DOCTORAL DISSERTATION

博士論文

Minkowski Flux Vacua on CM-type K3 $\times$ K3 Orbifolds and their Particle Physics Implications

（CM型 K3 $\times$ K3 オービフォルド上のミンコフスキーフラックス真空解と素粒子物理への示唆）

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Abstract

As a starting point for describing our Universe in string theory, it is important to find flux configurations with exponentially small superpotential W, so that one can enjoy the tremendous success of the standard cosmology including the Big-Bang Nucleosynthesis, together with the supersymmetric gauge coupling unification in particle physics. While such vacua are strongly believed to abound as we have exponentially many choices of fluxes, it is extremely difficult to construct such vacua explicitly by specifying integer fluxes. In this work, we focus on a special class of complex structure, called CM-type, and work out the condition for the complex structure to support flux vacua with vanishing superpotential $W = 0$ in F-theory. We find that the moduli space of the orbifolds of products of two K3 surfaces contains infinitely many such vacua. Possible gauge groups and matter representations in those vacua are also worked out.

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Contents

Chapter 1

Introduction

This thesis is devoted to a part of efforts studying string theory[1] as the quantum gravity theory of our Universe. Soon after the introduction of string theory as a theory of the strong interaction, it was realized that it contains a spin-2 excitation, that serves as a graviton in the low-energy effective theory, and that it may be the quantum gravity theory of the Universe. Although it is not conclusive whether or not the gravitational theory of the Universe is actually quantized [Car01], it would be promising to explore the quantum gravity theory if there is a candidate, as an experimental result excluded at least the most naïve version of a classical gravity theory coupled to quantum matters [PG81].

The effort of describing our Universe in string theory is two-fold; the first direction is to show, by explicit constructions, that the Universe is described by string theory, so that one can use string theory to solve quantum gravitational problems, such as the black hole information problem or the initial singularity problem, with confidence. The second direction is to impose some constraints on the low-energy effective field theory, assuming that the Universe is described by string theory, to guide the bottom-up studies in phenomenology. We will explore mainly the first direction in this thesis, as we elaborate more in the following.

To achieve the ultimate goal of the first direction, a model of string theory that predicts all the observed physical quantities must be constructed. The two important ingredients of the Universe, namely the particle physics and the cosmology, are addressed in string theory, and a significant progress has been made, although a complete understanding is yet to be achieved [CHSW85], [AKT00, AKT01], [TW06, DW11, BHV09, HTT+09], [GKP02, KKLT03, CCQ08], [BM15]. Those studies, though, have been done separately most of the time, i.e. a model of string theory that gives the standard particle spectrum and the standard history of the Universe is yet to be achieved. In this thesis, we will present a class of models in string theory that has supersymmetry and small cosmological constant in 4-dimensional low-energy effective theory, with a rich particle spectrum. Although the match to the Standard Model is far from perfect, the result would be an important step for further investigation.

The second direction is in a sense opposite to the first direction, as one is now assuming, rather than testing, that the Universe is described by string theory, and also one seeks what is *not* possible, rather than what is possible in string theory. The effort is sometimes called the swampland program, introduced in [Vaf05]. The swampland is defined as the space of theories that is consistent in itself but does not appear as a low-energy effective theory of string theory, as opposed to the landscape, which

[1]Throughout the thesis, we call superstring theory simply as string theory.

consists of the theories that is an effective theory of string theory. The aim of the swampland program is to carve out the landscape by identifying the swampland in terms of the effective theory, so that model building in terms of effective field theory can be made less incompatible, if not perfectly compatible, with string theory. Note that, although some swampland conjectures are argued to be valid for any quantum gravity, others may be specific to string theory, and could be used to falsify string theory. This is an arena of active research [Pal19], and there are definitely much more to be explored, but we will not go further as it is beyond the scope of the thesis.

In this thesis, as already briefly mentioned, we explore mainly along the first direction. Among many intrinsic properties of the Universe, we will take the success of the standard scenario of the early Universe, combined with the Grand Unified Theories (GUTs) with the celebrated gauge coupling unification as observational inputs, and address the question of how to implement those in string theory. For a rich particle spectrum such as GUTs, F-theory would be the first choice as a framework, as it accommodates a variety of gauge groups, matter representations, and interactions among them, as we will review in Chapter 3. The $(3+1)$-dimensional low-energy effective theory[2] constructed in this way, however, has a problem: it generically has light scalar degrees of freedom, often called **moduli**. This is not a problem particular to F-theory, but rather a generic property of string theory compactified down to lower dimensions. The moduli is problematic mainly in the early part of the standard cosmological scenario, as we will review in Chapter 2. We will see in Chapter 4 that a certain kind of moduli, called **complex structure moduli**, can be stabilized, i.e. can obtain masses of the order of the string scale, by turning on **fluxes**. This is still problematic for phenomenology, as such solutions will generically result in the gravitino mass of the order of the string scale, at the same time; this means a high-scale SUSY breaking, and we will lose the control of low-energy theory in string theory, and also the gauge coupling unification.

The difficulty of finding a model with stabilized moduli and light gravitino originates in the fact that the fluxes are quantized. We thus cannot continuously tune the fluxes so that the vacuum expectation value (vev) of superpotential W, to which the gravitino mass is proportional, generated by the fluxes, are small. Hinted by the integrality of the fluxes, there are ideas of considering Calabi-Yau manifolds with certain arithmetic properties to compactifying the string theory on [Moo07, DGKT05a, KW17a, KNY20]. We will explore the direction further in this thesis, and study the F-theory on CM-type Calabi-Yau fourfolds Y. **CM-type** is a certain property of the complex structure of Y, introduced in Chapter 5. We work exclusively on orbifolds of the product of two K3 surfaces[3], in this thesis; we can find CM-type complex structures most easily in those cases, as we will briefly review in Section 5.3.

The main part of the thesis starts at Chapter 6, where we analyze the flux vacua with vanishing superpotential from fluxes (3.12), in the simplest case of $\mathbb{Z}_2$-orbifolds of $\mathrm{K3} \times \mathrm{K3}$. We will extend the analysis in a more general orbifold in Chapter 7. We work out the criteria for $W = 0$ vacua with non-trivial fluxes, and also examine the mass terms, interactions and symmetries of the complex structure moduli. These analyses should be understood as those of 3-dimensional vacua of M-theory, as we will not care if there is an F-theory limit for each vacuum. In Chapter 8, we will consider the F-theory limit of those vacua, focusing on their consequences for gauge symmetries and particle spectra in 4 dimensions.

[2]We will say that a spacetime with 3 spatial dimensions and 1 temporal dimension is 4-dimensional.
[3]K3 surface is the Calabi-Yau twofold, and will be introduced in Section 5.2.1

The thesis is based on the following article [KW20]

- Keita Kanno and Taizan Watari, "$W = 0$ Complex Structure Moduli Stabilization on CM-type K3 $\times$ K3 Orbifolds:—Arithmetic, Geometry and Particle Physics—," arXiv:2012.01111 [hep-th]

and the original work is presented in the main part, Chapters 6, 7, and 8.

Part I

Review

Chapter 2

Phenomenology

2.1 Cosmological history and the moduli problem

This thesis is a part of the effort towards describing our Universe as a solution of string theory, and thus as a quantum gravity theory, as we have already discussed in Introduction. Such a model should match all the physical quantities that can be observed. There is another big challenge of addressing the "uniqueness" of the solution in some proper sense, but it is beyond the scope of the thesis.

As the observed fundamental physical quantities, one has two kinds of sources; telescopes, and ground experiments like colliders. The ground experiments revealed that the Standard Model of particle physics explains the fundamental physics very well, although some hints exist, such as muon $g-2$-anomaly [MdRRS12]. The Standard Model itself has problems of "naturalness", such as the hierarchy between the electroweak and the Planck scale, the smallness of the neutrino masses, and the lack of the theta-term in QCD[1], but they do not spoil the fact that the Standard Model predicts what happens in the ground experiments perfectly.

The hierarchy of the electroweak and the Planck scale may be relaxed by introducing the supersymmetry. Then the gauge coupling unifies at around 10^{16} GeV, which strongly suggests a new physics at the scale, such as $SU(5)$ Grand Unified Theory (GUT). The smallness of the neutrino mass can be explained by the seesaw mechanism, i.e. introduction of heavy right-handed neutrinos. Finally the strong CP problem may be solved by introducing the QCD axion.

The telescopes, in a broad sense, offer much more about the unknowns of the Universe, including its history. The observation of the expansion rate of the Universe throughout the history since the recombination strongly suggests that there exist non-relativistic matter that we cannot observe by light, sometimes called the dark matter, and an energy component that behaves like the cosmological constant, called the dark energy. The dark matter could be made up of axion-like particles, or the superpartners of the Standard Model particles, in the supersymmetric case. The dark energy is often understood as the cosmological constant in string theory, although a convincing explanation for its smallness is still missing.

Although the history before the recombination is ambiguous, an inflationary period right after the beginning of the Universe beautifully explains the observational data of the Cosmic Microwave Background (CMB). There are numerous scenarios of what

[1] Among those naturalness problems, the last one, known as the strong CP problem, may be the worst because it is argued that even the anthropic principle cannot explain the smallness of the theta-term [BDG04, Don04, Uba10, DSHUX18].

has happened in that period, even in string theory [BM15], and it would be necessary to wait for additional observational data. After the inflationary period, by the reheating process the energy of the inflaton is converted to other particles, and then the Universe experienced the electroweak transition and the QCD phase transition, though the details of the scenario is yet to be understood. After such transitions, though, there is a scenario that goes very well, called the Big-Bang Nucleosynthesis (BBN), which tells us that if we have a certain amount of baryon number n_b compared to the total radiation density n_γ at around $1\,\mathrm{s}$ after the inflation with the temperature around $1\,\mathrm{MeV}$, then the abundance of light nuclei D, ^{3}He, ^{4}He, and ^{7}Li of the present Universe is beautifully explained. Surprisingly, the expected ratio $n_b/n_\gamma \sim 6 \times 10^{-10}$ perfectly matches the one that is observed in the CMB; see [Z$^+$20, Chapter 24] for the state-of-the-art analysis.

The task for us is, then, to construct a model in string theory that contains all the particles described above, and that can accommodate the standard inflation-BBN scenario. However, when one constructs a 4-dimensional model in string theory, it is in general expected that there are light scalar degrees of freedom in the low-energy effective theory, which may spoil the standard scenario.

One might worry about the consistency between the presence of the light degrees of freedom and the fifth force experiments. In the case of string compactifications, though, this is not a serious problem. The real scalars will get a mass of the order of the SUSY breaking scale, and the pseudo scalars will have almost vanishing contributions to the fifth force experiment, because its coupling to the Standard Model sector involves derivatives.

The more serious problem is the consistency with the BBN scenario. The moduli may upset the standard BBN scenario mainly in two ways. Firstly, if the moduli decay after the BBN into high energy particles, they may break the successfully produced nuclei apart. Secondly, an over-production of the moduli may dilute the baryon number density, which is generated by a mechanism like leptogenesis [FY86], as we will see briefly in the following. Note that similar problems also applies to gravitino, and is known as the gravitino problem [Wei82, KL84].

Firstly, let us analyze the decay of moduli after the BBN. For example, assuming the decay rate of a modulus ϕ with mass m_ϕ as

$$\Gamma_\phi = \frac{\alpha}{2\pi}\frac{m_\phi^3}{M_{\mathrm{Pl}}^2}, \tag{2.1}$$

with a dimensionless parameter α, the mass m_ϕ should satisfy $m_\phi \gtrsim 30\,\mathrm{TeV}$ for $\Gamma_\phi \gtrsim 1\,\mathrm{s}^{-1}$ when $\alpha = 1$.

This problem may be avoided if the gravitino has the mass, say $m_{3/2} \sim 30\,\mathrm{TeV}$, in which case the moduli will get similar masses and satisfy $m_\varphi \gtrsim 30\,\mathrm{TeV}$ to escape from the above problem. There is, however, the second problem; the dilution of the baryon asymmetry. After the inflation, moduli are produced thermally and non-thermally and will dilute the baryon number created during the reheating process; especially, non-thermal production of moduli from the coherent oscillation around the shifted minimum after the inflation is problematic [CFK$^+$83, BKN94, dCCQR93]. There are two ways out: keep the moduli mass heavy so that it will decay at latest during the reheating process, or recover the baryon asymmetry by a low-energy mechanism such as the Affleck-Dine mechanism [AD85, MYY95].

In conclusion, if one assumes that there is no mechanism above the SUSY breaking scale to give moduli masses[2], then one should make sure that the moduli are not over-produced, especially non-thermally, and the baryon asymmetry is recovered by some mechanism, if necessary. Some models along this line may survive, and it is actually very interesting to explore in this direction as some of the moduli is harder to give masses than others, but we will proceed to the safest way, giving the masses of the string scale to as many of the moduli as possible.

2.2 Grand Unified Theories and the gravitino mass

An obvious solution to the above-mentioned problems is to set the gravitino mass very heavy, and make the moduli heavy as a result. This is not desirable, both theoretically and phenomenologically. Firstly, since this is almost equivalent to take the high energy SUSY breaking scenario, we will lose any control of the IR, i.e. low-energy theory from the UV theory, i.e. the string theory in our case, because nothing protects the physical parameters from quantum corrections. Secondly, we will lose the explanation of the electroweak hierarchy and the gauge coupling unification, as there is no mechanism that protects Higgs boson to acquire mass, and the gaugino will acquire the mass contribution proportional to the gravitino mass from the anomaly mediation [RS99, GLMR98, BMP00].

It is thus desirable to construct a model with a small gravitino mass. The gravitino mass is described by the Kähler potential K and the superpotential W as

$$\frac{m_{3/2}}{M_{\rm Pl}} = e^{\frac{K}{2M_{\rm Pl}^2}} \frac{W}{M_{\rm Pl}^3}, \tag{2.2}$$

normalized by the reduced Planck scale $M_{\rm Pl} \simeq 2.4 \times 10^{18}\,\mathrm{GeV}$. For a small gravitino mass, there are basically two strategies: take the vev of the Kähler potential K largely negative, or take the vev of the superpotential W small. The former choice was taken by the so-called Large Volume Scenario [CCQ08], where the volume of the compact manifold is taken to be large, resulting in a large negative vev of K. This is, however, hardly compatible with a standard F-theory GUT scenario. Using the expressions given in [TTW09, eq. (21)-(23)], we can determine the compactification scale $M_6 := (\mathrm{Vol}(X))^{-1/6}$,

$$M_6 = (4\pi)^{1/6} c^{-4/3} M_{\rm Pl} \left(\frac{M_{\rm GUT}}{M_{\rm Pl}}\right)^{4/3} \alpha_{\rm GUT}^{-1/3} \tag{2.3}$$

$$\simeq \frac{2\times 10^{16}\,\mathrm{GeV}}{c^{4/3}} \left(\frac{M_{\rm GUT}}{2\times 10^{16}\,\mathrm{GeV}}\right)^{4/3} \left(\frac{1/24}{\alpha_{\rm GUT}}\right)^{1/3}, \tag{2.4}$$

where c is a $\mathcal{O}(1)$ constant relating GUT scale $M_{\rm GUT}$ to the volume of the 7-brane supporting the GUT gauge group, and we have put a typical value of $M_{\rm GUT}$ and the coupling constant $\alpha_{\rm GUT}$ at the GUT scale in the parentheses. The Large Volume Scenario typically requires $M_6 \sim \mathcal{O}(10^9\,\mathrm{GeV})$, and is, at least naïvely, hardly compatible with the standard F-theory GUT scenario. Thus in what follows, we will explore models with small superpotential vev and in particular, models with vanishing contribution to the superpotential from the fluxes (3.12); any correction to the

[2]Note that we are assuming the unstable gravitino scenario, i.e. gravitini are expected to decay before BBN. There is another scenario where the gravitino is extremely light and stable, but since this scenario is much more difficult to achieve in our setup, we will not take it into account.

superpotential (3.12) will give rise to $W \neq 0$ contribution, resulting in non-zero gravitino mass, in accord with the current observation. Throughout the thesis, especially in the main part, we will often denote by W (the vev of) the term (3.12) in the entire superpotential, ignoring other terms.

Let us finally comment on the relation of the requirement of small superpotential vev and the cosmological constant problem. We are concerned with the F-term potential

$$V_F = e^{\frac{K}{M_{\rm Pl}^2}} \left(K^{i\bar{j}} (D_i W)(\overline{D_j W}) - 3 \frac{|W|^2}{M_{\rm Pl}^2} \right), \tag{2.5}$$

where K is the Kähler potential, $K^{i\bar{j}}$ is the inverse of the Kähler metric, and D_i denotes the covariant derivative $\partial_i + (\partial_i K)/M_{\rm Pl}^2$ with respect to a chiral multiplet. Obviously a supersymmetric solution $W = 0$ and $D_i W = 0$ for all i results in a supersymmetric Minkowski solution $V_F = 0$, and thus the title of the thesis. In some compactifications of supergravity theories, though, a Kähler potential of special kind, called "no-scale type" arises, and in that case, $V_F = 0$ is automatically satisfied regardless of the vev of W, because of a cancellation between a part of an F-term $|(\partial_i K) W|^2$ and $|W|^2$ for some i, often the overall volume moduli of the internal compact space. While one has to make sure that such a special form is maintained even after Kaluza-Klein, string, quantum and non-perturbative corrections are taken into account, we cannot say that W being small is a necessary condition to solve the cosmological constant problem. This is why we have motivated the small W solution using the gravitino mass and its consequence to the gauge coupling unification, although small W will definitely help to keep the cosmological constant small, even in the presence of any kind of corrections.

Chapter 3

Effective field theory of F-theory

In this thesis, we will explore the possibility that our Universe is described as a solution of *F-theory*. F-theory is a framework that can generate solutions of string theory. It can produce the Type IIB orientifold solutions, and also the "fibrations" of those solutions. F-theory can also be understood as a certain limit of M-theory, and the two pictures are thought to be equivalent, at least in non-exotic cases.

In this chapter, we will review how to construct a 4-dimensional field theory as a solution of F-theory. This is by no means an exhaustive review of 4-dimensional effective theory of F-theory; we will mainly focus on what is relevant to the main part, and omit several important aspects entirely, in addition to the details of the reviewed topics. For a pedagogical review of the F-theory, see e.g. [Wei18].

3.1 F-theory

For constructing a consistent model of our Universe in string theory, F-theory is considered as one of the best frameworks; it has a large solution space containing the whole Type IIB orientifolds, it accommodates a variety of matter representations such as the spinor representation of $SO(10)$, and it provides the couplings for those matters, which is sometimes absent in the perturbative string theories. One of the best things about F-theory is that most of those features are embedded in the geometry; by simply specifying a geometry, one can construct an F-theory model with an intrinsic configuration of 7-branes, which is guaranteed to satisfy at least part of[1] the consistency conditions.

In this section, we will introduce F-theory [Vaf96] as an extension of Type IIB superstring theory and then see in the next section that it corresponds to a certain limit of M-theory. We will assume that the two pictures are consistent, and use the two pictures interchangeably for convenience. See [Wei18] for details of the discussion below.

As is well known, the Type IIB supergravity action is invariant under an $SL(2;\mathbb{R})$ group action [Sch83], which breaks down to the $SL(2;\mathbb{Z})$ subgroup in the perturbative Type IIB superstring theory [GS84]. Assuming that the $SL(2;\mathbb{Z})$ duality group holds in the full Type IIB superstring theory, it would be natural to seek solutions with non-trivial $SL(2;\mathbb{Z})$ background configurations. Such non-trivial configurations indeed exist, with non-trivial monodromies generated by 7-branes. 7-branes can generate any of the $SL(2;\mathbb{Z})$ monodromies because there is a corresponding $[p,q]$ 7-brane for any of the monodromies.

[1] See e.g. [GG98] for additional requirements.

There is another object that is acted on by $\mathrm{SL}(2;\mathbb{Z})$: the modular parameter τ of an elliptic curve[2]. F-theory can be described as an ambitious identification of the axio-dilaton ϕ in the Type IIB theory with the modular parameter τ of an elliptic curve; both of them are at least acted on by $\mathrm{SL}(2;\mathbb{Z})$ in the same way. More formally, let B be a complex n-dimensional[3] variety[4], on which we will compactify our Type IIB theory. Let ϕ vary holomorphically over B, so that a part of the supersymmetry is preserved. Then one can define an **elliptic fibration** (Y, B, π), where Y is complex $(n+1)$-dimensional variety, $\pi : Y \to B$ is a regular[5] map and the inverse image of a generic point $p \in B$, i.e. the **fiber** over the point p, $\pi^{-1}(p)$, is an elliptic curve. B is called the **base** of the elliptic fibration. We say that Y is **elliptically-fibered** or **elliptic** when such set (B, π) exists. **F-theory** on an elliptically-fibered variety Y with base B is defined to be a theory that is equivalent to the Type IIB superstring theory compactified on B with its axio-dilaton varying in exactly the same way as in Y. It can be also shown that the Einstein equation is satisfied if the total space Y has the vanishing first Chern class [BCM11], which can be understood as a consequence of preserved supersymmetry, in M-theoretic picture.

Note that, although F-theory compactified on complex $(n+1)$-dimensional variety gives rise to $(12-2(n+1))$-dimensional low-energy effective theory, it is not a theory that has 12-dimensional spacetime. The two additional dimensions should be considered as an auxiliary geometry that represents the value of axio-dilaton ϕ. The variety Y will be, though, part of the spacetime in M-theoretic picture, in some sense; see the discussion in the next section.

In Y, the fiber torus enjoys a monodromy as the axio-dilaton does around a 7-brane. At the center of the monodromy, the fiber degenerates; the center in the base is called the **discriminant locus**; the reason is explained later. The possible singular fiber over a (complex) codimension-1 discriminant locus in the base, after the resolution[6], is classified in [Kod63, Nér64]; see Table 3.1. The classification is similar to the ADE classification of Lie algebra, with some additional cases. The relation to the gauge symmetry can be inferred by looking at the monodromy action in $\mathrm{SL}(2;\mathbb{Z})$, or by M-theory, as we review in the next section.

Before moving on, though, let us introduce the **Weierstrass model** of an elliptically-fibered variety. In general, an elliptic curve can be defined as a sub-variety of the

[2] An **elliptic curve** E_τ with complex structure τ is defined as

$$E_\tau := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\langle\tau\rangle) = \{z \in \mathbb{C} \mid z \sim z + (n + m\tau) \text{ for } n, m \in \mathbb{Z}\}\,, \tag{3.1}$$

which is a torus with a marked point $z = 0$.

[3] One often counts the complex-dimensions, which is half of the real-dimensions, when a compact space in F-theory is concerned, as the compact spaces we deal with in F-theory often have well-defined complex structures. B is $2n$-dimensional in the usual sense.

[4] We often use the term *variety* to specify a geometry, especially when the geometry is not guaranteed to be smooth, because the term *manifold* sometimes implies that the geometry is non-singular.

[5] π is said to be **regular** when the coordinates of B can be expressed as polynomials of coordinates of Y.

[6] A **resolution** of a singular variety Y is to find a smooth, or at least less singular, variety $\tilde{Y}$ with a regular map $\nu : \tilde{Y} \to Y$, that is an isomorphism except the singular locus in Y.

weighted projective space[7] [8] $W\mathbb{P}^2_{2,3,1}$ with coordinates (x, y, z), satisfying

$$y^2 = x^3 + axz^4 + bz^6 \tag{3.2}$$

for some $a, b \in \mathbb{C}$. We often take the open patch where $z \neq 0$, where one can take $z = 1$ by the scaling relation. One can uplift the coordinates x, y, z and the parameters a, b to be functions, or more precisely sections of line bundles, over a complex variety B, in which case the geometry is an elliptically-fibered variety Y with base B. For example, a K3 surface, which will be introduced in detail in Section 5.2.1, has always the 1-dimensional complex projective space $\mathbb{P}^1$, which is a 2-sphere S^2 as a real manifold, as a base, when elliptically-fibered. It is defined by functions f and g, which are degree-8 and -12 polynomials, respectively, in the homogeneous coordinates $[s : t]$ of $\mathbb{P}^1$. One can take, say $s = z = 1$ patch, where the defining equation is

$$y^2 = x^3 + f(t)x + g(t), \tag{3.3}$$

which covers the whole geometry except $s = 0$ and $z = 0$. The fiber degenerates where the discriminant of the right hand side

$$\Delta := 4f^3 + 27g^2 \tag{3.4}$$

vanishes, and thus the name discriminant locus. The vanishing orders of f, g and Δ is used to classify the singular fibers; the f, g and Δ columns in Table 3.1 shows the vanishing orders of f, g and Δ for each singular fiber type. The definition of vanishing order involves some subtleties in higher dimensions, but it is simply the degree of the leading term in the case of elliptic complex surfaces. In the case of K3 surface, Δ is a degree-24 polynomial over the base, and thus has 24 I_1 fibers generically. Note that the Weierstrass model is not smooth in general. One can tune f and g so that the K3 surface contains more non-trivial singularities and one can get non-trivial singular fibers after resolution, as shown in Table 3.1.

3.2 F/M-duality and non-abelian gauge symmetry

In this section, we introduce another formalism of F-theory, which will be used mainly in this thesis. Take a Calabi-Yau manifold Y with an elliptic fibration. The F-theory on $Y \times \mathbb{R}^{3,1}$ can be formulated as an M-theory on $Y \times \mathbb{R}^{1,2}$ with vanishing fiber volume. The basic idea is the following. Consider the fiber torus as a product of two circles, S^1_A and S^1_B. One can go to the Type IIA picture by taking the radius of S^1_A small. Then, take the T-dual along the S^1_B. The resulting theory is a Type IIB theory with 4-dimensional non-compact space when the radius of S^1_B is taken to be zero. We will elaborate the correspondence in the following.

Firstly, at a generic point in the base, i.e. a base point with a smooth fiber torus, one can find an explicit correspondence; see [Den08, pp.23-24] for details. Start with

[7] A **weighted projective space** $W\mathbb{P}^n_{w_1,\dots,w_{n+1}}$ is defined as $\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$ with a scaling relation $(x_1, \dots, x_{n+1}) \sim (\lambda^{w_1} x_1, \dots, \lambda^{w_{n+1}} x_{n+1})$, where $(x_1, \dots, x_{n+1})$ is an element of $\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$. A **projective space** is a weighted projective space with all the weights are equal, $w_1 = \dots = w_{n+1} = 1$. We denote the n-dimensional projective space by $\mathbb{P}^n$ following the convention of algebraic geometry, but it always denotes the *complex* projective space, which is sometimes denoted by $\mathbb{CP}^n$ elsewhere.

[8] One can relate this construction to the previous definition, $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\langle\tau\rangle)$; for a concise presentation, see [Wei18, §2.2].

M-theory on $T^2 \times B_6 \times \mathbb{R}^{1,2}$ with some 6-dimensional manifold B_6 with metric

$$ds_M^2 = \frac{v}{\tau_2}\left((dx + \tau_1 dy)^2 + \tau_2^2 dy^2\right) + ds_6^2, \tag{3.5}$$

where T^2 has total area v and modular parameter $\tau = \tau_1 + i\tau_2$, x and y is periodic with periodicity 1, and ds_9^2 is the metric of $B_6 \times \mathbb{R}^{1,2}$. By the correspondence to Type IIA theory using the A-cycle (x-direction) and further T-dualizing in the B-cycle (y-direction), we get a Type IIB theory with metric, RR 0-form field, and the coupling constant

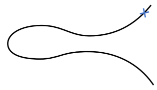
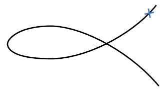
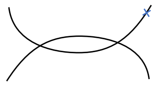
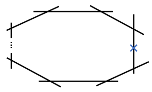
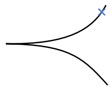
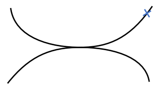
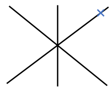
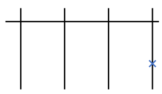
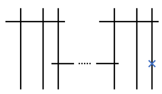
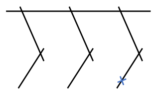
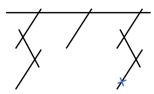
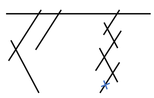
$$ds_{\mathrm{IIB}}^2 = \frac{l_s^4}{v} dy^2 + ds_9^2, \qquad C_0 = \tau_1, \qquad \frac{1}{g_{\mathrm{IIB}}} = \tau_2. \tag{3.6}$$

Now it is visible that, by taking $v \to 0$ while keeping the string length l_s constant, one can recover $\mathbb{R}^{1,3} \times B_6$ as the spacetime. The argument holds even when τ varies over B_6, and we have successfully obtained a Type IIB theory with τ being the axio-dilaton $\phi = C_0 + ie^{-\Phi}$ with $\langle e^{\Phi} \rangle = g_{\mathrm{IIB}}$, as we have formulated previously, at least when the fiber torus is non-degenerate. Note that the Calabi-Yau condition imposed on Y is more apparent, as Y itself consists the spacetime in M-theory and Y needs to be Calabi-Yau for supersymmetry. It also makes sense to take $v \to 0$, as we were only concerned with the complex structure of the fiber torus, and not with the volume of it, in the previous Type IIB formalism.

The discussion above assumed that the fiber torus is not degenerate. Although it is hard to discuss the duality at the discriminant locus for all cases, let us at least see what happens at a codimension-1 discriminant locus. We focus on the I_n singular fiber, which corresponds to the resolution of A_{n-1} singularity and is related to $\mathrm{SU}(n)$ gauge symmetry, as we will see. The I_n fiber is a chain of $\mathbb{P}^1$'s, as shown in Table 3.1. It can be argued [Tow95], at least in the supergravity theories, that the geometrical configuration in M-theory and a 6-brane solution in Type IIA theory are equivalent, when the direction normal to the chain is taken as the IIA-cycle S_A^1; the "pinching" points, i.e. the intersection points of a pair of $\mathbb{P}^1$'s, in M-theory corresponds to the 6-brane locus. There are n 6-branes, which become 7-branes extending to the T-dualized direction in addition to the $(6+1)$ spacetime directions. In Type IIA, taking the T-dual cycle infinitely small corresponds to making the 6-branes coincident, resulting in $\mathrm{SU}(n)$ gauge symmetry, while keeping the cycle finite enables us to explore the Coulomb branch of the theory, which is parametrized by the separations between the 6-branes along the S^1. See [Sen97] for details and the discussion for D-type singularities. It is also possible to understand the appearance of the gauge symmetry in purely M-theoretic language [Wit95]. The gauge boson in the effective theory is an M2-brane wrapping some of the $\mathbb{P}^1$'s for non-Cartan elements, and is a zero-mode of C_3 field corresponding to a harmonic 2-form corresponding to one of the $\mathbb{P}^1$'s. This picture is also powerful in analyzing the matter representation at codimension-2 discriminant loci [MT12, EY13], although the wrapping rule is ambiguous in some cases [AKM00].

It is in general not true, however, that an I_n fiber corresponds to $\mathrm{SU}(n)$ gauge symmetry [BIK+96, AKM00], when the codimension-1 loci is not 0-dimensional, i.e. the base is complex n-dimensional with $n > 1$. In those cases, the singular fiber may experience a monodromy around a codimension-2 locus in the codimension-1 locus,

TABLE 3.1: Possible singular fibers at codimension-1 discriminant locus. The columns f, g, Δ shows the vanishing order of them, **singularity** shows the singularity that is present in the total space, G is for the possible gauge groups in the low-energy effective theory, and **components** shows the number of irreducible curves in the fiber, and the last column shows the fiber structure; each line represents a irreducible complex curve, and a blue cross means an intersection with the zero section.

type	f	g	Δ	singularity	G	components	fiber
I_0	≥ 0	≥ 0	0	none	–	1	
I_1	0	0	1	none	–	1	
I_2	0	0	2	A_1	$\mathrm{SU}(2)$	2	
I_n $(n \geq 3)$	0	0	n	A_{n-1}	$\mathrm{Sp}(\lfloor \frac{n}{2} \rfloor)$ or $\mathrm{SU}(n)$	n	
II	≥ 1	1	2	none	–	1	
III	1	≥ 2	3	A_1	$\mathrm{SU}(2)$	2	
IV	≥ 2	2	4	A_2	$\mathrm{Sp}(1)$ or $\mathrm{SU}(3)$	3	
I_0^*	≥ 2	≥ 3	6	D_4	G_2 or $\mathrm{SO}(7)$ or $\mathrm{SO}(8)$	5	
I_n^* $(n \geq 1)$	2	3	$n+6$	D_{n+4}	$\mathrm{SO}(2n+7)$ or $\mathrm{SO}(2n+8)$	$n+5$	
IV^*	≥ 3	4	8	E_6	F_4 or E_6	7	
III^*	3	≥ 5	9	E_7	E_7	8	
II^*	≥ 4	5	10	E_8	E_8	9	

and the M2-branes wrapped around the 2-cycles that are exchanged by the monodromy are in a sense "identified" to each other. A singular fiber that enjoys a non-trivial monodromy is said to be of **non-split type**, and **split type** otherwise. The resulting gauge symmetry is $\mathrm{Sp}(k)$ for I_n with $n = 2k$, and it is argued in [AKM00] that it is also $\mathrm{Sp}(k)$ for $n = 2k + 1$. There are also other reductions [BIK+96]

$$D_n \to B_{n-1}, \qquad E_6 \to F_4, \qquad D_4 \to G_2; \tag{3.7}$$

see the column G of Table 3.1. The reduction of the gauge symmetry often results in matter fields, as the adjoint representation of the original gauge group becomes reducible under the reduced gauge group [AKM00, GM00].

3.3 Particle spectrum

3.3.1 Charged matter

In the last section, we have seen how the non-abelian gauge symmetry arises from the codimension-1 degenerations of the elliptic fibration. When these codimension-1 loci in the base intersect, the singularity enhances at the codimension-2 intersection locus, and localized matters arises in the effective theory, in general. The classification of those enhancements is yet to be achieved, both in mathematics and in physics [Mir83, GM00, MT12, EY13].

However, we will be only interested in the case of normal-crossings in our study, i.e. the codimension-1 loci can be defined as $x = 0$ and $y = 0$ in an open patch with coordinates $(x, y, \dots)$, and two loci intersect, or "collide", along $x = y = 0$; this case is relatively well-understood. Among the ADE-type singular fibers, in the case of $A \times A$ collision or $A \times D$ collision, it is well known that the collision results in a bi-fundamental matter, e.g. chiral multiplets in 4 dimensions, in the effective field theory; this can be understood as open-strings stretched between the two intersecting 7-branes [BVS96, BDL96, BCLS05], or M2-brane wrapping the extra $\mathbb{P}^1$ cycles which are present in the fiber of the codimension-2 locus of the base [KV97, MT12, EY13].

When the collision is among D-types and E-types, i.e. $D \times D$, $D \times E$, or $E \times E$, there is no known flat[9] and crepant[10] resolution of the codimension-2 singularity, at least to the author's knowledge. One can, however, change not only the fiber of Y, but also its base, to get a resolution of Y. The procedure is introduced in [Mir83] and applied to F-theory in [BJ97a].

The procedure can be described locally as follows. Consider, for example, a Weierstrass model Y with base $\mathbb{C}^2$ with coordinates s, t

$$y^2 = x^3 + s^2t^2x + s^3t^3, \tag{3.8}$$

where x, y can be thought of as non-homogeneous coordinates of $W\mathbb{P}_{2,3,1}$ with the third homogeneous coordinate set to 1. This is the collision of $D_4 \times D_4$. By blowing-up[11] at $s = t = 0$, the base is now covered by two patches, with coordinates (s_t, t)

[9]A **flat** resolution is a resolution of elliptically-fibered variety Y, which has an elliptic fibration where the fiber is always 1-dimensional; see Section 3.4.2 for detail.

[10]A resolution $\nu : X \to X_0$ is said to be **crepant** when the canonical divisor of X is equal to the pullback of the canonical divisor of X_0.

[11]**Blow-up** is an algebro-geometric prescription to obtain a resolution of a singular variety. Roughly speaking, one replaces a singular point with a projective space, and leaves the non-singular locus as it

TABLE 3.2: Possible collisions [BJ97a]. Each entry corresponds to the number of blow-ups required to get a flat resolution, and "−" means that the blow-up process is non-crepant. Only the collisions with same j-invariant is considered.

	I_0^*	$\mathrm{I}_{n>0}^*$	I_n	II^*	II	IV^*	IV	III^*	III
I_0^*, J =any	1	1	0	4−	0	1	0	2	0
$\mathrm{I}_{n>0}^*$, $J=\infty$	1	1	0						
I_n, $J=\infty$	0	0	0						
II^*, $J=0$	4−			13−	1	6−	3−		
II, $J=0$	0			1	3−	0	1−		
IV^*, $J=0$	1			6−	0	3	1		
IV, $J=0$	0			3−	1−	1	3−		
III^*, $J=1$	2							5	1
III, $J=1$	0							1	1

and (s, t_s). The regular map to the original base $\mathbb{C}^2$ is defined by $s = s_t t$ and $t = t_s s$, while t in the first patch and s in the second patch is mapped identically. The pull-back of Y to the new base is, in, say, the first patch,

$$y^2 = x^3 + s_t^2 t^4 x + s_t^3 t^6. \tag{3.9}$$

Now the model is more singular than before, but by considering the coordinate change $y' = y/t^3$ and $x' = x/t^2$, one can get, as one of the irreducible components,

$$y'^2 = x'^3 + s_t^2 x' + s_t^3, \tag{3.10}$$

which is less singular and has a flat and crepant resolution. For a more global treatment of the procedure, including the $(x, y) = (\infty, \infty)$ locus of Y, can be found in Appendix B.3. One can find there that, for example, the resulting geometry is not a resolution of Y because the map from the resulting geometry to Y is not regular but rational. The Calabi-Yau condition for the resulting geometry is also addressed in [BJ97a] and summarized in Table 3.2. Note that, although the procedure is found to be extremely effective for constructing non-trivial superconformal field theories when the volume of the exceptional locus in the base is taken to be zero [HMV14, HR19], we will keep the volume finite so that the effective theory is more suitable for phenomenology.

3.3.2 Moduli

In addition to the charged matters, there are neutral matters, i.e. moduli, in the 4-dimensional effective field theory of F-theory compactified on a Calabi-Yau fourfold Y_4. In particular, there are $h^{3,1}(Y_4)$ of complex scalars, which accounts for the complex structure of Y_4, and corresponds to complex structure moduli, D7-brane deformation moduli, and axio-dilaton in Type IIB theory. These **complex structure moduli** in F-theory can be stabilized using 4-form fluxes G_4, as we will see later in

is. For more details, see standard textbooks of algebraic geometry, or [BJ97b] for a presentation specific to our setup. The projective space is called the **exceptional locus** of the blow-up.

this chapter. In addition, there are $h^{1,1}(Y_4) - 1$ of real scalars called **Kähler moduli**, and $6N_{\mathrm{D3}}$ of real scalars, accounting for the position of spacetime-filling M2-brane (or D3-brane in F-theory.) There are also axions arising from the dimensional reduction of C_3 and C_6 gauge fields, but we will not consider the moduli other than the complex structure moduli in this thesis.

3.3.3 Grand Unified Theories in F-theory

F-theory is of particular interest for phenomenology, as it can accommodate varieties of Grand Unified Theories (GUTs); it can generate up-type Yukawa couplings in the $\mathrm{SU}(5)$ GUT model, which is perturbatively vanishing in Type IIB theory [TW06], and spinor representation in $\mathrm{SO}(10)$ GUT models. Implementations of $\mathrm{SU}(5)$ GUT in F-theory is addressed in [DW11, BHV09, HTT+09] and many subsequent papers. As we have already reviewed how gauge symmetry and matter spectrum arises from codimension-1 and -2 singularities in the base of the elliptic fourfold Y_4, there remains one more ingredient: interactions among matters. Yukawa couplings are actually known to be generated at each codimension-3 singularity. At a codimension-3 singularity, three 7-branes intersect and three matter curves, i.e. codimension-2 singularities where matters are localized, collide. The structure of the geometry can be used to explain some of the flavor structures [HKTW10]. In this work, though, there is no such intrinsic structure, and thus we will not review the details of the topic here. There is, though, some chance to generate Yukawa couplings from gauge interaction [TW06].

3.4 Requirements on Calabi-Yau fourfolds for F-theory

The definition in terms of M-theory enables us to explore more exotic situations. In this section, we review two of such situations; genus-one fibration and non-flat fibration. As we will not consider such exotic situations in the main part, the review here will remind ourselves what we may loose by excluding such cases.

3.4.1 Genus-one fibration

In the physics literature, when we say that a Calabi-Yau manifold Y has an elliptic fibration, it is often assumed that Y has a **section**, i.e. a divisor[12] that is isomorphic to the base; in other words, the origin of the fiber torus is globally well-defined. Mathematicians, on the other hand, say that such manifolds have an elliptic fibration *with section*, and do not assume that the manifold has a section when simply state that the manifold has an elliptic fibration. Manifolds with an elliptic fibration without section is said to have a genus-one fibration in the physics literature, and we will follow the convention.

Let us consider the case of an F-theory on a genus-one fibered Calabi-Yau manifold. Although this is well-defined at least in the M-theoretic picture, the situation is not that exciting at first sight, as any genus-one Calabi-Yau manifold Y has a Jacobian fibration Y^J that has an elliptic fibration (with section), identical base as Y, and the same τ parameter at generic points in the base, which indicates that F-theory on Y gives exactly the same physics as the elliptic Calabi-Yau manifold Y^J, in a naïve Type IIB picture. However, it is possible that for certain Y^J a standard analysis is not applicable because of singularities, while Y is smooth. It is actually often the case,

[12]A **divisor** is a codimension-1 subvariety that is algebraically defined.

as pointed out in [BM14]. It is argued that a genus-one Calabi-Yau threefold with 2-section, i.e. has two points in the fiber that may be interchanged by monodromy but are globally well-defined as two points, gives rise to a $\mathbb{Z}_2$ gauge symmetry in the 6-dimensional effective field theory [BM14], which can also be understood as a Higgsing of $\mathrm{U}(1)$ gauge symmetry by a charge-2 matter [MT14]. A similar statement holds for Calabi-Yau fourfolds [MT14], and examples with 3-section leading to $\mathbb{Z}_3$ discrete gauge symmetry is also known [KMPO+15, CDK+15]. Phenomenological application of the discrete gauge symmetry is explored in [GEGK14, MPTW14] and subsequent papers.

The $\mathbb{Z}_2$ discrete gauge symmetry can actually be understood in the Jacobian fibration Y^J; it is shown in [MPTW15] that for a genus-one fibered Calabi-Yau threefold Y with 2-section, its Jacobian fibration Y^J has a torsional cohomology $\mathrm{Tor}(H^3(Y;\mathbb{Z})) \simeq \mathbb{Z}_2$, which results in $\mathbb{Z}_2$ discrete gauge symmetry in the effective theory, as expected. In this sense, we may keep generality even if we focus on elliptic fibration with section.

In this thesis, though, we will focus on *non-singular* elliptic Calabi-Yau fourfolds Y with section, and it is important to keep in mind that we may loose some generality in this case.

3.4.2 Non-flat fibration

In this thesis, we will also require that the elliptic fibration of the Calabi-Yau fourfold Y is **flat**, i.e. the fiber is always (complex) 1-dimensional. One often faces a non-flat fibration when exploring a general elliptic Calabi-Yau manifolds [CDF+02, LSN13, BGK13, BMPW14, AZGEM19].

In the Type IIB picture, it is not clear what we should expect from theories with non-flat fibration, but in the M-theoretic picture, one can get the idea of what will happen; at a point where the fiber is a (complex) surface, an M5-brane can wrap around the 4-cycle, leaving one spatial dimension. The remaining dimension will be seen as a tensionless string, when the F-theory limit is taken, i.e. the volume of the 4-cycle becomes zero. There is no inconsistency as a theory, but such objects are not acceptable for a phenomenological model.

However, non-flat geometry does not always imply the existence of tensionless strings, as there is a situation where an M5-brane cannot wrap the 4-cycle due to the existence of flux [FW99, AZGEM19] for example. As we will require that the elliptic fibration of Calabi-Yau fourfold Y is flat in the main part, we may lose some phenomenologically viable models.

3.5 Flux compactification

Up to this point, we have dealt with the geometry of the F-theory model, and seen how it encodes a rich structure, containing gauge symmetries and matter spectrum. There are, however, some additional ingredients that must be specified to identify a vacuum. Among them, we are interested in the background value of the 4-form flux G_4 in M-theory. The flux G_4 can be, when appropriately chosen, present in F-theory limit, and a part of the fluxes can be thought of as a generalization of 3-form fluxes H_3 and F_3 in Type IIB theory, giving masses of the order of the string scale to the complex structure moduli in F-theory. The complex structure moduli in F-theory may be interpreted as a composition of complex structure moduli, the axio-dilaton, and 7-brane moduli in Type IIB language.

To begin with, the flux G_4 is half-integer quantized such that

$$G_4 + \frac{1}{2}c_2(Y_4) \in H^4(Y_4; \mathbb{Z}) \tag{3.11}$$

as argued in [Wit97, CS12a, CS12b], but we will simply assume, in the main work of the thesis, that[13] $G_4 \in H^4(Y_4; \mathbb{Q})$.

The flux G_4 induce several terms in the action of the 3-dimensional effective theory. Firstly, there is Gukov-Vafa-Witten superpotential [GVW00, HL01]

$$W_{\mathrm{GVW}} \propto \int_{Y_4} G_4 \wedge \Omega_{Y_4}, \tag{3.12}$$

where Ω_{Y_4} is the $(4,0)$-form of Y_4. Let us decompose G_4 according to the Hodge decomposition,

$$G_4 = G_4^{4,0} + G_4^{3,1} + G_4^{2,2} + G_4^{1,3} + G_4^{0,4}, \qquad G_4^{p,q} \in H^{p,q}(Y_4; \mathbb{C}). \tag{3.13}$$

By the F-term conditions[14] $D_\alpha W = 0$ with respect to the complex structure moduli z_α, together with the supersymmetric condition $W = 0$, we obtain the condition

$$G_4 = G_4^{2,2}, \tag{3.14}$$

i.e. G_4 is purely of $(2,2)$-type in the Hodge decomposition, $G_4 \in H^{2,2}(Y_4; \mathbb{C}) \cap H^4(Y_4; \mathbb{Q})$. This is because $W = 0$ requires $G_4^{0,4} = 0$, and then $D_\alpha W = \int G_4 \wedge (D_\alpha \Omega_{Y_4}) = 0$ forces $G^{1,3} = 0$, because $D_\alpha \Omega_{Y_4}$ is a sum of $(4,0)$-component and $(3,1)$-component in general. When the Kähler potential is in the form of the large volume limit, it is actually purely $(3,1)$-type, and we will assume this hereafter. Finally, the reality of G_4 requires that it is purely of $(2,2)$-type. We will investigate this term further in this thesis. In particular, when we say a flux G_4 satisfies $W = 0$, it means that the term (3.12) vanishes, i.e. $G_4^{0,4} = 0$ and when we say a flux G_4 satisfies the F-term conditions, or simply $DW = 0$, it means that the flux satisfies $D_\alpha W = 0$ for all the complex structure moduli z_α, i.e. $G_4^{1,3} = 0$. Note that, for a rational 4-form $G_4 \in H^4(Y_4; \mathbb{Q})$, it is extremely non-trivial to satisfy such constraints concerning a Hodge decomposition. When one takes a random complex structure for Y_4, then there will be no non-trivial horizontal fluxes allowed.

The complex structure moduli are stabilized by this term (3.12); the Hodge decomposition is defined by the complex structure on Y_4, and when one deforms the complex structure, the $DW = 0$ condition is no longer satisfied, which implies that there is a potential against the complex structure deformation. The flux in fact generates mass terms in the superpotential, because a second order fluctuation of Ω_{Y_4} with

[13] $G_4 \in H^4(Y_4; \mathbb{Q})$ roughly means that G_4 is assumed to be parametrized by rational numbers, rather than integers. Note that, if one finds a vacuum with $G_4 \in H^4(Y_4; \mathbb{Q})$, then one can multiply the solution by an integer and get $G_4 \in H^4(Y_4; \mathbb{Z})$, although the solution may be excluded by the tadpole cancellation condition (3.15). The shift by $c_2(Y_4)$ does not spoil the argument in our case; $c_2(Y_4)$ is purely of Hodge $(2,2)$-type, so one can turn on additional flux in $W_{(20|02)} \subset H^4(Y_4; \mathbb{Q})$ to cancel the shift in the horizontal part; the notion of the horizontal part will be introduced later in this section, and the component $W_{(20|02)}$ will be introduced in the main part of the thesis.

[14] Recall that the covariant derivative D_α is defined as $\partial_\alpha + (\partial_\alpha K)/M_{\mathrm{Pl}}^2$, where K is the Kähler potential.

respect to the complex structure deformation contains Hodge $(2,2)$-component, and will have non-trivial inner-product with $G_4^{2,2}$, which is nothing but the mass term for the fluctuation. We will see explicitly the mass terms generated in our setup, in the main part. Conversely, when one specifies a flux configuration G_4 first and lets the dynamics find the vacuum, then the complex structure moduli will adjust themselves to satisfy the F-term conditions; it is non-trivial if they can find such configuration or not, but at least the number of constraints is equal to the number of degrees of freedom. When the dynamics find the vacuum in that way, the vev of W is expected to be random; i.e. the vev of W is expected to be $\mathcal{O}(1)$ in string scale for a generic flux vacuum, implying that the gravitino mass is also of order of the string scale. This is the problem that we will attack in this thesis.

Secondly, the Chern-Simons term in the 11d supergravity action implies the tadpole cancellation condition

$$-\frac{1}{2}\int_{Y_4} G_4 \wedge G_4 + \frac{1}{24}\chi(Y_4) - N_{\mathrm{D3}} = 0. \tag{3.15}$$

N_{D3} denotes the number of spacetime-filling M2-branes, which is spacetime-filling D3-branes in F-theory picture, and its positivity poses an upper-bound on the flux G_4. In the main part, though, we will not consider the tadpole condition.

Finally, there is a term corresponding to the primitivity condition $J_{Y_4} \wedge G_4 = 0$, where J_{Y_4} is the Kähler form of Y_4, corresponding to D-term conditions in the effective field theory of F-theory. This term constrains the Kähler structure of the geometry Y_4, rather than the complex structure, and it has little relevance to the main result of the thesis.

The cohomology $H^4(Y_4;\mathbb{C})$ can be decomposed into three orthogonal components [Str90, GMP95, BW15]:

$$H^4(Y_4;\mathbb{C}) = H^4_{\mathrm{hor}}(Y_4;\mathbb{C}) \oplus H^{2,2}_{\mathrm{ver}}(Y_4;\mathbb{C}) \oplus H^{2,2}_{\mathrm{rem}}(Y_4;\mathbb{C}). \tag{3.16}$$

The first term is called the horizontal component, and is spanned by the unique $(4,0)$-form $\Omega_{Y_4}(z+\delta z)$ of Y_4 with deformed complex structures $z+\delta z$ but with the Hodge decomposition at z. The second term is the vertical component, which is generated by the product of $(1,1)$-forms, i.e.

$$H^{2,2}_{\mathrm{ver}}(Y_4;\mathbb{C}) = \left\{ x \wedge y \mid \forall x, y \in H^{1,1}(Y_4;\mathbb{C}) \right\}, \tag{3.17}$$

and the third term is the remaining component, which is known to exist in some cases.

Physically, the horizontal part has a non-vanishing inner-product with the $(4,0)$-form and its fluctuations, while others are not. We are thus mainly concerned with the horizontal part of the flux, as our motivation is to fix the complex structure moduli, and other will not contribute to the job, and will not contribute to the vev of W_{GVW} either. Although other parts are important, e.g. to give the chirality to the matter spectrum, or to compensate the difference of $\chi(Y_4)$ before and after Higgsing involving topology change in the condition (3.15), these topics are beyond the scope of the thesis, and we will focus on the horizontal part in this work. Note also that the primitivity condition is irrelevant to our study in this sense, as the condition constrains the vertical part of the flux.

In order to make the flux G_4 compatible with the F-theory limit, there are the transversality conditions [DRS99]

$$[G_4] \cdot [S_0] \cdot \pi^*[D^B_\alpha] = 0, \qquad [G_4] \cdot \pi^*[D^B_\alpha] \cdot \pi^*[D^B_\beta] = 0, \tag{3.18}$$

For every $[D^B_\alpha], [D^B_\beta] \in H^{1,1}(B_3; \mathbb{Z})$, where B_3 is the base of the elliptic fibration of Y_4, and $[S_0]$ denotes the class of the section of the elliptic fibration[15]. This constraint mainly constrains the vertical part of the flux, so it has little relevance to our main analysis.

[15] or the class of the multi-section in the case of genus-one fibrations [LMTW16]

Chapter 4

Flux landscape problem and arithmetic approaches

As we have seen in Chapter 2, it would be nice to have all, or at least most, of the moduli to acquire masses of the order of the Planck scale, so that we can keep the standard scenario involving the Big-Bang Nucleosynthesis (BBN) and the gauge coupling unification. Fluxes will fix the complex structure moduli, while giving a large gravitino mass in generic cases, as explained in Section 3.5. In this chapter, we will review some trials to reconcile the apparent contradiction, and their problems.

4.1 Flux landscape problem

Since we are trying to construct one explicit model of our Universe, it is actually not a problem in principle that we have a heavy gravitino in *generic* cases, if one can construct a non-generic solution with a small gravitino mass. One can also argue that there could be a solution for $|W| < \epsilon$ with ϵ sufficiently small for phenomenology [Dou03, DD04, Den08]. The idea is very simple; as a Calabi-Yau fourfold typically has more than $\mathcal{O}(1000)$ 4-cycles[1] where 4-form fluxes can be turned on, i.e. $b_4 := \dim_{\mathbb{Z}} H^4(Y_4;\mathbb{Z}) \gtrsim \mathcal{O}(1000)$ typically, and the flux for each 4-cycle is parametrized by an integer, there are exponentially many flux configurations, even under the tadpole condition (3.15). Since $|W|$ cannot be exponentially large, we expect that there is a good chance to find solutions with exponentially small W. More subtle questions, such as if the number of solutions satisfying F-term conditions is exponentially suppressed, is addressed in the original articles with some assumptions, and turned out not to bother the naïve argument.

Constructing an explicit example is, though, not easy, essentially due to the integrality of the fluxes. Since the fluxes are parametrized by integers, it is not possible to solve the equations $DW = 0$ and $W = 0$ analytically, in terms of fluxes. Also, it is difficult to solve the problem numerically; it is actually shown in [DD07] that the problem *without moduli*, i.e. without $DW = 0$ conditions, is classified as NP-Complete, which is a class of problems which are believed, but not proved, to require exponential time to solve[2]. Even a quantum computer cannot solve the problem efficiently, i.e. in polynomial time, at least with known algorithms [DD07]. The problem *with*

[1] A Calabi-Yau fourfold with $b_4 = 1,819,942$ and the Euler character $\chi = 1,820,448$ is reported in [TW15].

[2] In [BBJL17] it is argued that a heuristic algorithm can find a vacuum with small cosmological constant with some probability when there is no moduli. It would be interesting to work along this line to take the moduli into account.

moduli is even harder; for a given flux configuration, one needs to solve the F-term conditions to obtain the complex structure, which determines the value of W.

4.2 Arithmetic idea

As we have seen in previous sections, although vacua with $|W| < \epsilon$ is expected to abound in the flux landscape, it is quite difficult to explicitly construct such vacua by specifying a flux G_4; after deforming the complex structure so that it satisfy the F-term conditions, one will end up with a generically $\mathcal{O}(1)$ vev of W.

For constructing an explicit example of vacua with small $\langle W\rangle$, it is useful to think in the other way around [DGKT05a]: fix the complex structure first, and then check if there is a flux configuration that satisfies the F-term conditions. This idea will fail if one chooses a generic point in the complex structure moduli space, but there is a chance if one can properly select "special points" in the moduli space. There will be a variety of choice for the "special points", and there are indeed numerous works on the subject [Moo07, DGKT05b, DGKT05a, DeW05, KW17a], [Dim08, GLV20, DKMM20].

The authors of [DGKT05a] defines[3] the "special points" in terms of the complex numbers called periods. Periods of Ω_{Y_4} in the basis $\{\gamma^i\}_i$ of $H_4(Y_4;\mathbb{Z})$ is defined as

$$\Pi_i = \int_{\gamma^i} \Omega_{Y_4} \in \mathbb{C} \quad \text{with} \quad i = 1, \dots, b_4(Y_4), \tag{4.1}$$

where $b_4(Y_4)$ is the dimension of $H^4(Y_4;\mathbb{Z})$. We will assume the normalization of Ω_{Y_4} such that $\Pi_1 = 1$. The authors of [DGKT05a] chose the "special points" where the periods take special values that we will specify below.

Before going into the details of [DGKT05a], let us also express the flux G_4 in the basis $\{e_i\}_i$ of $H^4(Y_4;\mathbb{Q})$ where each e_i is Poincaré dual to γ^i, i.e. $\int_{Y_4} \alpha \wedge e_i = \int_{\gamma^i} \alpha$,

$$G_4 = \sum_i n^i e_i, \qquad\qquad n^i \in \mathbb{Q}. \tag{4.2}$$

Then we can rewrite the superpotential as a linear combination of complex numbers with rational coefficients,

$$\int_{Y_4} G_4 \wedge \Omega_{Y_4} = \sum_i n^i \Pi_i. \tag{4.3}$$

One can also define the periods of $(3,1)$-forms, such that

$$\xi_i^a = \int_{\gamma^i} \Xi_{Y_4}^a, \quad a = 1, \dots, h^{3,1}(Y_4), \tag{4.4}$$

where $\left\{\Xi_{Y_4}^a\right\}_a$ is a basis of $H^{3,1}(Y_4;\mathbb{C})$.

To solve the problem, the authors of [DGKT05a] considered Calabi-Yau manifolds[4] with the following property: for all periods of the $(4,0)$-form and $(3,1)$-forms, $\{\Pi_i\}_i$ and $\{\xi_i^a\}_i^a$,

$$\Pi_i \in F, \quad \xi_i^a \in F \quad \text{for some number field } F \subset \mathbb{C}. \tag{4.5}$$

[3] The authors of [DGKT05a] deal with the vacuum counting problem in Type IIB theory, rather than in F-theory, but we will review their ideas applied to F-theory.

[4] threefolds in the paper, but fourfolds in our review

We will introduce the notion of *fields*[5] in a more mathematical fashion in Appendix A.1, but let us introduce them briefly.

A **field** F is defined to be, roughly speaking, a set on which the four arithmetic operations are defined consistently. For example, the set of rational numbers $\mathbb{Q}$ is a field, but the set of integers $\mathbb{Z}$ is not a field because the division operation sometimes leads to a number which is outside of $\mathbb{Z}$. In particular, a **number field** is a field that contains $\mathbb{Q}$ as a subset, with consistently extended four arithmetic operations of $\mathbb{Q}$. We are particularly interested in number fields that are sub-fields of $\mathbb{C}$ in this section, but a number field is a much broader concept, as we will see in the next chapter; see also Appendix A.1. The best thing about the number fields in our problem is that most of the number fields can be regarded as a finite-dimensional vector space over $\mathbb{Q}$. For example, $\mathbb{Q}(i\sqrt{2})$ defined as

$$\mathbb{Q}(i\sqrt{2}) := \left\{ p_0 + p_1 i\sqrt{2} \mid p_0, p_1 \in \mathbb{Q} \right\} \tag{4.6}$$

is a number field and is a 2-dimensional vector space over $\mathbb{Q}$ with $\left\{1, i\sqrt{2}\right\}$ being the basis; one can check that it is actually closed under the four arithmetic operations. In general, if α is a root of degree-n polynomial with $\mathbb{Q}$-coefficients, $\mathbb{Q}(\alpha)$, defined as

$$\mathbb{Q}(\alpha) := \left\{ p_0 + p_1\alpha + \cdots + p_{n-1}\alpha^{n-1} \mid \forall i, \quad p_i \in \mathbb{Q} \right\}, \tag{4.7}$$

is a number field and is an n-dimensional vector space over $\mathbb{Q}$. The dimension as a vector space is called the **extension degree** of the **extension** $\mathbb{Q}(\alpha)/\mathbb{Q}$ and is denoted by $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. Furthermore, any finite-dimensional number field can be expressed in the form of $\mathbb{Q}(\alpha)$ for some α, as formally stated in Lemma A.1.

Now let us get back to our problem. Our starting point is a Calabi-Yau fourfold Y_4 with the periods of the basis Ω_{Y_4} and $\Xi^a_{Y_4}$ of $H^{4,0}(Y_4;\mathbb{C})$ and $H^{3,1}(Y_4;\mathbb{C})$ taking their values in a number field $F = \mathbb{Q}(\alpha) \subset \mathbb{C}$. Denote the extension degree by $n_F := [F : \mathbb{Q}]$. Now we can demand the $DW = 0$ conditions for complex structure moduli, and also the $W = 0$ condition, in a form that we can solve in terms of fluxes, in the following sense. Consider, as an example, the $W = 0$ condition

$$\sum_i \Pi_i n^i = \sum_i \sum_{k=0}^{n_F-1} \pi_{i,k}\alpha^k n^i = 0. \tag{4.8}$$

The first equality expands Π_i with respect to the basis of $\mathbb{Q}(\alpha)$, which is regarded as a vector space over $\mathbb{Q}$. Now the $W = 0$ condition is equivalent to demanding

$$\sum_i \pi_{i,k} n^i = 0 \tag{4.9}$$

for all k. This equality involves rational numbers only, and we can solve each equality in terms of a flux $n^i \in \mathbb{Q}$ for some i, if there is enough degrees of freedom for fluxes. The counting goes as follows: for each condition $DW = 0$ or $W = 0$, we need n_F fluxes to satisfy the equality, and there are $h^{3,1} + 1$ conditions. The number of total fluxes equals $b_4(Y_4)$, so the degrees of freedom left after demanding $DW = 0$

[5] In Japanese, fields are called "体（たい）". French, German and many other languages also call the object "body" in their own words.

and $W = 0$ $N_{DW,W}$ is

$$N_{DW,W} = b_4^H - n_F(h^{3,1}+1) = (2-n_F)(h^{3,1}+1) + h_H^{2,2}, \tag{4.10}$$

where the sub/superscript H signifies that we are focusing on the horizontal part of the flux. For example, if $n_F = 2$, which is minimal for having $\Omega_{Y_4} \neq \overline{\Omega}_{Y_4}$, there are nontrivial solutions if $h_H^{2,2} > 0$. In the examples described in [BW15, Tables 1-3], one can observe that $h_H^{2,2}/h_H^{3,1} \sim 4$, so in that case, larger n_F up to $n_F = 6$ may have $W = 0$ vacua.

Although this direction seems promising, there is a caveat in the assumption; it is in general difficult to engineer a Calabi-Yau fourfold with its periods Π_i and ζ_i^a taking values in a number field F; we will elaborate on this later in Section 5.3, but the reason is roughly because the space of periods has larger dimension $b_4 - 1$ than that of the complex structure moduli space $h^{3,1}$ and thus one cannot choose the periods freely. The situation does not change in the situation of the original article [DGKT05a], where the Type IIB counterpart of what we have discussed is considered, although they present some examples including the mirror quintic.

4.3 Type IIB theory on CM-type Calabi-Yau threefolds

There is, however, a class of Calabi-Yau fourfolds that satisfies the condition (4.5), which is called CM-type[6] Calabi-Yau fourfolds. We will introduce CM-type Calabi-Yau manifolds in Chapter 5, and the flux counting on CM-type Calabi-Yau fourfolds in F-theory is one of our main topic in this thesis, but there is a previous work of flux counting on CM-type Calabi-Yau manifolds in Type IIB theory, done by the author and his supervisor [KW17a]. There, it was found that one can construct infinitely many $W = 0$ vacua of the form $(\mathrm{K3} \times T^2)/\mathbb{Z}_2$. The caveat is that, as the analysis is restricted to the Type IIB theory, it is difficult to construct a model with a realistic particle spectrum. The aim of the main part of the thesis is to generalize the analysis of [KW17a] to F-theory, and work out the possible gauge groups and matter representations in such a setup. Since our F-theory analysis contains the orientifold analysis of [KW17a], we review the relation in Appendix C.

[6] CM stands for Complex Multiplication, as will be explained later.

Chapter 5

CM-type Calabi-Yau manifolds

5.1 CM-type Calabi-Yau manifolds

In this section, we introduce CM-type Calabi-Yau n-folds. Let us start with one of the simplest Calabi-Yau manifolds, the elliptic curve, to see why it is called CM (Complex Multiplication)-type.

Definition 5.1 (CM-type elliptic curve)**.** Take an elliptic curve E with a modular parameter τ. E is **of CM-type** if and only if $\tau = p + i\sqrt{q}$ for some $p, q \in \mathbb{Q}$. We will also call E a **CM-type elliptic curve**.

CM-type elliptic curves are special because of the following property. Take $\tau = p + i\sqrt{q}$ for some $p, q \in \mathbb{Q}$. τ satisfies

$$\tau^2 - 2p\tau + p^2 + q = 0 \tag{5.1}$$

and the $(1,0)$-form Ω_E of the elliptic curve E, expressed in an integral basis, has a *rational* map $\mathcal{T}$

$$\mathcal{T} : \begin{pmatrix} 1 \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -p^2 - q & 2p \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix} = \tau \begin{pmatrix} 1 \\ \tau \end{pmatrix}. \tag{5.2}$$

$\mathcal{T}$ acts as a multiplication by $\tau \in \mathbb{C}$ on Ω_E, i.e. $\Omega_E \mapsto \tau\Omega_E$, whereas it is defined as a *rational* map[1] $\mathcal{T} : H^1(E;\mathbb{Q}) \to H^1(E;\mathbb{Q})$; The action of $\mathcal{T}$ is called **complex multiplication** and thus the name CM-type. Note that the action of $\mathcal{T}$, combined with a scalar multiplication by $\mathbb{Q}$, generates a field

$$K := \{p_0 I + p_1 \mathcal{T} \mid p_0, p_1 \in \mathbb{Q}\}\,, \tag{5.3}$$

where I is an identity operator. This field turns out to be equal to the endomorphism algebra of $H^1(E;\mathbb{Q})$ that preserves the Hodge decomposition

$$K = \mathrm{End}_{\mathrm{Hdg}}(H^1(E;\mathbb{Q})) := \left\{\varphi \in \mathrm{End}_{\mathbb{Q}}(H^1(E;\mathbb{Q})) \mid \varphi(H^{p,q}(E;\mathbb{C})) \subset H^{p,q}(E;\mathbb{C})\right\}. \tag{5.4}$$

We will extend this final form to general CM-type Calabi-Yau manifolds, after several definitions. The concepts that will be introduced in the following are important not only for defining general CM-type Calabi-Yau manifolds, but also in the main analysis of flux counting.

We first define the Hodge structure on a general vector space over $\mathbb{Q}$, and its simpleness.

[1]We will assume that, for a Calabi-Yau n-fold Y_n, $H^n(Y_n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq H^n(Y_n;\mathbb{Q})$ throughout the thesis.

Definition 5.2 (rational Hodge structure)**.** Let $V_{\mathbb{Q}}$ be a vector space over $\mathbb{Q}$. A decomposition of the vector space $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ over $\mathbb{C}$ into the form of

$$V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \cong \oplus_{p+q=n} V_{\mathbb{C}}^{p,q} \qquad \left(\overline{V_{\mathbb{C}}^{p,q}} = V_{\mathbb{C}}^{q,p}\right) \tag{5.5}$$

of vector subspaces $V_{\mathbb{C}}^{p,q}$ for non-negative integers p, q, and n, is called a **rational Hodge structure of weight** n. For a smooth compact Kähler manifold M, the cohomology group $H^n(M;\mathbb{Q})$ has a rational Hodge structure of weight n given by the complex structure of the Kähler manifold M, for example.

Definition 5.3 (simple rational Hodge structure)**.** A rational Hodge structure on a vector space $V_{\mathbb{Q}}$ is said to be **simple**, if there is no vector proper subspace $W_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ over $\mathbb{Q}$ so that $\oplus_{p,q}(V_{\mathbb{C}}^{p,q} \cap (W_{\mathbb{Q}} \otimes \mathbb{C}))$ reproduces $(W_{\mathbb{Q}} \otimes \mathbb{C})$. When such a proper subspace $W_{\mathbb{Q}}$ exists, $V_{\mathbb{Q}}$ [resp. $W_{\mathbb{Q}}$] is said to have [resp. to support] a **rational Hodge sub-structure**. When a vector space $V_{\mathbb{Q}}$ with a rational Hodge structure is decomposed into vector subspaces over $\mathbb{Q}$, $V_{\mathbb{Q}} \cong \oplus_{a \in A} W_a$, and each W_a supports a rational Hodge sub-structure that is simple, we say that it is a **simple component decomposition of the rational Hodge structure**.

Definition 5.4 (level of simple rational Hodge structure)**.** A simple component W_a in such a decomposition is said to be **level-ℓ**, when $\ell := \mathrm{Max}(|p-q|; V^{p,q} \cap (W_a \otimes \mathbb{C}) \neq 0)$.

We will see an example of such decomposition when we introduce the K3 surface.

We now define a rational Hodge structure of CM-type, which will be used to define CM-type Calabi-Yau manifolds.

Definition 5.5 (rational Hodge structure of CM-type)**.** When a vector space $V_{\mathbb{Q}}$ over $\mathbb{Q}$ has a rational Hodge structure that is simple, then the algebra of endomorphisms of the simple rational Hodge structure

$$L := \mathrm{End}_{\mathrm{Hdg}}(V_{\mathbb{Q}}) := \{\varphi \in \mathrm{End}_{\mathbb{Q}}(V_{\mathbb{Q}}) \mid \varphi(V^{p,q}) \subset V^{p,q}\} \tag{5.6}$$

is always a division algebra[2]. When L contains a number field K such that $[K:\mathbb{Q}] = \dim_{\mathbb{Q}} V_{\mathbb{Q}}$, we say that the simple rational Hodge structure is **of CM-type**. We will call the field K the **endomorphism field** of $V_{\mathbb{Q}}$. A rational Hodge structure that is not necessarily simple is said to be **of CM-type** if all of its simple rational Hodge components are of CM-type.

We finally define CM-type Calabi-Yau manifolds.

Definition 5.6 (CM-type Calabi-Yau manifold)**.** Let Y be a Calabi-Yau manifold with a complex structure $z \in \mathcal{M}^{[Y]}_{\text{cpx str}}$ in the complex structure moduli space $\mathcal{M}^{[Y]}_{\text{cpx str}}$ of Y. Y is said to be **of CM-type** if every simple Hodge component of $H^n(Y;\mathbb{Q})$ is of CM-type. The corresponding point z in the moduli space is called a **CM point**.

The definition turns out to be equivalent to the previous one, Definition 5.1, in the case of elliptic curves, i.e. Calabi-Yau 1-folds.

[2]Roughly speaking, a **division algebra** is an algebra over a field in which division is always possible, except by zero. A rigorous definition is not necessary for our purpose.

5.1.1 A frequently used property

We will now show that any CM-type Calabi-Yau manifold satisfies the condition (4.5). It actually satisfies a much more non-trivial condition, which will be frequently used in this thesis.

We first introduce several notions of number field theory to state the afore-mentioned non-trivial condition. An **embedding** of a number field K over $\mathbb{Q}$ is an injective homomorphism $K \hookrightarrow \mathbb{C}$. As is shown in Lemma A.1, K is always of the form $K = \mathbb{Q}(\alpha)$ for some $\alpha \in K$, so an embedding ρ can be specified by the image $\rho(\alpha)$ of α, which should be one of the roots in $\mathbb{C}$ of the minimal polynomial[3] of α over $\mathbb{Q}$. As there are $[K : \mathbb{Q}]$ different roots for the minimal polynomial of α, there are also $[K : \mathbb{Q}]$ distinct embeddings. We consider the minimal field $K^{\mathrm{nc}} \subset \mathbb{C}$ that contains the image of any of the embeddings of K, and call it the **normal closure**[4] of K. Now we are ready to state the non-trivial property of CM-type Hodge structure.

Proposition 5.1. Let $V_{\mathbb{Q}}$ be a vector space over $\mathbb{Q}$, and $K \subset \mathrm{End}_{\mathbb{Q}}(V_{\mathbb{Q}})$ is a field with extension degree $n_K := [K : \mathbb{Q}] = \dim_{\mathbb{Q}} V_{\mathbb{Q}}$. Then, firstly, the action of $K \subset \mathrm{End}_{\mathbb{Q}}(V_{\mathbb{Q}})$ on $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} K^{\mathrm{nc}}$ can be diagonalized simultaneously; to be more specific, $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} K^{\mathrm{nc}}$ has a diagonalization basis

$$V_{\mathbb{Q}} \otimes_{\mathbb{Q}} K^{\mathrm{nc}} = \mathrm{Span}_{K^{\mathrm{nc}}}\{v_a \mid a = 1, \cdots, \dim_{\mathbb{Q}} V_{\mathbb{Q}}\}, \tag{5.7}$$

there is a 1-to-1 correspondence between those $\dim_{\mathbb{Q}} V_{\mathbb{Q}}$ basis elements and the set of all the n_K embeddings $\Phi_K := \mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$, and

$$x \cdot v_a = v_a \rho_a(x), \qquad \rho_a \in \Phi_K, \quad {}^\forall x \in K. \tag{5.8}$$

Moreover, when we express the eigenvectors v_a as K^{nc}-coefficient linear combinations of a $\mathbb{Q}$-basis $\{e_i \mid i = 1, \cdots, \dim_{\mathbb{Q}} V_{\mathbb{Q}}\}$ of $V_{\mathbb{Q}}$, $v_a = \sum_i e_i c_a^i$, there exists, up to a normalization of each v_a, a basis $\{y_i \mid i = 1, \cdots, n_K\}$ of the vector space K over $\mathbb{Q}$ so that

$$c_a^i = \rho_a(y_i). \tag{5.9}$$

Proof. To show the first part, recall that K can be regarded as a simple extension of $\mathbb{Q}$, i.e. $K = \mathbb{Q}(\alpha)$ for some $\alpha \in \mathrm{End}_{\mathbb{Q}}(V_{\mathbb{Q}})$. α has n_K distinct eigenvalues, which are the roots in $\mathbb{C}$ of the minimal polynomial of α over $\mathbb{Q}$. As the roots are in K^{nc}, eigenvectors $\{v_a\}_a$ of α are in $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} K^{\mathrm{nc}}$. The n_K roots are in 1-to-1 correspondence with the n_K embeddings in Φ_K, so the eigenvectors are also in 1-to-1 with the embeddings. Note that any element in $x \in K$ is diagonalized using the eigenvectors, as x can be written as sum of powers of α.

For the second statement, let $\{x_{p=1,\cdots,n_K}\}$ be a $\mathbb{Q}$-basis[5] of K. Denoting the matrix representation of $x \in K$ in the $\mathbb{Q}$-basis $\{e_i\}_i$ of $V_{\mathbb{Q}}$ by $A_{ji}(x)$,

$$A_{ji}(x_p) c_a^i = c_a^j \rho_a(x_p). \tag{5.10}$$

[3]The minimal polynomial of $x \in K$ over $\mathbb{Q}$ is the $\mathbb{Q}$-coefficient polynomial of which x is a root. See Definition A.7.

[4]See Appendix A.1.4 for details.

[5]One can take $x_p = \alpha^{p-1}$.

This relation holds for any j, so one can pick some $j = j_a$ for a fixed a such that $c_a^{j_a} \neq 0$ and reorganize the relations; one can write

$$\left[A_{j_a i}(x_p)\right]_{pi} \left[\frac{c_a^i}{c_a^{j_a}}\right]_i = \left[\rho_a(x_p)\right]_p, \tag{5.11}$$

where a $\mathbb{Q}$-valued matrix $[\cdots]_{pi}$ is multiplied to a $\mathbb{C}$-valued vector $[\ \ ]_i$ to be a $\mathbb{C}$-valued vector $[\ \ ]_p$ on the right-hand side. We see that the $\mathbb{Q}$-valued matrix must be invertible; this is because x_p's in K (and hence $\rho_a(x_p)$'s in $\rho_a(K)$) should be linearly independent over $\mathbb{Q}$. We replace the $\mathbb{Q}$-basis of K $\{x_{p=1,\cdots,n_K}\}$ by the one, denoted by $\{y^{(a)}_{i=1,\cdots,n_K}\}$, obtained by multiplying the inverse of the $\mathbb{Q}$-valued matrix $[A_{j_a i}(x_p)]_{pi}$ on x_p's. In this new basis, we have the relation

$$\left[c_b^i\right]_i = \left[\rho_b(y_i^{(a)})c_b^{j_a}\right]_i \tag{5.12}$$

for any $b = 1, \ldots, n_K$. This implies that $c_b^{j_a} \neq 0$ for any b; otherwise $v_b = 0$ will follow. After normalizing the vectors such that $c_b^{j_a} = 1$ for all b, we have the relation (5.9), denoting the basis $\left\{y_i^{(a)}\right\}_i$ for the fixed a simply by $\{y_i\}_i$.

□

In the context of this thesis, we will use the property above for $V_{\mathbb{Q}}$ as a rational Hodge component of CM-type. In that context, each one of the eigenvectors, say, v_a, belongs to a definite Hodge (p_a, q_a) component, as the endomorphism field K preserves the Hodge decomposition. This means that, when a Calabi-Yau fourfold Y is of CM-type, the condition (4.5) holds in a stronger form (5.9); Eq. (5.9) implies[6] not only that each of Π_i and ξ_i^a in Eq. (4.5) takes values in some number field $F = K^{\mathrm{nc}}$, where K is the endomorphism field, but also that the difference of any pair of them entirely comes from the difference of the embeddings ρ_a, if both of the pair are contained in the same simple rational Hodge component.

Let us make the relation among v_a's more precise; one can map v_a to v_b for any a, b, by an action of a Galois group, in the following sense. When the field $F := \rho_1(K)$ is Galois, i.e. $F \cong K^{\mathrm{nc}}$, then $\rho_a(K) = F$ for any a and $\rho_a \circ (\rho_b)^{-1} \in \mathrm{Gal}(F/\mathbb{Q})$ maps the algebraic number $c_b^i \in F \subset \overline{\mathbb{Q}}$ to $c_a^i \in F \subset \overline{\mathbb{Q}}$ for all $i = 1, \cdots, n_K$ simultaneously; this phenomenon is observed in [DGKT05a]. Even when the field F is not a Galois extension over $\mathbb{Q}$, an isomorphism $\rho_a \circ (\rho_b)^{-1} : \rho_b(F) \to \rho_a(F)$ extends to an isomorphism from $\overline{\mathbb{Q}}$ to itself (Thm. 2.19, [Fuj91]), which can be restricted to an automorphism of a normal extension F^{nc} over $\mathbb{Q}$. Thus, $\rho_a \circ (\rho_b)^{-1} : \rho_b(F) \to \rho_a(F)$ can be realized by restricting some elements in $\mathrm{Gal}(F^{\mathrm{nc}}/\mathbb{Q})$. Therefore, the simultaneous map of algebraic numbers $c_b^i \in \rho_b(F)$ to $c_a^i \in \rho_a(F)$ can be regarded as an action in $\mathrm{Gal}(F^{\mathrm{nc}}/\mathbb{Q})$.

[6] c_a^i is related to Π_i by a multiplication by the intersection matrix. Let us fix some a such that v_a is a $(4,0)$-form, i.e. $\Omega_Y = v_a$. Then

$$\Pi_i = \int_{\gamma^i} \Omega_Y = \int_{\gamma^i} c_a^j e_j = \int_Y c_a^i e_j \wedge e_i =: c_a^i M_{ji}, \tag{5.13}$$

where M_{ji} is the intersection matrix and we have assumed that $\left\{\gamma^i\right\}_i$ is Poincaré dual to $\{e_i\}_i$. Because M_{ji} is a $\mathbb{Q}$-valued matrix, Π_i being contained in a number field is equivalent to c_a^i being contained in a number field. The same argument applies for ξ_a^i.

It can be shown that, when[7] $\dim V_{\mathbb{Q}} > 1$ and the Hodge structure is non-trivial, then K is always a **CM field**. The rigorous definition and some properties of CM fields are reviewed in Appendix A.2, but a CM field K is a field on which complex conjugation is consistently defined, its the extension degree $[K : \mathbb{Q}]$ is even, and the embeddings of K into $\mathbb{C}$ are in pairs, such that the complex conjugation maps an embedding to its partner within the pair.

5.2 CM-type K3 surface

In the subsequent chapters, we will use CM-type K3 surfaces to construct our F-theory vacua. In this section, we introduce the K3 surface, its CM-type complex structure, and their properties that is relevant to our analyses.

5.2.1 K3 surface

We introduce one of the simplest examples of Calabi-Yau manifolds, K3 surface. K3 surface is defined to be a Ricci-flat 2-dimensional complex manifold which has $h^{1,0} = 0$. The integral cohomology $H^2(X;\mathbb{Z})$ of a K3 surface X endowed with the intersection form $(\ ,\ )$ is known to be isomorphic to the lattice

$$H^2(X;\mathbb{Z}) \cong E_8 \oplus E_8 \oplus U \oplus U \oplus U. \tag{5.14}$$

Here, E_8 is the negative definite root lattice[8] of E_8, and U is the hyperbolic plane lattice. This is the signature $(3,19)$ unimodular lattice, often denoted by $\mathrm{II}_{3,19}$. In this sense, the K3 surface is unique. An intuitive presentation of each 2-cycle is given in [Asp98].

Although unique in the homological sense, K3 surface has a rich structure, when one considers its complex structure. The complex structure of a K3 surface X is specified by the $(2,0)$-form Ω_X, which is unique up to rescaling. We can use Ω_X to define the Néron-Severi lattice S_X

$$S_X := \left\{x \in H^2(X;\mathbb{Z}) \mid (x, \Omega_X) = 0\right\} \tag{5.15}$$

and the transcendental lattice T_X as its orthogonal complement

$$T_X := \left\{x \in H^2(X;\mathbb{Z}) \mid (x, y) = 0 \text{ for all } y \in S_X\right\}, \tag{5.16}$$

where T_X has a signature $(2, 20-\rho)$ and S_X a signature $(1, \rho-1)$. As can be seen explicitly, the decomposition into S_X and T_X is determined by the $(2,0)$-form Ω_X, and it turns out that each element in S_X represents algebraic complex curves, i.e. curves defined by algebraic equation(s), related by the Poincaré duality. For example, a 2-dimensional sub-lattice isomorphic to U in S_X can be thought of as a set of elliptic fiber and a section, and if there is ADE negative root lattices orthogonal to U in S_X, it signifies that there is a singular fiber of corresponding ADE type. Taking T_X large, i.e. Ω_X generic, corresponds to deforming the singularity that gives rise to the singular fibers, which results in S_X with lower rank.

It can be further shown that there is one-to-one correspondence between the choice of $\Omega_X \in H^2(X;\mathbb{C})$ and a complex structure of K3 surface. More precisely, let us

[7] If $V_{\mathbb{Q}}$ is 1-dimensional, then its Hodge structure is trivial, i.e. $V_{\mathbb{Q}} = V^{p,p}$ for some p, and its endomorphism field is $\mathbb{Q}$.

[8] The lattice is sometimes denoted by $E_8[-1]$.

define the **period domain** $D(\Lambda)$ of a lattice Λ:

$$D(\Lambda) := \left\{\Omega \in \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C}) \mid (\Omega, \Omega) = 0,\ (\Omega, \overline{\Omega}) > 0\right\}. \tag{5.17}$$

The conditions in the brace is the properties that the $(n, 0)$-form of a Calabi-Yau n-fold Y should satisfy, where Λ is taken to be $H^n(Y; \mathbb{Z})$ or its sub-lattice. Let us choose T_X as a sub-lattice of $H^2(X; \mathbb{Z})$ with signature $(2, 20 - \rho)$. Then, it turns out that the space $D(T_X)$ parametrizes the complex structure of K3 surfaces which has T_X as the transcendental lattice [PSS71]; see [BKW14, §3.1], [Huy16] for review. Note that this is not true for Calabi-Yau n-folds with $n \geq 3$, as the dimension of period domain is larger than that of the complex structure moduli space.

5.2.2 CM-type K3 surface

Now let us apply the definition of CM-type to K3 surfaces. Note first that S_X is purely of Hodge $(1, 1)$-type, and thus it is decomposed to 1-dimensional Hodge sub-structures. It has an endomorphism field $\mathbb{Q}$ and thus it is, in a sense, a trivial CM-type simple Hodge structure. There is one non-trivial simple Hodge component, T_X. If T_X were not simple, then one of the simple Hodge components that does not contain the $(2, 0)$-form should be contained in S_X, thus T_X must have a simple Hodge structure. We will simply say that a K3 surface X is of CM-type with the endomorphism field K, meaning that the simple Hodge component T_X is of CM-type with the endomorphism field K.

5.3 CM-type Calabi-Yau fourfolds and Borcea-Voisin orbifolds

In this section, we review how the CM-type manifolds are distributed in the complex structure moduli space, and discuss how to construct CM-type Calabi-Yau fourfolds.

In the case $Y = E$ is an elliptic curve, a one-dimensional Calabi-Yau manifold, The set of CM points $\mathcal{M}^{[E]}_{\mathrm{CM}}$ in the moduli space of complex structure of elliptic curves $\mathcal{M}^{[E]}_{\text{cpx str}} \cong \mathcal{H}/\mathrm{SL}(2; \mathbb{Z})$ is completely understood; CM points in the upper complex half-plane $\mathcal{H}$ are the set of the roots of any quadratic polynomial of one variable with coefficients in $\mathbb{Q}$. They are labeled by the imaginary quadratic fields K; the CM points sharing the same imaginary quadratic field form an orbit under the action of $\mathrm{GL}(2; \mathbb{Q}) = \mathbb{G}Sp(2; \mathbb{Q})$. In the case $Y = X$ is a K3 surface with a transcendental lattice T_X, it is also known that any CM point in the moduli space $\mathcal{M}^{[X(T_X)]}_{CM}$ is associated with a CM field K of degree $[K : \mathbb{Q}] = \mathrm{rank}(T_X)$; the CM points sharing the same CM field K form orbits under the action of the group[9] $\mathbb{G}O(T_X; \mathbb{Q})$ on $\mathcal{M}^{[X(T_X)]}_{CM} \subset \mathcal{M}^{[X(T_X)]}_{\text{cpx str}} = \mathrm{Isom}(T_X) \backslash D(T_X)$; here, $\mathrm{Isom}(T_X)$ is the group of integral isometries of T_X. In particular, we know that there are infinitely many CM points in the moduli space of the complex structure of elliptic curves and K3 surfaces.

When it comes to the case $Y = M$ is a Calabi-Yau threefold, or a Calabi-Yau fourfold Y, however, much less is known. It is believed that the Calabi-Yau threefolds M realized by rational CFT's have complex structure of CM-type [GV04a], but they are nothing more than a small number of isolated points in the moduli space. Although the group $\mathrm{Sp}(b_3(M))$ is a symmetry of some of the relations that the Hodge structure of a Calabi-Yau threefold M satisfies, yet the action of the group takes a point in

[9]The groups $\mathbb{G}Sp$ and $\mathbb{G}O$ consist of linear transformations that preserve skew-symmetric and symmetric bilinear forms, respectively, up to overall scalar multiplications.

$\mathcal{M}^{[M]}_{\text{cpx str}}$ outside of $\mathcal{M}^{[M]}_{\text{cpx str}}$ in general; the latter observation also holds true in the case Y is a Calabi-Yau fourfold, when the group $\mathrm{Sp}(b_3(M))$ is replaced by the isometry group of the lattice $H^4(Y;\mathbb{Z})$. So, in particular, we do not have an argument in the case Y is a threefold or a fourfold that infinitely many CM points $\mathcal{M}^{[Y]}_{\text{CM}}$ show up in the form of orbits of $\mathbb{G}Sp(b_3)$ or $\mathbb{G}O(b_4(Y))$.[10] Indeed, the André-Oort conjecture and Coleman-Oort conjecture hint that there are not so many CM points available in $\mathcal{M}^{[Y]}_{\text{cpx str}}$ in those cases. For more information, see [KW17b, §2.2].

For a special class of topological types of Calabi-Yau threefolds $[Y = M]$ or of fourfolds $[Y]$, however, it is possible to identify systematically a set of points ($\langle z\rangle$'s) of $\mathcal{M}^{[Y]}_{\text{cpx str}}$ where $H^3(Y_{\langle z\rangle};\mathbb{Q})$ or $H^4(Y_{\langle z\rangle};\mathbb{Q})$ has a CM-type rational Hodge sub-structure.[11] An idea, originally given in [Bor92, Voi02], is to take a product of a CM-type elliptic curve E and a CM-type K3 surface, or of a pair of CM-type K3 surfaces, first, and then to take an orbifold that preserves the Calabi-Yau condition. Not all the topological types available for a Calabi-Yau three/four-fold will be realized in this construction. The moduli space $\mathcal{M}^{[Y]}_{\text{cpx str}}$ of a three/four-fold Y constructed in that way contains an orbifold locus $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$ where the orbifold singularity of $Y_{\langle z\rangle}$ is not deformed in complex structure;[12] as long as the building block E or K3 surfaces are of CM-type, and the vacuum choice $\langle z\rangle$ of the complex structure of Y_z is in the orbifold locus $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$, then $H^3(Y_{\langle z\rangle};\mathbb{Q})$ or $H^4(Y_{\langle z\rangle};\mathbb{Q})$ has a rational Hodge sub-structure of CM-type indeed.

The simplest class of Calabi-Yau fourfolds Y being K3 $\times$ K3 orbifolds is of the form $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$. Both of the K3 surfaces $X^{(1)}$ and $X^{(2)}$ are assumed to have a non-symplectic automorphism of order two, $\sigma_{(1)}$ and $\sigma_{(2)}$, respectively; an automorphism is said to be **non-symplectic** when the holomorphic (2,0)-forms $\Omega_{X^{(1)}}$ or $\Omega_{X^{(2)}}$ acted non-trivially, $\sigma^*_{(i)}(\Omega_{X^{(i)}}) = -\Omega_{X^{(i)}}$ for $i = 1, 2$. By choosing the generator σ of the orbifold group $\mathbb{Z}_2$ to be $(\sigma_{(1)}, \sigma_{(2)})$, the orbifold Y becomes Calabi-Yau because $(\Omega_{X^{(1)}} \wedge \Omega_{X^{(2)}})$ is invariant under the generator σ, yet $\Omega_{X^{(i)}}$'s are not. We call such fourfolds $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ as Borcea-Voisin K3 $\times$ K3 orbifolds, or simply as Borcea-Voisin orbifolds, in this thesis; more general orbifolds will be called generalized Borcea-Voisin K3 $\times$ K3 orbifolds. Until the end of Chapter 6, we deal with M-theory compactifications on a Borcea-Voisin fourfold with an F-theory application in mind.

Reference [Nik81] provides a theory of topological classification of a pair (X, σ) of a K3 surface X and an automorphism $\sigma \in \mathrm{Aut}(X)$ of order two ($\sigma^2 = \mathbf{Id}_X$) acting non-symplectically ($\sigma^*\Omega_X \neq \Omega_X$) on the holomorphic (2, 0)-form Ω_X. To be more precise, it classifies (S_0, T_0, σ) modulo isometry of $H^2(X;\mathbb{Z})$, where S_0 and T_0 are mutually orthogonal primitive sub-lattices of $H^2(X;\mathbb{Z})$ of signature $(1, r-1)$ and $(2, 20-r)$, respectively, and σ an isometry of $H^2(X;\mathbb{Z})$ that acts trivially on S_0 and as $(-1)\times$ on T_0. The lattice is completely classified by three integers (r, a, δ); r is

[10]This argument still does not rule out infinitely many CM points; in fact infinitely many CM points are contained in the 101-dimensional moduli space of the quintic Calabi-Yau threefolds (e.g., see [KW17b, footnote 18] for references). The Fermat sextic fourfold [BV20] is CM-type (e.g., [Yui03]).

[11]It is a stronger condition for a rational Hodge structure on $H^4(Y;\mathbb{Q})$ to be of CM-type than for it to have a rational Hodge sub-structure that is of CM-type. See the discussion at the end of Section 6.2. Whether the Coleman-Oort conjecture is relevant in the current context (whether supersymmetric flux is available for moduli stabilization) should also be reconsidered along this line.

[12]We will not discuss the choice of Kähler moduli in this thesis. Whether the orbifold singularity is resolved or not, discussions in this work are valid.

the rank of S_0, as already used, and the other two also characterizes the sub-lattices. See Figure 5.1. This lattice-theory classification of (S_0, T_0, σ) is regarded as that of non-symplectic automorphisms of order two, because one may choose $\mathbb{C}\Omega_X$ from $D(T_0)/\mathrm{Isom}(T_0)$. For such a complex structure, the transcendental lattice T_X is contained within T_0, and $\sigma^*\Omega_X = -\Omega_X$; the Néron-Severi lattice S_X contains S_0. The list of [Nik81] consists of 75 choices of (S_0, T_0, σ). Therefore, we have 75 choices of $(S_0^{(i)}, T_0^{(i)}, \sigma_{(i)})$ for each one of $i = 1, 2$; for a given choice, a topological family of Borcea-Voisin orbifolds is available for M-theory compactification. In the next chapter, supersymmetric flux configurations are studied for a vacuum complex structure in

$$\mathcal{M}_{\mathrm{CM}}^{[X(T_0^{(1)})]} \times \mathcal{M}_{\mathrm{CM}}^{[X(T_0^{(2)})]} \subset \mathcal{M}_{\text{cpx str}}^{[X(T_0^{(1)})]} \times \mathcal{M}_{\text{cpx str}}^{[X(T_0^{(2)})]} = \mathcal{M}_{\text{cpx str}}^{[Y]BV}. \tag{5.18}$$

Here is a remark before moving on. One may also construct a Calabi-Yau fourfold as an orbifold of two elliptic curves E_ϕ, E_τ, and a K3 surface $X^{(2)}$, instead of a pair of K3 surfaces:[13]

$$\begin{aligned} Y &= \left(E_\phi \times \left(E_\tau \times X^{(2)}\right)/\mathbb{Z}_2\right)/\mathbb{Z}_2 =: \left(E_\phi \times M\right)/\mathbb{Z}_2, \\ &= \left(\left(E_\phi \times E_\tau\right)/\mathbb{Z}_2 \times X^{(2)}\right)/\mathbb{Z}_2 =: \left(X^{(1)} \times X^{(2)}\right)/\mathbb{Z}_2. \end{aligned} \tag{5.19}$$

This is for Type IIB Calabi-Yau orientifold compactification, where the Calabi-Yau threefold is $M = (E_\tau \times X^{(2)})/\mathbb{Z}_2$. This construction is nothing more than a special case of the Borcea-Voisin K3 $\times$ K3 orbifolds; we can see the combination $X^{(1)} = (E_\phi \times E_\tau)/\mathbb{Z}_2 =: \mathrm{Km}(E_\phi \times E_\tau)$ as the K3 surface $X^{(1)}$; along with an involution $\sigma_{(1)}$ that multiplies (-1) to E_τ, the pair $(X^{(1)}, \sigma_{(1)})$ becomes one of the 75 topological types classified by Nikulin,the one with $T_0 = U[2] \oplus U[2]$. For this reason, we do not loose generality at all by thinking only of K3 $\times$ K3 orbifolds.

[13]In Chapters 6 and 7, we do not distinguish a pair of fourfolds that are mutually birational and have the same number of complex structure and Kähler deformations. configuration and complex structure moduli effective field theory.

For example, an orbifold $(E_\phi \times E_\tau \times X^{(2)})/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ has $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity along a curve $Z_{(2)} \subset X^{(2)}$; the fourfold $(E_\phi \times [(E_\tau \times X^{(2)})/\mathbb{Z}_2])/\mathbb{Z}_2$ in the first line and $([(E_\phi \times E_\tau)/\mathbb{Z}_2] \times X^{(2)})/\mathbb{Z}_2$ in the second line are regarded as different resolutions of the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity (cf [DDF+05]). Two flops convert one to the other. For this reason, we do not even make a clear distinction between an orbifold with singularity and a non-singular manifold obtained as a crepant resolution of the orbifold in Chapters 6 and 7.

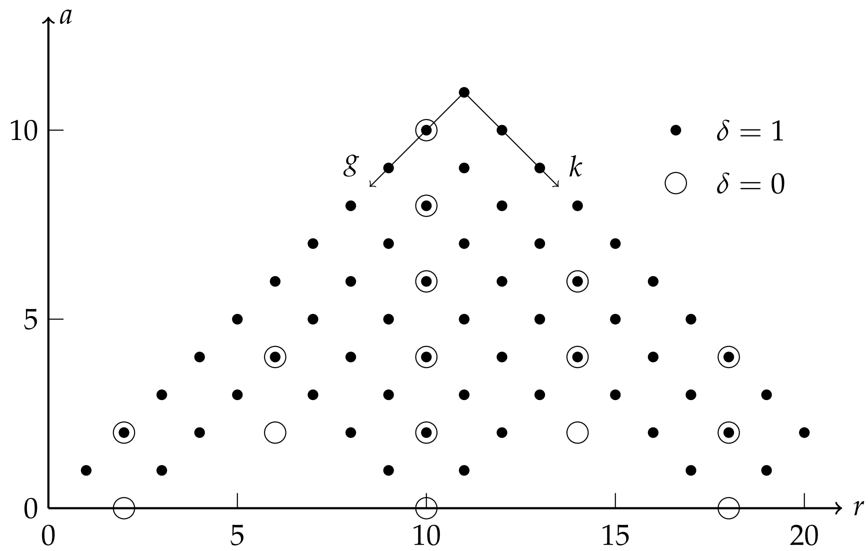


FIGURE 5.1: Classification of K3 surfaces with non-symplectic involution by Nikulin [Nik83]. Dots and circles corresponding to a possible combination of (r, a, δ), with black dots representing $\delta = 1$ and circles $\delta = 0$.

Part II

Original work

Chapter 6

Supersymmetric flux vacua on CM-type $(\mathrm{K3} \times \mathrm{K3})/\mathbb{Z}_2$ orbifolds

As we have reviewed in the previous part, F-theory is a promising framework to construct a model of our Universe in string theory, while it is hard to construct a model with $W \simeq 0$, which is phenomenologically favored. CM-type Calabi-Yau fourfolds is a well-motivated candidate for such solution, and exists densely in the moduli space of K3 $\times$ K3 orbifolds. In this part, we will work out the condition for $W = 0$ vacua, and discuss its implications to the particle spectrum in the 4-dimensional theory. We will start in this chapter by working out the condition for $W = 0$ flux vacua in the case of K3 $\times$ K3$/\mathbb{Z}_2$.

6.1 The conditions of supersymmetric fluxes for CM-type

As we have already reviewed in the Chapter 4, there are two different perspectives in describing the way a topological flux $G \in H^4(Y;\mathbb{Q})$ in a Calabi-Yau fourfold Y stabilizes the complex structure moduli of Y. One is more physical, and the other more mathematical, as we repeat them shortly. Either way, the condition for supersymmetry is stated concisely by the F-term condition[1]

$$DW = 0: \qquad G^{(1,3)} = 0 \tag{6.1}$$

and the additional condition for the Minkowski spacetime and $m_{3/2} = 0$ after compactification,

$$W = 0: \qquad G^{(0,4)} = 0. \tag{6.2}$$

In the more physical perspective, we think that a topological flux[2] G is specified as a part of data of compactification first, and then the superpotential (3.12) gives rise to non-trivial scalar potential of the complex structure moduli fields of Y; the expectation value of those fields adjust themselves in the early period of time in the universe to arrive at a potential minimum, where the resulting complex structure of Y is such that the Hodge $(1,3)$ component of the topological $G \in H^4(Y;\mathbb{Q})$ must be

[1] Here, we use the superpotential (3.12) and the Kähler potential obtained in the large volume limit, as we have already stated in Section 3.5. All kinds of corrections expected in an effective theory of four supersymmetry charges are not taken into account. Also, the F-term conditions are considered only for complex structure moduli.

[2] In this part, we will denote the 4-form flux by G, rather than G_4, as we will later introduce something called G_k.

absent when measured in that complex structure. For such a topological G and the complex structure of Y so determined, it is a non-trivial question whether the Hodge $(0,4)$ component of G vanishes (the condition (6.2) is satisfied) or not.

In the more mathematical perspective, on the other hand, we pose questions that are concerned about classification of flux vacua, forgetting about cosmological time evolution before the complex structure moduli fields come down the potential to their vacuum value. We pick up one point in the complex structure moduli space $z \in \mathcal{M}^{[Y]}_{\text{cpx str}}$, and ask if there is any topological flux $G \in H^4(Y_z;\mathbb{Q})$ whose $H^{1,3}(Y_z;\mathbb{C})$ component vanishes; here, $Y_z = Y$ is the fourfold of the topological type $[Y]$ with the complex structure corresponding to the point $z \in \mathcal{M}^{[Y]}_{\text{cpx str}}$, emphasizing the z-dependence. The condition (6.2) can also be phrased in the same way. At a generic point $z \in \mathcal{M}^{[Y]}_{\text{cpx str}}$, only the trivial flux $G = 0 \in H^4(Y_z;\mathbb{Q})$ satisfy the conditions (6.1). Points in $\mathcal{M}^{[Y]}_{\text{cpx str}}$ where non-trivial fluxes $G \in H^4(Y;\mathbb{Q})$ satisfy the condition (6.1) form a special sub-locus of $\mathcal{M}^{[Y]}_{\text{cpx str}}$. This is a Noether-Lefschetz problem in a Calabi-Yau fourfold $[Y]$. In this work, we exploit the latter perspective.

In the rest of this Section 6.1, we will state the conditions of supersymmetric flux on CM-type Calabi-Yau fourfolds, using the notion of simple components of the Hodge structure, which we have introduced in Definition 5.3.

First, let us assume that Y_z is of CM-type, and let

$$H^4(Y_z;\mathbb{Q}) \cong \oplus_{k\in A} \left(H^4(Y_z;\mathbb{Q})\right)_k \tag{6.3}$$

be the simple component decomposition of the rational Hodge structure of $H^4(Y_z;\mathbb{Q})$ at $z \in \mathcal{M}^{[Y]}_{\text{cpx str}}$. Accordingly, one can decompose a flux G as $G = \sum_{k\in A} G_k$ such that $G_k \in \left(H^4(Y_z;\mathbb{Q})\right)_k$. Whether G satisfies the $DW = 0$ condition or the $DW = W = 0$ condition for the complex structure z can be discussed separately for individual components G_k.

Let us further decompose G_k as $G_k = \sum_a g_a^{(k)} v_a^{(k)}$ by the diagonalization basis $\left\{v_a^{(k)}\right\}_a$ of $\left(H^4(Y_z;\mathbb{Q})\right)_k \otimes_{\mathbb{Q}} \mathbb{C}$, which diagonalizes the action of the endomorphism field $\mathrm{End}_{\mathrm{Hdg}}\left(\left(H^4(Y_z;\mathbb{Q})\right)_k\right)$ as in Eq. (5.7). As presented in the last part of Section 5.1.1, any Galois group action σ acts on the embeddings $\{\rho_a\}_a$ and the basis elements $\left\{v_a^{(k)}\right\}_a$ as permutation and transitively; $\sigma\cdot\rho_a =: \rho_{\sigma(a)}$, and $\sigma(v_a^{(k)}) = v_{\sigma(a)}^{(k)}$. G_k should be invariant under any of the Galois group action σ as it is a rational element $G_k \in \left(H^4(Y_z;\mathbb{Q})\right)_k$, so we obtain $\sigma(g_a^{(k)}) = g_{\sigma(a)}^{(k)}$. In particular, if $g_a^{(k)} = 0$ for some a, then $g_b^{(k)} = 0$ for all $b = 1, \cdots, \dim_{\mathbb{Q}}((H^4(Y_z;\mathbb{Q}))_k)$, i.e. $G_k = 0$.

Therefore, the condition that a topological flux $G \in H^4(Y_z;\mathbb{Q})$ does not have the Hodge $(1,3)$-component is translated as follows:

$$\begin{aligned} {}^{\forall}G_k \in \left(H^4(Y_z;\mathbb{Q})\right)_k \quad &\text{if } \left(\left(H^4(Y_z:\mathbb{Q})\right)_k \otimes_{\mathbb{Q}} \mathbb{C}\right) \cap H^{1,3} = 0, \\ G_k = 0 \quad &\text{if } \left(\left(H^4(Y_z:\mathbb{Q})\right)_k \otimes_{\mathbb{Q}} \mathbb{C}\right) \cap H^{1,3} \neq 0. \end{aligned} \tag{6.4}$$

In particular, if all the simple components have non-zero Hodge $(1,3)$-components, then only the trivial flux $G = 0$ is consistent with the $DW = 0$ condition at $z \in \mathcal{M}^{[Y]}_{\text{CM}}$.

Similarly, the condition that the topological flux $G \in H^4(Y_z;\mathbb{Q})$ has neither the $(1,3)$-component nor $(0,4)$-component is translated as:

$$\begin{aligned} {}^{\forall}G_k \in \left(H^4(Y_z;\mathbb{Q})\right)_k \quad & \text{if } \left(\left(H^4(Y_z:\mathbb{Q})\right)_k \otimes_{\mathbb{Q}} \mathbb{C}\right) \cap \left(H^{1,3} \oplus H^{0,4}\right) = 0, \\ G_k = 0 \quad & \text{if } \left(\left(H^4(Y_z:\mathbb{Q})\right)_k \otimes_{\mathbb{Q}} \mathbb{C}\right) \cap \left(H^{1,3} \oplus H^{0,4}\right) \neq 0. \end{aligned} \tag{6.5}$$

The $DW = W = 0$ condition, and hence this last condition, is further translated as follows: $G_k = 0$ in all the simple components with the level $\ell > 0$.

Note that the condition (6.5) is still necessary, not only sufficient, for a flux with $DW = W = 0$ on a Calabi-Yau fourfolds not necessarily of CM-type; if there is a $G_k \neq 0$ with $G_k = G_k^{(2,2)}$, it generates a 1-dimensional simple Hodge structure, and thus is automatically in the first line of Eq. (6.5). By exactly the same argument, the condition (6.4) is necessary for a level-2 simple Hodge substructure of $H^4(Y_z;\mathbb{Q})$, as $G_k^{(1,3)} = 0$ implies $G_k = G_k^{(2,2)}$ in this case.

The D-term condition, or equivalently the primitivity, also needs to be satisfied for a flux on Y to be supersymmetric. The F-term condition (6.1) and the D-term condition are almost[3] independent, however, because the F-term [resp. D-term] condition constrains the purely horizontal [resp. purely vertical] part of the flux (cf. [GMP95], [BCV12, MSN11, KMW12], [BW15]) as we have briefly mentioned in the review part. In Chapters 6 and 7, we do not deal with the purely vertical part of the flux (or the D-term condition), because they are not relevant to the gravitino mass.

6.2 $H^4((X^{(1)} \times X^{(2)})/\mathbb{Z}_2;\mathbb{Q})$ and complex structure deformations

Having stated the conditions for supersymmetric flux configuration in the case of general CM-type Calabi-Yau fourfolds, we now apply this thinking framework to a Borcea-Voisin orbifold $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ with both $X^{(1)}$ and $X^{(2)}$ being a generic CM-type K3 surface in the moduli space $D(T_0^{(1)})$ and $D(T_0^{(2)})$, respectively, i.e. $T_X^{(i)} = T_0^{(i)}$ for $i = 1,2$. To start off, however, we need to remind ourselves of a bit of math of the cohomology group of this fourfold Y.

The fourfold $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ would remain singular[4], if it stays precisely at the orbifold locus without complex structure deformation or Kähler parameter resolution. Because we do not assume anything about the vacuum value of the Kähler parameter, we do not need to think that Y is singular, and moreover, we can always take a limit from non-zero resolution to the orbifold limit, if we wish. Thus the topology of the fourfold Y is well-defined.[5]

To describe the topology of Y, we need one more preparation. The non-symplectic automorphism $\sigma_{(i)} : X^{(i)} \to X^{(i)}$ may have fixed points (for $i = 1,2$ individually), and the locus of fixed points are denoted by $Z^{(i)}$ for $i = 1,2$. The set $Z^{(i)}$ of fixed

[3] As we have already stated, we treat fluxes in this thesis as elements in the $\mathbb{Q}$-coefficient cohomology, not in the $\mathbb{Z}$-coefficient, and the upper bound on the D3-brane charge is not imposed. At this level of analysis, fluxes in the purely vertical part and purely horizontal parts can be regarded completely independent.

[4] Y has $\mathbb{C}^2/\mathbb{Z}_2$ singularities at the fixed locus

[5] In Chapter 8, we will use Y^{BV} for the non-singular fourfold after resolution, and Y_0 the orbifold without deformation or resolution of the $\mathbb{C}^2/\mathbb{Z}_2$ singularity.

points in $X^{(i)}$ consists of curves whose irreducible components are disjoint from one another, when $\sigma_{(i)}$ acts non-symplectically and is order 2 [Nik81]. Among the 75 choices of[6] (S_0, T_0, σ) in [Nik81], this subset Z of fixed points is empty in just one choice, where[7] $S_0 = U[2] \oplus E_8[2]$. The subset Z consists of two disjoint elliptic curves in the choice with $S_0 = U \oplus E_8[2]$. For all other 73 choices,[8] the set Z consists of one curve $C_{(g)}$ of genus $g = (22 - r - a)/2$ in addition to $k = (r - a)/2$ rational curves $\mathbb{P}^1$ [Nik81]:

$$Z = C_{(g)} \amalg \cup_{p=1}^{k} L_p; \qquad g(C_{(g)}) = (22 - r - a)/2, \qquad L_p \simeq \mathbb{P}^1; \tag{6.6}$$

see also g and k axis in Figure 5.1. The subset $Z_{(4)} \subset X^{(1)} \times X^{(2)}$ of fixed points under the action of $\sigma = (\sigma_{(1)}, \sigma_{(2)})$ is $Z_{(4)} = Z^{(1)} \times Z^{(2)}$. The topological cohomology group $H^4(Y; \mathbb{Q})$ of $[Y]$ is [Voi02, Thm. 7.31], as an abelian group,

$$H^4(Y; \mathbb{Q}) \simeq \left[H^4(X^{(1)} \times X^{(2)}; \mathbb{Q}) \right]^{\sigma} \oplus H^2(Z_{(4)}; \mathbb{Q}); \tag{6.7}$$

the superscript $^{\sigma}$ in the first term extracts the part invariant under the action of σ. A 2-form on $Z_{(4)}$ has a corresponding 4-form in Y; the 2-form on $Z_{(4)}$ is pulled back to the exceptional divisor of the resolved Y, and then is taken a wedge product with the Poincaré dual of the exceptional divisor, i.e. mapped by the Gysin homomorphism.

In the family of fourfolds $[Y]$, the horizontal component of $H^4(Y; \mathbb{Q})$ is

$$H^4_{\mathrm{hor}}(Y; \mathbb{Q}) = \left(T_0^{(1)} \otimes T_0^{(2)} \right) \otimes \mathbb{Q} \oplus H^1(Z^{(1)}; \mathbb{Q}) \otimes H^1(Z^{(2)}; \mathbb{Q}), \tag{6.8}$$

where the first term is from $[H^4(X^{(1)} \times X^{(2)}; \mathbb{Q})]^{\sigma}$, and the second term from $H^2(Z_{(4)}; \mathbb{Q})$. The vertical component is

$$\begin{aligned} H^4_{\mathrm{ver}}(Y; \mathbb{Q}) \;=\; & \left(S_0^{(1)} \otimes S_0^{(2)} \right) \otimes \mathbb{Q} \\ & \oplus H^4(X^{(1)}; \mathbb{Q}) \otimes H^0(X^{(2)}; \mathbb{Q}) \oplus H^0(X^{(1)}; \mathbb{Q}) \otimes H^4(X^{(2)}; \mathbb{Q}) \\ & \oplus H^2(Z^{(1)}; \mathbb{Q}) \otimes H^0(Z^{(2)}; \mathbb{Q}) \oplus H^0(Z^{(1)}; \mathbb{Q}) \otimes H^2(Z^{(2)}; \mathbb{Q}), \end{aligned} \tag{6.9}$$

where the first to second line and the third line come from $[H^4(X^{(1)} \times X^{(2)})]^{\sigma}$ and $H^2(Z_{(4)})$, respectively. The entire cohomology group $H^4(Y; \mathbb{Q})$ is covered by the direct sum of the horizontal component and the vertical component in the case of the family of $[Y]$ over $\mathcal{M}^{[Y]}_{\text{cpx str}}$. The holomorphic 4-form Ω_{Y_z} varies for $z \in \mathcal{M}^{[Y]}_{\text{cpx str}}$, but it does so only within $H^4_{\mathrm{hor}}(Y; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. When the point z is in the subvariety $\mathcal{M}^{[Y]BV}_{\text{cpx str}} \subset \mathcal{M}^{[Y]}_{\text{cpx str}}$, Ω_{Y_z} remains within $(T_0^{(1)} \otimes T_0^{(2)}) \otimes_{\mathbb{Z}} \mathbb{C}$.

At any point $z \in \mathcal{M}^{[Y]}_{\text{cpx str}}$, a ($z$-dependent) Hodge structure is introduced in the

[6] Recall the definition of S_0, T_0 in Section 5.3.

[7] In this thesis, negative definite root lattices of A_n, D_n and E_n type are denoted by A_n, D_n and E_n. For a lattice L, $L[n]$ stands for a lattice where $L \cong L[n]$ as free abelian groups, and the intersection form of the latter is n times that of the former.

[8] The discriminant group $G_0 := T_0^{\vee}/T_0 \cong S_0^{\vee}/S_0$ is always isomorphic to $(\mathbb{Z}_2)^{\oplus a}$ for some $a \in \mathbb{Z}_{\geq 0}$, because the order-2 non-symplectic automorphism σ is assumed to act trivially on S_0 in [Nik81]. The pair of integers a and $r = \mathrm{rank}(S_0)$ capture the geometry of the set Z of (X, σ) associated with (S_0, T_0, σ) [Nik81]. In this work, the values of a, r, k and g for $i = 1, 2$ are denoted by $a_{(i)}$, $r_{(i)}$, k_i and $g_{(i)}$, respectively.

vector space $H^4_{\rm hor}(Y;\mathbb{Q})$; the vertical subspace $H^4_{\rm ver}(Y;\mathbb{Q})$ contains only the level-0 Hodge structure. For a vacuum complex structure $\langle z \rangle$ within $\mathcal{M}^{[Y]BV}_{\rm cpx\ str}$, the vector subspace $H^1(Z^{(1)};\mathbb{Q}) \otimes H^1(Z^{(2)};\mathbb{Q})$ supports a rational Hodge sub-structure of level 2, and $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ a rational Hodge sub-structure of level-4. Linear fluctuation δz in the complex structure from $\langle z \rangle$ are in the Hodge (3, 1) component of $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{C}$ (there are $(20 - r_{(1)}) + (20 - r_{(2)})$ such deformations) and also in the vector space $H^{1,0}(Z^{(1)};\mathbb{C}) \otimes H^{1,0}(Z^{(2)};\mathbb{C})$ (there are $g_{(1)}g_{(2)}$ of them); the former group of fluctuations are within $\mathcal{M}^{[Y]BV}_{\rm cpx\ str}$ and the latter group ventures out from $\mathcal{M}^{[Y]BV}_{\rm cpx\ str}$ into $\mathcal{M}^{[Y]}_{\rm cpx\ str}$. At the quadratic order in the deformation of complex structure, $\Omega_{Y_z} \simeq \Omega_{Y_{\langle z \rangle}} + (\delta z)^a \psi_a + (\delta z)^a (\delta z)^b \psi_{ab}$ for $z = \langle z \rangle + \delta z$, the quadrature of the $(40 - r_{(1)} - r_{(2)})$ complex structure deformations do not bring Ω_{Y_z} out of $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{C}$. The quadrature involving $g_{(1)}g_{(2)}$ complex structure deformations, however, may be in the entire $H^4_H(Y_{\langle z \rangle};\mathbb{C})$.

The observation above on the Hodge sub-structures on $H^4(Y_{\langle z \rangle};\mathbb{Q})$ and finitely perturbed Ω_{Y_z} on them indicates that a non-trivial flux is necessary at least in the $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ component in order to generate mass terms of the $(40 - r_{(1)} - r_{(2)})$ moduli fields.[9] The $g_{(1)}g_{(2)}$ moduli fields may also acquire mass terms from a flux in $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$, or they may not. We take it out of the scope of this thesis to study Ω_{Y_z} at the quadratic order in δz including those $g_{(1)}g_{(2)}$ moduli. THerefore it is not a necessary condition. at this moment. for all the complex structure moduli stabilization that $H^1(Z^{(1)};\mathbb{Q}) \otimes H^1(Z^{(2)};\mathbb{Q})$ contains a level-0 rational Hodge sub-structure.[10] In this work, therefore, we assume that the Hodge structure on $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ is of CM-type, and study when and how supersymmetric flux is admitted in this component; we do not ask whether the Hodge structure on $H^1(Z^{(1)};\mathbb{Q}) \otimes H^1(Z^{(2)};\mathbb{Q})$ is CM-type, or has a level-0 Hodge sub-structure.

6.3 Cases with a generic CM point in $D(T_0)$

In Sections 6.3 and 6.4, we will work out the conditions (6.4, 6.5) for existence of a non-trivial supersymmetric flux in the $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ component for a vacuum complex structure in (5.18). It is done by translating the conditions (6.4, 6.5) into arithmetic characterizations on the vacuum complex structure.

In this Section 6.3, we deal with the cases where complex structure of $X^{(1)}$ and $X^{(2)}$ are CM-type but otherwise generic in the period domains $D(T_0^{(1)})$ and $D(T_0^{(2)})$; this means that $T_X^{(i)} = T_0^{(i)}$. Analysis in Sections 6.3.1 and 6.3.2 reveals that the condition (6.4) for a $DW = 0$ flux is translated to Eq. (6.39), and the condition (6.5) for a $DW = W = 0$ flux to Eq. (6.38); readers may choose to skip the analysis and proceed to the recap in p. 54 at the end of Section 6.3.2. The effective field theory (including mass matrices and symmetries) of complex structure moduli fields is studied in Section 6.3.3.

[9] Comments on the $r_{(1)} = r_{(2)} = 20$ case, where there is no complex structure moduli, will be found later.

[10] This condition is equivalent to existence of an algebraic curve in $Z^{(1)} \times Z^{(2)}$ other than a copy of $Z^{(1)} \times {\rm pt}$ or ${\rm pt} \times Z^{(2)}$.

6.3.1 Tensor product of a pair of CM-type Hodge structures

For a complex structure in (5.18) generic enough to have $T_0^{(i)} = T_X^{(i)}$, the rational Hodge structure on $V_i := T_0^{(i)} \otimes \mathbb{Q}$ is simple and CM-type (by assumption) for both $i = 1, 2$; let $K^{(i)}$ denote their endomorphism fields. It is then known [Bor92, Prop. 1.2] that the rational Hodge structure on $V_1 \otimes V_2 = (T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ is also of CM-type. The rational Hodge structure on $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ is not necessarily simple, however.

In fact, the non-simple nature of a rational Hodge structure of $H^4(Y;\mathbb{Q})$ (or its three-fold counterpart $H^3(M;\mathbb{Q})$) is an essential ingredient for $\langle W \rangle = 0$ [DGKT05a]. In Ref. [KW17a], for example, $M = (E_\tau \times X^{(2)})/\mathbb{Z}_2$ with a CM elliptic curve E_τ and a CM-type K3 surface $X^{(2)}$ is used for a Type IIB orientifold; the authors of [KW17a] found $DW = W = 0$ fluxes by exploiting a case the rational Hodge structure is not simple on $V_1 \otimes V_2$ with $V_1 = H^1(E;\mathbb{Q})$ and $V_2 = T_X^{(2)} \otimes \mathbb{Q}$. We will also do so on $V_1 \otimes V_2 = (T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ in this work.

It was not difficult to work out the simple component decomposition of $(V_1 \otimes V_2)$ in [KW17a], when $V_1 = H^1(T^2;\mathbb{Q})$ is 2-dimensional, and we know that $K^{(1)}$ is an imaginary quadratic field, i.e. $\mathbb{Q}(\sqrt{-q})$ for some $q \in \mathbb{Q}$. For a general $V_1 = T_X^{(1)} \otimes \mathbb{Q}$ and $K^{(1)}$ of a CM-type K3 surface $X^{(1)}$, however, we need to be equipped with an understanding on general structure of the simple component decomposition of $V_1 \otimes V_2$. That is what we do in Section 6.3.1 (by exploiting [ST61, §5]), and we will arrive at Eqs. (6.11), (6.16), (6.18) and (6.27).

Step 1: The endomorphism fields $K^{(1)} \subset \mathrm{End}_{\mathrm{Hdg}}(V_1)$ and $K^{(2)} \subset \mathrm{End}_{\mathrm{Hdg}}(V_2)$ give rise to an algebra of Hodge endomorphisms of the vector space $(V_1 \otimes_{\mathbb{Q}} V_2)$; $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)} \hookrightarrow \mathrm{End}_{\mathrm{Hdg}}(V_1 \otimes_{\mathbb{Q}} V_2)$. Similarly to the fact that the rational Hodge structure on $(V_1 \otimes_{\mathbb{Q}} V_2)$ is not necessarily simple, the algebra $(K^{(1)} \otimes_{\mathbb{Q}} K^{(2)})$ of endomorphisms of $(V_1 \otimes_{\mathbb{Q}} V_2)$ is not necessarily a field. The first step is to look at the structure of the algebra $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}$.

First, let us introduce[11] the decomposition (6.11). Recall that the field extension $K^{(1)}$ over $\mathbb{Q}$ is always expressed in the form of $K^{(1)} = \mathbb{Q}(\alpha)$ for some $\alpha \in K^{(1)}$. Let $f_{\alpha/\mathbb{Q}} \in \mathbb{Q}[x]$ be a minimal polynomial of $\alpha \in K^{(1)}$ over $\mathbb{Q}$, which means that[12] $K^{(1)} = \mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(f_{\alpha/\mathbb{Q}})$, and

$$K^{(2)} \otimes_{\mathbb{Q}} K^{(1)} \cong K^{(2)} \otimes_{\mathbb{Q}} \mathbb{Q}[x]/(f_{\alpha/\mathbb{Q}}) \cong K^{(2)}[x]/(f_{\alpha/\mathbb{Q}}). \tag{6.10}$$

Although the minimal polynomial $f_{\alpha/\mathbb{Q}}$ is irreducible in the ring $\mathbb{Q}[x]$, it may in principle be factorizable in the ring $K^{(2)}[x]$; let $f_{\alpha/\mathbb{Q}}(x) = \prod_{i=1}^r g_i(x)$ be an irreducible factorization, where $g_i(x) \in K^{(2)}[x]$. The Chinese remainder theorem is used to obtain

$$(K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}) \cong K^{(2)}[x]/(f_{\alpha/\mathbb{Q}}) \cong \oplus_{i=1}^r K^{(2)}[x]/(g_i) =: \oplus_{i=1}^r L_i. \tag{6.11}$$

[11] Any introductory textbook on field theory, such as [Fuj91, Rom05], will be useful in following the discussions in Sections 6.3.1 and 6.3.2.

[12] see Theorem A.3

The algebra $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}$ is decomposed into a direct sum of number fields $K^{(2)}[x]/(g_i)$; each component is a degree $[L_i : K^{(2)}] = \deg(g_i)$ extension field over $K^{(2)}$.

Second, let us spell out the relation between the sets of embeddings of $K^{(1)}$ and $K^{(2)}$,

$$\Phi_{K^{(1)}} := \mathrm{Hom}_{\mathbb{Q}}(K^{(1)}, \overline{\mathbb{Q}}) \quad \text{and} \quad \Phi_{K^{(2)}} := \mathrm{Hom}_{\mathbb{Q}}(K^{(2)}, \overline{\mathbb{Q}}), \tag{6.12}$$

respectively, and those of the number fields L_i; remember that the set of embeddings of the CM fields play an important role in describing a Hodge structure of CM-type (Section 5.1.1). The set of embeddings $\Phi_{K^{(1)}} \times \Phi_{K^{(2)}}$ of the algebra $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}$ is decomposed into

$$\Phi_{K^{(1)}} \times \Phi_{K^{(2)}} = \amalg_{i=1}^{r} \Phi_{L_i}, \quad \Phi_{L_i} = \left\{ (\rho^{(1)}, \rho^{(2)}) \mid \rho^{(1)}(\alpha) \text{ is a root of } (\rho^{(2)}(g_i))(x) \right\}; \tag{6.13}$$

obviously individual Φ_{L_i} consist of $\deg(g_i) \times [K^{(2)} : \mathbb{Q}]$ distinct embeddings of the number field L_i (so the notation Φ_{L_i} is appropriate), and the subsets Φ_{L_i} for $i = 1, \cdots, r$ are mutually exclusive in $\Phi_{K^{(1)}} \times \Phi_{K^{(2)}}$, because the polynomial $f_{\alpha/\mathbb{Q}}$ is separable. Now, both the algebra $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}$ and its set of embeddings $\Phi_{K^{(1)}} \times \Phi_{K^{(2)}}$ have decompositions, (6.11) and (6.13), respectively. The two decompositions are compatible in fact, in that the embeddings in Φ_{L_i} are trivial on the other direct sum components, L_j with $j \neq i$, as follows. As a part of the Chinese remainder theorem, there exists $a_i \in K^{(2)}[x]/(g_i)$ for $i = 1, \cdots, r$ so that

$$1 = \sum_i a_i f_i' \in K^{(2)}[x]/(f_{\alpha/\mathbb{Q}}), \qquad f_i' := \prod_{j \neq i} g_j, \tag{6.14}$$

in line with the decomposition $K^{(2)}[x]/(f_{\alpha/\mathbb{Q}}) \cong \oplus_i K^{(2)}[x]/(g_i)$. An element in L_j can thus be regarded as a polynomial in $K^{(2)}[x]$ times $a_j f_j'$ (mod $f_{\alpha/\mathbb{Q}}$), whose image by any embedding in Φ_{L_i} with $i \neq j$ vanishes because f_j' contains the factor g_i.

It is useful to note that the Galois group $\mathrm{Gal}((K^{(1)}K^{(2)})^{\mathrm{nc}}/\mathbb{Q})$ acts on the set $\Phi_{K^{(1)}} \times \Phi_{K^{(2)}}$; a Galois transformation $\sigma \in \mathrm{Gal}((K^{(1)}K^{(2)})^{\mathrm{nc}}/\mathbb{Q})$ converts an embedding $\rho^{(1)} \otimes \rho^{(2)} \in \Phi_{K^{(1)}} \times \Phi_{K^{(2)}}$ to another embedding given by $\sigma \cdot (\rho^{(1)} \otimes \rho^{(2)}) : K^{(1)} \otimes_{\mathbb{Q}} K^{(2)} \to \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$. The decomposition (6.13) can be regarded as the orbit decomposition under this group action. This observation further indicates that the decomposition (6.13) is independent of the choice of the primitive element α of $K^{(1)} \cong \mathbb{Q}(\alpha)$.

Instead of exploiting the structure of $K^{(1)}$ as $\mathbb{Q}(^{\exists}\alpha)$, we could have exploited the same structure of $K^{(2)}$; $K^{(2)}$ is regarded as $K^{(2)} \cong \mathbb{Q}(\alpha')$ for an appropriate choice of $\alpha' \in K^{(2)}$; find its minimal polynomial over $\mathbb{Q}$, and factorize the polynomial over $K^{(1)}$ to find another decomposition of $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}$ into a direct sum of number fields, so yet another decomposition of the set $\Phi_{K^{(1)}} \times \Phi_{K^{(2)}}$ also follows. This decomposition must be the orbit decomposition of the action of $\mathrm{Gal}((K^{(1)}K^{(2)})^{\mathrm{nc}}/\mathbb{Q})$ on $\Phi_{K^{(1)}} \times \Phi_{K^{(2)}}$, where the same group acts on the same set precisely in the same way as before. Thus the decomposition of the embeddings should be independent of whether we exploit $K^{(1)} \cong \mathbb{Q}(\alpha)$ or $K^{(2)} = \mathbb{Q}(\alpha')$, and so is the decomposition of the algebra $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)} \cong \oplus_{i=1}^{r} L_i$. It also follows that $[L_i : \mathbb{Q}]$ is divisible by both $[K^{(2)} : \mathbb{Q}]$ and $[K^{(1)} : \mathbb{Q}]$.

Step 2: In general, when a simple rational Hodge structure V is of CM-type with endomorphism field K, there is an isomorphism $i : V \cong K$ as vector spaces over

$\mathbb{Q}$ which is compatible with the action of K on V, that is, for $v \in V$ and $x \in K$, $(i \circ x)(v) = x \cdot i(v)$. The choice of $v_1 \in V$ such that $i(v_1) = 1 \in K$ is arbitrary, so the isomorphism is not unique. In other words, V_1 can be regarded as a 1-dimensional vector space over K_1, and the isomorphism between V_1 and K_1 is unique up to a scalar multiplication. Choose such isomorphisms $i_1 : K^{(1)} \cong V_1$ and $i_2 : K^{(2)} \cong V_2$. Then the isomorphism

$$i_1 \otimes i_2 : (K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}) \cong V_1 \otimes_{\mathbb{Q}} V_2 \tag{6.15}$$

combined with Eq. (6.11) introduces a decomposition of the vector space

$$V_1 \otimes_{\mathbb{Q}} V_2 \cong \oplus_{i=1}^{r} W_i. \tag{6.16}$$

Individual components W_i in $V_1 \otimes V_2$ are vector subspaces over $\mathbb{Q}$; by the definition of the isomorphisms between V_i and $K^{(i)}$ for $i = 1, 2$, the number field L_i acts on W_i, $[L_i : \mathbb{Q}] = \dim_{\mathbb{Q}} W_i$, and each one of the simultaneous eigenstates $v_a \in W_i \otimes_{\mathbb{Q}} (K^{(1)}K^{(2)})^{\mathrm{nc}}$ of the action of L_i is in a definite Hodge (p, q) component (Section 5.1.1), so all the elements in L_i are in $\mathrm{End}_{\mathrm{Hdg}}(W_i)$. Thus, the decomposition (6.16) over $\mathbb{Q}$ is compatible with the rational Hodge sub-structure, and each W_i has a rational Hodge structure of CM-type[13].

Step 3: We have not done yet, because each component W_i of the decomposition (6.16) is not guaranteed to be a simple Hodge component; in fact, as we will see in a discussion later, W_i can be of level-0, and in that case W_i is not simple unless it is 1-dimensional, which is not the case in our setup. In this Step 3, we will discuss the general property of CM-type Hodge structures, and then apply the discussion in Step 4 to show a relation (6.27), that will be used to show that the decomposition (6.16) is sufficiently fine for our discussion.

One may consider a simple component decomposition of a not necessarily CM-type rational Hodge structure of $V_{\mathbb{Q}}$:

$$V_{\mathbb{Q}} \cong \oplus_{k \in A} V_k. \tag{6.17}$$

Combining this structure (6.17) and the structure theorem of semi-simple algebras, one can state—as we do in the following—the structure of the entire algebra $\mathrm{End}_{\mathrm{Hdg}}(V_{\mathbb{Q}})$; as a reminder, $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}$ is a part of $\mathrm{End}_{\mathrm{Hdg}}(V_{\mathbb{Q}})$.

For any pair of simple components V_k and V_l in Eq. (6.17), any $\phi \in \mathrm{Hom}_{\mathrm{Hdg}}(V_k, V_l)$ is either a zero map or an invertible Hodge morphism.[14] One can think of grouping the simple components $\{V_k \mid k \in A\}$ into Hodge-isomorphism classes based on whether the set $\mathrm{Hom}_{\mathrm{Hdg}}(V_k, V_l)$ is non-trivial (i.e., a Hodge isomorphism exists). The set of Hodge isomorphism classes of the simple components in $V_{\mathbb{Q}}$ is denoted by $\mathcal{A}$, and one can think of the decomposition

$$V_{\mathbb{Q}} \cong \oplus_{\kappa \in \mathcal{A}} \left(\oplus_{k \in A; [k] = \kappa} V_k \right) =: \oplus_{\kappa \in \mathcal{A}} V_{\kappa}. \tag{6.18}$$

[13] This is because $V_1 \otimes V_2$ is of CM-type.

[14] If ϕ is not surjective, then V_l has a rational Hodge sub-structure, which contradicts against the assumption that the rational Hodge structure on V_l is simple. If ϕ has a non-trivial kernel, that implies that V_k has a rational Hodge sub-structure, which is a contradiction once again. Thus, ϕ must be an isomorphism between the vector spaces V_k and V_l over $\mathbb{Q}$.

The algebra of Hodge endomorphisms of $V_\mathbb{Q}$ has the structure

$$\mathrm{End}_{\mathrm{Hdg}}(V_\mathbb{Q}) \cong \oplus_{\kappa\in\mathcal{A}} M_{n_\kappa\times n_\kappa}(D_\kappa), \qquad D_{\kappa=[k]} = \mathrm{End}_{\mathrm{Hdg}}(V_k), \tag{6.19}$$

where n_κ is the number of simple components V_k that fall into a given Hodge-isomorphism class κ, $[k] = \kappa$. D_κ is a division algebra over $\mathbb{Q}$, because all the non-zero element is invertible. Therefore, $\mathrm{End}_{\mathrm{Hdg}}(V_\mathbb{Q})$ is a semi-simple algebra over $\mathbb{Q}$, and the factor $M_{n_\kappa\times n_\kappa}(D_\kappa)$ for a Hodge-isomorphism class $\kappa \in \mathcal{A}$ is a simple algebra.[15]

Now, we can invoke a few known facts about semi-simple algebras. One is that V_k is regarded as an irreducible module of $D_{\kappa=[k]}$, and further

$$\dim_\mathbb{Q} V_k = \dim_\mathbb{Q} D_\kappa. \tag{6.20}$$

As another fact ([TN87, Thm. II.4.11] or [GS06, Cor.2.2.3]),

$$\dim_\mathbb{Q} D_\kappa = q_\kappa^2[K_\kappa : \mathbb{Q}] \tag{6.21}$$

for some $q_\kappa \in \mathbb{N}_{>0}$, where K_κ is the center of the division algebra D_κ.

Step 4: The general structure of $\mathrm{End}_{\mathrm{Hdg}}(V_\mathbb{Q})$ in Step 3 is for a general rational Hodge structure not necessarily of CM-type, whereas the CM-type nature of the Hodge structure on $V_1 \otimes V_2$ has been exploited in Steps 1 and 2. Let us see in the following (by following [ST61, §5]) how the decomposition (6.16) is related to (6.18) in Step 3, and how $K^{(1)} \otimes K^{(2)}$ with the structure (6.11) fits into the general structure (6.19) of $\mathrm{End}_{\mathrm{Hdg}}(V_\mathbb{Q})$ in Step 3, when $V_\mathbb{Q} = V_1 \otimes_\mathbb{Q} V_2$.

First observation is that one $\kappa \in \mathcal{A}$ is assigned to each label $i \in \{1, \cdots, r\}$ in the decomposition (6.11, 6.16); the corresponding κ is denoted by $\kappa(i)$. To see this correspondence, think of

$$L_i \hookrightarrow (K^{(1)} \otimes K^{(2)}) \hookrightarrow \mathrm{End}_{\mathrm{Hdg}}(V_\mathbb{Q}) \to M_{n_\kappa\times n_\kappa}(D_\kappa) \tag{6.22}$$

for a given $i \in \{1, \cdots, r\}$ and an arbitrary $\kappa \in \mathcal{A}$. The image of L_i must be non-trivial at least for one $\kappa \in \mathcal{A}$; now we wish to see that that is the case for only one Hodge isomorphism class κ in $\mathcal{A}$, so that $\kappa(i)$ is consistently defined.

Suppose that the image of L_i is non-zero for $\kappa_0 \in \mathcal{A}$. Then the vector space V_{κ_0} contains a vector subspace isomorphic to L_i, and the algebra $L_i \hookrightarrow M_{n_{\kappa_0}\times n_{\kappa_0}}(D_{\kappa_0})$ is represented on this copy of the vector space L_i as a full set of Φ_{L_i}. If there were distinct $\kappa_0, \kappa_0' \in \mathcal{A}$ where L_i is embedded non-trivially, then the set of representations Φ_{L_i} would appear more than once in $V_{\kappa_0} \oplus V_{\kappa_0'} \subset V_\mathbb{Q} = (V_1 \otimes V_2)$; that contradicts against the fact that all the representations in $\Phi_{K^{(1)}} \times \Phi_{K^{(2)}}$ appear just once on $V_\mathbb{Q}$. We have thus established a claim that there is just one $\kappa \in \mathcal{A}$ where the image of L_i in $M_{n_\kappa\times n_\kappa}(D_\kappa)$ is non-trivial.

Second, we will see how L_i fits into the algebra $M_{n_\kappa\times n_\kappa}(D_\kappa)$ with $\kappa = \kappa(i)$ by exploiting the CM nature of the Hodge structure on V_κ. The following argument (built on Step 3) is almost[16] a copy of the logic of §5 of [ST61].

[15] It is a simple algebra in the sense that it does not have a non-trivial two-sided ideal.

[16] The original version [ST61, §5] is for $V_\mathbb{Q} = H^1(A;\mathbb{Q})$ for an abelian variety A. cf [Zar83, vG08, Huy16] for $V_\mathbb{Q} = T_X$ of a K3 surface X.

For a given $\kappa \in \mathcal{A}$, now consider a set of the label i in $\{1,\cdots,r\}$ with $\kappa(i)=\kappa$. Due to the CM nature, the relation

$$\sum_{i \text{ s.t. } \kappa(i)=\kappa} [L_i:\mathbb{Q}] = \dim_{\mathbb{Q}} V_\kappa \tag{6.23}$$

holds for individual κ's in $\mathcal{A}$. Furthermore, general arguments in Step 3—(6.20) and (6.21)— implies that

$$\dim_{\mathbb{Q}} V_\kappa = n_\kappa \dim_{\mathbb{Q}} V_{k\ ([k]=\kappa)} = n_\kappa q_\kappa^2 [K_\kappa:\mathbb{Q}]. \tag{6.24}$$

On the other hand, the algebra

$$L' = \left(\bigoplus_{i \text{ s.t. } \kappa(i)=\kappa} L_i \right) \cdot (K_\kappa \mathbf{1}_{n_\kappa\times n_\kappa}) \subset M_{n_\kappa\times n_\kappa}(D_\kappa) \tag{6.25}$$

remains to be a commutative sub-algebra, and any commutative sub-algebra of a central simple algebra $M_{n_\kappa\times n_\kappa}(D_\kappa)$ is bounded in its dimension by

$$\sum_{i \text{ s.t. } \kappa(i)=\kappa} [L_i:\mathbb{Q}] \le \dim_{\mathbb{Q}} L' \le n_\kappa \times q_\kappa \times [K_\kappa:\mathbb{Q}]. \tag{6.26}$$

Therefore, by combining Eqs. (6.23) and (6.24) against Eq. (6.26), we can see that $q_\kappa = 1$ (which means that $D_\kappa = K_\kappa$), and also that $K_\kappa \mathbf{1}_{n_\kappa\times n_\kappa}$ is contained in $\oplus_{i;\kappa(i)=\kappa} L_i$. The latter statement further indicates[17] that those L_i can be regarded as an extension of $K_{\kappa(i)}$. For the field L_i to be a non-trivial extension of $K_{\kappa(i)}$, at least some of the endomorphisms in $L_i \subset M_{n_\kappa\times n_\kappa}(K_\kappa)$ must mix multiple different simple components V_k with $[k]=\kappa$.

To summarize,

$$V_\kappa \cong \bigoplus_{i \text{ s.t. } \kappa(i)=\kappa} W_i. \tag{6.27}$$

K_κ is the endomorphism field of the CM-type simple rational Hodge structure of V_k (such that $[k]=\kappa$), L_i is an extension of $K_{\kappa(i)}$, and $\oplus_{i \text{ s.t.}}^{(\kappa(i)=\kappa)} L_i \hookrightarrow M_{n_\kappa\times n_\kappa}(K_\kappa) = \mathrm{End}_{\mathrm{Hdg}}(V_\kappa)$.

Note that the vector space $W_i \otimes \overline{\mathbb{Q}}$ has a well-motivated basis; basis elements are in one-to-one with the embeddings $\rho^{(1)} \otimes \rho^{(2)}$ in Φ_{L_i}. This is just a special case of Section 5.1.1 with $F = L_i$ and $V_{\mathbb{Q}} = W_i$. Each one of the basis elements are also associated with a particular Hodge (p,q) type, so each embedding $\rho^{(1)} \otimes \rho^{(2)}$ of L_i has its corresponding Hodge type (p,q). This correspondence will be exploited in the following analysis.

6.3.2 $DW=0$ flux and $DW=W=0$ flux, assuming $T_X = T_0$

Toward the end of Section 6.1, we used the language of the simple Hodge component decomposition to write down the conditions for the presence of a non-trivial supersymmetric flux. Whether a non-trivial flux with those conditions exists or not can be

[17] This is because $K_\kappa \mathbf{1} \ni 1\cdot \mathbf{1} = \sum_i \epsilon_i \in \oplus_i L_i$; $K_\kappa \mathbf{1}\cdot(0,\cdots,\epsilon_i,\cdots,0) \subset L_i \subset \oplus_j L_j$ is a sub-field of L_i and is isomorphic to K_κ.

studied for individual simple rational Hodge components. For simple Hodge components that are mutually Hodge-isomorphic, say, $\phi : V_k \cong V_l$, $[k] = [l] = \kappa \in \mathcal{A}$,

$$h^{p,q}[V_k] = h^{p,q}[V_l] \tag{6.28}$$

holds for all (p,q). We can thus talk of the level of individual Hodge-isomorphism classes, $\kappa \in \mathcal{A}$, and we can also study whether fluxes with $DW = 0$ and/or $W = 0$ exists for individual Hodge-isomorphism classes.

- In a V_κ of level 0, there are only Hodge (2,2) components (by definition). Any rational flux here satisfies both of the $DW = 0$ and $W = 0$ conditions.
- In a V_κ of level 2, any non-zero rational flux breaks the $DW = 0$ condition (although $W = 0$ would be satisfied).
- There is just one level-4 simple rational Hodge component of $H^4(Y;\mathbb{Q})$ of a Calabi-Yau fourfold Y, so this simple component alone forms one Hodge-isomorphism class of the simple components in $H^4(Y;\mathbb{Q})$. This simple component admits a rational flux with $DW = 0$ if and only if $h^{3,1} = 0$ holds in this simple component. Let us say that a simple component is $(3,1)$-**free** if the component has $h^{3,1} = 0$. Even when this condition is satisfied, such a flux does not satisfy the $W = 0$ condition.

Let us continue to focus on a Borcea-Voisin orbifold $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ of a pair of CM-type K3 surfaces with $T_X^{(1)} = T_0^{(1)}$ and $T_X^{(2)} = T_0^{(2)}$. We have seen in Section 6.3.1 that the Hodge structure on $V_1 \otimes V_2$, where $V_1 \cong T_X^{(1)} \otimes \mathbb{Q}$ and $V_2 \cong T_X^{(2)} \otimes \mathbb{Q}$, has the decomposition (6.16), which is compatible with the Hodge-isomorphism-class decomposition (6.18), although Eq. (6.16) may be a finer classification[18] than (6.18). Therefore, we can rephrase the criteria for the existence of non-trivial supersymmetric fluxes, which is stated above, by simply replacing Hodge-isomorphism classes of simple components by individual components W_i in Eq. (6.16), thanks to the relation (6.27).

In the decomposition (6.16) of the Hodge structure on $V_1 \otimes_{\mathbb{Q}} V_2$, the individual components W_i are either level-4, level-2, or level-0. We will see, first, that there are at most only two W_i's that are not level-2 (so, a $DW = 0$ flux is possible only in those at most two W_i's); this is the Step 1 below. In Step 2, we work out the conditions on the CM fields $K^{(1)}$ and $K^{(2)}$ for those one or two component(s) to be $(3,1)$-free, so that a $DW = 0$ flux is indeed available. A physics recap (Step 3) comes at the end of this Section 6.3.2.

Step 1: To show that there are at most two W_i's, let us introduce some notations. We denote the extension degrees of $K^{(1)}$ and $K^{(2)}$ over $\mathbb{Q}$ by $n_1 := [K^{(1)} : \mathbb{Q}]$ and $n_2 := [K^{(2)} : \mathbb{Q}]$, respectively. The embeddings of $K^{(i)}$ with $i = 1,2$ are denoted by $\mathrm{Hom}(K^{(i)}, \overline{\mathbb{Q}}) = \left\{\rho^{(i)}_{(20)}, \rho^{(i)}_{(02)}, \rho^{(i)}_3, \ldots, \rho^{(i)}_{n_i}\right\}$, where $\rho^{(i)}_{(20)}$ and $\rho^{(i)}_{(02)}$ correspond to the $(2,0)$ component and $(0,2)$ component of $H^2(X^{(i)})$, respectively, in the sense of a remark at the end of Section 6.3.1; the action of $x \in K^{(i)}$ on the $(2,0)$-form $\Omega_X^{(i)}$ of $X^{(i)}$ is $x : \Omega_X^{(i)} \mapsto \rho^{(i)}_{(20)}(x) \cdot \Omega_X^{(i)}$ for any $x \in K^{(i)}$. Let us denote by $L_{(20|20)}$ the number field L_i for which Φ_{L_i} contains $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}$, and by $L_{(20|02)}$ the number field L_j for which Φ_{L_j} contains $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$. For those i and j, the vector spaces W_i and W_j are denoted

[18] because of Eq. (6.27)

by $W_{(20|20)}$ and $W_{(20|02)}$, respectively. Note that both $i \neq j$ and $i = j$ are possible. We claim that these (at most) two components have a chance to be different from level-2, and that all other W_k's in Eq. (6.16) are level-2.

Obviously, $W_{(20|20)}$ is always the unique level-4 component. $\Phi_{L_{(20|20)}}$ contains $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}$.

To see that all other W_k's except $W_{(20|02)}$ are level-2, note first that every Hodge $(3,1)$ component in $(V_1 \otimes V_2) \otimes \mathbb{C}$ corresponds to an embedding of the form

$$\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{b} \text{ with } 3 \leq b \leq n_2 \qquad \text{or} \tag{6.29}$$

$$\rho^{(1)}_{a} \otimes \rho^{(2)}_{(20)} \text{ with } 3 \leq a \leq n_1, \tag{6.30}$$

because a $(3,1)$-form in $(V_1 \otimes V_2) \otimes \mathbb{C}$ is always a product of a $(2,0)$-form in $V_1 \otimes \mathbb{C}$ and a $(1,1)$-form in $V_2 \otimes \mathbb{C}$, or vice versa. On the other hand, each set of embeddings Φ_{L_k} contains at least one element of the form $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{\beta}$ for some β in $\{(20),(02),3,\ldots,n_2\}$, because Φ_{L_k} forms an orbit under the Galois group action. Therefore, Φ_{L_k} for $k \neq (20|20),(20|02)$ contains $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{\beta}$ with $\beta \in \{3,\cdots,n_2\}$, and the corresponding W_k is of level 2. We conclude that a $DW = 0$ flux is possible only within $W_{(20|20)}$ and $W_{(20|02)}$.

Step 2: Now let us work out the conditions for non-trivial fluxes to exist in $W_{(20|20)}$ and $W_{(20|02)}$ in terms of the CM fields $K^{(1)}$, $K^{(2)}$, and their actions on $T^{(1)}_X$ and $T^{(2)}_X$. The analysis will take several pages, but the conclusion can be summarized quite simply; a non-trivial flux with $DW = 0$ exists if and only if Eq. (6.39) is satisfied. A stronger condition (6.38) is necessary and sufficient for a non-trivial $DW = W = 0$ flux.

We first study the level-4 component $W_{(20|20)}$. Recall that a non-trivial flux in a level-4 component preserves the $DW = 0$ condition if and only if the component is $(3,1)$-free, i.e. free of Hodge $(3,1)$ components.[19] We are thus interested in when the component is $(3,1)$-free. Since we know all the elements in $\Phi_{K^{(1)}} \times \Phi_{K^{(2)}}$ that correspond to Hodge $(3,1)$ components, (6.29) and (6.30), our task reduces to finding out whether or not $\Phi_{L_{(20|20)}}$ contains such embeddings. This is equivalent to working out whether or not there exists an action of $\mathrm{Gal}((K^{(1)}K^{(2)})^{\mathrm{nc}}/\mathbb{Q})$ that maps $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}$ to one of such embeddings that correspond to $(3,1)$ components, since $\Phi_{L_{(20|20)}}$ is generated by $\mathrm{Gal}((K^{(1)}K^{(2)})^{\mathrm{nc}}/\mathbb{Q})$ acting on $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}$ (see p. 45). Such a map must be contained in $G^{(1)}_{(20)} := \mathrm{Gal}((K^{(1)}K^{(2)})^{\mathrm{nc}}/\rho^{(1)}_{(20)}(K^{(1)}))$ or $G^{(2)}_{(20)} := \mathrm{Gal}((K^{(1)}K^{(2)})^{\mathrm{nc}}/\rho^{(2)}_{(20)}(K^{(2)}))$, since either $\rho^{(1)}_{(20)}$ or $\rho^{(2)}_{(20)}$ must be held fixed by the map. Thus the component $W_{(20|20)}$ is $(3,1)$-free, if and only if the following two conditions are satisfied simultaneously:

(i) There is no element $\sigma^{(1)} \in G^{(1)}_{(20)}$ such that $\sigma^{(1)} \circ \left(\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}\right) = \rho^{(1)}_{(20)} \otimes \rho^{(2)}_{b}$, for any $3 \leq b \leq n_2$.

(ii) There is no element $\sigma^{(2)} \in G^{(2)}_{(20)}$ such that $\sigma^{(2)} \circ \left(\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}\right) = \rho^{(1)}_{a} \otimes \rho^{(2)}_{(20)}$, for any $3 \leq a \leq n_1$.

[19] Note that any non-trivial flux in a level-4 component violates the $W = 0$ condition.

Let us work out in turn when each one of these conditions is satisfied. We first focus on the condition (i). We define N_1 to be the extension degree[20] $N_1 := [L_{(20|20)} : K^{(1)}]$. There are $N_1 - 1$ non-trivial actions in $G^{(1)}_{(20)}$, which map $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}$ to $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{\beta}$ for some $\beta = (02), 3, \ldots, n_2$. Except for the one element that maps to $\beta = (02)$, which may or may not exist, each one of them will violate the condition (i). We thus immediately conclude that the condition (i) is violated whenever $N_1 > 2$.

There are two ways to satisfy the condition (i) when $N_1 \leq 2$, i.e. $W_{(20|20)}$ does not contain Hodge $(3,1)$ components of the form (6.29):

(i-1) When $N_1 = 1$, the condition (i) is always satisfied. This is because there are no non-trivial action in $G^{(1)}_{(20)}$. Note that this also means that $W_{(20|20)} \neq W_{(20|02)}$ in this case, because $\Phi_{L_{(20|20)}}$ does not contain $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$.

(i-2) When $N_1 = 2$ and $\Phi_{L_{(20|20)}}$ contains $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$, then the condition (i) is satisfied. This is because the only non-trivial action of $G^{(1)}_{(20)}$ maps $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}$ to $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$. At the same time this means that $W_{(20|20)} = W_{(20|02)}$.

Note that, even when $N_1 = 2$, if $\Phi_{L_{(20|20)}}$ does not contain $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$, the condition (i) is violated; the only non-trivial element in $G^{(1)}_{(20)}$ will map $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}$ to an embedding of the form (6.29).

Similarly, defining $N_2 := [L_{(20|20)} : K^{(2)}]$, one can argue that there are only two ways to satisfy the condition (ii), i.e. $W_{(20|20)}$ does not contain Hodge $(3,1)$ components of the form (6.30):

(ii-1) When $N_2 = 1$, the condition (ii) is satisfied, since there are no non-trivial action in $G^{(2)}_{(20)}$. This means that $W_{(20|20)} \neq W_{(20|02)}$ in this case, because $\Phi_{L_{(20|20)}}$ does not contain $\rho^{(1)}_{(02)} \otimes \rho^{(2)}_{(20)}$, which is the complex conjugate of $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$ and must be contained in $\Phi_{L_{(20|02)}}$.

(ii-2) When $N_2 = 2$ and $\Phi_{L_{(20|20)}}$ contains $\rho^{(1)}_{(02)} \otimes \rho^{(2)}_{(20)}$, then the condition (ii) is again satisfied. At the same time this means that $W_{(20|20)} = W_{(20|02)}$.

Now we are ready to see when the conditions (i) and (ii) are simultaneously satisfied. In order to satisfy both (i) and (ii), there seems to be four choices for not having a Hodge $(3,1)$ component in $W_{(20|20)}$, i.e. two choices for the condition (i) and another pair of choices for the condition (ii). However, two of them, (i-1)-(ii-2) and (i-2)-(ii-1) cannot happen; (i-1) or (ii-1) imply $W_{(20|20)} \neq W_{(20|02)}$, whereas (i-2) or (ii-2) imply $W_{(20|20)} = W_{(20|02)}$, thus contradiction.

In summary, there are two cases, (i-1)-(ii-1) and (i-2)-(ii-2), where the level-4 component $W_{(20|20)}$ is $(3,1)$-free. For these two cases, let us leave the list of embeddings in $\Phi_{L_{(20|20)}}$ for clarification, and also rephrase these conditions on the $\mathrm{Gal}((K^{(1)}K^{(2)})^{\mathrm{nc}}/\mathbb{Q})$ action in terms of $K^{(1)}$, $K^{(2)}$ and their actions.

[20] Strictly speaking, $L_{(20|20)}$ is defined to be an abstract extension field of $K^{(2)}$, $L_{(20|20)} = K^{(2)}[x]/g(x)$ for some $g \in K^{(2)}[x]$ and the endomorphism field $K^{(1)}$ is not a sub-field of it, so the extension degree $[L_{(20|20)} : K^{(1)}]$ does not make sense. However, since we know that $L_{(20|20)}$ is isomorphic to $K^{(1)}[x]/h(x)$ with some $h \in K^{(1)}[x]$, we abuse the notation and define $[L_{(20|20)} : K^{(1)}] := [\varphi(L_{(20|20)}) : K^{(1)}]$ with an isomorphism $\varphi : K^{(2)}[x]/g(x) \to K^{(1)}[x]/h(x)$.

- Firstly, let us choose (i-2) and (ii-2), i.e. $[L_{(20|20)}:K^{(1)}]=[L_{(20|20)}:K^{(2)}]=2$ (so, $n_1=n_2=:n$) and $W_{(20|20)}=W_{(20|02)}$, to satisfy the conditions (i) and (ii). The contents of $\Phi_{L_{(20|20)}}$ are

$$\begin{aligned}\Phi_{L_{(20|20)}} &= \left\{\rho^{(1)}_{(20)}\otimes\rho^{(2)}_{(20)},\ \rho^{(1)}_{(20)}\otimes\rho^{(2)}_{(02)},\ \rho^{(1)}_{(02)}\otimes\rho^{(2)}_{(20)},\ \rho^{(1)}_{(02)}\otimes\rho^{(2)}_{(02)}\right\}\\ &\qquad\cup\left\{\rho^{(1)}_a\otimes\rho^{(2)}_b \mid 3\le a\le n,\ 3\le b\le n,\ \text{each } a,b \text{ appears twice}\right\}.\end{aligned}\tag{6.31}$$

The second line means that, for any fixed a, there are two corresponding b such that $\rho^{(1)}_a\otimes\rho^{(2)}_b$ is contained in $\Phi_{L_{(20|20)}}$, and vice versa. This is so because $L_{(20|02)}$ is a degree-2 extension of $K^{(1)}$, so when the embedding of $K^{(1)}$ is fixed, there are still two choices left to embed $L_{(20|02)}$ into $\mathbb{C}$. There are $4+2\times(n-2)=2n$ embeddings of $L_{(20|20)}=L_{(20|02)}$; $2n-2$ embeddings among them correspond to Hodge (2,2) components, and the other two are the (4,0) and (0,4) Hodge components; indeed there are no Hodge (3,1) or (1,3) components. This case turns out to happen if and only if

$$\rho^{(1)}_{(20)}(K^{(1)}_0)=\rho^{(2)}_{(20)}(K^{(2)}_0)\subset\overline{\mathbb{Q}}\quad\text{and}\quad\rho^{(1)}_{(20)}(K^{(1)})\neq\rho^{(2)}_{(20)}(K^{(2)}),\tag{6.32}$$

where $K^{(1)}_0$ and $K^{(2)}_0$ are the maximal totally real sub-fields of $K^{(1)}$ and $K^{(2)}$, respectively.[21]

- Alternatively, we can choose (i-1) and (ii-1) to satisfy the conditions (i) and (ii). In this case, $[L_{(20|20)}:K^{(1)}]=[L_{(20|20)}:K^{(2)}]=1$ and $W_{(20|20)}\neq W_{(20|02)}$. This happens if and only if

$$\rho^{(1)}_{(20)}(K^{(1)})=\rho^{(2)}_{(20)}(K^{(2)})\subset\overline{\mathbb{Q}}.\tag{6.33}$$

[21] Let us formally state the claim and prove it here. The claim is that if and only if $[L_{(20|20)}:K^{(2)}]=2$ and $\Phi_{L_{20|20)}}$ contains both $\rho^{(1)}_{(20)}\otimes\rho^{(2)}_{(20)}$ and $\rho^{(1)}_{(02)}\otimes\rho^{(2)}_{(20)}$, then $\rho^{(1)}_{(20)}(K^{(1)}_0)=\rho^{(2)}_{(20)}(K^{(2)}_0)$ and $\rho^{(1)}_{(20)}(K^{(1)})\neq\rho^{(2)}_{(20)}(K^{(2)})$.

Recall that $L_{(20,20)}=K^{(2)}[x]/g(x)$, where $g\in K^{(2)}[x]$ and is a degree-2 polynomial. The two roots of $\rho^{(2)}_{(20)}(g(x))$, α_+ and α_-, must correspond to a simple generator α of $K^{(1)}$, i.e. $K^{(1)}=\mathbb{Q}(\alpha)$, such that $\alpha_+=\rho^{(1)}_{(20)}(\alpha)$ and $\alpha_-=\rho^{(1)}_{(02)}(\alpha)$. This means that α_+ and α_- are complex conjugate to each other. Let us explicitly define $g(x)=x^2+a_1x+a_0$ with $a_1,a_0\in K^{(2)}$. Then from the explicit form of the roots, one can conclude that $a_0,a_1\in\mathbb{R}$. This implies that $\mathbb{Q}(\alpha_+)=\rho^{(1)}_{(20)}(K^{(1)})$ is a degree-2 extension of a totally real field $\rho^{(2)}_{(20)}(K^{(2)}_0)$, which must equal to $\rho^{(1)}_{(20)}(K^{(1)}_0)$. Note that $\rho^{(2)}_{(20)}(K^{(2)})\neq\rho^{(2)}_{(20)}(K^{(1)})$ because $\alpha_+\notin\rho^{(2)}_{(20)}(K^{(2)})$.

Conversely, let us assume $\rho^{(2)}_{(20)}(K^{(2)}_0)=\rho^{(1)}_{(20)}(K^{(1)}_0)$ and $\rho^{(2)}_{(20)}(K^{(2)})\neq\rho^{(2)}_{(20)}(K^{(1)})$. Denoting the totally real field $\rho^{(2)}_{(20)}(K^{(2)}_0)=\rho^{(1)}_{(20)}(K^{(1)}_0)$ by K_0, the composite field $\rho^{(1)}_{(20)}(K^{(1)})\rho^{(2)}_{(20)}(K^{(2)})$ can be rewritten as $\rho^{(1)}_{(20)}(K^{(1)})\rho^{(2)}_{(20)}(K^{(2)})=K_0(\eta^{(1)},\eta^{(2)})$ for some $\eta^{(1)}\in\rho^{(1)}_{(20)}(K^{(1)})$ and $\eta^{(2)}\in\rho^{(2)}_{(20)}(K^{(2)})$. The Galois action that maps $\eta^{(1)}$ to its complex conjugate and leaves everything else will map $\rho^{(1)}_{(20)}\otimes\rho^{(2)}_{(20)}$ to $\rho^{(1)}_{(02)}\otimes\rho^{(2)}_{(20)}$, so the latter is also contained in $\Phi_{L_{(20|20)}}$. One can also see that $[\rho^{(1)}_{(20)}(K^{(1)})\rho^{(2)}_{(20)}(K^{(2)}):\rho^{(2)}_{(20)}(K^{(2)})]=2$, so $[L_{(20|20)}:K^{(2)}]=2$.

The contents of $\Phi_{L_{(20|20)}}$ are

$$
\begin{aligned}
\Phi_{L_{(20|20)}} &= \left\{\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)},\ \rho^{(1)}_{(02)} \otimes \rho^{(2)}_{(02)}\right\} \\
&\quad \cup \left\{\rho^{(1)}_{a} \otimes \rho^{(2)}_{b} \mid 3 \leq a \leq n,\ 3 \leq b \leq n,\ \text{each } a, b \text{ appears once}\right\},
\end{aligned}
\tag{6.34}
$$

where the second line means that for each a, there is a corresponding b; $L_{(20|20)}$ is isomorphic to $K^{(1)}$. There are $2 + (n-2) = n$ embeddings of $L_{(20|20)}$; two of them correspond to the Hodge $(4,0)$ and $(0,4)$ components, and the rest correspond to $(2,2)$ components. There are no $(3,1)$ or $(1,3)$ components.

Let us move on to the $W_{(20|02)}$ component. This component is level-4 when $W_{(20|20)} = W_{(20|02)}$, and is level-0 or level-2 otherwise. We are interested in how $K^{(1)}, K^{(2)}$ and their actions on $T^{(1)}_X$ and $T^{(2)}_X$ controls whether this component is (3,1)-free, especially whether it is level-0, or not. Almost the same analysis as above can be carried out, and there turn out to be only two cases where the component becomes $(3,1)$-free:

- The first case is where $[L_{(20|02)} : K^{(1)}] = [L_{(20|02)} : K^{(2)}] = 2$ and $W_{(20|20)} = W_{(20|02)}$ holds. The component is level-4, and this case has already been considered in the analysis of $W_{(20|20)}$ as the (i-2)-(ii-2) case.
- The component is also $(3,1)$-free when $[L_{(20|02)}; K^{(1)}] = [L_{(20|02)} : K^{(2)}] = 1$ holds. This is equivalent to

$$
\rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(02)}(K^{(2)}) \subset \overline{\mathbb{Q}}
\tag{6.35}
$$

 and in this case, $W_{(20|02)} \neq W_{(20|20)}$ holds, which means that the $W_{(20|02)}$ component is level-0. The contents of $\Phi_{L_{(20|02)}}$ are

$$
\begin{aligned}
\Phi_{L_{(20|02)}} &= \left\{\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)},\ \rho^{(1)}_{(02)} \otimes \rho^{(2)}_{(20)}\right\} \\
&\quad \cup \left\{\rho^{(1)}_{a} \otimes \rho^{(2)}_{b} \mid 3 \leq a \leq n,\ 3 \leq b \leq n,\ \text{each } a, b \text{ appears once}\right\},
\end{aligned}
\tag{6.36}
$$

 and all the embeddings are associated with Hodge $(2,2)$ components, and thus the component $W_{(20|02)}$ is indeed level-0. Note that $\rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(02)}(K^{(2)})$ is equivalent to $\rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(20)}(K^{(2)})$, since $\rho^{(2)}_{(02)}(K^{(2)})$ is the complex conjugate of $\rho^{(2)}_{(20)}(K^{(2)})$, which is $\rho^{(2)}_{(20)}(K^{(2)})$ itself because it is a CM field. This means that, when $W_{(20|02)} \neq W_{(20|20)}$, $W_{(20|20)}$ is (3,1)-free if and only if $W_{(20|02)}$ is level-0.

The analysis of the Hodge structures of $W_{(20|20)}$ and $W_{(20|02)}$ in the last pages can be summarized in a very simple way:

A). When[22]

$$
\rho^{(1)}_{(20)}(K^{(1)}_0) = \rho^{(2)}_{(20)}(K^{(2)}_0) \subset \overline{\mathbb{Q}} \quad \text{and} \quad \rho^{(1)}_{(20)}(K^{(1)}) \neq \rho^{(2)}_{(20)}(K^{(2)}),
\tag{6.37}
$$

[22]Let us remind ourselves that $K^{(i)}_0$ is defined to be the maximal totally real sub-field of $K^{(i)}$ for $i = 1, 2$.

the component $W_{(20|20)} = W_{(20|02)}$ in Eq. (6.16) is level-4 and $(3,1)$-free. All other W_i components in Eq. (6.16) are level-2.

B). When
$$\rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(20)}(K^{(2)}) \subset \overline{\mathbb{Q}}, \tag{6.38}$$
then there are two components in Eq. (6.16) that are not level-2: $W_{(20|20)}$ is level-4 and $(3,1)$-free, and $W_{(20|02)}$ is level-0. All other components are level-2.

C). When neither Eq. (6.37) nor Eq. (6.38) is satisfied, none of the component W_i in Eq. (6.16) is (3,1)-free.

Before getting into the recap of the physical consequences, it is worth while to take a slightly different view on cases A) and B): A non-trivial flux satisfying $DW = 0$ condition is possible only when the condition (6.37) or (6.38) is satisfied. Noticing that $\rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(20)}(K^{(2)})$ implies $\rho^{(1)}_{(20)}(K^{(1)}_0) = \rho^{(2)}_{(20)}(K^{(2)}_0)$, it is clear that such flux configurations exist if and only if
$$\rho^{(1)}_{(20)}(K^{(1)}_0) = \rho^{(2)}_{(20)}(K^{(2)}_0) =: K_0 \tag{6.39}$$
holds. Let us call the situation case A+B), since it combines the cases A) and B). Introducing some generators $\eta^{(1)}, \eta^{(2)} \in \overline{\mathbb{Q}}$ such that $K_0(\eta^{(1)}) = \rho^{(1)}_{(20)}(K^{(1)})$ and $K_0(\eta^{(2)}) = \rho^{(2)}_{(20)}(K^{(2)})$, one can see that the case B) is a non-generic situation where $K_0(\eta^{(1)})$ coincides with $K_0(\eta^{(2)})$, and the case A) is the generic situation complementary to it. The condition (6.4) for a $DW = 0$ flux has been translated into an arithmetic characterization (6.39), and a stronger condition (6.5) for a $DW = W = 0$ flux into a stronger characterization (6.38).

Step 3 (a physics recap): Now let us discuss the physical consequences, although most of what follows is included in the discussion so far. As a first physical consequence, one can see that there is no topological flux satisfying the $DW = 0$ condition, if $n_1 := [K^{(1)} : \mathbb{Q}]$ is not equal to $n_2 := [K^{(2)} : \mathbb{Q}]$; as we have been assuming $T^{(1)}_X = T^{(1)}_0$ and $T^{(2)}_X = T^{(2)}_0$ in this Section 6.3.2, this condition is equivalent to $\mathrm{rank}(T^{(1)}_0) = \mathrm{rank}(T^{(2)}_0)$. Furthermore, a non-trivial supersymmetric flux exists only in either one of these:

- In case A), with the condition (6.37), the component $W_{(20|20)} = W_{(20|02)}$ is level-4 and a $2 \times (n = n_1 = n_2)$-dimensional subspace of the n^2-dimensional vector space $V_1 \otimes_{\mathbb{Q}} V_2$. Any flux in this component satisfies the $DW = 0$ condition but always violates the $W = 0$ condition.
- In case B), with the condition (6.38), $W_{(20|02)}$ is an n-dimensional subspace of $V_1 \otimes_{\mathbb{Q}} V_2$ and is level-0. One has n-dimensional degrees of freedom to turn on the flux in this component without violating $DW = 0$ or $W = 0$ conditions. Another n-dimensional subspace $W_{(20|20)}$ is also free of $(3,1)$-components, but since it contains the $(4,0)$ component by definition, turning on any flux in this component violates the $W = 0$ condition. In summary, non-trivial flux vacua with $DW = 0$ and $W = 0$ is possible, if and only if (6.38) is satisfied.

As a reminder, we did not study arithmetic characterization for the $H^1(Z^{(1)};\mathbb{Q}) \otimes H^1(Z^{(2)};\mathbb{Q})$ component of Eq. (6.8) to support a $DW = 0$ flux, while $W = 0$ is automatic.

The conclusion above is similar[23] to, and also a generalization of the study by Aspinwall-Kallosh [AK05]. They chose the pair of K3 surfaces $X^{(1)}$ and $X^{(2)}$ to be attractive, that is, $\mathrm{rank}(T_X^{(1)}) = \mathrm{rank}(T_X^{(2)}) = 2$, and studied topological fluxes satisfying the $DW = 0$ condition as well as ones satisfying both of the $DW = 0$ and $W = 0$ conditions. Note that attractive K3 surfaces are always of CM-type with endomorphism fields being imaginary quadratic fields. The condition (6.39) follows immediately from their set-up because $K_0^{(1)} = K_0^{(2)} = \mathbb{Q}$ with $K^{(1)} = \mathbb{Q}(\sqrt{-d_1})$, $K^{(2)} = \mathbb{Q}(\sqrt{-d_2})$ in this case. The condition (6.39) for non-trivial fluxes with $DW = 0$ is regarded as a generalization of the $\mathrm{rank}(T_X^{(1)}) = \mathrm{rank}(T_X^{(2)}) = 2$ setup in [AK05]. For fluxes with $DW = W = 0$ to exist, [AK05] concluded that $K^{(1)} = \mathbb{Q}(\sqrt{-d_1})$ should be isomorphic to $K^{(2)} = \mathbb{Q}(\sqrt{-d_2})$; we have seen that this condition should be generalized to Eq. (6.38). See also footnote 39 in Section 6.4.

6.3.3 Complex structure moduli masses with $W = 0$

Now that we have worked out the conditions for non-trivial supersymmetric flux to exist in terms of arithmetic of the endomorphisms fields $K^{(1)}$ and $K^{(2)}$, let us move on to see whether such fluxes generate mass of complex structure moduli of M-theory compactification on $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$. In this Section 6.3.3, we assume that the vacuum complex structure of the pair of K3 surfaces $X^{(1)}$ and $X^{(2)}$ are of CM-type, generic enough in $D(T_0^{(i)})$ so that $T_X^{(i)} = T_0^{(i)}$, and satisfy the condition (6.38); low-energy effective field theory (including the the mass matrix) of the fluctuation fields around the vacuum complex structure is studied in the following.[24]

At the vacuum, a holomorphic (2,0) form $\Omega_{X^{(i)}}$ of the K3 surface $X^{(i)}$ can be chosen to be $v_{(20)}^{(i)}$ (by choosing the normalization of $\Omega_{X^{(i)}}$). Over the moduli space $\mathcal{M}_{\text{cpx str}}^{[X(T_0^{(i)})]}$ around the vacuum $\left\langle z_{(i)} \right\rangle$, the 2-form $\Omega_{X^{(i)}}(z_{(i)})$ that is holomorphic and purely of Hodge (2,0)-type in the complex structure of $z_{(i)} \in \mathcal{M}_{\text{cpx str}}^{[X(T_0^{(i)})]}$ is parameterized by

$$\Omega_{X^{(i)}} = v_{(20)}^{(i)} + t^{(i)} - \frac{(t^{(i)}, t^{(i)})_{T_X^{(i)} \otimes \mathbb{C}}}{2C^{(i)}} v_{(02)}^{(i)} \tag{6.40}$$

where $t^{(i)}$ collectively denotes[25] the local coordinates of the moduli space $\mathcal{M}_{\text{cpx str}}^{[X(T_0^{(i)})]}$ around the vacuum $\left\langle z_{(i)} \right\rangle$, i.e. the moduli field fluctuations around the vacuum, and is regarded as an element of $[T_0^{(i)} \otimes \mathbb{C}]^{(1,1)}$—the (1,1) Hodge component with respect to $\left\langle z_{(i)} \right\rangle$; $v_{(20)}^{(i)}$ and $v_{(02)}^{(i)}$ are also fixed against $T_X^{(i)} \otimes \mathbb{Q}$ and provide a fixed frame[26]

[23] References [AK05, BKW14] considered compactification by $Y = X^{(1)} \times X^{(2)}$, but since they did not take an orbifold, their set-up is different from the one in this work. When it comes to the study of supersymmetric fluxes within $(T_X^{(1)} \otimes T_X^{(2)}) \otimes \mathbb{Q}$, however, their case can be regarded as a special case of the study in this section.

[24] We restrict our attention to the fields of complex structure deformation within $\mathcal{M}_{\text{cpx str}}^{[Y]BV}$, not to the full deformation in $\mathcal{M}_{\text{cpx str}}^{[Y]}$. Nothing is lost when $g_{(1)}g_{(2)} = 0$, because there is no complex structure moduli deforming away from the orbifold limit then. For cases $g_{(1)}g_{(2)} \neq 0$, we do not have something to add to what we have already written in Section 6.2.

[25] See below (6.46) for a component description of the moduli fluctuation fields.

[26] We could use an integral basis of the lattices $T_0^{(i)}$ for a fixed frame, but the choice in the main text is obviously much more convenient for the discussion here.

with which we describe deformation of complex structure of $X^{(i)}$; finally, $C^{(i)} = (v^{(i)}_{(20)}, v^{(i)}_{(02)})$. The 4-form $\Omega_Y = \Omega_{X^{(i)}} \wedge \Omega_{X^{(2)}}$ to be fed into the flux superpotential (3.12) is

$$\begin{aligned}\Omega_Y &= v^{(1)}_{(20)} v^{(2)}_{(20)} + \left(v^{(1)}_{(20)} t^{(2)} + t^{(1)} v^{(2)}_{(20)} \right) \\ &- v^{(1)}_{(20)} v^{(2)}_{(02)} (2C^{(2)})^{-1} (t^{(2)}, t^{(2)})_{T^{(2)}_X} - v^{(1)}_{(02)} v^{(2)}_{(20)} (2C^{(1)})^{-1} (t^{(1)}, t^{(1)})_{T^{(1)}_X} + t^{(1)} t^{(2)} + \mathcal{O}(t^3).\end{aligned} \tag{6.41}$$

Suppose that a non-trivial flux is in the $W_{(20|02)}$ component; the condition (6.38) is implicit now. Then the contributions to Ω_Y in the first line of Eq. (6.41) do not yield any terms in the effective superpotential, so both of the $DW = 0$ and $W = 0$ conditions are satisfied at the vacuum, as designed. A flux in $W_{(20|02)}$ gives rise to terms that are quadratic in the fluctuation of the $2(n_{1,2} - 2) = 2(20 - r_{(i=1,2)})$ moduli fields—$t^{(i=1,2)}$—that would deform the complex structure of Y (within $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$) from the CM-type vacuum complex structure $\langle z \rangle$. Note also that a flux in $W_{(20|02)}$ does not generate a cubic or quartic terms of those moduli fields $t^{(i=1,2)}$, but just yields the mass terms.

With a closer look, one finds that the mass matrix is Dirac type, and that the product of the mass eigenvalues is real. As a first step to see this, we write down the mass terms using the following notations, so that we can keep track of Galois-conjugate relations among the coefficients in the effective superpotential. Under the condition (6.38), we can fix one isomorphism from $K^{(2)}$ to $K^{(1)}$,

$$(\rho^{(1)}_{(20)})^{-1} \circ \rho^{(2)}_{(02)} : K^{(2)} \to \overline{\mathbb{Q}} \to K^{(1)}; \tag{6.42}$$

this isomorphism can be used to set up a 1-to-1 correspondence between the embeddings of $K^{(1)}$ and those of $K^{(2)}$;

$$\rho^{(2)}_{\beta(\alpha)} := \rho^{(1)}_\alpha \circ \left((\rho^{(1)}_{(20)})^{-1} \circ \rho^{(2)}_{(02)} \right), \qquad \alpha \in \{(20), (02), 3, \cdots, n\}. \tag{6.43}$$

Then $\Phi_{L_{(20|02)}} = \{\rho^{(1)}_\alpha \otimes \rho^{(2)}_{\beta(\alpha)} \mid \alpha = (20), (02), 3, \cdots, n\}$ in this notation. The fact that a flux must be in the $\mathbb{Q}$-coefficient cohomology, rather than in the $\mathbb{R}$ or $\mathbb{C}$-coefficient cohomology groups, is translated into the condition $n^{ij} \in \mathbb{Q}$ defined below, when we use $v^{(1)}_\alpha \otimes v^{(2)}_\beta$'s for a basis of the cohomology:

$$\int_Y G \wedge (v^{(1)}_\alpha \otimes v^{(2)}_{\beta(\alpha)}) =: \sum_{i,j} n^{ij} \rho^{(1)}_\alpha (y^{(1)}_i)\, \rho^{(2)}_{\beta(\alpha)} (y^{(2)}_j) =: G_{\alpha\beta(\alpha)}, \quad \alpha \in \{(20), (02), 3, \cdots, n\}, \tag{6.44}$$

where $\{y^{(1)}_i \mid i = 1, \cdots, n\}$ and $\{y^{(2)}_j \mid j = 1, \cdots, n\}$ are the basis of $K^{(1)}$ and $K^{(2)}$, respectively, over $\mathbb{Q}$, as introduced in Section 5.1.1. $G_{(20)(02)}$ is an algebraic number within $\rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(02)}(K^{(2)})$. Other $G_{\alpha\beta(\alpha)}$'s are Galois conjugate of $G_{(20)(02)}$:

$$\sigma_\alpha(G_{(20)(02)}) = G_{\alpha\beta(\alpha)}, \tag{6.45}$$

where $\sigma_\alpha \in \mathrm{Gal}((\rho^{(1)}_{(20)}(K^{(1)}))^{\mathrm{nc}}/\mathbb{Q})$ that brings $\rho^{(1)}_{(20)}$ to $\sigma_\alpha \cdot \rho^{(1)}_{(20)} = \rho^{(1)}_\alpha$ and $\rho^{(2)}_{(02)}$ to $\sigma_\alpha \cdot \rho^{(2)}_{(02)} = \rho^{(2)}_{\beta(\alpha)}$. Thus the mass matrix is of the form

$$W \propto -\frac{G_{(20)(02)}}{2C^{(2)}}\,(t^{(2)}, t^{(2)})_{T_X^{(2)}} - \frac{(G_{(20)(02)})^{\mathrm{c.c.}}}{2C^{(1)}}\,(t^{(1)}, t^{(1)})_{T_X^{(1)}} + \sum_{a=3}^{n} \sigma_a(G_{(20)(02)}) t^{(1)}_a t^{(2)}_a, \tag{6.46}$$

when we parametrize the moduli by $t^{(1)} = \sum_{a=3}^n t^{(1)}_a v^{(1)}_a$ and $t^{(2)} = \sum_{a=3}^n t^{(2)}_a v^{(2)}_{\beta(a)}$.

The Dirac structure of the mass matrix becomes manifest only after examining the mass terms $\propto (t^{(2)}, t^{(2)})$ and $\propto (t^{(1)}, t^{(1)})$ that are apparently Majorana. A key observation is that $(v^{(i)}_{(20)}, v^{(i)}_\gamma) = 0$ for any $\gamma \in \{(20), 3, \cdots, n\}$. Applying the Galois transformations,[27] we see that

$$(v^{(i)}_\alpha, v^{(i)}_{\overline{\alpha}}) = \sigma_\alpha(C^{(i)}), \qquad (v^{(i)}_\alpha, v^{(i)}_\gamma) = 0 \quad \text{for } \gamma \neq \overline{\alpha}. \tag{6.47}$$

Using this property, the moduli effective superpotential is written in the following form:[28]

$$\sum_{a' \in}^{\{2,\cdots,n/2\}} (t^{(1)}_{a'}, t^{(2)}_{a'}) \begin{pmatrix} -(G_{(20)(02)})^{\mathrm{c.c.}} \frac{\sigma_{a'}(C^{(1)})}{C^{(1)}} & \sigma_{\overline{a'}}(G_{(20)(02)}) \\ \sigma_{a'}(G_{(20)(02)}) & -G_{(20)(02)} \frac{\sigma_{\overline{a'}}(C^{(2)})}{C^{(2)}} \end{pmatrix} \begin{pmatrix} t^{(1)}_{\overline{a'}} \\ t^{(2)}_{\overline{a'}} \end{pmatrix}. \tag{6.48}$$

This mass matrix is obviously Dirac type, and is furthermore split into $(n/2 - 1)$ blocks of 2×2 matrices.

The product of all the mass eigenvalues is in $\mathbb{R}$. This is so even at the level of the individual 2×2 mass matrices; the product is the determinant of the mass matrix above, which is

$$(C^{(1)}C^{(2)})^{-1}\,\left(|G_{(20)(02)}|^2 \sigma_{a'}(C^{(1)}C^{(2)}) - |\sigma_{a'}(G_{(20)(02)})|^2 C^{(1)}C^{(2)} \right), \tag{6.49}$$

which takes values in $\mathbb{R}$ because $C^{(1)}, C^{(2)} \in \mathbb{R} \cap \rho^{(1)}_{(20)}(K^{(1)})$. Also note that the product is not expected to vanish generically. To see this, one only needs to check if there is a cancellation between the two terms, as each of the two terms should be non-zero generically. Let us focus on the case where $K^{(1)}$ is Galois, and the Galois action $\sigma_{a'}$ is an order-two automorphism, where a cancellation seems to be most likely to be present; in this case,

$$\sigma_{a'}\left(|G_{(20)(02)}|^2 \sigma_{a'}(C^{(1)}C^{(2)}) \right) = |\sigma_{a'}(G_{(20)(02)})|^2 C^{(1)}C^{(2)}, \tag{6.50}$$

[27] Note that the map $\Phi_{K^{(i)}} \ni \rho^{(i)}_\gamma \mapsto \sigma_\alpha \cdot \rho^{(i)}_\gamma \in \Phi_{K^{(i)}}$ is one-to-one map, and that the basis vectors v_γ have a component description (5.9) for a $\mathbb{Q}$-basis of $T_0^{(i)}$.

[28] Here is a little more set of notations. The n embeddings $\Phi_{K^{(i)}}$ form $n/2$ pairs under the complex conjugations in $\overline{\mathbb{Q}}$ (and also in the CM fields $K^{(i)}$); $\rho^{(i)}_{\alpha'}$ is paired with $\mathrm{cc} \cdot \rho^{(i)}_{\alpha'} = \rho^{(i)}_{\alpha'} \cdot \mathrm{conj.}$, which is denoted by $\rho^{(i)}_{\overline{\alpha'}}$; the set $\Phi_{K^{(i)}}$ can be grouped into two $\{\rho^{(i)}_{\alpha'} \mid \alpha' \in \{(20), 2, \cdots, n/2\}\}$ and $\{\rho^{(i)}_{\overline{\alpha'}} \mid \alpha' \in \{(20), 2, \cdots, n/2\}\}$; a separation into two in this way is not unique. Note also that $\overline{\beta(\alpha)} = \beta(\overline{\alpha})$.

so the first term is mapped to the second one by the Galois action. This means that the cancellation occurs if and only if the equality

$$\sigma_{a'}\left(|G_{(20)(02)}|^2\sigma_{a'}(C^{(1)}C^{(2)})\right) = |G_{(20)(02)}|^2\sigma_{a'}(C^{(1)}C^{(2)}) \tag{6.51}$$

holds. We will see that the cancellation does not happen in generic cases under our assumptions. Note first that $|G_{(20)(02)}|$, $C^{(1)}$ and $C^{(2)}$ are generic in $\mathbb{R} \cap \rho^{(1)}_{(20)}(K^{(1)})$; $G_{(20)(02)}$ takes generic values in $\rho^{(1)}_{(20)}(K^{(1)})$ as it is defined to be an inner-product with the $(2,0) \otimes (0,2)$-form, which is a linear combination of the basis elements of $\rho^{(1)}_{(20)}(K^{(1)})$ because of the CM-type nature of the $W_{(20|02)}$ component, and also there is no structure to constrain the value $C^{(i)} = (v^{(i)}_{(20)}, v^{(i)}_{(02)})$ to a smaller field than $\mathbb{R} \cap \rho^{(1)}_{(20)}(K^{(1)})$. Then the quantity $|G_{(20)(02)}|^2\sigma_{a'}(C^{(1)}C^{(2)})$ also takes generic values in $\mathbb{R} \cap \rho^{(1)}_{(20)}(K^{(1)})$, and if the quantity is invariant under the action of $\sigma_{a'}$, i.e. if $\mathbb{R} \cap \rho^{(1)}_{(20)}(K^{(1)})$ is invariant under the action of $\sigma_{a'}$, then the action is unique, because $\rho^{(1)}_{(20)}(K^{(1)})$ is a degree-2 extension of $\mathbb{R} \cap \rho^{(1)}_{(20)}(K^{(1)})$. The unique action is actually the complex conjugation on $\rho^{(1)}_{(20)}(K^{(1)})$, because the unique Galois action of imaginary degree-2 extension of a sub-field of $\mathbb{R}$ always is the complex conjugation[29]. The action is denoted by $\sigma_{(02)}$ in our notation, but $\sigma_{a'}$ in Eq. (6.49) cannot be $\sigma_{(02)}$; $\sigma_{(02)}$ does not appear in Eq. (6.46). Thus the product of masses (6.49) does not vanish generically, under our assumptions of $K^{(1)}$ being Galois and $\sigma_{a'}$ being order-two. One can confirm the latter part of the argument explicitly[30], and one would not expect the product of masses to vanish in more complicated setups.

To summarize, for a given vacuum complex structure in $\mathcal{M}^{[X(T_0^{(1)})]}_{\mathrm{CM}} \times \mathcal{M}^{[X(T_0^{(2)})]}_{\mathrm{CM}}$ satisfying the condition (6.38), each choice of a flux from $W_{(20|02)} \simeq \mathbb{Q}^n$ is consistent with the $DW = 0$ and $W = 0$ conditions, and the $(n-2)$ Dirac mass eigenvalues (all the values in $\overline{\mathbb{Q}}$) can be computed systematically. As a reminder, $n = [K^{(1)} : \mathbb{Q}] = [K^{(2)} : \mathbb{Q}] = \mathrm{rank}(T^{(i)}_X) = 22 - r_{(1,2)}$. The Dirac type mass matrix and the real nature of the product of the mass eigenvalues are a common (and unexpected!)[31] consequence the class of flux vacua under consideration.

[29] In other words, the two roots of a quadratic polynomial with coefficients in $\mathbb{R}$ are always complex conjugate to each other, when they have non-zero imaginary parts.

[30] For a very explicit example, take $K^{(1)} \cong K^{(2)} \cong \mathbb{Q}(i\sqrt{p}, i\sqrt{q})$. Assume that

$$G_{(20)(02)} = g_1 + g_2 i\sqrt{p} + g_3 i\sqrt{q} + g_4\sqrt{pq}, \tag{6.52}$$
$$C^{(1)}C^{(2)} = c_1 + c_4\sqrt{pq}. \tag{6.53}$$

Also assume that σ_1 acts as a multiplication by -1 on $i\sqrt{p}$ and acts trivially on $i\sqrt{q}$. Then the determinant of the mass matrix is given by

$$-2\sqrt{pq}\left(c_4\left(g_1^2 + g_2^2 p + g_3^2 q + g_4^2 pq\right) - 2c_1(g_1 g_4 + g_2 g_3)\right). \tag{6.54}$$

There are several cases where the determinant vanishes, e.g. $c_4 = 0$ and $g_0 = g_1 = 0$, but generically it does not vanish.

[31] The vacuum complex structure of the pair of K3 surfaces being CM-type does not imply at all such things as period integrals being real, or the field of moduli having embedding into $\mathbb{R}$. Here, we pay attention to the product of the mass eigenvalues as an exercise problem for potential applications to the strong CP problem.

The moduli stabilization discussed above (with $\mathrm{rk}(T_X^{(1)}) = \mathrm{rk}(T_X^{(2)}) = 2$) appears similar to the one in [AK05, BKW14].[32] Direct comparison with [AK05, BKW14] is easier in the cases we discuss in Section 6.4 (see footnote 39). When we take $T_X^{(i)} = T_0^{(i)}$ (as in this Section 6.3) and set $\mathrm{rank}(T_X^{(i)}) = 2$ (as in [AK05, BKW14]), all the complex structure moduli fields of $Y = (X^{(1)} \times X^{(2)})$ whose mass discussed in [AK05, BKW14] are now projected out in the orbifold $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ here.

The moduli mass from fluxes (with $\langle W \rangle = 0$) above is closer to the one in [KW17a]. Discussion there corresponds to a special case of the above result in this thesis; Ref. [KW17a] was for $X^{(1)} = \mathrm{Km}(E_\phi \times E_\tau)$, the Type IIB orientifold set-up. Although [KW17a] only argued that the complex structure moduli of Type IIB Calabi-Yau threefold $M = (E_\tau \times X^{(2)})/\mathbb{Z}_2$ and the axi-dilaton chiral multiplets are stabilized along with $\langle W \rangle = 0$, the discussion above shows that the moduli fields of D7-brane positions are also stabilized along with $\langle W \rangle = 0$.

Having studied the mass terms in (6.46) and (6.48), let us now have a look at the whole low energy effective theory superpotential of the complex structure moduli fields $t^{(1,2)}$ from the perspective of symmetry. The superpotential have $\mathrm{U}(1)^{\frac{n}{2}-1} \times \mathrm{U}(1)_R$ symmetry. All the moduli fields $t_a^{(i)}$ have $+1$ charge under the R-symmetry; there is also one U(1) symmetry for each one of $a' \in \{2, \cdots, n/2\}$, where the chiral multiplets $t_{a'}^{(1)}$ and $t_{\overline{a'}}^{(2)}$ have charge $+1$, and the chiral multiplets $t_{\overline{a'}}^{(1)}$ and $t_{a'}^{(2)}$ charge -1. This symmetry is a part of the symmetry of the Kähler potential,[33] which is

$$\begin{aligned} K &= -\sum_{i=1}^{2} \ln\left((\Omega_{X^{(i)}}, \overline{\Omega}_{X^{(i)}}) \right), && (6.56) \\ &= -\sum_{i=1}^{2} \ln\left(C^{(i)} + \sum_{a}^{3\sim n} \sigma_a(C^{(i)}) t_a^{(i)} (t_a^{(i)})^\dagger + \sum_{a',b'}^{2\sim n/2} \frac{\left(\sigma_{a'}(C^{(i)}) t_{a'}^{(i)} t_{\overline{a'}}^{(i)}\right)\left(\sigma_{b'}(C^{(i)}) (t_{b'}^{(i)} t_{\overline{b'}}^{(i)})^\dagger\right)}{C^{(i)}} \right). \end{aligned}$$

To understand the origin and nature of the $\mathrm{U}(1)^{\frac{n}{2}-1}$ symmetry in the moduli effective field theory, it helps to reflect more upon the symmetry of the Kähler potential, the non-linear sigma model metric (6.56). The target space $\mathcal{M}_{\text{cpx str}}^{[Y]BV}$ is a homogeneous space with the symmetry group

$$\mathbb{G}U\mathrm{Isom}(T_X^{(1)}; \mathbb{C}) \times \mathbb{G}U\mathrm{Isom}(T_X^{(2)}; \mathbb{C}), \tag{6.57}$$

where

$$\mathbb{G}U\mathrm{Isom}(L; \mathbb{C}) := \left\{ g \in \mathrm{Isom}(L \otimes \mathbb{C}) \mid (g^\dagger \bar{x}, gx)_L = {}^\exists c_g \times (\bar{x}, x)_L \text{ for } {}^\forall x \in L \otimes \mathbb{C} \right\} \tag{6.58}$$

[32] The relation to [DDF+05] is discussed in Section 8.3. For other references dealing with a similar setup, see [BHLV09, BBGnL20]

[33] Recall that we use the Kähler potential obtained in the large volume limit, i.e.

$$K = -\ln\left(\int_Y \Omega_Y \wedge \overline{\Omega}_Y \right), \tag{6.55}$$

as already stated in Section 3.5.

for a lattice L; c_g can be any constant independent of x. For any (not necessarily CM) point in $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$ chosen as a vacuum, the isotropy group—the symmetry group linearly realized on the fluctuations fields—is

$$\mathrm{SO}(n_1-2)\times\mathrm{SO}(n_2-2); \tag{6.59}$$

we have seen that $n_1 = n_2 =: n$ under the condition (6.38). The $\mathrm{U}(1)^{n/2-1}$ symmetry of the full moduli effective theory (including the moduli mass terms due to fluxes) is the Cartan part of the diagonal $\mathrm{SO}(n-2)$. They are global symmetries.[34]

It is worth noting that the presence of the symmetry $\mathrm{U}(1)^{n/2-1}$ in the non-linear sigma model in $\mathbb{R}^{2,1}$ in M-theory ($\mathbb{R}^{3,1}$ in F-theory) is essentially due to the nature of the period domain of a K3 surface[35] rather than that of a Calabi-Yau manifold of higher dimensions. As already briefly referred to[36] in Section 5.3, one could think of the space of rational Hodge structures on $H^3(M;\mathbb{Q})$ for a family of Calabi-Yau threefolds $[M]$, which is a homogeneous space just like $D(T_0^{(i)})$'s are; the complex structure moduli space of $[M]$ is only a subspace of the homogeneous space, so the symmetry of the non-linear sigma model of the complex structure moduli of $[M]$ cannot be simply stated by just referring to the vector space $H^3(M;\mathbb{Q})$ and the skew-symmetric intersection form on it. The same discussion applies also to Calabi-Yau fourfolds.

Furthermore, those continuous symmetries—either $\mathrm{U}(1)^{n/2-1}$ or $\mathrm{SO}(n-2)\times\mathrm{SO}(n-2)$—of the non-linear sigma model in $\mathbb{R}^{2,1}$ or $\mathbb{R}^{3,1}$ cannot be attributed to a symmetry of the geometry $X^{(1)}\times X^{(2)}$. A symmetry transformation on $X^{(i)}$ would manifest itself as a symmetry action on $H^2(X^{(i)};\mathbb{Q})$; a transformation on $H^2(X^{(i)};\mathbb{C})$ that cannot be derived from one on $H^2(X^{(i)};\mathbb{Q})$ does not have an interpretation as an $X^{(i)}\to X^{(i)}$ map. Those continuous symmetries are not symmetries of the geometry $X^{(1)}\times X^{(2)}$, but are symmetries of their moduli spaces. They are accidental symmetry in the low-energy effective theory.

The continuous $\mathrm{U}(1)^{\frac{n}{2}-1}\times\mathrm{U}(1)_R$ symmetry in the moduli effective theory are likely not to be an exact symmetry apart from its possible non-trivial discrete subgroup (cf footnote 34). This expectation is from general arguments in quantum gravity; as for the $\mathrm{U}(1)_R$ part,[37] one may also argue this by computing triangle anomalies against the Standard Model gauge groups (e.g., [EIKY12]). The source of explicit

[34] A non-trivial discrete subgroup in $\mathrm{U}(1)^{\frac{n}{2}-1}$ may be gauged, in the sense that a discrete subgroup of the symmetry of a vacuum complex structure may be regarded as a part of the isotropy group of the form $\mathrm{Isom}(T_X^{(1)})^{\text{Hdg Amp}}\times\mathrm{Isom}(T_X^{(2)})^{\text{Hdg Amp}}$, which induces automorphisms (unphysical difference) of $X^{(1)}\times X^{(2)}$.

[35] The complex structure moduli effective superpotential (6.46) is of very specific—purely quadratic—form also essentially due to this.

There is an argument on the ground of genericity that $\langle W\rangle = 0$ must be associated with some discrete R-symmetry (more than $\mathbb{Z}_2$), and then the moduli superpotential must be of the form $W\sim\sum_i X_i f_i(\phi)$, where X_i are chiral fields that transform the same way as W under the R-symmetry, and ϕ's other moduli fields that are neutral under the R-symmetry [DOS05, Sun12]. It then follows in this regime when all those fields have masses. The moduli superpotential on K3 × K3 orbifolds in this thesis is not within this genericity regime.

[36] Interested readers are referred to [Bor92, Ker15, GGK12].

[37] There are tight constraints on how R-symmetry charge is assigned on the particles in supersymmetric Standard Models. On the other hand, we need to know how the moduli fields $t_a^{(i)}$ couple to the Standard Model particles to find out how (or whether) the $\mathrm{U}(1)^{n/2-1}$ symmetry can be extended to the whole low-energy effective theory.

breaking of the symmetry may be the anomalies with gauge fields, stringy non-perturbative effects, or just stringy perturbative corrections to the approximation $K = -\ln(\int_Y \Omega_Y \wedge \overline{\Omega}_Y)$ and $W \propto \int_Y G \wedge \Omega_Y$. Better understanding on the source of explicit breaking[38] will give us better hint on a discrete exact symmetry in the effective theory containing all of moduli, the supersymmetric Standard Model and anything else. In the case the discrete exact symmetry is larger than the symmetry acceptable at TeV scale (such as a subgroup of $\mathrm{U}(1)_R$ larger than $\mathbb{Z}_2$ R symmetry), the domain wall problem sets constraints on inflation and the thermal history after that. If the explicit breaking leaves only the $\mathbb{Z}_2$ subgroup of the $\mathrm{U}(1)_R$ symmetry, then the source of the explicit breaking also determines the gravitino mass.

6.4 Cases with $T_X \subsetneq T_0$

Think of a case where the vacuum complex structure of $X^{(i)}$ is still of CM-type, but not generic enough to have $T_X^{(i)} = T_0^{(i)}$ for at least one of $i = 1, 2$; $T_X^{(i)} \subsetneq T_0^{(i)}$ and $S_0^{(i)} \subsetneq S_X^{(i)}$ then. Put differently, the vacuum complex structure of $X^{(i)}$ is in a Noether-Lefschetz locus of $\overline{D(T_0^{(i)})}$, where there must be an element of $H^2(\mathrm{K3};\mathbb{Z})$ (Poincaré dual of a 2-cycle) that becomes algebraic. Now, the rank of $T_X^{(i)}$ is still even (because of its CM nature), but $T_0^{(i)}$ in Nikulin's list may be of odd rank. We will see below that much the same story unfolds for a $DW = 0$ flux, and also for a $DW = W = 0$ flux; one difference, though, is that there is one more way (without the relation (6.38)) to stabilize moduli in $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$ by a $DW = W = 0$ flux, when $T_X^{(i)} \subsetneq T_0^{(i)}$ for both $i = 1, 2$.

Let $\overline{T}_0^{(i)}$ be the negative definite lattice $[(T_X^{(i)})^\perp \subset T_0^{(i)}]$, so that

$$T_0^{(i)} \otimes \mathbb{Q} \cong (T_X^{(i)} \otimes \mathbb{Q}) \oplus (\overline{T}_0^{(i)} \otimes \mathbb{Q}). \tag{6.60}$$

The $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ component of $H^4_{\text{hor}}(Y;\mathbb{Q})$ is then expanded as follows:

$$(T_0^{(1)} \otimes T_0^{(2)}) = (T_X^{(1)} \otimes T_X^{(2)}) \oplus (T_X^{(1)} \otimes \overline{T}_0^{(2)}) \oplus (\overline{T}_0^{(1)} \otimes T_X^{(2)}) \oplus (\overline{T}_0^{(1)} \otimes \overline{T}_0^{(2)}). \tag{6.61}$$

The two components $(T_X^{(1)} \otimes \overline{T}_0^{(2)})$ and $(\overline{T}_0^{(1)} \otimes T_X^{(2)})$ with rational Hodge sub-structure are always level-2, and the $(\overline{T}_0^{(1)} \otimes \overline{T}_0^{(2)})$ component always level-0, if it is non-zero. The component $T_X^{(1)} \otimes T_X^{(2)}$ contains a level-4 component; whether the rational Hodge structure on this component is simple or not depends.

Suppose that the condition (6.38) is satisfied. Then any rational flux in $\overline{T}_0^{(1)} \otimes \overline{T}_0^{(2)} \otimes \mathbb{Q}$ and the $i = (20|02)$ component of $T_X^{(1)} \otimes T_X^{(2)} \otimes \mathbb{Q}$ satisfies the $DW = 0$ and $W = 0$ conditions.[39] When a flux is non-zero only in the $W_{(20|02)}$ component within $T_X^{(1)} \otimes T_X^{(2)}$, the moduli effective theory superpotential (6.46) remains as it is if it is interpreted as follows; the third term of Eq. (6.46) only involves the moduli fluctuation fields within $D(T_X^{(1)}) \times D(T_X^{(2)})$, while $(t^{(2)}, t^{(2)})$ and $(t^{(1)}, t^{(1)})$ in the first two terms

[38] Alternatively, one may focus on the common subset of $\mathrm{Isom}(T_X^{(i)}) \times \mathrm{Isom}(T_X^{(2)})$ and $\mathrm{U}(1)^{n/2-1}$, which will be a more mathematical study, to infer what the discrete gauged symmetry is.

[39] This mechanism is quite close to the one in [AK05, BKW14]; the flux in $(\overline{T}_0^{(1)} \otimes \overline{T}_0^{(2)}) \otimes \mathbb{Q}$ corresponds to a part of G_0-flux, and the one in $(T_X^{(1)} \otimes T_X^{(2)}) \otimes \mathbb{Q}$ to G_1-flux in [AK05, BKW14].

of Eq. (6.46) are meant to include all the fluctuations fields in $D(T_0^{(1)}) \times D(T_0^{(2)})$. When a non-zero flux is in the $\overline{T}_0^{(1)} \otimes \overline{T}_0^{(2)} \otimes \mathbb{Q}$, there is one more term in the effective superpotential, which is the Dirac-type mass term of the moduli fluctuation fields in $N_{D(T_X^{(1)})|D(T_0^{(1)})}$ (normal directions) and $N_{D(T_X^{(2)})|D(T_0^{(2)})}$. Because the (stabilizing) mass terms for the fluctuations within $D(T_X^{(1)}) \times D(T_X^{(2)})$ rely on the flux in $W_{(20|02)}$, the condition (6.38) is necessary (apart from the caveat mentioned below). This moduli effective theory has an U(1) R-symmetry, where all the moduli fluctuation fields have $+1$ R-charge; to see this, we almost have to repeat the argument in Section 6.3.3, and the fact that a flux in $\overline{T}_0^{(1)} \otimes \overline{T}_0^{(2)}$ also generates only the mass term. There is no additional non-R U(1) symmetry where the moduli fluctuation fields in $N_{D(T_X^{(1)})|D(T_0^{(1)})}$ and $N_{D(T_X^{(2)})|D(T_0^{(2)})}$ are charged, however. This is because there is no Dirac-like structure for those moduli fields in the $(t^{(2)}, t^{(2)})$ and $(t^{(1)}, t^{(1)})$ in the first two terms in the superpotential (6.46).

One caveat in the argument above is the case there is no moduli fluctuation fields within $D(T_X^{(1)}) \times D(T_X^{(2)})$, which is when both $X^{(1)}$ and $X^{(2)}$ are attractive ($\mathrm{rank}(T_X^{(i)}) = 2$, $\mathrm{rank}(S_X^{(i)}) = 20$) K3 surfaces. Even when the condition (6.38) is not satisfied, a flux in $\overline{T}_0^{(1)} \otimes \overline{T}_0^{(2)}$ provide a mass term for all the moduli fluctuation fields in $D(T_0^{(1)}) \times D(T_0^{(2)})$ if the condition

$$\mathrm{rank}(\overline{T}_0^{(1)}) = \mathrm{rank}(\overline{T}_0^{(2)}) \tag{6.62}$$

is satisfied. The mass matrix is Dirac type then. This is because the mass matrix from the flux here is always Dirac type.

Chapter 7

General K3 $\times$ K3 orbifolds

The Borcea-Voisin orbifold $(X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ in the previous chapter is a way to construct a Calabi-Yau variety of higher dimensions by using K3 surfaces (and/or elliptic curves), and there is an obvious generalization; think of any supersymmetry-preserving orbifold of a product of K3 surfaces (and/or elliptic curves); the orbifold group Γ is not necessarily $\mathbb{Z}_2$ [Dil12]. In this Chapter 7, we bring known materials together from the literature, to have a broad brush picture of possible variety in the construction, to identify open math problems for a complete classification, and to repeat the same study as in Sections 6.3 and 6.4 for the cases with $\Gamma \neq \mathbb{Z}_2$.

7.1 K3 $\times$ K3 orbifold

Consider an orbifold $Y_0 = (X^{(1)} \times X^{(2)})/\Gamma$, where $X^{(1)}$ and $X^{(2)}$ is a pair of K3 surfaces. The orbifold group Γ should be a subgroup of $\mathrm{Aut}(X^{(1)}) \times \mathrm{Aut}(X^{(2)})$, first of all. For the action of the orbifold group Γ to preserve supersymmetry, one more condition needs to be imposed. To state the condition, we prepare some notations.

Under the projection $p_i : \mathrm{Aut}(X^{(1)}) \times \mathrm{Aut}(X^{(2)}) \to \mathrm{Aut}(X^{(i)})$, let $G_i := p_i(\Gamma)$. Let $\alpha'_i : \mathrm{Aut}(X^{(i)}) \to \mathrm{Isom}(T_X^{(i)})^{\text{Hdg Amp}}$ be the projection that fits into the exact sequence[1]

$$1 \to \mathrm{Aut}_N(X^{(i)}) \to \mathrm{Aut}(X^{(i)}) \to \mathrm{Isom}(T_X^{(i)})^{\text{Hodge Amp}} \to 1. \tag{7.1}$$

Because the elements of $\mathrm{Isom}(T_X^{(i)})^{\text{Hdg Amp}}$ acts on the holomorphic (2,0) form $\Omega_{X^{(i)}}$ faithfully, $\alpha'_i(\sigma_{(i)}) \in \mathrm{Isom}(T_X^{(i)})^{\text{Hdg Amp}}$ for $\sigma_{(i)} \in \mathrm{Aut}(X^{(i)})$ may well be identified with the complex phase $\alpha_i(\sigma_{(i)})$ in $\sigma_{(i)}^* \Omega_{X^{(i)}} = \alpha_i(\sigma_{(i)})\Omega_{X^{(i)}}$. With those preparations, the supersymmetry condition is written as

$$^{\forall}\sigma \in \Gamma, \qquad \alpha_1(p_1(\sigma))\, \alpha_2(p_2(\sigma)) = 1 \in \mathbb{C}. \tag{7.2}$$

We will discuss only the cases that the group Γ has a finite number of elements.[2]

[1] In this section, we use the same notation as in [BKW13, BKW14] without spelling out their definitions. Reviews in [Asp96, Huy16, BKW13, BKW14] will also be useful.

[2] It sounds like an orbifold $(X^{(1)} \times X^{(2)})/\Gamma$ with $|\Gamma| = \infty$ would yield a pathological "Calabi-Yau fourfold", although we are not absolutely sure if such possibilities should be completely ruled out.

An equivalent way to state the supersymmetry condition is that there is a group Δ, so that[3,4]

$$\alpha'_i(G_i) \cong \Delta, \qquad \Gamma \subset G_1 \times_\Delta G_2. \tag{7.3}$$

When we impose the Calabi-Yau condition ($h^{p,0}(Y_0) = 0$ for $p = 1,2,3$ in addition to $h^{4,0}(Y_0) = 1$), the group Δ needs to be something other than[5] the trivial group $\{1\}$.

The two K3 surfaces $X^{(1)}$ and $X^{(2)}$ for an M-theory/F-theory compactification come with one Kähler form for each one of them. The orbifold group action by G_i for $i = 1,2$ should preserve the Kähler form on $X^{(i)}$ (so the orbifold defines a consistent theory).

7.2 K3 surfaces with non-symplectic automorphisms

7.2.1 Discrete classification

Just like we used Nikulin's classification in the previous section, one can think of a similar classification problem whose answer can be used for this generalized form of the Borcea-Voisin orbifolds. Here is how we formulate the problem: how many different choices of $(S \oplus T, G)$ there are modulo $\mathrm{Isom}(\mathrm{II}_{3,19})$, subject to the conditions

- G is a finite subgroup of $\mathrm{Isom}(\mathrm{II}_{3,19})$, and S and T are mutually orthogonal primitive sub-lattices of $\mathrm{II}_{3,19} \cong H^2(K3;\mathbb{Z})$ such that $(S \oplus T) \otimes \mathbb{Q} \cong \mathrm{II}_{3,19} \otimes \mathbb{Q}$,
- $g(T) = T$ (and also $g(S) = S$) for any $g \in G$,
- for any $g \in G$ whose $g|_T$ is non-trivial, $g|_T$ is not identity on any vector subspace of $T \otimes \mathbb{Q}$.
- S and T have signature $(1, r-1)$ and $(2, 20-r)$ (with $1 \leq r \leq 20$),
- for any $g \in G$, the sub-lattice $S^g := \{x \in S \mid g \cdot x = x\}$ contains 1 signature-positive direction.

The first three conditions characterize G, T and S as a set of automorphism group, transcendental lattice and Néron-Severi lattice of a K3 surface.[6] The last two conditions reflect[7] the Calabi-Yau condition of Y_0 ($\Delta \neq \{1\}$) and the Γ-invariance of the Kähler parameter discussed in Section 7.1.

For one choice $(S \oplus T, G)$, we can determine two groups

$$G_s := \mathrm{Ker}\,(G \to \mathrm{Isom}(T))\,, \tag{7.4}$$
$$\Delta := \mathrm{Im}\,(G \to \mathrm{Isom}(T))\,. \tag{7.5}$$

[3] The isomorphism $\alpha'_1(G_1) \cong \alpha'_2(G_2)$ should be such that their representations α_1 and α_2 are complex conjugate.

[4] Complete classification of $(S^{(i)}, T^{(i)}, G_i; G_{s,i}, \Delta)$ for $i = 1,2$ and $\Gamma \subset G_1 \times_\Delta G_2$ will be redundant for classification of variety in the generalized Borcea-Voisin fourfolds for compactification. For example, in a case Γ has a structure of $\Gamma \cong \Gamma_0 \times G'_1 \times G'_2$ with $\Gamma_0 \subset \mathrm{Aut}(X^{(1)}) \times_\Delta \mathrm{Aut}(X^{(2)})$ and $G'_i \subset \mathrm{Aut}_s(X^{(i)})$ for $i = 1,2$, one may replace a compactification by $(X^{(1)} \times X^{(2)})/\Gamma$ with a compactification by $(X^{(1)}_{\rm cr} \times X^{(2)}_{\rm cr})/\Gamma_0$, where $X^{(i)}_{\rm cr}$ is a crepant resolution of $X^{(i)}/G'_i$.

[5] It can be shown ([Nik80] Thm. 0.1 (a), Thm. 3.1 (a) and Cor 3.2) that a K3 surface with $\Delta \neq \{1\}$ is always algebraic.

[6] See [Nik80, Thm. 3.1 (b)] for the third condition.

[7] See [Nik80, Thm. 0.1 (a) and 3.1 (a)] for the fourth condition, and [Nik80, Lemma 4.2 (a)] for the fifth condition.

1_1	2_1	4_2	8_4	16_8	32_{16}
3_2	6_2	12_4	24_8	48_{16}	
9_6	18_6	36_{12}			
27_{18}	54_{18}				

1_1	5_4	7_6	11_{10}	13_{12}	17_{16}	19_{18}	25_{20}
2_1	10_4	14_6	22_{10}	26_{12}	34_{16}	38_{16}	50_{20}
3_2	15_8	21_{12}	33_{20}				
4_2	20_8	28_{12}	44_{20}				
6_2	30_8	42_{12}	66_{20}				
8_4	40_{16}						
12_4	60_{16}						

TABLE 7.1: The list of $m := |\Delta|$ and the corresponding $\varphi(m)$. The $m_{\varphi(m)}$'s with an m of the form of $m = 2^p 3^q$ are shown in the table on the left, while the table on the right lists up m's that are not divisible by 2^4, 3^2, or 24. $m = 1, 2, 3, 4, 6, 8, 12$ overlap in the two Tables.

Therefore, the classification of $(S \oplus T, G)$ may well be regarded as classification of the data $(S \oplus T, G; G_s, \Delta)$. Furthermore, one may state the result of the classification by listing up all possible choices $1 \to G_s \to G \to \Delta \to 1$ first, and then by listing up of all possible lattice pairs $S \oplus T$ for $(G; G_s, \Delta)$. Nikulin's classification implies that the case $G_s \cong \{1\}$ and $G \cong \Delta \cong \mathbb{Z}_2$ contains 75 different choices of $S \oplus T$, as we have referred to in Chapters 5 and 6.

There are at most 41 different choices of the group Δ including $\Delta \cong \{1\}$ ([Nik80, Cor 3.2] and [MO98]); all the possible group Δ's are cyclic groups $\mathbb{Z}_m$ for some $m \in \mathbb{N}_{>0}$ ([Nik80, Thm. 3.1 (b,c)], [Ste85, Lemma 2.1], [Huy16, Cor 3.3.4]), and the list of 41 m's (Table 1 of [MO98]) are reprinted explicitly in Table 7.1 here for convenience of the readers. The rank $(22 - r)$ of the lattice T should be divisible by $\varphi(m)$.

There are at most 82 different choices of G_s including $G_s \cong \{1\}$. They should be a subgroup of 11 different finite groups listed in [Muk88] (two of them are $\mathfrak{A}_6$ and $\mathfrak{S}_5$; see [Muk88] for nine others). See [Xia96] for the list of those all those 82 finite groups.[8] It is also known that if G_s has an element g of order $n > 1$, then it must be[9] that $n \leq 8$ first of all, and secondly, $\mathrm{rank}(S) \geq 9$ if $n = 2$, $\mathrm{rank}(S) \geq 13$ if $n = 3$, $\mathrm{rank}(S) \geq 15$ if $n = 4$, $\mathrm{rank}(S) \geq 17$ if $n = 5, 6$, and $\mathrm{rank}(S) \geq 19$ if $n = 7, 8$ [Huy16, Cor 15.1.8].

Therefore, there can be at most $[82 \times 41]$ different choices of the finite groups G_s and Δ; the choice $G_s \cong \{1\}$ and $\Delta \cong \mathbb{Z}_2$ in Chapter 6 is one of this $[82 \times 41]$ choices. In fact, not all the 82×41 choices can be realized. A group $\Delta \cong \mathbb{Z}_m$ with larger m requires that the lattice T has a larger rank because $\varphi(m) | \mathrm{rk}(T)$, whereas a larger group G_s requires S with a larger rank (and T with a smaller rank). Using the data available in [Xia96], the range of $m_{\varphi(m)}$ can be narrowed down for each choice of G_s; Table 7.2 is the summary (cf. also [Keu15, Keu16]).

For $\Delta \cong \mathbb{Z}_m$ with $m = 66, 44, 33, 50, 25, 40$, and 60, for which $G_s = \{1\}$ (and $G \cong \Delta$) is the only option, all the possible $S \oplus T$'s have been worked out by using lattice theory and a bit of geometry [MO98, Lemma (1.2)]. It turns out that there is just one choice of $S \oplus T$ for each one of $m = 66, 44, 33, 50, 25, 40$, and that $G \cong \Delta$ happens to

[8] Among the 82 of them are 15 abelian groups (including $\{1\}$) worked out by [Nik80, Thm 4.5]. If we are to demand that $|\mathrm{Aut}(X)| < \infty$ (not just $|G| < \infty$), then just the three in the list, $G_s = \{1\}$, $\mathbb{Z}_2$ and S_3, are possible [Kon89, Kon86].

[9] An immediate consequence of this fact is that, if we are to choose a finite subgroup $G \subset \mathrm{Aut}(X)$, then any element in G has an order not larger than $8 \cdot 66$; in fact there is no such an element of order 8×66 because the unique K3 surface admitting $\Delta \cong \mathbb{Z}_{66}$ does not have symplectic automorphisms. The true upper bound is known to be 66 [Keu16].

G_s in [Xia96]	c	$\Delta \cong \mathbb{Z}_m$
$\{1\}$		any 41 m's
#1	$(c=8)$	26 $m_{\varphi(m)}$'s with $\varphi(m) \leq 12$
#2, 3	$(c=12)$	18 $m_{\varphi(m)}$'s with $\varphi(m) \leq 8$
# 4,6,9,10,21	$c = 14, 15$	13 $m_{\varphi(m)}$'s with $\varphi(m) \leq 6$
17 G_s's	$c = 16, 17$	9 $m_{\varphi(m)}$'s with $\varphi(m) \leq 4$
56 G_s's	$c = 18, 19$	$m_{\varphi(m)} \in \{1_1, 2_1, 3_2, 4_2, 6_2\}$

TABLE 7.2: The range of $m_{\varphi(m)}$ such that $\Delta \cong \mathbb{Z}_m$ can be combined with a given G_s to form G. The 82 choices of G_s are grouped into six by their value of some integer c listed in Table 2 of [Xia96]. For those six groups of G_s's, the possible range of $m_{\varphi(m)}$ is determined by the condition $\varphi(m) \leq 21 - c$, shown on the right.

act on S trivially. There is no choice of $S \oplus T$ where $m = 60$ [Zha05]. For $\Delta \cong \mathbb{Z}_m$ with $m = 17$ and 19, see [OZ00].

For general $(G; G_s, \Delta)$, complete classification of the choices of $S \oplus T$ is not available yet. For cases with $G_s = \{1\}$ (so $G \cong \Delta$), all the possibilities of $(S \oplus T)$ with $G \cong \Delta \cong \mathbb{Z}_m$ acting trivially on S have been classified, however. For cases with an m that is divisible by two (or more) prime numbers (such as $m = 6, 10, 15, \cdots$), it turns out that both S and T have to be unimodular; see [Kon92] for the list of $S \oplus T$ for the m's that are not in the form of $m = p^k$ for a single prime number p. For cases with $m = p^k$, this is an immediate generalization of the classification of Nikulin [Nik81]. See [Tak12, Sch10, TST14] for the $m = 2^2$ case (there are 12 $S \oplus T$), the $m = 2^3$ case (there are 3 $S \oplus T$), and the $m = 2^4$ case ($S = UD_4$ unique), while there is no choice of $S \oplus T$ for the case $m = 2^5$ [Vor83, Kon92]. For the cases with $m = 3^k$, and $m = 5, 7, 11, 13, 17, 19$, see [AS08, Tak11, Tak10] and [AST11], respectively.

For the cases with $G_s = \{1\}$ (so $G \cong \Delta \cong \mathbb{Z}_m$), one may think of the classification of $(S \oplus T)$'s where G may act on S non-trivially. Only partial results are known. Results for $m = 17, 19, 40, 25, 50, 33, 44, 66, 60$ have been quoted earlier already. The $m = 2^5$ case has just one choice of $S \oplus T$, where $G \cong \mathbb{Z}_{32}$ acts on the rank-6 $S = UD_4$ (the same as the unique choice for the $m = 2^4$ case) non-trivially on a 2-dimensional subspace through a quotient $\mathbb{Z}_{32} \to \mathbb{Z}_4$ (and trivially on a 4-dimensional subspace) [Ogu93, Tak14]. For a similar study in the case of $G_s = \{1\}$ and $G \cong \Delta \cong \mathbb{Z}_m$ with $m = 2^4$, $m = 2^3$, and $m = 2^2$, see [TST14, AT15, TS16, AS15].

Just like there is only small number of choices of $(S \oplus T)$ is available for a large Δ, it is also known that there are tight constraints on the possible choices of $(S \oplus T)$ when G_s is large. See such references as [Nik80, Thm. 4.7], [Has12], [GS07], [GS09], [Kon99], and [OZ02].

This Section 7.2.1 is a literature survey, relying mostly on [Huy16] as a guide. We wished to learn what is known as well as what has not been known about how much the $\mathbb{Z}_2$ orbifold in Chapter 6 can be generalized.

7.2.2 Period Domains for K3 Surfaces with Automorphisms

The period integrals (complex structure) of a K3 surface X should be in the period domain $D(T)$ for one of the choice $(S \oplus T)$, when X has an automorphism $(G; G_s, \Delta)$, but the converse is not true. For a complex structure to be consistent with the automorphism group $(G; G_s, \Delta)$ with $\Delta \neq \{1\}$, $G|_T = \Delta \cong \mathbb{Z}_m$ needs to be a Hodge isometry.

The subspace in $D(T)$ consistent with such a non-symplectic automorphism group is specified as follows. Note that the action of $\Delta \cong \mathbb{Z}_m$ on $T \otimes \mathbb{Q}$ is always of the form of

$$T \otimes \mathbb{Q} \cong (N_m)^{\oplus \ell} \qquad \text{with} \quad \ell := \mathrm{rk}(T)/\varphi(m) \tag{7.6}$$

where $\mathbb{Z}_m$ acts as $\mathbb{Q}$-valued matrices on a $\varphi(m)$-dimensional vector space N_m over $\mathbb{Q}$; the generator $[[\sigma]]$ of $\mathbb{Z}_m$ has the set of eigenvalues $\{\zeta_m^a \mid a \in [\mathbb{Z}_m]^\times\}$. Thus $T \otimes \mathbb{C}$ is divided into $\varphi(m)$ distinct eigenspaces of $\mathbb{Z}_m$, $\oplus_{a\in[\mathbb{Z}_m]^\times} V_a$, where $[[\sigma]]|_{V_a} = \zeta_m^a$. Individual V_a's are ℓ-dimensional over $\mathbb{C}$. Thus, the complex structure should be in

$$D(V_a) := \mathbb{P}[V_a] \cap D(T) \tag{7.7}$$

for some $a \in [\mathbb{Z}_m]^\times$. The subvariety $D(V_a)$ of $D(T)$ is determined only by $\Delta \cong \mathbb{Z}_m$, independent of the symplectic subgroup of the automorphism G_s.

This extra condition on the complex structure moduli space was absent in the case of $\mathbb{Z}_2$ orbifold in the previous section, because $\varphi(m = 2) = 1$, and $T \otimes \mathbb{Q} \cong V_{a=1}$. For the cases with $m > 2$, however, $V_a \subsetneq T \otimes \mathbb{C}$, and $D(V_a) \subsetneq D(T)$. In fact, there is just one pair of $D(V_a)$ and $D(V_{a'})$ with $a, a' \in [\mathbb{Z}_m]^\times$ and $a' = -a \in \mathbb{Z}_m$; that is because the intersection matrix remains non-zero only between V_b–$V_{b'}$ pairs with $b' = -b \in \mathbb{Z}_m$ (remember that $[[\sigma]]$ is an isometry of T), and the 2-dimensional positive signature directions must be contained only in one of those pairs. In that non-zero pair $D(V_{a0})$ and $D(V_{-a0})$, the $\Omega^2 = 0$ condition is automatically satisfied in $\mathbb{P}(V_{a0})$ and $\mathbb{P}(V_{-a0})$, so $D(V_{a0})$ and $D(V_{-a0})$ are open subspace of $\mathbb{P}^{\ell-1}$ specified by the $(\Omega, \overline{\Omega}) > 0$ condition [DK07, AST11].

In the cases with $\varphi(m) = \mathrm{rk}(T)$, so $\ell = 1$, the subvarieties $D(V_a)$ are 0-dimensional, so they are isolated points.[10] This is consistent with the fact that a CM-type K3 surface corresponds to an isolated point on the moduli space, as discussed later.

7.2.3 K3 surfaces of CM-type and with non-symplectic automorphisms

Not all the points in the moduli space $D(V_{a0})$ of K3 surfaces with non-symplectic automorphisms correspond to K3 surfaces of CM-type as one may expect, but there is some correlation. The subspace of $D(V_{a0})$ corresponding to CM-type K3 surfaces is characterized as in the discussion in the following. We focus on the cases with $\Delta \cong \mathbb{Z}_m$ for $m > 2$, but some parts of the discussion applies to the cases of involution, $m = 2$.

In the cases with $\ell = 1$ and $\varphi(m) = \mathrm{rk}(T)$, the one point $D(V_{a0})$ corresponds to a CM-type K3 surface (cf [LSY10]). This is because the algebra $\mathrm{Span}_{\mathbb{Q}}\{[[\sigma]] \in \Delta\}$ is a part of the endomorphism algebra $\mathrm{End}(T)^{\mathrm{Hdg}}$, and already $\dim_{\mathbb{Q}}(\mathrm{Span}_{\mathbb{Q}}\{[[\sigma]]\}) = \mathrm{rk}(T)$. The endomorphism field is isomorphic to $\mathbb{Q}(\zeta_m)$.

In the cases with $\ell := \mathrm{rk}(T)/\varphi(m) > 1$, if a CM point is contained $D(V_{a0})$ outside of Noether-Lefschetz loci, then the CM field K must be an extension of $\mathbb{Q}([[\sigma]]) \cong$

[10] Those isolated points are further subject to identification by a certain finite index subgroup of $\mathrm{Isom}(T)$. More is known in the literature about the identification of those isolated points (e.g., [Kon92, §5]).

$\mathbb{Q}(\zeta_m)$. Beyond that, however, the author have not been able to find a comprehensive and concise statement about how to find out all possible K's for a given lattice T.

7.2.4 Bonus symmetry

By construction, the K3 surfaces $X^{(1)}$ and $X^{(2)}$ to be used in the orbifold construction have certain amount of automorphisms, G_1 and G_2, respectively. It happens to be the case for some $(S, T, G; G_s, \Delta)$, though, that a K3 surface X with a generic complex structure in $D(T)$ has $\mathrm{Aut}(X)$ larger than G.

For example, think of $(G; G_s, \Delta) = (\mathbb{Z}_m; \{1\}, \mathbb{Z}_m)$ with an $m_{\varphi(m)}$ in Table 7.1 such that $\varphi(m)$ divides either one of $4, 12, 20$. Then (S, T) can be unimodular lattices of rank $(18, 4)$, $(10, 12)$ and $(2, 20)$. For a unimodular T, a K3 surface with a generic complex structure in $D(T)$ has a $\mathbb{Z}_2$ purely non-symplectic[11] automorphism (generated by the combination of (-1) multiplication on T and id. on S) [Kon92]. This automorphism is a part of the symmetry $\Delta \cong \mathbb{Z}_m$ that we imposed, if m is even. For an odd m, namely $m = 5, 7, 11, 13, 25, 3, 21, 33, 9$, however, we have more automorphisms ($\mathbb{Z}_{2m} \subset \mathrm{Aut}(X)$) than we imposed for orbifold construction ($G = \Delta = \mathbb{Z}_m$). See also [GS13], where similar enhancement of automorphism groups are discussed in the case (S, T) are not necessarily unimodular.

As another class of examples, we may think of a case of $(G; G_s, \Delta)$ with $G_s \neq \{0\}, \mathbb{Z}_2, S_3$. It is then known that $|\mathrm{Aut}_s(X^{(i)})| = \infty$ (see footnote 8 and also [Nik14]). Thus there are more automorphisms than we impose in this class of examples.

Those bonus automorphisms are available for any point in the period domain $D(T)$; this means that they act trivially on $D(T)$. The automorphisms can be realized linearly in the effective theory (not a broken symmetry), because a choice of a complex structure in $D(T)$ is not shifted by the automorphisms. This observation also indicates that the fluctuation fields of complex structure within $D(T)$ are neutral under these symmetry transformation.[12] Because those bonus automorphisms[13] act on $X^{(i)}$, and are non-trivial transformation on the orbifold geometry $Y_0 = (X^{(1)} \times X^{(2)})/\Gamma$, they are still non-trivial information on the effective field theory.[14] When one considers F-theory applications (where the orbifold Y_0 and its crepant resolutions are replaced by a birationally equivalent fourfold $\widetilde{Y}$ and some Kähler parameters are brought to zero (see Chapter 8)), one will be interested in working out how the symmetry acts[15] on fluctuation fields other than the complex structure moduli in $D(T)$. That is beyond the scope of this thesis, however.

[11] An automorphism is said to be **purely non-symplectic** when $G \simeq \Delta$.

[12] Discussion here focuses on automorphisms available for a generic point in $D(T)$, once $(G; G_s, \Delta)$ and (S, T) are given. For special loci in $D(T)$, there can be larger group of automorphisms.

We will see in Section 7.3 in the case of $\Delta \cong \mathbb{Z}_m$ with $m > 2$ that $D(T)$ moduli are stabilized by a $DW = W = 0$ flux only in the case the vacuum complex structure minimizes the rank of the transcendental lattice T_X to $\varphi(m)$. Thus the question of real interest is not necessarily about a generic point in $D(T)$.

[13] Besides the bonus automorphisms $(\mathrm{Aut}(X^{(1)}) \times \mathrm{Aut}(X^{(2)}))/(G_1 \times G_2)$, there are also automorphisms $(G_1 \times G_2)/\Gamma$ acting non-trivially on the orbifold $Y_0 = (X^{(1)} \times X^{(2)})/\Gamma$ by construction. Note that $\Gamma \subset G_1 \times_\Delta G_2 \subsetneq G_1 \times G_2$ (because $\Delta \neq \{1\}$). Both of the bonus automorphisms and the by-construction automorphisms present themselves as symmetries of the low-energy effective theory.

[14] Some of those bonus automorphisms (symmetries) may be broken by a non-trivial flux in $H^4(Y; \mathbb{Q})$. It is the symmetry respected by the flux that matters in the low-energy effective theory and cosmology after inflation.

[15] Choices of configuration of metric and other fields that become equivalent under automorphisms are regarded as one and the same point in the space of path integral. Therefore, an automorphism may be regarded as a gauge symmetry.

7.3 Complex structure moduli masses with $W = 0$

For a general choice of the orbifold group $\Gamma \subset G_1 \times_\Delta G_2$, we do not try to say what the cohomology $H^4(Y;\mathbb{Q})$ is like, e.g. a statement analogous to Eq. (6.7) or Eq. (6.8), for Y, a minimal crepant resolution, if exists, of the orbifold $(X^{(1)} \times X^{(2)})/\Gamma$. In the cases $\Gamma \cong G_1 \cong G_2 \cong \Delta = \mathbb{Z}_m$, the cohomology group $H^4(Y;\mathbb{Q})$ contains [CR04]

$$[(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}]^{[[\sigma]]} \oplus \left(H^2(Z^{(1)} \times Z^{(2)};\mathbb{Q})\right)^{\oplus(m-1)} \oplus (\cdots), \tag{7.8}$$

where $Z^{(1)}$ and $Z^{(2)}$ are curve loci of points in $X^{(1)}$ and $X^{(2)}$, respectively, fixed under the group Δ. The last term stands for possible contributions from fixed loci $Z^{(1)} \times$ (isolated pts), (isolated pts) $\times Z^{(2)}$, and (isolated pts) $\times$ (isolated pts) in $X^{(1)} \times X^{(2)}$. The second term has a Hodge structure of level 2 (for a vacuum complex structure within $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$). The $H^2(Z^{(1)} \times Z^{(2)};\mathbb{Q})$ component may contain a level-0 rational Hodge structure in some cases (see footnote 10 in Section 6.2). Possible contributions $(\cdots)$ are also level 0. Thus $W = DW = 0$ flux are available in those level-0 components. Those fluxes may (or may not) give rise to the complex structure moduli fluctuation fields that move away from $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$ into $\mathcal{M}^{[Y]}_{\text{cpx str}}$, but they do not generate a mass term or interaction term of moduli fluctuation fields within $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$.

Let us now focus on supersymmetric fluxes available within the $[(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}]^{[[\sigma]]}$ component; only such fluxes can stabilize (generate mass terms) of the $(\ell^{(1)} - 1) + (\ell^{(2)} - 1)$ moduli fluctuation fields[16] in $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$. In the case of $\langle[[\sigma]]\rangle = \Delta \cong \mathbb{Z}_2$, we have nothing to modify[17] in the discussions in Sections 6.3 and 6.4. In a case of $\Delta \cong \mathbb{Z}_m$ with $m > 2$, let us start our discussion with an assumption that $T_X^{(i)} = T_0^{(i)}$, i.e., a generic CM complex structure available within $D(V_{a0}) \times D(V_{-a0})$. A few observations to be added to the discussions in Sections 6.3.1 and 6.3.2 are the following.

First, only a proper subspace of $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ survives the orbifold projection, as we consider a case with $m > 2$ now. Second, the decomposition (6.16) of the vector space $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ is still useful; keeping in mind the fact that individual components W_i in Eq. (6.16) are in one-to-one with the orbits Φ_{L_i} under the action of $\mathrm{Gal}((K^{(1)}K^{(2)})^{\text{nc}}/\mathbb{Q})$, and also the fact that both $K^{(1)}$ and $K^{(2)}$ contain a sub-field $\mathbb{Q}([[\sigma]])$, one concludes that $[[\sigma]]$ acts on each one of W_i's either trivially entirely or non-trivially entirely on that W_i. We thus have

$$[(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}]^{[[\sigma]]} \cong \bigoplus_{i \in \Delta\text{neut}) \subset \{1,\cdots,r\}} W_i, \tag{7.9}$$

where only the subset of $\{1, \cdots, r\}$ where W_i is neutral under Δ—denoted by Δneut—is retained on the right hand side. Finally, the component with $i = (20|20)$ is in the subset (Δneut), but the component $i = (20|02)$ is not.[18]

It makes sense to study non-trivial representations of those automorphisms (gauge symmetries) on field fluctuations instead of throwing away all the modes in non-trivial representations, because two particle excitation state can be gauge-symmetry neutral, while each particle is not.

[16] $(\ell^{(i)} - 2)$ instead of $\ell^{(i)} - 1$ in the case of $m = 2$.

[17] The cohomology group $H^4(Y;\mathbb{Q})$ outside of the $[(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}]^{[[\sigma]]}$ should be modified, when Γ acts non-trivially on $S_0^{(i)} := [(T_0^{(i)})^\perp \subset H^2(X^{(i)};\mathbb{Z})]$, for $i = 1$ or 2.

[18] Note that $\zeta_m^{2a_0} = 1$ and $a_0 \in [\mathbb{Z}_m]^\times$, only when $m = 2$ and $a_0 = 1$.

We can review the conclusions in Section 6.3.2 with those three observations in mind. Now (for $m > 2$), the case-A in page 53 is not logically possible. Besides the case-C, where there is no flux with the $DW = 0$ condition available, the only possibility for a supersymmetric flux is the case-B in page 53, where all but one components W_i in Eq. (7.9) are level-2, and the remaining $W_{(20|20)}$ is simple and level-4. Therefore, to conclude (when $m > 2$ and $T_X^{(i)} = T_0^{(i)}$), a $DW = 0$ flux is possible if and only if the condition (6.38) is satisfied; such a flux is in the level-4 $W_{(20|20)}$ component, so $\langle W \rangle \neq 0$. There is no chance for a $DW = W = 0$ flux in $[(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}]^{[[\sigma]]}$ when $m > 2$ and $T_X^{(i)} = T_0^{(i)}$, because the level-0 $W_{(20|02)}$ component does not survive the orbifold projection when $m > 2$.

A $DW = W = 0$ flux is possible within $[(T_0^{(1)} \otimes T_0^{(2)})]^{[[\sigma]]}$ if and only if $T_x^{(i)} \subsetneq T_0^{(i)}$ for both $i = 1, 2$; it is not enough to have $T_X^{(i)} \subsetneq T_0^{(i)}$ for just one of $i = 1, 2$. To see this, remember that

$$
\begin{aligned}
& [(T_0^{(1)} \otimes T_0^{(2)})]^{[[\sigma]]} \\
\cong\ & [(T_X^{(1)} \otimes T_X^{(2)})]^{[[\sigma]]} \oplus [(T_X^{(1)} \otimes \overline{T}_0^{(2)})]^{[[\sigma]]} \oplus [(\overline{T}_0^{(1)} \otimes T_X^{(2)})]^{[[\sigma]]} \oplus [(\overline{T}_0^{(1)} \otimes \overline{T}_0^{(2)})]^{[[\sigma]]};
\end{aligned}
\tag{7.10}
$$

the middle two components on the right-hand side are level-2, and the first component consists only of level-2 and level-4 Hodge components; the latter statement is obtained by repeating the argument above (for $T_0^{(1)} \otimes T_0^{(2)}$ with $T_X^{(i)} = T_0^{(i)}$ there). Thus a $DW = W = 0$ flux can only be in the last component. Such a flux cannot generate a mass term for the moduli field fluctuations in $D(T_X^{(1)}) \cap D(V_{a0})$ and $D(T_X^{(2)}) \cap D(V_{-a_0})$, however.

Therefore, the only possibility for a $DW = W = 0$ flux stabilizing all the complex structure moduli, if $m > 2$, is when

$$
\mathrm{rank}(T_X^{(1)}) = \mathrm{rank}(T_X^{(2)}) = \varphi(m) \tag{7.11}
$$

so that there is no moduli within $[(T_X^{(1)} \otimes T_X^{(2)}) \otimes \mathbb{C}]^{[[\sigma]]}$. The condition (6.38) is satisfied automatically then ($K^{(1)} \cong K^{(2)} \cong \mathbb{Q}(\zeta_m)$), but there is no $W_{(20|02)}$ component to support a $DW = W = 0$ flux when $m > 2$. The $(\ell^{(1)} - 1) + (\ell^{(2)} - 1)$ moduli field fluctuations have Dirac type mass terms from a flux in the $(\overline{T}_0^{(1)} \otimes \overline{T}_0^{(2)}) \otimes \mathbb{Q}$ component. Thus, for all those moduli fields to have masses,

$$
\ell^{(1)} = \ell^{(2)} \tag{7.12}
$$

is also necessary, just like the condition (6.62) in Section 6.4. Just like in Sections 6.3.3 and 6.4, this moduli effective field theory has an approximate $\mathrm{U}(1) \times \mathrm{U}(1)_R$ symmetry.

Chapter 8

F-theory applications and particle physics aspects

In the earlier chapters, we have discussed the supersymmetry conditions, (6.1) and (6.2), of fluxes on CM-type Borcea-Voisin Calabi-Yau fourfolds $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$, and also stabilization of the complex structure moduli. The analysis in Chapters 6 and 7 can be read in the context of M-theory compactification on such fourfolds down to $(2+1)$ dimensions; the orbifold geometry $Y_0 = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ is singular, but the study in Chapters 6 and 7 in that M-theory context should be read[1] as that for a fourfold Y^{BV} which is the minimal and crepant resolution of Y_0, with positive values of Kähler parameters for the exceptional cycles.

To think of an F-theory compactification down to 4 dimensions, we need a Calabi-Yau fourfold $\widetilde{Y}$ that has a flat[2] elliptic fibration.[3] When F-theory is compactified on $\widetilde{Y}$ such that $\mathcal{M}^{[\widetilde{Y}]}_{\text{cpx str}}$ is contained in $\mathcal{M}^{[Y]BV}_{\text{cpx str}}$, the analysis for presence of a non-trivial supersymmetric flux and stabilization of moduli in $\mathcal{M}^{[\widetilde{Y}]}_{\text{cpx str}}$ is still valid.

In a large fraction of this chapter, we will be concerned about when and how one can find such $\widetilde{Y}$ birational to Y^{BV}. When a $\widetilde{Y}$ is available, its geometry should determine gauge groups and possible matter representations in the effective theory on 4 dimensions, motivated by the constraint $\langle W \rangle \simeq 0$. We will take steps to read out those implications.

8.1 Elliptically-fibered K3 surface with a non-symplectic involution

One of the technical problems that we face in the context of F-theory compactification is to find, for a given Calabi-Yau variety Y for an M-theory compactification, a set of (Y, B, π), where $\pi : Y \to B$ is a flat elliptic fibration and B a base manifold. For our setup, we will assume that the elliptic fibration of $\tilde{Y}$ is inherited from that of $X^{(1)}$. In that case, nearly a complete classification of $(Y = X^{(1)}, B = \mathbb{P}^1_{(1)}, \pi^f_{X^{(1)}})$ is available ([CG15, GS18] and references therein) for K3 surfaces $X^{(1)}$ associated with

[1]For example, Eq. (6.7) is justified for smooth manifolds $Y = Y^{BV}$.

[2]As we have reviewed in Section 3.4.2, this condition is for absence of an exotic particle spectrum on $\mathbb{R}^{3,1}$. This is thus a phenomenological constraint.

[3]Recall that, as we have already seen in Eq. (3.18), we also need to take the limit in the Kähler moduli so that the volume of the elliptic fiber vanishes, and to keep some part of the purely vertical part of $H^4(Y;\mathbb{Q})$ free from fluxes, in order to restore the SO(3,1) symmetry.

the 75 choices of $(S_0^{(1)}, T_0^{(1)}, \sigma_{(1)})$ of Nikulin, as we will review briefly in this Section 8.1. Such an elliptic fibration $(X^{(1)}, \mathbb{P}^1_{(1)}, \pi^{[f]}_{X^{(1)}})$ is used in Sections 8.3 and 8.4 to construct a fourfold $\widetilde{Y}$ birational to a Borcea-Voisin orbifold $Y_0 = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ where there is a flat elliptic fibration morphism $\pi : \widetilde{Y} \to B_3$.

We begin with recalling known facts about how we find elliptic fibration morphisms from an algebraic K3 surface to $\mathbb{P}^1$. There exists a genus-one fibration[4] morphism from a K3 surface $X^{(1)}$ to $\mathbb{P}^1$ if and only if there exists a divisor class $[f] \in S_X^{(1)}$ with $[f]^2 = 0$ [PSS71]. The corresponding fibration morphism is denoted by $\pi^{[f]}_{X^{(1)}} : X^{(1)} \to \mathbb{P}^1_{(1)}$. For a genus-one fibration morphism $\pi^{[f]}_{X^{(1)}} : X^{(1)} \to \mathbb{P}^1_{(1)}$ to be an elliptic fibration morphism,[5] there must exist another divisor class $[s] \in S_X^{(1)}$ satisfying $(s, f) = +1$ and $(s, s) = -2$. The primitive sub-lattice generated by $[f]$ and $[s]$ within $S_X^{(1)}$ is isomorphic to U then. To repeat, existence of an elliptic fibration is equivalent to existence of a factor U in $S_X^{(1)}$.

In the context of F-theory applications, when we write $S_X^{(1)} = U \oplus W$, the lattice W contains the information of non-Abelian 7-brane gauge groups, the number of U(1) gauge fields, and also the spectrum of charges under those gauge groups in 8 dimensions. Therefore, a well-motivated classification of elliptic fibration morphisms of $X^{(1)}$ is equivalent to classifying[6] primitive embeddings of U into $S_X^{(1)}$ modulo isometry of the lattice $S_X^{(1)}$. One and the same K3 surface $X^{(1)}$ (with a common $S_X^{(1)}$ and $T_X^{(1)}$) may have multiple different types of elliptic fibration morphisms in this classification; one of the most famous examples is the case $S_X = U \oplus E_8^{\oplus 2} \cong \mathrm{II}_{1,17} \cong U \oplus (D_{16}; \mathbb{Z}_2)$. An F-theory limit takes the volume of a fiber elliptic curve class $[f] \in U$ to zero, so different choices of $U \subset S_X^{(1)}$ correspond to different F-theory limits.

In general, $S_0^{(1)} \subset S_X^{(1)}$; when the complex structure of the K3 surface $X^{(1)}$ is not fully generic in the period domain of the lattice $T_0^{(1)}$, $S_X^{(1)}$ is strictly larger than $S_0^{(1)}$ (and $T_X^{(1)}$ is strictly smaller than $T_0^{(1)}$). Although it is enough to find a factor U within $S_X^{(1)}$ in constructing an elliptic fibration $\pi_{X^{(1)}} : X^{(1)} \to \mathbb{P}^1_{(1)}$, we wish to use the elliptic fibration morphism to construct an elliptic fibration $\pi_Y : Y \to B_3$ with some threefold B_3 (which is to be constructed in the following). We thus need to be concerned with how the elliptic fibration morphism $\pi_{X^{(1)}} : X^{(1)} \to \mathbb{P}^1_{(1)}$ behaves under the generator σ of the $\mathbb{Z}_2$ orbifold. We stick to the simplest case where the U sub-lattice is within $S_0^{(1)} \subset S_X^{(1)}$, which means that

$$\sigma^*_{(1)} : [f] \mapsto [f], \qquad \sigma^*_{(1)} : [s] \mapsto [s]. \tag{8.1}$$

There are two types in the way the involution $\sigma_{(1)}$ acts on a K3 surface with elliptic fibration $(X^{(1)}, \mathbb{P}^1_{(1)}, \pi^f_{X^{(1)}})$ [GS18, Prop. 2.3]. It always maps the zero section s to itself, but it may be either an identity $\sigma_{(1)}|_s = \mathrm{id}_s$ (Type 1 (referred to as type b in [CG15])), or a non-trivial holomorphic involution (Type 2 (referred to as type a in [CG15])). An involution of Type 1 acts on individual fiber elliptic curves, while

[4] Recall the definition in Section 3.4.1.

[5] As we have stated in Section 3.4.1, we will constrain ourselves to the case with an elliptic fibration.

[6] A review addressed to string theorists is found in [BKW13].

an involution of Type 2 exchanges two fiber curves (except the fibers over the two $\sigma_{(1)}|_s$-fixed points in the base $\mathbb{P}^1_{(1)}$).

Let us take a few examples from the 75 choices of $(S_0^{(1)}, T_0^{(1)}, \sigma_{(1)})$ of [Nik81]. For $S_0^{(1)} = U[2] \oplus E_8[2]$, there is no primitive embedding of U into $S_0^{(1)}$, so there is no elliptic fibration [GS18, Thm. 2.6.(i)]. For $S_0^{(1)} = U$, there is unique elliptic fibration, which is Type 1. Think of the case $S_0^{(1)} = U \oplus E_8[2]$, next. An obvious primitive embedding of U into $S_0^{(1)}$ corresponds to an elliptic fibration of Type 2; this embedding is actually the only one available for this $S_0^{(1)}$ [GS18, Thm. 2.6.(ii)]. For the choice $(S_0^{(1)}, T_0^{(1)})$ for $T_0 = U[2]^{\oplus 2}$, which is for $X^{(1)} = \mathrm{Km}(E \times E')$ for mutually non-isogenous elliptic curves E and E', there are 11 different elliptic fibrations (modulo $\mathrm{Isom}(S_0^{(1)})$) [Ogu89]; three out of the 11 elliptic fibrations ($\mathcal{J}_{1,2,3}$ in [Ogu89]) are Type 2, and the remaining eight ($\mathcal{J}_{4,\cdots,11}$) are Type 1. In the study of [CG15, GS18], it turns out that more than 60 choices out of the 75 in [Nik81] admit at least one elliptic fibration; choices with larger [resp. smaller] $g_{(1)} = (22 - r_{(1)} - a_{(1)})/2$ tend to have less [resp. more] inequivalent primitive embeddings $U \hookrightarrow S_0^{(1)}$ and inequivalent elliptic fibrations consequently. A pair $(\pi^f_{X^{(1)}}, \sigma_{(1)})$ of Type 2 is rare relatively to one of Type 1, and is possible only for the choices with $g_{(1)} \leq 1$. For more information, see [CG15, GS18] and references therein.

8.2 Borcea-Voisin manifold and Weierstrass model

For an F-theory compactification, we need a Calabi-Yau fourfold Y that has an elliptic fibration $\pi : Y \to B_3$ and its section $\sigma : B_3 \to Y$. It is not obvious in F-theory (due to the lack of its theoretical formulation) which one of Y and Y' should be regarded as input data of compactification, when there is a birational pair of Calabi-Yau varieties Y and Y' with no difference in cycles of finite volume or the number of complex structure deformation parameters. This work deals with F-theory compactification on such an equivalence class[7] of Calabi-Yau fourfolds that is represented by a non-singular model $\widetilde{Y}$ with a flat elliptic fibration, $\widetilde{Y} \to B_3$. Although the Borcea-Voisin manifold Y^{BV}—the minimal resolution of the Borcea-Voisin orbifold $Y_0 = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$—is non-singular, it is hard, or even seems to be impossible for some choices[8] of $(S_0^{(1)}, T_0^{(1)}, [f])$, to find a flat elliptic fibration on Y^{BV}. Therefore, for F-theory applications, let us find $\widetilde{Y}$ that is birational to Y^{BV}, along with a threefold B_3 so that there is a flat elliptic fibration[9] $\widetilde{Y} \to B_3$.

[7] Although we attempted to write down the equivalence relation explicitly above, the choice of the relations may have to be refined or corrected from the version written there.

[8] Suppose that a singular fiber of $\pi^f_{X^{(1)}} : X^{(1)} \to \mathbb{P}^1_{(1)}$ contains both an irreducible component in $Z^{(1)}$ and also $\mathbb{P}^1$ not in $Z^{(1)}$——(**). Then the fibration $\mathrm{Bl}_{\sigma-\text{fixed}}(X^{(1)} \times X^{(2)}) \to \mathbb{P}^1_{(1)}$ is not flat, where Bl_Z denotes a blow-up along Z. Apart from the choice of $S_0^{(1)} = U \oplus E_8[2]$, which has an elliptic fibration of Type 2, all other Type 2 elliptic fibrations available in K3 surfaces with a non-symplectic involutions fall into the category (**). Elliptic fibrations of Type 1 that stay out of the category (**) is when $S_0^{(1)} = U \oplus W_0$, with W_0 containing only A_1's and the Mordell-Weil group, but no other ADE-type lattices. Such $S_0^{(1)} = U \oplus W_0$ constitutes a small fraction of the tables in [GS18].

[9] $\tilde{Y}$ may be different from Y^{BV} in the sense of the equivalent class discussed above, in order to avoid exotic spectrum, as we will discuss in this chapter.

We will find such $\widetilde{Y}$ and B_3 in Sections 8.3 and 8.4 as a resolution of a Weierstrass model fourfold Y^W; see (8.6) and (8.8). As a first step for that purpose, consider an orbifold[10] $Y_0^W = (X^{(1)W} \times X^{(2)})/\mathbb{Z}_2$. $X^{(1)W}$ is the Weierstrass model of a non-singular K3 surface $X^{(1)}$, which is obtained from $(X^{(1)}, \mathbb{P}^1_{(1)}, \pi^{[f]}_{X^{(1)}})$ discussed in Section 8.1 by collapsing (-2)-curves in the singular fibers of $\pi^{[f]}_{X^{(1)}}$ except those that intersect the section s of $\pi^{[f]}_{X^{(1)}}$. The $\mathbb{Z}_2$ quotient is by $(\sigma_{(1)W}, \sigma_{(2)})$, where $\sigma_{(1)W}$ is described below.

A K3 surface $X^{(1)}$ of interest in this thesis is in a family parameterized by the (CM points in the) period domain $D(T_0^{(1)})$ characterized by the pair $(S_0^{(1)}, T_0^{(1)})$, where $\sigma_{(1)}$ acts identically on $S_0^{(1)}$ and by $[(-1)\times]$ on $T_0^{(1)}$. Its Weierstrass model $X^{(1)W}$, however, can be regarded as $X^{(1)}$ with $S_0^{(1)W} = U$ in the Type 1 case[11], and the period domain $D(T_0^{(1)})$ as a special subspace in $\overline{D(T_0^{(1)W})}$; $T_0^{(1)W} = U^{\oplus 2} \oplus E_8^{\oplus 2}$ now. The involution $\sigma_{(1)W}$ on $X^{(1)W}$ is that of $X^{(1)}$ with[12] $S_0^{(1)} = U$, which multiplies (-1) to the y coordinate of the Weierstrass equation $y^2 = x^3 + f(t)x + g(t)$ [CG15, GS18].

In the Type 2 case, its Weierstrass model $X^{(1)W}$ is regarded as $X^{(1)}$ with $S_0^{(1)W} = U \oplus E_8[2]$, and the period domain $D(T_0^{(1)})$ as a special subspace in $\overline{D(T_0^{(1)W})}$; $T_0^{(1)W} = U^{\oplus 2} \oplus E_8[2]$ now. The involution $\sigma_{(1)W}$ on $X^{(1)W}$ is that of $X^{(1)}$ with $S_0^{(1)} = U \oplus E_8[2]$, which multiplies (-1) to the inhomogeneous coordinate t of the base $\mathbb{P}^1_{(1)}$, where the Weierstrass equation is $y^2 = x^3 + f(t^2)x + g(t^2)$ [CG15, GS18].

The orbifold Y_0^W is now well-defined; we claim now that there is a regular map $Y_0 \to Y_0^W$, and that this map is birational. To see that they are birational, note first that there is a field isomorphism $\mathbb{C}(X^{(1)W})\mathbb{C}(X^{(2)}) \cong \mathbb{C}(X^{(1)})\mathbb{C}(X^{(2)})$ because of the birationality between $X^{(1)}$ and $X^{(1)W}$. The action of $(\sigma_{(1)W}, \sigma_{(2)})$ on the left and that of $(\sigma_{(1)}, \sigma_{(2)})$ on the right are compatible with this field isomorphism, so we have

$$\mathbb{C}(Y_0^W) \cong \left[\mathbb{C}(X^{(1)W})\mathbb{C}(X^{(2)})\right]^{\mathbb{Z}_2} \cong \left[\mathbb{C}(X^{(1)})\mathbb{C}(X^{(2)})\right]^{\mathbb{Z}_2} \cong \mathbb{C}(Y_0). \tag{8.2}$$

They are birational indeed. The regularity of the map $Y_0 \to Y_0^W$ follows from

$$(\mathbb{C}[U_i]\mathbb{C}[V])^{\mathbb{Z}_2} \hookrightarrow \left(\mathbb{C}[X_i^{(1)}]\mathbb{C}[V]\right)^{\mathbb{Z}_2} \tag{8.3}$$

for open patches V of $X^{(2)}$; here, U_i's are open patches of $X^{(1)W}$ and $X_i^{(1)}$'s those of $X^{(1)}$ so that $X_i^{(1)}$'s are mapped to U_i's under the regular map $X^{(1)} \to X^{(1)W}$.

Construction of Y_0^W from Y_0 or Y^{BV}, the resolution of Y_0, is essentially the same for both Type 1 and Type 2. From this point on, however, we need to deal with the Type

[10] It is likely that the constructions of $(\widetilde{Y}, B_3, \pi)$ starting from here are not the most general ones with a moduli space containing $D(T_X^{(1)}) \times D(T_X^{(2)})$. The author is not yet ready to write down a broader class of constructions, however.

[11] One may convince oneself by thinking that the singularities of $X^{(1)W}$ are infinitesimally deformed, rather than resolved.

[12] Complex structure can be tuned continuously from the bulk of $D(T_0^{(1)W})$ to $D(T_0^{(1)})$, but the process of blowing-up singularity of $X^{(1)W}$ (or collapsing (-2)-curves of $X^{(1)}$ in the other way around) is not a continuous process; the involution on the cohomology group $H^2(K3;\mathbb{Z})$ changes in this discontinuous process.

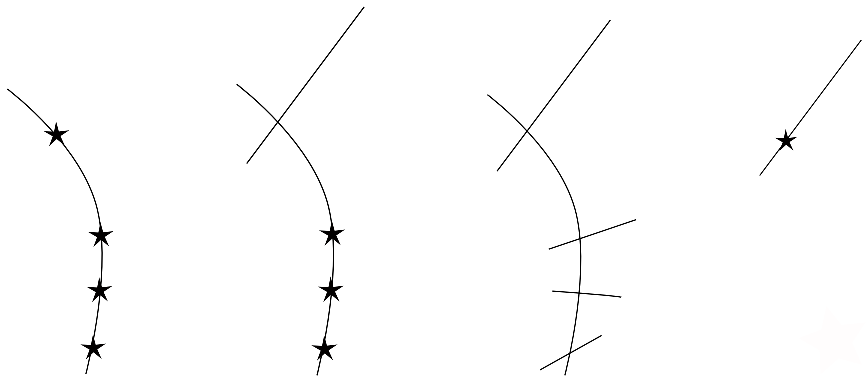


FIGURE 8.1: Schematic picture of the singular fiber geometry for a generic point in $[Z^{(2)}]$ in (a) Y_0^W, (b) $Y^{W'}$, (c) $Y^{W''}$, and (d) Y^W.

1 and 2 cases separately in the construction of a non-singular model $\widetilde{Y}$ with a flat fibration over a base threefold B_3, which are what we are after.

8.3 Fibration and involution of Type 1

8.3.1 Construction of $\widetilde{Y}$, and gauge group and matter representations

In the case of a pair of fibration and involution of Type 1, a Weierstrass model Y^W is obtained by once blowing up Y_0^W ($Y^{W'} \to Y_0^W$), and then blowing it down ($Y^{W'} \to Y^W$), as we elaborate a bit more in the following. The procedure is very much similar to the case of [Sen96]; see Figure 8.1.

Construction of $Y^{W'}$ from Y_0^W is as follows. The $\mathbb{Z}_2$-orbifold Y_0^W has a two-dimensional locus of singularity that is A_1-type for each isolated component of $[Z^{(2)}] \subset B^{(2)}$, where $B^{(2)}$ denotes $X^{(2)}/\mathbb{Z}_2$. The two transverse directions are the transverse direction of $[Z^{(2)}]$ in $B^{(2)}$ and also the elliptic fiber direction. For a generic point in $[Z^{(2)}] \subset B^{(2)}$, the locus of A_1-singularity consists of two pieces of curves, one of which is a three-fold cover over $\mathbb{P}^1_{(1)}$ and the other a one-fold cover. The proper transform of Y_0^W in a blow-up centered along the latter singular locus (the one-fold covering one) is denoted by $Y^{W'}$; one may also think of the blow-up along both of the singular loci, where the proper transform is denoted by $Y^{W''}$. See (8.6) and Fig. 8.1.

The Weierstrass model Y^W is obtained from $Y^{W'}$ by collapsing the divisors over $[Z^{(2)}] \times \mathbb{P}^1_{(1)}$ that are non-exceptional in the blow-up $Y^{W'} \to Y_0^W$ (see Fig. 8.1). This variety Y^W has a projection $\pi : Y^W \to B_w := (\mathbb{P}^1_{(1)} \times B^{(2)})$, and is given by

$$\tilde{y}^2 = \tilde{x}^3 + V^2 f(t)\tilde{x} + V^3 g(t) \tag{8.4}$$

in one of its Affine patch. See Appendix B.1.1 for the detail. The Affine coordinates $(\tilde{x}, \tilde{y}, t, V, u)$ of Y^W are related with the coordinates (x, y, t, v, u) of $X^{(1)W}$ and $X^{(2)}$ through

$$t = t, \quad u = u, \quad V = v^2, \quad \tilde{x} = xv^2, \quad \tilde{y} = yv^3, \tag{8.5}$$

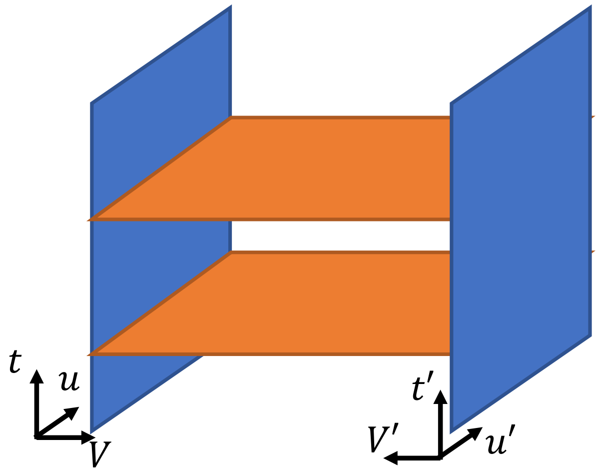


FIGURE 8.2: A schematic figure of the base B_w of $\widetilde{Y}_{A_n}$. Each vertical blue plane represents the I_0^* locus Δ_b associated with an irreducible curve in $Z^{(2)}$, while the horizontal orange planes represents the discriminant locus associated with the singular fiber in $X^{(1)}$. We should have different coordinate patch for each irreducible curve in $Z^{(2)}$, (t, u, V) and (t', u', V') in the figure, where t and t' denote the coordinates of the base of $X^{(1)}$, V and V' the coordinates normal to $Z^{(2)}$ in $B^{(2)}$, and u and u' tangent to $Z^{(2)}$.

and the involution $\sigma_{(1)W}$ acts trivially on t, u, and x, and by $[(-1)\times]$ on y and v. Here is a summary (all the arrows between Y's are regular and birational):

$$\begin{tikzcd}
Y^{BV} \arrow[d] & & Y^{W''} \arrow[d] & \widetilde{Y}_{A_n} \arrow[d] & & & \widetilde{Y}_{D_n,E_6} \arrow[d] \\
Y_0 \arrow[r] & Y_0^W & Y^{W'} \arrow[l] \arrow[r] & Y^W \arrow[d] & \nu^*(Y^W) \arrow[l] \arrow[d] & \overline{\nu^*(Y^W)} \arrow[l] \arrow[r] & Y \arrow[dll] \\
& & & B_w & \widetilde{B}_w \arrow[l] & &
\end{tikzcd} \tag{8.6}$$

See the following discussions for $\widetilde{B}_w = \mathrm{Bl}_{\mathrm{pt}_* \times [Z^{(2)}]}(B_w)$, $\nu^*(Y^W)$, Y, $\widetilde{Y}_{A_n}$ and $\widetilde{Y}_{D_n,E_6}$.

So long as complex structure of $X^{(1)}$ is that of a generic one in $D(T_0^{(1)W}) = D(\mathrm{II}_{2,18})$, which means that $S_X^{(1)} = S_0^{(1)} = U$, there is no difference between Y_0 and Y_0^W; $Y^{W''}$ is nothing but Y^{BV}; the projection $\pi : Y^{W''} = Y^{BV} \to (\mathbb{P}^1_{(1)} \times B^{(2)})$ yields a flat elliptic fibration, so one can take $\widetilde{Y} = Y^{W''}$ and $B_3 = B_w$ in this case. The discriminant locus Δ_{discr} of the elliptic fibration $Y^W \to B_w$ is of the form[13]

$$\Delta_{\mathrm{discr}} = \Delta_f + \Delta_b, \qquad \Delta_f = (24\mathrm{pts}) \times B^{(2)}, \quad \text{and} \quad \Delta_b = 6(\mathbb{P}^1_{(1)} \times [Z^{(2)}]). \tag{8.7}$$

On a generic point in Δ_b, the singular fiber in $\widetilde{Y} = Y^{W''}$ is the I_0^*-type in the Kodaira classification [CGP19], and Eq. (8.4) is completely in the non-split type[14] over $\mathbb{P}^1_{(1)}$

[13]The multiplicity 6 of Δ_b comes from the vanishing order 6 for I_0^* fiber.
[14]Recall the definition in Section 3.2. See Appendix B.1.2 for detail.

(and also over $\mathbb{P}^1_{(1)} \times [Z^{(2)}]$) [BIK+96]. The $\mathcal{N} = 1$ supersymmetric effective theory on $\mathbb{R}^{3,1}$ thus has one vector multiplet with the gauge group G_2 for each one of the isolated components[15] of $[Z^{(2)}]$. The 7-branes Δ_f do not yield a massless vector multiplet on the effective theory on $\mathbb{R}^{3,1}$. There may be massless $\mathcal{N} = 1$ chiral multiplets (matter fields) charged under those G_2 gauge groups, possibly in the adjoint representation, and also possibly in the **7**-representation, because matter hypermultiplets in those two representations can be present in F-theory compactifications to 6 dimensions [BIK+96, AKM00]. None of them must be charged under multiple G_2's, because all the irreducible components of $[Z^{(2)}] \subset B^{(2)}$ are disjoint from each other [Nik81]. All those matter fields are in self-real representations, so there is no such things as a formula for the net chirality. Although the $(20 - r_{(1)}) + (20 - r_{(2)}) = 38 - r_{(2)}$ complex structure moduli of $Y^{W''} = Y^{BV}$ remain to be gauge-group neutral moduli chiral multiplets in the effective theory on $\mathbb{R}^{3,1}$, the $g_{(1)}g_{(2)} = 10g_{(2)}$ complex structure moduli of Y^{BV} are likely to be part of G_2-charged matter chiral multiplets; an evidence for this statement can be provided by studying F-theory compactification on a threefold M^{BV} as the crepant resolution of $(X^{(1)} \times E_\tau)/\mathbb{Z}_2$.

Now, let us turn to cases where $X^{(1)}$ and $X^{(1)W}$ are not isomorphic. In terms of the lattice, let $S_X^{(1)} =: U \oplus W$, and let R denote the sub-lattice of W generated by the norm-(-2) elements of W; $X^{(1)}$ and $X^{(1)W}$ are not mutually isomorphic if and only if R is non-trivial. Y^{BV} and $Y^{W''}$ are not mutually isomorphic either in such cases. A flat elliptic fibration $(\widetilde{Y}, B_3, \pi)$ is constructed as reviewed below by starting from $Y^W \to B_w$, or from $Y^{W''} \to B_w$.

Suppose first that the lattice R contains only A_n's, not D_n's or E_n's; see Figure 8.2. It is then known that we can take B_w as B_3; Y^W has singularity of A_n type over the 7-brane $(n+1)(\mathrm{pt}_* \times B^{(2)}) \subset \Delta_f$, so those codimension-2 singularity from the I_0^*–I_{n+1} collision is resolved canonically; after a small resolution, a non-singular Calabi-Yau fourfold $\widetilde{Y}$ is obtained in this case [Mir83, BJ97b][16]. The fourfold $\widetilde{Y}$ in this case is denoted by $\widetilde{Y}_{A_n}$ in (8.6). The gauge group on the 7-brane $\mathrm{pt}_* \times B^{(2)}$ becomes Sp-type[17] in the effective theory on $\mathbb{R}^{3,1}$ (and a product of G_2's is from $\mathbb{P}^1_{(1)} \times [Z^{(2)}]$ as before). The matter fields must be in the bifundamental representation of G_2 and Sp [KV97], besides those in the adjoint representations, G_2-**7**, and the Sp rank-2 antisymmetric representation [BJ97b, AKM00] (consistent with the Type IIB brane constructions). All the fourfolds Y^{BV}, Y_0, $Y^{W'}$, $Y^{W''}$ and $\widetilde{Y}$ are Calabi-Yau and are birational, and no cycles of finite volume are added or removed. The $(20 - r_{(1)} - \mathrm{rk}(R)) + (20 - r_{(2)})$ complex structure moduli remain neutral chiral multiplets on $\mathbb{R}^{3,1}$; other complex structure moduli of Y^{BV} will remain massless chiral multiplets on $\mathbb{R}^{3,1}$, but as a part of gauge-charged matter fields (they are the $g_{(1)}g_{(2)} = 10g_{(2)}$ moduli deforming the

[15]There are $k_2 + 1$ isolated components for all the $(75-2)$ choices of $(S_0^{(2)}, T_0^{(2)})$ from Nikulin's list.

[16] In the construction of $\widetilde{Y}$ in the main text, we consider choosing a complex structure of $X^{(1)}$ from $D(T_0^{(1)}) = D(\mathrm{II}_{2,18})$ in such a way that $S_X^{(1)}$ is enhanced from $S_0^{(1)} = U$ to $U \oplus W$ with W containing A_n's. When we consider a complex structure so that $S_0^{(1)} = U \oplus W_0$, $W_0 = W$, and $R \subset W_0$ contains only A_1's (cf footnote 8), however, one may think of another construction of (Y, B_3, π). That is to choose Y^{BV} as the fourfold, and $B_3 = \mathbb{P}^1_{(1)} \times B^{(2)}$; this is a flat elliptic fibration [CGP19, Prop. 3.1] in such a case. It is a question of interest whether Y^{BV} is isomorphic to $\widetilde{Y}$ in the main text and whether the matter spectra are the same or not.

[17]The gauge symmetry is reduced to Sp because the collision with the G_2 7-brane induces a monodromy for the I_{n+1} fiber.

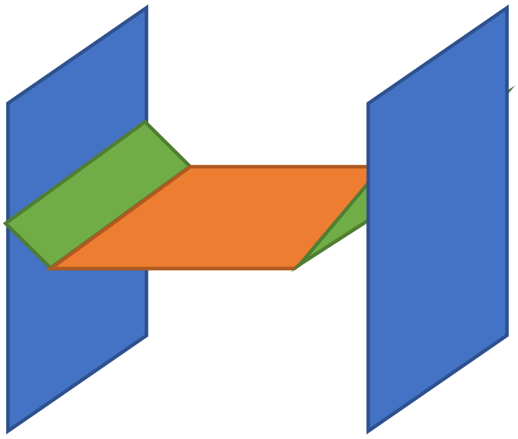


FIGURE 8.3: A schematic figure of the base $\widetilde{B}_w$ of $\widetilde{Y}_{D_n,E_6}$. The meaning of the vertical and horizontal components are the same as that in Figure 8.2, except that they are not intersecting to each other anymore because we have blown-up the intersection locus. The green components represent the exceptional loci, on which a singular fiber may or may not exist.

$\mathbb{C}^2/\mathbb{Z}_2$ singularity of Y_0 and the $\mathrm{rk}(R)$ moduli of $X^{(1)}$ that reduces[18] $S_X^{(1)}$ back to $S_0^{(1)}$).

Suppose next that the lattice R contains a factor D_n or E_6, corresponding to a singular fiber of I^*_{n-4} type or IV^* in $X^{(1)}$ over $\mathrm{pt}_* \in \mathbb{P}^1_{(1)}$. The known prescription[19] is, as already introduced in Section 3.3.1, to set[20] [21] $B_3 = \mathrm{Bl}_{\mathrm{pt}_* \times [Z^{(2)}]} B_w =: \widetilde{B}_w$, and think of $\nu^*(Y^W)$ with a Weierstrass fibration over B_3 for a moment; $\nu : \widetilde{B}_w \to B_w$ is the blow-up map. The fourfold $\nu^*(Y^W)$, which corresponds to Eq. (3.9), has a parabolic singularity at $\{\tilde{y} = \tilde{x} = 0\}$ in the fiber of the exceptional locus E of $B_3 = \widetilde{B}_w$. The ambient space of $\nu^*(Y^W)$ is blown-up three times with the center in the fiber of E, and now the proper transform $\overline{\nu^*(Y^W)}$ has only A_{n-5} singularity (assuming an even $n > 4$; none for I^*_0 or IV^*). The fourfold $\overline{\nu^*(Y^W)}$ is not Calabi-Yau due to the non-trivial morphisms $\overline{\nu^*(Y^W)} \to \nu^*(Y^W) \to Y^W$, but there is a morphism $\overline{\nu^*(Y^W)} \to Y$ to a fourfold Y, which corresponds to Eq. (3.10), ramified along the canonical divisor of $\overline{\nu^*(Y^W)}$, so Y is a Calabi-Yau fourfold. There is also a projection morphism $Y \to B_3$ (see (8.6)). The fourfold Y has D_4 singularity in the fiber of $\overline{\Delta_b}$ (the proper transform of Δ_b under $\nu : B_3 \to B_w$). See also Appendix B.3 for the detail of the process.

In the case of I^*_{n-4}, the fourfold Y also has D_{4+n} singularity in the fiber of $\overline{\mathrm{pt}_* \times B^{(2)}}$, there is also A_{n-5} singularity (if $n > 4$) in the fiber of the exceptional divisor E

[18] It is desirable to carry out the Higgs cascade analysis [MV96b, BIK+96] of all those kinds of constructions in Sections 8.3 and 8.4, where F-theory prediction including matter multiplicity information is compared against symmetry breaking processes in the effective field theory on 4 dimensions (or on 6 dimensions). Such a study will uncover much more aspects of F-theory compactification on K3 × K3 orbifolds (or on Borcea-Voisin orbifolds) than those presented in Sections 8.3 and 8.4.

[19] The prescription of Ref. [BJ97b] is to replace $\nu^*(Y^W)$ by the Affine part of Y and then to add the zero section by hand, without discussing birational map between them. In that prescription, the Calabi-Yau condition of Y had to be tested independently from the Calabi-Yau nature of Y^W.

[20] We have in mind that the Kähler parameter is such that the exceptional divisor in the blow-up $\widetilde{B}_w \to B_w$ has a non-zero positive volume.

[21] $\mathrm{Bl}_Z X$ denotes a blow-up of X along $Z \subset X$.

(statements in the rest of this paragraph is for an even n); see Figure 8.3. Those singularities in Y should be resolved canonically; further small resolution in the fiber of codimension-2 loci in B_3 yields $\widetilde{Y}$ that has a flat elliptic fibration over B_3 [Mir83] and $\widetilde{Y}$ in this case is denoted by $\widetilde{Y}_{D_n,E_6}$ in (8.6). The 7-brane $\overline{\mathrm{pt}_* \times B^{(2)}}$ yields $\mathrm{SO}(2n)$ gauge group in the effective theory on $\mathbb{R}^{3,1}$; the effective theory also has an $(\mathrm{Sp}((n-4)/2))^{k_2+1}$ gauge group (for an even $n > 4$).[22] A I_0^*–I_{n-4}^* collision may yield chiral multiplets of 4D $\mathcal{N} = 1$ supersymmetry in the G_2–$\mathrm{Sp}((n-4)/2)$ bifundamental, and in the $\mathrm{Sp}((n-4)/2)$–$\mathrm{SO}(2n)$ bifundamental representations (in the case $n = 4$ there is no matter fields associated particularly with the I_0^*–I_0^* collision) [BJ97b]. Cases with an odd $n > 4$ are less trivial, but remain similar [BIK+96, AKM00].

In the case of IV^*, we have an F_4 vector multiplet on $\mathbb{R}^{3,1}$ from the brane $\overline{\mathrm{pt}_* \times B^{(2)}}$ [BIK+96, BJ97b]; see also Appendix B.2.2. Chiral multiplets may arise from the I_0^*–IV^* collision, which are in the fundamental representations of G_2 and F_4, but there is no matter in a mixed representation [BJ97b, AKM00]. The types of matter representations available are the same for all (k_2+1) singularity collisions along the (k_2+1) disjoint components of $\mathrm{pt}_* \times [Z^{(2)}]$. Details of the massless spectrum may be different due to a choice of a 4-form flux in the non-horizontal part of $H^4(\widetilde{Y};\mathbb{Q})$.

In the case the lattice $R \subset W$ contains a factor E_7, B_3 is obtained by blowing-up B_w *twice*; $\widetilde{Y}$ is also obtained in a similar procedure, although it is not contained in the diagram (8.6). The gauge group on $\mathbb{R}^{3,1}$ becomes $(G_2 \times \mathrm{SU}(2))^{k_2+1} \times E_7$. There can be matter chiral multiplets charged under $(G_2 \times \mathrm{SU}(2))^{k_2+1}$ [BJ97b], but it is unlikely that there is a localized matter charged under the E_7 at the collision locus, as there seems to be no enhancement of singularity there after the two blow-ups.

8.3.2 More consequences in physics

In all those cases[23] where R involves D_n, E_6 or E_7, birational morphisms between the two Calabi-Yau's Y^{BV} and Y in (8.6) can be constructed for any choice of moduli in $D([(S_X^{(1)})^\perp]) \times D(T_0^{(2)})$. Therefore, those deformation degrees of freedom and their corresponding cohomology groups (i.e., $T_X^{(1)} \otimes T_0^{(2)}$) will remain to be there for Y and $\widetilde{Y}$. The $g_{(1)}g_{(2)}$ complex structure moduli of Y^{BV} (and the $H^2(Z^{(1)} \times Z^{(2)};\mathbb{Q})$ component in $H^4(Y^{BV};\mathbb{Q})$) may or may not be present in $\widetilde{Y}$, but even when they are present, they will be part of G_2-charged matter fields. The $\mathrm{rk}(R)$ moduli fields necessary in enhancing the D_n, E_6 or E_7 singularity in Y^W may either be part of gauge-charged matter fields, or be absent as massless degrees of freedom in the F-theory compactification.

If there is any chance of accommodating grand unification of the Standard Model in this Type 1 framework, a GUT gauge group such as $\mathrm{SU}(5)$, $\mathrm{SO}(10)$, E_6, and E_7 at the level of 8 dimensions should be from Δ_f, because those gauge groups do not fit within G_2, or even within $\mathrm{SO}(8)$. We have seen above that implementing $A_4 = \mathrm{SU}(5)$ or E_6 in $R \subset W$ of $X^{(1)}$ does not result in an $\mathrm{SU}(5)$ or E_6 gauge group on 4 dimensions due to the monodromy at the I_0^*–R collision.[24] Even when we require D_5

[22] $\mathrm{Sp}(n) = \mathrm{USp}(2n)$ in this notation.

[23] One may think of a case a complex structure is tuned in $D(T_0^{(1)} = \mathrm{II}_{2,18})$ so that $(S_X^{(1)}, T_X^{(1)})$ just happens to be identical to one of (S_0, T_0) in Nikulin's list. It is a question of interest whether there is an isomorphism between Y^{BV} and $\widetilde{Y}$ constructed as in the main text.

[24] An exception is when $S_0^{(2)} = U[2] \oplus E_8[2]$ in the list of Nikulin, because $Z^{(2)}$ is empty and there is no I_0^*–R collision; this scenario is still not suitable for GUT, however, because there is no massless matter chiral multiplets charged under R.

within $R \subset W$, there is no chance having a matter field in the spinor representation of SO(10). The remaining option is to assume E_7 within $R \subset W$. A gauge flux in E_7 must be non-trivial on the divisor $\overline{\mathrm{pt}_* \times B^{(2)}}$, so the gauge symmetry is broken down to that of the Standard Model. As there is no massless adjoint chiral multiplets of E_7 because $h^{0,1}(B^{(2)}) = h^{0,2}(B^{(2)}) = 0$, the origin of quarks and leptons[25] should be the E_7-**56** representation, which is also not likely to be present in our setup, at least generically.

The conditions for a $DW = W = 0$ flux (or a $DW = 0$ flux) and study of complex structure moduli stabilization in Sections 6.3 and 6.4 can be recycled without modifications for F-theory, as we see below. We stick to the Type 1 case available for $S_0^{(1)} = U$ and $T_0^{(1)} = \mathrm{II}_{2,18}$. For a CM-type vacuum complex structure such that $T_X^{(1)} = T_0^{(1)}$, then $T_X^{(2)}$ should[26] also be of rank 20, so that $T_X^{(2)}$ has a CM point in $D(T_X^{(2)})$ with the CM field $K^{(2)}$ satisfying the condition (6.38) (or (6.37)). This means that $\mathrm{rank}(T_0^{(2)})$ is either 20 (when $\mathrm{rank}(S_0^{(2)}) = 2$) or 21 (when $\mathrm{rank}(S_0^{(2)}) = 1$); there are three such pairs $(S_0^{(2)}, T_0^{(2)})$ in Nikulin's list ($S_0^{(2)} = \langle +2 \rangle, U, U[2]$). For any one of the three choices of $(S_0^{(2)}, T_0^{(2)})$, all the 18+(19 or 18) complex structure fluctuation fields in $D(T_0^{(1)}) \times D(T_0^{(2)})$ are valid Calabi-Yau deformations of $\widetilde{Y} = Y^{W''}$, not just of Y^{BV}. A $DW = W = 0$ flux provides large supersymmetric masses to all those complex structure moduli fluctuations.

For a CM-type vacuum complex structure with $T_X^{(1)} \subsetneq T_0^{(1)}$, for example, when $S_X^{(1)} \supset U \oplus E_7$, the CM field $K^{(1)}$ has a degree $[K^{(1)} : \mathbb{Q}] = \mathrm{rank}(T_X^{(1)}) < 20$. Therefore the necessary condition $\mathrm{rank}(T_X^{(1)}) = \mathrm{rank}(T_X^{(2)})$ for (6.38), which is also for a non-trivial $DW = W = 0$ flux, allows a choice of $(S_0^{(2)}, T_0^{(2)})$ from a broader subset of Nikulin's list, Figure 5.1. The complex structure deformation fields in $D(T_X^{(1)}) \times D(T_0^{(2)})$ obtain large supersymmetric masses by a $DW = W = 0$ flux, which one can see by repeating the same discussion as in Section 6.4.

The complex structure moduli stabilization in [DDF+05] can be regarded as a special case of the general discussion above. Our interpretation is that the fourfolds for F-theory in [DDF+05] correspond to $(S_0^{(1)}, T_0^{(1)}) = (U, \mathrm{II}_{2,18})$ as stated above, $(S_0^{(2)}, T_0^{(2)})$ that of a Kummer surface ($r_{(2)} = 18$, $a_{(2)} = 4$, $k_2 = 7$ and $g_{(2)} = 0$), $T_X^{(1)} = U[2]^{\oplus 2} \subsetneq T_0^{(1)}$ and $T_X^{(2)} = T_0^{(2)} = U[2]^{\oplus 2}$. The discussion above further indicates that there should be a flux with the vev $\langle W \rangle = 0$, when we choose the vacuum complex structure of all the tori in $X^{(1)} \sim (T^2 \times T^2)/\mathbb{Z}_2$ and $X^{(2)} \sim (T^2 \times T^2)/\mathbb{Z}_2$ so that they all have complex multiplication, and the condition (6.38) is satisfied.

[25] If we are to assume that the Yukawa couplings of quarks and leptons are from the perturbative E_7 gauge interaction [TW06], then all those quarks and leptons should be on just one of the $(k_2 + 1)$ irreducible components of the curve $\overline{\mathrm{pt}_* \times [Z^{(2)}]}$. This implies that the contrast between the small mixing angles of q_L's and the large mixing angles of ℓ_L's cannot be attributed to geometry of their matter curves [HKTW10].

[26] It is known that this $\mathrm{rank}(T_X^{(2)}) = [K^{(1)} : \mathbb{Q}]$ condition is only a necessary condition for an existence of such a CM point. At least in the case of $T_0^{(2)} = T_X^{(1)}$, we are sure that $D(T_0^{(2)})$ contains a CM point whose CM field is isomorphic to the CM field $K^{(1)}$ of a CM point in $D(T_X^{(1)})$.

8.4 Fibration and involution of Type 2

In the Type 2 case, we start from the projection map $Y_0^W \to B_{w0}$, which is in between singular varieties; $B_{w0} := \left(\mathbb{P}^1_{(1)} \times X^{(2)}\right)/\mathbb{Z}_2$. Consider the canonical resolution of the A_1-singularity of B_{w0}, $\nu : B_w := \widetilde{B}_{w0} \to B_{w0}$, and set $Y^W := \nu^*(Y_0^W)$. Now the projection $Y^W \to B_w$ is a Weierstrass model over a non-singular threefold B_w. The fourfold Y^W satisfies the Calabi-Yau condition because $\nu : B_w \to B_{w0}$ is crepant; see Figure 8.4 and also Appendix B.2 for detail.

$$\begin{tikzcd}
Y^{BV} \arrow[d] & & \widetilde{Y}_{A_{2n-1},D_4} \arrow[d] & & & \widetilde{Y}_{E_6} \arrow[d] \\
Y_0 \arrow[r] & Y_0^W \arrow[d] & Y^W \arrow[l] \arrow[d] & \nu'^*(Y^W) \arrow[l] \arrow[d] & \overline{\nu'^*(Y^W)} \arrow[l] \arrow[r] & Y \arrow[dll] \\
& B_{w0} = (\mathbb{P}^1_{(1)} \times X^{(2)})/\mathbb{Z}_2 & B_w = \widetilde{B}_{w0} \arrow[l] & \widetilde{B}_w \arrow[l] & &
\end{tikzcd} \tag{8.8}$$

See the following discussions for ν', Y, $\widetilde{Y}_{A_{2n-1},D_4}$, $\widetilde{Y}_{E_6}$ and $\widetilde{B}_w$.

So long as complex structure of $X^{(1)}$ corresponds to a generic point in $D(T_0^{(1)W}) = D(U^{\oplus 2} \oplus E_8[2])$, which means that $S_X^{(1)} = S_0^{(1)} = U \oplus E_8[2]$, the Weierstrass model Y^W is already non-singular; the projection $Y^W \to B_w$ is a flat elliptic fibration, so we can set $\widetilde{Y} = Y^W$ and $B_3 = B_w$. The base threefold B_3 is a $\mathbb{P}^1$-fibration over[27] $B^{(2)}$; the $\mathbb{P}^1$-fiber degenerates into three irreducible pieces ($\mathbb{P}^1 + 2\mathbb{P}^1 + \mathbb{P}^1$) over $[Z^{(2)}] \subset B^{(2)}$. Note that there is no difference between Y_0 and Y_0^W, and that Y^{BV} and Y^W are identical in this generic complex structure. The discriminant locus $\Delta_{\rm discr}$ of the elliptic fibration $Y^W \to B_w$ consists of 12 isolated components; there are 24 I_1 fibers in $X^{(1)}$, consisting 12 pairs of them, and within each pair, two fibers are exchanged by the involution to each other; see Figure 8.4. Each one of the 12 components is a double cover over $B^{(2)}$ ramified over $[Z^{(2)}]$, i.e. each piece is isomorphic to the K3 surface $X^{(2)}$. Here, we assume on the ground of genericity that the 12 pairs of I_1 fibers of $X^{(1)}$ stay away from the two fixed points of $\mathbb{P}^1_{(1)}$ under the action of $\sigma_{(1)}$. There is no non-abelian gauge group in the effective theory on $\mathbb{R}^{3,1}$ then.

When the vacuum complex structure of $X^{(1)}$ is tuned so that some of the 12 pairs of I_1 fiber come on top of each other (but remain distant from the $\sigma_{(1)}$-fixed locus), $S_X^{(1)}$ may be different from $S_0^{(1)} = U \oplus E_8[2]$, and in particular, the sub-lattice R of W in $S_X^{(1)} =: U \oplus W$ may contain a pair of copies of an ADE-type root lattice. Because the discriminant locus of the ADE-type fiber forms a single irreducible component, the effective theory on $\mathbb{R}^{3,1}$ will have a gauge group of that ADE type, with one chiral multiplet in the adjoint representation (because $h^{0,2}(X^{(2)}) = 1$). A non-trivial gauge flux on these 7-branes may reduce the ADE symmetry further down to a smaller non-abelian gauge group, but we cannot obtain a chiral spectrum on $\mathbb{R}^{3,1}$ in this way; note that $c_1(X^{(2)}) = 0$, to which the chirality is proportional.

Consider instead an $X^{(1)}$ that has a singular fiber at a fixed point of $\sigma_{(1)}$ in $\mathbb{P}^1_{(1)}$, represented by blue and dashed line in Figure 8.4. Suppose that the singular fiber

[27] Recall the definition $B^{(2)} := X^{(2)}/\mathbb{Z}_2$.

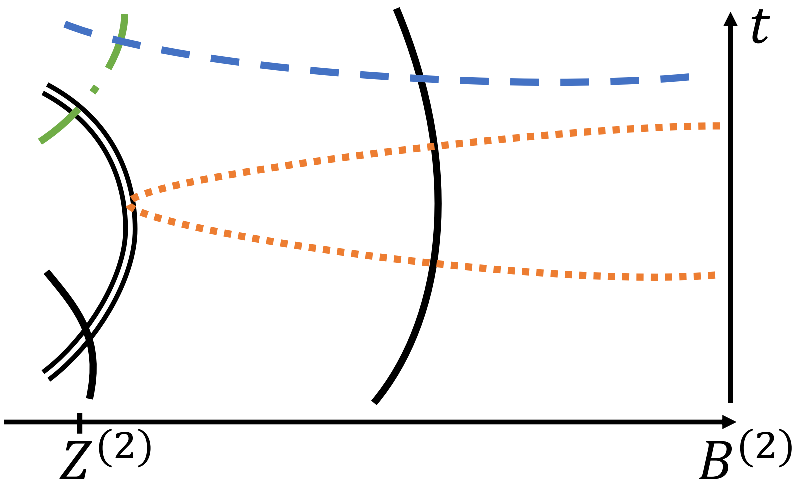


FIGURE 8.4: A schematic figure of the base B_w, which is a $\mathbb{P}^1$-fibration over $B^{(2)}$. The horizontal axis is the coordinate in $B^{(2)}$ normal to $Z^{(2)}$ and the vertical axis corresponds to the base of $X^{(1)}$. A generic $\mathbb{P}^1$-fiber is represented as a vertical curve, while over $Z^{(2)}$ the fiber is orbifolded by $\mathbb{Z}_2$ and becomes reducible after the resolution. Among the three irreducible components over $Z^{(2)}$, the top and the bottom component is from the blow-up of the A_1 singularity dut to the orbifolding, while the middle component is denoted by double line to signify that it is multiplicity 2 compared to the generic fiber. There are three types of discriminant loci, originated in $X^{(1)}$. The orange and dotted component comes from a generic point in the base of $X^{(1)}$, which must be paired because the involution acts on the base non-trivially. The pair is connected over $Z^{(2)}$, so they consist one component in B_w. There may be multiple of these pairs, up to 12. The other two component comes from the fixed point of the involution $\sigma^{(1)}$ in the base of $X^{(1)}$; the blue and dashed component has the same singular fiber as in $X^{(1)}$, while the green and dash-dotted component has a reduced singular fiber.

is I_{2n} [resp. IV* or I_0^*],[28] and all the other singular fibers of $X^{(1)}$ are of I_1 type and are away from the $\sigma_{(1)}$-fixed points. The discriminant Δ_{discr} consists of three distinct groups of components; see Figure 8.4. One of them, depicted by a blue dashed curve in the figure, consists of a section of the $\mathbb{P}^1$-fibration over $B_{(2)}$, which yields the SU$(2n)$ [resp. E_6, or SO(7) (due to monodromy)] gauge group on $\mathbb{R}^{3,1}$. Another group of 7-branes, shown in green in the figure, is the (k_2+1) isolated pieces of the exceptional divisors associated with the $\sigma_{(1)}$-fixed point in $\mathbb{P}^1_{(1)}$ where $X^{(1)}$ has the singular fiber. The last group, shown as a dotted orange curve in the figure, consists of $(12-n)$ [resp. 8 or 9] copies of $X^{(2)}$ that do not yield a non-abelian gauge group on $\mathbb{R}^{3,1}$. Each one of those 7-branes yields a gauge group SU(n) [resp. SU(3) or SU(2) (monodromy is absent)] on $\mathbb{R}^{3,1}$.

In the case of I_{2n} [resp. I_0^*], we can set $B_3 = B_w$, and $\widetilde{Y}$ as the canonical resolution of Y^W for its codimension-2 singularities followed by a small resolution in the fiber of I_{2n}–I_n collision [resp. I_0^*–III collision], as we have denoted by $\widetilde{Y}_{A_{2n-1},D_4}$ in (8.8). The projection $\widetilde{Y} \to Y^W \to B_3$ is flat [Mir83]. In the case of IV*, on the other hand, we need to blow up the base for a flat fibration; we can use as B_3 the blow-up of B_w, $\widetilde{B}_w$, centered at the intersection of the E_6 (Kodaira type IV*) 7-brane and the SU(3) (Kodaira type IV) 7-branes. Y^W is pulled backed to be $\nu'^*(Y^W)$ fibered over B_3; it is possible to construct birational and regular maps $\overline{\nu'^*(Y^W)} \to \nu'^*(Y^W)$ and $\overline{\nu'^*(Y^W)} \to Y$, as in Section 8.3, where Y is Calabi-Yau [BJ97b], and $\widetilde{Y}$ is obtained as a canonical resolution of the codimension-2 singularities of Y.

If there is any chance of accommodating a GUT gauge group, one might first consider SU(5) as a part of SU(6). In this case, there may be 4D $\mathcal{N}$=1 chiral multiplets in the SU(6)–SU(3) bifundamental representation localized at the I_6–I_3 collision matter curves. But, there is no matter in the rank-2 anti-symmetric representation. The other possibility is E_6, but there seems to be no matter fields in the E_6-**27** representation, as there is no singular enhancement point over the E_6 7-brane, at least generically. There may be E_6-adjoint chiral multiplets from the E_6 7-brane, but its irreducible decomposition to SU(5) subgroup cannot yield a reasonably successful phenomenology [TW06]. Also note that, even if there are matter fields in **27** representation, it is not possible to generate Yukawa couplings [TW06]. To summarize, it is not possible to implement GUT phenomenology in any one of the constructions considered in this Section 8.4.

There is not much to add particularly for the Type 2 case on the flux-induced supersymmetric mass terms of the complex structure moduli fields. The discussion at the end of Section 8.3 can be repeated with minimal changes;[29] the only difference from the Type 1 case is that $(S_0^{(1)}, T_0^{(1)}) = (U \oplus E_8[2], U^{\oplus 2} \oplus E_8[2])$ rather than $(U, U^{\oplus 2} \oplus E_8^{\oplus 2})$.

For a K3 surface $X^{(1)}$ that corresponds to $S_0 = U \oplus E_8[2]$, there automatically exist two non-symplectic involutions. One acts on the base, and the other on the fiber,

[28]There is a rule on the Kodaira type of a singular fiber that can appear over the base point of $\mathbb{P}^1_{(1)}$ fixed by the Type 2 involution [KLO06, CG15, GS18]. Those Kodaira types are consistent with the rule.

[29]The $\mathbb{C}^2/\mathbb{Z}_2$-deforming moduli of Y^{BV} and corresponding deformations in $\widetilde{Y}$ are localized in the fiber of non-abelian 7-branes in the case of Type 1, but that is not the case generically for a Type 2 fibration and involution. Therefore, there are gauge-neutral moduli fields whose stabilization / mass term is not discussed in this work (cf. discussion at the end of Section 6.2).

so their combination also yields a non-trivial symplectic subgroup of the automorphisms. This means that all the compactifications in a Type 2 case has an extra $\mathbb{Z}_2$ symplectic (=non-R) symmetry in the effective theory (unless the flux breaks it).

Chapter 9

Conclusion

This work is motivated by the tremendous success of the standard cosmology including the Big-Bang Nucleosynthesis, the Standard Model and Grand Unified Theories of particle physics, in combination with the needs for a quantum gravity theory for deeper understanding of our Universe. We investigated a scenario in F-theory where we can construct a quantum gravity theory while preserving all these ingredients.

In particular, we have studied flux vacua in F-theory on CM-type Calabi-Yau fourfolds of the form $\mathrm{K3} \times \mathrm{K3}/\mathbb{Z}_m$. The case of $m = 2$ is intensively studied in Chapter 6, where we found that, when there is at least one modulus for one of the K3 surfaces, the condition (6.38) is necessary and sufficient for non-trivial $DW = W = 0$ vacua with all the moduli having masses, with an additional option (6.62) when the complex structure is non-generic. The condition is not at all restrictive, as any two identical copies of a CM-type K3 surface will satisfy the conditions. We have found families of non-trivial flux vacua with $DW = W = 0$, i.e. vacua with massless gravitino, supersymmetry, and vanishing cosmological constant, when no correction to the superpotential (3.12) or to the Kähler potential is taken into account[1]. In the case of $m > 2$, it is found in Chapter 7 that conditions (7.11) and (7.12) are necessary and sufficient for non-trivial and stabilized $DW = W = 0$ flux vacua.

The low-energy effective theory of such F-theory vacua with $m = 2$ is addressed in Chapter 8. We intensively studied the case where one of the K3 surfaces has an elliptic fibration, and the fourfold inherits the elliptic fibration. There are two types of involutions, i.e. $\mathbb{Z}_2$-actions on the K3 surfaces, namely that non-trivially acts on the fiber (Type 1) and on the base (Type 2). We have found a varieties of gauge groups and (bi)fundamental matters charged under those groups in those vacua, but it seems to be unlikely that a Grand Unified Theory that explains our Universe is in this setup.

There are several ways to explore further along the line of this work. The first would be to investigate the effective theory in the case of $m > 2$, as we have found $DW = W = 0$ vacua there too. Although there is an issue concerning singularities with no crepant resolution [Dil12], there is no obvious reason that we should exclude those geometries from our consideration, as we can think of M-theory compactification on such geometries and we can keep the volume of a cycle that breaks the Calabi-Yau condition to be infinitesimally small; see e.g. [AGW18] for a related analysis. The cases with $m = 3, 4$ are of particular interest, as the resolution of $(E \times \mathbb{C})/\mathbb{Z}_m$ where E is an elliptic curve yields IV^*- and III^*-type fibers, which means

[1]Note, though, that the case $DW = W = 0$ holds if there is no correction to the superpotential (3.12), regardless of the corrections to the Kähler potential.

that in a case analogous to Type 1 in $m = 2$, the G_2 gauge group may replaced by E_6 or E_7, which can accommodate Grand Unified Theories. Secondly, one may look for another moduli space that contains many, hopefully infinitely many, CM points. In the case of threefolds, the famous quintic has such moduli space, so there may be a good chance of finding such moduli space for a Calabi-Yau fourfold. It is in particular desirable to develop a systematic way to find a CM-type complex structure for arbitrary Calabi-Yau fourfolds constructed as a toric hypersurface, because countless numbers of interesting models are constructed in this way. The condition for general CM-type fourfolds, Eqs. (6.4) and (6.5), will be useful in those studies. Finally, CM-type manifolds are known to be related to rational conformal field theories [Moo98, GV04b]. The current and future works along the line (see [KW19b, KW19a] and references therein) may shed light on what we have assumed in this work, in terms of the string worldsheet theory.

Appendix A

Number field theory

A.1 Basics

In this section, we review some of the basic definitions and properties of fields. See standard textbooks such as [Rom05, Fuj91] for more information.

A.1.1 Rings and fields

Definition A.1 (Ring)**.** A **ring** is a nonempty set R, with two binary operations on it, called **addition** and **multiplication**, which satisfy the following properties.

1. R is an abelian group under the addition.
2. $(xy)z = x(yz)$ for all $x, y, z \in R$.
3. For all $x, y, z \in R$,

$$(x + y)z = xz + yz \quad \text{and} \quad x(y + z) = xy + xz. \tag{A.1}$$

On a ring R, there are three operations: $+$ and $\times$ are included by definition, and $-$ operation is always possible because any element in R has an inverse of the additional operation. A ring with a $\div$ operation is called a field:

Definition A.2 (Field)**.** A ring F is called a **field** if the nonzero element of F form an abelian group under multiplication, and if the multiplicative identity $1 \in F$ and the additional identity $0 \in F$ are different, $1 \neq 0$.

The second condition excludes the trivial field $F = \{0\}$. In the following, rings and fields are assumed to be of **characteristic zero**, i.e. $nr := r + r + \cdots + r \neq 0$ for any positive integer n.

A.1.2 Algebraic extension and algebraic number

Definition A.3. A nonempty subset F of a field E is called a **sub-field** of E if it is a field with the same operations as in E.
If F is a sub-field of a field E, we call E an **extension field** of F. This **extension** is denoted by E/F.

Definition A.4. Let E/F be an extension and S be a subset of E. The smallest sub-field of E containing both F and S is denoted by $F(S)$. If $S = [\alpha_1, \ldots, \alpha_n]$ is a finite set, then the extension $F(S)/F$ is said to be **finitely generated** and denoted by $F(\alpha_1, \ldots, \alpha_n)$. An extension of the form $F(\alpha)/F$ is said to be **simple** and α is called a **primitive element**.

Definition A.5. Let F be a field and E be an extension field of F. If an element $x \in E$ is a root of some polynomial with all the coefficients in F, then x is said to be **algebraic** over F. Otherwise x is said to be **transcendental** over F. An extension E/F is called an **algebraic extension** if every element in E is algebraic over F.

Definition A.6. An extension field E of a field F can be viewed as a vector space over F. The dimension of the vector space is called the **degree** of the extension and denoted by $[E : F]$. If $[E : F]$ is finite, then E/F is called a **finite extension**.

Theorem A.1. Let a field K be an extension field of a field F. Then the following conditions are equivalent:

1. K is a finite extension field of F, i.e., $[K : F] < \infty$.
2. K is a finitely generated algebraic extension field of F.

Theorem A.2. Let E/F and K/E be extensions. Then

$$[K : F] = [K : E][E : F]. \tag{A.2}$$

If $\{\alpha_i \mid i = 1, \dots, [E : F]\}$ is a basis of the vector space E over F, and $\{\beta_j \mid j = 1, \dots, [K : E]\}$ that of K regarded as a vector space over E, then the set of products $\{\alpha_i\beta_j \mid i = 1, \dots, [E : F],\ j = 1, \dots, [K : E]\}$ is a basis of the vector space K over F.

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Definition A.7. For a field F, $F[x]$ denotes the ring of polynomials in a single variable x with all the coefficients in F. For a finite algebraic extension E/F, and for an element $\alpha \in E$, non-zero polynomials $p_\alpha(x) \in F[x]$ satisfying $p_\alpha(\alpha) = 0 \in E$ with the smallest degree possible are called **minimal polynomials of α over** F. Such a polynomial always exist (because α is algebraic over F), and is unique up to overall multiplication of elements in $F^\times$. Minimal polynomials are always irreducible in $F[x]$.

Theorem A.3. Let K/F be an extension and let $\alpha \in K$ be algebraic over F. Then the sub-field $F(\alpha)$ of K has a structure

$$F(\alpha) \simeq F[x]/(p_\alpha(x)), \tag{A.3}$$

where p_α is a minimal polynomial of α over F.

In fact, a finite algebraic extension K/F—not just a sub-field $K(\alpha) \subset K$—always has a structure like that, when $\mathrm{char}(F) = 0$; this useful property is stated as follows:

Lemma A.1. When $\mathrm{char}(F) = 0$, any finite extension K/F is a simple extension; that is, there exists an element $\theta \in K$ so that $K = F(\theta)$. Using a minimal polynomial of θ over F, therefore, the field K has a structure $K \cong F[x]/(p_\theta(x))$. It is always possible to take $\{1, \theta, \theta^2, \cdots, \theta^{[K:F]-1}\}$ as a basis, when K is regarded as a $[K : F]$-dimensional vector space over F.

Example A.1. This theorem states that even a field that is generated by multiple elements can be thought of as a simple extension. For example, $\mathbb{Q}(i\sqrt{2}, i\sqrt{3}) = \mathbb{Q}(i\sqrt{2} + i\sqrt{3})$.

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All the definitions and theorems on algebraic extension so far are for fields that are defined abstractly by the laws of addition and multiplication among their elements.

We may sometimes have a little more specific interest, however, in a field K that is defined as a sub-field of $\mathbb{C}$. For such a field K, $\mathrm{char}(K) = 0$ by definition.

Definition A.8. A complex number $\alpha \in \mathbb{C}$ is called an **algebraic number**, if there is a non-zero polynomial $p_\alpha(x) \in \mathbb{Q}[x]$ satisfying $p_\alpha(x = \alpha) = 0 \in \mathbb{C}$. It is known that all the algebraic numbers form a sub-field of $\mathbb{C}$; this sub-field is denoted by $\overline{\mathbb{Q}}$. Any finite extension field K of $\mathbb{Q}$ that is defined as a sub-field of $\mathbb{C}$ is called a **number field**.

Any number field is always a sub-field of $\overline{\mathbb{Q}}$. While $\overline{\mathbb{Q}}/\mathbb{Q}$ is an algebraic extension, it is not a finite extension. Thus, $\overline{\mathbb{Q}}$ itself is not a number field.

A.1.3 Embeddings into $\mathbb{C}$

Here is a summary of results on embeddings of a finite extension field K over $\mathbb{Q}$ into a sub-field of $\mathbb{C}$. We begin, however, with the following preparation.

Theorem A.4. Let K be an algebraic extension over $\mathbb{Q}$, and $\alpha \in K$. For a minimal polynomial $p_\alpha(x)$ of α over $\mathbb{Q}$, there are $\deg(p_\alpha)$ solutions to $p_\alpha(x) = 0$ in $\mathbb{C}$. It is known that all the roots of $p_\alpha(x) = 0$ come with multiplicity 1.

This property is valid for any algebraic extension over F, in place of $\mathbb{Q}$, as long as $\mathrm{char}(F) = 0$, and is called **separability**.

Theorem A.5. Let K be a finite extension over $\mathbb{Q}$. Then there are $[K : \mathbb{Q}]$ distinct embeddings (isomorphism onto the image) $\rho : K \hookrightarrow \mathbb{C}$ over $\mathbb{Q}$. Since all the elements in K are algebraic, the image of such an embedding is always contained within $\overline{\mathbb{Q}}$; $\rho(K) \subset \overline{\mathbb{Q}} \subset \mathbb{C}$.

This is because K can always be regarded as a simple extension over $\mathbb{Q}$ by a primitive element $\theta \in K$ (Lemma A.1); let $p_\theta(x)$ be its minimal polynomial over $\mathbb{Q}$, and $\{\xi_{i=1,\cdots,[K:\mathbb{Q}]}\} \subset \mathbb{C}$ be the roots of $p_\theta(x) = 0$ in $\mathbb{C}$. Then $\rho_i : K \hookrightarrow \mathbb{C}$ is given by $\rho_i : K \ni \theta \mapsto \rho_i(\theta) = \xi_i \in \mathbb{C}$ for $i = 1, \cdots, [K : \mathbb{Q}]$. Note that all the $[K : \mathbb{Q}]$ roots $\{\xi_i\}$ are distinct from one another (separable), and hence the corresponding embeddings are distinct from one another.

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Definition A.9. Now let K/F be a finite extension with degree $m = [K : F]$. For any element $x \in K$, then, $A(x) : y \mapsto x \cdot y$ for $y \in K$ is an F-linear transformation on the vector space K over F. $\mathrm{Tr}_{K/F}(x)$ denotes the trace of the F-valued $m \times m$ matrix representation of $A(x)$, and is called the **trace** of $x \in K$.

A.1. Let $\{\omega_{i=1,\cdots,m}\}$ be a basis of K as a vector space over F. Then

$$x \cdot \omega_i = \omega_j [A(x)]_{ji}, \tag{A.4}$$

where $[A(x)]_{ji}$ is the F-valued $m \times m$ matrix representation of $A(x)$. Now, let us take $F = \mathbb{Q}$. The relation (A.4) among elements in K still holds as one among their images under the embeddings of K into $\overline{\mathbb{Q}} \subset \mathbb{C}$.

$$\rho_a(x)\rho_a(\omega_i) = \rho_a(\omega_j)[A(x)]_{ji}. \tag{A.5}$$

Since there are m distinct embeddings $\rho_{a=1,\cdots,m} : K \to \overline{\mathbb{Q}} \subset \mathbb{C}$, $\rho_a(\omega_i)$, $\rho_a(\omega_j)$ and $\rho_a(x)$ can be regarded as $\mathbb{C}$-valued $m \times m$ matrices (the matrix $\rho_a(x)$ is diagonal), and the following relation is obtained:

$$\mathrm{Tr}_{K/\mathbb{Q}}(x) = \mathrm{tr}_{m\times m}\left[A(x)\right] = \sum_{a=1}^{m} \rho_a(x); \tag{A.6}$$

each contribution on the right-hand side is an algebraic number in $\overline{\mathbb{Q}} \subset \mathbb{C}$, but their sum should be in $\mathbb{Q}$, because the left-hand side is, by definition.

A.1.4 Normal closure

Definition A.10. Let K be a number field, i.e., a sub-field of $\overline{\mathbb{Q}} \subset \mathbb{C}$ that is a finite extension over $\mathbb{Q}$. Let θ be a primitive element (i.e., $K = \mathbb{Q}(\theta)$), $p_\theta(x)$ be its minimal polynomial over $\mathbb{Q}$, and $\{\xi_1 = \theta, \xi_2, \cdots, \xi_{[K:\mathbb{Q}]}\}$ be the roots of $p_\theta(x) = 0$ in $\mathbb{C}$. The field $\mathbb{Q}(\xi_1, \cdots, \xi_{[K:\mathbb{Q}]}) \subset \overline{\mathbb{Q}}$ is called the **smallest splitting field of $p_\theta(x) \in \mathbb{Q}[x]$ in $\overline{\mathbb{Q}}$**.

A.2. Thinking of a number field K as an abstract finite extension field over $\mathbb{Q}$, we see that there must be $[K : \mathbb{Q}]$ embeddings $\rho_i : K \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$, $i = 1, \cdots, [K : \mathbb{Q}]$ (Thm A.5). The embedding $\rho_{i=1} : K \hookrightarrow \overline{\mathbb{Q}}$ is a trivial identification, and $\rho_1(K) = K \subset \overline{\mathbb{Q}}$. For other ρ_i's, however, it is not guaranteed that $\rho_i(K) = K$.

A.3. The field $\mathbb{Q}(\xi_1, \cdots, \xi_{[K:\mathbb{Q}]})$ can be regarded as the minimal sub-field of $\overline{\mathbb{Q}}$ that contains all the images $\cup_{i=1,\cdots,[K:\mathbb{Q}]}\rho_i(K)$ of the $[K : \mathbb{Q}]$ embeddings from K to $\overline{\mathbb{Q}}$. Because of this characterization, the smallest splitting field $\mathbb{Q}(\xi_1, \cdots, \xi_{[K:\mathbb{Q}]}) \subset \overline{\mathbb{Q}}$ of $p_\theta(x)$ in $\overline{\mathbb{Q}}$ does not depend on the choice of a primitive element θ.

Theorem A.6. For a sub-field $K^{\mathrm{nc}} := \mathbb{Q}(\xi_1, \cdots, \xi_{[K:\mathbb{Q}]})$ of $\overline{\mathbb{Q}}$ for a number field K, any one of the embeddings $\rho : K^{\mathrm{nc}} \hookrightarrow \overline{\mathbb{Q}}$ over $\mathbb{Q}$ maps K^{nc} to $K^{\mathrm{nc}} \subset \overline{\mathbb{Q}}$, not outside of K^{nc} (though not necessarily as a trivial map on K^{nc})—(*). This is because such an embedding ρ has to send ξ_i's to ξ_i's, possibly with a permutation among them, and cannot do anything more than that.

A.4. A sub-field E of $\mathbb{C}$ is said to be a **normal extension of** $\mathbb{Q}$, if it has the property (*) referred to above. The minimum sub-field in $\mathbb{C}$ of a number field K that is a normal extension over $\mathbb{Q}$ is called the **normal closure of** $K/\mathbb{Q}$ **in** $\mathbb{C}$, and is denoted by K^{nc}, as we have done already above. For a number field K, therefore, the smallest splitting field $\mathbb{Q}(\xi_1, \cdots, \xi_{[K:\mathbb{Q}]}) \subset \overline{\mathbb{Q}}$ of a primitive element θ such that $K = \mathbb{Q}(\theta)$ is the normal closure of K.

For a finite extension field E over $\mathbb{Q}$ that is defined as an abstract field, one can pick any one of embeddings $\rho : E \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$. The normal closure of $\rho(E)$ in $\overline{\mathbb{Q}}$ does not depend on which one of $[E : \mathbb{Q}]$ embeddings is used. So, we use a notation E^{nc} for $(\rho(E))^{\mathrm{nc}}$ in this article.

The definition of a normal extension $E \subset \overline{\mathbb{Q}}$ over $\mathbb{Q}$ is generalized to extensions $E \subset \overline{\mathbb{Q}}$ over an arbitrary number field F by replacing $\mathbb{Q}$ in A.6 and A.4 with F.

Definition A.11. An algebraic extension E/F is said to be **Galois**, if it is a separable and normal extension. Note that the separability is always guaranteed, when F has $\mathrm{char}(F) = 0$.

Example A.2. The normal closure K^{nc} of a number field K is always a Galois extension over $\mathbb{Q}$, by definition. Not all the number fields K are Galois over $\mathbb{Q}$, however.

Cyclotomic fields $K = \mathbb{Q}(\zeta_N)$ are examples of Galois extensions over $\mathbb{Q}$; the number field $K = \mathbb{Q}[x]/(x^3 - 2)$, on the other hand, is not Galois; another example of non-Galois extension is found in Example A.5.

A.2 CM fields

We introduce CM fields in this section, which plays a crucial role in the main text.

Definition A.12. A finite extension field K over $\mathbb{Q}$ is said to be **totally real** if $\rho_i(K) \subset \mathbb{R}$ for all the $[K:\mathbb{Q}]$ embeddings $\rho_{i=1,\cdots,[K:\mathbb{Q}]} : K \hookrightarrow \mathbb{C}$. On the other hand, a finite extension field K over $\mathbb{Q}$ is said to be **totally imaginary** if $\rho_i(K)$ is not contained within $\mathbb{R}$ for any one of the $[K:\mathbb{Q}]$ embeddings $\rho_{i=1,\cdots,[K:\mathbb{Q}]} : K \hookrightarrow \mathbb{C}$.

Example A.3. Let $n \in \mathbb{Z}$ and suppose that $|n|$ is not the square of an integer. $K = \mathbb{Q}[x]/(x^2 - n)$ is totally real [resp. totally imaginary] if $n > 0$ [resp. $n < 0$]. This field K has two embeddings into $\mathbb{C}$; $\rho_\pm : x \mapsto \pm\sqrt{n} \in \mathbb{C}$. On the other hand, $K = \mathbb{Q}[x]/(x^3 - 2)$ is neither totally real nor totally imaginary; the three embeddings of K to $\mathbb{C}$ send $x \in K$ to one of the three roots of $x^3 - 2 = 0$ in $\mathbb{C}$.

Now, here is the definition of a CM field.

Definition A.13. A finite extension field K over $\mathbb{Q}$ is said to be a **CM field**, if (i) it contains a sub-field K_0 that is totally real, (ii) K is a degree-2 extension of K_0, and (iii) K itself is totally imaginary. Therefore, $[K:\mathbb{Q}] = [K:K_0][K_0:\mathbb{Q}] = 2[K_0:\mathbb{Q}]$ is always an even integer.

Proposition A.1. Let K be a CM field with $[K:\mathbb{Q}] = 2n$. Its $2n$ embeddings to $\mathbb{C}$ can be grouped into n pairs, $(\rho_i, \bar{\rho}_i)$ for $i = 1, \cdots, n$, so that $\bar{\rho}_i(x) = (\rho_i(x))^{\rm cc}$, where the superscript cc is the complex conjugation operation in $\mathbb{C}$. To see this, let $K = \mathbb{Q}(\theta)$ for some primitive element $\theta \in K$. For a minimal polynomial $p_\theta(x) \in \mathbb{Q}[x]$ for θ, all the $2n$ roots of $p_\theta(x) = 0$ have non-zero imaginary parts, and are grouped into n pairs, $(\xi_i, \xi_i^{\rm cc})$ with $i = 1, \cdots, n$. The embedding $\rho_i : \theta \mapsto \xi_i$ forms a pair with $\bar{\rho}_i : \theta \mapsto \xi_i^{\rm cc}$.

Example A.4. Because the extension degree of a CM field is always even, the simplest CM field is a quadratic extension over $\mathbb{Q}$; quartic extensions come next.
CM fields K with $[K:\mathbb{Q}] = 2$ are always in the form of $K \cong \mathbb{Q}[x]/(x^2 + d) \cong \mathbb{Q}(\sqrt{-d})$, where d is a positive integer that is not divisible by the square of an integer. Fields defined by $K = \mathbb{Q}[x]/(ax^2 + bx + c)$ for $a, b, c \in \mathbb{Q}$ with $4ac - b^2 > 0$ can always be brought into the form of $\mathbb{Q}[x]/(x^2 + d)$ by redefining x. Such fields are called **quadratic imaginary fields**. Two embeddings $\rho_\pm$ send x to $\pm i\sqrt{d} \in \mathbb{C}$. For quadratic imaginary fields, $K^{\rm nc} \cong K$.

Example A.5. A CM field K with $[K:\mathbb{Q}] = 4$ is always in the form of $K \cong K_0[x]/(x^2 - p - q\eta)$, $K_0 = \mathbb{Q}[\eta]/(\eta^2 - d)$ for a positive square free integer d, and $p, q \in \mathbb{Q}$, satisfying $p \pm q\sqrt{d} < 0$. The last condition needs to be imposed both for $+$ and $-$, because the condition (iii) would not be satisfied if $p + q\sqrt{d} < 0$ but $p - q\sqrt{d} > 0$ (or vice versa).

CM fields K with $[K:\mathbb{Q}] = 4$ are not always Galois over $\mathbb{Q}$. It is Galois (i.e., $K^{\rm nc} \cong K$) if and only if $(p^2 - dq^2) = r^2$ for some ${}^\exists r \in \mathbb{Q}$, or $(p^2 - dq^2) = ds^2$ for some ${}^\exists s \in \mathbb{Q}$. When $q = 0$, in particular, K is Galois, $K = \mathbb{Q}(\sqrt{-p}, \sqrt{d})$ and $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z})$. See Ex. 8.4 (2) of [ST61, Shi98] for more information.

Example A.6. Any cyclotomic field $K = \mathbb{Q}(\zeta_m)$ is a CM field. $K_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$.

A.5. When K is a CM field, its normal closure K^{nc} is also a CM field (Prop. 5.12, [Shi71]).

Appendix B

Details of geometries

Details of geometrical analysis in the main text are gathered in this Appendix B.

B.1 Type 1

B.1.1 Derivation of the Weierstrass equation

In this section, we derive the Weierstrass equation (8.4) for Y^W. Let us assume that $X^{(1),W}$ is given by a Weierstrass equation

$$P_W = -y^2 + x^3 + xf(t) + g(t) = 0, \tag{B.1}$$

where t is a coordinate of the base $\mathbb{P}^1$ of $X^{(1),W}$. We will focus on an affine patch $U_0 \in X^{(1),W} \times X^{(2)}$, where $X^{(1),W}$ is given by the above Weierstrass equation and the involution on $X^{(2)}$ acts as $(u, v) \mapsto (u, -v)$, where u and v are the coordinates of a patch of $X^{(2)}$. The ring of regular functions on U_0 is given by

$$\mathbb{C}[U_0] := \mathbb{C}[x, y, v, t, u]/P_W. \tag{B.2}$$

After the $\mathbb{Z}_2$ quotient that acts as $(y, v) \mapsto (-y, -v)$ and identity for others, the ring on $U_1 \in Y_1$, where Y_1 is the singular geometry after the quotient, is given by

$$\mathbb{C}[x, Y = y^2, V = v^2, W = yv, t, u]/\left\{(-Y + x^3 + xf(t) + g(t)), YV = W^2\right\}, \tag{B.3}$$

which is equivalent to

$$\mathbb{C}[U_1] = \mathbb{C}[x, V, W, t, u]/(W^2 = V(x^3 + xf(t) + g(t))). \tag{B.4}$$

There are three A_1 singularities along $W = V = 0$ at the three roots of the cubic polynomial in x. The other A_1 singularity is not visible in this patch. To construct $Y^{W'}$, we blow-up the invisible A_1 singularity, so nothing happens in this patch. In order to get the Weierstrass model Y^W, let us "blow-down" the geometry. Namely, we consider a geometry with the following ring of functions

$$\mathbb{C}[\tilde{x}, \tilde{y}, V, t, u]/(\tilde{y}^2 = \tilde{x}^3 + \tilde{x}V^2 f(t) + V^3 g(t))). \tag{B.5}$$

Let us denote the new geometry by U_2. The geometry U_1 can then be regarded as a (proper transform of) a blow-up at $\tilde{x} = \tilde{y} = V = 0$ of the new geometry U_2 with the map $(\tilde{x}, \tilde{y}, V, t, u) = (xV, WV, V, t, u)$. We have successfully derived a Weierstrass model that is birational to Y_0^W, which should be called as Y^W.

	A_1	A_2	A_3	A_4	A_6	Δ
I_0^{*ns}	1	1	2	2	3	6
$\mathrm{I}_0^{*ss}, \mathrm{I}_0^{*s}$	1	1	2	2	4	6

TABLE B.1: Conditions on the order of vanishing along C of the coefficients in the generalized Weierstrass equation (B.6).

B.1.2 Note on split, semi-split, and non-split fibers

In this section, we will see that the Δ_f of Y^W of Type 1 induces a generically G_2 gauge symmetry. To this end, let us first consider a three-fold X that is an elliptic fibration over a surface B. The elliptic fiber may degenerate on a curve C in the base surface B; we review what kind of X over a generic point of C is of Kodaira I_0^* type, and how the generically-I_0^* degenerate fiber behave over the curve C, depending on more detailed conditions on X. After that, we will apply the analysis to Y^W. Let us start from X given by

$$y^2 + A_1(u,v)xy + A_3(u,v)y = x^3 + A_2(u,v)x^2 + A_4(u,v)x + A_6(u,v) \tag{B.6}$$

where the order of vanishing of the coefficients along C (whose normal coordinate is v) is those in the first row of the Table B.1. u is the tangential coordinate along C. The condition in the first row of Table B.1 is weaker (more general) than the one in the second row, so we can study what happens when the condition in the second row is satisfied as a special case of analysis that assumes only the condition in the first row.

After the first blow-up, let us have a look at an open affine patch with the coordinates (v,ξ,η); $(x,y,v) = (\xi v, \eta v, v)$ is the blow-up map. The proper transform of X is given by

$$\eta^2 + b_5 v\eta\xi + b_3 v\eta \; = v\xi^3 + b_4 v\xi^2 + b_2 v\xi + b_0 v, \tag{B.7}$$
$$\eta(\eta + b_5 v\xi + b_3 v) \; = v(\xi^3 + b_4\xi^2 + b_2\xi + b_0). \tag{B.8}$$

where the normal coordinates have been factorized from the coefficients $A_r(u,v)$'s by $A_r(u,v) = v^{\mathrm{ord}_v(A_r)} b_{6-r}(u,v)$ (for $r = 1,2,3,4,6$). The exceptional divisor is the $s = 0$ locus, where $\eta = 0$ also has to hold. In the three dimensional space (s,ξ,η) (with an irrelevant u direction), the equation above still leaves singularity at $\eta = s = 0$ and any one of the three roots of $\xi^3 + b_4(u)\xi^2 + b_2(u)\xi + b_0 = 0$. By resolving those codimension-2 singularities, we obtained a three-fold that has Kodaira-I_0^*-type singular fiber at a generic point in C. Note that the three roots of the cubic polynomial $\xi^3 + b_4(u)\xi^2 + b_2(u)\xi + b_0(u) = 0$ are in one to one correspondence with the three non-central nodes of the Dynkin diagram of D_4.

Along the curve C, where u is the tangential coordinate, the value of $b_4(u)$, $b_2(u)$, and $b_0(u)$ change, and three roots of the cubic polynomial mix up as we track them over C. The 7-brane gauge group in the effective theory of F-theory compactification on X is therefore reduced from D_4 to G_2; the three nodes of the Dynkin diagram of D_4 merge into one node, and the three edges of the diagram turn into three-fold edge between the two nodes that correspond to the D_4-central and D_4-non-central ones. For just the two of the nodes to mix up over C and one node stay distinct from the two others, the cubic polynomial just need to factorize into a linear polynomial and a quadratic polynomial. When this condition is satisfied, then we can always redefine ξ by shifting it to ξ' to turn the linear piece into just ξ'. So, the cubic polynomial must look like $\xi(\xi^2 + b_4\xi + b_2) = 0$. This is equivalent to A_6 having order of vanishing higher, as in the second row of Table B.1. The two nodes out of the three nodes of the Dynkin diagram now merge into one node, and the edge joining this merged node and the D_4-central node become two-fold. That is for $SO(7)$. For all the three non-central nodes of the Dynkin diagram of D_4 to stay distinct without mixing up over the entire curve C, the cubic polynomial $\xi^3 + b_4(u)\xi^2 + b_2(u)\xi + b_0(u) = 0$ just needs to factorize completely. So, it is better to say that the distinction between I_0^{*ns}, I_0^{*ss} and I_0^{*s} depend on whether the cubic polynomial $[X^3 + (A_2/v)X^2 + (A_4/v^2)X + (A_6/v^3)]_{v=0}$ factorize globally over C, or not.

Now let us go back to our problem, Y^W. By blowing-up Y^W once, we get the geometry U_1 in (B.4). There are three A_1 singularities, away from the zero section, as roots of the cubic polynomial $x^3 + xf(t) + g(t)$. As the cubic cannot be factorized in general, one expects that the three roots are interchanged among them, which implies that the gauge symmetry is reduced to G_2. Note that there can be cases where f and g are non-generic and there are not enough monodromies to reduce the gauge symmetry to G_2.

B.2 Type 2

B.2.1 Definition of the geometry

We first define the geometry. Locally, it can be defined using the coordinates (v, w) of $X^{(2)}$ and the Weierstrass equation $y^2 = x^3 + f(t^2)x + g(t^2)$ of $X^{(1)W}$. The involution on $X^{(2)}$ acts as $(v, w) \mapsto (-v, w)$. f and g can only depend on the even powers of the base coordinate t, since the involution acts as $t \mapsto -t$. Y_0^W can be defined by the following two equations

$$y^2 = x^3 + f(T)x + g(T), \qquad TV = S^2. \tag{B.9}$$

Here, the relation to the original coordinate (t, v, w) is $T = t^2, V = v^2, S = tv$. We blow-up the A_1 singularity canonically to get

$$y^2 = x^3 + f(T)x + g(T), \qquad V_T = S_T^2 \qquad (T, V_T, S_T) \tag{B.10}$$
$$y^2 = x^3 + f(S_V^2 V)x + g(S_V^2 V), \qquad T_V = S_V^2 \qquad (T_V, V, S_V) \tag{B.11}$$
$$y^2 = x^3 + f(T_S S)x + g(T_S S), \qquad T_S V_S = 1 \qquad (T_S, V_S, S), \tag{B.12}$$

where we have listed the coordinates on the right and notation such as $V_T = V/T$ is used. Note that in the last patch the fiber is only singular along $S = 0$ and not $T_S = 0$, since $T_S \neq 0$ from the second equation. The blue component in Figure 8.4 corresponds to $S_V = 0$ in (B.11) and the green component corresponds to $V = 0$ in the same patch.

B.2.2 Monodromy

The singular fiber of $X^{(1)}$ at $t = 0$ must be either I_{2n}, I_0^*, IV, or IV*, which are the only ones that can be made out of even order f and g [1]. We will analyse the monodromy

[1]One can understand geometrically why $\mathrm{I}_{2n+1}, \mathrm{I}_{n>0}^*, \mathrm{III}^*, \mathrm{II}^*$ are not allowed. For an illuminating example, let us first see why III* is excluded from the list. The III* fiber has eight rational curves. Let us denote the one that intersects with zero-section by C_0, the branched one by C_7, and others so that $C_i.C_{i+1} = 1$ for $i = 0, \ldots, 5$. There can be no nontrivial automorphism for E_7, so each curve class must be mapped to itself by the involution of $X^{(1)}$, and for each neighboring pair of curves, exactly one of them must be fixed (point-wisely) by the involution. This is roughly because the involution is

around $S_V = V = 0$ for the four cases in turn.

I_{2n} fiber: We consider the general collision $\mathrm{I}_n \times \mathrm{I}_m$ so that one can see the monodromy of not only I_{2n} along $S_V = 0$ but also that of I_n along $V = 0$. The model is

$$y^2 = z^2(z + 3c) + t^n s^m, \tag{B.13}$$

where c is a constant near $t = s = 0$ and $z = x - c$. Blow-up at $z = y = t = 0$ to get

$$y_t^2 = z_t^2(z_t t + 3c) + t^{n-2}s^m. \tag{B.14}$$

The exceptional locus is $y_t = \pm z_t\sqrt{3c}$ if $n > 2$ and $y_t^2 = 3z_t^2 c + s^m$ if $n = 2$. There seems to be no monodromy in any case. Note that in the literature one can see statements like $\mathrm{I}_1 \times \mathrm{I}_{2k+1}$ does not admit a small resolution [GM00, AGW18], although those cases do not occur in this work.

I_0^* fiber: The model is

$$y^2 = x^3 + s^2tx + s^4t^2. \tag{B.15}$$

Blow-up once at $x = y = s = 0$ to get

$$y_s^2 = x_s^3 s + stx_s + s^2t^2 = s(x_s^3 + tx_s + st^2). \tag{B.16}$$

Around $x_s = y_s = s = 0$, the rhs reduces to

$$sx_s(x_s^2 + t). \tag{B.17}$$

So it has three A_1 singularities along $s = y_s = 0$, at $x_s = 0, \pm i\sqrt{t}$. The last two A_1 singularities are exchanged around $t = 0$, so there is a monodromy for the I_0^* fiber. III fiber does not admit any monodromy.

IV^* **fiber:** The model would be

$$y^2 = x^3 + s^4t^2x + s^4t^2. \tag{B.18}$$

In this case, one should further blow-up the base according to [Mir83, BJ97a]. Over the new exceptional locus, the vanishing order of (f, g, Δ) is $(2, 0, 0)$, so there is no singularity over the exceptional locus. Thus there is no chance that the IV or IV^* have monodromies.

Still, we can analyze the monodromy. To analyze the monodromy of IV^*, blow-up twice along $y = x = s = 0$ to get

$$y_{ss}^2 = x_{ss}^3 s^2 + t^2 x_{ss} s^2 + t^2. \tag{B.19}$$

non-symplectic; see [GS18, Proposition 2.2]. Now, as we have assumed that the zero-section is acted by the involution non-trivially, C_0 must be a fixed curve. Then, C_2, C_4, and C_7 must be a fixed curve, but this leads to a contradiction because C_3 is acted by the involution non-trivially with at least three fixed points. The case of type $\mathrm{I}_{n>0}^*$ and II^* can be excluded by a similar reasoning, while I_0^* is allowed; let us denote the curve in I_0^* that intersects with the zero-section by C_0 and others so that $C_1.C_i = 1$ for $i = 0, 2, 3, 4$. There can be an involution that fixes C_0 and, say, C_4, but exchanges C_2 and C_3. C_1 has two fixed points at the intersections with C_0 and C_4, and this is a perfectly consistent involution, so I_0^* is allowed. I_{2n+1} has odd number of curves, so it has no consistent non-symplectic involution that fixes each curve class. The automorphism cannot be used in this case, since the automorphism is not compatible with the fact that the curve C_0 that intersects with the zero-section is a fixed curve. We thus have excluded $\mathrm{I}_{2n+1}, \mathrm{I}_{n>0}^*, \mathrm{III}^*$ and II^* type fibers.

The exceptional locus is given by $s = 0$, which corresponds to

$$y_{ss}^2 = t^2, \tag{B.20}$$

i.e. $y_{ss} = \pm t$. We do not have the monodromy around $t = 0$.

The monodromy of IV can be analyzed by blowing-up at $y = x = t = 0$. We get

$$y_t^2 = x_t^3 t + s^4 t x_t + s^4. \tag{B.21}$$

The exceptional locus is given by $y_t = \pm s^2$ and There is no monodromy for IV either.

IV **fiber:** In this case, the author is not sure if II $\times$ IV does not admit a flat resolution. Ref. [BJ97a] tries to find a resolution by blowing-up the base and concludes that the process is non-crepant.

B.3 Relations among $Y^W, \nu^*(Y^W), \overline{\nu^*(Y^W)}$ and Y

In this section, we will make the argument in the final part of Section 3.3.1 refined and global, in the sense that we take care of whole elliptic fiber including the $Z = 0$ locus, i.e. the origin of the elliptic curve. We will start from a model of $\nu^*(Y^W)$.

B.3.1 Blow-up of $\nu^*(Y^W)$

We start from the geometry Y_0, which corresponds to $\nu^*(Y^W)$ in the main text[2]. Y_0 is defined by

$$Y^2 = X^3 + f(t, s_t)t^4 X Z^4 + g(t, s_t)t^6 Z^6 \tag{B.22}$$

in the ambient space $A_0 = W\mathbb{P}^2_{2,3,1} \times \mathbb{C}^2$ with coordinates $[X : Y : Z] \times (t, s_t)$.

Let us perform three, blow-ups in turn:

	1st	2nd	3rd
X	$X \to tX$	$X \to tX$	
Y	$Y \to tY$	$Y \to tY$	$Y \to tY$
ambient space	A_1	A_2	A_3

We define $U_{\check{X}}$ by $X \neq 0$ and $U_{\check{Z}}$ by $Z \neq 0$. These two patches are sufficient to cover Y_0 entirely, as $X = Z = 0$ implies $X = Z = Y = 0$, which is absent in $W\mathbb{P}^2_{2,3,1}$. The singularity is along $X = Y = 0$, so we take $U_{\check{Z}}$ to perform a blow-up, leaving $U_{\check{X}}$. In what follows, we use the inhomogeneous coordinate in $U_{\check{Z}}$, $x = X/Z^2$ and $y = Y/Z^3$. Let us first blow-up at $x = y = t = 0$ to get A_1. We will use notations such that x_t in A_1 corresponds to x/t in A_0.

patch in A_1	coordinates	defining polynomials
$U_{\check{t}}$	(x_t, y_t, t)	$t^2(-y_t^2 + t x_t^3 + f x_t t^3 + g t^4)$
$U_{\check{x}}$	(x, y_x, t_x)	$x^2(-y_x^2 + x + f t_x^4 x^3 + g t_x^6 x^4)$
$U_{\check{y}}$	(x_y, y, t_y)	$y^2(-1 + y x_y^3 + f x_y t_y^4 y^3 + g t_y^6 y^4)$

[2] Y_0 is a model of $\nu^*(Y^W)$, in the sense that we are taking $\mathbb{C}^2$ as the base, rather than a compact threefold as in the main text. What will be called Y_3 corresponds to $\overline{\nu^*(Y^W)}$. We will eventually end up with Y', which will correspond to Y in the main text.

We define the proper transform to be Y_1. The canonical divisor is

$$K_{Y_1} = (K_{A_1} + Y_1)|_{Y_1} = (K^*_{A_0} + 2E_1 + Y^* - 2E_1)|_{Y_1} = K^*_Y|_{Y_1}. \tag{B.23}$$

The only singular locus is $x_t = y_t = t = 0$ in $U_{\check{t}}$, so we blow-up this locus. Note that the locus is completely invisible in other patches since $t_x = 1/x_t$ and $t_y = 1/y_t$ on overlaps.

We will keep $U_{\check{x}}$ and $U_{\check{y}}$ and blow-up $U_{\check{t}}$, which we will call A_2.

patch in A_2	coordinates	def. pol. of Y_1^*
$U_{t\check{t}}$	(x_{tt}, y_{tt}, t)	$t^2(-y_{tt}^2 + t^2 x_{tt}^3 + f x_{tt} t^2 + g t^2)$
$U_{t\check{x}}$	(x_t, y_{tx_t}, t_{x_t})	$x_t^2(-y_{tx_t}^2 + t_{x_t} x_t^2 + f t_{x_t}^3 x_t^2 + g t_{x_t}^4 x_t^2)$
$U_{t\check{y}}$	(x_{ty_t}, y_t, t_{y_t})	$y_t^2(-1 + t_{y_t} x_{ty_t}^3 y_t^2 + f x_{ty_t} t_{y_t}^3 y_t^2 + g t_{y_t}^4 y_t^2)$

We take $Y_2 = Y_1^* - 2E_2$. The canonical divisor is

$$K_{Y_2} = (K_{A_2} + Y_2)|_{Y_2} = (K^*_{A_1} + 2E_2 + Y_1^* - 2E_2)|_{Y_2} = K^*_{Y_1}|_{Y_2}, \tag{B.24}$$

so the process is crepant. Here, in $U_{t\check{t}}$, $y_{tt} = t = 0$ is singular. This locus corresponds to $y_{tx_t} = x_t = 0$ in $U_{t\check{x}}$. We cannot see the locus in $U_{t\check{y}}$ since $t_{y_t} = 1/y_{tt}$, so we leave the patch. Note that the singular locus is identical to $E_2|_{Y_2}$.

By blowing-up the singular locus, we get the geometry

patch in A_3	coordinates	def. pol. of Y_2^*
$U_{t\check{t}t}$	(x_{tt}, y_{ttt}, t)	$t^2(-y_{ttt}^2 + x_{tt}^3 + f x_{tt} + g)$
$(U_{t\check{t}y})$	$(x_{tt}, y_{tt}, t_{y_{tt}})$	$y_{tt}^2(-1 + t_{y_{tt}}^2 x_{tt}^3 + f x_{tt} t_{y_{tt}}^2 + g t_{y_{tt}}^2)$
$U_{t\check{x}x}$	$(x_t, y_{tx_t x_t}, t_{x_t})$	$x_t^2(-y_{tx_t x_t}^2 + t_{x_t} + f t_{x_t}^3 + g t_{x_t}^4)$
$(U_{t\check{x}y})$	$(x_{ty_x}, y_{tx_t}, t_{x_t})$	$y_{tx_t}^2(-1 + t_{x_t} x_{ty_x}^2 + f t_{x_t}^3 x_{ty_x}^2 + g t_{x_t}^4 x_{ty_x}^2)$

We define Y_3 as $Y_3 = Y_2^* - 2E_3$. We denoted the first coordinate in the last patch by x_{ty_x} because $x_{ty_{tx_t}} = x/(ty/x)$ at generic points. We have completely resolved the singularity, when f, g are generic. Note that $U_{t\check{t}y}$ and $U_{t\check{x}y}$ can be dropped, since $t_{y_{tt}} = 0$ and $x_{ty_x} = 0$ are, respectively, missing in Y_3. The canonical divisor is

$$K_{Y_3} = (K_{A_3} + Y_3)|_{Y_3} = (K^*_{A_2} + E_3 + Y_2^* - 2E_3)|_{Y_3} = (K^*_{Y_2} - E_3)|_{Y_3}, \tag{B.25}$$

so the process is not crepant. Since the previous two blow-ups were crepant, we have

$$K_{Y_3} = (K^*_Y - E_3)|_{Y_3}. \tag{B.26}$$

Summary so far

We have constructed a geometry Y_3

$$Y_3 = Y^* - 2(E_1^* + E_2^* + E_3)|_{Y_3}. \tag{B.27}$$

The canonical divisor of Y_3 is,

$$K_{Y_3} = (K^*_Y - E_3)|_{Y_3}. \tag{B.28}$$

Patches and defining polynomials are

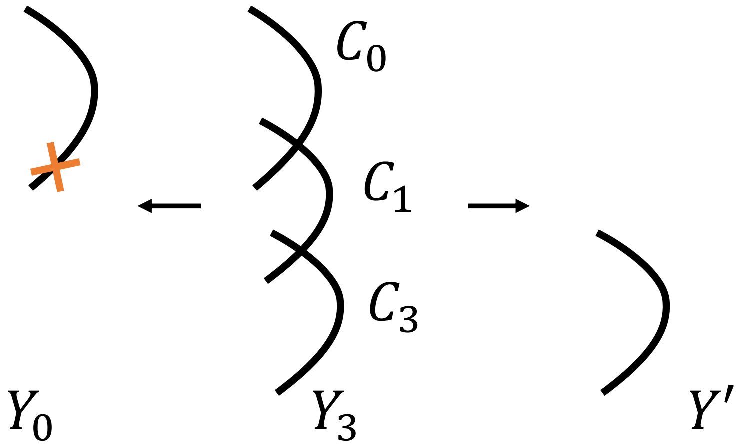

FIGURE B.1: A schematic figure of the fiber over $t = 0$, along the regular transformation denoted by the two arrows.

patches	coordinates	def. pol. of Y_3
$U_{\breve{X}}$	$(B = yz, C = z^2, t)$	$B^2 = C(1 + ft^4C^2 + gt^6C^3)$
$U_{\breve{x}}$	(x, y_x, t_x)	$-y_x^2 + x + ft_x^4x^3 + gt_x^6x^4$
$U_{\breve{y}}$	(x_y, y, t_y)	$-1 + yx_y^3 + fx_yt_y^4y^3 + gt_y^6y^4$
$U_{t\breve{y}}$	(x_{ty_t}, y_t, t_{y_t})	$-1 + t_{y_t}x_{ty_t}^3y_t^2 + fx_{ty_t}t_{y_t}^3y_t^2 + gt_{y_t}^4y_t^2$
$U_{t\breve{t}t}$	(x_{tt}, y_{ttt}, t)	$-y_{ttt}^2 + x_{tt}^3 + fx_{tt} + g$
$U_{t\breve{x}x}$	$(x_t, y_{tx_tx_t}, t_{x_t})$	$-y_{tx_tx_t}^2 + t_{x_t} + ft_{x_t}^3 + gt_{x_t}^4$

Fiber structure at $t = 0$

Let us study the fiber at $t = 0$. For a schematic figure of the $t = 0$ fiber, see Figure B.1

patches	coordinates	$t = 0$
$U_{\breve{X}}$	$(B = yz, C = z^2, t)$	$t = 0 \Rightarrow B^2 = C$: curve C_0
$U_{\breve{x}}$	(x, y_x, t_x)	$t_x = 0 \Rightarrow -y_x^2 + x = 0$: curve C_0
		$x = 0 \Rightarrow y_x^2 = 0$: curve C_1
$U_{\breve{y}}$	(x_y, y, t_y)	$t_y = 0 \Rightarrow 1 = yx_y^3$: curve C_0
		$y = 0 \Rightarrow$ empty
$U_{t\breve{y}}$	(x_{ty_t}, y_t, t_{y_t})	$t_{y_t} = 0 \Rightarrow$ empty
		$y_t = 0 \Rightarrow$ empty
$U_{t\breve{t}t}$	(x_{tt}, y_{ttt}, t)	$t = 0 \Rightarrow y_{ttt}^2 = x_{tt}^3 + fx_{tt} + g$: curve C_3
$U_{t\breve{x}x}$	$(x_t, y_{tx_tx_t}, t_{x_t})$	$t_{x_t} = 0 \Rightarrow y_{tx_tx_t}^2 = 0$: curve C_1
		$x_t = 0 \Rightarrow y_{tx_tx_t}^2 = t_{x_t} + ft_{x_t}^3 + gt_{x_t}^4$: curve C_3

The curves are identified by analyzing the overlaps of the patches. One can also see that $C_0 \cap C_1 = 1$, $C_1 \cap C_3 = 1$, and $C_0 \cap C_3 = 0$.

Regular map from Y_3 to Y'

Let us construct a regular map from Y_3 to Y'. Y' defined globally by the homogeneous equation

$$Y'^2 = X'^3 + f(t', s'_t)X'Z'^4 + g(t', s'_t)Z'^6. \tag{B.29}$$

with an ambient space isomorphic to A_0 with primes on the coordinates. The geometry is covered by two patches $V_{\check{X}'}$ and $V_{\check{Z}'}$, but we also use another patch $V_{\check{Y}'}$. We first try to construct a regular map patch by patch. We then check the consistency over the overlaps. The defining equations and coordinates for each patch is as follows[3]:

patches	coordinates	def. eq. of Y'
$V_{\check{X}'}$	$(B' = Y'Z', C' = Z'^2, t')$	$B^2 = C(1 + ft^4C^2 + gt^6C^3)$
$V_{\check{Y}'}$	$(E' = Z'^3, F' = X'Z', t')$	$E' = F'^3 + fE'F'^2 + gE'^3$
$V_{\check{Z}'}$	(x', y', t')	$y'^2 = x'^3 + fx' + g$

$U_{t\check{t}t}$ **patch** This is the easiest one. Take

$$x' = x_{tt}, \qquad y' = y_{ttt}, \qquad t' = t. \tag{B.30}$$

This is simply an isomorphism.

$U_{t\check{x}x}$ **patch** Take

$$B' = y_{tx_tx_t}, \qquad C' = t_{x_t}, \qquad t' = t_{x_t}x_t \tag{B.31}$$

Then (the pullback of) the defining equation of Y' is

$$(-y_{tx_tx_t}^2 + t_{x_t}(1 + ft_{x_t}^2 + gt_{x_t}^3)) = 0 \tag{B.32}$$

and the image of the regular map is in Y'.

$U_{\check{x}}$ **patch** Take

$$B' = y_x t_x, \qquad C' = t_x^2 x, \qquad t' = t_x x \tag{B.33}$$

Then the defining equation of Y' is

$$t_x^2(-y_x^2 + x(1 + ft_x^4x^2 + gt_x^6x^3)) = 0, \tag{B.34}$$

so the image is in Y'.

$U_{\check{X}}$ **patch** Take

$$B' = tB, \qquad C' = t^2C, \qquad t' = t, \tag{B.35}$$

then

$$-B'^2 + C'(fC'^2 + gC'^3) = t^2(-B^2 + C(ft^4C^2 + gt^6C^3) \tag{B.36}$$

so the image is in Y'.

$U_{t\check{y}}$ **patch** This patch is mapped to $V_{\check{Y}}$. Now take

$$E' = t_{y_t}^2 y_t, \qquad F' = x_{ty_t}t_{y_t}y_t, \qquad t' = t_{y_t}y_t \tag{B.37}$$

Then the defining equation of Y' is

$$t_{y_t}^2 y_t(-1 + t_{y_t}x_{ty_t}^3 y_t^2 + fx_{ty_t}t_{y_t}^3 y_t^2 + gt_{y_t}^4 y_t^2) = 0, \tag{B.38}$$

[3] To derive the equation for $V_{\check{X}'}$, for example, one needs to $\mathbb{Z}_2$ redundancy after setting $X = 1$. By setting $A' = Y'^2$ and B', C' as shown in the table, and by imposing $A'C' = B'^2$ in addition to the Weierstrass equation, we get the shown equation.

so the image is in Y'.

$U_{\check{y}}$ **patch** Take

$$E' = y^2 t_y^3, \qquad F' = x_y y t_y, \qquad t' = t_y y, \tag{B.39}$$

then the defining equation $E' = ...$ is

$$-y^2 t_y^3(-1 + x_y^3 y + f x_y y^3 t_y^4 + g y^4 t_y^6) = 0 \tag{B.40}$$

so the image is in Y'.

Now let us check the consistency. First, note that the map to $V_{\check{X}'}$, $V_{\check{Y}'}$, and $V_{\check{Z}'}$ are consistently defined in terms of x, y, t among the maps that maps to the same patch. Namely, they are defined so that

$$x' = x/t^2, \qquad y' = y/t^3, \qquad t' = t \tag{B.41}$$
$$B' = yt/x^2, \qquad C' = t^2/x, \qquad t' = t \tag{B.42}$$
$$E' = t^3/y, \qquad F' = xt/y, \qquad t' = t. \tag{B.43}$$

Next we check that they are consistent with the isomorphism between any two of the three patches of Y'; one can prove this using the isomorphisms like

$$x' = F'/E', \qquad y' = 1/E', \tag{B.44}$$
$$x' = 1/C', \qquad y' = B'/C'^2... \tag{B.45}$$

Now the consistency of the whole regular map is almost trivial, since the isomorphisms between the U patches can be deduced from the relation between their coordinates and the original (x, y, t) coordinates. The case of $U_{\check{X}}$ is a bit trickier, but still one can check that the regular map is consistent with the isomorphism between $U_{\check{X}}$ and $U_{\check{x}}$.

Regular map to Y' and pullback of the canonical divisor

It is not easy to describe the map $Y_3 \to Y'$ as a sequence of blow-ups, so we take different approach to compute its canonical divisor. Using the ramification divisor $R_{Y'}$ of the regular map $Y_3 \to Y'$, we have the equation

$$K_{Y_3} = K^*_{Y'} + R_{Y'}. \tag{B.46}$$

Let us compute $R_{Y'}$ and see if it is compatible with K_{Y_3}, computed from the sequence of blow-ups of Y, and $K_{Y'} = 0$. To this end, let us note that the above relation holds for the ambient spaces

$$K_{A_3} = K^*_{A'} + R_{A'}. \tag{B.47}$$

Here, the ambient space A' for Y' is defined for each patch, and not necessarily defined globally. Combined with the adjunction formula, we have

$$R_{A'} - R_{Y'} = (K_{A'_3} - K_{Y'_3}) - (K^*_{A'} - K^*_{Y'}) = Y'^* - Y_3 \tag{B.48}$$

when restricted to Y_3. We will compute

$$R_{Y'} = R_{A'} - (Y'^* - Y_3) \tag{B.49}$$

for each patch.

We summarize the regular map, the pullback of Y', the pullback of $K_{Y'}$, and $R_{Y'}$. Note that $R_{A'}$ can be read off from the coefficient of the pullback of the holomorphic 3-form in A'.

$U_{t\check{t}t}$ **patch**

$$x' = x_{tt}, \qquad y' = y_{ttt}, \qquad t' = t. \tag{B.50}$$

$$-y'^2 + x'^3 + fx' + g = -y_{ttt}'^2 + x_{tt}^3 + fx_{tt} + g \tag{B.51}$$

$$dx'dy'dt' = dx_{tt}dy_{ttt}dt, \qquad R_{Y'} = 0 \tag{B.52}$$

$R_{Y'}$ is empty in this patch. Note that this implies that $R_{Y'}$ does not contain C_3, since in this patch the curve is visible.

$U_{t\check{x}x}$ **patch**

$$B' = y_{tx_tx_t}, \qquad C' = t_{x_t}, \qquad t' = t_{x_t}x_t \tag{B.53}$$

$$-B'^2 + C'(fC'^2 + gC'^3) = -y_{tx_tx_t}^2 + t_{x_t}(1 + ft_{x_t}^2 + gt_{x_t}^3) \tag{B.54}$$

$$dB'dC'dt' = t_{x_t}(dx_tdy_{tx_tx_t}dt_{x_t}), \qquad R_{Y'} = (t_{x_t}) \tag{B.55}$$

$R_{Y'}$ is C_1 in this patch. C_3 is not contained, as expected.

$U_{\check{x}}$ **patch**

$$B' = y_xt_x, \qquad C' = t_x^2x, \qquad t' = t_xx \tag{B.56}$$

$$-B'^2 + C'(fC'^2 + gC'^3) = t_x^2(-y_x^2 + x(1 + ft_x^4x^2 + gt_x^6x^3)) \tag{B.57}$$

$$dB'dC'dt' = t_x^3x(dxdy_xdt_x), \qquad R_{Y'} = (t_x^3x) - (t_x^2) = (t_xx) \tag{B.58}$$

$R_{Y'}$ is C_0 and C_1 in this patch.

$U_{\check{X}}$ **patch**

$$B' = tB, \qquad C' = t^2C, \qquad t' = t, \tag{B.59}$$

$$-B'^2 + C'(fC'^2 + gC'^3) = t^2(-B^2 + C(ft^4C^2 + gt^6C^3)) \tag{B.60}$$

$$dB'dC'dt' = t^3x(dBdCdt), \qquad R_{Y'} = (t^3) - (t^2) = (t) \tag{B.61}$$

$R_{Y'}$ is C_0 in this patch.

$U_{t\check{y}}$ **patch**

$$E' = t_{y_t}^2y_t, \qquad F' = x_{ty_t}t_{y_t}y_t, \qquad t' = t_{y_t}y_t \tag{B.62}$$

$$-E' + F'^3 + fF'E'^2 + gE'^3 = t_{y_t}^2y_t(-1 + t_{y_t}x_{ty_t}^3y_t^2 + fx_{ty_t}t_{y_t}^3y_t^2 + gt_{y_t}^4y_t^2), \tag{B.63}$$

$$dE'dF'dt' = t_{y_t}^3y_t^2(dx_{ty_t}dy_tdt_{y_t}), \qquad R_{Y'} = (t_{y_t}^3y_t^2) - (t_{y_t}^2y_t) = (t_{y_t}y_t) \tag{B.64}$$

$R_{Y'}$ is empty in this patch when restricted to Y_3.

$U_{\check{y}}$ **patch**

$$E' = y^2 t_y^3, \qquad F' = x_y y t_y, \qquad t' = t_y y, \tag{B.65}$$

$$-E' + F'^3 + fF'E'^2 + gE'^3 = y^2 t_y^3(-1 + x_y^3 y + f x_y y^3 t_y^4 + g y^4 t_y^6) \tag{B.66}$$

$$dE'dF'dt' = t_y^4 y^3 (dx_y dy dt_y), \qquad R_{Y'} = (t_y^4 y^3) - (t_y^3 y^2) = (t_y y) \tag{B.67}$$

$R_{Y'}$ is $t_y = 0$, i.e. C_0 in this patch, when restricted to Y_3.

Up to here, everything is compatible with $R_{Y'} = C_0 + C_1$, globally.

Map to Y

The same procedure can be done with the map to Y to confirm (B.28).

$U_{t\check{t}t}$ **patch**

$$x = x_{tt} t^2, \qquad y = y_{ttt} t^3, \qquad t = t. \tag{B.68}$$

$$-y^2 + x^3 + ft^4 x + gt^6 = t^6(-y_{ttt}^2 + x_{tt}^3 + f x_{tt} + g) \tag{B.69}$$

$$dxdydt = t^5 dx_{tt} dy_{ttt} dt, \qquad R_Y = (1/t) \tag{B.70}$$

R_Y is $-C_3$ in this patch.

$U_{t\check{x}x}$ **patch**

$$x = x_t^2 t_{x_t}, \qquad y = y_{tx_t x_t} x_t^3 t_{x_t}, \qquad t = t_{x_t} x_t \tag{B.71}$$

$$-y^2 + x^3 + ft^4 x + gt^6 = x_t^6 t_{x_t}^2 (-y_{tx_t x_t}^2 + t_{x_t}(1 + f t_{x_t}^2 + g t_{x_t}^3) \tag{B.72}$$

$$dxdydt = x_t^5 t_{x_t}^2 (dx_t dy_{tx_t x_t} dt_{x_t}), \qquad R_Y = (1/x_t) \tag{B.73}$$

R_Y is $-C_3$ in this patch, and also implies that the coefficient of C_1 in R_Y is 0, since C_1 is visible in this patch.

$U_{\check{x}}$ **patch**

$$x = x, \qquad y = y_x x, \qquad t = t_x x \tag{B.74}$$

$$-y^2 + x^3 + ft^4 x + gt^6 = x^2(-y_x^2 + x(1 + f t_x^4 x^2 + g t_x^6 x^3)) \tag{B.75}$$

$$dxdydt = x^2(dx dy_x dt_x), \qquad R_Y = 0 \tag{B.76}$$

R_Y is empty in this patch, which implies that C_0 and C_1 are not contained in R_Y.

One needs to check the consistency, but these three computations supports that $R_Y = -C_3$.

B.3.2 Canonical divisor

Let us follow how the canonical divisor varies along the rational maps

$$Y^W \to \nu^*(Y^W) \to \overline{\nu^*(Y^W)} \to Y, \tag{B.77}$$

going back to the notation in the main text. Let us denote the discriminant locus of Y^W by its irreducible components D_i,

$$\Delta(Y^W) = 12\sum_i a_i D_i = 12(a_1 D_1 + a_2 D_2) + \widehat{\Delta}(Y^W). \tag{B.78}$$

Here, D_1 and D_2 are the support of the colliding singularities we are concerned.

Let us assume that the canonical divisor $K(Y^W)$ of Y^W vanishes, $K(Y^W) = 0$. Then we have[4]

$$K(B_w) = -\sum_i a_i D_i. \tag{B.79}$$

Moving on to $\nu^*(Y^W)$, we have

$$K(\nu^*(Y^W)) = \pi^*(K(B_3) + \sum_i a_i' D_i') = \pi^*(K^*(B_w) + E + aE + \sum_i a_i \widehat{D_i}), \tag{B.80}$$

where π always denotes the elliptic fibration morphism, $\widehat{D_i}$ is the irreducible component of D_i^* that does not contain E and $a_i D_i'$ denotes the irreducible components of $\Delta(\nu^*(Y^W))$. On the other hand, we have

$$K^*(Y^W) = \pi^*(K^*(B_w) + \sum_i a_i D_i^*), \tag{B.81}$$

so

$$K(\nu^*(Y^W)) = K^*(Y^W) + \pi^*((a+1)E - (a_1 + a_2)E) = \pi^*(E). \tag{B.82}$$

We used $a = a_1 + a_2$.

After the blow-up, if f, g of the Weierstrass equation is of the form

$$f(t, s_t) = t^4 f'(t, s_t), \qquad g(t, s_t) = t^6 g'(t, s_t), \tag{B.83}$$

in the patch spanned by (t, s_t) (or similarly in that of (t_s, s)), then by following the sequence of rational maps $\nu^*(Y^W) \to \overline{\nu^*(Y^W)} \to Y$, one can finally see that

$$K^*(Y) = K(\overline{\nu^*(Y^W)}) - C_0 - C_1 = \pi^*(E) - C_3 - C_0 - C_1 = 0, \tag{B.84}$$

i.e. Y is Calabi-Yau. Note that C_i are divisors in $\overline{\nu^*(Y^W)}$.

In the case of $\mathrm{I}_0^* \times \mathrm{I}_0^*$, $\mathrm{I}_0^* \times \mathrm{I}_n^*$ and $\mathrm{I}_0^* \times \mathrm{IV}^*$ collisions, this procedure yields Y, which is locally defined as the Weierstrass model

$$y^2 = x^3 + f'(t, s_t)x + g'(t, s_t), \tag{B.85}$$

which has a flat and crepant resolution, including at the new collision point $t = s_t = 0$.

[4]See, for example, [MV96a]. We will ignore the "error term" entirely.

In the case of $\mathrm{I}_0^* \times \mathrm{III}^*$, one needs to repeat the procedure since the new geometry Y has the collision $\mathrm{III} \times \mathrm{III}^*$, but after repeating the procedure once, one gets a Weierstrass model over a blow-up of B_3, which has a flat resolution.

The case of $\mathrm{I}_0^* \times \mathrm{II}^*$ is more subtle; it is said that by repeating the procedure, one gets a non-crepant resolution. This is because if one blows-up the $\mathrm{II} \times \mathrm{IV}$ collision, the factorization $f = f't^4$ and $g = g't^6$ is not possible over the exceptional locus, and thus the procedure is non-crepant. It is not clear how one can see that the collision has no flat fibration without changing the base.

Appendix C

Type IIB orientifold case

As a special case of the analysis of supersymmetric flux configurations for M/F-theory in Section 6, the case of Type IIB orientifold compactification on a Borcea-Voisin threefold $M = (E_\tau \times X^{(2)})/\mathbb{Z}_2$ is covered (see (5.19)); $X^{(1)} = \mathrm{Km}(E_\phi \times E_\tau)$ corresponds to a choice of $(S_0^{(1)}, T_0^{(1)}, \sigma_{(1)})$ from [Nik81] where $T_0^{(1)} = U[2]U[2]$, $r_{(1)} = 18$, $a_{(1)} = 4$ and $g_{(1)} = 0$. The conditions (6.37, 6.38) for the case of $X^{(1)} = \mathrm{Km}(E_\phi \times E_\tau)$ should therefore be equivalent[1] to the conditions worked out in [KW17a]. The two sets of conditions do not look similar at first sight (as reviewed below), but we confirm in the following that they are equivalent indeed. This appendix can thus be regarded as a supplementary note to [KW17a]; consistency check in this appendix also gives confidence in the study in Section 6 in this thesis.

Let us start off by recalling the Type IIB conditions in [KW17a] for a non-trivial supersymmetric flux. $K^{(2)}$ and K_E are the endomorphism fields of the CM-type Hodge structure on $T_X^{(2)}$ and $H^1(E_\tau;\mathbb{Q})$, respectively. $n := \mathrm{rank}(T_X^{(2)})$.

When the untwisted sector $T_X^{(2)} \otimes_\mathbb{Q} H^1(E_\tau;\mathbb{Q})$ is itself a simple component of the rational Hodge structure,[2] it is level-3 and $K^{(2)} \otimes_\mathbb{Q} K_E$ is the endomorphism field. A non-trivial $DW = 0$ flux exists if and only if

$$\left(K^{(2)} \otimes_\mathbb{Q} K_E\right)^r \cong \mathbb{Q}(\phi), \qquad [\mathbb{Q}(\phi) : \mathbb{Q}] = 2. \tag{C.1}$$

The half set[3]

$$\Phi = \left\{\rho^{(2)}_{(20)} \otimes \rho^\tau_{(10)},\ \rho^{(2)}_{a=1,\cdots,n-2} \otimes \rho^\tau_{(01)},\ \rho^{(2)}_{(02)} \otimes \rho^\tau_{(10)}\right\} \tag{C.2}$$

of all the $2n$ embeddings $K^{(2)} \otimes K_E \to \overline{\mathbb{Q}}$ is used in determining the reflex field.[4]

When $T_X^{(2)} \otimes_\mathbb{Q} H^1(E_\tau;\mathbb{Q})$ is not a simple component, instead, $K^{(2)}$ has a structure of $K_0\mathbb{Q}(\xi_S)$ for its totally real sub-field K_0 and an imaginary quadratic field $\mathbb{Q}(\xi_S)$

[1] In Section 3.3 of [KW17a], the authors worked out orientifold projection on the moduli of the threefold M, and found that the twisted sector moduli of the complex structure of M are projected out. In this work, the absence of such moduli is understood as the absence of the $H^1(Z^{(1)};\mathbb{Q}) \otimes H^1(Z^{(2)};\mathbb{Q})$ component; it is of $g_{(1)}g_{(2)} = 0$ dimension.

[2] Then there is no chance for a non-trivial flux with $W = 0$.

[3] Recall that we always consider the reflex field in the sense of Weil intermediate Jacobian, i.e. the Jacobian $J_W(M)$ associated with $H^{0,3}(M) \oplus H^{2,1}(M)$.

[4] Note that we started out in F/M-theory analysis in Section 6 in this thesis by assuming that $X^{(1)} = \mathrm{Km}(E_\phi \times E_\tau)$ is of CM type (that both E_τ and E_ϕ are CM elliptic curves). In the analysis of [KW17a], however, the CM nature of E_ϕ, namely $[\mathbb{Q}(\phi) : \mathbb{Q}] = 2$, follows from the CM nature of $X^{(2)}$ and E_τ and the supersymmetry conditions on a non-trivial flux.

isomorphic to K_E, and $T_X^{(2)} \otimes_{\mathbb{Q}} H^1(E_\tau;\mathbb{Q})$ has a structure $K^{(2)} \oplus K^{(2)}$ under the action of the algebra $K^{(2)} \otimes_{\mathbb{Q}} K_E$ (K_E acts through an isomorphisms $\mathbb{Q}(\zeta_S) \cong K_E$). For a non-trivial $DW = 0$ flux to exist, it is necessary and sufficient that

$$(K^{(2)})^r \cong \mathbb{Q}(\phi) \cong \mathbb{Q}(\tau). \tag{C.3}$$

A few more words are necessary for this condition to have a clear meaning. Let $\theta_{a=1,\cdots,n/2}$ be the embeddings $K_0 \to \overline{\mathbb{Q}}$, and $\theta_a^{\pm}$ those of $K^{(2)}$ so that their restriction on K_0 are θ_a, and $\theta_a^+(\zeta_S)$ [resp. $\theta_a^-(\zeta_S)$] is in the upper [resp. lower] complex half plane. The reflex field $(K^{(2)})^r$ in the condition (C.3) should be for the half set[5]

$$\left\{\theta_{a=1}^+,\ \theta_{a=2,\cdots,n/2}^-\right\}. \tag{C.4}$$

The case $T_X^{(2)} \otimes H^1(E_\tau;\mathbb{Q})$ is simple: Now, we begin with making the condition (C.1) more explicit. To this end, a set of notations is introduced in order to capture the structure of the fields $K^{(2)}$ and K_E. As a general property of CM fields, $K^{(2)}$ has a structure of $K_0(\underline{x})$ where K_0 is the totally real sub-field of $K^{(2)}$, and $\underline{x}$ an element of $K^{(2)}$ with the following properties: $\underline{x}^2 \in K_0$, and the element $Q := -\underline{x}^2$ in K_0 is mapped onto the real positive axis by all the $[K_0 : \mathbb{Q}] = n/2$ embeddings $K_0 \to \overline{\mathbb{Q}}$. Similarly, $K_E = \mathbb{Q}(\underline{\tau})$ for some $\underline{\tau} \in K_E$ such that $p := -\underline{\tau}^2 \in \mathbb{Q}_{>0}$. The vector space $K^{(2)} \otimes_{\mathbb{Q}} K_E$ is regarded 4-dimensional over K_0 generated by $\{1, \underline{x}, \underline{\tau}, \underline{x\tau}\}$; the totally real sub-field of $K^{(2)} \otimes K_E$—denoted by K_0^{tot}—is 2-dimensional over K_0 generated by $\{1, \underline{x\tau}\}$.

The condition that the reflex field in (C.1) is an imaginary quadratic extension of $\mathbb{Q}$ is equivalent to existence of $\underline{\eta} \in K^{(2)} \otimes K_E$ such that its images by the n embeddings in Φ are all identical $\eta \in \overline{\mathbb{Q}}$ which generates an imaginary quadratic field $\mathbb{Q}(\eta)$. For

$$\underline{\eta} = A + B\underline{x} + C\underline{\tau} + D\underline{x\tau} \in K^{(2)} \otimes K_E, \qquad A, B, C, D \in K_0, \tag{C.5}$$

the condition $\underline{\eta}^2 \in \mathbb{Q}$ is equivalent to

$$AC = QBD, \quad AB = pCD, \quad AD + BC = 0, \quad (A^2 - QB^2 - pC^2 + pQD^2) \in \mathbb{Q}. \tag{C.6}$$

This leaves five distinct possibilities: i) none of A, B, C, D is zero, ii-A) $A \neq 0$, and $B = C = D = 0$, ii-B) $B \neq 0$ and three others are zero, ii-C) $C \neq 0$ and three others are zero, and ii-D) $D \neq 0$ and three others are zero.

In fact, only the possibility ii-C) is viable. The possibility i) runs into a contradiction: (B/D) is a well-defined element of the totally real field K_0 in this possibility, and yet one can derive that $(B/D)^2 = -p \in \mathbb{Q}_{<0}$. In the possibilities ii-A) and ii-D), the element $\underline{\eta} = A$ or $\underline{\eta} = D\underline{x\tau}$ would not generate a totally imaginary extension over K_0^{tot}. The possibility ii-B) cannot be consistent with the condition that the images of $\underline{\eta} = B\underline{x}$ under the n embeddings in Φ should be all identical; $\rho^{(2)}_{(20)}(B\underline{x}) = -\rho^{(2)}_{(02)}(B\underline{x}) \neq 0$.

[5]It was not clearly stated in [KW17a] which half set of the n embeddings of $K^{(2)}$ should be used in determining the reflex field in (C.3).

Let us focus on the remaining ii-C) possibility. The condition (C.6) implies that $C^2 \in \mathbb{Q}$. There are two cases, (*1) $C_{\neq 0} \in \mathbb{Q}$, and (*2) $C \notin \mathbb{Q}$ whose square is a positive rational number $r \in \mathbb{Q}_{>0}$ that is not a square.

In the case (*1), the condition that all the n images of $\underline{\eta} = C\underline{\tau}$ are identical is satisfied if and only if $n = 2$; if $n > 2$, then $\rho^{(2)}_{a>2}(\underline{\eta}) = C\rho^{\tau}_{(01)}(\underline{\tau})$ cannot be the same as the images $\rho^{(2)}_{(20)}(C)\rho^{\tau}_{(10)}(\underline{\tau})$ and $\rho^{(2)}_{(02)}(C)\rho^{\tau}_{(10)}(\underline{\tau})$. Therefore, $K^{(2)} = \mathbb{Q}(\underline{x})$ must be some imaginary quadratic field, and the reflex field $(K^{(2)} \otimes K_E)^r$ must be $\mathbb{Q}(\sqrt{-p}) \cong K_E$. It follows that E_ϕ also has the endomorphism field $\mathbb{Q}(\sqrt{-p})$, E_τ and E_ϕ are isogenous (and are both CM), and $X^{(1)} = \mathrm{Km}(E_\phi \times E_\tau)$ has a rank-20 Néron-Severi lattice. Therefore, to conclude, the case (*1) solution to the condition (C.6) implies that $T^{(1)}_X \subsetneq T^{(1)}_0$, $K^{(1)} \cong \mathbb{Q}(\phi) \cong K_E \cong \mathbb{Q}(\sqrt{-p})$, $K^{(2)}$ is an imaginary quadratic field (and is not isomorphic to $\mathbb{Q}(\sqrt{-p})$ as assumed before (C.1)), and the condition (6.37) is satisfied; both $\rho^{(1)}_{(20)}(K^{(1)}_0) = \rho^{(2)}_{(20)}(K^{(2)}_0) = \mathbb{Q}$.

In the case (*2), the totally real field K_0 must be a real quadratic field. To see this, note that K_0 contains $\mathbb{Q}(C) \cong \mathbb{Q}(\sqrt{r})$, which means that $n/2 \geq 2$. The condition that all the n images of $\underline{\eta} = C\underline{\tau}$ should be the same now implies that $\rho^{(2)}_{a>2}(C) = -\rho^{(2)}_{(20)}(C)$. Because the $n/2$ embeddings of K_0 should yield the same number of two different embeddings of the sub-field $\mathbb{Q}(C)$, $(n-2)/2$ must be equal to $2/2$; $n = 4$. Therefore, $K^{(2)} \cong \mathbb{Q}(\underline{x}, C)$, its totally real sub-field must be $K^{(2)}_0 \cong \mathbb{Q}(C)$, and $(K^{(2)} \otimes K_E)^r \cong \mathbb{Q}(\sqrt{-pr})$. Now, the remaining condition in (C.1) is $\mathbb{Q}(\phi) \cong \mathbb{Q}(\sqrt{-pr})$. Therefore, it turns out that $K^{(1)} \cong \mathbb{Q}(\sqrt{-p}, \sqrt{-pr})$, and $K^{(1)}_0 \cong \mathbb{Q}(\sqrt{r})$. Thus, to summarize, the case (*2) solution to the condition (C.1) implies that $T^{(1)}_X = T^{(1)}_0$, $K^{(1)}_0 \cong K^{(2)}_0 \cong \mathbb{Q}(\sqrt{r})$, and hence the condition (6.37), in particular.

The case $T^{(2)}_X \otimes_{\mathbb{Q}} H^1(E_\tau; \mathbb{Q})$ is not simple: Let us now turn to the case $T^{(2)}_X \otimes_{\mathbb{Q}} H^1(E_\tau; \mathbb{Q})$ is not itself a simple component of the rational Hodge structure. The condition that the reflex field $(K^{(2)})^r$ *with respect to the half set* (C.4) should be imaginary and quadratic implies in fact that $n/2 = 1$, so $K^{(2)}$ also needs to be an imaginary quadratic field. To conclude, the condition (C.4) implies $K^{(1)} \cong \mathbb{Q}(\phi) \cong K_E$, $T^{(1)}_X \subsetneq T^{(1)}_0$, and $K^{(2)}$ is also isomorphic to $\mathbb{Q}(\phi)$; the condition (6.38) is satisfied.

To wrap up, here is what we learned in this appendix, stated in a colloquial language. Although it is not apparent from the Type IIB conditions (C.1, C.4) in [KW17a], only small classes of CM fields $K^{(2)}$ can satisfy either one of those conditions; the analysis in this appendix left $[K^{(2)} : \mathbb{Q}] = 2, 4$ as the only possibilities, in particular. The M/F-theory condition (6.37, 6.38) in the main text of this thesis also imply $[K^{(2)} : \mathbb{Q}] = 2, 4$, because the CM field $K^{(1)}$ for $X^{(1)} = \mathrm{Km}(E_\tau \times E_\phi)$ can only be degree-4 or degree-2 extension over $\mathbb{Q}$. Thus both perspectives led us to the same result.

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