博士論文

Entropy of the Janus interface in superconformal field theories

（超共形場理論におけるヤヌス・インターフェースのエントロピー）

永野廉人

Abstract

In this thesis we study the entropy of the Janus interface in a $4d$ $\mathcal{N}=2$ superconformal field theory (SCFT). The Janus interface is a co-dimension one defect across which a coupling constant changes its value. We show that the entropy of the Janus interface in a $4d$ $\mathcal{N}=2$ SCFT can be written by a specific linear combination of analytically continued Kähler potentials on moduli space called Calabi's diastasis.

First we give a definition of an interface entropy as a contribution from an interface to an entanglement entropy across a spherical entangling surface. Then we derive the relation between the interface entropy and the sphere partition function via the conformal map introduced by Casini, Huerta, Meyers. Finally we evaluate the entropy of the Janus interface by using this relation and SUSY localization to show that it can be written by Calabi's diastasis.

Contents

Chapter 1

Introduction

Quantum field theories (QFTs) are very fundamental and important subjects in theoretical physics because they describe wide phenomena; particle physics, condensed matter physics, cosmology and so on. Morever QFTs themselves exhibit rich and fascinating structure. The main purpose of this thesis is to study a QFT non-perturbatively by using non-local objects called defects. In this chapter we briefly explain basic backgrounds to give motivations to the subjects of this thesis. Then we summarize main results and explain the organization of this thesis.

1.1 Background and motivation

1.1.1 Defects

What we mainly focus on in this paper are non-local objects in QFTs which are so called defects. Defects have many applications and thus are very important. We list some of them.

- The expectation value of a defect can be used as an order parameter of a phase transition. For example, a dimension one defect in quantum chromodynamics (QCD) called the Wilson loop is used to diagnose confinement.

- By inserting a defect into a QFT, we can extract data in the ambient QFT (bulk) via relations between data in the bulk and defect, *e.g.* [1].

- In a realistic setting in a laboratory, materials have boundaries and/or interfaces. This can be understood theoretically as co-dimension one defects in a QFT.

- Open string theory is formulated by a conformal field theories (CFT) on a world-sheet with a boundary.

In this paper we particularly focus on a co-dimension one defect, an interface.

1.1.2 Interface entropy

We particularly focus on an entropy associated to an interface which we call an interface entropy. Here we explain the motivation to study this quantity on a general background. A motivation to study the entropy for a specific interface (the Janus interface) will be explained in the next subsection. One of motivations to study an interface entropy is that it is conjectured to be a C-function in a CFT with an interface which we call an interface CFT or an ICFT [2]. Let me explain this point.

What is a C-function? Suppose that we have a conformal field thoery (CFT) at a ultraviolet (UV) fixed point, then flow to an infrared (IR) fixed point along with a renormalization group (RG) flow. A function which decreases monotonically along with a RG flow is called a C-function. If we can find a C-function, we can get nontrivial constraints on IR quantities from UV quantities. Furthermore this function characterizes the number of degrees of freedom since the value of this function at the UV theory where many modes fluctuate is bigger than the value at the IR theory where only few modes fluctuate.

We have some examples of such functions in CFTs without an interface. In two dimensions such a function was first constructed by [3] from a particular linear combination of two point functions of the stress-energy tensor. It reduces to the coefficient of the Weyl anomaly or the c-central charge at fixed points. In four dimensions the a-coefficient of the Weyl anomaly is shown to be a weak version [1] of C-function by [4]. In three dimensions it was proposed that the free energy which is the log of a partition function on a sphere is a C-function [5, 6]. This statement was later proved by [7].

[1]This means that the value of the function at an UV fixed point is bigger than that at an IR fixed point, not necessarily decreases along RG flows.

Then a natural question is whether we can find a C-function in ICFT. One of candidates is an entropy of an interface which we call an interface entropy [2]. So if this statement is proved we can use an interface entropy to classify/constraint theories with an interface. This motivates us to study an interface entropy.

1.1.3 Janus interface and its entropy

In the latter half of this thesis we focus on a specific interface, called the Janus interface [8, 9]. The direct motivation of our research is the fascinating relation [10] between the entropy of the Janus interface in a $2d$ $\mathcal{N} = (2, 2)$ superconformal field theory (SCFT) and the geometric quantity on the space of CFTs called Calabi's diastasis [11].

Let us consider the space of CFTs $\mathcal{S}$ called the conformal manifold or the moduli space which is parametrized by moduli parameters τ^I [2]. The Janus interface is an interface across which moduli parameters change their values. The author of the paper [10] showed that the entropy of the Janus interface in a $2d$ $\mathcal{N} = (2, 2)$ SCFT can be written by a specific linear combination of analytically continued Kähler potentials on moduli space:

$$S_{\mathcal{I}} \propto [K(\tau_+, \overline{\tau}_+) + K(\tau_-, \overline{\tau}_-) - K(\tau_+, \overline{\tau}_-) - K(\tau_-, \overline{\tau}_+)] =: \mathcal{D}. \tag{1.1}$$

Calabi named this quantity $\mathcal{D}$ diastasis [11] which is an ancient Greek word meaning "distance". It reduces to a geodesic distance defined from the Zamolodchikov metric on the moduli space [11], but does not satisfy the axiom of distance, *e.g.* the triangle inequality [10]. The relation (1.1) in a $2d$ $\mathcal{N} = (2, 2)$ SCFT was further confirmed by holography in [12], super-Weyl anomaly in [13], and supersymmetric localization in [14].

Then a natural question is whether the relation [15] holds for other cases. It was conjectured [16] that the relation (1.1) also holds in $4d$ $\mathcal{N} = 2$ SCFTs [3] from a holographic consideration. The main purpose of this thesis is to give a proof of this claim via supersymmetric localization based on our paper [15].

[2]The precise definition of the conformal manifold will be given in Section 2.3.1.

[3]The half-BPS superconformal Janus interface in a $4d$ $\mathcal{N} = 2$ SCFT was studied in the previous works [17, 18, 19, 20, 21].

1.2 Summary of this thesis

Now let us summarize an outline of the proof [15] of the claim (1.1) in $4d$ $\mathcal{N}=2$ SCFTs.

- We first define an interface entropy as a contribution to an entanglement entropy from an interface.
- We then derive the relation between an entropy of a conformal and superconformal interface and the partition function on a sphere by the conformal map introduced by [22].
- We evaluate the sphere partition function in the presence of the Janus interface in a $4d$ $\mathcal{N}=2$ SCFT via supersymmetric localization.
- Finally we combine this result and the above relation to show that the Janus interface entropy can be written by Calabi's diastasis.

The organization of this thesis is as follows. In Chapter 3 we define an interface entropy by using an entanglement entropy, and derive the relations between the interface entropy and partition functions in the presence of a conformal/superconformal interface. The results in Chapter 3 hold for general conformal/superconformal interfaces.

We focus on the Janus interface in $4d$ $\mathcal{N}=2$ SCFTs after Chapter 4. In Chapter 4 we give an off-shell construction of the Janus interface. In Chapter 5 we compute the sphere partition function via supersymmetric localization technique. By combining this result and the result in Chapter 3 we show that an interface entropy of the Janus interface in $4d$ $\mathcal{N}=2$ SCFTs can be written as Calabi's diastasis.

We give a brief review on CFT and SCFT in Chapter 2. Each chapter also has a brief review on relevant topics. Notations, conventions and technical issues are summarized in Appendix.

Chapter 2

Conformal and superconformal field theory

In this chapter we give a very brief review on conformal field theories (CFTs) and superconformal field theories (SCFTs). The main purpose is to explain symmetry and general structure of correlation functions and partition functions.

2.1 Conformal Field Theory

2.1.1 Conformal symmetry and conformal algebra

First let us explain about conformal field theories in general dimensions with Lorentzian signature [1]. We use a convention in which $\eta_{\mu\nu} = (-1, +1, \cdots, +1)$. The conformal transformation is a transformation which preserves a metric up to a factor,

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{2.1}$$

Under infinitesimal coordinate transformations $x^\mu \to x^\mu + \epsilon^\mu(x)$, the metric transforms as

$$g_{\mu\nu} \to g_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu. \tag{2.2}$$

Imposing the above metric transformation to be a conformal transformation (2.1), we have the following condition on ϵ:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\rho \epsilon^\rho g_{\mu\nu}. \tag{2.3}$$

[1] For more detail on CFTs, see for example [23, 24].

This equation is called the conformal Killing equation. Let us focus on the solution for $d \geq 3$ [2]. The general solutions for (2.3) are given by

$$\epsilon^\mu = a^\mu + \lambda^{\mu\nu} x_\nu + \lambda_D x^\mu + \left(x^2 \lambda_K^\mu - 2 x^\mu x^\nu \lambda_{K\nu} \right), \tag{2.4}$$

where $a^\mu, \lambda^{\mu\nu}, \lambda_D, \lambda_K^\mu$ are real parameters. We explain the transformations caused by each terms.

- The first term corresponds to translations $x'^\mu = x^\mu + a^\mu$ where a^μ are d real parameters.
- The second term corresponds to rotations $x'^\mu = \lambda^{\mu\nu} x_\nu$ where $\lambda^{\mu\nu}$ is real and antisymmetric tensor $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$ which has $\frac{1}{2}d(d-1)$ real parameters.
- The third term corresponds to dilatations $x'^\mu = \lambda_D x^\mu$ where λ_D is a real parameter.
- The forth term corresponds to special conformal transformations $x'^\mu = x^2 \lambda_K^\mu - 2 x^\mu x^\nu \lambda_{K\nu}$ where λ_K^μ are d real parameters.

Thus in total the solutions (2.4) are parametrized by $d + \frac{1}{2}d(d-1) + 1 + d = \frac{1}{2}(d+1)(d+2)$ real parameters. We denote the corresponding generators by $P_\mu, M_{\mu\nu}, D, K_\mu$ to write conformal transformations as

$$\delta_C(\epsilon) = a^\mu + \frac{1}{2}\lambda^{\mu\nu} M_{\mu\nu} + \lambda_D D + \lambda_K^\mu K_\mu. \tag{2.5}$$

The commutators of these generators are given by

$$[M_{\mu\nu}, M^{\rho\sigma}] = \eta_{\mu\rho} M_{\sigma\nu} - \eta_{\nu\rho} M_{\sigma\mu} - \eta_{\mu\sigma} M_{\rho\nu} + \eta_{\nu\sigma} M_{\rho\mu}, \tag{2.6}$$
$$[P_\mu, M_{\nu\rho}] = 2\eta_{\mu[\nu} P_{\rho]}, \quad [K_\mu, M_{\nu\rho}] = 2\eta_{\mu[\nu} K_{\rho]}, \tag{2.7}$$
$$[P_\mu, K_\nu] = 2\left(\eta_{\mu\nu} D + M_{\mu\nu}\right), \tag{2.8}$$
$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = K_\mu, \tag{2.9}$$

and the other commutators vanish [3]. The group generated by them are called the conformal group, which is equivalent to $SO(d,2)$. If we perform Wick rotation and consider the Euclidean signature, then the resulting conformal group is $SO(d+1,1)$.

[2] In $d = 2$ finite conformal symmetry is enhanced to infinite symmetry called Virasoro symmetry.

[3] We define $a_{[\mu} b_{\nu]} := \frac{1}{2}(a_\mu b_\nu - a_\nu b_\mu)$.

2.1.2 Primary and descendant operator

Instead of demonstrating a whole story about a representation of conformal group in this thesis, we quickly summarize the conclusion. A representation is specified by an operator $\mathcal{O}(x)$ called a primary operator which satisfies the following conditions:

$$[D, \mathcal{O}(0)] = \Delta \mathcal{O}(0)\,, \tag{2.10}$$

$$[M_{\mu\nu}, \mathcal{O}(0)] = \mathcal{S}_{\mu\nu} \mathcal{O}(0)\,, \tag{2.11}$$

$$[K_\mu, \mathcal{O}(0)] = 0\,, \tag{2.12}$$

where Δ is called a conformal dimension or a Weyl wight, and $\mathcal{S}_{\mu\nu}$ is a representation matrix of Lorentz group satisfying the commutation relation (2.6). The other operator in this representation can be obtained by acting the operator P_μ on the primary operator. They are called descendants.

2.1.3 Correlation function in conformal field thoery

Quantum field theories (QFTs) which are invariant under conformal transformations are called conformal field theories (CFTs). Correlation functions in CFTs are highly constrained by the conformal symmetry. As a result, n-point functions with $n = 1, 2, 3$ of primary operators are completely fixed up to constant factors. Let us explain this point. First, all one point functions vanish on $\mathbb{R}^d$. Next, the two point function of real scalar primary operators $\mathcal{O}$ with conformal dimension Δ on $\mathbb{R}^d$ is determined as

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \frac{k}{|x-y|^{2\Delta}}, \tag{2.13}$$

where k is a constant which can be normalized as $k = 1$ in an unitary CFT. Furthermore, the three point function of primary operators $\mathcal{O}_i$ with dimension Δ_i $(i = 1, 2, 3)$ is given by

$$\begin{aligned} &\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle \\ &= \frac{c_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1} |x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}}, \end{aligned} \tag{2.14}$$

where c_{123} is a dynamical coefficient called an OPE (operator product expansion) coefficient. Higher points functions can be determined by two and three point functions, *i.e.* spectrum and OPE coefficients in principle by using OPE. Thus these data are called CFT data [4].

2.1.4 Trace anomaly

A CFT is a scale invariant theory by definition. This means that the trace of a stress-energy tensor $T_{\mu\nu}$ vanishes in classical levels:

$$\langle T_\mu{}^\mu \rangle = 0, \quad \text{(classical)}. \tag{2.15}$$

However, this does not hold in quantum levels for generic CFTs in even dimensions. This fact is called the trace anomaly or the Weyl anomaly or the conformal anomaly. They can be written as

$$\langle T_\mu{}^\mu \rangle = -2(-)^{d/2} A E_d + \sum_n B_n I_n, \tag{2.16}$$

where I_n are the independent Weyl invariants with a Weyl weight $-d$, and E_d is the d-dimensional Euler density. The first and the other terms are called the A-type and the B-type anomaly, respectively. The coefficient A in two dimensions is proportional to the c-central charge and the coefficient in four dimensions is proportional to the a-central charge. They are very fundamental quantities in CFTs to characterize them partly because they have monotonic properties under RG flows in four and two dimensions, as explained in Introduction. They also appear the universal part of sphere partition functions as we will explain in Section 2.3.2 and an entanglement entropy as we will explain in Section 3.1.3. Note that in odd dimensions we do not have Weyl anomaly.

2.2 Supersymmetry and superconformal field theory

2.2.1 Supersymmetry

The supersymmetry (SUSY) is symmetry which exchanges bosons and fermions. It is an extension of the Poincaré group whose generators we denote by P_μ

[4] As we will explain later, in a CFT with an interface one point functions can have non-trivial values, so we have to include coefficients of one point functions to CFT data in this case.

and $M_{\mu\nu}$. Representations of the SUSY algebra depend on spacetime dimensions. In this thesis we mostly focus on the $4d$ $\mathcal{N}=2$ SUSY, so we here explain the SUSY in $d=4$. We use the conventions and notations in [25] which are summarized in Appendix A. The SUSY transformations are generated by chiral generators $Q^i_\alpha, Q_{i\alpha}, (i=1,\ldots,\mathcal{N})$:

$$Q_i = P_L Q_i, \quad Q^i = P_R Q^i, \tag{2.17}$$

where $P_{L/R} = \frac{1}{2}(1 \pm \gamma_*)$ are chiral projection operators. The Weyl conjugation of Q_i, Q^i which are denoted by $\overline{Q}_i, \overline{Q}^i$ are also chiral:

$$\overline{Q}_i = \overline{Q}_i P_L, \quad \overline{Q}^i = \overline{Q}^i P_R. \tag{2.18}$$

They satisfy the following commutation relations [5]:

$$\{Q_{i\alpha}, \overline{Q}^{j\beta}\} = -\frac{1}{2}\delta_i^j (P_L\gamma_\mu)_\alpha{}^\beta P^\mu, \quad \{Q^i_\alpha, \overline{Q}^\beta_j\}, = -\frac{1}{2}\delta^i_j (P_R\gamma_\mu)_\alpha{}^\beta P^\mu, \tag{2.19}$$

$$\{Q_{i\alpha}, \overline{Q}^\beta_j\} = 0, \qquad \{Q^i_\alpha, \overline{Q}^{j\beta}\} = 0 \tag{2.20}$$

$$[M_{[\mu\nu]}, Q_{i\alpha}], = -\mathrm{i}\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_{i\beta}, \qquad [M_{[\mu\nu]}, Q^i_\alpha] = -\mathrm{i}\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q^i_\beta, \tag{2.21}$$

$$[P_\mu, Q_{i\alpha}] = 0, \qquad [P_\mu, Q^i_\alpha] = 0. \tag{2.22}$$

In terms of these generators, general SUSY transformations can be written as

$$\delta_Q(\epsilon) = \bar{\epsilon}^i Q_i + \bar{\epsilon}_i Q^i, \tag{2.23}$$

where ϵ_i, ϵ^i are left/right handed SUSY parameters:

$$\epsilon_i = P_L \epsilon_i, \quad \epsilon^i = P_R \epsilon^i. \tag{2.24}$$

There is the symmetry which mixes $\mathcal{N}$-supercharges. This symmetry is called the R-symmetry.

2.2.2 Superconformal symmetry

A superconformal group contains conformal symmetry and R-symmetry as its bosonic subgroup. It also contains the symmetry generated by spinor generators Q^i, Q_i that are superpartners of translation generators P_μ and S^i, S_i that

[5] We can have more general algebras by including the so-called central charge which we do not refer to.

are superpartners of specialconformal generators K_μ. The former is called the Poincaré SUSY and the latter is called the special superconformal symmetry. Explicitly, the superconformal group in four dimensions is $SU(2,2|\mathcal{N})$ for $\mathcal{N}=1,2,3$ whose bosonic subgroup is $SO(4,2)\times SU(\mathcal{N})_R\times U(1)_R$. The first factor is the conformal symmetry in four dimensions, and the rest factors are R-symmetry as indicated in subscripts. For $\mathcal{N}=4$ case the superconformal group is $PSU(2,2|4)$.

The Poincaré SUSY transformations are generated by $\delta_Q(\epsilon)$ (2.23) while the special superconformal transformations are generated by

$$\delta_S(\eta)=\overline{\eta}^i S_i+\overline{\eta}_i S^i, \tag{2.25}$$

where η^i,η_i are right/left handed SUSY parameters:

$$\eta^i=P_R\eta^i, \quad \eta_i=P_L\eta_i. \tag{2.26}$$

CFTs which admit superconformal symmetry are called superconformal field thoeries (SCFTs).

2.3 Sphere partition function

2.3.1 Conformal manifold

Consider a d-dimensional CFT and deform it by a set of operators $\{\mathcal{O}_I\}_{I=1}^n$. If we have a Lagrangian description this means that we add

$$\frac{1}{\pi^{d/2}}\int \mathrm{d}^d x\tau^I\mathcal{O}_I(x) \tag{2.27}$$

to an original Lagrangian where τ^I are parameters. If we do not have a Lagrangian description we insert

$$\exp\left[\frac{1}{\pi^{d/2}}\int \mathrm{d}^d x\tau^I\mathcal{O}_I(x)\right] \tag{2.28}$$

to correlation functions. If operators $\mathcal{O}_I$ have a conformal dimension d, then this operators are called marginal operators and the corresponding deformation is called a marginal deformation. Furthermore, if the deformed theory is again conformal in all order in perturbation theory with respect to τ^I then the operators are called exactly marginal operators and the deformation is called an exactly marginal deformation. In this case, we have a family of CFTs parametrized by τ^I which we call the conformal manifold or the moduli space $\mathcal{S}$. The parameters τ^I are called moduli parameters or exactly marginal couplings. It is known that the conformal manifold $\mathcal{S}$ admits the Riemannian metric called the Zamolodchikov metric [3].

2.3.2 Divergent structure of sphere partition function

In this subsection we explain the general structure of partition functions of CFTs on a d-dimensional sphere. Since a sphere has a finite volume, there are no IR divergences, but there are still UV divergences. So we introduce an UV cut-off Λ_{UV} to regularize the partition function. Then sphere partition function has the following structure for even d [6]:

$$\begin{aligned}\log Z[\mathbb{S}^d] = & A_1(\tau)(r\Lambda_{UV})^{2n} + A_2(\tau)(r\Lambda_{UV})^{2n-2} + \cdots + A_n(\tau)(r\Lambda_{UV})^2 \\ & + A(\tau)\log(r\Lambda_{UV}) + F_{2n}(\tau),\end{aligned} \tag{2.29}$$

where $d = 2n$ and r is a radius of the sphere. On the other hand, for odd d it has the following structure:

$$\begin{aligned}\log Z[\mathbb{S}^d] = & B_1(\tau)(r\Lambda_{UV})^{2n+1} + B_2(\tau)(r\Lambda_{UV})^{2n-1} + \cdots + B_n(\tau)(r\Lambda_{UV}) \\ & + F_{2n+1}(\tau).\end{aligned} \tag{2.30}$$

The power law divergences correspond to the counterterms which are explicitly written as

$$\Lambda_{UV}^{2n-2k+2}\int d^{2n}x\sqrt{g}A_k(\tau)R^{k-1} \tag{2.31}$$

for even d and

$$\Lambda_{UV}^{2n-2k+3}\int d^{2n+1}x\sqrt{g}B_k(\tau)R^{k-1} \tag{2.32}$$

for odd d. So these divergent terms are ambiguous and can be removed in the continuum limit.

For even d one can show that the coefficient in front of the log divergent term does not depend on the exactly marginal parameter τ and is proportional to the Weyl anomaly coefficient. On the other hand, the finite term

[6] Without loss of generality we consider a single moduli parameter τ and omit indices.

$F_{2n}(\tau)$ is ambiguous and can be removed by local conterterms

$$\int d^{2n}x\sqrt{g}F_{2n}(\tau)E_{2n}. \tag{2.33}$$

As we will explain in the next subsecion, when we treat a supersymmetric theory the situation can be different.

For odd d one can also show that the finite term $F_{2n+1}(\tau)$ is unambiguous and does not depend on the exactly marginal parameter τ.

To recap, the universal term in the log of the sphere partition function is the log divergent term for even d and the constant term for odd d.

2.3.3 Supsersymmetric partition function

When we consider SCFTs the structure of sphere partition functions can be different [26]. In this case we often regularize a partition function with a part of SUSY preserved. In the absence of supersymmetry, we consider diffeomorphism invariant counterterms. On the other hand, in SUSY preserving scheme we consider SUSY invariant and diffeomorphism invariant counterterms given by supergravity. Thus the allowed counterterms become more restricted. As a result, a finite term $F_{2n}(\tau)$ for even d which is ambiguous in non-SUSY cases can be unambiguous. Indeed, for a $2d$ $\mathcal{N}=(2,2)$ SCFT and a $4d$ $\mathcal{N}=2$ SCFT it gives Kähler potentials on moduli space [27, 28, 26, 29] [7]:

$$Z_{\text{SUSY}}[\mathbb{S}^2](\tau,\overline{\tau}) = e^{-K(\tau,\overline{\tau})}, \quad Z_{\text{SUSY}}[\mathbb{S}^4](\tau,\overline{\tau}) = e^{K(\tau,\overline{\tau})/12}. \tag{2.34}$$

The $2d$ result was first conjectured in [27] and later proved by [28] via the SUSY localization technique [8] [9]. The authors of [26] further gave a new proof of this claim by using supersymmetric Ward identity which does not depend on a Lagrangian description nor SUSY localization. The $4d$ result was first derived in [26] via SUSY localization and further proved by supersymmetric Ward identity in [29].

A choice of SUSY preserving regularization schemes corresponds to the choice of Kähler potentials on the conformal manifold. In other words, chang-

[7] For a $4d$ $\mathcal{N}=1$ and a $2d$ $\mathcal{N}=(1,1)$ SCFT, the finite terms are ambiguous and thus not physical quantity [26].

[8] SUSY localization will be reviewed in Section 5.1.

[9] Here we focus only on the constant term in (2.29). Thus these equalities hold up to moduli independent factors due to the Weyl anomaly.

ing a regularization scheme causes the the Kähler transformations

$$K(\tau,\overline{\tau}) \to K(\tau,\overline{\tau}) + \mathcal{F}(\tau) + \overline{\mathcal{F}}(\overline{\tau}), \tag{2.35}$$

where $\mathcal{F}(\cdot)$ and $\overline{\mathcal{F}}(\cdot)$ are a holomorphic and an anti-holomorphic function, respectively. This means that the sphere partition function is not actually a function but a section. This difference can be explained by supergravity counterterms [26, 29]. We will explain this point more explicitly in the presence of the Janus interface in Section 5.2.4.

Chapter 3

Interface entropy

In this chapter we study an entropy of a general conformal/superconformal interface. Although we explained motivations to study an interface entropy in Chapter 1, here we want to recap points. One motivation to study an interface entropy is that it is conjectured to be a C-function in a CFT with an interface (ICFT) [2]. If this statement is true, then an interface entropy can be used to classify/constraints data in ICFTs. Thus it is very important to compute an interface entropy nonperturbatively for examining its property. The second motivation that is more directly related to this thesis is to study the relation between an entropy of the Janus interface and Calabi's diastasis. This topic will be explained in Chapter 4 and Chapter 5.

Let us summarize contents covered in this chapter. The goal of this chapter is to derive the relation between a sphere partition function and an interface entropy in the presence of a conformal/superconformal interface which are given in (3.36) and (3.42). The main tool to derive these relations is a conformal map introduced in the paper [22] which we call the CHM map. So we first review in Section 3.1 the CHM map and the derivation of the relation between a sphere partition function and an entanglement entropy across a spherical entangling surface in the absence of an interface according to [22]. Next we introduce a conformal interface and define its entropy. Then we apply the CHM map to derive the similar relation in the presence of a conformal interface in Section 3.2. In Section 3.3 we further derive the similar relation for a half-BPS superconformal interface in a SCFT after defining it. In an intermediate step we make an assumption. We also discuss this assumption is quite natural in terms of the supersymmetric Rényi entropy.

3.1 Entanglement entropy and sphere partition function in CFT

In this section we review the results [22] in general CFTs without an interface to apply them to CFTs with an interface.

3.1.1 CHM map

First we explain the original map which transforms the Minkowski space $\mathbb{R}^{1,d-1}$ into the static patch of the de Sitter space. We parametrize the d-dimensional Minkowski space by (t, r, Ω_{d-2}) where Ω_{d-2} parametrize $\mathbb{S}^{d-2}$. The metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{d-2}^2. \tag{3.1}$$

Next we consider the conformal transformation $(r, t, \Omega_{d-2}) \to (\theta, \tau, \Omega_{d-2})$

$$t = R \frac{\cos\theta \sinh(\tau/R)}{1 + \cos\theta \cosh(\tau/R)}, \tag{3.2}$$

$$r = R \frac{\sin\theta}{1 + \cos\theta \cosh(\tau/R)}, \tag{3.3}$$

where R is a constant and $0 \le \theta \le \pi/2$. We call this transformation as the (Lorentzian version of) CHM map [22]. Under this map, the metric (3.1) transforms as

$$\mathrm{d}s^2 = \Omega^2 \left(-\cos^2\theta \mathrm{d}\tau^2 + R^2 \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega_{d-2}^2 \right), \tag{3.4}$$

where a conformal factor Ω is given by

$$\Omega = \frac{1}{1 + \cos\theta \cosh(\tau/R)}. \tag{3.5}$$

The new coordinates $(\theta, \tau, \Omega_{d-2})$ are the static patch of the de Sitter space with a radius R. This fact can be seen more transparently when we write the metric (3.4) by $(\hat{r} = R\sin\theta, \tau, \Omega_{d-2})$ as

$$\mathrm{d}s^2 = -\left(1 - \frac{\hat{r}^2}{R^2}\right) \mathrm{d}\tau^2 + \frac{\mathrm{d}\hat{r}^2}{1 - \hat{r}^2/R^2} + \hat{r}^2 \mathrm{d}^2\Omega_{d-2}. \tag{3.6}$$

The surface $\hat{r} = 1$ corresponds to a cosmological horizon.

On the other hand, if we start with the Euclidean flat space $\mathbb{R}^d$ instead of Minkowski space in which the metric can be obtained by performing Wick rotation $t \to \mathrm{i}t$ in (3.1) as

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{d-2}^2. \tag{3.7}$$

In this case, the (Euclidean version of) CHM map is given by

$$t = R \frac{\cos\theta \sin(\tau/R)}{1 + \cos\theta \cos(\tau/R)}, \tag{3.8}$$

$$r = R \frac{\sin\theta}{1 + \cos\theta \cos(\tau/R)}, \tag{3.9}$$

and the metric after this transformation is

$$\mathrm{d}s^2 = \Omega^2 \left(\cos^2\theta \mathrm{d}\tau^2 + R^2 \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega_{d-2}^2 \right) = \Omega^2 \mathrm{d}s_{\mathbb{S}^d}^2, \tag{3.10}$$

where a conformal factor Ω is given by

$$\Omega = \frac{1}{1 + \cos\theta \cos(\tau/R)}, \tag{3.11}$$

and $0 \leq \theta \leq \pi/2$. So in this case we finally have a d-dimensional sphere $\mathbb{S}^d$.

3.1.2 Original derivation of the relation

By using the CHM map, the author of the paper [22] derived the relation between entanglement entropy with a spherical entangling surface and the sphere partition function. We review their results here to apply them to an ICFT later.

We start with the Lorentzian flat space $\mathbb{R}^{1,d-1}$, and take $\Sigma := \mathbb{S}^{d-2}$ on a constant time slice as an entangling surface. See Figure 3.1. We denote inside and outside of Σ as V and $\tilde{V}$ respectively. Then by tracing out the degree of freedom in $\tilde{V}$ we obtain a reduced density matrix ρ. An entanglement entropy S_E across Σ is given by the von Neumann entropy

$$S_E := -\mathrm{tr}_V \left[\rho \ln \rho \right], \tag{3.12}$$

where a trace is taken for the degree of freedom in V. Note that an entanglement entropy in a continuum quantum field theory has UV divergent which

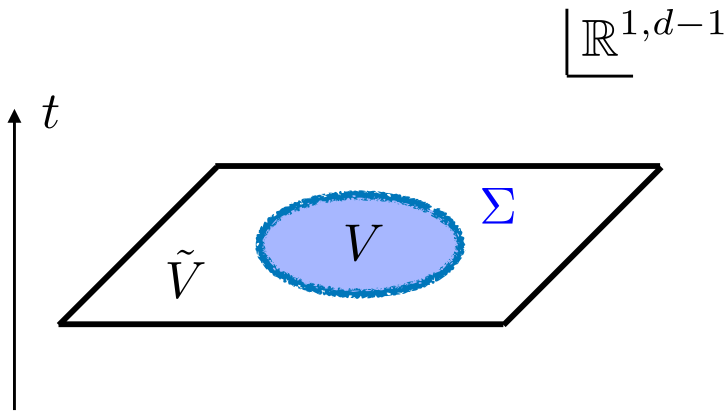

Figure 3.1: Entanglement entropy without an interface

we should regulate properly. We will explain this point in Section We will explain how to regularize UV divergent in Section 3.1.3.

Next, we use the CHM map (3.2). After this transformation, the modular flow induced by ρ^{is} inside the causal development of V is mapped to the time translation $\tau \to \tau + 2\pi R s$ in de Sitter space [1]. Then one can show that states in the static de Sitter space is thermal at a temperature $T = 1/(2\pi R)$ and the density matrix is given by

$$\rho = \frac{e^{-\beta H_\tau}}{\mathrm{tr} e^{-\beta H_\tau}}, \tag{3.13}$$

where H_τ is a operator generating the translation in τ-direction. Thus the entanglement entropy (3.12) can be written as

$$S_E = \beta E + \ln Z[\mathrm{dS}_d], \tag{3.14}$$

where $\beta := 2\pi R$, $E := \mathrm{tr}(\rho H_\tau)$, and $Z := \mathrm{tr} e^{-\beta H_\tau}$.

As explained in the next subsection, we are interested in the universal part of an entanglement entropy S_E which is a log divergent term for even d and a constant term for odd d. Since the energy term E is given by an integral of the one point function of the energy momentum tensor, it gives a finite result for even d and 0 for odd d, see Section 2.1.4. Therefore we do not need to consider this term for our purpose. Thus all we need is the

[1]The causal development of V is a set of points such that all causal curves through them intersect V.

second term in (3.14) which becomes $\ln Z[\mathbb{S}^d]$ after Wick rotation. Therefore we can finally write the entanglement entropy across a spherical entangling surface by free energy $W := -\ln Z[\mathbb{S}^d]$ as

$$S_E = -W = \ln Z[\mathbb{S}^d]. \tag{3.15}$$

3.1.3 UV divergence

Let us mention about UV divergence. In general, an entanglement entropy in a continuum QFT has the following UV structure:

$$S_E = (\text{power low divergences}) + \begin{cases} a_d \log(r\Lambda_{UV}) + f_d & (d : \text{even}) \\ f_d & (d : \text{odd}) \end{cases}. \tag{3.16}$$

The log divergent term in even dimensions and the constant term in odd dimensions are known to be universal, i.e. independent of UV regularization schemes.

On the other hand, the log of the sphere partition function also has the same divergence structure as (2.29) and (2.30). Then the precise version of the equation (3.15) is that

$$S_E|_{\log} = \ln Z[\mathbb{S}^d]|_{\log}, \qquad (d : \text{even}), \tag{3.17}$$

$$S_E|_{\text{const}} = \ln Z[\mathbb{S}^d]|_{\text{const}}, \qquad (d : \text{odd}). \tag{3.18}$$

3.1.4 Another derivation

In the previous section, we derive the relation (3.15) between the free energy and entanglement entropy. Here we introduce another equivalent derivation via replica methods according to [2]. Here we assume that we are in even dimensions and using dimensional regularization to avoid an extra care about the conformal anomaly in intermediate steps. It is automatically incorporated into a pole in even dimensions.

We first introduce the quantity called the Rényi Entropy S_n which is parametrized by a real parameter n called the Rény parameter as

$$S_n := \frac{1}{1-n} \ln \mathrm{tr}_V \rho^n. \tag{3.19}$$

This quantity reduces to an entanglement entropy (3.12) by taking $n \to 1$ limit:

$$\lim_{n\to 1} S_n = S_E. \tag{3.20}$$

In the replica trick we identify the Rényi entropy S_n with the partition function on a branched cover of the original manifold $\mathcal{M}$ which we denote by $\mathcal{M}_n$ [2]. With a proper normalization, this can be stated as

$$\mathrm{tr}_V \rho^n = \frac{Z[\mathcal{M}_n]}{Z[\mathcal{M}]^n}. \tag{3.21}$$

In this case $\mathcal{M} = \mathbb{R}^d$. We use the Euclidean version of the CHM map (3.8). The metric is conformally equivalent to the one of sphere, $\mathrm{d}s^2_{\mathbb{R}^d} = \Omega^2 \mathrm{d}s^2_{\mathbb{S}^d}$ via the CHM map. Moreover, the n-fold cover of $\mathbb{R}^d$ is mapped to $\mathbb{S}^d_n$ as

$$\mathrm{d}s^2_{\mathbb{R}^d_n} = \Omega^2 \mathrm{d}s^2_{\mathbb{S}^d_n}, \tag{3.22}$$

where

$$\mathrm{d}s^2_{\mathbb{S}^d_n} = \cos^2\theta \mathrm{d}\tau^2 + R^2 n^2 \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega^2_{d-2}. \tag{3.23}$$

So the partition functions on each manifolds are related as

$$Z[\mathbb{R}^d_n] = Z[\mathbb{S}^d_n], \quad Z[\mathbb{R}^d] = Z[\mathbb{S}^d] \tag{3.24}$$

Note that we do not need care about the conformal anomaly in this step since we use the dimensional regularization.

Thus we can obtain the Rényi entropy via

$$S_n = \frac{1}{1-n} \ln \mathrm{tr}_V \rho^n = \ln \frac{Z[\mathbb{S}^d_n]}{Z[\mathbb{S}^d]^n}. \tag{3.25}$$

To obtain the entanglement entropy, we evaluate the above equation around $n = 1$.

$$S_E = \lim_{n\to 1} \frac{1}{1-n} \left(\ln Z[\mathbb{S}^d_n] - \ln Z[\mathbb{S}^d] \right) + \ln Z[\mathbb{S}^d] \tag{3.26}$$

The first term around $n = 1$ can be written as

$$\ln Z[\mathbb{S}^d_n] - \ln Z[\mathbb{S}^d] = -\frac{1}{2} \int_{\mathbb{S}^d} \delta g_{\mu\nu} \left\langle T^{\mu\nu} \right\rangle_{\mathbb{S}^d} + \mathcal{O}\left((n-1)^2 \right), \tag{3.27}$$

[2]More precisely $\mathcal{M}_n$ is made as follows. First we prepare n-th copies of $\mathcal{M}$. Then we cut and glue them at entangling surfaces.

where $\delta g_{\mu\nu}$ is a deviation of the $\mathbb{S}_n^4$ metric from $\mathbb{S}^4$ up to $O(n-1)$. In the absence of the conformal anomaly, the one point function of the energy momentum tensor on sphere is proportional to that on flat space as

$$\langle T^{\mu\nu}\rangle_{\mathbb{S}^d} = (\text{Weyl factor})^2 \langle T^{\mu\nu}\rangle_{\mathbb{R}^d} = 0. \tag{3.28}$$

So we show that the first term in (3.26) vanishes to have $S_E = \ln Z[\mathbb{S}^d]$. This is exactly the same result in the previous subsection.

3.2 Entropy of a conformal interface

In this section we define a conformal interface and its entropy. Then we derive the relation between a sphere partition function and an interface entropy by generalizing the results reviewed in the previous section.

3.2.1 Conformal interface

We consider non-local objects in QFTs that are called defects. Let us consider a CFT in d-dimensional flat space $\mathbb{R}^d$ and insert a co-dimesion p defect. We are interested in defects which preserve a part of conformal symmetry $SO(d+1,1)$. We call defects which preserve the maximal subgroup $SO(p+1,1)\times SO(q)$ as conformal defects and CFTs with defects as defect conformal field theories (DCFTs). In particular co-dimension one defects are denoted as interfaces, domain-walls, and boundaries. An interface preserving $SO(d,1) \subset SO(d+1,1)$ is called as a conformal interface and a CFT with a conformal interface as an interface CFT (ICFT).

Although the full conformal invariance is violated by inserted defects, the correlation functions in DCFTs or ICFTs are strongly constrained by the remaining conformal symmetry. For example, one can show that one point functions in an ICFT of an operator with nonzero spin vanish [30]. On the other hand, the one point function of scalar operators $\mathcal{O}$ with a conformal dimension $\Delta_{\mathcal{O}}$ is given by

$$\langle \mathcal{O}(y)\rangle^{(\text{ICFT})} = \frac{a_{\mathcal{O}}}{|y_\perp|^{\Delta_{\mathcal{O}}}}, \tag{3.29}$$

where we introduce Cartesian coordinate y^μ and $y_\perp$ is a coordinate perpendicular to the interface. The nontrivial coefficient $a_{\mathcal{O}}$ which vanish in a CFT without an interface is also dynamical data in an ICFT.

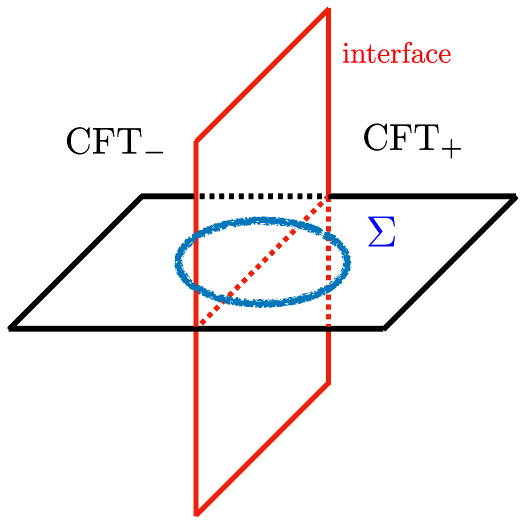

Figure 3.2: Entanglement entropy with an interface

3.2.2 Entropy of conformal interface

Let us explain setup depicted in Figure 3.2. We start with a flat space $\mathbb{R}^d$ and parametrize it by Cartesian coordinates $(y^0, y^1, \cdots, y^{d-1})$ [3]. A planar interface $\mathcal{I}$ is placed at $y^{d-1} = 0$. This preserves $SO(d,1)$ and therefore is a conformal interface. Two different CFTs live in both sides of the interface, and we denote it by CFT_+ and CFT_-, respectively.

Then we define an entropy associated to this interface. Frist we take $\Sigma := \mathbb{S}^{d-2}$ as an entangling surface. Then we have an entanglement entropy in the presence of an interface by (3.12) which we denote as $S_E^{(\text{ICFT})}$. We also have an entanglement entropy across the same surface Σ in the absence of an interface for a CFT_+ and CFT_-. We denote them by $S_E^{(\text{CFT}_\pm)}$. Then a definition of an interface entropy $S_\mathcal{I}$ is given as a contribution to an entanglement entropy from an interface:

$$S_\mathcal{I} := S_E^{(\text{ICFT})} - \frac{1}{2}\left(S_E^{(\text{CFT}_+)} + S_E^{(\text{CFT}_-)}\right). \tag{3.30}$$

In the absence of an interface we can use the result reviewed in Section 3.1:

$$S_E^{(\text{CFT}_\pm)} = \ln Z^{(\text{CFT}_\pm)}[\mathbb{S}^d]. \tag{3.31}$$

[3]From now on we denote Cartesian coordinates in a flat space by y^μ to distinguish them from stereographic coordinates on a sphere denoted by x^μ which will be introduced in the next chapter.

In the presence of an interface, we again use the CHM map to map $\mathbb{R}^d$ to $\mathbb{S}^d$ after the Wick rotation. By this map, a planar interface is mapped to spherical surface at an equator of $\mathbb{S}^d$. Similar as in a CFT without an interface, an entanglement entropy with an interface can be written as

$$S_E^{(\mathrm{ICFT})} = \lim_{n\to 1} \frac{1}{1-n} \left(\ln Z^{(\mathrm{ICFT})}[\mathbb{S}^d_n] - \ln Z^{(\mathrm{ICFT})}[\mathbb{S}^d]\right) + \ln Z^{(\mathrm{ICFT})}[\mathbb{S}^d] \quad (3.32)$$

via the CHM map. The first term around $n = 1$ can be written as

$$\ln Z^{(\mathrm{ICFT})}[\mathbb{S}^d_n] - \ln Z^{(\mathrm{ICFT})}[\mathbb{S}^d] = -\frac{1}{2}\int_{\mathbb{S}^d} \delta g_{\mu\nu} \langle T^{\mu\nu}\rangle^{(\mathrm{ICFT})}_{\mathbb{S}^d} + \mathcal{O}\left((n-1)^2\right). \quad (3.33)$$

As we explained in 3.2.1, one point functions of operators with nonzero spins vanish due to the remaining conformal symmetry. So we can drop the first term since the one point function on a sphere is proportional to it on a flat space:

$$\langle T^{\mu\nu}\rangle^{(\mathrm{ICFT})}_{\mathbb{S}^d} = (\text{Weyl factor})^2 \langle T^{\mu\nu}\rangle^{(\mathrm{ICFT})}_{\mathbb{R}^d} = 0. \quad (3.34)$$

So we can again show that the first term in (3.32) vanishes and thus

$$S_E^{(\mathrm{ICFT})} = \ln Z^{(\mathrm{ICFT})}[\mathbb{S}^d]. \quad (3.35)$$

Then by combining the relations (3.31) and (3.31) with the definition of an interface entropy (3.30) we have [4]

$$S_{\mathcal{I}} = \ln\left[\frac{Z^{(\mathrm{ICFT})}[\mathbb{S}^d]}{\left(Z^{(\mathrm{CFT}_+)}[\mathbb{S}^d] Z^{(\mathrm{CFT}_-)}[\mathbb{S}^d]\right)^{1/2}}\right]. \quad (3.36)$$

3.2.3 UV divergence

Let us here mention about UV divergence in the interface entropy. As explained in Section 3.1.3, each terms in (3.30) have UV divergence. We focus on the even d case which will be used later. In this case, the entanglement entropy has the structure given in (3.16). Thus the interface entropy has the

[4]This result is the special case of the result in [2] where they considered more general conformal defects.

structure as [5]

$$S_{\mathcal{I}} = (\text{power low divergence}) + f_d^{\mathcal{I}}. \tag{3.37}$$

We explained that the constant terms in an entanglement entropy are ambiguous and thus not physical. However, their difference $f_d^{\mathcal{I}}$ is considered to be universal [6]. So the formula (3.36) should be understood as the equality between finite contributions.

3.3 Entropy of a half-BPS superconformal interface

In this section, we define a half-BPS superconformal interface in a SCFT and derive the similar formula as (3.36) for a superconformal interface. We focus on CFTs with specific dimensions and SUSY, $4d$ $\mathcal{N}=2$ SCFTs.

3.3.1 Half-BPS superconformal interface

Let us consider a $\mathcal{N}=2$ superconformal field theory on a flat space. The superconformal algebra is $SU(2,2|2)$ whose bosonic subgroup is $SO(4,2)\times SU(2)_R\times U(1)_R$ as explained in Section 2.2.2. We are interested in an interface which maximally preserves a part of $SU(2,2|2)$ symmetry. Such an interface is called a half-BPS superconformal interface.

As explained in Section 2.2.2, the full symmetry $SU(2,2|2)$ is generated by Q^i, Q_i and S^i, S_i as

$$\delta_Q = \overline{\epsilon}^i Q_i + \overline{\epsilon}_i Q^i, \quad \delta_S = \overline{\eta}^i S_i + \overline{\eta}_i S^i. \tag{3.38}$$

A half-BPS superconformal interface is invariant under transformations by parameters satisfying

$$\epsilon_i = \rho_{ij}\gamma^3\epsilon^j, \quad \eta_i = -\rho_{ij}\gamma^3\eta^j, \tag{3.39}$$

[5]The log divergence has subtleties. This term arises from the conformal anomaly. The conformal anomaly in the presence of an interface has the contribution from bulk and the interface. One can show that the bulk contribution cancels out in (3.36), while we do not have a proof for vanishing of the contribution from the interface. However, holographic examples in [2] and Section 5.4 provide non-trivial evidences.

[6]For example the author of the paper [31] showed that this part is independent of the choice of UV regularization schemes by explicit calculation. This fact can be seen also in Section 5.4.

where ρ_{ij} satisfies $\rho_{ij} = \rho_{ji}, \rho_{ij}\overline{\rho}^{jk} = \delta_i^k$ with $\overline{\rho}^{ij} := (\rho_{ij})^*$. In other words, the preserved symmetry is generated by the following specific linear combination of supercharges:

$$Q_i - \rho_{ij}\gamma_3 Q^j, \quad S_i + \rho_{ij}\gamma_3 S^j. \tag{3.40}$$

The resulting symmetry is the $3d$ $\mathcal{N} = 2$ superconformal algebra denoted as $OSp(2|4)_{\mathrm{sc}}$ [7].

3.3.2 Entropy of a supefconformal interface

Now we want to derive similar relation as (3.36) for a half-BPS superconformal interface $\mathcal{I}$.

In this case, there are deferences from the previous argument. First, fields in SCFTs can couple to fields in a supergravity multiplet other than a metric. This makes a partition function of a SCFT a functional of a supergravity multiplet rather than a functional of a metric.

Second, we use a regularization scheme preserving the conformal symmetry in the previous derivation. In supersymmetric case, we have another choice; a regularization scheme which preserve a part of supersymmetry [8]. We denote a partition function on a spacetime manifold $\mathcal{M}$ in a SCFT in the presence of a half-BPS superconformal interface $\mathcal{I}$ as $Z^{\mathcal{I}}_{\mathrm{SUSY}}[\mathcal{M}]$. As we will see later in Chapter 5 it can be complex in general. If we simply replace $Z^{(\mathrm{ICFT})}[\mathbb{S}^d]$ in (3.36) with $Z^{\mathcal{I}}_{\mathrm{SUSY}}[\mathbb{S}^d]$, the resulting entropy becomes complex. So we here make the following assumption:

$$Z^{(\mathrm{ICFT})}[\mathbb{S}^4] = \left| Z^{\mathcal{I}}_{\mathrm{SUSY}}[\mathbb{S}^4] \right| . \tag{3.41}$$

Then we obtain a relation between a SUSY partition function and an interface entropy for a half-BPS interface $\mathcal{I}$ as

$$S_{\mathcal{I}} = \log \left[\frac{\left| Z^{\mathcal{I}}_{\mathrm{SUSY}}[\mathbb{S}^4] \right|}{\left(Z^{(\mathrm{CFT}_+)}_{\mathrm{SUSY}}[\mathbb{S}^4]\, Z^{(\mathrm{CFT}_-)}_{\mathrm{SUSY}}[\mathbb{S}^4] \right)^{1/2}} \right] , \tag{3.42}$$

where $Z^{(\mathrm{CFT}_\pm)}_{\mathrm{SUSY}}[\mathbb{S}^4]$ is a partition function for $\mathrm{CFT}_\pm$ computed in a SUSY

[7] We write the letter "sc" to distinguish the $3d$ $\mathcal{N} = 2$ superconformal algebra from $4d$ $\mathcal{N} = 2$ massive SUSY algebra which we will denote $OSp(2|4)_{\mathrm{m}}$. We will explain their correspondence later.

[8] One example is SUSY localization which is reviewed in Section 5.1.

preserving scheme. Unfortunately we do not have a proof for the assumption (3.41), but we have some evidences for it [9].

- The assumption (3.41) is equivalent to an assumption that an entanglement entropy with an interface $S_E^{(\mathrm{ICFT})}$ can be derived from the supersymmetric Rényi entropy. See the next subsection.

- The results obtained in holographic calculation are consistent with the formula (3.42). See Section 5.4.

- In two dimensional case, we can give a proof of similar relation (C.8) by using boundary super-Weyl anomaly. See Appendix C.

3.3.3 Supersymmetric Rényi entropy

Now we want to explain that the assumption (3.41) is natural in terms of the supersymmetric Rényi entropy. First let us define the supersymmetric Rényi entropy [33]. As we explained before, fields in SCFTs can couple to the supergravity background. Similar as the Rényi entropy, we consider the n-fold cover of a supergravity background on $\mathbb{S}^4$ which has singularity at $n = 1$. To define a supersymmetric Rényi entropy, we resolve this singularity and denote the resulting background as $\tilde{\mathbb{S}}^4_n$. The metric of the resolved n-fold cover $\tilde{\mathbb{S}}^4_n$ is given by

$$\mathrm{d}s^2_{\tilde{\mathbb{S}}^d_n} = \left(f(\theta)\right)^2 \mathrm{d}\theta^2 + n^2 \sin^2\theta \mathrm{d}\tau^2 + \cos^2\theta \mathrm{d}\Omega_2^2, \tag{3.43}$$

where we set $R = 1$ and a smooth function $f(\theta)$ satisfies

$$f(\theta \to 0) = n, \quad f(\theta \gg \delta) = 1, \tag{3.44}$$

for a small parameter $\delta \ll 1$. Then the supersymmetric Rényi entropy in the presence of a half-BPS interface $\mathcal{I}$ denoted by $S^{\mathcal{I}}_{\mathrm{SUSY}\ n}$ is given by

$$S^{\mathcal{I}}_{\mathrm{SUSY}\ n} := \frac{1}{1-n} \operatorname{Re}\log \frac{Z^{\mathcal{I}}_{\mathrm{SUSY}}[\widetilde{\mathbb{S}}^4_n]}{\left(Z^{\mathcal{I}}_{\mathrm{SUSY}}[\mathbb{S}^4]\right)^n} \ . \tag{3.45}$$

[9] In $d = 3$, it is usual to define the free energy by taking the absolute value of the partition function as $F = -\log\left|Z_{\mathrm{SUSY}}[\mathbb{S}^3]\right|$. See for example [6, 32].

Now we assume that an entanglement entropy with an interface can be derived from the supersymmetric Rényi entropy:

$$S_E^{(\mathrm{ICFT})} = \lim_{n\to 1} S^{\mathcal{I}}_{\mathrm{SUSY}\ n} \ . \tag{3.46}$$

Below we want to show that the formula (3.42) can be derived from this assumption and therefore the assumption (3.46) is equivalent to the assumption (3.41).

By substituting the equation (3.45) to the definition of an interface entropy (3.30), we have

$$S_E^{(\mathrm{ICFT})} = \log \left| Z^{\mathcal{I}}_{\mathrm{SUSY}}[\mathbb{S}^4] \right| - \partial_n \,\mathrm{Re}\, \log\, Z^{\mathcal{I}}_{\mathrm{SUSY}}[\widetilde{\mathbb{S}}^4_n] \Big|_{n=1} \ . \tag{3.47}$$

Let us proof that the second term in (3.47) vanish. The background Weyl mutiplet

$$\mathcal{W} = (g_{\mu\nu},\, \psi_\mu{}^i,\, V_\mu{}_i{}^j,\, A_\mu,\, D,\, \chi^i,\, T^{\pm}_{\mu\nu}) \ . \tag{3.48}$$

couples to the supercurrent multiplet $\mathcal{J}$ which contains a stress-energy tensor [10]:

$$\mathcal{J} = (T_{\mu\nu},\, S^i_\mu,\, j^{ij}_\mu,\, j_\mu,\, J,\, j^i,\, j^{\pm}_{\mu\nu}) \ . \tag{3.49}$$

Then the partition function $Z^{\mathcal{I}}_{\mathrm{SUSY}}[\widetilde{\mathbb{S}}^4_n]$ on the branched sphere is expanded around $n=1$ as

$$\begin{aligned} &- \mathrm{Re}\log Z^{\mathcal{I}}_{\mathrm{SUSY}}[\widetilde{\mathbb{S}}^4_n] + \mathrm{Re}\log Z^{\mathcal{I}}_{\mathrm{SUSY}}[\mathbb{S}^4] \\ &= \int_{\mathbb{S}^4} \mathrm{d}^4x\sqrt{g}\,\Big\langle \frac{1}{2}\,\delta g_{\mu\nu}T^{\mu\nu} + \delta\overline{\psi}_\mu{}^i S^\mu{}_i + \delta\overline{\psi}_{\mu i}S^{\mu i} + \delta V_\mu{}^{ij} j^\mu{}_{ij} + \delta A_\mu j^\mu \\ &\qquad + \delta D\, J + \delta\overline{\chi}^i\, j_i + \delta\overline{\chi}_i\, j^i + \delta T^{+\mu\nu} j_{+\mu\nu} + \delta T^{-\mu\nu} j_{-\mu\nu} \Big\rangle^{(\mathrm{ICFT})}_{\mathbb{S}^4} + \mathcal{O}((n-1)^2) \ . \end{aligned} \tag{3.50}$$

As in Section 3.2 contributions from operators with nonzero spins vanish. Thus we look into the one point function $\langle J \rangle^{(\mathrm{ICFT})}_{\mathbb{S}^4}$. For this purpose, let us consider the SUSY transformation of j_i in the flat space:

$$\delta j_i = -\frac{1}{2}(\partial\!\!\!/ J)\epsilon_i + \frac{1}{2} j_{\mu i}{}^j \gamma^\mu \epsilon_j + \frac{\mathrm{i}}{2} j_\mu \gamma^\mu \epsilon_i + j^-_{\mu\nu}\gamma^{\mu\nu}\varepsilon_{ij}\epsilon^j, \tag{3.51}$$

[10]The explicit form of the supercurrent multiplet for a vector multiplet is given in Appendix A.6.

where constant SUSY parameters ϵ^i, ϵ_i satisfy the half-BPS condition (3.39). Then we take vacuum expectation values of both sides. Again one point functions of operators with nonzero spins vanish. We also know that $\langle\, \delta j_i \,\rangle_{\mathbb{R}^4}^{(\mathrm{ICFT})} = 0$ from the Ward identities. Thus for constant ϵ^i, ϵ_i satisfying the condition (3.39) we have

$$0 = \partial_\mu \langle\, J \,\rangle_{\mathbb{R}^4}^{(\mathrm{ICFT})} \gamma^\mu \epsilon_i \ . \tag{3.52}$$

On the other hand, we know that the operator J has the Weyl weight 2 and thus [11]

$$\langle\, J(y) \,\rangle_{\mathbb{R}^4}^{(\mathrm{ICFT})} = \frac{a_J}{|y^3|^2}, \tag{3.53}$$

where a_J is a constant. Substituting this equation into (3.52), we show that $a_J = 0$. So we conclude that $\langle\, J \,\rangle_{\mathbb{S}^4}^{(\mathrm{ICFT})} = 0$ by the similar argument as (3.34).

Thus we show that the second term in (3.47) vanish and therefore

$$S_E^{(\mathrm{ICFT})} = \log \left| Z_{\mathrm{SUSY}}^{\mathcal{I}}[\mathbb{S}^4] \right| \ . \tag{3.54}$$

By comparing this result and (3.35) we reach the assumption (3.41). Conversely, if we assume (3.41) we obtain (3.54) which is same as the supersymmetric Rényi entropy (3.45), thus we arrive at the assumption (3.46). So we conclude that the assumption (3.45) isequivalent to the assumption (3.41).

[11] Remind that this form is determined by the remaining conformal symmetry. See Seciton 3.2.1.

Chapter 4

Construction of the Janus interface

So far we have considered general conformal/superconformal interfaces. From now on, we consider a specific interface called the Janus interface. The goal of the latter half is to show the relation between the Janus interface in $4d$ $\mathcal{N}=2$ SCFTs and a specific linear combination of analytically continued Kähler potentials on the moduli space called Calabi's diastasis.

In this Chapter we explain a construction of the Janus interface. In Section 4.1, we explain an off-shell construction of the Janus interface in the flat space. We introduce a crucial ingredient, the coupling multiplet whose bottom component is a position dependent coupling $\tau(x)$. We will check the resulting interface is indeed a half-BPS superconformal interface. In Section 4.2 we construct the Janus interface on the four sphere and discuss the relation to the Janus interface in the flat space. In Section 4.3, we explicitly construct the Janus interface in a $4d$ $\mathcal{N}=2$ gauge theory which will be used in the computation in the next chapter.

4.1 Off-shell Construction of Janus interface in flat space

First we explain an off-shell construction of the Janus interface in flat space [1]. We denote Cartesian coordinates in flat space as $y^\mu, (\mu=0,1,2,3)$ and place

[1] The similar method was used in [34, 35, 36, 14, 37] for various defects.

the Janus interface orthogonal to the y^3-direction, see Figure 3.2. As we explained in Introduction, the Janus interface is an interface across which an exactly marginal coupling τ changes its values. To construct such an object we promote a coupling constant τ to a position-depending coupling field $\tau(y)$. Further, to construct a half-BPS interface we promote the coupling field $\tau(y)$ to the bottom component of a $\mathcal{N}=2$ chiral multiplet

$$\mathcal{T} = (\tau, \Psi_i^{(\tau)}, B_{ij}^{(\tau)}, F_{\mu\nu}^{(\tau)-}, \Lambda_i^{(\tau)}, C^{(\tau)}). \tag{4.1}$$

We denote this multiplet by a coupling multiplet. As we will see soon later, by solving symmetry preserving conditions we can write all components in $\mathcal{T}$ in terms of $\tau(y)$. We also have the anti-chiral counterpart of $\mathcal{T}$ which we denote as $\overline{\mathcal{T}}$. The components of $\overline{\mathcal{T}}$ are given by

$$\overline{\mathcal{T}} = (\overline{\tau},\ \Psi^{(\tau)i},\ B^{(\tau)ij},\ F_{\mu\nu}^{(\tau)+},\ \Lambda^{(\tau)i},\ \overline{C}^{(\tau)}), \tag{4.2}$$

where we take $\overline{\tau}$ to be the complex conjugate of τ: $\overline{\tau} = \tau^*$.

Now we consider symmetry preserving conditions on $\mathcal{T}$. First, to preserve a part of Lorentz symmetry we set $F_{\mu\nu}^{(\tau)\pm} = 0$. Next, to preserve the half of supersymmetry, we impose that fermionic components in $\mathcal{T}, \overline{\mathcal{T}}$ and their SUSY variation vanish. Using the SUSY transformations of a chiral multiplet [2], the conditions are written as

$$\delta\Psi_i^{(\tau)} = (\partial_3\tau)\gamma^3\epsilon_i + \frac{1}{2}\,B_{ij}^{(\tau)}\epsilon^j = 0\ , \tag{4.3}$$

$$\delta\Lambda_i^{(\tau)} = -\frac{1}{2}\partial_3 B_{ij}^{(\tau)}\varepsilon^{jk}\gamma^3\epsilon_k + \frac{1}{2}C^{(\tau)}\varepsilon_{ij}\epsilon^j = 0\ , \tag{4.4}$$

$$\delta\Psi^{(\tau)i} = (\partial_3\overline{\tau})\gamma^3\epsilon^i + \frac{1}{2}\,B^{(\tau)ij}\epsilon_j = 0\ , \tag{4.5}$$

$$\delta\Lambda^{(\tau)i} = -\frac{1}{2}\partial_3 B^{(\tau)ij}\varepsilon_{jk}\gamma^3\epsilon^k + \frac{1}{2}\overline{C}^{(\tau)}\varepsilon^{ij}\epsilon_j = 0\ , \tag{4.6}$$

where ϵ^i, ϵ_i are constant SUSY parameters. By solving these equations, we have

$$B_{ij}^{(\tau)} = -2\rho_{ij}\,\partial_3\tau\ , \qquad C^{(\tau)} = -2e^{+2\mathrm{i}\alpha}\,\partial_3^2\tau\,, \tag{4.7}$$

$$B^{(\tau)ij} = -2\overline{\rho}^{ij}\,\partial_3\overline{\tau} \qquad \overline{C}^{(\tau)} = -2e^{-2\mathrm{i}\alpha}\,\partial_3^2\overline{\tau}\ , \tag{4.8}$$

[2]See appendix A.3.2.

for fields in the coupling multiplet and

$$\epsilon_i = \rho_{ij}\gamma^3\epsilon^j, \tag{4.9}$$

for SUSY parameters. These solutions are parametrized by matrices $\rho_{ij}, \overline{\rho}^{ij}$ given as

$$\rho_{ij} = e^{\mathrm{i}\alpha}\,\vec{n}\cdot\vec{\tau}_{ij}\ , \qquad \overline{\rho}^{ij} = e^{-\mathrm{i}\alpha}\,\vec{n}\cdot\vec{\tau}^{ij}\ , \tag{4.10}$$

where α is a real parameter and $\vec{n}$ is a three-dimensional unit vector. The condition on SUSY parameters is equivalent to the first condition in (3.39). So we construct an interface which preserves the half of the supersymmetry. We further confirm that this interface is also invariant under special superconformal transformations generated by SUSY parameters satisfying the second condition in (3.39) when taking a step function profile in the next subsection.

Note that the auxiliary fields satisfy reality conditions even for generic $\tau(y)$:

$$B^{(\tau)ij} = (B^{(\tau)}_{ij})^*, \quad \overline{C}^{(\tau)} = (C^{(\tau)})^*. \tag{4.11}$$

This situation is different from the Janus interface on $\mathbb{S}^4$ which we will construct in Section 4.2.

4.1.1 Step function profile

So far we have considered an interface for a general profile $\tau(y)$. We now consider a step function profile:

$$\tau(y^3) = \begin{cases} \tau_+ & \text{for} \quad y^3 > 0\ , \\ \tau_- & \text{for} \quad y^3 < 0\ . \end{cases} \tag{4.12}$$

In this case, the derivatives of $\tau(x)$ are given by a delta function as

$$\partial_3\tau = \Delta\tau\,\delta(y^3), \quad \partial_3^2\tau = \Delta\tau\,\delta'(y^3). \tag{4.13}$$

So by substituting these expressions into (4.7) and (4.8), we can obtain the auxiliary fields for a step function profile as

$$\begin{aligned} B^{(\tau)}_{ij} &= -2\rho_{ij}\Delta\tau\,\delta(y^3)\ , \quad C^{(\tau)} = -2e^{+2\mathrm{i}\alpha}\Delta\tau\,\delta'(y^3)\ , \\ B^{(\tau)ij} &= -2\overline{\rho}^{ij}\Delta\overline{\tau}\,\delta(y^3)\,, \quad \overline{C}^{(\tau)} = -2e^{-2\mathrm{i}\alpha}\Delta\overline{\tau}\,\delta'(y^3)\,, \end{aligned} \tag{4.14}$$

where $\Delta\tau := \tau_+ - \tau_-$.

Now we show that the Janus interface for a step function profile is invariant under special superconformal transformations. The special superconformal transformations δ_η are obtained by substitute $\epsilon^i \to y^\mu \gamma_\mu \eta^i$ in transformation rules in Appendix A.3.1. Then the special superconformal transformation of $\Psi_i^{(\tau)}$ automatically vanishes: $\delta_\eta \Psi_i^{(\tau)} = 0$. On the other hand, the special superconformal transformation of $\Lambda_i^{(\tau)}$ is given by

$$\delta_\eta \Lambda_i^{(\tau)} = -\frac{1}{2}\,\partial_3 B_{ij}^{(\tau)} \varepsilon^{jk} y^\mu \gamma^3 \gamma_\mu \eta_k + \frac{1}{2}\,C^{(\tau)} \varepsilon_{ij} y^\mu \gamma_\mu \eta^j - B_{ij}^{(\tau)} \varepsilon^{jk} \eta_k \tag{4.15}$$

$$\begin{aligned} &= 2\Delta\tau\,\partial_3(y^3 \delta(y^3))\rho_{ij}\,\varepsilon^{jk}\eta_k \\ &\quad - \Delta\tau\, e^{2\mathrm{i}\alpha}\,\delta'(y^3)\Big(y^3 + \sum_{a=0}^{2} y^a \gamma_a \gamma^3\Big)\varepsilon_{ij}\overline{\rho}^{jk}(\eta_k + \rho_{kl}\gamma^3\eta^l)\ . \end{aligned} \tag{4.16}$$

The first term in the final expression vanishes as a distribution, while the second term vanishes for SUSY parameters satisfying the condition (3.39). Thus we conclude that $\delta_\eta \Psi_i^{(\tau)} = \delta_\eta \Lambda_i^{(\tau)} = 0$ for a step function profile. Similarly, we can show that $\delta_\eta \Psi^{(\tau)i} = \delta_\eta \Lambda^{(\tau)i} = 0$. So in a step function limit, the Janus interface becomes a half-BPS superconformal interface which preserves the subalgebra $OSp(2|4)_{\mathrm{sc}}$ of the $4d$ $\mathcal{N} = 2$ superconformal algebra.

4.2 Off-shell construction of Janus interface in $\mathbb{S}^4$

In this section we explain an off-shell construction of the Janus interface on $\mathbb{S}^4$. Reality properties of auxiliary fields are different from them in flat space. We also explore the relation between the Janus interface on flat space and $\mathbb{S}^4$.

4.2.1 Massive subalgebra on $\mathbb{S}^4$

In Section 3.3.1 we explained that a half-BPS superconformal interface in a $4d$ $\mathcal{N} = 2$ preserves the subalgebra $OSp(2|4)_{\mathrm{sc}} \subset SU(2,2|2)$ which is a $3d$ $\mathcal{N} = 2$ superconformal algebra. Here we explain another relevant subalgebra of $SU(2,2|2)$, a $4d$ $\mathcal{N} = 2$ massive subalgebra which we denote as $OSp(2|4)_{\mathrm{m}}$. We will evaluate sphere partition function with this symmetry preserved.

The massive subalgebra $OSp(2|4)_{\rm m}$ is generated by Killing spinors satisfying

$$\nabla_m \epsilon^i = \frac{\rm i}{2r}\kappa^{ij}\epsilon_j, \quad \nabla_m \epsilon_i = \frac{\rm i}{2r}\overline{\kappa}_{ij}\epsilon^j, \tag{4.17}$$

with

$$\kappa^{ij} = e^{-{\rm i}\beta}(\vec{n}\cdot\vec{\tau})^{ij}, \quad \overline{\kappa}_{ij} = e^{+{\rm i}\beta}(\vec{n}\cdot\vec{\tau})_{ij}. \tag{4.18}$$

We can diagonalize the equations (4.17) as

$$\nabla_m \chi^i = \frac{\rm i}{2r}\gamma_m \chi^i, \tag{4.19}$$

by introducing spinors χ^i defined by

$$\chi^i = e^{{\rm i}\beta}\left(\epsilon^i + \kappa^{ij}\epsilon_j\right). \tag{4.20}$$

The above equation can be solved for ϵ^i and ϵ_i as

$$\epsilon^i = e^{-\frac{\rm i}{2}\beta}P_L\chi^i, \quad \epsilon^i = e^{\frac{\rm i}{2}\beta}(\vec{n}\cdot\vec{\tau})_{ij}P_R\chi^j. \tag{4.21}$$

Now we introduce the stereographic coordinates x^μ in which the metric is given by [3]

$$g_{\mu\nu} = f(x)^2\,\delta_{\mu\nu}\ , \qquad f(x) = \frac{1}{1+\frac{x^2}{4r^2}}\ , \tag{4.22}$$

where $x := \sqrt{x^\mu x_\mu}$. Then solutions of (4.19) in the stereographic coordinates are given as

$$\chi^j = \sqrt{f}\left(1+\frac{\rm i}{2r}x_\mu\Gamma^\mu\right)\chi_0^j\ , \tag{4.23}$$

where χ_0^j are constant spinors. By substituting this solution into (4.21), we finally obtain

$$\epsilon^i = e^{-\frac{\rm i}{2}\beta}\sqrt{f}\left(P_L\chi_0^i + \frac{\rm i}{2r}x_\mu\Gamma^\mu P_R\chi_0^i\right)\ , \tag{4.24}$$

$$\epsilon_i = e^{\frac{\rm i}{2}\beta}\sqrt{f}\,\vec{n}\cdot\vec{\tau}_{ij}\left(P_R\chi_0^j + \frac{\rm i}{2r}x_\mu\Gamma^\mu P_L\chi_0^j\right)\ . \tag{4.25}$$

[3] Although we use the same notation, this is different from the function $f(\cdot)$ in (3.43).

We further restrict the symmetry by imposing $P_L\chi_0^i = 0$ without loss of generality [4]. The resulting symmetry is $OSp(2|2)_{\rm m}$ [29].

4.2.2 Construction

Next, we consider a construction of the Janus interface in $\mathbb{S}^4$. The reality condition in this case is different from the one in flat space.

Same as in the construction in flat space, we introduce a chiral coupling multiplet $\mathcal{T} = (\tau, \Psi_i^{(\tau)}, B_{ij}^{(\tau)}, F_{\mu\nu}^{(\tau)-}, \Lambda_i^{(\tau)}, C^{(\tau)})$ and an anti-chiral coupling multiplet $\overline{\mathcal{T}} = (\overline{\tau}, \Psi^{(\tau)i}, B^{(\tau)ij}, F_{\mu\nu}^{(\tau)+}, \Lambda^{(\tau)i}, \overline{C}^{(\tau)})$. Next, we impose that $F_{ab}^{(\tau)+} = F_{ab}^{(\tau)-} = 0$ to preserve a part of Lorentz symmetry. We also impose that fermionic components in the chiral/antichiral coupling multiplet and their SUSY variations vanish to preserve a part of supersymmetry. The SUSY invariant conditions for the chiral coupling multiplet are written as

$$\delta\Psi_i^{(\tau)} = (\nabla\!\!\!\!/\,\tau)\,\epsilon_i + \frac{1}{2}\,B_{ij}^{(\tau)}\,\epsilon^j = 0\ , \tag{4.27}$$

$$\delta\Lambda_i^{(\tau)} = -\frac{1}{2}\,\nabla\!\!\!\!/\,B_{ij}^{(\tau)}\varepsilon^{jk}\epsilon_k + \frac{1}{2}\,C^{(\tau)}\varepsilon_{ij}\epsilon^j - B_{ij}^{(\tau)}\varepsilon^{jk}\eta_k = 0\ . \tag{4.28}$$

By solving the above equations for the auxiliary fields $B_{ij}^{(\tau)}$ and $C^{(\tau)}$, we have

$$B_{ij}^{(\tau)} = \frac{4\mathrm{i}e^{\mathrm{i}\beta}\,r}{xf(x)}\tau'(x)\vec{n}\cdot\vec{\tau}_{ij}\ , \tag{4.29}$$

$$C^{(\tau)} = \frac{8e^{2\mathrm{i}\beta}r^2}{x^2f(x)^2}\left(\tau''(x) - \frac{1}{x}\tau'(x)\right)\ . \tag{4.30}$$

Similarly, conditions for an anti-chiral coupling multiplet can be written as

$$\delta\Psi^i = (\nabla\!\!\!\!/\,\overline{\tau})\,\epsilon^i + \frac{1}{2}\,B^{(\tau)ij}\,\epsilon_j = 0\ , \tag{4.31}$$

$$\delta\Lambda^i = -\frac{1}{2}\,\nabla\!\!\!\!/\,B^{(\tau)\,ij}\varepsilon_{jk}\epsilon^k + \frac{1}{2}\,\overline{C}^{(\tau)}\varepsilon^{ij}\epsilon_j - B^{(\tau)ij}\varepsilon_{jk}\eta^k = 0\ . \tag{4.32}$$

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[4] Imposing this condition corresponds to choosing a special point (north pole). If we do not impose this condition, the solution of the equation

$$\epsilon^j \propto P_L\chi^j = 0 \tag{4.26}$$

gives a special point, *i.e.* north pole. This is a system of four equations for four unknowns x^μ, and thus at least generically has a solution.

Solving the conditions (4.31) and (4.32) for $B^{(\tau)ij}$ and $\overline{C}^{(\tau)}$ leads to

$$B^{(\tau)ij} = -\frac{\mathrm{i}\,e^{-\mathrm{i}\beta}\,x}{rf(x)}\,\overline{\tau}'(x)\vec{n}\cdot\vec{\tau}^{ij}\,, \tag{4.33}$$

$$\overline{C}^{(\tau)} = \frac{e^{-2\mathrm{i}\beta}\,x^2}{2r^2 f(x)^2}\left(\overline{\tau}''(x) + \frac{3}{x}\,\overline{\tau}'(x)\right)\,. \tag{4.34}$$

Note that the conditions for the anti-chiral coupling multiplet (4.31)-(4.32) are formally related to those for the chiral coupling multiplet (4.27)-(4.28) by charge conjugation in Minkowski signature, but the resulting auxiliary fields do not satisfy the reality condition as (4.11) for a generic profile $\tau(x)$.

To discuss reality properties of auxiliary fields and a relation between the Janus interface in flat space and $\mathbb{S}^4$, we introduce another coordinate θ which is related to stereographic coordinates by

$$x = 2r\tan\frac{\theta}{2}\,. \tag{4.35}$$

In this coordinate, the auxiliary fields (4.29)-(4.30) and (4.33)-(4.34) can be rewritten as

$$B^{(\tau)}_{ij} = \frac{2\,\mathrm{i}\,e^{\mathrm{i}\beta}}{r}\cot(\theta/2)\,\frac{\mathrm{d}\tau}{\mathrm{d}\theta}\,\vec{n}\cdot\vec{\tau}_{ij}\,, \tag{4.36}$$

$$B^{(\tau)ij} = -\frac{2\,\mathrm{i}\,e^{-\mathrm{i}\beta}}{r}\tan(\theta/2)\,\frac{\mathrm{d}\overline{\tau}}{\mathrm{d}\theta}\,\vec{n}\cdot\vec{\tau}^{ij}\,, \tag{4.37}$$

$$C^{(\tau)} = \frac{e^{2\mathrm{i}\beta}}{r^2}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\left[(\cos\theta - 2)\,\frac{\mathrm{d}\tau}{\mathrm{d}\theta} + \sin\theta\,\frac{\mathrm{d}^2\tau}{\mathrm{d}\theta^2}\right]\,, \tag{4.38}$$

$$\overline{C}^{(\tau)} = \frac{e^{-2\mathrm{i}\beta}}{r^2}\frac{\sin(\theta/2)}{\cos^3(\theta/2)}\left[(\cos\theta + 2)\,\frac{\mathrm{d}\overline{\tau}}{\mathrm{d}\theta} + \sin\theta\,\frac{\mathrm{d}^2\overline{\tau}}{\mathrm{d}\theta^2}\right]\,. \tag{4.39}$$

As noted before, these auxiliary fields do not satisfy reality conditions (4.11):

$$B^{(\tau)ij} \neq (B^{(\tau)}_{ij})^*, \quad \overline{C}^{(\tau)} \neq (C^{(\tau)})^*. \tag{4.40}$$

However, if we take a step function profile

$$\tau(\theta) = \begin{cases} \tau_+ & \text{for} \quad 0 \le \theta < \frac{\pi}{2}\,, \\ \tau_- & \text{for} \quad \frac{\pi}{2} < \theta \le \pi\,, \end{cases} \tag{4.41}$$

then they satisfy the reality conditions (4.11) as in a flat space. Indeed, the auxiliary fields in θ coordinates for a step function profile are given by

$$\begin{aligned} B_{ij}^{(\tau)} &= -\frac{2\,\mathrm{i}\,e^{\mathrm{i}\beta}}{r}\,\vec{n}\cdot\vec{\tau}_{ij}\,\Delta\tau\,\delta\left(\theta-\frac{\pi}{2}\right)\ , \\ C^{(\tau)} &= -\frac{2\,e^{2\mathrm{i}\beta}}{r^2}\,\Delta\tau\,\delta'\left(\theta-\frac{\pi}{2}\right)\ , \\ B^{(\tau)ij} &= \frac{2\,\mathrm{i}\,e^{-\mathrm{i}\beta}}{r}\,\vec{n}\cdot\vec{\tau}^{ij}\,\Delta\overline{\tau}\,\delta\left(\theta-\frac{\pi}{2}\right)\ , \\ \overline{C}^{(\tau)} &= -\frac{2\,e^{-2\mathrm{i}\beta}}{r^2}\,\Delta\overline{\tau}\,\delta'\left(\theta-\frac{\pi}{2}\right)\ , \end{aligned} \tag{4.42}$$

which satisfy the reality conditions.

Furthermore we can show that these auxiliary fields in $\mathbb{S}^4$ are related to them in a flat space (4.14) by the Weyl transformation and identification $e^{\mathrm{i}\beta} = -\mathrm{i}e^{\mathrm{i}\alpha}$. See appendix B for more details.

4.3 Janus interface in gauge theory

In this section, we introduce the Janus interface in a $4d$ $\mathcal{N}=2$ gauge theory by using the construction explained in the previous section. The main tool is $\mathcal{N}=2$ supergravity which is reviewed in Appendix A.

A general $\mathcal{N}=2$ supersymmetric gauge theory contains a vector multiplet for a gauge group G and a matter hypermultiplet. We take an appropriate representation for a hypermultiplet to make the theory superconformal. Since the Janus interface couples to a hypermultiplet indirectly we mostly focus on a vector multiplet. We will refer to this point later.

We consider a single gauge factor with a complexified coupling constant

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi\,\mathrm{i}}{g_{\mathrm{YM}}^2} \tag{4.43}$$

without loss of generality. The action of a vector multiplet $\mathcal{V} = (X, \Omega_i, A_\mu, Y_{ij})$ in the flat space is given as

$$\begin{aligned} I_{\text{vector}}^{\text{flat}} = \int \mathrm{d}^4x\,\mathrm{Tr}\Bigg[\frac{1}{g_{\mathrm{YM}}^2}\Big(4D_\mu X\,D^\mu\overline{X} - \frac{1}{2}\,\varepsilon^{ik}\,\varepsilon^{jl}\,Y_{ij}\,Y_{kl} \\ + 2\overline{\Omega}_i \not{D}\Omega^i + \frac{1}{2}\,F_{\mu\nu}F^{\mu\nu}\Big) + \mathrm{i}\,\frac{\vartheta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\Bigg]\ , \end{aligned} \tag{4.44}$$

where Tr[•] is a normalized inner product of Lie algebra and D_μ are covariant derivatives and $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$ is a dual field strength. For more details on notations and conventions, see Appendix A.3.

The action of a vector multiplet in $\mathbb{S}^4$ with a radius r can be obtained by performing Weyl transformation on the flat space action (4.44),

$$I_{\text{vector}} = \int \mathrm{d}^4x\,\sqrt{g}\,\mathrm{Tr}\Bigg[\frac{1}{g_{\text{YM}}^2}\Big(4D_\mu X\,D^\mu\overline{X} + \frac{8}{r^2}X\overline{X} - \frac{1}{2}\,\varepsilon^{ik}\,\varepsilon^{jl}\,Y_{ij}\,Y_{kl} + 2\overline{\Omega}_i\not{D}\Omega^i + \frac{1}{2}\,F_{\mu\nu}F^{\mu\nu}\Big) + \mathrm{i}\,\frac{\vartheta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\Bigg]\,, \tag{4.45}$$

where $\sqrt{g} = \sqrt{-\det(g_{\mu\nu})}$. For this action to be positive semi-difinite, we demand that

$$(Y^I_{ij})^* = -Y^{Iij}\,, \tag{4.46}$$

as in [38, 39]. Note that this condition is different from the physical reality condition in Minkowski signature.

Next we explain how to write down this action via supersupace formalism. The vector multiplet $\mathcal{V}$ can be embedded into the chiral multiplet with a Weyl weight $w = 1$ which we denote by $\mathcal{A}(\mathcal{V})$ as

$$\begin{aligned}
A|_{\mathcal{A}(\mathcal{V})} &= X,\\
\Psi_i|_{\mathcal{A}(\mathcal{V})} &= \Omega_i\,,\\
B_{ij}|_{\mathcal{A}(\mathcal{V})} &= Y_{ij}\,,\\
F^-_{ab}|_{\mathcal{A}(\mathcal{V})} &= \frac{1}{2}\Big(F_{ab} - \tilde{F}_{ab}\Big)\,,\\
\Lambda_i|_{\mathcal{A}(\mathcal{V})} &= -\varepsilon_{ij}\,\not{D}\Omega^j\,,\\
C|_{\mathcal{A}(\mathcal{V})} &= \Big(-2D_\mu D^\mu + \frac{4}{r^2}\Big)\,\overline{X}\,.
\end{aligned} \tag{4.47}$$

Then we can obtain the squared chiral multiplet $\mathcal{A}(\mathcal{V})^2$ via tensor calculus which is explained in Appendix A.4. By integrating the top components of $\mathcal{A}(\mathcal{V})^2$ and its antichiral counterpart, we finally obtain the action

$$I_{\text{vector}} = \frac{1}{8\pi\,\mathrm{i}}\int \mathrm{d}^4x\sqrt{g}\,\mathrm{Tr}\left[\tau\,C|_{\mathcal{A}(\mathcal{V})^2} - \overline{\tau}\,\overline{C}|_{\overline{\mathcal{A}}(\mathcal{V})^2}\right] \tag{4.48}$$

which is equivalent to (4.45).

Now we want to couple the gauge theory to the Janus interface defined by the coupling multiplet $\mathcal{T}$. The hypermultiplet does not have a explicit coupling with τ, so its interaction with the Janus interface enters indirectly via the coupling with the vector multiplet. Thus we can concentrate on the vector multiplet. To couple the Janus interface and the vector multiplet we obtain the product chiral multiplet $\mathcal{T}\mathcal{A}(\mathcal{V})^2$ from $\mathcal{T}$ and $\mathcal{A}(\mathcal{V})^2$ via tensor calculus. Then integrating out the top component of $\mathcal{A}(\mathcal{V})^2$ and its antichiral counterpart gives the action in the presence of the Janus interface as

$$I_{\text{Janus}} = \frac{1}{8\pi\,\mathrm{i}} \int \mathrm{d}^4x \sqrt{g}\, \mathrm{Tr}\left[C|_{\mathcal{T}\mathcal{A}(\mathcal{V})^2} - \overline{C}|_{\overline{\mathcal{T}}\,\overline{\mathcal{A}}(\mathcal{V})^2} \right] \,. \tag{4.49}$$

Note that if we take $\tau(x)$ as a constant function, then this action reduces to the action in the absence of the Janus interface (4.45).

Chapter 5

SUSY localization and Janus interface entropy

In this chapter we calculate the sphere partition function in the presence of the Janus interface via SUSY localization. By combining this result and the formula (3.42) we show that the entropy of the Janus interface in a $4d\ \mathcal{N}=2$ SCFT is proportional to Calabi's diastasis. We also have a result for the Janus interface in a $4d\ \mathcal{N}=4$ super-Yang-Mills (SYM) via the AdS/CFT correspondence which gives a non-trivial check of our fourmula (3.42).

The organization of this chapter is as follows. In Section 5.1 we review the main tool for our calculation, SUSY localization. In Section 5.2 we evaluate the partition function in the presence of the Janus interface by using SUSY localization. By substituting this result into the formula (3.42), we obtain the relation between entropy of the Janus interface and Calabi's diastasis in Section 5.3. In Section 5.4 we give another calculation via the AdS/CFT correspondence.

5.1 SUSY Localization

In general to perform a path integral exactly is very hard. However, in some supersymmetric QFTs we can use a technique called the SUSY localization to perform a path integral. Let us explain this technique in this section.

Consider a supersymmetric quantum field theory with an action I_{phys}. Assume that the action is invariant under SUSY transformation: $\delta I_{\text{phys}} = 0$.

The partition function is defined by

$$Z = \int [\mathrm{d}\Phi] e^{-I_{\text{phys}}[\Phi]}, \tag{5.1}$$

where we formally denote fields in this theory by Φ. Then we deform the original action as

$$Z(\mathrm{t}) = \int [\mathrm{d}\Phi] e^{-I_{\text{phys}} - \mathrm{t}\delta V}, \tag{5.2}$$

where t is a real deformation parameter and δV is the SUSY variation of a functional V. We can show that $Z(\mathrm{t})$ does not depend on t. So instead of evaluating the original path integral $Z = Z(0)$, we can evaluate it in $\mathrm{t} \to \infty$ limit:

$$Z(0) = Z(\mathrm{t}) = \lim_{\mathrm{t}\to\infty} Z(\mathrm{t}). \tag{5.3}$$

By taking $\mathrm{t} \to \infty$ we can reduce infinite dimensional integrals reduce to finite dimensional ones. This technique is called the SUSY localization.

Let us explain the result in a $4d$ $\mathcal{N} = 2$ gauge theory [39]. SUSY localization reduces a path integral to the discrete sum over saddle points of δV. There are smooth saddle points and point-like (anti)instanton configurations localized at a north pole ($x = 0$) and a south pole ($x = \infty$). Smooth saddle points are parametrized by a continuous variable $a \in \mathrm{Lie}\, G$, while point-like (anti)instanton configurations are parametrized by two integers $k, \overline{k}$. The resulting integral has the form

$$Z = \int [\mathrm{d}a] e^{-I_{\text{cl}}(a)} Z_{\text{1-loop}}(a) Z_{\text{inst}}(a, q) Z_{\text{inst}}(a, \overline{q}), \tag{5.4}$$

where

$$Z_{\text{inst}}(a, q) = \sum_k q^k Z_k(a), \quad Z_{\text{inst}}(a, q) = \sum_{\overline{k}} \overline{q}^{\overline{k}} Z_{\overline{k}}(a), \tag{5.5}$$

and $q = e^{2\pi \mathrm{i}\tau}, \overline{q} = e^{-2\pi \mathrm{i}\tau}$. The first factor in (5.4) $e^{-I_{\text{cl}}(a)}$ is an action evaluated at a saddle point which we denote as the classical action. The factor $Z_{\text{1-loop}}(a)$ is a contribution from fluctuations around the localization locus. Note that this factor is determined only by δV. Finally Z_{inst} are contributions from instanton/anti-instanton configurations.

5.2 Janus partition function

In this section we compute the partition function in the presence of the Janus interface via SUSY localization. The result for the partition function in the absence of the Janus interface is reviewed in the previous section. Now let us consider effects by introducing the Janus interface. Since the one-loop determinant depends only on δV, it is not affected by introducing the Janus interface. So we can focus on the classical action and the instanton partition function.

5.2.1 Classical action

Let us evaluate the classical action in the presence of the Janus interface. We denote the vector multiplet evaluated on the localization locus by $\mathcal{V}_{\mathrm{cl}}$. The values of fields in $\mathcal{V}_{\mathrm{cl}}$ are given by

$$A|_{\mathcal{A}(\mathcal{V}_{\mathrm{cl}})} = X \ , \tag{5.6}$$

$$B_{ij}|_{\mathcal{A}(\mathcal{V}_{\mathrm{cl}})} = -\frac{2\,\mathrm{i}\,e^{\mathrm{i}\beta}X}{r}\,\vec{n}\cdot\vec{\tau}_{ij} \ , \tag{5.7}$$

$$C|_{\mathcal{A}(\mathcal{V}_{\mathrm{cl}})} = \frac{4\,e^{2\mathrm{i}\beta}\,X}{r^2} \ , \tag{5.8}$$

where the value of scalar component X is a constant. From this multiplet, we can compute the squared chiral multiplet $\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2$ via tensor calculus as

$$A|_{\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2} = X^2 \ , \tag{5.9}$$

$$B_{ij}|_{\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2} = -\frac{4\,\mathrm{i}\,e^{\mathrm{i}\beta}\,X^2}{r}\,\vec{n}\cdot\vec{\tau}_{ij} \ , \tag{5.10}$$

$$C|_{\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2} = \frac{12\,e^{2\mathrm{i}\beta}\,X^2}{r^2} \ . \tag{5.11}$$

Then we can couple them to the Janus interface as explained in Section 4.3 to evaluate the classical action. The top component of $\mathcal{T}\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2$ is given by

$$C|_{\mathcal{T}\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2} = \frac{12\,e^{2\mathrm{i}\beta}\,X^2\,\tau(x)}{r^2} + X^2\,C^{(\tau)} + \frac{2\,\mathrm{i}\,e^{\mathrm{i}\beta}\,X^2}{r}\,\vec{n}\cdot\vec{\tau}^{\,ij}\,B^{(\tau)}_{ij} \tag{5.12}$$

$$= e^{2\mathrm{i}\beta}X^2\left[\frac{12}{r^2}\,\tau(x) + q^{(1)}(x)\,\tau'(x) + q^{(2)}(x)\,\tau''(x)\right] \ , \tag{5.13}$$

where

$$q^{(1)}(x) = -\frac{8\,r^2}{x^3 f(x)^2} - \frac{16}{x f(x)}\ , \qquad q^{(2)} = \frac{8\,r^2}{x^2 f(x)^2}\ . \tag{5.14}$$

Thus the chiral part of the classical action is evaluated as

$$\int \mathrm{d}^4x\,\sqrt{g}\,C|_{\mathcal{T}\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2} = 2\pi^2 \int_0^\infty \mathrm{d}x\,x^3\,f^4\,C|_{\mathcal{T}\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2} \tag{5.15}$$

$$= 32\pi^2\,e^{2\mathrm{i}\beta}\,X^2\,r^2\,\tau(0)\ . \tag{5.16}$$

The important point is that the final result only depends on the value of $\tau(x)$ at a north pole $x = 0$.

On the other hand, the anti-chiral multiplet from $\mathcal{V}_{\mathrm{cl}}$ is given as

$$\overline{A}|_{\overline{\mathcal{A}}(\mathcal{V}_{\mathrm{cl}})} = \overline{X}\ , \tag{5.17}$$

$$B^{ij}|_{\overline{\mathcal{A}}(\mathcal{V}_{\mathrm{cl}})} = -\frac{2\,\mathrm{i}\,e^{-\mathrm{i}\beta}\,\overline{X}}{r}\,\vec{n}\cdot\vec{\tau}^{ij}\ , \tag{5.18}$$

$$\overline{C}|_{\overline{\mathcal{A}}(\mathcal{V}_{\mathrm{cl}})} = \frac{4\,e^{-2\mathrm{i}\beta}\,\overline{X}}{r^2}\ . \tag{5.19}$$

By using tensor calculus we obtain

$$\int \mathrm{d}^4x\,\sqrt{g}\,C|_{\overline{\mathcal{T}\mathcal{A}}(\mathcal{V}_{\mathrm{cl}})^2} = 32\pi^2 e^{-2\mathrm{i}\beta}\,\overline{X}^2 r^2\,\tau(\infty)\ . \tag{5.20}$$

This is again written only by the value of the coupling field $\tau(x)$ at a south pole.

The auxiliary fields in the chiral and anti-chiral multiplet from a same vector multiplet are given by a same auxiliary field $\vec{Y}$ as

$$B_{ij} = \vec{Y}\cdot\vec{\tau}_{ij}, \quad B^{ij} = \vec{Y}\cdot\vec{\tau}^{ij}. \tag{5.21}$$

So by comparing (5.10) and (5.18) we can write $X, \overline{X}$ as

$$X = \frac{1}{2}e^{-\mathrm{i}\beta}\,a\ , \qquad \overline{X} = \frac{1}{2}e^{\mathrm{i}\beta}\,a, \tag{5.22}$$

with a real constant a. The normalization for a is chosen to be consistent with [39].

Finally the classical action is given by a sum of the chiral term (5.16) and the antichiral term (5.20) with the relation (5.22):

$$I_{\text{cl}}^{(\text{Janus})} = -\mathrm{i}\,\pi\, r^2\,(\tau_+ - \overline{\tau}_-)\,\mathrm{Tr}\,a^2\ , \tag{5.23}$$

where $\tau_+ := \tau(0)$ and $\tau_- := \tau(\infty)$. This classical action in the presence of the Janus interface can be obtained by analytically continuing moduli parameters $(\tau, \overline{\tau}) \to (\tau_+, \overline{\tau}_-)$ in the classical action in the absence of the Janus interface:

$$I_{\text{cl}}^{(\text{Janus})} = I_{\text{cl}}(\tau_+, \overline{\tau}_-)\ . \tag{5.24}$$

5.2.2 Instanton and anti-instanton parition functions

Next we consider the instanton partition function. The instanton/anti-instanton partition functions in the absence of the Janus interface are given by (5.5). Remind that they arise from the contribution from instantons localized at a south and north pole. So an effect by introducing the Janus interface is to modify the parameters $q, \overline{q}$ to the values of them at each poles: $q \to q_+ = e^{2\pi \mathrm{i}\tau_+}$ and $\overline{q} \to \overline{q}_- = e^{-2\pi\mathrm{i}\overline{\tau}_-}$. So the instanton partition function in the presence of the Janus interface are given by $Z_{\text{inst}}(a, q_+) Z_{\text{inst}}(a, \overline{q}_-)$. This result is also obtained by analytically continuing $(\tau, \overline{\tau}) \to (\tau_+, \overline{\tau}_-)$ in the result without the Janus interface similar as the classical action (5.24).

5.2.3 Partition function with the Janus interface

In summary the classical action and the instanton partition functions are obtained by an analytic continuation $(\tau, \overline{\tau}) \to (\tau_+, \overline{\tau}_-)$ in the results without the Janus interface, while the one-loop determinant is same as one in the absence of the Janus interface. Thus the resulting partition function on $\mathbb{S}^4$ in the presence of the Janus interface can be obtained by the same analytic continuation. On the other hand, the partition function of a $4d\ \mathcal{N} = 2$ SCFT on $\mathbb{S}^4$ can be written by the Kähler potential on moduli space:

$$Z_{\text{SUSY}}[\mathbb{S}^4](\tau, \overline{\tau}) = e^{K(\tau, \overline{\tau})/12}\ , \tag{5.25}$$

as explained in Section 2.3.3. So by analytically continuing this expression we finally obtain the partition function with the Janus interface on $\mathbb{S}^4$ as

$$Z_{\text{SUSY}}^{\mathcal{I}}[\mathbb{S}^4] = e^{K(\tau_+, \overline{\tau}_-)/12}\ . \tag{5.26}$$

5.2.4 Supergravity conterterm and Kähler ambiguity

As explained in Section 2.3.3, different UV regularization schemes cause the Kähler transformations

$$K(\tau,\overline{\tau}) \to K(\tau,\overline{\tau}) + \mathcal{F}(\tau) + \overline{\mathcal{F}}(\overline{\tau}), \tag{5.27}$$

and they are explained by supergravity counterterms [26, 29] in the absence of the Janus interface. In this subsection we evaluate the supergravity counterterm which causes Kähler transformations in the presence of the Janus interface.

We can put a $4d\,\mathcal{N}=2$ SCFT on $\mathbb{S}^4$ by coupling it to an off-shell Poincaré supergravity. An off-shell Poincaré supergravity can be obtained by gauge fixing an off-shell conformal supergravity using the so called compensating vector multiplet. We construct a SUSY invariant supergravity counterterm from the compensating vector multiplet according to [29]. The compensating vector multiplet on $\mathbb{S}^4$ which we denote by $\mathcal{V}_\mathrm{c}$ is given by

$$X|_{\mathcal{V}_\mathrm{c}} = \mu e^{-\mathrm{i}\beta}\ , \qquad Y_{ij}|_{\mathcal{V}_\mathrm{c}} = -\frac{2\,\mathrm{i}\,\mu}{r}\,(\vec{n}\cdot\vec{\tau})_{ij}\ , \qquad \Omega_i|_{\mathcal{V}_\mathrm{c}} = F^-_{\mu\nu}|_{\mathcal{V}_\mathrm{c}} = 0\ , \tag{5.28}$$

$$\overline{X}|_{\mathcal{V}_\mathrm{c}} = \mu e^{+\mathrm{i}\beta}\ , \qquad Y^{ij}|_{\mathcal{V}_\mathrm{c}} = -\frac{2\,\mathrm{i}\,\mu}{r}\,(\vec{n}\cdot\vec{\tau})^{ij}\ , \qquad \overline{\Omega}^i|_{\mathcal{V}_\mathrm{c}} = F^+_{\mu\nu}|_{\mathcal{V}_\mathrm{c}} = 0\ , \tag{5.29}$$

where $\mu > 0$ is an arbitrary mass scale. This vector multiplet can be embedded into the anti-chiral multiplet $\overline{\Phi} := \overline{\mathcal{A}}(\mathcal{V}_\mathrm{c})$ with Weyl weight one. We can further construct a chiral multiplet $\mathbb{T}(\log\overline{\Phi})$ with Weyl weight two from $\overline{\Phi}$. The definition of $\mathbb{T}(\log\overline{\Phi})$ is given in Appendix A.5. In this case its components are evaluated as

$$A|_{\mathbb{T}(\log\overline{\Phi})} = \frac{2\,e^{-2\mathrm{i}\beta}}{r^2}, \tag{5.30}$$

$$B_{ij}|_{\mathbb{T}(\log\overline{\Phi})} = -\frac{8\,\mathrm{i}\,e^{-\mathrm{i}\beta}}{r^3}\,(\vec{n}\cdot\vec{\tau})_{ij}, \tag{5.31}$$

$$C|_{\mathbb{T}(\log\overline{\Phi})} = \frac{24}{r^4}. \tag{5.32}$$

Next we compute the chiral multiplet $\mathcal{F}(\mathcal{T})$ for an arbitrary holomorphic function $\mathcal{F}(\cdot)$ via the tensor calculus explained in Appendix A.4. Its components are evaluated as

$$A|_{\mathcal{F}(\mathcal{T})} = \mathcal{F}(\tau) \ , \tag{5.33}$$

$$B_{ij}|_{\mathcal{F}(\mathcal{T})} = \frac{\mathrm{d}\mathcal{F}(\tau)}{\mathrm{d}x} \frac{\mathrm{i}\, r\, e^{\mathrm{i}\beta}}{x f(x)} (\vec{n}\cdot\vec{\tau})_{ij} \ , \tag{5.34}$$

$$C|_{\mathcal{F}(\mathcal{T})} = \frac{8\, r^2\, e^{2\mathrm{i}\beta}}{x^2 f^2} \left(\frac{\mathrm{d}^2\mathcal{F}(\tau)}{\mathrm{d}x^2} - \frac{1}{x}\frac{\mathrm{d}\mathcal{F}(\tau)}{\mathrm{d}x} \right) \ . \tag{5.35}$$

According to [29] we compute the SUSY invariant counterterm from the top component of $\mathcal{F}(\mathcal{T})\mathbb{T}(\log\overline{\Phi})$. Note that the components of $\mathcal{F}(\mathcal{T})$ are obtained by replacing τ with $\mathcal{F}(\tau)$ in those of $\mathcal{T}$ given in (4.30). Similarly the components of $\mathbb{T}(\log\overline{\Phi})$ are obtained by replacing X^2 with $\frac{2e^{-2\mathrm{i}\beta}}{r^2}$ in those of $\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2$ given in (5.9), (5.10), and (5.11). Therefore the top component of $\mathcal{F}(\mathcal{T})\mathbb{T}(\log\overline{\Phi})$ can be obtained from $C|_{\mathcal{T}\mathcal{A}(\mathcal{V}_{\mathrm{cl}})^2}$ given in (5.13) by the same substitutions:

$$C\big|_{\mathcal{F}(\mathcal{T})\mathbb{T}(\log\overline{\Phi})} = \frac{2}{r^2}\left[\frac{2}{r^2}\mathcal{F}(\tau) + q^{(1)}(x)\,\frac{\mathrm{d}\mathcal{F}(\tau)}{\mathrm{d}x} + q^{(2)}(x)\,\frac{\mathrm{d}^2\mathcal{F}(\tau)}{\mathrm{d}x^2} \right] \ . \tag{5.36}$$

Thus the chiral part of SUSY invariant supergravity counterterm is given by

$$\int \mathrm{d}^4x\,\sqrt{g}\, C|_{\mathcal{F}(\mathcal{T})\mathbb{T}(\log\overline{\Phi})} = 64\pi^2\,\mathcal{F}(\tau_+) \ . \tag{5.37}$$

Similarly we can compute the anti-chiral counterterm constructed from the anti-chiral coupling multiplet $\overline{\mathcal{T}}$ and the compensating vector multiplet $\mathcal{V}_{\mathrm{c}}$:

$$\int \mathrm{d}^4x\,\sqrt{g}\,\overline{C}|_{\overline{\mathcal{F}}(\overline{\mathcal{T}})\mathbb{T}(\log\Phi)} = 64\pi^2\,\overline{\mathcal{F}}(\overline{\tau}_-) \ , \tag{5.38}$$

where the anti-holomorphic $\overline{\mathcal{F}}(\overline{\tau})$ is the complex conjugate of the holomorphic function $\mathcal{F}(\tau)$ when $\overline{\tau} = \tau^*$. The supergravity conterterm is given by a sum of (5.37) and (5.38). They give rise to the Kähler transformations

$$K(\tau_+,\overline{\tau}_-) \to K(\tau_+,\overline{\tau}_-) + \mathcal{F}(\tau_+) + \overline{\mathcal{F}}(\overline{\tau}_-) \ , \tag{5.39}$$

in the Janus partition function $Z^{\mathcal{I}}_{\mathrm{SUSY}}[\mathbb{S}^4]$ (5.26).

5.3 Entropy of Janus interface

By substituting the Janus partition function (5.26) into the formula (3.42), we finally obtain the entropy of the Janus interface:

$$S_{\mathcal{I}} = -\frac{1}{24}\left[K(\tau_+,\overline{\tau}_+) + K(\tau_-,\overline{\tau}_-) - K(\tau_+,\overline{\tau}_-) - K(\tau_-,\overline{\tau}_+)\right] \ . \qquad (5.40)$$

The linear combination in the bracket is Calabi's diastasis defined in Introduction. The important point is that this result does not have ambiguity due to the Kähler transformations given in (5.27) and (5.39), although the sphere partition functions given in (5.25) and (5.26) have such ambiguity.

5.4 Holographic example

In this section we focus on a theory with more supersymmetries, a $4d\ \mathcal{N}=4$ super Yang-Mills (SYM) theory. A $4d\ \mathcal{N}=4$ SYM theory is conjectured to be a string theory (or its supergravity limit) on a $AdS_5 \times \mathbb{S}^5$ background [40]. This correspondence is called the AdS/CFT correspondence or holography. Although there is no proof for AdS/CFT correspondence, there are many non-trivial results supporting this mysterious conjecture. This correspondence is considered to still hold in the presence of defects. In this section we compute an interface entropy and sphere partition function via holography. It provides a non-trivial check for our formula (3.42).

5.4.1 The Janus solution in supergravity

A $4d\ \mathcal{N}=4$ SYM has 32 supersymmetries and an internal symmetry $SO(6)_R$. We here use the dual supergravity solution [41] to the Janus interface in the Janus interface in a $4d\ \mathcal{N}=4$ SYM. The solution is given by type IIB supergravity on $AdS_4 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \Sigma_2$ where Σ_2 is a two-dimensional Rimeann surface. This solution preserves the $SO(1,3) \times SO(3) \times SO(3)$ subgroup in $SO(2,3) \times SO(6)$ The metric is given by

$$\mathrm{d}s^2 = f_4^2\, \mathrm{d}s^2_{AdS_4} + \rho^2\, \mathrm{d}v \mathrm{d}\overline{v} + f_1^2\, \mathrm{d}s^2_{\mathbb{S}^2} + f_2^2\, \mathrm{d}s^2_{\mathbb{S}^2} \ , \qquad (5.41)$$

where $\mathrm{d}s^2_{\mathbb{S}^2}$ is the metric of a unit 2-sphere and $v = x + \mathrm{i}y$ is a complex coordinate on a strip with $x \in \mathbb{R}$ and $0 \le y \le \pi/2$. The functions f_1, f_2, f_4, ρ

in the above metric are given as

$$
\begin{aligned}
f_4^8 &= 16\,\frac{F_1F_2}{W^2}\ , & \rho^8 &= \frac{2^8F_1F_2W^2}{h_1^4h_2^4}\ ,\\
f_1^8 &= 16h_1^8\,\frac{F_2W^2}{F_1^3}\ , & f_2^8 &= 16h_2^8\,\frac{F_1W^2}{F_2^3}\ ,
\end{aligned} \tag{5.42}
$$

where

$$
F_i = 2h_1h_2|\partial_v h_i|^2 - h_i^2 W \quad (i=1,2)\ , \qquad W = \partial_v\partial_{\overline{v}}(h_1h_2)\ . \tag{5.43}
$$

The two real functions $h_1(v,\overline{v})$ and $h_2(v,\overline{v})$ above are defined as

$$
h_1(v,\overline{v}) = -\mathrm{i}\,\alpha_1 \sinh\left(v - \frac{\Delta\phi}{2}\right) + \text{c.c.}\ , \tag{5.44}
$$

$$
h_2(v,\overline{v}) = \alpha_2 \cosh\left(v + \frac{\Delta\phi}{2}\right) + \text{c.c.}. \tag{5.45}
$$

By taking $x \to \pm\infty$ we recover the dual CFT on the right/left side of the Janus interface in which we assume to have different couplings $g^{\pm}_{\text{YM}}$. The dilaton ϕ in the SUGRA solution takes different values across the interface corresponding to different couplings in the dual CFT, and they are related by

$$
(g^{\pm}_{\text{YM}})^2 = 4\pi\left|\frac{\alpha_2}{\alpha_1}\right| e^{\pm\Delta\phi}, \tag{5.46}
$$

where $\Delta\phi$ is a difference of dilaton values and constants α_1, α_2 are related to AdS radius L as

$$
L^4 = 16|\alpha_1\alpha_2| \cosh\Delta\phi. \tag{5.47}
$$

5.4.2 Sphere free energy

First let us evaluate the free energy on a four-sphere. The free energy is defined as the log of a partition function. This quantity in the current setting is obtained by evaluating corresponding the on-shell supergravity action via the AdS/CFT correspondence as [42] [1]

$$
I = -\frac{3\cdot 2^6\,\text{Vol}(\mathbb{S}^2)^2}{16\pi G_N}\int_{AdS_4} \mathrm{d}^4x\sqrt{g_{(4)}}\int \mathrm{d}x\,\mathrm{d}y\,W h_1 h_2\ , \tag{5.48}
$$

[1] They evaluated this quantity in another setting.

where G_N is the Newton constant in ten dimensions and $g_{(4)}$ is a four dimensional metric of AdS_4. To evaluate this integral, we introduce the coordinates for AdS_4 in which the metric is given by

$$\mathrm{d}s^2_{AdS_4} = \frac{1}{\cos^2\lambda}\left[\mathrm{d}\lambda^2 + \sin^2\lambda\,\mathrm{d}s^2_{\mathbb{S}^3}\right] \ , \tag{5.49}$$

with $0 \le \lambda \le \pi/2$. Then by using these coordinates we can write the on-shell action (5.48) as

$$\begin{aligned} I = &\frac{3\,\mathrm{Vol}(\mathbb{S}^2)^2\,\mathrm{Vol}(\mathbb{S}^3)\,L^8}{2^6\pi G_N}\int_0^{\pi/2}\mathrm{d}\lambda\,\frac{\sin^3\lambda}{\cos^4\lambda} \\ &\times\int_0^{\pi/2}\mathrm{d}y\,\sin^2(2y)\int_{-\infty}^{\infty}\mathrm{d}x\left(1+\frac{\cosh(2x)}{\cosh(\Delta\phi)}\right) \ . \end{aligned} \tag{5.50}$$

This integral is divergent since the AdS space has an infinite volume. So we have to regularize it. In the absence of an interface we can simply regularize the off-shell action by using Fefferman-Graham coordinates. However, in the presence of an interface, it can be difficult to construct Fefferman-Graham coordinates to cover all of the interface. One way to overcome this difficulty is to use the single cut-off regularization [43, 44] [2]. In this approach we introduce the hypersuface defined by

$$\frac{f_4}{Z} = \frac{L}{\delta} \tag{5.51}$$

with $Z := \cos\lambda$ and eliminate the region between this hypersurface and the boundary of the AdS spacetime. In other words, the range of x is restricted to the region between $x_\pm(Z,y)$ which are determined by $f_4(x_\pm, y) = LZ/\delta$ for Z, y fixed, and the range of Z is also restricted from $Z_* := f_4(v=0)\delta/L$ to 1. We expand $x_\pm(Z,y)$ in Z/δ:

$$\begin{aligned} x_\pm(Z,y) = &\pm\frac{1}{2}\log\left(4\cosh(\Delta\phi)\,\frac{Z^2}{\delta^2}\right) \\ &-\frac{\cos(2y)\,\tanh(\Delta\phi)\pm 2}{8}\left(\frac{\delta}{Z}\right)^2 + \mathcal{O}\left(\frac{\delta^4}{Z^4}\right) . \end{aligned} \tag{5.52}$$

[2] Another approach is to use two Fefferman-Graham coordinates and connect them to cover all space [31]. They showed that the universal part of an entanglement entropy does not depend on the detail of the curve connecting two patches. Instead one can introduce double cut-offs. This scheme was studied for the Janus interface in [44].

Thus we can expand the integral over x as

$$\begin{aligned}&\int_{x_-(Z,y)}^{x_+(Z,y)} \mathrm{d}x \left(1+\frac{\cosh(2x)}{\cosh(\Delta\phi)}\right)\\ &= \log\left(4\cosh(\Delta\phi)\,\frac{Z^2}{\delta^2}\right) + 2\frac{Z^2}{\delta^2} - 1 + \mathcal{O}\left(\frac{\delta^2}{Z^2}\right) \ . \end{aligned} \tag{5.53}$$

By substituting this expansion into the on-shell action (5.50), we have

$$\begin{aligned} I = &\frac{3\,\mathrm{Vol}(\mathbb{S}^2)^2\,\mathrm{Vol}(\mathbb{S}^3)\,L^8}{2^6\pi G_N} \int_0^{\pi/2} \mathrm{d}y\, \sin^2(2y) \int_{Z_*}^1 \mathrm{d}Z\, \frac{1-Z^2}{Z^4} \\ &\times \left[\log\left(4\cosh(\Delta\phi)\,\frac{Z^2}{\delta^2}\right) + 2\frac{Z^2}{\delta^2} + 1 + \mathcal{O}\left(\frac{\delta^2}{Z^2}\right)\right] \end{aligned} \tag{5.54}$$

$$\begin{aligned} = &\frac{\mathrm{Vol}(\mathbb{S}^2)^2\,\mathrm{Vol}(\mathbb{S}^3)\,L^8}{2^7 G_N} \\ &\times \left[\frac{c_3}{\delta^3} + \frac{c_2}{\delta^2} + \frac{c_1}{\delta} + \log\left(\frac{4\cosh(\Delta\phi)}{\delta^2}\right) + \frac{5}{3} + \mathcal{O}(\delta^2)\right] \ , \end{aligned} \tag{5.55}$$

where $c_i, (i=1,2,3)$ are constants whose explicit forms are not needed for our discussion here.

To extract a contribution from the Janus interface, we subtract a free energy in the absence of the Janus interface which is obtained by setting $\Delta\phi = 0$ to have

$$\Delta I = I - I|_{\Delta\phi=0} = \frac{\mathrm{Vol}(\mathbb{S}^2)^2\,\mathrm{Vol}(\mathbb{S}^3)\,L^8}{2^7 G_N} \log\cosh(\Delta\phi) \ . \tag{5.56}$$

Note that we use the same regularization scheme for calculations of the free energy with and without the interface. Then we have the universal result, that is, the result which does not depend on the regularization scheme [3].

The free energy evaluated in the AdS space can be mapped to one in the dual CFT. The dictionary between the deference of the value of a dilaton $\Delta\phi$ in the AdS and that of a coupling g_{YM} in the CFT is given in (5.46). Moreover the relation between the Newton constant and the rank of the gauge theory is given by

$$G_N = \frac{\mathrm{Vol}(\mathbb{S}^2)^2\,\mathrm{Vol}(\mathbb{S}^3)\,L^8}{2^6 N^2}. \tag{5.57}$$

[3] Remind that when we define an interface entropy as (3.30) we subtract the vacuum (*i.e.* no interface) contribution from the net entanglement entropy.

By using these relations, we can translate ΔI obtained from supergravity (5.56) into the CFT language as

$$\Delta I = \frac{N^2}{2} \log \left[1 + \frac{(g_{\rm YM}^+ - g_{\rm YM}^-)^2}{2 g_{\rm YM}^+ g_{\rm YM}^-} \right] . \tag{5.58}$$

So far we considered the Janus interface across which only a coupling $g_{\rm YM}$ changes its value whereas a theta angle ϑ does not. The result for the Janus interface changing a theta angle is obtained by acting $SL(2,\mathbb{R})$ transformations of type IIB supergravity on the previous result as

$$\Delta I = \frac{1}{24} \left[K(\tau_+, \overline{\tau}_+) + K(\tau_-, \overline{\tau}_-) - K(\tau_+, \overline{\tau}_-) - K(\tau_-, \overline{\tau}_+) \right] , \tag{5.59}$$

where K is the Kähler potential given by

$$K(\tau, \overline{\tau}) = -6 N^2 \log \left[{\rm i} \left(\overline{\tau} - \tau \right) \right] . \tag{5.60}$$

5.4.3 Entanglement entropy

In this section we compute an entanglement entropy across a spherical entangling surface in the presence of the Janus interface via the AdS/CFT correspondence. This computation was done by using the Fefferman-Graham regularization in [31] and the single and double cut-off regularization in [44]. We here review the computation using single cut-off regularization [44] and compare it to the result in the previous subsection.

An entanglement entropy in a CFT in a holographic setup can be calculated by the Ryu-Takayanagi (RT) formula [45] [4]. The RT formula connects an entanglement entropy S_Σ across an entangling surface Σ in a CFT with the geometrical quantity in the AdS space, the area of the minimal surface anchored on the entangling surface Σ as

$$S_\Sigma = \min_{\gamma_\Sigma} \frac{\text{Area}(\gamma_\Sigma)}{4 G_N} , \tag{5.61}$$

where γ_Σ is a surface anchored on Σ. So to compute an entanglement entropy we first write down the area of a surface γ_Σ and then minimize it. For

[4] The formula for a CFT with a boundary was first proposed in [46].

this purpose, we here move to the Lorentzian signature and introduce the Poincaré patch for AdS_4 in which the metric is given by

$$\mathrm{d}s^2_{AdS_4} = \frac{1}{z^2}\left[\mathrm{d}z^2 - \mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2\right] \ . \tag{5.62}$$

In these coordinates, the spherical entangling surface we have considered is given by

$$\Sigma = \{t = 0,\, r = R,\, z = 0\} \ . \tag{5.63}$$

Then the area of a surface γ_Σ in the coordinates is given by [45, 47]

$$S = \frac{\mathrm{Vol}(\mathbb{S}^2)^2\,\mathrm{Vol}(\mathbb{S}^1)}{4G_N}\left(\int \mathrm{d}x\,\mathrm{d}y\,(f_1\,f_2\,f_4\,\rho)^2\right)\int \mathrm{d}z\,\frac{r}{z^2}\sqrt{1+(\partial_z r)^2}. \tag{5.64}$$

It is specified by the function $r(z)$. Note that $r(z)$ does not depend on (t, ϕ) because of a spherical symmetry.

Minimizing (5.64) gives the equation of motion. By solving this equation of motion we have [48]

$$r = \sqrt{R^2 - z^2} \ . \tag{5.65}$$

Similar as in the previous subsection, the integral (5.64) has divergence and we need to regularize it. To regulate this divergence due to an infinite volume of the AdS spacetime we cut it at a hypersurface specified by

$$\frac{f_4}{z} = \frac{L}{\varepsilon}. \tag{5.66}$$

Then the integration range for x is restricted to $x_-(z, y) \le x \le x_+(z, y)$. The values of $x_\pm(z, y)$ are obtained by replacing (Z, δ) with (z, ε) in (5.52). The z integral is also restricted to $z_* \equiv f_4(0)\varepsilon/L \le z \le R$. Then evaluating the regularized minimal area leads to

$$\begin{aligned} S = \frac{2^4\pi\,\mathrm{Vol}(\mathbb{S}^2)^2\,L^8}{2^4 G_N}&\int_0^{\pi/2}\mathrm{d}y\,\sin^2(2y) \\ &\times\int_{z_*/R}^{1}\frac{\mathrm{d}z}{z^2}\int_{x_-(z,y)}^{x_+(z,y)}\mathrm{d}x\left(1+\frac{\cosh(2x)}{\cosh(\Delta\phi)}\right) \ . \end{aligned} \tag{5.67}$$

Similar as the calculation of the free energy we extract a contribution from the Janus interface by subtracting an ambient contribution:

$$S_{\mathcal{I}}|_{\text{univ}} = -\frac{N^2}{2} \log\cosh(\Delta\phi). \tag{5.68}$$

This again gives the universal part of the entanglement entropy. Note that this result agrees with the result obtained from another regularization [31].

Thus we show via holography that the minus of the interface entropy (5.68) is equal to the contribution to the free energy from the interface (5.56). This is consistent with our formula (3.42) derived from the CFT consideration.

Chapter 6

Conclusion and outlook

In this thesis we study the entropy of the Janus interface in a $4d\,\mathcal{N}=2$ SCFT and show that it can be written by Calabi's diastasis. The former half of this thesis is on a general conformal/superconformal interface. We define an entropy of an interface as its contribution to an entanglement entropy with a spherical entangling surface. Then we derive the formula (3.42) relating the interface entropy to the sphere partition function in the presence of a half-BPS superconformal interface via the CHM map. In the intermediate step we make the assumption (3.41). We do not have a proof for it, but we give some evidences:

- It is natural in terms of the supersymmetric Rényi entropy. See Section 3.3.3.
- The results in a holographic example are consistent with the resulting formula (3.42). See Section 5.4.
- We provide a proof for the similar relation (C.8) in a $2d\ \mathcal{N}=(2,2)$ SCFT by using boundary super-Weyl anomaly [13]. See Appendix C.

The latter half of this thesis we focus on the Janus interface. We give an off-shell construction of the Janus interface by introducing the coupling multiplet. We then evaluate the sphere partition function in the presence of the Janus interface. By combining this result and the formula (3.42) we show that the Janus interface entropy is given by Calabi's diastasis.

There are interesting open problems. We list some of them:

- It is very important and interesting to prove (3.41). One possible way to achieve this is to use similar argument in $2d$ by extending the results for the boundary super-Weyl anomaly [13] to $4d$.
- It is interesting to show that the contribution to the conformal anomaly from an interface indeed vanishes in the right hand side of (3.42). See Section 3.2.3.
- It can be possible to construct the Janus interface in a theory with other dimensions and SUSY. It is nice to study its entropy and investigate whether it can be written as Calabi's diastasis.
- General properties of Calabi's diastasis is interesting to study. Can we use it to investigate the structure of conformal manifold?
- General properties of an interface entropy is also interesting. Can we confirm whether it is C-function or not? Can we get interesting constraints on ICFTs from it?

Ultimately we want to understand general properties and structure of the space of conformal/quantum field theories. We hope that study of interface entropies can be a strong tool for this purpose.

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Appendix A

Supersymmetry and supergravity

A.1 Notations and conventions

Here We mostly use conventions and notations used in [49, 25]. The imaginary unit is denoted by i. Complex conjugation is indecated by $*$ while hermitian conjugation by $\dagger$.

A.1.1 Signature and coordinate index

We denote coordinates indices on a general manifold as $\mu, \nu, \cdots$ while coordinates indices on a flat space as $a, b, \cdots$. Indices run within $0, 1, 2, 3$ in the Lorentzian signature, while $1, 2, 3, 4$ in the Euclidean signature. A flat space metric is given by $\eta^{ab} = \mathrm{diag}(-1, +1, +1, +1)$ in the Lorentzian signature and $\eta^{ab} = \mathrm{diag}(+1, +1, +1, +1)$ in the Euclidean signature. The vielbein and its inverse are denoted by ${e_\mu}^a$ and ${e_a}^\mu$ respectively.

A.1.2 Gamma matrix

The gamma matrices γ^μ with a Greek alphabet satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ , \tag{A.1}$$

while the gamma matrices γ^μ with a Latin alphabet satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \ . \tag{A.2}$$

We sometimes denote γ^a as Γ^a. We can take a specific matrix representation to have more explicit expressions. For example, in the Weyl representation, γ^a are given by

$$\gamma^a = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \tag{A.3}$$

where $\sigma^\mu = (\sigma^1, \sigma^2, \sigma^3, \mathrm{i}), \overline{\sigma}^a = (\sigma^1, \sigma^2, \sigma^3, -\mathrm{i})$, and $\sigma^i\,(i = 1, 2, 3)$ are Pauli matrices.

The gamma matrices with Greek and Latin indices are related by vielbein as

$$\gamma^\mu = \gamma^a e_a{}^\mu \ . \tag{A.4}$$

The gamma matrix γ^a is hermitian if $a = 0$ and anti-hermitian otherwise. The gamma matrices in the Lorentzian signature and in the Euclidean signature are related by

$$\gamma^{a=0} = -\mathrm{i}\gamma^{a=4}, \tag{A.5}$$

and the other components are same.

We define the chirality matrix by $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_1\gamma_2\gamma_3\gamma_4$. By using the matrix γ_*, we define chiral projection operators as

$$P_L = \frac{1}{2}(1 + \gamma_*) \ , \qquad P_R = \frac{1}{2}(1 - \gamma_*) \ . \tag{A.6}$$

A.1.3 Charge and Weyl conjugation

The charge conjugate matrix C satisfies

$$CC^\dagger = 1 \quad C^T = -C \ , \quad C\gamma_\mu C^{-1} = -\gamma_\mu^T \ . \tag{A.7}$$

Then the charge conjugation of a four-component spinor Ψ which we denote as Ψ^C is defined by

$$\Psi^C = B^{-1}\Psi^* \ , \tag{A.8}$$

where $B = \mathrm{i}\,C\gamma^0$. By definition the matrix B satisfies the relation

$$B^{-1}(\gamma^\mu)^* B = \gamma^\mu \ . \tag{A.9}$$

We list some properties of the charge conjugation:

$$(\Psi^C)^C = \Psi, \quad (\gamma_{\mu_1} \ldots \gamma_{\mu_N}\Psi)^C = \gamma_{\mu_1} \ldots \gamma_{\mu_N}\Psi^C, \tag{A.10}$$

and

$$(\bar{\epsilon}\gamma_{\mu_1}\dots\gamma_{\mu_N}\eta)^* = \pm\overline{\epsilon^C}\gamma_{\mu_1}\dots\gamma_{\mu_N}\eta^C \ . \tag{A.11}$$

We denote the Weyl conjugate of Ψ as $\overline{\Psi}$ which is defined by

$$\overline{\Psi} := \Psi^T C \ . \tag{A.12}$$

Note that this is equivalent to the Dirac conjugate when the spinor satisfies the certain condition. See Appendix A.2.

A.1.4 $SU(2)_R$ multiplets

We denote $SU(2)_R$ doublet indices by $i, j, \dots$. We can regard $SU(2)_R$ triplet as a tree-component vector $\vec{Y}$. We can construct the 2×2 matrix $Y_i{}^j$ from $\vec{Y}$ as

$$Y_i{}^j = \vec{\tau}_i{}^j \cdot \vec{Y} \ , \tag{A.13}$$

where $\vec{\tau}_i{}^j = \mathrm{i}\,\vec{\sigma}_i{}^j$. $SU(2)_R$ indices are raised or lowered by anti-symmetric tensors ε^{ij} and ε_{ij} which satisfy $\varepsilon^{12} = \varepsilon_{12} = 1$ as

$$\vec{\tau}^{ij} = \varepsilon^{ik}\vec{\tau}_k{}^j = (\vec{\tau}_{ij})^* = \varepsilon^{ik}\varepsilon^{jl}\vec{\tau}_{kl} \ . \tag{A.14}$$

By using τ^{ij} we can also construct a symmetric matrix Y^{ij} from $\vec{Y}$ as

$$Y^{ij} = \vec{\tau}^{ij} \cdot \vec{Y} \ . \tag{A.15}$$

A.2 Supersymmetry parameters

The Poincaré SUSY parameters ϵ^i, ϵ_i in the Lorentzian signature satisfy

$$(\epsilon^i)^C = \epsilon_i \ . \tag{A.16}$$

For spinors satisfying the above condition, the Weyl conjugate is equivalent to the Dirac conjugate:

$$\overline{\epsilon}^i = (\epsilon_i)^\dagger \mathrm{i}\gamma^0 \ . \tag{A.17}$$

The special conformal SUSY parameters η^i, η_i satisfy the same condition as

$$(\eta^i)^C = \eta_i \ . \tag{A.18}$$

According to [25], these parameters are chiral in the Lorentzian and Euclidean signature :

$$\epsilon^i = P_L\epsilon^i \ , \quad \epsilon_i = P_R\epsilon_i \ , \quad \eta^i = P_R\eta^i \ , \quad \eta_i = P_L\eta_i \ . \tag{A.19}$$

A.3 $\mathcal{N} = 2$ supermultiplets

In the rest of Appendix A, we assume that the values of the fields in the Weyl multiplet vanish except for a metric and vielbein. We also assume that the SUSY parameters satisfy the following conditions [29]:

$$\eta^i = \frac{1}{4}\gamma^\mu \nabla_\mu \epsilon^i, \quad \eta_i = \frac{1}{4}\gamma^\mu \nabla_\mu \epsilon_i. \tag{A.20}$$

Under these assumptions, we give the SUSY transformations of a vector and chiral multiplet.

A.3.1 Vector multiplet

A vector multiplet has $(X, \Omega_i, A_\mu, Y_{ij})$ as its components. The spinor Ω_i is left-chiral and its conjugate Ω_i is right-chiral. They are related by $(\Omega_i^I)^C = \Omega^{Ii}$ in Minkowski space. We expand fields in terms of hermitian generators T_I as $X = T_I X^I$, $A_\mu = T_I A^I_\mu$, and so on. Then we give the SUSY transformations for them as [49]

$$\delta X^I = \frac{1}{2}\,\bar{\epsilon}^i \Omega^I_i, \tag{A.21}$$

$$\delta\Omega^I_i = \not{D} X^I \epsilon_i + \frac{1}{4}\,\gamma^{\mu\nu} F^I_{\mu\nu}\,\varepsilon_{ij}\epsilon^j + \frac{1}{2}\,Y^I_{ij}\,\epsilon^j + X^J \overline{X}^K f_{JK}{}^I \varepsilon_{ij}\,\epsilon^j + 2X^I \eta_i\ , \tag{A.22}$$

$$\delta A^I_\mu = \frac{1}{2}\,\varepsilon^{ij}\,\bar{\epsilon}_i \gamma_\mu \Omega^I_j + \frac{1}{2}\,\varepsilon_{ij}\,\bar{\epsilon}^i \gamma_\mu \Omega^{jI}\ , \tag{A.23}$$

$$\delta \vec{Y}^I = \frac{1}{2}\vec{\tau}^{\,ij}\,\bar{\epsilon}_i \not{D}\Omega^I_j - f_{JK}{}^I\,\vec{\tau}_i{}^j\,\bar{\epsilon}_j\,X^J \Omega^{iK} + \text{h.c.}\ . \tag{A.24}$$

A.3.2 Chiral multiplet

A chiral multiplet $\mathcal{A}$ has $(A, \Psi_i, B_{ij}, F^-_{ab}, \Lambda_i, C)$ as its components. Their SUSY transformations are [50, 49]

$$\delta A = \frac{1}{2}\,\overline{\epsilon}^i \Psi_i \ , \tag{A.25}$$

$$\delta \Psi_i = \nabla\!\!\!\!/ (A\,\epsilon_i) + \frac{1}{2}\,B_{ij}\,\epsilon^j + \frac{1}{4}\,\Gamma^{ab} F^-_{ab}\,\varepsilon_{ij}\,\epsilon^j + (2w-4)\,A\,\eta_i \ , \tag{A.26}$$

$$\delta B_{ij} = \overline{\epsilon}_{(i}\,\nabla\!\!\!\!/\,\Psi_{j)} - \overline{\epsilon}^k\,\Lambda_{(i}\,\varepsilon_{j)k} + 2(1-w)\,\overline{\eta}_{(i}\,\Psi_{j)} \ , \tag{A.27}$$

$$\delta F^-_{ab} = \frac{1}{4}\,\varepsilon^{ij}\,\overline{\epsilon}_i\,\nabla\!\!\!\!/\,\Gamma_{ab}\,\Psi_j + \frac{1}{4}\,\overline{\epsilon}^i\,\Gamma_{ab}\Lambda_i - \frac{1}{2}\,(1+w)\,\varepsilon^{ij}\,\overline{\eta}_i\,\Gamma_{ab}\Psi_j \ , \tag{A.28}$$

$$\begin{aligned}\delta \Lambda_i = &-\frac{1}{4}\,\Gamma^{ab}\,\nabla\!\!\!\!/ (F^-_{ab}\,\epsilon_i) - \frac{1}{2}\,\nabla\!\!\!\!/\,B_{ij}\,\varepsilon^{jk}\,\epsilon_k + \frac{1}{2}\,C\varepsilon_{ij}\,\epsilon^j \\ &- (1+w)\,B_{ij}\,\varepsilon^{jk}\,\eta_k + \frac{1}{2}\,(3-w)\,\Gamma^{ab} F^-_{ab}\,\eta_i \ ,\end{aligned} \tag{A.29}$$

$$\delta C = -\nabla_\mu(\varepsilon^{ij}\,\overline{\epsilon}_i\,\gamma^m \Lambda_j) + (2w-4)\,\varepsilon^{ij}\,\overline{\eta}_i\,\Lambda_j \ , \tag{A.30}$$

where w is the Weyl weight of the multiplet.

An anti-chiral multiplet $\overline{\mathcal{A}}$ has $(\overline{A}, \Psi^i, B^{ij}, F^+_{ab}, \Lambda^i, \overline{C})$ as its components. The SUSY transformations in Lorentzian signature are obtained by taking complex/Weyl conjugate of SUSY transformations of a chiral multiplet. The transformations in Euclidean signature are formally given by the same procedure.

A.4 Tensor calculus for chiral multiplets

Let us consider two chiral multiplets $\mathcal{A}, \mathcal{B}$ whose fermionic components vanish

$$\mathcal{A} = (A|_\mathcal{A},\ \Psi_i|_\mathcal{A} = 0,\ B_{ij}|_\mathcal{A},\ F^-_{ab}|_\mathcal{A},\ \Lambda_i|_\mathcal{A} = 0,\ C|_\mathcal{A}) \ , \tag{A.31}$$

$$\mathcal{B} = (A|_\mathcal{B},\ \Psi_i|_\mathcal{B} = 0,\ B_{ij}|_\mathcal{B},\ F^-_{ab}|_\mathcal{B},\ \Lambda_i|_\mathcal{B} = 0,\ C|_\mathcal{B}) \ . \tag{A.32}$$

Then we can make the product of these chiral multiplet which is also a chiral multiplet as [51]

$$A|_{\mathcal{AB}} = A|_\mathcal{A}\,A|_\mathcal{B} \ , \tag{A.33}$$

$$B_{ij}|_{\mathcal{AB}} = A|_\mathcal{A}\,B_{ij}|_\mathcal{B} + A|_\mathcal{B}\,B_{ij}|_\mathcal{A} \ , \tag{A.34}$$

$$F^-_{ab}|_{\mathcal{AB}} = A|_\mathcal{A}\,F^-_{ab}|_\mathcal{B} + A|_\mathcal{B}\,F^-_{ab}|_\mathcal{A} \ , \tag{A.35}$$

$$C|_{\mathcal{AB}} = A|_\mathcal{A}\,C|_\mathcal{B} + C|_\mathcal{A}\,A|_\mathcal{B} - \frac{1}{2}\,\varepsilon^{ik}\varepsilon^{jl} B_{ij}|_\mathcal{A}\,B_{kl}|_\mathcal{B} + F^-_{ab}|_\mathcal{A}\,F^{-ab}|_\mathcal{B} \ . \tag{A.36}$$

By iterating the above product, we obtain the n-th product of a chiral multiplet as

$$A|_{\mathcal{A}^n} = (A|_{\mathcal{A}})^n \ , \tag{A.37}$$
$$B_{ij}|_{\mathcal{A}^n} = n\,(A|_{\mathcal{A}})^{n-1}\,B_{ij}|_{\mathcal{A}} \ , \tag{A.38}$$
$$F^-_{ab}|_{\mathcal{A}^n} = n\,(A|_{\mathcal{A}})^{n-1}\,F^-_{ab}|_{\mathcal{A}} \ , \tag{A.39}$$
$$C|_{\mathcal{A}^n} = n\,(A|_{\mathcal{A}})^{n-1}\,C|_{\mathcal{A}} - \frac{1}{4}\,n(n-1)\,(A|_{\mathcal{A}})^{n-2}\left[\varepsilon^{ik}\varepsilon^{jl}B_{ij}|_{\mathcal{A}}B_{kl}|_{\mathcal{A}} - 2\left(F^-_{ab}|_{\mathcal{A}}\right)^2\right] \ . \tag{A.40}$$

We have to apply the above formulae to the coefficients of generators T_I when fields are in the adjoint representation.

A.5 Definition of $\mathbb{T}(\log\overline{\Phi})$

In this appendix we give the definition of the chiral multiplet $\mathbb{T}(\log\overline{\Phi})$ made from an anti-chiral multiplet $\overline{\Phi}$ with vanishing fermionic and field strength components. First, the components of $\log\overline{\Phi}$ are given by [52]

$$\overline{A}|_{\log\overline{\Phi}} = \log\left(\overline{A}|_{\overline{\Phi}}\right) , \tag{A.41}$$
$$B^{ij}|_{\log\overline{\Phi}} = \frac{B^{ij}|_{\overline{\Phi}}}{\overline{A}|_{\overline{\Phi}}}, \tag{A.42}$$
$$\overline{C}|_{\log\overline{\Phi}} = \frac{\overline{C}|_{\overline{\Phi}}}{\overline{A}|_{\overline{\Phi}}} + \frac{1}{4\left(\overline{A}|_{\overline{\Phi}}\right)^2}\varepsilon_{ik}\varepsilon_{jl}\left(B^{ij}|_{\overline{\Phi}}\right)\left(B^{kl}|_{\overline{\Phi}}\right). \tag{A.43}$$

The chiral multiplet $\mathbb{T}$(anti-chiral multiplet) is the so-called $\mathcal{N}=2$ kinetic multiplet [53]. The components of the kinetic multiplet made from $\log\overline{\Phi}$ are then given as [52]

$$A|_{\mathbb{T}(\log\overline{\Phi})} = \overline{C}|_{\log\overline{\Phi}} \ , \tag{A.44}$$
$$B_{ij}|_{\mathbb{T}(\log\overline{\Phi})} = -2\varepsilon_{ik}\varepsilon_{jl}\Box_C B^{kl}|_{\log\overline{\Phi}} \ , \tag{A.45}$$
$$C|_{\mathbb{T}(\log\overline{\Phi})} = 4\Box_C\Box_C\overline{A}|_{\log\overline{\Phi}} \ , \tag{A.46}$$

where $\Box_C$ is the so-called conformal d'Alembertian.

A.6 Supercurrent multiplet

The components of the supercurrent multiplet for an Abelian vector multiplet can be obtained by linearizing the superconformal action (20.89) of [25] in terms of the Weyl multiplet. They are given by

$$\begin{aligned}
T^{\mu\nu} &= 8\partial^{(\mu}\overline{X}\partial^{\nu)}X - 4g^{\mu\nu}\left|\partial_\rho X\right|^2 + \frac{4}{3}(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\left|X\right|^2 \\
&\quad - g^{\mu\nu}(\overline{X}\partial^2 X + X\partial^2\overline{X}) + \overline{\Omega}^i\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\Omega_i - \frac{1}{4}g^{\mu\nu}\overline{\Omega}^i\overleftrightarrow{\not{\partial}}\Omega_i \\
&\quad + 2F^\mu{}_\rho F^{\nu\rho} - \frac{1}{2}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \qquad \text{(A.47)}\\
S_i^\mu &= -\frac{1}{2}F_{\rho\sigma}\gamma^{\rho\sigma}\gamma^\mu\varepsilon_{ij}\Omega^j - 2\overline{X}\overleftrightarrow{\partial}^\mu\Omega_i + 2\overline{X}\gamma^\mu\not{\partial}\Omega_i - \frac{2}{3}\gamma^{\mu\nu}\partial_\nu\left(\overline{X}\Omega_i\right), \qquad \text{(A.48)}\\
j_\mu{}^i{}_j &= -2\overline{\Omega}^i\gamma_\mu\Omega_j + \delta^i_j\overline{\Omega}^k\gamma_\mu\Omega_k, \quad j_\mu = -4\mathrm{i}\overline{X}\overleftrightarrow{\partial}_\mu X + \mathrm{i}\overline{\Omega}^i\gamma_\mu\Omega_i, \qquad \text{(A.49)}\\
J &= -4\overline{X}X, \quad j_i = 4\overline{X}\Omega_i, \quad j^+_{\mu\nu} = XF^+_{\mu\nu}, \quad j^-_{\mu\nu} = \overline{X}F^-_{\mu\nu}. \qquad \text{(A.50)}
\end{aligned}$$

They are equivalent to the supercurrent multiplet written in [54] in terms of two component notations.

Appendix B

Weyl transformation between $\mathbb{S}^4$ and flat space

In this appendix we show that the auxiliary fields in the coupling multiplet in a flat space and $\mathbb{S}^4$ are related by a Weyl transformation.

We denote Cartesian coordinates in a flat space by y^μ while stereographic coordinates for $\mathbb{S}^4$ by x^μ. For this purpose, we introduce the five-dimensional embedding coordinates Y^M ($M = 1, \cdots, 5$). They satisfy

$$\sum (Y^M)^2 = r^2 \ , \qquad \mathrm{d}s^2_{\mathbb{S}^4} = \sum (\mathrm{d}Y^M)^2 \ . \tag{B.1}$$

They can be parametrized by x and y-coordinates as

$$\begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \\ Y^4 \\ Y^5 \end{pmatrix} = \begin{pmatrix} f(x) \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \\ g(x) \end{pmatrix} = \begin{pmatrix} g(y) \\ f(y) \begin{pmatrix} y^4 \\ y^1 \\ y^2 \\ y^3 \end{pmatrix} \end{pmatrix} \ , \tag{B.2}$$

where the function $f(\cdot)$ is defined in (4.22) and $g(\cdot)$ is defined by

$$g(z) := r \, \frac{1 - \frac{z^2}{4r^2}}{1 + \frac{z^2}{4r^2}} \ . \tag{B.3}$$

By using these relations, we can relate the delta function in θ and y-coordinates as

$$\frac{1}{r} \delta\left(\theta - \frac{\pi}{2}\right) = \frac{1}{f(y)} \, \delta(y^3) \ , \qquad \frac{1}{r^2} \delta'\left(\theta - \frac{\pi}{2}\right) = -\frac{1}{f(y)^2} \, \delta'(y^3) \ . \tag{B.4}$$

By substituting these relations into (4.14) and (4.42) we obtain the relations between them as

$$B_{ij}^{(\tau)}|_{\mathbb{S}^4} = \frac{1}{f(y)} B_{ij}^{(\tau)}|_{\mathbb{R}^4}, \quad B^{(\tau)ij}|_{\mathbb{S}^4} = \frac{1}{f(y)} B^{(\tau)ij}|_{\mathbb{R}^4}, \tag{B.5}$$

$$C^{(\tau)}|_{\mathbb{S}^4} = \frac{1}{(f(y))^2} C^{(\tau)}|_{\mathbb{R}^4}, \quad \overline{C}|_{\mathbb{S}^4} = \frac{1}{(f(y))^2} \overline{C}|_{\mathbb{R}^4}. \tag{B.6}$$

Note that the function $f(y)$ is the Weyl factor of transformations between the flat space and $\mathbb{S}^4$ and the powers of $f(y)$ in the above relations correspond to the Weyl weights of the auxiliary fields $B_{ij}^{(\tau)}, B^{(\tau)ij}, C^{(\tau)}, \overline{C}$ which are $1, 1, 2, 2$, respectively. Thus we have shown that the auxiliary fields in a flat space and $\mathbb{S}^4$ are related by the Weyl transformation.

Appendix C

Boundary super-Weyl anomaly in $2d$ $\mathcal{N} = (2, 2)$ SCFT

In this appendix we give a proof for the similar statement as (3.41) in $2d$ $\mathcal{N} = (2, 2)$ SCFTs by using the boundary super-Weyl anomaly [13].

C.1 Super-Weyl anomaly on closed manifold

First we review the result in [55]. The authors of the paper considered the supersymmetric version of the Weyl anomaly called the super-Weyl anomaly in $2d$ $\mathcal{N} = (2, 2)$ SCFTs on a closed Riemann manifold.

The Weyl anomaly is defined by rescaling a metric $g_{\mu\nu}$. Similarly, the super-Weyl anomaly is defined by varying values of the components in a graviton multiplet. We take a conformal gauge $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$ and introduce Cartesian coordinates (x^1, x^2). We then consider the so-called $U(1)_V$ supergravity in which the graviton multiplet is a twisted chiral multiplet Σ. The lowest component of Σ is the combination of a metric $g_{\mu\nu}$ and a gauge field V^μ that couples to the vector-like R-symmetry:

$$\Sigma(y^\mu) = \sigma(y^\mu) + \mathrm{i}\, a(y^\mu) + \theta^+\overline{\chi}_+(y^\mu) + \overline{\theta}^-\chi_-(y^\mu) + \theta^+\overline{\theta}^- w(y^\mu), \qquad \text{(C.1)}$$

where $y^\pm = x^\pm \mp \theta^\pm\overline{\theta}^\pm$ and $x^\pm = x^1 \pm x^2$. See Chapter 12 of [56] for the superfield conventions in $2d$ $\mathcal{N} = (2, 2)$ SCFTs. Then the anomalous dependence of the partition function on the graviton multiplet Σ for a closed manifold $\mathcal{M}$ is given by [55]

$$A_{\rm closed} := -{\rm i}\delta_\Sigma \log Z[\mathcal{M}], \tag{C.2}$$

$$= -\frac{\rm i}{4\pi}\int_{\mathcal{M}} {\rm d}^2x \int {\rm d}^4\theta \left[\frac{c}{6}(\delta\Sigma\overline{\Sigma} + \delta\overline{\Sigma}\Sigma) - (\delta\Sigma + \delta\overline{\Sigma})K(\mathcal{T},\overline{\mathcal{T}})\right], \tag{C.3}$$

where $\mathcal{T}$ is a twisted chiral multiplet whose lowest component is a moduli parameter τ and the function $K(\tau,\overline{\tau})$ is identified with the Kähler potential on the moduli space. By integrating (C.2), we have

$$\log Z[\mathcal{M}] \supset \frac{1}{4\pi}\int_{\mathcal{M}} {\rm d}^2x \int {\rm d}^4\theta \left[\frac{c}{6}\Sigma\overline{\Sigma} - (\Sigma + \overline{\Sigma})K(\mathcal{T},\overline{\mathcal{T}})\right]. \tag{C.4}$$

The first term gives the contribution from the usual Weyl anomaly, while the second term gives the contribution as (2.34).

C.2 Super-Weyl anomaly on open manifold and hemisphere partition function

The author of the paper [13] extend the result reviewed in the previous section to the super-Weyl anomaly for $2d$ $\mathcal{N} = (2,2)$ SCFTs with a supersymmetric boundary. We here review their results and then prove the similar statement as (3.41) for $2d$ $\mathcal{N} = (2,2)$ SCFTs by using their result.

The author of the paper [13] showed that the super-Weyl anomaly on a manifold $\mathcal{M}$ with a supersymmetric boundary is given by

$$A_{\rm open} := -{\rm i}\delta_\Sigma \log Z[\mathcal{M}], \tag{C.5}$$

$$\supset \delta\Big[\frac{\rm i}{4\pi}\int_{\mathcal{M}} {\rm d}^2x \left(\Box(\sigma - {\rm i}\, a)h^\Omega + \Box(\sigma + {\rm i}\, a)\overline{h}^\Omega\right) + \frac{1}{4\pi}\int_{\partial\mathcal{M}} {\rm d}x^2(\overline{w}h^\Omega - w\overline{h}^\Omega)\Big], \tag{C.6}$$

where dropped moduli independent terms and terms that contain derivatives of moduli parameters. The function $h^\Omega(\tau)$ is related to the holomorphic central charge $c^\Omega(\tau)$ which gives the hemisphere partition function [57, 58, 59] as

$$c^\Omega = \exp h^\Omega. \tag{C.7}$$

We call the anomaly (C.5) as the boundary super-Weyl anomaly.

Now, we consider the boundary super-Weyl anomaly on a two-dimensional hemisphere D^2 and prove that the hemisphere partition function computed in a conformal scheme $Z^{(\mathrm{BCFT})}[D^2]$ and in a supersymmetric scheme $Z^{\mathcal{B}}_{\mathrm{SUSY}}[D^2]$ are related by

$$Z^{(\mathrm{BCFT})}[D^2] = \left|Z^{\mathcal{B}}_{\mathrm{SUSY}}[D^2]\right| . \tag{C.8}$$

This is a $2d$ boundary CFT version of the statement (3.41).

The D^2 background is given by

$$\sigma = -\log(1+z\bar{z}), \quad a = 0, \tag{C.9}$$

where $z = x^1 + \mathrm{i}x^2$. First we want to evaluate the partition function in a conformal scheme which instead violates supersymmetry. For this purpose, we turn off auxiliary fields in a Weyl multiplet: $w = \overline{w} = 0$. Then integrating (C.5) leads to

$$Z^{(\mathrm{BCFT})}[D^2] = Z_0 \exp\left[\frac{1}{2}(h^{\Omega}(\tau) + \overline{h}^{\Omega}(\overline{\tau}))\right], \tag{C.10}$$

where Z_0 is a moduli independent factor due to the Weyl anomaly [1].

On the other hand, if we want to evaluate the partition function in a SUSY preserving scheme which instead violates the conformal symmetry, we take the background as

$$w = \overline{w} = -\frac{2\mathrm{i}}{1+z\overline{z}}. \tag{C.11}$$

Then the resulting partition function is

$$Z^{\mathcal{I}}_{\mathrm{SUSY}}[D^2] = Z_0 \exp\left[h^{\Omega}(\tau)\right] = Z_0 c^{\Omega}(\tau). \tag{C.12}$$

This is the hemisphere partition function obtained by [57, 58, 59]. Thus we prove the similar statement as (3.41) in a $2d$ $\mathcal{N} = (2,2)$ SCFT with a supersymmetric boundary.

[1] See (2.34) and the footnote 9 in Section 2.3.3.

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