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Stiffness Optimization of FRP Laminated Plates by Use of the Hessian Matrix

FRP積層平板のヘッセ行列に基づく剛性最適化

Shigeru NAKAGIRI*, Keiko SUZUKI* and Hideyuki TAKABATAKE*

中 桐　滋・鈴 木 敬 子・高 畑 秀 行

1. Introduction

Fiber Reinforced Plastic (FRP) laminated plates have the feature of anisotropy and heterogeneity so that the so-called tailoring design has been made possible by means of choosing proper stacking sequence, that is, fiber orientation and layer thickness. Optimum design of FRP laminated plate has been attempted so far in problems of buckling eigenvalue[1]. It seems that the Hessian matrix, the constituents of which are second order derivatives of the behavior variable with respect to the design variables, has not been in wide use[2] probably because of the difficulty in evaluating it efficiently.

It has been demonstrated that the second order derivatives of eigenvalue with respect to random variables can be evaluated by the stochastic finite element method on the basis of the second order perturbation technique[3]. It turns out that the Hessian matrix for the purpose of eigenvalue optimization can be obtained easily by use of the said methodology while the random variables are read as design variables.

This note discusses the possibility of optimization with the aid of the Hessian matrix in regard to a simple example in which the largest eigenvalue of the stress-strain matrix of FRP plate in maximized through the stacking sequence. This will be followed by the examples for buckling eigenvalue problems.

2. Stress-strain matrix of FRP plate and its derivatives

The stiffness of FRP plate consisting of N layers is evaluated on the basis of the k-th layer stiffness matrix $[Q_{ij}]_k$ and stacking sequence, as given below [4].

* Dept. of Applied Physics and Applied Mechanics, Institute of Industrial Science, University of Tokyo.

$$[S]=\begin{bmatrix} A & B \\ B & D \end{bmatrix} \tag{1}$$

$$[A]=\sum_{k=1}^{N}[Q_{ij}]_k(z_{k+1}-z_k) \tag{1-a}$$

$$[B]=\frac{1}{2}\sum_{k=1}^{N}[Q_{ij}]_k(z_{k+1}^2-z_k^2) \tag{1-2}$$

$$[D]=\frac{1}{3}\sum_{k=1}^{N}[Q_{ij}]_k(z_{k+1}^3-z_k^3) \tag{1-3}$$

The stress-strain matrix $[S]$ determines the relationship between the force term vector consisting of three stress resultants and three stress couples and the strain term vector of the corresponding mid-plane strains and curvatures. The origin of the thickness coordinate z is taken at the mid-plane. The matrices $[A]$, $[B]$ and $[D]$ are called extensional, coupling and bending stiffness respectively.

In the above, the layer stiffness matrix $[Q_{ij}]_k$ is constituted as

$$[Q_{ij}]_k=[T]^{-1}[E_{ij}][T] \tag{2}$$

$$[E_{ij}]=\begin{bmatrix} E_{11} & E_{12} & 0 \\ & E_{22} & 0 \\ \text{SYM.} & & G_{LT} \end{bmatrix} \tag{2-a}$$

$$[T]=\begin{bmatrix} l^2 & m^2 & 2lm \\ m^2 & l^2 & -2lm \\ -lm & lm & l^2-m^2 \end{bmatrix} \tag{2-b}$$

where $E_{11}=E_L/(1-\nu_{LT}\nu_{TL})$, $E_{12}=\nu_{TL}E_{11}$, $E_{22}=E_T/(1-\nu_{LT}\nu_{TL})$ and $l=\cos\theta_k$ and $m=\sin\theta_k$. θ_k stands for the angle between the material axes and plate axes of the k-th layer. Equation (2) is derived under the plane-stress condition with four independent elastic constants of E_L, E_T, G_{LT} and ν_{LT} ($\nu_{TL}=\nu_{LT}E_T/E_L$), and the Kirchhoff-Love's hypothesis is employed.

When the stacking sequence is changed from the current values of $\bar{\theta}_k$ and $\bar{z}_k$ as given by Eq. (3), the matrix $[S]$ after the change is summarized in form of the Taylor series expansion truncated at the second order regarding the design variables $\varepsilon_m(m=1$ to $M)$ as given by Eq. (4).

$$\theta_k=\bar{\theta}_k+\varepsilon_k \quad z_k=\bar{z}_k+\varepsilon_{N+k} \qquad (3)$$

$$[S]=[\bar{S}]+\sum_{m=1}^{M}[S_m^{I}]\varepsilon_m+\frac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{M}[S_{mn}^{II}]\varepsilon_m\varepsilon_n \qquad (4)$$

The first and second order derivatives $[S_m^{I}]$ and $[S_{mn}^{II}]$ are derived by the differentiation of $[A]$, $[B]$ and $[D]$ with respect to ε_m. M denotes the total number of design variables taken and the superfixes I and II denote the order of differentiation.

3. Eigenvalues of $[S]$ and its related Hessian matrix

Six eigenvalues of $[S]$, especially the largest one, are used as index of the stiffness in this note. When the stacking sequence is varied as is in Eq. (3), the changes in the eigenvalue and eigenvector, around their current values of $\bar{\lambda}$ and $\{\bar{\phi}\}$, are expressed by Eqs. (5) and (6) within the limit of second order truncation.

$$\lambda=\bar{\lambda}+\sum_{m=1}^{M}\lambda_m^{I}\varepsilon_m+\frac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{M}\lambda_{mn}^{II}\varepsilon_m\varepsilon_n \qquad (5)$$

$$\{\phi\}=\{\bar{\phi}\}+\sum_{m=1}^{M}\{\phi_m^{I}\}\varepsilon_m+\frac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{M}\{\phi_{mn}^{II}\}\varepsilon_m\varepsilon_n \qquad (6)$$

These hold for any of the i-th mode ($i=1$ to 6). Substituting Eqs. (4), (5) and (6) into usual eigenvalue problem, we have the following equations which determine the current values, first order derivatives and second order ones of the eigenvalue under interest,

$$([\bar{S}]-\bar{\lambda}[I])\{\bar{\phi}\}=\{0\} \qquad (7)$$

$$([\bar{S}]-\bar{\lambda}[I])\{\phi_m^{I}\}=-([S_m^{I}]-\lambda_m^{I}[I])\{\bar{\phi}\} \qquad (8)$$

$$([\bar{S}]-\bar{\lambda}[I])\{\phi_{mn}^{II}\}=-([S_m^{I}]-\lambda_m^{I}[I])\{\phi_n^{I}\} -([S_n^{I}]-\lambda_n^{I}[I])\{\phi_m^{I}\}-([S_{mn}^{II}]-\lambda_{mn}^{II}[I])\{\bar{\phi}\} \qquad (9)$$

where $[I]$ is unit matrix. Taking advantage of symmetricity of $[\bar{S}]-\bar{\lambda}[I]$, we have the first order derivatives λ_m^{I} as Eq. (10) easily based on Eq. (8). The second order derivatives λ_{mm}^{II} are dependent on the first order derivatives of eigenvector $\{\phi_m^{I}\}$. After the substitution of $\bar{\lambda}$ which is determined by Eq. (7), $[\bar{S}]-\bar{\lambda}[I]$ is made singular so that $\{\phi_m^{I}\}$ cannot be evaluated by solving Eq. (8) as simultaneous equations. This drawback is overcome by means of setting a constituent of $\{\phi_m^{I}\}$ zero, which corresponds to any but large constituent of $\{\bar{\phi}\}$[5]. Use is made of the result, and then λ_{mn}^{II} is calculated by Eq. (11).

$$\lambda_m^{I}=\{\bar{\phi}\}^T[S_m^{I}]\{\bar{\phi}\}/\{\bar{\phi}\}^T[I]\{\bar{\phi}\} \qquad (10)$$

$$\lambda_{mn}^{II}=\{\bar{\phi}\}^T\{([S_m^{I}]-\lambda_m^{I}[I])\{\phi_n^{I}\} +([S_n^{I}]-\lambda_n^{I}[I])\{\phi_m^{I}\}+[S_{mn}^{II}]\{\bar{\phi}\}\}/\{\bar{\phi}\}^T[I]\{\bar{\phi}\} \qquad (11)$$

In the above the superfix T denotes transpose. The Hessian matrix related with the eigenvalue is obtained by arranging λ_{mn}^{II} in form of matrix in size of M x M.

4. Optimization of the largest eigenvalue

Now that the derivatives λ_m^{I} and λ_{mn}^{II} are determined on the basis of $\bar{\lambda}$ and $\{\bar{\phi}\}$, the change of λ due to the variation in the stacking sequence can be approximated in form of Eq. (5) within the accuracy afforded by the second order perturbation. Then the largest eigenvalue λ can be assumed to take an extreme (not exact but approximated) value by the change in the design variables ε_m determined as solution of the following equation (12).

$$\begin{bmatrix} \lambda_{11}^{II} & \lambda_{12}^{II}\ldots\lambda_{1M}^{II} \\ & \lambda_{22}^{II}\ldots\lambda_{2M}^{II} \\ & \ddots\ \vdots \\ \text{SYM.} & \lambda_{MM}^{II} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_M \end{Bmatrix} = -\begin{Bmatrix} \lambda_1^{I} \\ \lambda_2^{I} \\ \vdots \\ \lambda_M^{I} \end{Bmatrix} \quad \cdots\cdots (12)$$

In order to approach the exact extreme, it is necessary to reiterate the procedure of changing the design variables of Eq. (3) and calculating the renewed λ_m^{I} and Hessian matrix.

It should be noted that the nature of the Hessian matrix is unknown beforehand the iteration. If it is positive-definite, the iteration is expected to result in decrease of λ, and tends towards the maximization of λ in negative-definite case. To check the definiteness of the Hessian matrix is one way, and there can be devised simpler way for the maximization to form condensed Hessian matrix by means of picking up only the constituents corresponding negative diagonal components of λ_{mm}^{II} (<0).

5. Numerical examples

Figure 1 shows the six eigenvalues of a three layered FRP plate with respect to the change in the orientation of the first layer θ_1, while those of θ_2 and θ_3 are set zero. Each layer thickness is taken equal to 0.254 mm, and the elastic constants used are $E_L=21100$, $E_T=2110$, $G_{LT}=703$(kgf/mm^2) and $\nu_{LT}=0.3$ for boron/epoxy[1]. Except the case of $\theta_1=0°$ or 180°, coupling stiffness is non-zero. The stretching mode or bending mode in the figure is judged by taking the eigenvectors into account. The eigenvalues change rather simply, showing the periodicity of $\cos 2\theta_1$ and $\cos 4\theta_1$. The change in the eigenvalues due to layer thickness is so small, provided that the total plate thickness remains constant, that the orientations alone are taken as design variables.

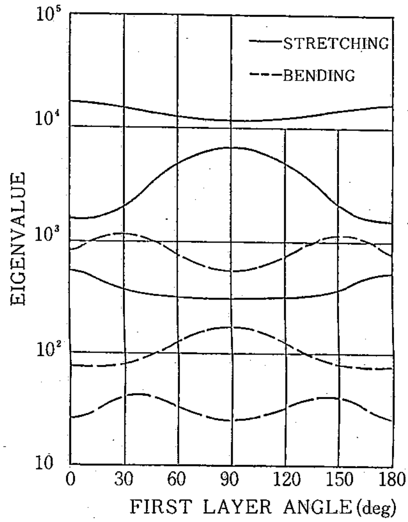

Figure 1 Eigenvalue change due to first layer angle

The largest eigenvalue is well converged by use of the full Hessian matrix and faster by the condensed one after three iterations. There takes place dependency of the converged value on initial variables as listed in Tab.1 (only each layer thickness changed to 0.2 mm). The range of the initial variables, which result in a same converged value, is estimated as 22.5° =(180°/4)/2 and evidenced in Tab. 1.

When the eigenvalue decreases with the iteration, the design variables is changed by 22.5° in case if the first order derivative is positive and by −22.5° in case if negative (this method is marked by B in Tab.2 and the mark A means use of the condensation stated above), for the purpose of maximization. Even when a converged value is attained by the methods A and B, it is necessary to repeat the iteration after changing all the design variables by 45° (marked by C in Tab.2), in order to examine the maximum in global. Table 2 shows that the maximal value is attained by the combination of the said methods, in case the initial variables are chosen rather at random for six

Table 1 Initial value dependency of converged behavior variable (initial : $\theta_1=\theta_3=0°$)

θ_2	−25	−15	15	21.25	25
λ_{max} : INITIAL	0.11752E5	0.12381E5	0.12381E5	0.12011E5	0.11752E5
λ_{max} : CONVERGED	0.09915E5	0.12788E5	0.12788E5	0.12788E5	0.09915E5
θ_i : CONVERGED	-45/59/-49	0/ 0/ 0/	0/ 0/ 0	0/ 0/ 0	45/59/45

Table 2 Maximization of λmax of six layered FRP plate (initial : 0°/30°/60°/70°/80°/90°)

ITERATION	METHOD	$\theta_1/\theta_2/\theta_3/\theta_4/\theta_5/\theta_6$	λ_{max}
1	A	0 /30 /60 /70 /80 /90	0.17867E5
2	A	0 /30 /-7.9/16.9/37.4/56.4	0.19491E5
3	B	54.9/21.4/10.0/141 /352 /56.4	0.16412E5
4	A	22.5/7.5 /82.5/47.5/57.5/67.5	0.17304E5
5	B	22.5/7.5 /-0.8/70.6/63.5/58.3	0.16937E5
6	A	45 /30 /60 /70 /80 /45	0.18860E5
7	A	38.3/30 /38.6/42.2/44.4/45.7	0.19860E5
8	B	45.7/44.9/45.8/45.8/45.7/45.5	0.19832E5
9	A	22.5/52.5/37.5/47.5/57.5/22.5	0.18873E5
10	A	55.1/54.5/32.2/48.8/32.3/56.1	0.19513E5
11	A	43.4/53.9/53.5/43.7/53.1/43.4	0.19754E5
12	A	45.0/55.0/45.0/45.0/44.5/45.0	0.19830E5
13	A	45.0/55.0/45.0/45.0/45.0/45.0	0.19831E5
14	C	0 / 0 / 0 / 0 / 0 / 0	0.25575E5
15	A	0 / 0 / 0 / 0 / 0 / 0	0.25575E5

layered FRP plate.

6. Concluding remarks

As summarized in Tab.2, optimal behavior variable can be chased by the scheme mainly based on the Hessian matrix. The usefulness of the Hessian matrix becomes more clear in case of the finite element buckling analysis of FRP plate where the number of design variables are large but still small compared with degrees of freedom employed in the finite element analysis.

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