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博士論文

Combinatorial Realization of Minimal Models for Gaiotto-Rapcak's W-algebras
(Gaiotto-RapcakによるW代数のミニマル模型の組み合わせ論的実現)

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Abstract

In this thesis, we study the minimal models' representation theory for a particular class of W-algebras proposed by Gaiotto and Rapcak. They constructed these W-algebras from supersymmetric gauge theories associated with type IIB string theory. One of their interesting features is that we can construct various W-algebras based on the most fundamental ones, which are called Y-algebras. We first study the Y-algebras' representation theory when their parameters agree with those of W_N minimal models. We use the plane partition representation of the affine Yangian of $\mathfrak{gl}_1$, which provides a uniform description for all Y-algebras. We find that a certain constraint on the plane partition matches the Hilbert space of the W_N minimal models. Using this result, we propose a way to describe the Hilbert space for the minimal models of more general W-algebras. As an example, we study the $\mathcal{N} = 2$ unitary minimal models in detail and reproduce the correct primary fields and characters. We also discuss Bershadsky-Polyakov algebra and check the consistency with the literature.

Contents

1 Introduction

Two-dimensional conformal field theories (CFTs) belong to a special class of quantum field theories (QFTs). Unlike the other QFTs, they possess infinite dimensional symmetry, which allows us to extract much information on the physical quantities without knowing the system's detail. The infinite dimensional algebra which describes this symmetry is known as Virasoro algebra, and we can study the 2d CFTs through its representation theory. In most cases, the symmetry does not determine the physical quantity completely, but we can sometimes solve the model exactly. The typical example is a minimal model that describes the system with only a finite number of fields.

Historically, the study of exactly solvable 2d CFTs was initiated by Belavin-Polyakov-Zamolodchikov (BPZ) in the pioneering work [1]. Because the conformal symmetry is a generalization of the scaling symmetry, the minimal models played a crucial role in the study of a phase transition. Two-dimensional CFTs also play a fundamental role in the string theory because they describe the theory on a world sheet. Since the BPZ's seminal work, 2d CFTs have been intensively studied, and it has become clear that some models possess larger symmetries than Virasoro symmetry. The algebras which describe such symmetries are usually referred to as W-algebras[1]. The W-algebra discovered for the first time is W_3 algebra [2]. While Virasoro algebra is generated by the current $T(z)$, W_3 algebra contains an extra current $W^{(3)}(z)$. One can also generalize it to W_N algebra which consists of the currents with spin $2, 3, 4, \cdots, N$ [3]. Unlike Virasoro algebra, W-algebras' commutation relation contains nonlinear terms and it is hard to write down the explicit commutation relation. Several techniques to deal with W-algebras have been developed so far, but there are still many points that have not been understood well.

The application of W-algebras is not limited to 2d CFTs. It has recently become clear that W-algebras play a crucial role in higher dimensional field theories. Especially, significant progress has been made in $\text{AdS}_3/\text{CFT}_2$ and 4d supersymmetric gauge theories. In these areas, various W-algebras appear and it has been gradually recognized that they can be described uniformly by a much larger W-algebra. For example, it was proposed that W_N minimal models in a large N limit are dual to the higher spin theories on AdS_3 and the underlying symmetry was identified with $W_\infty[\mu]$ [4, 5], which consists of the currents $W^{(s)}(z)$ with $s = 2, 3, 4, \cdots \infty$. One can obtain W_N algebra as its truncation for arbitrary $N \geq 2$.

Another important example is 4d $\mathcal{N} = 2$ supersymmetric gauge theories. In [6], a remarkable duality between $\mathcal{N} = 2$ SU(2) gauge theory and Liouville field theory was discovered. It was immediately generalized in [7] to the duality between SU(N) gauge theories and Toda theories with W_N symmetry. These dualities were proved by mathematicians in

[1]In this thesis, W-algebras denote the algebras generated by the energy-momentum tensor and other currents whose OPEs are local (i.e. the conformal dimension of the currents is integer or half-integer).

[8, 9] and are now called Alday-Gaiotto-Tachikawa (AGT) correspondence. In the proof process, it has become clear that the underlying symmetry can be understood well by a certain quantum group rather than W-algebras themselves. This quantum group has been studied in several contexts and has several names. In this thesis, we focus on the affine Yangian of $\mathfrak{gl}_1$ [10, 11]. One of the main advantages of using this algebra is that it has an explicit representation theory. This feature shed new light on the representation theory of W-algebras. The affine Yangian enables us to explicitly describe the Hilbert space of W_N algebras as opposed to the conventional approaches.

Interestingly, the affine Yangian of $\mathfrak{gl}_1$ is equivalent to $W_\infty[\mu]$ except that the former contains an extra U(1) free boson. This difference has a significant meaning. Adding the "U(1) factor" to W_∞ makes it easier to deal with the algebra. We refer to it as $W_{1+\infty}$[2]. The main advantage of using $W_{1+\infty}$ instead of W_∞ is that there is a one-to-one correspondence between its vacuum representation and a plane partition. Here, a plane partition is a two-dimensional array of non-negative integers $\pi_{i,j}$ satisfying $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$. One can interpret a plane partition as a 3d Young diagram as follows:

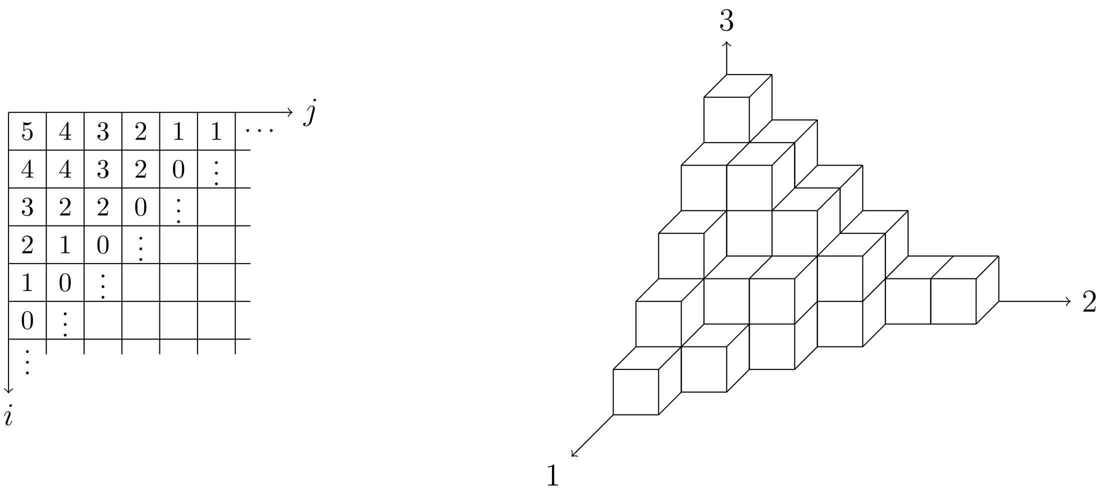


Figure 1: Left: An example of $\pi_{i,j}$. Right: The corresponding plane partition.

One may wonder why a plane partition appears in the study of W-algebras. The best way to see the relation between them is to consider the vacuum character of $W_{1+\infty}$. Here, the character is the quantity that counts the number of the states. It is defined by

$$\chi(q) = \text{Tr}\ q^{L_0 - \frac{c}{24}}, \tag{1.1}$$

where L_0 is the zero mode of Virasoro algebra and c is the central charge. It enables us to express the complicated structure of the Hilbert space simply by a power series. It also plays a crucial role as a physical quantity because we can interpret it as the torus partition

[2] It may be precise to express $W_{1+\infty}[\mu]$, but we denote it by $W_{1+\infty}$ just for simplicity.

function. In the construction of consistent theories, the study of its modular property is indispensable. Let us see how the character provides us crucial information. The vacuum character of $W_{1+\infty}$ serves as a good example. We can generate all the states by the creation operators $W^{(s)}_{-n_s}$ $(s \geq 1,\ n_s \geq s)$ and the (normalized) character is given by

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + \cdots . \tag{1.2}$$

It is precisely the MacMahon function, which counts the number of the boxes in a plane partition [12]. The following figure demonstrates the configurations at low levels:

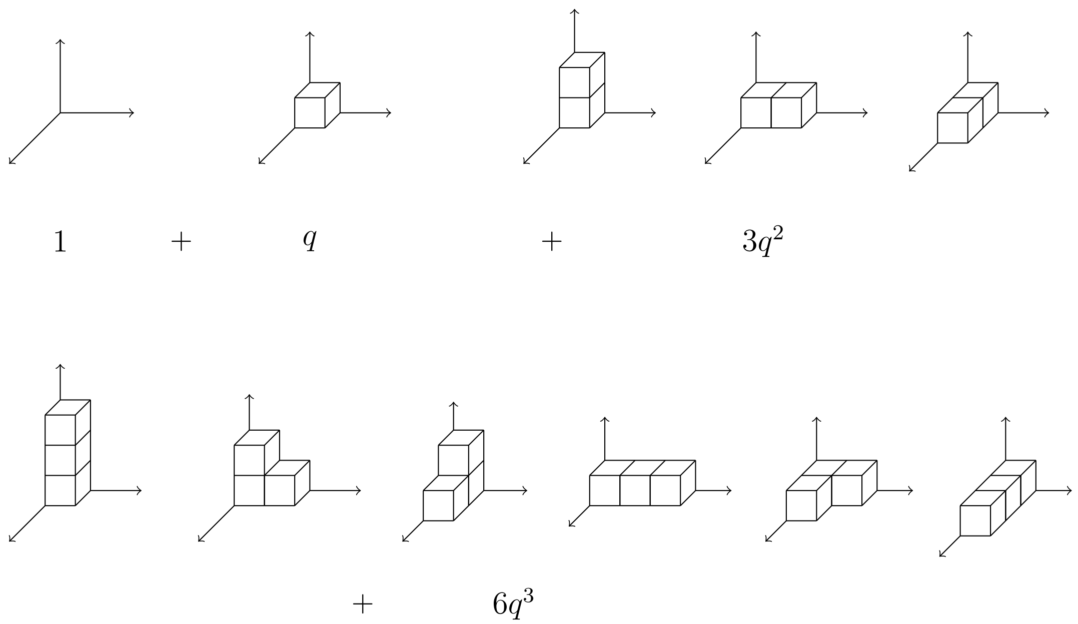


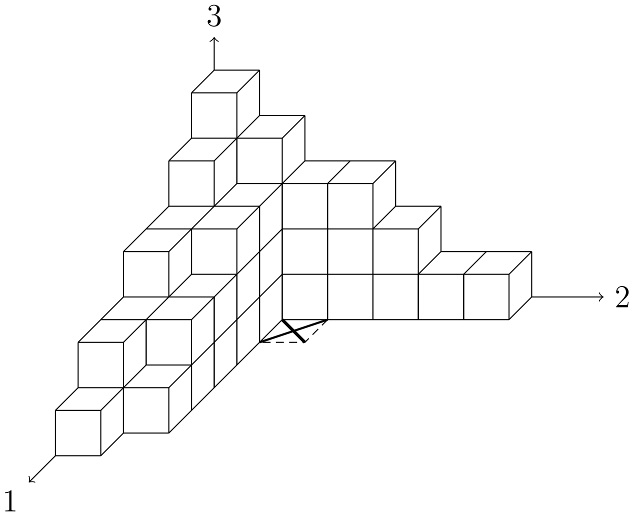
Figure 2: The configurations of a plane partition at low levels.

One can also describe the Hilbert space of W_N algebra (and an extra U(1) factor) by restricting the height of the plane partition to N. In this way, we can interpret the Hilbert space of the W_N-algebra simply and graphically. Furthermore, the affine Yangian enables us to understand explicitly how the algebra acts on the plane partition. That is quite different from the conventional technique to deal with W-algebras. This difference comes from the fact that the affine Yangian is described by the generators whose algebraic relations are explicitly defined. In other words, the affine Yangian realizes W-algebras in a totally different way from the conventional approaches. As a result, we can study the representation theory of W_N-algebra through manipulation (adding a box or removing a box) to a plane partition [11].

So far, we have seen that the affine Yangian provides a useful way to study the representation theory of $W_{1+\infty}$ or its truncation, W_N algebra. While W_N algebras are the most important and fundamental W-algebras, we often need to deal with other W-algebras.

In AdS_3/CFT_2 and AGT correspondence, many generalizations have been proposed and a variety of W-algebras appeared up until now. For AdS_3/CFT_2, the correspondence was generalized to supersymmetric cases [13, 14, 15], matrix-valued higher spin fields [16], and so on. For AGT correspondence, it has been generalized to various cases, for example, the asymptotically locally Euclidean (ALE) spaces [17, 18], insertion of surface operators [19, 20] and the other gauge groups [21]. A variety of W-algebras other than W_N algebras showed up in these works. We also note that W-algebras play an important role in other fields such as topological string [22] and 4d/2d duality proposed in [23].

Recently, Gaiotto and Rapcak brought a new development in [24]. They constructed a new family of W-algebras called Y-algebras $Y_{L,M,N}[\Psi]$ from the 4d supersymmetric gauge theories associated with type IIB string theory. These W-algebras are labeled by three non-negative integers L, M, N and reduce to W_N algebra (and a U(1) factor) when $L = M = 0$. It turned out that they were truncations of the affine Yangian[3]. The key observation which leads to this statement is that there is a one-to-one correspondence between the Hilbert space of $Y_{L,M,N}[\Psi]$ and the plane partition with a "pit" at $(L+1, M+1, N+1)$[4]. Here, a pit means a place where we cannot put a box [25]. For example, the following figure shows a plane partition with a pit at (3,2,1):



It is worth mentioning that a plane partition not only provides a technically useful tool but also captures the important property of the Y-algebra; $Y_{L,M,N}[\Psi]$ has a nontrivial symmetry which permutes L, M, N. This property is called triality and sometimes plays an important role in the study of W-algebras[5]. One can see that a plane partition manifestly respects the triality. In that sense, we can say that a plane partition provides a natural framework to describe the Hilbert space of the Y-algebra or $W_{1+\infty}$.

[3]We note that the same W-algebras were also proposed in a different context [25, 26].

[4]We set the coordinate of the origin to $(1, 1, 1)$.

[5]We note that the triality plays an important role also in AdS_3/CFT_2 [5].

One of the remarkable and desirable features in Gaiotto-Rapcak's construction is that one can obtain various W-algebras by considering more general 4d supersymmetric gauge theories. Prochazka and Rapcak studied this construction in detail in [27, 28]. Because various 4d theories can be easily constructed from the brane construction in string theory, we can obtain a vast class of W-algebras. Indeed, many familiar algebras such as the affine Kac-Moody algebras $\widehat{U}(N)$ and $\mathcal{N} = 2$ super W-algebras belong to this class. The important point in this construction is that these W-algebras are constructed from the combination of several Y-algebras. In other words, the W-algebra contains several Y-algebras as subalgebras. Therefore, we can describe the Hilbert space by using several plane partitions. In the following, we refer to the W-algebras constructed in this way as generalized Y-algebras.

In this thesis, we study the minimal models' representation theory for the generalized Y-algebras. We first study W_N minimal models in terms of a single plane partition. There are several related works [29, 30, 31, 32], where N-tuple Young diagrams are used as a basis of the Hilbert space. The authors of these works obtained the N-Burge condition, which imposes restrictions on Young diagrams' shape. In [11], Prochazka used a plane partition to study the W_N minimal models and checked that the result is compatible with the Young diagram approach when the central charge takes the particular value. We further study the plane partition description of W_N minimal models and find that the result is consistent with the N-Burge condition in all cases. We also find that all primary fields in the minimal model can be reproduced from a plane partition. As the next step, we study the Hilbert space of the minimal models for the generalized Y-algebras. We mainly focus on $\mathcal{N} = 2$ unitary minimal models. Fortunately, many physicists have intensively studied them, motivated by their applications to string compactification. Especially, the character formula has already been derived [33, 34, 35], and we can examine whether our approach is correct. We found that the correct character is obtained by counting the number of boxes in a plane partition.

This thesis is organized as follows. In Section 2, we review the W-algebras and minimal models. This section provides the background knowledge for this thesis. We first recall the basics of two-dimensional conformal field theories. Next, we explain several techniques to deal with W-algebras. We also review the minimal models for W_N algebra and $\mathcal{N} = 2$ super Virasoro algebra. These minimal models are the main subjects of this thesis. In Section 3, we review the Y-algebra and its generalization. The Y-algebra is the W-algebra constructed from a supersymmetric gauge theory associated with type IIB string theory. We review its construction and several algebraic properties. We also explain the generalization of the Y-algebras which provides various W-algebras. In Section 4, we review the affine Yangian of $\mathfrak{gl}_1$. We explain several properties, the plane partition representation and the relation to the Y-algebra. In Section 5, we study the plane partition representation for the particular value of the parameter. That is necessary to study minimal models. Next, we study the representation theory for W_N minimal model in terms of a plane partition. We find that the plane partition with a certain constraint provides the correct Hilbert space of W_N minimal models. In Section 6, we propose the way to describe the Hilbert space of minimal models for generalized Y-algebras by using the result obtained in Section 5. We focus on $\mathcal{N} = 2$ unitary minimal models and check that we can reproduce the known character and primary fields from the plane partition approach. In Section 7, we discuss Bershadsky-Polyakov algebra, which is a simplest non-principal W-algebra. We find that our approach reproduces the correct modules.

The original part of this thesis is provided in Section 5, Section 6 and Section 7 (more precisely, Section 7.2). This thesis is based on the paper [36], [37].

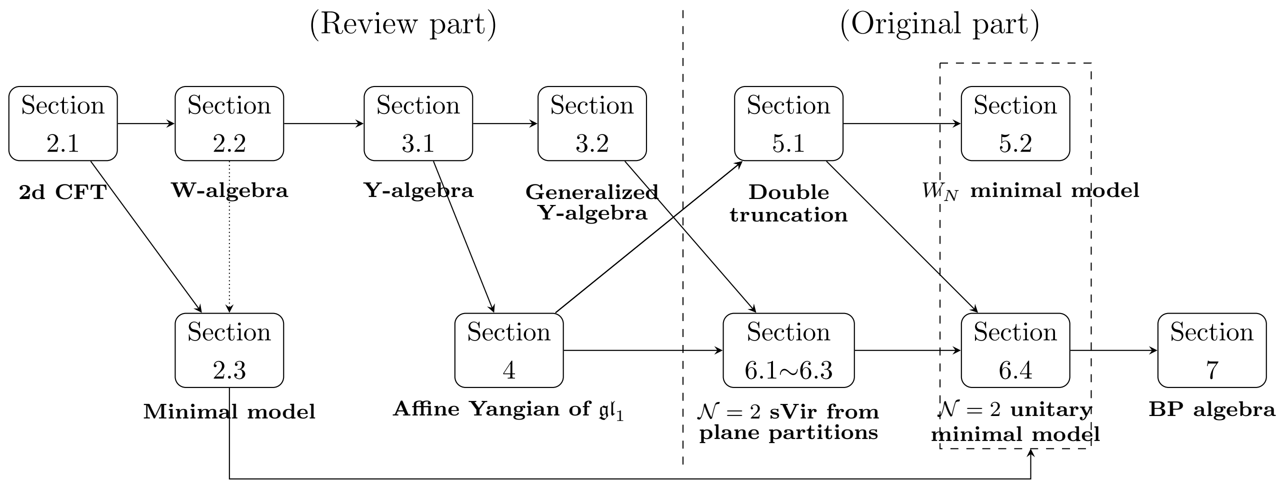


Figure 3: The flowchart of this thesis.

2 Review of W-algebras and minimal models

In this section, we review W-algebras and minimal models. We first recall the basics of two-dimensional conformal field theories in Section 2.1. In Section 2.2, we review some techniques to deal with W-algebras. Concretely, we explain the quantum Drinfeld-Sokolov reduction and coset construction. We use these techniques in the definition of the Y-algebra, which we will discuss in Section 3. In section 2.3, we review the minimal models for W_N algebra and $\mathcal{N}=2$ super Virasoro algebra. For W_N minimal models, we discuss the construction from free field realization. For $\mathcal{N}=2$ unitary minimal models, we use the coset construction.

2.1 Basics of two-dimensional conformal field theories

Before reviewing W-algebras, we recall the basics of 2d CFTs. They are governed by the infinite dimensional symmetry generated by Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n+1)n(n-1) \qquad (n, m \in \mathbb{Z}), \tag{2.1}$$

where c is called the central charge which commutes with all the other generators. These generators are obtained from the mode expansion of the holomorphic field called an energy-momentum tensor:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{2.2}$$

There is also an anti-holomorphic field $\overline{T}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{L}_n \overline{z}^{-n-2}$ and it commutes with the holomorphic part. In the following, we focus on the holomorphic part.

In 2d CFTs, it is often convenient to use the holomorphic field instead of individual modes to describe the algebraic relation. Let us consider the two fields $X(z), Y(z)$. In general, we cannot place them at the same point because their product diverges. In 2d CFTs, we express such behavior as follows:

$$X(z)Y(w) \sim \sum_{n=1}^{N} \frac{(XY)_n(w)}{(z-w)^n}. \tag{2.3}$$

Here, the symbol "$\sim$" indicates that this relation focuses on the singular behavior when we take the limit of $z \to w$. The order N of the pole depends on $X(z)$ and $Y(w)$. The field $(XY)_n$ in each term can be sometimes written by $X(w), Y(w)$ or their derivatives, but in general, gives a new field. This kind of relation is called operator product expansion (OPE) and provides us enough information. To see that, we consider the case for Virasoro algebra. The OPE relation between two energy-momentum tensors is given as follows:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{2.4}$$

where ∂ denotes a partial derivative. Let us check that this relation reproduces the commutation relation of Virasoro algebra. We can do that as follows:

$$\begin{aligned}[L_n, L_m] &= \oint_0 \frac{dw}{2\pi\mathrm{i}} \oint_w \frac{dz}{2\pi\mathrm{i}}\ z^{n+1} w^{m+1} T(z) T(w) \\ &= \oint_0 \frac{dw}{2\pi\mathrm{i}} \oint_w \frac{dz}{2\pi\mathrm{i}}\ z^{n+1} w^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right) \\ &= (n-m) L_{n+m} + \frac{c}{12}(n+1)n(n-1).\end{aligned} \tag{2.5}$$

It indeed agrees with (2.1).

In 2d CFTs, we usually deal with the fields $\Phi(z)$ which transform covariantly under the action of Virasoro algebra as

$$T(z)\Phi(w) \sim \cdots + \frac{h\Phi(w)}{(z-w)^2} + \frac{\partial\Phi(w)}{z-w}. \tag{2.6}$$

This field is called a quasi-primary field and especially called a primary field when the singular terms $\frac{1}{(z-w)^n}$ with $n \geq 3$ vanish. By setting $\Phi(z) = \sum_{n\in\mathbb{Z}} \Phi_n z^{-n-h}$, we see that this relation leads to $[L_0, \Phi_n] = -n\Phi_n$. The parameter h is called conformal dimension and corresponds to scaling dimension. One can see that the conformal dimension of $T(z)$ is $h = 2$. In Section 2.2, we also consider the additional current with conformal dimension greater than 2. The OPE between such currents involves singular terms at the higher order and becomes more complicated.

Let us consider the representation theory of Virasoro algebra. The highest weight representation is defined as follows:

$$L_0 \left|h\right\rangle = h \left|h\right\rangle, \quad L_n \left|h\right\rangle = 0 \ (n > 0). \tag{2.7}$$

When $h = 0$, the corresponding state $|0\rangle$ is called vacuum state. By considering the action of the raising operator L_n $(n < 0)$ on $|h\rangle$, we can obtain the descendant states $\prod_{n_1 \geq n_2 \geq \cdots > 0} L_{-n_1} L_{-n_2} \cdots |h\rangle$. The conformal dimension for this state is given by $h + \sum_i n_i$. The number $\sum_i n_i$ is often called level. The highest weight state $|h\rangle$ can be realized by inserting the primary field $\Phi(z)$ with conformal dimension h at the origin,

$$\left|h\right\rangle = \lim_{w\to 0} \Phi(w) \left|0\right\rangle. \tag{2.8}$$

We can see it by considering the action of $T(z)$ on the right hand side, which can be read off from (2.6) as $T(z)\,|h\rangle = (\frac{h}{z^2} + \cdots)\,|h\rangle$. Thus, there is a one-to-one correspondence between the state and the field in CFTs.

For generic values of the central charge and the conformal dimension, the highest weight representation is irreducible, but sometimes becomes reducible. That happens when there appears a state $|\xi\rangle$ satisfying

$$L_n \left|\xi\right\rangle = 0 \quad (n > 0) \tag{2.9}$$

other than the highest state $|h\rangle$. Such a state is called singular or null. Because the state $|\xi\rangle$ and its descendant states form subrepresentation, we have to remove them to obtain the irreducible representation. Such representation is called degenerate representation. For a generic value of the central charge, there is at most one null state and it is easy to remove it. However, there appear two null states $|\xi_1\rangle, |\xi_2\rangle$ simultaneously for the particular value of the central charge. In such cases, the degree of freedom is extremely reduced, which allows us to construct the exactly solvable models called minimal models. On the other hand, the Hilbert space structure becomes complicated; it is not adequate to remove two subrepresentation spaces independently because some states belong to both of them. Therefore, we need to study the representation theory carefully in such cases.

To express the conformal dimension of the degenerate fields, it is convenient to parametrize the central charge as

$$c = 1 + 6Q^2, \quad Q = b + b^{-1}. \tag{2.10}$$

Then the highest weight state with the conformal dimension

$$h_{r,s} = \frac{Q^2}{4} - \left(\frac{br + b^{-1}s}{2}\right)^2 \quad (r, s \in \mathbb{Z}_{\geq 1}) \tag{2.11}$$

has a null vector at level $N = rs$. The important property is that a set of degenerate fields are closed under the OPE (or fusion rule). However, there are still infinitely many fields. Meanwhile, when the central charge is parametrized by coprime integers $p, q \geq 2$ as

$$b^2 = -\frac{p}{q}, \quad c = 1 - 6\frac{(p-q)^2}{pq}, \tag{2.12}$$

a set of the degenerate fields labeled by

$$1 \leq r \leq q-1, \quad 1 \leq s \leq p-1. \tag{2.13}$$

is closed under the OPE. Because of $h_{r,s} = h_{q-r,p-s}$, there are null vectors at level rs and $(q-r)(p-s)$. Then we can construct the model which contains only a finite number of fields. It is the simplest example of minimal models. In general, we can also consider the algebra larger than Virasoro algebra and it is also possible to construct minimal models for such algebras. We will discuss them in more detail in Section 2.3.

In the representation theory, the character is one of the most important quantities because it captures the structure of the Hilbert space. It is defined as

$$\chi(\tau) = \mathrm{Tr}\; q^{L_0 - \frac{c}{24}}, \tag{2.14}$$

where q is the expansion parameter and the trace is taken over all the states. It counts the number of the states in the Hilbert space and provides crucial information for the null states. Furthermore, one can also interpret it as a torus partition function. To see that, let us consider the 2d CFT on the torus with the periodic coordinate $w \simeq w + 2\pi \simeq w + 2\pi\tau$, where $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$. While the cylinder $w \simeq w + 2\pi$ describes the time evolution from $t = -\infty$ to $t = \infty$, the torus describes the system with the periodicity $t \simeq t + \tau_2$ imposed on the time direction. Therefore, the partition function $Z(\tau, \overline{\tau})$ is given by

$$
\begin{aligned}
Z(\tau, \overline{\tau}) =& \mathrm{Tr}\ e^{-2\pi\tau_2 H + 2\pi i \tau_1 P} \\
=& \mathrm{Tr}\ e^{-2\pi\tau_2 (L_0 + \overline{L}_0 - \frac{c}{24}) + 2\pi i \tau_1 (L_0 - \overline{L}_0)} \\
=& \mathrm{Tr}\ q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}},
\end{aligned}
\tag{2.15}
$$

where $q = e^{2\pi i \tau}$. Here, we use the fact that the Hamiltonian H and the momentum P are given by $H = L_0 + \overline{L}_0 - \frac{c}{24}$ and $P = L_0 - \overline{L}_0$. Thus, we can compute the partition function purely from the representation theory. For generic parameters, the character can be computed as

$$
\chi(\tau) = \frac{q^{h - \frac{c}{24}}}{\prod_{n=1}^{\infty} (1 - q^n)},
\tag{2.16}
$$

where each factor in the denominator comes from the contribution from L_{-n}. When there is a null state at level N, we need to subtract all of its descendant states, and the character changes as

$$
\chi(\tau) = \frac{q^{h - \frac{c}{24}}}{\prod_{n=1}^{\infty} (1 - q^n)} - \frac{q^{h + N - \frac{c}{24}}}{\prod_{n=1}^{\infty} (1 - q^n)} = \frac{q^{h - \frac{c}{24}} (1 - q^N)}{\prod_{n=1}^{\infty} (1 - q^n)}.
\tag{2.17}
$$

For the case that there are two null states, the character becomes much more complicated. The character for minimal models is one of the most important objects in our study.

Finally, we explain the affine Kac-Moody algebra (also called current algebra), which plays an important role in the next section. It is realized from the currents $J^i(z)$ with conformal dimension $h = 1$. The possible form of the OPE is given as

$$
J^i(z) J^j(w) \sim \frac{\kappa^{ij}}{(z - w)^2} + \frac{f^{ij}_k J^k(w)}{z - w}.
\tag{2.18}
$$

This OPE must satisfy associativity (Jacobi-identity). That requires f^{ij}_k to be a structure constant of some Lie algebra $\mathfrak{g}$. Namely, the current $J^i(z)$ is associated with the element J^i of $\mathfrak{g}$ satisfying

$$
[J^i, J^j] = f^{ij}_k J^k \qquad (i, j, k = 1, 2, \cdots, \dim \mathfrak{g}),
\tag{2.19}
$$

where $\dim \mathfrak{g}$ denotes the number of the elements in $\mathfrak{g}$. For κ^{ij}, it must be proportional to a symmetric invariant bilinear form of $\mathfrak{g}$. Therefore, we can set

$$
\kappa^{ij} = k (J^i | J^j).
\tag{2.20}
$$

Here, $(\cdot|\cdot)$ is an invariant norm of $\mathfrak{g}$ and we usually set $(J^i | J^j) \propto \mathrm{Tr}\ J^i J^j$. For later convenience, we introduce the dual element J_i by $(J^i | J_j) = \delta^i_j$. The element k commutes with all of the other elements and its value is called level[6]. The algebra formed by $J^i(z)$ is usually denoted by $\widehat{\mathfrak{g}}$ or $\widehat{G}$, where G is the corresponding Lie group. When we want to express the level explicitly, we also denote them by $\widehat{\mathfrak{g}}_k$ or $\widehat{G}_k$.

We note that (2.18) is simplified when $\mathfrak{g} = \mathfrak{u}(1)$ because the second term disappears. Therefore, the level k can be absorbed into the normalization of the current and does not have an essential meaning. We often refer to $\mathfrak{u}(1)$ current as U(1) boson because it appears in the free boson theory.

One can construct the energy-momentum tensor $T(z)$ from the current algebra. Because the conformal dimension of $T(z)$ is two, it is expected that we can obtain it as the product of two currents $J^i(z), J_i(z)$. Here, we need to note that we have to remove the divergence from the product. That is possible by defining the normal ordered product as

$$: X(w)Y(w) :\equiv \oint_w \frac{dz}{z-w} X(z)Y(w). \tag{2.21}$$

One can see that the divergent part in (2.3) is indeed removed. After some computation, one can see that the current

$$T(z) = \frac{1}{2(k+h^\vee_{\mathfrak{g}})}(J^i J_i)(z) \tag{2.22}$$

satisfies the OPE (2.4) with the central charge

$$c = \frac{k \text{ dim } \mathfrak{g}}{k+h^\vee_{\mathfrak{g}}}. \tag{2.23}$$

Here, we use the dual Coxeter number $h^\vee_{\mathfrak{g}}$ defined by

$$\text{Tr}(\text{ad } J^i)(\text{ad } J^j) = 2h^\vee_{\mathfrak{g}}(J^i|J^j), \tag{2.24}$$

where ad J^i means the adjoint representation of J^i. The above way to obtain the energy-momentum tensor is called Sugawara construction.

One can also generalize the above construction to super Lie algebras. The super Lie algebra consists of bosonic elements and fermionic elements. They are graded by $\mathbb{Z}_2$ and decomposed by its charge as $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The representation space V is also decomposed as $V = V_0 \oplus V_1$. The $\mathbb{Z}_2$ grading is preserved under the product. The different point from Lie algebras is that the commutator is given by

$$[x, y] = xy - (-1)^{p(z)p(y)}yx, \tag{2.25}$$

where we use the notation

$$p(x) = \begin{cases} 0 & (x \in \mathfrak{g}_0) \\ 1 & (x \in \mathfrak{g}_1). \end{cases} \tag{2.26}$$

[6] We note that this terminology was already used in a different meaning. We need to be careful not to get confused.

We also need to replace dim $\mathfrak{g}$ with sdim $\mathfrak{g} \equiv \dim \mathfrak{g}_0 - \dim \mathfrak{g}_1$ and the trace with the super trace $\mathrm{STr}_V\ x \equiv \mathrm{Tr}_V\ \Gamma x$, where Γ acts as $(-1)^i$ on V_i. In this thesis, we consider only $\mathfrak{g} = \mathfrak{u}(N|M)$, which is defined by $(N+M) \times (N+M)$ as follows:

$$\begin{array}{c} \\ N\updownarrow \\ M\updownarrow \end{array} \begin{array}{c} \overset{N}{\longleftrightarrow} \quad \overset{M}{\longleftrightarrow} \\ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \end{array}$$

In this case, we have $\mathfrak{g}_0 = A \oplus D, \quad \mathfrak{g}_1 = B \oplus C.$

2.2 W-algebras

Two-dimensional CFTs always possess the Virasoro symmetry, but there are sometimes larger symmetries. To describe them, we have to introduce new currents in addition to the energy-momentum tensor so that they will satisfy the Jacobi-identity and form a closed algebra. Such an algebra is usually called W-algebra. The typical example is W_N algebra, which consists of the currents with the conformal dimension $2, 3, 4, \cdots, N$. One can see Virasoro algebra as W_2 algebra. Apart from Virasoro algebra, the simplest example is W_3 algebra, which was constructed in [2]. It is generated by the energy-momentum tensor $T(z)$ and an additional current $W^{(3)}(z)$. The OPE between two $W^{(3)}(z)$ currents is given as follows:

$$\begin{aligned} W^{(3)}(z)W^{(3)}(w) \sim \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{1}{(z-w)^2}\left[\frac{32}{22+5c}\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right] \\ + \frac{1}{z-w}\left[\frac{16}{22+5c}\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right], \end{aligned} \tag{2.27}$$

where

$$\Lambda(w) =: T(w)T(w) : -\frac{3}{10}\partial^2 T(w). \tag{2.28}$$

The relation is complicated compared with Virasoro algebra. For general W-algebras, the OPE relations become much more complicated and it is impossible to write down the explicit OPE relations among currents. Then we have to find an alternative way to deal with W-algebras. In the following, we review some systematic ways to construct and study W-algebras. We refer to the comprehensive reviews [38, 39] for a more detailed explanation of W-algebras.

2.2.1 Classical case [7]

One of the most powerful ways to construct W-algebras is the quantum Drinfeld-Sokolov reduction. As is clear from the name, it has a classical counterpart. The motivation of the construction comes from the classical integrable system known as KdV system. Before considering the quantum case, let us get an overview of the classical story, based on [40].

The KdV equation is a nonlinear differential equation for the function $u(x,t)$ given by

$$\dot{u} - 6uu' - u''' = 0. \tag{2.29}$$

Here, we set $\dot{u} = \partial_t u$ and $u' = \partial_x u$. It is known that the KdV system is an integrable system and possesses infinitely many conserved charges. That is guaranteed by the fact that the equation (2.29) can be rewritten into the Lax form as follows:

$$\partial_t L = [P, L], \tag{2.30}$$

$$L = \partial_x^2 + u, \quad P = 4\partial_x^3 + 6u\partial_x + 3u_x. \tag{2.31}$$

The Lax form allows us to construct infinitely many flow equations which are compatible with each other in a systematic way.[8]

The KdV equation can also be described as a Hamiltonian system. The Poisson bracket is given by

$$\{u(x), u(y)\} = \delta'''(y-x) + 2u'(x)\delta(y-x) + 4u\delta'(y-x) \tag{2.32}$$

and the equation (2.29) can be rewritten as follows:

$$\dot{u} = \{H, u\}, \tag{2.33}$$

where $H = \int \frac{u^2}{2} dx$. Let us define the component of $u(x)$ as follows:

$$h_k = \frac{1}{2\pi \mathrm{i}} \int x^{k+1} u(x) dx. \tag{2.34}$$

Then the Poisson bracket (2.32) implies

$$\{h_k, h_l\} = (k-l)h_{k+l} + \frac{1}{2}\delta_{k+l,0}(k^2-1)k. \tag{2.35}$$

This is precisely the classical Virasoro algebra. Here, we mean by "classical" that the algebraic relation is defined by not a commutator but a Poisson bracket.

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[7]The reader can skip this section, but it helps us to understand the motivation of the method explained in the next section.

[8]Concretely, the flow equations are given by $\partial_{t_i} L = [P_i, L]$ $(i = 1, 2, 3, \cdots)$, where $P_i = (L^{\frac{2i+1}{2}})_+$. Here, the fractional power of a differential operator contains negative power of derivative ∂^{-n} and the subscript "+" means the projection to the part without nonnegative derivative. These equations are compatible, namely, $\partial_{t_l}\partial_{t_m} = \partial_{t_m}\partial_{t_l}$, which leads to the existence of infinitely many conserved charges.

One can generalize the KdV system by replacing the Lax operator L defined in (2.31) with the higher differential operator,

$$L = \partial^n + u_{n-2}\partial^{n-2} + u_{n-3}\partial^{n-3} + \cdots + u_0. \tag{2.36}$$

The generalized KdV system can also be expressed in a Lax formalism similar to (2.30) and written as a Hamiltonian system. The classical W_n algebra is defined as the algebra formed by $u_i(x)$ $(i = 0, 1, \cdots, n-2)$ whose algebraic structure is given by the Poisson bracket.

The crucial observation is that one can reduce a higher order differential equation to a first order differential equation. To see that, we introduce n functions $p_0 = p$, $p_1 = p'$, $p_2 = p'_1, \cdots, p_{n-1} = p'_{n-2}$. Then the equation $Lp = 0$ is rewritten as

$$(\partial - J + U)\vec{p} = \vec{0}, \tag{2.37}$$

where J and U are $n \times n$ matrices whose nonzero components are $J_{i,i+1} = 1$ and $U_{n-1,i} = u_i$ $(i = 0, 1, \cdots, n-2)$, namely,

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ u_0 & u_1 & \cdots & u_{n-2} & 0 \end{pmatrix}, \quad \vec{p} = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-2} \\ p_{n-1} \end{pmatrix}. \tag{2.38}$$

More generally, any n-th order differential equation can be written in the form of

$$l_q\vec{p} = \vec{0}, \qquad l_q \equiv \partial - J + q, \tag{2.39}$$

where q is an $n \times n$ lower triangular matrix. Here, we need to note that the different matrices q_1 and q_2 may give the same L. That happens iff they are related by the similarity transformation $l_{q_1} = g^{-1}l_{q_2}g$ for $g \in \mathcal{N}$. Here, $\mathcal{N}$ denotes a set of the matrices in the form of $I + \nu$ where I is the unit matrix and ν is a strictly lower triangular matrix. Then we need to fix this redundancy. We have already seen one possible choice in (2.38), which is sometimes called U-gauge. On the other hand, we can also fix the gauge without setting a matrix to a particular form. One can consider the whole of $n \times n$ matrices by identifying the elements related by the gauge transformation. Namely, one can obtain the classical W_n algebra as the reduction of $\mathfrak{sl}_n$. One can also apply this method to the other simple Lie algebras [41]. In the next section, we discuss the quantum counterpart of the above story by using Becchi-Rouet-Stora-Tyutin (BRST) formalism which is known as the gauge fixing procedure.

2.2.2 Quantum Drinfeld-Sokolov reduction

The quantum Drinfeld-Sokolov reduction is the systematic method to construct W-algebras from affine Kac-Moody algebras [42, 43, 44]. The idea is the same as the classical case.

Namely, the W-algebra is defined through the gauge fixing procedure. In this construction, the data which characterizes the W-algebra are given by a simple Lie superalgebra $\mathfrak{g}$, the level k of affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ and $\mathfrak{sl}_2$ subalgebra $\{e, x, f\}$ in $\mathfrak{g}$[9]. The last one determines the type of reduction, as we will see below; in the previous discussion, we focused on the matrix in the form of (2.38), but the other reductions are also possible. We give the detailed prescription below:

1. $\mathfrak{g}$ is decomposed by the eigenvalues of x as follows:

$$\mathfrak{g} = \oplus_{i\in\frac{1}{2}\mathbb{Z}}\ \mathfrak{g}_i \tag{2.40}$$

   We note that the eigenvalues are half-integer due to the representation theory of $\mathfrak{sl}_2$. We set

$$\mathfrak{g}_+ = \oplus_{i>0}\ \mathfrak{g}_i, \qquad \mathfrak{g}_- = \oplus_{i<0}\ \mathfrak{g}_i. \tag{2.41}$$

   For later convenience, we introduce index sets S, S' such that

$$\mathfrak{g}_+ = \oplus_{i\in S}\ \mathfrak{g}_i, \qquad \mathfrak{g}_{\frac{1}{2}} = \oplus_{i\in S'}\ \mathfrak{g}_i. \tag{2.42}$$

2. We introduce a pair of the ghost fields $(\varphi^i(z), \varphi^*_i(z))$ called charged ghosts for $i \in S$[10]. Their statistics (i.e. fermionic or bosonic) are opposite to those of $\mathfrak{g}_i$. Their OPEs are given by

$$\varphi^i(z)\varphi^*_j(w) \sim \frac{\delta^i_j}{z-w}. \tag{2.43}$$

   We assign the charge 1 (resp. -1) to $\varphi^*(z)$ (resp. $\varphi(z)$). For the other fields, we set the charge to zero.

3. We introduce another ghost field $\Phi_i(z)$ called neutral ghost for $i \in S'$ whose statistics are the same as $\mathfrak{g}_i$. The OPE is given by

$$\Phi^i(z)\Phi^j(w) \sim \frac{(f|[J^i, J^j])}{z-w}, \tag{2.44}$$

   where $(\cdot|\cdot)$ is the non-degenerate invariant norm of $\mathfrak{g}$.

4. We define the operator d as follows:

$$d = \oint dz\ d(z), \tag{2.45}$$

[9] Precisely speaking, we can construct W-algebras from a nilpotent element f and a diagonizable element x satisfying some conditions. The elements f, x do not need to be a part of $\mathfrak{sl}_2$-triplet [44].

[10] They are often called bc ghosts for the fermionic fields and $\beta\gamma$ ghosts for the bosonic fields.

$$d(z) = \sum_{i\in S}\Big((-1)^{p(J^i)}J^i(z) + (f|J^i)\Big)\varphi_i^*(z) - \frac{1}{2}\sum_{i,j,l\in S}(-1)^{p(J^i)p(J^l)}f_l^{ij} :\varphi^l(z)\varphi_i^*(z)\varphi_j^*(z): + \sum_{i\in S'}\varphi_i^*(z)\Phi^i(z). \tag{2.46}$$

It is obvious that this operator is fermionic and its charge is 1. Because $d(z)d(w) \sim O((z-w)^0)$, we find $d^2 = 0$. Then we can see d as a BRST operator and define the BRST complex $(C_k^\bullet, d)$ graded by the charge introduced above. Here, the subscript k denotes the level of affine Kac-Moody algebra. The W-algebra is defined as the zeroth cohomology.

We note that the first two terms in (2.46) give the standard BRST operator which classically sets $J^i(z)$ to the constant $-(-1)^{p(J^i)}(f|J^i)$ for $i \in S$. The last term may be unfamiliar, but it is necessary in order to impose $e(z) = -1$ consistently; if we set all the elements of $\mathfrak{g}_{\frac{1}{2}}$ to zero, $e(z)$ is also set to zero because the commutator of the elements in $\mathfrak{g}_{\frac{1}{2}}$ becomes zero.

Let us first check that the W-algebra defined above always contains Virasoro algebra. The total Virasoro algebra should consist of three factors: the affine Lie algebra, the charged ghosts and the neutral ghosts. Let us introduce the following energy-momentum tensor to each sector:

$$T^{\mathfrak{g}}(z) = \frac{1}{2(k+h_{\mathfrak{g}}^\vee)}\sum_{i\in\text{all}} :J^i(z)J_i(z): + \partial x(z), \tag{2.47}$$

$$T^{ch}(z) = -\sum_{i\in S} m_i :\varphi_i^*(z)\partial\varphi^i(z): + \sum_{i\in S}(1-m_i):(\partial\varphi_i^*(z))\varphi^i(z):, \tag{2.48}$$

$$T^{ne}(z) = \frac{1}{2}\sum_{i\in S'} :(\partial\Phi^i(z))\Phi_i(z):, \tag{2.49}$$

where the parameters m_i are defined by

$$[x, J^i] = m_i J^i. \tag{2.50}$$

These currents are defined so that the conformal dimension of $d(z)$ will be equal to 1; each field behaves as a primary field whose conformal dimension is given as follows[11]:

$$J^i(z) : 1-m_i, \quad \varphi_i(z) : 1-m_i, \quad \varphi_i^*(z) : m_i, \quad \Phi_i(z) : \frac{1}{2} \quad . \tag{2.51}$$

The central charges of these energy-momentum tensors are given by

$$c^{\mathfrak{g}} = \frac{k\ \text{sdim}\mathfrak{g}}{k+h_{\mathfrak{g}}^\vee} - 12k(x|x), \tag{2.52}$$

$$c^{ch} = -\sum_{i\in S}(-1)^{p(\varphi^i)}(12m_i^2 - 12m_i + 2), \tag{2.53}$$

$$c^{ne} = -\frac{1}{2}\ \text{sdim}\mathfrak{g}. \tag{2.54}$$

[11]Precisely, the Cartan element $J^i(z)$ satisfying $(x|J^i) \neq 0$ behaves as a quasi-primary field.

The total energy-momentum tensor and its central charge are given by

$$T(z)=T^{\mathfrak{g}}(z)+T^{ch}(z)+T^{ne}(z), \tag{2.55}$$

$$c=c^{\mathfrak{g}}+c^{ch}+c^{ne}=\mathrm{sdim}\mathfrak{g}_0-\frac{1}{2}\mathrm{sdim}\mathfrak{g}_{\frac{1}{2}}-12\left|\frac{\rho}{(k+h^\vee_{\mathfrak{g}})^{1/2}}-x(k+h^\vee_{\mathfrak{g}})^{1/2}\right|^2, \tag{2.56}$$

where ρ is the Weyl vector which satisfies $(\rho,\alpha)=1$ for all the simple roots α. Because the conformal dimension of $d(z)$ is one, it satisfies $[d,T(z)]=0$, which implies that $T(z)$ is the energy-momentum tensor for $W_k(\mathfrak{g},x,f)$.

As an example, let us consider the case of $\mathfrak{g}=\mathfrak{sl}_N$[12] and $f=J^{-\theta}$, where θ is the longest root. We note that this case corresponds to (2.38). This $\mathfrak{sl}_2$ embedding is often called principal. In this case, we can identify S with the positive root system Δ^+. The BRST charge is given by

$$\begin{aligned} d&=d^{(0)}+d^{(1)},\\ d^{(0)}&=\int dz\Big(\sum_{\alpha\in\Delta^+}J^\alpha(z)\varphi^*_\alpha(z)-\frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta^+}f^{\alpha\beta}_\gamma:\varphi^\gamma(z)\varphi^*_\alpha(z)\varphi^*_\beta(z):\Big),\\ d^{(1)}&=\int dz\ \chi(J^\alpha)\varphi^*_\alpha(z)\end{aligned} \tag{2.57}$$

where we set

$$\chi(J^\alpha)=\begin{cases}1 & \text{if } \alpha \text{ is simple.}\\ 0 & \text{otherwise.}\end{cases} \tag{2.58}$$

We note that the neutral ghost does not appear in this case. For the computation of the cohomology, we introduce the current,

$$\widehat{J}^i(z)=J^i(z)+\sum_{\beta,\gamma\in\Delta^+}f^{i\beta}_\gamma:\varphi_\gamma(z)\varphi^*_\beta(z):. \tag{2.59}$$

We split the complex $C^\bullet_k$ into the two parts, $C^\bullet_k=C^\bullet_{k,1}\otimes C^\bullet_{k,2}$. Here, $C^\bullet_{k,1}$ is spanned by $\widehat{J}^\alpha(z)$ and $\varphi^\alpha(z)$ while $C^\bullet_{k,2}$ is spanned by $\widehat{J}^a(z)$ ($a\in\Delta^-\cup\mathfrak{n}$) and $\varphi^*_\alpha(z)$. We note that this decomposition holds not only at the level of the vector space but also at the level of the complex. The cohomology of $C^\bullet_{k,1}$ can be easily computed. From the relation

$$[d,\widehat{J}^\alpha]=0,\quad \{d,\varphi^\alpha(z)\}=\widehat{J}^\alpha(z)+\chi(J^\alpha), \tag{2.60}$$

we see that the identification $\widehat{J}^\alpha(z)\sim-\chi(J^\alpha)$ holds in the cohomology, which implies that we can identify the zeroth cohomology of $C^\bullet_{k,1}$ as constant $\mathbb{C}$. This result corresponds to the gauge fixing in (2.38) where the strictly upper triangular part of the matrix is fixed. We note that the cohomological degree (ghost charge) can now take non-negative values.

[12] In this thesis, we do not distinguish $\mathfrak{sl}_N$ and $\mathfrak{su}(N)$.

The remaining task is to compute the cohomology of $C^\bullet_{k,2}$. The important observation is that we can associate the structure of a double complex to $C^\bullet_{k,2}$ because of the relation $d^{(0)} \cdot d^{(0)} = d^{(1)} \cdot d^{(1)} = \{d^{(0)}, d^{(1)}\} = 0$. Then we can use the technique of the spectral sequence where the zeroth differential and the first differential are given by $d^{(1)}$ and $d^{(0)}$, respectively. Let us consider the cohomology for $d^{(1)}$. From the relation

$$[d^{(1)}, \widehat{J}^a(z)] = \sum_{\beta \in \Delta^+} ([f, J^a] | J^\beta) \varphi^*_\beta(z), \tag{2.61}$$

we see that $\widehat{J}^a(z)$ commutes with $d^{(1)}$ if and only if $[f, J^a] = 0$ holds, or in other words, J^a is the lowest weight vector of $\mathfrak{sl}_2$ multiplet. Because $\mathfrak{sl}_N$ is decomposed into the irreducible representations of $\mathfrak{sl}_2$ as $\mathfrak{sl}_N = \underline{3} \oplus \underline{5} \oplus \underline{7} \oplus \cdots \oplus \underline{2N-1}$ [13], we have $N-1$ lowest vector. We denote by $P^{(i)}(z)$ $(i = 1, 2, \cdots, N-1)$ the linear combination of the corresponding $\widehat{J}^a(z)$s. We note that these currents correspond to $u_0, u_1, \cdots, u_{n-2}$ in (2.38). One may also see that the fields $\varphi^*_\alpha(z)$ also commute with $d^{(1)}$, but they are exact elements, which can be seen from (2.61). That implies the identification $\varphi^*_\alpha(z) \sim 0$. Thus one can show that the cohomology for $d^{(1)}$ is generated by $P^{(i)}(z)$ if the cohomological degree (i.e. ghost charge) is 0. For the other degree, it vanishes. Next, we need to take the cohomology for $d^{(0)}$. That can be implemented by adding to $P^{(i)}(z)$ the correction terms so that it will commute with the differential $d = d^{(0)} + d^{(1)}$. They can be determined recursively by solving $d^{(1)} X_{j+1}(z) = -d^{(0)} X_j(z)$, $X_0 = P^{(i)}(z)$. Then the generator is given by $W^{(i)}(z) = P^{(i)}(z) + \sum_{j=1}^n X_j(z)$, where n is an integer such that $d^{(0)} X_n(z) = 0$ holds. One can see from (2.51) that the conformal dimension of these generators are $2, 3, 4, \cdots, N$. Thus this W-algebra is W_N algebra.

For generic $\mathfrak{g}$ and $\mathfrak{sl}_2$ embedding, it is difficult to compute the cohomology, but it is expected that the corresponding W-algebra is generated by the currents in the form of $P^{(i)}(z) + \cdots$, where $P^{(i)}$ is again the current associated with the lowest vector of $\mathfrak{sl}_2$ multiplet. When $\mathfrak{g}$ is decomposed by $\mathfrak{sl}_2$ into $\mathfrak{g} = \underline{n_1} \oplus \underline{n_2} \oplus \cdots \oplus \underline{n_M}$, the W-algebra is generated by the currents $W^{(i)}(z)$ $(i = 1, 2, \cdots, M)$ with the conformal dimension $\frac{n_1+1}{2}, \frac{n_2+1}{2}, \cdots, \frac{n_M+1}{2}$. One can easily check that this claim is true at least for a singlet of $\mathfrak{sl}_2$. Indeed, one can immediately check that the following current $\widehat{J}^i(z)$ satisfies $[d, \widehat{J}^i(z)] = 0$ if $\mathfrak{g}_i$ is a $\mathfrak{sl}_2$ singlet:

$$\widehat{J}^i(z) = J^i(z) + J^i_{ch}(z) + J^i_{ne}(z), \tag{2.62}$$

where

$$J^i_{ch}(z) = -\sum_{j,k \in S} (-1)^{p(\varphi_\alpha)} f^{ik}_j : \varphi^j(z) \varphi^*_k(z) :, \tag{2.63}$$

$$J^i_{ne}(z) = -\frac{1}{2} \sum_{j,k \in S'} (-1)^{p(\Phi_i)} f^{ik}_j : \Phi^j(z) \Phi_k(z) : . \tag{2.64}$$

[13] We denote by $\underline{n}$ the n-dimensional $\mathfrak{sl}_2$ irreducible representation.

We note that the $\mathfrak{sl}_2$ embedding of $\mathfrak{sl}_N$ is labeled by the partition of N as follows:

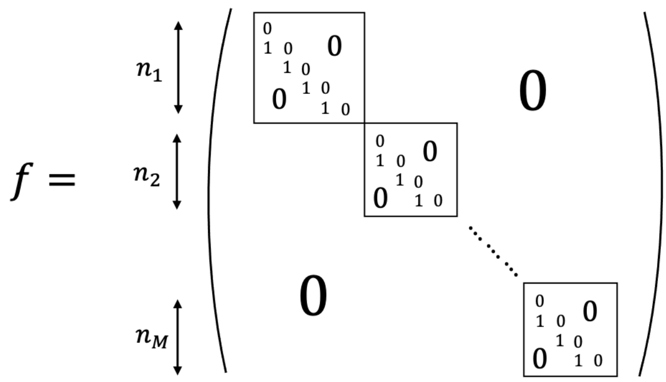


Figure 4: This figure corresponds to $\mathfrak{sl}_2$ embedding for the partition $N = n_1 + n_2 + \cdots n_M$ $(n_1, n_2, \cdots, n_M \geq 1.\ )$

For later convenience, we denote by $DS_n(\widehat{\mathfrak{sl}}_N)$ the DS reduction for the partition $N = n + 1 + 1 + 1 + \cdots + 1$.

Finally, we mention to the case of $\mathfrak{g} = \mathfrak{sl}_{2|1}$. We first fix the notation. We denote the elements of $\mathfrak{sl}_{2|1}$ by

$$\left(\begin{array}{cc|c} \frac{2h_1+h_2}{3} & e_{12} & e_1 \\ f_{12} & \frac{-h_1-2h_2}{3} & f_2 \\ \hline f_1 & e_2 & \frac{h_1-h_2}{3} \end{array}\right). \tag{2.65}$$

In this case, the $\mathfrak{sl}_2$ embedding is uniquely determined; it is given by $(e_{12}, h_1+h_2, f_{12})$. Then $\mathfrak{sl}_{2|1}$ is decomposed into $\underline{1} + \underline{2} + \underline{2} + \underline{3}$, which corresponds to $(h_1 - h_2)$, (e_1, f_2), (e_2, f_1) and $(e_{12}, h_1 + h_2, f_{12})$, respectively. According to the previous discussion, it is expected that we have BRST-closed currents $G^+(z), G^-(z), T(z)$ with conformal dimension $\frac{3}{2}, \frac{3}{2}, 2$ in addition to $J(z) = \widehat{J}^{(h_1-h_2)}(z)$. This is indeed true and it is not difficult to find an explicit form of these currents by hand [14]. The OPE between these currents are given as follows:

$$\begin{aligned} &T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \quad T(z)G^\pm(w) \sim \frac{\frac{3}{2}G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{z-w}, \\ &T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \quad G^\pm(z)G^\pm(w) \sim 0, \quad J(z)G^\pm(w) \sim \frac{G^\pm(w)}{z-w}, \\ &G^+(z)G^-(w) \sim \frac{\frac{2c}{3}}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w}, \end{aligned} \tag{2.66}$$

where the central charge is $c = -3(2k+1)$. This algebra is known as $\mathcal{N} = 2$ super Virasoro algebra. We note that the currents $T(z), J(z)$ are bosonic while $G^\pm(z)$ are fermionic.

[14] The explicit form can be found in [44].

2.2.3 Coset construction

There is another way to define W-algebras from affine Kac-Moody algebras. It is called coset construction [45]. Let $\widehat{G}$ be an affine Kac-Moody algebra and $\widehat{H}$ be its subalgebra. One can construct a W-algebra from them as the set of the elements of $\widehat{G}$ commuting with $\widehat{H}$. We denote it by $\widehat{G}/\widehat{H}$. The energy-momentum tensor which commutes with the $\widehat{H}$ currents is given by

$$T(z) = T_G(z) - T_H(z), \tag{2.67}$$

where $T_G(z)$ and $T_H(z)$ are the Sugawara energy-momentum tensors for $\widehat{G}$ and $\widehat{H}$, respectively. The central charge is given by

$$c = c_G - c_H. \tag{2.68}$$

One of the examples which is important in this thesis is

$$\frac{\widehat{SU}(2)_k}{\widehat{U}(1)}, \tag{2.69}$$

which is known as SU(2) parafermion [46]. Let us consider the Hilbert space when k is a positive integer. Here, we fix the normalization of $\widehat{SU}(2)_k$ as

$$J^3(z)J^3(w) \sim \frac{k/2}{(z-w)^2}, \quad J^3(z)J^\pm(w) \sim \frac{\pm J^\pm(w)}{z-w}, \quad J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w}. \tag{2.70}$$

By definition, the integrable representation $\mathcal{L}_l$ of $\widehat{SU}(2)_k$ is decomposed into the representation of the parafermion and that of $\widehat{U}(1)$:

$$\mathcal{L}_l = \oplus_{\substack{m\in\mathbb{Z} \\ l-m\in 2\mathbb{Z}}} \mathcal{M}^{\rm PF}_{l,m} \otimes \mathcal{M}^{\widehat{U}(1)}_m, \tag{2.71}$$

where $\mathcal{M}^{\widehat{U}(1)}_m$ is the $\widehat{U}(1)$ module whose eigenvalue of J^0_3 is $\frac{m}{2}$. We note that not all of $\mathcal{M}^{\rm PF}_{l,m}$ give different modules. To see that, let us consider the affine Weyl group. It includes the transformation subgroup labeled by the coroot. In the case of $\widehat{SU}(2)$, it changes J^3_0 to $J^3_0 + nk$ ($n \in \mathbb{Z}$). Because the representation space of the integrable representation is invariant under the affine Weyl group, the RHS of (2.71) should be also invariant. Therefore, we find the identification

$$\mathcal{M}^{\rm PF}_{l,m} \sim \mathcal{M}^{\rm PF}_{l,m+2jk} \quad (j \in \mathbb{Z}) \tag{2.72}$$

and we can rewrite (2.71) as

$$\mathcal{L}_l = \oplus_{\substack{m\in\mathbb{Z}_{2k} \\ l-m\in 2\mathbb{Z}}} \mathcal{M}^{\rm PF}_{l,m} \otimes V^{\widehat{U}(1)}_m, \qquad V^{\widehat{U}(1)}_m \equiv \oplus_{j\in\mathbb{Z}} \mathcal{M}^{\widehat{U}(1)}_{m+2jk}. \tag{2.73}$$

There is still a redundant symmetry which transforms $\mathcal{L}_l$ into $\mathcal{L}_{k-l}$ and J^3_0 into $J^3_0 + \frac{k}{2}$. It corresponds to the permutation of the two simple roots for $\widehat{SU}(2)$. Therefore, the following identification also holds:

$$\mathcal{M}^{\rm PF}_{l,m} \sim \mathcal{M}^{\rm PF}_{k-l,m+k}. \tag{2.74}$$

From this, one can see that the independent module lies in the region

$$-l \leq m \leq l. \tag{2.75}$$

The conformal dimension $h^{\rm PF}_{l,m}$ of $\mathcal{M}^{\rm PF}_{l,m}$ can be computed from (2.67) as

$$h^{\rm PF}_{l,m} = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k}. \tag{2.76}$$

The character $\chi^{\rm PF}_{l,m}(\tau)$ of the parafermion can be read off from the relation

$$\chi^{\widehat{SU}(2)}_{l,k}(\tau,z) = \sum_{\substack{m\in\mathbb{Z}_{2k} \\ l-m\in 2\mathbb{Z}}} \chi^{\rm PF}_{l,m}(\tau)\frac{\Theta_{m,k}(\tau,z)}{\eta(\tau)}, \tag{2.77}$$

where $\chi(\tau,z) = {\rm Tr}\ q^{L_0-\frac{c}{24}}y^{J^3_0}$ ($q=e^{2\pi i\tau}$, $y=e^{2\pi i z}$) measures both L_0 and J^3_0 and we set

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{i=1}^{\infty}(1-q^i), \quad \Theta_{m,n}(\tau,z) = \sum_{j=-\infty}^{\infty} q^{n(j+\frac{m}{2n})^2}y^{n(j+\frac{m}{2n})}. \tag{2.78}$$

The character of $\widehat{SU}(2)_k$ is obtained as $\chi^{\widehat{SU}(2)}_{l,k}(\tau,z) = \frac{\Theta_{l+1,k+2}(\tau,z)-\Theta_{-(l+1),k+2}(\tau,z)}{\Theta_{1,2}(\tau,z)-\Theta_{-1,2}(\tau,z)}$ from Weyl-Kac formula. The function $c^{(k)}_{l,m}(\tau) = \chi^{\rm PF}_{l,m}(\tau)/\eta(\tau)$ is called string function and we will use this function later. Because of (2.72) and (2.74), it satisfies

$$c^{(k)}_{l,m+2k}(\tau) = c^{(k)}_{k-l,m+k}(\tau) = c^{(k)}_{l,m}(\tau). \tag{2.79}$$

We note that the string function becomes $1/\eta(\tau)$ when $k=1$.

Finally, we briefly mention the relation between the coset construction and the BRST approach, based on [47]. In a Lagrangian description, the coset CFT $\widehat{G}/\widehat{H}$ is realized from a gauged Wess-Zumino-Witten (WZW) model, which is obtained by gauging H-symmetry of the G WZW mode with level k. One can implement the gauge fixing procedure by the standard Fadeev-Popov procedure. As a result, the physical observables are given by BRST-closed operators within

$$\widehat{G}_k \otimes \widehat{H}_{-k-2h^\vee_H} \otimes \text{ghost fields}, \tag{2.80}$$

where the ghost fields are introduced for every element of H. The total central charge is

$$\begin{aligned} c &= \frac{k\ \dim G}{k+h^\vee_G} + \frac{(-k-2h^\vee_H)\dim H}{(-k-2h^\vee_H)+h^\vee_H} - 2\dim H \\ &= \frac{k\ \dim G}{k+h^\vee_G} - \frac{k\ \dim H}{k+h^\vee_H}, \end{aligned} \tag{2.81}$$

which indeed matches the coset construction. We will use this relation in the next section.

2.3 W_N minimal models and $\mathcal{N}=2$ unitary minimal models

In this section, we review the minimal models for W_N algebra and $\mathcal{N}=2$ super Virasoro algebra. As we mentioned, a minimal model describes the system which consists of a finite number of degenerate primary fields. There are several ways to obtain such fields. In Section 2.3.1, we consider the W_N minimal models by using free boson representation. In Section 2.3.2, we consider the unitary $\mathcal{N}=2$ minimal models by using the coset construction.

2.3.1 W_N minimal models from free boson realization

In Section 2.2.1, we have seen a close connection between the classical W_N algebra and the N-th order differential operator L given in (2.36). We discussed the particular gauge (2.38), but there is another simple gauge as follows:

$$L = (\partial + v_1)(\partial + v_2)\cdots(\partial + v_N). \tag{2.82}$$

Motivated by this observation, Fateev and Lukyanov constructed W_N algebra using $N-1$ free bosons [48]. They introduced the differential operator

$$R^{(N)}(z) =: (Q\partial + \vec{\xi}_N \cdot \partial\vec{\phi}(z))(Q\partial + \vec{\xi}_{N-1} \cdot \partial\vec{\phi}(z))\cdots(Q\partial + \vec{\xi}_1 \cdot \partial\vec{\phi}(z)) : . \tag{2.83}$$

Here, $\vec{\phi}$ is the $(N-1)$-component free bosons

$$\vec{\phi} = (\phi^{(1)}(z), \phi^{(2)}(z), \cdots, \phi^{(N-1)}(z)), \quad \phi^{(i)}(z) = q^{(i)} + a_0^{(i)} \log z - \sum_{n\neq 0} \frac{a_n^{(i)}}{n} z^{-n} \tag{2.84}$$

with the OPEs

$$\phi^{(i)}(z)\phi^{(j)}(w) \sim -\delta_{i,j}\log(z-w) \quad (i,j = 1,2,\cdots N-1). \tag{2.85}$$

$\vec{\xi}_i$ is an $(N-1)$-dimensional vector[15] satisfying

$$\sum_{i=1}^{N} \vec{\xi}_i = 0, \quad \vec{\xi}_i \cdot \vec{\xi}_j = \delta_{i,j} - \frac{1}{N} \quad (i,j = 1,2,\cdots N). \tag{2.86}$$

The parameter Q is related to the central charge, as we will see below.

The current $W^{(s)}(z)$ with conformal dimension s is obtained by expanding (2.83) as follows:

$$R^{(N)}(z) = \sum_{s=0}^{N} W^{(s)}(z)(Q\partial)^{n-s}. \tag{2.87}$$

[15] These vectors are the bases of the hyperplane in $\mathbb{R}^N$. Letting $\vec{e}_i$ $(i = 1,2,\cdots,N)$ be the unit vector along the i-th axis, one can see that the vector $\vec{\xi}_i$ can be expressed as $\vec{\xi}_i = \vec{e}_i - \frac{1}{N}\sum_{j=1}^{N}\vec{e}_j$.

For example, the explicit form of $W^{(1)}(z)$ and $W^{(2)}(z)$ is given by

$$W^{(1)}(z) = \sum_{i=1}^{N} \vec{\xi}_i \cdot \partial\vec{\phi}(z) = 0, \tag{2.88}$$

$$\begin{aligned} W^{(2)}(z) &= \sum_{1\le l<m\le N} :(\vec{\xi}_l \cdot \partial\vec{\phi}(z))(\vec{\xi}_m \cdot \partial\vec{\phi}(z)): + Q\sum_{k=1}^{N-1}(N-k)\vec{\xi}_k \cdot \partial^2\vec{\phi}(z) \\ &= -\frac{1}{2} :(\partial\vec{\phi}(z))^2: + Q\vec{\rho}\cdot\partial^2\vec{\phi}, \end{aligned} \tag{2.89}$$

where $\vec{\rho} = \frac{1}{2}\sum_{k=1}^{N}(N+1-2k)\vec{\xi}_k$ corresponds to the Weyl vector of $\mathfrak{su}(N)$. The current $W^{(2)}(z)$ is the energy-momentum tensor with the central charge

$$c = N - 1 + Q^2(N^2-1)N. \tag{2.90}$$

The explicit form of the higher spin currents is more complicated. Instead, one can characterize them as the operators commuting with a set of the vertex operators as follows:

$$S_i^{\pm} = \oint dz : e^{b^{\pm 1}\vec{\alpha}_i\cdot\vec{\phi}(z)} : \qquad (i = 1, 2, \cdots, N-1), \tag{2.91}$$

$$[W^{(s)}(z), S_i^{\pm}] = 0 \qquad (s = 2, 3, \cdots, N). \tag{2.92}$$

Here, b and $\vec{\alpha}_i$ are defined by

$$Q = b + b^{-1}, \quad \vec{\alpha}_i = \vec{\xi}_i - \vec{\xi}_{i+1} \tag{2.93}$$

and the vectors $\vec{\alpha}_i$ correspond to the simple roots of $\mathfrak{su}(N)$. The vertex operators $S_i^{\pm}$ are called screening charges. We note that W_N algebra is invariant under the transformation

$$b \leftrightarrow b^{-1}. \tag{2.94}$$

This duality is called Feigin-Frenkel duality.

One can construct the highest weight state of W_N algebra by using the Fock spaces given by

$$a_n^{(i)}|\vec{u}\rangle = 0 \ (n>0), \qquad a_0^{(i)}|\vec{u}\rangle = -u_i|\vec{u}\rangle \qquad (i = 1, 2, \cdots, N-1). \tag{2.95}$$

The conformal dimension h for this state can be computed from (2.89) as

$$h = -\frac{1}{2}\vec{u}^{\,2} + Q\vec{\rho}\cdot\vec{u} \tag{2.96}$$

The screening charges play an important role in the construction of the null vectors. For simplicity, let us consider the $N=2$ case. In this case, we have only one boson and can simply express (2.91) and (2.96) as

$$S^{\pm} = \oint dz : e^{\sqrt{2}b^{\pm 1}\phi(z)} :, \quad h = -\frac{u^2}{2} + \frac{Qu}{\sqrt{2}}. \tag{2.97}$$

Let us consider the following state[16]:

$$
\begin{aligned}
&(S^+)^r\,|u-\sqrt{2}rb\rangle \\
\equiv &\int dz_1 \int dz_2 \cdots \int dz_r \; S^+(z_1)\cdots S^+(z_r)\,|u-\sqrt{2}rb\rangle \\
= &\int dz_1 \int dz_2 \cdots \int dz_r \prod_{1\le i<j\le r}(z_i-z_j)^{-2b^2} : S^+(z_1)\cdots S^+(z_r) : |u-\sqrt{2}rb\rangle \\
= &\int dz_1 \int dz_2 \cdots \int dz_r \prod_{1\le i<j\le r}(z_i-z_j)^{-2b^2} \prod_{k=1}^{r} z_k^{-\sqrt{2}b(u-\sqrt{2}rb)} \prod_{l=1}^{r} \exp\left(b\sum_{n>0}\frac{a_{-n}}{n}z_l^n\right)|u\rangle\,.
\end{aligned}
\tag{2.98}
$$

Because of (2.92), one can see that this state also satisfies the condition for the highest weight state. However, there is a possibility that it becomes zero. To obtain a non-vanishing state, we have to discuss the integration contour carefully, but we take a shortcut here. The integrand in (2.98) consists of infinitely many descendant states, but the non-vanishing terms must be invariant under the transformation $z_i \to \lambda z_i$. That requires

$$
r - r(r-1)b^2 - \sqrt{2}rb(u-\sqrt{2}rb) + N = 0 \tag{2.99}
$$

for some $N \in \mathbb{N}$. If we set $N = rs$, it can be rewritten as

$$
\sqrt{2}u = (1+r)b + (1+s)b^{-1}. \tag{2.100}
$$

Using (2.97), one can check that the conformal dimension of $|u\rangle$ is equal to $h_{r,s}$ given in (2.11). Here, we note that (2.100) is a necessary condition and does not require s to be a positive integer. One can obtain the sufficient condition by considering the order of Kac determinant. We explain that along the line of [49]. One can read off the information of the null vectors from the determinant of the matrix

$$
\langle u|a^{\{m_i\}}L^{\{n_i\}}|u\rangle\,, \tag{2.101}
$$

where we set $L^{\{n_i\}}|u\rangle = \prod_{i=1}^{\infty}(L_{-i})^{n_i}|u\rangle$ and $\langle u|\,a^{\{m_i\}} = \langle u|\prod_{i=1}^{\infty}(a_i)^{m_i}$. Because the null state appears in the zero-locus of the determinant, we can find the number of the null states from its order. Let us denote by d_N the order of the determinant at level N. Then we have

$$
\begin{aligned}
\sum_{N=1}^{\infty} d_N x^N &= \sum_{N=1}^{\infty} \sum_{\substack{\{n_i\} \\ \sum kn_k = N}} n_k x^N \\
&= \sum_{r=1}^{\infty} \frac{x^r}{1-x^r}\,\frac{1}{\prod_{j=1}^{\infty}(1-x^j)} \\
&= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} x^{rs} \sum_{n=0}^{\infty} p(n)x^n \\
&= \sum_{N=1}^{\infty} x^N \sum_{\substack{r,s\ge 1 \\ 1\le rs\le N}} p(N-rs),
\end{aligned}
\tag{2.102}
$$

[16] One can also construct a similar state by using S^-.

where $p(n)$ denotes the number of the partition of an integer n. This implies that there is a null vector at level rs for each $r, s \geq 1$. Thus, we see that we should set s to a positive integer.

As we mentioned in Section 2.1, the degenerate primary fields labeled by r, s are not independent for the particular central charges. When $b^2 = -\frac{p}{q}$ for coprime integers p, q, the relation $h_{r,s} = h_{r+q,s+p}$ holds. Furthermore, the null state which we should set to zero can be expressed as $h_{r,s} + rs = h_{r+q,p-s} = h_{q-r,p+s}$. Therefore, we can construct the model which contains only a finite number of the degenerate primary fields with

$$1 \leq r \leq q-1, \quad 1 \leq s \leq p-1. \tag{2.103}$$

This is the Virasoro minimal model.

In the same way, we can construct the W_N minimal model. We can express the degenerate module as

$$S_i^{m_i} |\vec{u}\rangle \quad (i = 1, 2, \cdots N-1). \tag{2.104}$$

Here, we note that there can be at most $N-1$ null vectors in the case of W_N algebra. Such modules are called maximally degenerate modules. In this case, the condition (2.100) becomes

$$\vec{\alpha}_i \cdot \vec{u} = (1 + m_i)b + (1 + n_i)b^{-1} \tag{2.105}$$

for $m_i, n_i \in \mathbb{N}$. In this module, there are null states at level $m_i n_i$ for $i = 1, 2, \cdots, N-1$. By introducing the fundamental weight $\vec{\omega}_j$ which satisfies $\vec{\alpha}_i \cdot \vec{\omega}_j = \delta_{i,j}$, we can express

$$\vec{u} = \sum_{i=1}^{N-1} \Big((1 + m_i)b + (1 + n_i)b^{-1} \Big) \vec{\omega}_i. \tag{2.106}$$

Similarly to the previous case, we can construct the minimal models when $b^2 = -\frac{p}{q}$. The degenerate primary fields contained in this model are restricted to the following region:

$$\sum_{i=1}^{N-1} m_i < q, \qquad \sum_{i=1}^{N-1} n_i < p. \tag{2.107}$$

We note that we need $p, q \geq N$. These modules contain additional null states at level $(q - \sum_{i=1}^{N-1} m_i)(p - \sum_{i=1}^{N-1} n_i)$. We can compute the conformal dimension h from (2.96) as

$$\begin{aligned} h &= -\frac{1}{2}(\vec{u} - Q\vec{\rho})^2 + \frac{Q^2}{2}\vec{\rho}^{\,2} \\ &= \frac{12(\sum_{i=1}^{N-1}(pm_i - qn_i)\vec{\omega}_i)^2 - N(N^2-1)(p-q)^2}{24pq}, \end{aligned} \tag{2.108}$$

where we use $\vec{\rho}^{\,2} = \frac{N(N^2-1)}{12}$ and $\vec{\rho} = \sum_{i=1}^{N-1} \vec{\omega}_i$.

2.3.2 $\mathcal{N}=2$ unitary minimal models

In Section 2.2.2, we discuss $\mathcal{N}=2$ super Virasoro algebra (2.66) in terms of the quantum Drinfeld-Sokolov reduction. On the other hand, it is also possible to realize it by the coset construction as follows [50]:

$$\frac{\widehat{SU}(2)_k \otimes \mathrm{Ff}}{\widehat{U}(1)}. \tag{2.109}$$

Here, Ff denotes the free fermions $\psi^{\pm}(z)$ with the OPE

$$\psi^{+}(z)\psi^{-}(w) \sim \frac{1}{z-w}, \quad \psi^{\pm}(z)\psi^{\pm}(w) \sim 0. \tag{2.110}$$

They form a U(1) current $J^f(z) =: \psi^+(z)\psi^-(z) :$. The $U(1)$ current in the denominator of (2.109) is given by the sum of $J^3(z)$ in $\widehat{SU}(2)_k$ and $J^f(z)$. One can explicitly construct the currents of $\mathcal{N}=2$ super Virasoro algebra as the elements commuting with this $U(1)$ current. For example, the U(1) current $J(z)$ in $\mathcal{N}=2$ super Virasoro algebra is given by

$$J(z) = \frac{k}{k+2}J^f(z) - \frac{2}{k+2}J^3(z). \tag{2.111}$$

One can construct the representation space from that of $\widehat{SU}(2)_k$ and the free fermion. For the free fermion, we only have to determine the eigenvalue of J_0^f for the highest weight state. The Neveu-Schwarz (NS) sector corresponds to $J_0^f = 0$ while the Ramond (R) sector corresponds to $J_0^f = \pm\frac{1}{2}$. For $\widehat{SU}(2)_k$, we may need to remove null states to obtain the irreducible representation. Especially, one can obtain the unitary representation called integrable representation when $k \in \mathbb{Z}_{\geq 1}$. These modules realize the unitary minimal model. The central charge is given by

$$c = \frac{3k}{k+2} \quad (k \in \mathbb{Z}_{\geq 1}). \tag{2.112}$$

In the following, we focus on the NS sector.

As in the case of the parafermion discussed in (2.73), we have the following branching relation:

$$\mathcal{L}_l \otimes V^{\mathrm{Ff}} = \oplus_{\substack{m\in\mathbb{Z}_{2(k+2)} \\ l-m\in 2\mathbb{Z}}} \mathcal{M}^{\mathrm{sVir}}_{l,m} \otimes V_m^{\widehat{U}(1)}, \qquad V_m^{\widehat{U}(1)} \equiv \oplus_{j\in\mathbb{Z}} \mathcal{M}^{\widehat{U}(1)}_{-m+2j(k+2)}. \tag{2.113}$$

Here, V^{Ff} is the Hilbert space of the free fermion. Because the energy-momentum tensor and the U(1) current are defined as (2.67) and (2.111), the conformal dimension $h_{l,s}$ and the U(1) charge $J_{l,m}$ of the primary fields are given as

$$h_{l,m} = \frac{l(l+2)-m^2}{4(k+2)}, \quad J_{l,m} = \frac{m}{k+2}, \tag{2.114}$$

where $0 \leq l \leq k$, $-l \leq m \leq l$ and $l-m \in 2\mathbb{Z}$. The character was derived in [35] as follows[17]:

$$\chi_{l,m}(\tau,z) := \mathrm{Tr}\, q^{L_0 - \frac{c}{24}} y^{J_0} = \sum_{r\in\mathbb{Z}_{2n}} c^{(n)}_{l,m+2r}(\tau)\Theta_{2m+2r(n+2),2n(n+2)}(\tau, \frac{z}{n+2}). \tag{2.115}$$

[17]Precisely speaking, the character was first derived in [33, 34] by studying the embedding relation of the null states. We note that one can also construct the null states by using the screening charge [49].

3 Y-algebra and its generalization

In this section, we review the Y-algebra, a new family of W-algebras proposed by Gaiotto and Rapcak [24]. In Section 3.1, we review the construction from the gauge theory. We also explain the triality symmetry, the modules and the relation to $W_{1+\infty}$. These properties play an important role in Section 4. In Section 3.2, we discuss the generalization of the Y-algebra studied by Prochazka and Rapcak [27]. We will see that we can construct various W-algebras in this framework.

3.1 Y-algebras

The Y-algebra is constructed from the 4d supersymmetric gauge theory with the 3d Y-shaped domain wall (see Figure 6). It is the W-algebra which appears at the 2d corner of this domain wall. We can find its explicit expression by a step-by-step analysis; the 2d theory at the corner can be specified by first reducing the 4d theory to the 3d theory and then studying its boundary. The final result is given in (3.6).

3.1.1 Construction from the gauge theory

Gaiotto and Rapcak constructed a family of W-algebras from the gauge theory setup which naturally arises from type IIB string theory. In type IIB string theory, there are $(1 + 5)$-dimensional objects labeled by two charges (p, q), where p, q are coprime integers. It is believed that type IIB string theory possesses $SL(2, \mathbb{Z})$ symmetry, under which the charges (p, q) transform as a doublet. In our notation, the (1,0) brane and (0,1) brane correspond to D5-brane and NS5-brane, respectively. As discussed in [51, 52], one can consider a web of 5-branes which preserves 1/4 supersymmetry. The following figures show some examples:

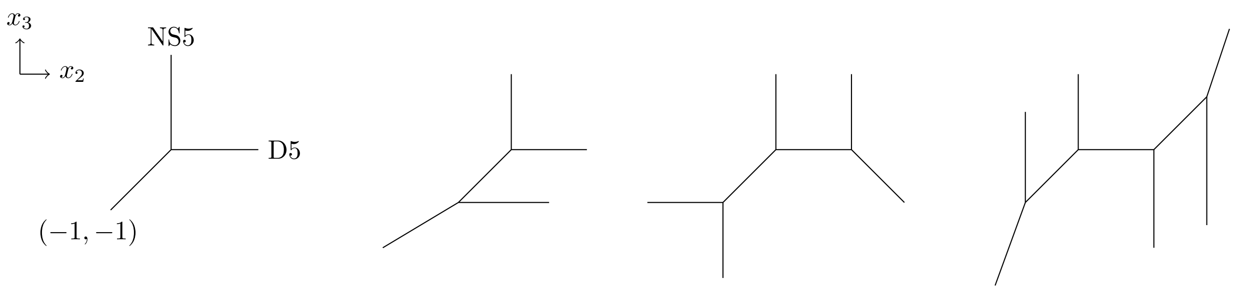
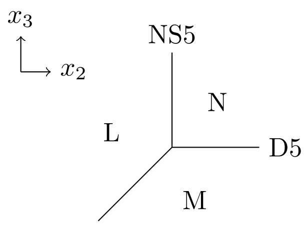


Figure 5: These figures show some examples for (p, q)-webs. The most fundamental one is the trivalent vertex formed by D5-brane, NS5-brane and (-1,-1) brane.

Let us explain how to interpret these figures. We denote by $x_0, x_1, \cdots, x_9$ the coordinate of ten-dimensional spacetime. Let us consider the situation that all of the 5-branes share (1+4)-dimension, for example, x_0, x_1, x_4, x_5, x_6 direction. Then each brane should fill one more dimension. We can encode the information about the configuration in a web-diagram in a two-dimensional plane, for example, (x_2, x_3)-plane. The web-diagram must satisfy two properties. First, the charge conservation must be satisfied at every junction. That means that each trivalent vertex consists of three (p_i, q_i)-branes $(i = 1, 2, 3)$ satisfying $\sum_{i=1}^{3} p_i = 0$ and $\sum_{i=1}^{3} q_i = 0$. Second, each brane must have an appropriate slope to keep a part of supersymmetry. For simplicity, we draw the (p, q)-fivebrane by the line with slope q/p. The diagrams drawn in this way are called (p, q)-web. The most fundamental diagram is the left one in Figure 5. We can obtain any other trivalent vertex via $SL(2, \mathbb{Z})$ transformation and any other (p, q)-web by gluing trivalent vertices.

One can also add (1+3)-dimensional objects called D3-branes to (p, q)-web, keeping a part of supersymmetry. In [24], Gaiotto and Rapcak considered the system consisting of the trivalent vertex and several numbers of D3-branes as follows:



	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D5	—	—	—		—	—	—			
NS5	—	—		—	—	—	—			
D3	—	—	—	—						

Figure 6: The trivalent vertex consisting of D5-brane, NS5-brane and (-1,-1)-fivebrane divides (x_2, x_3)-plane into the three regions, where L, M, N D3-branes are inserted, respectively. In the right table, the directions along which the branes extend are denoted by bars.

Here, D3-branes are inserted in each region separated by the 5-branes. They extend to both x_2 direction and x_3 direction. The spacetime in which the branes live is summarized in the above table. As is well known, the effective theory on the D3-brane is described by $\mathcal{N} = 4$ super Yang-Mills theory (SYM). In the low-energy limit, the above system is described by the four-dimensional $\mathcal{N} = 4$ SYMs whose gauge groups are U(L), U(M) and U(N), respectively. These three gauge theories are separated by three-dimensional interfaces. Our concern is the two-dimensional theory living at the corner in the (x_2, x_3)-plane.

In principle, the two-dimensional theory at the corner can be determined by carefully analyzing the boundary condition, but it is much simpler to consider the topologically twisted $\mathcal{N} = 4$ SYM [53]. The advantage of considering the topologically twisted $\mathcal{N} = 4$ SYM is that we can use the result of [54]. It was discussed that the path integral of Chern-Simons (CS) theory on the three-dimensional manifold M_3 can be interpreted as that of the topologically twisted $\mathcal{N} = 4$ SYM on $M_3 \times \mathbb{R}^+$ with a certain boundary condition. One can also think of it as the result of the fact that the degree of freedom in the topological field theory localizes to its boundary. In [24], this result was applied to Figure 6, and M_3 was identified with the intersection between D3-brane and $(q, 1)$-brane. The relation between the coupling constant Ψ of SYM and the level k of the CS theory is given as

$$\Psi + q = k + h^\vee, \tag{3.1}$$

where $h^\vee$ is the dual Coxeter number of the gauge group.

From the above discussion, the system reduces to the CS theories, which meet at the two-dimensional corner. As is well known, a two-dimensional CFT lives on the boundary of the CS theory. The W-algebra realized at the corner depends on the boundary condition. If we fix the form of the gauge field as in Figure 4 at the boundary, the W-algebra is given by the corresponding DS reduction. Let us first consider the simple case, $L = M = 0$. In this case, we have the CS theory on $M_3 = \mathbb{R}^2 \times \mathbb{R}^+$, where $\mathbb{R}^2$ corresponds to (x_0, x_1)-plane while $\mathbb{R}^+$ to $x_3 \geq 0$. This CS theory is mapped to the topologically twisted $\mathcal{N} = 4$ SYM on $M_3 \times \mathbb{R}^+$, where $\mathbb{R}^+$ corresponds to $x_2 \geq 0$ (see the following figure):

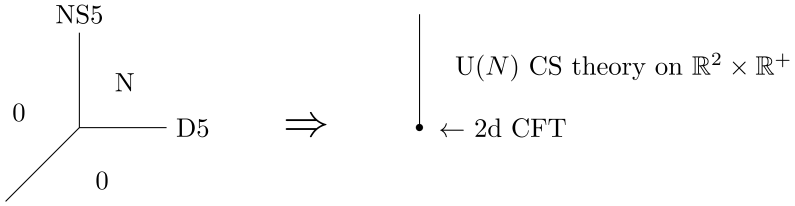
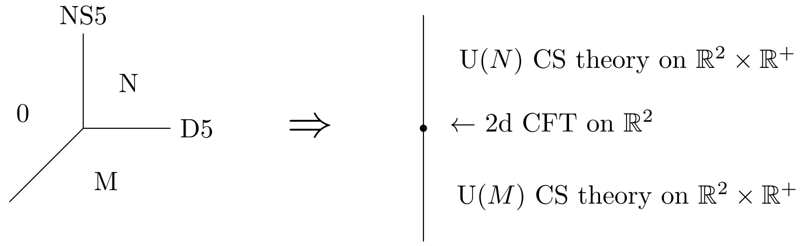


Figure 7: Left: topologically twisted $\mathcal{N} = 4$ U(N) SYM on $\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+$. Right: U($N$) CS theory on $\mathbb{R}^2 \times \mathbb{R}^+$.

The boundary condition at $\mathbb{R}^+(x_2 \geq 0)$ is in principle determined by that of the CS theory, and we want to set it so that it will correspond to the D5-boundary. We can guess it from S-duality of string theory. Because NS5-brane and D5-brane are related by $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z})$, the 2d CFT living at the corner should be invariant under this transformation and the flip of the diagram. Because it changes Ψ to Ψ^{-1}, the W-algebra realized at the corner should possess duality which exchanges Ψ and Ψ^{-1}. That is satisfied if we chose the principal embedding because the corresponding W-algebra $DS_N[\widehat{U}(N)_\Psi] = W_N \times \widehat{U}(1)$[18] possesses such a duality (2.94).

Next, let us consider the case of $L = 0$. It is enough to consider the case of $N \geq M$. In this case, there are two CS theories as follows:

[18]We fix the normalization for the level of $\widehat{U}(N)$ so that the level of its subalgebra $\widehat{SU}(N)$ will become $\Psi - N$.



It is known that the $U(N) \times U(M)$ gauge symmetry in the bulk is broken to the diagonal part of $U(M) \times U(M)$ by the D5-brane. Then the gauge field of the U(N) CS theory is decomposed at the corner into $(N-M) \times (N-M)$ block and $M \times M$ block. Namely, the $(N-M) \times (N-M)$ part of the gauge symmetry is broken at the corner. Thus, it gives $DS_{N-M}[\widehat{U}(N)_\Psi]$ current[19] at the corner. On the other hand, U(M) CS theory gives $\widehat{U}(M)_{1-\Psi}$ at the corner. The W-algebra realized at the corner is obtained after fixing the redundancy of the remaining U(M) gauge symmetry by the BRST procedure. We note that the level of the total current $\widehat{U}(M)$ must be zero because otherwise it would be anomalous. If the level of $\widehat{U}(M)$ in $DS_{N-M}[\widehat{U}(N)_\Psi] \otimes \widehat{U}(M)_{1-\Psi} \otimes \mathrm{bc}^{U(M)}$ is not zero, that must be canceled out by some matter fields living at the corner. Recalling that the level of $\widehat{SU}(M)$ in $\widehat{U}(M)_\Psi$ is $\Psi - M$ in our normalization, we see that we need to add $\widehat{U}(1) \otimes \widehat{SU}(N)_{-1}$ if and only if $N = M$. The matter field which realizes this current is given by $U(N)$ symplectic bosons $\mathrm{Sb}^{U(N)}$ which consist of N pairs of a symplectic boson $X^a(z), Y_b(z)$ $(a, b = 1, 2, \cdots, N)$ with the following OPEs:

$$X^a(z) Y_b(w) \sim \frac{\delta^a_b}{z-w}, \quad X^a(z) X^b(w) \sim 0, \quad Y_a(z) Y_b(w) \sim 0. \tag{3.2}$$

The $\widehat{U}(N)$ current is given by $J^a_b = X^a(z) Y_b(z)$. Summarizing, the W-algebra at the corner is given by the BRST reduction of

$$\begin{cases} \widehat{U}(N)_\Psi \otimes \widehat{U}(N)_{1-\Psi} \otimes \mathrm{Sb}^{U(N)} \otimes \mathrm{bc}^{U(N)} & (N = M) \\ DS_{N-M}[\widehat{U}(N)_\Psi] \otimes \widehat{U}(M)_{1-\Psi} \otimes \mathrm{bc}^{U(M)} & (N > M) \\ DS_{M-N}[\widehat{U}(M)_{1-\Psi}] \otimes \widehat{U}(N)_\Psi \otimes \mathrm{bc}^{U(N)} & (N < M). \end{cases} \tag{3.3}$$

We can do a similar analysis in the case of $L \neq 0$. The difference from the previous case is that D3-branes are inserted in both sides of $(q, 1)$-brane. Then we only have to replace the gauge group $U(N), U(M)$ with the super Lie group $U(N|L), U(M|L)$, respectively [55]. For the matter fields, we need to add $U(L)$ symplectic fermion $\mathrm{Sf}^{U(L)}$ as well as $\mathrm{Sb}^{U(N)}$, where $\mathrm{Sf}^{U(L)}$ consists of L pairs of a symplectic fermion $\psi^a(z), \chi^b(z)$ $(a, b = 1, 2, \cdots, L)$ with the following OPEs:

$$\psi^a(z) \chi_b(w) \sim \frac{\delta^a_b}{z-w}. \tag{3.4}$$

[19] See the comment below Figure 4 for the notation.

We denote by $\mathrm{Sb}^{U(N|L)}$ the combination of $\mathrm{Sb}^{U(N)}$ and $\mathrm{Sf}^{U(L)}$. Then we can generalize (3.3) to generic cases as

$$\begin{cases} \widehat{U}(N|L)_\Psi \otimes \widehat{U}(N|L)_{1-\Psi} \otimes \mathrm{Sb}^{U(N|L)} \otimes \mathrm{bc}^{U(N|L)} & (N = M) \\ DS_{N-M}[\widehat{U}(N|L)_\Psi] \otimes \widehat{U}(M|L)_{1-\Psi} \otimes \mathrm{bc}^{U(M|L)} & (N > M) \\ DS_{M-N}[\widehat{U}(M|L)_{1-\Psi}] \otimes \widehat{U}(N|L)_\Psi \otimes \mathrm{bc}^{U(N|L)} & (N < M), \end{cases} \tag{3.5}$$

where the off-diagonal part of the ghost fields $\mathrm{bc}^{U(N|L)}$ should be interpreted as $\beta\gamma$ ghost. We note that the relation between the parameters is given by (3.1), where $h^\vee = N - L$ for $U(N|L)$. By using (2.80), we can rewrite (3.5) into a simpler form as

$$Y_{L,M,N}[\Psi] = \begin{cases} \dfrac{\widehat{U}(N|L)_\Psi \otimes \mathrm{Sb}^{U(N|L)}}{\widehat{U}(N|L)_{\Psi-1}} & (N = M) \\ \dfrac{DS_{N-M}[\widehat{U}(N|L)_\Psi]}{\widehat{U}(M|L)_{\Psi-1}} & (N > M) \\ \dfrac{DS_{M-N}[\widehat{U}(M|L)_{-\Psi+1}]}{\widehat{U}(N|L)_{-\Psi}} & (N < M). \end{cases} \tag{3.6}$$

This is the final result for the W-algebra realized at the corner.

Let us see some examples. According to the above definition, we have $Y_{0,0,N}[\Psi] = DS_N[\widehat{U}(N)_\Psi]$. As we explained in Section 2.2.2, it gives W_N algebra with an extra U(1) boson. The next example is $Y_{0,1,2}[\Psi] = \frac{\widehat{U}(2)_\Psi}{\widehat{U}(1)}$. As we have seen in 2.2.3, it gives SU(2) parafermion with an extra U(1) boson. In this way, one can obtain various W-algebras by changing the value of L, M, N.

3.1.2 Triality

The Y-algebra possesses a nontrivial automorphism inherited from $SL(2,\mathbb{Z})$ symmetry of type IIB string theory. While the D3-branes are invariant under $SL(2,\mathbb{Z})$, 5-branes and the coupling constant Ψ are transformed nontrivially. We will give the detail in Section 3.2. Here, we describe the main point. For some $SL(2,\mathbb{Z})$, 5-branes are exchanged with each other and the corner configuration is kept invariant. That implies that the Y-algebra has S_3 automorphism under which L, M, N are permuted. This automorphism is called triality.

Let us consider $Y_{0,0,N}[\Psi]$ as an example. The triality implies the following relation:

$$Y_{0,0,N}[\Psi] = Y_{0,N,0}[1 - \Psi^{-1}] = Y_{N,0,0}[\frac{1}{1-\Psi}]. \tag{3.7}$$

According to (3.6), this relation can be rewritten as

$$DS_N[\widehat{U}(N)_\Psi] = DS_N[\widehat{U}(N)_{\Psi^{-1}}] = \frac{\widehat{U}(N)_{-\Psi} \otimes \mathrm{Sf}^{U(N)}}{\widehat{U}(N)_{1-\Psi}}. \tag{3.8}$$

The first equation is exactly Feigin-Frenkel duality. The second equation is more interesting. The last one is indeed known as the coset realization of W_N algebra[20]. In this way, the triality provides us nontrivial relations among different realizations of W-algebras.

3.1.3 Modules of the Y-algebra

Let us discuss the module of the Y-algebra. It is realized by the local operators at the corner. The characteristic ones are those induced from the line operators in the 3d interfaces. Concretely, we consider the line operator such that one of its endpoints attaches to the corner while the other one extends to the infinity as follows:

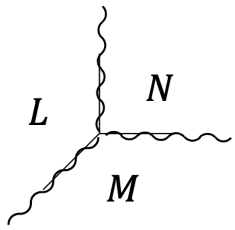


Figure 8: Line operators are inserted in the three-dimensional interfaces.

It is characterized by the weight of the gauge group. As an example, let us consider $Y_{0,0,N}[\Psi] = W_N \times \widehat{U}(1)$. In this case, we have two 3d interfaces and the module is labeled by two U(N) weights. We have already seen such modules in Section 2.3.1. They are expected to give the maximally degenerate modules. For general cases, we can insert three line operators and they are expected to give the maximally degenerate modules for $Y_{L,M,N}[\Psi]$. We will give a more detailed explanation in Section 4.

3.1.4 The relation to $W_{1+\infty}$ and a plane partition

As we have seen, the Y-algebra realizes various W-algebras. Then it is natural to expect that there is a unified description of these algebras. The vacuum character gives crucial information for that[21]. In [24], it was conjectured that the vacuum character of $Y_{L,M,N}$ matches the generating function of the plane partition with a pit at $(L+1, M+1, N+1)$[22]. Here, we mean by a "pit" the place in which we cannot place a box. This claim was checked for several cases. The observation suggests that the Y-algebra can be seen as a truncation of $W_{1+\infty}$; recall that $W_{1+\infty}$ consists of the currents whose conformal dimensions are $1, 2, 3, 4, \cdots, \infty$ and its vacuum character coincides with the MacMahon function (=

[20] This relation was proved in [56]. Further, the triality has been proved in [57] when one of L, M, N is zero.

[21] In this section, we do not include the factor $q^{-\frac{c}{24}}$ in the character.

[22] A plane partition is a 3d version of Young diagram. See the left of Figure 9 or Introduction.

generating function of a plane partition)

$$\chi(q) = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + \cdots . \tag{3.9}$$

For the explanation of a plane partition, see Figure 1. One of the simplest examples is $Y_{0,0,N}$. It is well known that the generating function of the plane partition with height at most N matches with the vacuum character of $W_N \times \widehat{U}(1)$. In the following, we consider $Y_{0,N,N}$ as a nontrivial example[23]. The algebra is given by the BRST reduction of

$$\widehat{U}(N)_{\Psi} \times \widehat{U}(N)_{1-\Psi} \otimes \mathrm{Sb}^{U(N)} \times \mathrm{bc}^{U(N)}. \tag{3.10}$$

Its character is obtained by taking the $U(N)$-invariant part of (3.10). Because $\widehat{U}(N) \times \widehat{U}(N)$ cancels out $\mathrm{bc}^{U(N)}$ except the zero modes of c, the character is given by

$$\chi_{0,N,N}(q) = \oint \prod_{i=1}^{N} \frac{dx_i}{x_i} \frac{\prod_{i\neq j}(1-\frac{x_j}{x_i})}{\prod_{n=0}^{\infty}\prod_{i=1}^{N}(1-x_i q^{n+\frac{1}{2}})}. \tag{3.11}$$

Here, the denominator comes from the symplectic boson while the numerator comes from c_0. We note that a Vandermonde determinant gives a measure of $U(N)$. Because the orthogonal basis under this measure are given by the character of $U(N)$ representation (= Schur polynomial), it is convenient to expand the integrand as

$$\frac{1}{\prod_{i=1}^{N}\prod_{n=0}^{\infty}(1-x_i q^{n+\frac{1}{2}})} = \sum_{\lambda} d_{\lambda}(q)\chi_{\lambda}(x_i), \tag{3.12}$$

where $\chi_\lambda(x_i)$ is the character of the $U(N)$ representation labeled by the Young diagram $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$. Then (3.11) becomes $\chi_{0,N,N}(q) = \sum_\lambda d_\lambda(q) d_\lambda(q)$.

One can give a combinatorial interpretation to $d_\lambda(q)$ by using the Pieri rule

$$\chi_\lambda(x_i)\,\frac{1}{\prod_{i=1}^{N}(1-x_i t)} = \sum_{\mu\succ\lambda} \chi_\mu(x_i) t^{|\mu|-|\lambda|}, \tag{3.13}$$

where $\mu \succ \lambda$ means

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_N \geq \lambda_N. \tag{3.14}$$

By applying (3.13) repeatedly to the LHS of (3.12), we see

$$\begin{aligned} d_{\lambda_0}(q) &= \sum_{\lambda_0\succ\lambda_1\succ\lambda_2\succ\lambda_3\succ\cdots} q^{\frac{1}{2}\left(|\lambda_0|-|\lambda_1|\right)+\frac{3}{2}\left(|\lambda_1|-|\lambda_2|\right)+\frac{5}{2}\left(|\lambda_2|-|\lambda_3|\right)+\cdots} \\ &= \sum_{\lambda_0\succ\lambda_1\succ\lambda_2\succ\lambda_3\succ\cdots} q^{\frac{1}{2}|\lambda_0|+|\lambda_1|+|\lambda_2|+|\lambda_3|+\cdots}. \end{aligned} \tag{3.15}$$

One can interpret it in terms of a plane partition as in Figure 9,:

[23] We consider the plane partition with a pit at $(N+1, N+1, 1)$, not $(1, N+1, N+1)$. This is merely because that is convenient to draw a figure.

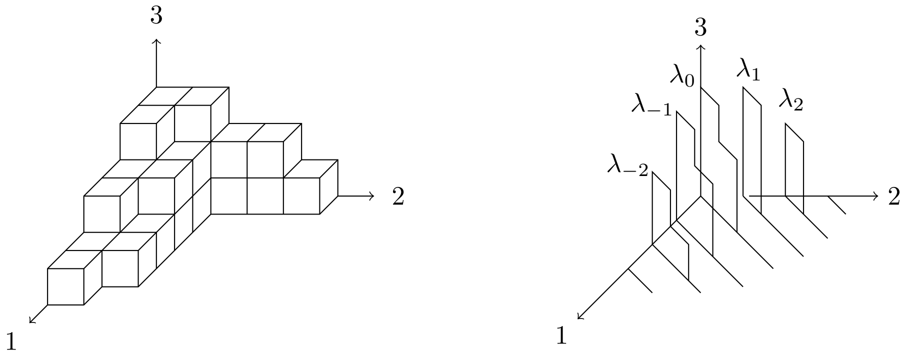


Figure 9: Left: An example of a plane partition. Right: Diagonal slicing of a plane partition. It satisfies $\lambda_0 \succ \lambda_1 \succ \lambda_2 \succ \cdots$ and $\lambda_0 \succ \lambda_{-1} \succ \lambda_{-2} \succ \cdots$.

From the right of Figure 9, we can interpret $\sum_\lambda d_\lambda(q) d_\lambda(q)$ as the generating function of the plane partition with a pit at $(N+1, N+1, 1)$. Thus, we see that the Hilbert space of $Y_{0,N,N}$ is indeed described by a plane partition with a pit at $(N+1, N+1, 1)$. The relation to $W_{1+\infty}$ plays an important role in Section 4. We note that the triality is manifestly realized in a plane partition because it can be interpreted just as the permutation of three axes.

3.2 Generalization of the Y-algebra

One of the important features in Gaiotto-Rapcak's construction is that one can obtain infinitely many W-algebras from (p, q)-webs [27]. In Figure 5, the internal lines with finite length shrink in the low-energy limit and the web-diagram reduces to a single junction. The W-algebra associated with a (p, q)-web is formed by the local operators living there. Because any (p, q)-webs can be constructed by gluing trivalent vertices, one can obtain the associated W-algebras as the extension of the tensor product of several Y-algebras. In other words, we can obtain various W-algebras by using the Y-algebra as a building block.

We have considered the trivalent vertex which consists of D5-brane, NS5-brane and $(-1, -1)$-brane so far, but it is necessary to consider the other trivalent vertices to discuss (p, q)-webs. We can obtain them by applying $SL(2, \mathbb{Z})$ transformation to the original vertex. Under the transformation, the charge (a, b) of the fivebrane transforms as

$$\begin{pmatrix} a \\ b \end{pmatrix} \to X \begin{pmatrix} a \\ b \end{pmatrix}, \tag{3.16}$$

where X is $SL(2, \mathbb{Z})$ matrix,

$$X = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad ps - qr = 1. \tag{3.17}$$

At the same time, the coupling constant Ψ transforms as

$$\Psi \to \frac{p\Psi + q}{r\Psi + s}. \tag{3.18}$$

The D3-branes are invariant under the transformation. Let $Y^{A_1,A_2,A_3}_{L,M,N}[\Psi]$ be the Y-algebra associated with the junction formed by $A_i = (p_i, q_i)$-fivebranes ($i = 1, 2, 3$). From the above discussion, we have

$$Y^{A_1,A_2,A_3}_{L,M,N}[\Psi] = Y_{L,M,N}\left[\frac{q_2\Psi - p_2}{-q_1\Psi + p_1}\right]. \tag{3.19}$$

Now we are ready to consider the W-algebras associated with various (p, q)-webs. In the following, we refer to them as generalized Y-algebras. Let us see some examples. We first consider the following diagram:

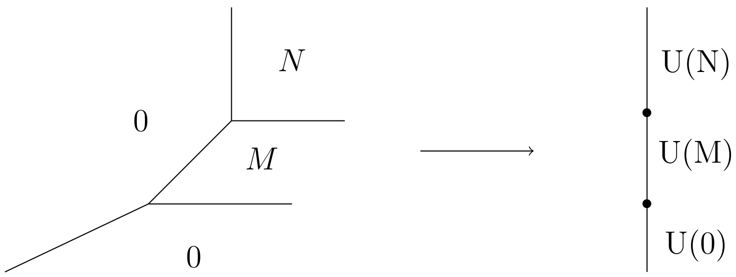


Figure 10: We focus on the case of $N > M$. The right figure shows the reduction to the Chern-Simons theories. The gauge group of the top CS theory is U(N), which is broken to U(M) in the middle line and completely broken at the bottom.

In the low-energy limit, the diagram reduces to a single junction, and our interest is in the W-algebra formed by the local operators there. It is obvious that the Y-algebra at each corner is the component of the W-algebra. There are also additional elements. Because we consider the low-energy limit where the internal line shrinks, it is natural to expect that the line operators connecting the two corners also behave as the local operators. Thus, it is considered that the W-algebra associated with Figure 10 consists of the two Y-algebras and the line operators. One can read off the parameters of the Y-algebras from (3.19) as $Y_{0,1,2}[\Psi]$ and $Y_{0,0,1}[\Psi - 1]$. The line operators are interpreted as the modules for both $Y_{0,1,2}[\Psi]$ and $Y_{0,0,1}[\Psi - 1]$, as we have seen in 3.1.3. They are characterized by the weight of gauge group $U(M)$ and we need to sum up all representations. The process that several W-algebras and their modules lift to a larger W-algebra is called conformal extension or module extension. In general, it is a highly nontrivial process. Therefore, the above brane construction provides us a powerful tool to construct various W-algebras.

One can also define the above W-algebra by the BRST reduction as in the case of the Y-algebra. Because there are two corners, the reduction is implemented twice. The gauge group is broken from $U(N)$ to $U(M)$ at the upper corner and completely broken at the lower corner. Therefore, the W-algebra under consideration is given by

$$DS_M[DS_{N-M}[\widehat{U}(N)_\Psi]]. \tag{3.20}$$

Here, DS_{N-M} means the DS reduction with SU(2) embedding in $(N-M)\times(N-M)$ block while DS_M means the DS reduction with SU(2) embedding in $M\times M$ block. The simplest example is when $N=2, M=1$. In this case, we have affine Kac-Moody algebra $\widehat{U}(2)$.

It is nontrivial that the above two descriptions give the same W-algebra. In [27], several consistency checks were provided. For example, it was checked that the conformal dimension of the module realized by the line operator is always integer or half-integer. This fact supports the claim that the two Y-algebras and their modules form a new W-algebra consistently. As an example, let us consider $\widehat{U}(2)$. In this case, the conformal dimension of the module with fundamental weight or anti-fundamental weight is 1. They can be interpreted as the current $J^{\pm}(z)$ of $\widehat{U}(2)$. In Section 4, we explain how to compute the conformal dimension of these modules in detail.

Next, let us consider the following diagram:

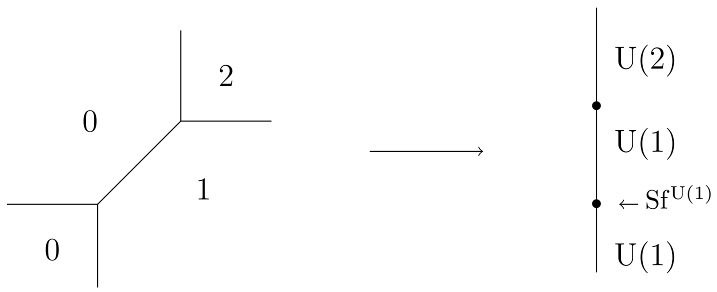


Figure 11: The left figure shows the web-diagram associated with $\mathcal{N}=2$ super Virasoro $\otimes$ U(1) boson. The right figure shows the reduction to the Chern-Simons theories. The U(1) symplectic fermion $\mathrm{Sf}^{\mathrm{U}(1)}$ lives at the lower junction.

The corresponding W-algebra is given by

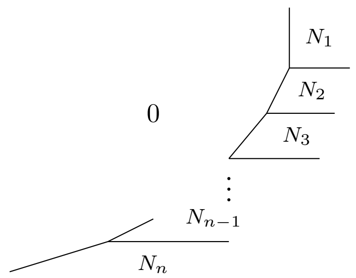
$$\frac{\widehat{U}(2)_{\Psi}\times \mathrm{Ff}}{\widehat{U}(1)_{\Psi}}, \tag{3.21}$$

which realizes $\mathcal{N}=2$ super Virasoro algebra and an extra $\widehat{U}(1)$ as we have seen in Section 2.3.2. In terms of the Y-algebras, it is realized as the conformal extension of

$$Y_{0,1,2}[\Psi]\otimes Y_{0,0,1}[1-\frac{1}{\Psi}]. \tag{3.22}$$

Here, we mention why it is convenient to express the algebra (3.21) in terms of the Y-algebras. As we have seen in Section 3.1.4, we can describe the Hilbert space of the Y-algebra by a plane partition. Therefore, we can describe the Hilbert space of (3.22) by two plane partitions. We also need to take into account the elements coming from the line operators. As we will explain in Section 4.5, they can also be interpreted in terms of plane partitions. Thus, we can describe the total Hilbert space by using two plane partitions. The same discussion can be applied to any (p,q)-webs. Therefore, we can describe the representation theory of any generalized Y-algebras by using plane partitions. We will discuss the plane partition representation in detail in the next section.

Finally, let us consider the following diagram.



When $N_1 > N_2 > N_3 \cdots > N_{n-1} > N_n$, this diagram realizes

$$\frac{Ds_{N_{n-1}-N_n}[DS_{N_{n-2}-N_{n-1}}[\cdots DS_{N_1-N_2}[\widehat{U}(N_1)_\Psi]]]}{\widehat{U}(N_n)_{\Psi-n+1}}. \tag{3.23}$$

Especially, when $N_n = 0$, it provides all the W-algebras obtained from the DS reduction of $\widehat{U}(N)_\Psi$. On the other hand, we can also see it as the conformal extension of

$$Y_{0,N_2,N_1}[\Psi] \otimes Y_{0,N_3,N_2}[\Psi - 1] \otimes \cdots \otimes Y_{0,N_n,N_{n-1}}[\Psi - n + 2]. \tag{3.24}$$

In this way, the generalized Y-algebra realizes various W-algebras. Before ending this section, we make a technical comment. The W-algebra (3.23) is constructed from not only (3.24) but the line operators connecting two neighboring corners. They are characterized by the weight of gauge group $U(N_2)$, $U(N_3), \cdots U(N_{n-1})$, respectively. Let us recall that the weight of $U(N)$ is labeled by the Young diagram $\mu = (\mu_1, \mu_2, \cdots, \mu_N)$. As was claimed in [27], we have to sum up all the Young diagrams $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots \geq \mu_N$. In the following, we refer to them as intermediate Young diagrams. Here, we note that the length of the row is not necessarily nonnegative. In other words, we have to sum up negative Young diagrams. It may seem unimportant, but it causes a subtle problem when we try to interpret the above construction in terms of a plane partition. We will discuss this point in Section 6.1.

4 Affine Yangian of $\mathfrak{gl}_1$

In this section, we review the affine Yangian of $\mathfrak{gl}_1$, mainly based on [11]. We start from a description of its definition in Section 4.1. It is generated by three currents $e(u), f(u), \psi(u)$ and centers. Because these currents are completely different from those of 2d CFTs, the affine Yangian is seemingly unrelated to W-algebras at first sight. However, it is actually isomorphic to $W_{1+\infty}$, as we will see in Section 4.2. In Section 4.3, we show the explicit relation between the generators of the affine Yangian and U(1) and Virasoro subalgebras of $W_{1+\infty}$. In Section 4.4, we discuss the representation theory. One of the advantages of the affine Yangian is that the defining relation between the generating currents can be explicitly written. That allows us to construct an explicit representation theory as opposed to the conventional technique to deal with W-algebras. The representation which is of particular interest in this thesis is a plane partition representation. It provides us a new way to study the representation theory of W-algebras. In Section 4.5, we consider the truncation of the affine Yangian and its connection to the Y-algebra. As we mentioned in Section 3.1.4, the Hilbert space of the Y-algebra is described by a plane partition with a pit. We discuss such a plane partition in terms of the affine Yangian and show the relation between the affine Yangian and the Y-algebra.

4.1 Definition

The affine Yangian of $\mathfrak{gl}_1$ is generated by the following elements,

$$e_i, \ f_i, \ \psi_i \quad (i \geq 0). \tag{4.1}$$

The algebra has two parameters, but it is convenient to introduce three parameters $h_1, h_2, h_3 \in \mathbb{C}$ with a constraint,

$$h_1 + h_2 + h_3 = 0. \tag{4.2}$$

We also introduce the parameters

$$\sigma_2 = h_1 h_2 + h_2 h_3 + h_3 h_1, \qquad \sigma_3 = h_1 h_2 h_3. \tag{4.3}$$

The defining relations are given as follows:

$$[\psi_i, \psi_j] = 0, \tag{4.4}$$

$$\begin{aligned}&[e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}]\\ &\qquad +\sigma_2[e_{i+1}, e_j] - \sigma_2[e_i, e_{j+1}] - \sigma_3\{e_i, e_j\} = 0,\end{aligned} \tag{4.5}$$

$$\begin{aligned}&[f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}]\\ &\qquad +\sigma_2[f_{i+1}, f_j] - \sigma_2[f_i, f_{j+1}] + \sigma_3\{f_i, f_j\} = 0,\end{aligned} \tag{4.6}$$

$$\begin{aligned}&[\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}]\\ &\qquad +\sigma_2[\psi_{i+1}, e_j] - \sigma_2[\psi_i, e_{j+1}] - \sigma_3\{\psi_i, e_j\} = 0,\end{aligned} \tag{4.7}$$

$$\begin{aligned}&[\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}]\\ &\qquad +\sigma_2[\psi_{i+1}, f_j] - \sigma_2[\psi_i, f_{j+1}] + \sigma_3\{\psi_i, f_j\} = 0,\end{aligned} \tag{4.8}$$

$$[e_i, f_j] = \psi_{i+j}, \tag{4.9}$$

$$[\psi_0, e_j] = [\psi_1, e_j] = [\psi_0, f_j] = [\psi_1, f_j] = 0, \quad [\psi_2, e_j] = 2e_j, \quad [\psi_2, f_j] = -2f_j. \tag{4.10}$$

where $i, j \geq 0$ and $\{x, y\} \equiv xy + yx$. The Serre relations are also imposed, but we omit it because we do not use it. By introducing the Drinfeld currents

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \qquad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \qquad \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}, \tag{4.11}$$

the above relations except (4.10) can be rewritten as follows:

$$[\psi(u), \psi(v)] = 0, \tag{4.12}$$

$$\phi(u-v)e(u)e(v) + \phi(v-u)e(v)e(u) \sim 0, \tag{4.13}$$

$$\phi(v-u)f(u)f(v) + \phi(u-v)f(v)f(u) \sim 0, \tag{4.14}$$

$$\phi(u-v)\psi(u)e(v) + \phi(v-u)e(v)\psi(u) \sim 0, \tag{4.15}$$

$$\phi(v-u)\psi(u)f(v) + \phi(u-v)f(v)\psi(u) \sim 0, \tag{4.16}$$

$$[e(u), f(v)] = -\frac{1}{\sigma_3}\frac{\psi(u) - \psi(v)}{u - v}, \tag{4.17}$$

where

$$\phi(u) = \prod_{i=1}^{3}(u - h_i). \tag{4.18}$$

Here, the notation "$\sim$" implies that the both hand sides are equal up to the regular terms at $u = 0$ or $v = 0$. One can see from the above relations that ψ_0 commutes with all of the other elements. Such an element is called center.

The affine Yangian of $\mathfrak{gl}_1$ possesses several automorphisms. One can see that the above defining relations are invariant under the following transformation,

$$h_i \to \alpha h_i, \quad u \to \alpha u, \quad e(u) \to \alpha^{-2}e(u), \quad f(u) \to \alpha^{-2}f(u), \quad \psi(u) \to \psi(u), \tag{4.19}$$

where α is an arbitrary parameter. In terms of the modes, it is rewritten as

$$e_j \to \alpha^{j-1} e_j, \quad f_j \to \alpha^{j-1} f_j, \quad \psi_j \to \alpha^{j-2} \psi_j \qquad (j \geq 0). \tag{4.20}$$

Due to this automorphism, the parameters h_i and the center ψ_0 do not have meanings by themselves. It is convenient to introduce the parameters λ_i $(i = 1, 2, 3)$ which are invariant under the automorphism as follows:

$$\lambda_i = -\frac{\psi_0 \sigma_3}{h_i}. \tag{4.21}$$

Because of the constraint (4.2), they are not independent but constrained by the condition,

$$\sum_{i=1}^{3} \lambda_i^{-1} = 0. \tag{4.22}$$

The second automorphism is called spectral shift and given by

$$e(u) \to e(u-q), \quad f(u) \to f(u-q), \quad \psi(u) \to \psi(u-q), \tag{4.23}$$

where q is an arbitrary parameter. In terms of the modes, it can be rewritten as

$$e_j \to \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} q^{j-k} e_k, \quad f_j \to \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} q^{j-k} f_k, \quad \psi_j \to \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} q^{j-k} \psi_k. \tag{4.24}$$

Finally, we mention that the algebra is obviously symmetric under the permutation of h_1, h_2, h_3. We will see that it is related to the triality of Y-algebra.

4.2 The relation to $W_{1+\infty}$

The relation between affine Yangian and $W_{1+\infty}$ can be presented in the following figure:

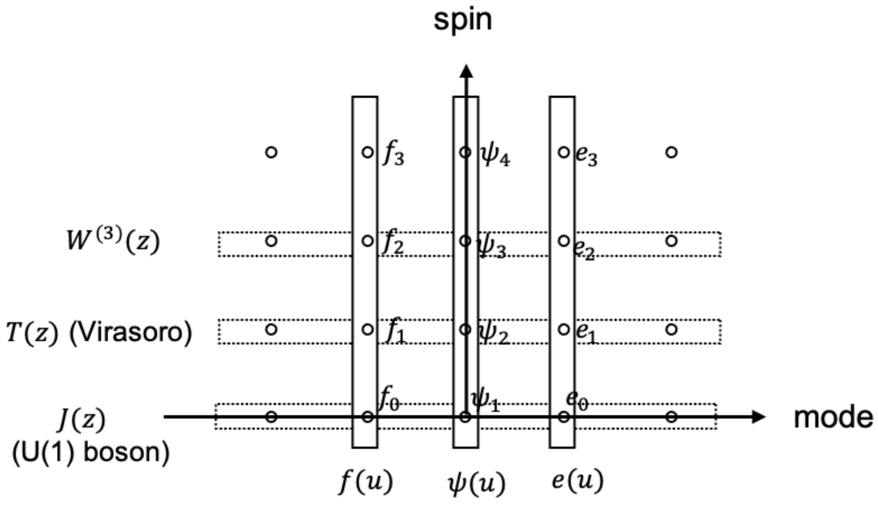


Figure 12: A rough sketch of the relation between the affine Yangian and $W_{1+\infty}$. The elements of the algebra are depicted in the two dimensional lattice. The concrete relation for U(1) boson and Virasoro algebra is given in the next section.

Before explaining the relation, we mention the linear $W_{1+\infty}$ [58]. Let us recall that the Virasoro algebra is obtained as the central extension of the differential operator $l_n = -z^n D_z$, where we set $D_z = z\partial_z$. As its generalization, we can consider the extended algebra formed by all differential operators $z^n D^m$ ($n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}$). One can see that these differential operators are closed under the commutator as follows:

$$[z^n f(D_z), z^m g(D_z)] = z^{n+m} f(D_z + m) g(D_z) - z^{n+m} f(D_z) g(D_z + n), \tag{4.25}$$

where f and g are arbitrary polynomials. The linear $W_{1+\infty}$ algebra is obtained as the central extension of (4.25). Let us denote by $W(z^n f(D_z))$ the corresponding element. Then the commutation relation after the central extension is given by

$$\begin{aligned} &[W(z^n f(D_z)), W(z^m g(D_z))] \\ &= W\big(z^{n+m} f(D_z + m) g(D_z)\big) - W\big(z^{n+m} f(D_z) g(D_z + n)\big) + C\Psi(z^n f(D_z), z^m g(D_z)), \end{aligned} \tag{4.26}$$

where the extension term Ψ is fixed by Jacobi-identity as follows:

$$\begin{aligned} &\Psi(z^n f(D_z), z^m g(D_z)) \\ &= \delta_{n+m,0} \left(\theta(n > 0) \sum_{j=1}^{n} f(-j) g(n-j) - \theta(m > 0) \sum_{j=1}^{m} f(m-j) g(-j) \right). \end{aligned} \tag{4.27}$$

The representation theory of $W_{1+\infty}$ depends on the value of C. It was found that it realizes W_N algebra (and an extra U(1) boson) when $C = N$. However, it has become clear that it cannot realize W_N algebra for generic central charges. Therefore, we need to add one more parameter to $W_{1+\infty}$ in order to realize W_N algebra with generic central charges. The affine Yangian gives such an algebra.

To see that the affine Yangian is one-parameter deformation of the linear $W_{1+\infty}$, let us consider the case of $h_3 = 0$. For the other two parameters, we can fix them as $h_1 = -h_2 = 1$ by using (4.2) and (4.19). In this case, the elements of $Y(\widehat{\mathfrak{gl}_1})$ can be realized by the differential operators as follows [11]:

$$e_i \to (-1)^i W(z^{-1} D_z^i), \quad f_i \to (-1)^{i+1} W(D_z^i z), \quad \psi_i \to (-1)^{i+1} W\big((D_z+1)^i - D_z^i\big). \tag{4.28}$$

Thus, the affine Yangian of $\mathfrak{gl}_1$ degenerates to the linear $W_{1+\infty}$ if we set $h_3 = 0$. We note that the currents $e(u), f(u), \psi(u)$ are totally different from the currents in CFT. For example, the current $\psi(u)$ involves only the zero modes of the CFT currents. On the other hand, it contains all the zero-modes of the currents with conformal dimension $1, 2, 3, \cdots, \infty$. The parameter u is called spectral parameter in the context of integrable models and plays a different role from z.

4.3 U(1) and Virasoro subalgebra

Although the relation between the elements of the affine Yangian and those of $W_{1+\infty}$ is highly nontrivial, we can find it for the U(1) current and Virasoro algebra [11]. If we set

$$\begin{aligned} &J_0=\psi_1,\quad J_{-1}=e_0,\quad J_1=-f_0,\quad L_{-1}=e_1,\quad L_1=-f_1,\quad L_0=\frac{1}{2}\psi_2,\\ &L_{-2}=\frac{1}{2}[e_2,e_0]-\frac{\psi_0\sigma_3}{2}[e_1,e_0],\quad L_2=-\frac{1}{2}[f_2,f_0]+\frac{\psi_0\sigma_3}{2}[f_1,f_0], \end{aligned} \tag{4.29}$$

they satisfy the commutation relations of U(1) current and Virasoro algebra,

$$[J_n,J_m]=\psi_0 n\delta_{n+m,0}, \tag{4.30}$$

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{c}{12}(n+1)n(n-1)\delta_{n+m,0}, \tag{4.31}$$

$$[L_n,J_m]=-mJ_{n+m}, \tag{4.32}$$

where the central charge is given by

$$c=-\sigma_2\psi_0-\sigma_3^2\psi_0^3=1+(\lambda_1-1)(\lambda_2-1)(\lambda_3-1). \tag{4.33}$$

As is expected, the central charge is expressed by the parameters λ_i which is invariant under the automorphism. We note that the normalization of the U(1) current is not the standard one due to the factor ψ_0. We also note that the first term in the last expression of (4.33) comes from the U(1) current. We can remove its contribution from Virasoro algebra by subtracting the Sugawara current as follows:

$$L_n^{\text{decouple}}=L_n-\frac{1}{2\psi_0}\sum_{m\in\mathbb{Z}}:J_mJ_{n-m}:. \tag{4.34}$$

For the zero-mode, we can express

$$L_0^{\text{decouple}}=\frac{1}{2}\psi_2-\frac{\psi_1^2}{2\psi_0}+\cdots, \tag{4.35}$$

where the terms other than the first two terms vanish when they act on the highest weight state. The other modes of Virasoro algebra can be obtained from the commutator of the elements in (4.29). For example, we have $L_{-3}=[L_{-1},L_{-2}]=\frac{1}{2}[e_1,[e_2,e_0]]-\frac{\psi_0\sigma_3}{2}[e_1,[e_1,e_0]]$.

4.4 Plane partition representation of the affine Yangian

In this section, we discuss the representation theory of the affine Yangian. As we mentioned, the vacuum representation of $W_{1+\infty}$ can be described by a plane partition and the level of the state corresponds to the number of boxes. In the following, we see how we can determine the action of $e(u),f(u),\psi(u)$ on the plane partition [11].

Because of $[\psi_i,\psi_j]=0$, we can diagonalize ψ_is. Let us denote by Λ configuration of the plane partition, by $|\Lambda\rangle$ the corresponding state and by $\psi_\Lambda(u)=1+\sigma_3\sum_{j=0}^{\infty}\psi_{\Lambda,i}u^{-j-1}$ the eigenvalue of $\psi(u)$ for the state $|\Lambda\rangle$. We also denote by $|\emptyset\rangle$ the highest weight state, which satisfies

$$f_j\,|\emptyset\rangle = 0. \tag{4.36}$$

Here, we think of f_j (resp. e_j) as the lowering (resp. raising) operators. We also introduce the dual state $\langle\Lambda|$ and the highest weight state $\langle\emptyset|$ which satisfies

$$\langle\emptyset|\,e_j = 0. \tag{4.37}$$

Because the affine Yangian has infinitely many elements, there are generically infinitely many states even at level 1. To obtain the representation with a finite number of states at level 1, we have to impose some conditions on $\psi_\emptyset(u)$. We can obtain this condition by computing the Shapovalov form,

$$\langle\emptyset|f_j e_i|\emptyset\rangle = \langle\emptyset|[f_j, e_i]|\emptyset\rangle = -\,\langle\emptyset|\psi_{\emptyset,i+j}|\emptyset\rangle\,. \tag{4.38}$$

If the rank of this Shapovalov form is finite, the number of the independent states becomes finite. In particular, we are interested in the rank 1 case because there is one configuration at level 1 in the plane partition. This condition leads to

$$\psi_{\emptyset,j} = q^j\psi_0 \tag{4.39}$$

for some $q \in \mathbb{C}$[24]. It can be rewritten into

$$\psi_\emptyset(u) = \frac{u - q + \psi_0\sigma_3}{u - q}. \tag{4.40}$$

Because of the spectral shift (4.23), we can fix q to an arbitrary value without loss of generality. In the following, we set $q = 0$.

Next, let us consider the action of $e(u)$ and $f(u)$. Because $e(u)$ (resp. $f(u)$) increases (resp. decreases) the level by one, it plays a role of adding to (resp. removing from) the plane partition a box. Let us put the following ansatz:

$$e_j\,|\Lambda\rangle = \sum_{\square\in\Lambda^+} h_\square^j\; E(\Lambda\to\Lambda+\square)\,|\Lambda+\square\rangle\,, \tag{4.41}$$

$$f_j\,|\Lambda\rangle = \sum_{\square\in\Lambda^-} h_\square^j\; F(\Lambda\to\Lambda-\square)\,|\Lambda-\square\rangle\,, \tag{4.42}$$

which can be checked at the lower level[25]. Here, we assign the value

$$h_\square = h_1 x + h_2 y + h_3 z \tag{4.43}$$

[24]We note that ψ_0 is center and takes the same value in the representation space.

[25]For more detail, see Appendix A.

to each box at (x,y,z) and we denote by $\Lambda^{\pm}$ the places where we can add or remove a box. By considering the action of (4.7) on $|\Lambda\rangle$, we have

$$\frac{\psi_{\Lambda+\Box}(u)}{\psi_\Lambda(u)} = \varphi(u-h_\Box), \tag{4.44}$$

where we set

$$\varphi(u) = \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}. \tag{4.45}$$

Then we have

$$\psi_\Lambda(u) = \frac{u+\psi_0\sigma_3}{u}\prod_{\Box\in\Lambda}\varphi(u-h_\Box). \tag{4.46}$$

The remaining task is to determine $E(\Lambda\to\Lambda+\Box)$ and $F(\Lambda\to\Lambda-\Box)$. From (4.9), we see

$$\begin{aligned}&\psi_\Lambda(u)\\ &= 1+\sigma_3\sum_{\Box\in\Lambda^-}\frac{E(\Lambda-\Box\to\Lambda)F(\Lambda\to\Lambda-\Box)}{u-h_\Box} - \sigma_3\sum_{\Box\in\Lambda^+}\frac{E(\Lambda\to\Lambda+\Box)F(\Lambda+\Box\to\Lambda)}{u-h_\Box}.\end{aligned} \tag{4.47}$$

We note that this expression makes sense because $\psi_\Lambda(u)$ has a pole of order one only at $u=h_\Box$ $(\Box\in\Lambda^{\pm})$ as was checked in [11]. Because $e(u)$ and $f(u)$ are conjugate to each other, it is natural to set

$$E(\Lambda\to\Lambda+\Box) = F(\Lambda+\Box\to\Lambda). \tag{4.48}$$

Combining it with (4.47), we have

$$E(\Lambda\to\Lambda+\Box) = \sqrt{-\frac{1}{\sigma_3}\mathrm{res}_{u\to h_\Box}\psi_\Lambda(u)}. \tag{4.49}$$

Summarizing the above, the action of the currents on the state $|\Lambda\rangle$ is given by

$$\psi(u)|\Lambda\rangle = \psi_\Lambda(u)|\Lambda\rangle, \tag{4.50}$$

$$e(u)|\Lambda\rangle = \sum_{\Box\in\Lambda^+}\frac{1}{u-h_\Box}\sqrt{-\frac{1}{\sigma_3}\mathrm{res}_{u\to h_\Box}\psi_\Lambda(u)}\,|\Lambda+\Box\rangle, \tag{4.51}$$

$$f(u)|\Lambda\rangle = \sum_{\Box\in\Lambda^-}\frac{1}{u-h_\Box}\sqrt{-\frac{1}{\sigma_3}\mathrm{res}_{u\to h_\Box}\psi_{\Lambda-\Box}(u)}\,|\Lambda-\Box\rangle, \tag{4.52}$$

where "$\mathrm{res}_{u\to h_\Box}$" means the residue at $u=h_\Box$. This is the final result for the plane partition representation of the affine Yangian. Strictly speaking, we have to check that this formula is compatible with the other defining relations such as (4.5), but we omit it for simplicity. For a more complete derivation of the plane partition representation, see Appendix A.

In the plane partition representation, we can introduce an asymptotic Young diagram to each axis as in the following figure:

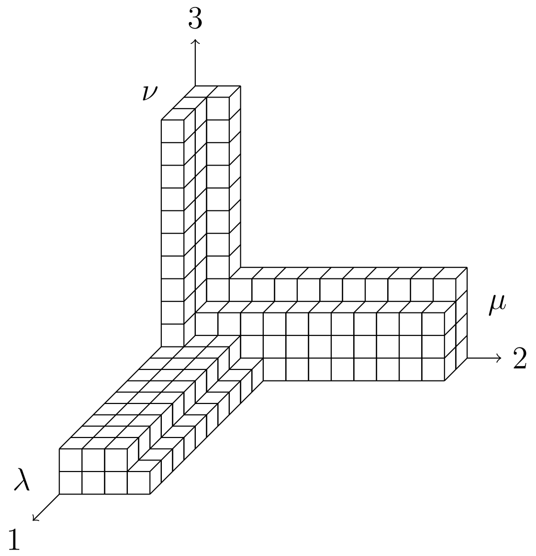


Figure 13: The asymptotic Young diagrams λ, μ, ν extend to infinity. They are expressed as $\lambda = (2,2,2,1)$, $\mu = (4,3)$ and $\nu = (3,1)$ if we choose the direction of the row and the column appropriately.

The asymptotic Young diagrams provide the module other than the vacuum module. In other words, it plays a role of the primary fields. According to (4.46), the eigenvalue of $\psi(u)$ should be obtained by multiplying the vacuum eigenvalue $\psi_\emptyset(u)$ by the contribution $\varphi(u - h_\square)$ from each box. Naively, it seems to diverge because there are infinitely many boxes. However, it actually converges because a lot of cancellations occur. As an example, let us consider the asymptotic Young diagram $\mu = \square$ inserted along the second axis. The eigenvalue $\psi_{\mu=\square}(u)$ of $\psi(u)$ can be computed as follows:

$$\begin{aligned}
\psi_{\mu=\square}(u) &= \psi_\emptyset(u) \prod_{i=0}^{\infty} \varphi(u - ih_2) \\
&= \psi_\emptyset(u) \prod_{i=0}^{\infty} \frac{\big(u + h_1 - ih_2\big)\big(u - (i-1)h_2\big)\big(u + h_3 - ih_2\big)}{\big(u - h_1 - ih_2\big)\big(u - (i+1)h_2\big)\big(u - h_3 - ih_2\big)} \\
&= \psi_\emptyset(u) \prod_{i=0}^{\infty} \frac{\big(u - h_1 - (i+1)h_2\big)\big(u - (i-1)h_2\big)\big(u - h_3 - (i+1)h_2\big)}{\big(u - h_1 - ih_2\big)\big(u - (i+1)h_2\big)\big(u - h_3 - ih_2\big)} \\
&= \frac{(u + \psi_0\sigma_3)(u + h_2)}{(u - h_1)(u - h_3)},
\end{aligned} \tag{4.53}$$

where we use (4.2) in the third line. We note that a lot of cancellations occur in the third line, which leads to the last line. Then we have

$$\psi_{\mu=\square}(u) = 1 + \sigma_3 \left(\frac{\psi_0}{u} - \frac{1}{h_2 u^2} + \frac{1 + \lambda_2}{u^3} + \cdots \right). \tag{4.54}$$

Comparing it with (4.29),we can read off the conformal dimension h and the U(1) charge j'

as

$$h = \frac{1+\lambda_2}{2}, \qquad j' = -\frac{1}{h_2}. \tag{4.55}$$

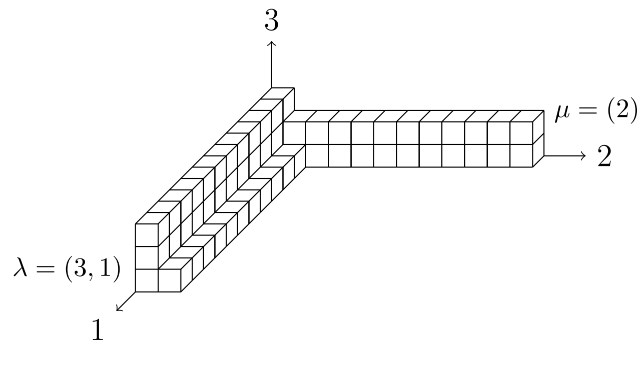
In a similar way, we can obtain the conformal dimension and U(1) charge for generic asymptotic Young diagrams and the formula is given in [27]. For example, when we insert the asymptotic Young diagram $\mu = (\mu_1, \mu_2, \mu_3, \cdots)$ in the second axis[26], the U(1) charge $j'(M^2_\mu)$ and the conformal dimension $h(M^2_\mu)$ are given as

$$\begin{aligned} j'(M^2_\mu) &= -\frac{1}{h_2}\sum_j \mu_j, \\ h(M^2_\mu) &= -\frac{\lambda_2}{2\lambda_3}\sum_j \mu_j^2 - \frac{\lambda_2}{2\lambda_1}\sum_j (2j-1)\mu_j + \frac{\lambda_2}{2}\sum_j \mu_j \\ &= -\frac{\lambda_2}{2\lambda_3}\sum_j \mu_j^2 - \frac{\lambda_2}{2\lambda_1}\sum_j (\mu^T)_j^2 + \frac{\lambda_2}{2}\sum_j \mu_j. \end{aligned} \tag{4.56}$$

where μ^T is a transposition of μ. The formula for the asymptotic Young diagram inserted in another axis is obtained just by replacing h_2, λ_2 with h_i, λ_i $(i = 1, 3)$. When we insert asymptotic Young diagrams in more than one axis, we have to be careful not to count some boxes twice. In the case that we insert the asymptotic Young diagrams λ and μ in the first axis and the second axis, respectively, the conformal dimension h and U(1) charge j' are given as follows:

$$\begin{aligned} j' &= j'(M^1_\lambda) + j'(M^2_\mu), \\ h &= h(M^1_\lambda) + h(M^2_\mu) - \#(\lambda \cap \mu), \end{aligned} \tag{4.57}$$

where $\#(\lambda \cap \mu)$ denotes the number of the overlapping boxes. For example, there are three overlapping boxes at $(x_1, x_2, x_3) = (1,1,1), (1,2,1), (1,1,2)$ in the following plane partition:



4.5 Constrained plane partition and the Y-algebra

In the previous section, we have discussed the plane partition representation for generic parameters. In this section, we discuss the special cases where null states appear. In

[26]In this notation, each row μ_i of the Young diagram extends to the direction of the third axis.

such cases, some configuration of the plane partition is prohibited if it contains a box at a particular position. As we mentioned in Section 3.1.4, we refer to such a position as a "pit", following [25]. We have already seen that the Hilbert space of the Y-algebra is described by a plane partition with a pit and they are expected to be truncations of $W_{1+\infty}$ algebra. Therefore, it is expected that the plane partition representation of the affine Yangian describes the Hilbert space of the Y-algebra when the parameters satisfy a particular condition.

As we mentioned, $\psi_\Lambda(u)$ has a non-vanishing residue at $u = h_\square$ for generic parameters iff $\square \in \Lambda^\pm$. However, when the condition

$$\psi_0\sigma_3 = -(Lh_1 + Mh_2 + Nh_3) \qquad (L, M, N \geq 0) \tag{4.58}$$

is satisfied, the pole of $\psi_\Lambda(u)$ at $u = Lh_1+Mh_2+Nh_3$ is canceled out (see 4.46). Then (4.51) and (4.52) imply that we cannot place a box at $(x, y, z) = (L + 1, M + 1, N + 1)$ because the corresponding coefficients become zero. As a result, the Hilbert space is described by the plane partition with a pit at $(L + 1, M + 1, N + 1)$. For later convenience, we rewrite the condition (4.58) by using (4.21) into a simpler form,

$$\frac{L}{\lambda_1} + \frac{M}{\lambda_2} + \frac{N}{\lambda_3} = 1. \tag{4.59}$$

Summarizing the above, the affine Yangian truncates to $Y_{L,M,N}[\Psi]$ when the condition (4.59) is satisfied[27].

In the remaining part, we explain the relation between the affine Yangian and the Y-algebra in more detail. Let us identify the relation between the parameter of the affine Yangian and that of the Y-algebra. By comparing the central charges, we find

$$\Psi = -\frac{h_2}{h_1} = -\frac{\lambda_1}{\lambda_2}. \tag{4.60}$$

Combining (4.22), (4.59) and (4.60), we have

$$\begin{aligned} \lambda_1 &= L - \Psi M - (1 - \Psi)N, \\ \lambda_2 &= -\frac{L - \Psi M - (1 - \Psi)N}{\Psi}, \\ \lambda_3 &= \frac{L - \Psi M - (1 - \Psi)N}{\Psi - 1}. \end{aligned} \tag{4.61}$$

Next, let us consider the module of the Y-algebra which we discussed in Section 3.1.3 in terms of the affine Yangian. We have seen that the line operators in the three-dimensional edges give the modules of the Y-algebra. As shown in Figure 14, we can introduce three line operators and they are labeled by the weight of the gauge group $U(L|M)$, $U(M|N)$, $U(N|L)$, respectively:

[27] We note that one of L, M, N can be set to zero by using (4.2). In this thesis, we focus on the case that one of L, M, N is equal to zero.

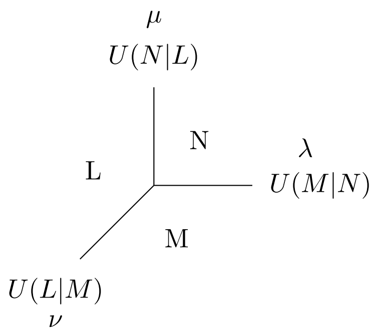


Figure 14: There are three edges where we can insert line operators. λ, μ, ν denote the weight of $U(L, M), U(M|N), U(N|L)$, respectively.

On the other hand, the affine Yangian has the module labeled by three asymptotic Young diagrams. Then it is natural to identify each line operator with the asymptotic Young diagram inserted in each axis of the plane partition. One can see the validity of this identification by considering the possible shape of the asymptotic Young diagram. As an example, let us consider $Y_{0,0,N}$. In this case, there is a pit at $(1, 1, N+1)$ and we can insert two asymptotic Young diagrams with height at most N. Such Young diagrams indeed correspond to the weight of $U(N)$ representation. For generic $Y_{L,M,N}$, we can insert hook-shaped Young diagrams, which describe the weight of super Lie algebra (see Figure 15).

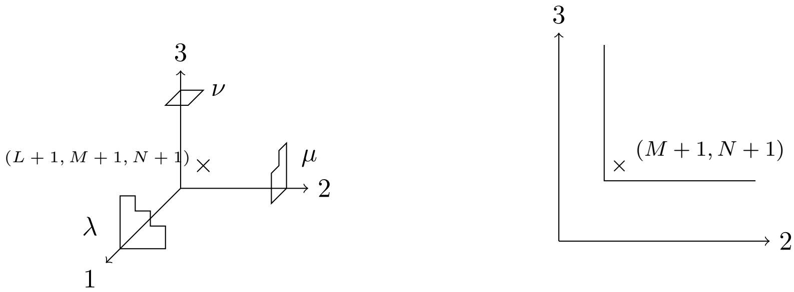


Figure 15: Left: The asymptotic Young diagrams λ, μ, ν. Right: The asymptotic Young diagram λ is restricted due to the pit. It is known that such Young diagram corresponds to the weight of $U(M|N)$.

5 Doubly constrained plane partition and W_N minimal models

In this section, we study the W_N minimal model in terms of the plane partition. In the previous section, we discussed the case that only one truncation condition (4.59) is satisfied. In Section 5.1, we consider the case that two truncation conditions are satisfied at the same time. In such cases, a further constraint is imposed on the plane partition. That describes the null state condition and plays an important role in the subsequent discussion. In Section 5.2, we study the W_N minimal model by using the result obtained in Section 5.1. The W_N minimal models have been studied in terms of N-tuple Young diagrams. We show the compatibility with our result. We will see that the plane partition provides much simpler description for the W_N minimal model's Hilbert space.

5.1 Doubly constrained plane partition

As we have seen, the affine Yangian of $\mathfrak{gl}_1$ provides a uniform description of a certain class of W-algebras. Especially, it includes all W_N algebras. Their representation theory is also described uniformly by a constrained plane partition. So far, we have discussed the cases for generic parameters. In this section, we consider the cases such that additional null states appear. In terms of a plane partition, the additional null states imply that some configuration is prohibited. Then we need to understand what happens when two truncation conditions (4.59) are satisfied at the same time as follows:

$$\frac{L_1}{\lambda_1}+\frac{M_1}{\lambda_2}+\frac{N_1}{\lambda_3}=1, \qquad \frac{L_2}{\lambda_1}+\frac{M_2}{\lambda_2}+\frac{N_2}{\lambda_3}=1, \tag{5.1}$$

where $(L_2-L_1, M_2-M_1, N_2-N_1)$ is not proportional to $(1,1,1)$ [28]. In the following, we focus on the case that one of the above conditions corresponds to the truncation to W_N algebra. Let us set $(L_2, M_2, N_2)=(0,0,N)$. For the other condition, we can set $N_1=0$ without loss of generality. Let us explain this point in more detail. As we mentioned, we can set one of L_1, M_1, N_1 to zero by using (4.22). When N_1 is the smallest, we can set $N_1=0$. The nontrivial case is when L_1 or M_1 is smaller than N_1. Let us consider the case that L_1 is the smallest. In this case, we can redefine $(L_1, M_1, N_1)=(0,m,n)$. When $n\geq N$, the pit at $(1,m+1,n+1)$ does not play any role because the pit at $(1,1,N+1)$ already restricts the height of the plane partition to N (see the left of Figure 16). We are not interested in such a case. Let us consider $n<N$. In this case, there is a pit on the (x_1,x_2) plane. To see that, we recall that the truncation condition (4.59) is satisfied by $(-N,-N,0)$ and $(-n, m-n, 0)$. That is also true for every integral point on the line connecting them. Thanks to $n<N$, such a point exists on the first quadrant of the (x_1,x_2) plane. Thus, we can set $(L_1,M_1,N_1)=(L,M,0)$ without loss of generality.

[28] If this condition is not satisfied, one of the two truncation conditions becomes trivial due to (4.22).

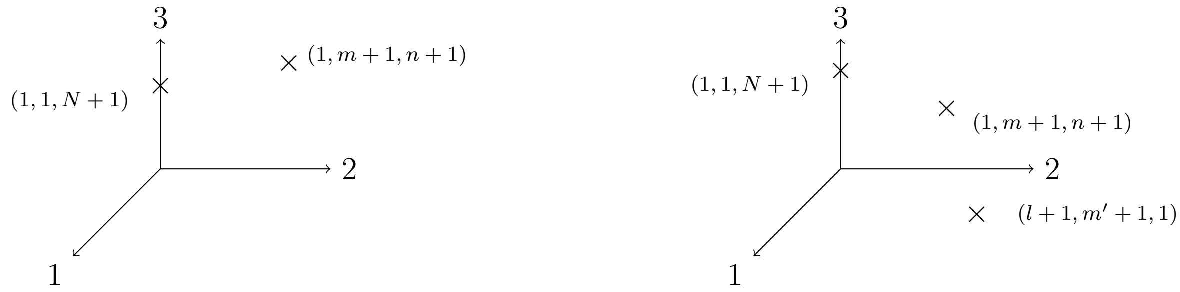


Figure 16: Left: The case for $n \geq N$. One can see that the pit at $(1, m+1, n+1)$ does not play any role. Right: The case that there are three pits.

In the following, we do not consider the case that there are more than two pits, as shown in the right of Figure 16. This assumption is necessary in the discussion below and we will see in Section 5.2 that all the W_N minimal models satisfy it. For the assumption to be satisfied, there should not be a pit in the (x_2, x_3) plane and (x_1, x_3) plane. To understand the condition for that, we recall that the truncation condition is satisfied by $(-N, -N, 0)$ and $(L, M, 0)$. The same is also true for the integral points on the line segment between them. One can see that the condition is satisfied when there is no pit on the line segment. To understand that, let us consider the case that there is an integral point on the line segment. If it is in the first quadrant of the (x_1, x_2) plane, that means we have to replace the value of L, M with the new ones. If it is not in the first quadrant of the (x_1, x_2) plane, we have a new pit in either of the (x_2, x_3) plane or (x_1, x_3) plane because we can shift the point in the direction of $(1, 1, 1)$. From the above, $L + N$ and $M + N$ must be coprime, namely,

$$\gcd(L + N, M + N) = 1, \tag{5.2}$$

where gcd means the greatest common divisor.

In the following, we consider what kind of constraint is imposed on the plane partition. For simplicity, we denote by $\boxed{1}$ (resp. $\boxed{2}$) the box at $(L+1, M+1, 1)$ (resp. $(1, 1, N+1)$). Because of (5.1), one can see that the values of (4.43) for these two boxes are the same. We note that the truncation condition was originally given in (4.58). We set $h = h_{\boxed{1}} = h_{\boxed{2}}$[29]. Before the analysis, let us recall why a single truncation condition produces a pit. When we have only one truncation condition $\frac{P}{\lambda_1} + \frac{Q}{\lambda_2} + \frac{R}{\lambda_3} = 1$, we have $\psi_0 \sigma_3 = -(Ph_1 + Qh_2 + Rh_3)$. Then $\psi_\Lambda(u)$ cannot have a pole at $u = Ph_1 + Qh_2 + Rh_3$ because the factor $u + \psi_0 \sigma_3$ in (4.46) cancels out the pole. According to (4.51), that implies we cannot place a box at $(P+1, Q+1, R+1)$.

However, the situation is different when two truncation conditions are satisfied. When both $\boxed{1}$ and $\boxed{2}$ belong to Λ^+, $\psi_\Lambda(u)$ has a pole at $u = h$ because both boxes contribute to

[29]We note that the symbol "h" does not mean a conformal dimension in this section.

the pole;

$$\psi_\Lambda(u) \propto \frac{u+\psi_0\sigma_3}{(u-h_{\boxed{1}})(u-h_{\boxed{2}})} = \frac{1}{u-h}. \tag{5.3}$$

Then we can add the boxes $\boxed{1}$, $\boxed{2}$ to the plane partition Λ. However, the two states, $|\Lambda+\boxed{1}\rangle$ and $|\Lambda+\boxed{2}\rangle$, are no longer independent. To see that, let us consider the action of $e(u)$ in more detail. From (4.51), we see that $|\Lambda+\boxed{1}\rangle$ and $|\Lambda+\boxed{2}\rangle$ are generated as follows:

$$e(u)\,|\Lambda\rangle \propto \frac{1}{u-h}\big(|\Lambda+\boxed{1}\rangle + |\Lambda+\boxed{2}\rangle\big) + \cdots . \tag{5.4}$$

It implies that any modes e_i of $e(u)$ generate only $|\Lambda+\boxed{1}\rangle + |\Lambda+\boxed{2}\rangle$ and cannot create the state $|\Lambda+\boxed{1}\rangle - |\Lambda+\boxed{2}\rangle$. Then this state becomes null and we should identify $|\Lambda+\boxed{1}\rangle$ with $|\Lambda+\boxed{2}\rangle$. The same is also true for their descendant states. These observation suggests that we should identify the box at $(x+L_1, y+M_1, z)$ with the box at $(x,y,z+N)$[30].

Due to the identification, there are two perspectives for the representation space; we can describe it by either a plane partition with a pit at $(L+1, M+1, 1)$ or a plane partition with a pit at $(1,1,N+1)$. One may think that the additional pit does not cause any effects, but the representation space is actually further constrained. That is because some configurations may not make sense as a plane partition in one of the two descriptions. Such states correspond to null states.

We demonstrate what we have explained above by using an example. We consider the case for $(L_1, M_1, N_1) = (1,2,0)$ and $(L_2, M_2, N_2) = (0,0,1)$. In this case, two boxes are identified if their coordinates are equal modulo $(1,2,-1)$. The following figure shows how the two plane partitions are identified:

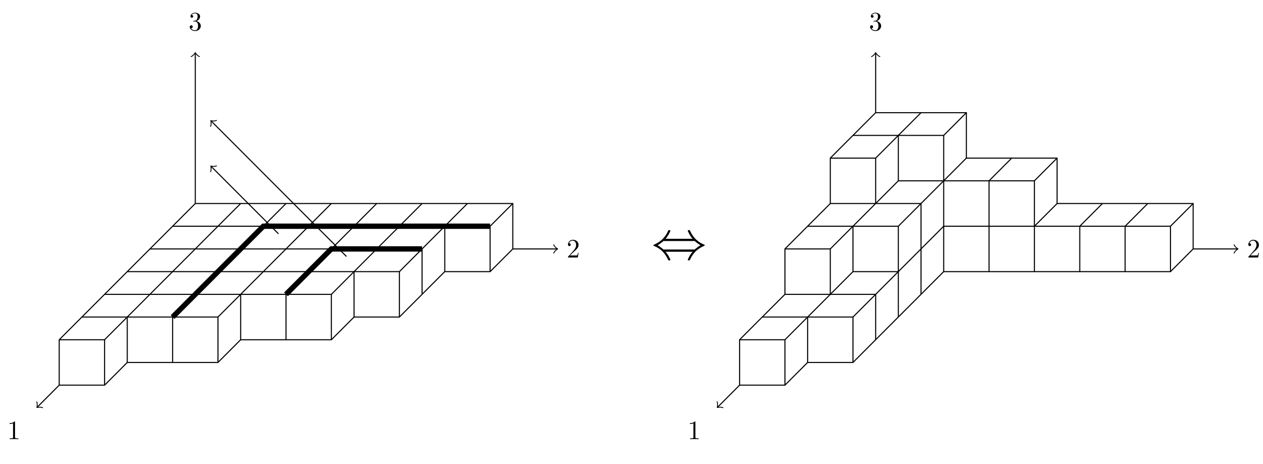


Figure 17: The left figure describes the plane partition with a pit at $(1,1,2)$ while the right figure describes the plane partition with a pit at $(2,3,1)$. This figure shows how they are identified.

[30]We note that the identification rule was also suggested in [11] when one of L and M is zero.

The left figure describes the plane partition with a pit at $(1,1,2)$, which is just a Young diagram. According to the identification rule, we can also interpret this state in terms of the plane partition with a pit at $(2,3,1)$. In order to switch to the latter description, we need to move the boxes along the direction of $(-1,-2,1)$. For that, we split the Young diagram into the hook-shaped regions with the corners located at $(k+1, 2k+1, 1)$ $(k = 0, 1, 2, \cdots)$. Then we shift the k-th region by $(-k, -2k, k)$. In other words, we pile the $(k+1)$-th piece on the k-th piece. After this procedure, we obtain the plane partition with a pit at $(2,3,1)$, as shown in the right figure. Because both figures make sense as a plane partition, this state is not null. One may see that any Young diagrams are transformed into consistent plane partitions with a pit at $(2,3,1)$. That implies that no restrictions are imposed on the Young diagram. This is not surprising because the left figure corresponds to $Y_{0,0,1} = \widehat{U}(1)$ and null states cannot arise in the free field. On the other hand, the second plane partition is indeed restricted. As an example, let us consider the following configuration:

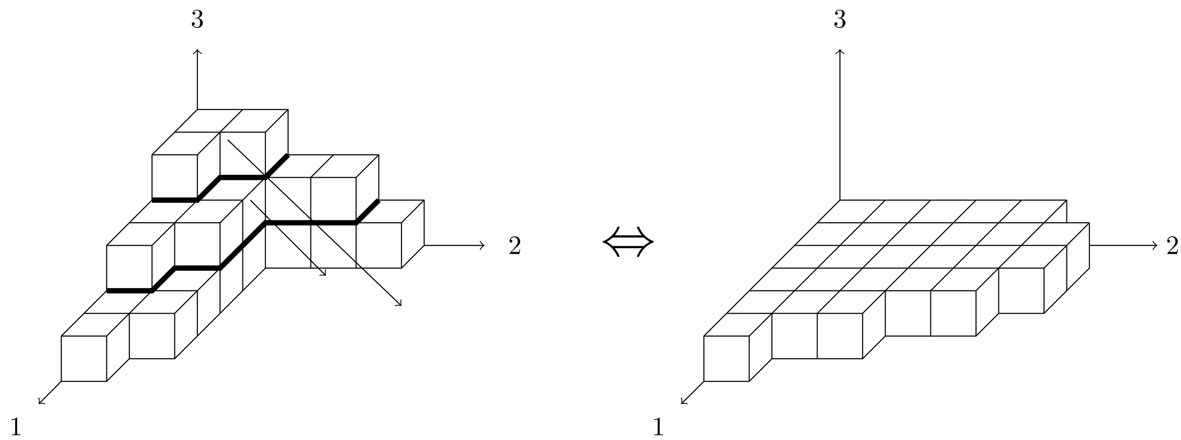


Figure 18: The left figure describes the plane partition with a pit at $(2,3,1)$ while the right figure shows the plane partition with a pit at $(1,1,2)$. While the left figure makes sense as a plane partition, the right figure does not. Therefore, this state is null.

While there is no problem with the left figure, it is clear that the right figure does not make sense. That means this state is null.

In the next section, we consider general N. We will see that both plane partitions are restricted in such cases and that describes the Hilbert space for the minimal model of W_N algebras correctly.

5.2 W_N minimal models from plane partitions

In this section, we discuss W_N minimal models in terms of doubly constrained plane partitions. The product of the W_N algebra and an extra U(1) boson are realized by the affine Yangian of $\mathfrak{gl}_1$ when $\lambda_3 = N$. The remaining parameters are λ_1 and λ_2, but they are subject to the relation (4.22). Then we can parametrize them as $\lambda_1 = N(\beta^{-1}-1)$ and $\lambda_2 = N(\beta-1)$. We see from (4.33) that the central charge is given as follows:

$$c = (N-1)(1 - Q^2 N(N+1)) + 1, \quad Q = \sqrt{\beta} - 1/\sqrt{\beta}. \tag{5.5}$$

We note that the second term comes from the extra U(1) boson. The minimal models are realized when $\beta = p/q$, where $p, q \geq N$ are coprime integers. For this parameter, the additional truncation condition is satisfied as follows:

$$\frac{p-N}{\lambda_1} + \frac{q-N}{\lambda_2} + \frac{0}{\lambda_3} = 1. \tag{5.6}$$

This is precisely the situation that we considered in the previous subsection. We note that the condition for p, q matches the assumption (5.2). As we explained in the previous section, the two positions $(x, y, z+N)$ and $(x+p-N, y+q-N, z)$ are identified and the plane partition must satisfy the identification rule as in Figure 17. The aim of this section is to confirm that the doubly constrained plane partition indeed provides the correct Hilbert space for the W_N minimal model. For that, it is convenient to decompose the plane partition into N-tuple Young diagrams because they are the standard basis of W_N algebra (and an extra U(1) boson)[31]. This decomposition is illustrated in Figure 19. One can see that the plane partition with a pit at $(1, 1, N+1)$ consists of N-tuple Young diagrams $Y^{(1)}, Y^{(2)}, \cdots, Y^{(N)}$ satisfying $Y^{(1)} \supset Y^{(2)} \supset \cdots \supset Y^{(N)}$.

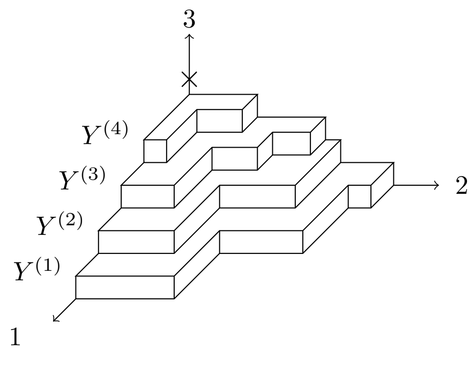


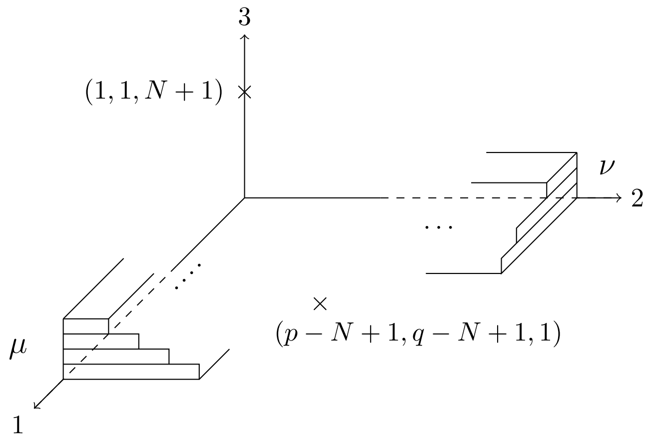
Figure 19: There is a Young diagram in each layer of the plane partition.

According to the identification rule, the plane partition is split into the hook-shaped regions and they are piled up one after another as in Figure 17. In this process, only the shape of $Y^{(1)}$ and $Y^{(N)}$ is relevant to whether the new plane partition is in a consistent shape. Summarizing, there are constraints not only between the neighboring Young diagrams but also between the first one and the last one.

Let us express the constraint more explicitly. In the following, we also consider the asymptotic Young diagrams. In the present case, we can insert them in the first axis and

[31]Recall that W_N algebra is constructed from $N-1$ free bosons as we discussed in Section 2.3.1.

the second axis. We denote by $\mu = (\mu_1, \mu_2, \cdots, \mu_N)$ (resp. $\nu = (\nu_1, \nu_2, \cdots, \nu_N)$) the first one (resp. the second one).



One can see that the Young diagram $(n, n, \cdots, n)$ does not play an essential role because it just shifts the origin of the plane partition. One may interpret it as the module for an extra U(1) boson. Therefore, we can set $\mu_N = \nu_N = 0$ without loss of generality. We also note that we have to impose

$$\mu_1 \leq q - N, \quad \nu_1 \leq p - N \tag{5.7}$$

because of the second pit at $(p - N + 1, q - N + 1, 1)$.

Now we can compare these asymptotic Young diagrams with the primary fields in the W_N minimal models. Let us define the parameters,

$$\begin{aligned} \tilde{m}_i &= \mu_i - \mu_{i+1}, & \tilde{n}_i &= \nu_i - \nu_{i+1}, \\ m_i &= \tilde{m}_i + 1, & n_i &= \tilde{n}_i + 1 \end{aligned} \tag{5.8}$$

for $i = 1, 2, \cdots, N-1$. The constraints $\mu_1 \leq q-N, \nu_1 \leq p-N$ are rewritten as $\sum_{i=1}^{N-1} m_i < q$ and $\sum_{i=1}^{N-1} n_i < p$. Such sets of positive integers are exactly the same as that of W_N minimal models. Indeed, one can compute the conformal dimension for the asymptotic Young diagrams by using the affine Yangian's formula (4.56) and see that it reproduces the correct conformal dimension

$$\Delta(\vec{m}, \vec{n}) = \frac{12(\sum_{i=1}^{N-1}(pm_i - qn_i)\vec{\omega}_i)^2 - N(N^2-1)(p-q)^2}{24pq}, \tag{5.9}$$

which is given in (2.108). The detail for the computation can be found in Appendix B.

Next, let us consider the constraint imposed on the Hilbert space. Due to the asymptotic Young diagrams μ, ν, the location of the N-tuple Young diagrams $Y^{(1)}, Y^{(2)}, \cdots, Y^{(N)}$ are shifted as follows:

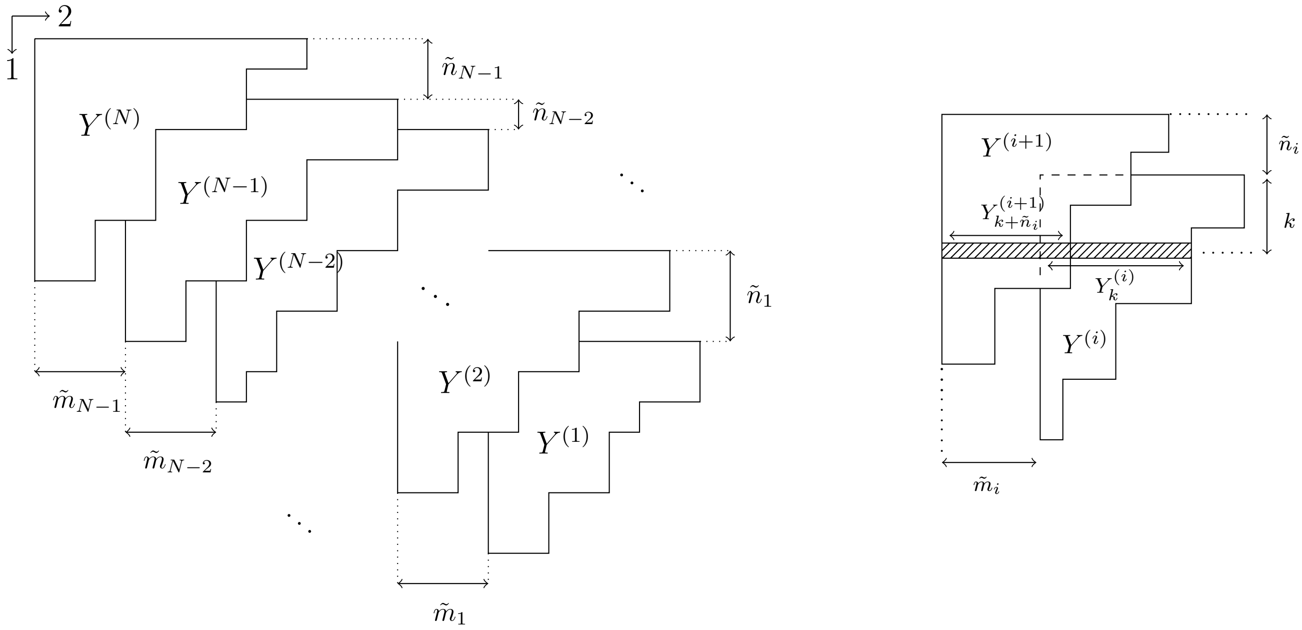


Figure 20: This figure is viewed from the top. The N-tuple Young diagrams must be in the shape such that they form a plane partition and satisfy the identification rule. This requirement correctly reproduces the Hilbert space of the W_N minimal model.

As we mentioned, we cannot determine the shape of the N-tuple Young diagrams independently because they must make sense as a plane partition. They must also satisfy the identification rule. These conditions can be written as follows:

$$Y^{(i)}_k - Y^{(i+1)}_{k+n_i-1} \geq -(m_i - 1) \qquad (i = 1, 2, \cdots, N,\ k \geq 1), \tag{5.10}$$

where $Y^{(i)}_k$ denotes the length of k-th row of $Y^{(i)}$. Here, we identify $Y^{(N+1)} = Y^{(1)}$ and set $m_N = q - \sum_{i=1}^{N-1} m_i$ and $n_N = p - \sum_{i=1}^{N-1} n_i$. We note that the condition for $i = N$ comes from the identification rule. See the right of Figure 20. The above condition agrees with the N-Burge condition [59, 29, 30, 31, 32], which is known to give the Hilbert space of the W_N minimal model[32]. Thus, the doubly constrained plane partition provides the correct Hilbert space of W_N minimal models.

Finally, we mention the level-rank duality, which is an equivalence between two different minimal models. So far, we have discussed the Hilbert space in terms of the plane partition with a pit at $(1, 1, N+1)$, but it is also possible to interpret it in terms of the plane partition with a pit at $(p-N+1, q-N+1, 1)$. In this perspective, the W-algebra realized in this space is $Y_{p-N,q-N,0}[\Psi]$. Thus we have two different descriptions for the same Hilbert space. In the case of $p = N$ or $q = N$, the other W-algebra is also W_M algebra ($M = q-N$ or $M = p-N$).

[32]It was proved in [29] that N-Burge condition matches with the character of the W_N minimal model found in [60]. The N-Burge condition is derived not only from the representation theory but also from the study of conformal blocks in the context of AGT [30].

The equivalence between two W_N minimal model is called level-rank duality [61, 62] and has been studied in terms of N-tuple Young diagrams in [32]. When $p \neq N$ and $q \neq N$, $Y_{p-N,q-N,0}[\Psi]$ is not a familiar W-algebra and the corresponding level-rank duality has not been discussed without a few exceptions; in the case of $p = N+1, q = N+M$ $(M \geq 2)$, the corresponding W-algebra was studied in [63] and the level-rank duality was also mentioned. We emphasize that the level-rank duality is manifestly realized in our construction.

6 $\mathcal{N}=2$ unitary minimal models from connected plane partitions

In the previous section, we studied the W_N minimal models by using the plane partition representation of the affine Yangian. In terms of Gaiotto-Rapcak's construction, a single plane partition corresponds to the W-algebra associated with a single trivalent vertex. Meanwhile, Gaiotto-Rapcak's construction can be generalized to various (p,q)-webs. Because an affine Yangian lives at each corner, the total Hilbert space is described by several plane partitions. We also need to take into account the modules associated with the line operators connecting the two corners. As we discussed in Section 4.5, we can interpret them as the asymptotic Young diagrams. Therefore, the Hilbert space consists of the plane partitions which share the common asymptotic Young diagrams. As we mentioned, we refer to them as intermediate Young diagrams.

One of the simplest but nontrivial example for the generalized Y-algebra is $\mathcal{N}=2$ super Virasoro algebra (with an extra U(1) boson). As we explained in Section 3.2, it is realized from the following diagram:

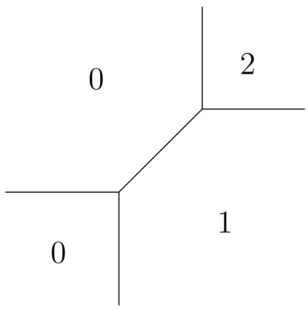


Figure 21: The diagram which realizes $\mathcal{N}=2$ super Virasoro algebra and U(1) boson.

In this section, we study the $\mathcal{N}=2$ unitary minimal models using plane partitions. We first clarify how the Hilbert space of $\mathcal{N}=2$ super Virasoro algebra for a generic central charge is described by plane partitions. In Section 6.1, we discuss the prescription for negative intermediate Young diagrams. In Section 6.2, we provide several necessary data such as conformal dimension and U(1) charge, which are used in the subsequent sections. In Section 6.3, we discuss the spectral flow, the automorphism of $\mathcal{N}=2$ super Virasoro algebra. It is obtained from the spectral shift of the affine Yangian. Because the spectral flow relates the R sector to the NS sector, it allows us to focus on the NS sector. Section 6.4 is the main part of this section. We study $\mathcal{N}=2$ unitary minimal models in terms of plane partitions. We find that the correct primary fields and characters are reproduced.

6.1 Interpretation of negative weights in terms of a plane partition

Before starting the analysis, we have to resolve a subtle problem. As we mentioned in the last part of Section 3.2, we need to consider the line operators with negative weights. However, it is not obvious how we should interpret the negative weight in terms of a plane partition. In the following, we focus on the situation that there is a pit at $(L+1, M+1, 1)$ and the asymptotic Young diagram $\mu = (\mu_1, \mu_2, \cdots, \mu_M)$ (resp. $\nu = (\nu_1, \nu_2, \cdots, \nu_L)$) is inserted in the first (resp. second) axis (see Figure 22):

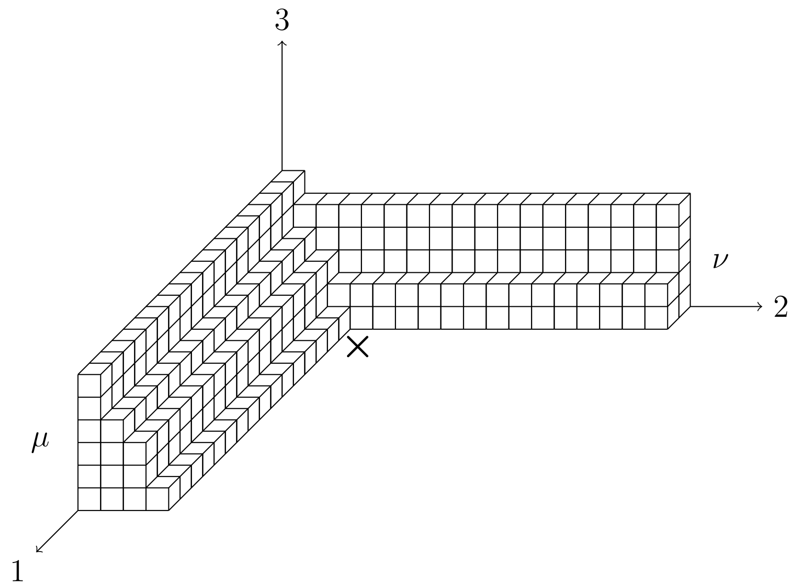


Figure 22: This figure shows the plane partition with a pit at $(3, 5.1)$. The asymptotic Young diagrams are inserted in the first and the second axis. In our notation, the first one is expressed as $\mu = (6, 4, 3, 1)$ while the other one is expressed as $\nu = (5, 2)$.

Let us consider the case of $\mu = (1, 1, \cdots, 1)$ and $\nu = (1, 1, \cdots, 1)$. Obviously, they just shift the origin of the plane partition by 1 in the third axis direction. It affects the representation theory through the factor $h_\square$ defined in (4.43). Compared with the case without the asymptotic Young diagrams, the value of $h_\square$ is shifted by h_3. We can take this effect into account by changing the spectral parameter u to $u - h_3$. Similarly, the asymptotic Young diagrams, $(\mu_1 + n, \mu_2 + n, \cdots, \mu_M + n)$ and $(\nu_1 + n, \nu_2 + n, \cdots, \nu_L + n)$, can be reduced to $(\mu_1, \mu_2, \cdots, \mu_M)$ and $(\nu_1, \nu_2, \cdots, \nu_L)$ by changing u to $u - nh_3$.

By applying this argument, we can deal with the negative weight. Let us consider the case for $\mu_M < 0$ and $\mu_M < \nu_L$. We can replace μ and ν with $\tilde{\mu} = (\mu_1 - \mu_M, \mu_2 - \mu_M, \cdots, \mu_{M-1} - \mu_M, 0)$ and $\tilde{\nu} = (\nu_1 - \mu_M, \nu_2 - \mu_M, \cdots, \nu_L - \mu_M)$ by changing u to $u - \mu_L h_3$.

Due to this shift, the mode of the current $\psi(u)$ changes as follows:

$$\psi(u) \to \psi(u-\mu_M h_3) = 1 + \frac{\psi_0\sigma_3}{u} + \frac{(\psi_1+\psi_0\mu_M h_3)\sigma_3}{u^2} + \frac{(\psi_2+2\psi_1\mu_M h_3 - \frac{\mu_M^2\lambda_1\lambda_2}{\lambda_3})\sigma_3}{u^3} + \cdots \quad (6.1)$$

Combining it with (4.29), we see that the U(1) charge j' and the conformal dimension h are modified as follows:

$$\begin{aligned} j' &\to j' + \psi_0\mu_M h_3, \\ h &\to h + j'\mu_M h_3 - \frac{\mu_M^2\lambda_1\lambda_2}{2\lambda_3}. \end{aligned} \quad (6.2)$$

Then the formula (4.57) for negative asymptotic Young diagrams is given as follows:

$$\begin{aligned} j' &= j'(M^1_{\tilde{\mu}}) + j'(M^2_{\tilde{\nu}}) + \psi_0\mu_M h_3 \\ &= j'(M^1_{\mu}) + j'(M^2_{\nu}), \\ h &= h(M^1_{\tilde{\mu}}) + h(M^2_{\tilde{\nu}}) + (j'(M^1_{\tilde{\mu}}) + j'(M^2_{\tilde{\nu}}))\mu_M h_3 - \frac{\mu_M^2\lambda_1\lambda_2}{2\lambda_3} - \#(\tilde{\mu}\cap\tilde{\nu}) \\ &= h(M^1_{\mu}) + h(M^2_{\nu}) - \#(\tilde{\mu}\cap\tilde{\nu}) - LM\mu_M. \end{aligned} \quad (6.3)$$

One can see that the formula for h becomes a bit complicated.

6.2 $\mathcal{N}=2$ super Virasoro algebra from plane partitions

Let us interpret the Hilbert space of $\mathcal{N}=2$ super Virasoro algebra in terms of plane partitions. We note that a similar discussion was made in [27], but we need to rewrite all of the results in terms of the affine Yangian for later analysis. Especially, we need to deal with the negative weight carefully, based on the prescription discussed in the previous section.

As we have seen in Section 3.2, there are two Y-algebras, $Y_{0,1,2}[\Psi]$ and $Y_{0,0,1}[1-\frac{1}{\Psi}]$, in Figure 21. By using (4.61), we can identify them with the affine Yangians, whose parameters are given as follows;

$$\lambda_1^{(1)} = -\frac{n}{n+2}, \quad \lambda_2^{(1)} = \frac{n}{n+1}, \quad \lambda_3^{(1)} = n, \quad (6.4)$$

$$\lambda_1^{(2)} = -\frac{1}{n+2}, \quad \lambda_2^{(2)} = \frac{1}{n+1}, \quad \lambda_3^{(2)} = 1, \quad (6.5)$$

where $n=\Psi-2$. Here, we put the subscripts (1) and (2) to distinguish two affine Yangians. In the following, we use a similar notation. We note that we have changed $\lambda_1,\lambda_2,\lambda_3$ to $\lambda_3,\lambda_1,\lambda_2$ in (6.4) just because that is convenient to draw figures. We can read off the central charge c from (4.33) as

$$c = c_{\text{sVir}} + 1, \quad c_{\text{sVir}} = \frac{3n}{n+2}, \quad (6.6)$$

where the second term in c comes from an extra U(1) boson.

The line operators connecting the two corners are labeled by the U(1) weight $\mu \in \mathbb{Z}$. In terms of the plane partitions, we can interpret them as the asymptotic Young diagrams inserted in the second axis. For the negative weight $\mu < 0$, we use the prescription discussed in the previous section. Then we can describe the total Hilbert space as follows:

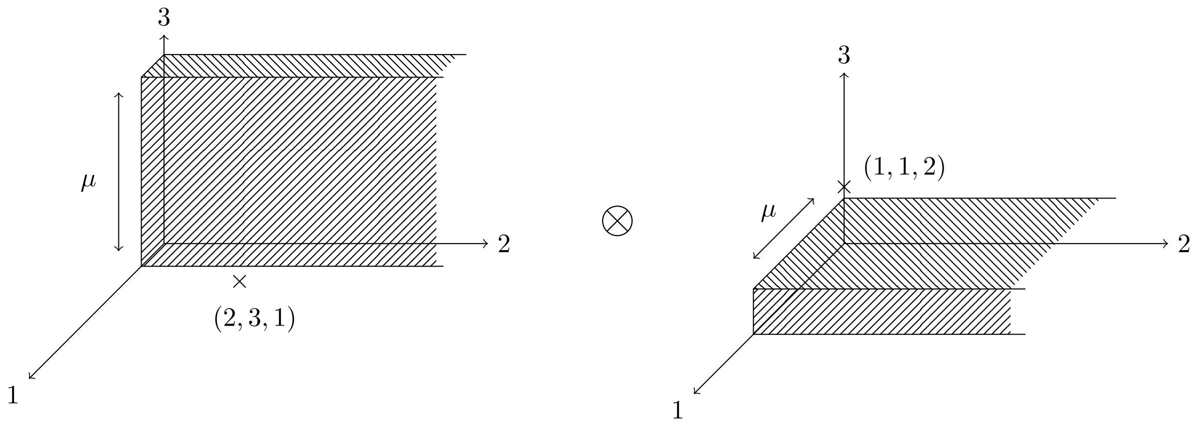


Figure 23: This figure shows the $\mu > 0$ sector where the asymptotic Young diagram is simply given by (μ).

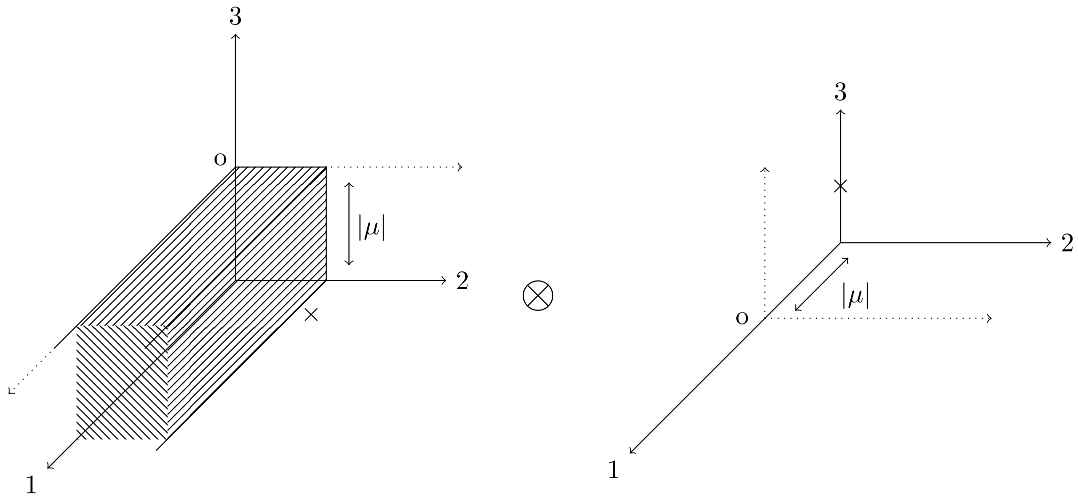


Figure 24: This figure shows the $\mu < 0$ sector. We shift the origin according to the prescription which we discussed in Section 6.1. The dashed axis means the axis before the shift. In the left plane partition, the Young diagram $(|\mu|, |\mu|)$ is inserted in the first axis instead of the second axis. In the right plane partition, we do not insert an asymptotic Young diagram. Actually, the asymptotic Young diagram in $Y_{0,0,1}$ can be always interpreted as the shift of the origin.

We can compute the conformal dimension h of these asymptotic Young diagrams by using (4.56). For $\mu > 0$, we have

$$\begin{aligned} h &= \big(-\frac{\lambda_2^{(1)}}{2\lambda_3^{(1)}}\mu^2 - \frac{\lambda_2^{(1)}}{2\lambda_1^{(1)}}\mu + \frac{\lambda_2^{(1)}}{2}\mu\big) + \big(-\frac{\lambda_2^{(2)}}{2\lambda_3^{(2)}}\mu - \frac{\lambda_2^{(2)}}{2\lambda_1^{(2)}}\mu^2 + \frac{\lambda_2^{(2)}}{2}\mu\big) \\ &= \frac{\mu(\mu+2)}{2} \end{aligned} \tag{6.7}$$

For $\mu < 0$, we need to use (6.3) and we have

$$h = \frac{\mu(\mu+2)}{2} - 2\mu = \frac{\mu(\mu-2)}{2} \tag{6.8}$$

We can summarize these results as

$$h = \frac{|\mu|(|\mu|+2)}{2}. \tag{6.9}$$

For the U(1) charge, we need to note that there are two U(1) currents. One of their linear combinations should correspond to the decoupled U(1) factor while the other one corresponds to the U(1) current in $\mathcal{N}=2$ super Virasoro algebra. We denote by $j(z)$ (resp. $J(z)$) the former (resp. latter) current. We also set their zero modes to j and J, respectively. Because $j(z)$ should give no charge to the asymptotic Young diagram μ, its zero mode j can be determined as

$$j = \psi_1^{(1)} h_2^{(1)} - \psi_1^{(2)} h_2^{(2)}. \tag{6.10}$$

We can determine the other zero mode J from the requirement that $J(z)$ commutes with $j(z)$. We fix the normalization so that the charge of the asymptotic Young diagram μ will become μ. Then we have

$$\begin{aligned} J &= -\frac{c_{\mathrm{sVir}}}{3}\left(\frac{\psi_1^{(1)}}{\psi_0^{(1)} h_2^{(1)}} + \frac{\psi_1^{(2)}}{\psi_0^{(2)} h_2^{(2)}}\right) \\ &= \frac{n}{n+2}\left(\psi_1^{(1)} h_2^{(1)} \cdot \frac{\lambda_2^{(1)}}{\lambda_1^{(1)}\lambda_3^{(1)}} + \psi_1^{(2)} h_2^{(2)} \cdot \frac{\lambda_2^{(2)}}{\lambda_1^{(2)}\lambda_3^{(2)}}\right) \\ &= -\frac{\psi_1^{(1)} h_2^{(1)} + n\psi_1^{(2)} h_2^{(2)}}{n+1}, \end{aligned} \tag{6.11}$$

where we use (4.21), (6.4) and (6.5). One can check that this normalization agrees with the standard normalization of $\mathcal{N}=2$ super Virasoro algebra in (2.66). Namely, it is compatible with both of the following two relations:

$$[J_l, J_m] = \frac{c_{\mathrm{sVir}}}{3} l \delta_{l+m,0}, \qquad [J_m, G_r^{\pm}] = \pm G_{m+r}^{\pm}. \tag{6.12}$$

Finally, we note that the state with the conformal dimension (6.9) and the U(1) charge μ is uniquely determined to be

$$\begin{cases} \prod_{i=1}^{\mu} G_{-i-1/2}^{+} |0\rangle & (\text{for } \mu > 0) \\ \prod_{i=1}^{-\mu} G_{-i-1/2}^{-} |0\rangle & (\text{for } \mu < 0), \end{cases} \tag{6.13}$$

where $|0\rangle$ denotes the vacuum state.

6.3 Spectral flow of $\mathcal{N}=2$ super Virasoro algebra

It is known that $\mathcal{N}=2$ super Virasoro algebra has an automorphism called spectral flow. It is given as follows [64]:

$$\begin{aligned} L_n &\to L_n - \eta J_n + \frac{c_{\text{sVir}}}{6}\eta^2\delta_{n,0}, \\ J_n &\to J_n - \frac{c_{\text{sVir}}}{3}\eta\delta_{n,0}, \\ G_r^{\pm} &\to G_{r\mp\eta}^{\pm}, \end{aligned} \tag{6.14}$$

where η is an arbitrary parameter. One can see that the mode of the supercurrents changes under the automorphism. In particular, the integer mode and half-integer mode are exchanged if we set $\eta = \frac{1}{2}$. The former one is known as the NS-sector while the latter one is known as the R-sector. Because of this correspondence, it is enough to consider the NS-sector as we have done. In this section, we check that we can reproduce the spectral flow of $\mathcal{N}=2$ super Virasoro algebra from the automorphism (4.24) of the affine Yangian.

Because we have two affine Yangians, we have two parameters α, β for the automorphisms as follows:

$$\begin{cases} \frac{1}{2}\psi_2^{(1)} \to \frac{1}{2}\psi_2^{(1)} + \alpha\psi_1^{(1)} + \alpha^2\psi_0^{(1)} \\ \psi_1^{(1)} \to \psi_1^{(1)} + \alpha\psi_0^{(1)} \end{cases} \qquad \begin{cases} \frac{1}{2}\psi_2^{(2)} \to \frac{1}{2}\psi_2^{(2)} + \beta\psi_1^{(2)} + \beta^2\psi_0^{(2)} \\ \psi_1^{(2)} \to \psi_1^{(2)} + \beta\psi_0^{(2)}. \end{cases} \tag{6.15}$$

By requiring the invariance of the decoupled U(1) boson (6.10), we have

$$\alpha\psi_0^{(1)}h_2^{(1)} = \beta\psi_0^{(2)}h_2^{(2)}. \tag{6.16}$$

The other U(1) current (6.11) changes as follows:

$$\begin{aligned} J &\to J - \frac{c_{\text{sVir}}}{3}\left(\frac{\alpha}{h_2^{(1)}} + \frac{\beta}{h_2^{(2)}}\right) \\ &= J - \frac{c_{\text{sVir}}}{3}\left(\frac{\alpha\psi_0^{(1)}h_2^{(1)}}{\psi_0^{(1)}(h_2^{(1)})^2} + \frac{\beta\psi_0^{(2)}h_2^{(2)}}{\psi_0^{(2)}(h_2^{(2)})^2}\right) \\ &= J - \alpha\psi_0^{(1)}h_2^{(1)}, \end{aligned} \tag{6.17}$$

where we use

$$\psi_0^{(i)}(h_2^{(i)})^2 = -\frac{\lambda_1^{(i)}\lambda_3^{(i)}}{\lambda_2^{(i)}} = \begin{cases} \frac{n(n+1)}{n+2} & (i=1) \\ \frac{n+1}{n+2} & (i=2). \end{cases} \tag{6.18}$$

Comparing it with (6.14), we see that we should set

$$\alpha\psi_0^{(1)}h_2^{(1)} = \frac{c_{\text{sVir}}}{3}\eta. \tag{6.19}$$

The element $h = \frac{1}{2}(\psi_2^{(1)} + \psi_2^{(2)})$ which corresponds to L_0 transforms as follows:

$$\begin{aligned} h &\to h + (\alpha\psi_1^{(1)} + \beta\psi_1^{(2)}) + (\frac{\alpha^2}{2}\psi_0^{(1)} + \frac{\beta^2}{2}\psi_0^{(2)}) \\ &= h + \frac{c_{\mathrm{sVir}}}{3}\eta\left(\frac{\psi_1^{(1)}}{\psi_0^{(1)}h_2^{(1)}} + \frac{\psi_1^{(2)}}{\psi_0^{(2)}h_2^{(2)}}\right) + \frac{c_{\mathrm{sVir}}^2}{9}\eta^2\left(\frac{1}{2\psi_0^{(1)}(h_2^{(1)})^2} + \frac{1}{2\psi_0^{(2)}(h_2^{(2)})^2}\right) \\ &= h - \eta J + \frac{c_{\mathrm{sVir}}}{6}\eta^2, \end{aligned} \tag{6.20}$$

which is consistent with (6.14). In a similar way, one can easily check that the transformation of the other modes is also consistent with (6.14).

6.4 $\mathcal{N} = 2$ unitary minimal models

In this section, we study the $\mathcal{N} = 2$ unitary minimal models. As can be seen from (2.112) and (6.6), they are realized when we set the parameter n to a positive integer. As we explained in (2.114), there are primary fields with the following conformal dimension and U(1) charge:

$$h_{l,m} = \frac{l(l+2) - m^2}{4(n+2)}, \quad J_{l,m} = \frac{m}{n+2}, \tag{6.21}$$

where the integers l, m are in the range of

$$0 \le l \le n, \quad -l \le m \le l, \quad l - m \equiv 0 \ (\mathrm{mod}\ 2). \tag{6.22}$$

For convenience, we write down the character (2.115) again:

$$\chi_{l,m}(\tau, z) = \sum_{r \in \mathbb{Z}_{2n}} c_{l,m+2r}^{(n)}(\tau)\Theta_{2m+2r(n+2),2n(n+2)}(\tau, \frac{z}{n+2}). \tag{6.23}$$

We also note $c_{l,m}^{(1)}(\tau) = \frac{1}{\eta(\tau)}$.

The aim of this section is to reproduce these results by using plane partitions. We have already studied the Hilbert space for generic parameters in the previous sections. We have seen that it consists of two plane partitions, one of which has a pit at $(2,3,1)$ while the other has a pit at $(1,1,2)$. In the following, we refer to them as the first plane partition and the second plane partition, respectively. We also use terminologies such as the first affine Yangian and the second affine Yangian. In the case of $n \in \mathbb{Z}_{\ge 1}$, the first affine Yangian satisfies an additional truncation condition,

$$\frac{0}{\lambda_1} + \frac{0}{\lambda_2} + \frac{n}{\lambda_3} = 1, \tag{6.24}$$

which is clear from (6.4). Then the further constraint is imposed on the first plane partition. Concretely, it must satisfy the identification rule which we discussed in section 5.1. For the second plane partition, there is no additional condition as we mentioned in the last part of Section 5.1.

In the following, we study the Hilbert space in detail. We first discuss the case of $n = 1$, and then move to the case for the other n.

6.4.1 $n = 1$ case

Let us start from the simplest case $n = 1$, which corresponds to $c = 1$. In this case, there are three states as follows:

1. $l = m = 0 \qquad (h = J = 0)$
2. $l = m = 1 \qquad (h = \frac{1}{6}, J = \frac{1}{3})$
3. $l = -m = 1 \quad (h = \frac{1}{6}, J = -\frac{1}{3})$.

The first state corresponds to the vacuum state. The other two states correspond to degenerate primary fields. In terms of plane partitions, they should be realized from the asymptotic Young diagrams. For the first plane partition, we can insert them only in the first axis. We note that we cannot insert them in the other two axes due to the pits. For the second plane partition, we do not need to consider asymptotic Young diagrams because we are not interested in $\widehat{U}(1)$ modules. In the following, we consider the three cases one by one.

$\boldsymbol{l = m = 0}$ This case corresponds to the vacuum state, which should be realized when we insert no asymptotic Young diagram. Because there are pits both at $(2, 3, 1)$ and at $(1, 1, 2)$, we have two descriptions for the first plane partition. We note that this is precisely the situation we considered in Figure 17. Because any Young diagrams satisfy the identification rule, it is better to see the Hilbert space in terms of the plane partition with a pit at $(1, 1, 2)$.

When the intermediate Young diagram μ is positive, the first plane partition is identified with the Young diagram as follows:

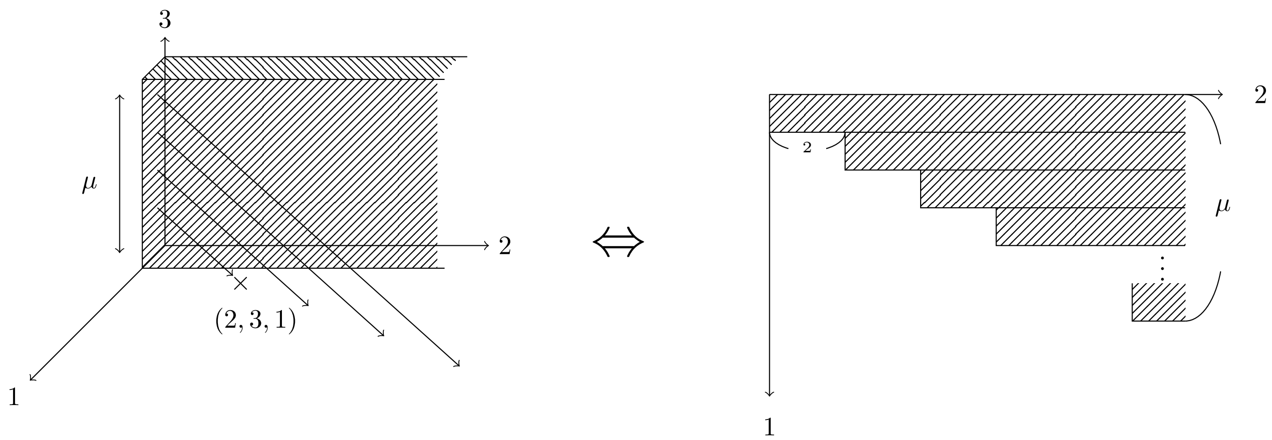


Figure 25: According to the identification rule, we can switch the two perspectives by translating boxes in the direction of $(1, 2, -1)$. It is obvious that the right figure does not make sense as a Young diagram. That means it is a null state.

This figure shows that the state is null because we cannot see the right figure as a Young diagram. That leads to the claim that the states (6.13) are null for $\mu > 1$ because the intermediate Young diagrams correspond to the primary states given in (6.13). One can check that this statement is indeed true by using the defining relation of $\mathcal{N} = 2$ super Virasoro algebra. For the detail of the calculation, see Appendix C.

To obtain the non-null states, we have to add at least $2 + 4 + \cdots + 2(\mu - 1)$ boxes as follows:

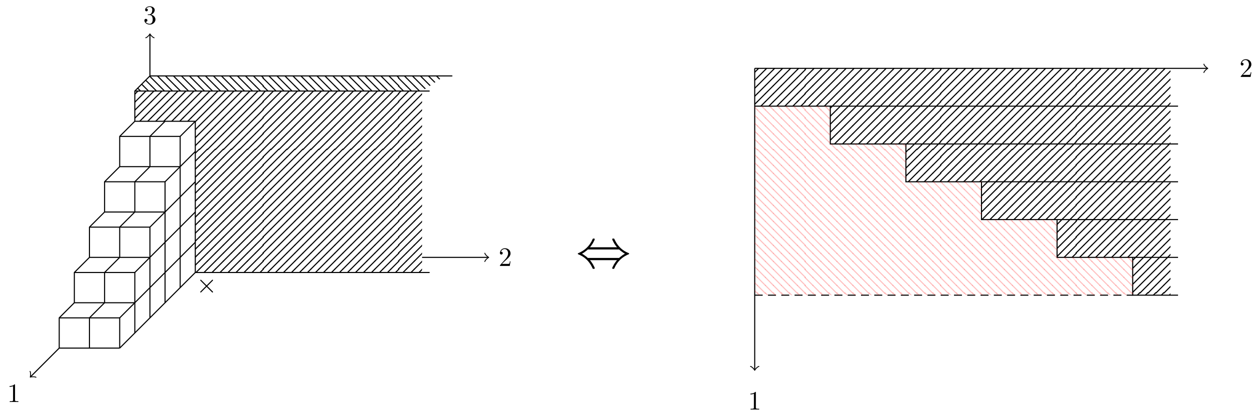


Figure 26: To obtain the non-null states, we need to fill in the red region with boxes. The left figure shows the corresponding configuration in terms of the first plane partition.

In a similar way, we have the following identification for the $\mu < 0$ sector:

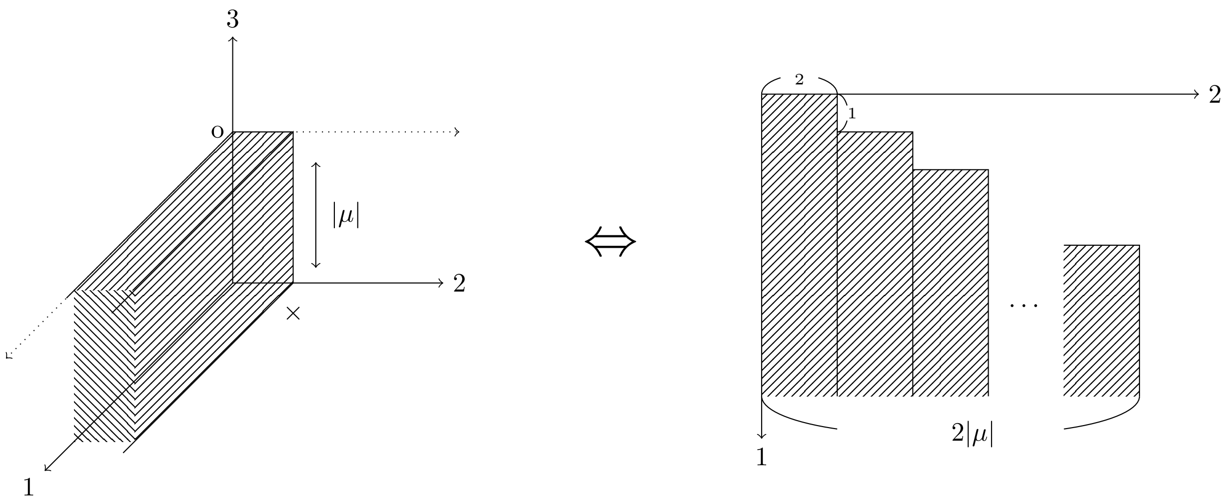


Figure 27: As in the case of $\mu > 0$, the right figure does not make sense as a Young diagram. To obtain non-null states, we need to add boxes.

We can obtain non-null states after we add $2 + 4 + \cdots + 2(|\mu| - 1)$ boxes as follows:

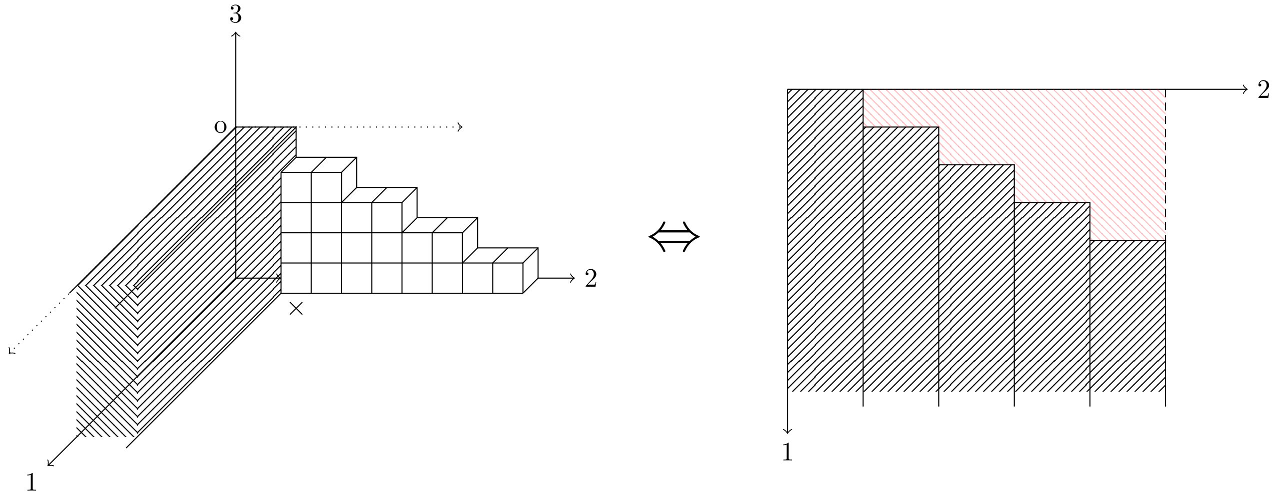


Figure 28: We need to fill the red region with boxes. The left figure shows the corresponding configuration in terms of the first plane partition.

Due to the above process, the conformal dimension increases by the number of the added boxes. After we add the boxes, the conformal dimension of the state changes from (6.9) to

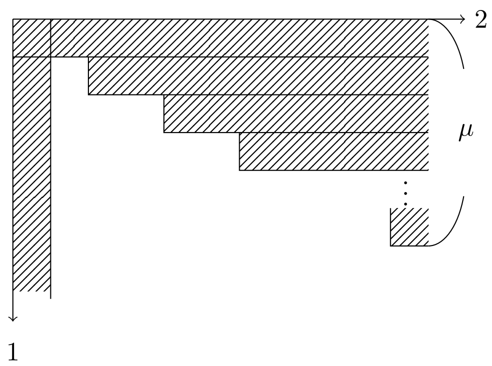
$$h = \frac{|\mu|(|\mu|+2)}{2} + \sum_{i=1}^{|\mu|-1} 2i = \frac{3\mu^2}{2}. \tag{6.25}$$

Once we add the boxes, the Hilbert space is described by the tensor product of the two Young diagrams. Thus, we have the character as follows:

$$\begin{aligned} \chi(\tau, z) &= \frac{\sum_{\mu=-\infty}^{\infty} q^{\frac{3\mu^2}{2}} y^{\mu}}{\eta(\tau)} \cdot \frac{1}{\eta(\tau)} \\ &= \frac{(\Theta_{0,6}(\tau, \frac{z}{3}) + \Theta_{6,6}(\tau, \frac{z}{3}))}{\eta(\tau)} \cdot \frac{1}{\eta(\tau)}. \end{aligned} \tag{6.26}$$

Up to the factor $\frac{1}{\eta(\tau)}$ coming from the extra U(1) factor, it is exactly the vacuum character of $c = 1$ $\mathcal{N} = 2$ unitary minimal models [35].

$\boldsymbol{l = 1,\ m = -1}$ Next, let us consider the primary field for $l = 1, m = -1$. To realize a non-vacuum state, we need to introduce an asymptotic Young diagram in the first axis of the first plane partition. Because of the pits, the possible shape is limited to □ or □□. Here, we consider the first one. Because of the asymptotic Young diagram, the configuration changes as follows:



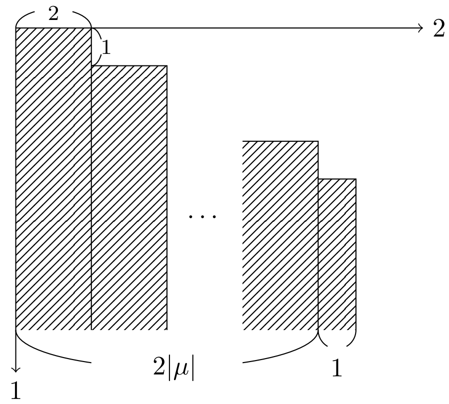


Figure 29: The asymptotic Young diagram is inserted in the first axis.

As in the case of the vacuum state, we have to add boxes. The number of the necessary boxes is

$$\sum_{i=1}^{\mu-1}(2i-1)=(\mu-1)^2 \tag{6.27}$$

for $\mu>0$ and

$$\sum_{i=1}^{|\mu|-1} 2i+|\mu|=\mu^2 \tag{6.28}$$

for $\mu<0$.

Let us compute the U(1) charge and the conformal dimension. We can compute the U(1) charge (6.11) from (4.56), (4.57). For the asymptotic Young diagram in the first axis, its contribution is evaluated as

$$\frac{h_2^{(1)}}{2h_1^{(1)}}=\frac{\lambda_1^{(1)}}{2\lambda_2^{(1)}}=-\frac{1}{3}, \tag{6.29}$$

where we use (6.4). Combining it with the contribution from the intermediate Young diagram μ, we have

$$J=\mu-\frac{1}{3}. \tag{6.30}$$

For the conformal dimension h, the computation is a bit more complicated. We first discuss the case of $\mu>0$. There are three factors contributing to the conformal dimension: the intermediate Young diagram, the asymptotic Young diagram and the added boxes. The first one gives (6.7). The second one can be evaluated from (4.56) and (6.4) as

$$h(M_{\square}^1)=-\frac{\lambda_1^{(1)}}{2\lambda_3^{(1)}}-\frac{\lambda_1^{(1)}}{2\lambda_2^{(1)}}+\frac{\lambda_1^{(1)}}{2}=\frac{1}{3}. \tag{6.31}$$

The third one gives (6.27). Because we count the box at the origin twice, we need to subtract 1 from the sum of these contributions. Then we have

$$\begin{aligned} h &= \frac{\mu(\mu+2)}{2} + \frac{1}{3} + (\mu-1)^2 - 1 \\ &= \frac{3}{2}\big(\mu - \frac{1}{3}\big)^2 + \frac{1}{6}. \end{aligned} \tag{6.32}$$

For the $\mu < 0$ case, there is no overlapping box and we just have to sum up the three factors. By using (6.8), (6.31) and (6.28), one can immediately check that the expression (6.32) is also true for $\mu < 0$.

We note that this is the conformal dimension for the total Virasoro algebra. Namely, it contains not only $\mathcal{N} = 2$ super Virasoro algebra but also an extra U(1) current $j(z)$. We have to remove the latter contribution. The energy-momentum tensor is given by the Sugawara construction and its zero-mode $L_0^{U(1)}$ is determined from (6.10) as

$$L_0^{U(1)} = \frac{\big(\psi_1^{(1)} h_2^{(1)} - \psi_1^{(2)} h_2^{(2)}\big)^2}{2\big(\psi_0^{(1)}(h_2^{(1)})^2 + \psi_0^{(2)}(h_2^{(2)})^2\big)} + \cdots . \tag{6.33}$$

We note that the denominator is not standard due to the normalization of U(1) current (4.30). We also note that we can ignore the terms other than the first term because they vanish when they act on the highest weight state. In order to compare (6.32) with the known result (6.21), we have to eliminate the contribution from (6.33). Because the intermediate Young diagram has no charge for j, we only have to consider the contribution from the asymptotic Young diagram in the first axis, which is evaluated as

$$\frac{\big(\psi_1^{(1)} h_2^{(1)} - \psi_1^{(2)} h_2^{(2)}\big)^2}{2\big(\psi_0^{(1)}(h_2^{(1)})^2 + \psi_0^{(2)}(h_2^{(2)})^2\big)} = \frac{(h_2^{(1)}/h_1^{(1)})^2}{-2\left(\frac{\lambda_1^{(1)}\lambda_3^{(1)}}{\lambda_2^{(1)}} + \frac{\lambda_1^{(2)}\lambda_3^{(2)}}{\lambda_2^{(2)}}\right)} = \frac{1}{6}. \tag{6.34}$$

Let us redefine h as the conformal dimension after subtracting this factor. Then we have

$$h = \frac{3}{2}\big(\mu - \frac{1}{3}\big)^2. \tag{6.35}$$

From this, we see that the highest weight state (= the state with the smallest conformal dimension) exists in the $\mu = 0$ sector. The conformal dimension and the U(1) charge is

$$h = \frac{1}{6}, \quad J = -\frac{1}{3}, \tag{6.36}$$

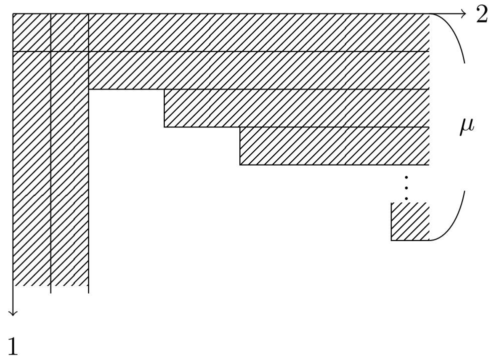
which agrees with the literature.

We can compute the character in the same way as the vacuum case as follows:

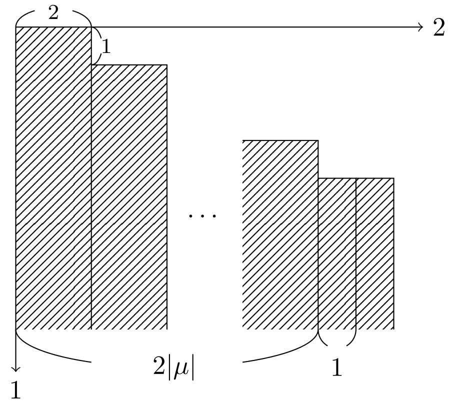
$$\begin{aligned} \chi(\tau, z) &= \frac{\sum_{\mu=-\infty}^{\infty} q^{\frac{3}{2}(\mu-\frac{1}{3})^2} y^{\mu-\frac{1}{3}}}{\eta(\tau)} \cdot \frac{q^{\frac{1}{6}}}{\eta(\tau)} \\ &= \frac{\big(\Theta_{-2,6}(\tau, \frac{z}{3}) + \Theta_{4,6}(\tau, \frac{z}{3})\big)}{\eta(\tau)} \cdot \frac{q^{\frac{1}{6}}}{\eta(\tau)}. \end{aligned} \tag{6.37}$$

Again, it agrees with the literature [35].

$l = m = 1$ Finally, we consider the case with the asymptotic Young diagram $\square\square$.



(i) $\mu > 0$



(ii) $\mu < 0$

By a similar calculation as in the previous case, we have

$$J = \mu - \frac{2}{3}, \qquad h = \frac{3}{2}\left(\mu - \frac{2}{3}\right)^2 + \frac{2}{3}. \tag{6.38}$$

The contribution from $L_0^{U(1)}$ is evaluated as $\frac{2}{3}$. We see that the highest weight state exists in the $\mu = 1$ sector. Its conformal dimension and the U(1) charge is

$$h = \frac{1}{6}, \quad J = \frac{1}{3}, \tag{6.39}$$

which agrees with the literature. The character is

$$\begin{aligned} \chi(\tau, z) &= \frac{\sum_{\mu=-\infty}^{\infty} q^{\frac{3}{2}(\mu-\frac{2}{3})^2} y^{\mu-\frac{2}{3}}}{\eta(\tau)} \cdot \frac{q^{\frac{2}{3}}}{\eta(\tau)} \\ &= \frac{(\Theta_{2,6}(\tau, \frac{z}{3}) + \Theta_{-4,6}(\tau, \frac{z}{3}))}{\eta(\tau)} \cdot \frac{q^{\frac{2}{3}}}{\eta(\tau)}, \end{aligned} \tag{6.40}$$

which agrees with the literature [35].

6.4.2 General n

We have seen that we can reproduce the $c = 1$ $\mathcal{N} = 2$ unitary minimal model from plane partitions. In this section, we discuss the case for the other integer n. In this case, the first plane partition has a pit at $(1, 1, n + 1)$ as well as at $(2, 3, 1)$. The asymptotic Young diagram $\nu = (\nu_1, \nu_2)$ in the first axis is constrained as $0 \le \nu_2 \le \nu_1 \le n$. Let us first compute the conformal dimension and the U(1) charge. When the intermediate Young diagram is empty, we can immediately obtain them from (4.56) as

$$h = \frac{-2\nu_1\nu_2 + \nu_1 + (2n + 3)\nu_2}{2(n + 2)}, \quad J = -\frac{\nu_1 + \nu_2}{n + 2}. \tag{6.41}$$

We note that we have already removed the contribution from the U(1) factor (6.33). In the following, we ignore the U(1) factor.

The highest weight state exists in the $\mu = 1$ sector if $\nu_2 \geq 1$ and in the $\mu = 0$ sector if $\nu_2 = 0$. This is because the intermediate Young diagram not only increases the conformal dimension by (6.9) but also decreases it by the number of the overlapping boxes. Then the conformal dimension and the U(1) charge for the highest weight state are summarized as follows:

$$\begin{cases} h = \frac{-2\nu_1\nu_2+\nu_1+(2n+3)\nu_2}{2(n+2)} - \frac{1}{2}, & J = 1 - \frac{\nu_1+\nu_2}{n+2} \qquad (\nu_2 \geq 1), \\ h = \frac{\nu_1}{2(n+2)}, & J = -\frac{\nu_1}{n+2} \qquad (\nu_2 = 0). \end{cases} \tag{6.42}$$

It should match with (6.21). Indeed, there is a one-to-one correspondence between (6.21) and (6.42) as follows:

$$\begin{cases} \nu_1 = n + 1 - \frac{l+m}{2}, \quad \nu_2 = \frac{l-m}{2} + 1 & (\nu_2 \geq 1), \\ \nu_1 = l = -m & (\nu_2 = 0). \end{cases} \tag{6.43}$$

Thus, the asymptotic Young diagrams correctly provide all the primary fields in the $\mathcal{N} = 2$ unitary minimal model.

In the remaining part, we check that we can reproduce the correct character from plane partitions. As in the case of $n = 1$, we study the Hilbert space in terms of the plane partition with a pit at $(1, 1, n + 1)$. When the intermediate Young diagram is given by $\mu = kn + r$ ($k \in \mathbb{Z}, 0 \leq r < n$), the configuration is identified as follows:

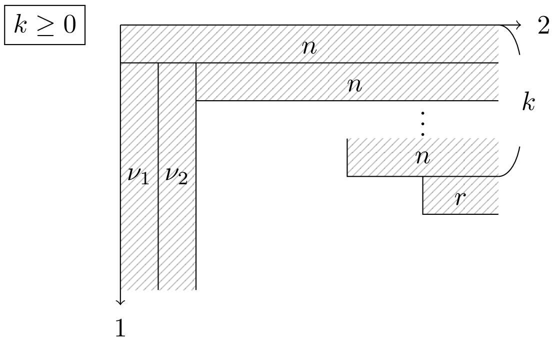
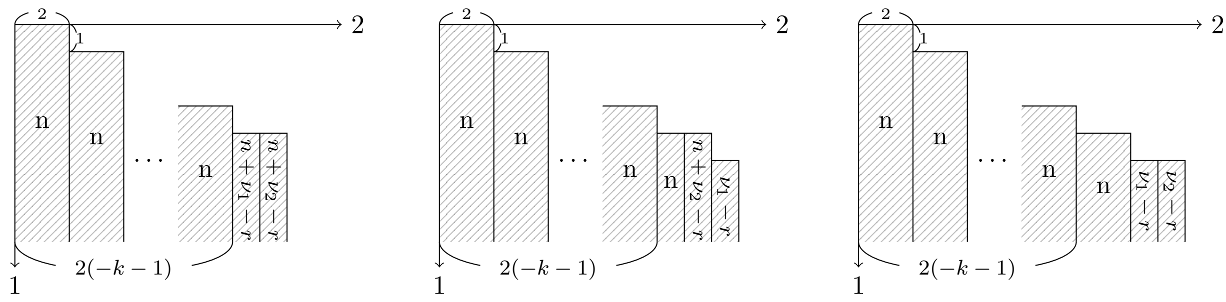


Figure 30: This is the figure for the case of $k \geq 0$. It is viewed from the top. The numbers written in the rectangles denote the heights to the direction of the third axis.

$\boxed{k<0}$



(i) $\nu_2 \leq \nu_1 \leq r$ (ii) $\nu_2 \leq r < \nu_1$ (iii) $r < \nu_2 \leq \nu_1$

Figure 31: For the case of $k < 0$, the configuration changes depending on the value of r.

As in the case of $n = 1$, we have to add boxes to obtain non-null states. Let us first consider the case of $k \geq 1$. In Figure 30, the number of the boxes we have to add to i-th row is as follows:

$$\begin{cases} 2n(i-1) - (\nu_1 + \nu_2) & (2 \leq i \leq k) \\ (2k-2)r + \theta & (i = k+1), \end{cases} \tag{6.44}$$

where we set

$$\theta = \begin{cases} 0 & (r \leq \nu_2) \\ r - \nu_2 & (\nu_2 < r \leq \nu_1) \\ (r - \nu_1) + (r - \nu_2) & (r > \nu_1). \end{cases} \tag{6.45}$$

We also need to remove the overlapping boxes at $x_1 = 1, x_2 = 1, 2$ so that we will not count them twice. For convenience, we subtract the number of these boxes from (6.44) in advance. Then we can evaluate their contribution to the conformal dimension as

$$\begin{aligned} &(\text{the number of the added boxes}) - (\text{the number of overlapping boxes}) \\ &= (k-1)kn - k(\nu_1 + \nu_2) + (2k-2)r + \theta. \end{aligned} \tag{6.46}$$

We note that this expression is also true for $k = 0$; we do not need to add boxes and the number of the overlapping boxes is indeed given by $2r - \theta$. After adding the boxes, the conformal dimension becomes the sum of (6.7), (6.41) and (6.46). Then the conformal dimension and the U(1) charge are given as

$$h = \frac{-2\nu_1\nu_2 + \nu_1 + (2n+3)\nu_2}{2(n+2)} + \frac{n(n+2)}{2}k^2 + \Big((n+2)r - (\nu_1 + \nu_2)\Big)k + \frac{r^2}{2} - r + \theta, \tag{6.47}$$

$$J = kn + r - \frac{\nu_1 + \nu_2}{n+2} \tag{6.48}$$

for $k \geq 0$. Actually, this expression is also true for $k < 0$. Let us check it in the case of $r < \nu_2$ (the third case in Figure 31). The number of the boxes we have to add is

$k(k+1)n - k(\nu_1+\nu_2-2r)$ and there are no overlapping boxes. Combining it with (6.8) and (6.41), we have (6.47). One can check the other two cases in a similar way. For the comparison with the literature, it is convenient to rewrite (6.47) and (6.48) in terms of the parameters l, m. From (6.43), we have

$$\begin{cases} h = \frac{n(n+2)}{2}\left(k + \frac{m+(r-1)(n+2)}{n(n+2)}\right)^2 + h^{\rm PF}_{l,m+2(r-1)} + \theta \\ J = \frac{m}{n+2} + kn + r - 1 \end{cases} \qquad (l \neq -m), \tag{6.49}$$

$$\begin{cases} h = \frac{n(n+2)}{2}\left(k + \frac{m+r(n+2)}{n(n+2)}\right)^2 + h^{\rm PF}_{l,m+2r} + \theta \\ J = \frac{m}{n+2} + kn + r \end{cases} \qquad (\, l = -m), \tag{6.50}$$

where $h^{\rm PF}_{l,m}$ is the conformal dimension of the SU(2) parafermion given in (2.76).

Once we have filled the blank with boxes, the shape of the plane partition makes sense and the corresponding state becomes non-null. We can obtain the descendant states by adding boxes to this plane partition so that the identification rule will be satisfied. We have already studied this Hilbert space in terms of $Y_{0,0,n} = W_n \times \widehat{U}(1)$ in Section 5. On the other hand, we can also see it as the Hilbert space of $Y_{1,2,0}$, namely the SU(2) parafermion (and an extra U(1) boson). Both descriptions are equivalent[33], but the latter one is more convenient for the comparison with (6.23) because the character of the parafermion can be expressed by the string function.

The remaining thing we have to do is to clarify the correspondence between the asymptotic Young diagram and the primary field for the parafermion system. Let us consider the $k \geq 0$ case. In this case, the asymptotic Young diagram in the first (resp. second) axis is (ν_1, ν_2) (resp. (r)). As we discussed in Section 6.1, one can shift ν_1, ν_2, r by the same value. That can also be seen from the cyclic symmetry of N-tuple Young diagrams in Figure 20. Namely, we can change their order cyclicly because of the identification rule. By using this property, we can set $r = 0$. The asymptotic Young diagram (ν_1, ν_2) changes into $(\nu_1 - r, \nu_2 - r)$ if $r < \nu_2$, $(n + \nu_2 - r, \nu_1 - r)$ if $\nu_2 \leq r \leq \nu_1$ or $(n + \nu_1 - r, n + \nu_2 - r)$ if $r \geq \nu_1$. This is exactly the same as $k < 0$ case, as we can see from Figure 31.

We can compute the conformal dimension from (4.56) and we have

$$h = \begin{cases} h^{\rm PF}_{l,m+2(r-1)} & (r \leq \nu_2) \\ h^{\rm PF}_{n-l,m+2(r-1)-n} & (\nu_2 < r \leq \nu_1) \\ h^{\rm PF}_{l,m+2(r-1)-2n} & (r > \nu_1) \end{cases} \tag{6.51}$$

[33]This equivalence has been known for a long time (for example, [65]) and was recently proved in [66].

for $l \neq -m$ and

$$h = \begin{cases} h^{\mathrm{PF}}_{l,m+2r} & (r=0) \\ h^{\mathrm{PF}}_{n-l,m+2r-n} & (0 < r \leq \nu_1) \\ h^{\mathrm{PF}}_{l,m+2r-2n} & (r > \nu_1) \end{cases} \tag{6.52}$$

for $l = -m$. One can rewrite this into a simpler form:

$$h = \begin{cases} h^{\mathrm{PF}}_{l,m+2(r-1)} + \theta & (l \neq -m) \\ h^{\mathrm{PF}}_{l,m+2r} + \theta & (l = -m), \end{cases} \tag{6.53}$$

which coincides with the factor in (6.49) and (6.50). Actually, these modules are the same because of the identification (2.72) and (2.74). Therefore, we can express the generating function of the plane partition by the string function $c_{l,m+2(r-1)}(\tau)$ for $l \neq -m$ and $c_{l,m+2r}(\tau)$ for $l = -m$.

From the above, we can compute the character as follows:

$$\begin{aligned} &\chi(\tau, z) \\ = &\sum_{r \in \mathbb{Z}_n} c^{(n)}_{l,m+2r}(\tau) \sum_{k=-\infty}^{\infty} q^{\frac{n(n+2)}{2}\left(k + \frac{m+r(n+2)}{n(n+2)}\right)^2} y^{kn+r+\frac{m}{n+2}} \\ = &\sum_{r \in \mathbb{Z}_n} c^{(n)}_{l,m+2r}(\tau) \Big(\Theta_{2m+2r(n+2),2n(n+2)}(\tau, \frac{z}{n+2}) + \Theta_{2m+2(r+n)(n+2),2n(n+2)}(\tau, \frac{z}{n+2}) \Big) \\ = &\sum_{r \in \mathbb{Z}_{2n}} c^{(n)}_{l,m+2r}(\tau) \Theta_{2m+2r(n+2),2n(n+2)}(\tau, \frac{z}{n+2}), \end{aligned} \tag{6.54}$$

which agrees with the known result (6.23).

7 Bershadsky-Polyakov algebra

As we discussed in Section 2.2.2, the W-algebra obtained from Drinfeld-Sokolov reduction varies depending on SU(2) embedding. The best understood W-algebras are those associated with principal embedding. Both algebras we studied in the previous sections correspond to such cases. On the other hand, not much is known for non-principal W-algebras. The simplest non-principal W-algebra is $DS_2[\widehat{SU}(3)_k]$, which is known as Bershadsky-Polyakov (BP) algebra [67, 68]. Because SU(3) is decomposed as $\underline{8} = \underline{1} + \underline{2} + \underline{2} + \underline{3}$, it is expected that BP algebra is generated by the currents with conformal dimension $1, \frac{3}{2}, \frac{3}{2}, 2$. That is indeed true. Unlike the other W-algebras, one can write down the OPE relation explicitly as follows:

$$\begin{aligned}
&J(z)J(w) \sim \frac{2k+3}{3(z-w)^2}, \qquad G^{\pm}(z)G^{\pm}(w) \sim 0, \qquad J(z)G^{\pm}(w) \sim \pm\frac{1}{z-w}G^{\pm}(w),\\
&T(z)T(w) \sim -\frac{(2k+3)(3k+1)}{2(k+3)(z-w)^4} + \frac{2}{(z-w)^2} + \frac{\partial T(w)}{z-w},\\
&T(z)G^{\pm}(w) \sim \frac{3}{2(z-w)^2}G\pm(w) + \frac{\partial G^{\pm}(w)}{z-w}, \quad T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w},\\
&G^{+}(z)G^{-}(w) \sim \frac{(k+1)(2k+3)}{(z-w)^3} + \frac{3(k+1)}{(z-w)^2}J(w)\\
&\qquad\qquad + \frac{1}{z-w}\left(3 : J(w)^2 : + \frac{3(k+1)}{2}\partial J(w) - (k+3)T(w)\right),\\
&G^{\pm}(z)G^{\pm}(w) \sim 0.
\end{aligned} \tag{7.1}$$

Here, $J(z), G^{\pm}(z), T(z)$ are the currents with conformal dimension $1, \frac{3}{2}, \frac{3}{2}, 2$, respectively. It may look similar to $\mathcal{N} = 2$ super Virasoro algebra, but $G^{\pm}(z)$ are bosonic. For later convenience, we redefine the energy-momentum tensor as

$$\tilde{T}(z) = T(z) + \frac{1}{2}\partial J(z), \tag{7.2}$$

which changes the conformal dimension of $G^{+}(z)$ and $G^{-}(z)$ to 1 and 2, respectively. We denote the mode expansion as follows:

$$\begin{aligned}
\tilde{T}(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1},\\
G^{+}(z) &= \sum_{n\in\mathbb{Z}} G^{+}_n z^{-n-1}, \qquad G^{-}(z) = \sum_{n\in\mathbb{Z}} G^{-}_n z^{-n-2}.
\end{aligned} \tag{7.3}$$

For later use, we rewrite the OPE relation between $G^{+}(z)$ and $G^{-}(w)$ into the commutator as

$$\begin{aligned}
[G^{+}_m, G^{-}_n] &= 3\sum_{l\in\mathbb{Z}} : J_l J_{m+n-l} : + \big(3(k+1)m - (2k+3)(m+n+1)\big)J_{m+n}\\
&\qquad - (k+3)L_{m+n} + \frac{(k+1)(2k+3)m(m+1)}{2}\delta_{m+n,0}.
\end{aligned} \tag{7.4}$$

In this section, we study the rationality of BP algebra discussed in [69]. Here, a W-algebra is called rational when it has a finite number of irreducible representations and the other properties such as modularity are satisfied. The typical example is a minimal model[34]. In the following, we check that we can reproduce the correct modules by plane partitions. We first summarize the result of [69] in Section 7.1. In Section 7.2, we discuss it in terms of plane partitions. The analysis is almost the same as that in Section 6.

7.1 Modules

The highest weight representation of BP algebra is defined as follows:

$$
\begin{aligned}
&L_0 |h,J\rangle = h|h,J\rangle , \quad J_0|h,J\rangle = J|h,J\rangle , \\
&L_n|h,J\rangle = 0, \quad J_n|h,J\rangle = 0, \quad G_n^+|h,J\rangle = 0 \quad (n>0), \\
&G_n^-|h,J\rangle = 0 \quad (n \geq 0).
\end{aligned} \tag{7.5}
$$

Because the action of G_0^+ does not change the conformal dimension, the highest weight state is degenerate. It may look similar to Ramond sector of $\mathcal{N}=2$ super Virasoro algebra, but there is a significant difference. Because $G^+(z)$ is bosonic, the multiplicity of the highest weight state can be more than 2. In the following, we focus on the modules such that the multiplicity is finite. We denote by $i \in \mathbb{N}$ the multiplicity, namely

$$
(G_0^+)^{i-1}|h,J\rangle = 0. \tag{7.6}
$$

BP algebra has an automorphism similar to the spectral flow of $\mathcal{N}=2$ super Virasoro algebra. The elements transform as follows:

$$
J_n \to J_n - \frac{2k+3}{3}\delta_{n,0}, \quad L_n \to L_n - J_n + \frac{2k+3}{3}\delta_{n,0}, \quad G_n^+ \to G_{n-1}^+, \quad G_n^- \to G_{n+1}^-. \tag{7.7}
$$

The conformal dimension and U(1) charge (h,J) for the highest weight state transform as

$$
(h,J) \to (h',J') \equiv (h-(J-i+1)+\frac{2k+3}{3}, J+i-1-\frac{2k+3}{3}). \tag{7.8}
$$

We set the multiplicity of this state to $j \in \mathbb{N}$, namely

$$
(G_0^+)^{j-1}|h',J'\rangle = 0. \tag{7.9}
$$

We note that the multiplicity is not finite in general, so the module satisfying (7.9) corresponds to the degenerate module. One may also think of it as an analogue of the integrable representation of affine Kac-Moody algebras. The condition (7.6) requires the norm to vanish as

$$
\langle h,J|(G_0^-)^{i-1}(G_0^+)^{i-1}|h,J\rangle = 0. \tag{7.10}
$$

[34]One may not need to distinguish rational CFTs and minimal models in this thesis.

The same is true for (7.9). Using (7.4), we have

$$\begin{aligned} J = J_{i,j} &\equiv \frac{1}{3}(-2i - j + 2k + 6), \\ h = h_{i,j} &\equiv \frac{i^2 + ji - ki - 3i + j^2 - 6j - 2jk + 3k + 6}{3(k+3)}. \end{aligned} \tag{7.11}$$

It was shown in [69] that one can construct the rational CFT when

$$k + 3 = \frac{p}{2} \qquad (p = 3, 5, 7, 9, \cdots). \tag{7.12}$$

In this case, we can see

$$(G^{+}_{-1})^{p-2} |0\rangle = 0 \tag{7.13}$$

by computing the norm. Here, we denote by $|0\rangle$ the vacuum state. This condition implies $: G^+(z)^{p-2} := 0$ by state-operator correspondence. Therefore, we have $1 \leq i, j \leq p - 2$. Furthermore, the conformal dimension and U(1) charge should be in the form of (7.11) after the transformation (7.8). That requires $J_{i,j} + i - 1 - \frac{2k+3}{3} \leq \frac{1}{3}(-2j - 1 + 2k + 6)$, which leads to $j \leq p - i - 1$. Thus, the parameters i, j lie in the range

$$1 \leq i \leq p - 2, \quad 1 \leq j \leq p - i - 1. \tag{7.14}$$

Summarizing, there are degenerate modules labeled by (7.14) and their conformal dimension and U(1) charge are given by (7.11).

7.2 Realization by plane partitions

Bershadsky-Polyakov algebra and an extra U(1) boson are realized by the following diagram:

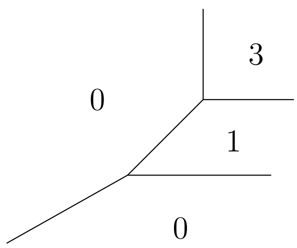


Figure 32: This diagram realizes $DS_2[\widehat{U}(3)_\Psi]$, which is the product of Bershadsky-Polyakov algebra and U(1) boson.

We have $Y_{0,1,3}[\Psi]$ at the upper corner and $Y_{0,0,1}[\Psi - 1]$ at the lower corner. We can interpret them in terms of the affine Yangians with the following parameters [27]:

$$\begin{aligned} &\lambda_1^{(1)} = 3\Psi^{-1} - 2, \quad \lambda_2^{(1)} = \frac{2\Psi - 3}{\Psi - 1}, \quad \lambda_3^{(1)} = 2\Psi - 3, \\ &\lambda_1^{(2)} = 1, \quad \lambda_2^{(2)} = -\frac{\Psi - 2}{\Psi - 1}, \quad \lambda_3^{(2)} = \Psi - 2. \end{aligned} \tag{7.15}$$

The central charge is given by

$$c = -\frac{(2\Psi - 3)(3\Psi - 8)}{\Psi} + 1, \tag{7.16}$$

where the second term comes from the decoupled U(1) boson. The relation between Ψ and k is given by

$$\Psi = k + 3. \tag{7.17}$$

For ease of understanding, we give a figure for the $\mu > 0$ sector:

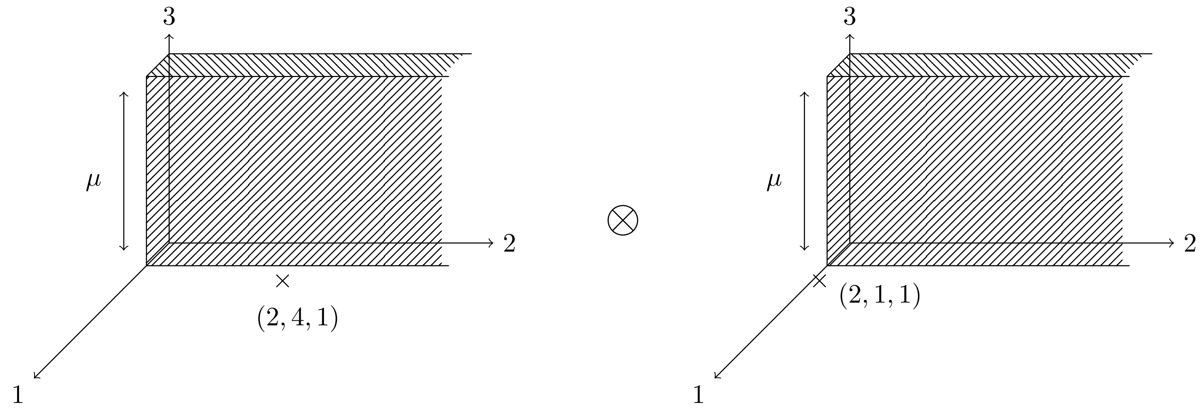


Figure 33: This figure shows the $\mu > 0$ sector. We omit the $\mu < 0$ case.

One can check that the conformal dimension of the intermediate Young diagram is $\frac{3}{2}|\mu|$. In terms of the modified energy-momentum tensor in (7.2), the conformal dimension is given by μ for $\mu > 0$ and $2|\mu|$ for $\mu < 0$. The configuration is identified with the state $(G^{+}_{-1})^{\mu} |0\rangle$ for $\mu > 0$ and $(G^{-}_{-2})^{|\mu|} |0\rangle$ for $\mu < 0$.

As in the case of $\mathcal{N} = 2$ super Virasoro algebra, we denote by j the zero mode of the decoupled U(1) current. We also set $J = J_0$. Then we have

$$j = \psi_1^{(1)} h_2^{(1)} - \psi_1^{(2)} h_2^{(2)}, \tag{7.18}$$

$$J = -\frac{2\Psi - 3}{3} \left(\frac{\psi_1^{(1)}}{\psi_0^{(1)} h_2^{(1)}} + \frac{\psi_1^{(2)}}{\psi_0^{(2)} h_2^{(2)}} \right). \tag{7.19}$$

Now we are ready to check whether we can reproduce the modules (7.11) from the affine Yangian. The condition (7.12) means $2\Psi = p$ ($p = 3, 5, 7, \cdots$). In this case, the additional pit appears at $(1, 1, p - 2)$. This is exactly the situation we discussed in Section 5.1. We note that the condition (5.2) is satisfied: $(p - 3) + 1$ and $(p - 3) + 3$ are coprime because p is an odd number. Then we can do the calculation in a similar way as in the case of $\mathcal{N} = 2$ super Virasoro algebra.

We can insert the asymptotic Young diagram $\nu = (\nu_1, \nu_2, \nu_3)$ in the first axis of the first plane partition[35]. Due to the pit at $(1, 1, p-2)$, the height of the Young diagram is at most $p-3$, namely $0 \leq \nu_3 \leq \nu_2 \leq \nu_1 \leq p-3$. Let us consider which configuration corresponds to the highest weight state. It is clear that the intermediate Young diagram $\mu < 0$ increases the conformal dimension. Let us focus on $\mu > 0$ sector. If we change μ to $\mu + 1$, the conformal dimension seems to increase by 1, but we need to take into account that we have to subtract the number of the overlapping boxes. As a result, the conformal dimension decreases when $\mu < \nu_2$, does not change when $\nu_2 \leq \mu < \nu_1$ and increases when $\mu \geq \nu_1$. Therefore, the highest weight state exists in the sector of $\mu = \nu_2$. Because the conformal dimension takes the same value in the region $\nu_2 \leq \mu \leq \nu_1$, the multiplicity of the highest weight state is $\nu_1 - \nu_2 + 1$. We will see soon that it is consistent with (7.6).

We can compute the conformal dimension h and the U(1) charge J by using (4.56). For the U(1) charge, we immediately have

$$\begin{aligned} J &= -\frac{\nu_1 + \nu_2 + \nu_3}{3} + \nu_2 \\ &= \frac{(\nu_2 - \nu_1) + (\nu_2 - \nu_3)}{3}, \end{aligned} \tag{7.20}$$

where the first term in the first line comes from the asymptotic Young diagram ν while the second one comes from the intermediate Young diagram. For the conformal dimension, we have

$$h = \frac{(\nu_1 - \nu_2)^2 + (\nu_2 - \nu_3)^2 + (\nu_3 - \nu_1)^2 + 6(1-\Psi)(\nu_1 - \nu_3) + 2\Psi\big((\nu_1 - \nu_2) + (\nu_1 - \nu_3)\big)}{6\Psi}. \tag{7.21}$$

See Appendix D for the detail of the computation. One can see that both (7.20) and (7.21) are invariant under the shift $(\nu_1, \nu_2, \nu_3) \to (\nu_1 + n, \nu_2 + n, \nu_3 + n)$. Comparing them with (7.11), we find the following one-to-one correspondence:

$$\nu_1 - \nu_2 = i - 1, \quad \nu_2 - \nu_3 = p - i - j - 1. \tag{7.22}$$

One can see that the condition $0 \leq \nu_3 \leq \nu_2 \leq \nu_1 \leq p-3$ leads to (7.14). We note that the multiplicity of the highest weight state is also compatible with (7.6).

[35] As mentioned earlier, we do not need to consider the second plane partition because there is an extra U(1) boson.

8 Conclusion

In this thesis, we studied the minimal models' representation theory by using plane partitions. We have first seen that the doubly constrained plane partition provides the correct Hilbert space of the W_N minimal model. By combining this result with Gaiotto-Rapcak's and Prochazka-Rapcak's construction, we proposed the prescription to describe the Hilbert space of the generalized Y-algebra. We mainly considered $\mathcal{N} = 2$ unitary minimal model and demonstrated that we can reproduce all the ingredients in the minimal models by plane partitions. More concretely, we saw that we can reproduce all the primary fields from the asymptotic Young diagrams and the character from the box-counting. We also showed that our construction reproduces the rational CFT for Bershadsky-Polyakov algebra.

The main advantage of using plane partitions is that we can compute the character for various W-algebras simply by counting the number of boxes. There are several applications of this technique to particle physics. First, it helps us understand higher spin theories on AdS_3 through the study of the 2d CFTs living at the boundary. In $\mathrm{Ads}_3/\mathrm{CFT}_2$, it is important to confirm that the partition function of the bulk theory matches that of the boundary theory. In particular, when the boundary is given by a torus, the partition function of the boundary 2d CFT must be expressed as the modular invariant combination of the characters. This constraint provides us nontrivial information for the consistency of the bulk theory. For example, the authors of [70] discussed the pure gravity in detail by using Virasoro minimal models. The similar analysis should be possible for higher spin theories as was studied for simple cases in [70]. The boundary theory of the higher spin theory is considered to be the minimal model for W-algebra, and our technique can be used to determine its partition function.

Another application is related to 4d superconformal field theories (SCFTs). According to the duality between 4d SCFT and 2d CFT [23], the character is mapped to the index in the 4d side, which counts the number of the particles keeping a part of the superconformal symmetry. This correspondence provides us a useful tool to understand the spectrum of 4d SCFTs. In particular, it enables us to study the strongly coupled SCFTs such as Argyres-Douglas theories. The counterparts of these theories are described by minimal models, and we expect that our technique helps calculate their characters.

Finally, we discuss several directions for the future work.

1. In Section 5, we considered the case that two truncation conditions are satisfied at the same time. To focus on the minimal model, we also imposed the condition (5.2). If it is not satisfied, there are more than two pits and the Hilbert space is expected to become more complicated. It may be interesting to explore its representation theory. One of the examples for such cases is $\widehat{U}(2)$ with a particular fractional level. These modules are called admissible and possess a good modular property. It is known that they appear in the logarithmic conformal field theory [71]. They also play an

important role in the context of 4d/2d duality [23]. We hope that we can classify the representation theory of the (generalized) Y-algebra from the affine Yangian.

2. In this thesis, we considered the W-algebras associated with Lie algebra of type A. It is natural to consider the generalization to B, C, D-type. In Gaiotto-Rapcak's construction, one can do that by introducing an orientifold [24]. The corresponding W-algebra is referred to as ortho-symplectic Y -algebras. If we can interpret them in terms of the affine Yangian, it may be possible to describe the Hilbert space explicitly using plane partitions. The related work has recently appeared [72]. By introducing a comodule algebra into the quantum toroidal algebra of $\mathfrak{gl}_1$, the authors of [72] reproduced the q-deformed W-algebras of type B, C, D [73] and obtained their supersymmetric generalization. In this construction, the connection to W-algebras can be seen through the screening charges. Meanwhile, the plane partition representation has not been understood. If we achieve that, we can do the same analysis as we have done in this thesis. It is interesting to explore whether we can reproduce the correct Hilbert space of the minimal models for the W-algebras of type B, C, D, $\mathcal{N} = 1$ super Virasoro algebra and so on.

3. $W_{1+\infty}$ appears also in the context of AdS_3/CFT_2 as the underlying symmetry [5]. The generalization to supersymmetric cases were also discussed. In particular, $\mathcal{N} = 2$ supersymmetric generalization of $W_{1+\infty}$ was proposed in [13] and its realization by supersymmetric affine Yangian has been recently discussed in [74, 75]. We expect the relation to our results.

4. The affine Yangian of $\mathfrak{gl}_1$ originally attracted the attention of physicists and mathematicians in the context of AGT correspondence. In terms of 4d $\mathcal{N} = 2$ supersymmetric gauge theories, the Young diagram basis describes the fixed points of the instanton moduli space under the torus action. Gaiotto-Rapcak's construction is expected to provide a more general framework of AGT. For example, it was shown in [76] that the Y-algebra is the underlying symmetry of spiked instantons [77, 78, 79]. Interestingly, closely related works have recently appeared [80, 81]. In these papers, the authors proposed new affine Yangians associated with counting of Bogomol'nyi-Prasad-Sommerfield (BPS) states in type IIA string theory compactified on a toric Calabi-Yau threefold. It was discussed that the BPS spectrum can be identified with the configuration of plane partitions and the affine Yangian serves as the underlying symmetry. It is interesting to explore the relation to our work. In particular, it is interesting to study the meaning of a doubly constrained plane partition we discussed in this thesis in the context of string theory or gauge theory.

We hope that these problems will be solved in the future.

Acknowledgement

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A Plane partition representation from the quantum toroidal algebra[36]

In this section, we briefly discuss the plane partition representation of the quantum toroidal algebra of $\mathfrak{gl}_1$ (also called Ding-Iohara-Miki algebra [82, 83]), which is the q-deformation of the affine Yangian of $\mathfrak{gl}_1$. It has two properties the affine Yangian does not have. First, we can construct the plane partition representation systematically. Second, it has free field representation, which makes the relation to W-algebra clear. In the following, we give a sketch for these points.

The quantum toroidal algebra $\mathcal{E}_1(q_1, q_2, q_3)$ of $\mathfrak{gl}_1$ is generated by the currents $E(z)$,$F(z)$, $K^{\pm}(z)$ and the centers $C, C^{\perp}$. It has three parameters q_1, q_2, q_3 with the constraint $q_1 q_2 q_3 = 1$. As we will see later, it degenerates to the affine Yangian in the limit of $q_1, q_2, q_3 \to 1$. As opposed to the affine Yangian, the currents have both positive modes and negative modes as follows:

$$E(z) = \sum_{n\in\mathbb{Z}} E_n z^{-n}, \quad F(z) = \sum_{n\in\mathbb{Z}} F_n z^{-n}, \quad K^{\pm}(z) = (C^{\perp})^{\mp 1} \exp\left(\sum_{r>0} \mp \frac{\kappa_r}{r} H_{\pm r} z^{\mp r}\right), \tag{A.1}$$

where

$$\kappa_r = \prod_{i=1}^{3} (q_i^{r/2} - q_i^{-r/2}). \tag{A.2}$$

The defining relations are given as follows:

$$\begin{aligned}
&g(z,w)E(z)E(w) + g(w,z)E(w)E(z) = 0, \qquad g(w,z)F(z)F(w) + g(z,w)F(w)F(z) = 0,\\
&K^{\pm}(z)K^{\pm}(w) = K^{\pm}(w)K^{\pm}(z), \qquad \frac{g(Cz,w)}{g(z,Cw)}K^{+}(z)K^{-}(w), = \frac{g(w,Cz)}{g(Cw,z)}K^{-}(w)K^{+}(z)\\
&g(z,w)K^{\pm}(C^{(-1\mp1)/2}z)E(w) + g(w,z)E(w)K^{\pm}(C^{(-1\mp1)/2}z) = 0,\\
&g(w,z)K^{\pm}(C^{(-1\pm1)/2}z)F(w) + g(z,w)F(w)K^{\pm}(C^{(-1\pm1)/2}z) = 0\,,\\
&[E(z), F(w)] = \frac{-1}{\kappa_1}\left(\delta\left(\frac{Cw}{z}\right)K^{+}(w) - \delta\left(\frac{Cz}{w}\right)K^{-}(z)\right),
\end{aligned} \tag{A.3}$$

where

$$g(z,w) = \prod_{i=1}^{3}(z - q_i w), \quad \delta(z) = \sum_{m\in\mathbb{Z}} z^m. \tag{A.4}$$

The algebra $\mathcal{E}_1$ is equipped with the coproduct

$$\begin{aligned}
&\Delta(E(z)) = E(z)\otimes 1 + K^{-}(z)\otimes E(z), \quad \Delta(F(z)) = F(z)\otimes K^{+}(z) + 1\otimes F(z),\\
&\Delta(K^{+}(z)) = K^{+}(z)\otimes K^{+}(C_1 z), \quad \Delta(K^{-}(z)) = K^{-}(C_2 z)\otimes K^{-}(z),\\
&\Delta(X) = X\otimes X \quad \text{for } X = C, C^{\perp},
\end{aligned} \tag{A.5}$$

[36] This section is a supplement for Section 4 and the reader can skip it.

where $C_1 = C \otimes 1,\ C_2 = 1 \otimes C$. One can rewrite some of the above relations in terms of H_r $(r \in \mathbb{Z})$ as follows:

$$
\begin{aligned}
&[H_r, H_s] = \delta_{r+s,0}\frac{C^r - C^{-r}}{\kappa_r},\\
&[H_r, E(z)] = -C^{(-r-|r|)/2}E(z)z^r, \quad [H_r, F(z)] = C^{(-r+|r|)/2}F(z)z^r,\\
&\Delta(H_r) = H_r \otimes C^{(-r+|r|)/2} + C^{(-r-|r|)/2} \otimes H_r.
\end{aligned}
\tag{A.6}
$$

When the centers act as $C = l_1^{1/2}$ and $C^{\perp} = l_2^{1/2}$, it is said that the representation has a level (l_1, l_2). When $l_1 = 1$, three types of the representations are known, which are called vector representation , Young diagram representation and plane partition representation [84, 85], respectively. The representation space can be graphically shown as follows:

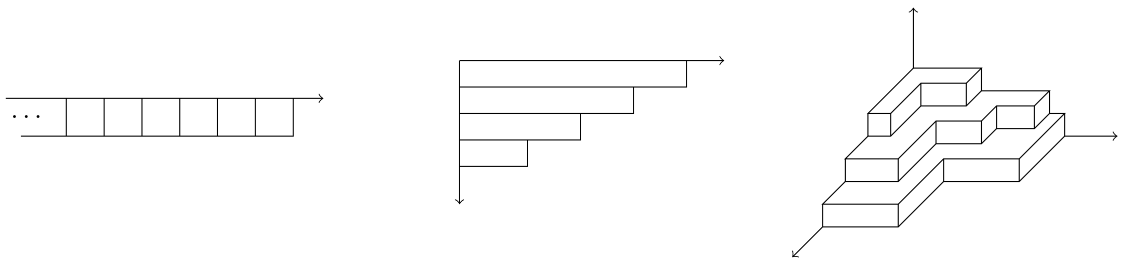

Figure 34: Left: vector representation. Middle: Young diagram representation. Right: plane partition representation.

As expected from this figure, we can construct the Young diagram representation from the tensor product of the vector representation and the plane partition representation from the Young diagram representation.

A.1 Construction of the plane partition representation

The bases of the vector representation are labelled by an integer $i \in \mathbb{Z}$ and the action of the currents are given as follows:

$$(1 - q_2)E(z)[u]_i = \delta(q_2^i u/z)[u]_{i+1}, \tag{A.7}$$

$$-(1 - q_2^{-1})F(z)[u]_i = \delta(q_2^{i-1} u/z)[u]_{i-1}, \tag{A.8}$$

$$K^+(z)[u]_i = \frac{(1 - q_2^i q_1 u/z)(1 - q_2^i q_3 u/z)}{(1 - q_2^i u/z)(1 - q_2^{i-1} u/z)}[u]_i, \tag{A.9}$$

$$K^-(z)[u]_i = \frac{(1 - q_2^{-i} q_1^{-1} z/u)(1 - q_2^{-i} q_3^{-1} z/u)}{(1 - q_2^{-i} z/u)(1 - q_2^{-i+1} z/u)}[u]_i. \tag{A.10}$$

Here, the parameter u is originated from the rescaling symmetry $z \to uz$. We denote by $V(u)$ the representation space. The eigenvalues of $K^{\pm}(z)$ are the same as rational functions.

For convenience, we set

$$K_i^V(u/z) = K^V(q_2^i u/z), \qquad K^V(z) = \frac{(1-q_1 u/z)(1-q_3 u/z)}{(1-u/z)(1-q_2^{-1}u/z)}. \tag{A.11}$$

We note that the vector representation is not a highest (or lowest) weight representation.

One can consider the tensor product of the vector representations. Let us consider $V(u)\otimes V(uq_3^{-1})\otimes V(uq_3^{-2})\otimes\cdots V(uq_3^{1-N})$. We set

$$|\lambda\rangle = [u]_{\lambda_1}\otimes[uq_3^{-1}]_{\lambda_2-1}\otimes\cdots\otimes[uq_3^{1-N}]_{\lambda_N+1-N}. \tag{A.12}$$

The action of the currents are determined from the coproduct (A.5). For example,

$$\Delta^{(N-1)}(E(z)) = E(z)\otimes 1\otimes\cdots\otimes 1 + K^-(z)\otimes E(z)\otimes 1\otimes\cdots\otimes 1+\cdots+K^-(z)\otimes\cdots\otimes K^-(z)\otimes E(z), \tag{A.13}$$

where we set $\Delta^{(i)} = (\underbrace{\Delta\otimes 1\otimes\cdots\otimes 1}_{i})\cdot\Delta^{(i-1)}$, $\Delta^{(1)} = \Delta$. From this, one can see that the state $|\lambda_1,\cdots,\lambda_{i-1},\lambda_i+1,\cdots,\lambda_N\rangle$ is not generated if $\lambda_{i-1}=\lambda_i$ because

$$K_{\lambda_{i-1}+2-i}(q_3^{2-i}u/z)\delta(q_2^{\lambda_i+1-i}q_3^{1-i}u/z) = 0. \tag{A.14}$$

In a similar way, we see that the state $|\lambda_1,\cdots,\lambda_{i-1}-1,\lambda_i,\cdots,\lambda_N\rangle$ is not generated by the action of $F(z)$ if $\lambda_i=\lambda_{i+1}$. Therefore, the subspace spanned by $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N$ is irreducible.

The above representation is not a highest (or lowest) weight representation for finite N, which does not match with the representation theory of W-algebras. We can resolve this by considering the semi-infinite tensor product as follows:

$$\mathcal{F}(u) = V(u)\otimes V(uq_3^{-1})\otimes V(uq_3^{-2})\otimes V(uq_3^{-3})\otimes\cdots. \tag{A.15}$$

In this case, we can bound the representation space from below. To see that, let us consider the state $|\emptyset\rangle\equiv|0,0,\cdots\rangle$. Then we have

$$F(z)\,|\emptyset\rangle = 0 \tag{A.16}$$

because we have $\lambda_i=\lambda_{i+1}$ for all $i\geq 1$. We note that this annihilation condition is satisfied thanks to the combination $\cdots\otimes F(z)\otimes K^-(z)\otimes\cdots$: for finite N, the last term in $\Delta^{(N-1)}(F(z))$ breaks this condition. As a result, the representation space consists of the states labelled by a set of integers $\lambda=(\lambda_1,\lambda_2,\lambda_3,\cdots)$ with $\lambda_1\geq\lambda_2\geq\lambda_3\geq\cdots\geq 0$ and $\lambda_N=0$ for a large N. This representation is the Young diagram representation.

In order to justify the infinite product, we need some regularization. The eigenvalues of $K^\pm(z)$ are given by the infinite product,

$$K^\pm(z)\,|\lambda\rangle = K_\lambda^Y(u/z)\,|\lambda\rangle\,,\quad K_\lambda^Y(u/z) = \prod_{i=1}^\infty K_{\lambda_i-i+1}^V(uq_3^{1-i}/z). \tag{A.17}$$

Because of the relation

$$K_\lambda^V(u/z) = \tilde{K}(u/z)\tilde{K}(q_1 u/z)^{-1}, \qquad \tilde{K}(u/z) = \frac{1 - q_3 u/z}{1 - u/z}, \tag{A.18}$$

it is natural to regularize the eigenvalue for the empty Young diagram as follows:

$$K_\emptyset^Y(u/z) = \prod_{i=0}^{\infty} \tilde{K}(q_3^i u/z)\tilde{K}(q_3^{i+1}u/z)^{-1} \sim \tilde{K}(u/z). \tag{A.19}$$

For general Young diagrams, we can determine the eigenvalues recursively as

$$K_\lambda^Y(u/z) = \quad K_\emptyset^Y(u/z)\prod_{(i,j)\in\lambda} G(q_1^{i-1}q_2^{j-1}u/z), \tag{A.20}$$

where we set

$$G(q_1^{i-1}q_2^{j-1}u/z) = K_{\lambda+1_i}^Y(u/z)/K_\lambda^Y(u/z), \quad G(w/z) = \prod_{i=1}^{3} \frac{1 - q_i^{-1}w/z}{1 - q_i w/z} \tag{A.21}$$

and $\lambda + 1_i = (\lambda_1, \lambda_2, \cdots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \cdots)$. The same infinite product also appears in the action of $F(z)$ and we can apply the same regularization to $F(z)$. We note that the level of the representation has changed to $(1, q_3)$.

We can construct the plane partition representation from the semi-infinite product of the Young diagram representation in the same way as the above. We set

$$\mathcal{M}(u, K) = \mathcal{F}(u) \otimes \mathcal{F}(uq_3) \otimes \mathcal{F}(uq_3^2) \otimes \mathcal{F}(uq_3^3) \otimes \cdots . \tag{A.22}$$

The parameter K is related to the regularization as we will explain later. The subspace spanned by $|\Lambda\rangle = |\Lambda^{(1)}\rangle \otimes |\Lambda^{(2)}\rangle \otimes |\Lambda^{(3)}\rangle \otimes \cdots$ with $\Lambda^{(1)} \supseteq \Lambda^{(2)} \supseteq \Lambda^{(3)} \supseteq \cdots$ is irreducible. We note that Λ corresponds to a configuration of a plane partition. As in the previous case, the eigenvalue of $K^\pm(z)$ is given as the infinite product

$$K^\pm(z)\,|\Lambda\rangle = K_\Lambda^P(u/z)\,|\Lambda\rangle\,, \quad K_\Lambda^P(u/z) = \prod_{i=1}^{\infty} K_{\Lambda^{(i)}}^Y(u/z). \tag{A.23}$$

Let us first consider the vacuum state, namely, $\Lambda = \vec{\emptyset} \equiv (\emptyset, \emptyset, \emptyset, \cdots)$. As opposed to the previous case, the infinite product does not converge:

$$K_{\vec{\emptyset}}^P(u/z) = \prod_{i=1}^{\infty} \frac{1 - q_3^i u/z}{1 - q_3^{i-1}u/z} = \lim_{N\to\infty} \frac{1 - q_3^N u/z}{1 - u/z}. \tag{A.24}$$

To regularize it, we formally replace q_3^N with an arbitrary parameter K:

$$K_{\vec{\emptyset}}^P(u/z) = \frac{1 - Ku/z}{1 - u/z}. \tag{A.25}$$

The eigenvalue for generic plane partitions is given by

$$K_\Lambda^P(u/z) = K_{\vec{\emptyset}}^P(u/z) \prod_{(i,j,k)\in\Lambda} G(q_1^{i-1}q_2^{j-1}q_3^{k-1}). \tag{A.26}$$

The action of $E(z), F(z)$ can be determined by the coproduct. For $F(z)$, we need to apply the same regularization. One can express the action of $E(z), F(z)$ into a simpler form by changing the normalization of the state $|\Lambda\rangle$ as follows:

$$
\begin{aligned}
&|\Lambda\rangle \to c_\Lambda |\Lambda\rangle, \\
&c_\Lambda = \prod_{(i,j,k)\in\Lambda} \left(-q_2\sqrt{\frac{(1-q_1)(1-q_3)}{(1-q_2)(1-u/z)}K^P_{\vec{\emptyset}}(u/z)\frac{1-u/z}{1-q_3^k u/z}K^Y_\emptyset(q_1^i q_3^{k-1}u/z)}\right)|_{z/u\to q_1^{i-1}q_2^{j-1}q_3^{k-1}} \\
&\qquad \times \sqrt{\prod_{\substack{(i_1,j_1,k_1),(i_2,j_2,k_2)\in\Lambda \\ (i_1,j_1,k_1)<(i_2,j_2,k_2)}} G(q_1^{i_2-1}q_2^{j_2-1}q_3^{k_2-1}u/z)|_{z/u\to q_1^{i_1-1}q_2^{j_1-1}q_3^{k_1-1}}},
\end{aligned}
\tag{A.27}
$$

where we define $(i_1, j_1, k_1) < (i_2, j_2, k_2)$ as $\begin{cases} k_1 < k_2 \\ k_1 = k_2, i_1 < i_2 \\ k_1 = k_2, i_1 = i_2, j_1 < j_2 \end{cases}$. Then we have

$$
E(z)|\Lambda\rangle = \sum_{\square\in\Lambda^+} \delta(q_\square u/z)\sqrt{\frac{(1-q_\square u/z)K^P_\Lambda(u/z)}{(1-q_1)(1-q_2)(1-q_3)}}|\Lambda+\square\rangle, \tag{A.28}
$$

$$
F(z)|\Lambda\rangle = \sum_{\square\in\Lambda^-} \delta(q_\square u/z)\sqrt{\frac{(1-q_\square u/z)K^P_{\Lambda-\square}(u/z)}{(1-q_1)(1-q_2)(1-q_3)}}|\Lambda-\square\rangle, \tag{A.29}
$$

where we assign the value $q_\square = q_1^i q_2^j q_3^k$ to each box at $(x, y, z) = (i, j, k)$.

A.2 Free field realization and Miki-automorphism

The algebra $\mathcal{E}_1$ has three types of free field realizations. The levels of these representations are $(q_i, 1)$ $(i = 1, 2, 3)$, respectively. For each i, we introduce the Heisenberg algebra a_r $(r \in \mathbb{Z})$ and Q with the commutation relations

$$
[a_r, a_s] = r\frac{(q_i^{r/2}-q_i^{-r/2})^3}{-\kappa_r}\delta_{r+s,0} \quad r,s\neq 0, \qquad [a_0, a_r] = 0, \quad [a_n, Q] = \frac{\epsilon_i^3}{\epsilon_1\epsilon_2\epsilon_3}\delta_{n,0}. \tag{A.30}
$$

Then we have the following representations:

$$
\rho_u^{(i)}(E(z)) = \frac{u(1-q_i)}{\kappa_1}\exp\left(\sum_{r=1}^\infty \frac{q_i^{-r/2}\kappa_r}{r(q_i^{r/2}-q_i^{-r/2})^2}a_{-r}z^r\right)\exp\left(\sum_{r=1}^\infty \frac{\kappa_r}{r(q_i^{r/2}-q_i^{-r/2})^2}a_r z^{-r}\right), \tag{A.31}
$$

$$
\rho_u^{(i)}(F(z)) = \frac{u^{-1}(1-q_i^{-1})}{\kappa_1}\exp\left(\sum_{r=1}^\infty \frac{-\kappa_r}{r(q_i^{r/2}-q_i^{-r/2})^2}a_{-r}z^r\right)\exp\left(\sum_{r=1}^\infty \frac{-q_i^{r/2}\kappa_r}{r(q_i^{r/2}-q_i^{-r/2})^2}a_r z^{-r}\right) \tag{A.32}
$$

$$
\rho_u^{(i)}(H_r) = \frac{a_r}{q_i^{r/2}-q_i^{-r/2}}. \tag{A.33}
$$

Here, $u \equiv \exp(-\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_i^2}a_0)$ is the parameter associated with the weight parameter. The representation space is the Fock module $\mathcal{F}_u^{(i)}$, which is spanned by the vectors $a_{-n_1}a_{-n_2}\cdots a_{-n_N}\left|u\right\rangle$ $(n_1 \geq n_2 \geq \cdots n_N \geq 0,\ N \in \mathbb{Z}_{\geq 0})$ where $\left|u\right\rangle$ is the the highest weight vector satisfying

$$a_n\left|u\right\rangle = 0,\ n > 0, \qquad u\left|u\right\rangle = u\left|u\right\rangle. \tag{A.34}$$

One can consider the tensor product of the Fock modules. In general, we can consider

$$\mathcal{F}_{u_1}^{(i_1)} \otimes \mathcal{F}_{u_2}^{(i_2)} \otimes \cdots \otimes \mathcal{F}_{u_N}^{(i_N)}, \tag{A.35}$$

where $i_1, i_2, \cdots i_N \in \{1, 2, 3\}$. The action of the generators is determined by the coproduct structure. One can see that the level of this representation is $\left(\prod_{j=1}^N q_{i_j}, 1\right)$. The simplest case is the tensor product of two Fock spaces of the same type. Let us consider $\mathcal{F}_{u_1}^{(3)} \otimes \mathcal{F}_{u_2}^{(3)}$. From (A.5), one can see that the current $E(z)$ acts on the representation space as the sum of two vertex operators:

$$E(z) \to \frac{u_1(1-q_3)}{\kappa_1}\Lambda_1(z) + \frac{u_2(1-q_3)}{\kappa_1}\Lambda_2(z), \tag{A.36}$$

$$\Lambda_1(z) = \exp\left(\sum_{r=1}^\infty \frac{q_3^{-r}\kappa_r}{r(q_3^{r/2}-q_3^{-r/2})^2}a_{-r}^{(1)}z^r\right)\exp\left(\sum_{r=1}^\infty \frac{q_3^{r/2}\kappa_r}{r(q_3^{r/2}-q_3^{-r/2})}\left(\frac{a_r^{(1)}}{q_3^r-q_3^{-r}} - a_r^{(2)}\right)z^{-r}\right) \tag{A.37}$$

$$\Lambda_2(z) = \exp\left(\sum_{r=1}^\infty \frac{q_3^{-r/2}\kappa_r}{r(q_3^{r/2}-q_3^{-r/2})^2}a_{-r}^{(2)}z^r\right)\exp\left(\sum_{r=1}^\infty \frac{\kappa_r}{r(q_3^{r/2}-q_3^{-r/2})^2}a_r^{(2)}z^{-r}\right). \tag{A.38}$$

Here, we set $a_n^{(1)} = a_n \otimes 1$ and $a_n^{(2)} = 1 \otimes a_n$. The current $F(z)$ is written in a similar form. For H_n, it acts as

$$H_r \to \frac{1}{q_3^{r/2}-q_3^{-r/2}}(a_r^{(1)} + q_3^{-r/2}a_r^{(2)}), \quad H_{-r} \to \frac{-1}{q_3^{r/2}-q_3^{-r/2}}(q_3^{r/2}a_{-r}^{(1)} + a_{-r}^{(2)}) \qquad r > 0, \tag{A.39}$$

which corresponds to the extra U(1) factor of $W_{1+\infty}$. The current $E(z), F(z)$ are exactly the current of q-Virasoro algebra [86] up to U(1) factor. Next, we consider the tensor product of N Fock modules of the same type, for example, $\mathcal{F}_{u_1}^{(3)} \otimes \mathcal{F}_{u_2}^{(3)} \otimes \cdots \otimes \mathcal{F}_{u_N}^{(3)}$. The level of the representation is $(q_3^N, 1)$. From the coproduct structure, we see that the current $E(z), F(z)$ are written by the sum of N vertex operators. As pointed out in [87], they are exactly the current of q-W_N algebra [88]. Thus, the algebra $\mathcal{E}_1$ realizes the product of the quantum W_N algebra and U(1) boson in the representation space $\mathcal{F}_{u_1}^{(3)} \otimes \mathcal{F}_{u_2}^{(3)} \otimes \cdots \otimes \mathcal{F}_{u_N}^{(3)}$. One can also consider the tensor product of the Fock modules of different types. See [25] for the detail.

The free boson representation and theYoung diagram representation are related by Miki-automorphism [83], under which the generators transform as follows:

$$E_0 \to H_1,\quad H_1 \to F_0,\quad F_0 \to H_{-1},\quad H_{-1} \to E_0,\quad C \to C^\perp,\quad C^\perp \to C^{-1}. \tag{A.40}$$

One can see that this transformation indeed preserves the following relation:

$$[E_0, F_0] = \frac{1}{\kappa_1}\big((C^\perp)^{-1} - C^\perp\big), \qquad [H_1, H_{-1}] = \frac{1}{\kappa_1}(C - C^{-1}) \tag{A.41}$$

Because the elements $E_0, F_0, H_{\pm 1}$ and the centers generate all of the elements in $\mathcal{E}_1$, the transformation rule (A.40) can be extended to the whole elements. Because the automorphism changes the centers, we find that the free boson representation with level $(q_i, 1)$ and the Young diagram representation with level $(1, q_i)$ are equivalent. By applying the same discussion to the tensor products, we find that the product of q-W_N algebra and U(1) boson is realized on the plane partition representation with $K = q_i^N$, which is given by the plane partition with a pit at $(1, 1, N + 1)$.

A.3 Degenerate limit of the quantum toroidal algebra

Let us go back to the affine Yangian of $\mathfrak{gl}_1$. The quantum toroidal algebra $\mathcal{E}_1(q_1, q_2, q_3)$ of $\mathfrak{gl}_1$ degenerates to the affine Yangian $Y_1(h_1, h_2, h_3)$ of $\mathfrak{gl}_1$ in the limit of $q_1, q_2, q_3 \to 1$. The explicit relation between $\mathcal{E}_1(q_1, q_2, q_3)$ and $Y_1(h_1, h_2, h_3)$ was discussed in [10], where the algebra homomorphism Υ from $\mathcal{E}_1(q_1, q_2, q_3)$ to the completion of $Y_1(\epsilon_1, \epsilon_2, \epsilon_3)$ was established. As we will explain later, this homomorphism is valid only when $C = 1$. The correspondence between the generators of these two algebras is complicated, but it becomes simple in the degenerate limit. For convenience, let us set $q_i = e^{Rh_i}$ $(i = 1, 2, 3)$ and take the limit of $R \to 0$. Then the algebra homomorphism under the limit is given as follows [10]:

$$\begin{aligned}
&\Upsilon_{R\to 0}:\ \mathcal{E}_1(e^{Rh_1}, e^{Rh_2}, e^{Rh_3}) \to Y_1(Rh_1, Rh_2, Rh_3)\\
&\quad E_j \mapsto \sum_{n=0}^{\infty}\frac{j^n}{n!}e_n, \quad F_j \mapsto \sum_{n=0}^{\infty}\frac{j^n}{n!}f_n, \quad H_m \mapsto -\frac{1}{R^3\sigma_3}\sum_{n=0}^{\infty}\frac{m^{n-2}}{n!}k_n \quad (j \in \mathbb{Z},\ m \in \mathbb{Z}_{\neq 0}),\\
&\quad C^\perp \mapsto e^{-R^3\psi_0\sigma_3},
\end{aligned} \tag{A.42}$$

where $k(z) = \sum_{n=0}^{\infty} k_n z^{-n-1} \equiv \log\psi(z)$[37]. We note that the limit of $R \to 0$ does not cause substantial effects on the affine Yangian of $\mathfrak{gl}_1$ because it has the automorphism (4.19) which scales h_1, h_2, h_3 simultaneously. By using this automorphism, we can rewrite (A.42) as

$$\begin{aligned}
&\Upsilon_{R\to 0}:\ \mathcal{E}_1(e^{Rh_1}, e^{Rh_2}, e^{Rh_3}) \to Y_1(h_1, h_2, h_3)\\
&\quad E_j \mapsto R^{-1}\sum_{n=0}^{\infty}\frac{(Rj)^n}{n!}e_n, \quad F_j \mapsto R^{-1}\sum_{n=0}^{\infty}\frac{(Rj)^n}{n!}f_n, \quad H_m \mapsto -\frac{1}{\sigma_3}\sum_{n=0}^{\infty}\frac{(Rm)^{n-2}}{n!}k_n,\\
&\quad C^\perp \to e^{-R\psi_0\sigma_3}.
\end{aligned} \tag{A.43}$$

[37]For the definition of the affine Yangian, see Section 4.1.

We can further simplify it in terms of Drinfeld currents. For that, we need the following formal relation,

$$\sum_{j=0}^{\infty}\frac{(Rj)^n}{n!}z^{-j}=\frac{1}{n!}(RD_w)^n\frac{1}{1-w}=\frac{1}{n!}(-\partial_u)^n\frac{1}{1-e^{-Ru}}\xrightarrow{R\to 0}\frac{1}{Ru^{n+1}},\quad (A.44)$$

$$\sum_{j=0}^{\infty}\frac{(-Rj)^n}{n!}z^{j}\xrightarrow{R\to 0}-\frac{1}{Ru^{n+1}},\quad (A.45)$$

where we set $z=w^{-1}=e^{Ru}$, $D_w=w\partial_w$. Using this, we have

$$\begin{aligned}\Upsilon_{R\to 0}:\ &\mathcal{E}_1(e^{Rh_1},e^{Rh_2},e^{Rh_3})\to Y_1(h_1,h_2,h_3)\\ &E^{\pm}(z)\mapsto \pm R^{-2}e(u),\quad F^{\pm}(z)\mapsto \pm R^{-2}f(u),\quad K^{\pm}(z)\mapsto\psi(u),\end{aligned}\quad (A.46)$$

where we set $E^{\pm}(z)=\sum_{j=0}^{\infty}E_{\pm j}z^{\mp j}, F^{\pm}(z)=\sum_{j=0}^{\infty}F_{\pm j}z^{\mp j}$. This is the final result which describes explicitly how the toroidal algebra degenerates to the affine Yangian.

As we mentioned, the above procedure is valid when $C=1$. That implies that we cannot tale the degenerate limit of the free boson representation of $\mathcal{E}_1$ in this way. Indeed, the right hand side in (A.43) consists of infinitely many terms, which does not make sense as long as they do not converge to finite terms on the representation space. Meanwhile, this condition is satisfied for the vector representation, Young diagram representation and plane partition representation. Thus, the plane partition representation of the affine Yangian (4.50), (4.51), (4.52) are obtained from (A.26), (A.28) and (A.29).

B The conformal dimension of the primary field in W_N minimal models from affine Yangian

In this section, we check that the asymptotic Young diagrams correspond to the primary fields with the conformal dimension (5.9) under the identification (5.8). We can implement that in the four steps as follows. First, we see from (4.56) that the asymptotic Young diagram μ has the conformal dimension

$$\begin{aligned}h_1&=-\frac{\lambda_1}{2\lambda_2}\sum_{i=1}^{N-1}\mu_i^2-\frac{\lambda_1}{2\lambda_3}\sum_{i=1}^{N-1}(2i-1)\mu_i+\frac{\lambda_1}{2}\sum_{i=1}^{N-1}\mu_i\\ &=\frac{\beta}{2}\sum_{i=1}^{N-1}\mu_i^2+\frac{1-\beta}{2}\sum_{i=1}^{N-1}(2i-1-N)\mu_i.\end{aligned}\quad (B.1)$$

In the same way, we can calculate the contribution from the asymptotic Young diagram ν as

$$h_2=\frac{1}{2\beta}\sum_{i=1}^{N-1}\nu_i^2-\frac{1-\beta}{2\beta}\sum_{i=1}^{N-1}(2i-1-N)\nu_i.\quad (B.2)$$

As we have seen in (4.57), we need to subtract the number of overlapping boxes

$$\#(\mu \cap \nu) = \sum_{i=1}^{N-1} \mu_i \nu_i. \tag{B.3}$$

Finally, we have to remove the contribution from the extra $U(1)$ boson, which is evaluated by $\frac{\psi_1^2}{2\psi_0}$. Using (4.56), we have

$$\begin{aligned} h^{U(1)} &= \frac{1}{2\psi_0}\Big(-\frac{\sum_{i=1}^{N-1}\mu_i}{h_1} - \frac{\sum_{i=1}^{N-1}\nu_i}{h_2}\Big)^2 \\ &= \frac{\beta}{2N}\big(\sum_{i=1}^{N-1}\mu_i\big)^2 + \frac{1}{2N\beta}\big(\sum_{i=1}^{N-1}\nu_i\big)^2 - \frac{1}{N}\big(\sum_{i=1}^{N-1}\mu_i\big)\big(\sum_{i=1}^{N-1}\nu_i\big). \end{aligned} \tag{B.4}$$

Here, we use the relations such as $\frac{1}{\psi_0 h_1^2} = \frac{h_2}{\psi_0\sigma_3}\frac{h_3}{h_1} = -\frac{\lambda_1}{\lambda_2\lambda_3}$. Combining the above, we have

$$\begin{aligned} h &= h_1 + h_2 - \#(\mu\cap\nu) - h^{U(1)} \\ &= \frac{N-1}{2N}\big(\beta\sum_{i=1}^{N-1}\mu_i^2 - 2\sum_{i=1}^{N-1}\mu_i\nu_i + \frac{1}{\beta}\sum_{i=1}^{N-1}\nu_i^2\big) - \frac{1}{2N}\big(\beta\sum_{i\neq j}\mu_i\mu_j - 2\sum_{i\neq j}\mu_i\nu_j + \frac{1}{\beta}\sum_{i\neq j}\nu_i\nu_j\big) \\ &\qquad + \frac{1-\beta}{2}\sum_{i=1}^{N-1}(2i-1-N)(\mu_i - \frac{1}{\beta}\nu_j) \\ &= \frac{1}{2pq}\Big(\frac{N-1}{N}\sum_{i=1}^{N-1}(p\mu_i - q\nu_i)^2 - \frac{1}{N}\sum_{i\neq j}(p\mu_i - q\nu_i)(p\mu_j - q\nu_j) \\ &\qquad + (p-q)\sum_{i=1}^{N-1}(N+1-2i)(p\mu_i - q\nu_i)\Big). \end{aligned} \tag{B.5}$$

By using the vectors introduced in Section 2.3.1, we can rewrite (B.5) as

$$\begin{aligned} h &= \frac{1}{2pq}\bigg(\Big(\sum_{i=1}^{N-1}(p\mu_i - q\nu_i)\vec{\xi}_i\Big)^2 + 2(p-q)\sum_{i=1}^{N-1}(p\mu_i - q\nu_i)\vec{\xi}_i\cdot\vec{\rho}\bigg) \\ &= \frac{1}{2pq}\bigg(\Big(\sum_{i=1}^{N-1}(p\mu_i - q\nu_i)\vec{\xi}_i + (p-q)\vec{\rho}\Big)^2 - (p-q)^2\frac{N(N^2-1)}{12}\bigg) \\ &= \frac{12(\sum_{i=1}^{N-1}(pm_i - qn_i)\vec{\omega}_i)^2 - N(N^2-1)(p-q)^2}{24pq}, \end{aligned} \tag{B.6}$$

which is consistent with (5.9). Here, we use the formula $\vec{\xi}_i\cdot\vec{\rho} = \frac{N+1}{2} - i$ and $\vec{\rho}\cdot\vec{\rho} = \frac{N(N^2-1)}{12}$.

C The norm of the primary states corresponding to the intermediate Young diagrams

In this section, we check that the state given in (6.13) becomes null when the central charge is $c = \frac{3n}{n+2}$ ($n \in \mathbb{Z}_{\geq 1}$). According to the discussion below Figure 25, the state should become null if $|\mu| > n$. Let us check that directly from the defining relation (2.66) of $\mathcal{N}=2$ super Virasoro algebra. For the computation, we rewrite a part of it in terms of the commutator as follows:

$$\begin{aligned} &[L_m, G^{\pm}_r] = (\frac{1}{2}m - r)G^{\pm}_{m+r}, \qquad [J_m, G^{\pm}_r] = \pm G^{\pm}_{m+r}, \qquad \{G^{\pm}_r, G^{\pm}_s\} = 0. \\ &\{G^{+}_r, G^{-}_s\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}. \end{aligned} \tag{C.1}$$

Using it, we can compute the norm of the state (6.13) as follows:

$$\begin{aligned} &\langle 0|\, G^{\mp}_{3/2} \cdots G^{\mp}_{|\mu|-1/2} G^{\mp}_{|\mu|+1/2} \;\cdot\; G^{\pm}_{-|\mu|-1/2} G^{\pm}_{-|\mu|+1/2} \cdots G^{\pm}_{-3/2}\, |0\rangle \\ =&\langle 0|\, G^{\mp}_{3/2} \cdots G^{\mp}_{|\mu|-1/2} \{G^{\mp}_{|\mu|+1/2}, G^{\pm}_{-|\mu|-1/2}\} G^{\pm}_{-|\mu|+1/2} \cdots G^{\pm}_{-3/2}\, |0\rangle \\ =&\langle 0|\, G^{\mp}_{3/2} \cdots G^{\mp}_{|\mu|-1/2} \Big(2L_0 \mp 2\big(|\mu| + \frac{1}{2}\big) J_0 + \frac{c}{3}\big((|\mu| + \frac{1}{2})^2 - \frac{1}{4}\big)\Big) G^{\pm}_{-|\mu|+1/2} \cdots G^{\pm}_{-3/2}\, |0\rangle \\ =&\frac{|\mu|^2 + |\mu|}{3}\Big(c - \frac{3(|\mu|-1)}{|\mu|+1}\Big)\, \langle 0|\, G^{\mp}_{3/2} \cdots G^{\mp}_{|\mu|-1/2} G^{\pm}_{-|\mu|+1/2} \cdots G^{\pm}_{-3/2}\, |0\rangle \\ =&\cdots \\ =&\prod_{i=1}^{|\mu|} \frac{i(i+1)}{3}\Big(c - \frac{3(i-1)}{i+1}\Big). \end{aligned} \tag{C.2}$$

We note that the terms containing $\{G^{\mp}_{|\mu|+1/2}, G^{\pm}_{-i-1/2}\}$ $(i = 1, 2, \cdots, |\mu|-1)$ do not appear in the second line because they become zero. This result shows that the state is indeed null if $|\mu| > n$.

D The computation for (7.21)

The conformal dimension (7.21) can be obtained as follows. First, the contribution from the asymptotic Young diagram $\nu = (\nu_1, \nu_2, \nu_3)$ can be evaluated from (4.56) as

$$h_\nu = \frac{\nu_1^2 + \nu_2^2 + \nu_3^2}{2\Psi} + \frac{1 - \Psi^{-1}}{2}(\nu_1 + 3\nu_2 + 5\nu_3) + \frac{3\Psi^{-1} - 2}{2}(\nu_1 + \nu_2 + \nu_3). \tag{D.1}$$

Second, the contribution from the intermediate Young diagram $\mu = \nu_2$ is given by

$$h_\mu = \nu_2. \tag{D.2}$$

Third, we have to subtract the contribution from U(1) factor which is evaluated as

$$h^{U(1)} = \frac{\left(\sum_{j=1}^{3} \nu_j\right)^2}{6\Psi}. \tag{D.3}$$

Finally, we have to remove the number of the overlapping boxes

$$\#(\nu \cap \mu) = 2\nu_2 + \nu_3. \tag{D.4}$$

Combining the above, the conformal dimension h can be obtained as

$$h = h_\nu + h_\mu - h^{U(1)} - \#(\nu \cap \mu), \tag{D.5}$$

which gives (7.21).

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