博士論文

論文題目

Bounded cohomology of volume-preserving diffeomorphism groups

（体積保存微分同相群の有界コホモロジー）

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Abstract

Since Gromov's seminal paper [34], bounded cohomology has been extensively studied by many authors. Although, the computation of bounded cohomology is difficult in general. We consider the real coefficient case. The second bounded cohomology has been relatively well studied by using a real-valued function on a group, which is called a quasimorphism. It seems that there have been few results on the third bounded cohomology for a while, but several results have appeared in the last few years. In this thesis, we study bounded cohomology and quasimorphisms on certain diffeomorphism groups. Let us discuss three main results in this thesis.

First, we introduce the notion of G-invariant quasimorphism and study its properties and applications. We prove Bavard's duality theorem for G-invariant quasimorphisms. We also study the extension problem of quasimorphisms. We show that Py's Calabi quasimorphism, which is a $\mathrm{Symp}_0(\Sigma_g, \omega)$-invariant quasimorphism on $\mathrm{Ham}(\Sigma_g, \omega)$, does not extend to $\mathrm{Symp}_0(\Sigma_g, \omega)$ if $g \geq 2$. As a corollary, if $g \geq 2$, we show that the flux homomorphism $\mathrm{Symp}_0(\Sigma_g, \omega) \to H^1(\Sigma_g, \mathbb{R})$ does not have a section homomorphism.

Next, we generalize the result of Brandenbursky and Marcinkowski [9]. They studied the third bounded cohomology $H_b^3(\mathcal{T}_M)$ of a certain transformation group $\mathcal{T}_M$ on a complete Riemannian manifold M of finite volume. They proved that $\dim_{\mathbb{R}} H_b^3(\mathcal{T}_M)$ is infinite if $\pi_1(M)$ is "complicated enough". We extend their results to the case where the volume of M can be infinite. To do this, we introduce the notion of norm controlled cohomology.

Finally, we study the third bounded cohomology $H_b^3(\mathcal{G}_\Sigma)$ of the area-preserving diffeomorphism group $\mathcal{G}_\Sigma$ on a compact surface Σ. We show that $\dim_{\mathbb{R}} H_b^3(\mathcal{G}_\Sigma)$ is infinite for every surface Σ. Although the case $\chi(\Sigma) < 0$ is covered by [9], the case $\chi(\Sigma) \geq 0$ remains. To deal with this case, we define a higher-degree version of Gambaudo–Ghys' construction [30] and prove the injectivity theorem, which is a generalization of Ishida's result [35].

Acknowledgment

I am very indebted to Professor Takashi Tsuboi who was my supervisor for a long time, from when I was a master's student until he retired. I would like to express my great appreciation to Professor Toshitake Kohno for accepting me into his lab and supervising me. I am deeply grateful to my current supervisor Professor Nariya Kawazumi for his instruction and consideration.

I would particularly like to thank Morimichi Kawasaki for frequent discussions and for his many suggestions and comments. The work in Chapter 3 and 4 would not have been done without him. The work of Michael Brandenbursky and Michał Marcinkowski [9] was the trigger for the study in Chapters 4 and 5. I am grateful to them for their comments on the webinar too. I would like to thank Takahiro Matsushita, Masato Mimura, and Ryuma Orita for their comments, advice, and encouragement. I thank the organizers and members of Shin Kyoto Seminar and Dosemi (Saturday Seminar) for the many opportunities they have given me to talk and their comments.

Finally, I would like to express my gratitude to my family for their understanding and support.

Contents

Chapter 1

Introduction and main results

1.1 Introduction

Geometric group theory studies geometric aspects of groups. One can regard a group as a metric space by its norm.

Definition 1.1.1. Let G be a group and e denotes the identity element of G. A function $\nu\colon G \to [0, \infty)$ is a *norm* if it satisfies

(1) $\nu(gh) \leq \nu(g) + \nu(h)$ for any $g, h \in G$,

(2) $\nu(g^{-1}) = \nu(g)$ for any $g \in G$,

(3) $\nu(e) = 0$,

(4) $\nu(g) > 0$ if $g \neq e$.

If one drops the condition (4), ν is said to be a *pseudo norm.*

For finitely generated groups, which are main objects in geometric group theory, a natural norm called the word norm is defined. In [33], Gromov introduced the notion of hyperbolic groups. A finitely generated group is said to be hyperbolic if the group is "negatively curved" in some sense with respect to its word norm (Section 2.1.2). This works played a role in establishing geometric group theory as a field of study. Various attempts have been made to extend the notion of hyperbolic groups, for example, acylindrically hyperbolic groups [47] (Section 2.1.3).

In [12], Burago, Ivanov, and Polterovich introduced the notion of conjugation invariant norms. A norm ν is said to be conjugation-invariant if it satisfies $\nu(hgh^{-1}) = \nu(g)$ for every $g, h \in G$. It gives a framework for geometric group theory for groups that are not necessarily finitely generated. The following are typical examples of conjugation-invariant norms.

Example 1.1.2. (1) Let G be a group and $[G, G]$ the commutator subgroup. The *commutator length* $\mathrm{cl}\colon [G, G] \to \mathbb{N}$ is defined by

$$\mathrm{cl}(h) = \min\{k \mid \exists f_1, \ldots, f_k, g_1, \ldots, g_k \in S, h = [f_1, g_1] \ldots [f_k, g_k]\}$$

for every $h \in [G, G]$.

(2) Let M be a connected orientable smooth manifold. For simplicity, assume that M is closed (i.e, compact and without boundary). Let $\mathrm{Diff}_0(M)$ denote the identity component of the group of diffeomorphisms on M. For a non-empty open subset U of M, we define the *fragmentation norm* $\nu_U\colon \mathrm{Diff}_0(M) \to \mathbb{N}$ with respect to U is defined by

$$\nu_U(h) = \min\left\{ k \;\middle|\; \begin{array}{c} \exists f_1, \ldots, f_k, g_1, \ldots, g_k \in \mathrm{Diff}_0(M), \\ h = (f_1 g_1 f_1^{-1}) \cdots (f_k g_k f_k^{-1}), \mathrm{supp}(g_i) \subset U \end{array} \right\}$$

for every $h \in \mathrm{Diff}_0(M)$.

A *quasimorphism* is a real-valued function ϕ on a group G such that

$$D(\phi) = \sup_{g,h \in G} |\phi(gh) - \phi(g) - \phi(h)| < \infty.$$

Quasimorphisms are useful to study conjugate invariant norms. Especially, quasimorphisms and commutator length are closely related by Bavard's duality [3] (Theorem 2.4.8). If $\mathrm{Diff}_0(M)$ admits a non-trivial quasimorphism, then the fragmentation norm on $\mathrm{Diff}_0(M)$ with respect to an open ball is unbounded [12]. Recently, for the closed surface Σ_g with genus $g \geq 1$, Bowden, Hensel and Webb [6] proved that the fragmentation norm on $\mathrm{Diff}_0(\Sigma_g)$ is unbounded by constructing non-trivial quasimorphisms. In contrast, it is known that the fragmentation norm on $\mathrm{Diff}_0(M)$ is bounded when $\dim(M) \neq 2, 4$ [12, 55, 56].

A quasimorphism can be thought of as a second bounded cohomology class. Bounded cohomology was introduced by Gromov [34] in his study of (simplicial) volume of Riemannian manifolds. From the viewpoint of geometric group theory, the non-triviality of bounded cohomology of groups reflects the negatively curved nature of groups. For example, the bounded cohomology of an acylindrically hyperbolic group is highly non-trivial [4, 28]. On the other hand, the bounded cohomology of an amenable group is trivial.

Bounded cohomology has been extensively studied, but its computation is difficult in general. For the real coefficient case, the first bounded cohomology is trivial. The second bounded cohomology has been relatively well studied by constructing non-trivial quasimorphisms. Except for the early works of Yoshida [58] and Soma [54], it seems that there had been no work on the third bounded cohomology for a while. However, there have been several works on the third bounded cohomology in the last few years [24, 25, 26, 28]. Recently, Brandenbursky and Marcinkowski proved the following theorem.

Theorem 1.1.3 ([9])**.** *Let M be a complete Riemannian manifold with finite volume. Let $\mathcal{T}_M$ denote $\mathrm{Homeo}_0(M,\mu)$ or $\mathrm{Diff}_0(M,\mathrm{vol})$ or $\mathrm{Symp}(M,\omega)$. Set $\pi_M = \pi_1(M)/Z(\pi_1(M))$. If either*

(1) π_M surjects onto F_2 or

(2) π_M is an acylindrically hyperbolic group,

then the (reduced exact) third bounded cohomology $\overline{EH}_b^3(\mathcal{T}_M)$ is uncountably infinite-dimensional.

1.2 Main results

In this thesis, we establish three main results. First, we introduce the notion of G-invariant quasimorphism and study its properties and applications. This part contains joint work with Morimichi Kawasaki. Let G be a group and H its normal subgroup. A quasimorphism ϕ on H is said to be *G-invariant* if $\phi(ghg^{-1}) = \phi(h)$ for every $g \in G$ and $h \in H$. Let $Q(H)^G$ denote the space of homogeneous G-invariant quasimorphisms. We consider a (G,H)-commutator ($g \in G$, $h \in H$) and define (G,H)-commutator subgroup $[G,H]$ and (G,H)-commutator length $\mathrm{cl}_{G,H}$ as the ordinary ones. We prove the following Bavard-type duality theorem for G-invariant quasimorphisms.

Theorem 1.2.1. *Assume that $H = [G,H]$. For any $x \in [G,H]$,*

$$\mathrm{scl}_{G,H}(x) = \frac{1}{2} \sup_{\phi \in Q(H)^G} \frac{|\phi(x)|}{D(\phi)}.$$

Here, $\mathrm{scl}_{G,H}(x) = \lim_{n\to\infty} \mathrm{cl}_{G,H}(x^n)/n$. Note that the assumption $H = [G,H]$ in the above theorem was removed in [38].

We also consider the extension problem of quasimorphisms. A homogeneous quasimorphism ϕ on H is *extendable to* G if there exists a homogeneous quasimorphism ψ on G such that $\psi|_H = \phi$. If a quasimorphism on H is extendable to G, it is necessary to be G-invariant. We find an example of non-extendable G-invariant quasimorphism.

Theorem 1.2.2. *Let Σ_g be an oriented closed surface wits genus $g \geq 2$ and ω a symplectic form on Σ_g. Py's Calabi quasimorphism $\mu_P \colon \mathrm{Ham}(\Sigma_g, \omega) \to \mathbb{R}$ is non-extendable to* $\mathrm{Symp}_0(\Sigma_g, \omega)$.

We obtain the following interesting corollary from the above theorem.

Corollary 1.2.3. *If $g \geq 2$, the flux homomorphism $\mathrm{Flux}_\omega \colon \mathrm{Symp}_0(\Sigma_g, \omega) \to H^1(\Sigma_g; \mathbb{R})$ does not have a section homomorphism.*

Note that if $g = 1$, the (descended) flux homomorphism $\mathrm{Flux}_\omega \colon \mathrm{Symp}_0(\Sigma_1, \omega) \to H^1(\Sigma_1; \mathbb{R})/H^1(\Sigma_1; \mathbb{Z})$ has a section homomorphism.

Next, we introduce the notion of norm controlled cohomology which is a generalization of bounded cohomology.

Definition 1.2.4. Let G be a group and ν a (pseudo) norm on G. An inhomogeneous cochains $\bar{c} \in \bar{C}^n(G)$ is a *level d norm controlled cochain* if there exist $C, D \geq 0$, for all $g_1, \dots, g_n \in G$,

$$|\bar{c}(g_1, \dots, g_n)| \leq C \cdot \min_{\substack{I \subset \{1,\dots,n\} \\ \#I = n-d}} \left\{ \sum_{i \in I} \nu(g_i) \right\} + D.$$

When $d \geq n$, let a norm controlled cochain mean a bounded cochain. Let $\bar{C}^n(G, \nu)$ denotes the set of level d norm controlled cochains. *The norm controlled cohomology*, denoted by $H^n_{(d)}(G, \nu)$, is defined to be the cohomology of the cochain complex $(\bar{C}^n_\nu(G), \bar{\delta})$. *The exact norm controlled cohomology* $EH^n_{(d)}(G, \nu)$ is the kernel of the comparison map $H^n_{(d)}(G, \nu) \to H^n(G)$.

If ν is a bounded norm, then the norm controlled cohomology $H^n_{(d)}(G, \nu)$ is nothing but the bounded cohomology $H^n_b(G)$. Our next main theorem is the following.

Theorem 1.2.5. *Let M be a complete Riemannian manifold. Let $\mathcal{T}_M$ denote* $\mathrm{Homeo}_0(M, \mu)$*,* $\mathrm{Diff}_0(M, \mathrm{vol})$ *or* $\mathrm{Symp}(M, \omega)$*. Set $\pi_M = \pi_1(M)/Z(\pi_1(M))$. Assume that there exists an open subset U of M with finite volume such that the fragmentation norm ν_U is well-defined on $\mathcal{T}_M$. If either*

(1) π_M surjects onto F_2 or

(2) π_M is an acylindrically hyperbolic group,

then $EH^3_{(d)}(\mathcal{T}_M)$ is uncountably infinite-dimensional for $d = 0, 1, 2$.

Note that ν_U is well-defined on $\mathcal{T}_M$ if the inclusion $U \to M$ is homotopy equivalent. If M has a finite volume and $U = M$, this implies (a weak version of) the result of Theorem 1.1.3.

Example 1.2.6. It is known that for most 3-manifolds, their fundamental groups are acylindrically hyperbolic [43]. On the other hand, if M is 3-dimensional and $\pi_1(M)$ is finitely generated, there exists a 3-dimensional compact submanifold C such that the inclusion $C \to M$ is homotopy equivalent by the Scott core theorem [52]. Hence we can find an open subset U of M which is of finite volume and homotopy equivalent to M. Thus most 3-manifolds enjoy the assumption of the above theorem.

Finally, we consider bounded cohomology of area-preserving diffeomorphism groups on surfaces. First, we consider the case of the 2-disk $\mathbb{D}$.

Let $\mathcal{G}$ denote the group of area-preserving diffeomorphisms $\mathrm{Diff}(\mathbb{D}, \partial\mathbb{D}, \mathrm{area})$ on $\mathbb{D}$ which are the identity near the boundary $\partial\mathbb{D}$. In [30], Gambaudo and Ghys constructed a linear map $\Gamma\colon Q(P_m) \to Q(\mathcal{G})$, where $Q(G)$ denotes the space of homogenous quasimorphisms on a group G and P_m denotes the pure braid group on m strands. Let B_m be the braid group on m strands and $i\colon P_m \to B_m$ be the standard inclusion. Ishida [35] proved that the composition map $\Gamma \circ i^*\colon Q(B_m) \to Q(\mathcal{G})$ is injective. He also proved that the map $EH^2_b(B_m) \to EH^2_b(\mathcal{G})$ induced by $\Gamma \circ i^*$ is also injective, where $EH^n_b(G)$ denotes the exact bounded cohomology of G.

We generalize Ishida's result to higher dimensional bounded cohomology for the case of three strands. We define a map $\overline{E\Gamma}_b\colon \overline{EH}^n_b(P_m) \to \overline{EH}^n_b(\mathcal{G})$ which generalizes Gambaudo–Ghys' construction and prove the following theorem.

Theorem 1.2.7. *The composition map $\overline{E\Gamma}_b \circ i^*\colon \overline{EH}^n_b(B_3) \to \overline{EH}^n_b(\mathcal{G})$ is injective.*

Here $\overline{EH}_b^n(G)$ denotes the reduced exact bounded cohomology of G. As a corollary, we obtain the following result.

Corollary 1.2.8. *The dimension of* $\overline{EH}_b^3(\mathcal{G})$ *is uncountably infinite.*

We also prove similar results for compact surfaces Σ with non-negative Euler characteristic $\chi(\Sigma) \geq 0$. Let $B_m(\Sigma)$ and $P_m(\Sigma)$ denote the braid group and the pure braid group on a surface Σ, respectively. Let $\mathcal{G}_\Sigma$ denote the identity component of the group of area-preserving diffeomorphisms $\mathrm{Diff}_0(\Sigma, \partial\Sigma, \mathrm{area})$ on Σ which are the identity near the boundary $\partial\Sigma$. The notation G^Z is used to denote the central quotient $G/Z(G)$ of a group G. We define a map $\overline{E\Gamma}_b^Z : \overline{EH}_b^n(P_m(\Sigma)^Z) \to \overline{EH}_b^n(\mathcal{G}_\Sigma)$ instead of on $\overline{EH}_b^n(P_m(\Sigma))$ because $\mathcal{G}_\Sigma$ is not contractible in general. Let $i^Z \colon P_m(\Sigma)^Z \to B_m(\Sigma)^Z$ denote the map induced by the standard inclusion $i \colon P_m(\Sigma) \to B_m(\Sigma)$.

Theorem 1.2.9. *Let* Σ *be a compact oriented surface such that* $\chi(\Sigma) \geq 0$. *The maps* $\overline{E\Gamma}_b^Z \circ (i^Z)^* \colon \overline{EH}_b^n(B_m(\Sigma)^Z) \to \overline{EH}_b^n(\mathcal{G}_\Sigma)$ *is injective for* $m = 2 + \chi(\Sigma)$.

As a corollary of Theorem 1.2.9, we obtain the following.

Corollary 1.2.10. *Let* Σ *be a compact oriented surface such that* $\chi(\Sigma) \geq 0$. *The dimension of* $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ *is uncountably infinite.*

We remark that Corollary 1.2.10 is not covered by the result of Brandenbursky and Marcinkowski. On the other hand, their result covers the case of surfaces with negative Euler characteristics. Therefore, in some sense, our results and theirs are complementary to each other in the case of 2-manifolds. Namely, we obtain the following.

Theorem 1.2.11. *For any compact oriented surface* Σ, *the dimension of* $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ *is uncountably infinite.*

Chapter 2

Preliminaries

2.1 Geometric group theory

2.1.1 Quasi-isometry

Definition 2.1.1. Let $K \geq 1$, $L \geq 0$, and $f\colon X \to Y$ be a map between metric spaces (X, d_X) and (Y, d_Y).

- The map f is called (K, L)-*quasi-isometric embedding* if

$$\frac{1}{K} d_X(x_1, x_2) - L \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + L$$

  for any $x_1, x_2 \in X$.

- The map f has a *quasi-dense image* if there exists a constant $C \geq 0$ such that for every $y \in Y$ there exists $x \in X$ such that

$$d_Y(f(x), y) \leq C.$$

- The map f is a (K, L)-*quasi-isometry* if it is (K, L)-quasi-isometric embedding with a quasi-dense image. We say that f is quasi-isometric if it is (K, L)-quasi-isometric for some K and L.

- The spaces (X, d_X) and (Y, d_Y) are said to be (K, L)-*quasi-isometric* if there exists a (K, L)-quasi-isometry between X and Y. We say that X and Y are quasi-isometric if they are (K, L)-quasi-isometric for some K and L.

The following *Švarc–Milnor lemma* is fundamental in geometric group theory.

Lemma 2.1.2. *Assume that a group G acts on a (K, L)-quasi-geodesic space (X, d) by isometries. If there exists a subset $B \subset X$ such that*

- *the diameter of B is finite,*
- $X = \bigcup_{g \in G} g \cdot B$, *and*
- *the set*
$$S = \{g \in G \mid g \cdot B' \cap B'\}$$
is finite, where B' is the $2L$-neighborhood of B,

then G is generated by S, and the map $(G, d_S) \to (X, d)$ defined by $g \mapsto g \cdot x$ is a quasi-isometry for all $x \in X$, where d_S denotes the word metric on G.

In many cases, this lemma is used in the following form (Corollary 2.1.3). A metric space is *proper* if any closed ball is compact. An action of a group G on a topological space X is *proper* if the set $\{g \in G \mid g \cdot B \cap B\}$ is finite for all compact sets $B \subset X$. A map $\gamma \colon I \to X$ on a closed segment $I \subset \mathbb{R}$ to a metric space is a *geodesic* if γ is an isometry. A metric space is a *geodesic space* if any two points are connected by a geodesic.

Corollary 2.1.3. *Let G be a group acting on a proper geodesic space X by isometry. Assume that this action is proper and cocompact. Then G is finitely generated and the map $G \to X$ defined by $g \mapsto g \cdot x$ is a quasi-isometry.*

2.1.2 Hyperbolic groups

We review a definition of Gromov hyperbolic space for metric spaces. Note that there are several equivalent definitions of hyperbolicity for geodesic spaces (see [31] for example). Let (X, d) be a metric space.

Definition 2.1.4. For $x, y, z \in X$, we define the Gromov product $(y|z)_x$ by

$$(y|z)_x = \frac{1}{2}\{d(y, z) + d(z, x) - d(y, z)\}.$$

Definition 2.1.5. Let (X, d) be a metric space and $x_0 \in X$ a base point. For $\delta \geq 0$, we say that (X, d) is *δ-hyperbolic with respect to x_0* if

$$(x|y)_{x_0} \geq \min\{(x|z)_{x_0}, (y|z)_{x_0}\} - \delta$$

for every $x, y, z \in X$. The space (X, d) is said to be *hyperbolic* if there exists δ such that (X, d) is δ-hyperbolic with respect to some base point $x_0 \in X$.

It is known that the existence of δ does not depend on the choice of the base point. The following proposition states that the hyperbolicity is quasi-isometric invariant.

Proposition 2.1.6. *Let X and Y be metric spaces. Assume that X and Y are quasi-isometric. If X is hyperbolic, then Y is also hyperbolic.*

We can define a natural metric d on a connected graph Γ by setting the length of each edge to 1. In particular, for each vertices v and w, the distance $d(v, w)$ between v and w is represented by the minimal length of a path in Γ which connects v and w.

Remark 2.1.7. A tree (i.e., a graph without cycles) is a 0-hyperbolic space.

For a finitely generated group (G, S), we can define a graph $\Gamma(G, S)$ which is called the Cayley graph.

Definition 2.1.8. For a group G with a generating set S, we define the *Cayley graph* $\Gamma = \Gamma(G, S)$ of G with respect to S as follows. The set of vertices $V(\Gamma)$ is the set of elements of G. The set of edges $E(\Gamma)$ is defined by

$$E(\Gamma) = \{\{g, h\} \mid g, h \in G, \exists s \in S, gs = h\}.$$

Definition 2.1.9. A finitely generated group G is *hyperbolic* if its Cayley graph $\Gamma(G, S)$ is Gromov hyperbolic for some (any) finite generating set S.

Since the quasi-isometry type of a Cayley graph does not depend on the choice of a finite generating set and Gromov hyperbolicity is a quasi-isometric invariant, the hyperbolicity of a group is a property of the group.

A finite group is hyperbolic. The group $\mathbb{Z}$ is hyperbolic since the Cayley graph $\Gamma(\mathbb{Z}, \{\pm 1\})$ is a tree and thus 0-hyperbolic. If a group G is virtually $\mathbb{Z}$ (i.e., G contains a finite index subgroup which is isomorphic to $\mathbb{Z}$), then G is hyperbolic since G is quasi-isometric to $\mathbb{Z}$.

Definition 2.1.10. A hyperbolic group G is *elementary* if G is finite or virtually $\mathbb{Z}$. Otherwise, G is called a *non-elementary hyperbolic group*.

2.1.3 Acylindrically hyperbolic groups

In [47], Osin introduced the notion of *acylindrically hyperbolic groups* which is a generalization of hyperbolic groups. We review one of the definitions of acylindrically hyperbolic groups.

Let G be a group, H a subgroup of G, and $\mathcal{X}$ a subset of G. Assume that $\mathcal{X} \cup H$ generates G. Let $\Gamma(G, \mathcal{X} \sqcup H)$ denote the Cayley graph of G with respect to the disjoint union $\mathcal{X} \sqcup H$. That is, for $g \in G$, if $s \in \mathcal{X} \cap H$ then g and gs in the vertex of $\Gamma(G, \mathcal{X} \sqcup H)$ are joined by two edges labeled by $s \in \mathcal{X}$ and $s \in H$.

We define a metric $\widehat{d}\colon H \times H \to [0, \infty]$. We say that a path p in $\Gamma(G, \mathcal{X} \sqcup H)$ is *admissible* if p does not contain the edge of the complete subgraph $\Gamma(H, H) \subset \Gamma(G, \mathcal{X} \sqcup H)$. For $h_1, h_2 \in H$, we define $\widehat{d}(h_1, h_2)$ to be the length of a shortest admissible path from h_1 to h_2. If no such path exists, we set $\widehat{d}(h_1, h_2) = \infty$.

Definition 2.1.11. We say that H is *hyperbolically embedded in G with respect to $\mathcal{X}$* (and write $H \hookrightarrow_h (G, \mathcal{X})$) if the graph $\Gamma(G, \mathcal{X} \sqcup H)$ is hyperbolic and the metric $\widehat{d}$ is proper. We also say that H is *hyperbolically embedded in G* (and write $H \hookrightarrow_h G$) if $H \hookrightarrow_h (G, \mathcal{X})$ for some $\mathcal{X}$.

For every group G, $G \hookrightarrow_h G$ for $\mathcal{X} = \emptyset$ since $\Gamma(G, G)$ has diameter 1 and $\widehat{d}(h_1, h_2) = \infty$ if $h_1 \neq h_2$. If H is a finite subgroup of G, then $H \hookrightarrow_h G$ for $\mathcal{X} = G$. Thus we are interested in the case where H is a proper infinite subgroup.

Definition 2.1.12. A group G is said to be *acylindrically hyperbolic* if there exists a proper infinite hyperbolically embedded subgroup of G.

Example 2.1.13. Examples of acylindrically hyperbolic groups include;

- non-elementary hyperbolic groups,
- mapping class group of hyperbolic surfaces,
- most 3-manifold groups [43].

We can choose a special form as a hyperbolically embedded subgroup.

Theorem 2.1.14 ([17])**.** *Suppose that G is acylindrically hyperbolic. Then there exists a subgroup H of G such that $H \hookrightarrow_h G$ and $H \cong F_2 \times K$, where F_2 is a free group of rank 2 and K is a finite group.*

2.1.4 Amenable groups

Amenable groups are introduced by von Neumann [57] in the study of Banach–Tarski paradox. See [14, 29] for more information.

Let $L^\infty(G)$ denote the set of all bounded real-valued functions on a group G. The set $L^\infty(G)$ has a structure of $\mathbb{R}$-vector space. The group G acts on $L^\infty(G)$ by

$$g \cdot f(h) = f(g^{-1}h)$$

for all $f \in L^\infty(G)$ and $g, h \in G$.

Definition 2.1.15. A group G is *amenable* if there exists a left-invariant mean $m\colon L^\infty(G) \to \mathbb{R}$. That is, the map m is $\mathbb{R}$-linear and satisfies the following:

- $m(1_G) = 1$, where $1_G\colon G \to \mathbb{R}$, $g \mapsto 1$ is the constant map,
- $m(f) \geq 0$ for all $f \in L^\infty(G)$ that satisfy $f \geq 0$,
- $m(g \cdot f) = m(f)$ for all $f \in L^\infty(G)$ and $g \in G$.

Example 2.1.16. A finite group is amenable. An abelian group is amenable [57]. Moreover, virtually solvable groups are amenable.

Remark 2.1.17. The class of amenable groups is closed under the operations of taking subgroups, forming quotients, forming extensions, and taking direct unions. The smallest class of groups which contains all finite and abelian groups, and is closed under these operations, is called *elementary amenable groups*. For example, virtually solvable groups are elementary amenable.

2.2 Group cohomology

2.2.1 Group cohomology

Throughout this thesis, we only consider the cohomology with real coefficients. In this section, we define inhomogeneous complex $\bar{C}^\bullet(-)$ and homogeneous complex $C^\bullet(-)$. Then we recall the correspondence between them.

Let G be a group. We consider the space of (inhomogeneous) n-cochains

$$\bar{C}^n(G) = \{\bar{c}\colon G^n \to \mathbb{R}\},$$

and the coboundary map $\bar{\delta} : \bar{C}^{n-1}(G) \to \bar{C}^n(G)$ defined by

$$\bar{\delta}\bar{c}(g_1, \ldots, g_n) = \bar{c}(g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i \bar{c}(g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^n \bar{c}(g_1, \ldots, g_{n-1})$$

for $\bar{c} \in \bar{C}^n(G)$ and $g_1, \ldots, g_n \in G$. The cohomology of this cochain complex is called the *(group) cohomology* of G and denoted by $H^n(G)$.

There is another definition of this cohomology. A map $c \colon G^{n+1} \to \mathbb{R}$ is said to be *homogeneous* if $c(g_0 h, \ldots, g_n h) = c(g_0, \ldots, g_n)$ for every $g_0, \ldots, g_n, h \in G$. The space of (homogeneous) n-cochains is

$$C^n(G) = \{c \colon G^{n+1} \to \mathbb{R} \mid c \text{ is homogeneous}\},$$

and the coboundary map $\delta : C^{n-1}(G) \to C^n(G)$ is defined by

$$\delta c(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i c(g_0, \ldots, \widehat{g_i}, \ldots, g_n)$$

for $c \in C^n(G)$ and $g_0, \ldots, g_n \in G$, where $\widehat{g_i}$ means that we omit the entry g_i. The cohomology of $(C^n(G), \delta)$ also defines $H^n(G)$.

The correspondence between an inhomogeneous cochain $\bar{c} \in \bar{C}^n(G)$ and homogeneous one $c \in C^n(G)$ is the following:

$$\bar{c}(g_1, g_2, \ldots, g_n) = c(1, g_1, g_1 g_2, \ldots, g_1 g_2 \ldots g_n), \tag{2.2.1}$$
$$c(g_0, g_1, \ldots, g_n) = \bar{c}(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n). \tag{2.2.2}$$

We call that a cochain $c \in C^n(G)$ is *alternating* if

$$c(g_{\sigma(0)}, \ldots, g_{\sigma(n)}) = \mathrm{sgn}(\sigma) c(g_0, \ldots, g_n)$$

for any $g_0, \ldots, g_n \in G$ and $\sigma \in \mathfrak{S}_{n+1}$, where $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ is the sign of σ. Let $C^n_{\mathrm{alt}}(G)$ denote the set of alternating n-cochains. Then $(C^n_{\mathrm{alt}}(G), \delta)$ is a subcomplex of $(C^n(G), \delta)$. It is known that the cohomology of $(C^n_{\mathrm{alt}}(G), \delta)$ coincides with $H^n(G)$.

2.2.2 Bounded cohomology

We review the definition of bounded cohomology. We only mention the inhomogeneous case but the homogeneous case is defined similarly. If we consider the subcomplex

$$\bar{C}^n_b(G) = \{\bar{c} \colon G^n \to \mathbb{R} \mid \bar{c} \text{ is bounded}\}$$

of $\bar{C}^n$, the homology of the complex $(\bar{C}_b^n(G), \bar{\delta})$ is called the *bounded cohomology* of G and is denoted by $H_b^n(G)$. The natural inclusion $\bar{C}_b^n(G) \to \bar{C}^n(G)$ induces the homomorphism $H_b^n(G) \to H^n(G)$ called the *comparison map*. The kernel of the comparison map $H_b^n(G) \to H^n(G)$ is called the *exact bounded cohomology* and is denoted by $EH_b^n(G)$.

For a cochain $\bar{c} \in \bar{C}_b^n(G)$, we define the *norm* $\|\bar{c}\|$ of $\bar{c}$ by

$$\|\bar{c}\| = \sup_{g_1,\dots,g_n \in G} |\bar{c}(g_1, \dots, g_n)|.$$

This norm induces a natural norm on $H_b^n(G)$ which is also denoted by $\|\cdot\|$. Let $N^n(G)$ denote the norm zero subspace of $H_b^n(G)$, i.e.,

$$N^n(G) = \{\alpha \in H_b^n(G) \mid \|\alpha\| = 0\}.$$

The *reduced cohomology* $\overline{H}_b^n(G)$ is defined by the quotient $H_b^n(G)/N^n(G)$. The *reduced exact cohomology* $\overline{EH}_b^n(G)$ is defined by $EH_b^n(G)/EN(G)$, where $EN^n(G) = N^n(G) \cap EH_b^n(G)$.

We can consider the homogeneous complex $C_b^\bullet(G)$, alternating homogeneous and inhomogeneous subcomplex $C_{b,\mathrm{alt}}^\bullet(G)$ and $\bar{C}_{b,\mathrm{alt}}^\bullet(G)$, and they also define the cohomology $H_b^\bullet(G)$.

We summarize several facts which we use later.

Lemma 2.2.1. *Let G be a group and H a normal subgroup of G of finite index. Then the inclusion map $H \to G$ induces an isomorphism $H^n(G) \cong H^n(H)^G$ and an isometric isomorphism $H_b^n(G) \cong H_b^n(H)^G$.*

The inverse maps of those isomorphisms are given by the transfer maps (see [11, 14]). We remark that $H_b^n(G) \to H_b^n(H)^G$ is an isometric isomorphism even if G/H is amenable [34].

The following theorem is known as the mapping theorem (for groups).

Theorem 2.2.2 ([34])**.** *If $\phi\colon G_1 \to G_2$ is a surjective group homomorphism with an amenable kernel, then $\phi^*\colon H_b^n(G_2) \to H_b^n(G_1)$ is an isometric isomorphism.*

It is known that the bounded cohomology of an amenable group is trivial in every degree. On the other hand, non-positively curved groups tend to have highly non-trivial bounded cohomology. For example, the following theorem is known.

Theorem 2.2.3 ([28]). *If G is an acylindrically hyperbolic group, then the dimension of $\overline{EH}_b^3(G)$ is uncountably infinite.*

In particular, the third bounded cohomology of a non-elementary hyperbolic group is infinite-dimensional.

2.3 Diffeomorphism and homeomorphism groups

Let M be a connected, oriented smooth manifold without boundary. Let $\mathrm{Diff}^c(M)$ denote the group of compactly supported diffeomorphisms on M and $\mathrm{Diff}_0^c(M)$ denote the subgroup of $\mathrm{Diff}^c(M)$ consisting of diffeomorphisms that are isotopic to the identity. If M is compact, groups $\mathrm{Diff}^c(M)$ and $\mathrm{Diff}_0^c(M)$ coincide with $\mathrm{Diff}(M)$ and $\mathrm{Diff}_0(M)$, respectively.

Let N be a compact, connected, oriented smooth manifold which might have boundary ∂N. Let $\mathrm{Diff}(N, \partial N)$ denotes the group of diffeomorphisms on N which is identity near the boundary and $\mathrm{Diff}_0(N, \partial N)$ denote the subgroup of $\mathrm{Diff}(N, \partial N)$ consisting of diffeomorphisms that are isotopic to the identity. Note that $\mathrm{Diff}(N, \partial N) = \mathrm{Diff}^c(\mathring{N})$ and $\mathrm{Diff}_0(N, \partial N) = \mathrm{Diff}_0^c(\mathring{N})$, where $\mathring{N}$ is the interior of N.

For a path-connected topological group G, let $\widetilde{G}$ denotes the universal cover of G, i.e., the group of path homotopy equivalent classes in the path space of G. Note that $\pi_1(G)$ can be regarded as a subgroup of $\widetilde{G}$.

2.3.1 Volume-preserving diffeomorphism groups

Let M be a manifold with a volume form $\mathrm{vol} \in \Omega^n(M)$, where $n = \dim(M)$. Let $\mathrm{Diff}^c(M, \mathrm{vol})$ denote the *volume-preserving diffeomorphism group*

$$\mathrm{Diff}^c(M, \mathrm{vol}) = \{f \in \mathrm{Diff}^c(M) \mid f^* \mathrm{vol} = \mathrm{vol}\}$$

and $\mathrm{Diff}_0^c(M, \mathrm{vol})$ denote its identity component, i.e.,

$$\mathrm{Diff}_0^c(M, \mathrm{vol}) = \{g \in \mathrm{Diff}^c(M, \mathrm{vol}) \mid \exists \{g^t\}_{0 \leq t \leq 1} \subset \mathrm{Diff}^c(M, \mathrm{vol}), g^0 = \mathrm{id}, g^1 = g\}.$$

Definition 2.3.1. We define the (volume) *flux homomorphism*

$$\widetilde{\mathrm{Flux}} \colon \widetilde{\mathrm{Diff}}_0^c(M, \mathrm{vol}) \to H_c^{n-1}(M; \mathbb{R})$$

on the universal cover $\widetilde{\mathrm{Diff}}_0^c(M, \mathrm{vol})$ of $\mathrm{Diff}_0^c(M, \mathrm{vol})$ by

$$\widetilde{\mathrm{Flux}}([\{\psi^t\}_{0\le t\le 1}]) = \int_0^1 [\iota_{X_t}\,\mathrm{vol}]dt,$$

where

- $\{\psi^t\}_{0\le t\le 1}$ is a path in $\mathrm{Diff}_0^c(M, \mathrm{vol})$ with $\psi^0 = \mathrm{id}$,
- $[\{\psi^t\}_{0\le t\le 1}]$ is an element of $\widetilde{\mathrm{Diff}}_0^c(M, \mathrm{vol})$ represented by the path $\{\psi^t\}_{0\le t\le 1}$,
- X_t is the (time-dependent) vector field induced by the flow $\{\psi^t\}_{0\le t\le 1}$,
- ι_X is the interior product with respect to a vector field X.

The group $\Gamma = \widetilde{\mathrm{Flux}}(\pi_1(\mathrm{Diff}_0^c(M, \mathrm{vol})))$ is called the (volume) *flux group*. The flux homomorphism $\widetilde{\mathrm{Flux}}$ induces the reduced flux homomorphism

$$\mathrm{Flux}\colon\ \mathrm{Diff}_0^c(M, \mathrm{vol}) \to H_c^{n-1}(M;\mathbb{R})/\Gamma.$$

2.3.2 Symplectomorphism groups

Let M be a manifold. A 2-form $\omega \in \Omega^2(M)$ is called a *symplectic form* if ω is non-degenerate and $d\omega = 0$. A *symplectic manifold* (M, ω) is a manifold M with a symplectic form ω.

Let $\mathrm{Symp}^c(M, \omega)$ denote the *symplectomorphism group*

$$\mathrm{Symp}^c(M, \omega) = \{f \in \mathrm{Diff}^c(M) \mid f^*\omega = \omega\}$$

and $\mathrm{Symp}_0^c(M, \omega)$ denote its identity component, i.e.,

$$\mathrm{Symp}_0^c(M, \omega) = \{g \in \mathrm{Symp}^c(M, \omega) \mid \exists \{g^t\}_{0\le t\le 1} \subset \mathrm{Symp}^c(M, \omega), g^0 = \mathrm{id}, g^1 = g\}.$$

Definition 2.3.2. We define the *flux homomorphism*

$$\widetilde{\mathrm{Flux}}_\omega\colon \widetilde{\mathrm{Symp}}_0^c(M, \omega) \to H_c^1(M;\mathbb{R})$$

on the universal covering $\widetilde{\mathrm{Symp}}_0^c(M, \omega)$ of $\mathrm{Symp}_0^c(M, \omega)$ by

$$\widetilde{\mathrm{Flux}}_\omega([\{\psi^t\}_{0\le t\le 1}]) = \int_0^1 [\iota_{X_t}\omega]dt,$$

where

- $\{\psi^t\}_{0\le t\le 1}$ is a path in $\mathrm{Symp}_0^c(M,\omega)$ with $\psi^0 = \mathrm{id}$,
- $[\{\psi^t\}_{0\le t\le 1}]$ is an element of $\widetilde{\mathrm{Symp}}_0^c(M,\omega)$ represented by the path $\{\psi^t\}_{0\le t\le 1}$,
- X_t is the (time-dependent) vector field induced by the flow $\{\psi^t\}_{0\le t\le 1}$,
- ι_X is the interior product with respect to a vector field X.

The group $\Gamma_\omega = \widetilde{\mathrm{Flux}}_\omega(\pi_1(\mathrm{Symp}_0^c(M,\mathrm{vol})))$ is called the *flux group*. The flux homomorphism $\widetilde{\mathrm{Flux}}_\omega$ induces the reduced flux homomorphism

$$\mathrm{Flux}_\omega\colon\ \mathrm{Symp}_0^c(M,\omega) \to H_c^1(M;\mathbb{R})/\Gamma.$$

The kernel of the flux homomorphism $\mathrm{Flux}_\omega\colon\ \mathrm{Symp}_0^c(M,\omega) \to H_c^1(M;\mathbb{R})/\Gamma$ is called the *Hamiltonian diffeomorphism group* and denoted by $\mathrm{Ham}(M,\omega)$.

We give another description of Hamiltonian diffeomorphism groups. For a Hamiltonian function $H\colon M \to \mathbb{R}$ with compact support, we define the *Hamiltonian vector field* X_H associated with H by

$$\omega(X_H, V) = -dH(V) \text{ for any } V \in \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ is the set of smooth vector fields on M.

Let S^1 denote $\mathbb{R}/\mathbb{Z}$. For a (time-dependent) Hamiltonian function $H\colon S^1\times M \to \mathbb{R}$ with compact support and for $t \in S^1$, we define a function $H_t\colon M \to \mathbb{R}$ by $H_t(x) = H(t,x)$. Let X_H^t denote the Hamiltonian vector field associated with H_t and let $\{\varphi_H^t\}_{t\in\mathbb{R}}$ denote the isotopy generated by X_H^t such that $\varphi^0 = \mathrm{id}$. Let φ_H denote φ_H^1 and φ_H is called *the Hamiltonian diffeomorphism generated by* H. For a symplectic manifold (M,ω), we define the group of Hamiltonian diffeomorphisms by

$$\mathrm{Ham}(M,\omega) = \{\varphi \in \mathrm{Diff}(M) \mid \exists H \in C^\infty(S^1\times M) \text{ such that } \varphi = \varphi_H\}.$$

2.3.3 Measure-preserving homeomorphism groups

Let M be a complete Riemannian manifold and μ the measure on M which is induced by the Riemannian structure. Let $\mathrm{Homeo}_0^c(M,\mu)$ denotes the group of compactly supported homeomorphisms that are isotopic to the identity and $\widetilde{\mathrm{Homeo}}_0^c(M,\mu)$ denotes its universal covering. Fathi [27] defined the *mass flow homomorphism* $\tilde\theta\colon \widetilde{\mathrm{Homeo}}_0^c(M,\mu) \to H_1(M;\mathbb{R})$. For the definition of the mass flow homomorphisms, see [27, 46]. Set $\Gamma = \tilde\theta(\pi_1(\mathrm{Homeo}_0^c(M,\mu)))$. The map $\tilde\theta$ induces the map $\theta\colon\ \mathrm{Homeo}_0^c(M,\mu) \to H_1(M;\mathbb{R})/\Gamma$.

2.4 Quasimorphisms

2.4.1 Definitions and Properties

Definition 2.4.1. A *quasimorphism* ϕ is a real-valued function on a group Γ such that there exists a constant $D \geq 0$ and

$$|\phi(xy) - \phi(x) - \phi(y)| \leq D$$

for any $x, y \in \Gamma$. Such a smallest D is called the *defect* of ϕ and denoted by $D(\phi)$. A quasimorphism ϕ is *homogeneous* if $\phi(x^n) = n\phi(x)$ for every $n \in \mathbb{Z}$ and $x \in \Gamma$.

Let $\widehat{Q}(\Gamma)$ and $Q(\Gamma)$ denote the set of quasimorphisms on Γ and the set of homogeneous quasimorphisms on Γ, respectively. The sets $\widehat{Q}(\Gamma)$ and $Q(\Gamma)$ are naturally regarded as $\mathbb{R}$-linear spaces.

Example 2.4.2.

- A bounded function is a quasimorphism. Thus the set of bounded functions $\bar{C}_b^1(\Gamma)$ on Γ is a linear subspace of $\widehat{Q}(G)$.
- A homomorphism is a homogeneous quasimorphism with defect zero. Thus the set of homomorphisms $H^1(\Gamma)$ is a linear subspace of $Q(\Gamma)$.

We define a linear map $\widehat{Q}(\Gamma) \to Q(\Gamma)$, $\phi \mapsto \overline{\phi}$, which is called the *homogenization*, by

$$\overline{\phi}(x) = \lim_{n\to\infty} \frac{\phi(x^n)}{n}.$$

The limit exists by Fekete's lemma. The kernel of the homogenization is the space of bounded functions. Thus the homogenization induces an isomorphism $\widehat{Q}(\Gamma)/\bar{C}_b^1(\Gamma) \cong Q(\Gamma)$.

Example 2.4.3.

- (Poincaré's rotation number [48])

  Let $\mathrm{Homeo}^+(S^1)$ be the group of orientation-preserving homeomorphisms on the circle and $\widetilde{\mathrm{Homeo}}^+(S^1)$ the preimage of $\mathrm{Homeo}^+(S^1)$ in $\mathrm{Homeo}^+(\mathbb{R})$ under the covering projection $\mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$, i.e.,

  $$\widetilde{\mathrm{Homeo}}^+(S^1) = \{\tilde{f} \in \mathrm{Homeo}^+(\mathbb{R}) \mid \forall x \in \mathbb{R}, n \in \mathbb{Z}, \tilde{f}(x+n) = \tilde{f}(x)+n\}$$

  We define the *rotation number* $\widetilde{\mathrm{rot}} \colon \widetilde{\mathrm{Homeo}}^+(S^1) \to \mathbb{R}$ by

  $$\widetilde{\mathrm{rot}}(\tilde{f}) = \lim_{n\to\infty} \frac{\tilde{f}^n(x) - x}{n}.$$

This definition does not depend on the choice of $x \in \mathbb{R}$. The rotation number $\widetilde{\mathrm{rot}}$ is a quasimorphism and not a homomorphism.

- (Brooks' counting quasimorphism on free groups [10])

  Let $F_2 = \langle x, y \rangle$ be a free group of rank 2 and w a reduced word in $\{x^{\pm 1}, y^{\pm 1}\}$. A *counting function* $c_w \colon F_2 \to \mathbb{Z}$ is defined as $c_w(g)$ being the maximal number of disjoint copies of w in the reduced representative of $g \in F_2$. A *counting quasimorphism* is a function of the form

$$h_w(g) = c_w(g) - c_{w^{-1}}(g).$$

  Let a and b be two generators of F_2. For $n \in \mathbb{Z}$, set $w_n = ab^n$. Then we can show that the set of quasimorphisms $\{\overline{h}_{w_n}\}_{n \in \mathbb{Z}}$ are linearly independent in $Q(F_2)$. Therefore, the space $Q(F_2)$ is infinite-dimensional.

These examples are fundamental in the sense that there are many generalizations of them. For example, Py's Calabi quasimorphisms we use later are constructed from the rotation number. The counting quasimorphisms are generalized to hyperbolic groups [21], mapping class groups [4], and surface diffeomorphism groups [6].

We will use the following commutator calculus several times.

Lemma 2.4.4. *For $x, y \in \Gamma$ and $n \in \mathbb{N}$, $(xy)^{2n} y^{-2n} x^{-2n}$ can be written as a product of n commutators.*

Proof. Since,

$$(xy)^{2n} y^{-2n} x^{-2n} = x \cdot (yx)^{2n-1} y^{-(2n-1)} x^{-(2n-1)} \cdot x^{-1},$$

it is sufficient to prove that $(yx)^{2n-1} y^{-(2n-1)} x^{-(2n-1)}$ can be written as a product of n commutators. We prove it by the induction of n. It is clear if $n = 1$. Set

$$\alpha_m = (xy)^{2m-1} x^{-(2m-1)} y^{-(2m-1)}$$

and

$$\beta_m = (yx)^{2m-1} y^{-(2m-1)} x^{-(2m-1)}$$

Assume that α_m and β_m can be written as a product of m commutators. Since

$$\begin{aligned} & yxy\alpha_m^{-1}(yxy)^{-1}\beta_{m+1} \\ =& yxy^{2m}x^{2m}y^{-(2m+1)}x^{-(2m+1)} \\ =& yx^{-1}[x^2y^{2m}x^{2m-1}, xy^{-1}x^{2m-1}](yx^{-1})^{-1}, \end{aligned}$$

β_{m+1} can be written as a product of $m+1$ commutators by the assumption. $\square$

It is known that the defect changes at most twice by homogenization.

Lemma 2.4.5. *For every $\phi \in Q(\Gamma)$, $D(\overline{\phi}) \leq 2D(\phi)$.*

Proof. For $a, b, D \in \mathbb{R}$, we write $a \sim_D b$ to mean $|a - b| \leq D$.

We define $\phi' : \Gamma \to \mathbb{R}$ by

$$\phi'(x) = \frac{1}{2}(\phi(x) - \phi(x^{-1})).$$

Then ϕ' is anti-symmetric, i.e., $\phi'(x^{-1}) = -\phi(x)$ for every $x \in \Gamma$. Since

$$\begin{aligned}&\phi'(xy) - \phi'(x) - \phi'(y)\\ =&\frac{1}{2}(\phi(xy) - \phi(x) - \phi(y)) - \frac{1}{2}(\phi(y^{-1}x^{-1}) - \phi(y^{-1}) - \phi(x^{-1})),\end{aligned}$$

$$D(\phi') = \sup_{x,y\in\Gamma} |\phi'(xy) - \phi'(x) - \phi'(y)| \leq \frac{1}{2}D(\phi) + \frac{1}{2}D(\phi) = D(\phi).$$

Thus ϕ' is also a quasimorphism and $D(\phi') \leq D(\phi)$. Since

$$\phi(x) - \phi'(x) = \frac{1}{2}(\phi(x) + \phi(x^{-1})) \sim_{\frac{1}{2}D(\phi)} \frac{1}{2}\phi(e)$$

for every $x \in \Gamma$, $\phi - \phi'$ is a bounded function. Especially, $\overline{\phi} = \overline{\phi'}$.

For $x, y \in \Gamma$, and $n \in \mathbb{N}$,

$$\phi'((xy)^{2n}) - \phi'(x^{2n}) - \phi'(y^{2n}) \sim_{2D(\phi')} \phi'((xy)^{2n}y^{-2n}x^{-2n}).$$

Thus,

$$\lim_{n\to\infty} \frac{\phi'((xy)^{2n}y^{-2n}x^{-2n})}{2n} = \lim_{n\to\infty} \frac{\phi'((xy)^{2n}) - \phi'(x^{2n}) - \phi'(y^{2n})}{2n} \tag{2.4.1}$$

For an anti-symmetric quasimorphism ϕ' and a commutator $c = [x, y]$,

$$\phi'(c) \sim_{3D(\phi')} \phi(x) + \phi(y) + \phi(x^{-1}) + \phi(y^{-1}) = 0.$$

By Lemma 2.4.4 that $(xy)^{2n}y^{-2n}x^{-2n}$ can be written as a product of n commutators $c_1c_2\cdots c_n$. Since

$$\phi'(c_1c_2\cdots c_n) \sim_{(n-1)D(\phi')} \phi'(c_1) + \phi'(c_2) + \cdots + \phi'(c_n) \sim_{3nD(\phi')} 0,$$

we obtain

$$\phi'((xy)^{2n}y^{-2n}x^{-2n}) \leq (4n-1)D(\phi')$$

Thus the left-hand side of (2.4.1) is less than or equal to $2D(\phi')$. On the other hand, the right-hand side of (2.4.1) equals $|\overline{\phi}'(xy) - \overline{\phi}'(x) - \overline{\phi}'(y)| = |\overline{\phi}(xy) - \overline{\phi}(x) - \overline{\phi}(y)|$. Therefore,

$$D(\overline{\phi}) = \sup_{x,y\in\Gamma} |\overline{\phi}(xy) - \overline{\phi}(x) - \overline{\phi}(y)| \leq 2D(\phi') \leq 2D(\phi) \qquad \square$$

An exact sequence of complexes

$$0 \to C_b^\bullet(\Gamma) \to C^\bullet(\Gamma) \to C^\bullet(\Gamma)/C_b^\bullet(\Gamma) \to 0$$

induces the exact sequence

$$0 \to H^1(\Gamma) \to Q(\Gamma) \to H_b^2(\Gamma) \to H^2(\Gamma)$$

since $H_b^1(\Gamma) = 0$ and $H^1(C^\bullet/C_b^\bullet) \cong Q$. Hence, the second exact bounded cohomology $EH_b^2(\Gamma) = \mathrm{Ker}(H_b^2(\Gamma) \to H^2(\Gamma))$ is isomorphic to $Q(\Gamma)/H^1(\Gamma)$.

The following lemma is well-known and fundamental.

Lemma 2.4.6. *For every $\phi \in Q(\Gamma)$ and $x, y \in \Gamma$,*

(1) $\phi(xyx^{-1}) = \phi(y)$,

(2) if $xy = yx$, then $\phi(xy) = \phi(x) + \phi(y)$.

Proof. For $a, b, D \in \mathbb{R}$, we write $a \sim_D b$ to mean $|a - b| \leq D$.

(1) For every $n \in \mathbb{N}$,

$$n\phi(yxy^{-1}) = \phi((yxy^{-1})^n) = \phi(yx^ny^{-1}) \sim_{2D(\phi)} \phi(y)+\phi(x^n)+\phi(y^{-1}) = n\phi(x).$$

Thus we obtain

$$|\phi(yxy^{-1}) - \phi(x)| \leq \frac{2D}{n}.$$

Since n can be taken arbitrarily large, we obtain $\phi(xyx^{-1}) = \phi(y)$.

(2) For every $n \in \mathbb{N}$,

$$n\phi(xy) = \phi((xy)^n) = \phi(x^ny^n) \sim_D \phi(x^n) + \phi(y^n) = n\phi(x) + n\phi(y).$$

Thus we obtain

$$|\phi(xy) - \phi(x) - \phi(y)| \leq \frac{D}{n}.$$

Since n can be taken arbitrarily large, we obtain $\phi(xy) = \phi(x) + \phi(y)$.

$\square$

2.4.2 Bavard's duality

Quasimorphisms are closely related to the commutator length. We define the *stable commutator length* $\mathrm{scl}\colon [\Gamma,\Gamma] \to [0,\infty)$ by

$$\mathrm{scl}(x) = \lim_{n\to\infty} \frac{\mathrm{cl}(x^n)}{n}.$$

By Fekete's lemma, the limit exists.

Lemma 2.4.7. *For any $x \in [\Gamma,\Gamma]$ and $\phi \in Q(\Gamma)$,*

$$\mathrm{scl}(x) \geq \frac{1}{2}\frac{|\phi(x)|}{D(\phi)}.$$

Proof. Note that $|\phi([x,y])| = |\phi([x,y]) - \phi(xyx^{-1}) - \phi(y^{-1})| \leq D(\phi)$ for any commutator $[x,y] \in [\Gamma,\Gamma]$. If x^n is a product of commutators $c_1,\ldots,c_m$, then we obtain an inequality

$$n|\phi(x)| = |\phi(x^n)| \leq (m-1)D(\phi) + \sum_{k=1}^{k} |\phi(c_k)| < 2mD(\phi).$$

and the lemma follows from it. □

Moreover, following *Bavard's duality theorem* holds.

Theorem 2.4.8 ([3]). *For $x \in [\Gamma,\Gamma]$*

$$\mathrm{scl}(x) = \frac{1}{2}\sup_{\phi\in Q(\Gamma)} \frac{|\phi(x)|}{D(\phi)}.$$

Remark that we regard the right-hand side as 0 if $Q(\Gamma) = H^1(\Gamma)$. As a corollary, we can see that scl on $[\Gamma,\Gamma]$ is identically zero if and only if $EH_b^2(\Gamma) \cong Q(\Gamma)/H^1(\Gamma) = 0$ (i.e., the comparison map $H_b^2(\Gamma) \to H^2(\Gamma)$ is injective). There are several applications ([18, 15, 42] for example) and generalizations [16, 37] of Bavard's duality theorem.

Chapter 3

G-invariant quasimorphisms and their applications

3.1 G-invariant quasimorphisms

Throughout this section, let G be a group and H a normal subgroup of G.

Definition 3.1.1. A quasimorphism ϕ on H is *G-quasi-invariant* if there exist a constant $C \geq 0$ such that

$$|\phi(ghg^{-1}) - \phi(h)| \leq C$$

for any $g \in G$ and $h \in H$. If the constant C can be taken by 0, ϕ is called *G-invariant*.

Let $\widehat{Q}(H)^G$ denote the set of G-quasi-invariant quasimorphisms on G and $Q(H)^G$ denote the set of G-(quasi-)invariant homogeneous quasimorphisms on H.

Remark 3.1.2.

- Every G-quasi-invariant homogeneous quasimorphisms are G-invariant; if ϕ is G-quasi-invariant and homogeneous, then for every $n \in \mathbb{N}$,

  $$n|\phi(ghg^{-1}) - \phi(h)| = |\phi((ghg^{-1})^n) - \phi(h^n)| = |\phi(gh^n g^{-1}) - \phi(h^n)| \leq C.$$

  Hence $|\phi(ghg^{-1}) - \phi(h)| \leq C/n$ for arbitrarily large n and thus ϕ is G-invariant.

- If $G = H$, then $\widehat{Q}(G)^G = \widehat{Q}(G)$ and $Q(G)^G = Q(G)$.

As is the case of ordinary quasimorphism, the (G-invariant) homogenization $\widehat{Q}(H)^G \to Q(H)^G$ induces an isomorphism $\widehat{Q}(H)^G/\bar{C}_b^1(H) \cong Q(H)^G$.

We define the notion of a (G,H)-commutator (or a mixed commutator). A (G,H)*-commutator* is an element of the form $[g,h]$ with $g \in G$ and $h \in H$. Note that $[h,g] = [hg, h^{-1}]$ is also a (G,H)-commutator. The (G,H)-*commutator subgroup* $[G,H]$ is the group generated by (G,H)-commutators. We remark that $[G,H] \subset H$ since H is a normal subgroup of G. The (G,H)-*commutator length* $\mathrm{cl}_{G,H} \colon [G,H] \to \mathbb{N}$ is defined by

$$\mathrm{cl}_{G,H}(x) = \left\{ k \;\middle|\; \begin{array}{c} \exists g_1, \ldots, g_k \in G, \exists h_1, \ldots, h_k \in H, \\ x = [g_1, h_1] \cdots [g_k, h_k] \end{array} \right\}.$$

We define the *stable* (G,H)*-commutator length* $\mathrm{scl}_{G,H} \colon [G,H] \to [0,\infty)$ by

$$\mathrm{scl}_{G,H}(x) = \lim_{n\to\infty} \frac{\mathrm{cl}_{G,H}(x^n)}{n}.$$

Lemma 3.1.3. *For any $x \in [G,H]$ and $\phi \in Q(H)^G$,*

$$\mathrm{scl}_{G,H}(x) \geq \frac{1}{2}\frac{|\phi(x)|}{D(\phi)}.$$

Proof. Note that $|\phi([g,h])| = |\phi([g,h]) - \phi(ghg^{-1}) - \phi(h^{-1})| \leq D(\phi)$ for any (G,H)-commutator $[g,h] \in [G,H]$. If x^n is a product of (G,H)-commutators $c_1, \ldots, c_m$, then we obtain an inequality

$$n|\phi(x)| = |\phi(x^n)| \leq (m-1)D(\phi) + \sum_{k=1}^{k} |\phi(c_k)| < 2mD(\phi).$$

and the lemma follows from it. □

As is the ordinary case, we can prove the following Bavard-type duality theorem.

Theorem 3.1.4. *Assume that $H = [G,H]$. For any $x \in [G,H]$,*

$$\mathrm{scl}_{G,H}(x) = \frac{1}{2} \sup_{\phi \in Q(H)^G} \frac{|\phi(x)|}{D(\phi)}$$

Proof of Theorem 3.1.4 is given in the next subsection. We note that the assumption $H = [G,H]$ in Theorem 3.1.4 can be removed [38].

3.2 Proof of G-invariant Bavard duality

In this section, we give the proof of Theorem 3.1.4. We use the method of Kawasaki [36] and his idea came from [16].

For a group Γ, we define a set

$$A_\Gamma = \bigsqcup_{k=0}^{\infty} (\Gamma \times \mathbb{R})^k.$$

We denote elements of A_Γ by $x_1^{s_1} \cdots x_k^{s_k}$, where $x_1, \ldots, x_k \in \Gamma$ and $s_1, \ldots, s_k \in \mathbb{R}$. Set $\Gamma = [G, H]$. We define a function $\| \cdot \|_\Gamma : A_\Gamma \to \mathbb{R}_{\geq 0}$ by

$$\|x_1^{s_1} \cdots x_k^{n_k}\|_\Gamma = \lim_{n \to \infty} \frac{1}{n} \mathrm{cl}_{G,H}(x_1^{\lfloor s_1 n \rfloor} \cdots x_k^{\lfloor n_k n \rfloor}),$$

where $\lfloor t \rfloor$ denotes the integer part of $t \in \mathbb{R}$.

Proposition 3.2.1. $\| \cdot \|_\Gamma : A_\Gamma \to \mathbb{R}_{\geq 0}$ *is well-defined.*

We prove Proposition 3.2.1 in Section 3.5.

We define some operations on A_Γ. For elements $\mathsf{x} = x_1^{s_1} \ldots x_k^{s_k}$, $\mathsf{y} = y_1^{t_1} \ldots y_l^{t_l}$ in A_Γ and $\lambda \in \mathbb{R}$, we define $\mathsf{x} \star \mathsf{y}$, $\bar{\mathsf{x}}$ and $\mathsf{x}^{(\lambda)}$ by

$$\begin{aligned} \mathsf{x} \star \mathsf{y} &= x_1^{s_1} \ldots x_k^{s_k} y_1^{t_1} \ldots y_l^{t_l}, \\ \bar{\mathsf{x}} &= x_k^{-s_k} \ldots x_1^{-s_1}, \\ \mathsf{x}^{(\lambda)} &= x_1^{\lambda s_1} \ldots x_k^{\lambda s_k}. \end{aligned}$$

Since $\mathrm{cl}_{G,H}$ is a conjugation-invariant norm, we can confirm that for any $\mathsf{x}, \mathsf{y} \in A_\Gamma$,

$$\begin{aligned} \|\mathsf{x}\mathsf{y}\|_\Gamma &\leq \|\mathsf{x}\|_\Gamma + \|\mathsf{y}\|_\Gamma, \\ \|\bar{\mathsf{y}}\mathsf{x}\mathsf{y}\|_\Gamma &= \|\mathsf{x}\|_\Gamma, \\ \|\bar{\mathsf{x}}\|_\Gamma &= \|\mathsf{x}\|_\Gamma. \end{aligned}$$

We define the equivalent relation $\sim$ on A_Γ by $\mathsf{x} \sim \mathsf{y}$ if and only if $\|\mathsf{x}\bar{\mathsf{y}}\|_\Gamma = 0$ for $\mathsf{x}, \mathsf{y} \in A_\Gamma$. We denote the set $A_\Gamma / \sim$ by A, and the function $\| \cdot \|_\Gamma : A_\Gamma \to \mathbb{R}_{\geq 0}$ on A_Γ induce the function $\| \cdot \| : A \to \mathbb{R}_{\geq 0}$ on A. Let $[\mathsf{x}] \in A$ denote the equivalent class of $\mathsf{x} \in A_\Gamma$. For $\mathbf{x} = [\mathsf{x}]$ and $\mathbf{y} = [\mathsf{y}]$ in A and $\lambda \in \mathbb{R}$, we define $\mathbf{x}+\mathbf{y}$ and $\lambda\mathbf{x}$ by

$$\mathbf{x}+\mathbf{y} = [\mathsf{x} \star \mathsf{y}],$$
$$\lambda\mathbf{x} = [\mathsf{x}^{(\lambda)}].$$

Proposition 3.2.2. *The above operators are well-defined.*

Proposition 3.2.3. $(A, \|\cdot\|)$ *is a normed vector space.*

We prove Proposition 3.2.2 and 3.2.3 in Section 3.5.

By the Hahn-Banach theorem, we obtain the following:

Proposition 3.2.4. *For any* $\mathsf{x} \in A$,

$$\|\mathsf{x}\| = \sup_{\hat{\phi} \in A^*} \frac{\hat{\phi}(\mathsf{x})}{\|\hat{\phi}\|^*},$$

where A^* *is the dual space of* A *and* $\|\cdot\|^*$ *is the dual norm on* A^*.

For $\hat{\phi} \in A^*$, we define the function $\phi : \Gamma \to \mathbb{R}$ by $\phi(x) = \hat{\phi}([x^1])$.

Proposition 3.2.5. ϕ *is a* G*-invariant homogeneous quasimorphism.*

Proof. • (ϕ is a quasimorphism)

For $x, y \in \Gamma$,

$$\begin{aligned}
&|\phi(xy) - \phi(x) - \phi(y)| \\
&= |\hat{\phi}([(xy)^1]) - \hat{\phi}([x^1]) - \hat{\phi}([y^1])| \\
&= |\hat{\phi}([(xy)^1] + (-1)[x^1] + (-1)[x^1])| \\
&\leq \|\hat{\phi}\|^* \|(xy)^1 \star x^{-1} \star y^{-1}\| \\
&= \|\hat{\phi}\|^* \cdot \lim_{n\to\infty} \frac{1}{n} \mathrm{cl}_{G,H}((xy)^n x^{-n} y^{-n}).
\end{aligned}$$

Since $(xy)^{2n}x^{-2n}y^{-2n}$ is a product of n commutators (see [14, Lemma2.24.] for example),

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{cl}_{G,H}((xy)^n x^{-n} y^{-n}) \leq \frac{1}{2}.$$

Thus

$$|\phi(xy) - \phi(x) - \phi(y)| \leq \frac{1}{2}\|\hat{\phi}\|^*.$$

- (ϕ is homogeneous)

  Since $(x^n)^1 \sim x^n$ for $x \in \Gamma$ and $\hat{\phi}: A \to \mathbb{R}$ is a linear map,
$$\phi(x^n) = \hat{\phi}([(x^n)^1]) = \hat{\phi}([x^n]) = \hat{\phi}(n[x^1]) = n\hat{\phi}([x^1]) = n\phi(x)$$
  for $x \in \Gamma$ and $n \in \mathbb{Z}$.

- (ϕ is G-invariant)

  For $g \in G$, $x \in \Gamma$,
$$\begin{aligned}
&|\phi(gxg^{-1}) - \phi(x)| \\
&= |\hat{\phi}([(gxg^{-1})^1]) - \hat{\phi}([x^1])| \\
&= |\hat{\phi}([(gxg^{-1})^1] + (-1)[x^1])| \\
&\le \|\hat{\phi}\|^* \|((gxg^{-1})^1) \star x^{-1}\| \\
&= \|\hat{\phi}\|^* \cdot \lim_{n\to\infty} \frac{1}{n} \mathrm{cl}_{G,H}((gxg^{-1})^n \cdot x^{-n}) \\
&= \|\hat{\phi}\|^* \cdot \lim_{n\to\infty} \frac{1}{n} \mathrm{cl}_{G,H}([g, x^n]) \\
&= 0. \qquad \square
\end{aligned}$$

Proof of Theorem 3.1.4. By Proposition 3.2.4 and 3.2.5, since $D(\phi) \le \frac{1}{2}\|\phi\|^*$,
$$\mathrm{scl}_{G,H}(x) = \|x^1\| = \sup_{\hat{\phi}\in A^*} \frac{\hat{\phi}([x^1])}{\|\hat{\phi}\|^*} \le \sup_{\phi} \frac{\phi(x)}{2D(\phi)}.$$
Lemma 3.1.3 states the opposite direction. $\square$

3.3 Extension problem

In this section, we consider the extension problem of quasimorphisms.

Definition 3.3.1. A G-invariant homogeneous quasimorphism $\phi\colon H \to \mathbb{R}$ is *extendable to* G if there exists a homogeneous quasimorphism $\psi\colon G \to \mathbb{R}$ such that $\psi|_H = \phi$.

If a homogeneous quasimorphism ϕ is extendable, then ϕ is G-invariant. Hence, G-invariance is a necessary condition to extend. Shtern [53] studied the conditions under which a quasimorphism can be extended. By applying his result, we obtain the following sufficient conditions to extend.

Proposition 3.3.2 ([53, Theorem 3])**.** *Suppose that one of the following conditions are satisfied:*

- *the group homomorphism $G \to G/H$ has a section homomorphism.*
- *H is a finite index subgroup of G.*

Then any homogeneous G-invariant quasimorphism ϕ is extendable.

Later we will give another proof of Proposition 3.3.2, including the evaluation of defect (Proposition 3.4.6).

We provide a convenient lemma for proving non-extendability.

Lemma 3.3.3. *Let ϕ be a G-invariant quasimorphism on H. Let f and g be elements of G satisfying*

- $f(gf^{-1}g^{-1}) = (gf^{-1}g^{-1})f$,
- $[f, g] \in H$,
- $\phi([f, g]) \neq 0$.

Then, ϕ is non-extendable to G.

Proof. Assume that ϕ is extendable to G. Let ψ be a homogeneous quasimorphism on G such that $\psi|_H = \phi$. Then, by Lemma 2.4.6,

$$\phi([f, g]) = \psi([f, g]) = \psi(f) + \psi(gf^{-1}g^{-1}) = 0$$

and this contradicts the assumption. Hence ϕ is non-extendable to G. □

We give an example of non-extendable quasimorphism. Namely, we observe that Py's Calabi quasimorphism is non-extendable.

Theorem 3.3.4. *Let Σ be an oriented closed surface whose genus is greater than one and ω a symplectic form on Σ_g. Py's Calabi quasimorphism $\mu_P \colon \mathrm{Ham}(\Sigma, \omega) \to \mathbb{R}$ is non-extendable to $\mathrm{Symp}_0(\Sigma, \omega)$.*

We review the notion of Calabi quasimorphism. A symplectic manifold (M, ω) is *exact* if there exists a 1-form $\lambda \in \Omega^1(M)$ such that $\omega = -d\lambda$. A subset X of a symplectic manifold (M, ω) is *displaceable* if there exists $\phi \in \mathrm{Ham}(M, \omega)$ such that $\phi(X) \cap \bar{X} = \emptyset$, where $\bar{X}$ is the topological closure of X. For an exact symplectic manifold (M, ω), we recall that the *Calabi homomorphism* is a function $\mathrm{Cal}_M\colon \mathrm{Ham}(M, \omega) \to \mathbb{R}$ defined by

$$\mathrm{Cal}_M(\varphi_F) = \int_0^1 \int_M F_t \omega^n \, dt.$$

The Calabi homomorphism is known to be well-defined and a group homomorphism. (see [13, 1, 2, 41]).

Definition 3.3.5. Let $\mu\colon \mathrm{Ham}(M, \omega) \to \mathbb{R}$ be a homogeneous quasimorphism. An open subset U of M has *the Calabi property with respect to* μ if $\omega|_U$ is exact and the restriction of μ to $\mathrm{Ham}(U, \omega)$ coincides with the Calabi homomorphism Cal_U.

Definition 3.3.6 ([19, 49])**.** A *Calabi quasimorphism* is a homogeneous quasimorphism $\mu\colon \mathrm{Ham}(M, \omega) \to \mathbb{R}$ such that any displaceable open subset of M has the Calabi property with respect to U.

The following properties of Py's Calabi quasimorphism is important to prove Theorem 3.3.4. See [50, 51] for the definition of Py's Calabi quasimorphism.

Proposition 3.3.7 ([50])**.** *Let Σ be a closed orientable surface whose genus is larger than one, ω a symplectic form on Σ and U an open subset of Σ which is homeomorphic to an annulus. Then U has the Calabi property with respect to Py's Calabi quasimorphism μ_P.*

Let Σ be a closed orientable surface of a positive genus and ω a symplectic form on Σ. We set $H = \mathrm{Ham}(\Sigma, \omega)$ and $G = \mathrm{Symp}_0(\Sigma, \omega)$. In order to prove Theorems 3.3.4, we prepare the following elements of $G = \mathrm{Symp}_0(\Sigma, \omega)$.

Since the genus of Σ is positive, we can take a non-separating simple closed curve C in Σ. Then, there are a positive number r and a symplectic embedding $\iota\colon (-1, 1) \times \mathbb{R}/r\mathbb{Z} \to \Sigma$ such that $\iota(\{0\} \times \mathbb{R}/r\mathbb{Z}) = C$. Here, the symplectic form on $(-1, 1) \times \mathbb{R}/r\mathbb{Z}$ is defined by $dx \wedge dy$, where (x, y) is the coordinate on $(-1, 1) \times \mathbb{R}/r\mathbb{Z}$.

Let $\epsilon \in (0, 1)$ and let $\chi\colon (-1, 1) \to [0, 1]$ be a function satisfying the following conditions.

- $\chi(x) = 0$ for any $x \in (-1, -1 + \epsilon) \cup (1 - \epsilon, 1)$,
- $\chi(x) + \chi(1 + x) = 1$ for any $x \in (-1, 0)$.

By the above conditions, we see that $\chi(x) = 1$ for any $x \in (-\epsilon, \epsilon)$. Define a function $F\colon \Sigma \to \mathbb{R}$ by

$$F(z) = \begin{cases} \chi(x) & (\text{if } z = \iota(x,y) \text{ for some } (x,y) \in (-1,1) \times \mathbb{R}/r\mathbb{Z}), \\ 0 & (\text{if } z \notin \mathrm{Im}(\iota)). \end{cases}$$

Since C is non-separating, $\Sigma \setminus \mathrm{Im}(\iota)$ is path-connected. Thus, there exists $g_0 \in G = \mathrm{Symp}_0(\Sigma, \omega)$ such that $g_0(\iota(x,y)) = \iota(x+1, y)$ for any $(x,y) \in (-1,0) \times \mathbb{R}/r\mathbb{Z}$.

Define a map $f_0\colon \Sigma \to \Sigma$ by

$$f_0(z) = \begin{cases} \varphi_F(z) & (\text{if } z \in \iota((-1,0) \times \mathbb{R}/r\mathbb{Z})), \\ z & (\text{otherwise}). \end{cases}$$

Since $f_0(z) = z$ for any $z \in \iota((-1,-1+\epsilon) \cup (-\epsilon,\epsilon)) \times \mathbb{R}/r\mathbb{Z})$, f_0 is well-defined and $f_0 \in G = \mathrm{Symp}_0(\Sigma, \omega)$. Since $\chi(x) + \chi(1+x) = 1$ for any $x \in (-1,0)$, by the definition of g_0,

$$g_0 f_0^{-1} g_0^{-1}(z) = \begin{cases} \varphi_F(z) & (\text{if } z \in \iota((0,1) \times \mathbb{R}/r\mathbb{Z})), \\ z & (\text{otherwise}). \end{cases}$$

Thus, we obtain $\varphi_F = f_0 g_0 f_0^{-1} g_0^{-1}$. Since $\mathrm{supp}(f_0) \subset \iota((-1,0) \times \mathbb{R}/r\mathbb{Z})$ and $\mathrm{supp}(g_0 f_0^{-1} g_0^{-1}) \subset \iota((0,1) \times \mathbb{R}/r\mathbb{Z})$, $f_0(g_0 f_0^{-1} g_0^{-1}) = (g_0 f_0^{-1} g_0^{-1}) f_0$.

Proof of Theorem 3.3.4. Since $f_0(g_0 f_0^{-1} g_0^{-1}) = (g_0 f_0^{-1} g_0^{-1}) f_0$ and $\varphi_F = [f_0, g_0] \in \mathrm{Ham}(\Sigma, \omega)$, by Lemma 3.3.3, it is sufficient to prove that $\mu_P([f_0, g_0]) \neq 0$.

By the definition of F, $\int_\Sigma F\omega > 0$. By Proposition 3.3.7, $\mathrm{Im}(\iota)$ has the Calabi property with respect to μ_P. Since $\varphi_F = f_0 g_0 f_0^{-1} g_0^{-1}$ and $\mathrm{Supp}(F) \subset \mathrm{Im}(\iota)$,

$$\mu_P([f_0, g_0]) = \mu_P(\varphi_F) = \int_\Sigma F\omega > 0. \qquad \square$$

3.4 Comparison of commutator lengths

We compare the (G,H)-commutator length $\mathrm{cl}_{G,H}$ with the ordinary commutator lengths cl_G of G and cl_H of H. By definition, $\mathrm{cl}_G \leq \mathrm{cl}_{G,H}$ on $[G,H]$, and $\mathrm{cl}_{G,H} \leq \mathrm{cl}_H$ on $[H,H]$.

3.4.1 $\mathrm{scl}_{G,H}$ vs scl_G

We consider a sufficient condition under which $\mathrm{scl}_{G,H}$ and scl_G are bi-Lipschitz.

Proposition 3.4.1. *Suppose that $H = [G, H]$ and one of the following conditions are satisfied:*

- *The group homomorphism $G \to G/H$ has a section homomorphism.*
- *H is a finite index subgroup of G.*

Then for every $x \in [G, H]$,

$$\mathrm{scl}_G(x) \leq \mathrm{scl}_{G,H}(x) \leq 2\,\mathrm{scl}_G(x).$$

Remark 3.4.2. We can remove the assumption $H = [G, H]$ in Proposition 3.4.1 since the assumption $H = [G, H]$ in Theorem 3.1.4 is removed in [38].

Example 3.4.3. Let G be the braid group B_n of n strands and H its commutator subgroup $[B_n, B_n]$. For any integer $n > 4$, H is a perfect group [32], especially $H = [G, H]$. It is known that $G/H \cong \mathbb{Z}$ and the abelianization map $G \to G/H$ is given by the index sum homomorphism $G \to \mathbb{Z}$ defined by $\sigma_i \mapsto 1$ for $i = 1, 2, \ldots, n-1$, where σ_i is the ith Artin generator. Since there is a section homomorphism $s\colon \mathbb{Z} \to G$, the pair (G, H) satisfies the assumptions of Proposition 3.4.1 if $n > 4$.

Example 3.4.4. Let (M, ω) be an exact symplectic manifold. Let G be the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms and H the commutator subgroup of $\mathrm{Ham}(M, \omega)$. Let $\mathrm{Cal}\colon \mathrm{Ham}(M, \omega) \to \mathbb{R}$ denote the Calabi homomorphism.

It is known that $G/H \cong \mathbb{R}$ and the abelianization map $G \to G/H$ is given by the Calabi homomorphism [1]. We can take a time-independent Hamiltonian function $H\colon M \to \mathbb{R}$ such that $\mathrm{Cal}(H) = 1$ (for instance, consider a function supported on a Darboux ball). Then, the map $s\colon \mathbb{R} \to \mathrm{Ham}(M, \omega)$ defined by $s(t) = \varphi_{tH}$ is a section homomorphism of Cal. Since it is known that H is a perfect group [1], the pair (G, H) satisfies the assumptions of Proposition 3.4.1.

Example 3.4.5. Let T^2 be a 2-dimensional torus and ω a symplectic form on T^2. Let G be the identity component $\mathrm{Symp}_0(T^2, \omega)$ of the group of symplectomorphisms of (T^2, ω) and H the group $\mathrm{Ham}(T^2, \omega)$ of Hamiltonian diffeomorphisms of (T^2, ω). Let $\mathrm{Flux}_\omega\colon \mathrm{Symp}_0(T^2, \omega) \to H^1(T^2; \mathbb{R})/\Gamma_\omega$

be the (descended) flux homomorphism. Then, $\mathrm{Ker}(\mathrm{Flux}_\omega) = H$ and H is known to be perfect [1]. Thus, since there exists a section homomorphism of $\mathrm{Flux}_\omega \colon \mathrm{Symp}_0(T^2, \omega) \to H^1(T^2; \mathbb{R})/\Gamma_\omega$, G and H satisfy the assumption of Proposition 3.4.1.

To prove Proposition 3.4.1, we prove a precise version of Proposition 3.3.2.

Proposition 3.4.6. *Suppose that one of the following conditions are satisfied:*

(1) the group homomorphism $G \to G/H$ has a section homomorphism,

(2) H is a finite index subgroup of G.

Then, for every homogeneous G-invariant quasimorphism ϕ, there exists a homogeneous quasimorphism ψ on H such that $\psi|_H = \phi$ and $D(\psi) \leq 2D(\phi)$.

Proof of Proposition 3.3.2. Let $\pi \colon G \to G/H$ be the quotient map.

(1) Let $\sigma \colon G/H \to G$ be a section homomorphism. For $g \in G$, we set $q_g = \sigma(\pi(g))$ and $h_g = q_g^{-1} g \in H$. We define the function $\phi' \colon G \to \mathbb{R}$ by $\phi'(g) = \phi(h_g)$. Since $\sigma \circ \pi$ is a homomorphism, $q_{g_1 g_2} = q_{g_1} q_{g_2}$ for $g_1, g_2 \in G$. Thus

$$\begin{aligned}
&|\phi'(g_1 g_2) - \phi'(g_1) - \phi'(g_2)| \\
&= |\phi(h_{g_1 g_2}) - \phi(h_{g_1}) - \phi(h_{g_2})| \\
&= |\phi(q_{g_2}^{-1} q_{g_1}^{-1} g_1 g_2) - \phi(q_{g_1}^{-1} g_1) - \phi(q_{g_2}^{-1} g_2)| \\
&= |\phi(q_{g_1}^{-1} g_1 g_2 q_{g_2}^{-1}) - \phi(q_{g_1}^{-1} g_1) - \phi(g_2 q_{g_2}^{-1})| \\
&\leq D(\phi).
\end{aligned}$$

Hence, ϕ' is a quasimorphism with $D(\phi') \leq D(\phi)$.

(2) We choose a representative $Q \subset G$ of the coset G/H, i.e., $G = \bigsqcup_{q \in Q} qH$. Assume that $e \in Q$. For $g \in G$, we can represent uniquely as a form $g = q_g h_g$, where $q_g \in Q$ and $h_g \in H$. We define the function $\phi' \colon G \to \mathbb{R}$ by

$$\phi'(g) = \frac{1}{\# Q} \sum_{q \in Q} \phi(h_{gq}).$$

Since $q_{g_1g_2q}^{-1}g_1g_2q = (q_{g_1g_2q}^{-1}g_1q_{g_2q})(q_{g_2q}^{-1}g_2q)$,

$$
\begin{aligned}
&|\phi'(g_1g_2) - \phi'(g_1) - \phi'(g_2)| \\
&= \frac{1}{\#Q}\left|\sum_{q\in Q}\phi(h_{g_1g_2q}) - \phi(h_{g_1q}) - \phi(h_{g_2q})\right| \\
&= \frac{1}{\#Q}\left|\sum_{q\in Q}\phi(q_{g_1g_2q}^{-1}g_1g_2q) - \phi(q_{g_1q}^{-1}g_1q) - \phi(q_{g_2q}^{-1}g_2q)\right| \\
&= \frac{1}{\#Q}\left|\sum_{q\in Q}\phi((q_{g_1g_2q}^{-1}g_1q_{g_2q})(q_{g_2q}^{-1}g_2q)) - \phi(q_{g_1g_2q}^{-1}g_1q_{g_2q}) - \phi(q_{g_2q}^{-1}g_2q)\right| \\
&\leq D(\phi).
\end{aligned}
$$

□

Proof of Proposition 3.4.1. The inequality $\mathrm{scl}_G(x) \leq \mathrm{scl}_{G,H}(x)$ immediately follows from the definitions of norms. Thus, we prove $\mathrm{scl}_{G,H}(x) \leq 2\,\mathrm{scl}_G(x)$ below.

By Theorem 3.1.4, for any $\epsilon > 0$, there exists a G-invariant homogeneous quasimorphism ϕ such that

$$\mathrm{scl}_{G,H}(x) - \epsilon \leq \frac{1}{2}\frac{\phi(x)}{D(\phi)}.$$

By Proposition 3.3.2, there exists an extension $\hat{\phi}$ of ϕ which is homogeneous and $D(\hat{\phi}) \leq 2D(\phi)$. Therefore,

$$\frac{1}{2}\frac{\phi(x)}{D(\phi)} \leq \frac{\hat{\phi}(x)}{D(\hat{\phi})} \leq 2\,\mathrm{scl}_G(x).$$

Since ϵ can be taken arbitrarily small, we have finished the proof. □

On the other hand, there exist an example of a pair (G, H) of groups such that $\mathrm{scl}_{G,H}$ and scl_G are not bi-Lipschitz.

Proposition 3.4.7. *Let Σ be an oriented closed surface whose genus is greater than one and ω a symplectic form on Σ. Set $G = \mathrm{Symp}_0(\Sigma, \omega)$ and $H = \mathrm{Ham}(\Sigma, \omega)$. Then $\mathrm{scl}_{G,H}$ and scl_G are not bi-Lipschitz.*

Proof. Take $f_0, g_0 \in G$ as in Section 3.3. Let μ_P denote Py's Calabi quasimorphism. We observed that $\mu_P([f_0, g_0]) > 0$ in the proof of Theorem 3.3.4. Hence, by Lemma 3.1.3, $\mathrm{scl}_{G,H}([f_0, g_0]) > 0$. On the other hand,

$$[f_0, g_0]^n = (f_0(g_0 f_0^{-1} g_0^{-1}))^n = f_0^n (g_0 f_0^{-1} g_0^{-1})^n = f_0^n (g_0 f_0^{-n} g_0^{-1}) = [f_0^n, g_0]$$

for any integer n. Thus,

$$\mathrm{cl}_G([f_0, g_0]^n) = \mathrm{cl}_G([f_0^n, g_0]) = 1$$

and hence $\mathrm{scl}_G([f_0, g_0]) = 0$. □

3.4.2 $\mathrm{scl}_{G,H}$ vs scl_H

We give an example of a pair (G, H) of groups such that $\mathrm{scl}_{G,H}$ and scl_H are not bi-Lipschitz even if the quotient group G/H finite.

Let B_3 and P_3 denote the braid group and the pure braid group on 3 strands, respectively. Set $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, where σ_1 and σ_2 are the Artin generators. Note that Δ^2 is the full twist. Set $x = \sigma_1^2$, $y = \sigma_2^2$, and $z = \Delta^2$. Then P_3 has a presentation

$$P_3 = \langle x, y, z \mid xz = zx, yz = zy \rangle \cong F_2 \times \mathbb{Z}.$$

Proposition 3.4.8. *For $G = B_3$ and $H = P_3$, there exists an element $\alpha \in [H, H]$ such that $\mathrm{scl}_{G,H}(\alpha) = 0$ and $\mathrm{scl}_H(\alpha) > 0$.*

Proof of Proposition 3.4.8. We set $\alpha = [x, y] = [\sigma_1^2, \sigma_2^2] \in [H, H]$. Since $\Delta\alpha\Delta^{-1} = [\sigma_2^2, \sigma_1^2] = \alpha^{-1}$, $\phi(\alpha)$ is equal to zero for every G-invariant homogeneous quasimorphism ϕ on $[G, H]$. Thus, by Theorem 3.1.4, $\mathrm{scl}_{G,H}(\alpha) = 0$.

On the other hand, we can prove that $\mathrm{scl}_H(\alpha) > 0$ as follows. Set $\phi = \bar{h}_w \circ \mathrm{pr}_1$, where $\bar{h}_w$ is the homogenization of h_w for $w = xyx^{-1}y^{-1}$ and $\mathrm{pr}_1 \colon P_3 \cong F_2 \times \mathbb{Z} \to F_2$ is the first projection homomorphism. Since $c_w([x, y]^n) = n$ and $c_{w^{-1}}([x, y]^n) = 0$,

$$\bar{\phi}(\alpha) = \bar{h}_w([x, y]) = 1.$$

Therefore, by Theorem 2.4.8,

$$\mathrm{scl}_H(\alpha) \geq \frac{1}{2}\frac{1}{D(\bar{\phi})} > 0.$$ □

3.5 Appendix

In this subsection, we finish the proof of Proposition 3.2.1 and 3.2.3. For $x, y \in \Gamma$, let x^y denote the conjugation yxy^{-1}.

Lemma 3.5.1. *For any $x_1, \dots, x_k \in \Gamma$ and integers $n_1, \dots, n_k, m_1, \dots, m_k \in \mathbb{Z}$,*

$$\mathrm{cl}_{G,H}((x_1^{n_1} \dots x_k^{n_k})^{-1} x_1^{m_1} \dots x_k^{m_k}) \leq \sum_{i=1}^{k} |m_i - n_i| \mathrm{cl}_{G,H}(x_i)$$

Proof. Since

$$\begin{aligned}
&(x_1^{n_1} \dots x_k^{n_k})^{-1} x_1^{m_1} \dots x_k^{m_k} \\
=&x_k^{-n_k} \dots x_2^{-n_2} x_1^{-n_1} x_1^{m_1} x_2^{m_2} \dots x_k^{m_k} \\
=&x_k^{-n_k} \dots x_2^{-n_2} x_1^{m_1-n_1} x_2^{m_2} \dots x_k^{m_k} \\
=&x_k^{-n_k} \dots x_2^{-n_2} x_2^{m_2} \dots x_k^{m_k} (x_1^{m_1-n_1})^{y_1} \\
=&x_k^{-n_k} \dots x_2^{m_2-n_2} \dots x_k^{m_k} (x_1^{m_1-n_1})^{y_1} \\
=&\dots \\
=&x_k^{m_k-n_k} (x_{k-1}^{m_{k-1}-n_{k-1}})^{y_{k-1}} \dots (x_2^{m_2-n_2})^{y_2} (x_1^{m_1-n_1})^{y_1},
\end{aligned}$$

where $y_i = (x_{i+1}^{m_{i+1}} \dots x_k^{m_k})^{-1}$,

$$\mathrm{cl}_{G,H}((x_1^{n_1} \dots x_k^{n_k})^{-1} x_1^{m_1} \dots x_k^{m_k}) \leq \sum_{i=1}^{k} \mathrm{cl}_{G,H}(x_i^{m_i-n_i}) \leq \sum_{i=1}^{k} |m_i - n_i| \mathrm{cl}_{G,H}(x_i).$$

□

Lemma 3.5.2. *For $x_1, \dots, x_k \in \Gamma$ and integers $n_1, \dots, n_k, m_1, \dots, m_k \in \mathbb{Z}$,*

$$(x_1^{n_1} \dots x_k^{n_k})^{-1} (x_1^{m_1} \dots x_k^{m_k})^{-1} x_1^{n_1+m_1} \dots x_k^{n_k+m_k}$$

is a product of k commutators.

Proof. There exists elements $y_1, \dots y_k \in \langle x_1, \dots, x_k \rangle$ such that

$$\begin{aligned}
&(x_1^{n_1} \dots x_k^{n_k})^{-1}(x_1^{m_1} \dots x_k^{m_k})^{-1} x_1^{n_1+m_1} \dots x_k^{n_k+m_k} \\
=&x_k^{-n_k} \dots x_2^{-n_2} x_1^{-n_1} x_k^{-m_k} \dots x_2^{-m_2} x_1^{-m_1} x_1^{n_1+m_1} x_2^{n_2+m_2} \dots x_k^{n_k+m_k} \\
=&x_k^{-n_k} \dots x_2^{-n_2} x_1^{-n_1} x_k^{-m_k} \dots x_2^{-m_2} x_1^{n_1} x_2^{n_2+m_2} \dots x_k^{n_k+m_k} \\
=&x_k^{-n_k} \dots x_2^{-n_2} (x_1^{n_1})^{y_1} x_1^{-n_1} x_k^{-m_k} \dots x_2^{-m_2} x_2^{n_2+m_2} \dots x_k^{n_k+m_k} \\
=&\dots \\
=&(x_k^{n_k})^{y_k} x_k^{-n_k} \dots (x_1^{n_1})^{y_1} x_1^{-n_1} \\
=&[y_k, x_k^{n_k}] \dots [y_1, x_1^{n_1}].
\end{aligned}$$

□

Proof of Proposition 3.2.1. Fix an element $x_1^{s_1} \cdots x_k^{s_k} \in A_{\mathcal{G}}$. Define a sequence $(a_n)_{n \in \mathbb{N}}$ by $a_n = \mathrm{cl}_{G,H}(x_1^{\lfloor s_1 n \rfloor} \dots x_k^{\lfloor s_k n \rfloor})$. By Fekete's lemma, it is sufficient to prove that there exist a constant C such that $a_{m+n} \leq a_m + a_n + C$ for all $m, n \in \mathbb{N}$. By Lemma 3.5.1,

$$\begin{aligned}
a_{m+n} &= \mathrm{cl}_{G,H}(x_1^{\lfloor s_1(m+n) \rfloor} \dots x_k^{\lfloor s_k(m+n) \rfloor}) \\
&\leq \mathrm{cl}_{G,H}(x_1^{\lfloor s_1 m \rfloor + \lfloor s_1 n \rfloor} \dots x_k^{\lfloor s_k m \rfloor + \lfloor s_k n \rfloor}) \\
&\quad + \mathrm{cl}_{G,H}((x_1^{\lfloor s_1 m \rfloor + \lfloor s_1 n \rfloor} \dots x_k^{\lfloor s_k m \rfloor + \lfloor s_k n \rfloor})^{-1} x_1^{\lfloor s_1(m+n) \rfloor} \dots x_k^{\lfloor s_k(m+n) \rfloor}) \\
&\leq \mathrm{cl}_{G,H}(x_1^{\lfloor s_1 m \rfloor + \lfloor s_1 n \rfloor} \dots x_k^{\lfloor s_k m \rfloor + \lfloor s_k n \rfloor}) + \sum_{i=1}^{k} \mathrm{cl}_{G,H}(x_i).
\end{aligned}$$

Set $M_i = \lfloor s_i m \rfloor$ and $N_i = \lfloor s_i n \rfloor$. Therefore, by Lemma 3.5.2,

$$\begin{aligned}
&a_{m+n} - a_m - a_n \\
\leq\, &\mathrm{cl}_{G,H}(x_1^{M_1+N_1} \dots x_k^{M_k+N_k}) + \sum_{i=1}^{k} \mathrm{cl}_{G,H}(x_i) \\
&- \mathrm{cl}_{G,H}(x_1^{M_1} \dots x_k^{M_k}) - \mathrm{cl}_{G,H}(x_1^{N_1} \dots x_k^{N_k}) \\
\leq\, &\mathrm{cl}_{G,H}\left((x_1^{M_1} \dots x_k^{M_k})^{-1}(x_1^{N_1} \dots x_k^{N_k})^{-1}(x_1^{M_1+N_1} \dots x_k^{M_k+N_k})\right) + \sum_{i=1}^{k} \mathrm{cl}_{G,H}(x_i) \\
\leq\, &k + \sum_{i=1}^{k} \mathrm{cl}_{G,H}(x_i).
\end{aligned}$$

□

Lemma 3.5.3. *For any* $\mathsf{x} \in A_\Gamma$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\|\mathsf{x}^{(\lambda_1+\lambda_2)} \star \bar{\mathsf{x}}^{(\lambda_1)} \star \bar{\mathsf{x}}^{(\lambda_2)}\|_\Gamma = 0$$

Proof. Assume that x is represented by $x_1^{s_1}x_2^{s_2}\dots x_k^{s_k} \in A_{\mathcal{G}}$. Set $p_i = \lfloor n\lambda_1 s_i \rfloor$, $q_i = \lfloor n\lambda_2 s_i \rfloor$, and $r_i = \lfloor n\lambda_1 s_i + n\lambda_2 s_i \rfloor$. By Lemma 3.5.1 and 3.5.2,

$$\begin{aligned}
&\mathrm{cl}_{G,H}(x_1^{r_1}\dots x_k^{r_k}(x_1^{p_1}\dots x_k^{p_k})^{-1}(x_1^{q_1}\dots x_k^{q_k})^{-1})\\
\le\ &\mathrm{cl}_{G,H}(x_1^{r_1}\dots x_k^{r_k}(x_1^{p_1+q_1}\dots x_k^{p_k+q_k})^{-1})\\
&+\mathrm{cl}_{G,H}(x_1^{p_1+q_1}\dots x_k^{p_k+q_k}(x_1^{p_1}\dots x_k^{p_k})^{-1}(x_1^{q_1}\dots x_k^{q_k})^{-1})\\
\le\ &\sum_{i=1}^{k}\mathrm{cl}_{G,H}(x_i) + k < +\infty.
\end{aligned}$$

Here, we used that $|(p_i+q_i)-r_i| \le 1$. Therefore,

$$\begin{aligned}
&\|\mathsf{x}^{(\lambda_1+\lambda_2)} \star \bar{\mathsf{x}}^{(\lambda_1)} \star \bar{\mathsf{x}}^{(\lambda_2)}\|_{\mathcal{G}}\\
=\ &\lim_{n\to\infty}\frac{1}{n}\mathrm{cl}_{G,H}(x_1^{r_1}\dots x_k^{r_k}(x_1^{p_1}\dots x_k^{p_k})^{-1}(x_1^{q_1}\dots x_k^{q_k})^{-1})\\
=\ &0.
\end{aligned}$$

□

Lemma 3.5.4. *For $\mathsf{x} \in A_\Gamma$ and $\lambda \in \mathbb{R}$,*

$$\|\mathsf{x}^{(\lambda)}\|_\Gamma = |\lambda|\|\mathsf{x}\|_\Gamma$$

Proof. We set $\mathsf{x} = x_1^{s_1}\dots x_k^{s_k}$. If $\lambda = \frac{p}{q}$ is a positive rational number, where p, q are positive integers, then by considering subsequences,

$$\begin{aligned}
\|\mathsf{x}^{(\lambda)}\|_\Gamma &= \lim_{n\to\infty}\frac{1}{n}\mathrm{cl}_{G,H}(x_1^{\lfloor\lambda s_1 n\rfloor}\dots x_k^{\lfloor\lambda s_k n\rfloor})\\
&= \lim_{n\to\infty}\frac{1}{qn}\mathrm{cl}_{G,H}(x_1^{\lfloor p s_1 n\rfloor}\dots x_k^{\lfloor p s_k n\rfloor})\\
&= \lim_{n\to\infty}\frac{p}{qn}\mathrm{cl}_{G,H}(x_1^{\lfloor s_1 n\rfloor}\dots x_k^{\lfloor s_k n\rfloor})\\
&= \lambda\|\mathsf{x}\|_\Gamma.
\end{aligned}$$

We consider the case $\lambda = -1$. By Lemma 3.5.3 we obtain $\|\mathsf{x}^{(-1)}\mathsf{x}\| = \|\mathsf{x}^{(0)}\| = 0$ and it means that $[\mathsf{x}^{(-1)}] = [\bar{\mathsf{x}}]$. Therefore $\|\mathsf{x}^{(-1)}\| = \|\bar{\mathsf{x}}\| = \|\mathsf{x}\|$ and we complete the proof for the case when λ is a rational number.

Since Lemma 3.5.1 implies that the function $\lambda \mapsto \|\mathsf{x}^{(\lambda)}\|$ is continuous, we complete the proof. □

Proof of Proposition 3.2.2. Assume that $[\mathsf{x}_1] = [\mathsf{x}_2]$ and $[\mathsf{y}_1] = [\mathsf{y}_2]$ for $\mathsf{x}_1, \mathsf{x}_2, \mathsf{y}_1, \mathsf{y}_2 \in A_\Gamma$.

$$\begin{aligned}
&\|\mathsf{x}_1 \star \mathsf{y}_1 \star \overline{\mathsf{x}_2 \star \mathsf{y}_2}\|_\Gamma \\
&= \|\mathsf{x}_1 \star \mathsf{y}_1 \star \bar{\mathsf{y}}_2 \star \bar{\mathsf{x}}_2\|_\Gamma \\
&\leq \|\mathsf{x}_1 \star \mathsf{y}_1 \star \bar{\mathsf{y}}_2 \star \bar{\mathsf{x}}_1\|_\Gamma + \|\mathsf{x}_1 \star \bar{\mathsf{x}}_2\|_\Gamma \\
&= \|\mathsf{y}_1 \star \bar{\mathsf{y}}_2\|_\Gamma + \|\mathsf{x}_1 \star \bar{\mathsf{x}}_2\|_\Gamma = 0
\end{aligned}$$

Thus $[\mathsf{x}_1 \star \mathsf{y}_1] = [\mathsf{x}_2 \star \mathsf{y}_2]$.

Assume that elements $\mathsf{x}_1, \mathsf{x}_2 \in A_\Gamma$ satisfy $[\mathsf{x}_1] = [\mathsf{x}_2]$. For any $\lambda \in \mathbb{R}$, by Lemma 3.5.4,

$$\|\mathsf{x}_1^{(\lambda)} \star \bar{\mathsf{x}}_2^{(\lambda)}\|_\Gamma = \|(\mathsf{x}_1 \star \bar{\mathsf{x}}_2)^{(\lambda)}\|_\Gamma = |\lambda| \|\mathsf{x}_1 \star \bar{\mathsf{x}}_2\|_\Gamma = 0$$

Thus $[\mathsf{x}_1^{(\lambda)}] = [\mathsf{x}_2^{(\lambda)}]$. $\square$

Proof of Proposition 3.2.3. By Lemma 3.5.3 and 3.5.4, for any $\lambda_1, \lambda_2, \lambda \in \mathbb{R}$ and $\mathbf{x} \in A$,

$$(\lambda_1 + \lambda_2)\mathbf{x} = \lambda_1 \mathbf{x} + \lambda_2 \mathbf{x}, \ \|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|.$$

For any $\mathbf{x} = [\mathsf{x}]$ and $\mathbf{y} = [\mathsf{y}]$ in A, where $\mathsf{x}, \mathsf{y} \in A_\Gamma$,

$$\mathbf{x} + \mathbf{y} = [\mathsf{x} \star \mathsf{y}] = [\bar{\mathsf{x}} \star \mathsf{x} \star \mathsf{y} \star \mathsf{x}] = [\mathsf{y} \star \mathsf{x}] = \mathbf{y} + \mathbf{x}.$$

The other axioms of a normed space can be confirmed easily. $\square$

Chapter 4

Norm controlled cohomology of transformation groups

4.1 Norm controlled cohomology

4.1.1 Definition

We introduce the notion of norm controlled cohomology which is a generalization of bounded cohomology. Note that a similar generalization of bounded cohomology is studied for finitely generated groups and its word length, which is called the polynomially bounded cohomology (see [45] for example).

Definition 4.1.1. For a cochain $\bar{c} \in \bar{C}^n(G)$ and a function $\mu\colon G^n \to [0, \infty)$, we say that $\bar{c}$ is *Lipschitz with respect to* μ if there exist constants $C, D \geq 0$ such that for every $g_1, \ldots, g_n \in G$

$$|\bar{c}(g_1, \ldots, g_n)| \leq C \cdot \mu(g_1, \ldots, g_n) + D.$$

Definition 4.1.2. A *normed group* (G, ν) is a pair of a group G and a norm ν on G. For a normed group (G, ν) and non-negative integers n and d, we define $\bar{C}^n_{(d)}(G, \nu)$ as follows.

- If $n > d$, we define $\bar{C}^n_{(d)}(G, \nu)$ as the set of Lipschitz cochains $\bar{c} \in \bar{C}^n(G)$

with respect to $\nu_{(n,d)}$, where $\nu_{(n,d)} : G^n \to [0, \infty)$ is defined by

$$\begin{aligned}\nu_{(n,d)}(g_1, \dots, g_n) &= \min_{\substack{I \subset \{1,\dots,n\} \\ \#I = n-d}} \left\{ \sum_{i \in I} \nu(g_i) \right\} \\ &= \min_{1 \le i_i < \cdots < i_d \le n} \left\{ \nu(g_1) + \cdots + \widehat{\nu(g_{i_1})} + \cdots + \widehat{\nu(g_{i_d})} + \cdots + \nu(g_n) \right\}.\end{aligned}$$

- If $d \geq n$, we define $\bar{C}^n_{(d)}(G, \nu) = \bar{C}^n_b(G)$.

Note that $\bar{c} \in \bar{C}^n_{(0)}(G, \nu)$ implies

$$|\bar{c}(g_1, \dots, g_n)| \le C \cdot \{\nu(g_1) + \cdots + \nu(g_n)\} + D$$

and $\bar{c} \in \bar{C}^n_{(n-1)}(G, \nu)$ implies

$$|\bar{c}(g_1, \dots, g_n)| \le C \cdot \min\{\nu(g_1), \dots, \nu(g_n)\} + D.$$

Lemma 4.1.3. *For any integer $d \geq 0$, $(\bar{C}^n_{(d)}(G, \nu), \bar{\delta})$ is a subcomplex of $(\bar{C}^n(G), \bar{\delta})$.*

Proof. It is sufficient to prove that $\bar{\delta}(\bar{C}^{n-1}_{(d)}(G, \nu)) \subset \bar{C}^n_{(d)}(G, \nu)$ for the case $n - 1 > d$.

Let $g_1, \dots, g_n$ be elements in G. It is easy to see that

$$\begin{aligned}\nu_{(n-1,d)}(g_2, \dots, g_n) &\le \nu_{(n,d)}(g_1 \dots, g_n), \\ \nu_{(n-1,d)}(g_1, \dots, g_{n-1}) &\le \nu_{(n,d)}(g_1, \dots, g_n).\end{aligned}$$

Since $\nu(g_i g_{i+1}) \le \nu(g_i) + \nu(g_{i+1})$,

$$\nu_{(n-1,d)}(g_1, \dots, g_i g_{i+1}, \dots, g_n) \le \nu_{(n,d)}(g_1, \dots, g_i, g_{i+1}, \dots, g_n)$$

for $i = 1, 2, \dots, n-1$.

Therefore, for $\bar{c} \in \bar{C}^{n-1}_{(d)}(G, \nu)$,

$$\begin{aligned}&|\bar{\delta}\bar{c}(g_1, \dots, g_n)| \\ &\le |\bar{c}(g_2, \dots, g_n)| + \sum_{i=1}^{n-1} |\bar{c}(g_1, \dots, g_i g_{i+1}, \dots, g_n)| + |\bar{c}(g_1, \dots, g_{n-1})| \\ &\le (n+1)\{C \cdot \nu_{(n,d)}(g_1, \dots, g_n) + D\}.\end{aligned}$$

□

Definition 4.1.4. For a normed group (G,ν) and an integer $d \geq 0$, we define the *norm controlled cohomology* $H^\bullet_{(d)}(G,\nu)$ *of level* d to be the cohomology of the cochain complex $(\bar{C}^n_{(d)}(G,\nu),\bar{\delta})$.

Note that the complexes $\{\bar{C}^n_{(d)}(G,\nu)\}_{n,d}$ can be seen as a filtered complex, i.e., $\bar{C}^n_{(d)}(G,\nu) \subset \bar{C}^n_{(d')}(G,\nu)$ if $d \geq d'$.

By the correspondence (2.2.2), we can define the homogeneous norm controlled cochain complex $C^n_{(d)}(G,\nu)$ as the set of cochain $c \in C^n(G)$ which satisfies the following: there exist constants $C, D \geq 0$ such that

$$|c(g_1,\ldots,g_n)| \leq C \cdot \min_{\substack{I\subset\{1,\ldots,n\}\\ \#I=n-d}} \left\{ \sum_{i\in I} \nu(g_{i-1}^{-1}g_i) \right\} + D.$$

We can also define the inhomogeneous (resp. homogeneous) alternating subcomplex $\bar{C}^\bullet_{(d),\mathrm{alt}}(G,\nu)$ (resp. $C^\bullet_{(d),\mathrm{alt}}(G,\nu)$) and they also define the cohomology $H^\bullet_{(d)}(G,\nu)$.

Example 4.1.5. Let $\mathbb{Z}^n$ be the free abelian group of rank n. For a positive integer $l \leq n$, define a (pseudo) norm ν_l on $\mathbb{Z}^n$ by

$$\nu_l(m_1,\ldots,m_l,\ldots,m_n) = |m_1| + \cdots + |m_l|$$

for $m_1,\ldots,m_n \in \mathbb{Z}$. Now we compute $H^1_{\nu_l}(\mathbb{Z}^n)$. Note that $H^1_\nu(G) = H^1_{(0)}(G,\nu) = \mathrm{Ker}(\bar{\delta}\colon \bar{C}^1_{(d)} \to \bar{C}^2_{(d)})$ is the set of Lipschitz homomorphisms with respect to ν.

We define a homomorphism $\phi_i\colon \mathbb{Z}^n \to \mathbb{R}$ by $\phi_i(m_1,\ldots,m_l,\ldots,m_n) = m_i$. $\mathrm{Hom}(\mathbb{Z}^n,\mathbb{R}) \cong \mathbb{R}^n$ is generated by $\phi_1,\ldots,\phi_n$. It is easy to see that ϕ_i is Lipschitz with respect to ν_l for $i \leq l$ and not for $i > l$. Thus, $H^1_{\nu_l}(\mathbb{Z}^n)$ is generated by $\phi_1,\ldots,\phi_l$ and isomorphic to $\mathbb{R}^l$.

Norm controlled cohomology provides a framework for relative quasimorphisms (Figure 4.1). Let (G,ν) be a normed group. A *relative quasimorphism with respect to* ν is a real-valued function ϕ on G such that there exist constants $C, D \geq 0$ with

$$|\phi(gh) - \phi(g) - \phi(h)| \leq C \cdot \min\{\nu(g),\nu(h)\} + D$$

for all $g,h \in G$. Relative quasimorphisms appear in the context of symplectic topology (see [20] for example). Let $\widehat{Q}(G,\nu)$ denote the space of relative quasimorphisms on (G,ν). An exact sequence of complexes

$$0 \to C^\bullet_{(1)}(G,\nu) \to C^\bullet(G) \to C^\bullet(\Gamma)/C^\bullet_{(1)}(G,\nu) \to 0$$

$$\begin{tikzcd}
\text{quasimorphisms} \arrow[r, "\text{coboundary}"] \arrow[d, "\text{generalize}"] & \text{bounded cohomology} \arrow[d, "\text{generalize}"] \\
\text{relative quasimorphisms} \arrow[r, "\text{coboundary}"'] & \text{norm controlled cohomology}
\end{tikzcd}$$

Figure 4.1: Relationship between norm controlled cohomology and other notions

induces the exact sequence

$$0 \to H^1(G) \to \widehat{Q}(G,\nu)/C_b^1(G) \to H^2_{(1)}(G,\nu) \to H^2(G)$$

since $H^1_{(1)}(G,\nu) = H^1_b(G) = 0$. Hence, $EH^2_{(1)}(G) = \mathrm{Ker}(H^2_{(1)}(G,\nu) \to H^2(G))$ is isomorphic to $\widehat{Q}(G,\nu)/(C_b^1(G) + H^1(G))$.

Example 4.1.6. For the following cases, $EH_b^2(G)$ is trivial but $EH^2_{(1)}(G,\nu)$ is non-trivial for a certain norm ν.

- G is the identity component of the group of symplectomorphisms $\mathrm{Symp}_0^c(\mathbb{R}^{2n},\omega_0)$ of the standard symplectic space $(\mathbb{R}^{2n},\omega_0)$ with compact support [36].
- G is the infinite braid group B_∞ [40].
- G is the Hamiltonian diffeomorphism group $\mathrm{Ham}(T^*\Sigma_g \times \mathbb{R}^{2n})$ of $T^*\Sigma_g \times \mathbb{R}^{2n}$, where Σ_g is a closed surface of genus $g > 1$ [7].

4.1.2 Functoriality

We show that our cohomology is a functor for a certain category.

Definition 4.1.7. Let (G,ν_G) and (H,ν_H) be normed groups. A homomorphism $\phi\colon G \to H$ is said to be *Lipschitz* if there exist $C, D \geq 0$ such that for all $g \in G$,

$$\nu_H(\phi(g)) \leq C \cdot \nu_G(g) + D.$$

Definition 4.1.8. We define the *category* **NGrp** *of normed groups* as follows.

- The objects $Ob(\mathbf{NGrp})$ are normed groups.
- The morphisms $Mor(\mathbf{NGrp})$ are Lipschitz homomorphisms $\phi\colon (G,\nu_G) \to (H,\nu_H)$ between normed groups (G,ν_G) and (H,ν_H).

The composition of morphisms is the composition of group homomorphisms, and hence the associativity holds. For every $(G,\nu) \in Ob(\mathbf{NGrp})$, there exists the identity $\mathrm{id}_G : (G,\nu) \to (G,\nu)$ in $Mor(\mathbf{NGrp})$. Hence $\mathbf{NGrp}$ is a category.

Let $H^n_{(d)}$ denote the correspondence from a norm group (G,ν) to $H^n_{(d)}(G,\nu)$.

Proposition 4.1.9. *The correspondence $H^n_{(d)}$ is a contravariant functor from the category of normed groups $\mathbf{NGrp}$ to the category of real vector spaces $\mathbf{Vect}_{\mathbb{R}}$.*

Proof. Let $\phi\colon (G,\nu_G) \to (H,\nu_H)$ be a Lipschitz homomorphism. It induces the linear map $\phi^*\colon \bar{C}^n_{(d)}(H,\nu_H) \to \bar{C}^n_{(d)}(G,\nu_G)$ by

$$\phi^*\bar{c}(g_1,\ldots,g_n) = \bar{c}(\phi(g_1),\ldots,\phi(g_n))$$

since $\nu_{H(n,d)}(\phi(g_1),\ldots,\phi(g_n)) \leq C\cdot\nu_{G(n,d)}(g_1,\ldots,g_n) + D$, where C and D are the Lipschitz constants of ϕ.

Let $\bar{B}^n_{(d)}(G)$ denote $\mathrm{Im}(\bar{\delta}\colon \bar{C}^{n-1}_{(d)}(G,\nu_G) \to \bar{C}^n_{(d)}(G,\nu_G))$ and $\bar{Z}^n_{(d)}(G)$ denote $\mathrm{Ker}(\bar{\delta}\colon \bar{C}^n_{(d)}(G,\nu_G) \to \bar{C}^{n+1}_{(d)}(G,\nu_G))$. Note that $H^n_{(d)}(G,\nu_G) = \bar{Z}^n_{(d)}(G)/\bar{B}^n_{(d)}(G)$.

We have to show that $\phi^*(\bar{Z}^n_{(d)}(H)) \subset \bar{Z}^n_{(d)}(G)$ and $\phi^*(\bar{B}^n_{(d)}(H)) \subset \bar{B}^n_{(d)}(G)$. The former follows immediately. The latter is proved as follows. For $\bar{c} \in \bar{B}^n_{(d)}(H)$, there exists $\bar{c}' \in \bar{C}^{n-1}_{(d)}(H,\nu_H)$ such that $\bar{\delta}\bar{c}' = \bar{c}$. For $g_1,\ldots,g_n \in G$,

$$\begin{aligned}
&\phi^*\bar{c}(g_1,\ldots,g_n)\\
&= \bar{c}(\phi(g_1),\ldots,\phi(g_n))\\
&= \bar{\delta}\bar{c}'(\phi(g_1),\ldots,\phi(g_n))\\
&= \bar{c}'(\phi(g_2),\ldots,\phi(g_n)) + \sum_{i=1}^{n-1}\bar{c}'(\phi(g_1),\ldots,\phi(g_i)\phi(g_{i+1}),\ldots,\phi(g_n))\\
&\qquad\qquad\qquad\qquad +(-1)^n\bar{c}'(\phi(g_1),\ldots,\phi(g_{n-1}))\\
&= \bar{c}'(\phi(g_2),\ldots,\phi(g_n)) + \sum_{i=1}^{n-1}\bar{c}'(\phi(g_1),\ldots,\phi(g_ig_{i+1}),\ldots,\phi(g_n))\\
&\qquad\qquad\qquad\qquad +(-1)^n\bar{c}'(\phi(g_1),\ldots,\phi(g_{n-1}))\\
&= \bar{\delta}(\phi^*\bar{c}')(g_1,\ldots,g_n).
\end{aligned}$$

Therefore, ϕ induces the linear map $\phi^*\colon H^n_{(d)}(H,\nu_H) \to H^n_{(d)}(G,\nu_G)$. $\square$

If two norms on the same group are bi-Lipschitz, then they defines the same norm controlled cohomology.

Corollary 4.1.10. *Let G be a group with (pseudo) norms ν_1 and ν_2. If both $\mathrm{id}_G\colon (G,\nu_1) \to (G,\nu_2)$ and $\mathrm{id}_G\colon (G,\nu_2) \to (G,\nu_1)$ are Lipschitz, then $\mathrm{id}_G^*\colon H^n_{(d)}(G,\nu_2) \to H^n_{(d)}(G,\nu_1)$ is an isomorphism.*

Proof. Let ϕ denote $\mathrm{id}_G\colon (G,\nu_1) \to (G,\nu_2)$ and ψ denote $\mathrm{id}_G\colon (G,\nu_2) \to (G,\nu_1)$. Note that $\psi \circ \phi$ is the identity morphism $\mathrm{id}_{(G,\nu_1)}$ for $(G,\nu_1) \in Ob(\mathbf{NGrp})$. Thus, $\phi^* \circ \psi^*\colon H^n_{(d)}(G,\nu_1) \to H^n_{(d)}(G,\nu_1)$ is the identity. Similarly, $\psi^* \circ \phi^*\colon H^n_{(d)}(G,\nu_2) \to H^n_{(d)}(G,\nu_2)$ is also the identity. Therefore, $\mathrm{id}_G^* = \phi^*\colon H^n_{(d)}(G,\nu_2) \to H^n_{(d)}(G,\nu_1)$ is an isomorphism. $\square$

4.2 Norm controlled cohomology of transformation groups

4.2.1 Brandenbursky–Marcinkowski's construction

We briefly review the construction of Brandenbursky and Marcinkowski [9]. Let M be a complete Riemannian manifold with a finite volume and μ the measure on M associate to the Riemannian structure. Fix a base point $z \in M$. Let $\mathrm{Homeo}_0^c(M,\mu)$ denotes the group of homeomorphisms of M with compact support that are isotopic to the identity and preserve the measure μ. Recall that π_M denotes the quotient group $\pi_1(M,z)/Z(\pi_1(M,z))$, where $Z(G)$ denotes the center of a group G.

For a subgroup $\mathcal{T}_M$ of $\mathrm{Homeo}_0^c(M,\mu)$, they constructed a map $\Gamma_b^\bullet\colon H_b^\bullet(\pi_M) \to H_b^\bullet(\mathcal{T}_M)$ as follows. Let C denote the cut locus of z. For $x \in M$ and $g \in \mathcal{T}_M$ such that $x \notin C$ and $g(x) \notin C$, we define $\gamma(g,x) \in \pi_1(M,z)$ by the concatenation of the geodesic between z and x, the path defined by $\{g^t(x)\}_{0\le t\le 1}$, where $\{g^t\}_{0\le t\le 1}$ is an isotopy of g with $g^0 = \mathrm{id}$ and $g^1 = g$, and the geodesic between $g(x)$ and z. Then $\gamma(g,x)$ is defined uniquely up to center for any choice of isotopies. Thus it defines an element of π_M.

Since the measure $\mu(C)$ of the cut locus C is zero and the map $\gamma(f,\cdot)\colon M \to \pi_M$ has essentially finite image for $f \in \mathcal{T}_M$, we can define the map $\Phi_{BM}\colon C^n(\pi_M) \to$

$C^n(\mathcal{T}_M)$ by

$$\Phi_{BM}(c)(g_0,\ldots,g_n) = \int_M c(\gamma(g_0,x),\ldots,\gamma(g_n,x))d\mu(x)$$

for $c \in C^n(\pi_M)$ and $g_0,\ldots,g_n \in \mathcal{T}_M$. The map $\Gamma^n\colon H^n(\pi_M) \to H^n(\mathcal{T}_M)$ is defined as the induced map from Φ_{BM}. If $c \in C_b^n(\pi_M)$ is a bounded cochain, then $\Phi_{BM}(c)$ is also a bounded cochain since

$$|\Phi_{BM}(c)(g_0,\ldots,g_n)| \leq \|c\| \cdot \mathrm{vol}(M) < +\infty$$

for any $g_0,\cdots,g_n \in \pi_M$. Hence Φ_{BM} induces the map $\Gamma_b^n\colon H_b^n(\pi_M) \to H_b^n(\mathcal{T}_M)$. We also obtain the map of the exact part $E\Gamma_b^n\colon EH_b^n(\pi_M) \to EH_b^n(\mathcal{T}_M)$.

4.2.2 Infinite volume case

We consider the above construction for the case that M has infinite volume. In this case, the map Φ_{BM} is well-defined on $C_{b,\mathrm{alt}}(\pi_M)$ since we consider compactly supported homeomorphisms. On the other hand, the image $\Phi_{BM}(c)$ of a bounded cochain $c \in C_{b,\mathrm{alt}}(\pi_M)$ might not be a bounded cochain. We prove that, however, the image is a norm controlled cochain with respect to a fragmentation norm (Proposition 4.2.5).

Let $H_\nu^n(G)$ denote the norm controlled cohomology of level zero $H_{(0)}^n(G,\nu)$. In this section, we prove the following theorem.

Theorem 4.2.1. *Let $\mathcal{T}_M$ be $\mathrm{Homeo}_0^c(M,\mu)$, $\mathrm{Diff}_0^c(M,\mathrm{vol})$ or $\mathrm{Symp}_0^c(M,\omega)$. Assume that there exists an open subset U of M with finite volume such that ν_U is well-defined on $\mathcal{T}_M$. If either*

(1) π_M surjects onto F_2 or

(2) π_M is an acylindrically hyperbolic group,

then

$$\dim EH_{\nu_U}^n(\mathcal{T}_M) \geq \dim \overline{EH}_b^n(F_2).$$

We give the definition of fragmentation norm.

Definition 4.2.2. Let U be an open subset of M. Let $\mathcal{S}_U$ denote the set of elements $h \in \mathcal{T}_M$ that satisfy the following condition: there exists an isotopy $\{h^t\}_{0 \leq t \leq 1}$ of h such that $\mathrm{supp}(h^t) \subset U$ for every $t \in [0,1]$. We define the *fragmentation norm* ν_U *with respect to* U on $\mathcal{T}_M$ by

$$\nu_U(g) = \min \left\{ k \;\middle|\; \begin{matrix} \exists f_i \in \mathcal{T}_M, \exists h_i \in \mathcal{S}_U, (i = 1, \dots, k) \\ g = (f_1^{-1} h_1 f_1) \cdots (f_k^{-1} h_k f_k) \end{matrix} \right\}$$

for $g \in \mathcal{T}_M$. If no such decomposition of g exists, we define $\nu_U(g) = +\infty$. We call that ν_U is *well-defined on* $\mathcal{T}_M$ if $\nu_U(g) < +\infty$ for all $g \in \mathcal{T}_M$.

Example 4.2.3. Let M be a manifold, U a non-empty open subset of M, and $i \colon U \to M$ the inclusion.

- Let $\mathcal{T}_M$ be $\mathrm{Homeo}_0^c(M, \mu)$ and $\widetilde{\mathcal{T}}_M$ its universal covering. In [27], Fathi defined the homomorphism $\tilde{\theta} \colon \widetilde{\mathcal{T}}_M \to H_1(M; \mathbb{R})$ and $\tilde{\theta}$ induces the mass flow homomorphism $\theta \colon \mathcal{T}_M \to H_1(M; \mathbb{R})/\Gamma$, where $\Gamma = \tilde{\theta}(\pi_1(\mathcal{T}_M))$. Since $\mathrm{Ker}(\theta)$ has the fragmentation property [27], ν_U is well-defined on $\mathrm{Ker}(\theta)$.

- Let $\mathcal{T}_M$ be $\mathrm{Diff}_0^c(M, \mathrm{vol})$ and $\mathrm{Flux} \colon \mathcal{T}_M \to H_c^{n-1}(M; \mathbb{R})/\Gamma$ denotes the volume flux homomorphism, where Γ is the volume flux group. Since $\mathrm{Ker}(\mathrm{Flux})$ has the fragmentation property (an unpublished result of W. Thurston, see Banyaga's book [2]), ν_U is well-defined on $\mathrm{Ker}(\mathrm{Flux})$.

- Let $\mathcal{T}_M$ be $\mathrm{Symp}_0^c(M, \omega)$ and let $\mathrm{Flux}_\omega \colon \mathcal{T}_M \to H_c^1(M; \mathbb{R})/\Gamma_\omega$ denote the symplectic flux homomorphism, where Γ_ω is the symplectic flux group. Since $\mathrm{Ker}(\mathrm{Flux}_\omega)$ has the fragmentation property [1], ν_U is well-defined on $\mathrm{Ker}(\mathrm{Flux}_\omega)$.

Note that $H_c^\bullet$ denotes the (de Rham) cohomology with compact support and $H_c^\bullet$ defines a covariant functor.

Example 4.2.4. Let M, U, and $i \colon U \to M$ be as above.

- Let $\mathcal{T}_M = \mathrm{Homeo}_0^c(M, \mu)$. If $i_* \colon H_1(U; \mathbb{R}) \to H_1(M; \mathbb{R})$ is surjective, we can see that ν_U is well-defined on $\mathcal{T}_M$ as follows. For $g \in \mathcal{T}_M$, there exists $h \in \mathrm{Homeo}_0^c(U, \mu)$ such that $\theta(g) = \theta(h)$. Thus $g = (gh^{-1})h$ is written as a product of the conjugation of the elements of $\mathcal{S}_U$ since $h \in \mathcal{S}_U$ and $gh^{-1} \in \mathrm{Ker}(\theta)$.

- Let $\mathcal{T}_M = \mathrm{Diff}_0^c(M, \mathrm{vol})$. If $i^* \colon H_c^{n-1}(U; \mathbb{R}) \to H_c^{n-1}(M; \mathbb{R})$ is surjective, we can see that ν_U is well-defined on $\mathcal{T}_M$ by the same argument.

- Let $\mathcal{T}_M = \mathrm{Symp}_0^c(M, \omega)$. If $i^* \colon H_c^1(U; \mathbb{R}) \to H_c^1(M; \mathbb{R})$ is surjective, we can see that ν_U is well-defined on $\mathcal{T}_M$ by the same argument.

Now we prove that we obtain norm controlled cochains by Brandenbursky–Marcinkowski's construction.

Proposition 4.2.5. *For $c \in C^n_{b,\mathrm{alt}}(\pi_M)$, there exists $C \geq 0$ such that*

$$|\Phi_{BM}(c)(g_0, \dots, g_n)| \leq C \cdot \min_{0 \leq i < j \leq n} \{\nu_U(g_i^{-1} g_j)\}.$$

for all $g_0, \dots, g_n \in \mathcal{T}_M$. In particular, $\Phi_{BM}(c) \in C^n_{(n-1),\mathrm{alt}}(\mathcal{T}_M, \nu_U)$.

Proof. We fix i and j ($0 \leq i < j \leq n$). Assume that $\nu_U(g_i^{-1} g_j) = m$. Then we can write $g_i^{-1} g_j = (f_1^{-1} h_1 f_1) \dots (f_m^{-1} h_m f_m)$, where $h_k \in \mathcal{S}_U$ and $f_k \in \mathcal{T}_M$ for $k = 1, \dots, m$. Take an isotopy $\{g_i^t\}_t$ of g_i and isotopies $\{h_1^t\}_t, \dots, \{h_m^t\}_t$ for $h_1, \dots, h_m$ such that $\mathrm{supp}(h_k^t) \subset U$ for every $t \in [0,1]$ and $k = 1, \dots, m$. We define $g_j^t = g_i^t (f_1^{-1} h_1^t f_1) \cdots (f_m^{-1} h_m^t f_m)$. Then $\{g_j^t\}_t$ is an isotopy of g_j. Set

$$U_{ij} = \bigcup_{0 \leq t \leq 1} \mathrm{supp}\left((g_i^t)^{-1} g_j^t\right) = \bigcup_{0 \leq t \leq 1} \mathrm{supp}\left((f_1^{-1} h_1^t f_1) \cdots (f_m^{-1} h_m^t f_m)\right).$$

Note that $U_{ij} \subset f_1(U) \cup \dots \cup f_m(U)$. If $x \notin U_{ij}$, $g_i^t(x) = g_j^t(x)$ for every $t \in [0,1]$. Thus $\gamma(g_i, x) = \gamma(g_j, x) \in \pi_M$. Since c is alternating, $c(\gamma(g_0, x), \dots, \gamma(g_n, x)) = 0$. Therefore,

$$|\Phi_{BM}(c)(g_0, \dots, g_n)| \leq \mathrm{vol}(U_{ij}) \cdot \|c\| \leq m \cdot \mathrm{vol}(U) \cdot \|c\|.$$

Since we can arbitrarily take i and j, the inequality holds for $C = \mathrm{vol}(U) \cdot \|c\|$. ∎

Remark 4.2.6. For $d \leq n-1$, the map $\Phi_{BM} \colon C^n_{b,\mathrm{alt}}(\pi_M) \to C^n_{(d),\mathrm{alt}}(\mathcal{T}_M, \nu_U)$ is well-defined. However, if $d = n-1$, Φ_{BM} does not induce the map $H_b^n(\pi_M) \to H^n_{(n-1)}(\mathcal{T}_M, \nu_U)$ because the image of $\bar{B}^n_{(n-1)}(\pi_M)$ might not be in $\bar{B}^n_{(n-1)}(\mathcal{T}_M)$. On the other hand, if $d < n-1$, Φ_{BM} induces $H_b^n(\pi_M) \to H^n_{(d)}(\mathcal{T}_M, \nu_U)$. Especially, if $d = 0$, then $\Phi_{BM} \colon C^n_{b,\mathrm{alt}}(\pi_M) \to C^n_{\nu_U,\mathrm{alt}}(\mathcal{T}_M)$ induces $H_b^n(\pi_M) \to H^n_{\nu_U}(\mathcal{T}_M)$ for any $n \geq 2$.

We prove the following key lemma which corresponds to [9, Lemma 3.3].

Lemma 4.2.7. *Let $U \subset M$ be an open subset such that ν_U is well-defined on $\mathcal{T}_M$. Assume that there exists an injection $i \colon F_2 \to \pi_M$. Let a and b be generators of F_2. Let α and β be two loops in M representing $i(a)$ and $i(b)$. Suppose that α and β are contained in U. Then there exists a family of Lipschitz homomorphisms $\rho_\epsilon \colon (F_2, \nu_0) \to (\mathcal{T}_M, \nu_U)$ for $\epsilon \in (0,1)$ such that there exists $\Lambda > 0$, for every $c \in EH^n_{\nu_U}(\pi_M)$,*

$$\lim_{\epsilon \to +0} \|\rho_\epsilon^*(E\Gamma_{\nu_U}(c)) - \Lambda i^*(c)\| = 0.$$

Here, $\nu_0 \colon F_2 \to [0, \infty)$ is the trivial norm defined by

$$\nu_0(w) = \begin{cases} 0 & (w = 1_{F_2}), \\ 1 & (w \neq 1_{F_2}). \end{cases}$$

The maps i^* and ρ_ϵ^* represent the induced maps $i^* \colon EH^n_b(\pi_M) \to EH^n_b(F_2)$ and $\rho_\epsilon^* \colon EH^n_{\nu_U}(\pi_M) \to EH^n_{\nu_0}(F_2) = EH^n_b(F_2)$.

Proof. We can prove in the same way as [9, Lemma 3.3]. Let $N(\alpha)$ denote a tubular neighborhood of α in U and take a diffeomorphism $n_\alpha \colon N(\alpha) \to S^1 \times B^{n-1}(1)$. Here $B^{n-1}(r)$ denotes the $(n-1)$-ball in $\mathbb{R}^n$ with radius r. Let $A_\epsilon(\alpha)$ denote $n_\alpha^{-1}(S^1 \times B^{n-1}(1-\epsilon))$. We define an element $\rho_\epsilon(a) \in \mathcal{T}_M$ which "rotates" every point in A_ϵ one lap in the direction of S^1 and fixes outside of $N(\alpha)$ (see [9] for more details). Similarly, we define $N(\beta) \subset U$, B_ϵ and $\rho_\epsilon(b) \in \mathcal{T}_M$. Thus we obtain the representation $\rho_\epsilon \colon F_2 \to \mathcal{T}_M$. Since $\mathrm{supp}(\rho_\epsilon(w))$ is contained in U for any $w \in F_2$, the map $\rho_\epsilon \colon (F_2, \nu_0) \to (\mathcal{T}_M, \nu_U)$ is a Lipschitz homomorphism. By the functoriality of the correspondence $H^n_{(0)}$ (Proposition 4.1.9), the map $\rho_\epsilon^* \colon EH^n_{\nu_U}(\mathcal{T}_M) \to EH^n_b(\pi_M)$ is induced.

For $w_0, \ldots, w_n \in F_2$, we have

$$\rho_\epsilon^*(E\Gamma_{\nu_U}(c))(w_0, \ldots, w_n) = \int_M c(\gamma(\rho_\epsilon(w_0), x), \ldots, \gamma(\rho_\epsilon(w_n), x)) d\mu(x).$$

Let $B_\epsilon(\alpha)$ and $B_\epsilon(\beta)$ denote $N(\alpha) - A_\epsilon(\alpha)$ and $N(\beta) - A_\epsilon(\beta)$ respectively. We calculate this integral by decomposing M into 5 parts; $A_\epsilon := A_\epsilon(\alpha) \cap A_\epsilon(\beta)$, $A_\epsilon^a := A(\alpha) - N(\beta)$, $A_\epsilon^b := A_\epsilon(\beta) - N(\alpha)$, $B_\epsilon := B_\epsilon(\alpha) \cup B_\epsilon(\beta)$, and their exterior $M - (N(\alpha) \cup N(\beta))$.

The exterior part is 0 and it turns out that A_ϵ^a and A_ϵ^b part are also 0. The A_ϵ part is calculated to be $\mu(A_\epsilon) i^*(c)$ and the B_ϵ is bounded by

$\mu(B_\epsilon)\|c\|$. Hence the claim follows from $\mu(A_\epsilon) \xrightarrow{\epsilon\to+0} \mu(N(\alpha)\cap N(\beta)) > 0$ and $\mu(B_\epsilon) \xrightarrow{\epsilon\to+0} 0$. $\square$

We give the proof of Theorem 4.2.1. The proof is inspired by [9].

Proof of Theorem 4.2.1. First, we prove for the case (1). Let $p\colon \pi_M \to F_2$ be a surjection. Assume that $\dim(M) \geq 3$. Then there exists an injection $i\colon F_2 \to \pi_M$ such that $p \circ i = \mathrm{id}_{F_2}$. If $\dim(M) = 2$, we can find an injection $i\colon F_2 \to \pi_M$ and there exists a retraction $p\colon \pi_M \to F_2$, we use this p instead of the given p. If necessary we retake U to be containing α and β in Lemma 4.2.7.

$$
\begin{tikzcd}
EH_b^n(\pi_M) \arrow[r, "E\Gamma_{\nu_U}"] \arrow[d, shift left, "i^*"] & EH^n_{\nu_U}(\mathcal{T}_M) \arrow[dl, "\rho_\epsilon^*"] \\
EH_b^n(F_2) \arrow[u, shift left, "p^*"] &
\end{tikzcd}
$$

Note that $EH^n_{\nu_U}(\mathcal{T}_M) \supset \mathrm{Im}(E\Gamma_{\nu_U} \circ p^*) \cong H_b^n(F_2)/\operatorname{Ker}(E\Gamma_{\nu_U} \circ p^*)$. For $d \in \operatorname{Ker}(E\Gamma_{\nu_U} \circ p^*)$, set $c = p^*(d) \in EH_b^n(\pi_M)$. Since $i^* \circ p^* = \mathrm{id}$, $i^*(c) = i^* \circ p^*(d) = d$.

By Lemma 4.2.7, there exist $\Lambda > 0$ and a family of representation $\{\rho_\epsilon\}$ such that

$$\lim_{\epsilon\to+0} \|\rho_\epsilon^*(E\Gamma_{\nu_U}(c)) - \Lambda i^*(c)\| = 0.$$

Since $E\Gamma_{\nu_U}(c) = E\Gamma_{\nu_U} \circ p^*(d) = 0$, $\|i^*(c)\| = \|d\| = 0$. Hence $\operatorname{Ker}(E\Gamma_{\nu_U} \circ p^*) \subset EN^n(F_2)$. Therefore,

$$\dim_{\mathbb{R}} \left(H_b^n(F_2) \Big/ \operatorname{Ker}(E\Gamma_{\nu_U} \circ i^*) \right) \geq \dim_{\mathbb{R}} \left(EH_b^n(F_2) \Big/ EN^n(F_2) \right) = \dim_{\mathbb{R}} \overline{EH}_b^n(F_2)$$

and we complete the proof for (1).

Next, we prove for case (2). If $\dim(M) = 2$, we can use the argument in the proof of (1). Thus we can assume that $\dim(M) \geq 3$. Let $j\colon F_2 \times K \to \pi_M$ be a hyperbolic embedding. We define $s\colon F_2 \to F_2 \times K$ by $r(x) = (x, \mathrm{id})$ for $x \in F_2$ and $i\colon F_2 \to \pi_M$ by $i = j \circ s$. Since we assumed that $\dim(M) \geq 3$, i is injective. If necessary we retake U to be containing α and β in Lemma 4.2.7.

The induced map $j^*\colon EH_b^n(\pi_M) \to EH_b^n(F_2 \times K)$ is surjective [28]. Since $s^*\colon EH_b^n(F_2 \times K) \to EH_b^n(F_2)$ induces an isomorphism, $i^* = j^* \circ s^*$ is also surjective.

$$\begin{tikzcd}
EH_b^n(\pi_M) \arrow[r, "E\Gamma_{\nu_U}"] \arrow[d, "i^*"] & EH_{\nu_U}^n(\mathcal{T}_M) \arrow[dl, "\rho_\epsilon^*"] \\
EH_b^n(F_2) &
\end{tikzcd}$$

Note that $EH_{\nu_U}^n(\mathcal{T}_M) \supset \operatorname{Im}(E\Gamma_{\nu_U}) \cong H_b^n(\pi_M)/\operatorname{Ker}(E\Gamma_{\nu_U})$. Let $c \in EH_b^n(\pi_M)$. If $E\Gamma_{\nu_U}(c) = 0$, then $\|i^*(c)\| = 0$ by Lemma 4.2.7. Thus $\operatorname{Ker}(E\Gamma_{\nu_U}) \subset \operatorname{Ker}(q \circ i^*)$, where $q\colon EH_b^n(F_2) \to \overline{EH}_b^n(F_2)$ is the quotient map. Therefore,

$$\dim_{\mathbb{R}} \left(H_b^n(\pi_M) \Big/ \operatorname{Ker}(E\Gamma_{\nu_U}) \right) \geq \dim_{\mathbb{R}} \left(EH_b^n(F_2) \Big/ \operatorname{Ker}(q \circ i^*) \right).$$

Since $q \circ i^*$ is surjective, $EH_b^n(F_2)/\operatorname{Ker}(q \circ i^*) \cong \overline{EH}_b^n(F_2)$ and we complete the proof. □

Corollary 4.2.8. *Suppose M and U satisfy the assumption in Theorem 4.2.1. Then $EH_{(d)}^3(\mathcal{T}_M, \nu_U)$ is uncountably infinite-dimensional for $d = 0, 1, 2$.*

Proof. Since the dimension of $\overline{EH}_b^3(F_2)$ is uncountably infinite [54], by Theorem 4.2.1, $EH_{\nu_U}^3(\mathcal{T}_M) = EH_{(0)}^3(\mathcal{T}_M, \nu_U)$ is also uncountably infinite-dimensional. For $d = 1, 2$, There is the natural map $EH_{(d)}^3(\mathcal{T}_M, \nu_U) \to EH_{(0)}^3(\mathcal{T}_M, \nu_U)$ induced by the inclusion $C_{(d)}^3(\mathcal{T}_M, \nu_U) \to C_{(0)}^3(\mathcal{T}_M, \nu_U)$. Since $\Phi_{BM}(c) \in C_{(d)}^3(\mathcal{T}_M, \nu_U)$ for $c \in C_b^3(\pi_M)$ by Proposition 4.2.5, this map surjects onto $\operatorname{Im}(E\Gamma_{\nu_U}) \subset EH_{\nu_U}^3(\mathcal{T}_M)$. We can see that the dimension of $\operatorname{Im}(E\Gamma_{\nu_U})$ is uncountably infinite in the proof of Theorem 4.2.1, thus $EH_{(d)}^3(\mathcal{T}_M, \nu_U)$ is also uncountably infinite-dimensional. □

Chapter 5

Bounded cohomology of area-preserving diffeomorphism groups

5.1 Gambaudo–Ghys' construction

In this section, we define a generalized Gambaudo–Ghys' construction. See [9, 30, 35] for more information about Gambaudo–Ghys' construction.

Let M be a manifold. Let $X_m(M)$ denote the *configuration space* of m points in M, i.e.,

$$X_n(M) = \{(x_1, \ldots, x_m) \in M^m \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Note that $X_m(M)$ is a codimension 0 submanifold of M^m. The fundamental group of $X_m(M)$ is called the *pure braid group on m strands* on M and denoted by $P_m(M)$. Let $\mathfrak{S}_m$ denote the symmetric group of m symbols. We consider the action of $\mathfrak{S}_m$ on $X_m(M)$ by the permutation. The fundamental group of $X_m(M)/\mathfrak{S}_m$ is called the *braid group on m strands* on M and denoted by $B_m(M)$. There exists a short exact sequence

$$1 \to P_m(M) \to B_m(M) \to \mathfrak{S}_m \to 1.$$

If $\dim M \geq 3$, it is known that the inclusion $X_m(M) \to M^m$ induces an isomorphism $P_m(M) \to \pi_1(M^m) \cong \pi_1(M) \times \cdots \times \pi_1(M)$ [5, Theorem 1.5]. Thus we are especially interested in the case of $\dim M = 2$. Note that $B_m(\mathbb{D})$ is the ordinary Artin braid group B_m and $P_m(\mathbb{D})$ is the pure braid group P_m.

Set $\mathcal{G} = \mathrm{Diff}(\mathbb{D}, \partial\mathbb{D}, \mathrm{area})$ and fix a base point $\bar{z} = (z_1, \ldots, z_n) \in X_n(\mathbb{D})$. For simplicity, we assume that $\mathbb{D}$ is equipped with the standard area form (i.e., geodesics are straight lines). For every $g \in \mathcal{G}$ and almost every $\bar{x} = (x_1, \ldots, x_m) \in X_m(\mathbb{D})$, we define a pure braid $\gamma(g, \bar{x}) \in P_n$ as follows. We take an isotopy $\{g^t\}_{0\leq t\leq 1}$ of g such that $g^0 = \mathrm{id}_{\mathbb{D}}$ and $g^1 = g$. We define a loop $l(\{g^t\}, \bar{x})\colon [0,1] \to X_m(\mathbb{D})$ in $X_m(\mathbb{D})$ as follows.

$$l(\{g^t\}, \bar{x})(t) = \begin{cases} \{(1-3t)z_i + 3tx_i\}_{i=1,\ldots,m} & (0 \leq t \leq 1/3) \\ \{g^{3t-1}(x_i)\}_{i=1,\ldots,m} & (1/3 \leq t \leq 2/3) \\ \{(3-3t)g(x_i) + (3t-2)z_i\}_{i=1,\ldots,m} & (2/3 \leq t \leq 1) \end{cases}$$

We define $\gamma(g, \bar{x})$ as the element of $\pi_1(X_m(\mathbb{D}), \bar{z})$ represented by the loop $l(\{g^t\}, \bar{x})$. The above definition of $\gamma(g, \bar{x})$ does not depend on the choice of an isotopy $\{g^t\}_{0\leq t\leq 1}$ since $\mathcal{G}$ is contractible. If there exist i and j $(1 \leq i < j \leq m)$ such that

$$(1-3s)z_i + 3sx_i = (1-3s)z_j + 3sx_j$$

for some $s \in [0, 1/3]$ or

$$(3-3s)g(x_i) + (3s-2)z_i = (3-3s)g(x_j) + (3s-2)z_j$$

for some $s \in [2/3, 1]$, then $\gamma(g, \bar{x})$ is not defined. Although, for any $g \in \mathcal{G}$, such points $\bar{x} \in X_m(\mathbb{D})$ consist a measure zero subset in $X_m(\mathbb{D})$. Here, $X_m(\mathbb{D})$ is equipped with the volume form induced by $\mathbb{D}^m$.

For $c \in C_b^n(P_m)$, we define a map $\widehat{\Gamma}_b(c)\colon \mathcal{G}^{n+1} \to \mathbb{R}$ by

$$\widehat{\Gamma}_b(c)(g_0, \ldots, g_n) = \int_{\bar{x}\in X_m(\mathbb{D})} c(\gamma(g_0, \bar{x}), \ldots, \gamma(g_n, \bar{x}))d\bar{x} \tag{5.1.1}$$

for $g_0, \ldots, g_n \in \mathcal{G}$. Since c is bounded and the map $\bar{x} \mapsto c(\gamma(g_0, \bar{x}), \ldots, \gamma(g_n, \bar{x}))$ is defined on a full measure subset in $X_m(\mathbb{D})$, a map $\widehat{\Gamma}_b(c)$ is well-defined.

Lemma 5.1.1. *For every $c \in C_b^n(P_m)$, $\widehat{\Gamma}_b(c)$ is a bounded homogeneous cochain. Moreover, the map $\widehat{\Gamma}_b\colon C_b^n(P_m) \to C_b^n(\mathcal{G})$ is a cochain map.*

Proof. Since

$$|\widehat{\Gamma}_b(c)(g_0, \ldots, g_n)| \leq \mathrm{vol}(X_m(\mathbb{D})) \cdot \|c\|_\infty,$$

for every $g_0, \ldots, g_n \in \mathcal{G}$, $\widehat{\Gamma}_b(c)$ is bounded. Since $\gamma(gh, \bar{x}) = \gamma(g, h \cdot \bar{x})\gamma(h, \bar{x})$ for $g, h \in \mathcal{G}$ (where $\mathcal{G}$ acts on $X_m(\mathbb{D})$ by the diagonal action),

$$\begin{aligned}\widehat{\Gamma}_b(c)(g_0h, \ldots, g_nh) &= \int_{\bar{x} \in X_m(\mathbb{D})} c(\gamma(g_0h, \bar{x}), \ldots, \gamma(g_nh, \bar{x}))d\bar{x} \\ &= \int_{\bar{x} \in X_m(\mathbb{D})} c(\gamma(g_0, h \cdot \bar{x})\gamma(h, \bar{x}), \ldots, \gamma(g_n, h \cdot \bar{x})\gamma(h, \bar{x}))d\bar{x} \\ &= \int_{\bar{x} \in X_m(\mathbb{D})} c(\gamma(g_0, h \cdot \bar{x}), \ldots, \gamma(g_n, h \cdot \bar{x}))d\bar{x}.\end{aligned}$$

Since the action by h preserves the volume form, $\widehat{\Gamma}(c)(g_0h, \ldots, g_nh) = \widehat{\Gamma}(c)(g_0, \ldots, g_n)$ and hence $\widehat{\Gamma}(c)$ is homogeneous. By definition, the map $\widehat{\Gamma}$ and the coboundary map δ are commutative. Thus $\widehat{\Gamma}$ is a cochain map. □

By Lemma 5.1.1, the map $\widehat{\Gamma}_b \colon C_b^n(P_m) \to C_b^n(\mathcal{G})$ induces the homomorphism

$$\Gamma_b \colon H_b^n(P_m) \to H_b^n(\mathcal{G}).$$

We also define a map $\widehat{\Gamma} \colon C^n(P_m) \to C^n(\mathcal{G})$ on the ordinary cochain complex by the equation (5.1.1). In this case, the well-definedness of the map $\widehat{\Gamma}(c) \colon \mathcal{G}^{n+1} \to \mathbb{R}$ is not trivial since $c \in C^n(P_m)$ is not necessarily bounded.

Lemma 5.1.2. *For $c \in C^n(P_m)$, the map $\widehat{\Gamma}(c) \colon \mathcal{G}^{n+1} \to \mathbb{R}$ is well-defined.*

Proof. Fix $g \in \mathcal{G}$ and an isotopy $\{g^t\}_{0 \leq t \leq 1}$ of g. Let $g_\Delta \in \mathrm{Diff}(\mathbb{D}^m)$ denote the diffeomorphism on $\mathbb{D}^m$ induced by the diagonal action of g on $\mathbb{D}^m$. The length $L(l)$ of the loop $l = l(\{g^t\}, \bar{x})$ is represented as

$$L(l) = d(\bar{x}, \bar{z}) + d(g \cdot \bar{x}, \bar{z}) + \int_0^1 \|(\mathfrak{X}^t)_{g_\Delta^t(\bar{x})}\| dt,$$

where d is the metric on $\mathbb{D}^m$ and $\mathfrak{X}^t$ denotes the time-depended vector field on $\mathbb{D}^m$ generated by the isotopy $\{g_\Delta^t\}_{0 \leq t \leq 1}$ of g_Δ. The continuous map $\mathbb{D}^m \times [0,1] \to \mathbb{R}$ defined by $(\bar{x}, t) \mapsto \|(\mathfrak{X}^t)_{\bar{x}}\|$ has a maximum value M since $\mathbb{D}^m \times [0,1]$ is compact. Thus we obtain

$$L(l) \leq 2\,\mathrm{diam}(\mathbb{D}^m) + M \tag{5.1.2}$$

and hence $L(l)$ has a uniform upper bound for a fixed isotopy $\{g^t\}$ (i.e., the function $\bar{x} \mapsto L(l(\{g^t\}, \bar{x}))$ is bounded).

Set $X = X_m(\mathbb{D})$ and $R = \mathrm{diam}(X)$ $(= \mathrm{diam}(\mathbb{D}^m))$. Let $\widetilde{X}$ be the universal cover of X. The group $P_m \cong \pi_1(X, \bar{z})$ acts on $\widetilde{X}$ by the deck transformation. Since the covering map $\widetilde{X} \to X$ is an immersion, $\widetilde{X}$ inherits a Riemannian metric $\tilde{d}$. The metric space $(\widetilde{X}, \tilde{d})$ is a $(1, \epsilon)$-quasi-geodesic space for any $\epsilon > 0$. Let $\tilde{z} \in \widetilde{X}$ be a base point and set

$$B = \{\tilde{x} \in \widetilde{X} \mid \tilde{d}(\tilde{x}, \tilde{z}) \leq R\}.$$

By definition, B has a finite diameter and $\widetilde{X} = \bigcup_{\gamma \in P_m} \gamma \cdot B$. Since the action of P_m on $\widetilde{X}$ is discrete, the set $\{\gamma \in P_m \mid \gamma \cdot B' \cap B'\}$ is finite, where B' is the 2ϵ-neighborhood of B. Hence, by Lemma 2.1.2, the space $(\widetilde{X}, \tilde{d})$ and the group P_m with the word length (with respect to a finite generating set S) are quasi-isometric. Thus there exist constants $K \geq 1$ and $C \geq 0$ such that

$$\|\gamma(g, \bar{x})\|_S \leq K \cdot L(l(\{g^t\}, \bar{x})) + C, \tag{5.1.3}$$

where $\|\cdot\|_S$ denotes the word length with respect to S. By (5.1.2) and (5.1.3), the function $\bar{x} \mapsto \|\gamma(g, \bar{x})\|_S$ is bounded. This means that there are a finite number of possible patterns of elements that $\gamma(g, \bar{x})$ can take, i.e., the map $\gamma(g, \cdot)\colon X_m(\mathbb{D}) \to P_m$ has a finite image. Therefore, the map $c(\ldots, \gamma(g_i, \cdot), \ldots)\colon X_m(\mathbb{D}) \to \mathbb{R}$ is integrable and the map $\widehat{\Gamma}(c)$ is well-defined. $\square$

The map $\widehat{\Gamma}\colon C^n(P_m) \to C^n(\mathcal{G})$ induce the map $\Gamma\colon H^n(P_m) \to H^n(\mathcal{G})$. The maps $E\Gamma_b\colon EH_b^n(P_m) \to EH_b^n(\mathcal{G})$ and $\overline{E\Gamma}_b\colon \overline{EH}_b^n(P_m) \to \overline{EH}_b^n(\mathcal{G})$ are also induced.

Remark 5.1.3. Let $\mathcal{H}_M$ denote the identity component of the group of measure-preserving homeomorphisms $\mathrm{Homeo}_0(M, \mu)$ on a complete Riemannian manifold M with the measure μ induced by the Riemannian metric. In [9], Brandenbursky and Marcinkowski also considered maps $\Gamma_b\colon H_b^n(\pi_1 M) \to H_b^n(\mathcal{H}_M)$ and $\Gamma\colon H^n(\pi_1 M) \to H^n(\mathcal{H}_M)$ and proved that $\overline{EH}_b^3(\mathcal{H}_M)$ is infinite-dimensional if $\pi_1(M)$ is complicated enough. In our setting, we cannot prove the well-definedness of $\Gamma\colon H^n(P_m) \to H^n(\mathcal{H}_{\mathbb{D}})$ as Lemma 5.1.2. However, we can define the map $\Gamma_b\colon H_b^n(P_m) \to H_b^n(\mathcal{H}_{\mathbb{D}})$ and prove that $\overline{H}_b^3(\mathcal{H}_{\mathbb{D}})$ is infinite-dimensional, in the same way as in Corollary 5.2.3.

5.2 Generalized Ishida's theorem

In this section, we prove the following theorem which is a generalization of the result of Ishida [35].

Theorem 5.2.1. *The composition map* $\overline{E\Gamma}_b \circ i^* \colon \overline{EH}_b^n(B_3) \to \overline{EH}_b^n(\mathcal{G})$ *is injective. Equivalently, the restriction map* $\overline{E\Gamma}_b \colon \overline{EH}_b^n(P_3)^{B_3} \to \overline{EH}_b^n(\mathcal{G})$ *is injective.*

Here $\overline{EH}_b^n(G)$ denotes the reduced exact bounded cohomology of G and $\overline{EH}_b^n(P_3)^{B_3}$ denotes the subspace of $\overline{EH}_b^n(P_3)$ which is invariant under the conjugation of B_3. To prove this theorem, we use the following key lemma.

Lemma 5.2.2. *There exist a constant* $\Lambda > 0$ *and a family of homomorphisms* $\{\rho_\epsilon \colon P_3 \to \mathcal{G}\}_{0<\epsilon<1}$ *such that*

$$\lim_{\epsilon \to +0} \|\rho_\epsilon^*(\overline{E\Gamma}_b \circ i^*(u)) - \Lambda \cdot i^*(u)\| = 0$$

for any $u \in \overline{EH}_b^n(B_3)$.

Before we prove Lemma 5.2.2, we give the proof of Theorem 5.2.1 from Lemma 5.2.2.

Proof of Theorem 5.2.1. By Lemma 2.2.1, the inclusion $i \colon P_3 \to B_3$ induces an isomorphism $i^* \colon \overline{EH}_b^n(B_3) \to \overline{EH}_b^n(P_3)^{B_3}$. In particular, $i^* \colon \overline{EH}_b^n(B_3) \to \overline{EH}_b^n(P_3)$ is injective.

Let $u \in \overline{EH}_b^n(B_3)$ be a non-trivial class. It means that $\|u\| > 0$ and $\|i^*(u)\| > 0$ since i^* is injective. By Lemma 5.2.2, $\|\overline{E\Gamma}_b \circ i^*(u)\| > 0$ and it means that $\overline{E\Gamma}_b \circ i^*$ is injective. This argument also implies that the restriction map $\Gamma_b \colon \overline{EH}_b^n(P_3)^{B_3} \to \overline{EH}_b^n(\mathcal{G})$ is also injective. □

As a corollary of Theorem 5.2.1, we obtain the following result.

Corollary 5.2.3. *The dimension of* $\overline{EH}_b^3(\mathcal{G})$ *is uncountably infinite.*

Proof. By Theorem 2.2.3, the dimension of $\overline{EH}_b^3(B_3/Z(B_3))$ is uncountably infinite since $B_3/Z(B_3) \cong PSL(2,\mathbb{Z})$ is non-elementary hyperbolic. The quotient map $B_3 \to B_3/Z(B_3)$ induces isomorphism $H_b^n(B_3) \to H_b^n(B_3/Z(B_3))$ by Theorem 2.2.2. Since $H^3(B_3) = 0$ and $H^3(PSL(2,\mathbb{Z})) = 0$, $EH_b^3(B_3)$ and $EH_b^3(B_3/Z(B_3))$ are also isomorphic. Therefore, by Theorem 5.2.1, $\overline{EH}_b^3(\mathrm{Diff}(\mathbb{D}, \mathrm{area}))$ is also uncountably infinite-dimensional. □

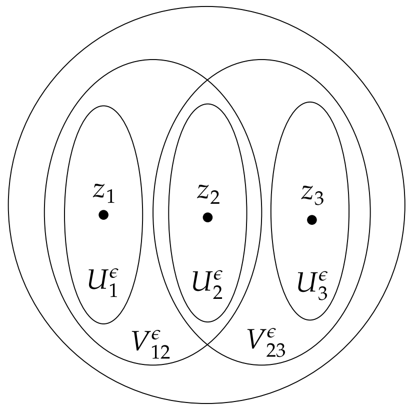

Figure 5.1: Open subsets in $\mathbb{D}$

Now we prove the key lemma. The strategy of the proof comes from [9] and the method is inspired by [35].

Proof of Lemma 5.2.2. Recall that $\bar{z} = (z_1, z_2, z_3)$ denotes the base point of $X_3(\mathbb{D})$. For simplicity, we assume that $\mathrm{area}(\mathbb{D}) = 1$. For each ϵ, we take open subsets U_i^ϵ $(i = 1, 2, 3)$ in $\mathbb{D}$ such that

- $z_i \in U_i^\epsilon$,
- $U_i^\epsilon \cap U_j^\epsilon = \emptyset$ if $i \neq j$, and
- $\mathrm{area}(U^\epsilon) = 1 - \epsilon$, where $U^\epsilon = U_1^\epsilon \cup U_2^\epsilon \cup U_3^\epsilon$.

Moreover, we take open subsets W_{12}^ϵ and V_{12}^ϵ of $\mathbb{D}$ which are diffeomorphic to a disk such that

- $U_1^\epsilon \cup U_2^\epsilon \subset W_{12}^\epsilon \subset V_{12}^\epsilon$ and
- $V_{12}^\epsilon \cap U_3^\epsilon = \emptyset$.

We also take W_{23}^ϵ and V_{23}^ϵ similarly (see Figure 5.1). Finally, we take open disks W_{123}^ϵ and V_{123}^ϵ to be $V_{12}^\epsilon \cup V_{23}^\epsilon \subset W_{123}^\epsilon \subset V_{123}^\epsilon$.

We define $\rho_\epsilon \colon P_3 \to \mathcal{G}$ as follows. Set $a_1 = {\sigma_1}^2$, $a_2 = {\sigma_2}^2$ and $a_3 = \Delta^2$. Then P_3 has a presentation

$$P_3 = \langle a_1, a_2, a_3 \mid a_1 a_3 = a_3 a_1, a_2 a_2 = a_3 a_2 \rangle \cong F_2 \times \mathbb{Z}.$$

For open disks V and W such that $W \subset V$, let $g_{V,W} \in \mathcal{G}$ denote a diffeomorphism which rotates W once such that $\mathrm{supp}(g_{V,W}) \subset V$. We define $\rho_\epsilon \colon P_3 \to \mathcal{G}$ by $\rho_\epsilon(a_1) = g_{V_{12}^\epsilon, W_{12}^\epsilon}$, $\rho_\epsilon(a_2) = g_{V_{23}^\epsilon, W_{23}^\epsilon}$ and $\rho_\epsilon(a_3) = g_{V_{123}^\epsilon, W_{123}^\epsilon}$. Note that $\rho_\epsilon(a_3)|_{W_{123}^\epsilon} = \mathrm{id}_{W_{123}^\epsilon}$. Since $\mathrm{supp}(\rho_\epsilon(a_1)) \subset V_{12}^\epsilon \subset W_{123}^\epsilon$, $\rho_\epsilon(a_1)$ and $\rho_\epsilon(a_3)$ are commutative. Similarly, $\rho_\epsilon(a_2)$ and $\rho_\epsilon(a_3)$ are also commutative. Thus ρ_ϵ is well-defined.

For $u = [c] \in \overline{EH}_b^n(B_3)$, $\rho_\epsilon^*(\overline{E\Gamma}_b \circ i^*(u)) \in \overline{EH}_b^n(P_3)$ is the cohomology class of a cochain defined by

$$(\alpha_0, \ldots, \alpha_n) \mapsto \int_{\bar{x} \in X_3(\mathbb{D})} c(\gamma(\rho_\epsilon(\alpha_0), \bar{x}), \ldots, \gamma(\rho_\epsilon(\alpha_n), \bar{x})) d\bar{x}$$

for $\alpha_0, \ldots, \alpha_n \in P_3$.

We calculate $\gamma(\rho_\epsilon(\alpha), \bar{x}) \in P_3$ for $\alpha \in P_3$ and $\bar{x} = (x_1, x_2, x_3) \in X_3(\mathbb{D})$. To describe it, we prepare several notions. We call that $x \in X_3(\mathbb{D})$ is an *ϵ-good point* if all of x_1, x_2 and x_3 are in U^ϵ. Otherwise, we call that $\bar{x}$ is an *ϵ-bad point*. We say that an ϵ-good point $\bar{x}$ is *of type* (p, q, r) if U_1^ϵ has p points, U_2^ϵ has q points and U_3^ϵ has r points out of x_1, x_2 and x_3. For example, if $x_1, x_2 \in U_1^\epsilon$ and $x_3 \in U_3^\epsilon$, then $\bar{x}$ is of type $(2, 0, 1)$.

We define homomorphisms $s_i \colon P_3 \to \mathbb{Z}$ $(i = 1, 2, 3)$ by $s_i(a_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$, where δ_{ij} is the Kronecker delta. For each type (p, q, r), we define a homomorphism $\phi_{pqr} \colon P_3 \to P_3$ by

$$\phi_{pqr}(\alpha) = \begin{cases} \alpha & \text{type } (1,1,1), \\ (\Delta^2)^{s_1(\alpha)+s_3(\alpha)} & \text{type } (3,0,0) \text{ or } (2,1,0), \\ (\Delta^2)^{s_2(\alpha)+s_3(\alpha)} & \text{type } (0,0,3) \text{ or } (0,1,2), \\ (\Delta^2)^{s_1(\alpha)+s_2(\alpha)+s_3(\alpha)} & \text{type } (0,3,0), \\ (\sigma^2)^{s_1(\alpha)}(\Delta^2)^{s_3(\alpha)} & \text{type } (2,0,1), \\ (\sigma^2)^{s_2(\alpha)}(\Delta^2)^{s_3(\alpha)} & \text{type } (1,0,2), \\ (\sigma^2)^{s_1(\alpha)}(\Delta^2)^{s_2(\alpha)+s_3(\alpha)} & \text{type } (0,2,1), \\ (\sigma^2)^{s_2(\alpha)}(\Delta^2)^{s_1(\alpha)+s_3(\alpha)} & \text{type } (1,2,0), \end{cases} \tag{5.2.1}$$

where σ denotes σ_1 or σ_2 and Δ^2 denotes the full twist.

Our main observation is the following. For any ϵ-good point $\bar{x} \in X_m(\mathbb{D})$ of type (p, q, r), there exists a braid $\beta(\bar{x}) \in B_3$ such that

$$\gamma(\rho_\epsilon(\alpha, \bar{x})) = \beta(\bar{x})\phi_{pqr}(\alpha)\beta(\bar{x})^{-1}$$

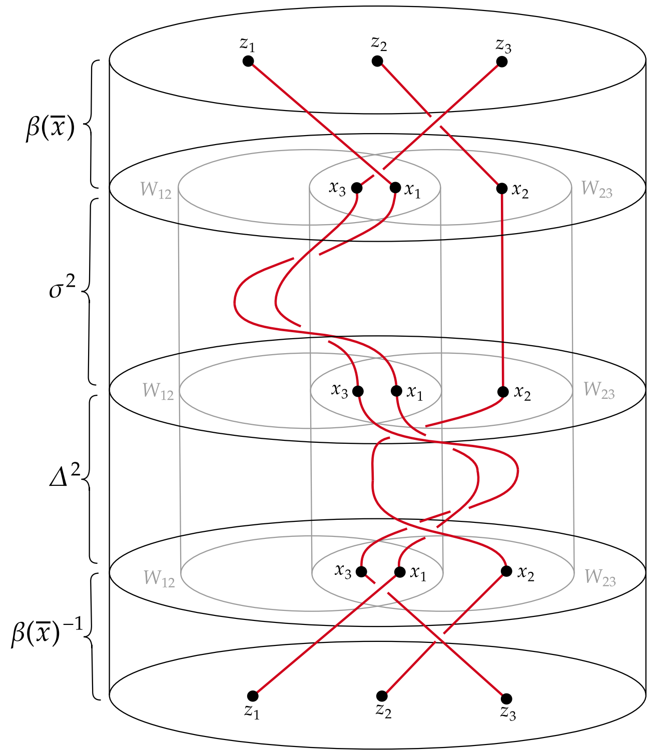

Figure 5.2: A braid $\gamma(\rho_\epsilon(a_1a_2), \bar{x})$ when $\bar{x}$ is of type $(0, 2, 1)$

for every $\alpha \in P_3$. We can see this as follows. Let $\bar{x}$ be of type (p, q, r). If $p + q \leq 1$, $\gamma(\rho_\epsilon(a_1), \bar{x})$ is trivial. If $p + q = 2$, $\gamma(\rho_\epsilon(a_1), \bar{x})$ is a conjugate of σ^2. If $p + q = 3$, $\gamma(\rho_\epsilon(a_1), \bar{x})$ is a conjugate of Δ^2. We can apply the same argument for $\gamma(\rho_\epsilon(a_2), \bar{x})$ by changing $p+q$ to $q+r$. For any type, $\gamma(\rho_\epsilon(a_3), \bar{x})$ is a conjugate of Δ^2. By noting that Δ^2 commutes with any braid, we obtain (5.3.1). See also Figure 5.2.

Let X^ϵ_{pqr} denote the set of ϵ-good points in $X_3(\mathbb{D})$ of type (p, q, r) and Y^ϵ denote the set of ϵ-bad points. We define cochains $c^\epsilon_{pqr}, c^\epsilon_Y \in C^n_b(P_3)$ by

$$c^\epsilon_{pqr}(\alpha_0, \ldots, \alpha_n) = \int_{\bar{x} \in X^\epsilon_{pqr}} c(\gamma(\rho_\epsilon(\alpha_0), \bar{x}), \ldots, \gamma(\rho_\epsilon(\alpha_n), \bar{x}))d\bar{x},$$

$$c^\epsilon_Y(\alpha_0, \ldots, \alpha_n) = \int_{\bar{x} \in Y^\epsilon} c(\gamma(\rho_\epsilon(\alpha_0), \bar{x}), \ldots, \gamma(\rho_\epsilon(\alpha_n), \bar{x}))d\bar{x}$$

for $\alpha_0, \ldots, \alpha_n \in P_3$. Note that

$$\rho_\epsilon^*(\overline{E\Gamma}_b \circ i^*(u)) = \sum_{p,q,r} [c_{pqr}^\epsilon] + [c_Y^\epsilon] \in \overline{EH}_b^n(P_3).$$

For $c \in C^n(B_3)$ and $\beta \in B_3$, let $\beta \cdot c \in C^n(B_3)$ denote the cochain defined by

$$(\beta \cdot c)(\gamma_0, \ldots, \gamma_n) = c(\beta\gamma_0\beta^{-1}, \ldots, \beta\gamma_n\beta^{-1}).$$

for $\gamma_0, \ldots, \gamma_n \in B_3$. For any type (p, q, r),

$$\begin{aligned} c_{pqr}^\epsilon(\alpha_0, \ldots, \alpha_n) &= \int_{X_{pqr}^\epsilon} c(\beta(\bar{x})\phi_{pqr}(\alpha_0)\beta(\bar{x})^{-1}, \ldots, \beta(\bar{x})\phi_{pqr}(\alpha_n)\beta(\bar{x})^{-1}) d\bar{x} \\ &= \sum_{\beta \in B_3} \mathrm{vol}\left(\{\bar{x} \in X_3(\mathbb{D}) \mid \beta(\bar{x}) = \beta\}\right)(\beta \cdot c)(\phi_{pqr}(\alpha_0), \ldots, \phi_{pqr}(\alpha_n)) \end{aligned}$$

for $\alpha_0, \ldots, \alpha_n \in P_3$. Since $[\beta \cdot c] = [c] = u$ for any $\beta \in B_3$,

$$[c_{pqr}^\epsilon] = \mathrm{vol}(X_{pqr}^\epsilon) \cdot \phi_{pqr}^*(i^*(u)). \tag{5.2.2}$$

If $(p, q, r) = (1, 1, 1)$, since $\phi_{111} = \mathrm{id}$ and by (5.2.2),

$$[c_{111}^\epsilon] = \mathrm{vol}(X_{111}^\epsilon) \cdot i^*(u) = 3! \cdot \mathrm{area}(U_1^\epsilon)\,\mathrm{area}(U_2^\epsilon)\,\mathrm{area}(U_3^\epsilon) \cdot i^*(u).$$

If $(p, q, r) \neq (1, 1, 1)$, by (5.3.1), the homomorphism ϕ_{pqr} factors through the abelian subgroup $\langle \sigma^2, \Delta^2 \rangle \cong \mathbb{Z}^2$ of P_3. Since $\mathbb{Z}^2$ is amenable, $\overline{EH}_b^n(\mathbb{Z}^2) = 0$. Thus $\phi_{pqr}^* = 0$ and hence $[c_{pqr}^\epsilon] = 0$ by (5.2.2).

By the definition of c_Y^ϵ,

$$|c_Y^\epsilon(\alpha_0, \ldots, \alpha_n)| \leq \mathrm{vol}(Y^\epsilon)\|c\|_\infty.$$

Since $\mathrm{vol}(Y^\epsilon) = \mathrm{vol}(X_3(\mathbb{D})) - \mathrm{vol}(U^\epsilon \times U^\epsilon \times U^\epsilon) = 1 - (1-\epsilon)^3$, $\lim_{\epsilon \to +0} \|[c_Y^\epsilon]\| = 0$.

Therefore, by setting $\Lambda = \lim_{\epsilon \to +0} 3! \cdot \mathrm{area}(U_1^\epsilon)\,\mathrm{area}(U_2^\epsilon)\,\mathrm{area}(U_3^\epsilon)$,

$$\lim_{\epsilon \to +0} \|\rho_\epsilon^*(\overline{E\Gamma}_b \circ i^*(u)) - \Lambda \cdot i^*(u)\| = 0. \qquad \square$$

5.3 The case of other surfaces

In this section, we apply the argument in the previous section to the other surface cases.

Let Σ be a compact surface with an area form. For simplicity, we assume that $\mathrm{area}(\Sigma) = 1$. We set $\mathcal{G}_\Sigma = \mathrm{Diff}_0(\Sigma, \partial\Sigma, \mathrm{area})$ and fix a base point $\bar{z} \in X_m(\Sigma)$. For an isotopy $\{g^t\}_{0\le t\le 1}$ of $g \in \mathcal{G}_\Sigma$ and $\bar{x} \in X_m(\Sigma)$, we can define the loop $l(\{g^t\}, \bar{x})\colon [0,1] \to X_m(\Sigma)$ as in the case of the disk but we should use geodesics in Σ instead of straight lines in $\mathbb{D}$. Since the measure of the cut locus of $\bar{z}$ is zero, the loop $l(\{g^t\}, \bar{x})$ is defined for almost every $\bar{x} \in X_m(\Sigma)$. Let $\gamma(\{g^t\}, \bar{x})$ denote an element of $\pi_1(X_m(\Sigma), \bar{z}) \cong P_m(\Sigma)$ represented by the loop $l(\{g^t\}, \bar{x})$. In general, $\gamma(\{g^t\}, \bar{x})$ depends on the choice of an isotopy $\{g^t\}$. However, $\gamma(\{g^t\}, \bar{x})$ is determined up to center since the image of the map $e_{\bar{z}}^*\colon \pi_1(\mathcal{G}_\Sigma, \mathrm{id}_\Sigma) \to \pi_1(X_m(\Sigma), \bar{z})$ induced by the evaluation map $e_{\bar{z}}\colon \mathcal{G} \to X_m(\Sigma), g \mapsto g \cdot \bar{z}$ is contained in the center $Z(P_m(\Sigma))$. Thus it defines an element of $P_m(\Sigma)^Z$ and we write this element as $\gamma(g, \bar{x})$. Recall that G^Z denotes the central quotient $G/Z(G)$. In this way, we can define maps $\widehat{\Gamma}_b^Z\colon C_b^n(P_m(\Sigma)^Z) \to C_b^n(\mathcal{G}_\Sigma)$ and $\widehat{\Gamma}^Z\colon C^n(P_m(\Sigma)^Z) \to C^n(\mathcal{G}_\Sigma)$ as in the case of the disk since the arguments in Lemma 5.1.1 and 5.1.2 also go well for Σ instead of $\mathbb{D}$. Hence they induce the map $\overline{E\Gamma}_b^Z : \overline{EH}_b^n(P_m(\Sigma)^Z) \to \overline{EH}_b^n(\mathcal{G}_\Sigma)$.

In this setting, we can prove the following injectivity theorem.

Theorem 5.3.1. *Let Σ be a compact oriented surface such that $\chi(\Sigma) \ge 0$. The maps $\overline{E\Gamma}_b^Z \circ (i^Z)^*\colon \overline{EH}_b^n(B_m(\Sigma)^Z) \to \overline{EH}_b^n(\mathcal{G}_\Sigma)$ is injective for $m = 2 + \chi(\Sigma)$.*

For the sphere case, Ishida [35] proved a similar result of Theorem 5.3.1 for $n = 2$ not only for four strands but also for m strands ($m \ge 4$). For the torus case, Brandenbursky, Kędra, and Shelukhin [8] proved Theorem 5.3.1 for $n = 2$. As in the case of the disk, we obtain the following.

Corollary 5.3.2. *Let Σ be a compact oriented surface such that $\chi(\Sigma) \ge 0$. The dimension of $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ is uncountably infinite.*

By combining Corollary 5.3.2 and the result of Brandenbursky–Marcinkowski [9] (Theorem 1.1.3), we obtain the following.

Theorem 5.3.3. *For any compact oriented surface Σ, the dimension of $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ is uncountably infinite.*

Proof. If $\chi(\Sigma) \geq 0$, by Corollary 5.3.2, $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ is infinite-dimensional. If $\chi(\Sigma) < 0$, $\pi_1(\Sigma)$ is a non-elementary hyperbolic group. Therefore, by Theorem 1.1.3, $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ is infinite-dimensional. □

5.3.1 For a disk

We prove the central quotient version of Lemma 5.2.2. We remark that $P_3^Z = \langle a_1, a_2 \rangle \cong F_2$. We define $s_i \colon P_3^Z \to \mathbb{Z}$ $(i = 1, 2)$ by $s_i(a_j) = \delta_{ij}$.

Lemma 5.3.4. *There exist a constant $\Lambda > 0$ and a family of homomorphisms $\{\rho_\epsilon \colon P_3^Z \to \mathcal{G}_{\mathbb{D}}\}_{0<\epsilon<1}$ such that*

$$\lim_{\epsilon \to +0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^Z \circ (i^Z)^*(u)) - \Lambda \cdot (i^Z)^*(u)\| = 0$$

for any $u \in \overline{EH}_b^n(B_3^Z)$.

Proof. We define open subsets $U_\bullet^\epsilon$, $V_\bullet^\epsilon$ and $W_\bullet^\epsilon$ as in Lemma 5.2.2. We define $\rho_\epsilon \colon P_3^Z \to \mathcal{G}_{\mathbb{D}}$ by $\rho_\epsilon(a_1) = g_{V_{12}, W_{12}}$ and $\rho_\epsilon(a_2) = g_{V_{23}, W_{23}}$. We define $s_i \colon P_3^Z \to \mathbb{Z}$ $(i = 1, 2)$ by $s_1({\sigma_1}^2) = 1$, $s_1({\sigma_2}^2) = 0$, $s_2({\sigma_1}^2) = 0$, and $s_2({\sigma_1}^2) = 1$.

For any type (p, q, r), we define $\phi_{pqr} \colon P_3^Z \to P_3^Z$ by

$$\phi_{pqr}(\alpha) = \begin{cases} \alpha & \text{type } (1,1,1), \\ (\sigma^2)^{s_1(\alpha)} & \text{type } (2,0,1) \text{ or type } (0,2,1), \\ (\sigma^2)^{s_2(\alpha)} & \text{type } (1,0,2) \text{ or type } (1,2,0), \\ e & \text{otherwise} \end{cases} \tag{5.3.1}$$

for $\alpha \in P_3^Z$, where σ denotes σ_1 or σ_2. The rest part of the proof the is the same with the proof of Lemma 5.2.2. □

5.3.2 For a sphere

Let $\mathbb{S}$ denote the 2-sphere. We summarize some facts on the braid group on $\mathbb{S}$ we use later. See [5, 22, 30, 44] for more details.

The inclusion $\mathbb{D} \to \mathbb{S}$ induces the projection $B_m \to B_m(\mathbb{S})$ and let δ_i denote the image of σ_i by this projection. It is known that the kernel of this projection is normally generated by $\sigma_1\sigma_2 \cdots \sigma_{m-2}\sigma_{m-1}^2\sigma_{m-2} \cdots \sigma_2\sigma_1$. The natural map $X_{m-1}(\mathbb{D}) \to X_m(\mathbb{S})$ induces the map $P_{m-1} \to P_m(\mathbb{S})$ and it is known to be surjective. The center $Z(P_m(\mathbb{S}))$ of $P_m(\mathbb{S})$ is generated by the full twist $\xi^2 = (\delta_1\delta_2\cdots\delta_{m-1})^m$ and ξ^2 has order two.

We consider in particular the case $m = 4$. Then $P_4(\mathbb{S}) \cong F_2 \times \mathbb{Z}/2\mathbb{Z}$. In particular, $P_4(\mathbb{S})^Z$ is isomorphic to a free group of rank 2 and generated by ${\delta_1}^2$ and ${\delta_2}^2$. Note that full twists of three strands are also in the center, i.e., $(\delta_1\delta_2)^3 = (\delta_2\delta_3)^3 = \xi^2 \in Z(P_4(\mathbb{S}))$.

Lemma 5.3.5. *There exist a constant $\Lambda > 0$ and a family of homomorphisms $\{\rho_\epsilon \colon P_4(\mathbb{S})^Z \to \mathcal{G}_\mathbb{S}\}_{0<\epsilon<1}$ such that*

$$\lim_{\epsilon\to+0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^Z \circ (i^Z)^*(u)) - \Lambda \cdot (i^Z)^*(u)\| = 0$$

for any $u \in \overline{EH}_b^n(B_4(\mathbb{S})^Z)$.

Proof. For each ϵ, we take open subsets U_i^ϵ $(i = 1, 2, 3, 4)$ in $\mathbb{S}$ so that

- $z_i \in U_i^\epsilon$,
- $U_i^\epsilon \cap U_j^\epsilon = \emptyset$ if $i \neq j$, and
- $\mathrm{area}(U^\epsilon) = 1 - \epsilon$, where $U^\epsilon = U_1^\epsilon \cup U_2^\epsilon \cup U_3^\epsilon \cup U_4^\epsilon$.

Moreover, we take open subsets W_{12}^ϵ and V_{12}^ϵ of $\mathbb{S}$ which are diffeomorphic to a disk so that

- $U_1^\epsilon \cup U_2^\epsilon \subset W_{12}^\epsilon \subset V_{12}^\epsilon$,
- $V_{12}^\epsilon \cap U_3^\epsilon = \emptyset$, and
- $V_{12}^\epsilon \cap U_4^\epsilon = \emptyset$.

We also take W_{23}^ϵ and V_{23}^ϵ similarly (see Figure 5.3). We define $\rho_\epsilon \colon P_4(\mathbb{S})^Z \to \mathcal{G}$ as in the case of the disk for generators ${\delta_1}^2$ and ${\delta_2}^2$ of $P_4(\mathbb{S})^Z$. We define $s_1, s_2 \colon P_4(\mathbb{S})^Z \to \mathbb{Z}$ by $s_1({\delta_1}^2) = 1$, $s_1({\delta_2}^2) = 0$, $s_2({\delta_1}^2) = 0$ and $s_2({\delta_2}^2) = 1$.

We calculate $\gamma(\rho_\epsilon(\alpha), \bar{x}) \in P_4(\mathbb{S})^Z$ for $\alpha \in P_4(\mathbb{S})^Z$ and $\bar{x} \in X_4(\mathbb{S})$. We call that $\bar{x} = (x_1, x_2, x_3, x_4) \in X_4(\mathbb{S})$ is an *ϵ-good point* if all of x_1, x_2, x_3 and x_4 are in U^ϵ. We say that an ϵ-good point $\bar{x}$ is *of type* (p, q, r, s) if U_1^ϵ has p points, U_2^ϵ has q points, U_3^ϵ has r points and U_4^ϵ has s points out of x_1, x_2, x_3, and x_4.

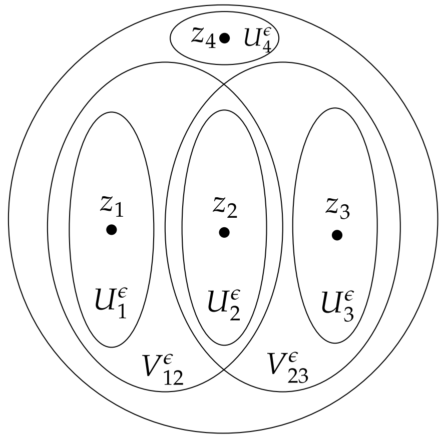

Figure 5.3: Open subsets in $\mathbb{S}$

Let X^ϵ_{pqrs} denotes the set of ϵ-good points $\bar{x}$ is of type (p, q, r, s). We define a cochain $c^\epsilon_{pqrs} \in C^n_b(P_4(\mathbb{S})^Z)$ by

$$c^\epsilon_{pqrs}(\alpha_0, \ldots, \alpha_n) = \int_{X^\epsilon_{pqrs}} c(\gamma(\rho_\epsilon(\alpha_0), \bar{x}), \ldots, \gamma(\rho_\epsilon(\alpha_n), \bar{x}))d\bar{x}$$

for $\alpha_0, \ldots, \alpha_n \in P_4(\mathbb{S})^Z$. In order for $[c^\epsilon_{pqrs}]$ to be non-zero, by an argument similar to the proof of Lemma 5.2.2, both W^ϵ_{12} and W^ϵ_{23} must contain exactly two points since the full twist of three or four strands is in the center $Z(P_4(\mathbb{S}))$. Thus, if (p, q, r, s) is not $(1, 1, 1, 1)$, $(0, 2, 0, 2)$ or $(2, 0, 2, 0)$, then $[c^\epsilon_{pqrs}] = 0$.

Let $\bar{x} \in X_4(\mathbb{S})$ be an ϵ-good point of type $(1, 1, 1, 1)$, $(0, 2, 0, 2)$ or $(2, 0, 2, 0)$. For $\gamma_1, \gamma_2 \in P_4(\mathbb{S})^Z$ and $\beta \in B_4(\mathbb{S})^Z$, we write $\gamma_1 \sim_\beta \gamma_2$ if $\gamma_1 = \beta\gamma_2\beta^{-1}$. For $\alpha \in P_4(\mathbb{S})^Z$,

$$\gamma(\rho_\epsilon(\alpha), \bar{x}) \sim_\beta \begin{cases} \alpha & \text{type } (1,1,1,1), \\ ({\delta_1}^2)^{s_1(\alpha)+s_2(\alpha)} & \text{type } (0,2,0,2), \\ ({\delta_1}^2)^{s_1(\alpha)}({\delta_3}^2)^{s_2(\alpha)} & \text{type } (2,0,2,0), \end{cases}$$

where $\beta = \beta(\bar{x}) \in B_4(\mathbb{S})^Z$ is a braid which depends only on $\bar{x}$. Hence, we can prove $[c^\epsilon_{0202}] = [c^\epsilon_{2020}] = 0$ and

$$[c^\epsilon_{1111}] = \mathrm{vol}(X^\epsilon_{1111}) \cdot (i^Z)^*(u)$$

by an argument similar to the proof of Lemma 5.2.2. Therefore,

$$\lim_{\epsilon \to +0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^Z \circ (i^Z)^*(u)) - \Lambda \cdot (i^Z)^*(u)\| = 0$$

by setting

$$\Lambda = \lim_{\epsilon \to +0} 4! \cdot \operatorname{area}(U_1^\epsilon) \operatorname{area}(U_2^\epsilon) \operatorname{area}(U_3^\epsilon) \operatorname{area}(U_4^\epsilon). \qquad \square$$

5.3.3 For a torus

Let $\mathbb{T}$ denote the 2-torus. We only mention the case of two strands. See [8, 44] for more details. Recall that $\bar{z} = (z_1, z_2)$ denotes the base point of $X_2(\mathbb{T})$. We define a braid a_1 so that it moves z_1 to the meridian direction and rotates once and does not move z_2. We define a braid b_1 so that it moves z_1 to the longitude direction and rotates once and does not move z_2. We define a_2 and b_2 similarly by exchanging the role of z_1 and z_2. It is known that $P_2(\mathbb{T}) \cong F_2 \times \mathbb{Z}^2$ and $P_2(\mathbb{T})^Z \cong F_2$. Namely, the set $\{a_1, b_1\}$ generates $P_2(\mathbb{T})^Z$ and $\{a_1a_2, b_1b_2\}$ generates $Z(P_2(\mathbb{T}))$.

Lemma 5.3.6. *There exist a constant $\Lambda > 0$ and a family of homomorphisms $\{\rho_\epsilon \colon P_2(\mathbb{T})^Z \to \mathcal{G}_\mathbb{T}\}_{0<\epsilon<1}$ such that*

$$\lim_{\epsilon \to +0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^Z \circ (i^Z)^*(u)) - \Lambda \cdot (i^Z)^*(u)\| = 0$$

for any $u \in \overline{EH}_b^n(B_2(\mathbb{T})^Z)$.

Proof. For each ϵ, we take open subsets U_i^ϵ $(i = 1, 2)$ in $\mathbb{T}$ so that

- $z_i \in U_i^\epsilon$,
- $U_1^\epsilon \cap U_2^\epsilon = \emptyset$, and
- $\operatorname{area}(U^\epsilon) = 1 - \epsilon$, where $U^\epsilon = U_1^\epsilon \cup U_2^\epsilon$.

Moreover, we take open subsets W_a^ϵ and V_b^ϵ of $\mathbb{T}$ which are diffeomorphic to an annulus so that

- $U_1^\epsilon \subset W_a^\epsilon \subset V_a^\epsilon$,
- $U_2^\epsilon \cap V_a^\epsilon = \emptyset$, and

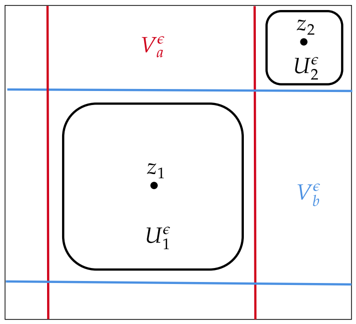

Figure 5.4:

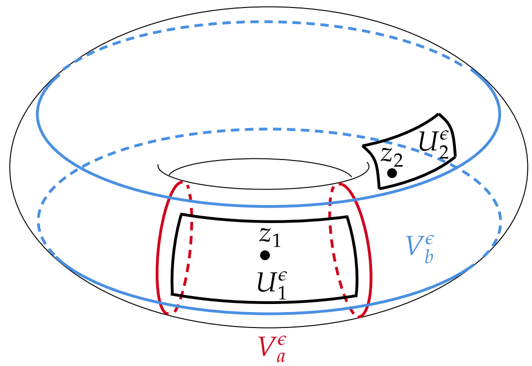

Figure 5.5: Open subsets in $\mathbb{T}$

- W_a^ϵ and V_a^ϵ contain a meridian.

We also take W_b^ϵ and V_b^ϵ similarly but to contain a longitude (see Figure 5.6 and 5.5).

We define $\rho_\epsilon\colon P_2(\mathbb{T})^Z \to \mathcal{G}_\mathbb{T}$ as follows. We take an isotopy $\{g_a^t\}$ which rotates W_a^ϵ once and whose support is contained in V_a^ϵ. For the generator $a_1 \in P_2(\mathbb{T})^Z$, we define $\rho_\epsilon(a_1) = g_a^1$. We also define $\rho_\epsilon(b_1)$ similarly.

We call that $\bar{x} = (x_1, x_2) \in X_2(\mathbb{T})$ is an *ϵ-good point* if both x_1 and x_2 are in U^ϵ. We say that an ϵ-good point $\bar{x}$ is *of type* (p, q) if U_1^ϵ has p points and U_2^ϵ has q points out of x_1 and x_2.

Let $\bar{x} \in X_2(\mathbb{T})$ be an ϵ-good point of type (p, q). We take an isotopy $\{g_a^t\}$ defined above. For $\gamma_1, \gamma_2 \in P_2(\mathbb{T})^Z$ and $\beta \in B_2(\mathbb{T})^Z$, we write $\gamma_1 \sim_\beta \gamma_2$ if $\gamma_1 = \beta\gamma_2\beta^{-1}$. Then $\gamma(\{g_a^t\}, \bar{x}) \in P_2(\mathbb{T})$ is calculated as follows.

$$\gamma(\{g_a^t\}, \bar{x}) \sim_\beta \begin{cases} e & (p = 0), \\ a_1 & (p = 1), \\ a_1 a_2 & (p = 2), \end{cases}$$

where $\beta = \beta(\bar{x}) \in B_2(\mathbb{T})$ is a braid which depends only on $\bar{x}$. Thus we can see that $\gamma(\rho_\epsilon(a_1), \bar{x}) \in P_2(\mathbb{T})^Z$ to be

$$\gamma(\rho_\epsilon(a_1), \bar{x}) \sim_\beta \begin{cases} a_1 & (p = 1), \\ e & (\text{otherwise}). \end{cases}$$

Similarly, we can see that

$$\gamma(\rho_\epsilon(b_1), \bar{x}) \sim_\beta \begin{cases} b_1 & (q = 1), \\ e & (\text{otherwise}). \end{cases}$$

Hence, for $\alpha \in P_2(\mathbb{T})^Z$, $\gamma(\rho_\epsilon(\alpha), \bar{x}) \sim_\beta \alpha$ if $\bar{x}$ is of type $(1, 1)$. By an argument similar to the proof of Lemma 5.2.2, we can prove that

$$\lim_{\epsilon \to +0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^Z \circ (i^Z)^*(u)) - \Lambda \cdot (i^Z)^*(u)\| = 0$$

by setting $\Lambda = \lim_{\epsilon \to +0} 2! \cdot \operatorname{area}(U_1^\epsilon) \operatorname{area}(U_2^\epsilon)$. □

5.3.4 For an annulus

Let $\mathbb{A}$ denote the annulus $S^1 \times [0, 1]$. The braid group $B_m(\mathbb{A})$ on $\mathbb{A}$ is isomorphic to the inverse image $\pi^{-1}(\mathfrak{S}_m)$ of the subgroup $\mathfrak{S}_m \subset \mathfrak{S}_{m+1}$ of $\mathfrak{S}_{m+1}$ by the projection $\pi\colon B_{m+1} \to \mathfrak{S}_{m+1}$ [39] since the "pillar" in $\mathbb{A} \times [0, 1]$ can be seen as a "fixed" strand (Figure 5.3.4). Namely, the pure braid group $P_m(\mathbb{A})$ on $\mathbb{A}$ is isomorphic to the ordinary pure braid group P_{m+1} thus we identity them.

Lemma 5.3.7. *There exist a constant $\Lambda > 0$ and a family of homomorphisms $\{\rho_\epsilon\colon P_2(\mathbb{A})^Z \to \mathcal{G}_{\mathbb{A}}\}_{0<\epsilon<1}$ such that*

$$\lim_{\epsilon \to +0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^Z \circ (i^Z)^*(u)) - \Lambda \cdot (i^Z)^*(u)\| = 0$$

for any $u \in \overline{EH}_b^n(B_2(\mathbb{A})^Z)$.

Proof. For each ϵ, we take open subsets U_i^ϵ $(i = 1, 2)$ in $\mathbb{A}$ so that

- $z_i \in U_i^\epsilon$,
- $U_1^\epsilon \cap U_2^\epsilon = \emptyset$ and
- $\operatorname{area}(U^\epsilon) = 1 - \epsilon$, where $U^\epsilon = U_1^\epsilon \cup U_2^\epsilon$.

Moreover, we take open subsets W_1^ϵ and V_1^ϵ of $\mathbb{A}$ which are diffeomorphic to an annulus so that

- $U_1^\epsilon \subset W_a^\epsilon \subset V_a^\epsilon$,

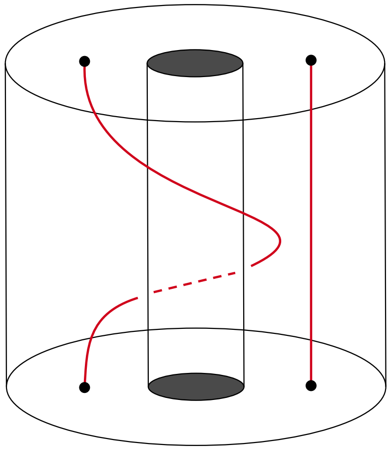

Figure 5.6: The 2-braid ${\sigma_1}^2$ on $\mathbb{A}$

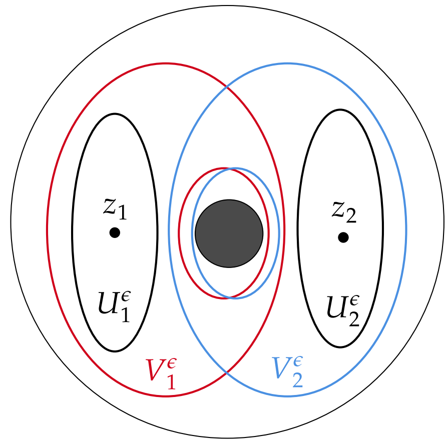

Figure 5.7: Open subsets in $\mathbb{A}$

- $U_2^\epsilon \cap V_1^\epsilon = \emptyset$ and
- the inclusion map $W_1^\epsilon \to \mathbb{A}$ induces an isomorphism $\pi_1(W_1^\epsilon) \to \pi_1(\mathbb{A})$.

We also take W_2^ϵ and V_2^ϵ similarly (Figure 5.7).

We define $\rho_\epsilon \colon P_2(\mathbb{A})^Z \to \mathcal{G}_{\mathbb{A}}$ as follows. Recall that $P_2(\mathbb{A})^Z \cong P_3^Z$ is freely generated by ${\sigma_1}^2$ and ${\sigma_2}^2$. We take an isotopy $\{g_1^t\}$ which rotates W_1^ϵ once and whose support is contained in V_1^ϵ. For ${\sigma_1}^2 \in P_2(\mathbb{A})^Z$, we define $\rho_\epsilon({\sigma_1}^2) = g_1^1$. We also define $\rho_\epsilon({\sigma_2}^2)$ similarly.

We call that $\bar{x} = (x_1, x_2) \in X_2(\mathbb{A})$ is an *ϵ-good point* if both x_1 and x_2 are in U^ϵ. We say that an ϵ-good point $\bar{x}$ is *of type* (p, q) if U_1^ϵ has p points and U_2^ϵ has q points out of x_1 and x_2.

Let $x \in X_2(\mathbb{A})$ be an ϵ-good point pf type (p, q). If $(p, q) \neq (1, 1)$, we can see that $\gamma(\rho_\epsilon(\alpha), \bar{x}) = e$ for any $\alpha \in P_2(\mathbb{A})^Z$. By an argument similar to the proof of Lemma 5.2.2, we can prove that

$$\lim_{\epsilon \to +0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^Z \circ (i^Z)^*(u)) - \Lambda \cdot (i^Z)^*(u)\| = 0$$

by setting $\Lambda = \lim_{\epsilon \to +0} 2! \cdot \mathrm{area}(U_1^\epsilon)\,\mathrm{area}(U_2^\epsilon)$. $\square$

5.3.5 Proof of the injectivity theorem

We complete the proof of Theorem 5.3.1 and Lemma 5.3.2.

Proof of Theorem 5.3.1. By Lemmas 5.3.4, 5.3.5, 5.3.6, and 5.3.7, we can prove Theorem 5.3.1 by the same argument in the proof of Theorem 5.2.1. $\square$

Proof of Corollary 5.3.2. As we saw in the proof of Corollary 5.2.3, $\overline{EH}_b^3(B_3^Z)$ is uncountably infinite-dimensional. Since $B_2(\mathbb{A})$ is a finite index subgroup of B_3 and $Z(B_2(\mathbb{A})) = Z(B_3)$, $\overline{EH}_b(B_2(\mathbb{A})^Z)$ is also uncountably infinite-dimensional.

It is known that $B_4(\mathbb{S})^Z$ is isomorphic to the mapping class group $MCG(\Sigma_{0,4})$ of the four-punctured sphere $\Sigma_{0,4}$ (see [5]). It is also known that $MCG(\Sigma_{0,4})$ surjects onto $PSL(2,\mathbb{Z})$ and its kernel is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (see [23]). Thus $MCG(\Sigma_{0,4})$ is quasi-isometric to $PSL(2,\mathbb{Z})$. Since $PSL(2,\mathbb{Z})$ is non-elementary hyperbolic, $MCG(\Sigma_{0,4})$ is also. Hence, by Theorem 2.2.3, $\overline{EH}_b^3(B_4(\mathbb{S})^Z) \cong \overline{EH}_b^3(MCG(\Sigma_{0,4}))$ is also uncountably infinite-dimensional.

Set $G = B_2(\mathbb{T})^Z$. Then G has a presentation

$$G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$$

[44, Exercise 6.3]. Since the Cayley graph of G is quasi-isometric to a trivalent tree, G is a non-elementary hyperbolic group. Hence $\overline{EH}_b^3(G)$ is uncountably infinite-dimensional. Therefore, we can prove as with Corollary 5.2.3. $\square$

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