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Some Aspects of Boundary and Defect Conformal Field Theories

(境界または欠損付き共形場理論の諸相)

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Abstract

In this thesis, we explore various theoretical aspects of conformal field theories with boundaries and defects to better classify and understand these theories. We would like to compute conformal anomaly coefficients, the sphere free energy, and the entanglement entropy, all of which contains considerable dynamical information of theories.

In the former half of the thesis, we specifically consider the scalar $O(N)$ symmetric field theory with $(\phi^2)^3$ interactions in $\mathbb{R}^2 \times \mathbb{R}^+$. This theory exhibits an approximate conformal symmetry in $N \to \infty$ limit where we can do explicit calculations. We first study possible phases associated with boundary conditions and argue their relative stabilities. We then compute stress tensors correlation function in Dirichlet boundary condition, further decomposing it into bulk and boundary conformal blocks to gain the operator spectrum underlying this theory. We finally elucidate boundary conformal anomaly coefficients. We find all of these quantities depend on a quasi-marginal coupling.

In the latter half of the thesis, we investigate the sphere free energy and the entanglement entropy of conformal field theories in the presence of the boundary or the defect, since these two numbers can be means to count effective degree of freedom under RG flows. Establishing the universal relation between them, we find the sphere free energy and the entanglement entropy are equivalent with a suitable ultraviolet regularization in the case of codimension-1 defects and boundaries, however, they differ due to the contribution from the defect when we consider higher codimensional defects. We then propose the monotonicity theorem (termed C-theorem) stating that the sphere free energy, not the entanglement entropy, should monotonically decrease under any RG flows. This proposal can unify all known C-theorems in the literature and also provides the new series of them in general dimensions of the spacetime and defects. We confirm that our proposal holds in various models. In some holographic models, we are able to prove our conjecture with the assumption of the null energy condition.

Contents

Chapter 1

Introduction

1.1 Space of quantum field theories

A quantum field theory (QFT) is a powerful mean to study a number of aspects in physics. One of the successful triumphs is the well known standard model in particle physics, but it's been widely used to describe other interesting systems from in cosmology to in condensed matter physics. Despite the fact that QFTs have a lot of applications and successfully find new phenomena, we know next to nothing about its definition. Instead, we have a collection of facts that provide us how QFTs should look like.

Among them is the renormalization group (RG) flow as one of the characteristics in QFTs, meaning that every QFT is defined at some energy or length scales, and flow into an effective theory by coarse graining the microscopic degree of freedom which have higher scales than that we are interested in [1, 2]. It is equivalent to say that varying the energy scale causes a QFT to change into another QFT. In the long distance limit, or equivalently low energy limit, the theory exhibits scale invariance in general [1]. It is also widely expected that such scale invariant QFT becomes conformal invariant rather than scale invariant [3] under the assumptions of unitarity and Poincaré invariance. This class of field theories is called *conformal field theories* (CFTs). As this emergent symmetry requires no additional information, CFTs appear in many physical systems. For instance, critical phenomena in statistical mechanics are often described as CFTs because at a second-order phase transition the correlation length becomes infinity, which is of course much bigger than the typical scale of the system. A CFT does not necessarily have microscopic description, since such information is coarse grained along the RG flows. Because of this, it is almost always the case that the same CFT can describe different physical systems, which gives rise to a notion of *universality*. As a specific example, let us consider three-dimensional Ising model defined by the following hamiltonian,

$$\mathcal{H} = -\sum_{<ij>} J s_i s_j \ , \tag{1.1}$$

where $s_i = \pm 1$ is a spin variable defined at each site and $< ij >$ means site i and site j are the nearest neighbors. Remarkably, for a special value of the coupling J, in the long distance limit the Ising model is equivalent to the scalar field theory with ϕ^4 interaction with the following

[1] There are in principle two other possibilities: One is a theory with a mass gap, e.g. Yang-Mills theory. The other is a theory with massless particles, e.g. QED. We will focus on scale invariant theories in this thesis.

lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + m^2\phi^2 + g\phi^4 \ . \tag{1.2}$$

In addition, the same CFT arise at the vapour-liquid transition in the water. Hence these examples show that CFT appears in many different contexts and is of very importance in QFTs.

Getting back to RG flows, since the set of coupling constants of a QFT is also affected by such renormalization transformations, we can regard the RG flow as a trajectory in the QFT parameter space where the coordinate is characterized by a set of coupling constants. Looking into such a parameter space, we may realize there are special points characterized by invariant points under scale variance, namely fixed points where CFTs can live. As for long distance limit we mentioned earlier, it corresponds to the infrared (IR) fixed point. Then in the RG flow we can begin with a certain ultraviolet (UV)-complete QFT. Mostly, we consider a CFT at the UV fixed point as a starting point[2]. We can trigger the RG flow by perturbating the UV CFT, which cause the theory to end up with some other CFT at the IR fixed point. In general, it is possible to have multiple RG flows depending on which kind of perturbations we consider, meaning that the UV CFT can flow into several IR CFTs (see fig 1.1). We want to ask whether we have to consider all the possible RG flows, or whether there is a way to prevent some of RG flows from existing. Finding such a constraint enables us to carve out some domains of space of QFTs, providing us a part of ways how to classify them. The idea to make this statement concrete comes from the intuition that along RG flows the effective degree of freedom must be decreasing because we integrate out massive degrees of freedom once the energy scale of RG flow becomes below the scale set by their masses.

In 1986, Zamolodchikov established a very well known theorem [4] stating that in two-dimensional QFTs there is a function, which depends on coupling constants and the energy scale, such that it monotonically decrease under any RG flows. Such a function is broadly named a C-function. Similarly, the corresponding theorem is named a C-theorem. In two dimensions, C-function is constructed from the two-point function of stress tensors $T_{\mu\nu}$, and its monotonicity follows from positivity conditions of the Hilbert space. At fixed points, this C-function corresponds to central charges of associated two-dimensional CFTs. After this celebrated discovery, many researchers have attempted to extend his theorem into other spacetime dimensions [5–10]. A physical guiding principle to construct such C-functions is that the anomalous trace of stress tensors characterizes CFTs in even dimensions. In a flat space, the trace of the stress tensor vanishes; however, putting a CFT on a curved background causes nonzero trace of the stress tensor quantum mechanically. Actually in two dimensions, the coefficient of the stress tensor trace corresponds to the central charge. Similarly, in four dimensions there are two terms in the trace and one of them serves as a C-function [6, 7, 11]. On the other hand, as there is no such an anomaly in odd dimensions, the universal term of the sphere free energy was proposed as a C-function [8, 9].

Another candidate for C-functions is the entanglement entropy, which measures the quantum entanglement between two subsystems where we have to specify. The entanglement entropy provides not only an alternative proof of Zamolodchikov's theorem [12] but also the nonperturbative proof of C-theorem in three dimensions [13]. Even when C-functions can be well established only at the endpoints in RG flows, these functions offer a promising way to better understand the space of QFTs.

[2]Of course it does not need to be CFT, but we assume it as a useful framework to study.

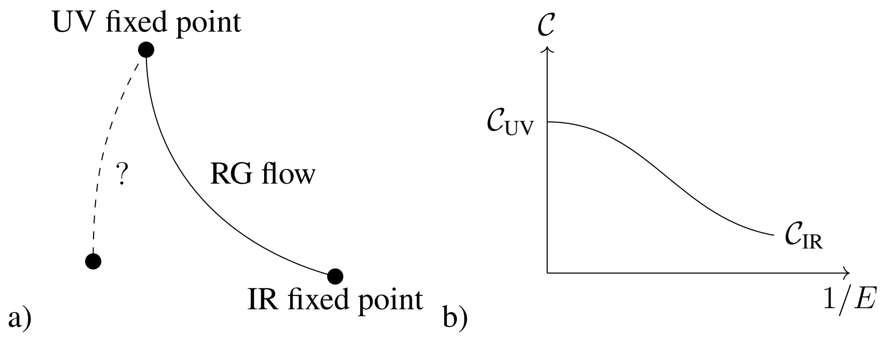

Figure 1.1: a)Schematic picture of RG flow. The UV CFT can flow into several IR theories. b) The idea of monotonically decreasing function along the RG flow. The degree of freedom should be the case.

1.2 QFT meets boundary and defect

A part of the reason why it is hard to understand the definition of QFT comes from a fact that we haven't been fully aware of the roles of boundaries and defects in QFTs. Defects are in general extended objects, such as lines, surfaces, branes inserted in the spacetime. Let us first realize boundaries play central roles in many aspects of physics. In experimental physics, a boundary allows us to model finite size effects of a sample. In theoretical physics, it offers a number of applications. In string theory, D-branes are described by two-dimensional conformal field theories with boundary [14]. In particular, boundary conditions on such theories describe associated D-branes, from which we may classify possible D-brane configurations. Furthermore, string theory provides a striking hypothesis, namely holographic duality stating that $(d+1)$-dimensional quantum theories of gravity should be equivalent to d-dimensional quantum field theories which live on the boundary of $(d+1)$-dimensional gravity [15]. A concrete example of holographic duality is the AdS/CFT correspondence where $(d+1)$-dimensional quantum theories of gravity in Anti de Sitter spacetime should be equivalent to d-dimensional CFT, which strongly motives us to study CFT in order to understand quantum gravity. As for condensed matter physics, we encounter topological insulators where Weyl fermions live on the boundary of metals. In particular, in recent developments about 't Hooft anomaly the boundary plays a cornerstone role.

As for defects, condensed matter physicists do not need convincing how defects are of importance: there are impurities, domain walls, vortices... For high energy physicists, significant examples are Wilson line or 't Hooft line operators in gauge theories. From their expectation values, we can extract the information which we do not access without such objects. In case of Wilson line operators, we can diagnose whether the gauge theory is confined or deconfined. The entanglement entropy in QFT is another essential application of defects, in which we have to specify entangling regions to measure the entanglement between in and out, causing the introduction of boundary conditions.

So boundaries and defects are ubiquitous in QFTs, and can be regarded as probes to classify them, but how can these topics be trivial once we fully understand the roles of boundaries and defects? This question may sound subtle, so we make it clear as follows: What characterizes QFTs with boundary and defect in order to classify them? To answer such a question, this thesis is dedicated to exploring theoretical aspects of quantum field theories with boundaries and defects. In particular, we are interested in fixed point theories in RG flows, where QFTs become

CFTs. In the presence of boundaries or defects, such fixed point CFTs are called *boundary conformal field theories* (BCFTs) or *defect conformal field theories* (DCFTs).

As for BCFTs, while it provides us ways of classification of D-branes in two dimensions, it is also used to describe critical phenomena in the presence of a boundary, named surface critical phenomena in general dimensions. When a boundary exists, the relevant coupling on the boundary causes boundary RG flows, resulting in the rich phase structure compare to only bulk interactions. These boundary relevant couplings also control the boundary conditions associated with corresponding phase transitions. In CFT point of view, uncovering boundary conditions can be seen as the way of classifying BCFTs. Mostly studied cases of surface critical models are scalar $O(N)$ field theory with a quartic interaction in the bulk when we consider ϵ expansion in $d = 4 - \epsilon$ or large N limit. They successfully estimated surface critical exponents and matched the experimental data of some materials. The simple variant of this quartic theory is a scalar field theory with ϕ^6 interaction in three dimensions, which can be described tricritical fixed points that appear in various physical systems, such as ^{3}He-^{4}He mixture [16], NH_4Cl [17] and polymer physics [18, 19]. Despite the variety of its application, the analysis of ϕ^6 theory with a boundary was less investigated and, to our knowledge, there is no literature about large N limit in the presence of a boundary. It is the first goal of this thesis that revealing the large N structure of the tricritical theory with the boundary, which provides us the controllable BCFT and allows to extract its dynamical information.

The next topic that we would like to study is the more broad dynamics of BCFTs and DCFTs under RG flows, though there are several difficulties in studying these theories. One is that CFT itself is in general strongly coupled: we may have no Feynman diagram calculations, or no lagrangian description. Another is that apart from free field theories it is not easy to find tractable examples of interacting BCFTs and DCFTs in higher dimensions than $d = 2$ as interactions may violate such superior properties of them. Therefore, instead of studying particular theories, we explore the very quantities to classify RG flows of BCFTs and DCFTs in general settings. Given Zamolodchikov's celebrated theorem and its extensions, one way to characterize RG flows of BCFTs and DCFTs is again to find monotonically decreasing functions under RG flows. Here we mean RG flows of BCFTs and DCFTs by the RG flows triggered by perturbations localized on boundaries and defects. Compared to the case without boundaries or defects, it is less explored to establish such monotonicity theorems in BCFTs and DCFTs [20–27]. In analogous to CFTs, there can be two types of quantities, which contain dynamical information about these RG flows: One is an anomalous trace of the stress tensor in even dimensions, or a sphere free energy in odd dimensions. The other is the entanglement entropy across the spherical entangling region. While they are equivalent to each other [28] in the absence of boundaries or defects, it is less clear if it still holds in the case of BCFTs and DCFTs. Our task is to make it clear what is the relation between the sphere free energy and the entanglement entropy as well as to find C-functions in BCFTs and DCFTs.

1.3 Organization and summary of the thesis

The main body of this thesis begins with the introductory review of (Euclidean) CFTs in $d > 2$ dimensions in chapter 2. After collecting basic properties of CFTs, such as radial quantization, state/operator correspondence, reflection positivity, and operator product expansion, we then briefly argue the implication of introducing boundaries and defects in the spacetime of CFTs.

In chapter 3, we next study the specific example of BCFTs. The model we study is the very simple one: $O(N)$ symmetric scalar field theory with the sextic interaction in three dimensions with a planar boundary. The key feature of this theory is that it exhibits approximate conformal symmetry at large N limit, which allows us to compute physical quantities analytically as a function of a coupling constant. We compute the large N effective potential and analyze possible phase structures of this theory, depending on which boundary condition we take. We then compute the two-point function of the stress tensors. Having these results, we try to elucidate the anomalous trace coefficients of the stress tensor. In our model, these numbers can be easily calculated from the effective potential and the stress tensor two-point function. We find they rely on a quasi-marginal coupling. We also see that our model can be a counterexample of a pair of conjectures proposed in [29, 30].

Chapter 4 is to study dynamical behaviors of BCFTs and DCFTs under RG flows. To make it precise, we explore the sphere free energy and the entanglement entropy in BCFTs and DCFTs as possible means to measure the physical degree of freedom. Using a specific conformal transformation, we can map the entanglement entropy into the free energy, then find universal relations between these two quantities. With this formula in our hands , we propose the monotonicity theorem valid in BCFTs and DCFTs, stating that the sphere free energy, not the entanglement entropy, should serve as a C-function. Various examples pass our conjecture and using holographic duality we also present its proof in some models. Among these examples we find the very models that the entanglement entropy does not monotonically decrease in, while in all the examples the sphere free energy does decrease. The thesis ends with some closing remarks and interesting open questions.

Chapter 3 is mainly based on the paper [31] in a collaboration of the author with Christopher Herzog. Chapter 4 is also based on the work [32] in a collaboration of the author with Tatsuma Nishioka, Yoshiki Sato, and Kento Watanabe.

1.4 Terminology and notation

Before going to the main body, We summarize our terminology and notation used in this thesis.

- A bulk space where the CFT lives is d-dimensional and is labeled by the Greek letters $\mu, \nu, \cdots$.

- A defect is p-dimensional, whose worldvolume coordinates are labeled by the Roman letters $a, b, \cdots$. The same labeling applies for boundary transverse coordinates. The quantities on the defect or the boundary are hatted to distinguish from the ambient ones. For example, a scalar operator localized on the defect is denoted by $\hat{\mathcal{O}}(\hat{x})$.

- The transverse directions to the defect are labeled by the Roman indices $i, j, \cdots$, while for BCFT we use n which stands for the normal direction.

- A bulk space holographically dual to DCFT is $(d+1)$-dimensional whose coordinates are labeled by the capital Roman letters $M, N, \cdots$.

- In some holographic models, the defect is introduced by a brane in the bulk. The coordinates on the branes are labeled by the capital Roman letters $A, B, \cdots$.

Coordinate For BCFTs, we use $x^\mu = (\boldsymbol{x}, z)$where

- x^μ : the bulk space coordinate,
- $\boldsymbol{x}$: the boundary tangential coordinate,
- z : the coordinate transverse to the boundary.

Similarly, in DCFTs we split the coordinates $x^\mu = (\hat{x}^a, x_\perp^i)$ where

- x^μ : the bulk space coordinate,
- $\hat{x}^a$: the defect worldvolume coordinate,
- $x_\perp^i$: the coordinates transverse to the defect.

In the holography side, we use

- X^M : the bulk (AdS) space coordinates,
- η^A : the brane coordinates.

Metric To distinguish the metrics in the bulk, defect worldvolume, bulk and brane worldvolume coordinates, we use

- $g_{\mu\nu}$: the bulk space metric,
- $\hat{g}_{ab} = \frac{\partial x^\mu}{\partial x^a}\frac{\partial \hat{x}^\nu}{\partial \hat{x}^b} g_{\mu\nu}$: the defect worldvolume metric (the induced metric on the defect), same goes to boundary induced metric.
- G_{MN} : the bulk (AdS) space metric,
- $\hat{G}_{AB}$: the induced metric on the brane in holographic models

Chapter 2

Basics of Conformal Field Theories and its Extension

In this chapter, we briefly review the basic properties of conformal field theories in $d > 2$ dimension. Our discussion mainly focuses on Euclidean signature. We follow several nice references [33, 34]. First of all, we introduce conformal symmetry as a spacetime symmetry and construct the associated algebra it forms. Next, we argue its representations in terms of local operators. Having these in mind, conformal symmetry enables us to exploit special features in CFTs and further to constrain correlation functions as we show in section 2.4, 2.5, and 2.6. We then present the implication of boundaries in section 2.7. Section 2.8 explains significant objects in CFTs, namely conformal anomalies. We also examine such anomalies in the presence of a boundary. Finally in section 2.9 we introduce extended objects, defects in spacetime of CFTs.

2.1 Conformal Transformation and its algebra

Let us consider a d-dimensional Euclidean space $\mathbb{R}^d$ with a metric $g_{\mu\nu} = \delta_{\mu\nu}$. Conformal transformation is defined as a coordinate transformation $x^\mu \to x'^\mu$ such that

$$g'_{\mu\nu}(x') = \frac{x'^\alpha}{x^\mu}\frac{x'^\beta}{x^\nu} g_{\alpha\beta}(x) = \Omega(x) g_{\mu\nu}(x), \tag{2.1}$$

which preserves the metric up to scaling factor $\Omega(x)$.

The set of conformal transformations forms a group. To see this, we consider the infinitesimal conformal transformation with a vector field $\epsilon = \epsilon^\mu(x)\partial_\mu$,

$$x'^\mu = x^\mu + \epsilon^\mu(x), \tag{2.2}$$

where $|\epsilon(x)| \ll 1$. In order to satisfy (2.1), $\epsilon^\mu(x)$ must obey the following equation,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = c(x)\delta_{\mu\nu} \tag{2.3}$$

with $c(x) = 2(\partial\cdot\epsilon)/d$. It turns out that in $d > 2$ above equation has only four classes of solutions and each corresponds to associated the generator of the group:

$$\begin{aligned}
\text{translation: } \epsilon^\mu &= a^\mu, & &\to P_\mu, & (2.4)\\
\text{rotation: } \epsilon^\mu &= \omega^{\mu\nu} x_\nu, & &\to M_{\mu\nu}, & (2.5)\\
\text{scale transform (dilatation): } \epsilon^\mu &= \lambda x^\mu, & &\to D, & (2.6)\\
\text{special conformal transform: } \epsilon^\mu &= x^2 b^\mu - 2x^\alpha b_\alpha x^\mu, & &\to K_\mu. & (2.7)
\end{aligned}$$

Since rotation and dilatation manifestly preserve an angle between vectors, it motivates us to call the transformation (2.1) "conformal", whose meaning is preserving angles.

We are now able to show that the generators in (2.4) satisfy the following commutation relations:

$$\begin{aligned}[M_{\mu\nu}, M_{\rho\sigma}] &= -\mathrm{i}(\delta_{\mu\rho}M_{\nu\sigma} \pm \text{permutations}) \\ [M_{\mu\nu}, P_\rho] &= \mathrm{i}(\delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu)\end{aligned} \tag{2.8}$$

$$\begin{aligned}[D, P_\mu] &= -\mathrm{i}P_\mu, \\ [D, K_\mu] &= \mathrm{i}K_\mu, \\ [P_\mu,\ K_\nu] &= 2i(\delta_{\mu\nu}D - 2M_{\mu\nu}).\end{aligned} \tag{2.9}$$

All other commutators vanish. (2.8) represents usual Poincaré algebra. We find here new relations (2.9). The net commutation relations form the conformal group, which is isomorphic to $SO(d+1,1)$ that we can show. We define a conformal field theory as a quantum field theory which is classically invariant under the conformal group.

2.2 Representations of conformal group

The aim of this section is to classify local operators into representations of the conformal group. As usual, an operator $\mathcal{O}(x)$ transforms under an infinitesimal conformal transformation for $U = e^{\mathrm{i}\sum gT_g}$ as follows,

$$\mathcal{O}(x') = U\mathcal{O}(x)U^{-1} \sim \mathcal{O}(x) + \mathrm{i}\sum g[T_g, \mathcal{O}(x)] + O(g^2), \tag{2.10}$$

where g is a parameter and T_g represents the generator of the conformal group. We would like to derive the form of $[T_g, \mathcal{O}(x)]$. Since $x = 0$ is invariant under dilatation, rotation, and special conformal transformation, it is good for us to start with these actions on an operator at the origin.

First, we define transformation of local operators at the origin in irreducible representations of rotation group $SO(d)$,

$$[M_{\mu\nu}, \mathcal{O}(0)] = -\mathrm{i}(S_{\mu\nu})^a_b\, \mathcal{O}^b(x), \tag{2.11}$$

where $S_{\mu\nu}$ is a spin matrix and a, b are labels for $SO(d)$ representation of $\mathcal{O}$. Noticing that D and $M_{\mu\nu}$ commute, by Schur's lemma, we find the action of dilatation,

$$[D, \mathcal{O}(0)] = -\mathrm{i}\Delta\mathcal{O}(0)\,, \tag{2.12}$$

which is characterized by a c-number Δ that we call a scaling dimension. We come to know here that local operators in CFT can be classified by a pair of (Δ, ρ) where ρ represents some $SO(d)$ representation.

Next let us examine the remaining part of actions of the conformal group. Using (2.9), one may find

$$\begin{aligned}[D, [K_\mu, \mathcal{O}(0)]] &= [[D, K_\mu], \mathcal{O}(0)] + [K_\mu, [D, \mathcal{O}(0)]]\,, \\ &= -\mathrm{i}(\Delta - 1)\,[K_\mu, \mathcal{O}(0)]\,,\end{aligned} \tag{2.13}$$

$$\begin{aligned}[D, [P_\mu, \mathcal{O}(0)]] &= [[D, P_\mu], \mathcal{O}(0)] + [P_\mu, [D, \mathcal{O}(0)]]\,, \\ &= -\mathrm{i}(\Delta + 1)\,[P_\mu, \mathcal{O}(0)]\,.\end{aligned} \tag{2.14}$$

These relations show that $[K_\mu, \mathcal{O}(0)]$ decrease the scaling dimension of $\mathcal{O}(0)$ by 1, while $[P_\mu, \mathcal{O}(0)]$ increase it by 1. They are analogue of raising/lowering operators in harmonic oscillator in quantum mechanics. Acting K_μ on $\mathcal{O}(0)$ repeatedly, one can construct an arbitrary lower dimensional operator. However, we expect (and later we will show) that scaling dimensions of local operators should be bounded below in physically meaningful theories, because of which, we demand that there must be operators annihilated by K_μ,

$$[K_\mu, \mathcal{O}(0)] = 0. \tag{2.15}$$

We call such operators *primary*. We also define a finite conformal transformation on primary operators of the dimension Δ and the irreducible representation ρ as follows:

$$\mathcal{O}^a(x') = \Omega(x)^{-\Delta} D(\mathcal{R}(x))^a_b \mathcal{O}(x)^b \,, \tag{2.16}$$

where a, b are labels of ρ and $D(\mathcal{R}(x))^a_b$ represents finite rotation of $\mathcal{O}(x)$ with

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x)\mathcal{R}^\mu_\nu(x), \quad \mathcal{R}^\mu_\nu(x) \in O(d) \tag{2.17}$$

Given primary operators, one can build higher dimensional operators by acting P_μ,

$$\mathcal{O}(x) \to [P_{\mu_1}, \cdots, [P_{\mu_n}, \mathcal{O}(x)] \cdots], \qquad \Delta \to \Delta + n \tag{2.18}$$

which we call *descendants*, and the set of operators consisting of

$$\{\mathcal{O}(x)\} = \{\mathcal{O}(x), [P_{\mu_1}, \mathcal{O}(0)], [P_{\mu_1}, [P_{\mu_2}, \mathcal{O}(x)] \cdots\} \tag{2.19}$$

form the *conformal multiplet* of $\mathcal{O}$. Notice that descendant operators are just given by derivatives acting on $\mathcal{O}(x)$ because[1]

$$[P_\mu, \mathcal{O}(x)] = -\mathrm{i}\partial_\mu \mathcal{O}(x). \tag{2.20}$$

Actions of conformal group on local operators at $x \neq 0$ can be derived by using the fact that $\mathcal{O}(x) = e^{\mathrm{i}x_\mu P^\mu} \mathcal{O}(0) e^{-\mathrm{i}x_\mu P^\mu}$ and commutation relations thereof.

2.3 Radial Quantization and State/Operator correspondence

In every QFTs, one must specify the foliation of the spacetime to define the Hilbert space and states on it. In particular, it is in general convenient to choose the foliation with respect to the symmetries that the theory has. For example, in Poincaré invariant theory we can choose the foliation by surfaces of equal time. The states are defined on every time slices. Thanks to the symmetry, we can move on to different surfaces by the time translation. In this foliation a state $|\Phi\rangle$ on some time slice are evolved by the time evolution operator $U = e^{\mathrm{i}P_0 t}$.

Since (euclidean) conformal field theories are rotationally invariant, here we have another choice of the foliation. Namely, we can foliate the spacetime by S^{d-1} spheres centered at the origin. One can consider "time evolution" as moving from a smaller sphere to a larger sphere by dilatation operator, which plays the role of the hamiltonian. The radii of spheres represent the time component. This scheme of the foliation is called *radial quantization*.

[1]This action is easily derived by expanding (2.10) for infinitesimal translation $x'^\mu = x^\mu + a^\mu$.

One can make heavy use of radial quantization in CFTs, leading to the remarkable feature called the *state/operator correspondence*. Such correspondence states that there is an isomorphism between states and local operators. To make this statement precise, let us consider an *initial* state and its evolution to a *final* state in path integral formulation. The wavefunction of the initial state $\Psi_i[\phi_i(x), r_i] \equiv \langle \phi_i(x), r_i | initial \rangle$ is evolved to the wavefunction of the final state $\Psi_f[\phi_f(x), r_f] \equiv \langle \phi_f(x), r_f | final \rangle$ in radial quantization as follows,

$$\Psi_f[\phi_f(x), r_f] = \int \mathcal{D}\phi_i \int_{\phi(r_i)=\phi_i}^{\phi(r_f)=\phi_f} \mathcal{D}\phi e^{-S[\phi]} \Psi_i[\phi_i(x), r_i], \tag{2.21}$$

where we integrate over all fields ϕ_i with fixed boundary conditions $\phi(r_i) = \phi_i$ and $\phi(r_f) = \phi_f$ on the two edges of the annulus which has radii r_i and r_f, respectively. Let us see if we take the initial state "far past", which means $r_i \to 0$. In this case we integrate over the whole interior of sphere $r \leq r_f$ and the initial state is defined at a point of the origin. Since the wave functional of origin is equivalent to some insertion of local operator, we obtain a state by inserting a local operator $\mathcal{O}(x)$ at the origin:

$$\Psi[\phi_f, r] = \int^{\phi(r)=\phi_f} \mathcal{D}\phi e^{-S[\phi]} \mathcal{O}(0), \tag{2.22}$$

which manifestly gives us the map between states and local operators. $\mathcal{O}(0)$ may be an elementary field or some composite operators. There exists a unique state that we can create in this map. Such a state can be obtained by inserting the identity operator and named *vacuum*. We denote the vacuum state as $|0\rangle$. To sum up, we can identify the local operators and the states as following:

$$\mathcal{O}(0) \leftrightarrow \mathcal{O}(0) |0\rangle \equiv |\mathcal{O}\rangle \tag{2.23}$$

Along with a primary operator, we define a primary state such that

$$[K_\mu, \mathcal{O}(0)] = 0 \qquad \longleftrightarrow \qquad K_\mu |\mathcal{O}\rangle = 0 \tag{2.24}$$

$$[D, \mathcal{O}(0)] = -\mathrm{i}\Delta\mathcal{O}(0) \qquad \longleftrightarrow \qquad D|\mathcal{O}\rangle = -\mathrm{i}\Delta |\mathcal{O}\rangle \tag{2.25}$$

$$[M_{\mu\nu}, \mathcal{O}(0)] = -\mathrm{i}\mathcal{S}_{\mu\nu}\mathcal{O}(0) \qquad \longleftrightarrow \qquad M_{\mu\nu} |\mathcal{O}\rangle == \mathrm{i}\mathcal{S}_{\mu\nu} |\mathcal{O}\rangle \tag{2.26}$$

where we suppress labels for the irreducible representation of $SO(d)$ for simplicity.

2.4 Conformal constraints on correlation functions

In quantum field theories, fundamental observables are correlation functions of local operators. Below this section, we will see that conformal symmetry enables us to significantly constrain correlation functions of primary operators.

2.4.1 Scalar primaries

Let us first consider the correlation function of scalar primary operators. Using transformation rules, n-point function of scalar primaries transforms under a conformal transformation as below:

$$\langle \mathcal{O}_1(x_1') \cdots \mathcal{O}_n(x_n') \rangle = \left(\prod_i^n \Omega^{-\Delta_k}(x_k) \right) \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle , \tag{2.27}$$

where Δ_k represents scaling dimension of $\mathcal{O}_k$. One may notice that Poincaré invariance (which means $\Omega = 1$) implies scalar n-point function only depends on mutual distance $x_{ij}^2 \equiv (x_i - x_j)^2$ between n points. The trivial consequence of the above relation is the case of $n = 1$, where we end up with the fact that one-point functions in CFT must be zero except the identity operator.

The first nontrivial constrain is the two-point function. If we consider dilatation $x \to \lambda x$, we must have

$$\langle \mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)\rangle = \lambda^{-\Delta_1-\Delta_2}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \tag{2.28}$$

which can be reduced to

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\kappa}{x_{12}^{\Delta_1+\Delta_2}} , \tag{2.29}$$

with the normalization constant κ. Here we denote $x_{12} = |x_1 - x_2|$. What is more, considering special conformal transformations, we can conclude that two-point functions are non-zero only when $\Delta_1 = \Delta_2$:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \begin{cases} \dfrac{\kappa}{x_{12}^{2\Delta_1}} & (\Delta_1 = \Delta_2) \\ 0 \quad (\text{otherwise}) \end{cases} \tag{2.30}$$

By redefining the field, we can always take $\kappa = 1$. Similarly, three-point functions is highly constrained as well. In summary, we have

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x)\rangle = \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}} , \tag{2.31}$$

where c_{123} is called three-point coefficient, or OPE coefficient. The reason why it is named OPE coefficient will be explained later.

Unfortunately higher-point functions are not uniquely determined only by conformal invariance. Instead, they are functions of so called conformal cross ratios:

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} . \tag{2.32}$$

When we, for example, consider four-point functions, we have two cross ratios,

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} , \tag{2.33}$$

and for identical scalars, the correlation function is written as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\mathcal{F}(u,v) , \tag{2.34}$$

where the form of $\mathcal{F}(u,v)$ is theory-dependent.

2.4.2 Spinning primaries

Next we would like to extend the results we obtain in the previous subsection to the case of spinning operators. In such a case, we have a number of difficulties in computing correlation functions due to complicated transformation laws on spinning primaries. The novel approach to overcome them is so called embedding space formalism, where we embed the d-dimensional spacetime into $d+2$-dimensional Minkowski spacetime in which the conformal transforms act linearly [34–38]. In this thesis we do not use this formalism, but take a more straightforward way.

Let us consider the two-point function of spin-1 symmetric traceless tensors $\mathcal{O}_\mu$ of $SO(d)$. Following similar procedures as in scalar primaries, we find

$$\langle \mathcal{O}_\mu(x_1)\mathcal{O}_\nu(x_2)\rangle = C\frac{I_{\mu\nu}(x_{12})}{x_{12}^{2\Delta}}\,, \tag{2.35}$$

where $I_{\mu\nu}(x)$ is a bilocal tensor defined as

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\,. \tag{2.36}$$

Two-point functions of higher spin primaries can be computed similarly, because there appears no additional covariant tensor except $I_{\mu\nu}$. For $l=2$, we find

$$\langle \mathcal{O}_{\mu\nu}(x_1)\mathcal{O}_{\sigma\rho}(x_2)\rangle = C\frac{I_{\mu\sigma}(x_{12})I_{\nu\rho}(x_{12}) + (\mu\leftrightarrow\nu) - (\text{trace})}{x_{12}^{2\Delta}} \tag{2.37}$$

2.5 Reflection positivity and unitarity bound

Along the way to the definition of primary operators, we mentioned that scaling dimensions Δ should be bounded below. Let us hereby make the statement clearer. In quantum mechanics, we believe the theory must be unitary, otherwise it suffers from negative probabilities or the violation of preserving them. The same is true for any QFTs, but the statement is for Lorentzian signature. Since we are interested in Euclidean CFTs, we have to translate unitarity into a Euclidean statement. Taking the vacuum state, unitarity means that correlation functions made out of $\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)$ and their Hermitian conjugations with the following form must be positive:

$$\left\langle \left(\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\right)^\dagger \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\right\rangle \geq 0\,, \tag{2.38}$$

since it can be regarded as a norm in the Hilbert space. In Euclidean signature, Hermitian conjugations are equivalent to (Euclidean) time reflections along the $t=0$ plane, thus we call unitarity in Euclidean signature *reflection positivity*. The immediate consequence of reflection positivity in CFTs is the positivity of conformal multiplets, meaning that

$$\langle\mathcal{O}|\,K_{\mu_n}\cdots K_{\mu_1}P_{\mu_1}\cdots P_{\mu_n}\,|\mathcal{O}\rangle \geq 0\,. \tag{2.39}$$

For scalar primaries, by taking $n=1$ we learn that $\Delta\geq 0$. This condition can be further restricted when considering $n=2$. As for spin-l primaries, we can constrain scaling dimensions in a similar manner at the first level $n=1$. The net result is given by [39–42]

$$\Delta \geq \begin{cases} \frac{d-2}{2} & (l=0)\,, \\ l+d-2 & (l>0)\,, \end{cases} \tag{2.40}$$

which are called unitarity bounds. It turns out that the bounds are saturated when the conformal multiplet becomes short in the sense that at some level descendants of the primary vanish, resulting zero norm. As for $l = 0$, the saturation occurs in a free massless field due to the equation of motion, $\partial^2\phi(x) = 0$, while spinning bounds are saturated by conserved currents $\partial_{\mu_1}\mathcal{O}^{\mu_1\cdots\mu_l} = 0$. Though one may wonder whether we can get additional constraints by taking n large, there is no possibility for such a case, which one may convince themselves by straightforward computations. A beautiful proof exists in Lorentzian signature by moving onto momentum space and using the positivity condition for two-point functions. See, for example, [43].

2.6 Operator Product Expansion and conformal blocks

The idea of operator product expansion (OPE) has a long history. At first Wilson considered that given two local operators sitting at points x and y respectively in any relativistic QFTs, in the limit where $x \to y$ the product of them could be approximated by a superposition of other local operators at y [44]:

$$\mathcal{O}(x)\mathcal{O}(y) \sim \sum_{\chi} C_{\chi}(x-y, y)\mathcal{O}_{\chi}(y) \,, \tag{2.41}$$

in correlation functions. Here $C_{\chi}(x-y, y)$ represent some functions and χ labels local operators. Of course such a expansion is generally asymptotic, however, in CFTs OPE has a finite radius of convergence [45, 46].

We shall derive OPE in CFTs by using radial quantization. We consider the insertion of two operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$. We take the sphere centered at some point x and it has radius r such that containing both of points x_1 and x_2 without any other operators. If we evolve radially from x outwards, we can get some state at the sphere

$$|\psi_{12}(r)\rangle = \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\,|0\rangle \tag{2.42}$$

because state is in the vacuum at x (figure 2.1). This state would be some linear combination of all of the states in the Hilbert space. By state/operator correspondence, there exists some local operator such that

$$\mathcal{O}_{\psi_{12}}(x)\,|0\rangle = |\psi_{12}(r)\rangle = \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\,|0\rangle \tag{2.43}$$

$\mathcal{O}_{\psi_{12}}(x)$ cannot have definite dimension or representation of $SO(d)$, however, we can express it as the sum over all of the conformal multiplet and this gives the OPE,

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{\chi} \lambda_{\chi} C_{\chi}(x_1 - x, x_2 - x, \partial_x)\mathcal{O}_{\chi}(x) \,, \tag{2.44}$$

where λ_{χ} is a real number. We can fix the function $C_{\chi}(x_1 - x, x_2 - x, \partial_x)$ by inserting $\mathcal{O}_{\chi}(x')$ in both sides and evaluating the expecting values:

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_{\chi}(y)\rangle = \lambda_{\chi} C_{\chi}(x_1 - x, x_2 - x, \partial_x)\langle\mathcal{O}_{\chi}(x)\mathcal{O}_{\chi}(y)\rangle \,, \tag{2.45}$$

where we used the orthogonality of two-point functions. Since both forms of three-point function and two-point functions are fixed by conformal symmetry, from above equation we can

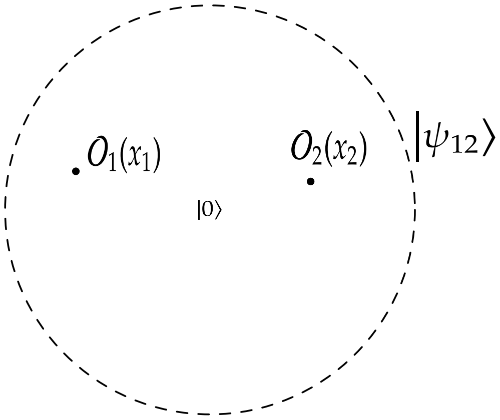

Figure 2.1: state $|\psi_{12}\rangle$ is represented by two operator insertion from state/operator correspondence.

completely determine the form of $C_\chi(x_1 - x, x_2 - x, \partial_x)$ for the case of two scalars. And we usually choose the expansion coefficient λ_χ so that it corresponds to the three-point coefficient, $\lambda_\chi = c_{12\chi}$, which is the reason why the three-point coefficient is called the OPE coefficient. It is often the case that we take the sphere centered at x_1 or x_2 and the expansion take the form as follows,

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_\chi \lambda_\chi C_\chi(x_{12}, \partial_2)\mathcal{O}_\chi(x_2) \tag{2.46}$$

Using OPE we can decompose higher point functions into lower point functions. The central objects to study are four-point functions. For identical scalars, we find,

$$\begin{aligned}\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle &= \sum_\chi \lambda_\chi C_\chi(x_{12}, \partial_2)\langle \mathcal{O}_\chi(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \\ &= \sum_\chi \sum_{\chi'} \lambda_\chi \lambda_{\chi'} C_\chi(x_{12}, \partial_2) C_{\chi'}(x_{34}, \partial_4)\langle \mathcal{O}_\chi(x_2)\mathcal{O}_{\chi'}(x_4)\rangle \\ &= \sum_\chi \lambda_\chi^2 C_\chi(x_{12}, \partial_2) C_\chi(x_{34}, \partial_4)\langle \mathcal{O}_\chi(x_2)\mathcal{O}_\chi(x_4)\rangle\end{aligned} \tag{2.47}$$

where we used the orthogonality of two-point functions. Compared with (2.34), we may notice

$$\mathcal{F}(u, v) = x_{12}^{2\Delta} x_{34}^{2\Delta} \sum_\chi \lambda_\chi^2 C_\chi(x_{12}, \partial_2) C_\chi(x_{34}, \partial_4)\langle \mathcal{O}_\chi(x_2)\mathcal{O}_\chi(x_4)\rangle \ , \tag{2.48}$$

which gives us so called *conformal block decompositions* where each term represents the conformal block $g_\chi(u, v) = C_\chi(x_{12}, \partial_2) C_\chi(x_{34}, \partial_4)\langle \mathcal{O}_\chi(x_2)\mathcal{O}_\chi(x_4)\rangle$. Notice also the right hand side of above equation is completely fixed by conformal invariance. Thus conformal blocks are purely kinematical objects. It is advantageous to realize that $g_\chi(u, v)$ is the eigenfunction of the Casimir equation of $SO(d + 1, 1)$ in order to derive its form. Unfortunately we can write down the closed form of conformal blocks only in $d = 2, 4$ [47].

2.7 Boundary Conformal Field Theories

In this section we would like to consider CFT with a planar boundary and its implication. The immediate effect of introducing a boundary is that the full d-dimensional conformal group breaks down into its subgroup. To see this, let us denote the coordinate $x^\mu = (\boldsymbol{x}, z)$ and define a planar boundary along $z = 0$. Recall $\boldsymbol{x}$ represents tangential components to the boundary while z represents the normal component. One may immediately realize that, for instance, translation invariance along z direction breaks down, while on the boundary, conformal invariance remains, resulting in a situation that only $(d-1)$-dimensional conformal group $SO(d,1)$ preserves.

Under such restricted conformal transformations, one may find that two points x, x' transform as

$$(x-x')^2 \to \frac{(x_1 - x')^2}{\Omega(x)\Omega(x')} \quad z \to \frac{z}{\Omega(x)} \quad z' \to \frac{z'}{\Omega(x')}\,, \tag{2.49}$$

from which we can construct conformal invariant variables,

$$\xi = \frac{(x-x')^2}{4zz'}\,, \quad v^2 = \frac{(x-x')^2}{(x-x')^2 + 4zz'} = \frac{\xi}{\xi+1}\,. \tag{2.50}$$

These two variables are conformal cross ratios for BCFT, which are analogous to usual cross ratios in CFT. We can build up correlation functions out of ξ, or equivalently v in similar fashion. In physical region, we have $0 \le \xi \le \infty$ and $0 \le v \le 1$.

2.7.1 General properties of correlation functions with boundary

In CFT without a boundary, conformal symmetry completely fixes the form of two- and three-point functions. On the other hand, the presence of a boundary makes the situation more complicated. a one-point function of scalar primary $\mathcal{O}(x)$ can have non zero expectation value:

$$\langle \mathcal{O}(x) \rangle = \frac{a_\Delta}{(2z)^\Delta} \tag{2.51}$$

while for spinning operators one-point functions must be zero owing to conformal invariance. For two-point functions, the story is similar to four point functions in CFT without a boundary. The two-point function of scalar primaries can be written as a function of the cross ratios,

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(x') \rangle = \frac{1}{(2z)^{\Delta_1}(2z')^{\Delta_2}} f_{12}(\xi) = \frac{(2z')^{\Delta_1 - \Delta_2}}{(x-x')^{2\Delta_1}} F_{12}(v)\,, \tag{2.52}$$

where $\xi_1^\Delta f_{12}(\xi) = F_{12}(v)$. One may notice that two-point functions of operators which have different scaling dimensions or representations of $SO(d)$ do not vanish unlike CFT without a boundary. The computation of two-point function involving operators with spin becomes more complicated. One may also use the embedding space formalism for BCFT to systematically construct correlation functions, but here we follow the approach in [48] and take an alternative steps to analyze them. The idea is to construct weight zero bilocal vector under $O(d,1)$ transformations. For later convenience, we introduce a useful notation

$$s = x - x'\,, \quad \boldsymbol{s} = \boldsymbol{x} - \boldsymbol{x}'\,. \tag{2.53}$$

Then one may find that

$$\begin{aligned} X_\mu &\equiv z\frac{v}{\xi}\partial\xi = v\left(\frac{2z}{s^2}s_\mu - n_\mu\right), \\ X'_\mu &\equiv z'\frac{v}{\xi}\partial'_\mu\xi = v\left(-\frac{2z'}{s^2}s_\mu - n_\mu\right), \end{aligned} \tag{2.54}$$

where $n_\mu = (\mathbf{0}, 1)$ is a unit normal vector to the boundary. These bilocal vector transform nicely under restricted conformal group as $X_\mu \to R_{\mu\nu}(x)X_\nu$, $X'_\nu \to R_{\mu\nu}(x')X'_\nu$ with $X'_\mu = I_{\mu\nu}(s)X_\nu$. Now that we have conformally covariant tensors $\delta_{\mu\nu}$, $I_{\mu\nu}(x)$, X_μ, X'_μ, it is a straightforward exercise to compute two-point functions for spinning fields. For our purpose, we present various two-point functions involving conserved currents with dimension $d-1$, stress energy tensors with dimension d and scalar operators with dimension Δ:

$$\langle J_\mu(x)J_\nu(x')\rangle = \frac{1}{s^{2(d-1)}}\left(I_{\mu\nu}(s)P(v) + X_\mu X'_\nu Q(v)\right) \tag{2.55}$$

$$\langle J_\mu(x)\mathcal{O}(x')\rangle = \frac{\xi^{1-d}}{(2z)^{d-1}(2z')^{\Delta}}X_\mu f_{J\mathcal{O}}(v) \tag{2.56}$$

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(x')\rangle = \frac{\xi^{-d}}{(2z)^d(2z')^d}\left[\alpha_{\mu\nu}\alpha'_{\sigma\rho}A(v) + \beta_{\mu\nu,\sigma\rho}B(v) + I_{\mu\nu,\sigma\rho}C(v)\right] \tag{2.57}$$

$$\langle T_{\mu\nu}(x)\mathcal{O}(x')\rangle = \frac{\xi^{-d}}{(2z)^d(2z')^{\Delta}}\alpha_{\mu\nu}f_{T\mathcal{O}}(v) \tag{2.58}$$

where we define

$$\alpha_{\mu\nu} = \left(X_\mu X_\nu - \frac{1}{d}\delta_{\mu\nu}\right), \quad \alpha'_{\mu\nu}\left(X'_\mu X'_\nu - \frac{1}{d}\delta_{\mu\nu}\right), \tag{2.59}$$

$$\begin{aligned} \beta_{\mu\nu,\sigma\rho} = \Big(&X_\mu X'_\sigma I_{\nu\rho}(s) + X_\nu X'_\sigma I_{\mu\rho}(s) + X_\mu X'_\rho I_{\nu\sigma}(s) + X_\nu X'_\rho I_{\mu\sigma}(s) \\ &-\frac{4}{d}\delta_{\sigma\rho}X_\mu X_\nu - \frac{4}{d}\delta_{\mu\nu}X'_\sigma X'_\rho + \frac{4}{d^2}\delta_{\mu\nu}\delta_{\sigma\rho}\Big), \end{aligned} \tag{2.60}$$

$$I_{\mu\nu,\sigma\rho}(s) = \frac{1}{2}\left(I_{\mu\sigma}(s)I_{\nu\rho}(s) + I_{\mu\rho}(s)I_{\nu\sigma}(s)\right) - \frac{1}{d}\delta_{\mu\nu}\delta_{\sigma\rho}. \tag{2.61}$$

We have implicitly used the fact that the stress tensor is traceless $T^\mu_\mu = 0$, while we haven't yet imposed conservation laws, which gives us further constraints between conformal invariant functions. Using $\partial_\mu J^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$ on (2.56) and (2.58) respectively, one can fix the form of functions $f_{J\mathcal{O}}$ and $f_{T\mathcal{O}}$:

$$f_{J\mathcal{O}}(v) = C_{J\mathcal{O}}v^{d-1}, \quad f_{T\mathcal{O}}(v) = C_{T\mathcal{O}}v^d, \tag{2.62}$$

whose normalization constants $C_{J\mathcal{O}}$ and $C_{T\mathcal{O}}$ are fixed by Ward identities. If we apply same strategy for (2.55) and (2.57), we instead obtain following relations:

$$v\frac{d}{dv}(P+Q) = (d-1)Q, \tag{2.63}$$

$$\left(v\frac{d}{dv} - d\right)(C+2B) = -\frac{2}{d}(A+2B) - dC, \tag{2.64}$$

$$\left(v\frac{d}{dv} - d\right)\left[(d-1)A + 2(d-2)B\right] = 2A - 2(d^2-4)B. \tag{2.65}$$

With these relations in mind, we come to realize that two-point functions of conserved currents, or stress tensors are fixed by single function of the cross ratio; other functions can be constructed from (2.63),(2.64) and (2.65).

Though structures in (2.55) and (2.57) seems to be natural with respect to bilocal tensors $I_{\mu\nu}(x)$ and $X_\mu(x)$, it is sometimes useful to introduce another basis for $\langle J_\mu(x)J_\nu(x')\rangle$ and $\langle T_{\mu\nu}(x)T_{\sigma\rho}(x')\rangle$. Such a new basis for the stress tensor two-point function can be given by [49]

$$\begin{aligned}\alpha(v) &= \frac{d-1}{d^2}[(d-1)(A+4B)+dC]\ ,\\ \gamma(v) &= -B-\frac{1}{2}C\ ,\\ \epsilon(v) &= \frac{1}{2}C\ .\end{aligned} \tag{2.66}$$

which become transparent when we consider a special configuration $x=(\mathbf{0},z)$ and $x'=(\mathbf{0},z')$ because we may find,

$$\langle T_{nn}(\mathbf{0},z)T_{nn}(\mathbf{0},z')\rangle = \frac{\alpha(v)}{s^{2d}}\ , \tag{2.67}$$

$$\langle T_{an}(\mathbf{0},z)T_{bn}(\mathbf{0},z')\rangle = \frac{\gamma(v)}{s^{2d}}\delta_{ab}\ , \tag{2.68}$$

$$\langle T_{ab}(\mathbf{0},z)T_{cd}(\mathbf{0},z')\rangle = \frac{\delta(v)\delta_{ab}\delta_{cd}+\epsilon(v)(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})}{s^{2d}}\ . \tag{2.69}$$

For the current two-point function, one can find

$$\langle J_n(\mathbf{0},z)J_n(\mathbf{0},z')\rangle = \frac{\pi(v)}{s^{2(d-1)}}\ , \qquad \pi(v)=P(v)+Q(v) \tag{2.70}$$

$$\langle J_a(\mathbf{0},z)J_b(\mathbf{0},z')\rangle = \frac{P(v)}{s^{2(d-1)}}\ \delta_{ab}\ . \tag{2.71}$$

Away from the boundary, we expect that they should be reduced to the usual two-point functions in CFT without boundary. In terms of the cross ratio, we have the limit $v\to 0$, which corresponds to the usual OPE limit $s\to 0$. In this limit operators are much closer to each other than to the boundary; we call $v\to 0$ the coincident or bulk limit. Let us consider the coincident limit of the two-point function for two scalars $\mathcal{O}_1$ and $\mathcal{O}_2$. Since we know that two-point functions for different scalars must vanish in usual CFT, we demand that

$$F_{12}(0) = \kappa\delta_{\Delta_1,\Delta_2}\ . \tag{2.72}$$

As for correlators involving conserved currents and stress tensors, referring to (2.35) and (2.37), we also demand $A(0)=B(0)=Q(0)$. $P(0)$ and $C(0)$ are fixed by the normalization constants of two-point functions of J_μ or $T_{\mu\nu}$, which we denote $P(0)=C_J$ and $C(0)=C_T$.

Finally we comment on the other limit of the cross ratio, which is governed by $v\to 1$. This limit is archived by letting operators be much closer to the boundary and called the boundary limit. The general behavior of $F_{12}(1)$ or other functions relies on the boundary condition of fields, which we have to specify.

2.7.2 Boundary Operator Expansion and Conformal block decompositions

We have written down the various structures of two-point functions in BCFTs. These two-point functions are built up from conformal invariant functions of the cross ratio v. Like four-point functions in CFT without a boundary, we can decompose these functions into conformal blocks, which are the eigenfunctions of conformal Casimir. In the presence of a boundary, we have two types of decompositions concerning the coincident limit and the boundary limit of two-point functions.

First let us consider the decomposition for the coincident limit. As is the case with four-point functions in CFT, conformal block decompositions can be derived by using OPE in two-point functions. We remark that even with a boundary, OPE still remains valid in $s \to 0$ because OPE is a local property of CFT. Thus for two identical scalars, we have

$$\mathcal{O}(x)\,\mathcal{O}(x') = \frac{\kappa}{s^{2\Delta}} + \sum_{\Delta\neq 0} \lambda_\Delta C_\Delta(x - x'\,, \partial')\mathcal{O}_\Delta(x') \tag{2.73}$$

where the first term reflects the fact that in BCFT one-point functions of scalars are non zero. For later convenience, we denote this expansion the *bulk OPE*. Conformal block decompositions of two identical scalars for bulk OPE can be obtained by substituting (2.73) into (2.52),

$$F_{\mathcal{O}\mathcal{O}}(v) = \kappa + \sum_{\Delta\neq 0} \lambda_\Delta a_\Delta G_{\text{bulk}}(\Delta, v) \tag{2.74}$$

where we used (2.51) and $G_{\text{bulk}}(\Delta, v)$ are bulk conformal blocks. The form of $G_{\text{bulk}}(\Delta, v)$ can be fixed by expanding

$$C_\Delta(x - x', \partial')\frac{1}{(2z')^\Delta}\,, \tag{2.75}$$

or by solving bulk conformal Casimir equation for $SO(d+1,1)$ [50]. The result is written by a hypergeometric function,

$$G_{\text{bulk}}(\Delta, v) = \xi^{\Delta/2}{}_2F_1\left(\frac{\Delta}{2}\,, \frac{\Delta}{2}\,, 1 - \frac{d}{2} + \Delta\,; -\xi\right) \tag{2.76}$$

One may find analogous expressions for spinning correlators. For $Q(v)$ in (2.55) and $A(v)$ in (2.57), the decompositions take the following forms [30]:

$$\begin{aligned} Q(v) &= \sum_{\Delta\neq 0} a_\Delta \lambda_\Delta Q_{\text{bulk}}\left(\Delta, v\right)\,, \\ A(v) &= \sum_{\Delta\neq 0} a_\Delta \lambda_\Delta A_{\text{bulk}}\left(\Delta, v\right)\,, \end{aligned} \tag{2.77}$$

where conformal blocks are given by

$$Q_{\text{bulk}}\left(\Delta, v\right) = \xi^{\frac{\Delta}{2}}{}_2F_1\left(1 + \frac{\Delta}{2}, 1 + \frac{\Delta}{2}, 1 - \frac{d}{2} + \Delta; -\xi\right)(1+\xi)\,, \tag{2.78}$$

$$A_{\text{bulk}}\left(\Delta, v\right) = \xi^{\frac{\Delta}{2}}{}_2F_1\left(2 + \frac{\Delta}{2}, 2 + \frac{\Delta}{2}, 1 - \frac{d}{2} + \Delta; -\xi\right)(1+\xi)^2\,. \tag{2.79}$$

As for other conformal invariant functions, the decompositions can be obtained by (2.63),(2.64) and (2.65).

Next let us investigate the decomposition for the boundary limit. In the presence of a boundary, one can expand a bulk scalar operator $\mathcal{O}_\Delta$ by summing up boundary local operators,

$$\mathcal{O}_\Delta(x) = \frac{a_\Delta}{(2z)^\Delta} + \sum_{\Delta\neq 0} \tilde{\mu}_\Delta \hat{C}_\Delta(z, \boldsymbol{\partial}) \hat{\mathcal{O}}_\Delta(\boldsymbol{x}) \,, \tag{2.80}$$

which is called the *boundary OPE* in contrast to the bulk OPE. $\tilde{\mu}_\Delta$ is a boundary OPE coefficient, which contains information of this expansion. Like the bulk OPE, the form of $\hat{C}_\Delta(z, \boldsymbol{\partial})$ is completely fixed by conformal invariance. From the viewpoint of boundary local primaries, they have $(d-1)$-dimensional conformal symmetry, which implies the correlation functions of them are constrained as same as ones in d-dimensional CFT. Therefore boundary conformal invariance forbids one-point functions. The two-point function of identical scalars is fixed as

$$\langle \hat{\mathcal{O}}_\Delta(\boldsymbol{x}) \hat{\mathcal{O}}_\Delta(\boldsymbol{x}') \rangle = \frac{\hat{c}}{(\boldsymbol{x} - \boldsymbol{x}')^{2\Delta}} \,, \tag{2.81}$$

up to normalization constant. In order to obtain boundary conformal block decompositions, we use (2.80) twice in (2.52) and (2.81), then we find

$$G_{\mathcal{O}\mathcal{O}} = \xi^\Delta \left(a_\Delta^2 + \sum_{\Delta\neq 0} \mu_\Delta^2 G_{\text{bry}}(\Delta, v) \right) \,, \tag{2.82}$$

where we redefine $\mu_\Delta^2 = \tilde{\mu}_\Delta^2 \hat{c}$. Similarly, we can obtain the form of $G_{\text{bry}}(\Delta, v)$ by expanding

$$\hat{C}_\Delta(z, \boldsymbol{\partial}) \hat{C}_\Delta(z', \boldsymbol{\partial}') \frac{1}{(\boldsymbol{x} - \boldsymbol{x}')^{2\Delta}} \tag{2.83}$$

or solving the boundary conformal Casimir equation for $SO(d, 1)$. One may find

$$G_{\text{bry}}(\Delta, v) = \xi^{-\Delta} {}_2F_1 \left(\Delta, 1 - \frac{d}{2} + \Delta, 2 - d + 2\Delta; -\frac{1}{\xi} \right) \,. \tag{2.84}$$

Regarding decompositions of spin-l conserved currents, there are extra subtleties that boundary expansions contain spin s fields with $s \leq l$ due to angular momentum conservations. However, thanks to restricted form of (2.56), (2.58) and $\langle T_{\mu\nu}(x) V_\rho(x) \rangle$ where $V_\mu(x)$ represents some vector field, it turns out that the number of exchanging fields with spin less than l are few. Additional discussions constrain it further, see [30] for a lengthy explanation. For $\langle J_\mu(x) J_\nu(x') \rangle$ and $\langle T_{\mu\nu}(x) T_{\sigma\rho}(x') \rangle$, we find the following decompositions:

$$\begin{aligned} \pi(v) &= \xi^{d-1} \left(\textstyle\sum_{\Delta \geq d-2} \mu_\Delta^2 \pi_{\text{bry}}^{(1)}(\Delta, v) \right) \,, \\ \alpha(v) &= \xi^d \left(\mu_{(0)}^2 \alpha_{\text{bry}}^{(0)}(v) + \textstyle\sum_{\Delta \geq d-1} \mu_\Delta^2 \alpha_{\text{bry}}^{(2)}(\Delta, v) \right) \,, \end{aligned} \tag{2.85}$$

where superscripts (0), (2) denote spins of exchanging fields. We tabulate formulae for corresponding boundary blocks:

$$\pi_{\text{bry}}^{(1)}(\Delta, v) = \xi^{-\Delta-1} {}_2F_1 \left(1 + \Delta, 1 - \frac{d}{2} + \Delta, 2 - d + 2\Delta; -\frac{1}{\xi} \right) \,, \tag{2.86}$$

$$\alpha_{\text{bry}}^{(0)}(v) = \frac{1}{4(d-1)} \left(v^{-1} - v \right)^d \left(d \left(v^{-1} + v \right)^2 - 4 \right) \,, \tag{2.87}$$

$$\alpha_{\text{bry}}^{(2)}(\Delta, v) = \xi^{-\Delta-2} {}_2F_1 \left(2 + \Delta, 1 - \frac{d}{2} + \Delta, 2 - d + 2\Delta; -\frac{1}{\xi} \right) \,. \tag{2.88}$$

Here $\alpha^{(0)}_{\rm bry}$ is a conformal block corresponding to so called displacement operator, i.e. a scalar operator conjugate to the location of the boundary. No other scalar operators contribute. The $\alpha^{(2)}_{\rm bry}(\Delta, v)$ are spin two boundary operators with scaling dimension Δ. There is no spin one contribution to the decomposition.

2.8 On trace of stress tensor

Stress tensor plays a special role in QFTs. For a while we consider a theory without boundaries. The existence of stress tensor reflects spacetime symmetry, since stress tensor is one of conserved currents. According to famous Noether's theorem, One can construct conserved charges, for example,

$$P_\mu = \int_{S^{d-1}} d\Sigma n_\nu T^{\nu\mu} \,, \tag{2.89}$$

which is associated with translation. In similar manner one can write down other conserved charges for the rest part of conformal group. For our purpose, let us consider the conserved current for dilatation, $J^\mu = x_\nu T^{\nu\mu}$. The conservation law for this current implies that

$$\partial_\mu J^\mu = 0 \Rightarrow \delta_{\mu\nu} T^{\nu\mu} + x_\nu \partial_\mu T^{\nu\mu} = 0 \tag{2.90}$$

$$\Rightarrow T^\mu_\mu = 0 \tag{2.91}$$

where we used $\partial_\mu T^{\nu\mu} = 0$. We may naively think that in QFTs with conformal invariance the trace of stress tensor vanishes, however, this statement holds true only for classical theories. If we go on to quantum theories, stress tensor trace may not be zero, which is an example of quantum *anomalies*.

Let us make the above statement more concrete. When a QFT has some symmetries, one can couple conserved currents to classical background fields. For spin-1 current J_μ, we can archive it by adding $\int d^d x J_\mu(x) A^\mu(x)$ in the action where A_μ is a gauge field. For the case of the stress tensor, which we are interested in, we put the theory on nontrivial background metric $g_{\mu\nu}$ coupling to the stress tensor. In general, anomaly is a phenomenon of symmetry breaking on nontrivial background fields. As for classically conformal invariant theories, one cannot preserve diffeomorphism invariance and Weyl invariance simultaneously in some nontrivial metrics. There are subtleties which invariance to preserve, but we always take diffeomorphism invariance as we would like to have the conserved stress tensor. As a result, Weyl invariance is broken, which we call conformal (Weyl) anomaly.

Conformal anomaly has a long history, but it has been still getting a certain attention for its properties, providing us how to classify quantum field theories, and to constrain possible RG flows. Before discussing its dynamical properties, we shall present formulae of conformal anomaly. See [51–55] for details. A partition function on a background metric is given by

$$Z = \int \mathcal{D}\phi e^{-S[g_{\mu\nu},\phi]} = e^{-W[g_{\mu\nu}]} \,, \tag{2.92}$$

where ϕ represents a matter field. Integrating out ϕ, we can obtain an effective action $W[g_{\mu\nu}]$ as a functional of the metric. As we mentioned above, conformal anomaly measures how non-invariant a QFT becomes under Weyl transformation. More precisely, under an infinitesimal

Weyl transformation,

$$\delta g_{\mu\nu} = 2\omega(x) g_{\mu\nu} \,, \tag{2.93}$$

we define the conformal anomaly $\mathcal{A}$ as a local functional,

$$\delta_\omega W = -\int \mathrm{d}^d x \sqrt{g}\,\omega \mathcal{A} \,. \tag{2.94}$$

Recall that (an expectation value of) stress tensor is as always given by[2]

$$\langle T^{\mu\nu}(x)\rangle = \frac{2}{\sqrt{g}} \frac{\delta W[g_{\alpha\beta}]}{\delta g_{\mu\nu}} \,, \tag{2.95}$$

one may find

$$\langle T^\mu_\mu(x)\rangle = -\mathcal{A} \,, \tag{2.96}$$

which relates trace of stress tensor to conformal anomaly. The general form of $\mathcal{A}$ can be derived by solving so called the Wess-Zumino consistency condition [58], which states that two repeated Weyl transformations must commute since the Weyl group is Abelian. We do not show derivations for the explicit forms of these anomalies, but present final results. For conformal field theories in even dimensions $d = 2n+2$ with $n = 0, 1, 2, \cdots$, we find that

$$\left\langle T^\mu_\mu \right\rangle^{d=2n+2} = -\frac{4}{d!\,\mathrm{Vol}\left(\mathbb{S}^d\right)} \left[\sum_i c_i I_i - (-1)^{\frac{d}{2}} a_d E_d\right] \,. \tag{2.97}$$

Here E_d is the d-dimensional Euler density, whose integration over $\mathbb{S}^d$ yields $d!\,\mathrm{Vol}\left(\mathbb{S}^d\right)$, and I_i denote local Weyl invariants of weight $-d$. The number of I_i counts on spacetime dimensions. The first terms in (2.97) are called type-B anomalies and the second term type-A anomaly [59]. In odd dimensions, on the other hand, it turns out that we are not able to write down non-zero invariants at all, resulting that we do not have trace anomalies in odd dimensions.

Let us consider explicit examples. for $d = 2$, we have

$$\langle T^\mu_\mu\rangle^{d=2} = -\frac{a}{2\pi} R \tag{2.98}$$

where there is no type-B anomaly contributions. The coefficient a is related to the central charge c of Virasoro algebra by $a = c/12$. For $d = 4$, we find

$$\langle T^\mu_\mu\rangle^{d=4} = -\frac{1}{16\pi^2}\left(c W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - a E_4\right) \,, \tag{2.99}$$

where $W_{\mu\nu\rho\sigma}$ is a Weyl tensor. Both coefficients a and c can be extracted from correlation functions of stress tensors. Similarly, we can obtain higher dimensional results.

Among anomaly coefficients in (2.97), it is revealed that type-A anomaly coefficient a_d plays a significant role in understanding RG flows of QFTs. In two-dimensional QFTs, it is proven that the central charge c, or in our formula a, must monotonically decrease under RG flow, which is

[2]Remark that this definition of stress tensor may deffer up to the sign from standard reference, e.g. [30, 56, 57]

celebrated Zamolodchikov's c-theorem [60]. Specifically if we only look at the UV CFT with the central charge c_{UV} and the IR CFT with the central charge c_{IR}, this theorem tells us that

$$c_{UV} \geq c_{IR} \,. \tag{2.100}$$

Therefore c-theorem forbids any RG flows that central charge does not monotonically decrease.

One may wonder if we can extend this theorem to higher-dimensional QFTs. Four-dimensional version of c-theorem was proposed in [5] and further studied in [61,62], however it had not been proven until [6,7] gave a physically beautiful proof, which provides the inequality between UV and IR a coefficients,

$$a_{UV} \geq a_{IR} \,. \tag{2.101}$$

This is referred to a-theorem [3]. Though six- or higher-dimensional extension has not been proven yet, we expect in general dimensions the type A-anomaly coefficients should monotonically decrease under RG flows.

2.8.1 Boundary conformal anomalies

When there is a boundary, the situation changes drastically. In contrast to the case without the boundary, the conformal anomaly can contain boundary localized contributions even in odd dimensions and its geometric structure becomes much rich.

According to [30,63], conformal anomaly in even dimensional CFT with a boundary is given by,

$$\left\langle T^{\mu}_{\mu}\right\rangle^{d=2n+2} = -\frac{4}{d!\,\mathrm{Vol}\left(\mathbb{S}^{d}\right)}\left[\sum_{i} c_i I_i + \delta\left(x_{\perp}\right)\sum_{j} b_j I^{\text{bry}}_j - (-1)^{\frac{d}{2}} a_d\left(E_d + \delta\left(x_{\perp}\right)E^{\text{bry}}\right)\right] , \tag{2.102}$$

where we denote the normal coordinate to boundary $x_{\perp}$. There are bunch of boundary localized terms including boundary Weyl invariant terms I^{bry} and the boundary term of Euler characteristic E^{bry}. As for CFT in odd dimensions $d = 2n + 1$, we find that

$$\left\langle T^{\mu}_{\mu}\right\rangle^{d=2n+1} = -\frac{2}{(d-1)!\,\mathrm{Vol}\left(\mathbb{S}^{d-1}\right)}\delta\left(x_{\perp}\right)\left(\sum_{i} b_i I_i + (-1)^{\frac{(d+1)}{2}} a_d \hat{E}_{d-1}\right) , \tag{2.103}$$

where $\hat{E}_{d-1}$ is the $(d-1)$-dimensional Euler density on the boundary. Here a_d is the type-A anomaly coefficient associated with the boundary. As an explicit example, we can write down the conformal anomaly in three dimensions as follows:

$$\langle T^{\mu}{}_{\mu}\rangle = -\frac{\delta(x_{\perp})}{4\pi}\left(a\,\hat{R} + b\,\tilde{K}_{ab}\tilde{K}^{ab}\right) , \tag{2.104}$$

where $\tilde{K}_{ab}$ is the traceless part of the extrinsic curvature defined as

$$\tilde{K}_{ab} = K_{ab} - \frac{\hat{g}_{ab}}{d-1}K \,, \tag{2.105}$$

[3] In the viewpoint of (2.97), the terminologies, c-theorem and a-theorem, may be confusing, which is because c-theorem meant first for the central charge, not for the anomaly coefficient. Though one may complain that we should unify both theorems as a-theorems, we will stick to using the original terminologies.

with the induced metric $\hat{g}_{ab}$ and $\hat{R}$ is the boundary Ricci scalar. With knowledge of c-theorem or a-theorem in even dimensional CFTs, it is natural to ask whether a similar statement exists for the boundary type-A anomaly coefficient. The authors in ref [20] showed by using similar technique as in [6, 7] that the coefficient a in (2.104) monotonically decrease under RG flows:

$$a_{UV} \geq a_{IR} \ . \tag{2.106}$$

Remark that they refer to this theorem as b-theorem because in their notation the type-A anomaly coefficient is represented by b, which may bring about some confusion.

2.9 Conformal defects

Defects collectively stand for non-local operators in QFT as exemplified by Wilson-'t Hooft line operators. A certain class of defects has realizations by fundamental fields in a given QFT (e.g., Wilson lines) while some are rather defined by specifying boundary conditions around them on the fundamental fields (e.g., 't Hooft lines). One can also couple a lower-dimensional theory to a higher-dimensional theory (e.g., the mixed-dimensional QED and the D3/D5 brane model). Thus there are at least three different ways to introduce defects[4] [23]:

1. Localize the bulk fields at the location of the defect.
2. Impose a boundary condition on the bulk fields around the defect [64, 65].
3. Introduce new degrees of freedom localized on the defect and couple them to the bulk fields.

Its dimensions or codimensions also characterize a defect. We denote the former one as p, and the latter as q with $p + q = d$. When $p = d - 1$, we can instead introduce a boundary or an interface by gluing two different theories along a boundary.

In this thesis we restrict our attention to a special class of defects, called conformal defects, which are planar or spherical to preserve the conformal symmetry on and the rotational symmetry around the worldvolumes. Conformal defects of dimension p break the full d-dimensional conformal symmetry $SO(d + 1, 1)$ to the subgroup $SO(p + 1, 1) \times SO(d - p)$ as the case of boundaries. We then define bulk conformal field theories with conformal defects as defect conformal field theories (DCFTs).

Recalling that restricted conformal group allows us to have non-vanishing (bulk) one-point functions in boundary CFTs, the same thing happens in DCFTs. What is more, one-point functions of spinning primaries do not vanish, while they do in BCFTs. To illustrate this point in detail, let us consider a (p-dimensional) planar defect in $\mathbb{R}^d$ and the stress-energy tensor one-point function for later usage. The metric is then divided into the parallel and orthogonal components:

$$\mathrm{d}s^2 = \mathrm{d}\hat{x}^a\, \mathrm{d}\hat{x}^a + \mathrm{d}x_\perp^i\, \mathrm{d}x_\perp^i \ , \qquad (a = 0, \cdots, p-1,\ i = p, \cdots, d-1) \ . \tag{2.107}$$

First, we deal with the cases 1 and 3 in the classification we mentioned earlier. Assuming DCFT has a Lagrangian description, the Lagrangian consists of the bulk part and the defect part,

$$I_{\text{DCFT}} = \int d^d x \sqrt{g}\, \mathcal{L}_{\text{CFT}} + \int d^p \hat{x} \sqrt{\hat{g}}\, \hat{\mathcal{L}}_{\text{defect}} \ . \tag{2.108}$$

[4] These constructions may be equivalent in certain cases while we are not aware of their precise relations.

In the case 1, the defect part is absent, but a defect operator $\mathcal{D}^{(p)}$ should be inserted in evaluating correlation functions [56]

$$\langle\, \mathcal{O}\cdots\mathcal{O}\,\rangle_{\mathcal{D}^{(p)}} \equiv \frac{\langle\, \mathcal{O}\cdots\mathcal{O}\,\mathcal{D}^{(p)}\,\rangle}{\langle\, \mathcal{D}^{(p)}\,\rangle}\ . \tag{2.109}$$

We are then allowed to regard $-\log\,\mathcal{D}^{(p)}$ as the defect part in the action. In either case the stress-energy tensor follows from the partition function $Z^{(\mathrm{DCFT})}$[5]

$$T^{\mu\nu}_{\mathrm{DCFT}} = -\frac{2}{\sqrt{g}}\,\frac{\delta\,\log Z^{(\mathrm{DCFT})}[g_{\mu\nu}]}{\delta g_{\mu\nu}}\ . \tag{2.110}$$

It will be useful to split it into the bulk part $T^{\mu\nu}_{\mathrm{CFT}}$ and the defect localized part $t^{\mu\nu}$

$$T^{\mu\nu}_{\mathrm{DCFT}} = T^{\mu\nu}_{\mathrm{CFT}} + t^{\mu\nu}\ . \tag{2.111}$$

$t^{\mu\nu}$ contains the contribution from the response to the induced metric [56]

$$t^{\mu\nu} = \delta_{\mathcal{D}}(x_\perp)\,\left[\delta^\mu_a\delta^\nu_b\,B^{ab} + \cdots\right] + \frac{1}{2}\,\partial_i\delta_{\mathcal{D}}(x_\perp)\,\delta^\mu_a\delta^\nu_b\,C^{abi} + \cdots\ , \tag{2.112}$$

where $\delta_{\mathcal{D}}(x_\perp)$ is the delta function localized on the worldvolume of the defect and B^{ab} and C^{abi} are defined by the variation of the defect action (see [49, 56] for the detail). In what follows, we ignore the higher derivative terms of the delta function as they vanish for the planar defect. While the conservation and tracelessness of the bulk stress tensor are violated in the presence of the defect, $T^{\mu\nu}_{\mathrm{DCFT}}$ is traceless and partially conserved

$$\begin{aligned} \partial_\mu T^{\mu a}_{\mathrm{DCFT}} &= 0\ , \\ \partial_\mu T^{\mu i}_{\mathrm{DCFT}} &= -\delta_{\mathcal{D}}(x_\perp)\,\mathrm{D}^i\ , \\ (T_{\mathrm{DCFT}})^\mu{}_\mu &= 0\ . \end{aligned} \tag{2.113}$$

where D^i are called displacement operators for each orthogonal directions against the defect. These relations hold as the operator identities in DCFT.

In contrast to the cases 1 and 3, the Lagrangian and the stress-energy tensor in the case 2 are the same as those in CFTs without a defect. However, careful treatment is required in evaluating the one-point function of the stress-energy tensor as we will discuss later on.

Now consider the one-point function of the bulk stress-energy tensor, T_{CFT}. T_{CFT} is a symmetric traceless tensor of dimension d and spin 2, hence the residual conformal symmetry completely fixes the form of the correlator[6]

$$\begin{aligned} \langle\, T^{ab}_{\mathrm{CFT}}(x)\,\rangle &= \frac{d-p-1}{d}\,\frac{a_T}{|x_\perp|^d}\,\delta^{ab}\ , \\ \langle\, T^{ij}_{\mathrm{CFT}}(x)\,\rangle &= -\frac{a_T}{|x_\perp|^d}\left(\frac{p+1}{d}\delta^{ij} - \frac{x^i_\perp x^j_\perp}{|x_\perp|^2}\right)\ , \\ \langle\, T^{ai}_{\mathrm{CFT}}(x)\,\rangle &= 0\ , \end{aligned} \tag{2.114}$$

[5]The definition differs in the sign from the one used in [56], so T_{DCFT} equals $-T_{\mathrm{tot}}$ there.
[6]Our a_T is $-a_T$ in [56].

where a_T is a constant characterizing the defect. While this one-point function does not vanish in general, we recover the result in BCFTs, $\langle T^{\mu\nu}_{\text{CFT}} \rangle = 0$ by setting $p = d - 1$ in (2.114). More generically, the one-point function of the bulk operator with non-zero spin vanish in BCFT and DCFT with a defect of dimension $d - 1$ [49, 50, 66].

Furthermore, the one-point function of $t^{\mu\nu}$ vanishes

$$\langle t^{\mu\nu}(x) \rangle = 0 \ . \tag{2.115}$$

This is seen by writing $t^{\mu\nu}$ as

$$t^{\mu\nu}(x) = \delta_{\mathcal{D}}(x_\perp) \frac{\partial x^\mu}{\partial \hat{x}^a} \frac{\partial x^\nu}{\partial \hat{x}^b} \hat{t}^{ab}(\hat{x}) \ , \tag{2.116}$$

and define the defect stress-energy tensor $\hat{t}^{ab}(\hat{x})$, which is a defect local operator of dimension p whose vev must be zero due to the invariance under the translation, rotation and scale transformation on the defect.

When p is even, there exist conformal anomalies as we discussed in section 2.8 (the Graham-Witten anomaly [67]), while we do not touch them for a while. In fact we can avoid these anomalies by using dimensional regularization for both d and p.

Chapter 3

Tractable models of Boundary CFT: the case of scalar ϕ^6 theory at large N

This chapter is heavily based on the author's publication [31] in collaboration with Christopher Herzog.

3.1 Opening remarks

So far we have discussed the basic properties of conformal field theories and their extensions by introducing boundaries. In this chapter, we attempt to examine a specific model of BCFTs. Despite its long history, it is not easy to find tractable models of BCFT away from free theories, because interactions may violate conformal invariance. The possible simplest model of BCFT is the $O(N)$ model with the quartic interaction ϕ^4 in d dimensions. We can perturbatively study this model by using ϵ expansion in $d = 4 - \epsilon$ or large N expansion when N goes very large as in [48, 49]. The motivation to study $O(N)$ ϕ^4 model comes from its connection to surface critical phenomena in statistical mechanics as well. Researchers investigated estimations of surface critical exponents and found successful correspondence to some experimental data. See [18, 68, 69] for reviews.

There are of course other examples of BCFT: free theories in the bulk, supersymmetric ones, and models constructed as gravitational theories via AdS/CFT correspondence. Though the latter two parts seem to be artificial at first glance, they have rich and exciting properties in their own right. There is much literature about them, see e.g., [22, 26, 27, 70–73]. As for theories which are free in the bulk, we have those of a scalar field which interacts only through a boundary. See refs. [74, 75] for recent investigations although such a theory provides an important cross check already in [76]. Another important class of boundary CFTs that are free in the bulk are graphene like: They have a 4d photon and 3d charged matter coupled on the boundary (see e.g. [30] and its supersymmetric extensions [77, 78]).

Beside these examples, we are here interested in a simple extension of the scalar ϕ^4 theory. Namely, we consider a $O(N)$ scalar model with the classically marginal $(\phi^2)^3$ interaction in three dimensions, which is given by the following lagrangian:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{bulk}} + \delta(z)\mathcal{L}_{\text{bry}} \,, \\
\mathcal{L}_{\text{bulk}} &= \frac{N}{2}\left[(\partial_\mu \vec{\phi})^2 + m^2(\vec{\phi}^2) + r\,(\vec{\phi}^2)^2 + \frac{g}{3}(\vec{\phi}^2)^3\right] \,, \\
\mathcal{L}_{\text{bry}} &= N\left[h_0\,\vec{\phi}\partial_z\vec{\phi} + h_1\,\vec{\phi}^2 + \frac{h_2}{2}\,(\vec{\phi}^2)^2\right] \,,
\end{aligned} \tag{3.1}$$

where ϕ is a scalar field with N components and m, r, g, and h_i are coupling constants. As in section 2.7, we place a boundary along $z = 0$. We include all classically relevant and marginal couplings to preserve such that they preserve $O(N)$symmetry both in bulk and boundary lagrangians.[1,2] Since we are interested in conformal fixed points, we finely tune m and r to be zero. For a while, we do not touch h_i and regard them as free parameters, which control boundary behaviors of ϕ.

The beta function for the marginal coupling g in absence of a boundary was calculated in eighties [83] (see also [84, 85]):

$$\beta(g) = \Lambda\frac{dg}{d\Lambda} = \frac{3g^2}{2\pi^2 N}\left(1 - \frac{g}{192}\right) + O(N^{-2}) \,, \tag{3.2}$$

which implies that in the large N limit, $\beta(g) \sim 0$. We shall take advantage of the fact that the beta function approximately vanishes, and treat g as a marginal coupling, to leading order in $1/N$. The full story is much more involved and not completely settled.[3] One might naively think that in $N \to \infty$ limit, there is a flow from an interacting UV fixed point with $g = 192$ to a free IR fixed point, which actually, turns out that the theory appears to be unstable for $g > 16\pi^2 \approx 158$ [88–90]. We will find some additional evidence for this instability in the following section.

The main subject in this chapter is to elucidate trace of stress tensor for this theory. As discussed in section 2.8.1, boundary conformal anomaly contains rich dynamical information, providing us the better way of classifying and understanding BCFTs. For our three-dimensional theory, we again present the formula,

$$\langle T^\mu{}_\mu\rangle = -\frac{\delta(x_\perp)}{4\pi}\left(a\,\hat{R} + b\,\tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu}\right) \,. \tag{3.3}$$

The coefficient a must be monotonically decreasing function under boundary RG flows [20], while the coefficient b is related to two-point function of displacement operators. As in (2.113), for BCFT we have the modified conservation law,

$$\begin{aligned}
\partial_\mu T^{\mu n} &= \mathrm{D}\delta\left(x_\perp\right) \,, \\
\partial_\mu T^{\mu a} &= -\partial_b \hat{T}^{ab}\delta\left(x_\perp\right) \,,
\end{aligned} \tag{3.4}$$

[1] Refs. [79, 80] argue that the $\vec{\phi}\partial_z\vec{\phi}$ term is in some sense redundant, that having fixed a boundary condition for the field, the coefficient of $\vec{\phi}\partial_z\vec{\phi}$ becomes scheme dependent and limited in effect to renormalizing the wavefunction of the boundary field $\phi|_{z=0}$.

[2] The particular form of the large N limit we consider here may not be unique. Researchers have speculated about the existence of other large N limits of this theory [81, 82].

[3] See [86, 87] for recent work about this subject.

where scalar primary D is the displacement operator sourced by small perturbation around the boundary. Using as same pill box argument as Gauss's law, we find

$$T^{nn}(\mathbf{x}, z)|_{z=0} = \mathrm{D}(\mathbf{x}) \ . \tag{3.5}$$

Thus in flat space with the planar boundary, the displacement operator is defined by the boundary limit of the stress tensor.

To hold them in our hands, we adopt two different approaches. For coefficient a, we compute it by placing the theory on three-dimensional hyperbolic space, which is related to flat space with a boundary by conformal transformations, and by computing its partition function. Large N limit allows us to evaluate the effective potential to leading order in $1/N$ explicitly, and as a by-product we also find in section 3.2 a bunch of boundary ordered and disorderd phases, which are separated by first and second order phase transitions.[4] Thanks to approximate conformal invariance, we see that effective potential is a function of quasi-marginal coupling g.

As for the number b, we compute it by taking the boundary limit of the two-point function of stress tensors. Recall (3.5), we can compute the two-point function of the displacement operators from that of the stress tensors.

This chapter is organized as follows. In section 3.2 we investigate the effective potential of this theory by using the large N technique, finding the interesting collection of possible phases. For later usage, we also examine propagators and Feynman rules of them. Section 3.3 is devoted to computing the two-point function of stress tensors. Then in section 3.4 we extract anomaly coefficients.

3.2 $O(N)$ model with planar boundary at large N

Let us begin with the classification of boundary conditions of this theory. The relevant term $h_1\vec{\phi}^2$ establishes the boundary conditions dominantly. The other two operators $\vec{\phi}\partial_z\vec{\phi}$ and $(\vec{\phi}^2)^2$ are marginal. In the low energy limit, the effective value of h_1/Λ becomes $\pm\infty$ or zero. The case $h_1 \to \infty$ imposes Dirichlet (or named "ordinary" in the context of surface critical phenomena) conditions on the field $\vec{\phi}$ while the finely tuned $h_1 = 0$ imposes Neumann (or "special"). The case $h_1 \to -\infty$ allows for the so-called extraordinary boundary conditions where $\phi_\alpha \sim z^{-1/2}$. Given the Coleman-Mermin-Wagner Theorem, fluctuations should destroy this $\phi_\alpha \sim z^{-1/2}$ ordering behavior on our two-dimensional surface. We presumably see this behavior because we are working in a large N limit where the fluctuations are suppressed.

As discussed in [19, 76], the Neumann case here is more subtle than in ϕ^4 theory. At this critical value, the marginal coupling h_2 can become important. These references demonstrated that there is a nonzero beta function for h_2, proportional to g, in the $3-\epsilon$ expansion. We do not have much to say about this special case $h_1 = 0$ in the present thesis, but it would be interesting to examine it more thoroughly in the future.

Our starting point is (3.1). In order to do large N analysis, we follow Hubbard-Stratonovich type transform [88] to rewrite the bulk lagrangian introducing two auxiliary fields χ and σ:

$$\mathcal{L}_{\text{bulk}} = \frac{N}{2}\left[(\partial_\mu\vec{\phi})^2 + \frac{1}{3}g\chi^3 + \sigma(\vec{\phi}^2 - \chi)\right] \ . \tag{3.6}$$

[4] Given the Coleman-Mermin-Wagner Theorem, it may seem surprising that we find boundary ordered phases in our set-up. From the point of view of the, in general, nonlocal effective two dimensional field theory living on the boundary, this theorem should prohibit surface ordering phase transitions. Presumably, we find such phases because we are looking in a large N limit.

It is easy to see that we can recover the original lagrangian (3.1) by integrating out χ and σ.

To proceed the large N computation, we divide these fields up into background plus fluctuations:

$$\phi_\alpha = \delta_{\alpha 1}\frac{\Phi}{z^{1/2}} + \delta\phi_\alpha \,, \tag{3.7}$$

$$\sigma = \frac{\Sigma}{z^2} + \delta\sigma \,, \tag{3.8}$$

$$\chi = \frac{\Xi}{z} + \delta\chi \,. \tag{3.9}$$

We are taking advantage of the presence of a boundary at $z = 0$ to allow for a coordinate dependence in the background values of the fields. To find a scale invariant solution, we are assuming that at leading order in N, the scaling dimensions of ϕ_α, σ, and χ are given by their classical values, and that Φ, Σ, and Ξ are constants. Note that a boundary ordered phase corresponds to $\Phi \neq 0$, i.e. an extraordinary transition, while a boundary disordered phase to $\Phi = 0$. In this section we consider general boundary conditions in order to study the phase structure of the theory. In the latter half of this chapter, however, we focus on correlation functions for Dirichlet boundary conditions, where $\Phi = 0$.

Next we find an effective action for the fluctuations $\delta\phi_\alpha$:

$$\frac{N}{2}\left[(\partial\delta\phi_\alpha)^2 + \frac{1}{z^2}\Sigma\,\delta\phi_\alpha^2\right] \,. \tag{3.10}$$

There is a cross term proportional to $\Phi\,\sigma\,\delta\phi_1$ which involves fluctuations only in the direction in which ϕ_α is turned on, and thus is down by a power of $1/N$ compared to the expression above; we ignore this cross term.

3.2.1 Feynman rules at large N

In what follows we analyze the Lagrangian density (3.10) which describes the behavior of a free scalar field with a position dependent mass. The $O(N)$ symmetry restricts the form of two-point functions to be $\langle\delta\phi_\alpha(x)\delta\phi_\beta(x')\rangle = \delta_{\alpha\beta}G_\phi(x,x')$, and then G_ϕ can be determined by

$$\left[\Box - \frac{\mu^2 - \frac{1}{4}}{z^2}\right] G_\phi(x,x') = \delta(x-x') \,, \quad \Sigma \equiv \mu^2 - \frac{1}{4} \,. \tag{3.11}$$

As it is not more difficult, let us for a while work in general dimension d; We can set $d = 3$ at the end of calculations. The symmetry implies that G_ϕ must take the form (see section 2.7.1),

$$G_\phi(x,x') = \frac{F(v)}{|x-x'|^{d-2}} \,, \tag{3.12}$$

where v is the conformal cross ratio (2.50), and we used the fact that at leading order the bulk scaling dimension of $\delta\phi_\alpha$ is given by $\Delta_\phi = d/2 - 1$. Substituting (3.12) into (3.11), we realize that $F(v)$ satisfies the differential equation,

$$(1-v^2)^2 vF''(v) - (d-3)(1-v^2)^2 F'(v) - \left(4\mu^2 - 1\right) vF(v) = 0 \,. \tag{3.13}$$

To have a well defined problem, we need to fix the boundary conditions in the coincident $v = 0$ and boundary $v = 1$ limits. In the coincident limit, we expect to recover the usual two-point function for a massless free field,

$$G_\phi(x, x') \sim \frac{\kappa}{|x - x'|^{d-2}} , \quad \kappa \equiv \frac{1}{N(d-2)\,\mathrm{Vol}(\mathbb{S}^{d-1})} , \tag{3.14}$$

where the value of κ follows from the normalization of the kinetic term for $\delta\phi_\alpha$. That the Lagrangian has an over-all factor of N means the propagators must all scale with $1/N$.

In the boundary limit where $v \to 1$, there are two possible behaviors $F(v) \sim (1 - v^2)^{\frac{1}{2} \pm \mu}$. We keep the $+\mu$ behavior and set the other scaling behavior to zero; a linear combination would force us to introduce a scale and break the conformal symmetry. Note the choice $\mu = \frac{1}{2}$ is the usual Dirichlet boundary condition while $\mu = -\frac{1}{2}$ is Neumann. With these boundary conditions, the unique solution of (3.13) is

$$F(v) = \kappa \frac{\Gamma(\frac{1}{2} + \mu)\Gamma(\frac{d-1}{2} + \mu)}{\Gamma(\frac{d}{2} - 1)\Gamma(1 + 2\mu)} \xi^{-\frac{1}{2} - \mu} {}_2F_1\left(\frac{1}{2} + \mu, \frac{d-1}{2} + \mu, 1 + 2\mu, -\frac{1}{\xi}\right) , \tag{3.15}$$

where ξ is a different expression of the cross ratio related to v as in (2.50). We can of course recover the other boundary condition at $z = 0$ by changing the sign $\mu \to -\mu$.

Finally we comment on the propagators of auxiliary fields σ and χ. The equation of motion for σ states that $\vec{\phi}^2 - \chi = 0$. Employing the Schwinger-Dyson equations, any correlation function involving this equation of motion should vanish up to contact terms. In particular, we have

$$\frac{N}{2} \langle \sigma(x)(\vec{\phi}^2(x') - \chi(x')) \rangle = \delta(x - x') . \tag{3.16}$$

We expect in the large N limit that the $\langle \sigma(x)\vec{\phi}^2(x') \rangle$ piece of the expression dominates as there are N identical components of $\vec{\phi}$. Furthermore, we can re-express this three point function in terms of the corresponding propagators and the three point vertex $\frac{N}{2}\sigma\vec{\phi}^2$ in the effective Lagrangian.

$$\langle \phi_\alpha(x_1)\phi_\beta(x_2)\sigma(x_3) \rangle = -\delta_{\alpha\beta} N \int_{\mathbb{R}^d_+} \mathrm{d}^d r\, G_\phi(x_1, r) G_\phi(x_2, r) G_\sigma(r, x_3) . \tag{3.17}$$

In particular, we learn that

$$\int_{\mathbb{R}^d_+} \mathrm{d}^d x'' G^2_\phi(x, x'') G_\sigma(x'', x') = -\frac{2}{N^3} \delta^d(x - x') . \tag{3.18}$$

Given that G_ϕ is $O(1/N)$, we conclude that G_σ is also $O(1/N)$. We do not need the explicit form of G_σ, but will make heavy use of (3.18) later.

The diagrams in figure 3.1 show the leading contributions to the stress tensor correlation functions of our interest. We employ rules where every propagator comes with a factor of $1/N$, every vertex and every loop with a factor of N. The black dots correspond to the inserted operators and may influence the N counting. In section 3.3, we will use these Feynman rules to compute the $\langle T^{\mu\nu}(x) T^{\sigma\rho}(x') \rangle$ at leading order in N.

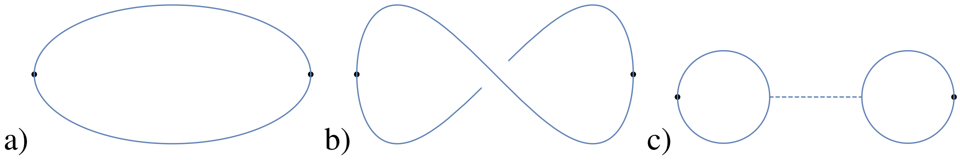

Figure 3.1: The Feynman diagrams needed for computing the $\langle T^{\mu\nu}(x)T^{\lambda\rho}(x')\rangle$ correlation functions at leading order in N. All three diagrams contribute an amount proportional to N. The solid lines are ϕ_α propagators while the dashed line is a σ propagator.

3.2.2 Effective potential

Since the effective action for $\delta\phi_\alpha$ is quadratic, which we can integrate out, the lagrangian gets the one loop contribution:

$$\mathcal{L} \to \mathcal{L} + \frac{N}{2}\,\mathrm{tr}\log\left(-\Box + \frac{\Sigma}{z^2}\right) . \tag{3.19}$$

The trace log factor is the integral of the one-point function of the operator $\langle\delta\phi_\alpha^2\rangle$. Such a one-point function can be calculated by the regularizing coincident limit of the Green's function $G_\phi(x,x')$. By a hypergeometric identity, the result (3.15) can be rewritten as

$$\begin{aligned} F(v) =& \kappa(1-v^2)^{\frac{1}{2}-\mu}{}_2F_1\left(\frac{1}{2}-\mu, \frac{3-d}{2}-\mu, 2-\frac{d}{2}, v^2\right) \\ &+ cv^2(1-v^2)^{\frac{1}{2}-\mu}{}_2F_1\left(\frac{1}{2}-\mu, \frac{d-1}{2}-\mu, \frac{d}{2}, v^2\right) , \end{aligned} \tag{3.20}$$

where

$$c = \kappa\frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}+\mu\right)}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{3-d}{2}+\mu\right)} .$$

The first hypergeometric function has singularities that must be removed in the coincident limit $v \to 0$. The one-point function is then fixed essentially by the constant c:

$$\langle\delta\phi_\alpha^2(x)\rangle = \frac{\kappa}{N}\frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}+\mu\right)}{2^{d-2}\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{3-d}{2}+\mu\right)}\frac{1}{z^{d-2}} , \tag{3.21}$$

summation on α not implied.

Integrating this one-point function over μ gives the difference in effective potential between theories with different values of μ:

$$\frac{N\kappa}{z^d}\int_0^\mu \frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}+x\right)}{2^{d-2}\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{3-d}{2}+x\right)}x\,\mathrm{d}x . \tag{3.22}$$

We followed [91] in this derivation but see also [48,92].[5] Note we are using $\mu = 0$ as a reference value around which to compute the change in the potential.

[5]For (3.22) to be consistent with scale invariance, we must either be in $d = 3$ dimensions or the integral must vanish. We are in $d = 3$, but it is useful to compare with the general d results of other authors. The large N results of Bray and Moore [93] and later McAvity and Osborn [48] correspond to setting the integrand to zero which happens when $\mu = \frac{d-3}{2}, \frac{d-5}{2}, \frac{d-7}{2}$, etc. The first two cases are the "ordinary" (Dirichlet) and "special" (Neumann) phase transitions close to $d = 4$. In general, the scaling μ means there is an operator on the boundary with scaling dimension $\mu + \frac{d-1}{2}$. The condition the integrand vanishes gives the series of dimensions $d-2$, $d-3$, $d-4$, etc. The unitarity bound cuts off this series at $d-3$ in $d = 4$ and at $d-2$ in $d = 3$.

	$\mu > 0$		$\mu < 0$	
Φ	0	$\frac{1}{2\sqrt{2\pi}}\sqrt{1 \pm \frac{2\pi\sqrt{3}}{\sqrt{g}}}$	0	$\frac{1}{2\sqrt{2\pi}}\sqrt{-1 + \frac{2\pi\sqrt{3}}{\sqrt{g}}}$
Σ	$\frac{g}{4(16\pi^2-g)}$	$\frac{3}{4}$	$\frac{g}{4(16\pi^2-g)}$	$\frac{3}{4}$
Ξ	$-\frac{1}{2\sqrt{16\pi^2-g}}$	$\pm\sqrt{\frac{3}{4g}}$	$\frac{1}{2\sqrt{16\pi^2-g}}$	$\sqrt{\frac{3}{4g}}$
$\frac{2}{N}V$	$-\frac{1}{12\sqrt{16\pi^2-g}}$	$-\frac{1}{6\pi} \pm \sqrt{\frac{3}{16g}}$	$\frac{1}{12\sqrt{16\pi^2-g}}$	$\frac{1}{6\pi} - \sqrt{\frac{3}{16g}}$

Figure 3.2: The various solutions to the equations (3.23). The potential V is calculated from (3.25).

For us, setting $d = 3$, the expression (3.22) reduces to $-\frac{N\mu^3}{12\pi z^3}$. Variational principle tells us the following conditions on Φ, Σ, and Ξ:

$$\begin{aligned} \Phi(3 - 4\Sigma) &= 0 \,, \\ \pm\sqrt{1 + 4\Sigma} - 8\pi(\Phi^2 - \Xi) &= 0 \,, \\ \Xi^2 g - \Sigma &= 0 \,, \end{aligned} \tag{3.23}$$

where the $\pm$ in the second line corresponds to a choice of sign for μ. The boundary ordered and disordered solutions to these three equations are summarized in figure 3.2. We will discuss how to compute the potential V in this figure shortly.

There are boundary ordered phases with $\phi_\alpha \neq 0$. There are two such solutions with $\mu > 0$. The solution associated with negative Ξ exists only for $g > 12\pi^2$ and corresponds to a local maximum of the effective potential, as we will see shortly. For negative μ, there is only a single ordered solution, and it exists only for $g < 12\pi^2$. Note $\Sigma = 3/4$ corresponds to $\mu = \pm 1$. The value $g = 12\pi^2$ is special for another reason, for here two of the three boundary ordered phases become disordered, with $\phi_\alpha = 0$.

There are a pair of disordered solutions with $\phi_\alpha = 0$ for more general values of g, one for each sign choice of μ. Note $\mu^2 = (4 - g/4\pi^2)^{-1}$ for these solutions. The dependence of Ξ on g in the disordered phase, in particular that Ξ becomes imaginary for $g > 16\pi^2$, suggests the theory becomes sick for $g > 16\pi^2$, consistent with the results [88–90] in absence of a boundary.

Moving onto the curved background

The fact that this theory has conformal symmetry at large N further gives us an advantage that we can work in as conformally equivalent spacetime as $\mathbb{R}^2 \times \mathbb{R}^+$. For our purpose, we take three dimensional hyperbolic space $\mathbb{H}^3$ with radius of curvature L. It is very straightforward to see that $\mathbb{H}^3$ is related to the flat space only up to Weyl factor:

$$ds^2 = \frac{L}{z^2}(dz^2 + \delta_{ab} dx^a dx^b) \,. \tag{3.24}$$

A nice feature on this curved background is that the explicit z dependence in the lagrangian disappears due to Weyl rescaling. On the other hand, what we have to pay in order to work in $\mathbb{H}^3$ is that we must include the conformal coupling of ϕ_α to the curvature $\mathcal{L} \to \mathcal{L} + \frac{N}{2}\frac{d-2}{4(d-1)} R\phi_\alpha^2$

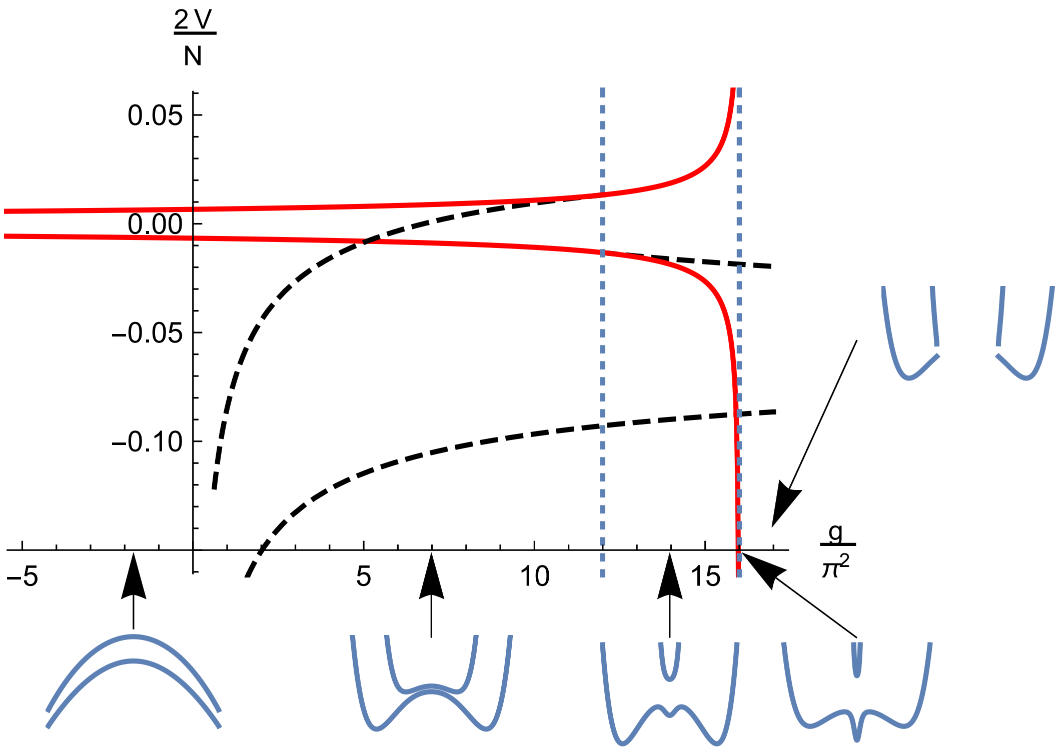

Figure 3.3: Potential vs. coupling. The solid curves are disordered ($\phi_\alpha = 0$) and the dashed curves are boundary ordered ($\phi_\alpha \neq 0$). The disordered phases cease to exist for $g > 16\pi^2$ while the ordered phases require $g > 0$. The disordered phases can join with the ordered phases at $g = 12\pi^2$. Dotted vertical lines are placed at $g = 12\pi^2$ and $g = 16\pi^2$ as a guide to the eye. The inset plots show the qualitative shape of the potential as a function of Φ in the different regions of the larger plot. There are two different branches of $V(\Phi)$: the upper branch corresponds to $\mu < 0$ and the lower branch to $\mu > 0$.

where $R = -\frac{3}{4L^2}$ for our $\mathbb{H}^3$ background with the radius of curvature L. We find the following effective potential for the fields

$$V = \frac{N}{2}\left[\frac{1}{3}g\Xi^3 + \Sigma(\Phi^2 - \Xi) - \frac{3}{4}\Phi^2 \mp \frac{(\Sigma + \frac{1}{4})^{3/2}}{6\pi}\right], \tag{3.25}$$

which gives rise to the same conditions (3.23). The choice in sign refers to the choice of sign of μ. From this hyperbolic viewpoint, we should keep the mass of the scalar field above the Breitenlohner-Freedman bound [94], $\Sigma - \frac{3}{4} > -1$. In the disordered phase, for g in the allowed range $-\infty < g < 16\pi^2$, Σ satisfies the bound, while for $g > 16\pi^2$, the fluctuations in the scalar field will have a mass below the BF bound, which causes the theory to be unstable.

We would like to then understand relative stability of the different phases from the effective potential V (see figure 3.3). The analysis has some familiar Landau-Ginzburg features, but is complicated by the dependence of the phases on boundary conditions. One can form an effective potential $V(\Phi)$ of a single variable by first extremizing $V(\Phi, \Sigma, \Xi)$ with respect to Σ and Ξ. We find that for $g < 0$, the potential has a single maximum, albeit with a curvature below the BF bound. For $0 < g < 12\pi^2$, the potential has a classical Mexican hat shape, with minima corresponding to the ordered phase and a maximum corresponding to the disordered phase. Then for $12\pi^2 < g < 16\pi^2$, there is a qualitative difference between the $\mu > 0$ and $\mu < 0$ cases. For $\mu > 1$, the maximum at $\Phi = 0$ develops a dimple that grows deeper and eventually overtakes the minima associated with the ordered phase. In contrast, for $\mu < -1$, the disordered and ordered phases coalesce into a single minimum associated with a stable disordered phase. Given that $\mu < -1$ leads to a boundary primary below the unitarity bound, we could discard this portion of the $\mu < 0$ disordered phase based on unitarity. For $g > 16\pi^2$ and either choice of sign for μ, the effective potential $V(\Phi)$ is not defined for Φ close to the origin although there are still critical points associated with the disordered phases.

Recall that we impose a boundary condition on the field $\vec{\phi}$ by adding the relevant boundary deformation $h_1 \vec{\phi}^2 \delta(z)$. For $h_1 > 0$, $\vec{\phi}$ must vanish on the boundary. To be consistent with this Dirichlet condition, the critical exponent for the fluctuation $\delta\phi_\alpha$ must satisfy $\mu > -1/2$. The only phases that are consistent with these restrictions are the lower solid (red) curve in figure 3.3 and the portion of the upper solid (red) curve satisfying $g < 0$. As the lower curve has lower potential V, it should represent the stable phase.

We next consider the choice $h_1 < 0$, for which $\vec{\phi}$ can blow up at the boundary – extraordinary boundary conditions. In this case, as $\vec{\phi}$ is already infinite, there is no restriction on μ of the fluctuation field $\delta\phi_\alpha$. All of the curves in figure 3.3 are allowed. Based on energetic considerations, the lower dashed (black) curve, corresponding to a boundary ordered phase, is preferred in the range $0 < g < \frac{3}{2}(7 + \sqrt{13})\pi^2 \approx 15.9\pi^2$. There is a first-order phase transition to a boundary disordered phase at the upper end of the range. For $\frac{3}{2}(7 + \sqrt{13})\pi^2 < g < 16\pi^2$, the boundary disordered phase is preferred. In the regime $g < 0$, there are only boundary disordered phases, while in the regime $g > 16\pi^2$ there are only boundary ordered phases. (Given the Coleman-Mermin-Wagner Theorem, we should of course keep in mind that we are likely only seeing boundary ordered phases because of the large N limit.)

The last case is "Neumann" boundary conditions $h_1 = 0$. In reality, at this point the marginal couplings h_0 and h_2 become important, and the system needs a more thorough examination. For this reason, we put "Neumann" in parentheses because the actual boundary conditions will be determined by h_0 and h_2. We leave a more thorough examination of this case to the future.

We note before moving on that it is not clear to us that the theory makes sense outside the range $0 \leq g < 16\pi^2$. The potential is unbounded for $g < 0$ and missing pieces for $g \geq 16\pi^2$.

3.3 Two-point function of stress tensor at large N

In this section we compute the stress tensor two point function $\langle T^{\mu\nu}(x) T^{\sigma\rho}(x')\rangle$ in the Dirichlet boundary condition where $\Phi = 0$. From (3.10), the stress tensor for a ϕ field at large N is given by

$$\begin{aligned}\frac{1}{N}T_{\mu\nu} =&(\partial_\mu\vec{\phi})\cdot(\partial_\nu\vec{\phi}) - \frac{\delta_{\mu\nu}}{2}\left((\partial\vec{\phi})^2 + \left(\mu^2 - \frac{1}{4}\right)\frac{\vec{\phi}^2}{z^2}\right) - \frac{d-2}{4(d-1)}\left(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2\right)\vec{\phi}^2 \\ =& -\vec{\phi}\cdot\mathcal{D}_{\mu\nu}\vec{\phi} + \frac{d}{4(d-1)}\mathcal{D}_{\mu\nu}\vec{\phi}^2 - \frac{\delta_{\mu\nu}}{d}\left(\mu^2 - \frac{1}{4}\right)\frac{\vec{\phi}^2}{z^2}\,, \end{aligned} \tag{3.26}$$

where in the last line we used the equation of motion and introduced $\mathcal{D}_{\mu\nu} \equiv \partial_\mu\partial_\nu - \frac{1}{d}\delta_{\mu\nu}\partial^2$. The overall factor of N comes from the normalization of the Lagrangian. The last term represents position depending coupling to the background σ field.

Using the usual Feynman rules adapted to this large N boundary situation, we divide up the calculation of the stress-tensor two point function into a free part and an interaction part:

$$\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle = \langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle_{\text{free}} + \langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle_{\text{int}}\,. \tag{3.27}$$

For the free part, we use the stress tensor (3.26) and Wick's Theorem, albeit with the propagator $G_\phi(x,x')$ involving a nonzero μ. The two different ways of contracting the ϕ fields give the t and u channel diagrams in figure 3.1. $\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle_{\text{free}}$ can be further decomposed into a trace free part

$$\begin{aligned}\frac{1}{N^3}\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle'_{\text{free}} =& G_\phi\mathcal{D}_{\mu\nu}\mathcal{D}'_{\sigma\rho}G_\phi + (\mathcal{D}_{\mu\nu}G_\phi)\mathcal{D}'_{\sigma\rho}G_\phi \\ &- \frac{d}{2(d-1)}\left(\mathcal{D}_{\mu\nu}(G_\phi\mathcal{D}'_{\sigma\rho}G_\phi) + \mathcal{D}'_{\sigma\rho}(G_\phi\mathcal{D}_{\mu\nu}G_\phi)\right) \\ &+ \frac{d^2}{8(d-1)^2}\mathcal{D}_{\mu\nu}\mathcal{D}'_{\sigma\rho}G_\phi^2\,, \end{aligned} \tag{3.28}$$

and a remainder

$$\begin{aligned}&\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle_{\text{free}} - \langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle'_{\text{free}} = \\ &-\frac{2}{d}\left(\mu^2 - \frac{1}{4}\right)\left(\frac{\delta_{\mu\nu}\hat{t}_{\sigma\rho}(x',x)}{z^2} + \frac{\delta_{\sigma\rho}\hat{t}_{\mu\nu}(x,x')}{z'^2}\right) + \frac{2N^3}{d^2}\delta_{\mu\nu}\delta_{\sigma\rho}\frac{\left(\mu^2 - \frac{1}{4}\right)^2}{(zz')^2}G_\phi^2(x,x')\,, \end{aligned} \tag{3.29}$$

where we have defined

$$\frac{1}{N^3}\hat{t}_{\mu\nu}(x,x') \equiv -G_\phi(x,x')\mathcal{D}'_{\mu\nu}G_\phi(x,x') + \frac{d}{4(d-1)}\mathcal{D}_{\mu\nu}(G_\phi(x,x'))^2\,. \tag{3.30}$$

The interaction contribution to the stress-tensor is dominated at leading order in N by exchange of a σ field:

$$\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle_{\text{int}} = \int_{\mathbb{R}^d_+}\mathrm{d}^d r\int_{\mathbb{R}^d_+}\mathrm{d}^d r'\, t_{\mu\nu}(x,r)t_{\sigma\rho}(x',r')G_\sigma(r,r')\,, \tag{3.31}$$

where the unhatted $t_{\mu\nu}(x,x')$ has a trace part,

$$t_{\mu\nu}(x,x') = \hat{t}_{\mu\nu}(x,x') - \frac{\delta_{\mu\nu}}{d}\frac{\mu^2 - \frac{1}{4}}{z^2}N^3G_\phi(x,x')^2\,. \tag{3.32}$$

Thanks to the identity (3.18), the trace parts of the free contribution and the interaction contribution cancel out and one is left with

$$\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle = \langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle'_{\text{free}} + \langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle'_{\text{int}} \,, \tag{3.33}$$

where

$$\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle'_{\text{int}} = \int_{\mathbb{R}^d_+} \mathrm{d}^d r \int_{\mathbb{R}^d_+} \mathrm{d}^d r' \, \hat{t}_{\mu\nu}(x,r)\hat{t}_{\sigma\rho}(x',r')G_\sigma(r,r') \,. \tag{3.34}$$

It is not necessary to have an explicit form for $G_\sigma(r,r')$ to proceed. Instead, we recognize the two-point function

$$\langle T_{\mu\nu}(x)\sigma(x')\rangle = -N\int_{\mathbb{R}^d_+} \mathrm{d}^d r \, t_{\mu\nu}(x,r)G_\sigma(r,x') \,, \tag{3.35}$$

is fixed by conformal symmetry and a Ward identity to have the form [48]

$$\langle T_{\mu\nu}(x)\sigma(x')\rangle = -N\frac{2d(4\mu^2-1)}{(d-1)\,\mathrm{Vol}(\mathbb{S}^{d-1})}\frac{(2z')^{d-2}}{s^{2d}}\left(X_\mu X_\nu - \frac{1}{d}\delta_{\mu\nu}\right)v^d \,. \tag{3.36}$$

Changing between the hatted $\hat{t}_{\mu\nu}$ and the unhatted $t_{\mu\nu}$ alters $\langle T_{\mu\nu}(x)\sigma(x')\rangle$ by a contact term proportional to $\langle\sigma\rangle\delta(x-x')$, as can be seen from (3.18). In fact the two point function $\langle T_{\mu\nu}(x)\sigma(x')\rangle$ more generally is arbitrary up to contact terms of this form [48]. The stress tensor itself is ambiguous up to a shift $T_{\mu\nu} \to T'_{\mu\nu} = T_{\mu\nu} + c\lambda\sigma\delta_{\mu\nu}$ where λ is a position dependent source for σ and c is an arbitrary constant. The stress tensor one point function is untouched when $\lambda = 0$. Through this shift, however, we can adjust the contact term in the two point function at will. We choose to regulate the two point function such that $\langle T^\mu_\mu(x)\sigma(x')\rangle = 0$, including distributional contributions of the form $\delta(x-x')$. Through the identification (3.35), we can then be sure that the stress-tensor two-point function $\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle$ is traceless.

We can also write $\hat{t}_{\mu\nu}$ itself in terms of the X_μ. Inserting the form of G_ϕ into the definition (3.30), we obtain

$$\frac{1}{N^3}\hat{t}_{\mu\nu} = \frac{(2z')^2}{s^{2d}}\left(X_\mu X_\nu - \frac{1}{d}\delta_{\mu\nu}\right)f(v) \,, \tag{3.37}$$

$$f(v) = -\frac{2}{d-1}\xi(\xi+1)\left((d-2)F(v)\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}F(v)\right) - d\xi^2\left(\frac{\mathrm{d}}{\mathrm{d}\xi}F(v)\right)^2\right) \,. \tag{3.38}$$

Assembling the pieces, we can write the trace free part of the interaction contribution to the stress tensor two point function as

$$\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle'_{\text{int}} = \tag{3.39}$$
$$N^3\frac{2d(4\mu^2-1)}{(d-1)\,\mathrm{Vol}(\mathbb{S}^{d-1})}\int_{\mathbb{R}^d_+}\mathrm{d}^d r\left(\frac{2z\tilde{v}}{\tilde{s}^2\tilde{s}'^2}\right)^d f(\tilde{v}')\left(\tilde{X}_\mu\tilde{X}_\nu - \frac{\delta_{\mu\nu}}{d}\right)\left(\tilde{X}'_\rho\tilde{X}'_\sigma - \frac{\delta_{\sigma\rho}}{d}\right) ,$$

where we denote $r = (\boldsymbol{r}, y)$, $\tilde{s} = (x-r)^2$, $\tilde{v}^2 = \tilde{s}^2/(\tilde{s}^2+4zy)$ and $\tilde{s'}^2$, $\tilde{v'}^2$ similarly defined with $x \to x'$.

To organize the information in the stress-tensor two-point function, we again take advantage of the conformal symmetry. Recall basis functions $\alpha(v)$, $\gamma(v)$ and $\epsilon(v)$ as defined in (2.67), (2.68) and (2.69). We further impose tracelessness, which is reduced to $\alpha = (d-1)((d-1)\delta + 2\epsilon)$. In addition, the conservation law reduces the information further, to a single function of a cross ratio:

$$v\alpha'(v) - d\alpha(v) = 2(d-1)\gamma(v) \,, \tag{3.40}$$

$$v\gamma'(v) - d\gamma(v) = \frac{d}{(d-1)^2}\alpha(v) + \frac{(d-2)(d+1)}{d-1}\epsilon(v) \,. \tag{3.41}$$

That $\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle'_{\text{free}}$ and $\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle'_{\text{int}}$ are independently traceless means that we can completely specify their form by computing the functions α, γ and ϵ for each structure. However, they are not independently conserved. Only the total is conserved.

We first compute $\langle T^{\mu\nu}(x)T^{\sigma\rho}(x')\rangle'_{\text{free}}$, restricting to the case $d = 3$. From the definitions (2.67), (2.68), (2.69), and plugging in the explicit form of G_ϕ into (3.28), we establish

$$\begin{aligned}\alpha_{\text{free}}(v) = \frac{N^3\kappa^2}{9}\frac{(1-v)^{2\mu-1}}{(1+v)^{2\mu+1}}\big\{v\big[9\mu + v\big(32\mu^4v^2 + 48\mu^3\left(v^2+1\right)v + 44v^2 + \\ + 4\mu^2\left(9v^4+8v^2+9\right) + 3\mu\left(3v^4+5v^2+5\right)v + 9\left(v^2-3\right)v^4 - 27\big)\big] + 9\big\} \,,\end{aligned} \tag{3.42}$$

$$\begin{aligned}\gamma_{\text{free}}(v) = -\frac{1}{4}N^3\kappa^2\left(\frac{1-v}{1+v}\right)^{2\mu} \\ \times\big\{v\big[6\mu + v\big(3v^4 + 8\mu^2\left(v^2+1\right) + 2\mu\left(3v^2-1\right)v - 2v^2 + 8\mu^3v - 2\big)\big] + 3\big\} \,,\end{aligned} \tag{3.43}$$

$$\epsilon_{\text{free}}(v) = \frac{1}{8}N^3\kappa^2\frac{(1-v)^{2\mu+1}}{(1+v)^{2\mu-1}}\left(v\left(21\mu + v\left(20\mu^2+6v^2+21\mu v+10\right)\right)+6\right) \,. \tag{3.44}$$

One can confirm when $\mu = \pm 1/2$, they reproduce results [49] for the free scalar with Dirichlet and Neumann boundary conditions.

Our next task is to calculate the interaction part (3.39). In our setup, with (3.15) in $d = 3$, $f(v)$ becomes relatively simple:

$$f(v) = \frac{1}{2}\kappa^2v^3(v+1)^{-4\mu}\left(1-v^2\right)^{2\mu-1}\left(3\mu + v\left(4\mu^2+3\mu v+2\right)\right) \,. \tag{3.45}$$

The integral in (3.39) is generally organized into

$$\int_0^\infty \mathrm{d}y \int \mathrm{d}^{d-1}\boldsymbol{r}\frac{1}{(2y)^d}\, f_1(\tilde\xi)f_2(\tilde\xi')\left(\tilde X_\mu\tilde X_\nu - \frac{1}{d}\delta_{\mu\nu}\right)\left(\tilde X'_\rho\tilde X'_\sigma - \frac{1}{d}\delta_{\sigma\rho}\right) \,, \tag{3.46}$$

where we can identify $f_1 = [\xi(\xi+1)]^{-d/2}$ and $f_2 = N^2f(v)\xi^{-3}$. In Appendix D of [48], the authors investigate a method how to compute (3.46) for the case that $f_2 \sim [\xi(\xi+1)]^{-n}$. We review some aspects of their method in our Appendix A and further generalize it. The final result is

$$\begin{aligned}\epsilon_{\text{int}}(v) &= c\xi^d 4\mathcal{G}''(\xi) \,, \\ \gamma_{\text{int}}(v) &= c\xi^d\left[4(1+2\xi)\mathcal{G}''(\xi) + 8(1+\xi)\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{G}''\right] \,, \\ \alpha_{\text{int}}(v) &= c\xi^d\Big[-\frac{8}{d}(d-1)^2(1+\xi)\xi\left((2\xi+1)\frac{\mathrm{d}}{\mathrm{d}\xi}+2\right)\mathcal{G}''(\xi) \\ &\qquad + \frac{8}{d}(d-1)\mathcal{G}''(\xi) - \frac{(d-1)^2}{d^3}\operatorname{Vol}(\mathbb{S}^{d-1})f_2(\xi)\Big] \,,\end{aligned} \tag{3.47}$$

where

$$c = N\frac{2d(4\mu^2-1)}{(d-1)\,\mathrm{Vol}(\mathbb{S}^{d-1})}$$

is a constant of proportionality. The function $\mathcal{G}''(\xi)$ for given f_2 is a solution of the second order differential equation (A.17).

Our strategy for finding $\mathcal{G}''(\xi)$ is somewhat different than [48]. Rather than pursuing a solution via integral transforms, we solve the differential equation (A.17). In $d = 3$, defining $\mathcal{F}(v) \equiv \mathcal{G}''(\xi)$, the differential equation takes the form

$$\mathcal{F}''(v) + \frac{2(3+2v^2)}{v(1-v^2)}\mathcal{F}'(v) + \frac{20}{(1-v^2)^2}\mathcal{F}(v) = S(v)\,, \tag{3.48}$$

where the source term is

$$S(v) = -\frac{(1+v)^{-2\mu+2}(1-v)^{2\mu+2}(3\mu + v(2+3v\mu+4\mu^2))}{96\pi v^5}\,. \tag{3.49}$$

The expression (3.48) has two homogeneous solutions:

$$\mathcal{F}_1(v) = \frac{(1-v^2)^5}{v^5}\,, \tag{3.50}$$

$$\mathcal{F}_2(v) = \frac{-3v + 14v^3 - 14v^7 + 3v^9 + 3(1-v^2)^5\tanh^{-1}(v)}{128v^5}\,, \tag{3.51}$$

with Wronskian

$$\mathcal{W} = \mathcal{F}_1(v)\mathcal{F}_2'(v) - \mathcal{F}_1'(v)\mathcal{F}_2(v) = \frac{(1-v^2)^5}{v^6}\,. \tag{3.52}$$

Our boundary conditions are that $\mathcal{F}(v)$ is less singular than v^{-5} in the coincident $v \to 0$ limit and vanishes faster than $(v-1)$ in the boundary $v \to 1$ limit, leading to the solution of interest

$$\mathcal{F}(v) = -\mathcal{F}_2\int_v^1 \frac{\mathcal{F}_1(v')S(v')}{\mathcal{W}(v')}\mathrm{d}v' - \mathcal{F}_1\int_0^v \frac{\mathcal{F}_2(v')S(v')}{\mathcal{W}(v')}\mathrm{d}v'\,. \tag{3.53}$$

These boundary conditions are consistent with the behavior of the integral (A.11) in the $v \to 0$ and $v \to 1$ limits.

As the two point function satisfies a conservation Ward identity, all of the information in the two point function is encoded in the single function $\alpha(v)$. With a solution for $\mathcal{F}(v)$ in hand, we can plug it into (3.47) to obtain $\alpha_{\text{int}}(v)$ and add to that the "free" contribution $\alpha_{\text{free}}(v)$ (3.42) to obtain the net result. The remaining functions $\gamma(v)$ and $\epsilon(v)$ can then be constructed from the conservation relation (3.40) and (3.41). Alternatively and as a cross check, one can obtain $\gamma(v)$ and $\epsilon(v)$ from (3.47), (3.43) and (3.44). The result is the same.

We have not been able to find a closed form expression for the integral (3.53), but nevertheless, this presentation of the solution is very convenient. We will use it to analyze the limits $\alpha(0)$ and $\alpha(1)$ next. In the subsections to come, we present closed form expressions in four special cases $\mu = \pm\frac{1}{2}$, 0, and 1. Figure 3.4 presents a graph of $\alpha(v)$ in these four cases. Finally in section 3.3.4, we decompose $\alpha(v)$ into bulk and boundary conformal blocks for general μ, which will give us some information about the spectrum of bulk and boundary conformal primaries in this theory.

The value of $\alpha(v)$ in the coincident limit is universal, $\alpha(0) = N/16\pi^2$ regardless of μ. The interaction part $\alpha_{\text{int}}(0)$ vanishes, and the answer is given just by the free part $\alpha_{\text{free}}(0)$, which is equal to $N/16\pi^2$. Without a boundary, the two-point function is fixed up to not just a function but a constant. In the coincident limit of our theory with a boundary, we expect to recover this constant, or central charge, $\alpha(0)$, also sometimes called C_T. This number should be independent of boundary conditions. Here we find it is also independent of the quasi-marginal coupling g.

On the other hand, $\alpha(1)$ is very sensitive to μ and through μ, to the coupling g. It is known that $\alpha(1)$ gives the normalization of the displacement operator two-point function and thus is also related to a boundary central charge in the trace anomaly [30], a fact whose consequences we will investigate in section 3.4. It is straightforward to analyze

$$\alpha(1) = -\frac{64(4\mu^2 - 1)N}{\pi} \int_0^1 \frac{\mathcal{F}_2(v)S(v)}{\mathcal{W}(v)} \mathrm{d}v \,, \tag{3.54}$$

numerically for $\mu > 1/2$ and also via saddlepoint approximation in the large μ limit. With a little bit of effort, we can extend the region of validity of this formula to $\mu > -1$ through a minimal subtraction procedure, removing the power law divergences at the upper range of the integral $v \to 1$. Beyond $\mu = -1$ (the unitarity bound for the boundary operators), the subtraction procedure becomes ambiguous because of the presence of logarithms.

We provide plots of $\alpha(1)$ in figure 3.5. The saddlepoint approximation yields

$$\frac{\alpha(1)}{\alpha(0)} \sim \mu \frac{8}{15} e^{\frac{1-\sqrt{13}}{2}} \sqrt{\pi \left(50 + \frac{172}{\sqrt{13}}\right)} \sim 2.54\mu \,. \tag{3.55}$$

Numerically, we see that for $g < 0$ (equivalently $-\frac{1}{2} < \mu < \frac{1}{2}$), $\alpha(1)$ satisfies the inequality $\alpha(1) < 2\alpha(0)$ while for the coupling in the domain $0 < g < 16\pi^2$ (equivalently $|\mu| > \frac{1}{2}$), we have instead $\alpha(1) > 2\alpha(0)$. It is unclear to us whether the $g < 0$ cases are physical. On the one hand, they correspond to an unbounded ϕ^6 potential. On the other, from the point of view of a Weyl equivalent hyperbolic space, the curvature at the maximum of the potential is above the BF bound. In ref. [30], it was found that $\alpha(1) < 2\alpha(0)$ in the case of a theory with interactions confined to the boundary. Thus our "less physical" case agrees with the previous study. Interestingly, the $\alpha(v)$ we find for the $\mu = 0$ case is the same as that found in [48] for $d = 3$ ϕ^4 theory at large N with Dirichlet boundary conditions.

3.3.1 Perturbative expansion by small coupling

We begin with the small coupling limit. Recalling $\mu^2 = (4 - g/4\pi^2)^{-1}$, in the small g limit, μ can be expanded as

$$\mu = \pm \left(\frac{1}{2} + \frac{g}{64\pi^2}\right) + O\left(g^2\right) \,. \tag{3.56}$$

To leading order, we are allowed to set $\mu = \pm 1/2$ in $f(v)$ due to the overall coefficient in the interaction part (3.39). In these cases, we find

$$S(v) = \mp \frac{(1 - v^2)^3}{64\pi v^5} \,. \tag{3.57}$$

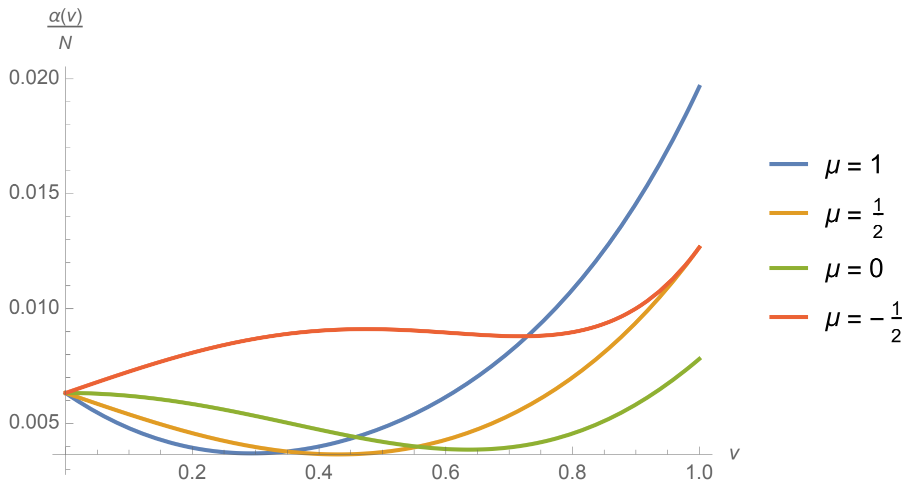

Figure 3.4: We plot $\alpha(v)$ for various values of μ. All curves start with the same value at $v = 0$ while they end with different values at $v = 1$.

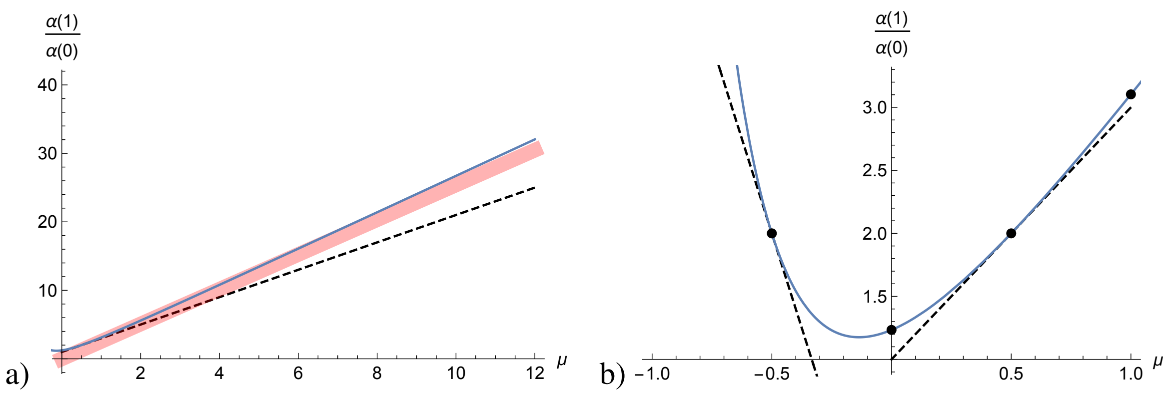

Figure 3.5: A plot of $\alpha(1)/\alpha(0)$ vs. μ. The solid blue line was computed numerically. The dashed black lines are tangents at $\mu = \pm 1/2$. The thick red line is the large μ saddle point approximation. The black dots are analytically computed points. (b) zooms in on the small μ region of (a). There is a minimum at approximately $\mu = -0.136$.

Enforcing the boundary conditions described above, we find a solution that in fact vanishes at $v = 0$ and 1. We obtain

$$\mathcal{F}(v) = \mp \frac{(1-v)^5(1+v)(v(3+v(8+3v)) - 3(1+v)^4 \tanh^{-1}(v))}{1536\pi v^5} . \tag{3.58}$$

Writing $\mathcal{F}(v) = \mathcal{G}''(\xi)$ in terms of ξ and using the relations (3.47), the interaction parts are given as follows:

$$\alpha_{\text{int}}(v) = \pm \frac{gN}{64\pi^2} \frac{v\left((9v^4+6v^2+9)\tanh^{-1}(v) + v((4-9v)v-9)\right)}{48\pi^2} + O(g^2) , \tag{3.59}$$

$$\gamma_{\text{int}}(v) = \pm \frac{gN}{64\pi^2} \frac{(v-1)v\left(3(v+1)^2(v^2+1)\tanh^{-1}(v) - v(v(3v+4)+3)\right)}{32\pi^2(v+1)} + O(g^2) , \tag{3.60}$$

$$\epsilon_{\text{int}}(v) = \pm \frac{gN}{64\pi^2} \frac{(v-1)^2 v\left(3(v+1)^4\tanh^{-1}(v) - v(v(3v+8)+3)\right)}{128\pi^2(v+1)^2} + O(g^2) . \tag{3.61}$$

To combine with the free part, we also expand (3.42)-(3.44) in the small coupling limit. The net result for $\alpha(v)$ is

$$\begin{aligned}\alpha(v) = {} & N\left(\frac{1+v^6}{16\pi^2} + \frac{g}{512\pi^4} v\left(v+v^3+3(1-v^2)^2\tanh^{-1}(v)\right)\right) \\ & \pm N\Bigg(-\frac{3v(1-v^2)^2}{32\pi^2} + \frac{g}{1024\pi^4}\Big(v(1+(v-3)v)(1+v^2)+ \\ & + (-4+3v+2v^3+3v^5-4v^6)\tanh^{-1}(v)\Big)\Bigg) + O(g^2) ,\end{aligned} \tag{3.62}$$

where the plus sign corresponds to Dirichlet boundary conditions and the minus sign to Neumann. As the total result satisfies the conservation Ward identities (3.40) and (3.41), we can easily construct $\gamma(v)$ and $\epsilon(v)$ from $\alpha(v)$.

The boundary limit of $\alpha(v)$ is interesting because it represents the normalization of the displacement operator two point function. We find

$$\alpha(1) = \frac{N}{8\pi^2}\left(1 + \frac{2g \mp g}{64\pi^2}\right) + O(g^2) , \tag{3.63}$$

which suggests that $\alpha(1)$ starts as an increasing function of the coupling g. We also see that the bulk limit of α is $\alpha(0) = N/16\pi^2$, which implies that $\alpha(1) > 2\alpha(0)$ when $g > 0$.

3.3.2 $\mu = 0$: strong coupling limit

The next example is the $\mu = 0$ case, which corresponds to the $g \to -\infty$ limit. It is not clear that the theory is stable in this limit, as the ϕ^6 potential is unbounded below. We can nevertheless naively proceed with the same analysis of the stress tensor two point function. In this case, we have

$$S(v) = -\frac{(1-v^2)^2}{48\pi v^4} . \tag{3.64}$$

We discover a solution

$$\begin{aligned}\mathcal{F}(v) = \frac{1-v^2}{6144\pi v^5}\{&6(1-v^2)^4\tanh^{-1}(v)\log(v)\\ &+2v(3-v^2+v^4-3v^6-(1+v^2)(3-14v^2+3v^4)\log(v))\\ &+3(1-v^2)^4(\mathrm{Li}_2(-v)-\mathrm{Li}_2(v))\}\ ,\end{aligned} \tag{3.65}$$

where $\mathrm{Li}_n(x)$ is a polylogarithm. This solution scales as v^{-3} in the coincident limit and $(1-v)^4$ in the boundary limit.

Adding α_{int} and α_{free} together, the information in the stress tensor two point function is encapsulated in the single function

$$\begin{aligned}\alpha(v) = \frac{N}{512\pi^2}\{&v\left(3v^4+2v^2+3\right)\left(4\mathrm{Li}_2(v)-\mathrm{Li}_2\left(v^2\right)\right)\\ &+4\left(8-8v^6+19v^4-19v^2+v\log(v)\left(3\left(v^3+v\right)-\left(3v^4+2v^2+3\right)\tanh^{-1}(v)\right)\right)\}\ ,\end{aligned} \tag{3.66}$$

from which we may construct $\gamma(v)$ and $\epsilon(v)$ using the conservation equations (3.40) and (3.41). We observe that

$$\alpha(1) = \frac{N}{128}\,. \tag{3.67}$$

As usual we find $\alpha(0) = N/16\pi^2$, and so it follows $\alpha(1) < 2\alpha(0)$. While the inequality $\alpha(1) < 2\alpha(0)$ is consistent with results that were found for a theory with only boundary interactions [30], it is not clear that the $\mu = 0$ case studied here is physical – because of the unbounded potential.

As mentioned already, this case was studied in [48]. The authors computed the two-point function of the stress tensor in ϕ^4 theory at large N, for general dimension $d \leq 4$. In the particular case $d = 3$ with "Dirichlet" boundary conditions, their two-point function reduces to ours. Their answer, valid for general d, is written in term of a hypergeometric function ${}_3F_2$. With some effort, one can demonstrate that in fact the two solutions are the same at $d = 3$.

3.3.3 One more special case: $\mu = 1$

For $\mu = 1$, we have

$$S(v) = -\frac{(1-v)^4(1+v)^2}{32\pi v^5}\,, \tag{3.68}$$

and find that

$$\begin{aligned}\mathcal{F}(v) = -\frac{1-v^2}{6144\pi v^5}\Bigg(&-2v(1-v^2)(-9+v(16+v(-6+v(-32+v(-9+16v)))))\\ &+6(-3v+11v^3+11v^5-3v^7+3(1-v^2)^4\tanh^{-1}(v))\log(v)\\ &+9(1-v^2)^4(\mathrm{Li}_2(-v)-\mathrm{Li}_2(v))\Bigg)\ ,\end{aligned} \tag{3.69}$$

with the same boundary conditions as before. This function diverges as v^{-3} in the coincident limit and vanishes as $(v-1)^5$ in the boundary limit. Inserting the result for $\mathcal{F}(v)$ into (3.47) and adding the free result, we obtain

$$\begin{aligned}\alpha(v) = -\frac{N}{256\pi^2} &\left\{9v\left(3v^4+2v^2+3\right)\left(\mathrm{Li}_2(-v)-\mathrm{Li}_2(v)\right)\right.\\ &+18v\log(v)\left(\left(3v^4+2v^2+3\right)\tanh^{-1}(v)-3\left(v^3+v\right)\right)\\ &\left.+2v(v(v(v(8v(v+3)+21)+16)-21)+24)-16\right\},\end{aligned} \tag{3.70}$$

from which we may construct $\gamma(v)$ and $\epsilon(v)$ using the conservation relations (3.40) and (3.41). Taking the boundary and bulk limit, we end up with

$$\alpha(1) = N\left(\frac{9}{128}-\frac{1}{2\pi^2}\right), \quad \alpha(0) = \frac{N}{16\pi^2}, \tag{3.71}$$

which implies $\alpha(1) > 2\alpha(0)$ in the case at hand $\mu = 1$, and $\alpha(1)|_{\mu=1} > \alpha(1)|_{\mu=\frac{1}{2}}$.

3.3.4 Conformal block decomposition

Thus far we have calculated the two-point function of the stress tensor. By using the operator product expansion, we can re-express these correlation functions as sums over exchanged operators. Given conformal symmetry, these sums naturally arrange themselves into conformal blocks, where each block compactly represents the exchange of a conformal primary operator and all its descendants, see section 2.7.2. Our task is to determine the OPE coefficients μ^2_Δ, $a_\Delta\lambda_\Delta$, and scaling dimensions Δ in our theory.

Boundary decomposition

Let us begin with the boundary decomposition of $\alpha(v)$ in $\langle T_{\mu\nu}(x_1)T_{\sigma\rho}(x_2)\rangle$. The decomposition has the form

$$\alpha(v) = \xi^d\left(\mu^2_{(0)}\alpha^{(0)}_{\mathrm{bry}}(v) + \sum_{\Delta\geq d-1}\mu^2_\Delta\alpha^{(2)}_{\mathrm{bry}}(\Delta, v)\right), \tag{3.72}$$

where $\alpha^{(0)}_{\mathrm{bry}}(v)$ and $\alpha^{(2)}_{\mathrm{bry}}(\Delta, v)$ are given in (2.87) and (2.88).

Before giving the general solution, let us recall what happens in the free case $g = 0$ [30,50]. In the free theory, one can make use of the following identity

$$\frac{1}{2}\left(1+v^{2d}\right) = \xi^d\left(\alpha^{(0)}_{\mathrm{bry}}(v) + \sum_{j\in 2\mathbb{Z}^*}\mu^2_j\alpha^{(2)}_{\mathrm{bry}}(d+j, v)\right), \tag{3.73}$$

where $\mathbb{Z}^*$ is the set of non-negative integers and

$$\mu^2_j = \frac{2^{-d-2j}\sqrt{\pi}\Gamma(d+j-1)\Gamma(d+j+2)}{\Gamma(d)\Gamma\left(\frac{d}{2}-1\right)\Gamma(j+3)\Gamma\left(\frac{d+1}{2}+j\right)}. \tag{3.74}$$

The full result is then obtained by tweaking the series representation of $\frac{1}{2}(1+v^{2d})$ slightly. For Dirichlet conditions $\alpha^{(2)}_{\mathrm{bry}}(d, v)$ is removed while for Neumann conditions, its contribution is doubled.

While we cannot find a general closed form solution for $\alpha(v)$, it is straightforward to expand (3.53) near $v=1$ and from this integral representation, construct the first few terms in a series expansion for $\alpha(v)$ near the boundary.

We find the dimensions of the spin-two boundary blocks are $4+2\mu+2j$ where j is a non-negative integer and $\alpha(v)$ is expanded as

$$\alpha(v)=\xi^3\left(\alpha(1)\alpha^{(0)}(v)+\sum_{j=0}^{\infty}\mu_j^2\alpha_{\rm bry}^{(2)}(4+2\mu+2j,v)\right)\,. \tag{3.75}$$

The spectrum of dimensions is natural if we can associate the boundary limit of the field ϕ_α with an operator $\hat{\mathcal{O}}_\alpha$ of dimension $\mu+1$. we find spin-two operators of the form $\Box^j(\partial_\mu\hat{\mathcal{O}}_\alpha)(\partial_\nu\hat{\mathcal{O}}_\alpha)$ with scaling dimension of $4+2\mu+2j$. The first few coefficients in this sum are

	$16\pi^2N^{-1}\mu_j^2$				
j	μ	$\mu=-\frac{1}{2}$	$\mu=0$	$\mu=\frac{1}{2}$	$\mu=1$
0	$\frac{(1+\mu)(2+\mu)}{2^{4\mu-2}(3+2\mu)^2}$	3	$\frac{8}{9}$	$\frac{15}{64}$	$\frac{3}{50}$
1	$\frac{3(1+\mu)(3+\mu)}{2^{4\mu+2}(5+2\mu)^2}$	$\frac{15}{64}$	$\frac{9}{100}$	$\frac{7}{256}$	$\frac{3}{392}$
2	$\frac{3(2+\mu)(4+\mu)(3+2\mu)}{2^{4\mu+5}(5+2\mu)(7+2\mu)^2}$	$\frac{7}{256}$	$\frac{9}{980}$	$\frac{45}{16384}$	$\frac{25}{32{,}256}$
3	$\frac{5(2+\mu)(5+\mu)(5+2\mu)}{2^{4\mu+9}(7+2\mu)(9+2\mu)^2}$	$\frac{45}{16{,}384}$	$\frac{125}{145{,}152}$	$\frac{33}{131{,}072}$	$\frac{35}{495{,}616}$
4	$\frac{15(3+\mu)(6+\mu)(5+2\mu)}{2^{4\mu+14}(9+2\mu)(11+2\mu)^2}$	$\frac{33}{131{,}072}$	$\frac{75}{991{,}232}$	$\frac{91}{4{,}194{,}304}$	$\frac{735}{121{,}831{,}424}$

(3.76)

while $\alpha(1)$ was given in (3.54). The $\mu=\pm\frac{1}{2}$ columns agree with the $1+v^{2d}$ decomposition discussed above.

Bulk decomposition

In the case of bulk conformal decomposition, we consider $A(v)$ instead of $\alpha(v)$. From (2.66), we have

$$A(v)=\frac{d^2}{(d-1)^2}\alpha(v)+4\gamma(v)+\frac{2(d-2)}{d-1}\epsilon(v)\,. \tag{3.77}$$

$A(v)$ can be organized into the following form,

$$A(v)=\sum_{\Delta\neq 0}a_\Delta\lambda_\Delta A_{\rm bulk}(\Delta,v)\,, \tag{3.78}$$

where $A_{\rm bulk}(\Delta,v)$ is given by (2.79). We find first few coefficients as below

	$16\pi^2N^{-1}\,a_\Delta\lambda_\Delta$				
Δ	μ	$\mu=-\frac{1}{2}$	$\mu=0$	$\mu=\frac{1}{2}$	$\mu=1$
1	$-\frac{9\mu}{8}$	$\frac{9}{16}$	0	$-\frac{9}{16}$	$-\frac{9}{8}$
2	$2(4\mu^2-1)$	0	-2	0	6
3	$-5\mu(4\mu^2-1)$	0	0	0	-15
4	$2(4\mu^2-1)^2$	0	2	0	18
5	$-\frac{7}{3}\mu(4\mu^2-1)^2$	0	0	0	-21
6	$a_6\lambda_6$	6	$-\frac{16}{525}(29+105\log(v))$	6	$\frac{16}{175}(123-315\log v)$

(3.79)

where

$$a_6\lambda_6 = N\frac{-29+1632\mu^2-3474\mu^4+2240\mu^6-105(4\mu^2-1)^2\log(v)}{525\pi^2} \,. \tag{3.80}$$

We can see that for general μ the bulk blocks with dimension $\Delta \geq 6$ have logs in their expansion. The appearance of a logarithm is a problem as it introduces a scale to what is supposed to be a scale invariant theory. We can gain some insight from the $\mu = 0$ case, where our expression matches a result from [48]. In this older paper, the authors computed the stress tensor two-point function for ϕ^4 theory in a large N limit and general dimension. In the specific case $d = 3$, their expression matches ours, and so we see that their conformal block expansion must also involve logarithms. Using their result to move away from $d = 3$, a scalar operator of dimension $2d$ and a second of dimension 6 contribute to the conformal block decomposition. The coefficients of these conformal blocks are equal and opposite in the $d \to 3$ limit and scale as $1/(d-3)$. The collision and mixing of these two operators in $d = 3$ produces the logarithm.[6] A similar degeneracy happens in integer dimensions $d > 4$ for operators of dimension $2d$, but not in $d = 4$ where the theory is free. Interestingly, the lack of positivity of the bulk conformal block expansion allows these two diverging coefficients to cancel. We want to study how this mixing is affected by $1/N$ corrections although it is essential to note that in our context at least, there may be a problem that the theory is no longer conformal at subleading order in $1/N$. (A similar log in a one-point function was pointed out in [91], where it was likely related to an anomaly in the trace of the stress tensor.)

3.4 Boundary anomaly coefficients

In this section we finally elucidate boundary anomaly coefficients. The remarkable feature of our theory is that these two quantities a and b can be easily extracted. These quantities are thus far known only in a few examples. One is the conformally coupled scalar. There are two types of Weyl invariant boundary conditions: Dirichlet and Robin. The central charges for these two choices are $a^{(D)} = -\frac{1}{96}$ [22], $a^{(R)} = \frac{1}{96}$ [20], and $b^{(D)} = b^{(R)} = \frac{1}{64}$ [95]. The Robin boundary condition involves an extrinsic curvature, and for a planar boundary reduces to the Neumann condition. What motivates us to study this ϕ^6 theory is to collect other examples of these numbers.

The boundary type-A anomaly coefficient a can be extracted from the effective action of the theory on hyperbolic space $\mathbb{H}_3$ as we mentioned in section 3.1. Given the potential density V computed in section 3.2 (see figure 3.2), we can obtain the effective action at leading order in N by integrating V over $\mathbb{H}_3$, resulting in $W = V\,\mathrm{Vol}(\mathbb{H}_3)$ because V does not depend on spacetime coordinates. To compute the volume of $\mathbb{H}_3$, let us take the following line element

$$\mathrm{d}s^2 = L^2[\mathrm{d}\tau^2 + \sinh^2\tau(\mathrm{d}\theta^2 + \sin^2\theta d\phi^2)]\,, \quad \tau > 0\,, 0 \leq \theta \leq \pi\,, 0 \leq \phi \leq 2\pi \tag{3.81}$$

where L is the radius of curvature. We have the conformal boundary S^2 at $\tau \to \infty$. Although this volume diverges, we can regularize the integral by introducing a cutoff Λ at very close to boundary such that $e^{\tau_{\max}} = L\Lambda$. Now that we integrate over $0 \leq \tau \leq \tau_{\max}$, we find that

$$\mathrm{Vol}(\mathbb{H}_3) = -2\pi\log(L\Lambda)\,. \tag{3.82}$$

[6] We would like to thank H. Osborn for discussion on this point. For readers interested in duplicating the result, there is a typo in (5.34) [48]. A factor of v^d multiplying a ${}_2F_1$ hypergeometric function should be v^{2d}.

Since we introduced the scale Λ in the theory, the trace of stress tensor can be obtained by scale variation of the partition function. For our $\mathbb{H}_3$ with $\mathbb{S}^2$ boundary, the second term in (3.3) does not contribute and we obtain $\Lambda\partial_\Lambda W = -2a \log L\Lambda$, indicating that

$$a = \pi V \ . \tag{3.83}$$

Taking $g \to 0$, which corresponds to the free Neumann and Dirichlet cases, we recover the free field results $a = \pm N\frac{1}{96}$. More generally, for nonzero g and $\phi_\alpha = 0$, we find the simple scaling $a = -N\frac{\mu}{48}$. The result for the extraordinary boundary condition can be read off from figure 3.2. Boundary unitarity ($\mu > -1$) constrains the value of a as $a < N/48$.

The other coefficient b can be extracted from the two-point function of stress tensor, since the displacement operator is the boundary limit of the T^{nn} component of the stress tensor. From [29], we have

$$b = \frac{\pi^2}{8}\alpha(1) \ . \tag{3.84}$$

With these results (3.83) and (3.84) in mind, we can somehow falsify a pair of conjectures about these coefficients a and b.

In ref. [29], it was proposed that a could be extracted from the stress tensor two point function, in particular

$$a = \frac{\pi^2}{9}\left(\epsilon(1) - \frac{3}{4}\alpha(1) + 3C\right) \ , \tag{3.85}$$

where C is the central charge of a decoupled 2d CFT living on the boundary. In our case, there is no such decoupled CFT and C is zero. Moreover, $\epsilon(1)$ vanishes except in the special cases $\mu = \pm\frac{1}{2}$. Thus the conjecture boils down to the statement that $a = -\pi^2\alpha(1)/12$, which is manifestly not true in the disordered case, comparing the actual result $a = -\mu N/48$ with figure 3.5, which is not linear in μ. Thus the conjecture appears to be wrong.

Another conjecture, this time concerning b and $\alpha(1)$, was discussed in ref. [30]. The authors speculated that perhaps $\alpha(1)$ was bounded above by $2\alpha(0)$ because that is what they observed in a graphene like theory where the interaction was confined to the boundary. The value $\alpha(0)$ is related to the coincident limit of the stress tensor two point function. From figure 3.5, it is clear that this bound is satisfied only in the range $|\mu| < \frac{1}{2}$, or equivalently $g < 0$. For $g > 0$, on the other hand, $\alpha(1) > 2\alpha(0)$.

3.5 Summary

In this chapter we consider a scalar $O(N)$ field theory with ϕ^6 interaction, which provides us a tractable example of boundary conformal field theories. The primary motivation for studying this model comes from the fact that we can extract the information of boundary conformal anomaly in a simple setting. Though these anomaly coefficients are promising to give us a better understanding boundary CFTs, the quantities are far known only in a few examples. We here exploit them in a new playground. We also compute the effective potential and find surprisingly rich phase structures. We see that all of these quantities are very sensitive to the quasi-marginal coupling g.

Chapter 4

Towards understanding RG dynamics of Boundary and defect CFT

This chapter is heavily based on the author's publication [32] in collaboration with Tatsuma Nishioka, Yoshiki Sato, and Kento Watanabe.

4.1 Opening remarks

We have studied conformal field theories with boundaries. Saying that a boundary is a codimension-one object, we can consider higher-codimensional ones in quantum field theories, such as strings, membranes, vortexes, all of which in general we call defects. Since we are mostly interested in fixed point conformal theories, we would like to consider a defect preserving conformal invariance on it, named conformal defects.

Of course, it is true that very properties of conformal defects rely on specific details of theories, for instance, lagrangians. However we expect that restricted conformal invariance enables us to extract some universal information in defect CFTs, hoping that it provides us a certain answer to "what is defect CFT?" or "How defect CFT should behave?" Primarily, we would like to understand the dynamical behaviors of defect CFTs in order to classify them.

4.1.1 C-theorems in general dimensions without defects

Without defects, we partially argue such a problem by considering c-theorem in two dimensions, or a-theorem in four dimensions as in section 2.8. In both cases we find nice monotonically decreasing functions under RG flows from the stress tensor trace. These theorems severely constrain possible RG dynamics, providing us a theoretical tool to classify the space of QFTs. Then what happens when we consider odd-dimensional theories where there is no conformal anomaly? Can we find C-functions, which we recall are monotonically decreasing ones? In fact it was proposed that in three-dimensions a sphere free energy defined as

$$F = -\log Z[\mathbb{S}^3] \tag{4.1}$$

should be monotonic under RG flows [8, 9],[1] which is known as F-theorem. More broadly, the way to interpolate type-A anomaly in even dimensions to sphere free energy in odd dimensions

[1] Of course $Z[\mathbb{S}^3]$ has a collection of divergent pieces due to short distance effects, but there is a scheme-independent term, which we call the universal part and we mean by a monotonic function.

was proposed in [10] by introducing the following quantity:

$$\tilde{F} \equiv \sin\left(\frac{\pi\, d}{2}\right) \log\, Z[\mathbb{S}^d]\;, \tag{4.2}$$

where d is continuous dimensions. One can convince themselves that in setting $d = 2n + \epsilon$ with $\epsilon \to 0$, $\tilde{F}$ is reduced to the type-A anomaly coefficient. Then the authors of [10] claimed that $\tilde{F}$ is positive and decrease under any RG flows:

$$\tilde{F}_{\text{UV}} \geq \tilde{F}_{\text{IR}}\;. \tag{4.3}$$

This is a most general form of proposed C-functions, which, however, have not yet proven.

There is another way to propose C-theorems in general dimensions. The idea is to use the entanglement entropy as a C-function. Let us briefly review what the entanglement entropy is. Suppose the Hilbert space can be splitted into two parts as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and the entire system is described by the density matrix ρ. Then we define the reduced density matrix by tracing out $\mathcal{H}_B$,

$$\rho_A = \text{tr}_{\mathcal{H}_B}\, \rho\;, \tag{4.4}$$

and entanglement entropy for the subsystem $\mathcal{H}_A$ is given by the von Neumann entropy of ρ_A:

$$S_A = -\,\text{tr}_{\mathcal{H}_A}\, \rho_A \log \rho_A\;. \tag{4.5}$$

In quantum field theories, the density matrix can be written in the path integral and the subsystem corresponds to some spatial region on a fixed time slice. We call such a domain the entangling region and denote Σ. For later usage, readers may refer figure 4.1 as our setup.

Instead of the entanglement entropy, it is useful to consider the Rényi entropy

$$S_n = \frac{1}{1-n} \log \rho_A^n\;, \tag{4.6}$$

which is reduced to the entanglement entropy in $n \to 1$ limit. This is because by using path integral formalism and so called replica trick, we find that

$$S_n = \frac{1}{1-n} \log \frac{Z_n}{(Z_1)^n}\;, \tag{4.7}$$

where Z_n is a partition function on the n-fold cover $\mathcal{M}_n$ of the original manifold $\mathcal{M}$. It is often the case that the computation of (4.7) is easier than that of entanglement entropy itself. In d-dimensional QFTs, the entanglement entropy generally take the following form,

$$S^{(\text{QFT})} = \frac{A_{d-2}}{\epsilon^{d-2}} + \frac{A_{d-4}}{\epsilon^{d-4}} + \cdots + \begin{cases} a' \log\left(\frac{R}{\epsilon}\right)\;, & (d = \text{even})\;, \\ a_0\;, & (d = \text{odd})\;, \end{cases} \tag{4.8}$$

where R is a typical size of the entangling region and ϵ is the UV cutoff. When the theory is conformal and Σ is a sphere, a' and a_0 becomes universal quantities, meaning that they are independent of regularization schemes. These numbers were conjectured to be C-functions each in even and odd dimensions [96, 97] in the literature of holographic duality.

Even though (4.8) and (4.2) looks very different, in case of CFT with spherical entangling region, we have a remarkable relation between them as follows:

$$S^{(CFT)} = \log Z[\mathbb{S}^d] \ , \tag{4.9}$$

up to UV divergence [28]. This map enables us to use the quantum information technique to prove F-theorem [13] in three dimensions. Extending the proof in $d = 3$ (and $d = 2$ [12]) to higher dimensions was attempted in [98], which amounts to a different monotonicity theorem from (4.3) in $d > 4$ (see also [99]). It still remains open whether the F-theorem (4.3) in higher dimensions follows from the quantum inequality of a certain entanglement measure.

4.1.2 C-theorems in general dimensions with defects

Having the intuition about C-functions in generic spacetime dimensions, let us switch gears to our main interests, where quantum field theories contain defects. As discussed in section 2.8, we have the monotonic theorem in case of three-dimensional QFT with boundary, or d-dimensional QFT with codimension-two defect [20]. Again we remark that though it is referred to b-theorem, the monotonic function is the type-A anomaly coefficient a in (2.104).

For other dimensions, it is known that in two-dimensional CFTs with boundaries the constant term of the thermal entropy should be monotonic under boundary RG flows, which is so called g-theorem. As is the case without defects, one can establish an alternative proof of g-theorem by using the boundary entropy, which is difference between entanglement entropies in BCFT and in CFT,

$$S_{\text{bry}} = S^{\text{BCFT}} - \frac{1}{2} S^{\text{CFT}} \ , \tag{4.10}$$

which measures an increment of entanglement due to the boundary. The corresponding theorem follows from the positivity of the relative entropy [21]. Conformal transformation allows us to show that S_{bry} is equivalent to thermal entropy, resulting in the same g-theorem as the former one.

To establish g-theorem in general dimensional BCFTs, there are several proposals for monotonic quantities, the hemisphere partition function [22, 23], the boundary entropy [24] and the holographic consideration [25–27] with varying degrees of evidence.[2] The first two proposals are not independent but the same statement as an analogous identity to (4.9) holds for BCFT.[3]

It is still unclear whether one can propose the generic C-functions in d-dimensional CFTs with p-dimensional defects as in (4.2). See table 4.1 for the summary of the current status. The aim of this chapter is to investigate possible candidates for C-functions for DCFTs in generic dimensions. An important guiding principle is that the candidate C-function should reproduce all the known conjectures and theorems by setting the appropriate dimensions, which gives us logically sufficient conditions. We are then left with two possibilities: one is the defect free energy, the additional contribution to the sphere free energy from the spherical defect,

$$\log \langle \mathcal{D}^{(p)} \rangle = \log Z^{(\text{DCFT})} - \log Z^{(\text{CFT})} \ . \tag{4.11}$$

[2]The holographic g-theorems are proven to be correct under any holographic boundary RG flow satisfying the null energy condition, but their physical meanings are unclear unless the theory is at the fixed point as in the case of the holographic c-theorem [100].

[3]The partition function of BCFT is defined on a hemisphere $\mathbb{HS}^d$, so $Z^{(\text{BCFT})} \equiv Z[\mathbb{HS}^d]$.

And the other is the defect entropy, the increment of the entanglement entropy across a sphere due to the planar defect[4]

$$S_{\text{defect}} = S^{(\text{DCFT})} - S^{(\text{CFT})} . \tag{4.12}$$

One might ask if we can connect these two quantities as (4.9). When $p = d - 1$ they are indeed equivalent up to UV divergences. On the contrary, they differ by a term fixed by the one-point function of the stress-energy tensor for $p < d - 1$. Their precise relation will be derived in (4.54) by using the conformal transform known as the Casini-Huerta-Myers (CHM) map [28, 57], which is one of main results in this chapter.

This discrepancy suggests that either of them may not be a good candidate for the C-function generally. We try to check their behaviors under defect RG flows in various models, including holographic ones. We find several examples where the defect free energy decreases while the defect entropy increases along the RG flow. On the other hand, both of them always decreases in all the holographic models that we study. These observations therefore lead us to propose a C-theorem in DCFTs stating that the universal part of the defect free energy[5]

$$\tilde{D} \equiv \sin\left(\frac{\pi p}{2}\right) \log \langle \mathcal{D}^{(p)} \rangle , \tag{4.13}$$

decreases along any defect RG flow

$$\tilde{D}_{\text{UV}} \geq \tilde{D}_{\text{IR}} . \tag{4.14}$$

The more precise statement is presented around (4.65). Note that this should be seen as the counterpart to the generalized F-theorem (4.3) in CFTs. Our proposal here unifies known proposals of C-functions as well as asserts a new family of C-theorems in defect CFTs (see 4.1 for the summary).

This chapter is organized as follows. In section 4.2 we study the CHM map in our setting for DCFTs and calculate defect entropy as the thermal entropy, finding the universal formula which relates defect entropy to the defect free energy. In the following section 4.3, we propose our conjecture of C-theorem in DCFT and also study several field theoretic examples to confirm our proposal true. Section 4.4 is devoted to the study of our proposal in holographic models.

4.2 Sphere free energy and entanglement entropy in DCFT

In this section we consider two quantities: the defect free energy and the defect entropy. The former is the increment of the sphere free energy from a conformal defect in DCFT while the latter is the additional contribution to the entanglement entropy of a spherical region. To argue them, we first specify our configuration for the entanglement entropy and describe the conformal transformation known as the CHM map which relates the entanglement entropy of a spherical region to the thermal entropy of DCFT on a hyperbolic space in section 4.2.1. Establishing the map between the flat space and the hyperbolic space, section 4.2.2 is devoted to computing the defect entropy from the Rényi entropy. Along the way, we find the universal formula which

[4] The defect entropy has been conjectured to be a C-function for interface CFTs in [24] based on the studies of several holographic models.

[5] Recall that the free energy in QFTs contains a number of UV divergences, from which we have to extract the universal part independent of regularization scheme to make a meaningful statement.

	$d=2$	$d=3$	$d=4$	$d=5$
$p=1$	g-theorem Proof [21, 101]			
$p=2$		b-theorem Proof [20]		
$p=3$			bdy F-theorem Proposal [22, 23]	
$p=4$				

Table 4.1: Summary of the conjectured and proved C-theorems in BCFTs and DCFTs. Our proposal reduces to the known ones in the shaded regions and provides new ones in the region colored in light blue.

expresses the defect entropy by the defect free energy and the one-point function of the stress tensor. As conclusion remarks of this section, we investigate UV divergence structures of the defect entropy and the defect free energy in section 4.2.3, which is necessary to study in order to extract the universal numbers from them.

4.2.1 Setup and CHM map

The aim of this subsection is to make clear our setup to compute the entanglement entropy and review the conformal map, named CHM map, which relates the entanglement entropy across a sphere to thermal entropy in hyperbolic space. We begin with specifying our coordinate system. we adopt the polar coordinates for $\mathbb{R}^{1,d-1}$ in Lorentzian signature,

$$\begin{aligned} ds^2_{\mathbb{R}^{1,d-1}} &= \eta_{\mu\nu}\, dX^\mu\, dX^\nu \\ &= -dt^2 + dr^2 + r^2\, ds^2_{\mathbb{S}^{d-2}} \ , \end{aligned} \tag{4.15}$$

where $\eta_{\mu\nu} = \text{diag}(-,+,\cdots,+)$. We take the entangling surface Σ as a $(d-2)$-dimensional hypersphere of radius R located at $t=0$ time slice:

$$\Sigma = \{X^0 = t = 0,\ r = R\}\ . \tag{4.16}$$

We want to introduce a conformal defect $\mathcal{D}^{(p)}$ of dimension-p respecting the subgroup $SO(p,2)\times SO(d-p)$ of the conformal group $SO(d,2)$. Since conformal defects are either planar or spherical, and we choose $\mathcal{D}^{(p)}$ to be a hyperplane,

$$\mathcal{D}^{(p)} = \{X^p = \cdots = X^{d-1} = 0\}\ . \tag{4.17}$$

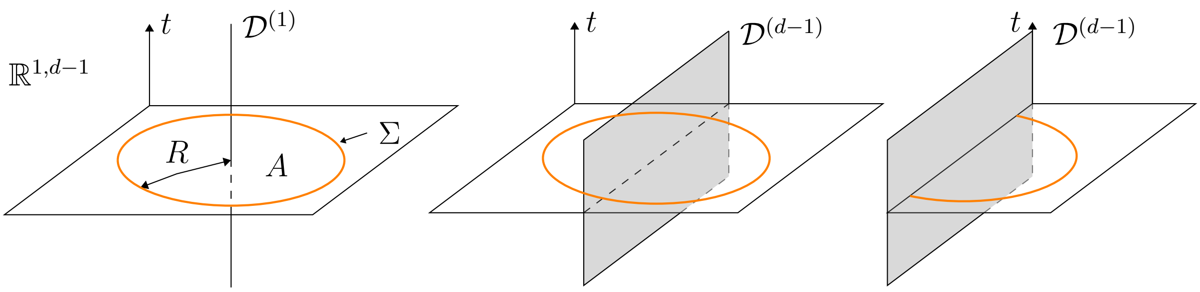

Figure 4.1: (Left) A dimension-one conformal defect $\mathcal{D}^{(1)}$ in Lorentzian flat spacetime. The spherical subsystem A of radius R surrounds the defect. (Center, Right) A codimension-one defects $\mathcal{D}^{(d-1)}$ as an interface(Center) and a boundary (Right). The subsystem A intersects with the defect in these cases.

Figure 4.1 shows our setups for $p = 1$ and $p = d - 1$.

We now turn to the discussion about the conformal map by which we can move into the hyperbolic space from the flat space. As [28, 57] introduced, let us consider the following coordinate transformation,[6]

$$x^\mu(X) = 4\left[\frac{X^\mu - |X|^2 C^\mu}{1 - 2X\cdot C + |X|^2 |C|^2} + \frac{R^2}{2} C^\mu\right] , \qquad C^\mu \partial_\mu = -\frac{1}{R}\partial_1 . \tag{4.18}$$

The resulting space is conformally flat with the metric,

$$\mathrm{d}s^2_{\mathbb{R}^{1,d-1}} = \Omega(x)^2\, \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu , \tag{4.19}$$

with the conformal factor,

$$\begin{aligned}\Omega &= \frac{1}{4}\left(1 - 2X\cdot C + |X|^2 |C|^2\right) \\ &= \frac{1}{1 + x\cdot C + |x|^2 |C|^2/4} .\end{aligned} \tag{4.20}$$

Under this conformal transformation, the causal domain $r \pm t \le R$ for the entangling region is mapped to the (right) Rindler wedge $x^\pm \equiv x^1 \pm x^0 \ge 0$ denoted $\mathcal{R}$. The light cones $r + t = R$ and $r - t = R$ on the boundary of the causal domain are mapped to the Rindler horizons,

$$\begin{aligned} r + t = R \quad &\Rightarrow \quad x^+ = 0 , \\ r - t = R \quad &\Rightarrow \quad x^- = 0 , \end{aligned} \tag{4.21}$$

The entangling surface is mapped to the origin in the x^0-x^1 plane,

$$\Sigma = \{x^0 = x^1 = 0\} , \tag{4.22}$$

while the defect is mapped to the hyperplane,

$$\mathcal{D}^{(p)} = \{x^p = \cdots = x^{d-1} = 0\} . \tag{4.23}$$

[6] We will focus on the case with $1 \le p < d - 1$ so that this map works, but the following results hold for $p = d - 1$ with a slight change [57].

We see that the vacuum state in the causal diamond of Σ is mapped to the vacuum state in the Rindler wedge $\mathcal{R}$, which results in the fact that the reduced density matrix for Σ is reduced to the density matrix of the theory on $\mathcal{R}$.

The point of this (inverse of) CHM map is that by famous Unruh's argument [102] an uniformly accelerator in the $\mathcal{R}$ should experience an effective thermal temperature. In our case, it is easy to see this by introducing the new coordinates,

$$x^{\pm} = z\, \mathrm{e}^{\pm\tau} \, . \tag{4.24}$$

The Rindler spacetime becomes

$$\begin{aligned} \mathrm{d}s^2_{\text{Rindler}} &= \mathrm{d}x^+\, \mathrm{d}x^- + \sum_{i=2}^{d-1} (\mathrm{d}x^i)^2 \\ &= z^2 \left[-\mathrm{d}\tau^2 + \frac{\mathrm{d}z^2 + \sum_{i=2}^{d-1}(\mathrm{d}x^i)^2}{z^2} \right] \, , \end{aligned} \tag{4.25}$$

which is conformally equivalent to $\mathbb{R}\times\mathbb{H}^{d-1}$ parametrized by τ and a hyperbolic space of unit radius,

$$-y_0^2 + y_1^2 + y_2^2 + \cdots + y_{d-1}^2 = -1 \, , \tag{4.26}$$

in the Poincaré coordinates,

$$\begin{aligned} y_0 &= \frac{z}{2}\left[1 + \frac{1+\sum_{i=2}^{d-1}(x^i)^2}{z^2}\right] \, , \\ y_1 &= \frac{z}{2}\left[1 + \frac{-1+\sum_{i=2}^{d-1}(x^i)^2}{z^2}\right] \, , \\ y_i &= \frac{x^i}{z} \, , \qquad (i = 2, \cdots, d-1) \, . \end{aligned} \tag{4.27}$$

In these new coordinates, the entangling surface and the defect are located at[7]

$$\Sigma = \{ z = 0 \, , \tau = 0 \} \, , \qquad \mathcal{D}^{(p)} = \{ x^p = \cdots = x^{d-1} = 0 \} \, . \tag{4.28}$$

Now we can convince ourselves that the vacuum state in $\mathcal{R}$ is then corresponds to the thermal state with temperature $T = 1/2\pi$ by the Unruh effect.[8] One may be able to realize this statement by Wick rotation $\tau \to \mathrm{i}\tau$ and compactification of τ direction with periodicity 2π.

For later convenience, we introduce the global coordinates of $\mathbb{H}^{d-1}$ by

$$\begin{aligned} y_a &= \cosh x\, f_a \, , \qquad (a = 0, \cdots, p-1) \, , \\ y_i &= \sinh x\, e_i \, , \qquad (i = p, \cdots, d-1) \, . \end{aligned} \tag{4.29}$$

[7]The position of Σ in the τ direction is ambiguous as the τ circle shrinks at $z = 0$. We thus choose a reference point at $\tau = 0$.

[8]For our case with boundaries or defects, it is not obvious to hold this argument true. See [57] for detailed proofs.

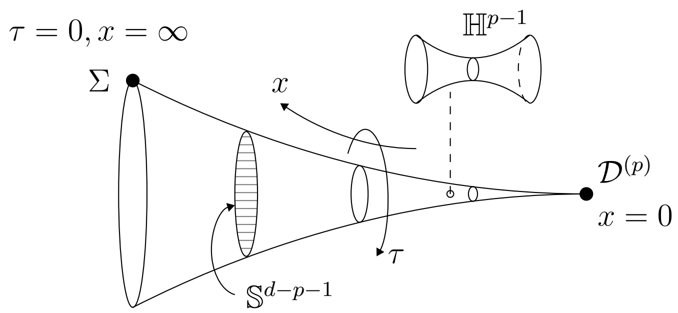

Figure 4.2: The locations of the entanglement surface Σ and the conformal defect $\mathcal{D}^{(p)}$ in the hyperbolic coordinates (4.34). The hyperbolic space $\mathbb{H}^{p-1}$ is fibered on each point of the base.

where $-f_0^2 + \sum_{i=1}^{p-1} f_a^2 = -1$ and $\sum_{i=p}^{d-1} e_i^2 = 1$. The resulting metric for $\mathbb{R} \times \mathbb{H}^{d-1}$ takes the form

$$ds^2_{\mathbb{R}\times\mathbb{H}^{d-1}} = -d\tau^2 + dx^2 + \cosh^2 x \, ds^2_{\mathbb{H}^{p-1}} + \sinh^2 x \, ds^2_{\mathbb{S}^{d-p-1}} \ , \tag{4.30}$$

where the entangling surface and the defect are situated at

$$\Sigma = \{x = \infty \, , \tau = 0\} \ , \qquad \mathcal{D}^{(p)} = \{x = 0\} \ . \tag{4.31}$$

4.2.2 Sphere free energy and defect entropy

We have shown the flat (Lorentzian) spacetime is conformally equivalent to $\mathbb{R} \times \mathbb{H}^{d-1}$ as in (4.30). Significantly, we concluded that the reduced density matrix for the spherical entangling region Σ is equivalent to the thermal density matrix in the theory on $\mathbb{R}\times\mathbb{H}^{d-1}$, implying that the entanglement entropy with the spherical entangling region is equivalent to the thermal entropy on the hyperbolic space. With this relation in our hands, we are now in a position to compute the entanglement entropy in the presence of a conformal defect.

Recalling the definition of the Rényi entropy in the replica trick,

$$S_n = \frac{1}{1-n} \log \frac{Z_n}{(Z_1)^n} \ , \tag{4.32}$$

the calculation of entanglement entropy ends up with deriving the partition function $Z_n \equiv Z[\mathcal{M}_n]$ on the n-fold cover $\mathcal{M}_n$ of the original manifold. Given equivalent between the entanglement entropy with Σ and the thermal entropy at temperature $T = T_0 = 1/2\pi$, we can also write down the Rényi entropy in terms of *free energy* $f(T)$:

$$S_n = \frac{n}{1-n} \frac{1}{T_0} \left(f(T_0) - f(T_0/n) \right) \ , \tag{4.33}$$

where $f(T) \equiv -T \log Z(T)$ with the thermal partition function $Z(T) = \mathrm{tr}[e^{-H/T}]$. We here adopt an alternative description for such a thermal theory by considering the Wick-rotated Euclidean theory on the compactified manifold:

$$ds^2_{\mathbb{S}^1\times\mathbb{H}^{d-1}} = d\tau^2 + dx^2 + \cosh^2 x \, ds^2_{\mathbb{H}^{p-1}} + \sinh^2 x \, ds^2_{\mathbb{S}^{d-p-1}} \ , \tag{4.34}$$

where $\mathbb{S}^1$ has a unit radius. The entangling surface and the defect are located at

$$\Sigma = \{x = \infty \,, \tau = 0\} \,, \qquad \mathcal{D}^{(p)} = \{x = 0\} \,, \tag{4.35}$$

in (see figure 4.2). In our setup, it is not difficult to see that the n-fold cover of the original flat space is conformally equivalent to the the n-fold cover $\mathbb{S}^1_n \times \mathbb{H}^{d-1}$ along the τ coordinate of the space (4.34),

$$\mathrm{d}s^2_{\mathbb{S}^1_n \times \mathbb{H}^{d-1}} = n^2\, \mathrm{d}\tau^2 + \mathrm{d}x^2 + \cosh^2 x\, \mathrm{d}s^2_{\mathbb{H}^{p-1}} + \sinh^2 x\, \mathrm{d}s^2_{\mathbb{S}^{d-p-1}} \,, \tag{4.36}$$

with the range $0 \le \tau < 2\pi$, which corresponds to the thermal theory with temperature $T = T_0/n$. Let us first discuss the Rényi entropy without defects. If there are no conformal anomalies, the partition function of CFT is invariant under the conformal map,

$$Z[\mathcal{M}_n] = Z[\mathbb{S}^1_n \times \mathbb{H}^{d-1}] \,. \tag{4.37}$$

Hence the Rényi entropy across a sphere in CFT is given by,

$$S_n^{(\mathrm{CFT})} = \frac{1}{1-n} \log \frac{Z^{(\mathrm{CFT})}[\mathbb{S}^1_n \times \mathbb{H}^{d-1}]}{\left(Z^{(\mathrm{CFT})}[\mathbb{S}^1 \times \mathbb{H}^{d-1}]\right)^n} \,. \tag{4.38}$$

In order to obtain (4.9), we have to use additional transformations, taking $\sinh x = \cot\theta$, whose metric change from (4.34) to another coordinates,

$$\mathrm{d}s^2_{\mathbb{S}^1 \times \mathbb{H}^{d-1}} = \frac{1}{\sin^2\theta}\, \mathrm{d}s^2_{\mathbb{S}^{d-p+1} \times \mathbb{H}^{p-1}} \,, \tag{4.39}$$

where resulting metric is just a d-dimensional sphere up to the conformal factor,

$$\begin{aligned} \mathrm{d}s^2_{\mathbb{S}^{d-p+1} \times \mathbb{H}^{p-1}} &= \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\tau^2 + \cos^2\theta\, \mathrm{d}s^2_{\mathbb{S}^{d-p-1}} + \mathrm{d}s^2_{\mathbb{H}^{p-1}} \,, \\ &= \mathrm{d}s_{\mathbb{S}^{d-p+1}} + \mathrm{d}z^2 + \sinh^2 z \mathrm{d}s_{\mathbb{S}^{p-2}} \,, \\ &= \frac{1}{\cos^2\phi}\left(\mathrm{d}\phi^2 + \cos^2\phi\, \mathrm{d}s_{\mathbb{S}^{d-p+1}} + \sin^2\phi\, \mathrm{d}s_{\mathbb{S}^{p-2}}\right) . \end{aligned} \tag{4.40}$$

Here we introduced a global coordinate for $\mathbb{H}^{p-1}$ and again use the map $\sinh z = \tan\phi$. In fact what we have shown is nothing but the well-known fact of the conformal equivalence between the flat space and a sphere.[9] With such transformations in our mind, we can derive the fact that

[9] One can more directly find a conformal transformation [28],

$$t = R\,\frac{\sin\tau\, \sin\theta}{1 + \sin\theta\, \cos\tau} \,, \qquad r = R\,\frac{\cos\theta}{1 + \sin\theta\, \cos\tau} \,, \tag{4.41}$$

which maps the Euclidean flat space to the spherical coordinates (4.40),

$$\mathrm{d}s^2_{\mathbb{R}^d} = \Omega^2(\theta, \tau)\, \mathrm{d}s^2_{\mathbb{S}^d} \,, \tag{4.42}$$

with the conformal factor,

$$\Omega(\theta, \tau) = \frac{R}{1 + \sin\theta\, \cos\tau} \,. \tag{4.43}$$

for CFT the sphere entanglement entropy equals to the free energy on a conformally flat space up to UV divergences [28],[10]

$$S^{\text{(CFT)}} = \log\, Z^{\text{(CFT)}}[\mathbb{S}^d] = \log\, Z^{\text{(CFT)}}[\mathbb{S}^1 \times \mathbb{H}^{d-1}] \,. \tag{4.44}$$

In what follows we would like to then consider the situation with a conformal defect $\mathcal{D}^{(p)}$ of dimension-p for $p \le d-2$, and separately discuss the case for $p = d-1$ at the end of this subsection. Specifically, We are interested in the *defect entropy*, which is given by the additional entanglement entropy due to the existence of $\mathcal{D}^{(p)}$ and defined from Rényi entropies as follows:

$$S_{\text{defect}} \equiv \lim_{n\to 1} \left(S_n^{\text{(DCFT)}} - S_n^{\text{(CFT)}} \right) \,. \tag{4.45}$$

The Rényi entropy $S_n^{\text{(DCFT)}}$ in DCFT is defined in a similar manner to that in CFT, which takes the same form as (4.38) for a spherical entangling region:

$$S_n^{\text{(DCFT)}} = \frac{1}{1-n} \log \frac{Z^{\text{(DCFT)}}[\mathbb{S}_n^1 \times \mathbb{H}^{d-1}]}{\left(Z^{\text{(DCFT)}}[\mathbb{S}^1 \times \mathbb{H}^{d-1}]\right)^n} \,. \tag{4.46}$$

It would be beneficial to rewrite the defect entropy as

$$S_{\text{defect}} \equiv \lim_{n\to 1} \frac{1}{1-n} \log \frac{\langle\, \mathcal{D}^{(p)} \,\rangle_n}{\langle\, \mathcal{D}^{(p)} \,\rangle^n} \,, \tag{4.47}$$

where $\langle\, \mathcal{D}^{(p)} \,\rangle_n$ is the expectation value of the conformal defect operator $\mathcal{D}^{(p)}$ of dimension p on $\mathbb{S}_n^1 \times \mathbb{H}^{d-1}$,

$$\langle\, \mathcal{D}^{(p)} \,\rangle_n \equiv \frac{Z^{\text{(DCFT)}}[\mathbb{S}_n^1 \times \mathbb{H}^{d-1}]}{Z^{\text{(CFT)}}[\mathbb{S}_n^1 \times \mathbb{H}^{d-1}]} \,. \tag{4.48}$$

We also denote $\langle\, \mathcal{D}^{(p)} \,\rangle \equiv \langle\, \mathcal{D}^{(p)} \,\rangle_1$ to simplify the notation.

In order to derive a similar relation for the defect entropy as (4.9), we expand the partition function $Z^{\text{(DCFT)}}[\mathbb{S}_n^1 \times \mathbb{H}^{d-1}]$ on the n-covered space (4.36) around $n=1$,

$$\begin{aligned} \log\, Z^{\text{(DCFT)}}[\mathbb{S}_n^1 \times \mathbb{H}^{d-1}] &= \log\, Z^{\text{(DCFT)}}[\mathbb{S}^1 \times \mathbb{H}^{d-1}] \\ &\quad - \frac{1}{2} \int_{\mathbb{S}^1 \times \mathbb{H}^{d-1}} \delta g_{\tau\tau}\, \langle\, (T_{\text{DCFT}})^{\tau\tau} \,\rangle^{\text{(DCFT)}}_{\mathbb{S}^1 \times \mathbb{H}^{d-1}} + \cdots \,, \end{aligned} \tag{4.49}$$

where $\delta g_{\tau\tau} = (n^2 - 1)$. Because $\cdots$ represent terms of higher order than $(n-1)$ and do not contribute to the entanglement entropy, we obtain

$$S_{\text{defect}} = \log \langle\, \mathcal{D}^{(p)} \,\rangle + \int_{\mathbb{S}^1 \times \mathbb{H}^{d-1}} \langle\, (T_{\text{DCFT}})^{\tau}{}_{\tau} \,\rangle^{\text{(DCFT)}}_{\mathbb{S}^1 \times \mathbb{H}^{d-1}} \,. \tag{4.50}$$

We call the first term in the right hand side the *defect free energy*, which can be written by the sphere free energy,

$$\log \langle\, \mathcal{D}^{(p)} \,\rangle = \log \frac{Z^{\text{(DCFT)}}[\mathbb{S}^d]}{Z^{\text{(CFT)}}[\mathbb{S}^d]} \,, \tag{4.51}$$

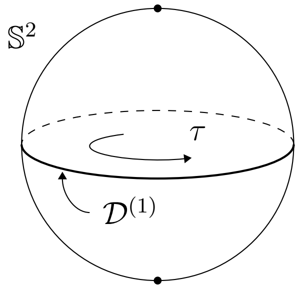

Figure 4.3: The conformal defect $\mathcal{D}^{(1)}$ on $\mathbb{S}^d$. For $d = 2$, it winds along the equator of $\mathbb{S}^2$ (τ-direction).

thanks to a further conformal transformation explained around (4.40) with the assumption that there are no conformal anomalies. After such transformations, the defect which was originally planar on flat space is mapped to a spherical defect on $\mathbb{S}^d$ (see figure 4.3).

Even if we are in $\mathbb{S}^1 \times \mathbb{H}^{d-1}$, the one-point function of the stress-energy tensor is fixed merely by imposing the conformal symmetry, tracelessness and conservation law and take the following form:

$$\begin{aligned}&\langle\, (T_{\text{DCFT}})_{\mu\nu}\,\rangle^{\text{(DCFT)}}_{\mathbb{S}^1\times\mathbb{H}^{d-1}}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \\ &\quad = \frac{a_T}{\sinh^d x}\left[\frac{d-p-1}{d}\left(\mathrm{d}\tau^2 + \mathrm{d}x^2 + \cosh^2 x\, \mathrm{d}s^2_{\mathbb{H}^{p-1}}\right) - \frac{p+1}{d}\sinh^2 x\, \mathrm{d}s^2_{\mathbb{S}^{d-p-1}}\right] ,\end{aligned} \tag{4.52}$$

where a_T is the same as the one in (2.114). In principle the term localized on the defect could appear in (4.52), but such a term vanishes due to (2.115) when DCFT has a Lagrangian description. If the defect is defined as a boundary condition for the bulk local operators, there is no defect localized part in the stress tensor, but the boundary condition still affects the bulk stress tensor in the same way as (4.52). Inserting (4.52) into (4.50), with the following equation are we eventually left:

$$\begin{aligned}S_{\text{defect}} &= \log \langle\, \mathcal{D}^{(p)}\,\rangle \\ &\quad + 2\pi\, \mathrm{Vol}(\mathbb{H}^{p-1})\, \mathrm{Vol}(\mathbb{S}^{d-p-1}) \int_0^\infty \mathrm{d}x\ \cosh^{p-1} x \sinh^{d-p-1} x\ \frac{d-p-1}{d}\,\frac{a_T}{\sinh^d x}\end{aligned} \tag{4.53}$$

where the factor $\cosh^{p-1} x \sinh^{d-p-1} x$ comes from the volume form of $\mathbb{S}^1 \times \mathbb{H}^{d-1}$ (4.34).

Remark that when evaluating the integral in (4.53), one has to adopt some regularization for the UV divergence arising from the integration near the defect. We follow the prescription employed by [64, 104] and remove the tubular neighborhood of a defect in flat space whose boundary is $\mathbb{R}^p \times \mathbb{S}^{d-p-1}$, on which we impose a boundary condition for the bulk fields. After performing the CHM map, this regularization means that we restrict the integration range of x to $\epsilon \leq x < \infty$ for a small parameter ϵ and send $\epsilon \to 0$ in the end. One may realize that such a prescription makes it manifest that we eliminate contributions from the defect localized term in the stress tensor. By expanding in the small ϵ, one can read off the constant part of the integral, but a more illuminating way is to employ the dimensional regularization in d after setting $\epsilon = 0$. Of course the two methods agree on giving the same universal constant.

[10] See, however, [103] where a variant of conformal anomalies was observed even in odd d dimensions.

No matter which we may adopt, it is possible to evaluate the integral in (4.53) and to derive the universal formula for the defect entropy:

In a DCFT with a conformal defect of dimension $p \leq d-2$ the defect entropy of a spherical entangling surface is given, up to UV divergence, by

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(p)} \rangle - \frac{2(d-p-1)\,\pi^{d/2+1}}{\sin(\pi p/2)\; d\,\Gamma(p/2+1)\;\Gamma((d-p)/2)}\, a_T \ , \tag{4.54}$$

which is one of our main results in this chapter. This formula is seen as a generalization of the result for $p=1$ [104]. For clarifying the validity of this formula, a few comments are in order:

- In deriving (4.54), we assume
  - (i) there are no conformal anomalies, i.e., we regard d as a continuous parameter,
  - (ii) the n-dependence is only through the metric,
  - (iii) the metric is coupled to the conformal stress-energy tensor.

  The last two assumptions should be regarded as the "choice" of the Rényi entropy in QFTs, and may vary depending on the situation. For instance, one can choose a boundary condition around the entangling surface to respect supersymmetry [105]. The n-dependence is not only through the metric, but also from the background fields of supergravity.

- The one-point function of the stress tensor in (4.52) is renormalized and the identity holds up to UV divergences that can be removed by counterterms to a background gravitational theory. Hence (4.54) holds only up to UV divergences.

- There are Graham-Witten type conformal anomalies [67] for even p, which we can manifestly see in (4.54) as a pole of the sine function and produces the logarithmic divergence.

Next let us consider the case for $p=d-1$, where there can be two types of theories, BCFTs and the others. For BCFTs we shall define the *boundary entropy* by

$$S_{\text{bdy}} \equiv \lim_{n\to 1}\left(S_n^{(\text{BCFT})} - \frac{1}{2}\,S_n^{(\text{CFT})}\right) . \tag{4.55}$$

As we saw in section 2.7.1, or just from (2.114), the residual conformal symmetry $SO(d,1)$ restricts the one-point function of the bulk primary operators of non-zero spin to be zero. It is also seen in (4.54) that for $p=d-1$ there doesn't exit the second term. It is straightforward to repeat the same argument as before for BCFTs, and we are led to the results:

In a DCFT with a conformal defect of dimension $d-1$ the defect entropy is given, up to UV divergence, by

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(d-1)} \rangle \ . \tag{4.56}$$

In a BCFT, the boundary entropy is given, up to UV divergence, by

$$S_{\text{bdy}} = \log\, Z^{(\text{BCFT})} - \frac{1}{2}\,\log\, Z^{(\text{CFT})} \ . \tag{4.57}$$

These are the special cases of the universal formula (4.54) for the defect entropy. In BCFT, the defect free energy is given by

$$\log \langle \mathcal{D}^{(d-1)} \rangle \big|_{\text{BCFT}} = \log Z^{(\text{BCFT})} - \frac{1}{2} \log Z^{(\text{CFT})} \,, \tag{4.58}$$

while in an interface CFT consisting of two theories CFT_+ and CFT_- we define

$$\log \langle \mathcal{D}^{(d-1)} \rangle \big|_{\text{ICFT}} = \left(\log Z^{(\text{CFT}_+)} + \log Z^{(\text{CFT}_-)}\right) - \frac{1}{2} \log Z^{(\text{CFT})} \,. \tag{4.59}$$

4.2.3 Remarks on UV divergence

We have gotten to the point where we derive the universal formula (4.54). Since this equation holds up to UV divergence as mentioned above, we need to specify the structure of the UV divergences in the defect free energy and defect entropy to obtain better understanding of the formula.

Recall that the defect free energy is a functional of the induced metric on a defect as well as the background bulk metric. In a local QFT, the UV divergent terms in the vev of a defect operator should consist of local diffeomorphism invariant functionals of the metrics on the worldvolume of the defect. From the dimensional ground, the most general effective action for the defect vev should take the following form (see e.g. [106, 107])

$$\log \langle \mathcal{D}^{(p)} \rangle = \int_{\mathcal{D}^{(p)}} \mathrm{d}^p \hat{x} \sqrt{\hat{g}} \left[\frac{a_p}{\epsilon^p} + \frac{a_{p-2}}{\epsilon^{p-2}} \hat{R} + \cdots \right] + (\text{UV finite non-local terms}) \,, \tag{4.60}$$

where $\hat{R}$ is the Ricci scalar of the induced metric $\hat{g}$, ϵ is the UV cutoff, and a_i are dimensionless constants. The $\cdots$ terms are subleading UV divergent terms built out of the Riemann curvature of the induced metric and the even power of the extrinsic curvatures.[11] For instance, the order of $1/\epsilon^{p-2i}$ divergent term roughly takes the form

$$\frac{a_{p-2i}}{\epsilon^{p-2i}} \sum_{l+m=i} \hat{R}^l\, K^{2m} \,, \tag{4.61}$$

where $\hat{\mathcal{R}}^l\, \mathcal{K}^{2m}$ are scalar polynomials of the Riemann curvatures and the extrinsic curvatures on the defect of order l and $2m$ respectively. There are only power law divergences in odd p dimensions while one can construct dimension p invariants out of $\hat{\mathcal{R}}$ and $\mathcal{K}$ such as the Euler density and there is an additional logarithmically divergent term $\log \epsilon$.

Applying (4.60) to the defect free energy on a sphere, we find the structure of the UV divergences depending on the dimensionality of the defect,

$$\log \langle \mathcal{D}^{(p)} \rangle = \frac{c_p}{\epsilon^p} + \frac{c_{p-2}}{\epsilon^{p-2}} + \cdots + \begin{cases} (-1)^{p/2}\, B \, \log \epsilon + \cdots \,, & (p:\text{even}) \,, \\ (-1)^{(p-1)/2}\, D \,, & (p:\text{odd}) \,. \end{cases} \tag{4.62}$$

Here the sign factors in front of B and D are chosen so that they are non-negative. It follows from this structure that the coefficients c_i ($i = p, p-2, \cdots$) of the power law divergences depend on

[11] The odd powers of the extrinsic curvatures are absent as the vev of a defect operator does not depend on the direction of the normal vectors.

the choice of the UV cutoff and are regularization scheme dependent while the constants B and D are invariant under the rescaling of ϵ, hence be scheme independent. The universal constant B is an analog of the type-A central charge of the conformal anomaly which can be read off from the sphere partition function in CFT. It is also known as the Graham-Witten anomaly [67]. Similarly D is an analog of the sphere partition function that is expected to measures the degrees of freedom in odd-dimensional CFT [10, 108].

On the other hand, the UV divergent terms of the defect entropy is also inferred from the generic structure (4.60) with the standard argument of the replica trick [107, 109], resulting in milder divergences than the defect free energy:

$$S_{\text{defect}} = \frac{c'_{p-2}}{\epsilon^{p-2}} + \frac{c'_{p-4}}{\epsilon^{p-4}} + \cdots + \begin{cases} (-1)^{p/2}\, B' \log \epsilon + \cdots \ , & (p : \text{even}) \ , \\ (-1)^{(p-1)/2}\, D' \ , & (p : \text{odd}) \ , \end{cases} \tag{4.63}$$

where B' and D' are universal constants different from B and D in general. The same UV structure was also observed in a few holographic calculations in [24], where the universal constants B' and D' were speculated to be C-functions in DCFTs.

4.3 Proposal for a C-theorem in DCFT

Now we have two candidates for a C-function in DCFT, the defect entropy and the defect free energy, both of which are natural counterparts of the C-theorem in CFT employing the entanglement entropy across a sphere or equivalently the sphere free energy as a C-function [8, 9, 96, 97]. To make it more concrete, we should only look at the universal parts of them because the other parts reckon on UV cutoff, or in other words, the regularization scheme. The universal constants of the defect free energy B, D in (4.62) should be regarded as analogs of the type-A central charge and the sphere free energy in CFT while the universal constants of the defect entropy B', D' in (4.63) differ from B, D due to the relation (4.54). In order to be consistent with b-theorem correctly, we hereby propose that the universal part of the defect free energy be a C-function in DCFT:

Conjecture. *In DCFT$_d$ with a defect of dimension p, the universal part of the defect free energy* (4.51) *defined by*

$$\tilde{D} \equiv \sin\left(\frac{\pi p}{2}\right) \log |\langle \mathcal{D}^{(p)} \rangle| \ , \tag{4.64}$$

does not increase along any defect RG flow

$$\tilde{D}_{\text{UV}} \geq \tilde{D}_{\text{IR}} \ . \tag{4.65}$$

Note that we take the absolute value $|\langle \mathcal{D}^{(p)} \rangle|$ to define the universal part. This is because there is a phase ambiguity appeared in $\langle \mathcal{D}^{(p)} \rangle$ such as the framing anomaly in the Chern-Simons theory which should be removed to extract the universal part as we will encounter in section 4.3.2. As seen from the relations (4.57) and (4.58), our conjecture includes, as a special case, the statement that the universal part of the boundary entropy defined by

$$\tilde{D} \equiv \sin\left(\frac{\pi (d-1)}{2}\right) S_{\text{bdy}} \ , \tag{4.66}$$

does not increase along any boundary RG flow in BCFT_d. Our conjecture is the most general one because it is consistent with all the proposals stated in the literature as we will show in what follows.

Let us clarify our conjecture in specific dimensions of the spacetime and the defect. We will see that the claim is reduced to known statements and consequently produces new ones. We multiply $\sin(\pi p/2)$ to the defect free energy to interpolate between B for even p and D for odd p smoothly in the dimensional regularization as in the generalized F-theorem [10]. Compared with the UV divergent structure (4.62), $\tilde{D}$ is nothing but the universal part of the defect free energy for odd p

$$\tilde{D} = D \,, \tag{4.67}$$

while one finds a more nontrivial relation for even p

$$\tilde{D} = \frac{\pi}{2} B \,. \tag{4.68}$$

When p is odd, our conjecture states the monotonicity of the constant universal term,

$$D_{\mathrm{UV}} \geq D_{\mathrm{IR}} \,. \tag{4.69}$$

For BCFT_2, this is just a weak form of the g-theorem [21, 101, 110], while as for BCFT_d with $d \geq 3$, a similar conjecture was proposed by [22, 23] and examined holographically in [24, 27]. Accordingly our proposal states a new one for $d \geq 3$ and $p \leq d-2$,.

For even p, our claim expresses

$$B_{\mathrm{UV}} \geq B_{\mathrm{IR}} \,, \tag{4.70}$$

which was speculated to hold in $d = 3$ based on the studies of the holographic models of BCFTs and ICFTs [22, 24]. For $p = 2$, this is equivalent to the b-theorem [20] stating the monotonicity[12]

$$b_{\mathrm{UV}} \geq b_{\mathrm{IR}} \,, \tag{4.71}$$

of the universal coefficient b of DCFT appearing in the trace of the stress-energy tensor on the defect[13] [14]

$$\langle T^\mu{}_\mu \rangle = -\frac{1}{24\pi} \left[b\,\hat{\mathcal{R}} + d_1\, \tilde{K}^{(\alpha)}_{ab} \tilde{K}^{(\alpha)\,ab} + d_2\, W_{abcd}\, \hat{g}^{ac}\, \hat{g}^{bd} \right] \delta^{d-2}(x_\perp) \,, \tag{4.72}$$

where $\tilde{K}^{(\alpha)}_{ab}$ denotes similarly the traceless part of the extrinsic curvature for unit normal vectors $n^{(\alpha)}_a$ ($\alpha = 1, \cdots, d-2$), and W_{abcd} is the pullback of the bulk Weyl tensor. Remark that in $d = 3$ the Weyl tensor vanish identically. B is proportional to b up to a positive constant. To fix the proportional constant one may consider a spherical defect of radius l and see how the defect free energy changes under the Weyl rescaling. Since $\tilde{K}^{(\alpha)}_{ab}$ and W_{abcd} vanish on a sphere[15] the Weyl rescaling reads

$$l\, \frac{\mathrm{d}}{\mathrm{d}l} \log \langle \mathcal{D}^{(2)} \rangle = -\int \mathrm{d}^d x\, \sqrt{g}\, \langle T^\mu{}_\mu \rangle = \frac{b}{3} \,, \tag{4.73}$$

[12]For comparison, we use their original notation only around here.

[13]Our convention of the stress tensor differs from the one in [20] up to the sign.

[14]The following trace anomaly has different normalization compared to (2.104).

[15]One can show $\tilde{K}^{(\alpha)}_{ab} = 0$ by mapping the bulk sphere and the two-sphere to flat space and a two-sphere and computing the extrinsic curvatures as $\tilde{K}^{(\alpha)}_{ab}$ is conformal covariant [111].

which fixes the logarithmic divergent term

$$\log \langle \mathcal{D}^{(2)} \rangle = \cdots + \frac{b}{3} \log \frac{l}{\epsilon} + \cdots \,, \tag{4.74}$$

where we recover the UV cutoff ϵ to make the argument of the logarithm dimensionless. Hence compared with (4.62) we find

$$B = \frac{b}{3} \,. \tag{4.75}$$

In conclusion, our conjecture not only unifies all the previous ones known to us, but also generates a new family of C-theorems in DCFTs with higher-codimensional defects. In the rest of this section to come, we test our conjecture in various concrete examples, including conformal perturbation theory of DCFT and Wilson loops in several gauge theories. We will find a number of evidence to support the conjecture that the sphere free energy be the monotonically decreasing function, while we will see that the defect entropy does not decrease under RG flows in a variety of Wilson loops.

4.3.1 Conformal perturbation theory on defect

To examine the validity of our conjecture, we first consider the conformal perturbation theory of DCFT on a sphere, which is a straightforward extension of the works for CFT on a sphere [5,9] and BCFT on a hemisphere [22,23]. Since the calculation is exactly the same as the bulk case, just replacing the bulk dimension d with the defect dimension p, we will only give the outline.

We locate DCFT on a sphere of a radius R and perturb the theory by a defect relevant operator $\hat{\mathcal{O}}$,

$$I = I_{\text{DCFT}} + \hat{\lambda}_0 \int \mathrm{d}^p \hat{x} \, \sqrt{\hat{g}} \, \hat{\mathcal{O}}(\hat{x}) \,. \tag{4.76}$$

Let the conformal dimension of $\hat{\mathcal{O}}$ be $\hat{\Delta} = p - \epsilon$ and take ϵ be very small so that a nontrivial fixed point can be reliably studied within the perturbation theory. Introducing the dimensionless renormalized coupling $\hat{\lambda}$ that is related to the bare coupling $\hat{\lambda}_0$ by

$$\hat{\lambda}_0 \, (2R)^{\epsilon} = \hat{\lambda} + \frac{\pi^{p/2}}{\epsilon \, \Gamma(p/2)} \, \hat{C} \, \hat{\lambda}^2 + O(\hat{\lambda}^3) \,, \tag{4.77}$$

the beta function is given by [5,9]

$$\beta(\hat{\lambda}) = -\epsilon \, \hat{\lambda} + \frac{\pi^{p/2}}{\Gamma(p/2)} \, \hat{C} \, \hat{\lambda}^2 + O(\hat{\lambda}^3) \,, \tag{4.78}$$

where $\hat{C}$ is the coefficient appearing in the three-point function of defect local operators evaluated at the unperturbed DCFT

$$\langle \, \hat{\mathcal{O}}(\hat{x}_1) \, \hat{\mathcal{O}}(\hat{x}_2) \, \hat{\mathcal{O}}(\hat{x}_3) \, \rangle_0 = \frac{\hat{C}}{|\hat{x}_1 - \hat{x}_2|^{\hat{\Delta}} |\hat{x}_2 - \hat{x}_3|^{\hat{\Delta}} |\hat{x}_3 - \hat{x}_1|^{\hat{\Delta}}} \,. \tag{4.79}$$

Hence if $\hat{C} > 0$ the theory flows to a nontrivial IR fixed point at

$$\hat{\lambda}_* = \frac{\Gamma(p/2)}{\pi^{p/2}\,\hat{C}}\,\epsilon + O(\epsilon^2)\ . \tag{4.80}$$

The difference of the sphere partition function is calculated perturbatively

$$\delta \log\, Z(\hat{\lambda}) \equiv \log\, Z(\hat{\lambda}_0) - \log\, Z(\hat{\lambda}_0 = 0) = \frac{\hat{\lambda}_0^2}{2}\, I_2 - \frac{\hat{\lambda}_0^3}{6}\, I_3 + O(\hat{\lambda}_0^4)\ , \tag{4.81}$$

where

$$I_2 = \int \mathrm{d}^p\hat{x}_1\sqrt{\hat{g}}\int \mathrm{d}^p\hat{x}_2\sqrt{\hat{g}}\,\langle\,\hat{\mathcal{O}}(\hat{x}_1)\,\hat{\mathcal{O}}(\hat{x}_2)\,\rangle_0 = \frac{\pi^{p+1/2}\,(2R)^{2\epsilon}}{2^{p-1}}\frac{\Gamma(-p/2+\epsilon)}{\Gamma\left((p+1)/2\right)\,\Gamma(\epsilon)}\ , \tag{4.82}$$

$$\begin{aligned} I_3 &= \int \mathrm{d}^p\hat{x}_1\sqrt{\hat{g}}\int \mathrm{d}^p\hat{x}_2\sqrt{\hat{g}}\int \mathrm{d}^p\hat{x}_3\sqrt{\hat{g}}\,\langle\,\hat{\mathcal{O}}(\hat{x}_1)\,\hat{\mathcal{O}}(\hat{x}_2)\,\hat{\mathcal{O}}(\hat{x}_3)\,\rangle_0 \\ &= \frac{8\pi^{3(p+1)/2}R^{3\epsilon}}{\Gamma(p)}\frac{\Gamma((-p+3\epsilon)/2)}{\Gamma((1+\epsilon)/2)^3}\,\hat{C}\ . \end{aligned} \tag{4.83}$$

Written in terms of the renormalized coupling, one finds [9]

$$\delta \log\, Z(\hat{\lambda}) = \frac{2\pi^{p+1}}{\sin(\pi p/2)\,\Gamma(p+1)}\left[-\frac{1}{2}\,\epsilon\,\hat{\lambda}^2 + \frac{1}{3}\,\frac{\pi^{p/2}}{\Gamma(p/2)}\,\hat{C}\,\hat{\lambda}^3 + O(\hat{\lambda}^4)\right]\ . \tag{4.84}$$

Thus the difference between the universal part of the defect free energy at the IR fixed point (4.80) and that at the UV fixed point is

$$\tilde{D}(\hat{\lambda}_*) - \tilde{D}(0) = -\frac{1}{3}\,\frac{\pi\,\Gamma(p/2)^2}{\Gamma(p+1)}\,\frac{\epsilon^3}{\hat{C}^2} + O(\epsilon^4)\ , \tag{4.85}$$

which is negative as consistent with our conjecture.

4.3.2 Wilson loop as a defect operator

Our next example to test the proposal is a circular Wilson loop operator,

$$W_{\mathfrak{R}}[A] = \mathrm{tr}_{\mathfrak{R}}\, \exp\left[\mathrm{i}\int \mathrm{d}x^\mu A_\mu\right]\ , \tag{4.86}$$

which can be regarded as a $p = 1$ defect. We assume that the gauge group is $SU(N)$ and $\mathfrak{R}$ is a representation of $SU(N)$ for a moment. The Wilson loop can be regarded as an action localized on the defect in the following way [112, 113].[16] First we consider fermions localized on the defect and coupled to the gauge field,

$$I_\chi = \int \mathrm{d}t\,\chi^\dagger\,(\mathrm{i}\,\partial_t - A(t))\,\chi\ , \tag{4.87}$$

where χ_a is in the fundamental representation of $SU(N)$. Then, the partition function on the defect,

$$Z_q[A] \equiv \frac{1}{q!}\int \mathcal{D}\chi^\dagger\mathcal{D}\chi\,\chi_{a_1}(+\infty)\cdots\chi_{a_q}(+\infty)\,\chi^{\dagger,a_1}(-\infty)\cdots\chi^{\dagger,a_q}(-\infty)\,\mathrm{e}^{-I_\chi}\ , \tag{4.88}$$

[16] See also a recent work [114] for a different formulation of a defect theory on Wilson loops.

is equivalent to the Wilson loop up to a normalization factor

$$\frac{Z_q[A]}{Z_q[0]} = W_{\mathfrak{R}}[A] \,, \tag{4.89}$$

where the representation $\mathfrak{R}$ in the Wilson loop relies on whether χ are fermions or bosons. When χ are fermions (bosons), $\mathfrak{R}$ is the q^{th} anti-symmetric (symmetric) representation of $SU(N)$.

Given this description, the defect theory can be flowed to the trivial theory without fermions, or equivalently

$$W_{\mathfrak{R}}[A] \to 1 \,, \tag{4.90}$$

under the mass deformation

$$I_M = -\int \mathrm{d}t \, M \, \chi^\dagger \chi \,, \tag{4.91}$$

by sending M to the infinity.

In what follows, we assume that any Wilson loop has a realization as a defect theory of the fermion and there exists a defect RG flow whose IR fixed point is a trivial theory without loops. Under this assumption, our conjecture amounts to the following inequality

$$\log \langle \, W_{\mathfrak{R}} \, \rangle|_{\text{UV}} \geq \log \langle \, W_{\mathfrak{R}} \, \rangle|_{\text{IR}} = 0 \,. \tag{4.92}$$

We will provide shreds of evidence for our assertion by working out several examples of Wilson loops.

$U(1)$ gauge theory in 4d

Our first example is the Wilson loop in a four-dimensional $U(1)$ gauge theory

$$W = \exp\left[\mathrm{i}\, e \oint \mathrm{d}x^\mu A_\mu\right] \,, \qquad e \in \mathbb{R} \,. \tag{4.93}$$

The defect free energy is given by

$$\log \langle \, W \, \rangle = \frac{e^2}{4} \,, \tag{4.94}$$

which is seen to be positive while the defect entropy vanishes [104]

$$S_{\text{defect}} = 0 \,. \tag{4.95}$$

It is expected that the Wilson loop becomes trivial under a defect RG flow,

$$\log \langle \, W \, \rangle \to 0 \,, \tag{4.96}$$

so this is consistent with our conjecture. On the other hand, the defect entropy vanishes at both the UV and IR fixed points. Hence, the defect entropy does not appear to capture degrees of freedom on the defect.

Free scalar field in 4d

Instead of gauge theories, one can construct a Wilson loop from a free scalar field in four dimensions [64],

$$W = \exp\left[\lambda \oint \mathrm{d}t\, \phi\left(x^\mu(t)\right)\right] , \qquad \lambda \in \mathbb{C} . \tag{4.97}$$

The defect entropy is computed by evaluating the Gaussian integral, and shown to vanish

$$\log \langle W \rangle = 0 . \tag{4.98}$$

Reassuringly this result does not contradict with our assertion. On the other hand, the defect entropy is given by [104]

$$S_{\text{defect}} = -\frac{\lambda^2}{12} , \tag{4.99}$$

which can be negative for real λ at the UV fixed point while it is supposed to be zero at the IR fixed point. Thus this is a counterexample for the defect entropy being a C-function.

Chern-Simons theory

Next we move onto a more nontrivial example, namely Wilson loops in the Chern-Simons theory in three-dimensions[17]

$$W_{\mathfrak{R}} = \mathrm{tr}_{\mathfrak{R}}\, \mathcal{P} \exp\left[\mathrm{i} \oint \mathrm{d}x^\mu A_\mu\right] . \tag{4.100}$$

For $SU(2)$ with level k, the Wilson loop in the representation $\mathfrak{R}_j$ is labeled by the dimension $j = 1, \cdots, k+1$, whose vev on $\mathbb{S}^3$ is [116, 117]

$$\langle W_{\mathfrak{R}_j} \rangle = \frac{\sin\left(\pi\, j/(k+2)\right)}{\sin\left(\pi/(k+2)\right)} , \tag{4.101}$$

which is greater than or equal to one.

More generally the vev of a Wilson loop in an arbitrary representation $\mathfrak{R}_j$ on $\mathbb{S}^3$ is given by [116]

$$\langle W_{\mathfrak{R}_j} \rangle = \frac{S_{0,j}}{S_{0,0}} \equiv d_j , \tag{4.102}$$

where $S_{i,j}$ is the matrix element of the modular group S-matrix. The vev or d_j is called the quantum dimension of $\mathfrak{R}_j$, which is known to be greater than or equal to one [118] (and see also Appendix C in [119]),

$$d_j \geq 1 , \tag{4.103}$$

which is consistent with our conjecture. Note that the defect entropy is also given by

$$S_{\text{defect}} = \log \langle W_{\mathfrak{R}_j} \rangle = \log\, d_j , \tag{4.104}$$

as the stress tensor vanishes in Chern-Simons theory.[18]

[17] Note that our normalization for Wilson loops are different from the one in [115] where the operators are divided by the dimension of the representation.

[18] This result was previously obtained by [120–122].

$1/2$-BPS Wilson loop in $4d$ $\mathcal{N}=4$ SYM

We then consider $1/2$-BPS Wilson loops in four-dimensional $\mathcal{N}=4$ super Yang-Mills theory with gauge group $U(N)$

$$W_{\mathfrak{R}} = \mathrm{tr}_{\mathfrak{R}}\, \mathcal{P}\, \exp\left[\oint \mathrm{d}t\, (\mathrm{i}\, A_\mu\, \dot{x}^\mu + \phi_I\, \dot{y}^I)\right] . \tag{4.105}$$

For the fundamental representation, the exact result of the defect free energy is known [123]

$$\log \langle\, W\, \rangle = \frac{\lambda}{8N} + \log L^1_{N-1}\left(-\frac{\lambda}{4N}\right) , \tag{4.106}$$

where $L^m_n(x)$ is the associated Laguerre polynomial. In the small λ region we find the expansion

$$\log \langle\, W\, \rangle = \log N + \frac{\lambda}{8} - \frac{1}{384}\left(1 - \frac{1}{N^2}\right)\lambda^2 + O(\lambda^3) , \tag{4.107}$$

which is seen to be positive for any N and small λ. One can indeed check numerically that it is always positive for any N and λ (see the left panel in figure 4.4).

On the other hand, the defect entropy can be calculated from the defect free energy through the relation [104]

$$S_{\text{defect}} = \left(1 - \frac{4}{3}\lambda\, \partial_\lambda\right) \log \langle\, W\, \rangle . \tag{4.108}$$

Then we find that the entropy is not necessarily positive in the small λ limit (see also the right panel in figure 4.4)

$$S_{\text{defect}} = \log N - \frac{\lambda}{24} + \frac{5}{1152}\left(1 - \frac{1}{N^2}\right)\lambda^2 + O(\lambda^3) . \tag{4.109}$$

This example also serves as a supporting evidence for our conjecture and a nontrivial counterexample for the defect entropy being a C-function.

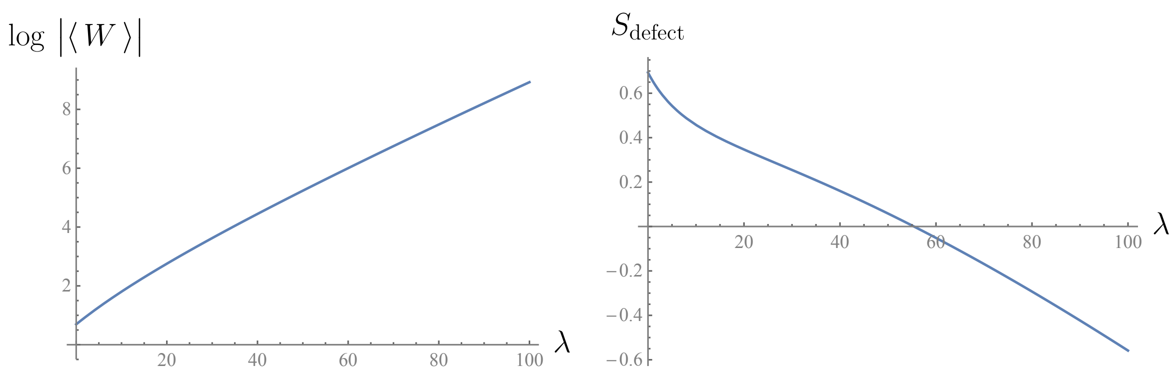

Figure 4.4: The defect free energy (Left) and the defect entropy (Right) of the 1/2-BPS Wilson loop in $4d$ $\mathcal{N}=4$ SYM. The $N=2$ cases are shown. The defect free energy is positive for any λ while the defect entropy can be negative.

1/6-BPS Wilson loop in ABJM

In the fundamental representation of the ABJM theory in three dimensions with gauge groups $U(N)_k \times U(N)_{-k}$ is the 1/6-BPS Wilson loop,

$$W = \mathrm{tr}\, \mathcal{P} \exp \left[\oint \mathrm{d}t \, (\mathrm{i}\, A_\mu \, \dot{x}^\mu + \frac{2\pi}{k} M_J^I \, C_I \, C^J \, |\dot{x}|) \right] \, , \tag{4.110}$$

where $C^I (I = 1, 2, 3, 4)$ are the scalar fields in the bi-fundamental chiral multiplets and M_J^I is a constant matrix whose diagonalized form is $\mathrm{diag}(1, 1, -1, -1)$. The localization technique for supersymmetric theories allows us to compute the vev of the m multiply-winding Wilson loop $W^{(m)}$ by the matrix model [115]

$$\begin{aligned} \langle \, W^{(m)} \, \rangle = \frac{1}{Z} \frac{1}{(N!)^2} \int \prod_{i=1}^{N} \frac{\mathrm{d}\mu_i \, \mathrm{d}\nu_i}{(2\pi)^2} \, \mathrm{e}^{\mathrm{i}\, k (\mu_i^2 - \nu_i^2)/4\pi} \\ \cdot \frac{\prod_{i<j} \left[4 \sinh\left((\mu_i - \mu_j)/2 \right) \, \sinh\left((\nu_i - \nu_j)/2 \right) \right]^2}{\prod_{i,j} \left[2 \cosh\left((\mu_i - \nu_j)/2 \right) \right]^2} \sum_i \mathrm{e}^{m \mu_i} \, , \end{aligned} \tag{4.111}$$

where Z is the partition function

$$Z = \frac{1}{(N!)^2} \int \prod_{i=1}^{N} \frac{\mathrm{d}\mu_i \, \mathrm{d}\nu_i}{(2\pi)^2} \, \mathrm{e}^{\mathrm{i}\, k (\mu_i^2 - \nu_i^2)/4\pi} \, \frac{\prod_{i<j} \left[4 \sinh\left((\mu_i - \mu_j)/2 \right) \, \sinh\left((\nu_i - \nu_j)/2 \right) \right]^2}{\prod_{i,j} \left[2 \cosh\left((\mu_i - \nu_j)/2 \right) \right]^2} \, . \tag{4.112}$$

Though the exact computation of the integral is quite difficult in general, it is straightforward for $N = 1$,

$$\langle \, W^{(m)} \, \rangle = \cos^{-2} \left(\frac{\pi m}{k} \right) \, , \qquad (N = 1) \, . \tag{4.113}$$

which is seen to be greater than or equal to one for $m = 1$ and any k, hence consistent with our conjecture.

The defect entropy can be read off from the vev of the winding Wilson loop by the formula

$$S_{\text{defect}} = \lim_{m \to 1} \left(1 - \frac{1}{2} m \partial_m \right) \log |\langle \, W^{(m)} \, \rangle| \, , \tag{4.114}$$

which is derived in [104] using the supersymmetric Rényi entropy [105]. Substituting (4.113) into (4.114) we find

$$S_{\text{defect}} = - \log \cos^2 \left(\frac{\pi}{k} \right) - \frac{\pi}{k} \tan \left(\frac{\pi}{k} \right) \, , \tag{4.115}$$

which is always negative for positive integer k.

In the large N limit, the matrix model reduces to the integral [124]

$$\langle \, W^{(m)} \, \rangle = \frac{N}{2\pi^2 \, \mathrm{i}\, \lambda} \int_{-a}^{a} \mathrm{d}x \, \mathrm{e}^{mx} \, \arctan \sqrt{\frac{\alpha - 2 \cosh x}{\beta + 2 \cosh x}} \, , \tag{4.116}$$

where

$$e^a = \frac{2 + \mathrm{i}\,\kappa + \sqrt{\kappa(4\mathrm{i} - \kappa)}}{2}\,, \qquad \alpha = 2 + \mathrm{i}\,\kappa\,, \qquad \beta = 2 - \mathrm{i}\,\kappa\,, \tag{4.117}$$

and

$$\lambda = \frac{N}{k} = \frac{\kappa}{8\pi}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\frac{\kappa^2}{16}\right)\,. \tag{4.118}$$

In the small λ limit, we find the vev of the fundamental Wilson loop [124]

$$\log\,|\langle\, W\,\rangle| = \log N + \frac{5\pi^2\lambda^2}{6} + O(\lambda^4)\,, \tag{4.119}$$

and the defect entropy [104]

$$S_{\text{defect}} = \log N - \frac{\pi^2\lambda^2}{6} + O(\lambda^4)\,, \tag{4.120}$$

both of which are dominated by $\log N$, hence positive. They are also increasing function for λ large enough.

$U(N)\,\mathcal{N}=4$ **SYM with** N_f **hypermultiplets in** $3d$

Our final example is the Wilson loop in three-dimensional $\mathcal{N}=2$ supersymmetric theories defined by

$$W_{\mathfrak{R}} \equiv \mathrm{tr}_{\mathfrak{R}}\,\mathcal{P}\,\exp\left[\oint \mathrm{d}t\,(\mathrm{i}\,A_\mu\,\dot{x}^\mu + \sigma\,|\dot{x}|)\right]\,, \tag{4.121}$$

where σ is the adjoint scalar field in the vector multiplet.

To be more concrete, let us consider the Wilson loop in $U(N)\,\mathcal{N}=4$ supersymmetric gauge theory with N_f hypermultiplets. The partition function in this theory is given by

$$Z = \frac{1}{N!}\int\prod_{i=1}^{N}\frac{\mathrm{d}\mu_i}{2\pi}\,\frac{\prod_{i<j}4\sinh^2\left((\mu_i-\mu_j)/2\right)}{\prod_i\left[2\cosh\left(\mu_i/2\right)\right]^{N_f}}\,, \tag{4.122}$$

and the expectation value of the Wilson loop in the representation labeled by the Young diagram of the partition λ is

$$\langle\, W_\lambda\,\rangle = \frac{1}{Z}\frac{1}{N!}\int\prod_{i=1}^{N}\frac{\mathrm{d}\mu_i}{2\pi}\,\frac{s_\lambda(e^{\mu_1},\cdots,e^{\mu_N})}{\prod_i\left[2\cosh\left(\mu_i/2\right)\right]^{N_f}}\prod_{i<j}4\sinh^2\left((\mu_i-\mu_j)/2\right)\,, \tag{4.123}$$

where s_λ is the Schur polynomial. This integral can be performed exactly, resulting in the simple formula [125],

$$\langle\, W_\lambda\,\rangle = \frac{s_\lambda(1_{N_f/2})\,s_\lambda(1_N)}{s_{\lambda'}(1_{N_f/2-N})}\,, \tag{4.124}$$

where λ' is the conjugate representation of λ and

$$s_\lambda(1_n) \equiv s_\lambda(1,1,\cdots,1) = \prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j + j - i}{j-i} \,. \tag{4.125}$$

Then the Wilson loop in the fundamental representation becomes

$$\langle\, W_{(1)}\,\rangle = \frac{N_f}{N_f/N - 2} \,, \tag{4.126}$$

which is greater than one when $N_f > 2N$. This regime corresponds to "good" or "ugly" theories while $N \le N_f < 2N$ corresponds to "bad" theories with unitarity violating monopole operators. In the latter parameter region, the theory is proposed to be dual to the "good" theory of $U(N_f - N)$ gauge group with N_f hypermultiplet and $2N - N_f$ additional free (twisted) hypermultiplets [126].

For a multiply-winding Wilson loop with winding number m, we replace e^{μ_i} with $\mathrm{e}^{m\mu_i}$ in the argument of the Schur polynomial

$$\langle\, W_\lambda^{(m)}\,\rangle = \frac{1}{Z}\frac{1}{N!}\int \prod_{i=1}^N \frac{\mathrm{d}\mu_i}{2\pi} \frac{s_\lambda(\mathrm{e}^{m\mu_1},\cdots,\mathrm{e}^{m\mu_N})}{\prod_i \left[2\cosh\left(\mu_i/2\right)\right]^{N_f}} \prod_{i<j} 4\sinh^2\left((\mu_i-\mu_j)/2\right) \,. \tag{4.127}$$

This expression can be expanded by a linear combination of singly winding Wilson loops. For instance, the Wilson loop with winding number m in the fundamental representation

$$\langle\, W_{(1)}^{(m)}\,\rangle = \sum_{l=0}^m (-1)^l \,\langle\, W^{(m-l,1^l)}\,\rangle \,, \tag{4.128}$$

which follows from the identity

$$s_{(1)}(x_1^m,\cdots,x_N^m) = \sum_{l=0}^m (-1)^l\, s_{(m-l,1^l)}(x_1,\cdots,x_N) \,. \tag{4.129}$$

With the aid of the formula (4.124) we find

$$\langle\, W_{(m-l,1^l)}\,\rangle = \frac{\Gamma(N_f/2+m-l)\,\Gamma(N+m-l)\,\Gamma(N_f/2-N-m+l+1)}{m\,\Gamma(m-l)\,\Gamma(l+1)\,\Gamma(N_f/2-l)\,\Gamma(N-l)\,\Gamma(N_f/2-N+l+1)} \,. \tag{4.130}$$

It follows that the vev of the winding Wilson loop is given exactly for $N=1$ by

$$\langle\, W_{(1)}^{(m)}\,\rangle = \frac{\Gamma\left(N_f/2-m\right)\Gamma\left(m+N_f/2\right)}{\Gamma\left(N_f/2\right)^2} \,, \tag{4.131}$$

and for $N=2$ by

$$\langle\, W_{(1)}^{(m)}\,\rangle = \frac{(N_f+2m^2-2)\,\Gamma\left(N_f/2-m-1\right)\Gamma\left(N_f/2+m-1\right)}{\Gamma\left(N_f/2-1\right)\Gamma\left(N_f/2\right)} \,. \tag{4.132}$$

Using the expression (4.114) for the defect entropy we obtain for $N=1$

$$S_{\text{defect}} = \frac{N_f^2 - 4N_f + 2}{(N_f-2)^2} \,, \qquad (N=1) \,, \tag{4.133}$$

which is negative for $N_f = 3 > 2N$, and

$$S_{\text{defect}} = \frac{2(N_f^3 - 10N_f^2 + 26N_f - 16)}{(N_f-4)^2(N_f-2)} \,, \qquad (N=2) \,, \tag{4.134}$$

which is also negative for $N_f = 5, 6 > 2N$. We thus conclude that the defect entropy does not necessarily decrease under the defect RG flow to the trivial fixed point in this example.

4.4 Holographic construction of DCFT

So far we have examined our conjecture in field theoretic examples. In this section we take a detour to check if it holds by using exotic models of DCFTs – employing holographic duality, or more precisely, AdS/CFT correspondence [15].

AdS/CFT correspondence states that a particular class of weakly coupled theories of quantum gravity on AdS$_{d+1}$ spacetime is equivalent to specific strongly coupled d-dimensional CFTs which live on a boundary of AdS spacetime. While string/M theory gives top-down constructions of such correspondence, one may assume it exit and establish bottom-up examples of holographic theories dual to certain CFTs.[19] Here we take the latter approach, in particular consider holographic models that can be written in terms of Einstein theories of gravity. The precise relation between the gravity theory in AdS and the dual CFT is given by so called GKP-W relation [129, 130]:

$$Z_{\text{AdS}_{d+1}}[\phi_0] = \langle e^{-\int \mathrm{d}^d x \phi_0(x)\mathcal{O}(x)} \rangle_{\text{CFT}_d} , \tag{4.135}$$

where $Z_{\text{AdS}_{d+1}}$ is a partition function of the theory on AdS$_{d+1}$ with the bulk fields ϕ, which have classical configurations ϕ_0 at the boundary of AdS$_{d+1}$,

$$Z_{\text{AdS}_{d+1}}[\phi_0] = \int_{\phi|_{\text{bry}}=\phi_0} \mathcal{D}\phi e^{-S_{\text{AdS}}[\phi]} . \tag{4.136}$$

The right hand side of (4.135) denotes a generating functional of CFT with a operator $\mathcal{O}$ sourced by ϕ_0. (4.135) enables us to compute CFT correlation functions by varying $Z_{\text{AdS}_{d+1}}$ with respect to ϕ_0:

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_{\text{CFT}_d} = \frac{\delta}{\delta\phi_0(x_1)}\cdots\frac{\delta}{\delta\phi_0(x_n)} Z_{\text{AdS}_{d+1}}[\phi_0]|_{\phi_0=0} . \tag{4.137}$$

In the following subsection, realizing the CHM map as a coordinate transformation in the AdS spacetime [57] we calculate the defect entropy as the black hole entropy in the mapped spacetime. Along the way we point out the difference between the defect entropy and the defect free energy that is holographically given by minus the on-shell action. We then perform the holographic calculations of the defect free energy and the defect entropy in these models. In the viewpoint of holography, a defect RG flow can be triggered geometrically by a deformation of the spacetime. We then establish the holographic C-theorem in DCFT by imposing the null energy condition on the gravity duals of field theories, which provides further evidence for our conjecture.

4.4.1 CHM map and defect entropy in holography

In usual AdS/CFT correspondence, a gravity dual of CFT is in general an asymptotically AdS spacetime which manifests conformal isometry $SO(d,2)$ in Lorentzian signature. Since we would like to study holographic model of CFT with defect, we consider general metric of an

[19] It isn't always the case that every CFT has such a gravity description. The condition for CFT to have holographic dual was investigated, for example in [127, 128].

asymptotically AdS space preserving the $SO(p,2)\times SO(d-p)$ isometry which is the symmetry of DCFT. Such a metric take the following form,

$$\mathrm{d}s^2 = L^2\left[\mathrm{d}\rho^2 + A(\rho)^2\,\mathrm{d}s^2_{\mathrm{AdS}_{p+1}} + B(\rho)^2\,\mathrm{d}s^2_{\mathbb{S}^{d-p-1}}\right] . \tag{4.138}$$

where $\mathrm{d}s^2_{\mathrm{AdS}_{p+1}}$ and $\mathrm{d}s^2_{\mathbb{S}^{d-p-1}}$ are the metrics of AdS_{p+1} and $\mathbb{S}^{d-p-1}$ of unit radius respectively. Here we denote the radius of curvature for the spacetime as L. For $p<d-1$ the range of ρ is $0\le\rho<\infty$. $A(\rho)$ and $B(\rho)$ are arbitrary positive definite functions that have the asymptotic forms near the boundary ($\rho\to\infty$)

$$A(\rho),\, B(\rho)\ \to\ \frac{\exp(\rho-c_p)}{2}\, . \tag{4.139}$$

For $p=d-1$, $\rho\in(-\infty,\infty)$ and the conformal boundary sits at $\rho\to\pm\infty$.

It is easy to see that the boundary spacetime of (4.138) reached by the $\rho\to\infty$ limit is $\mathrm{AdS}_{p+1}\times\mathbb{S}^{d-p-1}$, which is conformally equivalent to $\mathbb{R}\times\mathbb{H}^{d-1}$ by the CHM map as expected. We can tell this map directly in the AdS spacetime by taking the topological black hole coordinates for the AdS_{p+1} subspace,

$$\mathrm{d}s^2_{\mathrm{AdS}_{p+1}} = -f(V)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}V^2}{f(V)} + V^2\,\mathrm{d}s^2_{\mathbb{H}^{p-1}}\, ,\quad f(V)=V^2-1\, , \tag{4.140}$$

which is easily archived by changing slicing of AdS coordinates from Poincaré to hyperbolic. The resulting net metric is an asymptotically AdS black hole solution with the horizon located at $V=1$ and with the Hawking temperature $T_0=1/2\pi$,

$$\mathrm{d}s^2 = L^2\,A(\rho)^2\left[-f(V)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}V^2}{f(V)} + V^2\,\mathrm{d}s^2_{\mathbb{H}^{p-1}}\right] + L^2\left(\mathrm{d}\rho^2 + B(\rho)^2\,\mathrm{d}s^2_{\mathbb{S}^{d-p-1}}\right)\, , \tag{4.141}$$

whose asymptotic boundary at $\rho\to\infty$ becomes $\mathbb{R}\times\mathbb{H}^{d-1}$ as in (4.30) up to a conformal factor

$$\begin{aligned}\mathrm{d}s^2\ &\to \frac{L^2}{4}\,\mathrm{e}^{2(\rho-c_p)}\,f(V)\,\left[-\mathrm{d}\tau^2 + \frac{\mathrm{d}V^2}{f(V)^2} + \frac{V^2}{f(V)}\,\mathrm{d}s^2_{\mathbb{H}^{p-1}} + \frac{1}{f(V)}\,\mathrm{d}s^2_{\mathbb{S}^{d-p-1}}\right]\\ &= \frac{L^2}{4}\,\mathrm{e}^{2(\rho-c_p)}\,f(V)\,\left[-\mathrm{d}\tau^2 + \mathrm{d}x^2 + \cosh^2 x\,\mathrm{d}s^2_{\mathbb{H}^{p-1}} + \sinh^2 x\,\mathrm{d}s^2_{\mathbb{S}^{d-p-1}}\right]\, ,\end{aligned} \tag{4.142}$$

where we introduced the new coordinate x by $V=\coth x$. Note that since the topological black hole is just a patch of the AdS_{p+1} subspace, the horizon at $V=1$ is artificial. There is neither real singularity nor horizon.

Now let us evaluate the entanglement entropy of a spherical entangling region considered in section 4.2 holographically. We have two options to calculate the entanglement entropy, yielding the same answer: (1) use the Ryu-Takayanagi formula of the holographic entanglement entropy [109], (2) use the CHM map and equate the entanglement entropy with the thermal entropy.

We shall first explain the first method. Ryu-Takayanagi originally conjectured the holographic dual of the entanglement entropy with time-independent entangling region Σ in CFTs by considering the time-independent codimension-two minimal surface that approaches Σ along the boundary of AdS,

$$S^{(\mathrm{CFT})} = \frac{\mathcal{A}_{min}}{4G_{\mathrm{N}}}\, , \tag{4.143}$$

where $\mathcal{A}_{min}$ is the area of the minimal surface and $G_{\rm N}$ the $(d+1)$-dimensional Newton constant. Many researchers accumulated a number of evidence to hold the formula true, and also tried to extend it in more general settings. Readers who are interested in the whole story may refer to the textbook [131] and references therein for details.

We apply this procedure for our setting. In the original coordinate (4.138), the Ryu-Takayanagi minimal surface satisfies [57]

$$Z(r_\parallel)^2 + r_\parallel^2 = R^2 \, . \tag{4.144}$$

Moving into the topological black hole coordinates (4.141), one can realize this surface coincides with the black hole horizon, meaning that the entanglement entropy is just given by the same form of the Bekenstein entropy of the black hole,

$$S^{\rm (DCFT)} = \frac{\mathcal{A}_{\rm H}}{4G_{\rm N}} \, , \tag{4.145}$$

where $\mathcal{A}_{\rm H}$ is the area of the black hole horizon

$$\begin{aligned} \mathcal{A}_{\rm H} &= L^{d-1}\, \text{Vol}(\mathbb{S}^{d-p-1})\, \text{Vol}(\mathbb{H}^{p-1}) \int_0^\infty {\rm d}\rho\, A(\rho)^{p-1}\, B(\rho)^{d-p-1} \\ &= L^{d-1} \frac{2\pi^{d/2}}{\sin\left(\pi p/2\right)\Gamma\left(p/2\right)\, \Gamma\left((d-p)/2\right)} \int_0^\infty {\rm d}\rho\, A(\rho)^{p-1}\, B(\rho)^{d-p-1} \, , \end{aligned} \tag{4.146}$$

and we used the sphere volume and the regularized volume of the hyperbolic space[20]

$$\text{Vol}(\mathbb{S}^{d-p-1}) = \frac{2\pi^{(d-p)/2}}{\Gamma\left((d-p)/2\right)} \, , \qquad \text{Vol}(\mathbb{H}^{p-1}) = \frac{\pi^{p/2}}{\sin\left(\pi p/2\right)\Gamma\left(p/2\right)} \, . \tag{4.147}$$

With these in mind, the holographic defect entropy can be easily obtained. As the holographic entanglement entropy without defect is given by (4.145) with $A(\rho) = \cosh\rho$ and $B(\rho) = \sinh\rho$, one finds the holographic defect entropy as follows,

$$\begin{aligned} S_{\rm defect} = &\frac{L^{d-1}}{4G_{\rm N}} \frac{2\pi^{d/2}}{\sin\left(\pi p/2\right)\Gamma\left(p/2\right)\, \Gamma\left((d-p)/2\right)} \\ &\cdot \int_0^\infty {\rm d}\rho\, \left(A(\rho)^{p-1} B(\rho)^{d-p-1} - \cosh^{p-1}\rho\, \sinh^{d-p-1}\rho\right) \, . \end{aligned} \tag{4.148}$$

Next let us consider the second approach, namely using the CHM map and computing the thermal entropy holographically. To this end, we want to have solutions that are dual to DCFTs with finite temperature T, not only at $T = T_0$ as in (4.140). One may notice that (4.140) is just one of a family of black brane solutions, thus we can replace the function $f(V)$ with

$$f(V) = V^2 - 1 - \frac{V_{\rm H}^{p-2}}{V^{p-2}}\,(V_{\rm H}^2 - 1) \, . \tag{4.149}$$

The resulting geometry is an asymptotically AdS black hole whose boundary is $\mathbb{R}\times\mathbb{H}^{d-1}$ that the dual DCFT lives on at temperature

$$T = \frac{1}{4\pi}\left(p\, V_{\rm H} - \frac{p-2}{V_{\rm H}}\right) \, . \tag{4.150}$$

[20] Remark that in section 3.4 we use IR cutoff regularization for the volume of three-dimensional hyperbolic space, while here we use dimensional regularization.

The thermal entropy is obviously given by the black hole entropy

$$S_{\text{thermal}}(T) = V_{\text{H}}^{p-1} \frac{\mathcal{A}_{\text{H}}}{4G_{\text{N}}} , \tag{4.151}$$

from which we can recover (4.145) in $T \to T_0$ ($V_{\text{H}} \to 1$). The advantage of this approach is that one can calculate the Rényi entropy holographically from the thermal entropy [132] as well:

$$\begin{aligned} S_n^{(\text{DCFT})} &= \frac{n}{n-1} \frac{1}{T_0} \int_{T_0/n}^{T_0} \mathrm{d}T \, S_{\text{thermal}}(T) \\ &= \frac{n}{n-1} \left(2 - v^{p-2} - v^p\right) \frac{\mathcal{A}_{\text{H}}}{8G_{\text{N}}} , \end{aligned} \tag{4.152}$$

where $v \equiv \left(1 + \sqrt{1 + p\, n^2 (p-2)}\right) / p\, n$.

The thermal geometry also allows us to consider the difference between the defect entropy and the on-shell action in a similar fashion as the case of field theories. Suppose the holographic models of DCFT and CFT are described by the actions $I_{\text{DCFT}}[G_{MN}]$ and $I_{\text{CFT}}[G^{(0)}_{MN}]$ respectively. Here G_{MN} is the backreacted metric of the form (4.141) with (4.149) and $G^{(0)}_{MN}$ is the one with $A(\rho) = \cosh\rho$ and $B(\rho) = \sinh\rho$. The thermodynamic relation tells us that the defect contribution to the thermal entropy gives the defect entropy holographically,

$$\begin{aligned} S_{\text{defect}} &= \lim_{T \to T_0} \left[-\frac{\partial}{\partial T} (T\, \Delta I) \right] \\ &= \lim_{T \to T_0} \left[-\Delta I - T \frac{\partial}{\partial T} \Delta I \right] . \end{aligned} \tag{4.153}$$

The first term in the right hand side is the difference of the on-shell actions

$$\Delta I \equiv I_{\text{DCFT}}[G_{MN}] - I_{\text{CFT}}[G^{(0)}_{MN}] . \tag{4.154}$$

Compared with the CFT result on the defect entropy (4.54), we find that the first term in (4.153) should be identified with the defect free energy while the second term corresponds to the integrated one-point function $\int \langle\, (T_{\text{DCFT}})^{\tau}{}_{\tau} \,\rangle$ in the dual DCFT through the GKP-W relation [129, 130]. We note that there are some cases where $I_{\text{DCFT}} = I_{\text{CFT}}$. For example, a holographic dual of a Janus interface CFT is described by the type IIB supergravity where the Janus interface is implemented by a nontrivial profile of the dilaton field that backreacts to the metric. Hence ΔI is the difference between the same actions evaluated on the nontrivial and trivial profiles [133].

Domain wall defect RG flow

Having the holographic formula for the defect entropy, next let us consider a holographic defect RG flow, which interpolate between two fixed points described by deforming the metric (4.138) with the defining functions $A_{\text{UV}}(\rho)$, $B_{\text{UV}}(\rho)$ at the UV fixed point obeying the boundary conditions

$$A_{\text{UV}}(\rho),\; B_{\text{UV}}(\rho) \;\to\; \frac{\exp(\rho - c_{\text{UV}})}{2} , \tag{4.155}$$

while we impose $A_{\text{IR}}(\rho)$, $B_{\text{IR}}(\rho)$ at the IR fixed point obeying

$$A_{\text{IR}}(\rho),\ B_{\text{IR}}(\rho)\ \to\ \frac{\exp(\rho - c_{\text{IR}})}{2}\ . \tag{4.156}$$

This is the most general situation, but we restrict our attention to the RG flow with the IR fixed point characterized by

$$A_{\text{IR}}(\rho) = \ell\, A_{\text{UV}}(\rho)\ , \qquad B_{\text{IR}}(\rho) = \ell\, B_{\text{UV}}(\rho)\ , \tag{4.157}$$

for a positive dimensionless constant ℓ. In this case the interpolating metric between the two fixed points must respect the Poincaré symmetry on and the rotational symmetry around the defect, resulting in the domain wall type ansatz

$$\mathrm{d}s^2 = L^2\left[\mathrm{d}\rho^2 + \frac{A_{\text{UV}}(\rho)^2}{f(w)}\,\mathrm{d}s^2_{\text{AdS}_{p+1}} + \frac{B_{\text{UV}}(\rho)^2}{f(w)}\,\mathrm{d}s^2_{\mathbb{S}_{d-p-1}}\right]\ , \tag{4.158}$$

where w is the radial direction in Poincaré coordinate of the sliced AdS space,

$$\mathrm{d}s^2_{\text{AdS}_{p+1}} = \frac{\mathrm{d}w^2 - \mathrm{d}t^2 + \sum_{a=1}^{p-1}\mathrm{d}\hat{x}_a^2}{w^2}\ . \tag{4.159}$$

We regard w as the holographic renormalization scale ranging from the UV at $w = 0$ to the IR at $w = \infty$, and impose the boundary condition

$$f(w)\ \to\ 1\ , \qquad w\ \to\ 0\ , \tag{4.160}$$

at the UV fixed point and

$$f(w)\ \to\ \ell^{-2}\ , \qquad w\ \to\ \infty\ , \tag{4.161}$$

at the IR fixed point.

In order to make the ansatz physically sensible in the Einstein gravity coupled to matters we impose the null energy condition for the matters

$$T_{MN}\zeta^M\zeta^N \geq 0\ , \tag{4.162}$$

for any null vector ζ^M. Choosing ζ^M to be $\zeta^w = 1$, $\zeta^t = 1$ and $\zeta^{M\neq w,t} = 0$ and using the Einstein equation

$$8\pi G_{\text{N}}\, T_{MN} = R_{MN} - \frac{1}{2}\, G_{MN}\, R\ , \tag{4.163}$$

we find

$$8\pi G_{\text{N}}(T_{ww} + T_{tt}) = \frac{d-2}{2w^2\sqrt{f(w)}}\left(\frac{w^2 f'(w)}{\sqrt{f(w)}}\right)' \geq 0\ . \tag{4.164}$$

Since $f(w) > 0$ for $w > 0$, we obtain the inequality

$$f'(w) \geq 0\ , \tag{4.165}$$

which implies $f(w) \geq 1$ for $w > 0$ or equivalently

$$\ell < 1\,. \tag{4.166}$$

In this model, it follows from (4.145) and (4.146) that the difference of the defect entropies between the UV and IR fixed points is

$$S_{\text{defect}}|_{\text{UV}} - S_{\text{defect}}|_{\text{IR}} = (1 - \ell^{d-2})\, S_{\text{defect}}|_{\text{UV}}\,, \tag{4.167}$$

which suggests the monotonicity of the regularized defect entropy if the regularized value at the UV fixed point is positive. On the other hand, one fails to calculate the defect free energy without specifying the AdS action that allows the domain wall metric as a solution. Hence in what follows, we consider more explicit models and examine our proposal for the monotonicity of the defect free energy.

4.4.2 Probe brane model

As a concrete holographic model of DCFT, we consider a brane system embedded in the AdS space. In Euclidean signature, the action of the system becomes

$$I_{d,p} = I_{\text{EH}} + I_{\text{brane}}\,, \tag{4.168}$$

where I_{EH} is Einstein-Hilbert action with a cosmological constant

$$I_{\text{EH}} = -\frac{1}{16\pi G_{\text{N}}} \int_{\mathcal{B}} \mathrm{d}^{d+1}X\, \sqrt{G} \left(\mathcal{R} + \frac{d(d-1)}{L^2}\right)\,, \tag{4.169}$$

and I_{brane} is a brane action

$$I_{\text{brane}} = T_p \int_{\mathcal{Q}} \mathrm{d}^{p+1}\eta \sqrt{\hat{G}}\,. \tag{4.170}$$

with the brane tension T_p and the induced metric $\hat{G}_{AB}$ on the brane. The bulk spacetime $\mathcal{B}$ is fixed by solving the Einstein equation with the source from the brane on $\mathcal{Q}$ which is anchored on the defect of dimension p on the boundary $\mathcal{M} \equiv \partial\mathcal{B}$.

When the tension is small, $T_p L^{p+1} \ll 1$, the brane can be treated as a probe in the sense that the defect free energy is given by minus the on-shell action of the brane

$$\log \langle\, \mathcal{D}^{(p)}\, \rangle = -I_{\text{brane}}\,. \tag{4.171}$$

The on-shell action is simply the volume of the brane times the brane tension

$$\begin{aligned} I_{\text{brane}} &= \text{Vol}(\mathbb{H}^{p+1})\, T_p L^{p+1} \\ &= -\frac{1}{\sin(\pi p/2)} \frac{\pi^{p/2+1}}{\Gamma(p/2+1)}\, T_p L^{p+1}\,. \end{aligned} \tag{4.172}$$

We can similarly compute the leading contribution to the defect entropy in the probe limit. For the spherical entangling region one finds [57]

$$S_{\text{defect}} = \frac{1}{\sin(\pi p/2)} \frac{p}{d-1+\delta_{pd}} \frac{\pi^{p/2+1}}{\Gamma(p/2+1)}\, T_p L^{p+1}\,, \tag{4.173}$$

which is valid for any d and p even when $p = d$, meaning that the spacetime is filled by the brane. It is worthwhile to pointing out that the defect entropy is proportional to the on-shell action

$$S_{\text{defect}} = -\frac{p}{d-1+\delta_{pd}}\, I_{\text{brane}} \, . \tag{4.174}$$

Moreover they coincide up to the sign when $p = d-1$. The result (4.173) should be compared with our field-theoretical result (4.54) relating the defect entropy to the on-shell action

$$S_{\text{defect}} = -I_{\text{brane}} - \frac{1}{\sin(\pi p/2)} \frac{2(d-p-1)}{d\,\Gamma((d-p)/2)} \frac{\pi^{d/2+1}}{\Gamma(p/2+1)} a_T \, . \tag{4.175}$$

Comparing (4.174) with (4.175) we can read off the stress tensor one-point function constant a_T for $p < d-1$ in the probe brane model

$$a_T = \frac{d}{2(d-1)\,\pi^{(d-p)/2}}\, \Gamma\left(\frac{d-p}{2}\right) T_p L^{p+1} \, . \tag{4.176}$$

In the case of a codimension-one defect ($p = d-1$), we can include the backreaction. The backreacted metric takes the same form as (4.138) with the range $-\infty < \rho < \infty$ and [57, 134]

$$A(\rho) = \cosh(|\rho| - \rho_*) \, , \qquad \rho_* \equiv \text{arctanh}\left(\frac{4\pi G_{\text{N}}\, T_{d-1}\, L}{d-1}\right) \, . \tag{4.177}$$

The defect entropy is given exactly by

$$S_{\text{defect}} = \frac{L^{d-1}}{2G_{\text{N}}} \frac{\pi^{(d-1)/2}}{\sin(\pi(d-1)/2)\,\Gamma((d-1)/2)} \tanh\rho_* \cdot {}_2F_1\left(\frac{1}{2}, \frac{d}{2}, \frac{3}{2}; \tanh^2\rho_*\right) \, . \tag{4.178}$$

It reproduces (4.173) in the probe limit $\rho_* \to 0$ ($T_{d-1}L^d \ll 1$) as expected.[21]

Finally One can read off the universal part of this defect free energy in the probe brane model

$$\begin{aligned} \tilde{D}_{\text{brane}} &\equiv -\sin(\pi p/2)\, I_{\text{brane}} \\ &= \frac{\pi^{p/2+1}}{\Gamma(p/2+1)}\, T_p L^{p+1} \, , \end{aligned} \tag{4.179}$$

which is seen to be positive for $T_p > 0$. Hence our conjecture (4.65) asserts that the brane tension must decrease under any defect RG flow. This conforms with an intuition that the smaller the brane tension is, the less the degrees of freedom live on the defect (as there are no defects when $T_p = 0$). We will show the brane tension monotonically decreases under a defect RG flow described by a holographic model generalizing the probe brane model in the following.

Triggering defect RG flow

We adopt a simple holographic model of defect CFT described by the same type of the action as (4.168) with I_{brane} replaced by the action of a single real scalar field ϕ [25]

$$I_{\text{brane}} = \int \mathrm{d}^{p+1}\eta \sqrt{\hat{G}} \left[\frac{1}{2}\, \hat{G}^{AB}\partial_A\phi\, \partial_B\phi + V(\phi)\right] \, , \tag{4.180}$$

[21] This is twice the boundary entropy (4.205) calculated in the holographic model of BCFT in a later subsection.

on a $(p+1)$-dimensional hyperbolic space anchored on a p-dimensional defect at the boundary of the Euclidean AdS_{d+1} space. We assume that the potential $V(\phi)$ is bounded from below and allows a few critical points satisfying

$$\frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \,. \tag{4.181}$$

At each critical point ϕ_0 this model reduces to the probe brane model with the brane tension

$$T_p = V(\phi_0) \,, \tag{4.182}$$

and the defect RG flow is triggered by letting ϕ roll off from a local maximum to a local minimum of $V(\phi)$.

Now we focus on a holographic dual of a planar defect on $\mathbb{R}^d$. In the Poincaré coordinates

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{e}^{-2r/L}\, \delta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu \,, \tag{4.183}$$

the brane action is localized at $x^p = x^{p+1} = \cdots = x^{d-1} = 0$. The worldvolume coordinates η^A can be chosen as

$$\eta^i = x^i \quad (i = 0, \cdots, p-1) \,, \qquad \eta^p = r \,. \tag{4.184}$$

Let us define a function

$$T(\phi) \equiv V(\phi) - \frac{1}{2}\,(\partial_r \phi)^2 \,, \tag{4.185}$$

then it is easy to show $T(\phi)$ is a monotonically decreasing function with respect to r [25],

$$\partial_r T(\phi) = -\frac{p}{L}\,(\partial_r \phi)^2 \leq 0 \,, \tag{4.186}$$

where we use the equation of motion of ϕ and the translation invariance of the solution along the defect. For the holographic RG flow interpolating between the UV fixed point ϕ_{UV} and the IR ϕ_{IR}, (4.186) implies that the critical value of the potential is non-increasing under the RG flow,

$$V(\phi_{\mathrm{UV}}) \geq V(\phi_{\mathrm{IR}}) \,, \tag{4.187}$$

which in turn yields the brane tension is non-increasing in the probe brane model

$$T_{p,\mathrm{UV}} \geq T_{p,\mathrm{IR}} \,. \tag{4.188}$$

With (4.179) in mind we find the monotonicity

$$\tilde{D}_{\mathrm{brane}}|_{\mathrm{UV}} \geq \tilde{D}_{\mathrm{brane}}|_{\mathrm{IR}} \,, \tag{4.189}$$

in accordance with our proposal (4.65).

4.4.3 AdS/BCFT model

Finally we examine the g-theorem in general dimensions stating the monotonicity of the hemisphere partition function of BCFTs under any boundary RG flow. The bulk AdS metric respecting the $SO(d,1)$ symmetry of BCFT$_d$ on a hemisphere is

$$\mathrm{d}s^2 = L^2 \left[\mathrm{d}\rho^2 + \cosh^2\rho \left(\mathrm{d}w^2 + \sinh^2 w \, \mathrm{d}s^2_{\mathbb{S}^{d-1}}\right)\right] \ , \tag{4.190}$$

where $\rho \in (-\infty, \infty)$ and $w \in (0, \infty)$. This metric is equivalent to the more familiar form of the global AdS space

$$\mathrm{d}s^2 = L^2 \left[\mathrm{d}u^2 + \sinh^2 u \left(\mathrm{d}\theta^2 + \cos^2\theta \, \mathrm{d}s^2_{\mathbb{S}^{d-1}}\right)\right] \ , \tag{4.191}$$

where $u \in (0, \infty)$ and $\theta \in [-\pi/2, \pi/2]$. They are related by the following coordinate transformation

$$\cot\theta = \coth\rho \, \sinh w \ , \qquad \cosh u = \cosh\rho \, \cosh w \ . \tag{4.192}$$

The hemisphere defined by $\theta \in [-\pi/2, 0]$ at $u = \infty$ is reached by the $\rho \to -\infty$ limit in the coordinates (4.190) while the other half defined by $\theta \in [0, \pi/2]$ at $u = \infty$ is reached by the $\rho \to \infty$ limit. The boundary of the hemisphere at $\theta = 0$ is reached by the $w \to \infty$ limit for any ρ.

We locate BCFT on the hemisphere covered by $\theta \in [-\pi/2, 0]$ and construct the gravity dual following Takayanagi's proposal [22, 26, 27] by introducing the AdS boundary $\mathcal{Q}$ with a brane of tension T,

$$\begin{aligned} I = &-\frac{1}{16\pi G_\mathrm{N}} \int_\mathcal{B} \sqrt{G} \left(R + \frac{d(d-1)}{L^2}\right) \\ &- \frac{1}{8\pi G_\mathrm{N}} \int_\mathcal{Q} \sqrt{\hat{G}} \, (K - T) - \frac{1}{8\pi G_\mathrm{N}} \int_\mathcal{M} \sqrt{\hat{G}} \, K \ , \end{aligned} \tag{4.193}$$

where $\mathcal{B}$ is the bulk AdS space and $\mathcal{M}$ is the boundary on which the dual BCFT lives. In the present case, $\mathcal{M}$ is the hemisphere, $\mathcal{B}$ is the bulk AdS space in the coordinates (4.190) with the restricted range $\rho \in (-\infty, \rho_*)$, and $\mathcal{Q}$ is the AdS boundary at $\rho = \rho_*$ (see figure 4.5). To make the variational problem well-define in the presence of the boundary, the Gibbons-Hawking term is introduced with the extrinsic curvature defined by

$$K_{MN} = \hat{G}_{ML} \hat{G}_{NK} \nabla^L n^K \ , \tag{4.194}$$

for the outward pointing normal vector n^M. The Dirichlet boundary condition is imposed on $\mathcal{M}$, but the Neumann boundary condition is chosen on $\mathcal{Q}$

$$K_{MN} - \hat{G}_{MN} \, K = -T \, \hat{G}_{MN} \ . \tag{4.195}$$

Since the extrinsic curvature is given by

$$K = \frac{d}{L} \, \tanh\rho \ , \tag{4.196}$$

for any constant ρ surface, the brane tension is fixed to be

$$T = \frac{d-1}{L} \, \tanh\rho_* \ . \tag{4.197}$$

On-shell action at critical points

We now calculate the on-shell action of this system. Without regularization, the on-shell action diverges,

$$I(\rho_*) = \frac{L^{d-1}}{8\pi G_{\rm N}} \mathrm{Vol}(\mathbb{H}^d) \left[d \int_{-\infty}^{\rho_*} \mathrm{d}\rho \, \cosh^d \rho - \frac{TL}{d-1} \cosh^d \rho_* + d \lim_{\rho \to -\infty} \tanh \rho \, \cosh^d \rho \right] . \tag{4.198}$$

To compare with the boundary entropy (4.57) we subtract half of the on-shell action $I_{\rm AdS}$ of the whole AdS space without branes, which equals the on-shell action (4.198) with $\rho = 0$ where the brane tension vanishes,

$$\frac{1}{2} I_{\rm AdS} = I(0) . \tag{4.199}$$

Hence the boundary entropy (4.57) reads

$$\begin{aligned} S_{\rm bdy} &= -I(\rho_*) + \frac{1}{2} I_{\rm AdS} \\ &= \frac{L^{d-1}}{4G_{\rm N}} \frac{\pi^{(d-1)/2}}{\sin(\pi(d-1)/2) \, \Gamma((d-1)/2)} \tanh \rho_* \cdot {}_2F_1\left(\frac{1}{2}, \frac{d}{2}, \frac{3}{2}; \tanh^2 \rho_*\right) . \end{aligned} \tag{4.200}$$

We can read off the universal part of the boundary entropy from (4.66),

$$\tilde{D}(\rho_*) = \frac{L^{d-1}}{4G_{\rm N}} \frac{\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \tanh \rho_* \cdot {}_2F_1\left(\frac{1}{2}, \frac{d}{2}, \frac{3}{2}; \tanh^2 \rho_*\right) , \tag{4.201}$$

which can be checked numerically to be a monotonically increasing function of ρ_*.

Holographic boundary entropy

As a crosscheck, we now holographically calculate the entanglement entropy of a half ball region in BCFTs following [57]. Let $r_\perp$ be the transverse coordinate to the boundary and introduce the metric in the flat space

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{d}r_{||}^2 + r_{||}^2 \, \mathrm{d}s^2_{\mathbb{S}^{d-3}} + \mathrm{d}r_\perp^2 . \tag{4.202}$$

BCFTs are defined in the domain $r_\perp \in [0, \infty)$ and the entangling surface is located at the hypersurface satisfying $r_\perp^2 + r_{||}^2 = R^2$ at $t = 0$.

The gravity dual of the BCFT is described by the hyperbolic slicing of the AdS$_{d+1}$ spacetime

$$\mathrm{d}s^2 = L^2 \left[\mathrm{d}\rho^2 + \cosh^2 \rho \, \frac{\mathrm{d}z^2 + \mathrm{d}t^2 + \mathrm{d}r_{||}^2 + r_{||}^2 \, \mathrm{d}s^2_{\mathbb{S}^{d-3}}}{z^2} \right] , \tag{4.203}$$

where $\rho \in (-\infty, \rho_*)$ and $z \in [0, \infty)$. The holographic entanglement entropy is given by the area of the Ryu-Takayanagi surface parametrized by $z(x, r_{||})$ at $t = 0$, but assuming the x-independence one finds the unique semi-circle solution [57] (see figure 4.5)

$$z(r_{||})^2 + r_{||}^2 = R^2 . \tag{4.204}$$

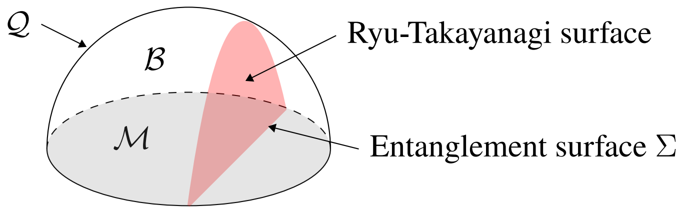

Figure 4.5: The bulk AdS space $\mathcal{B}$ is surrounded by the union $\mathcal{M} \cup \mathcal{Q}$ of the boundary and bulk hemispheres. It is bipartited by the RT surface anchored on the entanglement surface Σ.

Then we find the holographic boundary entropy

$$\begin{aligned} S_{\text{bdy}} &= \frac{L^{d-1}}{4G_{\text{N}}} \operatorname{Vol}(\mathbb{H}^{d-2}) \left(\int_0^{\rho_*} \mathrm{d}\rho \, \cosh^{d-2} \rho \right) \\ &= \frac{L^{d-1}}{4G_{\text{N}}} \frac{\pi^{(d-1)/2}}{\sin(\pi(d-1)/2)\, \Gamma((d-1)/2)} \tanh\rho_* \cdot {}_2F_1\left(\frac{1}{2}, \frac{d}{2}, \frac{3}{2}; \tanh^2 \rho_* \right) . \end{aligned} \tag{4.205}$$

This is equivalent to the boundary entropy (4.200) calculated by the formula (4.57) with the on-shell action as expected.

Holographic g-theorem in general dimensions

We shall prove the holographic g-theorem in the $\text{AdS}_{d+1}/\text{BCFT}_d$ model by adapting the setup of [26, 27] to the present case. Since the on-shell action (4.198) is a monotonic function of the brane tension the proof amounts to showing the monotonicity of the brane tension under boundary RG flows. Our strategy is in parallel with the holographic C-theorem [135] where the null energy condition is imposed on the bulk matter field to construct a monotonic function of a bulk metric component with respect to the holographic coordinate. In the present case, we would rather impose the null energy condition on the boundary $\mathcal{Q}$ for any null vector ζ^M

$$(K_{MN} - K\, \hat{G}_{MN})\, \zeta^M \zeta^N \geq 0 \,. \tag{4.206}$$

Since a boundary RG flow should respects the $SO(d)$ symmetry of the hemisphere, the brane configuration on $\mathcal{Q}$ is fixed by

$$\theta = \theta(u) \,, \tag{4.207}$$

in the global AdS coordinates (4.191). To impose the null energy condition, we analytically continue (4.191) to the Lorentzian signature by replacing the boundary sphere with the de Sitter space,

$$\mathrm{d}s^2_{\mathbb{S}^{d-1}} \;\longrightarrow\; -\,\mathrm{d}t^2 + \cosh^2 t\, \mathrm{d}s^2_{\mathbb{S}^{d-2}} \,. \tag{4.208}$$

The resulting metric becomes

$$\mathrm{d}s^2 = L^2 \left[\mathrm{d}u^2 + \sinh^2 u \left(\mathrm{d}\theta^2 - \cos^2\theta\, \mathrm{d}t^2 + \cos^2\theta\, \cosh^2 t\, \mathrm{d}s^2_{\mathbb{S}^{d-2}} \right) \right] \,, \tag{4.209}$$

in which the outward pointing unit normal vector to $\mathcal{Q}$ is given by

$$n^u = -\frac{L\,\theta'(u)}{\sqrt{\theta'(u)^2 + \text{csch}^2 u}}\,, \qquad n^\theta = \frac{L}{\sqrt{\theta'(u)^2 + \text{csch}^2 u}}\,, \qquad n^{M\neq u,\theta} = 0\,. \tag{4.210}$$

Choosing the null vector to be

$$\zeta^u = \text{const}\,, \qquad \zeta^\theta = \zeta^u\,\theta'(u)\,, \qquad \zeta^t = \zeta^u\,\frac{\sqrt{\theta'(u)^2 + \text{csch}^2 u}}{\cos\theta}\,, \qquad \zeta^{M\neq u,\theta,t} = 0\,, \tag{4.211}$$

the condition (4.206) becomes

$$\frac{L\,(\zeta^u)^2}{\sqrt{\theta'(u)^2 + \text{csch}^2 u}}\,g(u) \geq 0\,, \tag{4.212}$$

where

$$g(u) \equiv \tan\theta(u)\,\text{csch}^2 u - \theta'(u)\,\coth u + \theta'(u)^2\,\tan\theta(u) - \theta''(u)\,. \tag{4.213}$$

Hence the null energy condition yields the non-negativity of the function

$$g(u) \geq 0\,. \tag{4.214}$$

Next we want to show the monotonicity of the brane angle ρ_*. We switch to the coordinates (4.190) where the brane is located on the hypersurface

$$\rho(u) = \text{arcsinh}\left(\sinh u\,\sin\theta(u)\right)\,. \tag{4.215}$$

In what follows we show the derivative is non-negative

$$\rho'(u) = \frac{1}{\sqrt{\theta'^2 + \text{csch}^2 u}}\,f(u) \geq 0\,, \tag{4.216}$$

where

$$f(u) \equiv \sin\theta(u)\,\coth u + \theta'(u)\cos\theta(u)\,. \tag{4.217}$$

As long as the brane configuration satisfies $0 \leq \theta(u) \leq \pi/2$ the null energy condition implies

$$f'(u) = -\cos\theta(u)\,g(u) \leq 0\,. \tag{4.218}$$

As the boundary condition $\lim_{u\to\infty}\theta(u) = 0$ imposes $f(\infty) = 0$ we conclude $f(u) \geq 0$ and $\rho'(u) \geq 0$ for $u \in [0,\infty)$.

The inequality (4.216) means the brane angle ρ_* monotonically decreases under a boundary RG flow

$$\rho_{\text{UV}} \geq \rho_{\text{IR}}\,, \tag{4.219}$$

where $\rho_{\text{UV}} = \rho(\infty)$, $\rho_{\text{IR}} = \rho(0)$, and we interpret the coordinate u as the holographic renormalization scale as in [135]. Combined with (4.201) at the critical point, we prove the weak form of the holographic g-theorem

$$\tilde{D}_{\text{UV}} \geq \tilde{D}_{\text{IR}}\,. \tag{4.220}$$

4.5 Summary

In this chapter we have examined possible candidates for C-functions in BCFT and DCFT: one is the defect free energy, and the other is the defect entropy. We have derived the universal formula relating these two quantities and found that unlike CFT they have discrepancy coming from the term of the non-vanishing stress tensor one-point function. Having this new formula in our mind, we have proposed C-theorem for general dimensional DCFTs stating that the defect free energy, not the defect entropy, must be the monotonic decreasing function under the defect RG flows. We have checked our proposal in various models in field theories, most of which are one-dimensional defects, namely Wilson loops. Then employing AdS/CFT correspondence, we have further studied our proposal in bottom-up holographic models of DCFTs. In all examples that we study, we find that the defect free energies are monotonic, while in some examples the defect entropies are not.

Chapter 5

Conclusion

In this thesis, we presented various dynamical behaviors of BCFTs and DCFTs towards a better way of classifying and understanding them. The useful quantities to distinguish these theories are conformal anomalies in BCFTs. As for DCFTs, by using the dimensional regularization, we can unify conformal anomaly coefficients and sphere free energy in general dimensions. Another possible tool to count the effective degree of freedom is the entanglement entropy. It was less explored how these quantities behave under RG flows, guiding us to this work.

Even though the boundary conformal anomaly contains interesting dynamical information of BCFT, it is not obvious to present interacting models of them where we can extract these anomalies. In chapter 3, we find such a tractable model of BCFTs allowing us to calculate boundary conformal anomaly coefficients. We study the scalar $O(N)$ model with the sextic interaction in three dimensions. We take advantage of the fact that at large N theory becomes approximate conformal where we can easily obtain the anomaly coefficients as a function of the quasi-marginal coupling. However, we leave many interesting unanswered questions here:

- How do $1/N$ corrections change the story? It apparently breaks the conformal symmetry.
- Can we say more about the Neumann boundary case where the boundary marginal coupling becomes important?
- What happens when we try to compute the stress energy two-point function in the extraordinary boundary condition?
- Do any experimental models that can be described by our large N model exist?

In chapter 4, we turn to study general C-functions in BCFTs and DCFTs. Such monotonic functions were proposed in various forms, so one of our purposes is to organize them in the unified form to obtain general constraints on DCFTs (BCFTs). In particular we study the sphere free energy and the entanglement entropy in the presence of the defect, deriving the universal formula. We proposed that the defect free energy should be a desired C-function. To check our proposal, we study wilson loops as line defects and find supporting evidence. Along the way we also find the defect entropy, which is the increment of the entanglement entropy due to the defect, does not always decrease under RG flows. We further test our proposal in the holographic setting. At least in some holographic models, we can prove our conjecture. The fact that we have not fully know the examples of defect (boundary) RG flows makes it hard to collect more additional evidence and to prove our proposal directly. However, one may hope that supersymmetry serves as a useful tool to shed light on this direction in accordance with F-maximization [23, 136], or that conformal bootstrap technique can provide another perspective [137]. Given that the defect free energy is related to the defect entropy, one may also wonder if we can use the quantum information theoretic techniques to our end. We would like to return these problems at some point in the future.

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Appendix A

Conformal integral with boundary

In this appendix, we review the method to compute the integral (3.46), which was studied in Appendix D of [48]. More general integrals appearing in boundary CFT were investigated in [138]. Let us start with the following integral,

$$f(\xi) = \int_0^\infty \mathrm{d}z \int \mathrm{d}^{d-1}\mathbf{r} \frac{1}{(2z)^d} f_1(\tilde{\xi}) f_2(\tilde{\xi}') \,, \tag{A.1}$$
$$\tilde{\xi} = \frac{(x-r)^2}{4yz}, \quad \tilde{\xi}' = \frac{(x'-r)^2}{4y'z}, \quad r = (\boldsymbol{r}, z) \,.$$

To obtain the form of $f(\xi)$, we consider the problem backwards and perform the following invertible integral transform,

$$\begin{aligned}\hat{f}(\rho) &= \frac{1}{(4yy')^g} \int \mathrm{d}^{d-1}\boldsymbol{x}\, f(\xi) \\ &= \frac{\pi^g}{\Gamma(g)} \int_0^\infty \mathrm{d}u\, u^{g-1} f(u+\rho) \,,\end{aligned} \tag{A.2}$$

where $\rho = (y-y')^2/(4yy')$ and $g = (d-1)/2$. The inverse transform is given as follows,

$$f(\xi) = \frac{1}{\pi^g \Gamma(-g)} \int_0^\infty \mathrm{d}\rho\, \rho^{-g-1} \hat{f}(\rho+\xi) \,. \tag{A.3}$$

Employing the above transform, (A.2) can be recast as

$$\hat{f}(\rho) = \int_0^\infty \mathrm{d}z \frac{1}{2z} \hat{f}_1(\tilde{\rho}) \hat{f}_2(\tilde{\rho}') \;, \quad \tilde{\rho} = \frac{(y-z)^2}{4yz} \,, \quad \tilde{\rho}' = \frac{(y'-z)^2}{4y'z} \,. \tag{A.4}$$

Then if we can compute $\hat{f}(\rho)$ by (A.4), it enables us to obtain $f(\xi)$ by the inverse integral transform. To this end, we first change variables $z = e^{2\theta}$, $y = e^{2\theta_1}$ and $y' = e^{2\theta_2}$. (A.4) becomes

$$\hat{f}\left(\sinh^2(\theta_1-\theta_2)\right) = \int_{-\infty}^\infty \mathrm{d}\theta \hat{f}_1\left(\sinh^2(\theta-\theta_1)\right) \hat{f}_2\left(\sinh^2(\theta-\theta_2)\right) \,. \tag{A.5}$$

Taking the Fourier transform of (A.5),

$$\tilde{\hat{f}}(k) = \int_{-\infty}^\infty \mathrm{d}\theta\, e^{ik\theta} \hat{f}(\sinh^2\theta) \,, \tag{A.6}$$

the convolution property gives us the following simple relation,

$$\tilde{\hat{f}}(k) = \tilde{\hat{f}}_1(k)\tilde{\hat{f}}_2(k)\,, \tag{A.7}$$

which makes it possible to compute $f(\xi)$ from the given $f_1(\xi)$ and $f_2(\xi)$. One strategy is to perform the series of integral transforms that converts f_i to $\tilde{\hat{f}}_i$ and then to use the convolution property to obtain $\tilde{\hat{f}}$. Performing an inverse Fourier transform and (A.3), we obtain $f(\xi)$ in the end. The success of the method depends highly on the form of f_i, but for some of the f_i of interest, we can do these integral transforms.

The spin structures add another layer of complexity to the evaluation of (3.46). Let us introduce the differential operator

$$\tilde{\mathcal{D}}_{\mu\nu} \equiv \partial_\mu\partial_\nu + \frac{1}{y}\left(n_\mu\partial_\nu + n_\nu\partial_\mu\right) - \frac{1}{d}\delta_{\mu\nu}\left(\partial^2 + \frac{2}{y}n\cdot\partial\right). \tag{A.8}$$

This operator $\tilde{\mathcal{D}}_{\mu\nu}$ allows us to re-express the $X_\mu X_\nu - \frac{\delta_{\mu\nu}}{d}$ tensor structure in terms of derivatives acting on a function of a cross ratio:

$$\tilde{\mathcal{D}}_{\mu\nu}\mathcal{F}(\xi) = \frac{1}{z^2}\left(X_\mu X_\nu - \frac{1}{d}\delta_{\mu\nu}\right)\xi(1+\xi)\mathcal{F}''(\xi)\,, \tag{A.9}$$

which allows us to write (3.46) as

$$\mathcal{G}_{\mu\nu\sigma\rho} = (4zz')^2\tilde{\mathcal{D}}_{\mu\nu}\tilde{\mathcal{D}}'_{\sigma\rho}\mathcal{G}(\xi)\,, \tag{A.10}$$

where for $i = 1$ or 2

$$\mathcal{G}(\xi) = \int_0^\infty \mathrm{d}y \int \mathrm{d}^{d-1}\boldsymbol{r}\,\frac{1}{(2y)^d}\mathcal{F}_1(\tilde{\xi})\mathcal{F}_2(\tilde{\xi}')\,, \quad f_i(\xi) = 4\xi(1+\xi)\mathcal{F}_i''(\xi)\,. \tag{A.11}$$

In the above expression we can play the same game not for f_i, but for $\mathcal{F}_i$. The point is that we don't need to transform $\mathcal{F}$ itself because using integration by parts we can write

$$\hat{\mathcal{F}}(\rho) = \frac{\pi^g}{\Gamma(g+2)}\int_0^\infty \mathrm{d}u\, u^{g+1}\mathcal{F}''(u+\rho)\,. \tag{A.12}$$

In our setup, $f_1(\xi) = (\xi(1+\xi))^{-d/2}$ and corresponding integral transforms are easily done by

$$\hat{\mathcal{F}}_1\left(\sinh^2\theta\right) = \frac{1}{2}S_d\frac{1}{d(d+1)}e^{-(d+1)|\theta|}, \quad \tilde{\hat{\mathcal{F}}}_1(k) = \frac{1}{d}S_d\frac{1}{k^2+(d+1)^2}\,. \tag{A.13}$$

Now suppose we know $\tilde{\hat{\mathcal{F}}}_2$. Then we have

$$\hat{\mathcal{G}}(\sinh^2(\theta)) = \frac{S_d}{d}\frac{1}{2\pi}\int \mathrm{d}\theta\, e^{-ik\theta}\,\frac{1}{(d+1)^2+k^2}\tilde{\hat{\mathcal{F}}}_2(k)\,, \tag{A.14}$$

from which we find that

$$\left((d+1)^2 - \frac{\mathrm{d}^2}{\mathrm{d}^2\theta}\right)\hat{\mathcal{G}}(\sinh^2(\theta)) = \frac{S_d}{d}\hat{\mathcal{F}}_2(\sinh^2(\theta))\,. \tag{A.15}$$

Using the integral transform (A.2), we can pull this differential equation back to one involving $\mathcal{G}(\xi)$

$$\left(\xi(1+\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + d\left(\xi + \frac{1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\xi} - d\right)\mathcal{G}(\xi) = -\frac{1}{4d}S_d\mathcal{F}_2(\xi)\,, \tag{A.16}$$

or equivalently $\mathcal{G}''(\xi)$,

$$\left(\xi(1+\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + (d+4)\left(\xi + \frac{1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\xi} + d + 2\right)\mathcal{G}''(\xi) = -\frac{1}{d}S_d\frac{1}{16\xi(1+\xi)}f_2(\xi)\,. \tag{A.17}$$

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