epsilon_1^*, \dots, \epsilon_r^*)$  the local dual frame. Fixing these frames, it is enough to show that

$$\log(|u|_{h^*} |\epsilon_1^* \wedge \dots \wedge \epsilon_r^*|_{\det h^*} e^{-\frac{\Psi}{r+1}}) = \log |u|_{h^*} + \log |\epsilon_1^* \wedge \dots \wedge \epsilon_r^*|_{\det h^*} |e_1 \wedge \dots \wedge e_r|_{(\det h)^{\otimes r+1}}^{\frac{1}{r+1}}$$

is plurisubharmonic. Since  $h^*$  is Griffiths semi-negative,  $\log |u|_{h^*}$  is a plurisubharmonic function. We define a local holomorphic function  $f$  by  $f(\epsilon_1^* \wedge \dots \wedge \epsilon_r^*)^{\otimes r+1} = e_1^* \wedge \dots \wedge e_r^*$ .

Then we obtain

$$\begin{aligned}
(r+1) \log |\epsilon_1^* \wedge \cdots \wedge \epsilon_r^*|_{\det h^*} |e_1 \wedge \cdots \wedge e_r|^{\frac{1}{r+1}}_{(\det h)^{\otimes r+1}} &= \log |\epsilon_1^* \wedge \cdots \wedge \epsilon_r^*|_{\det h^*}^{r+1} |e_1 \wedge \cdots \wedge e_r|_{(\det h)^{\otimes r+1}} \\
&= \log \left( \frac{|(\epsilon_1^* \wedge \cdots \wedge \epsilon_r^*)^{r+1}|_{(\det h^*)^{\otimes r+1}}}{|e_1^* \wedge \cdots \wedge e_r^*|_{(\det h^*)^{\otimes r+1}}} \right) \\
&= \log |f|.
\end{aligned}$$

Since  $f \neq 0$ , this term is a harmonic function. Therefore, we complete the proof.  $\square$

If  $X$  admits a Kähler metric  $\omega_X$ , we can also prove the following theorem.

**THEOREM 16.6.** *Let  $\omega_X$  be a Kähler form on a Kähler manifold  $X$ . If  $(E, h)$  is strictly Griffiths  $\delta_{\omega_X}$ -positive, then  $(E \otimes \det E, h \otimes \det h)$  is strictly Nakano  $(r+1)\delta_{\omega_X}$ -positive.*

**PROOF.** We take an arbitrary open subset  $U$  and any Kähler potential  $\varphi$  of  $\omega_X$  on  $U$ . We also take a Stein coordinate  $(\Omega, \iota)$  of  $U$ . Then we use the same notation as in the proof of Theorem 16.4. By the definition of the strict Griffiths  $\delta_{\omega_X}$ -positivity, we have that  $he^{\delta\varphi}$  is Griffiths semi-positive. Hence, from Theorem 14.4, we get

$$he^{\delta\varphi} \otimes \det(he^{\delta\varphi}) = h \otimes \det he^{(r+1)\delta\varphi}$$

is Nakano semi-positive in the sense of singular Hermitian metrics on  $U$ . Thus we can conclude that  $h \otimes \det h$  is strictly Nakano  $(r+1)\delta_{\omega_X}$ -positive on  $X$ .  $\square$

## 17. $L^2$ -ESTIMATES AND VANISHING THEOREMS

In this section, we give a  $L^2$ -estimate and a vanishing theorem for holomorphic vector bundles with strictly Nakano positive singular Hermitian metrics. Then we prove Theorem 14.5, 14.6, and 14.7. In this section, we assume that  $X$  is a projective manifold and  $\omega_X$  is a Kähler form on  $X$ . First of all, we show Theorem 14.5.

**PROOF OF THEOREM 14.5.** Choose an arbitrary  $\bar{\partial}$ -closed  $f \in L^2_{(n,q)}(X, E; \omega_X, h)$  for  $q > 0$ . By Serre's GAGA, there exists a proper Zariski open subset  $Z \neq \emptyset$  such that  $E|_Z$  is trivial over  $Z$  and  $\omega$  is  $\partial\bar{\partial}$ -exact on  $Z$ . We can take  $Z$  as a Stein open subset. Then  $(Z, i)$  is a Stein coordinate of  $X$  such that  $E|_Z$  is trivial on  $Z$ , where  $i : Z \rightarrow X$  is the natural inclusion map. We fix a Kähler potential  $\varphi$  of  $\omega_X$  on  $Z$ , i.e.  $\varphi$  satisfies  $\sqrt{-1}\partial\bar{\partial}\varphi = \omega_X$ . Then we have that

$$\begin{aligned}
\langle [B_{\omega_X, \delta\varphi}, \Lambda_{\omega_X}]f, f \rangle_{(\omega_X, h)} &= \delta q |f|_{(\omega_X, h)}^2, \\
\langle [B_{\omega_X, \delta\varphi}^{-1}, \Lambda_{\omega_X}]f, f \rangle_{(\omega_X, h)} &= \frac{1}{\delta q} |f|_{(\omega_X, h)}^2,
\end{aligned}$$

respectively.

Thanks to the definition of the strict Nakano  $\delta_{\omega_X}$ -positivity, for any smooth strictly plurisubharmonic function  $\psi$  on  $Z$ , we can obtain  $u \in L^2_{(n, q-1)}(Z, E; \omega_X, he^{\delta\varphi-\psi})$  satisfying  $\bar{\partial}u = f$  and

$$\int_Z |u|_{(\omega_X, h)}^2 e^{\delta\varphi-\psi} dV_{\omega_X} \leq \int_Z \langle B_{\omega_X, \psi}^{-1} f, f \rangle_{(\omega_X, h)} e^{\delta\varphi-\psi} dV_{\omega_X}$$

if the right-hand side is finite. Taking  $\psi = \delta\varphi$ , we get a solution  $u \in L^2_{(n,q-1)}(Z, E; \omega_X, h)$  of  $\bar{\partial}u = f$  such that

$$\begin{aligned} \int_Z |u|_{(\omega_X, h)}^2 dV_{\omega_X} &\leq \int_Z \langle B_{\omega_X, \delta\varphi}^{-1} f, f \rangle_{(\omega_X, h)} dV_{\omega_X} \\ &= \frac{1}{\delta q} \int_Z |f|_{(\omega_X, h)}^2 dV_{\omega_X} \\ &\leq \frac{1}{\delta q} \int_X |f|_{(\omega_X, h)}^2 dV_{\omega_X} < +\infty. \end{aligned}$$

Letting  $u = 0$  on  $X \setminus Z$ , we have  $u \in L^2_{(n,q-1)}(X, E; \omega_X, h)$ ,  $\bar{\partial}u = f$ , and

$$\int_X |u|_{(\omega_X, h)}^2 dV_{\omega_X} \leq \frac{1}{\delta q} \int_X |f|_{(\omega_X, h)}^2 dV_{\omega_X}$$

from the following lemma. □

LEMMA 17.1. (cf. [B-book, Lemma 5.1.3]) *Let  $X$  be a complex manifold and let  $S$  be a complex hypersurface in  $X$ . Let  $u$  and  $f$  be (possibly bundle valued) forms in  $L^2_{loc}$  of  $X$  satisfying  $\bar{\partial}u = f$  on  $X \setminus S$ . Then the same equation holds on  $X$  (in the sense of distributions).*

REMARK 17.2. Lemma 17.1 holds when  $h$  is smooth. However, since we assume that  $h$  is Griffiths semi-positive, we can locally take a sequence of smooth Hermitian metrics increasing to  $h$  from Proposition 16.2. Thus, we have that  $f$  and  $u$  are  $L^2_{loc}$  forms with respect to some smooth Hermitian metric. Therefore, we can apply Lemma 17.1.

By using Theorem 14.5, we prove Theorem 14.6. Before proving Theorem 14.6, we state the following vanishing theorem for holomorphic line bundles, which was obtained by Nadel in [Nad90] and generalized by Demailly in [Dem93].

THEOREM 17.3. ([Nad90], [Dem93], and [Dem, (5.11)]) *Let  $(X, \omega_X)$  be a Kähler weakly pseudoconvex manifold, and  $L \rightarrow X$  be a holomorphic line bundle equipped with a singular Hermitian metric  $h$  of weight  $\varphi$ . We assume that  $\sqrt{-1}\Theta_{(L, h)} \geq \epsilon\omega$  for some continuous positive function  $\epsilon$  on  $X$ . Then*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0$$

for  $q > 0$ .

We also mention the following result related to the coherence of  $\mathcal{E}(h)$ .

PROPOSITION 17.4. (cf. Theorem 1.4) *Let  $h$  be a Nakano semi-positive singular Hermitian metric and  $\mathcal{E}(h)$  be the sheaf of germs of locally square integrable holomorphic sections of  $E$  with respect to  $h$ . Then  $\mathcal{E}(h)$  is a coherent subsheaf of  $\mathcal{O}(E)$ .*

In Part 1, we prove Proposition 17.4 in the case that  $h$  is positively curved in the sense of twisted Hörmander. The twisted Hörmander condition (cf. Definition 15.5) is slightly



different from the definition of Nakano semi-positivity. However, the proof of Proposition 17.4 is exactly the same as the proof in Part 1. Hence, we refrain from proving it here. Applying the above results, we can prove Theorem 14.6.

**PROOF OF THEOREM 14.6.** Let  $\mathcal{L}^q$  be the sheaf of germs of  $(n, q)$ -forms  $u$  with values in  $E$  and with square-integrable coefficients, such that  $|u|_{(\omega_X, h)}^2$  is locally integrable,  $\bar{\partial}u$  can be defined in the sense of currents with square-integrable coefficients, and  $|\bar{\partial}u|_{(\omega, h)}^2$  is locally integrable. Then  $(\mathcal{L}^\bullet, \bar{\partial})$  is a resolution of the sheaf  $K_X \otimes \mathcal{E}(h)$  for the reason that we can solve the  $\bar{\partial}$ -equation locally by applying Theorem 14.5 on any small polydisc. Hence, we have that  $\mathcal{L}^\bullet$  is a resolution by acyclic sheaves.

The compactness of  $X$  yields that locally integrable sections are also integrable on  $X$ . Hence, by using Theorem 14.5 globally, we also get that  $H^q(\Gamma(X, \mathcal{L}^\bullet)) = 0$  for  $q > 0$ . Consequently, we can conclude that  $H^q(X, K_X \otimes \mathcal{E}(h)) = 0$  for  $q > 0$ .  $\square$

**REMARK 17.5.** We see that the  $L^2$ -estimate in Theorem 14.5 also holds in the situation that the base manifold  $X$  is Stein. Hence, we can apply Theorem 14.5 on any small polydisc in the above proof.

As an application of Theorem 14.6 and 16.6, we obtain the following theorem, which generalizes the Griffiths vanishing theorem.

**THEOREM 17.6.** (= Theorem 14.7) *Let  $(X, \omega_X)$  be a projective manifold and a Kähler metric on  $X$ . If  $h$  is strictly Griffiths  $\delta_{\omega_X}$ -positive in the sense of Definition 15.15 on  $X$ , then*

$$H^q(X, K_X \otimes \mathcal{E}(h \otimes \det h)) = 0.$$

Here we introduce the notion of the Lelong number of a singular Hermitian metric on a holomorphic line bundle. Usually, the Lelong of a plurisubharmonic function of  $\varphi$  at a point  $x \in X$  is defined by

$$\liminf_{z \rightarrow x} \frac{\varphi(z)}{\log |z - x|}$$

for some coordinate  $(z_1, \dots, z_n)$  around  $x$ . We also denote by  $\nu(\varphi, x)$  the Lelong number of  $\varphi$  at  $x \in X$ . It is known that this number is independent of the choice of local coordinates.

For a semi-positive singular Hermitian metric  $g$  on a holomorphic line bundle  $L$ , we can also define the Lelong number  $\nu(g, x)$  of  $g$  at  $x$  such that

$$\nu(g, x) := \liminf_{z \rightarrow x} \frac{-\log g(z)}{\log |z - x|}.$$

Here we regard  $g(z)$  as a local semi-positive function. Since  $g$  is semi-positive,  $-\log g(z)$  is a plurisubharmonic function locally. Thus, the above definition is reasonable. We repeat that this definition is independent of the choice of local coordinates.

There is a relationship between the Lelong number of  $\varphi$  and the integrability of  $e^{-\varphi}$ . We introduce the following important result obtained by Skoda in [Sko72].

LEMMA 17.7. ([Sko72]) *Let  $\varphi$  be a plurisubharmonic function. If  $\nu(\varphi, x) < 1$ ,  $e^{-2\varphi}$  is integrable around  $x$ .*

We consider the strictly Nakano  $\delta_{\omega_X}$ -positive or strictly Griffiths  $\delta_{\omega_X}$ -positive singular Hermitian metric  $h$  again. We recall that  $\det h$  is a semi-positive singular Hermitian metric on  $\det E$  (cf. [Rau15, Proposition 1.3]). If the Lelong number of  $\det h$  satisfies some good inequalities, we have that  $\mathcal{E}(h) = \mathcal{O}(E)$  or  $\mathcal{E}(h \otimes \det h) = \mathcal{O}(E \otimes \det E)$ . These properties imply the following vanishing theorems.

THEOREM 17.8. *Let  $(X, \omega_X)$  be a projective manifold and a Kähler metric on  $X$ . We also let  $h$  be a strictly Nakano  $\delta_{\omega_X}$ -positive singular Hermitian metric on  $E$ . If  $\nu(\det h, x) < 2$  for any point  $x \in X$ , we have  $\mathcal{E}(h) = \mathcal{O}(E)$  and*

$$H^q(X, K_X \otimes E) = 0$$

for  $q > 0$ .

PROOF. By the definition of the Lelong number of a singular Hermitian metric on a holomorphic line bundle, we have  $\nu(\frac{1}{2} \log \det h^*, x) < 1$  for every  $x \in X$ . From Lemma 17.7,

$$e^{-\log \det h^*} = \frac{1}{\det h^*}$$

is locally integrable. Locally, we see that

$$h = \frac{1}{\det h^*} \widehat{h}^*,$$

where  $\widehat{h}^*$  is the adjugate matrix of  $h^*$ . Since  $h^*$  is Griffiths semi-negative, each element of  $\widehat{h}^*$  is locally bounded [PT18, Lemma 2.2.4]. Then it follows that  $|u|_h^2$  is locally integrable for any local holomorphic section  $u \in \mathcal{O}(E)$  of  $E$ . Therefore, we can conclude that  $\mathcal{E}(h) = \mathcal{O}(E)$  and  $H^q(X, K_X \otimes E) = 0$  for  $q > 0$  from Theorem 14.6.  $\square$

Repeating the above argument and using Theorem 14.7, we can also prove the following theorem.

THEOREM 17.9. (= Corollary 9.4) *Let  $(X, \omega_X)$  be a projective manifold and a Kähler metric on  $X$ . We also let  $h$  be a strictly Griffiths  $\delta_{\omega_X}$ -positive singular Hermitian metric on  $E$ . If  $\nu(\det h, x) < 1$  for any point  $x \in X$ , we have  $\mathcal{E}(h \otimes \det h) = \mathcal{O}(E \otimes \det E)$  and*

$$H^q(X, K_X \otimes E \otimes \det E) = 0$$

for  $q > 0$ .

## 18. PROPERTIES OF NAKANO SEMI-POSITIVITY

In this short section, we discuss the validity of the definition of Nakano semi-positive singular Hermitian metrics. We show the following results.

PROPOSITION 18.1. *Let  $L \rightarrow X$  be a holomorphic line bundle on a complex manifold  $X$ . We also let  $h$  be a (Griffiths) semi-positive singular Hermitian metric on  $L$ . Then  $h$  is Nakano semi-positive in the sense of singular Hermitian metrics.*

PROPOSITION 18.2. *Let  $S$  be a Riemann surface and  $E \rightarrow S$  be a holomorphic vector bundle on  $S$ . We also let  $h$  be a Griffiths semi-positive singular Hermitian metric on  $E$ . Then  $h$  is Nakano semi-positive in the sense of singular Hermitian metrics.*

If  $h$  is smooth, Griffiths semi-positivity is equivalent to Nakano semi-positivity in the setting of Proposition 18.1 and 18.2. These propositions imply that our definition of Nakano semi-positivity of singular Hermitian metrics is appropriate when we compare it with already-known positivity notions. Repeating the argument in the proof of Theorem 14.4, we can prove the above propositions. Here we use the same notation as in the proof of Theorem 14.4.

PROOF OF PROPOSITION 18.1. Let  $(\Omega, \iota)$  be a Stein coordinate of  $X$  such that  $L|_{\iota(\Omega)}$  is trivial on  $\iota(\Omega)$ . We simply write  $(\iota^*L, \iota^*h) = (L, h)$  on  $\Omega$ . We take an arbitrary Kähler metric  $\omega_\Omega$ , an arbitrary smooth plurisubharmonic function  $\psi$ , and a global holomorphic frame  $s$  of  $L$  on  $\Omega$ . We define the plurisubharmonic function  $\varphi$  on  $\Omega$  by

$$|s|_h = e^{-\varphi}.$$

By using a usual regularization technique of convolution or Proposition 16.2 and repeating the argument in the proof of Theorem 14.4, we get a sequence of smooth plurisubharmonic functions  $\{\varphi_\nu\}_{\nu=1}^\infty$  such that this sequence is decreasing to  $\varphi$  on any relatively compact subset of  $\Omega$ . Then, taking an exhaustion of  $\Omega$ , we can obtain the following estimate

$$\int_{\Omega} |u|_{\omega_\Omega}^2 e^{-(\varphi+\psi)} dV_{\omega_\Omega} \leq \int_{\Omega} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{\omega_\Omega} e^{-(\varphi+\psi)} dV_{\omega_\Omega}$$

for any  $\bar{\partial}$ -closed  $f \in L_{(n,q)}^2(\Omega, L; \omega_\Omega, h e^{-\psi})$  with the solution  $u \in L_{(n,q-1)}^2(\Omega, L; \omega_\Omega, h e^{-\psi})$  of  $\bar{\partial}u = f$ . Consequently, we complete the proof.  $\square$

PROOF OF PROPOSITION 18.2. We obtain a sequence of smooth Hermitian metrics, with Griffiths positive curvature, increasing to  $h$  on any relatively compact subset again. Since  $S$  is a Riemann surface,  $h_\nu$  is also Nakano semi-positive. Hence, repeating the argument in the proof of Theorem 14.4, we get

$$\int_{\Omega} |u|_{(\omega_\Omega, h)}^2 e^{-\psi} dV_{\omega_\Omega} \leq \int_{\Omega} \langle B_{\omega_\Omega, \psi}^{-1} f, f \rangle_{(\omega_\Omega, h)} e^{-\psi} dV_{\omega_\Omega}$$

for any  $\bar{\partial}$ -closed  $f \in L_{(1,1)}^2(\Omega, E; \omega_\Omega, h e^{-\psi})$  with the solution  $u \in L_{(1,0)}^2(\Omega, E; \omega_\Omega, h e^{-\psi})$  of  $\bar{\partial}u = f$ .  $\square$

## 19. RELATED PROBLEMS

In the last section, we propose important problems related to the main theorems.

First of all, we consider Proposition 16.2. This regularization technique is a fundamental tool to study Griffiths semi-positive singular Hermitian metrics. However, the way to

regularize a Nakano semi-positive singular Hermitian metric is not known. Then we propose the following problem.

QUESTION 19.1. Let  $E$  be a trivial vector bundle over a polydisc  $\Delta \subset \mathbb{C}^n$ . We also let  $h$  be a Nakano semi-positive singular Hermitian metric on  $E$ . Then, can we construct a sequence of smooth Hermitian metrics, with Nakano positive curvature, increasing to  $h$  on any smaller polydiscs?

Next, we think the Demailly-Nadel type vanishing theorem. In general, this vanishing theorem is established on weakly pseudoconvex manifolds. Then we can expect that the main theorems also hold on weakly pseudoconvex manifolds.

QUESTION 19.2. Let  $(E, h)$  be a holomorphic vector bundle and a strictly Nakano positive singular Hermitian metric over a weakly pseudoconvex manifold  $X$ . Then can we obtain  $L^2$ -estimates and vanishing theorems with coefficients in  $E$  on  $X$ ?

Next, we consider the definition of Nakano semi-positivity. In this article, we assume the Griffiths semi-positivity of Nakano semi-positive singular Hermitian metrics. In the smooth setting, it is clear that a Nakano semi-positive Hermitian metric is always Griffiths semi-positive. However, in the singular setting, we do not know whether Nakano semi-positivity yields Griffiths semi-positivity.

QUESTION 19.3. We let  $h$  satisfy the condition in Definition 14.2 without assuming the Griffiths semi-positivity of  $h$ . Can we say that  $h$  is Griffiths semi-positive?

We remark that there exists a result related to Question 19.3 (cf. [DNWZ20, Theorem 1.2] and Theorem 3.5).

At last, we consider the conditions  $\{(2-k)\}_{1 \leq k \leq n}$  in Remark 15.17. As already mentioned, these conditions are equivalent to each other when  $h$  is a smooth Hermitian metric. We expect that this equivalence is also valid even when  $h$  is a singular Hermitian metric.

QUESTION 19.4. Prove the equivalence of the conditions  $\{(2-k)\}_{1 \leq k \leq n}$  in the case that  $h$  is a singular Hermitian metric.

If we can verify Question 19.1, we can also prove Question 19.3 and 19.4 by using the regularization technique. In fact, Question 19.3 and 19.4 are correct if  $h$  is smooth. Then, if we can take a sequence of smooth Hermitian metrics with Nakano positive curvature, we verify these questions by repeating the argument in the proof of Theorem 14.4. Therefore, Question 19.1 is a crucial problem.

## Part 5. Curvature currents and Chern forms of singular Hermitian metrics on holomorphic vector bundles

ABSTRACT. We study curvature currents of singular Hermitian metrics on holomorphic vector bundles. It is known that curvature currents of singular Hermitian metrics on vector bundles are generally not defined with measure coefficients. In this paper, we give some sufficient conditions that curvature currents can be defined with measure coefficients. Moreover, we investigate Chern forms associated with singular Hermitian metrics.

### 20. INTRODUCTION

The results in Part 5 are based on the paper [Ina2]. We study curvature currents of singular Hermitian metrics on holomorphic vector bundles. One of the important notions of the geometry of a holomorphic vector bundle is the curvature of a Hermitian metric on it. However, in the singular setting, it is known that the curvature current of a singular Hermitian metric is not generally defined with measure coefficients [Rau15, Theorem 1.5]. Notions of Griffiths semi-positivity, Griffiths semi-negativity, and Nakano semi-negativity of singular Hermitian metrics can be defined without using the curvature currents (cf. [BP08], [PT18], [Rau15]). Nevertheless, Nakano positivity, strict positivity, or strict negativity has not been determined in the singular setting. For this reason, we have to investigate the condition that the curvature currents can be defined with measure coefficients.

Let  $X$  be a complex manifold,  $E \rightarrow X$  be a holomorphic vector bundle over  $X$ , and  $h$  be a singular Hermitian metric on  $E$ . In [Rau15], Raufi showed the following theorem. As the statement is local, we assume that  $X$  is a polydisk in  $\mathbb{C}^n$  and that  $E$  is trivial. Notations will be explained afterward. We let  $\xi$  be an arbitrary holomorphic vector field on  $X$ .

**THEOREM 20.1.** [Rau15, Theorem 1.6] *Let  $h$  be a singular Hermitian metric that is Griffiths semi-negative (see Definition 21.3),  $\{h_\nu\}_{\nu=1}^\infty$  be any approximating sequence of smooth Hermitian metrics with Griffiths semi-negative curvature, decreasing to  $h$ .*

*Suppose  $\det h > \epsilon$  for some positive constant  $\epsilon > 0$ . Then*

- (1)  $\tilde{\theta}_{h_\nu} := \theta_{h_\nu}(\xi) \in L_{loc}^2(X)$  uniformly in  $\nu$ , and  $\tilde{\theta}_h := \theta_h(\xi) \in L_{loc}^2(X)$ ,
- (2)  $\tilde{\theta}_{h_\nu} \in L_{loc}^2(X)$  weakly converges to  $\tilde{\theta}_h$  in  $L_{loc}^2(X)$ , and
- (3)  $\tilde{\Theta}_{h_\nu} := \Theta_{h_\nu}(\xi, \xi) \in L_{loc}^1(X)$  uniformly in  $\nu$ ,  $\tilde{\Theta}_h := \Theta_h(\xi, \xi)$  has measure coefficients, and  $\tilde{\Theta}_{h_\nu}$  weakly converges to  $\tilde{\Theta}_h$  in the sense of measures.

Here  $\tilde{\theta}_h$  denotes  $\theta_h(\xi)$ , and  $\tilde{\Theta}_h$  denotes  $\Theta_h(\xi, \xi)$ . The above notations mean that after representing  $\tilde{\theta}_{h_\nu}$ ,  $\tilde{\theta}_h$ ,  $\tilde{\Theta}_{h_\nu}$ , and  $\tilde{\Theta}_h$  as a matrix, each element of the matrix has measure coefficients, weakly converges, and so on. If the  $L^2$  norm of each element of  $\tilde{\theta}_{h_\nu}$  has

an upper bound which is independent of  $\nu$  on a fixed compact subset of  $X$ , we say  $\tilde{\theta}_{h_\nu} \in L_{loc}^2(X)$  uniformly in  $\nu$ .

In general, a Griffiths semi-negative singular Hermitian metric  $h$  possibly degenerates. In this paper, we find the condition that the curvature current can be defined with measure coefficients over a set including the degeneracy set  $\{\det h = 0\}$ . The Lelong number of  $\log \det h$  has an important role in our main theorems. We denote by  $\nu(\log \det h, x)$  the Lelong number of the local weight function  $\varphi$  of  $\det h = e^\varphi$  at  $x \in X$ . To be precise, we prove the following theorem.

**THEOREM 20.2.** *Let  $h$  be a Griffiths semi-negative singular Hermitian metric,  $\{h_\nu\}_{\nu=1}^\infty$  be a sequence of smooth Hermitian metrics, with Griffiths semi-negative curvature, decreasing pointwise to  $h$ , and  $0 \leq \epsilon < 1, \delta > 0$ . We assume that*

(i)  $\nu(\log \det h, x) < 1 - \epsilon$ , for all  $x \in X$ ,

(ii)  $\sqrt{-1}\partial\bar{\partial} \log \det h \in L_{loc}^{1+\delta}(X)$ .

*Then we can obtain*

(1)  $\tilde{\theta}_h := \theta_h(\xi) \in L_{loc}^{\frac{2}{2-\epsilon}}(X)$ , and  $\tilde{\theta}_{h_\nu} := \theta_{h_\nu}(\xi) \in L_{loc}^{\frac{2}{2-\epsilon}}(X)$  uniformly in  $\nu$ ,

(2)  $\tilde{\theta}_{h_\nu} = \theta_{h_\nu}(\xi)$  weakly converges to  $\tilde{\theta}_h = \theta_h(\xi)$  in the sense of distributions, and

(3) if  $(\epsilon + 1)(\delta + 1) \geq 2$ ,  $\tilde{\Theta}_h := \Theta_h(\xi, \xi)$  has measure coefficients, and  $\tilde{\Theta}_{h_\nu} := \Theta_{h_\nu}(\xi, \xi)$  weakly converges to  $\tilde{\Theta}_h = \Theta_h(\xi, \xi)$  in the sense of measures.

We denote by  $S_h$  the set of all points in  $X$  such that  $h$  satisfies the above assumptions (i), (ii), and  $(\epsilon + 1)(\delta + 1) \geq 2$  around there. Thanks to the above theorems, we can conclude that  $\Theta_h$  has measure coefficients on  $\bigcup_{\epsilon > 0} \{x \in X; \det h(x) > \epsilon\} \cup S_h$ . We will explain examples such that  $\Theta_h$  has measure coefficients over a set including the degeneracy set  $\{\det h = 0\}$  (see Example 22.4 and Example 22.5).

We can also prove a version of Theorem 20.2 in the case that  $h$  is Griffiths semi-positive. In the singular setting, we do not know whether  $\Theta_h = -{}^t\Theta_{h^*}$  holds. Therefore Theorem 20.2 does not immediately imply that the curvature current of the dual metric of  $h$  in Theorem 20.2 has measure coefficients.

We also investigate Chern forms associated with singular Hermitian metrics. Since the curvature current of a singular Hermitian metric has current coefficients, we cannot generally define Chern forms. Restricting codimension of the singular set of the metric, Lärkäng, Raufi, Ruppenthal, and Sera [LRRS18] defined Chern forms as closed currents of order 0. We propose another approach to Chern currents of singular Hermitian metrics. We obtain the following theorem.

**THEOREM 20.3.** *Let  $h$  be a Griffiths semi-negative singular Hermitian metric,  $0 \leq \epsilon < 1, \delta > 0$ , and  $k \in \mathbb{N}$ . We assume that*

(i)  $\nu(\log \det h, x) < 1 - \epsilon$ , for all  $x \in X$ ,

(ii)  $\sqrt{-1}\partial\bar{\partial} \log \det h \in L_{loc}^{1+\delta}(X)$ , and

(iii)  $(\epsilon - \frac{k-2}{k})(\delta + 1) \geq 2$ .

*Then we can define the Chern current  $c_k(E, h)$  with measure coefficients.*

Finally, in the last section, we show some applications of main theorems, and present related questions. We would like to consider a metric with minimal singularities in some sense on  $E$ . In the case that  $E$  is a line bundle, the metric with minimal singularities was obtained by Demailly, Peternell, and Schneider [DPS01]. However, if  $E$  is a vector bundle, it is difficult to construct the metric with minimal singularities. Instead we study a singular Hermitian metric whose determinant has minimal singularities. We denote it by  $h_{\text{dmin}}$  (see Definition 25.5). The metric  $h_{\text{dmin}}$  is also generally constructed only in the case that  $E$  is a line bundle. However, in the special case, we obtain an example of  $h_{\text{dmin}}$  (see Example 25.6). We also prove the following theorem.

**THEOREM 20.4.** *Let  $X$  be a projective manifold,  $T_X$  be the tangent bundle, and  $\omega$  be a Kähler form on  $X$ . We assume that*

- (i)  $(T_X, \omega)$  is semi-positive in the sense of Griffiths,
- (ii)  $E$  is nef and big.

*Then  $\Theta_{(h_{\text{dmin}})^*}$  and  $c_k(E^*, (h_{\text{dmin}})^*)$  have measure coefficients.*

The paper is organized as follows. In Section 21, we introduce some basic definitions and properties of singular Hermitian metrics on holomorphic vector bundles. In Section 22, we study Griffiths semi-negative singular Hermitian metrics and prove Theorem 20.2. In Section 23, we investigate various formulas for singular Hermitian metrics and show a version of Theorem 20.2 in the case that  $h$  is Griffiths semi-positive. In Section 24, we discuss Chern currents associated with singular Hermitian metrics and prove Theorem 20.3. In Section 25, we present some applications of main theorems.

## 21. PRELIMINARIES

In this section, we introduce basic notions and properties of a singular Hermitian metric on a vector bundle. Throughout this section,  $X$  denotes an  $n$ -dimensional complex manifold,  $E$  denotes a holomorphic vector bundle over  $X$ , and  $h$  denotes a singular Hermitian metric on  $E$ .

**21.1. Singular Hermitian metrics on vector bundles.** To begin with, we introduce the following definition of singular Hermitian metrics on vector bundles.

**DEFINITION 21.1.** (cf. [Rau15, Definition 1.1]) *A singular Hermitian metric  $h$  on  $E$  is a measurable map from the base manifold  $X$  to the space of non-negative Hermitian forms on the fibers. The Hermitian form possibly takes the value  $\infty$  at some points, and satisfies  $0 < \det h < +\infty$  almost everywhere.*

The Chern curvature current  $\Theta_h$  is locally defined as  $\Theta_h = \bar{\partial}(h^{-1}\partial h)$ . However, it is not always defined with measure coefficients. This example is discovered by Raufi [Rau15].

EXAMPLE 21.2. [Rau15, Theorem 1.5] Let  $\Delta \subset \mathbb{C}$  be the unit disk,  $E = \Delta \times \mathbb{C}^2$  be the trivial bundle of rank 2 over  $\Delta$ , and  $h$  be the singular Hermitian metric

$$h = \begin{pmatrix} 1 + |z|^2 & z \\ \bar{z} & |z|^2 \end{pmatrix}.$$

Then  $\Theta_h$  is not a current with measure coefficients.

For this reason, we cannot generally define the positivity or negativity of a singular Hermitian metric by using the curvature current. The definition of Griffiths positivity or negativity is as follows.

DEFINITION 21.3. ([BP08, Definition 3.1], [PT18, Definition 2.2.2], and [Rau15, Definition 1.2])

- (1) A singular Hermitian metric  $h$  is *Griffiths semi-negative* if  $\log |u|_h^2$  is a plurisubharmonic function for any local holomorphic section  $u$  of  $E$ .
- (1') A singular Hermitian metric  $h$  is *Griffiths semi-negative* if  $|u|_h^2$  is a plurisubharmonic function for any local holomorphic section  $u$  of  $E$ .
- (2) A singular Hermitian metric  $h$  is *Griffiths semi-positive* if the dual metric  $h^*$  is Griffiths semi-positive.

The above (1) is equivalent to (1') (cf. [Rau15, Section 2]).

**21.2. The Lelong number of plurisubharmonic functions.** Here we introduce the notion of the Lelong number of a plurisubharmonic function.

DEFINITION 21.4. (cf. [Dem, Theorem 2.8]) For a plurisubharmonic function  $\varphi$ , the Lelong number of  $\varphi$  at a point  $x \in X$  is defined to be the limit

$$\liminf_{z \rightarrow x} \frac{\varphi(z)}{\log |z - x|}.$$

We denote it by  $\nu(\varphi, x)$ . Here  $z$  is some local coordinate around  $x$ .

It is known that the Lelong number is invariant by holomorphic changes of local coordinates. The Lelong number of a singular Hermitian metric on a holomorphic line bundle is also defined. Let  $L$  be a holomorphic line bundle over  $X$ , and  $h$  be a singular Hermitian metric on  $L$ . Here a singular Hermitian metric  $h$  on  $L$  is a measurable metric with a locally integrable weight function, i.e.  $h$  has the form  $h = h_0 e^{-\varphi}$ , where  $h_0$  is a smooth metric on  $L$  and  $\varphi$  is an  $L^1_{loc}$  function on  $X$ . Locally,  $\log h$  (resp.  $-\log h$ ) is plurisubharmonic if  $h$  is semi-negative (resp. semi-positive). Hence we can define the Lelong number of a semi-negative (resp. semi-positive) singular Hermitian metric  $h = h_0 e^{-\varphi}$  at  $x \in X$  as  $\nu(\log h, x) = \nu(-\varphi, x)$  (resp.  $\nu(-\log h, x) = \nu(\varphi, x)$ ).

The above definition is independent of the choice of local coordinates. The Lelong number of a plurisubharmonic function is related to the integrability of  $e^{-\varphi}$ . We recall one of the important results of Skoda.



LEMMA 21.5. ([Dem, Lemma 5.6], [Sko72]) *Let  $\varphi$  be a plurisubharmonic function on  $X$ , and  $x \in X$ . If  $\nu(\varphi, x) < 1$ ,  $e^{-2\varphi}$  is integrable in a neighborhood of  $x$ .*

We will show a lemma that all plurisubharmonic functions belong to  $L_{loc}^p(X)$ .

LEMMA 21.6. *Let  $\varphi$  be a plurisubharmonic function on  $X$ . Then  $\varphi \in L_{loc}^p(X)$  for all  $p \geq 1$ .*

PROOF. We can assume that  $\varphi \leq 0$  without any loss of generality since  $\varphi$  is locally bounded from above. For any  $x \in X$ , the Lelong number  $\nu(\frac{\epsilon\varphi}{2}, x) < 1$  if we take  $\epsilon > 0$  small enough. We have

$$0 \leq \frac{(-\epsilon\varphi)^p}{p!} \leq 1 + (-\epsilon\varphi) + \cdots + \frac{(-\epsilon\varphi)^p}{p!} + \cdots = e^{-\epsilon\varphi}.$$

Since  $e^{-\epsilon\varphi}$  is integrable around  $x \in X$  from Lemma 21.5,  $\varphi$  is also an  $L_{loc}^p$  function around  $x \in X$ .  $\square$

## 22. CURVATURE CURRENTS OF GRIFFITHS SEMI-NEGATIVE SINGULAR HERMITIAN METRICS

In this section, we mainly prove Theorem 20.2. Throughout this section,  $X$  denotes an  $n$ -dimensional complex manifold,  $E$  denotes a holomorphic vector bundle of rank  $r$  over  $X$ , and  $h$  denotes a Griffiths semi-negative singular Hermitian metric on  $E$ .

We start by analyzing the condition  $\sqrt{-1}\partial\bar{\partial}\log\det h \in L_{loc}^{1+\delta}(X)$ . We remark that  $\det h$  is a semi-negative singular Hermitian metric on  $\det E$  since  $h$  is Griffiths semi-negative ([HPS18, Proposition 25.1], [Rau15, Proposition 1.3]). Taking a local trivialization, we have that  $\log\det h$  is plurisubharmonic, and  $\log\det h \in L_{loc}^p(X)$  for all  $p \geq 1$ .

In order to study analytic properties of  $h$ , we will take an approximating sequence  $\{h_\nu\}_{\nu=1}^\infty$  of  $h$ . We can locally take such a sequence, through convolution with an approximate identity, from the following Lemma.

LEMMA 22.1. [BP08, Proposition 3.1] *Let  $h$  be a Griffiths semi-negative singular Hermitian metric. If  $E$  is a trivial vector bundle over a polydisk, then there exists a sequence of smooth Hermitian metrics  $\{h_\nu\}_{\nu=1}^\infty$ , with Griffiths negative curvature, decreasing pointwise to  $h$  on any smaller polydisk.*

This section only deals with local properties of a singular Hermitian metric. Hence, we can assume that  $X$  is a domain in  $\mathbb{C}^n$ ,  $dV$  is a volume form on  $X$ ,  $E$  is the trivial bundle over  $X$ , and there is the above approximating sequence  $\{h_\nu\}_{\nu=1}^\infty$  on  $X$  without any loss of generality. We let  $\xi$  be a holomorphic vector field on  $X$ .

LEMMA 22.2. *Let  $p > 1$ . If  $\sqrt{-1}\partial\bar{\partial}\log\det h \in L_{loc}^p(X)$ ,  $\text{Tr}(\tilde{\Theta}_{h_\nu}) \in L_{loc}^p$  uniformly in  $\nu$ , where  $\tilde{\Theta}_{h_\nu} = \Theta_{h_\nu}(\xi, \xi)$  for a holomorphic vector field  $\xi$ .*

PROOF. Let  $U$  be an arbitrary relatively compact open subset. It is enough show that  $\text{Tr}(\tilde{\Theta}_{h_\nu}) \in L^p(U)$  uniformly in  $\nu$ . Let  $\chi \in C_c^\infty(U)$  be any test function. It is known

that  $\log \det h_\nu$  and  $\log \det h$  are plurisubharmonic functions for all  $\nu$ . We also see that  $\log \det h_\nu$  decreases to  $\log \det h$ . Therefore  $\sqrt{-1}\partial\bar{\partial} \log \det h_\nu(\xi, \xi)$  weakly converges to  $\sqrt{-1}\partial\bar{\partial} \log \det h(\xi, \xi)$ . Let  $q$  be a real number such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We obtain

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \left| \int_U (\sqrt{-1}\partial\bar{\partial} \log \det h_\nu(\xi, \xi)) \chi dV \right| &= \left| \int_U (\sqrt{-1}\partial\bar{\partial} \log \det h(\xi, \xi)) \chi dV \right| \\ &\leq C \|\chi\|_{L^q(U)} \end{aligned}$$

by the Hölder inequality, where  $C$  is some positive constant that is independent of  $\nu$ . Hence, we obtain

$$\sup_{\nu} \frac{\left| \int_U (\sqrt{-1}\partial\bar{\partial} \log \det h_\nu(\xi, \xi)) \chi dV \right|}{\|\chi\|_{L^q(U)}} \leq C'$$

for some positive constant  $C'$  that is independent of  $\nu$ . It follows that

$$\mathrm{Tr}(\tilde{\Theta}_{h_\nu}) \in L^p(U)$$

uniformly in  $\nu$  for the reason that

$$-\sqrt{-1}\partial\bar{\partial} \log \det h_\nu = \mathrm{Tr}(\Theta_{h_\nu}).$$

□

REMARK 22.3. [Rau15, Remark 4.1] Let  $h$  be a Griffiths semi-negative singular Hermitian metric. In general,  $\mathrm{Tr}(\tilde{\Theta}_{h_\nu}) \in L^1_{loc}(X)$  uniformly in  $\nu$ .

Then we show Theorem 20.2.

PROOF OF THEOREM 20.2. Firstly, since  $h$  is Griffiths semi-negative,  $\widetilde{\partial h} := \partial h(\xi)$  is an  $L^2_{loc}(X)$  valued matrix and  $\widetilde{\partial h_\nu} \in L^2_{loc}(X)$  uniformly in  $\nu$  [Rau15, Proposition 1.4, Lemma 5.1]. Let  $\hat{h}$  be the adjugate of  $h$ , i.e.  $h^{-1} = \frac{1}{\det h} \hat{h}$ . The entries of  $\hat{h}$  are locally bounded because they are polynomials of the entries of  $h$  and they are locally bounded [PT18, Lemma 2.2.7]. Hence, there exists a constant  $C > 0$  such that

$$\|\widetilde{\theta_h}\|_{HS} = \|h^{-1}\widetilde{\partial h}\|_{HS} \leq \frac{C}{\det h} \|\widetilde{\partial h}\|_{HS},$$

where  $\|\cdot\|_{HS}$  is the Hilbert-Schmidt norm. Let  $dV$  be a volume form on  $X$  and  $K$  be an arbitrary compact subset of  $X$ . We get

$$\begin{aligned} \left( \int_K \|\widetilde{\theta_h}\|_{HS}^{\frac{2}{2-\epsilon}} dV \right)^{\frac{2-\epsilon}{2}} &\leq C \left( \int_K \left( \frac{1}{\det h} \|\widetilde{\partial h}\|_{HS} \right)^{\frac{2}{2-\epsilon}} dV \right)^{\frac{2-\epsilon}{2}} \\ &\leq C \left( \int_K \left( \frac{1}{\det h} \right)^{\frac{2}{1-\epsilon}} dV \right)^{\frac{1-\epsilon}{2}} \left( \int_K \|\widetilde{\partial h}\|_{HS}^2 dV \right)^{\frac{1}{2}} \end{aligned}$$

by using the Hölder inequality. The Lelong number  $\nu(\log \det h, x)$  is less than  $1 - \epsilon$  for all  $x \in X$ . Therefore

$$e^{-\frac{2}{1-\epsilon} \log \det h} = \left( \frac{1}{\det h} \right)^{\frac{2}{1-\epsilon}}$$

is an  $L^1_{loc}$  function from Lemma 21.5. Hence the right-hand side of the above inequality is bounded.

The entries of  $h_\nu$  are locally bounded uniformly in  $\nu$  because  $\{h_\nu\}_{\nu=1}^\infty$  is a decreasing sequence. The entries of  $\widehat{h}_\nu$  are also locally bounded uniformly in  $\nu$ . Then we have

$$\begin{aligned} \int_K \|\widetilde{\theta}_{h_\nu}\|_{HS}^{\frac{2}{2-\epsilon}} dV &\leq C' \int_K \left( \frac{1}{\det h_\nu} \|\widetilde{\partial h_\nu}\|_{HS} \right)^{\frac{2}{2-\epsilon}} dV \\ &\leq C' \int_K \left( \frac{1}{\det h} \|\widetilde{\partial h_\nu}\|_{HS} \right)^{\frac{2}{2-\epsilon}} dV \end{aligned}$$

for some positive constant  $C' > 0$  since  $\{\det h_\nu\}_{\nu=1}^\infty$  is a decreasing sequence. Then we get

$$\begin{aligned} \left( \int_K \|\widetilde{\theta}_{h_\nu}\|_{HS}^{\frac{2}{2-\epsilon}} dV \right)^{\frac{2-\epsilon}{2}} &\leq C' \left( \int_K \left( \frac{1}{\det h} \|\widetilde{\partial h_\nu}\|_{HS} \right)^{\frac{2}{2-\epsilon}} dV \right)^{\frac{2-\epsilon}{2}} \\ &\leq C' \left( \int_K \left( \frac{1}{\det h} \right)^{\frac{2}{1-\epsilon}} dV \right)^{\frac{1-\epsilon}{2}} \left( \int_K \|\widetilde{\partial h_\nu}\|_{HS}^2 dV \right)^{\frac{1}{2}}. \end{aligned}$$

As the right-hand side of the above inequality is bounded independently of  $\nu$ , we can prove the part (1).

Secondly, let  $\chi \in C_c^\infty(X)$  be an arbitrary test function, and  $\widetilde{\theta}_{\lambda\mu}^h$  be the  $(\lambda, \mu)$  element of  $\widetilde{\theta}_h$ . It is enough to show that

$$\int_X (\widetilde{\theta}_{\lambda\mu}^{h_\nu} - \widetilde{\theta}_{\lambda\mu}^h) \chi dV \rightarrow 0$$

as  $\nu \rightarrow 0$ . We get

$$\left| \int_X (\widetilde{\theta}_{\lambda\mu}^{h_\nu} - \widetilde{\theta}_{\lambda\mu}^h) \chi dV \right| \leq \left| \int_X ((h_\nu^{-1} - h^{-1}) \widetilde{\partial h_\nu})_{\lambda\mu} \chi dV \right| + \left| \int_X (h^{-1} (\widetilde{\partial h_\nu} - \widetilde{\partial h}))_{\lambda\mu} \chi dV \right|.$$

For the first term, using the Cauchy-Schwarz inequality, we have

$$\left| \int_X ((h_\nu^{-1} - h^{-1}) \widetilde{\partial h_\nu})_{\lambda\mu} \chi dV \right| \leq C \left( \int_K \|h_\nu^{-1} - h^{-1}\|_{HS}^2 dV \right)^{\frac{1}{2}} \left( \int_K \|\widetilde{\partial h_\nu}\|_{HS}^2 dV \right)^{\frac{1}{2}},$$

for some  $C > 0$ , where  $K$  is a support of  $\chi$ . We have the inequality that

$$\begin{aligned} \|h_\nu^{-1} - h^{-1}\|_{HS}^2 &\leq 2(\|h_\nu^{-1}\|_{HS}^2 + \|h^{-1}\|_{HS}^2) \\ &\leq 2C' \left( \frac{1}{\det^2 h_\nu} + \frac{1}{\det^2 h} \right) \\ &\leq 4C' \frac{1}{\det^2 h}, \end{aligned}$$

for some positive constant  $C' > 0$ . We can take  $\frac{1}{\det h}$  as an  $L_{loc}^2$  function because  $\frac{1}{\det h} \in L_{loc}^{\frac{2}{1-\epsilon}}(X) \subset L_{loc}^2(X)$ . Since

$$\int_K \frac{1}{\det^2 h} dV, \quad \int_K \|\widetilde{\partial h_\nu}\|_{HS}^2 dV$$

are bounded independently of  $\nu$ , we obtain

$$\left| \int_X ((h_\nu^{-1} - h^{-1}) \widetilde{\partial h_\nu})_{\lambda\mu} \chi dV \right| \rightarrow 0$$

as  $\nu \rightarrow 0$  by the Lebesgue dominated convergence theorem.

For the second term, we get

$$\left| \int_X (h^{-1}(\widetilde{\partial h_\nu} - \widetilde{\partial h}))_{\lambda\mu} \chi dV \right| \leq C \int_K |(h^{-1}(\widetilde{\partial h_\nu} - \widetilde{\partial h}))_{\lambda\mu}| dV.$$

It is known that  $\widetilde{\partial h_\nu}$  weakly converges to  $\widetilde{\partial h}$  in  $L_{loc}^2(X)$  [Rau15, Lemma 5.1]. We also know that  $h^{-1} \in L_{loc}^2(X)$  since  $\frac{1}{\det^2 h}$  is locally integrable. Therefore it follows that

$$\left| \int_X (h^{-1}(\widetilde{\partial h_\nu} - \widetilde{\partial h}))_{\lambda\mu} \chi dV \right| \rightarrow 0$$

as  $\nu \rightarrow 0$ . Consequently, the part (2) follows.

Finally, we know that  $\Theta_h = \bar{\partial}\theta_h$  is well-defined in the sense of distributions since  $\theta_h$  has  $L_{loc}^{\frac{2}{2-\epsilon}} \subset L_{loc}^1$  coefficients. We begin to show that  $\widetilde{\Theta}_{h_\nu} \in L_{loc}^1(X)$  uniformly in  $\nu$ . Let  $(\cdot, \cdot)_e$  denote the Euclidean metric on  $\mathbb{C}^n$ . It is enough to show that  $|(\widetilde{\Theta}_{h_\nu} u, u)_e| \in L_{loc}^1(X)$  uniformly in  $\nu$  for any holomorphic section  $u \in \mathcal{O}(E)$ . The Griffiths semi-negativity of  $h_\nu$  implies that  $-{}^t\widetilde{\Theta}_{h_\nu} h_\nu$  is semi-positive metric on  $E$ . Hence, we have

$$\begin{aligned} |(\widetilde{\Theta}_{h_\nu} u, u)_e| &= |(u, \bar{h}_\nu^{-1} u)_{-{}^t\widetilde{\Theta}_{h_\nu} h_\nu}| \\ &\leq \sqrt{(u, u)_{-{}^t\widetilde{\Theta}_{h_\nu} h_\nu}} \sqrt{(\bar{h}_\nu^{-1} u, \bar{h}_\nu^{-1} u)_{-{}^t\widetilde{\Theta}_{h_\nu} h_\nu}} \\ &= \sqrt{(-\widetilde{\Theta}_{h_\nu} u, u)_{h_\nu}} \sqrt{(-\widetilde{\Theta}_{h_\nu} \bar{h}_\nu^{-1} u, \bar{h}_\nu^{-1} u)_{h_\nu}} \\ &\leq \sqrt{-\text{Tr}(\widetilde{\Theta}_{h_\nu})} \|u\|_{h_\nu} \sqrt{-\text{Tr}(\widetilde{\Theta}_{h_\nu})} \|\bar{h}_\nu^{-1} u\|_{h_\nu} \\ &= (-\text{Tr}(\widetilde{\Theta}_{h_\nu})) \|u\|_{h_\nu} \|u\|_{h_\nu^{-1}} \\ &\leq (-\text{Tr}(\widetilde{\Theta}_{h_\nu})) \|u\|_{h_0} \|u\|_{h^{-1}} \end{aligned}$$

by the Cauchy-Schwarz inequality (cf. [Rau15, Section 5]). We know that  $\|u\|_{h^{-1}} \in L_{loc}^{\frac{2}{1-\epsilon}}(X)$  for the reason that  $\frac{1}{\det h} \in L_{loc}^{\frac{2}{1-\epsilon}}(X)$ . We also see that  $-\text{Tr}(\widetilde{\Theta}_{h_\nu}) \in L_{loc}^{1+\delta}(X)$  uniformly in  $\nu$  from Lemma 22.2. Therefore, the right-hand side of the above inequality is in

$$L_{loc}^{\frac{1}{(\frac{1-\epsilon}{2} + \frac{1}{1+\delta})}}(X) = L_{loc}^{\frac{2(1+\delta)}{(1-\epsilon)(1+\delta)+2}}(X)$$

uniformly in  $\nu$  by Hölder's inequality. As the assumption  $(\epsilon + 1)(\delta + 1) \geq 2$  implies that

$$\frac{2(1 + \delta)}{(1 - \epsilon)(1 + \delta) + 2} \geq 1,$$

we obtain  $\tilde{\Theta}_{h_\nu} \in L^1_{loc}(X)$  uniformly. Moreover, we see that  $\tilde{\Theta}_{h_\nu}$  weakly converges to  $\tilde{\Theta}_h$  from the result of the part (2). Let  $\chi$  be any test function, and  $(\tilde{\Theta}_{h_\nu})_{\lambda,\mu}$  be the  $(\lambda, \mu)$  element of  $\tilde{\Theta}_{h_\nu}$ . We obtain

$$\left| \int_X (\tilde{\Theta}_h)_{\lambda,\mu} \chi dV \right| = \lim_{\nu \rightarrow \infty} \left| \int_X (\tilde{\Theta}_{h_\nu})_{\lambda,\mu} \chi dV \right| \leq C \sup_X \|\chi\|$$

for some positive constant  $C > 0$  since  $\tilde{\Theta}_{h_\nu} \in L^1_{loc}(X)$  uniformly in  $\nu$ . Consequently, we can prove the part (3).  $\square$

Theorem 20.1 and Theorem 20.2 imply that  $\Theta_h$  has measure coefficients on  $\bigcup_{\epsilon > 0} \{x \in X; \det h(x) > \epsilon\} \cup S_h$ . We show examples of Griffiths semi-negative singular Hermitian metrics whose curvature currents can be defined with measure coefficients over a set including the degeneracy set  $\{\det h = 0\}$ .

EXAMPLE 22.4. Let  $X$  be the unit polydisk in  $\mathbb{C}^2$ ,  $E$  be the trivial bundle  $X \times \mathbb{C}^2$ , and  $h$  be a singular Hermitian metric

$$h = \begin{pmatrix} |z_1|^2 + |z|^{\frac{1}{4}} & z_1 \\ \bar{z}_1 & 1 \end{pmatrix},$$

where  $z = (z_1, z_2)$  is a coordinate of  $\mathbb{C}^2$ . For any holomorphic section

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

we have

$$|u|_h^2 = |u_1 z_1 + u_2|^2 + |u_1|^2 |z|^{\frac{1}{4}}.$$

This is a plurisubharmonic function. Hence  $h$  is Griffiths semi-negative. We also have

$$\begin{aligned} \{\det h = 0\} &= \{(0, 0)\} \subset X, \\ \log \det h &= \frac{1}{4} \log |z|. \end{aligned}$$

It follows that

$$\nu(\log \det h, 0) = \frac{1}{4} < 1 - \epsilon.$$

We can take  $\epsilon = \frac{2}{3}$ . We also see that

$$\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} (\log \det h) = \frac{1}{8} \frac{|z_2|^2}{|z|^4} \leq \frac{1}{8|z|^2},$$

and  $(\frac{1}{|z|^2})^{\frac{3}{2}} = (\frac{1}{|z|^2})^{1+\frac{1}{2}}$  is locally integrable on  $X$ . Since other differential coefficients are also  $L_{loc}^{\frac{3}{2}}$ , we can take  $\delta = \frac{1}{2}$ . Then we get

$$(\epsilon + 1)(\delta + 1) = \frac{5}{3} \cdot \frac{3}{2} = \frac{5}{2} \geq 2,$$

and we can conclude that the curvature current associated with this  $h$  has measure coefficients on  $X$ , especially around  $\{\det h = 0\} = \{(0, 0)\} \subset X$ . We can also directly compute the curvature current and verify that it has measure coefficients.

EXAMPLE 22.5. Let  $X$  be the ball  $\{|z| < \frac{1}{\sqrt{e}}\}$  in  $\mathbb{C}^2$ ,  $E$  be the trivial bundle  $X \times \mathbb{C}^2$ , and  $h$  be a singular Hermitian metric

$$h = \begin{pmatrix} |z_1|^2 + \frac{1}{(-\log |z|^2)} & z_1 \\ \bar{z}_1 & 1 \end{pmatrix}.$$

For any holomorphic section

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

we have

$$|u|_h^2 = |u_1 z_1 + u_2|^2 + |u_1|^2 \frac{1}{(-\log |z|^2)}.$$

This is plurisubharmonic. Hence  $h$  is Griffiths semi-negative. We also have

$$\begin{aligned} \{\det h = 0\} &= \{(0, 0)\} \subset X, \\ \log \det h &= -\log(-\log |z|^2). \end{aligned}$$

It follows that

$$\nu(\log \det h, 0) = 0 < 1 - \epsilon.$$

We can take  $\epsilon = \frac{9}{10}$ . We also see that

$$\frac{\partial^2}{\partial z_1 \partial \bar{z}_1}(\log \det h) = \frac{|z_1|^2 + |z_2|^2(-\log |z|^2)}{|z|^4(-\log |z|^2)^2} \leq \frac{2}{|z|^2(-\log |z|^2)},$$

and

$$\frac{1}{|z|^3(-\log |z|^2)^{\frac{3}{2}}} = \left( \frac{1}{|z|^2(-\log |z|^2)} \right)^{\frac{3}{2}} = \left( \frac{1}{|z|^2(-\log |z|^2)} \right)^{1+\delta}$$

is  $L_{loc}^1$  on  $X$ . Since other differential coefficients are also  $L_{loc}^{\frac{3}{2}}$ , we can take  $\delta = \frac{1}{2}$ . We get

$$(\epsilon + 1)(\delta + 1) = \frac{19}{10} \cdot \frac{3}{2} = \frac{57}{20} \geq 2,$$

then we can conclude that the curvature current associated with this  $h$  has measure coefficients on  $X$ , especially around  $\{\det h = 0\} = \{(0, 0)\} \subset X$ . We can also directly compute the curvature current and verify that it has measure coefficients.

### 23. CURVATURE CURRENTS OF GRIFFITHS SEMI-POSITIVE SINGULAR HERMITIAN METRICS

In this section, we will prove a version of Theorem 20.2 for semi-positive curvature. If  $h$  is a Griffiths semi-negative singular Hermitian metric on  $E$ , the dual metric  $h^*$  is a Griffiths semi-positive singular Hermitian metric on  $E^*$ . If  $h$  is smooth,

$$\Theta_h = -{}^t\Theta_{h^*}.$$

However, in the singular setting, we do not know whether the above equation holds. To prove it, we begin to prepare some lemmas. Throughout this section, we assume that  $X$  is a domain in  $\mathbb{C}^n$ ,  $dV$  is a volume form on  $X$ ,  $\xi = \xi_1 \frac{\partial}{\partial z_1} + \cdots + \xi_n \frac{\partial}{\partial z_n}$  is a holomorphic vector field,  $E$  is a holomorphic trivial bundle over  $X$ , and  $h$  is a singular Hermitian metric on  $E$ .

It is known that the condition  $\sqrt{-1}\partial\bar{\partial}\log\det h \in L_{loc}^{1+\delta}(X)$  implies that first and second derivatives of  $\log\det h$  are also in  $L_{loc}^{1+\delta}(X)$  from the results of the Riesz transform [Ste70]. We use this property in this section.

LEMMA 23.1. *Let  $h$  be a Griffiths semi-negative singular Hermitian metric, and  $0 \leq \epsilon < 1, \delta > 0$ . We assume that*

- (i)  $\nu(\log\det h, x) < 1 - \epsilon$ , for all  $x \in X$ ,
- (ii)  $\sqrt{-1}\partial\bar{\partial}\log\det h \in L_{loc}^{1+\delta}(X)$ , and
- (iii)  $(\epsilon + 1)(\delta + 1) \geq 2$ .

*Then*

$$\frac{\partial\det h(\xi)}{\det h}$$

*can be defined as an  $L_{loc}^1$  function. Moreover, the equation*

$$\partial\log\det h(\xi) = \frac{\partial\det h(\xi)}{\det h}$$

*holds in the sense of distributions.*

PROOF. First of all,  $\det h$  is a locally bounded function for the reason that  $h$  is Griffiths semi-negative [PT18, Lemma 2.2.7]. Hence  $\partial\det h(\xi)$  can be defined as a distribution. As  $\log\det h_\nu$  is decreasing to  $\log\det h$ ,  $\partial\log\det h_\nu(\xi)$  weakly converges to  $\partial\log\det h(\xi)$ . Then the assumption  $\partial\log\det h(\xi) \in L_{loc}^{1+\delta}(X)$  implies that

$$\partial\log\det h_\nu(\xi) = \frac{\partial\det h_\nu(\xi)}{\det h_\nu} \in L_{loc}^{1+\delta}(X)$$

uniformly in  $\nu$ . We obtain

$$\partial\det h_\nu(\xi) \in L_{loc}^{1+\delta}(X)$$

uniformly in  $\nu$  since  $\det h_\nu \leq \det h_0 \leq C$  locally hold for some positive constant  $C > 0$ . We can conclude that  $\partial\det h(\xi) \in L_{loc}^{1+\delta}(X)$  because of the fact that  $\partial\det h_\nu(\xi)$  weakly

converges to  $\partial \det h(\xi)$ . Using the Hölder inequality as the proof of Theorem 20.2, we obtain

$$\frac{\partial \det h(\xi)}{\det h} \in L_{loc}^1(X).$$

Secondly, for any test function  $\chi \in C_c^\infty(X)$ , it is enough to show that

$$\int_X \partial \log \det h(\xi) \chi dV = \int_X \frac{\partial \det h(\xi)}{\det h} \chi dV.$$

For each  $\nu$ , we see that

$$\int_X \partial \log \det h_\nu(\xi) \chi dV = \int_X \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \chi dV \quad \dots (\diamond)$$

for the reason that  $h_\nu$  is a smooth metric. We have that

$$\begin{aligned} & \left| \int_X \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \chi dV - \int_X \frac{\partial \det h(\xi)}{\det h} \chi dV \right| \\ & \leq \left| \int_X \partial \det h_\nu(\xi) \left( \frac{1}{\det h_\nu} - \frac{1}{\det h} \right) \chi dV \right| + \left| \int_X (\partial \det h_\nu(\xi) - \partial \det h(\xi)) \frac{1}{\det h} \chi dV \right| \\ & \leq C \|\partial \det h_\nu(\xi)\|_{L^{1+\delta}(K)} \left\| \frac{1}{\det h_\nu} - \frac{1}{\det h} \right\|_{L^{\frac{1+\delta}{\delta}}(K)} \\ & \quad + \left| \int_X (\partial \det h_\nu(\xi) - \partial \det h(\xi)) \frac{1}{\det h} \chi dV \right|, \end{aligned}$$

where  $C$  denotes the supremum of  $\chi$  on  $X$ , and  $K$  denotes a support of  $\chi$ . Here we take  $\frac{1}{\det h}$  as an  $L^{\frac{1+\delta}{\delta}}$  function on  $K$  since the assumption  $(\epsilon + 1)(\delta + 1) \geq 2$  implies that

$$\frac{1+\delta}{\delta} \leq \frac{2}{1-\epsilon}$$

and  $\frac{1}{\det h}$  is an  $L_{loc}^{\frac{2}{1-\epsilon}}(X)$  function. For the first term, we have

$$\left| \frac{1}{\det h_\nu} - \frac{1}{\det h} \right|^{\frac{1+\delta}{\delta}} \leq \left( \frac{2}{\det h} \right)^{\frac{1+\delta}{\delta}} \in L^1(K)$$

and  $\partial \det h_\nu(\xi) \in L^{1+\delta}(K)$  uniformly in  $\nu$ . Therefore

$$\|\partial \det h_\nu(\xi)\|_{L^{1+\delta}(K)} \left\| \frac{1}{\det h_\nu} - \frac{1}{\det h} \right\|_{L^{\frac{1+\delta}{\delta}}(K)} \rightarrow 0$$

as  $\nu \rightarrow \infty$  by the Lebesgue convergence theorem. For the second term, it is known that  $\partial \det h_\nu(\xi)$  weakly converges to  $\partial \det h(\xi)$  in  $L_{loc}^{1+\delta}(X)$  (cf. [Rau15, Lemma 5.1]) and  $\frac{1}{\det h} \chi \in L_{loc}^{\frac{2}{1-\epsilon}}(X) \subset L_{loc}^{\frac{1+\delta}{\delta}}(X)$  has a compact support. Hence

$$\left| \int_X (\partial \det h_\nu(\xi) - \partial \det h(\xi)) \frac{1}{\det h} \chi dV \right| \rightarrow 0$$

as  $\nu \rightarrow \infty$ .

Finally, we conclude that  $\frac{\partial \det h_\nu(\xi)}{\det h_\nu}$  weakly converges to  $\frac{\partial \det h(\xi)}{\det h}$ . Then taking weak limits of  $(\diamond)$ , we obtain

$$\partial \log \det h(\xi) = \frac{\partial \det h(\xi)}{\det h}$$



in the sense of distributions. □

LEMMA 23.2. *Let  $h$  be a Griffiths semi-negative singular Hermitian metric, and  $0 \leq \epsilon < 1, \delta > 0$ . We assume that*

- (i)  $\nu(\log \det h, x) < 1 - \epsilon$ , for all  $x \in X$ ,
- (ii)  $\sqrt{-1}\partial\bar{\partial} \log \det h \in L_{loc}^{1+\delta}(X)$ , and
- (iii)  $(\epsilon + 1)(\delta + 1) \geq 2$ .

Then

$$\frac{\partial \det h(\xi)}{\det^2 h}$$

can be defined as an  $L_{loc}^1$  function, and the equation

$$\partial \left( \frac{1}{\det h} \right) (\xi) = - \frac{\partial \det h(\xi)}{\det^2 h}$$

holds in the sense of distributions.

PROOF. From Lemma 23.1, the equation

$$\partial \log \det h(\xi) = \frac{\partial \det h(\xi)}{\det h}$$

holds in the sense of distributions. The assumption  $\sqrt{-1}\partial\bar{\partial} \log \det h \in L_{loc}^{1+\delta}(X)$  implies that  $\frac{\partial \det h(\xi)}{\det h} \in L_{loc}^{1+\delta}(X)$ . Using Hölder's inequality, we see

$$\frac{\partial \det h(\xi)}{\det^2 h} \in L_{loc}^{\frac{2(1+\delta)}{(1-\epsilon)(1+\delta)+2}}(X) \subset L_{loc}^1(X)$$

since  $\frac{1}{\det h} \in L_{loc}^{\frac{2}{1-\epsilon}}(X)$ .

Then it is sufficient to show that

$$\int_X \partial \left( \frac{1}{\det h} \right) (\xi) \chi dV = \int_X - \frac{\partial \det h(\xi)}{\det^2 h} \chi dV$$

for any test function  $\chi \in C_c^\infty(X)$ . It is known that

$$\int_X \partial \left( \frac{1}{\det h_\nu} \right) (\xi) \chi dV = \int_X - \frac{\partial \det h_\nu(\xi)}{\det^2 h_\nu} \chi dV \quad \dots (\diamond)$$

for each  $\nu$ .

Firstly, it follows that  $\partial(\frac{1}{\det h_\nu})(\xi)$  weakly converges to  $\partial(\frac{1}{\det h})(\xi)$  for the reason that  $\frac{1}{\det h_\nu}$  is increasing to  $\frac{1}{\det h}$ .

Secondly, we begin to show that  $\frac{\partial \det h_\nu(\xi)}{\det^2 h_\nu}$  weakly converges to  $\frac{\partial \det h(\xi)}{\det^2 h}$ . We have

$$\begin{aligned} & \left| \int_X \frac{\partial \det h_\nu(\xi)}{\det^2 h_\nu} \chi dV - \int_X \frac{\partial \det h(\xi)}{\det^2 h} \chi dV \right| \\ & \leq \left| \int_X \left( \frac{1}{\det h_\nu} - \frac{1}{\det h} \right) \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \chi dV \right| + \left| \int_X \frac{1}{\det h} \left( \frac{\partial \det h_\nu(\xi)}{\det h_\nu} - \frac{\partial \det h(\xi)}{\det h} \right) \chi dV \right| \\ & \leq C \left\| \frac{1}{\det h_\nu} - \frac{1}{\det h} \right\|_{L^{\frac{1+\delta}{\delta}}(K)} \left\| \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \right\|_{L^{1+\delta}(K)} \\ & \quad + \left| \int_X \frac{1}{\det h} \left( \frac{\partial \det h_\nu(\xi)}{\det h_\nu} - \frac{\partial \det h(\xi)}{\det h} \right) \chi dV \right|, \end{aligned}$$

where  $C$  denotes the supremum of  $\chi$  on  $X$ , and  $K$  denotes a support of  $\chi$ . Replacing  $\partial \det h_\nu(\xi)$  with  $\frac{\partial \det h_\nu(\xi)}{\det h_\nu}$  and  $\partial \det h(\xi)$  with  $\frac{\partial \det h(\xi)}{\det h}$ , we can apply the proof of Lemma 23.1 to the proof of this lemma.

Finally, we can conclude that  $\frac{\partial \det h_\nu(\xi)}{\det^2 h_\nu}$  weakly converges to  $\frac{\partial \det h(\xi)}{\det^2 h}$ . Taking weak limits of  $(\diamond)$ , we obtain

$$\partial \left( \frac{1}{\det h} \right) (\xi) = - \frac{\partial \det h(\xi)}{\det^2 h}$$

in the sense of the distributions.  $\square$

LEMMA 23.3. (= Lemma 11.3) Let  $h$  be a Griffiths semi-negative singular Hermitian metric,  $\hat{h}$  be the adjugate matrix of  $h$ . Then we have that

- (1)  $\partial \hat{h}(\xi) \in L_{loc}^2(X)$ ,
- (2)  $\partial \hat{h}_\nu(\xi) \in L_{loc}^2(X)$  uniformly in  $\nu$ , and
- (3)  $\partial \hat{h}_\nu(\xi) \in L_{loc}^2(X)$  weakly converges to  $\partial \hat{h}(\xi) \in L_{loc}^2(X)$ .

LEMMA 23.4. Let  $h$  be a Griffiths semi-negative singular Hermitian metric,  $\hat{h}$  be the adjugate matrix of  $h$ , and  $0 \leq \epsilon < 1, \delta > 0$ . We assume that

- (i)  $\nu(\log \det h, x) < 1 - \epsilon$ , for all  $x \in X$ ,
- (ii)  $\sqrt{-1} \partial \bar{\partial} \log \det h \in L_{loc}^{1+\delta}(X)$ , and
- (iii)  $(\epsilon + 1)(\delta + 1) \geq 2$ .

Then it follows that

$$\begin{aligned} \partial \left( \frac{1}{\det h} \hat{h} \right) (\xi) &= - \frac{\partial \det h(\xi)}{\det^2 h} \hat{h} + \frac{1}{\det h} \partial \hat{h}(\xi), \\ \partial \left( \frac{1}{\det h} \hat{h} h \right) (\xi) &= - \frac{\partial \det h(\xi)}{\det^2 h} \hat{h} h + \frac{1}{\det h} \partial (\hat{h} h)(\xi) \end{aligned}$$

in the sense of distributions.

PROOF. The proof of the first part is almost same as the proof of the second part. It is enough to show that

$$\partial \left( \frac{1}{\det h} \hat{h} \right) (\xi) = - \frac{\partial \det h(\xi)}{\det^2 h} \hat{h} + \frac{1}{\det h} \partial \hat{h}(\xi).$$

First of all, we begin to prove that each term of the above equation can be defined as a distribution. Lemma 23.3 implies that  $\frac{1}{\det h} \widehat{h} \in L_{loc}^{\frac{2}{1-\epsilon}}(X)$ ,  $-\frac{\partial \det h(\xi)}{\det^2 h} \widehat{h} \in L_{loc}^{\frac{2(1+\delta)}{(1-\epsilon)(1+\delta)+2}}(X) \subset L_{loc}^1(X)$ , and  $\frac{1}{\det h} \partial \widehat{h}(\xi) \in L_{loc}^{\frac{2}{2-\epsilon}}(X) \subset L_{loc}^1(X)$ . Each term of the above equation is defined as a distribution. Hence, it is enough to show that

$$\int_X \partial \left( \frac{1}{\det h} \widehat{h} \right) (\xi) \chi dV = \int_X \left( -\frac{\partial \det h(\xi)}{\det^2 h} \widehat{h} + \frac{1}{\det h} \partial \widehat{h}(\xi) \right) \chi dV$$

for any test function  $\chi \in C_c^\infty(X)$ . It is known that

$$\int_X \partial \left( \frac{1}{\det h_\nu} \widehat{h}_\nu \right) (\xi) \chi dV = \int_X \left( -\frac{\partial \det h_\nu(\xi)}{\det^2 h_\nu} \widehat{h}_\nu + \frac{1}{\det h_\nu} \partial \widehat{h}_\nu(\xi) \right) \chi dV \quad \dots (\diamond)$$

for each  $\nu$ . For the left-hand side of the above equation, we will show that  $\partial(\frac{1}{\det h_\nu} \widehat{h}_\nu)(\xi)$  weakly converges to  $\partial(\frac{1}{\det h} \widehat{h})(\xi)$ . We have

$$\left| \int_X \left\{ \partial \left( \frac{\widehat{h}_\nu}{\det h_\nu} \right) (\xi) - \partial \left( \frac{\widehat{h}}{\det h} \right) (\xi) \right\} \chi dV \right| = \left| \int_K \left( \frac{\widehat{h}_\nu}{\det h_\nu} - \frac{\widehat{h}}{\det h} \right) \partial(\xi \chi) dV \right|,$$

where  $K$  denotes a support of  $\chi$  and  $\partial(\xi \chi) = \sum_{i,j} \frac{\partial^2(\xi_i \bar{\xi}_j \chi)}{\partial z_i \partial \bar{z}_j}$ . Here

$$\begin{aligned} \left| \left( \frac{\widehat{h}_\nu}{\det h_\nu} - \frac{\widehat{h}}{\det h} \right) \partial(\xi \chi) \right| &\leq C' \left( \frac{|\widehat{h}_\nu|}{\det h_\nu} + \frac{|\widehat{h}|}{\det h} \right) \\ &\leq \frac{2(r-1)! C^{r-1} C'}{\det h} \in L^{\frac{2}{1-\epsilon}}(K) \subset L^1(K), \end{aligned}$$

where  $C$  denotes the supremum of  $h_0$  on  $K$ , and  $C'$  denotes the supremum of  $\partial(\xi \chi)$  on  $K$ . The constant  $C$  satisfies the following inequalities

$$|h_\nu| \leq C, |h| \leq C$$

on  $K$ . Then we conclude that  $\partial(\frac{1}{\det h_\nu} \widehat{h}_\nu)(\xi)$  weakly converges to  $\partial(\frac{1}{\det h} \widehat{h})(\xi)$  by the Lebesgue convergence theorem.

For the right-hand side of the equation  $(\diamond)$ , we will prove that  $-\frac{\partial \det h_\nu(\xi)}{\det^2 h_\nu} \widehat{h}_\nu$  weakly converges to  $-\frac{\partial \det h(\xi)}{\det^2 h} \widehat{h}$  and  $\frac{1}{\det h_\nu} \partial \widehat{h}_\nu(\xi)$  weakly converges to  $\frac{1}{\det h} \partial \widehat{h}(\xi)$ .

Firstly, we have

$$\begin{aligned}
& \left| \int_X \left( \frac{\partial \det h_\nu(\xi)}{\det^2 h_\nu} \widehat{h}_\nu - \frac{\partial \det h(\xi)}{\det^2 h} \widehat{h} \right) \chi dV \right| \\
& \leq \left| \int_X \left( \frac{\partial \det h_\nu(\xi)}{\det^2 h_\nu} - \frac{\partial \det h(\xi)}{\det^2 h} \right) \widehat{h}_\nu \chi dV \right| + \left| \int_X \frac{\partial \det h(\xi)}{\det^2 h} (\widehat{h}_\nu - \widehat{h}) \chi dV \right| \\
& \leq \left| \int_X \frac{\partial \det h(\xi)}{\det h} \left( \frac{1}{\det h} - \frac{1}{\det h_\nu} \right) \widehat{h}_\nu \chi dV \right| \\
& \quad + \left| \int_X \left( \frac{\partial \det h(\xi)}{\det h} - \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \right) \frac{\widehat{h}_\nu}{\det h_\nu} \chi dV \right| + \left| \int_X \frac{\partial \det h(\xi)}{\det^2 h} (\widehat{h}_\nu - \widehat{h}) \chi dV \right| \\
& \leq \left| \int_X \frac{\partial \det h(\xi)}{\det h} \left( \frac{1}{\det h} - \frac{1}{\det h_\nu} \right) \widehat{h}_\nu \chi dV \right| \\
& \quad + \left| \int_X \left( \frac{\partial \det h(\xi)}{\det h} - \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \right) \left( \frac{\widehat{h}_\nu}{\det h_\nu} - \frac{\widehat{h}}{\det h} \right) \chi dV \right| \\
& \quad + \left| \int_X \left( \frac{\partial \det h(\xi)}{\det h} - \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \right) \frac{\widehat{h}}{\det h} \chi dV \right| + \left| \int_X \frac{\partial \det h(\xi)}{\det^2 h} (\widehat{h}_\nu - \widehat{h}) \chi dV \right|.
\end{aligned}$$

For the first term, the inequality

$$\left| \frac{\partial \det h(\xi)}{\det h} \left( \frac{1}{\det h} - \frac{1}{\det h_\nu} \right) \widehat{h}_\nu \chi \right| \leq 2(r-1)! C^{r-1} C''' \left| \frac{\partial \det h(\xi)}{\det^2 h} \right|$$

holds on  $K$ , where  $C'''$  denotes the supremum of  $\chi$  on  $K$ . Then the first term goes to zero by the Lebesgue convergence theorem.

For the second term, we have

$$\begin{aligned}
& \left| \int_X \left( \frac{\partial \det h(\xi)}{\det h} - \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \right) \left( \frac{\widehat{h}_\nu}{\det h_\nu} - \frac{\widehat{h}}{\det h} \right) \chi dV \right| \\
& \leq C''' \left\| \frac{\partial \det h(\xi)}{\det h} - \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \right\|_{L^{1+\delta}(K)} \left\| \frac{\widehat{h}_\nu}{\det h_\nu} - \frac{\widehat{h}}{\det h} \right\|_{L^{\frac{1+\delta}{\delta}}(K)}.
\end{aligned}$$

Lemma 23.1 and 23.2 imply that

$$\left\| \frac{\partial \det h(\xi)}{\det h} - \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \right\|_{L^{1+\delta}(K)} \leq \left\| \frac{\partial \det h(\xi)}{\det h} \right\|_{L^{1+\delta}(K)} + \left\| \frac{\partial \det h_\nu(\xi)}{\det h_\nu} \right\|_{L^{1+\delta}(K)}$$

is bounded uniformly in  $\nu$ . Moreover, we get

$$\left| \frac{\widehat{h}_\nu}{\det h_\nu} - \frac{\widehat{h}}{\det h} \right|^{\frac{1+\delta}{\delta}} \leq \left| \frac{2(r-1)! C^{r-1}}{\det h} \right|^{\frac{1+\delta}{\delta}} \in L^1(K).$$

The second term goes to zero by the Lebesgue convergence theorem.

The third term also goes to zero for the reason that  $\frac{\partial \det h_\nu(\xi)}{\det h_\nu}$  weakly converges to  $\frac{\partial \det h(\xi)}{\det h}$  in  $L^{1+\delta}(K)$  and  $\frac{\widehat{h}\chi}{\det h} \in L^{\frac{1+\delta}{\delta}}(K)$ .

For the fourth term, we obtain

$$\left| \frac{\partial \det h(\xi)}{\det^2 h} (\widehat{h}_\nu - \widehat{h}) \chi \right| \leq 2(r-1)! C^{r-1} C'' \left| \frac{\partial \det h(\xi)}{\det^2 h} \right|$$

on  $K$ . Hence it goes to zero by the Lebesgue convergence theorem. We can conclude that  $\frac{\partial \det h_\nu(\xi)}{\det^2 h_\nu} \widehat{h}_\nu$  weakly converges to  $\frac{\partial \det h(\xi)}{\det^2 h} \widehat{h}$ .

Secondly, we will prove that  $\frac{1}{\det h_\nu} \partial \widehat{h}_\nu(\xi)$  weakly converges to  $\frac{1}{\det h} \partial \widehat{h}(\xi)$ . We have

$$\begin{aligned} & \left| \int_X \left( \frac{1}{\det h_\nu} \partial \widehat{h}_\nu(\xi) - \frac{1}{\det h} \partial \widehat{h}(\xi) \right) \chi dV \right| \\ & \leq \left| \int_X \partial \widehat{h}_\nu(\xi) \left( \frac{1}{\det h_\nu} - \frac{1}{\det h} \right) \chi dV \right| + \left| \int_X \frac{1}{\det h} (\partial \widehat{h}_\nu(\xi) - \partial \widehat{h}(\xi)) \chi dV \right| \\ & \leq C'' \|\partial \widehat{h}_\nu(\xi)\|_{L^2(K)} \left\| \frac{1}{\det h_\nu} - \frac{1}{\det h} \right\|_{L^2(K)} + \left| \int_X \frac{1}{\det h} (\partial \widehat{h}_\nu(\xi) - \partial \widehat{h}(\xi)) \chi dV \right|. \end{aligned}$$

For the first term, it turns out that

$$\left| \frac{1}{\det h_\nu} - \frac{1}{\det h} \right|^2 \leq \frac{4}{\det^2 h} \in L^{\frac{1}{1-\epsilon}}(K) \subset L^1(K),$$

and  $\|\partial \widehat{h}_\nu(\xi)\|_{L^2(K)}$  is uniformly bounded in  $\nu$  from Lemma 23.3. Hence the first term goes to zero by the Lebesgue convergence theorem. The second term also goes to zero for the reason that  $\partial \widehat{h}_\nu(\xi)$  weakly converges to  $\partial \widehat{h}(\xi)$  in  $L^2(K)$  by Lemma 23.3 and  $\frac{1}{\det h} \chi \in L^{\frac{2}{1-\epsilon}}(K) \subset L^2(K)$ .

Finally, taking weak limits of  $(\diamond)$ , we get

$$\partial \left( \frac{1}{\det h} \widehat{h} \right) (\xi) = - \frac{\partial \det h(\xi)}{\det^2 h} \widehat{h} + \frac{1}{\det h} \partial \widehat{h}(\xi)$$

in the sense of distributions. □

Consequently, we can prove a version of Theorem 20.2 in the situation that  $h$  is Griffiths semi-positive.

**THEOREM 23.5.** *Let  $h$  be a Griffiths semi-positive singular Hermitian metric,  $\{h_\nu\}_{\nu=1}^\infty$  be a sequence of smooth Hermitian metrics, with Griffiths semi-positive curvature, increasing pointwise to  $h$ ,  $h^*$  be the dual metric of  $h$ , and  $0 \leq \epsilon < 1, \delta > 0$ . We assume that*

(i)  $\nu(\log \det h^*, x) < 1 - \epsilon$ , for all  $x \in X$ ,

(ii)  $\sqrt{-1} \partial \bar{\partial} \log \det h^* \in L_{loc}^{(1+\delta)}(X)$ .

Then we can obtain

(1)  $\widetilde{\theta}_h := \theta_h(\xi) \in L_{loc}^{\frac{2}{2-\epsilon}}(X)$ , and  $\widetilde{\theta}_{h_\nu} := \theta_{h_\nu}(\xi) \in L_{loc}^{\frac{2}{2-\epsilon}}(X)$  uniformly in  $\nu$ ,

(2)  $\widetilde{\theta}_{h_\nu} = \theta_{h_\nu}(\xi)$  weakly converges to  $\widetilde{\theta}_h = \theta_h(\xi)$  in the sense of distributions, and

(3) if  $(\epsilon + 1)(\delta + 1) \geq 2$ ,  $\widetilde{\Theta}_h := \Theta_h(\xi, \xi)$  has measure coefficients, and  $\widetilde{\Theta}_{h_\nu} := \Theta_{h_\nu}(\xi, \xi)$  weakly converges to  $\widetilde{\Theta}_h = \Theta_h(\xi, \xi)$  in the sense of measures.

PROOF. First of all, the dual metric  $h^*$  satisfies the assumption of Theorem 20.2. From Lemma 23.4, we get

$$\partial \left( \frac{1}{\det h^*} \widehat{h^* h^*} \right) (\xi) = - \frac{\partial \det h^* (\xi)}{\det^2 h^*} \widehat{h^* h^*} + \frac{1}{\det h^*} \partial (\widehat{h^* h^*}) (\xi).$$

Therefore we obtain

$$\begin{aligned} 0 &= \partial (h^{*-1} h^*) (\xi) \\ &= \partial \left( \frac{1}{\det h^*} \widehat{h^* h^*} \right) (\xi) \\ &= - \frac{\partial \det h^* (\xi)}{\det^2 h^*} \widehat{h^* h^*} + \frac{1}{\det h^*} (\partial \widehat{h^*}) (\xi) h^* + \frac{1}{\det h^*} \widehat{h^*} (\partial h^*) (\xi) \\ &= (\partial h^{*-1}) (\xi) h^* + h^{*-1} (\partial h^*) (\xi) \end{aligned}$$

in the sense of distributions, hence we have  $\theta_h = -{}^t \theta_{h^*}$ . For each  $\nu$ , we also have  $\theta_{h_\nu} = -{}^t \theta_{h_\nu^*}$ . Using Theorem 20.2, we can prove part (1) and (2). Part (3) also follows since

$$\Theta_h = \bar{\partial} \theta_h = -{}^t \bar{\partial} \theta_{h^*} = -{}^t \Theta_{h^*}.$$

□

## 24. CHERN CURRENTS OF SINGULAR HERMITIAN METRICS ON VECTOR BUNDLES

In this section, we study Chern forms associated with singular Hermitian metrics. Just because the Chern curvature of a singular Hermitian metric is well-defined with measure coefficients, does not mean the Chern form is well-defined.

Restricting codimension of the singular set of the metric, Lärkäng, Raufi, Ruppenthal, and Sera [LRRS18] define Chern forms as closed currents of order 0. The theorem is as follows. Throughout this section, we assume that  $X$  is an  $n$ -dimensional complex manifold,  $E$  is a holomorphic vector bundle over  $X$ , and  $h$  is a singular Hermitian metric on  $E$ .

**THEOREM 24.1.** [LRRS18, Theorem 1.5] *Let  $h$  be a Griffiths semi-positive or semi-negative singular Hermitian metric on  $E$ . We assume that there is a subvariety  $V$  of  $X$  with  $\text{codim}(V) \geq k$  such that  $h$  is continuous and non-degenerate on  $X \setminus V$ .*

*Then there exists a unique closed  $(k, k)$  current,  $c_k(E, h)$ , of order 0 with locally finite mass in  $X$  such that for any local regularization sequence  $\{h_\epsilon\}$  of  $h$ , with  $h_\epsilon \rightarrow h$  locally uniformly on  $X \setminus V$ , we get that*

$$c_k(E, h_\epsilon) \rightarrow c_k(E, h)$$

*in the sense of currents.*

We have another characterization of Chern forms associated with singular Hermitian metrics. We show Theorem 20.3.

PROOF OF THEOREM 20.3. We can without any loss of generality assume that  $X$  is a domain in  $\mathbb{C}^n$  since the setting is local. We take an approximating sequence  $\{h_\nu\}_{\nu=1}^\infty$  as in the case of Lemma 22.1. The inequality

$$(\epsilon - \frac{k-2}{k})(\delta + 1) \geq 2$$

is equivalent to

$$\frac{2(1+\delta)}{(1-\epsilon)(1+\delta)+2} \geq k.$$

As in the case of the proof of Theorem 20.2, we have  $\tilde{\Theta}_{h_\nu} := \Theta_{h_\nu}(\xi, \xi) \in L_{loc}^k(X)$  uniformly in  $\nu$  for any holomorphic vector field  $\xi$ . Then coefficients of each element of  $\Theta_{h_\nu}$  are  $L_{loc}^k$  uniformly in  $\nu$ . It follows that each element of  $\Theta_h$  is  $L_{loc}^k$ , and  $\Theta_{h_\nu}$  weakly converges to  $\Theta_h$  in  $L_{loc}^k(X)$  (cf. [Rau15, Lemma 5.1]). Therefore  $c_k(E, h)$  has  $L_{loc}^1$  coefficients, especially  $c_k(E, h)$  has measure coefficients. Moreover we obtain that  $c_k(E, h_\nu)$  weakly converges to  $c_k(E, h)$  in the sense of currents.  $\square$

EXAMPLE 24.2. Let  $X$  be the unit polydisk in  $\mathbb{C}^3$ ,  $E$  be the trivial bundle  $X \times \mathbb{C}^3$ , and  $h$  be a singular Hermitian metric

$$h = \begin{pmatrix} |z_1|^2 + |z|^{\frac{1}{4}} & z_1 \\ \bar{z}_1 & 1 \end{pmatrix},$$

where  $z = (z_1, z_2, z_3)$  is a coordinate of  $\mathbb{C}^3$ . This  $h$  is Griffiths semi-negative. We have

$$\begin{aligned} \log \det h &= \frac{1}{4} \log |z|, \\ \nu(\log \det h, 0) &= \frac{1}{4} < 1 - \epsilon. \end{aligned}$$

We also see that

$$\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} (\log \det h) = \frac{1}{8} \frac{|z_2|^2}{|z|^4} \leq \frac{1}{8|z|^2},$$

and  $\frac{1}{|z|^2}$  is  $L_{loc}^{\frac{47}{16}} = L_{loc}^{1+\delta}$  ( $\frac{47}{16} < 3$ ). Since other differential coefficients are also  $L_{loc}^{\frac{47}{16}}$ , we can take  $\epsilon = \frac{11}{16}$ ,  $\delta = \frac{31}{16}$ . For  $k = 2$ , we get

$$(\epsilon - \frac{k-2}{k})(\delta + 1) = \epsilon(\delta + 1) = \frac{11}{16} \cdot \frac{47}{16} = \frac{517}{256} \geq 2.$$

Therefore  $c_2(E, h)$  is the well-defined Chern current with measure coefficients.

We can also prove a version of Theorem 20.3 under curvature semi-positivity.

THEOREM 24.3. *Let  $h$  be a Griffiths semi-positive singular Hermitian metric,  $h^*$  be the dual metric of  $h$ , and  $0 \leq \epsilon < 1$ ,  $\delta > 0$ , and  $k \in \mathbb{N}$ . We assume that*

- (i)  $\nu(\log \det h^*, x) < 1 - \epsilon$ , for all  $x \in X$ ,
- (ii)  $\sqrt{-1} \partial \bar{\partial} \log \det h^* \in L_{loc}^{(1+\delta)}(X)$ , and
- (iii)  $(\epsilon - \frac{k-2}{k})(\delta + 1) \geq 2$ .

*Then we can define the Chern current  $c_k(E, h)$  with measure coefficients.*

## 25. APPLICATIONS OF MAIN THEOREMS AND RELATED QUESTIONS

In this section, we study singularities of singular Hermitian metrics on holomorphic vector bundles. We focus on a singular Hermitian metric whose determinant has minimal singularities. We also present an approach to the metric with minimal singularities on a vector bundle and show some related questions. Throughout this section,  $X$  denotes a compact complex manifold,  $\omega$  denotes a positive Hermitian  $(1, 1)$  form on  $X$ ,  $E$  denotes a holomorphic vector bundle of rank  $r$  over  $X$ .

We start by introducing the following definition. Let  $L$  be a holomorphic line bundle over  $X$ . We call  $L$  pseudo-effective if  $L$  can be equipped with a singular Hermitian metric  $h$  with  $\sqrt{-1}\Theta_h \geq 0$  as a current.

**DEFINITION 25.1.** ([DPS01, Definition 1.4]) Consider two semi-positive singular Hermitian metrics  $h_1, h_2$  on  $L$ .

- (1) We will write  $h_1 \preceq h_2$  if there exists a constant  $C > 0$  such that  $|u|_{h_1} \leq C|u|_{h_2}$  for all sections  $u$  of  $L$ .
- (2) We will write  $h_1 \sim h_2$  if  $h_1 \preceq h_2$  and  $h_1 \succeq h_2$  hold.

It is known that pseudo-effective line bundle has a Hermitian metric  $h_{min}$  with minimal singularities such that  $\sqrt{-1}\Theta_{h_{min}} \geq 0$ .

**THEOREM 25.2.** ([DPS01, Theorem 1.5]) *For every pseudo-effective line bundle  $L \rightarrow X$ , there exists up to equivalence of singularities a unique class of Hermitian metrics  $h_{min}$  with minimal singularities such that  $\sqrt{-1}\Theta_{h_{min}} \geq 0$ .*

We denote by  $h_{min} = h_L e^{-\varphi_{min}}$  the metric with minimal singularities on  $L$ . For a smooth closed  $(1, 1)$ -form  $\theta$ , we denote by  $PSH(X, \theta)$  the class of quasi-plurisubharmonic functions  $u$  on  $X$  such that  $dd^c u + \theta \geq 0$ . We also define a  $\theta$ -psh function with minimal singularities as

$$u_\theta(x) := \sup\{u(x) : u \leq 0, u \in PSH(X, \theta)\}.$$

We present a regularity theorem of the metric with minimal singularities. This type of theorem was observed by Berman and Demailly in [BD12]. Recently, in the case that the corresponding class in the Bott-Chern cohomology group  $[\theta] \in H^{(1,1)}(X, \mathbb{R})$  is nef and big, Berman proved  $u_\theta \in C_{loc}^{1,\alpha}(\Omega)$  for  $\alpha < 1$  in [Ber18] and Chu, Tosatti, and Weinkove showed that  $u_\theta \in C_{loc}^{1,1}(\Omega)$  in [CTW18] on the Zariski open set  $\Omega$  defined as the Kähler locus of  $[\theta]$ .

**THEOREM 25.3.** ([Ber18, Theorem 1.2, Proposition 2.13, Remark 2.14], [CTW18, Theorem 1.3]) *Let  $(X, \omega)$  be a compact Kähler manifold and  $[\theta]$  be nef and big. Then  $u_\theta \in C_{loc}^{1,1}(\Omega)$  on the Zariski open set  $\Omega$  defined as the Kähler locus of  $[\theta]$ , and the Laplacian of  $u_\theta$  is locally bounded on  $\Omega$ . More precisely, we obtain the following estimate*

$$|dd^c u_\theta + \theta|_\omega \leq C e^{-B\psi}.$$



Here  $\psi$  is a function on  $X$  with analytic singularities along the Zariski closed set  $X \setminus \Omega$  such that  $\theta + dd^c\psi \geq \delta\omega$  for some  $\delta > 0$ , the constant  $C$  only depends on an upper bound on  $|\theta|_\omega$ , and the constant  $B$  is proportional to the lower bound on the bisectional curvature of  $\omega$ .

If  $X$  is projective and a line bundle  $L$  is big and nef, the metric  $h_{\min}$  has good property.

**PROPOSITION 25.4.** [DPS01, Proposition 1.7] *Let  $X$  be projective, and  $L$  be nef and big. Then  $h_{\min} = h_L e^{-\varphi_{\min}}$  has zero Lelong numbers everywhere.*

Hereafter, we assume that  $E$  has a Griffiths semi-positive singular Hermitian metric  $h$ . We consider a Griffiths semi-positive singular Hermitian metric  $h$  on  $E$  such that  $\det h$  has minimal singularities.

**DEFINITION 25.5.** We denote by  $h_{\min}$  a Griffiths semi-positive singular Hermitian metric on  $E$  such that  $\det h_{\min}$  is a metric with minimal singularities on  $\det E$ .

In the special case, we obtain an example of  $h_{\min}$ .

**EXAMPLE 25.6.** Set  $E := L \oplus \mathcal{O}$ , where  $L$  is a pseudo-effective line bundle and  $\mathcal{O}$  is the trivial line bundle. Then the metric  $h_{\min}^L \oplus h_0$  on  $E$  satisfies Definition 25.5, where  $h_{\min}^L$  is a metric with minimal singularities on  $L$  and  $h_0$  is the standard Euclidean metric on  $\mathcal{O}$ .

Let  $\mathbb{P}(E)$  be the projectivized bundle of hyperplanes of  $E$ , and  $\mathcal{O}_{\mathbb{P}(E)}(1)$  be the tautological line bundle over  $\mathbb{P}(E)$ . The vector bundle  $E$  is called nef (resp. big) if  $\mathcal{O}_{\mathbb{P}(E)}(1)$  is nef (resp. big). In this case, we study the above metric  $h_{\min}$ .

**THEOREM 25.7** (= Theorem 20.4). *Let  $(X, \omega)$  be a projective manifold. We assume that*

- (i)  $(T_X, \omega)$  is semi-positive in the sense of Griffiths,
- (ii)  $E$  is nef and big.

*Then  $\Theta_{(h_{\min})^*}$  and  $c_k(E^*, (h_{\min})^*)$  have measure coefficients on  $X$ .*

Before showing this theorem, we prepare a lemma.

**LEMMA 25.8.** *We assume the above conditions (i) and (ii). Let  $\{h_\nu\}_{\nu=1}^\infty$  be a local approximating sequence of  $(h_{\min})^*$  with Griffiths semi-negative curvature (cf. Lemma 22.1). Then  $\text{Tr}(\tilde{\Theta}_{h_\nu}) \in L_{loc}^p(X)$  uniformly in  $\nu$  for all  $p > 1$ , where  $\text{Tr}(\tilde{\Theta}_{h_\nu}) = \text{Tr}(\Theta_{h_\nu})(\xi, \xi)$  for a holomorphic vector field  $\xi$ .*

**PROOF OF LEMMA 25.8.** If  $E$  is nef and big,  $\det E$  is nef and big (cf. [Yan17, Proposition 1.4]). Then we can take a singular Hermitian metric  $h = h_{\det E} e^{-\psi}$  such that  $h_{\det E}$  is smooth and  $\psi$  has analytic singularities along  $X \setminus \Omega$  and satisfies the following inequality

$$\frac{\sqrt{-1}}{\pi} \Theta_h = \frac{\sqrt{-1}}{\pi} \Theta_{h_{\det E}} + dd^c\psi \geq \delta\omega$$

for some  $\delta > 0$  (cf. Theorem 25.3). Let  $\Omega$  be the Zariski open set defined as the Kähler locus of  $[\frac{\sqrt{-1}}{\pi}\Theta_{h_{\det E}}]$ . Set  $Z := X \setminus \Omega$ . We also see that  $\log \det(h_{\min})^* = -\log \det h_{\min}$  satisfies the properties prescribed by Theorem 25.3. We can also take the constant  $B$  in Theorem 25.3 for 0. Hence, we obtain

$$|\frac{\sqrt{-1}}{\pi}\Theta_{h_{\det E}} + dd^c \log \det(h_{\min})^*|_{\omega} \leq C$$

on  $\Omega$  for some positive constant  $C > 0$ . The right-hand side of the above inequality is  $L^p_{loc}(X)$  for all  $p > 1$ . Repeating the proof of Lemma 22.2, we have  $\text{Tr}(\tilde{\Theta}_{h_{\nu}}) \in L^p_{loc}(X \setminus Z_0)$  uniformly on  $\nu$  for all  $p > 1$ . We will show that it also holds on  $X$ .

We fix a local coordinate chart  $U = (z_1, \dots, z_n)$  ( $U \Subset X$ ) around  $a \in Z$  and a volume form  $dV$  on  $U$ . It is enough to show that  $\text{Tr}(\tilde{\Theta}_{h_{\nu}}) \in L^p(U)$  uniformly in  $\nu$ . By using an induction on the dimension of  $Z$ , we can assume that  $a$  is a regular point and  $Z$  is contained in the hyperplane  $\{z_1 = 0\}$ , with  $a = 0$ . Let  $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})$  be a function such that  $\lambda(t) = 0$  for  $t \leq \frac{1}{2}$  and  $\lambda(t) = 1$  for  $t \geq 1$ . Set  $\lambda_\epsilon(z) = \lambda(\frac{|z_1|}{\epsilon})$ . For any test function  $\chi \in C_c^\infty(U)$ , we evaluate the following integral

$$\int_U \left( \frac{\sqrt{-1}}{\pi} \Theta_{h_{\det E}}(\xi, \xi) + dd^c \log \det h_{\nu}(\xi, \xi) \right) \lambda_\epsilon(z) \chi dV.$$

We obtain

$$\begin{aligned} & \lim_{\nu \rightarrow \infty} \left| \int_U \left( \frac{\sqrt{-1}}{\pi} \Theta_{h_{\det E}}(\xi, \xi) + dd^c \log \det h_{\nu}(\xi, \xi) \right) \lambda_\epsilon \chi dV \right| \\ &= \left| \int_U \left( \frac{\sqrt{-1}}{\pi} \Theta_{h_{\det E}}(\xi, \xi) + dd^c \log \det(h_{\min})^*(\xi, \xi) \right) \lambda_\epsilon \chi dV \right| \\ &= \left| \int_{U \setminus Z} \left( \frac{\sqrt{-1}}{\pi} \Theta_{h_{\det E}}(\xi, \xi) + dd^c \log \det(h_{\min})^*(\xi, \xi) \right) \lambda_\epsilon \chi dV \right| \\ &\leq C' \int_{U \setminus Z} |\lambda_\epsilon \chi| dV \\ &\leq C'' \|\lambda_\epsilon \chi\|_{L^q(U)} \end{aligned}$$

by Hölder's inequality, where  $C', C'' > 0$  are some positive constants independent of  $\nu$  and  $q = \frac{p}{p-1}$ . Since the right-hand side is independent of  $\nu$ , we have

$$\sup_{\nu} \frac{|\int_U dd^c \log \det h_{\nu}(\xi, \xi) \lambda_\epsilon \chi dV|}{\|\lambda_\epsilon \chi\|_{L^q(U)}} \leq C''$$

for some positive constant  $C'' > 0$ . Taking limits  $\epsilon \rightarrow 0$ , we get

$$\sup_{\nu} \frac{|\int_U dd^c \log \det h_{\nu}(\xi, \xi) \chi dV|}{\|\chi\|_{L^q(U)}} \leq C''$$

by the Lebesgue convergence theorem. Hence  $\text{Tr}(\tilde{\Theta}_{h_{\nu}}) \in L^p(U)$  uniformly in  $\nu$  for the reason that

$$-\sqrt{-1}\partial\bar{\partial} \log \det h_{\nu} = \text{Tr}(\Theta_{h_{\nu}}).$$

□

Then we will show Theorem 25.7.

PROOF OF THEOREM 25.7. We recall the inequality

$$|(\tilde{\Theta}_{h_\nu} u, u)_e| \leq (-\text{Tr}(\tilde{\Theta}_{h_\nu})) \|u\|_{h_0} \|u\|_{(h_{\text{dmin}})^{\star-1}}$$

in Proof of Theorem 20.2. From Proposition 25.4,  $\log \det(h_{\text{dmin}})^{\star}$  has zero Lelong numbers. Hence  $\|u\|_{(h_{\text{dmin}})^{\star-1}} \in L_{\text{loc}}^q(X)$  for all  $q > 1$ . Lemma 25.8 implies that  $\tilde{\Theta}_{h_\nu} \in L_{\text{loc}}^p(X)$  uniformly in  $\nu$  for all  $p > 1$ . Repeating the argument in Proof of Theorem 20.2 and Proof of Theorem 20.3, we can conclude that  $\Theta_{(h_{\text{dmin}})^{\star}}$  and  $c_k(E^{\star}, (h_{\text{dmin}})^{\star})$  have measure coefficients on  $X$ . □

We remark that the metric  $h_{\text{dmin}}$  is not necessarily a singular metric "with minimal singularities" on  $E$ .

We have another approach to the metric with minimal singularities on a vector bundle. If  $E$  is nef and big,  $\mathcal{O}_{\mathbb{P}(E)}(1)$  is a nef and big line bundle over  $\mathbb{P}(E)$ . Let  $L := \mathcal{O}_{\mathbb{P}(E)}(1)$ . As above, we obtain the singular Hermitian metric  $g_{\min}^L = g_0 e^{-\varphi_{\min}}$  on  $L$ , where  $g_0$  is a fixed smooth metric on  $L$  and  $\varphi_{\min} \in L_{\text{loc}}^1(\mathbb{P}(E))$  satisfies the properties prescribed by Theorem 25.3 and Proposition 25.4. Let  $\pi: \mathbb{P}(E) \rightarrow X$  be the canonical projection, and  $K_{\mathbb{P}(E)/X} = K_{\mathbb{P}(E)} \otimes \pi^*(K_X^{-1})$  be the relative canonical sheaf. Since  $\nu(g_{\min}^L, y) = 0$  for all  $y \in \mathbb{P}(E)$ , the multiplier ideal  $\mathcal{I}(g_{\min}^L) = \mathcal{O}_{\mathbb{P}(E)}$  by Lemma 21.5.

We have the isomorphism

$$\pi_*(K_{\mathbb{P}(E)/X} \otimes L^{r+1} \otimes \mathcal{I}((g_{\min}^L)^{r+1})) = \pi_*(K_{\mathbb{P}(E)/X} \otimes L^{r+1}) \cong E \otimes \det E.$$

We can define the singular Hermitian metric  $\tilde{g}_{\min}$  on  $E \otimes \det E$  by

$$\langle s, t \rangle_{\tilde{g}_{\min}} := \int_{\mathbb{P}(E)_x} \{s, t\}_{(g_{\min}^L)^{r+1}}$$

for  $s, t \in \pi_*(K_{\mathbb{P}(E)/X} \otimes L^{r+1})|_x$ . Since  $g_{\min}^L$  is a semi-positive singular Hermitian metric on  $L$ , it is known that  $\tilde{g}_{\min}$  is Griffiths semi-positive in the sense of singular Hermitian metrics on vector bundles (cf. [HPS18, Theorem 21.1]). Then we have the next question.

QUESTION 25.9. We assume that  $X$  is projective, and  $E$  is nef and big. Then we can construct the above Griffiths semi-positive singular Hermitian metric  $\tilde{g}_{\min}$  on  $E \otimes \det E$ . Can we say that  $\Theta_{\tilde{g}_{\min}}$  has measure coefficients on  $X$ ?

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