博士論文

論文題目　Two-dimensinal conformal field theory,

current-current deformation and mass formula

（二次元共形場理論のカレントカレント変形と重み公式）

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Two-dimensional conformal field theory, current-current deformation and mass formula

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Abstract

The main purpose of this thesis is a mathematical construction of a non-perturbative deformation of a two-dimensional conformal field theory.

We introduce a notion of a full vertex algebra which formulates a compact two-dimensional conformal field theory. Then, we construct a deformation family of a full vertex algebra which serves as a current-current deformation of conformal field theory in physics. The parameter space of the deformation is expressed as a double coset of an orthogonal group, a quotient of an orthogonal Grassmannian. As an application, we consider a deformation of chiral conformal field theories, vertex operator algebras. A current-current deformation of a "vertex operator algebra" may produce new vertex operator algebras. We give a formula for counting the number of the isomorphic classes of vertex operator algebras obtained in this way. We demonstrate it for some holomorphic vertex operator algebra of central charge 24.

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Contents

Introduction

In theoretical physics, quantum field theory is a conceptual framework that describes a wide range of objects from the world of elementary particles to the scale of the universe, and its mathematical basis is one of the most important problems in modern mathematics [We, PS, Ha, Wi]. In quantum field theory, deformations of theories are important since in the case of free field theories, their deformations give phenomenological predictions about the real world. A deformation is defined by adding a new term to the original Lagrangian $\mathcal{L}(O_i, \partial_\mu O_i) \mapsto \mathcal{L}(O_i, \partial_\mu O_i) + gO_k$. Here O_k is an additional field and $g \in \mathbb{R}$ is called a coupling constant (cf., [IZ, Sr]). A deformed *correlation function*, a physical quantity, can be obtained by perturbation theory, i.e., expanded as a power series in g by using the path-integral. In most cases, the deformation obtained in this way remains only an approximation. Therefore, it is not clear whether the deformed theory rigorously satisfies an axiom of quantum field theory. In fact, this is one of the difficulties in constructing new quantum field theory mathematically.

Quantum field theory in higher dimensions is difficult to construct, but *conformal field theory* (quantum field theory with conformal symmetry) in two-dimension has many mathematically rigorous and non-trivial examples [FMS]. It is noteworthy that two-dimensional conformal field theory is an interesting object in itself since it plays a very important role in statistical mechanics [He], condensed matter physics [Kitae] and string theory [Polc1] in physics and it is deeply related to elliptic genus [Ta], modular forms [Zh], infinite dimensional Lie algebras and sporadic finite simple groups [FLM, B2] in mathematics.

The purposes of this thesis are

(1) to introduce a notion of a *full vertex algebra* which is a mathematical formulation of two-dimensional conformal field theory;
(2) to construct a deformation of a full vertex algebra, which serves as a deformation of conformal field theory;
(3) to apply the deformation to the classification theory of vertex algebras.

(1) is based on [Mo2], (2) is on [Mo3] and (3) is on [Mo1, Mo3].

0.1. **Conformal field theory in physics and mathematics**

First, we briefly recall a formulation of quantum field theory in a general dimension from physics. One aim of quantum field theory is to calculate *n point correlation functions*, that is, the vacuum expectation value of an interaction of n particles. An interaction of n particles decomposes into subsequent interactions of three particles. Thus, an n point correlation function can be expressed in terms of three point correlation functions, together with a choice of decompositions. Quantum field theory requires that the resulting n point correlation functions are *independent* of the choice of decompositions. This principle is known as the *consistency of quantum field theory*. Although it is known to be difficult to construct mathematically rigorous quantum field theories, surprisingly many examples, especially conformal field theories have been constructed in two-dimension, in physics literatures (see [FMS]).

In (not necessarily two-dimensional) conformal field theories, it is believed in physics, that the whole consistency of n point correlation functions follows from the *bootstrap equations (or hypothesis)*, which are distinguished consistencies of four point correlation functions [FGG, Poly2]. This hypothesis was used successfully by Belavin, Polyakov and Zamolodchikov in [BPZ] where the modern study of two-dimensional conformal field theories was initiated.

Hereafter, we consider two-dimensional conformal field theory. A field of two-dimensional conformal field theory is an operator-valued real analytic function. A conformal field theory in which any field is holomorphic is called a *chiral conformal field theory.* It is noteworthy that the algebra of a chiral conformal field theory satisfies a purely algebraic axiom, which was introduced by Borcherds [B1], see also [Go]. It is called a *vertex algebra* or a *vertex operator algebra* [FLM] and has been studied intensively by many authors, see e.g., [LL, FHL, FB]. In contrast, a formulation of the algebra of a non-chiral conformal field theory needs analytic properties and seems impossible to describe in a purely algebraic way.

Moore and Seiberg constructed a non-chiral conformal field theory as an extension of a holomorphic and an anti-holomorphic vertex operator algebras by their modules [MS1, MS2]. The bootstrap equations in this case are translated as a monodromy invariant property of the four point correlation functions. In the physics literature, this property was reformulated later by Fuchs, Runkel and Schweigert in [FRS], which says that the algebra describing the conformal field theory is a Frobenius algebra object in the braided tensor category constructed from holomorphic and anti-holomorphic vertex operator algebras.

A mathematical approach in this direction is due to Huang and Kong [HK] based on the representation theory of a *regular vertex operator algebra* developed by Huang and Lepowsky in a series of papers [HL1, HL2, HL3, Hu1, Hu2]. A regular vertex operator algebra is a class of vertex operator algebras with a semisimple module category (all the representations are completely reducible). One of the prominent results is obtained by Huang, which states that the representation category of a regular vertex operator algebra (of strong CFT type) inherits a modular tensor category structure [Hu3, Hu4].

Based on this theory, Huang and Kong [HK] introduced a notion of a *full field algebra*, which is a mathematical axiomatization of the algebras describing non-chiral two-dimensional conformal field theory. They also constructed conformal field theories, called *diagonal theories* in physics, as finite module extensions of the tensor products of regular vertex operator algebras. Their theory basically assumes that the conformal field theory is a finite extension of a tensor product of holomorphic and anti-holomorphic regular vertex operator algebras. Such a conformal field theory is called a *rational conformal field theory*, and it is known that the energy spectrum of the theory becomes rational numbers. Unfortunately, when considering a deformation of a theory, the energies must change continuously and thus it is necessary to consider irrational conformal field theories.

0.2. **Full vertex algebra – a formulation of compact conformal field theory**

In this thesis, we introduce a notion of a full vertex algebra (and a full vertex operator algebra) which formulates compact two-dimensional conformal field theory on $\mathbb{C}P^1$. While the definition of a full field algebra by [HK] based on a part of the consistency of n point correlation functions for all $n \geq 1$, the definition of a full vertex algebra is based on "the bootstrap equations", which are expected to be sufficient to derive the whole consistency of the theory.

We note that in recent years, the bootstrap hypothesis has become more and more important in the study of conformal field theory including higher dimensional cases. An infinite number of inequalities can be obtained from the bootstrap equation for a unitary conformal field theory, which is a constraint on the existence of the theory. By numerically evaluating the constraint conditions, the critical exponents (physical quantities) of the three-dimensional critical Ising model are calculated with high accuracy (cf., [RRTV, EPPRSV]). In [Mo2], we prove that the axiom of a full vertex algebra is equivalent to the bootstrap equation under reasonable assumptions.

A crucial point of our definition is to introduce a class of real analytic functions on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ with certain possible singularities at $\{0, 1, \infty\}$, which we call *conformal singularities*. Roughly speaking, a function with a conformal singularity at 0 has the following expansion around $z = 0$,

$$\sum_{r\in\mathbb{R}}\sum_{n,m\geq 0} a^r_{n,m} z^n \bar{z}^m |z|^r, \tag{0.1}$$

where $|z| = z\bar{z}$, the square of the absolute value, and $a^r_{n,m} \in \mathbb{C}$. This series is assumed to be absolutely convergent in an annulus $0 < |z| < R$ (for the precise definition, see Section 1.3). A typical example of such a function on $\mathbb{C}P^1$ is $|z|^r$ ($r \in \mathbb{R}$), which has the conformal singularities at $\{0, \infty\}$. Another example is

$$f_{\mathrm{Ising}}(z) = \frac{1}{2}(|1 - \sqrt{1-z}|^{1/2} + |1 + \sqrt{1-z}|^{1/2}), \tag{0.2}$$

which appears as a four point function of the two-dimensional critical Ising model [FMS, Mo4]. The expansion of $f_{\mathrm{Ising}}(z)$ at $z = 0$ is

$$1 + |z|^{1/2}/4 - z/8 - \bar{z}/8 + |z|^{1/2}(z + \bar{z})/32 + z\bar{z}/64 - 5z^2/128 - 5\bar{z}^2/128 + \dots.$$

By using the notion of a conformal singularity, we introduce a space of real analytic functions on $Y_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq z_2, z_1 \neq 0, z_2 \neq 0\}$ which has possible similar singularities along $z_1 = 0, z_2 = 0, z_1 = z_2$ and denote it by GCor_2 (see Section 1.4).

Let us describe the precise definition of a full vertex algebra. For a vector space V, let $V[[z, \bar{z}, |z|^{\mathbb{R}}]]$ be a space of formal power series spanned by

$$\sum_{r\in\mathbb{R}}\sum_{n,m\geq 0} v^r_{n,m} z^n \bar{z}^m |z|^r,$$

where $v^r_{n,m} \in V$ and $V((z, \bar{z}, |z|^{\mathbb{R}}))$ a subspace of $V[[z, \bar{z}, |z|^{\mathbb{R}}]]$ consisting of formal power series which are bounded below and discrete (see Section 1.1). A full vertex algebra is an $\mathbb{R}^2$-graded vector space $F = \bigoplus_{h,\bar{h}\in\mathbb{R}} F_{h,\bar{h}}$ with a distinguished vector $\mathbf{1} \in F_{0,0}$ and a linear map

$$Y(-, \underline{z}) : F \to \mathrm{End}\, F[[z, \bar{z}, |z|^{\mathbb{R}}]],\ a \mapsto Y(a, \underline{z}) = \sum_{r,s\in\mathbb{R}} a(r, s) z^{-r-1} \bar{z}^{-s-1}$$

satisfying the following axioms:

FV1) For any $a, b \in F$, $Y(a, \underline{z})b \in F((z, \bar{z}, |z|^{\mathbb{R}}))$;
FV2) $F_{h,\bar{h}} = 0$ unless $h - \bar{h} \in \mathbb{Z}$;
FV3) For any $a \in F$, $Y(a, \underline{z})\mathbf{1} \in F[[z, \bar{z}]]$ and $\lim_{z\to 0} Y(a, \underline{z})\mathbf{1} = a(-1, -1)\mathbf{1} = a$;
FV4) $Y(\mathbf{1}, \underline{z}) = \mathrm{id}_F$;
FV5) For any $a, b, c \in F$ and $u \in F^\vee = \bigoplus_{h,\bar{h}\in\mathbb{R}} F^*_{h,\bar{h}}$, there exists $\mu(z_1, z_2) \in \mathrm{GCor}_2$ such that

$$\begin{aligned} u(Y(a, \underline{z}_1)Y(b, \underline{z}_2)c) &= \mu(z_1, z_2)|_{|z_1|>|z_2|}, \\ u(Y(Y(a, \underline{z}_0)b, \underline{z}_2)c) &= \mu(z_0 + z_2, z_2)|_{|z_2|>|z_0|}, \\ u(Y(b, \underline{z}_2)Y(a, \underline{z}_1)c) &= \mu(z_1, z_2)|_{|z_2|>|z_1|}, \end{aligned} \tag{0.3}$$

where $F^*_{h,\bar{h}}$ is the dual of $F_{h,\bar{h}}$ and $\mu(z_1, z_2)|_{|z_1|>|z_2|}$ is the expansion of $\mu(z_1, z_2)$ in $\{|z_1| > |z_2|\}$;
FV6) $F_{h,\bar{h}}(r, s)F_{h',\bar{h}'} \subset F_{h+h'-r-1,\bar{h}+\bar{h}'-s-1}$ for any $h, h', \bar{h}, \bar{h}', r, s \in \mathbb{R}$.

Let us explain a physical background of this definition. All the states of a conformal field theory form a vector space, which is F in our definition. The global conformal symmetry $\mathrm{SO}(3, 1)$ acts on F. The $\mathbb{R}^2$-grading on F is induced from this action and the assumptions (FV3), (FV4) and (FV6) are natural requirements which conformal field theory satisfies. For a vector $v \in F_{h,\bar{h}}$, the value $h + \bar{h}$ and $h - \bar{h}$ are physically the energy and the spin of a state v. A state v changes as $\exp(i\theta(h - \bar{h}))$ under the rotation group $\mathrm{SO}(2) \subset \mathrm{SO}(3, 1)$, which requires the assumption (FV2) (if the theory does not contain fermions). Although (FV1) and (FV5) are not satisfied by general conformal field theories, they are satisfied by a wide class of conformal field theories, called *compact conformal field theories*.

In this thesis, a compact conformal field theory is a conformal field theory whose state space F satisfies the following conditions:

C1) There exists $N \in \mathbb{R}$ such that $F_{h,\bar{h}} = 0$ for any $h \leq N$ or $\bar{h} \leq N$;
C2) For any $H \in \mathbb{R}$, $\sum_{h,\bar{h}\leq H} \dim F_{h,\bar{h}}$ is finite.

We also call a full vertex algebra F *compact* if it satisfies (C1) and (C2) (Note that this definition of compactness is a bit different from the definition used in physics).

For non-compact conformal field theory, the correlation functions are no longer power series of the form (0.1) but an integral over $\mathbb{R}^2$. Thus, in this thesis, we restrict ourselves on compact conformal field theory to avoid difficulties in analysis. There are important non-compact conformal field theories, e.g., the Liouville field theory and non-compact WZW conformal field theory [DO, ZZ]. We hope to come back to this point in the future.

Now, we explain a physical meaning of (FV1) and (FV5) from the compactness. (FV1) is a mathematical consequence of (C1), (C2), (FV2) and (FV6) (see Proposition 2.5), thus is satisfied for any compact conformal field theory. Furthermore, (FV1) and the bootstrap equation implies (FV5). Therefore, the notion of a compact full vertex algebra gives a mathematical formulation of two-dimensional compact conformal field theory on $\mathbb{C}P^1$. In particular, any correlation function of compact conformal field theory always has an expansion of the form (0.1), which is our motivation for the definition of a conformal singularity.

As we discussed in Section 0.1, rational conformal field theory contains many important conformal field theories, but it is too restrictive to consider deformations. Compact conformal field theory is a wider class of conformal field theory which includes rational conformal field theory, e.g., the WZW-model for a compact semisimple Lie group and the Virasoro minimal models. Furthermore, by the definition of the compactness, at least its small deformation seems to be compact. In particular, we prove under some mild assumption, compactness is preserved by the current-current deformation, constructed in this thesis (for the precise statement, see Proposition 5.3). We expect that (unitary) compact conformal field theory is stable under all exactly marginal deformations.

Finally, as expected in physics, a chiral conformal field theory (vertex algebra) naturally appears as a subalgebra despite the definition of a full vertex algebra is independent of the theory of a vertex algebra. In fact, the holomorphic subspace $\ker \partial_{\bar{z}}|_F$ of a full vertex algebra F forms a vertex algebra and F is a module on $\ker \partial_{\bar{z}}|_F$ (see Proposition 2.14). Hence the full vertex algebra F can be seen as an extension of the tensor product of holomorphic and anti-holomorphic vertex algebras $\ker \partial_{\bar{z}}|_F \otimes \ker \partial_z|_F$ (Proposition 2.18). This is an assumption in the study of Huang and Kong and actually the notions of a full field algebra [HK] and a full vertex algebra are equivalent if the algebra is an extension of a tensor product of regular vertex operator algebras. This follows from [Mo2, Proposition 4.3] and [HK, Theorem 2.11].

0.3. **Current-current deformation in physics and its formulation**

Now, we briefly review a deformation of two-dimensional conformal field theory in physics. The deformation of two-dimensional conformal field theory $F = \bigoplus_{h,\bar{h}\in\mathbb{R}} F_{h,\bar{h}}$ generated by a general field $O_k \in F_{h,\bar{h}}$ does not always preserve the conformal symmetry. This general deformation has been studied by many physicists, e.g., [Za, EY] to understand a structure of quantum field theories. Meanwhile, a deformation of a two-dimensional conformal field theory which preserves the conformal symmetry is known to be generated by a special field $O_k \in F_{1,1}$, called an (exactly) *marginal field* [DVV1].

Chaudhuri and Schwartz considered the deformation of a conformal field theory generated by a field in $F_{1,0} \otimes F_{0,1} \subset F_{1,1}$ (a sum of products of holomorphic currents and anti-holomorphic currents). They showed that the field is exactly marginal if and only if the holomorphic currents as well as the anti-holomorphic currents belong to *commutative current algebras* [CS]. The deformation generated by this $(1,1)$-field is called a *current-current deformation* in the physics literature (cf., [FR]). Those studies depend on the path integral method, which is not mathematically rigorous. The purpose of this thesis is to mathematically formulate and construct the current-current deformation of two-dimensional conformal field theory.

In terms of a full vertex algebra, the commutative current algebra which generates a current-current deformation corresponds to a subalgebra of a full vertex algebra which is isomorphic to the tensor product of holomorphic and anti-holomorphic Heisenberg vertex algebras.

It is convenient to introduce a notion of a *full $\mathcal{H}$-vertex algebra*. Let H_l and H_r be real vector spaces equipped with non-degenerate bilinear forms $(-,-)_l : H_l \times H_l \to \mathbb{R}$ and $(-,-)_r : H_r \times H_r \to \mathbb{R}$ and $M_{H_l}(0)$ and $M_{H_r}(0)$ be the affine Heisenberg vertex algebras associated with $(H_l, (-,-)_l)$ and $(H_r, (-,-)_r)$, respectively. Set $H = H_l \oplus H_r$ and let $p, \bar{p} \in \mathrm{End}\, H$ be the projections of H onto H_l and H_r, $(H, (-,-)_p) = (H_l \oplus H_r, (-,-)_l \oplus (-,-)_r)$ the orthogonal sum of vector spaces and

$$M_{H,p} = M_{H_l}(0) \otimes \overline{M_{H_r}(0)}$$

the tensor product of the vertex algebra $M_{H_l}(0)$ and the anti-holomorphic vertex algebra $\overline{M_{H_r}(0)}$ (see [Mo2]). A *full $\mathcal{H}$-vertex algebra* is a full vertex algebra F together with a full vertex algebra homomorphism $M_{H,p} \to F$. Since F is an $M_{H,p}$-module, F is a module of the affine Heisenberg Lie algebra $\hat{H}$ associated with $(H, (-,-)_l \oplus (-,-)_r)$. For $\alpha \in H$, set

$$\Omega_{F,H}^{\alpha} = \{v \in F \mid h(n)v = 0, h(0)v = (h,\alpha)_p v \text{ for any } h \in H \text{ and } n \geq 1\}$$

and $\Omega_{F,H} = \bigoplus_{\alpha \in H} \Omega^{\alpha}$. The lowest weight space $\Omega_{F,H}$ is called a *vacuum space* in [FLM]. We assume that a full $\mathcal{H}$-vertex algebra (F, H, p) is generated by the vacuum space as an $\hat{H}$-module, that is,

$$F \cong \bigoplus_{\alpha \in H} M_{H,p}(\alpha) \otimes \Omega_{F,H}^{\alpha}. \tag{0.4}$$

Then, as suggested by Förste and Roggenkamp in [FR], $\Omega_{F,H}$ inherits an algebra structure by modifying the full vertex algebra structure on F. More precisely, we introduce a notion of a *generalized full vertex algebra*, which is in fact a mathematical formulation of the above "structure of the lowest weight space". Then, we show that $\Omega_{F,H}$ is a generalized full vertex algebra (Theorem 4.3). Before stating the main results, we briefly explain the definition of a generalized full vertex algebra, which plays a crucial role in this thesis.

0.4. **Generalized full vertex algebras.**

The notion of a generalized full vertex algebra is a "full" analogy of the notion of a (chiral) generalized vertex algebra introduced by Dong and Lepowsky [DL], in order to study the affine vertex algebras and the parafermion vertex algebras [DL].

We first recall their results. Let $\mathfrak{g}$ be a simple Lie algebra and $L_{\mathfrak{g},k}$ the simple affine vertex algebra at level k. Then, $L_{\mathfrak{g},k}$ has a Heisenberg vertex subalgebra generated by a Cartan subalgebra of the Lie algebra, $H_{\mathfrak{g}} \subset \mathfrak{g}$. Thus, $(L_{\mathfrak{g},k}, H_{\mathfrak{g}})$ is a chiral full $\mathcal{H}$-vertex algebra, which we call a *$\mathcal{H}$-vertex algebra*. Dong and Lepowsky showed that if $k \in \mathbb{Z}_{\geq 0}$, called an integrable level, the vacuum space $\Omega_{L_{\mathfrak{g},k}, H_{\mathfrak{g}}}$ inherits a generalized vertex algebra structure [DL]. They also constructed a generalized vertex algebra from a pair of a real finite dimensional vector space H equipped with a non-degenerate symmetric bilinear form and an abelian subgroup $L \subset H$. They call it a *generalized lattice vertex algebra.*

We remark that our proof of the existence of a generalized full vertex algebra structure on $\Omega_{F,H}$ (Theorem 4.3) seems different from [DL]. In fact, since any $\mathbb{Z}$-graded vertex algebra is a full vertex algebra [Mo2, Proposition 2.2], Theorem 4.3 generalizes their results to any vertex algebras, in particular, to the affine vertex algebras at any level $k \in \mathbb{R}$. In fact, we prove that the category of generalized vertex algebras and the category of $\mathcal{H}$-vertex algebras are equivalent (Proposition 4.17).

A generalized full vertex algebra is, roughly, an H-graded vector space $\Omega = \bigoplus_{\alpha \in H} \Omega^\alpha$ equipped with a linear map

$$\hat{Y}(-,\underline{z}) : \Omega \to \operatorname{End}\Omega[[z^{\mathbb{R}}, \bar{z}^{\mathbb{R}}]],\ a \mapsto \hat{Y}(a,\underline{z}) = \sum_{r,s \in \mathbb{R}} a(r,s) z^{-r-1} \bar{z}^{-s-1},$$

where H is a finite dimensional vector space equipped with a non-degenerate symmetric bilinear form. The key point is that we allow the correlation function for $\alpha_i \in H$ and $a_i \in \Omega^{\alpha_i}$ to have a $U(1)$-monodromy of the form $\exp(2\pi(\alpha_i, \alpha_j))$ under the interchange of states a_i and a_j (for the precise definition see Section 3). Importantly, if the monodromy is trivial, then a generalized full vertex algebra is a full vertex algebra (Lemma 3.6).

Thus, a fundamental question is whether it is possible to cancel the monodromy for a given generalized full vertex algebra. The answer is yes. Let Ω be a generalized full vertex algebra graded by H and $P(H)$ the set of projections $p \in \operatorname{End} H$ such that the subspaces $\ker p$ and $\ker(1-p)$ is orthogonal. Then, for each $p \in P(H)$, we can construct a full vertex algebra by canceling the monodromy (Theorem 3.14). In fact, we have a family of full $\mathcal{H}$-vertex algebras parametrized by $P(H)$. Each element of $P(H)$ determines the charge of the decomposition (0.4).

0.5. **Main results**

Before stating the main result, we explain how the $U(1)$-monodromies on the vacuum space appear. Let (F, H, p) be a full $\mathcal{H}$-vertex algebra and $\alpha_1, \alpha_2 \in H$. Then, the conformal block (or the correlation function) of the affine Heisenberg full vertex algebra $M_{H,p}$ labeled by α_1, α_2 is of the form

$$(z_1 - z_2)^{(p\alpha_1, p\alpha_2)_l} (\bar{z}_1 - \bar{z}_2)^{(\bar{p}\alpha_1, \bar{p}\alpha_2)_r} = |z_1 - z_2|^{(\bar{p}\alpha_1, \bar{p}\alpha_2)_r} (z_1 - z_2)^{(p\alpha_1, p\alpha_2)_l - (\bar{p}\alpha_1, \bar{p}\alpha_2)_r},$$

where $|z_1 - z_2|$ is the square of the absolute value $(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$. The above $|z_1 - z_2|^r$ is a single-valued function for any $r \in \mathbb{R}$. Thus, the monodromy of the conformal block is controlled by the bilinear form $(-,-)_{\mathrm{lat}}$ on H defined by $(\alpha_1, \alpha_2)_{\mathrm{lat}} = (p\alpha_1, p\alpha_2)_l - (\bar{p}\alpha_1, \bar{p}\alpha_2)_r$. We denote the space $(H, (-,-)_{\mathrm{lat}})$ by $H_l \oplus -H_r$. Then, the first main result of this thesis is that the assignment $(F, H, p) \mapsto (\Omega_{F,H}, H_l \oplus -H_r, p)$ gives an equivalence between the category of full $\mathcal{H}$-vertex algebras and the category of generalized full vertex algebras with the charge structure p (Theorem 4.7).

The real orthogonal group $O(H_l \oplus -H_r; \mathbb{R})$ acts on the set of all the possible charge structures $P(H_l \oplus -H_r)$ and the orbit of the original projection p forms the orthogonal Grassmannian $O(H_l \oplus -H_r; \mathbb{R})/O(H_l; \mathbb{R}) \times O(-H_r; \mathbb{R})$, which is a connected component of $P(H_l \oplus -H_r)$. Thus, by using the inverse functor, we have a family of full $\mathcal{H}$-vertex algebras parametrized by the Grassmannian.

We note that for $h_l \in H_l$ and $h_r \in H_r$ with $(h_l, h_l) \neq 0$ and $(h_r, h_r) \neq 0$, we have a one-parameter subgroup $\{\sigma(g)\}_{g \in \mathbb{R}} \subset O(H_l \oplus -H_r)$ (see Section 5.1). The family of full $\mathcal{H}$-vertex algebras associated with $\{\sigma(g) p \sigma(g)^{-1}\}_{g \in \mathbb{R}} \subset P(H_l \oplus -H_r)$ is, in fact, the current-current deformation of a full $\mathcal{H}$-vertex algebra (F, H, p) associated with the exactly marginal field $Y(h_l(-1,-1)h_r, \underline{z}) = h_l(z) h_r(\bar{z})$. Thus, the above family gives a mathematical formulation of the non-perturbative current-current deformation associated with the commutative current algebras H_l and H_r.

Finally, we give the double coset description of the parameter space. The automorphism group of the generalized full vertex algebra $\Omega_{F,H}$ naturally acts on the grading $H_l \oplus -H_r$. Let $D_{F,H}$ be the image of the automorphism group in $O(H_l \oplus -H_r)$. Then, the isomorphism classes of the current-current deformation of a full $\mathcal{H}$-vertex algebra (F, H, p) is parametrized by the double coset (Theorem 5.5)

$$D_{F,H}\backslash O(H_l \oplus (-H_r))/O(H_l) \times O(-H_r), \tag{0.5}$$

which is conjectured in [FR]. Thus, $D_{F,H}$ is a mathematical formulation of the duality group, which in particular implies the T-duality of string theory (see below).

For example, let $F_{\mathrm{SU}(2)}$ be a full vertex algebra corresponding to the SU(2) WZW model at level one. Then, $F_{\mathrm{SU}(2)}$ is naturally a full $\mathcal{H}$-vertex algebra by one-dimensional Cartan subalgebras of SU(2). Since $O(1,1)/O(1) \times O(1) \cong \mathbb{R}_{>0}$, the current-current deformation of $F_{\mathrm{SU}(2)}$ is parametrized by $R \in \mathbb{R}_{>0}$. Let denote C_R the full $\mathcal{H}$-vertex algebra corresponding to $R \in \mathbb{R}_{>0}$. The algebra structure of C_R will be studied in detail in Section 5.3. As mentioned in Section 0.2, the holomorphic and the anti-holomorphic parts of C_R is a vertex operator algebra. If the square R^2 is irrational number, then both the holomorphic and the anti-holomorphic parts are Heisenberg vertex operator algebras and C_R defines an irrational conformal field theory. If $R^2 = p/q$ with $p, q \in \mathbb{Z}_{>0}$ are coprime integers, then both the holomorphic and the anti-holomorphic parts are isomorphic to the lattice vertex operator algebra $V_{\sqrt{2pq}\mathbb{Z}}$ and C_R is a finite extension of $V_{\sqrt{2pq}\mathbb{Z}} \otimes \overline{V_{\sqrt{2pq}\mathbb{Z}}}$, where $\sqrt{2pq}\mathbb{Z}$ is the rank one lattice generated by α with $(\alpha, \alpha) = 2pq$ and $\overline{V_{\sqrt{2pq}\mathbb{Z}}}$ is an anti-holomorphic vertex operator algebra (see Proposition 2.12). For example, the full vertex algebras C_R with $R = \sqrt{6}$ or $R = \sqrt{3/2}$ have the same underlying lattice vertex algebra $V_{\sqrt{12}\mathbb{Z}}$. However, $C_{\sqrt{6}}$ and $C_{\sqrt{3/2}}$ are non-isomorphic. In fact, the decomposition of $C_{\sqrt{6}}$ and $C_{\sqrt{3/2}}$ into irreducible $V_{\sqrt{12}\mathbb{Z}} \otimes \overline{V_{\sqrt{12}\mathbb{Z}}}$-modules are

$$C_{\sqrt{6}} = \bigoplus_{i \in \mathbb{Z}/12\mathbb{Z}} V_{\sqrt{12}\mathbb{Z}+\frac{i}{\sqrt{12}}} \otimes \overline{V_{\sqrt{12}\mathbb{Z}+\frac{i}{\sqrt{12}}}}$$

$$C_{\sqrt{3/2}} = \bigoplus_{i \in \mathbb{Z}/12\mathbb{Z}} V_{\sqrt{12}\mathbb{Z}+\frac{i}{\sqrt{12}}} \otimes \overline{V_{\sqrt{12}\mathbb{Z}+\frac{7i}{\sqrt{12}}}}.$$

Thus, while $C_{\sqrt{6}}$ is a diagonal sum of irreducible modules of $V_{\sqrt{12}\mathbb{Z}}$, $C_{\sqrt{3/2}}$ is twisted by $7 \in (\mathbb{Z}/\mathbb{Z}_{12})^\times$. The general twist $n_{p,q} \in (\mathbb{Z}/2pq\mathbb{Z})^\times$ for $R^2 = p/q$ is given in Proposition 5.7, which corresponds to an automorphism of the modular tensor category $\mathrm{Rep}V_{\sqrt{2pq}\mathbb{Z}}$. In this way, the rational conformal field theory C_R with $R^2 \in \mathbb{Q}$ is controlled by a number-theoretic discrete structure and the irrational conformal field theory connects them continuously.

It is noteworthy that C_R and $C_{R'}$ is isomorphic if and only if $R = R'$ or $R = \frac{1}{R'}$, which just corresponds to the action of the duality group $D_{F_{\mathrm{SU}(2)}} \cong D_4$ (the dihedral group) on $O(1,1)/O(1) \times O(1) \cong \mathbb{R}_{>0}$. The double coset $D_4\backslash \mathrm{O}(1,1;\mathbb{R})/\mathrm{O}(1;\mathbb{R}) \times \mathrm{O}(1;\mathbb{R})$ is a half line $[1, \infty)$. This corresponds to the horizontal line in the moduli space of conformal field theories of central charge $(c, \bar{c}) = (1,1)$ expected in physics (see Fig.1, [Gi, DVV1, DVV2]). The line also corresponds to a family of conformal field theories resulting from a compactification of string theory whose target space is the cycle $S^1_R = \mathbb{R}/R\mathbb{R}$ with a radius $R \in \mathbb{R}_{>0}$, and the group D_F generalizes the T-duality $R \leftrightarrow R^{-1}$ in string theory. We note that there is a conjectured central charge $(c, \bar{c}) = (1,1)$

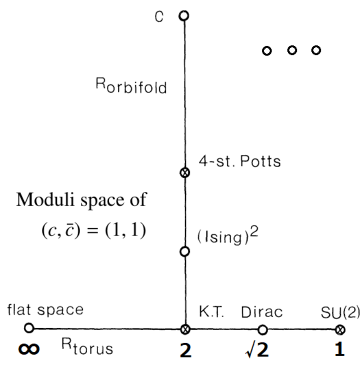

FIG. 1

conformal field theory which does not belong to Fig 1 [RW], however, if we restrict ourselves to compact conformal field theory, then such models seem to be excluded.

In Section 4.4, we construct a family of full $\mathcal{H}$-vertex operator algebras which corresponds to the toroidal compactification of string theory with N-dimensional target space, called a Narain moduli space [N, NSW], parameterized by the following double coset:

$$\mathrm{O}(N,N;\mathbb{Z})\backslash\mathrm{O}(N,N;\mathbb{R})/\mathrm{O}(N;\mathbb{R})\times\mathrm{O}(N;\mathbb{R}). \tag{0.6}$$

Thus, the double coset description (0.5) gives a global information about the moduli space of conformal field theories, which is important in the study of string theory. We remark that recently the moduli space of conformal field theories is also of interest in the context of three-dimensional gravity, where a random ensemble of conformal field theory seems to be important and Maloney and Witten considered an integral over the Narain moduli space (0.6) [MW]. We hope that our results will motivate further studies of the CFT moduli spaces.

0.6. **Application to vertex algebras.**

As an application, we consider a deformation of vertex algebras, which is the holomorphic part of a conformal field theory (chiral conformal field theory). Importantly, a vertex algebra does not admit any physical deformation since a general two point correlation function is of the form $C(z-w)^n$ for some $n\in\mathbb{Z}$ and $C\in\mathbb{C}$. In other words, the energies of the chiral part are equal to the spins and thus integers. In contrast, for a full vertex algebra, a general two point function is of the form $C|z-w|^r(z-w)^n(\bar{z}-\bar{w})^m$ for some $n,m\in\mathbb{Z}$ and $r\in\mathbb{R}$, where $|z-w|$ is the absolute value. Thus, we can deform the two point correlation function or the parameter $r\in\mathbb{R}$, continuously. So let us consider the tensor product of a $\mathbb{Z}$-graded vertex algebra V and the full $\mathcal{H}$-vertex algebra (C_R,H_R), the algebra of the toroidal compactification with the radius $R\in\mathbb{R}$ mentioned above. Assume that V is a full $\mathcal{H}$-vertex algebras, that is, V contains a (holomorphic) Heisenberg vertex algebra $M_{H_V}(0)$, which is called a VH pair in [Mo1]. Then, $(V\otimes C_R, H_V\oplus H_R)$ is naturally a full $\mathcal{H}$-vertex algebra. Thus, we can consider the current-current deformation of this algebra, which mixes V and C_R. In general, the deformed algebra does not split, that is, it cannot be expressed as $W\otimes C_r$ for some $\mathbb{Z}$-graded vertex algebra W and the radius $r\in\mathbb{R}$. But if it splits, then the $\mathbb{Z}$-graded vertex algebra W is not always isomorphic to V. Thus, the deformation may produce new $\mathbb{Z}$-graded vertex algebras and a fundamental question is how many $\mathbb{Z}$-graded vertex algebras are contained in the current-current deformation of $V\otimes C_R$.

The notion of a genus of vertex algebras introduced in [Mo1] gives us an answer. There, we introduce an equivalent relation on $\mathcal{H}$-vertex algebras, which we call a *genus of vertex algebras*. Two $\mathcal{H}$-vertex algebras (V,H_V) and (W,H_W) are said to be in the same genus (or equivalent) if $(V\otimes V_{II_{1,1}}, H_V\oplus H_{II_{1,1}})$ and $(W\otimes V_{II_{1,1}}, H_W\oplus H_{II_{1,1}})$ are isomorphic as $\mathcal{H}$-vertex algebras, where $II_{1,1}$ is the unique even unimodular lattice with the signature $(1,1)$ and $V_{II_{1,1}}$ is the lattice vertex algebra.

Then, one can show that $\mathcal{H}$-vertex algebras (V,H_V) and (W,H_W) are in the same genus if and only if there exists a current-current deformation between the full $\mathcal{H}$-vertex algebras $V\otimes C_R$ and $W\otimes C_R$ (Theorem 6.2). The weighted sum of the number of the isomorphism classes in a genus is called a mass of the genus. In [Mo1, Theorem 4.2], we gave a formula which computes the mass by using the mass of integral lattices [Si, CS] and the duality group $D_{V\otimes V_{II_{1,1}},H_V\oplus H_{II_{1,1}}}$ under some assumptions.

A non-trivial example of a genus is given by a modular invariant chiral conformal field theory (in mathematical literature it is called a *holomorphic vertex operator algebra*). In [LS], Lam and Shimakura constructed a modular invariant chiral conformal field theory of central charge 24 as an extension of a vertex operator algebra $L_{\mathrm{E}_{8,2}}\otimes L_{\mathrm{B}_{8,1}}$, where $L_{\mathrm{E}_{8,2}}$ and $L_{\mathrm{B}_{8,1}}$ are affine vertex algebras associated with simple Lie algebras E_8 and B_8 at level 2 and 1, respectively. We denote it by $L^{\mathrm{hol}}_{E_{8,2}B_{8,1}}$. In [Mo1], the duality group was identified as the automorphism group of some lattice $II_{17,1}(2_{II}^{+10})$. Thus, the current-current deformation of the full vertex operator algebra $L^{\mathrm{hol}}_{E_{8,2}B_{8,1}} \otimes C_R$ is parametrized by

$$\mathrm{Aut}\, II_{17,1}(2_{II}^{+10})\backslash O(17,1;\mathbb{R})/O(17;\mathbb{R})\times O(1;\mathbb{R}),$$

and there are 17 non-isomorphic vertex operator algebras contained in this family, all of which are modular invariant chiral conformal field theories (Proposition 6.10, see also [HS, Mo1]).

Outline.

In Section 1, we introduce a space of real analytic functions which serves as correlation functions. In Section 2, we introduce the notion of a full vertex algebra and study its properties and in Section 3, we introduce the notion of a generalized full vertex algebra, construct a standard example and tensor product and prove Theorem 3.14 by canceling the monodromies. The notion of a full $\mathcal{H}$-vertex algebra is introduced in Section 4. There we show that the vacuum space inherits a generalized full vertex algebra structure (Theorem 4.3) and the equivalence of the categories (Theorem 4.7). We also construct some adjoint functors which will be used latter. Combining the above results, the current-current deformation of a full $\mathcal{H}$-vertex algebra is defined and the double coset description of the parameter space is proved (Theorem 5.5) in Section 5. As an application, we study the relation between the current-current deformation of $\mathcal{H}$-vertex algebras and the genus of vertex algebras in Section 6.

1. Correlation functions and formal calculus

In this section, we introduce a notion of a conformal singularity which is a typical singularity appearing in correlation functions of conformal field theory as a consequence of a conformal invariance. We define a space of real analytic functions with possible conformal singularity, which is important to define a full vertex algebra.

1.1. **The space of formal power series.** In this section, we introduce certain space of formal variables, which will be used to define the conformal singularity. We assume that the base field is $\mathbb{C}$ unless otherwise stated. Let z and $\bar{z}$ be independent formal variables. We will use the notation $\underline{z}$ for the pair $(z,\bar{z})$ and $|z|$ for $z\bar{z}$.

For a vector space V, we denote by $V[[z,\bar{z},|z|^{\mathbb{R}}]]$ the set of formal sums

$$\sum_{s,\bar{s}\in\mathbb{R}} a_{s,\bar{s}} z^s \bar{z}^{\bar{s}}$$

such that $a_{s,\bar{s}}=0$ unless $s-\bar{s}\in\mathbb{Z}$. We also denote by $V((z,\bar{z},|z|^{\mathbb{R}}))$ the subspace of $V[[z,\bar{z},|z|^{\mathbb{R}}]]$ consisting of the series $\sum_{s,\bar{s}\in\mathbb{R}} a_{s,\bar{s}} z^s\bar{z}^{\bar{s}} \in V[[z,\bar{z},|z|^{\mathbb{R}}]]$ such that:

(1) For any $H\in\mathbb{R}$, $\#\{(s,\bar{s})\in\mathbb{R}^2 \mid a_{s,\bar{s}}\neq 0 \text{ and } s+\bar{s}\leq H\}$ is finite.
(2) There exists $N\in\mathbb{R}$ such that $a_{s,\bar{s}}=0$ unless $s\geq N$ and $\bar{s}\geq N$.

Let $f(\underline{z})\in V((z,\bar{z},|z|^{\mathbb{R}}))$. By the assumption, there exists $r_0,r_1,r_2,\dots\in\mathbb{R}$ such that

(1) $r_0<r_1<r_2<\cdots$;
(2) $r_i\to\infty$;
(3) $f(\underline{z})$ could be written as

$$\sum_{i=0}^{\infty}\sum_{n,m=0}^{\infty} a^i_{n,m} z^n\bar{z}^m|z|^{r_i},$$

where $a^i_{n,m}\in\mathbb{C}$.

Remark 1.1. *As seen above, $\mathbb{C}((z,\bar{z},|z|^{\mathbb{R}}))$ is a Novikov ring with polynomial coefficients.*

We will consider the following subspaces of $V[[z,\bar{z},|z|^{\mathbb{R}}]]$:

$$V[[z,\bar{z}]] = \{\sum_{s,\bar{s}\in\mathbb{Z}_{\geq 0}} a_{s,\bar{s}} z^s \bar{z}^{\bar{s}} \mid a_{s,\bar{s}} \in V\},$$
$$V[z^{\pm},\bar{z}^{\pm}] = \{\sum_{s,\bar{s}\in\mathbb{Z}} a_{s,\bar{s}} z^s \bar{z}^{\bar{s}} \mid a_{s,\bar{s}} \in V, \text{ all but finitely many } a_{s,\bar{s}} = 0\},$$
$$V[|z|^{\mathbb{R}}] = \{\sum_{r\in\mathbb{R}} a_r z^r \bar{z}^r \mid a_r \in V, \text{ all but finitely many } a_r = 0\}.$$

We will also consider their combinations, e.g., $V((y/x,\bar{y}/\bar{x},|y/x|^{\mathbb{R}}))[x^{\pm},\bar{x}^{\pm},|x|^{\mathbb{R}}]$, which is spanned by

$$\sum_{i=1}^{k} \sum_{n,m=-l}^{l} \sum_{s,\bar{s}\in\mathbb{R}} a^i_{n,m,r,s} x^{n+r_i} \bar{x}^{m+r_i} (y/x)^s (\bar{y}/\bar{x})^{\bar{s}}$$

for some $k,l \in \mathbb{Z}_{>0}$ and $r_i \in \mathbb{R}$ and $a^i_{n,m,s,\bar{s}} \in V$ such that $a^i_{n,m,s,\bar{s}} = 0$ unless $s-\bar{s} \in \mathbb{Z}$ and there exists N such that $a^i_{n,m,s,\bar{s}} = 0$ unless $s \geq N$ and $\bar{s} \geq N$ and $\{(s,\bar{s}) \in \mathbb{R} \mid a^i_{n,m,s,\bar{s}} \neq 0 \text{ and } s+\bar{s} \leq H\}$ is finite for any $H \in \mathbb{R}$.

Let $\frac{d}{dz}$ and $\frac{d}{d\bar{z}}$ be formal differential operators acting on $V[[z,\bar{z},|z|^{\mathbb{R}}]]$ by

$$\frac{d}{dz} \sum_{s,\bar{s}\in\mathbb{R}} a_{s,\bar{s}} z^s \bar{z}^{\bar{s}} = \sum_{s,\bar{s}\in\mathbb{R}} s a_{s,\bar{s}} z^{s-1} \bar{z}^{\bar{s}}$$
$$\frac{d}{d\bar{z}} \sum_{s,\bar{s}\in\mathbb{R}} a_{s,\bar{s}} z^s \bar{z}^{\bar{s}} = \sum_{s,\bar{s}\in\mathbb{R}} \bar{s} a_{s,\bar{s}} z^{s} \bar{z}^{\bar{s}-1}.$$

Since $\frac{d}{dz}|z|^s = s|z|^s z^{-1}$, the differential operators $\frac{d}{dz}$ and $\frac{d}{d\bar{z}}$ acts on all the above vector spaces.

Lemma 1.2. *If $f(\underline{z}) \in V((z,\bar{z},|z|^{\mathbb{R}}))$ satisfies $\frac{d}{d\bar{z}} f(\underline{z}) = 0$, then $f(\underline{z}) \in V((z))$, where $V((z))$ is a formal Laurent series with coefficients in V.*

1.2. **Convergence.** In this section, we discuss a convergence of a formal power series in $\mathbb{C}((z,\bar{z},|z|^{\mathbb{R}}))$ and the uniqueness of expansions. We will use $z,\bar{z}$ as both formal variables and the canonical coordinate of $\mathbb{C}$. For any $R \in \mathbb{R}_{>0}$, set $A_R = \{z \in \mathbb{C} \mid 0 < |z| < R\}$, an annulus.

Let $f(\underline{z}) \in \mathbb{C}((z,\bar{z},|z|^{\mathbb{R}}))$. Then, there exists $N \in \mathbb{R}$ such that

$$|z|^N f(\underline{z}) = \sum_{\substack{s,\bar{s}\in\mathbb{R} \\ s,\bar{s}\geq 0}} a_{s,\bar{s}} z^s \bar{z}^{\bar{s}}. \tag{1.1}$$

We say the series $f(\underline{z})$ is absolutely convergent around 0 if there exists $R \in \mathbb{R}_{>0}$ such that the sum $\sum_{s,\bar{s}\in\mathbb{R}} |a_{s,\bar{s}}| R^{s+\bar{s}}$ is convergent. In this case, $f(\underline{z})$ is compactly absolutely-convergent to a continuous function defined on the annulus A_R. We note that the definition of the convergence is independent of the choice of N.

Proposition 1.3. *If $f(\underline{z}) \in \mathbb{C}((z,\bar{z},|z|^{\mathbb{R}}))$ is absolutely convergent around 0, then both $\frac{d}{dz} f(\underline{z})$ and $\frac{d}{d\bar{z}} f(\underline{z})$ are absolutely convergent around 0.*

Proof. We may assume that $f(\underline{z}) = \sum_{\substack{s,\bar{s}\in\mathbb{R} \\ s,\bar{s}\geq 0}} a_{s,\bar{s}} z^s \bar{z}^{\bar{s}}$. Let $R > 0$ be a real number such that $\sum_{\substack{s,\bar{s}\in\mathbb{R} \\ s,\bar{s}\geq 0}} |a_{s,\bar{s}}| R^{s+\bar{s}} < \infty$. Then, $\sum_{\substack{s,\bar{s}\in\mathbb{R} \\ s,\bar{s}\geq 0}} |s a_{s,\bar{s}}| (R/2)^{s+\bar{s}} = \sum_{\substack{s,\bar{s}\in\mathbb{R} \\ s,\bar{s}\geq 0}} |s/2^{s+\bar{s}}| |a_{s,\bar{s}}| R^{s+\bar{s}} < \sum_{\substack{s,\bar{s}\in\mathbb{R} \\ s,\bar{s}\geq 0}} |a_{s,\bar{s}}| R^{s+\bar{s}} < \infty$. □

Remark 1.4. *In the above proof, the fact that the sum runs over $s, \bar{s} \geq 0$ is essential. In fact, $\sum_{n=1}^{\infty} \frac{1}{n^2}(z/\bar{z})^n \in \mathbb{C}[[z/\bar{z}]]$ is convergent, however, its derivative is not convergent well.*

Proposition 1.5. *If $f(\underline{z}) \in \mathbb{C}((z, \bar{z}, |z|^{\mathbb{R}}))$ is absolutely convergent around 0, then $f(\underline{z})$ is a real analytic function on the annulus A_R for some $R > 0$.*

For the proof, we use the following elementary lemma:

Lemma 1.6. *Let $s, r \in \mathbb{R}$. If $s \geq 0$ and $1 > |r|$, then $\sum_{n=0}^{\infty} |\binom{s}{n}| r^n < (1+r)^s + 2\frac{r^{1+s}}{1-r}$.*

proof of Proposition 1.5. We may assume that $f(\underline{z}) = \sum_{\substack{s,\bar{s}\in\mathbb{R}\\ s,\bar{s}\geq 0}} a_{s,\bar{s}} z^s \bar{z}^{\bar{s}}$. Let $R \in \mathbb{R}_{>0}$ such that $\sum_{s,\bar{s}\in\mathbb{R}} |a_{s,\bar{s}}| R^{s+\bar{s}}$ is convergent. Let $\alpha \in A_R$. We will show that $f(\underline{z})$ is a real analytic function around α. By the above lemma, for $w \in \mathbb{C}$ with $|w/\alpha| < 1$ and $|w| + |\alpha| < R$,

(1.2)
$$\sum_{\substack{s,\bar{s}\in\mathbb{R}\\ s,\bar{s}\geq 0}} \sum_{n,m=0}^{\infty} |\binom{s}{n}\binom{\bar{s}}{m}| |a_{s,\bar{s}}| |\alpha|^{s+\bar{s}} |w/\alpha|^{n+m} < \sum_{\substack{s,\bar{s}\in\mathbb{R}\\ s,\bar{s}\geq 0}} |a_{s,\bar{s}}| \Big((|\alpha|+|w|)^s + 2\frac{w^{s+1}}{|\alpha|-|w|}\Big)\Big((|\alpha|+|w|)^{\bar{s}} + 2\frac{w^{\bar{s}+1}}{|\alpha|-|w|}\Big)$$

Since the right-hand-side of (1.2) is convergent by the assumption, the sum

$$\sum_{\substack{s,\bar{s}\in\mathbb{R}\\ s,\bar{s}\geq 0}} \sum_{n,m=0}^{\infty} \binom{s}{n}\binom{\bar{s}}{m} a_{s,\bar{s}} \alpha^{s-n} \bar{\alpha}^{\bar{s}-m} w^n \bar{w}^m$$

is absolutely convergent to $\sum_{\substack{s,\bar{s}\in\mathbb{R}\\ s,\bar{s}\geq 0}} a_{s,\bar{s}} (\alpha+w)^s \overline{(\alpha+w)}^{\bar{s}}$. □

Let $\mathrm{Conv}((z, \bar{z}, |z|^{\mathbb{R}}))$ the subspace of $\mathbb{C}((z, \bar{z}, |z|^{\mathbb{R}}))$ consisting of $f(\underline{z}) \in \mathbb{C}((z, \bar{z}, |z|^{\mathbb{R}}))$ such that $f(\underline{z})$ is absolutely convergent around 0.

Let $\mathrm{St}_0^{\mathrm{real}}$ is a stalk of real analytic functions on the annuli, that is, the colimit of the space of real analytic functions on $\{z \in \mathbb{C} \mid 0 < |z| < R\}$ as $R \to 0$. Then, we have a map

$$\mathrm{Conv}((z, \bar{z}, |z|^{\mathbb{R}})) \to \mathrm{St}_0^{\mathrm{real}}.$$

Then, the following lemma is clear:

Lemma 1.7. *The above map $\mathrm{Conv}((z, \bar{z}, |z|^{\mathbb{R}})) \to \mathrm{St}_0^{real}$ is injective.*

The above lemma says the coefficients of convergent formal power series are uniquely determined.

We note that $\mathrm{Conv}((z, \bar{z}, |z|^{\mathbb{R}}))$ is a differential subalgebra of $\mathrm{St}_0^{\mathrm{real}}$ (closed under derivations and products).

Remark 1.8. *The product $(\sum_{n\in\mathbb{Z}} (z/\bar{z})^n) \cdot (\sum_{n\in\mathbb{Z}} (z/\bar{z})^n)$ is not well-defined.*

1.3. **Conformal singularity.** Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}P^1$ and f be a $\mathbb{C}$-valued real analytic function on $\mathbb{C}P^1 \setminus \{\alpha_1, \ldots, \alpha_n\}$. A chart (χ, α) of $\mathbb{C}P^1$ at a point $\alpha \in \mathbb{C}P^1$ is a biholomorphism χ from an open subset U of $\mathbb{C}P^1$ to an open subset of $\mathbb{C}$ such that $\alpha \in U$ and $\chi(\alpha) = 0$. We say that f has a *conformal singularity* at α_i if for any chart (χ, α_i) of $\mathbb{C}P^1$ at α_i, there exists a formal power series

(1.3)
$$\sum_{s,\bar{s}\in\mathbb{R}} a_{s,\bar{s}} z^s \bar{z}^{\bar{s}} \in \mathrm{Conv}((z, \bar{z}, |z|^{\mathbb{R}}))$$

such that it is compactly absolutely-convergent to $f \circ \chi^{-1}(z)$ in the annulus A_R for some $R \in \mathbb{R}_{>0}$. It is clear that the above condition is independent of a choice of a chart and by Lemma 1.7, the coefficients of the series is uniquely determined by the chart. Let f have a conformal singularity at α_i.

Denote by $j(\chi, f) \in \mathrm{Conv}((z, \bar{z}, |z|^{\mathbb{R}}))$ the formal power series which is compactly absolutely-convergent to $f \circ \chi^{-1}(z)$, and by $F_{0,1,\infty}$ the space of real analytic functions on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ with possible conformal singularities at $\{0, 1, \infty\}$.

Examples of functions belonging to $F_{0,1,\infty}$ are

$$|z|^r, |1-z|^r, z^n(1-z)^n, (1-\bar{z})^n \in F_{0,1,\infty},$$

where $r \in \mathbb{R}$ and $n \in \mathbb{Z}$. For instance, the expansions of $|1-z|^r$ are

$$\begin{aligned} j(z, |1-z|^r) &= \sum_{n,m=0}^{\infty} \binom{r}{n}\binom{r}{m} z^n \bar{z}^m, \\ j(1-z, |1-z|^r) &= |z|^r, \\ j(z^{-1}, |1-z|^r) &= \sum_{n,m=0}^{\infty} \binom{r}{n}\binom{r}{m} z^{n-r} \bar{z}^{m-r}, \end{aligned}$$

where $z, 1-z, z^{-1}$ are charts of $0, 1, \infty$, respectively. In fact, $F_{0,1,\infty}$ is a $\mathbb{C}[z^{\pm}, (1-z)^{\pm}, \bar{z}^{\pm}, (1-\bar{z})^{\pm}, |1-z|^{\mathbb{R}}]$-module.

A non-trivial example of a function in $F_{0,1,\infty}$ is

$$f_{\mathrm{Ising}}(z) = \frac{1}{2}\Big(|1 - \sqrt{1-z}|^{1/2} + |1 + \sqrt{1-z}|^{1/2}\Big), \tag{1.4}$$

which appears in a four point function of the 2 dimensional Ising model (see [Mo4]). The expansion of $f_{\mathrm{Ising}}(z)$ around 0 with the chart z is

$$2 + |z|^{1/2}/2 - z/4 - \bar{z}/4 + |z|^{1/2}(z+\bar{z})/16 + z\bar{z}/32 - 5z^2/64 - 5\bar{z}^2/64 + \dots. \tag{1.5}$$

Since $f_{\mathrm{Ising}}(z)$ satisfies the equations $f_{\mathrm{Ising}}(z) = f_{\mathrm{Ising}}(1-z) = (z\bar{z})^{1/4} f_{\mathrm{Ising}}(1/z)$, the expansions around 1 and ∞ are also of the form 1.3. Thus, $f_{\mathrm{Ising}}(z) \in F_{0,1,\infty}$.

More generally, a monodromy invariant combination of solutions of (holomorphic and anti-holomorphic) KZ-equations belongs to $F_{0,1,\infty}$.

Finally, we remark on the case that $f \in F_{0,1,\infty}$ is a holomorphic function. Recall that the ring of regular functions on the affine scheme $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ is $\mathbb{C}[z^{\pm}, (1-z)^{\pm}]$. It is easy to show that a function in $\mathbb{C}[z^{\pm}, (1-z)^{\pm}]$ has conformal singularities at $\{0, 1, \infty\}$. Thus, $\mathbb{C}[z^{\pm}, (1-z)^{\pm}] \subset F_{0,1,\infty}$. Conversely, let $f \in F_{0,1,\infty}$ satisfy $\frac{d}{d\bar{z}} f = \frac{1}{2}(\frac{d}{dx} - i\frac{d}{dy}) f = 0$. Then, by Lemma 1.2, f is a holomorphic function on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ with possible poles at $\{0, 1, \infty\}$, thus, a meromorphic function on $\mathbb{C}P^1$. Hence, $f \in \mathbb{C}[z^{\pm}, (1-z)^{\pm}]$.

Proposition 1.9. *If $f \in F_{0,1,\infty}$ is a holomorphic function on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$, then $f \in \mathbb{C}[z^{\pm}, (1-z)^{\pm}]$.*

1.4. **Generalized two-point Correlation function.** This section is devoted to defining and studying a space of generalized two-point functions. Set

$$U(y, z) = \mathbb{C}((z/y, \bar{z}/\bar{y}, |z/y|^{\mathbb{R}}))[y^{\pm}, \bar{y}^{\pm}, |y|^{\mathbb{R}}]$$

and

$$Y_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq z_2, z_1 \neq 0, z_2 \neq 0\}.$$

Let $\eta(z_1, z_2) : Y_2 \to \mathbb{C}P^1 \setminus \{0, 1, \infty\}$ be the real analytic function defined by $\eta(z_1, z_2) = \frac{z_2}{z_1}$. For $f \in F_{0,1,\infty}$, $f \circ \eta$ is a real analytic function on Y_2. Denote by GCor_2 the space of real analytic functions on Y_2 spanned by

$$z_1^{\alpha}\bar{z}_1^{\beta} f \circ \eta(z_1, z_2), \tag{1.6}$$

where $f \in F_{0,1,\infty}$ and $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha - \beta \in \mathbb{Z}$.

It is clear that GCor_2 is closed under the product and the derivations $\frac{d}{dz_1}, \frac{d}{d\bar{z}_1}, \frac{d}{dz_2}, \frac{d}{d\bar{z}_2}$. Since $(z_1\frac{d}{dz_1} + z_2\frac{d}{dz_2})z_1^{\alpha}\bar{z}_1^{\beta} f \circ \eta(z_1, z_2) = \alpha z_1^{\alpha}\bar{z}_1^{\beta} f \circ \eta(z_1, z_2)$ and $(\bar{z}_1\frac{d}{d\bar{z}_1} + \bar{z}_2\frac{d}{d\bar{z}_2})z_1^{\alpha}\bar{z}_1^{\beta} f \circ \eta(z_1, z_2) = \beta z_1^{\alpha}\bar{z}_1^{\beta} f \circ \eta(z_1, z_2)$, we have:

Lemma 1.10. *Let $\mu \in \mathrm{GCor}_2$ satisfy $(z_1\frac{d}{dz_1} + z_2\frac{d}{dz_2})\mu = \alpha\mu$ and $(\bar{z}_1\frac{d}{d\bar{z}_1} + \bar{z}_2\frac{d}{d\bar{z}_2})\mu = \beta\mu$ for some $\alpha, \beta \in \mathbb{R}$. Then, there exists unique $f \in F_{0,1,\infty}$ such that $\mu(z_1, z_2) = z_1^{\alpha}\bar{z}_1^{\beta} f(\frac{z_1}{z_2})$.*

Let $\mu(z_1, z_2) = z_1^{\alpha}\bar{z}_1^{\beta} f \circ \eta(z_1, z_2)$ in (1.6). The expansions of $\mu(z_1, z_2)$ in $\{|z_1| > |z_2|\}$ and $\{|z_2| > |z_1|\}$ are respectively given by

$$z_1^{\alpha}\bar{z}_1^{\beta} \lim_{z \to z_2/z_1} j(z, f)$$
$$z_1^{\alpha}\bar{z}_1^{\beta} \lim_{z \to z_1/z_2} j(z^{-1}, f),$$

which define maps

$$|_{|z_1|>|z_2|} : \mathrm{GCor}_2 \to U(z_1, z_2),\ \mu(z_1, z_2) \mapsto \mu(z_1, z_2)|_{|z_1|>|z_2|}$$

and

$$|_{|z_2|>|z_1|} : \mathrm{GCor}_2 \to U(z_2, z_1),\ \mu(z_1, z_2) \mapsto \mu(z_1, z_2)|_{|z_2|>|z_1|}.$$

Since $f(\frac{z_2}{z_1}) = f(\frac{z_2}{z_2 + (z_1 - z_2)})$, the expansions of μ in $\{|z_2| > |z_1 - z_2|\}$ is given by

$$z_2^{\alpha}\bar{z}_2^{\beta} \sum_{i,j \geq 0} \binom{\alpha}{i}\binom{\beta}{j} (z_0/z_2)^i (\bar{z}_0/\bar{z}_2)^j \lim_{z \to -z_0/z_2} j(1 - z^{-1}, f),$$

where $z_0 = z_1 - z_2$. We denote it by

$$|_{|z_2|>|z_1 - z_2|} : \mathrm{GCor}_2 \to U(z_2, z_0), \mu(z_1, z_2) \mapsto \mu(z_1, z_2)|_{|z_2|>|z_1 - z_2|}.$$

Then, we have:

Lemma 1.11. *For $f \in F_{0,1,\infty}$,*

$$f \circ \eta|_{|z_1|>|z_2|} = \lim_{z \to z_1/z_2} j(z, f),$$
$$f \circ \eta|_{|z_2|>|z_1|} = \lim_{z \to z_2/z_1} j(z^{-1}, f),$$
$$f \circ \eta|_{|z_2|>|z_1 - z_2|} = \lim_{z \to -z_0/z_2} j(1 - z^{-1}, f).$$

The following lemma connects a full vertex algebra (real analytic) and a vertex algebra (holomorphic):

Lemma 1.12. *Let $\mu(z_1, z_2) \in \mathrm{GCor}_2$ satisfies $\frac{d}{d\bar{z}_1}\mu = 0$, $(z_1\frac{d}{dz_1} + z_2\frac{d}{dz_2})\mu = \alpha\mu$ and $(\bar{z}_1\frac{d}{d\bar{z}_1} + \bar{z}_2\frac{d}{d\bar{z}_2})\mu = \beta\mu$ for some $\alpha, \beta \in \mathbb{R}$. Then, $\mu(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, (z_1 - z_2)^{\pm}, z_2^{\pm}, \bar{z}_2^{\pm}, |z_2|^{\mathbb{R}}]$. Furthermore, if $\frac{d}{d\bar{z}_2}\mu = 0$, then $\mu(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}, (z_1 - z_2)^{\pm}]$.*

Proof. By Lemma 1.10, there exits $f \in F_{0,1,\infty}$ such that $\mu(z_1, z_2) = z_2^{\alpha}\bar{z}_2^{\beta} f(z_1/z_2)$. By $\frac{d}{d\bar{z}_1}\mu = 0$, f is holomorphic and by Proposition 1.9, $f \in \mathbb{C}[z^{\pm}, (1-z)^{\pm}]$. Thus, $\mu \in \mathbb{C}[z_1^{\pm}, (z_1 - z_2)^{\pm}, z_2^{\pm}, \bar{z}_2^{\pm}, |z_2|^{\mathbb{R}}]$. If $\frac{d}{d\bar{z}_2}\mu = 0$, then $\beta = 0$ and $\alpha \in \mathbb{Z}$. Hence, the assertion holds. □

The space of holomorphic generalized two-point correlation functions is denoted by $\mathrm{GCor}_2^{\mathrm{chiral}}$, that is,

$$\mathrm{GCor}_2^{\mathrm{hol}} = \mathbb{C}[z_1^{\pm}, z_2^{\pm}, (z_1 - z_2)^{\pm}].$$

2. Full vertex algebra

In this section, we introduce the notion of a full vertex algebra, which is a generalization of a $\mathbb{Z}$-graded vertex algebra.

2.1. Definition of $\mathbb{Z}$-graded vertex algebra.

We first recall the definition of a $\mathbb{Z}$-graded vertex algebra.

For a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$, set $V^\vee = \bigoplus_{n\in\mathbb{Z}} V_n^*$, where V_n^* is the dual vector space of V_n.

A $\mathbb{Z}$-graded vertex algebra is a $\mathbb{Z}$-graded $\mathbb{C}$-vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ equipped with a linear map

$$Y(-,z) : V \to \mathrm{End}\,(V)[[z^{\pm}]],\ a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}$$

and an element $\mathbf{1} \in V_0$ satisfying the following conditions:

V1) For any $a, b \in F$, $Y(a,z)b \in V((z))$;
V2) For any $a \in V$, $Y(a,z)\mathbf{1} \in V[[z,\bar{z}]]$ and $\lim_{z\to 0} Y(a,z)\mathbf{1} = a(-1)\mathbf{1} = a$;
V3) $Y(\mathbf{1},z) = \mathrm{id} \in \mathrm{End}\,V$;
V4) For any $a,b,c \in V$ and $u \in V^\vee$, there exists $\mu(z_1,z_2) \in \mathrm{GCor}_2^{\mathrm{hol}}$ such that

$$u(Y(a,z_1)Y(b,z_2)c) = \mu|_{|z_1|>|z_2|},$$
$$u(Y(Y(a,z_0)b,z_2)c) = \mu|_{|z_2|>|z_1-z_2|},$$
$$u(Y(b,z_2)Y(a,z_1)c) = \mu|_{|z_2|>|z_1|},$$

where $z_0 = z_1 - z_2$;
V5) $V_n(r)V_m \subset V_{n+m-r-1}$ for any $n, m, r \in \mathbb{Z}$.

Remark 2.1. *A standard definition of a vertex algebra uses the Borcherds identity (the Jacobi identity). The above definition of a $\mathbb{Z}$-graded vertex algebra is slightly different from the standard one, but, is equivalent (see for example* [FLM, FB]*). We do not use the Borcherds identity since it seems difficult to obtain such an algebraic identity in the case of non-chiral conformal field theory in general.*

In the next section, we change $\mathbb{C}((z))$ into $\mathbb{C}((z,\bar{z},|z|^{\mathbb{R}}))$ and $\mathrm{GCor}_2^{\mathrm{hol}}$ into GCor_2, or, meromorphic functions with possible poles to real analytic functions with possible conformal singularities, and define a full vertex algebra.

2.2. Definition of full vertex algebra.

For an $\mathbb{R}^2$-graded vector space $F = \bigoplus_{h,\bar{h}\in\mathbb{R}^2} F_{h,\bar{h}}$, set $F^\vee = \bigoplus_{h,\bar{h}\in\mathbb{R}^2} F_{h,\bar{h}}^*$, where $F_{h,\bar{h}}^*$ is the dual vector space of $F_{h,\bar{h}}$. A full vertex algebra is an $\mathbb{R}^2$-graded $\mathbb{C}$-vector space $F = \bigoplus_{h,\bar{h}\in\mathbb{R}^2} F_{h,\bar{h}}$ equipped with a linear map

$$Y(-,\underline{z}) : F \to \mathrm{End}\,(F)[[z^{\pm},\bar{z}^{\pm},|z|^{\mathbb{R}}]],\ a \mapsto Y(a,\underline{z}) = \sum_{r,s\in\mathbb{R}} a(r,s) z^{-r-1}\bar{z}^{-s-1}$$

and an element $\mathbf{1} \in F_{0,0}$ satisfying the following conditions:

FV1) For any $a, b \in F$, $Y(a,\underline{z})b \in F((z,\bar{z},|z|^{\mathbb{R}}))$;
FV2) $F_{h,\bar{h}} = 0$ unless $h - \bar{h} \in \mathbb{Z}$;
FV3) For any $a \in F$, $Y(a,\underline{z})\mathbf{1} \in F[[z,\bar{z}]]$ and $\lim_{z\to 0} Y(a,\underline{z})\mathbf{1} = a(-1,-1)\mathbf{1} = a$;
FV4) $Y(\mathbf{1},\underline{z}) = \mathrm{id} \in \mathrm{End}\,F$;

FV5) For any $a, b, c \in F$ and $u \in F^\vee$, there exists $\mu(z_1, z_2) \in \mathrm{GCor}_2$ such that

$$u(Y(a, \underline{z}_1)Y(b, \underline{z}_2)c) = \mu|_{|z_1|>|z_2|},$$
$$u(Y(Y(a, \underline{z}_0)b, \underline{z}_2)c) = \mu|_{|z_2|>|z_1-z_2|},$$
$$u(Y(b, \underline{z}_2)Y(a, \underline{z}_1)c) = \mu|_{|z_2|>|z_1|},$$

where $z_0 = z_1 - z_2$;

FV6) $F_{h,\bar{h}}(r, s)F_{h',\bar{h}'} \subset F_{h+h'-r-1,\bar{h}+\bar{h}'-s-1}$ for any $r, s, h, h', \bar{h}, \bar{h}' \in \mathbb{R}$.

Remark 2.2. *Physically, the energy and the spin of a state in $F_{h,\bar{h}}$ are $h+\bar{h}$ and $h-\bar{h}$. Thus, the condition (FV2) implies that we only consider the particles whose spin is an integer, that is, we consider only bosons and not fermions. The notion of a full super vertex algebra can be defined by modifying (FV5) and (FV2).*

Remark 2.3. *Define the linear map $L(0), \overline{L}(0) \in \mathrm{End}\, F$ by $L(0)|_{F_{h,\bar{h}}} = h$ and $\overline{L}(0)|_{F_{h,\bar{h}}} = \bar{h}$ for any $h, \bar{h} \in \mathbb{R}$. Then, the condition (FV6) is equivalent to the the following condition: For any $h, \bar{h} \in \mathbb{R}$ and $a \in F_{h,\bar{h}}$,*

$$[L(0), Y(a, \underline{z})] = (z\frac{d}{dz} + h)Y(a, \underline{z}),$$
$$[\overline{L}(0), Y(a, \underline{z})] = (\bar{z}\frac{d}{d\bar{z}} + \bar{h})Y(a, \underline{z}).$$

Since $V((z)) \subset V((z, \bar{z}, |z|^{\mathbb{R}}))$ and $\mathrm{GCor}_2^{\mathrm{hol}} \subset \mathrm{GCor}_2$, we have:

Proposition 2.4. *A $\mathbb{Z}$-graded vertex algebra is a full vertex algebra.*

Let F be an $\mathbb{R}^2$-graded vector space. The set $\{(h, \bar{h}) \in \mathbb{R}^2 \mid F_{h,\bar{h}} \neq 0\}$ is called a spectrum. The spectrum of F is said to be *bounded below* if there exists $N \in \mathbb{R}$ such that $F_{h,\bar{h}} = 0$ for any $h \leq N$ or $\bar{h} \leq N$ and *discrete* if for any $H \in \mathbb{R}$, $\sum_{h+\bar{h}<H} \dim F_{h,\bar{h}}$ is finite and *compact* if it is both bounded below and discrete. A full vertex algebra with a compact spectrum is called a *compact full vertex algebra*. Many interesting models in conformal field theory (e.g., rational conformal field theory and its deformation) have a compact spectrum. The definition of $\mathbb{C}((z, \bar{z}, |z|^{\mathbb{R}}))$ is motivated by the following proposition:

Proposition 2.5. *Let F be an $\mathbb{R}^2$-graded vector space with a compact spectrum and $F_{h,\bar{h}} = 0$ unless $h - \bar{h} \in \mathbb{Z}$ and a linear map $Y(-, \underline{z}) : F \to \mathrm{End}\, F[[z, \bar{z}, |z|^{\mathbb{R}}]]$ satisfy the following condition:*

For any $h, \bar{h} \in \mathbb{R}$ and $a \in F_{h,\bar{h}}$,

$$[L(0), Y(a, \underline{z})] = (z\frac{d}{dz} + h)Y(a, \underline{z}),$$
$$[\overline{L}(0), Y(a, \underline{z})] = (\bar{z}\frac{d}{d\bar{z}} + \bar{h})Y(a, \underline{z}). \tag{2.1}$$

Then, $Y(a, \underline{z})b \in F((z, \bar{z}, |z|^{\mathbb{R}}))$ for any $a, b \in F$.

Proof. Set $Y(a, \underline{z})b = \sum_{r,s\in\mathbb{R}^2} v_{r,s} z^r \bar{z}^s$ where $v_{r,s} \in F$. Since the spectrum is bounded below, by (2.1), which is equivalent to (FV6), there exists $N \in \mathbb{R}$ such that $v_{r,s} = 0$ unless $r, s \geq N$. Similarly, Since the spectrum is discrete, $\#\{(r, s) \in \mathbb{R}^2 \mid v_{r,s} \neq 0 \text{ and } r + s \leq H\}$ is finite for any $H \in \mathbb{R}$. □

Remark 2.6. *By the above proposition, a full vertex algebra is a formulation of a compact two-dimensional conformal field theory on $\mathbb{R}^2$.*

Let $(F^1, Y^1, \mathbf{1}^1)$ and $(F^2, Y^2, \mathbf{1}^2)$ be full vertex algebras. A full vertex algebra homomorphism from F^1 to F^2 is a linear map $f : F^1 \to F^2$ such that

(1) $f(\mathbf{1}^1) = \mathbf{1}^2$

(2) $f(Y^1(a, \underline{z})-) = Y^2(f(a), \underline{z})f(-)$ for any $a \in F^1$.

The notions of a subalgebra and a left ideal are defined in the usual way. A simple full vertex algebra is a full vertex algebra which contains no proper left ideals.

A module of a full vertex algebra F is an $\mathbb{R}^2$-graded $\mathbb{C}$-vector space $M = \bigoplus_{h,\bar{h}\in\mathbb{R}^2} M_{h,\bar{h}}$ equipped with a linear map

$$Y_M(-, \underline{z}) : F \to \mathrm{End}\,(M)[[z^\pm, \bar{z}^\pm, |z|^\mathbb{R}]],\ a \mapsto Y_M(a, z) = \sum_{r,s\in\mathbb{R}} a(r, s)z^{-r-1}\bar{z}^{-s-1}$$

satisfying the following conditions:

FM1) For any $a \in F$ and $m \in M$, $Y(a, \underline{z})m \in M((z, \bar{z}, |z|^\mathbb{R}))$;

FM2) $Y_M(\mathbf{1}, \underline{z}) = \mathrm{id} \in \mathrm{End}\, M$;

FM3) For any $a, b \in F$, $m \in M$ and $u \in M^\vee$, there exists $\mu \in \mathrm{GCor}_2$ such that

$$u(Y_M(a, \underline{z}_1)Y_M(b, \underline{z}_2)m) = \mu|_{|z_1|>|z_2|},$$
$$u(Y_M(Y_M(a, \underline{z}_0)b, \underline{z}_2)m) = \mu|_{|z_2|>|z_1-z_2|},$$
$$u(Y_M(b, \underline{z}_2)Y_M(a, \underline{z}_1)m) = \mu|_{|z_2|>|z_1|};$$

FM4) $F_{h,\bar{h}}(r, s)M_{h',\bar{h}'} \subset M_{h+h'-r-1,\bar{h}+\bar{h}'-s-1}$ for any $r, s, h, h', \bar{h}, \bar{h}' \in \mathbb{R}$.

As a consequence of (FM1) and (FM3), we have:

Lemma 2.7. *Let $h_i, \bar{h}_i \in \mathbb{R}$, $a_i \in F_{h_i,\bar{h}_i}$, $(i = 1, 2)$, $m \in M_{h_3,\bar{h}_3}$ and $u \in M^*_{h_0,\bar{h}_0}$. Then, $u(Y(a_1, \underline{z}_1)Y(a_2, \underline{z}_2)m) \in z_1^{h_0-h_1-h_2-h_3}\bar{z}_1^{\bar{h}_0-\bar{h}_1-\bar{h}_2-\bar{h}_3}\mathbb{C}((z_2/z_1, \bar{z}_2/\bar{z}_1, |z_2/z_1|^\mathbb{R}))$.*

Proof. Set

$$\sum_{s_1,\bar{s}_1,\bar{s}_2,\bar{s}_2\in\mathbb{R}} c_{s_1,\bar{s}_1,\bar{s}_2,\bar{s}_2} z_1^{s_1}\bar{z}_1^{\bar{s}_1}z_2^{s_2}\bar{z}_2^{\bar{s}_2} = u(Y(a_1, \underline{z}_1)Y(a_2, \underline{z}_2)m).$$

Then,

$$c_{s_1,\bar{s}_1,s_2,\bar{s}_2} = u(a_1(-s_1 - 1, -\bar{s}_1 - 1)a_2(-s_2 - 1, -\bar{s}_2 - 1)m).$$

By (FM4), $a_1(-s_1 - 1, -\bar{s}_1 - 1)a_2(-s_2 - 1, -\bar{s}_2 - 1)m \in M_{h_1+h_2+h_3+s_1+s_2,\bar{h}_1+\bar{h}_2+\bar{h}_3+\bar{s}_1+\bar{s}_2}$. Hence, $c_{s_1,\bar{s}_1,s_2,\bar{s}_2} = 0$ unless $h_0 = h_1 + h_2 + h_3 + s_1 + s_2$ and $\bar{h}_0 = \bar{h}_1 + \bar{h}_2 + \bar{h}_3 + \bar{s}_1 + \bar{s}_2$. Thus, we have

$$u(Y(a_1, \underline{z}_1)Y(a_2, \underline{z}_2)m) = z_1^{h_0-h_1-h_2-h_3}\bar{z}_1^{\bar{h}_0-\bar{h}_1-\bar{h}_2-\bar{h}_3} \sum_{s_2,\bar{s}_2\in\mathbb{R}} c_{s_1,\bar{s}_1,s_2,\bar{s}_2}(z_2/z_1)^{s_2}(\bar{z}_2/\bar{z}_1)^{\bar{s}_2},$$

where $s_1 = h_0 - (h_1 + h_2 + h_3 + s_2)$ and $\bar{s}_1 = \bar{h}_0 - (\bar{h}_1 + \bar{h}_2 + \bar{h}_3 + \bar{s}_2)$. By (FM1), the assertion holds. □

By Lemma 2.7 and Lemma 1.10, we have:

Lemma 2.8. *Let $h_i, \bar{h}_i \in \mathbb{R}$, $a_i \in F_{h_i,\bar{h}_i}$ $(i = 1, 2)$, $m \in M_{h_3,\bar{h}_3}$ and $u \in M^*_{h_0,\bar{h}_0}$, there exists $f \in F_{0,1,\infty}$ such that*

$$z_2^{-h_0+h_1+h_2+h_3}\bar{z}_2^{-\bar{h}_0+\bar{h}_1+\bar{h}_2+\bar{h}_3}u(Y(a, \underline{z}_1)Y(b, \underline{z}_2)m) = \lim_{z\to z_1/z_2} j(z, f),$$
$$z_2^{-h_0+h_1+h_2+h_3}\bar{z}_2^{-\bar{h}_0+\bar{h}_1+\bar{h}_2+\bar{h}_3}u(Y(Y(a, \underline{z}_0)b, \underline{z}_2)m) = \lim_{z\to -z_0/z_2} j(1 - z^{-1}, f),$$
$$z_2^{-h_0+h_1+h_2+h_3}\bar{z}_2^{-\bar{h}_0+\bar{h}_1+\bar{h}_2+\bar{h}_3}u(Y(b, \underline{z}_2)Y(a, \underline{z}_1)m) = \lim_{z\to z_2/z_1} j(1/z, f).$$

Let M, N be a F-module. A F-module homomorphism from M to N is a linear map $f : M \to N$ such that $f(Y_M(a,\underline{z})-) = Y_N(a,\underline{z})f(-)$ for any $a \in F$.

Let M be a F-module. As an analogy of [L], a vector $v \in M$ is said to be a vacuum-like vector if $Y(a,\underline{z})v \in M[[z,\bar{z}]]$ for any $a \in F$.

Lemma 2.9. *Let $v \in M$ be a vacuum-like vector and $a, b \in F$ and $u \in M^\vee$ and $\mu \in \mathrm{GCor}_2$ satisfy $u(Y(a_1,\underline{z}_1)Y(a_2,\underline{z}_2)v) = \mu|_{|z_1|>|z_2|}$. Then, $\mu(z_1,z_2) \in \mathbb{C}[z_2^\pm, \bar{z}_2^\pm, (z_1-z_2)^\pm, (\bar{z}_1-\bar{z}_2)^\pm, |z_1-z_2|^\mathbb{R}]$. Furthermore, the linear function $F_v : F \to M$ defined by $a \mapsto a(-1,-1)v$ is a F-module homomorphism.*

Proof. By (FM3), $u(Y(Y(a_1,\underline{z}_0)a_2,\underline{z}_2)v) = \mu|_{|z_1|>|z_1-z_2|}$. Since v is a vacuum like vector, by Lemma 2.7 $p(z_0,z_2) = \mu|_{|z_1|>|z_1-z_2|} \in \mathbb{C}[z_0^\pm, \bar{z}_0^\pm, |z_0|^\mathbb{R}, z_2, \bar{z}_2] \subset U(z_2,z_0)$, which proves the first part of the lemma. It suffices to show that $F_v(Y(a_1,\underline{z}_0)a_2) = Y(a_1,\underline{z}_0)F_v(a_2)$. Since

$$\begin{aligned}u(Y(a_1,\underline{z}_1)Y(a_2,\underline{z}_2)v) &= \mu|_{|z_1|>|z_2|}\\ &= \lim_{z_0\to(z_1-z_2)|_{|z_1|>|z_2|}} p(z_0,z_2),\end{aligned}$$

we have

$$u(Y(a_1,\underline{z}_0)Y(a_2,\underline{z}_2)v) = \exp(-z_2\frac{d}{dz_0} - \bar{z}_2\frac{d}{d\bar{z}_0})u(Y(Y(a_1,\underline{z}_0)a_2,\underline{z}_2)v). \tag{2.2}$$

Thus,

$$\begin{aligned}Y(a_1,\underline{z}_0)F_v(a_2) &= \lim_{z_2\mapsto 0} u(Y(a_1,\underline{z}_0)Y(a_2,\underline{z}_2)v)\\ &= \lim_{z_2\mapsto 0}\exp(-z_2\frac{d}{dz_0} - \bar{z}_2\frac{d}{d\bar{z}_0})u(Y(Y(a_1,\underline{z}_0)a_2,\underline{z}_2)v)\\ &= F_v(Y(a_1,\underline{z}_0)a_2).\end{aligned}$$

□

Let F be a full vertex algebra and D and $\bar{D}$ denote the endomorphism of F defined by $Da = a(-2,-1)\mathbf{1}$ and $\bar{D}a = a(-1,-2)$ for $a \in F$, i.e.,

$$Y(a,z)\mathbf{1} = a + Daz + \bar{D}a\bar{z} + \dots.$$

Define $Y(a,-\underline{z})$ by $Y(a,-\underline{z}) = \sum_{r,s}(-1)^{r-s}a(r,s)z^r\bar{z}^s$, where we used $a(r,s) = 0$ for $r - s \notin \mathbb{Z}$, which follows from (FV2) and (FV6).

Proposition 2.10. *For $a \in F$, the following properties hold:*

(1) $Y(Da,\underline{z}) = \frac{d}{dz}Y(a,\underline{z})$ and $Y(\bar{D}a,\underline{z}) = \frac{d}{d\bar{z}}Y(a,\underline{z})$;
(2) $D\mathbf{1} = \bar{D}\mathbf{1} = 0$;
(3) $[D,\bar{D}] = 0$;
(4) $Y(a,\underline{z})b = \exp(zD + \bar{z}\bar{D})Y(b,-\underline{z})a$;
(5) $Y(\bar{D}a,\underline{z}) = [\bar{D}, Y(a,\underline{z})]$ and $Y(Da,\underline{z}) = [D, Y(a,\underline{z})]$.

Proof. Let $u \in F^\vee$ and $a, b \in F$ and $\mu_1, \mu_2 \in \mathrm{GCor}_2$ satisfy

$$u(Y(a,\underline{z}_1)Y(\mathbf{1},\underline{z}_2)b) = \mu_1|_{|z_1|>|z_2|}, u(Y(a,\underline{z}_1)Y(b,\underline{z}_2)\mathbf{1}) = \mu_2|_{|z_1|>|z_2|}.$$

By (FV4) and (FV5), $p_1(z_1) = \mu_1|_{|z_1|>|z_2|} \in \mathbb{C}[z_1^\pm, \bar{z}_1^\pm, |z_1|^\mathbb{R}]$. Then,

$$u(Y(Y(a,\underline{z}_0)\mathbf{1},\underline{z}_2)b) = \mu_1|_{|z_2|>|z_1-z_2|} = \lim_{z_1\to z_2}\exp(z_0\frac{d}{dz_1})\exp(\bar{z}_0\frac{d}{d\bar{z}_1})p_1(z_1).$$

Thus, $u(Y(Da,\underline{z}_2)b) = \lim_{z_1\to z_2}\frac{d}{dz_1}p_1(z_1) = \frac{d}{dz_2}u(Y(a,z_2)b)$, which implies that $Y(Da,\underline{z}) = \frac{d}{dz}Y(a,\underline{z})$ and similarly $Y(\bar{D}a,\underline{z}) = \frac{d}{d\bar{z}}Y(a,\underline{z})$.

By (FV4), $Y(D\mathbf{1},\underline{z}) = \frac{d}{dz}Y(\mathbf{1},\underline{z}) = 0$. Thus, by (FV3), $D\mathbf{1} = \bar{D}\mathbf{1} = 0$. Since $Y(D\bar{D}a,\underline{z}) = \frac{d}{dz}\frac{d}{d\bar{z}}Y(a,\underline{z}) = \frac{d}{d\bar{z}}\frac{d}{dz}Y(a,\underline{z}) = Y(\bar{D}Da,\underline{z})$, we have $[D,\bar{D}] = 0$.

By Lemma 2.9, $\mu_2|_{|z_2|>|z_1-z_2|} \in \mathbb{C}[z_2,\bar{z}_2][z_0^{\pm},\bar{z}_0^{\pm},|z_0|^{\mathbb{R}}]$. Set $p(z_0,z_2) = \mu_2|_{|z_2|>|z_1-z_2|} = u(Y(Y(a,\underline{z}_0)b,\underline{z}_2)\mathbf{1})$. Since $u(Y(Y(b,-\underline{z}_0)a,\underline{z}_1)\mathbf{1}) = p(z_0,z_1-z_0)|_{|z_1|>|z_0|}$, we have

$$
\begin{aligned}
u(Y(a,\underline{z}_0)b) = p(z_0,0) &= \lim_{z_1\to 0}\exp(z_0\frac{d}{dz_1}+\bar{z}_0\frac{d}{d\bar{z}_1})p(z_0,z_1-z_0)\\
&= \lim_{z_1\to 0}\exp(z_0\frac{d}{dz_1}+\bar{z}_0\frac{d}{d\bar{z}_1})u(Y(Y(b,-\underline{z}_0)a,\underline{z}_1)\mathbf{1})\\
&= \lim_{z_1\to 0}u(Y(\exp(z_0D+\bar{z}_0\bar{D})Y(b,-\underline{z}_0)a,\underline{z}_1)\mathbf{1})\\
&= u(\exp(z_0D+\bar{z}_0\bar{D})Y(b,-\underline{z}_0)a).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{d}{dz}Y(a,\underline{z})b &= \frac{d}{dz}\exp(Dz+\bar{D}\bar{z})Y(b,-\underline{z})a\\
&= D\exp(Dz+\bar{D}\bar{z})Y(b,-\underline{z})a - \exp(Dz+\bar{D}\bar{z})Y(Db,-\underline{z})a\\
&= DY(a,\underline{z})b - Y(a,\underline{z})Db.
\end{aligned}
$$

□

We will use the following lemma:

Lemma 2.11. *Let F be a simple full vertex algebra and $a,b \in F$. If $Y(a,\underline{z})b = 0$, then $a = 0$ or $b = 0$.*

Proof. Let (b) be the left ideal generated by b, that is, $(b) = \{c_1(r_1,s_1)c_2(r_2,s_2)\dots c_k(r_k,s_k)b\}$. We will show that $u(Y(a,z)c_1(r_1,s_1)c_2(r_2,s_2)\dots c_k(r_k,s_k)b) = 0$ for any $u \in F^\vee$, $k \in \mathbb{Z}_{\geq 0}$, $c_i \in F$ and $r_i,s_i \in \mathbb{R}$ $(i = 1,\dots,k)$ by induction on k. For $k = 0$, the assertion is clear. For $k \geq 1$, by the induction assumption, $u(Y(c_1,\underline{z}_2)Y(a,\underline{z}_1)c_2(r_2,s_2)\dots c_k(r_k,s_k)b) = 0$. Thus, by (FV5), $u(Y(a,\underline{z}_1)Y(c_1,\underline{z}_2)c_2(r_2,s_2)\dots c_k(r_k,s_k)b) = 0$. Hence, the assertion holds. Assume that $b \neq 0$. Then, since F is simple, $\mathbf{1} \in (b)$. Thus, a must be 0 by (FV3). □

Let $(F,Y,\mathbf{1})$ be a full vertex algebra. Set $\bar{F} = F$ and $\bar{F}_{h,\bar{h}} = F_{\bar{h},h}$ for $h,\bar{h} \in \mathbb{R}$. Define $\bar{Y}(-,\underline{z}) : \bar{F} \to \mathrm{End}(\bar{F})[[z,\bar{z},|z|^{\mathbb{R}}]]$ by $\bar{Y}(a,\underline{z}) = \sum_{s,\bar{s}\in\mathbb{R}} a(s,\bar{s})\bar{z}^{-s-1}z^{-\bar{s}-1}$. Let $C : Y_2 \to Y_2$ be the conjugate map $(z_1,z_2) \mapsto (\bar{z}_1,\bar{z}_2)$ for $(z_1,z_2) \in Y_2$. For $u \in \bar{F}^\vee$ and $a,b,c \in \bar{F}$, let $\mu \in \mathrm{GCor}_2$ satisfy $u(Y(a,\underline{z}_1)Y(b,\underline{z}_2)c) = \mu(z_1,z_2)|_{|z_1|>|z_2|}$. Then, $u(\bar{Y}(a,\underline{z})\bar{Y}(b,\underline{z})c) = \mu \circ C(z_1,z_2)$. Since $\mu \circ C \in \mathrm{GCor}_2$, we have:

Proposition 2.12. $(\bar{F},\bar{Y},\mathbf{1})$ *is a full vertex algebra.*

We call it a conjugate full vertex algebra of $(F,Y,\mathbf{1})$.

2.3. **Holomorphic vertex operators.** Let F be a full vertex algebra. A vector $a \in F$ is said to be a holomorphic vector (resp. an anti-holomorphic vector) if $\bar{D}a = 0$ (resp. $Da = 0$). Let $a \in \ker\bar{D}$. Then, since $0 = Y(\bar{D}a,\underline{z}) = \frac{d}{d\bar{z}}Y(a,\underline{z})$, we have $a(r,s) = 0$ unless $s = -1$. Hence, $Y(a,\underline{z}) = \sum_{n\in\mathbb{Z}} a(n,-1)z^{-n-1}$.

Lemma 2.13. *Let $a, b \in F$. If $\bar{D}a = 0$, then for any $n \in \mathbb{Z}$,*

$$[a(n,-1), Y(b,\underline{z})] = \sum_{i\geq 0} \binom{n}{i} Y(a(i,-1)b,\underline{z}) z^{n-i},$$

$$Y(a(n,-1)b,\underline{z}) = \sum_{i\geq 0} \binom{n}{i} (-1)^i a(n-i,-1) z^i Y(b,\underline{z}) - Y(b,\underline{z}) \sum_{i\geq 0} \binom{n}{i} (-1)^{i+n} a(i,-1) z^{n-i}.$$

Proof. For any $u \in F^\vee$ and $c \in F$, there exists $\mu \in \mathrm{GCor}_2$ such that (FV5) holds. Since $\bar{D}a = 0$, by Proposition 2.10, $d/d\bar{z}_1 \mu(z_1,z_2) = 0$. Then, by Lemma 1.12, $\mu \in \mathbb{C}[z_1^\pm, (z_1-z_2)^\pm, z_2^\pm, \bar{z}_2^\pm, |z_2|^\mathbb{R}]$. Thus, by the Cauchy integral formula, the assertion holds. □

By Proposition 2.10, $\bar{D}Y(a,\underline{z})b = Y(\bar{D}a,\underline{z})b + Y(a,\underline{z})\bar{D}b = 0$. Thus, the restriction of Y on $\ker \bar{D}$ define a linear map $Y(-,z) : \ker \bar{D} \to \mathrm{End} \ker \bar{D}[[z^\pm]]$. By the above Lemma and Lemma 1.12, we have:

Proposition 2.14. $\ker \bar{D}$ *is a vertex algebra and F is a* $\ker \bar{D}$*-module.*

Proof. In order to prove that $\ker \bar{D}$ is a vertex algebra, it suffices to show that $\ker \bar{D}$ satisfies the Goddard's axioms [LL]. Since $[D, \bar{D}] = 0$, D acts on $\ker \bar{D}$. By Proposition 2.10, it suffices to show that $Y(a,z)$ and $Y(b,w)$ are mutually local for any $a, b \in \ker \bar{D}$. Let $a, b \in \ker \bar{D}$ and $v \in F$, $u \in F^\vee$ and $\mu \in \mathrm{GCor}_2$ satisfy $u(Y(a,z_1)Y(a_2,z_2)v) = \mu|_{|z_1|>|z_2|}$. By Lemma 1.12, μ is a polynomial in $\mathbb{C}[z_1^\pm, z_2^\pm, (z_1-z_2)^{-1}]$. Since $\mu|_{|z_2|>|z_1-z_2|} = u(Y(Y(a_1,z_0)a_2,z_2)v)$ and $a_1(n,-1)a_2 = 0$ for sufficiently large $n \in \mathbb{Z}$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that $(z_1-z_2)^N \mu(z_1,z_2) \in \mathbb{C}[z_1^\pm, z_2^\pm]$. Thus, $(z_1-z_2)^N u(Y(a,z_1)Y(a_2,z_2)v) = (z_1-z_2)^N u(Y(a_2,z_2)Y(a_1,z_1)v)$ for any $v \in F$ and $u \in F^\vee$, which implies that $Y(a_1,z_1)$ and $Y(a_2,z_2)$ are mutually local and F is a $\ker \bar{D}$-module (see, for example, [LL, Proposition 4.4.3]). □

Lemma 2.15. *Let $a \in F$ be a holomorphic vector and $b \in F$ an anti-holomorphic vector. Then, $[Y(a,z), Y(b,\bar{w})] = 0$, that is, $[a(n,-1), b(-1,m)] = 0$ and $a(k,-1)b = 0$ for any $n, m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$.*

Proof. By Lemma 2.13, it suffices to show that $a(k,-1)b = 0$ for any $k \geq 0$. Since $DY(a,z)b = [D, Y(a,z)]b + Y(a,z)Db = \frac{d}{dz}Y(a,z)b$, we have $Da(n,-1)b = -na(n-1,-1)b$ for any $n \in \mathbb{Z}$. Thus, the assertion follows from (FV1). □

2.4. **Tensor product of full vertex algebras.** In this section, we define a tensor product of full vertex algebras and study the subalgebra of a full vertex algebra generated by holomorphic and anti-holomorphic vectors. Let $(F^1, Y^1, \mathbf{1}^1)$ and $(F^2, Y^2, \mathbf{1}^2)$ be full vertex algebras and assume that the spectrum of F^1 is discrete and the spectrum of F^2 is bounded below. Define the linear map $Y(-,\underline{z}) : F^1 \otimes F^2 \to \mathrm{End}\, F^1 \otimes F^2[[z,\bar{z},|z|^\mathbb{R}]]$ by $Y(a\otimes b,\underline{z}) = Y^1(a,\underline{z}) \otimes Y^2(b,\underline{z})$ for $a \in F^1$ and $b \in F^2$. Then, for $a, c \in F^1$ and $b, d \in F^2$,

$$Y(a\otimes b,\underline{z})c\otimes d = \sum_{s,\bar{s},r,\bar{r}\in\mathbb{R}} a(s,\bar{s})c \otimes b(r,\bar{r})d\, z^{-s-r-2}\bar{z}^{-\bar{s}-\bar{r}-2}.$$

By (FV1), the coefficient of $z^k\bar{z}^{\bar{k}}$ is a finite sum for any $k, \bar{k} \in \mathbb{R}$. Thus, $Y(-,\underline{z})$ is well-defined. For any $h_0, \bar{h}_0 \in \mathbb{R}$, set $(F^1\otimes F^2)_{h_0,\bar{h}_0} = \bigoplus_{a,\bar{a}\in\mathbb{R}} F^1_{a,\bar{a}} \otimes F^2_{h_0-a,\bar{h}_0-\bar{a}}$. Since the spectrum of F^2 is bounded below, there exists $N \in \mathbb{R}$ such that $(F^1\otimes F^2)_{h_0,\bar{h}_0} = \bigoplus_{a,\bar{a}\leq N} F^1_{a,\bar{a}} \otimes F^2_{h_0-a,\bar{h}_0-\bar{a}}$. Since the spectrum of F^1 is discrete, the sum is finite. Thus, $(F^1\otimes F^2)^*_{h_0,\bar{h}_0} = \bigoplus_{a,\bar{a}\in\mathbb{R}} (F^1_{a,\bar{a}})^* \otimes (F^2_{h_0-a,\bar{h}_0-\bar{a}})^*$, which implies that $F^\vee = (F^1)^\vee \otimes (F^2)^\vee$. Let $u_i \in (F^i)^\vee$ and $a_i, b_i, c_i \in F^i$ for $i = 1, 2$. Since

$$u_1\otimes u_2(Y(a_1\otimes a_2,\underline{z}_1)Y(b_1\otimes b_2,\underline{z}_2)c_1\otimes c_2) = u_1(Y(a_1,\underline{z}_1)Y(b_1,\underline{z}_2)c_1)u_2(Y(a_2,\underline{z}_1)Y(b_2,\underline{z}_2)c_2),$$

we have:

Proposition 2.16. *Let* $(F^1, Y^1, \mathbf{1}^1)$ *and* $(F^2, Y^2, \mathbf{1}^2)$ *be full vertex algebras. If the spectrum of* F^1 *is discrete and the spectrum of* F^2 *is bounded below, then* $(F^1 \otimes F^2, Y^1 \otimes Y^2, \mathbf{1}^1 \otimes \mathbf{1}^2)$ *is a full vertex algebra. Furthermore, if the spectrum of* F^1 *and* F^2 *are bounded below (resp. discrete), then the spectrum of* $F^1 \otimes F^2$ *is also bounded below (resp. discrete).*

By Proposition 2.4 and Proposition 2.12, we have:

Corollary 2.17. *Let* V, W *be a* $\mathbb{Z}_{\geq 0}$*-graded vertex algebras such that* $\dim V_n$ *and* $\dim W_n$ *are finite for any* $n \in \mathbb{Z}_{\geq 0}$*. Then,* $V \otimes \bar{W}$ *is a full vertex algebra with a discrete spectrum, where* $\bar{W}$ *is the conjugate full vertex algebra.*

Let F be a full vertex algebra. By Proposition 2.14, $\ker \bar{D}$ and $\ker D$ are subalgebras of F. Let $\ker \bar{D} \otimes \ker D$ be the tensor product full vertex algebra. Define the linear map $t : \ker \bar{D} \otimes \ker D \to F$ by $(a \otimes b) \mapsto a(-1, -1)b$ for $a \in \ker \bar{D}$ and $b \in \ker D$. Then, we have:

Proposition 2.18. *Let* F *be a full vertex algebra. Then,* $t : \ker \bar{D} \otimes \ker D \to F$ *is a full vertex algebra homomorphism.*

Proof. Let $a, c \in \ker \bar{D}$, $b, d \in \ker D$. By Lemma 2.15 and Lemma 2.13,

$$Y(a(-1, -1)b, \underline{z}) = Y(a, z)Y(b, \bar{z}) = Y(b, \bar{z})Y(a, z).$$

Thus, it suffices to show that $t(a \otimes b(n, m)c \otimes d) = t(a \otimes b)(n, m)t(c \otimes d)$ for any $n, m \in \mathbb{Z}$. By Lemma 2.13

$$\begin{aligned} t(a \otimes b(n, m)c \otimes d) &= t(a(n, -1)c \otimes b(-1, m)d) \\ &= (a(n, -1)c)(-1, -1)b(-1, m)d \\ &= \sum_{i=0} \binom{n}{i} (-1)^i (a(n-i, -1)c(-1+i, -1) + c(-1+n-i, -1)a(i, -1))b(-1, m)d. \end{aligned}$$

Since $b(-1, m)d \in \ker D$, by Lemma 2.15, $t(a \otimes b(n, m)c \otimes d) = a(n, -1)c(-1, -1)b(-1, m)d = a(n, -1)b(-1, m)c(-1, -1)d = t(a \otimes b)(n, m)t(c \otimes d)$. Thus, the assertion holds. □

We remark that if $\ker \bar{D} \otimes \ker D$ is simple, then the above map is injective.

2.5. **Full conformal vertex algebra.** In this section, we introduce a notion of a full conformal vertex algebra, which is a generalization of a conformal vertex algebra. An energy-momentum tensor of a full vertex algebra is a pair of vectors $\omega \in F_{2,0}$ and $\bar{\omega} \in F_{0,2}$ such that

(1) $\bar{D}\omega = 0$ and $D\bar{\omega} = 0$;
(2) There exist scalars $c, \bar{c} \in \mathbb{C}$ such that $\omega(3, -1)\omega = \frac{c}{2}\mathbf{1}$, $\bar{\omega}(-1, 3)\bar{\omega} = \frac{\bar{c}}{2}\mathbf{1}$ and $\omega(k, -1)\omega = \bar{\omega}(-1, k)\bar{\omega} = 0$ for any $k = 2$ or $k \in \mathbb{Z}_{\geq 4}$.
(3) $\omega(0, -1) = D$ and $\bar{\omega}(-1, 0) = \bar{D}$;
(4) $\omega(1, -1)|_{F_{t,\bar{t}}} = t$ and $\bar{\omega}(-1, 1)|_{F_{t,\bar{t}}} = \bar{t}$ for any $t, \bar{t} \in \mathbb{R}$.

We remark that $\{\omega(n, -1)\}_{n \in \mathbb{Z}}$ and $\{\bar{\omega}(-1, n)\}_{n \in \mathbb{Z}}$ satisfy the commutation relation of Virasoro algebra by Lemma 2.13. A full conformal vertex algebra is a pair of a full vertex algebra and its energy momentum tensor.

Let $(F, \omega, \bar{\omega})$ a full conformal vertex algebra and $a \in \ker \bar{D}$. Then, by Lemma 2.15, $\bar{\omega}(1)a = 0$. Thus, $\ker \bar{D} \subset \bigoplus_{n \in \mathbb{Z}} F_{n,0}$. Since $\omega \in \ker \bar{D}$, we have:

Proposition 2.19. *If* $(F, \omega, \bar{\omega})$ *is a full conformal vertex algebra, then* $(\ker \bar{D}, \omega)$ *is a* $\mathbb{Z}$*-graded conformal vertex algebra.*

3. Generalized full vertex algebras

In this section, we define and study a generalized full vertex algebra, which is a "full" analogy of the notion of a generalized vertex algebra introduced in [DL].

3.1. **Definition of generalized vertex algebra.** We first recall the notion of generalized vertex algebra introduced in [DL]. We remark that in the original definition in [DL] they use the Borcherds identity, however, in order to generalize it to non-chiral CFT we need to use generalized two point correlation function (see Remark 2.1).

For $\alpha_1, \alpha_2, \alpha_{12} \in \mathbb{R}$, set

$$z_1^{\alpha_1} z_2^{\alpha_2} (z_1 - z_2)^{\alpha_{12}}|_{|z_1|>|z_2|} = z_1^{\alpha_1+\alpha_{12}} z_2^{\alpha_2} \sum_{i\geq 0} (-z_2/z_1)^i,$$

$$z_1^{\alpha_1} z_2^{\alpha_2} (z_2 - z_1)^{\alpha_{12}}|_{|z_2|>|z_1|} = z_1^{\alpha_1} z_2^{\alpha_2+\alpha_{12}} \sum_{i\geq 0} (-z_1/z_2)^i,$$

$$(z_2 + z_0)^{\alpha_1} z_2^{\alpha_2} z_0^{\alpha_{12}}|_{|z_2|>|z_0|} = z_0^{\alpha_{12}} z_2^{\alpha_2+\alpha_1} \sum_{i\geq 0} (z_0/z_2)^i, \tag{3.1}$$

which are formal power series in $\mathbb{C}[[z_2/z_1]][z_1^{\mathbb{R}}, z_2^{\mathbb{R}}]$, $\mathbb{C}[[z_1/z_2]][z_1^{\mathbb{R}}, z_2^{\mathbb{R}}]$ and $\mathbb{C}[[z_0/z_2]][z_0^{\mathbb{R}}, z_2^{\mathbb{R}}]$, respectively.

Remark 3.1. *These notations do not conflict with the notation introduced in section 1.4, which represents series expansion in some regions. In fact, if $\alpha_1, \alpha_2, \alpha_{12} \in \mathbb{Z}$, then $z_1^{\alpha_1} z_2^{\alpha_2} (z_1 - z_2)^{\alpha_{12}} \in$ GCor_2 and both notations give the same formal power series. However, unless $\alpha_1, \alpha_2, \alpha_{12} \in \mathbb{Z}$, $z_1^{\alpha_1} z_2^{\alpha_2} (z_1 - z_2)^{\alpha_{12}}$ is not a single valued function. Thus, in order to expand it, we have to choose a branch. We decide to choose the branch given in (3.1).*

A generalized vertex algebra is a real finite dimensional vector space H equipped with a non-degenerate symmetric bilinear form

$$(-,-) : H \times H \to \mathbb{R}$$

and an $\mathbb{R} \times H$-graded $\mathbb{C}$-vector space $\Omega = \bigoplus_{t\in\mathbb{R},\alpha\in H} \Omega_t^\alpha$ equipped with a linear map

$$\hat{Y}(-,z) : \Omega \to \mathrm{End}\,\Omega[[z^{\mathbb{R}}]],\ a \mapsto \hat{Y}(a,z) = \sum_{r\in\mathbb{R}} a(r) z^{-r-1}$$

and an element $\mathbf{1} \in \Omega_0^0$ satisfying the following conditions:

- GV1) For any $\alpha, \beta \in H$ and $a \in \Omega^\alpha$, $b \in \Omega^\beta$, $z^{(\alpha,\beta)} \hat{Y}(a,z) b \in \Omega((z))$;
- GV2) $\Omega_t^\alpha = 0$ unless $(\alpha,\alpha)/2 + t \in \mathbb{Z}$;
- GV3) For any $a \in \Omega$, $\hat{Y}(a,\underline{z})\mathbf{1} \in \Omega[[z,\bar{z}]]$ and $\lim_{\underline{z}\to 0} \hat{Y}(a,\underline{z})\mathbf{1} = a(-1,-1)\mathbf{1} = a$;
- GV4) $\hat{Y}(\mathbf{1},\underline{z}) = \mathrm{id} \in \mathrm{End}\,\Omega$;
- GV5) For any $\alpha_i \in M_\Omega$ and $a_i \in \Omega^{\alpha_i}$ $(i = 1,2,3)$ and $u \in \Omega^\vee = \bigoplus_{t\in\mathbb{R},\alpha\in H} (\Omega_t^\alpha)^*$, there exists $\mu(z_1,z_2) \in \mathrm{GCor}_2^{\mathrm{hol}}$ such that

$$(z_1 - z_2)^{(\alpha_1,\alpha_2)} z_1^{(\alpha_1,\alpha_3)} z_2^{(\alpha_2,\alpha_3)}|_{|z_1|>|z_2|} u(\hat{Y}(a_1,z_1)\hat{Y}(a_2,z_2)a_3) = \mu(z_1,z_2)|_{|z_1|>|z_2|},$$

$$z_0^{(\alpha_1,\alpha_2)} (z_2 + z_0)^{(\alpha_1,\alpha_3)} z_2^{(\alpha_2,\alpha_3)}|_{|z_2|>|z_0|} u(\hat{Y}(\hat{Y}(a_1,z_0)a_2,z_2)a_3) = \mu(z_0 + z_2, z_2)|_{|z_2|>|z_0|},$$

$$(z_2 - z_1)^{(\alpha_1,\alpha_2)} z_1^{(\alpha_1,\alpha_3)} z_2^{(\alpha_2,\alpha_3)}|_{|z_2|>|z_1|} u(\hat{Y}(a_2,z_2)\hat{Y}(a_1,z_1)a_3) = \mu(z_1,z_2)|_{|z_2|>|z_1|};$$

- GV6) $\Omega_t^\alpha(r)\Omega_{t'}^\beta \subset \Omega_{t+t'-r-1}^{\alpha+\beta}$ for any $r,t,t' \in \mathbb{R}$ and $\alpha,\beta \in H$;
- GV7) For any $\alpha \in H$, there exists $N_\alpha \in \mathbb{R}$ such that $\Omega_t^\alpha = 0$ for any $t \leq N_\alpha$.

Remark 3.2. *As remarked in 3.1, the generalized correlation functions $u(\hat{Y}(a_1,z_1)\hat{Y}(a_2,z_2)a_3)$ is not single-valued real analytic function and no longer an analytic continuation of $u(\hat{Y}(a_2,z_2)\hat{Y}(a_1,z_1)a_3)$ along any path. But, the monodromy is controlled by the H-grading.*

Remark 3.3. *In the original definition in* [DL] *they use an $\mathbb{R}/2\mathbb{Z}$-valued bilinear form $H\times H\to\mathbb{R}/2\mathbb{Z}$ instead of $H\times H\to\mathbb{R}$. In fact, for the definition, we only need this $\mathbb{R}/2\mathbb{Z}$-valued bilinear form (see for example Lemma 3.6), however, for our purpose it is convenient to define a generalized vertex algebra in this way. We also remark that (GV7) is assumed for the sake of simplicity (It seems that all generalized vertex algebras which naturally arise satisfy (GV7)). We may drop it and it is not assumed in the original definition.*

3.2. **Definition of generalized full vertex algebra.** It is now straight forward to generalize the definition of a generalized vertex algebra to a (non-chiral) generalized full vertex algebra.

A generalized full vertex algebra is a real finite dimensional vector space H equipped with a non-degenerate symmetric bilinear form

$$(-,-):H\times H\to\mathbb{R}$$

and an $\mathbb{R}^2\times H$-graded $\mathbb{C}$-vector space $\Omega=\bigoplus_{t,\bar{t}\in\mathbb{R},\alpha\in H}\Omega^{\alpha}_{t,\bar{t}}$ equipped with a linear map

$$\hat{Y}(-,\underline{z}):\Omega\to\operatorname{End}\Omega[[z^{\mathbb{R}},\bar{z}^{\mathbb{R}}]],\ a\mapsto\hat{Y}(a,\underline{z})=\sum_{r,s\in\mathbb{R}}a(r,s)z^{-r-1}\bar{z}^{-s-1}$$

and an element $\mathbf{1}\in\Omega^0_{0,0}$ satisfying the following conditions:

GFV1) For any $\alpha,\beta\in H$ and $a\in\Omega^{\alpha}$, $b\in\Omega^{\beta}$, $z^{(\alpha,\beta)}\hat{Y}(a,z)b\in\Omega((z,\bar{z},|z|^{\mathbb{R}}))$;

GFV2) $\Omega^{\alpha}_{t,\bar{t}}=0$ unless $(\alpha,\alpha)/2+t-\bar{t}\in\mathbb{Z}$;

GFV3) For any $a\in\Omega$, $\hat{Y}(a,\underline{z})\mathbf{1}\in\Omega[[z,\bar{z}]]$ and $\lim_{z\to 0}\hat{Y}(a,\underline{z})\mathbf{1}=a(-1,-1)\mathbf{1}=a$;

GFV4) $\hat{Y}(\mathbf{1},\underline{z})=\mathrm{id}\in\operatorname{End}\Omega$;

GFV5) For any $\alpha_i\in H$ and $a_i\in\Omega^{\alpha_i}$ $(i=1,2,3)$ and $u\in\Omega^{\vee}=\bigoplus_{t,\bar{t}\in\mathbb{R},\alpha\in H}(\Omega^{\alpha}_{t,\bar{t}})^*$, there exists $\mu(z_1,z_2)\in\mathrm{GCor}_2$ such that

$$(z_1-z_2)^{(\alpha_1,\alpha_2)}z_1^{(\alpha_1,\alpha_3)}z_2^{(\alpha_2,\alpha_3)}|_{|z_1|>|z_2|}u(\hat{Y}(a_1,\underline{z}_1)\hat{Y}(a_2,\underline{z}_2)a_3)=\mu(z_1,z_2)|_{|z_1|>|z_2|},$$
$$z_0^{(\alpha_1,\alpha_2)}(z_2+z_0)^{(\alpha_1,\alpha_3)}z_2^{(\alpha_2,\alpha_3)}|_{|z_2|>|z_0|}u(\hat{Y}(\hat{Y}(a_1,\underline{z}_0)a_2,\underline{z}_2)a_3)=\mu(z_0+z_2,z_2)|_{|z_2|>|z_0|},$$
$$(z_2-z_1)^{(\alpha_1,\alpha_2)}z_1^{(\alpha_1,\alpha_3)}z_2^{(\alpha_2,\alpha_3)}|_{|z_2|>|z_1|}u(\hat{Y}(a_2,\underline{z}_2)\hat{Y}(a_1,\underline{z}_1)a_3)=\mu(z_1,z_2)|_{|z_2|>|z_1|};$$

GFV6) $\Omega^{\alpha}_{t,\bar{t}}(r,s)\Omega^{\beta}_{t',\bar{t}'}\subset\Omega^{\alpha+\beta}_{t+t'-r-1,\bar{t}+\bar{t}'-s-1}$ for any $r,s,t,\bar{t},t',\bar{t}'\in\mathbb{R}$ and $\alpha,\beta\in H$;

GFV7) For any $\alpha\in H$, there exists $N_\alpha\in\mathbb{R}$ such that $\Omega^{\alpha}_{t,\bar{t}}=0$ for any $t\leq N_\alpha$ or $\bar{t}\leq N_\alpha$.

Let (Ω,H) be a generalized full vertex algebra and set

$$\Omega^{\alpha}=\bigoplus_{t,\bar{t}\in\mathbb{R}}\Omega^{\alpha}_{t,\bar{t}}.$$

for $\alpha\in H$, and M_Ω be a subgroup of H generated by $\{\alpha\in H\mid\Omega^{\alpha}\neq 0\}$. Let D and $\bar{D}$ denote the endomorphism of Ω defined by $Da=a(-2,-1)\mathbf{1}$ and $\bar{D}a=a(-1,-2)\mathbf{1}$ for $a\in\Omega$, i.e.,

$$\hat{Y}(a,z)\mathbf{1}=a+Daz+\bar{D}a\bar{z}+\cdots\in\Omega[[z,\bar{z}]].$$

Let $a\in\Omega^{\alpha}$ and $b\in\Omega^{\beta}$ for $\alpha,\beta\in M_\Omega$. Since $z^{(\alpha,\beta)}\hat{Y}(a,\underline{z})b\in\Omega((z,\bar{z},|z|^{\mathbb{R}}))$, $\lim_{z\to -z}z^{(\alpha,\beta)}Y(a,\underline{z})b$ is well-defined. Then, similarly to the case of full vertex algebras, we have:

Proposition 3.4. *Let Ω be a generalized full vertex algebra. For $v\in\Omega$ and $\alpha,\beta\in M_\Omega$, $a\in\Omega^{\alpha}$, $b\in\Omega^{\beta}$, the following properties hold:*

(1) $\hat{Y}(Dv,\underline{z})=\frac{d}{dz}\hat{Y}(v,\underline{z})$ and $\hat{Y}(\bar{D}v,\underline{z})=\frac{d}{d\bar{z}}\hat{Y}(v,\underline{z})$;

(2) $D\mathbf{1} = \bar{D}\mathbf{1} = 0$*;*
(3) $[D, \bar{D}] = 0$*;*
(4) $z^{(\alpha,\beta)}\hat{Y}(a, \underline{z})b = \exp(zD + \bar{z}\bar{D}) \lim_{z\to -z} z^{(\alpha,\beta)}\hat{Y}(b, \underline{z})a$*;*
(5) $\hat{Y}(\bar{D}v, \underline{z}) = [\bar{D}, \hat{Y}(v, \underline{z})]$ *and* $\hat{Y}(Dv, \underline{z}) = [D, \hat{Y}(v, \underline{z})]$.

Proof. The proof of Proposition 2.10 also works for (1), (2), (3). Thus, we only prove (4) and (5).

Let $u \in \Omega^\vee$ and $a \in \Omega^\alpha$ and $b \in \Omega^\beta$ and $\mu(z_1, z_2) \in \mathrm{GCor}_2$ satisfy

$$(z_1 - z_2)^{(\alpha,\beta)} u(\hat{Y}(a, \underline{z}_1)\hat{Y}(b, \underline{z}_2)\mathbf{1}) = \mu(z_1, z_2)|_{|z_1|>|z_2|}.$$

Since

$$\mu(z_0 + z_2, z_2)|_{|z_2|>|z_0|} = z_0^{(\alpha,\beta)} u(\hat{Y}(\hat{Y}(a, \underline{z}_0)b, \underline{z}_2)\mathbf{1}) \in U(z_2, z_0)$$

and the right-hand-side contains only the positive power of z_2 and $\bar{z}_2$, $z_0^{(\alpha,\beta)} u(\hat{Y}(\hat{Y}(a, \underline{z}_0)b, \underline{z}_2)\mathbf{1}) \in \mathbb{C}[z_2, \bar{z}_2][z_0^\pm, \bar{z}_0^\pm, |z_0|^\mathbb{R}]$. Set $p(z_0, z_2) = z_0^{(\alpha,\beta)} u(\hat{Y}(\hat{Y}(a, \underline{z}_0)b, \underline{z}_2)\mathbf{1})$. By (GFV5) and setting $z_0' = z_2 - z_1$, we have

$$u(\hat{Y}(\hat{Y}(b, \underline{z}_0')a, \underline{z}_1)\mathbf{1}) = \mu(z_1, z_0' + z_1)|_{|z_1|>|z_0'|} = p(-z_0', z_1 + z_0')|_{|z_1|>|z_0'|}.$$

Thus,

$$\begin{aligned}
z_0^{(\alpha,\beta)} u(\hat{Y}(a, \underline{z}_0)b) &= p(z_0, 0) = \lim_{z_1\to 0} \exp(z_0 \frac{d}{dz_1} + \bar{z}_0 \frac{d}{d\bar{z}_1}) p(z_0, z_1 - z_0) \\
&= \lim_{z_1\to 0} \exp(z_0 \frac{d}{dz_1} + \bar{z}_0 \frac{d}{d\bar{z}_1}) \lim_{z_0'\to -z_0} z_0'^{(\alpha,\beta)} u(\hat{Y}(\hat{Y}(b, \underline{z}_0')a, \underline{z}_1)\mathbf{1}) \\
&= \lim_{\substack{z_1\to 0 \\ z_0'\to -z_0}} u(\hat{Y}(\exp(-z_0'D - \bar{z}_0'\bar{D}) z_0'^{(\alpha,\beta)} Y(b, z_0')a, \underline{z}_1)\mathbf{1}) \\
&= u(\exp(z_0 D + \bar{z}_0\bar{D}) \lim_{z_0'\to -z_0} z_0'^{(\alpha,\beta)} Y(b, \underline{z}_0')a).
\end{aligned}$$

Finally,

$$\begin{aligned}
&z^{(\alpha,\beta)}\hat{Y}(Da, \underline{z})b + (\alpha,\beta) z^{(\alpha,\beta)-1}\hat{Y}(a, \underline{z})b \\
&= \frac{d}{dz} z^{(\alpha,\beta)}\hat{Y}(a, \underline{z})b \\
&= \frac{d}{dz} \exp(Dz + \bar{D}\bar{z})(-z)^{(\alpha,\beta)}\hat{Y}(b, -\underline{z})a \\
&= D\exp(Dz + \bar{D}\bar{z})(-z)^{(\alpha,\beta)}\hat{Y}(b, -\underline{z})a \\
&+ (\alpha,\beta) z^{-1}\exp(Dz + \bar{D}\bar{z})(-z)^{(\alpha,\beta)} Y(b, -\underline{z})a - \exp(Dz + \bar{D}\bar{z})(-z)^{(\alpha,\beta)} Y(Db, -\underline{z})a \\
&= z^{(\alpha,\beta)} D\hat{Y}(a, \underline{z})b - z^{(\alpha,\beta)}\hat{Y}(a, \underline{z})Db + (\alpha,\beta) z^{(\alpha,\beta)-1}\hat{Y}(a, \underline{z})b.
\end{aligned}$$

Thus, the assertion holds. □

A homomorphism from a generalized full vertex algebra $(\Omega_1, \hat{Y}_1, \mathbf{1}_1, H_1)$ to a generalized full vertex algebra $(\Omega_2, \hat{Y}_2, \mathbf{1}_2, H_2)$ is a pair of a linear map $\psi : \Omega_1 \to \Omega_2$ and an $\mathbb{R}$-linear isomorphism $\psi' : H_1 \to H_2$ such that:

(1) ψ' is isometric;
(2) $\psi((\Omega_1)^\alpha_{t,\bar{t}}) \subset (\Omega_2)^{\psi'(\alpha)}_{t,\bar{t}}$ for any $t, \bar{t} \in \mathbb{R}$ and $\alpha \in M_{\Omega_1}$;
(3) $\psi(\mathbf{1}_1) = \mathbf{1}_2$;
(4) $\psi(\hat{Y}_1(a, \underline{z})b) = \hat{Y}_2(\psi(a), \underline{z})\psi(b)$ for any $a, b \in \Omega_1$.

A subalgebra of a generalized full vertex algebra Ω is an $\mathbb{R}^2 \times H$-graded subspace $\Omega' \subset \Omega$ such that $\mathbf{1} \in \Omega'$ and $a(r, s)b \in \Omega'$ for any $r, s \in \mathbb{R}$ and $a, b \in \Omega'$.

Lemma 3.5. *Let Ω be a generalized full vertex algebra. Then, for a subgroup $A \subset M_\Omega$, $\Omega^A = \bigoplus_{\alpha\in A} \Omega^\alpha$ is a subalgebra of Ω.*

The following lemma is clear from the definition:

Lemma 3.6. *Let $\alpha_i \in M_\Omega$ and $a_i \in \Omega_i^\alpha$ for $i = 1, 2, 3$. Suppose that $(\alpha_i, \alpha_j) \in \mathbb{Z}$ for $i \neq j$. Then, for any $u \in \Omega^\vee$ there exists $\mu(z_1, z_2) \in \mathrm{GCor}_2$ such that*

$$u(\hat{Y}(a_1, \underline{z}_1)\hat{Y}(a_2, \underline{z}_2)a_3) = \mu(z_1, z_2)|_{|z_1|>|z_2|}$$
$$u(\hat{Y}(\hat{Y}(a_1, \underline{z}_0)a_2, \underline{z}_2)a_3) = \mu(z_0 + z_2, z_2)|_{|z_2|>|z_0|}$$
$$(-1)^{(\alpha_1,\alpha_2)}u(\hat{Y}(a_2, \underline{z}_2)\hat{Y}(a_1, \underline{z}_1)a_3) = \mu(z_1, z_2)|_{|z_2|>|z_1|}.$$

In particular, if a subgroup $A \subset M_\Omega$ satisfies $(\alpha, \alpha') \in 2\mathbb{Z}$ for any $\alpha, \alpha' \in A$, then $\Omega^A = \bigoplus_{\alpha\in A} \Omega^\alpha$ is a full vertex algebra.

Let $a \in \Omega^0$ satisfy $\bar{D}a = 0$. Since $\hat{Y}(\bar{D}a, \underline{z}) = \frac{d}{d\bar{z}}\hat{Y}(a, \underline{z}) = 0$, $\hat{Y}(a, \underline{z}) = \sum_{n\in\mathbb{Z}} a(n, -1)z^{-n-1}$. Thus, similarly to the proof of Lemma 2.13 and Lemma 2.15, we have:

Lemma 3.7. *Let $a \in \Omega^0$ satisfy $\bar{D}a = 0$. Then, for any $b \in \Omega$,*

$$[a(n, -1), \hat{Y}(b, \underline{z})] = \sum_{i\geq 0} \binom{n}{i} \hat{Y}(a(i, -1)b, \underline{z})z^{n-i}.$$

Furthermore, if $Db = 0$, then $a(i, -1)b = 0$ for any $i \geq 0$.

A generalized full conformal vertex algebra is a generalized full vertex algebra Ω with distinguished vectors $\omega \in \Omega^0_{2,0}$ and $\bar{\omega} \in \Omega^0_{0,2}$ such that

1. $\bar{D}\omega = 0$ and $D\bar{\omega} = 0$;
2. There exist scalars $c, \bar{c} \in \mathbb{C}$ such that $\omega(3, -1)\omega = \frac{c}{2}\mathbf{1}$, $\bar{\omega}(-1, 3)\bar{\omega} = \frac{\bar{c}}{2}\mathbf{1}$ and $\omega(k, -1)\omega = \bar{\omega}(-1, k)\bar{\omega} = 0$ for any $k = 2$ or $k \in \mathbb{Z}_{\geq 4}$;
3. $\omega(0, -1) = D$ and $\bar{\omega}(-1, 0) = \bar{D}$;
4. $\omega(1, -1)|_{\Omega^\alpha_{t,\bar{t}}} = t$ and $\bar{\omega}(-1, 1)|_{\Omega^\alpha_{t,\bar{t}}} = \bar{t}$ for any $t, \bar{t} \in \mathbb{R}$ and $\alpha \in M_\Omega$.

We remark that $\{\omega(n, -1)\}_{n\in\mathbb{Z}}$ and $\{\bar{\omega}(-1, n)\}_{n\in\mathbb{Z}}$ satisfy the commutation relation of Virasoro algebra by Lemma 3.7. The pair $(\omega, \bar{\omega})$ is called an energy-momentum tensor of the generalized full conformal vertex algebra in this thesis.

3.3. **Locality of generalized full vertex algebra.** The most difficult part in the construction of a generalized full vertex algebra is to verify the condition (GFV5). In the following proposition, (GFV5) is replaced by the conditions (GFL1), (GFL2) and (GFL3), which are easier to prove.

Proposition 3.8. *Let $(\Omega, \hat{Y}, \mathbf{1}, H)$ satisfy (GFV1), (GFV2), (GFV3), (GFV4), (GFV6) and (GFV7) and $D, \bar{D} \in \mathrm{End}\ \Omega$ be linear operators. We assume that the following conditions hold:*

GFL1) $[D, \bar{D}] = 0$ *and* $D\mathbf{1} = \bar{D}\mathbf{1} = 0$*;*

GFL2) $[D, \hat{Y}(a, \underline{z})] = \frac{d}{dz}\hat{Y}(a, \underline{z})$ *and* $[\bar{D}, \hat{Y}(a, \underline{z})] = \frac{d}{d\bar{z}}\hat{Y}(a, \underline{z})$ *for any* $a \in \Omega$*;*

GFL3) For any $\alpha_i \in M_\Omega$ and $a_i \in \Omega^{\alpha_i}$ ($i = 1, \dots, 3$) and $u \in \Omega^\vee$, there exists $\mu(z_1, z_2) \in \mathrm{GCor}_2$ such that

$$(z_1 - z_2)^{(\alpha_1,\alpha_2)}z_1^{(\alpha_1,\alpha_3)}z_2^{(\alpha_2,\alpha_3)}|_{|z_1|>|z_2|}u(\hat{Y}(a_1, \underline{z}_1)\hat{Y}(a_2, \underline{z}_2)a_3) = \mu(z_1, z_2)|_{|z_1|>|z_2|},$$
$$(z_2 - z_1)^{(\alpha_1,\alpha_2)}z_1^{(\alpha_1,\alpha_3)}z_2^{(\alpha_2,\alpha_3)}|_{|z_2|>|z_1|}u(\hat{Y}(a_2, \underline{z}_2)\hat{Y}(a_1, \underline{z}_1)a_3) = \mu(z_1, z_2)|_{|z_2|>|z_1|}.$$

Then, Ω is a generalized full vertex algebra.

Proof. Let $a_i \in \Omega^{\alpha_i}_{t_i,\bar{t}_i}$ and $u \in (\Omega^{\alpha_0}_{t_0,\bar{t}_0})^*$ for $\alpha_i \in M_\Omega$, $t_i, \bar{t}_i \in \mathbb{R}$ and $i = 0, 1, 2$.

First, we prove the skew-symmetry, that is,

$$z^{(\alpha_1,\alpha_2)}Y(a_2, \underline{z})a_1 = \exp(Dz + \bar{D}\bar{z}) \lim_{z\to -z} z^{(\alpha_1,\alpha_2)}Y(a_1, \underline{z})a_2.$$

Since $DY(a_2, z)\mathbf{1} = \frac{d}{dz}Y(a_2, z)\mathbf{1}$, we have $Y(a_2, z)\mathbf{1} = \exp(Dz + \bar{D}\bar{z})a_2$, which implies that $Da_2 = a_2(-2, -1)\mathbf{1} \in F_{t_2+1,\bar{t}_2}$ and thus $D\Omega^\alpha_{t,\bar{t}} \subset \Omega^\alpha_{t+1,\bar{t}}$ and $\bar{D}\Omega^\alpha_{t,\bar{t}} \subset \Omega^\alpha_{t,\bar{t}+1}$ for any $t, \bar{t} \in \mathbb{R}$ and $\alpha \in M_\Omega$. Then,

$$\begin{aligned} u(\hat{Y}(a_1, \underline{z}_1)\hat{Y}(a_2, \underline{z}_2)\mathbf{1}) &= u(\hat{Y}(a_1, \underline{z}_1) \exp(Dz_2 + \bar{D}\bar{z}_2)a_2) \\ &= \lim_{z_{12}\to(z_1-z_2)|_{|z_1|>|z_2|}} u(\exp(Dz_2 + \bar{D}\bar{z}_2)\hat{Y}(a_1, \underline{z}_{12})a_2). \end{aligned}$$

Set $t = t_1 + t_2 - t_0$ and $\bar{t} = \bar{t}_1 + \bar{t}_2 - \bar{t}_0$. Then, by (GFV6),

$$\begin{aligned} u(\exp(Dz_2 + \bar{D}\bar{z}_2)\hat{Y}(a_1, \underline{z}_{12})a_2) &= \sum_{s_1,\bar{s}_1\in\mathbb{R}} \sum_{n,\bar{n}\in\mathbb{Z}_{\geq 0}} \frac{1}{n!\bar{n}!} u(D^n\bar{D}^{\bar{n}}a_1(s_1, \bar{s}_1)a_2)z_{12}^{-\bar{s}_1-1}\bar{z}_{12}^{-\bar{s}_1-1}z_2^n\bar{z}_2^{\bar{n}} \\ &= \sum_{n,\bar{n}\in\mathbb{Z}_{\geq 0}} \frac{1}{n!\bar{n}!} u(D^n\bar{D}^{\bar{n}}a_1(h + n - 1, \bar{h} + \bar{n} - 1)a_2)z_{12}^{-t-n}\bar{z}_{12}^{-\bar{t}-\bar{n}}z_2^n\bar{z}_2^{\bar{n}}. \end{aligned}$$

By (GFV1), there exists an integer N such that $a_1(t + n - 1, \bar{t} + \bar{n} - 1)a_2 = 0$ for any $n \geq N$ or $\bar{n} \geq N$. Thus, $z_{12}^{N+t}\bar{z}_{12}^{N+\bar{t}}u(\exp(Dz_2 + \bar{D}\bar{z}_2)a_2)\hat{Y}(a_1, \underline{z}_{12})a_2) \in \mathbb{C}[z_{12}, z_2, \bar{z}_{12}, \bar{z}_2]$. By (GFV1), we may assume that $(\alpha_1, \alpha_2) - t + \bar{t} \in \mathbb{Z}$. Set

$$p(z_{12}, z_2) = z_{12}^{(\alpha_1,\alpha_2)}u(\exp(Dz_2 + \bar{D}\bar{z}_2)a_2)\hat{Y}(a_1, \underline{z}_{12})a_2),$$

which is a polynomial in $\mathbb{C}[z_{12}^\pm, \bar{z}_{12}^\pm, |z_{12}|^\mathbb{R}, z_2, \bar{z}_2]$ by $z_{12}^t\bar{z}_{12}^{\bar{t}} = z_{12}^{t-\bar{t}}|z_{12}|^{\bar{t}}$. Then, by (GFL3), $p(z_{12}, z_2)$ satisfies

$$\begin{aligned} \lim_{z_{12}\to(z_1-z_2)|_{|z_1|>|z_2|}} p(z_{12}, z_2) &= (z_1 - z_2)^{(\alpha_1,\alpha_2)}u(\hat{Y}(a_1, \underline{z}_1)\hat{Y}(a_2, \underline{z}_2)\mathbf{1}) \\ \lim_{z_{12}\to(-z_2+z_1)|_{|z_2|>|z_1|}} p(z_{12}, z_2) &= (z_2 - z_1)^{(\alpha_1,\alpha_2)}u(\hat{Y}(a_2, \underline{z}_2)\hat{Y}(a_1, \underline{z}_1)\mathbf{1}). \end{aligned}$$

By taking $z_1 \to 0$, we have

$$z_2^{(\alpha_1,\alpha_2)}u(\hat{Y}(a_2, \underline{z}_2)a_1) = p(-z_2, z_2) = \lim_{z_{12}\to -z_2} z_{12}^{(\alpha_1,\alpha_2)}u(\exp(Dz_2 + \bar{D}\bar{z}_2)\hat{Y}(a_1, \underline{z}_{12})a_2).$$

Thus, the skew-symmetry holds.

Now, we will show (GFV5). By the assumption, there exists $\mu(z_1, z_2) \in \mathrm{GCor}_2$ such that

$$(z_1 - z_2)^{(\alpha_1,\alpha_2)}z_1^{(\alpha_1,\alpha_3)}z_2^{(\alpha_2,\alpha_3)}|_{|z_1|>|z_2|}u(\hat{Y}(a_1, \underline{z}_1)\hat{Y}(a_2, \underline{z}_2)a_3) = \mu(z_1, z_2)|_{|z_1|>|z_2|}.$$

By the skew-symmetry,

$$\begin{aligned} &(z_1-z_2)^{(\alpha_1,\alpha_2)}z_1^{(\alpha_1,\alpha_3)}z_2^{(\alpha_2,\alpha_3)}|_{|z_1|>|z_2|}u(\hat{Y}(a_1, \underline{z}_1)\hat{Y}(a_2, \underline{z}_2)a_3) \\ &\quad= (z_1 - z_2)^{(\alpha_1,\alpha_2)}z_1^{(\alpha_1,\alpha_3)}|_{|z_1|>|z_2|}u(\hat{Y}(a_1, \underline{z}_1) \exp(Dz_2 + \bar{D}\bar{z}_2) \lim_{\underline{z}'_2\to -z_2} z'^{(\alpha_2,\alpha_3)}_2\hat{Y}(a_3, \underline{z}'_2)a_2) \\ &\quad= \lim_{\substack{z_{12}\to(z_1-z_2)|_{|z_1|>|z_2|} \\ z'_2\to -z_2}} z_{12}^{(\alpha_1,\alpha_2)}(z_{12} - z'_2)^{(\alpha_1,\alpha_3)}u(\exp(-Dz'_2 - \bar{D}\bar{z}'_2)\hat{Y}(a_1, \underline{z}_{12})z'^{(\alpha_2,\alpha_3)}_2\hat{Y}(a_3, \underline{z}'_2)a_2). \end{aligned}$$

Since $\Omega^\alpha_{t,\bar{t}} = 0$ for sufficiently small t or $\bar{t}$, $u(\exp(-Dz'_2 - \bar{D}\bar{z}'_2)-)$ is in $\Omega^\vee[z'_2, \bar{z}'_2]$, i.e., a finite sum. Since

$$\mu(z_{12} - z'_2, -z'_2)|_{|z_{12}|>|z'_2|} = (z_{12} - z'_2)^{(\alpha_1,\alpha_3)}z_{12}^{(\alpha_1,\alpha_2)}u(\exp(-Dz'_2 - \bar{D}\bar{z}'_2)\hat{Y}(a_1, \underline{z}_{12})z'^{(\alpha_2,\alpha_3)}_2\hat{Y}(a_3, \underline{z}'_2)a_2),$$

by (GFL3) and the skew-symmetry, we have

$$
\begin{aligned}
&\mu(z_{12}-z_2', -z_2')|_{|z_2'|>|z_{12}|}\\
&\quad= (z_2' - z_{12})^{(\alpha_1,\alpha_3)} z_{12}^{(\alpha_1,\alpha_2)} u(\exp(-Dz_2' - \bar{D}\bar{z}_2')\hat{Y}(a_3, \underline{z}_2')\hat{Y}(a_1, \underline{z}_{12}) z'^{(\alpha_2,\alpha_3)}_2 a_2)\\
&\quad= (1 - z_{12}/z_2')^{(\alpha_1,\alpha_3)} z_{12}^{(\alpha_1,\alpha_2)} u(\exp(-Dz_2' - \bar{D}\bar{z}_2') z'^{(\alpha_2+\alpha_1,\alpha_3)}_2 \hat{Y}(a_3, \underline{z}_2')\hat{Y}(a_1, \underline{z}_{12}) a_2)\\
&\quad= (1 - z_{12}/z_2')^{(\alpha_1,\alpha_3)} z_{12}^{(\alpha_1,\alpha_2)} \lim_{\underline{z}_2 \to -\underline{z}_2'} u(z_2^{(\alpha_2+\alpha_1,\alpha_3)} \hat{Y}(\hat{Y}(a_1, \underline{z}_{12})a_2), \underline{z}_2)a_3)\\
&\quad= \lim_{\underline{z}_2 \to -\underline{z}_2'} (z_2 + z_{12})^{(\alpha_1,\alpha_3)} z_{12}^{(\alpha_1,\alpha_2)} z_2^{(\alpha_2,\alpha_3)}|_{|z_2|>|z_{12}|} u(\hat{Y}(\hat{Y}(a_1, \underline{z}_{12})a_2), \underline{z}_2)a_3).
\end{aligned}
$$

Thus, we have (GFV5). □

3.4. **Standard construction.** From a lattice, an example of a generalized vertex algebra is constructed in [DL]. They call it a generalized lattice vertex algebra. In this section, we generalize it to non-chiral setting, which plays an essential role in this thesis.

Let H be a real finite dimensional vector space equipped with a non-degenerate symmetric bilinear form

$$(-,-)_{\mathrm{lat}} : H \times H \to \mathbb{R}.$$

Let $P(H)$ be a set of $\mathbb{R}$-linear maps $p \in \operatorname{End} H$ such that:

P1) $p^2 = p$, that is, p is a projection;
P2) The subspaces $\ker(1-p)$ and $\ker(p)$ are orthogonal to each other.

Let $P_>(H)$ be a subset of $P(H)$ consisting of $p \in P(H)$ such that:

P3) $\ker(1-p)$ is positive-definite and $\ker(p)$ is negative-definite.

For $p \in P(H)$, set $\bar{p} = 1 - p$ and $H_l = \ker(\bar{p})$ and $H_r = \ker(p)$. We will construct a generalized full vertex algebra $G_{H,p}$ for each $p \in P(H)$.

Let $p \in P(H)$. Define the new bilinear forms $(-,-)_p : H \times H \to \mathbb{R}$ by

$$(h, h')_p = (ph, ph')_{\mathrm{lat}} - (\bar{p}h, \bar{p}h')_{\mathrm{lat}}$$

for $h, h' \in H$. By (P1) and (P2), $(-,-)_p$ is non-degenerate. Let $\hat{H}^p = \bigoplus_{n\in\mathbb{Z}} H \otimes t^n \oplus \mathbb{C}c$ be the affine Heisenberg Lie algebra associated with $(H, (-,-)_p)$ and $\hat{H}^p_{\geq 0} = \bigoplus_{n\geq 0} H \otimes t^n \oplus \mathbb{C}c$ a subalgebra of $\hat{H}^p$. Define the action of $\hat{H}^p_{\geq 0}$ on the group algebra of H, $\mathbb{C}[H] = \bigoplus_{\alpha\in H} \mathbb{C}e_\alpha$, by

$$
\begin{aligned}
ce_\alpha &= e_\alpha\\
h \otimes t^n e_\alpha &= \begin{cases} 0, & n \geq 1,\\ (h,\alpha)_p e_\alpha, & n = 0 \end{cases}
\end{aligned}
$$

for $\alpha \in H$. Let $G_{H,p}$ be the $\hat{H}^p$-module induced from $\mathbb{C}[H]$. Denote by $h(n)$ the action of $h \otimes t^n$ on $G_{H,p}$ for $n \in \mathbb{Z}$. For $h \in H$, set

$$\begin{aligned}
h(\underline{z}) &= \sum_{n\in\mathbb{Z}}((ph)(n)z^{-n-1} + (\bar{p}h)(n)\bar{z}^{-n-1}) \in \operatorname{End} G_{H,p}[[z^{\pm}, \bar{z}^{\pm}]] \\
h^+(\underline{z}) &= \sum_{n\geq 0}((ph)(n)z^{-n-1} + (\bar{p}h)(n)\bar{z}^{-n-1}) \\
h^-(\underline{z}) &= \sum_{n\geq 0}((ph)(-n-1)z^{n} + (\bar{p}h)(-n-1)\bar{z}^{n}). \\
E^+(h,\underline{z}) &= \exp\Big(-\sum_{n\geq 1}(\frac{ph(n)}{n}z^{-n} + \frac{\bar{p}h(n)}{n}\bar{z}^{-n})\Big) \\
E^-(h,\underline{z}) &= \exp\Big(\sum_{n\geq 1}(\frac{ph(-n)}{n}z^{n} + \frac{\bar{p}h(-n)}{n}\bar{z}^{n})\Big).
\end{aligned}$$

For $h_r \in H_r$ and $h_l \in H_l$, $h_r(\underline{z})$ and $h_l(\underline{z})$ are denoted by $h_l(z)$ and $h_r(\bar{z})$.

Then, similarly to the case of a lattice vertex algebra [FLM], we have:

Lemma 3.9. *For any $h_1, h_2 \in H$,*

$$E^+(h_1,\underline{z}_1)E^-(h_2,\underline{z}_2) = \Big(\sum_{n,\bar{n}\geq 0}\binom{(ph_1,ph_2)_p}{n}\binom{(\bar{p}h_1,\bar{p}h_2)_p}{\bar{n}}(z_2/z_1)^n(\bar{z}_2/\bar{z}_1)^{\bar{n}}\Big)E^-(h_2,\underline{z}_2)E^+(h_1,\underline{z}_1).$$

We remark that the formal power series $\sum_{n,\bar{n}\geq 0}\binom{(ph_1,ph_2)_p}{n}\binom{(\bar{p}h_1,\bar{p}h_2)_p}{\bar{n}}(z_2/z_1)^n(\bar{z}_2/\bar{z}_1)^{\bar{n}}$ is equal to $(1-z_2/z_1)^{(ph_1,ph_2)_p}(1-\bar{z}_2/\bar{z}_1)^{(\bar{p}h_1,\bar{p}h_2)_p}|_{|z_1|>|z_2|}$.

Let $\alpha \in H$. Denote by $l_{e_\alpha} \in \operatorname{End}\mathbb{C}[H]$ the left multiplication by e_α and define the linear map $z^{p\alpha}\bar{z}^{\bar{p}\alpha} : \mathbb{C}[H] \to \mathbb{C}[H][z^{\mathbb{R}}, \bar{z}^{\mathbb{R}}]$ by $z^{p\alpha}\bar{z}^{\bar{p}\alpha}e_\beta = z^{(p\alpha,p\beta)_p}\bar{z}^{(\bar{p}\alpha,\bar{p}\beta)_p}e_\beta$ for $\beta \in H$. Then, set

$$e_\alpha(\underline{z}) = E^-(\alpha,\underline{z})E^+(\alpha,\underline{z})l_{e_\alpha}z^{p\alpha}\bar{z}^{\bar{p}\alpha} \in \operatorname{End} G_{H,p}[[z^{\pm},\bar{z}^{\pm}]][z^{\mathbb{R}},\bar{z}^{\mathbb{R}}].$$

By Poincaré-Birkhoff-Witt theorem, $G_{H,p}$ is spanned by

$$\{h_l^1(-n_1-1)\dots h_l^l(-n_l-1)h_r^1(-\bar{n}_1-1)\dots h_r^k(-\bar{n}_k-1)e_\alpha\},$$

where $h_l^i \in H_l$, $n_i \in \mathbb{Z}_{\geq 0}$ and $h_r^j \in H_r$, $\bar{n}_j \in \mathbb{Z}_{\geq 0}$ for any $1 \leq i \leq l$ and $1 \leq j \leq k$ and $\alpha \in H$. Then, a map $\hat{Y} : G_{H,p} \to \operatorname{End} G_{H,p}[[z^{\pm},\bar{z}^{\pm}]][z^{\mathbb{R}},\bar{z}^{\mathbb{R}}]$ is defined inductively as follows: For $\alpha \in H$, define $\hat{Y}(e_\alpha,\underline{z})$ by $\hat{Y}(e_\alpha,\underline{z}) = e_\alpha(\underline{z})$. Assume that $\hat{Y}(v,\underline{z})$ is already defined for $v \in G_{H,p}$. Then, for $h_r \in H_r$ and $h_l \in H_l$ and $n,\bar{n} \in \mathbb{Z}_{\geq 0}$, $\hat{Y}(h_l(-n-1)v,\underline{z})$ and $\hat{Y}(h_r(-\bar{n}-1)v,\underline{z})$ is defined by

$$\begin{aligned}
\hat{Y}(h_l(-n-1)v,\underline{z}) &= \Big(\frac{1}{n!}\frac{d}{dz}^n h_l^-(z)\Big)\hat{Y}(v,\underline{z}) + \hat{Y}(v,\underline{z})\Big(\frac{1}{n!}\frac{d}{dz}^n h_l^+(z)\Big) \\
\hat{Y}(h_r(-\bar{n}-1)v,\underline{z}) &= \Big(\frac{1}{\bar{n}!}\frac{d}{d\bar{z}}^n h_r^-(\bar{z})\Big)\hat{Y}(v,\underline{z}) + \hat{Y}(v,\underline{z})\Big(\frac{1}{\bar{n}!}\frac{d}{d\bar{z}}^n h_r^+(\bar{z})\Big).
\end{aligned}$$

Set

$$\begin{aligned}
\mathbf{1} &= 1 \otimes e_0, \\
\omega_{H_l} &= \frac{1}{2}\sum_{i=1}^{\dim H_l} h_l^i(-1)h_l^i, \\
\bar{\omega}_{H_r} &= \frac{1}{2}\sum_{j=1}^{\dim H_r} h_r^j(-1)h_r^j,
\end{aligned}$$

where h_l^i and h_r^j is an orthonormal basis of $H_l \otimes_{\mathbb{R}} \mathbb{C}$ and $H_r \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the bilinear form $(-,-)_p$. Set $G = G_{H,p}$ and

$$G_{t,\bar{t}}^{\alpha} = \{v \in G \mid \omega(1,-1)v = tv, \bar{\omega}(-1,1)v = \bar{t}v, h(0)v = (\alpha, h)_p v \text{ for any } h \in H\}$$

for $t, \bar{t} \in \mathbb{R}$ and $\alpha \in H$.

For $\alpha \in H$ and $n, m \in \mathbb{Z}_{\geq 0}$, it is easy to show that $G^{\alpha}_{\frac{1}{2}(p\alpha,p\alpha)_p+n,\frac{1}{2}(\bar{p}\alpha,\bar{p}\alpha)_p+m}$ is spanned by $\{h_l^1(-i_1)\dots h_l^k(-i_k)h_r^1(j_1)\dots h_r^l(j_l)e_\alpha\}$, where $k, l \in \mathbb{Z}_{\geq 0}$, $h_l^a \in H_l$, $h_r^b \in H_r$, $i_a, j_b \in \mathbb{Z}_{\geq 1}$, $i_1 + \dots + i_k = n$ and $j_1 + \dots + j_l = m$ for any $a = 1, \dots, k$ and $b = 1, \dots, l$. Then,

$$G = \bigoplus_{\alpha \in H} \bigoplus_{n,m \in \mathbb{Z}_{\geq 0}} \Omega^{\alpha}_{\frac{1}{2}(p\alpha,p\alpha)_p+n,\frac{1}{2}(\bar{p}\alpha,\bar{p}\alpha)_p+m}$$

and

$$G^{\alpha}_{\frac{1}{2}(p\alpha,p\alpha)_p,\frac{1}{2}(\bar{p}\alpha,\bar{p}\alpha)_p} = \mathbb{C}e_\alpha.$$

Let $a^* \in \mathbb{C}[H]^\vee = \bigoplus_{\alpha \in H}(\mathbb{C}e_\alpha)^*$ and $\langle a^*, -\rangle$ be the linear map $\Omega \to \mathbb{C}$ defined by the composition of the projection $G = \mathbb{C}[H] \oplus \bigoplus_{\substack{n,m \in \mathbb{Z}_{\geq 0} \\ (n,m) \neq (0,0)}} G^{\alpha}_{\frac{1}{2}(p\alpha,p\alpha)_p+n,\frac{1}{2}(\bar{p}\alpha,\bar{p}\alpha)_p+m} \to \mathbb{C}[H]$ and $a^* : \mathbb{C}[H] \to \mathbb{C}$. Then, it is easy to verify $\langle a^*, -\rangle$ is a highest weight vector, that is, $\langle a^*, h(-n)-\rangle = 0$ for any $n \geq 1$ and $h \in H$. Thus, for any $\alpha \in H$, we have:

$$E^+(\alpha, \underline{z})\mathbf{1} = \mathbf{1},$$
$$\langle a^*, E^-(\alpha, \underline{z})-\rangle = \langle a^*, -\rangle.$$

Thus, by using the above fact and Lemma 3.9, for $\alpha_i \in H$ $(i = 1, 2, 3)$ and $a^* \in \mathbb{C}[H]^\vee$, we have

$$\langle a^*, Y(e_{\alpha_1}, \underline{z}_1)Y(e_{\alpha_2}, \underline{z}_2)e_{\alpha_3}\rangle = z_1^{(p\alpha_1,p\alpha_3)_p}\bar{z}_1^{(\bar{p}\alpha_1,\bar{p}\alpha_3)_p}z_2^{(p\alpha_2,p\alpha_3)_p}\bar{z}_2^{(\bar{p}\alpha_2,\bar{p}\alpha_3)_p}$$
$$(z_1 - z_2)^{(p\alpha_1,p\alpha_2)_p}(\bar{z}_1 - \bar{z}_2)^{(\bar{p}\alpha_1,\bar{p}\alpha_2)_p}|_{|z_1|>|z_2|}\langle a^*, e_{\alpha_1}e_{\alpha_2}e_{\alpha_3}\rangle.$$

Since

$$\begin{aligned}(z_i - z_j)^{(p\alpha_i,p\alpha_j)_p}(\bar{z}_i - \bar{z}_j)^{(\bar{p}\alpha_i,\bar{p}\alpha_j)_p} &= |(\bar{z}_i - \bar{z}_j)|^{(\bar{p}\alpha_i,\bar{p}\alpha_j)_p}(z_i - z_j)^{(p\alpha_i,p\alpha_j)_p-(\bar{p}\alpha_i,\bar{p}\alpha_j)_p}\\ &= |(\bar{z}_i - \bar{z}_j)|^{(\bar{p}\alpha_i,\bar{p}\alpha_j)_p}(z_i - z_j)^{(\alpha_i,\alpha_j)_{\mathrm{lat}}},\end{aligned}$$

the formal power series

$$z_1^{-(\alpha_1,\alpha_3)_{\mathrm{lat}}}z_2^{-(\alpha_2,\alpha_3)_{\mathrm{lat}}}(z_1 - z_2)^{-(\alpha_1,\alpha_2)_{\mathrm{lat}}}|_{|z_1|>|z_2|}\langle a^*, Y(e_{\alpha_1}, \underline{z}_1)Y(e_{\alpha_2}, \underline{z}_2)e_{\alpha_3}\rangle$$

is a single-valued real analytic function in GCor_2. Then, similarly to the proof of Proposition 5.1 in [Mo2] with Proposition 3.8, we have:

Proposition 3.10. *For $p \in P(H)$, $(G_{H,p}, \hat{Y}, \mathbf{1}, H, -(-,-)_{\mathrm{lat}}, \omega_{H_l}, \bar{\omega}_{H_r})$ is a generalized full conformal vertex algebra.*

We remark that the minus sign $-(-,-)_{\mathrm{lat}}$ appears in the above proposition in our convention.

We end this section by studying generalized full vertex algebra homomorphisms among $G_{H,p}$. Let $(H, (-,-))$ and $(H', (-,-)')$ be real finite dimensional vector spaces with non-degenerate symmetric bilinear forms and $p \in P(H)$ and $\sigma : H \to H'$ be an isometric isomorphism. Then, $\sigma \cdot p = \sigma \circ p \circ \sigma^{-1} \in P(H')$ and

$$\begin{aligned}(\sigma h_1, \sigma h_2)'_{\sigma\cdot p} &= ((\sigma \cdot p)\sigma h_1, (\sigma \cdot p)\sigma h_2)' - ((\sigma \cdot \bar{p})\sigma h_1, (\sigma \cdot \bar{p})\sigma h_2)'\\ &= (\sigma p h_1, \sigma p h_2)' - (\sigma \bar{p} h_1, \sigma \bar{p} h_2)'\\ &= (p h_1, p h_2) - (\bar{p} h_1, \bar{p} h_2) \qquad (3.2)\\ &= (h_1, h_2)_p.\end{aligned}$$

Thus, σ induces an isometry from $(H,(-,-)_p)$ to $(H',(-,-)'_{\sigma\cdot p})$ and an isomorphism of Lie algebras $\sigma_{Lie}:\hat{H}^p\to\hat{H'}^{\sigma\cdot p}$, where $\hat{H}^p$ (resp. $\hat{H'}^{\sigma\cdot p}$) is the Heisenberg Lie algebra associated with $(H,(-,-)_p)$ (resp. $(H',(-,-)'_{\sigma\cdot p})$). Let $\sigma_{alg}:\mathbb{C}[H]\to\mathbb{C}[H']$ be a linear map defined by $e_\alpha\to e_{\sigma(\alpha)}$ for $\alpha\in H$. Then, $\sigma_{alg}:\mathbb{C}[H]\to\mathbb{C}[H']$ is a $\hat{H}^p_{\geq 0}$-module homomorphism, where $\mathbb{C}[H']$ is regarded as a $\hat{H}^p_{\geq 0}$-module by σ_{Lie}. Thus, we have an $\hat{H}^p$-module homomorphism $\tilde{\sigma}:G_{H,p}\to G_{H',\sigma\cdot p}$. Since $\sigma_{alg}:C[H]\to\mathbb{C}[H']$ is a $\mathbb{C}$-algebra homomorphism, it is easy to show that $(\tilde{\sigma},\sigma)$ is a generalized full vertex algebra homomorphism. Thus, we have:

Lemma 3.11. *For $p\in P(H)$ and an isometry $\sigma:H\to H'$, $(\tilde{\sigma},\sigma):G_{H,p}\to G_{H',\sigma\cdot p}$ is an isomorphism of generalized full vertex algebras.*

3.5. **Tensor product.** Similarly to full vertex algebras, the spectrum of a generalized full vertex algebra Ω is said to be discrete if for any $\alpha\in M_\Omega$ and $H\in\mathbb{R}$, $\sum_{h+\bar{h}<H}\dim\Omega^\alpha_{t,\bar{t}}$ is finite. (The bounded below condition is already included in the definition).

Let $(\Omega_1,\hat{Y}_1,\mathbf{1}_1,H_1,(-,-)_1)$ and $(\Omega_2,\hat{Y}_2,\mathbf{1}_2,H_2,(-,-)_2,)$ be generalized full vertex algebras and assume that the spectrum of Ω_1 is discrete and the spectrum of Ω_2 is bounded below. Set $H=H_1\oplus H_2$ and $\Omega^{\alpha_1,\alpha_2}_{t,\bar{t}}=\bigoplus_{s,\bar{s}\in\mathbb{R}}(\Omega_1)^{\alpha_1}_{s,\bar{s}}\otimes(\Omega_2)^{\alpha_2}_{t-s,\bar{t}-\bar{s}}$ for $(\alpha_1,\alpha_2)\in M_{\Omega_1}\oplus M_{\Omega_2}\subset H_1\oplus H_2$ and $\Omega=\bigoplus_{\alpha\in H,t,\bar{t}\in\mathbb{R}}\Omega^\alpha_{t,\bar{t}}$ and $\mathbf{1}=\mathbf{1}_1\otimes\mathbf{1}_2$.

Define the linear map $\hat{Y}:\Omega\to\Omega[[z^\mathbb{R},\bar{z}^\mathbb{R}]]$ by $\hat{Y}(a\otimes b,\underline{z})=\hat{Y}_1(a,\underline{z})\otimes\hat{Y}_2(b,\underline{z})$ for $a\in\Omega_1$ and $b\in\Omega_2$. Then, for $a,c\in\Omega_1$ and $b,d\in\Omega_2$,

$$\hat{Y}(a\otimes b,\underline{z})c\otimes d=\sum_{s,\bar{s},r,\bar{r}\in\mathbb{R}}a(s,\bar{s})c\otimes b(r,\bar{r})dz^{-s-r-2}\bar{z}^{-\bar{s}-\bar{r}-2}.$$

By (GFV1), the coefficient of $z^k\bar{z}^{\bar{k}}$ is a finite sum for any $k,\bar{k}\in\mathbb{R}$. Thus, $\hat{Y}$ is well-defined. Since the spectrum of Ω_2 is bounded below, there exists $N\in\mathbb{R}$ such that $\Omega^{(\alpha_1,\alpha_2)}_{t_0,\bar{t}_0}=\bigoplus_{t,\bar{t}\leq N}(\Omega_1)^{\alpha_1}_{t,\bar{t}}\otimes(\Omega_2)^{\alpha_2}_{t_0-t,\bar{t}_0-\bar{t}}$. Since the spectrum of Ω_1 is discrete, the sum is finite. Thus, $(\Omega^{(\alpha_1,\alpha_2)}_{t_0,\bar{t}_0})^*=\bigoplus_{t,\bar{t}\in\mathbb{R}}((\Omega_1)^{\alpha_1}_{t,\bar{t}})^*\otimes((\Omega_2)^{\alpha_2}_{t_0-t,\bar{t}_0-\bar{t}})^*$. Define the bilinear form on H by $((\alpha_1,\alpha_2),(\beta_1,\beta_2))=(\alpha_1,\beta_1)_1+(\alpha_2,\beta_2)_2$ for $\alpha_i,\beta_i\in H_i$ $(i=1,2)$. Then, we have:

Proposition 3.12. $(\Omega,\hat{Y},\mathbf{1},H,(-,-))$ *defined above is a generalized full vertex algebra. Furthermore, if both Ω_1 and Ω_2 have energy-momentum tensors, then Ω is a generalized full conformal vertex algebra.*

The subalgebra of $\Omega_1\otimes\Omega_2$ associated with a subgroup $A\subset H_1\oplus H_2$ is denoted by $\Omega_1\otimes_A\Omega_2$ (see Lemma 3.5).

3.6. **Cancellation of monodromy.** Let $(\Omega,\hat{Y},\mathbf{1},H)$ be a generalized full vertex algebra and $p\in P(H)$. The following lemma follows from the construction:

Lemma 3.13. *The spectrum of the generalized full vertex algebra $G_{H,p}$ constructed in Proposition 3.10 is discrete and bounded below.*

Assume that the spectrum of Ω is bounded below. We consider the tensor product of generalized full vertex algebras Ω and $G_{H,p}$. Set $\Delta H=\{(\alpha,\alpha)\in H\oplus H\}_{\alpha\in H}$, which is a subgroup of $H\oplus H$. Then, by Lemma 3.5, $(G_{H,p}\otimes_{\Delta H}\Omega,H\oplus H)$ is a generalized full vertex algebra. We denote it by $F_{\Omega,H,p}$. Since the inner product of $(\alpha,\alpha),(\beta,\beta)\in\Delta H\subset H\oplus H$ is $((\alpha,\alpha),(\beta,\beta))=(\alpha,\beta)-(\alpha,\beta)=0$ by the minus sign in Proposition 3.10, $F_{\Omega,H,p}$ is a full vertex algebra by Lemma 3.6. Thus, we have:

Theorem 3.14. *For a generalized full vertex algebra $(\Omega,\hat{Y},\mathbf{1},H)$ and $p\in P(H)$, $F_{\Omega,H,p}$ is a full vertex algebra. Furthermore, if Ω has an energy-momentum tensor, then $F_{\Omega,H,p}$ is a full conformal vertex algebra.*

4. Categorical aspects

In this section, we introduce a notion of a full $\mathcal{H}$-vertex algebra and show that the vacuum space of a full $\mathcal{H}$-vertex algebra is a generalized full vertex algebra.

4.1. Full $\mathcal{H}$-vertex algebras to generalized full vertex algebra. Let H_l and H_r be real finite dimensional vector subspaces equipped with non-degenerate symmetric bilinear forms $(-,-)_l : H_l \times H_l \to \mathbb{R}$ and $(-,-)_l : H_r \times H_r \to \mathbb{R}$. Let $M_{H_l}(0)$ and $M_{H_r}(0)$ be affine Heisenberg vertex algebras associated with $(H_l,(-,-)_l)$ and $(H_r,(-,-)_r)$. Set $H = H_l \oplus H_r$ and let $p, \overline{p} : H \to H$ be projections on H_l and H_r and

$$M_{H,p} = M_{H_l}(0) \otimes \overline{M_{H_r}(0)},$$

the tensor product of the vertex algebra $M_{H_l}(0)$ and the conjugate vertex algebra $\overline{M_{H_r}(0)}$ (see Proposition 2.12 and Corollary 2.17).

In this section, we consider a class of a full vertex algebra which is an $M_{H,p}$-module (like an algebra over a ring). More precisely, let F be a full vertex algebra and we assume that $M_{H,p}$ is a subalgebra of F, $M_{H,p} \subset F$. Then, since $H_l \subset (M_{H,p})_{1,0}$ and $H_r \subset (M_{H,p})_{0,1}$, $F \subset F_{1,0}$ and $H_r \subset F_{0,1}$.

We note that the subspaces H_l and H_r satisfy the following conditions: For any $h_l, h_l' \in H_l$ and $h_r, h_r' \in H_r$,

- H1) $H_l \subset F_{1,0}$ and $H_r \subset F_{0,1}$;
- H2) $\bar{D}H_l = 0$ and $DH_r = 0$;
- H3) $h_l(1,-1)h_l' = (h_l, h_l')_l\mathbf{1}$, $h_r(-1,1)h_r' = (h_r, h_r')_r\mathbf{1}$;
- H4) $h_l(n,-1)h_l' = 0$, $h_r(-1,n)h_r' = 0$ for any $n = 0$ or $n \in \mathbb{Z}_{\geq 2}$.

In fact, these conditions characterize the existence of a homomorphism $M_{H,p} \subset F$:

Proposition 4.1. *If subspaces H_l and H_r of a full vertex algebra F satisfy (H1) – (H4), then H_l and H_r generates a subalgebra which is isomorphic to $M_{H,p}$ as a full vertex algebra.*

Proof. By Proposition 2.14, the full vertex algebra generated by $H_l \subset \ker \bar{D}$ (resp. $H_r \subset \ker D$) is isomorphic to $M_{H_l}(0)$ (resp. $M_{H_r}(0)$). By Proposition 2.18, the full vertex algebra generated by H_l and H_r is in the image of $M_{H,p} \subset \ker \bar{D} \otimes \ker D$. Since $M_{H,p}$ is simple, the assertion follows. □

Since $h_l \in H_l$ is a holomorphic vector, by Lemma 2.13, $Y(h_l, \underline{z}) = \sum_{n\in\mathbb{Z}} h_l(n,-1)z^{-n-1}$. Hereafter, we will use a shorthand notation for $h_l \in H_l$, $h_r \in H_r$ and $n \in \mathbb{Z}$, $h_l(n) = h(n,-1)$ and $h_r(n) = h_r(-1,n)$. Set $h_l(z) = Y(h_l, \underline{z}) = \sum_{n\in\mathbb{Z}} h_l(n)z^{-n-1}$ and $h_r(\bar{z}) = Y(h_r, \underline{z}) = \sum_{n\in\mathbb{Z}} h_r(n)\bar{z}^{-n-1}$. By Lemma 2.13 and Lemma 2.15,

$$[h_l(n), h_l'(m)] = (h_l, h_l')_l n\delta_{n+m,0}$$
$$[h_r(n), h_r'(m)] = (h_r, h_r')_r n\delta_{n+m,0}$$
$$[h_l(n), h_r(m)] = 0,$$

for any $n, m \in \mathbb{Z}$ and $h_l, h_l' \in H$ and $h_r, h_r' \in H_r$.

For $\alpha \in H$ and $h, \bar{h} \in \mathbb{R}$, we let $\Omega_{F,H}^{\alpha}$ be the set of all vectors $v \in F$ satisfying the following conditions:

1. $h_l(n)v = 0$ and $h_r(n)v$ for any $h_l \in H_l$ and $h_r \in H_r$ and $n \geq 1$.
2. $h_l(0)v = (h_l, p\alpha)_l v$ and $h_r(0)v = (h_r, \bar{p}\alpha)_r v$ for any $h_l \in H_l$ and $h_r \in H_r$.

Set

$$\Omega_{F,H} = \bigoplus_{\alpha\in H} \Omega_{F,H}^{\alpha}$$

and

$$(\Omega_{F,H})^{\alpha}_{t,\bar{t}} = F_{t+\frac{(p\alpha,p\alpha)_l}{2},\bar{t}+\frac{(\bar{p}\alpha,\bar{p}\alpha)_r}{2}} \cap \Omega^{\alpha}_{F,H}$$

for $\alpha \in H$ and $t, \bar{t} \in \mathbb{R}$.

A full $\mathcal{H}$-vertex algebra, denoted by (F, H, p), is a full vertex algebra F with a subalgebra $M_{H,p}$ such that

FH1) $h_l(0)$ and $h_r(0)$ are semisimple on F with real eigenvalues for any $h_l \in H_l$ and $h_r \in H_r$;
FH2) For any $\alpha \in H$, there exists $N \in \mathbb{R}$ such that $F^{\alpha}_{t,\bar{t}} = 0$ for $t \leq N$ or $\bar{t} \leq N$.

Let (F, H, p) be a full $\mathcal{H}$-vertex algebra. By (FH1) and (FH2) and the representation theory of an affine Heisenberg Lie algebra ([FLM, Theorem 1.7.3]), F is isomorphic to $\bigoplus_{\alpha\in H} M_{H,p}\otimes\Omega^{\alpha}_{F,H}$ as an $M_{H,p}$-module. In particular, F is generated by the subspace $\Omega_{F,H}$ as a module of the Heisenberg Lie algebra $\hat{H}$.

For $\alpha \in H$, define $z^{(p\alpha)(0)}\bar{z}^{(\bar{p}\alpha)(0)} \in \mathrm{End}\,\Omega_{F,H}[z^{\mathbb{R}}, \bar{z}^{\mathbb{R}}]$ by

$$z^{(p\alpha)(0)}\bar{z}^{(\bar{p}\alpha)(0)}v = z^{(p\alpha,p\beta)_l}\bar{z}^{(\bar{p}\alpha,\bar{p}\beta)_r}v$$

for $v \in \Omega^{\beta}_{F,H}$. For $\alpha \in H$, set

$$E^-(\alpha, \underline{z}) = \exp\Big(\sum_{n\geq 1} \frac{p\alpha(-n)}{n} z^n + \frac{\bar{p}\alpha(-n)}{n}\bar{z}^n\Big)$$
$$E^+(\alpha, \underline{z}) = \exp\Big(\sum_{n\geq 1} \frac{p\alpha(n)}{-n} z^{-n} + \frac{\bar{p}\alpha(n)}{-n}\bar{z}^{-n}\Big).$$

Then, for any $h_l \in H_l$ and $n > 0$,

$$[h_l(n), E^-(\alpha, \underline{z})] = (h_l, \alpha)_l z^n E^-(\alpha, \underline{z})$$
$$[h_l(-n), E^+(\alpha, \underline{z})] = (h_l, \alpha)_l z^{-n} E^-(\alpha, \underline{z})$$
$$[h_l(n), E^+(\alpha, \underline{z})] = 0$$
$$[h_l(-n), E^-(\alpha, \underline{z})] = 0$$
$$[h_l(0), E^{\pm}(\alpha, \underline{z})] = 0$$

hold (Similar results hold for $h_r \in H_r$.).

Let $v \in \Omega^{\alpha}_{F,H}$. Set

$$\hat{Y}(v, \underline{z}) = E^-(-\alpha, \underline{z})Y(v, \underline{z})E^+(-\alpha, \underline{z})z^{(-p\alpha)(0)}\bar{z}^{(-\bar{p}\alpha)(0)}.$$

By Lemma 2.13,

$$[h_l(n), Y(v, \underline{z})] = (h_l, \alpha)_l z^n Y(v, \underline{z})$$
$$[h_r(n), Y(v, \underline{z})] = (h_r, \alpha)_r \bar{z}^n Y(v, \underline{z})$$

for any $h_l \in H_l$ and $h_r \in H_r$ and $n \in \mathbb{Z}$. Hence, we have $[h(n), \hat{Y}(v, \underline{z})] = 0$ and $[\bar{h}(n), \hat{Y}(v, \underline{z})] = 0$ for any $0 \neq n \in \mathbb{Z}$ and $v \in \Omega_{H,H}$, $h_l \in H_l$, $h_r \in H_r$. Thus, $\hat{Y}(v, \underline{z})$ preserves $\Omega_{F,H}$, that is $\hat{Y}(v, \underline{z}) \in \mathrm{End}\,\Omega_{F,H}[[z^{\mathbb{R}}, \bar{z}^{\mathbb{R}}]]$, which defines a product on $\Omega_{F,H}$.

Set

$$\omega_{H_l} = \frac{1}{2}\sum_i h^i_l(-1, -1)h^i_l \in F_{2,0}$$
$$\omega_{H_r} = \frac{1}{2}\sum_i h^i_r(-1, -1)h^i_r \in F_{0,2},$$

where $\{h_l^i\}_i$ is an orthonormal basis of $H_l \otimes_{\mathbb{R}} \mathbb{C}$ and $\{h_r^i\}_i$ is an orthonormal basis of $H_r \otimes_{\mathbb{R}} \mathbb{C}$, and

$$D_\Omega = D - \omega_{H_l}(0,-1), \bar{D}_\Omega = \bar{D} - \omega_{H_r}(-1,0)$$

and

$$L_\Omega(0) = L_F(0) - \omega_{H_l}(1,-1), \overline{L}_\Omega(0) = \overline{L}_F(0) - \omega_{H_r}(-1,1),$$

where $L_F(0), \overline{L}_F(0) \in \mathrm{End}\, F$ are defined by $L_F(0)|_{F_{t,\bar{t}}} = t$ and $\overline{L}_F(0)|_{F_{t,\bar{t}}} = \bar{t}$ for $t, \bar{t} \in \mathbb{R}$. Then, we have:

Lemma 4.2. *For any $v \in \Omega^\alpha \cap F_{t,\bar{t}}$,*

$$[D_\Omega, \hat{Y}(v,\underline{z})] = \frac{d}{dz}\hat{Y}(v,\underline{z}),$$
$$[\bar{D}_\Omega, \hat{Y}(v,\underline{z})] = \frac{d}{d\bar{z}}\hat{Y}(v,\underline{z}),$$
$$[L_\Omega(0), \hat{Y}(v,\underline{z})] = (z\frac{d}{dz} + t - \frac{(p\alpha, p\alpha)_l}{2})\hat{Y}(v,\underline{z}),$$
$$[\overline{L}_\Omega(0), \hat{Y}(v,\underline{z})] = (\bar{z}\frac{d}{d\bar{z}} + \bar{t} - \frac{(\bar{p}\alpha, \bar{p}\alpha)_r}{2})\hat{Y}(v,\underline{z}).$$

Proof. It is easy to show that $D_\Omega, \bar{D}_\Omega, L_\Omega(0), L_\Omega(0)$ commute with the action of the Heisenberg Lie algebra $\hat{H}$. Since $[\omega_{H_l}(0), Y(v,\underline{z})] = Y(\omega_{H_l}(0)v,\underline{z})$ and $\omega_{H_l}(0) = \sum_i \sum_{k\geq 0} h_i(-k-1)h_i(k)$, we have $[\omega_{H_l}(0), Y(v,\underline{z})] = Y((p\alpha)(-1,-1)v,\underline{z})$. Since, by Lemma 2.13,

$$Y((p\alpha)(-1,-1)v,\underline{z}) = (p\alpha)^+(z)Y(v,\underline{z}) + Y(v,\underline{z})(p\alpha)^-(z),$$

we have

$$\begin{aligned}
[D_\Omega, \hat{Y}(v,\underline{z})] &= E^-(-\alpha,\underline{z})[D_\Omega, Y(v,\underline{z})]E^+(-\alpha,\underline{z})z^{(-p\alpha)(0)}\bar{z}^{(-\bar{p}\alpha)(0)}\\
&= E^-(-\alpha,\underline{z})(\frac{d}{dz}Y(v,\underline{z}) - Y((p\alpha)(-1,-1)v,\underline{z})E^+(-\alpha,\underline{z})z^{(-p\alpha)(0)}\bar{z}^{(-\bar{p}\alpha)(0)}\\
&= \frac{d}{dz}\hat{Y}(v,\underline{z})
\end{aligned}$$

Since $\omega_{H_l}(1,-1) = \sum_i\Big(1/2h_i(0)h_i(0) + \sum_{k\geq 1} h_i(-k)h_i(k)\Big)$, we have

$$\begin{aligned}
[\omega_{H_l}(1,-1), Y(v,\underline{z})] &= Y(\omega_{H_l}(0,-1)v,\underline{z})z + Y(\omega_{H_l}(1,-1)v,\underline{z})\\
&= zY((p\alpha)(-1,-1)v,\underline{z}) + \frac{(p\alpha,p\alpha)_l}{2}Y(v,\underline{z}).
\end{aligned}$$

Thus, similarly to the above,

$$\begin{aligned}
[L_\Omega(0), \hat{Y}(v,\underline{z})] &= E^-(-\alpha,\underline{z})[L_\Omega(0), Y(v,\underline{z})]E^+(-\alpha,\underline{z})z^{(-p\alpha)(0)}\bar{z}^{(-\bar{p}\alpha)(0)}\\
&= E^-(-\alpha,\underline{z})\Big((z\frac{d}{dz} + h - \frac{(p\alpha,p\alpha)_l}{2})Y(v,\underline{z}) - zY((p\alpha)(-1,-1)v,\underline{z})\Big)\\
&\quad E^+(-\alpha,\underline{z})z^{(-p\alpha)(0)}\bar{z}^{(-\bar{p}\alpha)(0)}\\
&= (z\frac{d}{dz} + (h - \frac{(p\alpha,p\alpha)_l}{2}))\hat{Y}(v,\underline{z}).
\end{aligned}$$

□

Define a new bilinear form $(-,-)_{\mathrm{lat}}$ on H by

$$(\alpha,\beta)_{\mathrm{lat}} = (p\alpha, p\beta)_l - (\bar{p}\alpha, \bar{p}\beta)_r$$

for $\alpha,\beta \in H$. The main result of this section is the following theorem:

Theorem 4.3. *For a full $\mathcal{H}$-vertex algebra (F,H,p), $(\Omega_{F,H},\hat{Y},\mathbf{1},H,(-,-)_{\mathrm{lat}})$ is a generalized full vertex algebra.*

Proof. We will show the assertion by using Proposition 3.8. (GFV2)-(GFV4) and (GFV7) is obvious. For $\alpha,\beta\in M_\Omega$ and $a\in\Omega^\alpha$ and $b\in\Omega^\beta$, $\hat{Y}(a,\underline{z})b=E^-(-\alpha,\underline{z})Y(a,\underline{z})z^{-(p\alpha,p\beta)_l}\bar{z}^{-(\bar{p}\alpha,\bar{p}\beta)_r}$. Since $z^{-(p\alpha,p\beta)_l}\bar{z}^{-(\bar{p}\alpha,\bar{p}\beta)_r}=z^{-(\alpha,\beta)_{\mathrm{lat}}}|z|^{-(\bar{p}\alpha,\bar{p}\beta)_r}$, (GFV1) holds. By Lemma 4.2, (GFV6) holds. (GFL1) and (GFL2) follow from Lemma 4.2. It suffices to show that (GFL3). Let $a_i\in\Omega^{\alpha_i}$ for $i=1,2,3$ and $u\in\Omega^\vee$. We remark that $M_{H,p}$ is graded by $\omega_{H_l}(1,-1)$ and $\omega_{H_r}(-1,1)$. Then, $(M_{H,p})_{0,0}=\mathbb{C}\mathbf{1}$ and $M_{H,p}=\bigoplus_{n,m\geq 0}(M_{H,p})_{n,m}$. Set $M_{H,p}^+=\bigoplus_{(n,m)\neq(0,0)}(M_{H,p})_{n,m}$. Denote by π the projection of $F=M_{H,p}\otimes\Omega=\mathbb{C}\mathbf{1}\otimes\Omega\oplus M_{H,p}^+\otimes\Omega$ to $\mathbb{C}\mathbf{1}\otimes\Omega$. Then, $u'=u\circ\pi\in F^\vee$. By the construction, $u'(h(-n)-)=u'(\bar{h}(-n)-)=0$ for any $n\in\mathbb{Z}_{\geq 1}$. Since

$$Y(a_i,\underline{z}_i)=E^-(\alpha_i,\underline{z}_i)\hat{Y}(a_i,\underline{z}_i)E^+(\alpha_i,\underline{z}_i)z_i^{(-p\alpha_i)(0)}\bar{z}_i^{(-\bar{p}\alpha_i)(0)}$$

for $i=1,2$, we have

$$\begin{aligned}
&u(Y(a_1,\underline{z}_1)Y(a_2,\underline{z}_2)a_3)\\
&=u(\hat{Y}(a_1,\underline{z}_1)E^+(\alpha_1,\underline{z}_1)z_1^{(p\alpha_1)(0)}\bar{z}_1^{(\bar{p}\alpha_1)(0)}E^-(\alpha_2,\underline{z}_2)\hat{Y}(a_2,\underline{z}_2)z_2^{(p\alpha_2)(0)}\bar{z}_2^{(\bar{p}\alpha_2)(0)}a_3)\\
&=z_1^{(p\alpha_1,p\alpha_2+p\alpha_3)_l}z_2^{(p\alpha_2,p\alpha_3)_l}\bar{z}_1^{(\bar{p}\alpha_1,\bar{p}\alpha_2+\bar{p}\alpha_3)_r}\bar{z}_2^{(\bar{p}\alpha_2,\bar{p}\alpha_3)_r}u(\hat{Y}(a_1,\underline{z}_1)E^+(\alpha_1,\underline{z}_1)E^-(\alpha_2,\underline{z}_2)\hat{Y}(a_2,\underline{z}_2)a_3).
\end{aligned}$$

By Lemma 3.9

$$E^+(\alpha_1,\underline{z}_1)E^-(\alpha_2,\underline{z}_2)=(1-z_2/z_1)^{(p\alpha_1,p\alpha_2)_l}(1-\bar{z}_2/\bar{z}_1)^{(\bar{p}\alpha_1,\bar{p}\alpha_2)_r}E^-(\alpha_2,\underline{z}_2)E^+(\alpha_1,\underline{z}_1).$$

Since $\{h(n),\bar{h}(n)\}_{n\neq 0,h\in H_l,\bar{h}\in H_r}$ commute with $\hat{Y}(a_i,\underline{z}_i)$, we have

$$\begin{aligned}
&u(Y(a_1,\underline{z}_1)Y(a_2,\underline{z}_2)a_3)\\
&z_1^{(p\alpha_1,p\alpha_3)_l}z_2^{(p\alpha_2,p\alpha_3)_l}\bar{z}_1^{(\bar{p}\alpha_1,\bar{p}\alpha_3)_r}\bar{z}_2^{(\bar{p}\alpha_2,\bar{p}\alpha_3)_r}(z_1-z_2)^{(p\alpha_1,p\alpha_2)_l}(\bar{z}_1-\bar{z}_2)^{(\bar{p}\alpha_1,\bar{p}\alpha_2)_r}u(\hat{Y}(a_1,\underline{z}_1)\hat{Y}(a_2,\underline{z}_2)a_3).
\end{aligned}$$

Since

$$\begin{aligned}
&z_1^{(p\alpha_1,p\alpha_3)_l}z_2^{(p\alpha_2,p\alpha_3)_l}\bar{z}_1^{(\bar{p}\alpha_1,\bar{p}\alpha_3)_r}\bar{z}_2^{(\bar{p}\alpha_2,\bar{p}\alpha_3)_r}(z_1-z_2)^{(p\alpha_1,p\alpha_2)_l}(\bar{z}_1-\bar{z}_2)^{(\bar{p}\alpha_1,\bar{p}\alpha_2)_r}\\
&=z_1^{(\alpha_1,\alpha_3)_{\mathrm{lat}}}z_2^{(\alpha_2,\alpha_3)_{\mathrm{lat}}}(z_1-z_2)^{(\alpha_1,\alpha_2)_{\mathrm{lat}}}|z_1|^{(\bar{p}\alpha_1,\bar{p}\alpha_3)_r}|z_2|^{(\bar{p}\alpha_2,\bar{p}\alpha_3)_r}|(z_1-z_2)|^{(\bar{p}\alpha_1,\bar{p}\alpha_2)_r}
\end{aligned}$$

and $|z_1|^{(\bar{p}\alpha_1,\bar{p}\alpha_3)_r}|z_2|^{(\bar{p}\alpha_2,\bar{p}\alpha_3)_r}|(z_1-z_2)|^{(\bar{p}\alpha_1,\bar{p}\alpha_2)_r}\in\mathrm{GCor}_2$, (GFL3) follows from (FV5). □

A full $\mathcal{H}$-conformal vertex algebra is a pair of a full $\mathcal{H}$-vertex algebra and an energy-momentum tensor $(\omega,\bar{\omega})$ such that $\omega(n+2,-1)H_l=0$ and $\bar{\omega}(-1,n+2)H_r=0$ for $n\in\mathbb{Z}_{\geq 0}$. By Lemma 4.2 and Theorem 4.3, we have:

Corollary 4.4. *Let $(F,H,p,\omega,\bar{\omega})$ be a full $\mathcal{H}$-conformal vertex algebra. Then, $(\omega-\omega_{H_l},\bar{\omega}-\omega_{H_r})$ is an energy-momentum tensor of the generalized full vertex algebra $\Omega_{F,H}$.*

4.2. **Equivalence between categories.** In this section, we show that Theorem 3.14 and Theorem 4.3 give an equivalence between a category of full $\mathcal{H}$-vertex algebras and a category of generalized full vertex algebras with an additional structure p.

We first define these categories. A morphism from a full $\mathcal{H}$-vertex algebra (F_1,H_1,p_1) to a full $\mathcal{H}$-vertex algebra (F_2,H_2,p_2) is a full vertex algebra homomorphism $\phi:F_1\to F_2$ such that $\phi(H_1)=H_2$. We denote the category of full $\mathcal{H}$-vertex algebras by <u>Full $\mathcal{H}$-VA</u>.

Let <u>G-full VAp</u> denote the following category. The objects are pairs of a generalized full vertex algebra (Ω,H) and $p\in P(H)$. A morphism from (Ω_1,H_1,p_1) to (Ω_2,H_2,p_2) is a generalized full vertex algebra homomorphism $(\psi,\psi'):(\Omega_1,H_1)\to(\Omega_2,H_2)$ satisfying $\psi'\circ p_1=p_2\circ\psi'$. We call $p\in P(H)$ *a charge structure* of a generalized full vertex algebra.

Let (F,H,p) be a full $\mathcal{H}$-vertex algebra. Then, $p \in P(H)$. Thus, $(\Omega_{F,H},H,p)$ is an object in $\underline{\text{G-full VAp}}$.

Lemma 4.5. *The assignment* $\Omega : \underline{\mathit{Full}\ \mathcal{H}\text{-}\mathit{VA}} \to \underline{\mathit{G\text{-}full\ VAp}}$, $(F,H,p) \mapsto (\Omega_{F,H},H,p)$ *is a functor.*

Proof. Let ϕ be a morphism from a full $\mathcal{H}$-vertex algebra (F_1,H_1,p_1) to a full $\mathcal{H}$-vertex algebra (F_2,H_2,p_2). Since ϕ preserves the vacuum vector, $\phi(\ker p_1) = \phi(H_1 \cap \ker D) = H_2 \cap \ker D = \ker p_2$. Since $\phi(h_l(1,-1)h_l') = \phi(h_l)(1,-1)\phi(h_l')$ for any $h_l, h_l' \in (H_1)_l$, ϕ is an isometric isomorphism between H_1 and H_2 and $\phi \circ p_1 = p_2 \circ \phi$. Since the restriction of ϕ on the vacuum spaces gives a linear map $\phi|_{\Omega_{F_1,H_1}} : \Omega_{F_1,H_1} \to \Omega_{F_2,H_2}$, the pair $(\phi|_{\Omega_{F_1,H_1}}, \phi|_{H_1})$ is a morphism of $\underline{\text{G-full VAp}}$. □

Let (Ω,H,p) be an object in $\underline{\text{G-full VAp}}$. Then, $F_{\Omega,H,p}$ is a full vertex algebra. Since $M_{H,p} = G_{H,p}^0 \otimes \mathbb{C}\mathbf{1} \subset G_{H,p}^0 \otimes \Omega^0 \subset F_{\Omega,H,p}$, $F_{\Omega,H,p}$ is naturally a full $\mathcal{H}$-vertex algebra.

Lemma 4.6. *The assignment* $F : \underline{\mathit{G\text{-}full\ VAp}} \to \underline{\mathit{Full}\ \mathcal{H}\text{-}\mathit{VA}}$, $(\Omega,H,p) \mapsto (F_{\Omega,H,p},H,p)$ *is a functor.*

Proof. Let (Ω_1,H_1,p_1) and (Ω_2,H_2,p_2) be objects in $\underline{\text{G-full VAp}}$ and (ψ,ψ') be a morphism from (Ω_1,H_1,p_1) to (Ω_2,H_2,p_2). Since ψ' is an isometric isomorphism, by Lemma 3.11, we have an isomorphism of generalized full vertex algebras

$$\tilde{\psi}' : G_{H_1,p_1} \to G_{H_2,p_2},$$

where we used $\psi' \circ p_1 = p_2 \circ \psi'$. Then, we have a generalized full vertex algebra homomorphism

$$\tilde{\psi}' \otimes \psi : G_{H_1,p_1} \otimes \Omega_1 \to G_{H_2,p_2} \otimes \Omega_2.$$

The restriction of the homomorphism on $G_{H_1,p_1} \otimes_{\Delta H_1} \Omega_1 \subset G_{H_1,p_1} \otimes \Omega_1$ gives us a full $\mathcal{H}$-vertex algebra homomorphism as desired. □

It is clear that the above functors are mutually inverse equivalences. Thus, we have:

Theorem 4.7. $\Omega : \underline{\mathit{Full}\ \mathcal{H}\text{-}\mathit{VA}} \to \underline{\mathit{G\text{-}full\ VAp}}$ *and* $F : \underline{\mathit{G\text{-}full\ VAp}} \to \underline{\mathit{Full}\ \mathcal{H}\text{-}\mathit{VA}}$ *gives an equivalence of categories.*

Corollary 4.8. *Let* (F,H,p) *be a full* $\mathcal{H}$*-vertex algebra. Then,* F *is isomorphic to* $F_{\Omega_{F,H},H,p} = G_{H,p} \otimes_{\Delta H} \Omega_{F,H}$ *as a full* $\mathcal{H}$*-vertex algebra.*

4.3. **Adjoint functor I – generalized full vertex algebra and associative algebra.** In this section, we construct an adjoint functor from the category of generalized full vertex algebras to some category of associative algebras.

We first recall that for a vertex algebra V, $\ker D_V = \{v \in V \mid v(-2)\mathbf{1} = 0\}$ is a commutative $\mathbb{C}$-algebra (see for example [Mo1]). Conversely, any commutative $\mathbb{C}$-algebra A is a vertex algebra, where the vertex operator is defined by $Y(a,z)b = ab$ (consisting of only the constant term). In fact, this correspondence gives an adjoint functor between the category of vertex algebras and the category of commutative $\mathbb{C}$-algebras.

In [Mo1], we show that if V is a simple vertex operator algebra, then $\ker D_V$ is $\mathbb{C}$.

Remark 4.9. *This fact is related to the notion of c-number in physics. That is any field which is independent on the position (formal variable) is a scalar.*

We generalize the above discussion to generalized full vertex algebra based on the discussion in [Mo1]. Since a generalized full vertex algebra has a monodromy, the $\mathbb{C}$-algebra is no longer commutative, which we call an AH-pair. We first recall the notion of AH pairs introduced in

[Mo1], which is a commutative algebra object of some braided tensor category (see [Mo2], Section 5.3).

Let H be a finite-dimensional vector space over $\mathbb{R}$ equipped with a non-degenerate symmetric bilinear form $(-,-)$ and A a unital associative algebra over $\mathbb{C}$ with the unity 1. Assume that A is graded by H as $A = \bigoplus_{\alpha\in H} A^\alpha$.

We will say that such a pair (A,H) is an *even AH pair* if the following conditions are satisfied:

AH1) $1 \in A^0$ and $A^\alpha A^\beta \subset A^{\alpha+\beta}$ for any $\alpha,\beta \in H$;

AH2) If $A^\alpha \neq 0$, then $(\alpha,\alpha) \in 2\mathbb{Z}$;

AH3) For $v \in A^\alpha, w \in A^\beta$, $vw = (-1)^{(\alpha,\beta)}wv$;

Remark 4.10. *Suppose that $A^\alpha A^\beta \neq 0$ for $\alpha,\beta \in H$. Then, by (AH1) and (AH2), $(\alpha,\alpha),(\beta,\beta),(\alpha+\beta,\alpha+\beta) \in 2\mathbb{Z}$ and thus $(\alpha,\beta) \in \mathbb{Z}$. Hence, $(-1)^{(\alpha,\beta)}$ is well-defined.*

Define an $\mathbb{R}^2 \times H$-grading on A by

$$\begin{cases} A^\alpha_{t,\bar{t}} = 0 \text{ if } (t,\bar{t}) \neq (0,0), \\ A^\alpha_{0,0} = A^\alpha \end{cases}$$

for any $\alpha \in H$ and set $\hat{Y}(a,\underline{z}) = l_a \in \mathrm{End}\, A$ for $a \in A$, where l_a is the left multiplication by a and $\mathbf{1} = 1$.

Proposition 4.11. *For an even AH pair (A,H), $(A,\hat{Y},\mathbf{1},H)$ is a generalized full vertex algebra. Furthermore, (A,H) is a generalized full conformal vertex algebra with the energy-momentum tensor $(0,0)$.*

Proof. Since $(\alpha,\alpha) \in 2\mathbb{Z}$ for any $\alpha \in M_{A,H}$, (GFV2) holds. Let $a_i \in A^{\alpha_i}$ and $u \in A^\vee$. Then, $u(a_1(a_2a_3)) = (-1)^{(\alpha_1,\alpha_2)}u(a_2(a_1a_3)) = u((a_1a_2)a_3)$ by (AH3), which implies that (GFV5) holds. The rest is obvious. □

For even AH pairs (A,H_A) and (B,H_B), a homomorphism of even AH pairs is a pair (f,f') of maps $f: A \longrightarrow B$ and $f': H_A \longrightarrow H_B$ such that f is an algebra homomorphism and f' an isometry such that $f(A^\alpha) \subset B^{f'(\alpha)}$ for all $\alpha \in H_A$. We denote by $\underline{\text{even AH pair}}$ the category of even AH pairs. Then, Proposition 4.11 gives a functor from the category of even AH pairs to the category of generalized full vertex algebras, denoted by $i: \underline{\text{even AH pair}} \to \underline{\text{G-full VA}}$. In the rest of this section, we construct an adjoint functor followed by [Mo1].

Let (Ω,H) be a generalized full vertex algebra. Set $A_\Omega = \ker D \cap \ker \bar{D} \cap \Omega_{0,0}$. By Proposition 3.4, D and $\bar{D}$ act as derivations of the algebra. Thus, $\ker D \cap \ker \bar{D}$ is a subalgebra of Ω. If $a \in \ker D \cap \ker \bar{D}$, by Proposition 3.4 again, $\hat{Y}(a,\underline{z}) = a(-1,-1) \in \mathrm{End}\,\Omega$, that is, the vertex operator is independent of the position. By (GFV6), A_Ω is a subalgebra of $\ker D \cap \ker \bar{D}$ and Ω. Set $A^\alpha_\Omega = A_\Omega \cap \Omega^\alpha_{0,0}$ for $\alpha \in H$. Define a product on A_Ω by

$$a \cdot b = a(-1,-1)b,$$

for $a,b \in A_\Omega$. Then, we have:

Proposition 4.12. *For a generalized full vertex algebra (Ω,H), (A_Ω,H) is an even AH-pair.*

Proof. By (GFV3) and (GFV4), $\mathbf{1}$ is unity and (AH1) holds. If $\Omega^\alpha_{0,0} \neq 0$, then by (GFV2) $(\alpha,\alpha) \in 2\mathbb{Z}$, which implies (AH2). Assume that $a\cdot b \neq 0$. Since $a\cdot b = \hat{Y}(a,\underline{z})b \in z^{(\alpha,\beta)}\Omega((z,\bar{z},|z|))$, $(\alpha,\beta) \in \mathbb{Z}$. By (GFV5), $a(bc) = (-1)^{(\alpha,\beta)}b(ac) = (ab)c$ for any $c \in A_\Omega$. Thus, A_Ω is an even AH pair. □

This correspondence

$$A : \underline{\text{G-full VA}} \to \underline{\text{even AH pair}}, \ (\Omega, H) \mapsto (A_\Omega, H)$$

is a functor since a morphism of generalized full vertex algebras preserves the vacuum vector $\mathbf{1}$, thus, commutes with $D, \bar{D}$.

Proposition 4.13. *The above functor A : $\underline{\text{G-full VA}} \to \underline{\text{even AH pair}}$ is right adjoint to the inclusion functor i : $\underline{\text{even AH pair}} \to \underline{\text{G-full VA}}$.*

Proof. Let (A, H) be an even AH pair and (Ω, H') a generalized full vertex algebra and $(f, f') : (A, H) \to (\Omega, H')$ a generalized full vertex algebra homomorphism. Since $DA = \bar{D}A = 0$ and $f(1) = \mathbf{1}$, thus, f commutes with $D, \bar{D}$, the image of f is in $\ker D \cap \ker \bar{D}$. Since f preserves the $\mathbb{R}^2$-grading of the generalized full vertex algebras, $f(A) \subset \ker D \cap \ker \bar{D} \cap \Omega_{0,0}$. Thus, the restriction gives a generalized full vertex algebra homomorphism $(f, f') : A \to A_\Omega$, which is an even AH pair homomorphism. Since the rest of the argument is completely similar to the proof of [Mo1, Theorem 3.1], the details are left to the reader. □

4.4. **Adjoint functor II – Lattice full vertex algebra revisit.** A structure of AH pairs is studied in [Mo1]. We briefly recall it. Let H be a finite dimensional real vector space equipped with a non-degenerate bilinear form $(-,-) : H \times H \to \mathbb{R}$.

A good AH pair is an even AH pair (A, H) such that:

GAH1) $A^0 = \mathbb{C}\mathbf{1}$;

GAH2) $ab \neq 0$ for any $\alpha, \beta \in H$ and $v \in A^\alpha \setminus \{0\}$, $w \in A^\beta \setminus \{0\}$.

A lattice pair is a good AH pair such that

LP) $A^{-\alpha} \neq 0$ if $A^\alpha \neq 0$ for $\alpha \in H$.

For a good AH pair (A, H), set $M_{A,H} = \{\alpha \in H \mid A^\alpha \neq 0\}$. Then, by (GAH1) and (GAH2), $0 \in M_{A,H}$ and $\alpha + \beta \in M_{A,H}$ for any $\alpha, \beta \in M_{A,H}$. Thus, $M_{A,H}$ is a submonoid of H. A good AH pair (A, H) is a lattice pair if and only if $M_{A,H}$ is a subgroup of H.

We also introduce a notion of an even H-lattice (see section 2.2 in [Mo1]). An even H-lattice is a subgroup $L \subset H$ such that $(\alpha, \alpha) \in 2\mathbb{Z}$ for any $\alpha \in L$. The subgroup $M_{A,H} \subset H$ for a lattice pair (A, H) is an example of an even H-lattice by (AH4).

Conversely, Let $L \subset H$ be an even H-lattice and $Z^2(L, \mathbb{C}^\times)$ the $\mathbb{C}^\times$-coefficient two-cocycles of the abelian group L. It is not hard to show that there exists $\epsilon \in Z^2(L, \mathbb{C}^\times)$ such that $\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1$ and $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}$ for any $\alpha, \beta \in L$ (see [Mo1]). Then, define a new product on the group algebra $\mathbb{C}[L] = \bigoplus_{\alpha \in L} \mathbb{C}e_\alpha$ by

$$e_\alpha e_\beta = \epsilon(\alpha, \beta) e_{\alpha+\beta}.$$

Since ϵ is a two-cocycle, the product is associative. Denote by $\mathbb{C}[\hat{L}]$ the associative algebra. By construction, $\mathbb{C}[\hat{L}]$ is a lattice pair, which is a generalization of the twisted group algebra constructed in [FLM]. In fact, any lattice pair (A, H) is isomorphic to $\mathbb{C}[\hat{M}_{A,H}]$. More precisely, we have (see section 2.1 and 2.2 in [Mo1]):

Proposition 4.14. *Let (A, H) be a lattice pair and $M_{A,H}$ be an even H-lattice associated with the lattice pair. Then, (A, H) is isomorphic to $(\mathbb{C}[\hat{M}_{A,H}], H)$ as even AH pairs.*

A category of good AH pairs (resp. a category of lattice pairs) is a full subcategory of $\underline{\text{even AH pair}}$ whose objects are good AH pairs (reps. lattice pairs), which is denoted by $\underline{\text{good AH pair}}$ (resp. $\underline{\text{Lattice pair}}$). Let i : $\underline{\text{Lattice pair}} \to \underline{\text{good AH pair}}$ be the inclusion functor. We will construct an adjoint functor lat : $\underline{\text{good AH pair}} \to \underline{\text{Lattice pair}}$. Let (A, H) be a good AH pair and set $L_{A,H} = \{\alpha \in M_{A,H} \mid -\alpha \in M_{A,H}\} = M_{A,H} \cap (-M_{A,H})$. Then, $L_{A,H}$ is a subgroup of H, thus, an even H-lattice. Set $A^{\mathrm{lat}} = \bigoplus_{\alpha \in L_{A,H}} A^\alpha$. Since $L_{A,H}$ is a subgroup, A^{lat} is a subalgebra of A as an AH pair. In fact, the correspondence $(A,H) \mapsto (A^{\mathrm{lat}}, H)$ define the functor $\mathrm{lat} : \underline{\text{good AH pair}} \to \underline{\text{Lattice pair}}$. Hence we have:

Proposition 4.15 (Proposition 2.5 in [Mo1])**.** *The functor* lat : *$\underline{\text{good AH pair}} \to \underline{\text{Lattice pair}}$ is right adjoint to the inclusion functor $i : \underline{\text{Lattice pair}} \to \underline{\text{good AH pair}}$.*

As an application, we give examples of full vertex algebras. Let L be an even non-degenerate lattice, that is, L is an abelian group of finite rank equipped with a symmetric bilinear form

$$(-,-) : L \times L \to \mathbb{Z}$$

such that $(\alpha, \alpha) \in 2\mathbb{Z}$ for any $\alpha \in L$ and the induced bilinear form on the real vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$ is non-degenerate. Since L is an even $L \otimes_{\mathbb{Z}} \mathbb{R}$-lattice, a lattice pair $\mathbb{C}[\hat{L}]$ can be constructed as above. Since $\mathbb{C}[\hat{L}]$ is an even AH pair, it is a generalized full conformal vertex algebra by Proposition 4.11. Thus, by Theorem 3.14, for any $p \in P(L \otimes_{\mathbb{Z}} \mathbb{R})$, $F_{\mathbb{C}[\hat{L}], L \otimes_{\mathbb{Z}} \mathbb{R}, p}$ is a full conformal vertex algebra. We denote it by $F_{L,p}$ and call *a lattice full vertex algebra.*

Remark 4.16. *It is natural in physics to choose the projection $p \in P(L \otimes_{\mathbb{Z}} \mathbb{R})$ such that* $\ker p$ *is a negative-definite subspace in $L \otimes_{\mathbb{Z}} \mathbb{R}$. If the signature of $L \otimes_{\mathbb{Z}} \mathbb{R}$ is (n,m), then such projections are parametrized by the orthogonal Grassmannian*

$$\mathrm{O}(n,m)/\mathrm{O}(n) \times \mathrm{O}(m),$$

where $\mathrm{O}(n,m)$ is an orthogonal group with the signature (n,m). It is noteworthy that, in this case, the spectrum of the lattice full vertex algebra is compact. Thus, we constructed a continuous family of a (compact) full vertex algebras. We study this algebras in more detail in section 5.3.

To summarize, we constructed two adjoint functors and one equivalence of categories in this section:

$$\underline{\text{Lattice pair}} \underset{\mathrm{lat}}{\overset{i}{\rightleftarrows}} \underline{\text{good AH pair}}, \quad (\top)$$

$$\underline{\text{even AH pair}} \underset{A_-}{\overset{i}{\rightleftarrows}} \underline{\text{G-full VA}}, \quad (\top)$$

$$\underline{\text{G-full VAp}} \underset{\Omega}{\overset{F_-}{\rightleftarrows}} \underline{\text{Full } \mathcal{H}\text{-VA}}. \quad (\cong)$$

4.5. **Remark on vertex algebras.** In this section, we discuss the equivalence of categories in the case that a full $\mathcal{H}$-vertex algebra consists of only holomorphic fields.

Let V be a $\mathbb{Z}$-graded vertex algebra and H a real subspace of V_1 such that:

HS1) $h(1)h' \in \mathbb{R}\mathbf{1}$ for any $h, h' \in H$;

HS2) For any $h, h' \in H$, $h(n)h' = 0$ if $n = 0$ or $n \geq 2$;

HS3) The bilinear form $(-,-)$ on H defined by $h(1)h' = (h,h')\mathbf{1}$ for $h, h' \in H$ is non-degenerate.

Then, as in Section 4.1, H generates a representation of the Heisenberg Lie algebra. Set

$$(\Omega_{V,H})_t^\alpha = \{v \in V_{t+(\alpha,\alpha)/2}^\alpha \mid h(0)v = (\alpha,h)v, h(n)v = 0 \text{ for any } h \in H \text{ and } n \in \mathbb{Z}_{\geq 1}\}$$

for $\alpha \in H$ and $t \in \mathbb{Z}$. The above pair (V, H) is said to be an $\mathcal{H}$-vertex algebra if the following conditions hold:

VH1) $h(0)$ is semisimple on V with real eigenvalues for any $h \in H$;
VH2) For any $\alpha \in H$, there exists $N \in \mathbb{Z}$ such that $V_t^\alpha = 0$ for any $t \leq N$.

By Proposition 2.4, an $\mathcal{H}$-vertex algebra is a full $\mathcal{H}$-vertex algebra. A category of $\mathcal{H}$-vertex algebras is a full subcategory of $\underline{\text{Full } \mathcal{H}\text{-VA}}$ whose objects are $\mathcal{H}$-vertex algebras. We denoted the category of $\mathcal{H}$-vertex algebras by $\underline{\mathcal{H}\text{-VA}}$. We also denote the category of generalized vertex algebras by $\underline{\text{G-VA}}$.

Let (V, H) be an $\mathcal{H}$-vertex algebra. Then, by the proof of Theorem 4.3, $(\Omega_{V,H}, H)$ is a generalized vertex algebra. Furthermore, the charge structure of $\Omega_{V,H}$ is the identical projection $\mathrm{id}_H \in \mathrm{End}\ H$ since all fields in V are holomorphic. Thus, (V, H) can be recovered from $\Omega_{V,H}$. Let $V : \underline{\text{G-VA}} \to \underline{\mathcal{H}\text{-VA}}$ be the functor defined by $V_{\Omega,H} = F_{\Omega,H,\mathrm{id}_H}$ for a generalized vertex algebra (Ω, H). Then, we have:

Proposition 4.17. *The restriction of the functor* $\Omega : \underline{\mathcal{H}\text{-VA}} \to \underline{\text{G-VA}}$ *gives an equivalence of the categories and the inverse functor is given by* $V : \underline{\text{G-VA}} \to \underline{\mathcal{H}\text{-VA}}$.

5. Current-current deformation

In this section, we define and study a current-current deformation of a full $\mathcal{H}$-vertex algebra.

Let (F, H, p_0) be a full $\mathcal{H}$-vertex algebra. For $p \in P(H)$, set $F_p = G_{H,p} \otimes_{\Delta H} \Omega_{F,H}$. Then, by Theorem 3.14, F_p is a full $\mathcal{H}$-vertex algebra. Thus, we have a family of full $\mathcal{H}$-vertex algebras parametrized by $P(H)$. By Corollary 4.8, F_{p_0} is isomorphic to F as a full $\mathcal{H}$-vertex algebra.

Let $O(H; \mathbb{R})$ be the orthogonal group of the real vector space $(H, (-,-)_{\mathrm{lat}})$. Then, $O(H; \mathbb{R})$ acts on $P(H)$ by $\sigma \cdot p = \sigma p \sigma^{-1}$ for $\sigma \in O(H; \mathbb{R})$ and $p \in P(H)$. From the elementary linear algebra, the following lemma follows:

Lemma 5.1. *For projections* $p, p' \in P(H)$*, the following conditions are equivalent:*

1. *There exits* $\sigma \in O(H; \mathbb{R})$ *such that* $\sigma \cdot p = p'$.
2. *The signature of the real spaces* $\ker p$ *and* $\ker p'$ *are the same.*

Thus, the $O(H; \mathbb{R})$ orbit of $p_0 \in P(H)$ is equal to the orthogonal Grassmannian

$$O(H; \mathbb{R}) / O(H_l; \mathbb{R}) \times O(H_r; \mathbb{R}),$$

which is the connected component of $P(H)$ containing p_0.

We call the family of full $\mathcal{H}$-vertex algebras $\{F_{\sigma \cdot p_0}\}_{\sigma \in O(H;\mathbb{R})}$ a *current-current deformation* of the full $\mathcal{H}$-vertex algebra (F, H, p_0).

By Corollary 4.4 and Theorem 3.14, we have:

Proposition 5.2. *If* F *is a full* $\mathcal{H}$*-conformal vertex algebra, then a current-current deformation of* F *also has an energy-momentum tensor.*

A full $\mathcal{H}$-vertex algebra is called positive if both $(H_l, (-,-)_l)$ and $(H_r, (-,-)_r)$ are positive-definite. The following proposition says that the compactness of conformal field theory is preserved by the current-current deformation under some mild assumption.

Proposition 5.3. *Let* (F, H, p_0) *be a full* $\mathcal{H}$*-vertex algebra such that* $(\Omega_{F,H})_{t,\bar{t}}^\alpha = 0$ *for any* $t \leq 0$ *or* $\bar{t} \leq 0$ *and any* $\alpha \in H$*. If* F *is positive and compact, then a current-current deformation of* F *is also positive and compact.*

Proof. Let $\sigma \in O(H;\mathbb{R})$. By Lemma 5.1, $F_{\sigma\cdot p_0}$ is a positive full $\mathcal{H}$-vertex algebra. Since for any $\alpha,\beta \in H$, $(\sigma p_0\sigma^{-1}\alpha, \sigma p_0\sigma^{-1}\beta)_{\mathrm{lat}} = (p_0\sigma^{-1}\alpha, p_0\sigma^{-1}\beta)_{\mathrm{lat}}$,

$$\begin{aligned} F_{\sigma\cdot p_0} &= G_{H,\sigma\cdot p_0} \otimes_{\Delta H} \Omega_{F,H} \\ &= \bigoplus_{\alpha\in H} M_{H,\sigma\cdot p_0}(\alpha) \otimes \Omega_{F,H}^{\alpha} \\ &= \bigoplus_{\alpha\in H} M_{H,p_0}(\sigma^{-1}\cdot\alpha) \otimes \Omega_{F,H}^{\alpha}. \end{aligned}$$

Thus, by the positivity and the assumption, $(F_{\sigma\cdot p_0})_{h,\bar{h}} = 0$ unless $h,\bar{h} \geq 0$, thus the spectrum of $F_{\sigma\cdot p_0}$ is bounded below.

Let $N \in \mathbb{R}$. It is easy to show that $\sum_{h+\bar{h}<N} \dim(F_{\sigma\cdot p_0})_{h,\bar{h}} < \infty$ if and only if $\sum_{t,\bar{t},\alpha} \dim(\Omega_{F,H})_{t,\bar{t}}^{\alpha} < \infty$, where in the sum $t,\bar{t} \in \mathbb{R}$ and $\alpha \in H$ satisfy $t+\bar{t}+\frac{1}{2}(\sigma^{-1}\alpha,\sigma^{-1}\alpha)_l + \frac{1}{2}(\sigma^{-1}\alpha,\sigma^{-1}\alpha)_r < N$. Set $\|\alpha\| = \frac{1}{2}(\alpha,\alpha)_l + \frac{1}{2}(\alpha,\alpha)_r$ for $\alpha \in H$. Since $\sigma \in \mathrm{GL}(H)$, by an elementary linear algebra, there exists $k_\sigma \in \mathbb{R}_{>0}$ such that $k_\sigma\|\alpha\| < \|\sigma^{-1}\alpha\|$ for any $\alpha \in H$. We may assume that $0 < k_\sigma < 1$. Then, for any $\alpha \in H$ and $t,\bar{t} \geq 0$,

$$\|\sigma^{-1}\alpha\| + t + \bar{t} > k_\sigma\Big(\|\alpha\| + \frac{1}{k_\sigma}(t+\bar{t})\Big)\Big) > k_\sigma\Big(\|\alpha\| + t + \bar{t}\Big).$$

Thus, the spectrum of $F_{\sigma\cdot p_0}$ is discrete since that of F_{p_0} is discrete. Hence, $F_{\sigma\cdot p_0}$ is compact. □

Remark 5.4. *It seems that for any unitary compact conformal field theory the assumption in the above proposition is satisfied. We conjecture that the unitary compact conformal field theory is stable under exactly marginal deformations.*

5.1. **Physical meaning of deformation.** In this section, we discuss a relation between a current-current deformation of a full $\mathcal{H}$-vertex algebra and an exactly marginal deformation in physics. Let (F,H,p) be a full $\mathcal{H}$-vertex algebra and $h_l \in \ker\bar{p}$ and $h_r \in \ker p$ satisfy $(h_l,h_l)_{\mathrm{lat}} = 1$ and $(h_r,h_r)_{\mathrm{lat}} = -1$. Set $H^{\perp} = \{h \in H \mid (h,h_l)_{\mathrm{lat}} = 0, (h,h_r)_{\mathrm{lat}} = 0\}$ and define a group homomorphism $\sigma : \mathbb{R} \to O(H;\mathbb{R})$ $g \mapsto \sigma(g)$ by

$$\begin{cases} \sigma(g)|_{H^{\perp}} &= \mathrm{id}, \\ \sigma(g)(h_l) &= \cosh(g)h_l + \sinh(g)h_r, \\ \sigma(g)(h_r) &= \cosh(g)h_r + \sinh(g)h_l. \end{cases}$$

It is believed that a quantum field theory can be deformed by adding a new field to the Lagrangian (see Introduction). We can show that the deformation family $\{F_{\sigma(g)\cdot p}\}_{g\in\mathbb{R}}$ corresponds to the deformation by the $(1,1)$-field $Y(h_l(-1,-1)h_r,\underline{z}) = h_l(z)h_r(\bar{z})$ by using the path-integral. This is why we call the deformation a current-current deformation.

5.2. **Double coset description.** In this section, we gives a double coset description of the parameter space of a current-current deformation. Let (F,H,p) be a full $\mathcal{H}$-vertex algebra and let (ψ,ψ') an automorphism of a generalized full vertex algebra $(\Omega_{F,H},H)$. Then, $\psi' \in O(H;\mathbb{R})$. Thus, we have a group homomorphism $\mathrm{Aut}\,(\Omega_{F,H},H) \to O(H;\mathbb{R})$ from the group of generalized full vertex algebra automorphisms to the orthogonal group. Denote the image of this map by $D_{F,H} \subset O(H)$, which we call a *duality group*. We note that $(\psi,\psi') \in \mathrm{Aut}\,(\Omega_{F,H},H)$ lifts to a full vertex algebra automorphism if and only if it preserves the charge structure, that is, $\psi'\cdot p = p$. The following theorem follows from Theorem 4.7:

Theorem 5.5. *For $p,p' \in P(H)$, F_p and $F_{p'}$ are isomorphic as full $\mathcal{H}$-vertex algebras if and only if there exists $\sigma \in D_{F,H}$ such that $\sigma\cdot p = p'$. In particular, there is a bijection between* *the isomorphism classes of a current-current deformation of* (F, H, p) *and the double coset* $D_{F,H}\backslash O(H;\mathbb{R})/O(H_l;\mathbb{R}) \times O(H_r;\mathbb{R})$.

5.3. **Example: Toroidal Compactification.** Let L be an even non-degenerate lattice of signature (n, m) and $H = L\otimes_{\mathbb{Z}}\mathbb{R}$. Then, we have a lattice full vertex algebra $F_{L,H,p}$ for any $p \in P_>(H)$. Since $D_{F_{L,H,p},H}$ is isomorphic to the lattice automorphism group, Aut L, the isomorphism classes is

$$\mathrm{Aut}\, L\backslash O(n, m)/O(n) \times O(m).$$

Let $II_{1,1} = \mathbb{Z}z\oplus\mathbb{Z}w$ be the rank two even lattice defined by $(z, z) = (w, w) = 0$ and $(z, w) = -1$. Then, $II_{1,1}$ is a unique even unimodular lattice of signature $(1, 1)$. Set $II_{k,k} = {II_{1,1}}^{\oplus k}$ for $k \in \mathbb{Z}_>$. The lattice full vertex algebras $\{F_{II_{k,k},II_{k,k}\otimes_{\mathbb{Z}}\mathbb{R},p}\}_{p\in P_>(II_{k,k}\otimes_{\mathbb{Z}}\mathbb{R})}$ appear in the toroidal compactification of string theory (see for example [Polc1]), which is parametrized by

$$O(k, k;\mathbb{Z})\backslash O(k, k)/O(k) \times O(k).$$

In the rest of this section, we explicitly describe the action of the duality group $O(k, k;\mathbb{Z})$ in detail in the case of $k = 1$. Set $H_{II_{1,1}} = II_{1,1}\otimes_{\mathbb{Z}}\mathbb{R}$. Let $p \in P_>(H)$. Since $\ker \bar{p}$ is positive-definite, there is a unique (up to the multiplication by $\pm 1 = O(1)$) vector $v \in \ker \bar{p}$ such that $(v, v) = 1$. It is clear that p is uniquely determined by this vector. Let $v = az + bw \in H_{II_{1,1}}$ be a norm 1 vector $(a, b \in \mathbb{R})$. Then, by $(v, v) = -2ab$, we may assume that $v = \frac{1}{\sqrt{2}}(Rz - R^{-1}w)$ for $R \in \mathbb{R}_{>0}$. Denote by p_R the corresponding projection in $P_>(H_{II_{1,1}})$. Thus, we have an isomorphism $\mathbb{R}_{>0} \to O(1, 1)/O(1)\times O(1)$, $R \mapsto p_R$. The lattice automorphism group $\mathrm{Aut}\, II_{1,1}$ is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, which is generated by the involutions σ, τ such that:

$$\sigma(z) = w, \sigma(w) = z,$$
$$\tau(z) = -z, \tau(w) = -w.$$

The action of σ on p_R is determined by

$$\sigma(\frac{1}{\sqrt{2}}(Rz - R^{-1}w)) = -\frac{1}{\sqrt{2}}(R^{-1}z - Rw).$$

Hence, $\sigma \cdot p_R = p_{R^{-1}}$. Since $\tau \in O(1) \times O(1) \subset O(1, 1)$,

$$\mathrm{Aut}\, II_{1,1}\backslash O(1, 1)/O(1) \times O(1) \cong \mathbb{R}_{\geq 1}.$$

In the string theory, R is a radius of the compactification of the target space. Denote by C_R the full vertex algebra $F_{II_{1,1},H_{II_{1,1}},p_R}$. The isomorphism $\tilde{\sigma} : C_R \to C_{R^{-1}}$ is called a *T-duality* of string theory. Let $R = e^s$ for $s \in \mathbb{R}$. Then, the action of a 1-parameter deformation $\sigma(g)$ associated with $h_l = \frac{1}{\sqrt{2}}(e^s z - e^{-s}w), h_r = \frac{1}{\sqrt{2}}(e^s z + e^{-s}w)$ is

$$\begin{aligned}\sigma(g)\Big(\frac{1}{\sqrt{2}}(e^s z - e^{-s}w)\Big) &= \frac{1}{\sqrt{2}}(\cosh(g)(e^s z - e^{-s}w) + \sinh(g)(e^s z + e^{-s}w))\\ &= \frac{1}{\sqrt{2}}(e^{s+g}z - e^{-s-g}w).\end{aligned}$$

Thus, $\sigma(g)$ changes the radius $R = e^s$ into $e^g R = e^{g+s}$.

We end this section by studying the chiral vertex algebra $\ker \bar{D}$ of a full vertex algebra C_R. It is easy to show that the conformal weight of $e_{nz+mw} \in \mathbb{C}[\hat{II}_{1,1}]$ is $(\frac{(nR^{-1}-mR)^2}{4}, \frac{(nR^{-1}+mR)^2}{4})$ for $n, m \in \mathbb{Z}$. The state e_{nz+mw} is in $\ker \bar{D}$ if and only if $R^2 = -\frac{n}{m}$. Thus, if $R^2 \in \mathbb{R} \setminus \mathbb{Q}$, $\ker \bar{D} \otimes \ker D$ is isomorphic to the affine Heisenberg full vertex algebras $M_{H_{II_{1,1}},p_R}$. We assume that $R^2 = \frac{p}{q}$ for some coprime intergers $p, q \in \mathbb{Z}_{>0}$. In this case,

$$\ker \bar{D} = M_{\ker \bar{p}_R} \otimes \bigoplus_{k \in \mathbb{Z}} \mathbb{C} e_{k(pz-qw)}.$$

Since the conformal weight of $e_{k(pz-qw)}$ is $(pqk^2, 0)$, $\ker \bar{D}$ is isomorphic to the lattice vertex algebra $V_{\sqrt{2pq}\mathbb{Z}}$ associated with the rank one lattice $\sqrt{2pq}\mathbb{Z}$. In particular, $C_{\sqrt{\frac{p}{q}}}$ is a finite extension of the lattice full vertex algebra $V_{\sqrt{2pq}\mathbb{Z}} \otimes \bar{V}_{\sqrt{2pq}\mathbb{Z}}$. We will determine the irreducible decomposition of $C_{\sqrt{\frac{p}{q}}}$ as a $V_{\sqrt{2pq}\mathbb{Z}} \otimes \overline{V}_{\sqrt{2pq}\mathbb{Z}}$-module. We recall that there are $2pq$ irreducible modules of $V_{\sqrt{2pq}\mathbb{Z}}$, denoted by $\{V_{\sqrt{2pq}\mathbb{Z}+\frac{i}{\sqrt{2pq}}}\}_{i \in \mathbb{Z}/2pq\mathbb{Z}}$, see for example [LL]. Since

$$\begin{aligned}\big(p_R(pz-qw), p_R(nz+mw)\big) &= nq - mp,\\ -\big(\bar{p}_R(pz+qw), \bar{p}_R(nz+mw)\big) &= nq + mp,\end{aligned}$$

e_{nz+mw} is contained in $V_{\sqrt{2pq}\mathbb{Z}+\frac{nq-mp}{\sqrt{2pq}}} \otimes \overline{V}_{\sqrt{2pq}\mathbb{Z}+\frac{nq+mp}{\sqrt{2pq}}}$.

We will use the following elementary lemma:

Lemma 5.6. *Let $(a, b) \in \mathbb{Z}^2$ satisfy $ap-bq = 1$. Then, $n_{p,q} = ap+bq$ satisfies $n_{p,q}^2 = 1 \in \mathbb{Z}/4pq\mathbb{Z}$, in particular, $n_{p,q} \in (\mathbb{Z}/2pq\mathbb{Z})^\times$. Furthermore, the value $n_{p,q} = ap+bq \in \mathbb{Z}/2pq\mathbb{Z}$ is independent of a choice of the solution.*

Since $n_{p,q} \in (\mathbb{Z}/2pq\mathbb{Z})^\times$, $\{kn_{p,q}\}_{k=0,1,\dots,2pq-1}$ runs through all the elements in $\mathbb{Z}/2pq\mathbb{Z}$. Thus, we have:

Proposition 5.7. *If $R^2 \in \mathbb{R} \setminus \mathbb{Q}$, then $\ker \bar{D} \otimes \ker D$ is isomorphic to the affine Heisenberg full vertex algebras $M_{H_{II_{1,1}}, p_R}$. If $R^2 = \frac{p}{q}$, then $\ker \bar{D} \otimes \ker D$ is isomorphic to the lattice full vertex algebra $V_{\sqrt{2pq}\mathbb{Z}} \otimes \overline{V}_{\sqrt{2pq}\mathbb{Z}}$ and the irreducible decomposition of $C_{\sqrt{\frac{p}{q}}}$ is*

$$C_{\sqrt{\frac{p}{q}}} = \bigoplus_{i \in \mathbb{Z}/2pq\mathbb{Z}} V_{\sqrt{2pq}\mathbb{Z}+\frac{i}{\sqrt{2pq}}} \otimes \bar{V}_{\sqrt{2pq}\mathbb{Z}+\frac{n_{p,q} i}{\sqrt{2pq}}}.$$

We remark that the condition $n_{p,q}^2 = 1 \in \mathbb{Z}/4pq\mathbb{Z}$ corresponds the condition (FV2). Thus, for $N \in \mathbb{Z}_{>0}$ and each order 2 element in $(\mathbb{Z}/4N\mathbb{Z})^\times$, there is an extension of the lattice full vertex algebra $V_{\sqrt{2N}\mathbb{Z}} \otimes \bar{V}_{\sqrt{2N}\mathbb{Z}}$.

For example, $C_{\sqrt{6}}$ is the diagonal model $\bigoplus_{i \in \mathbb{Z}/12\mathbb{Z}} V_{\sqrt{12}\mathbb{Z}+\frac{i}{\sqrt{12}}} \otimes \bar{V}_{\sqrt{12}\mathbb{Z}+\frac{i}{\sqrt{12}}}$, whereas $C_{\sqrt{\frac{2}{3}}}$ is twisted by 7, $\bigoplus_{i \in \mathbb{Z}/12\mathbb{Z}} V_{\sqrt{12}\mathbb{Z}+\frac{i}{\sqrt{12}}} \otimes \bar{V}_{\sqrt{12}\mathbb{Z}+\frac{7i}{\sqrt{12}}}$. We also remark that C_1 is isomorphic to the SU(2) WZW-model of level 1, which is the fixed point of the duality group.

Remark 5.8. *Let $q : \mathbb{Z}/2N\mathbb{Z} \to \mathbb{R}/2\mathbb{Z}$ be a norm defined by $a \mapsto \frac{a^2}{2}$. An element $n \in$ $\mathrm{Aut}\,\mathbb{Z}/2N\mathbb{Z} = (\mathbb{Z}/2N\mathbb{Z})^\times$ preserves the norm q if and only if $n^2 = 1$ in $\mathbb{Z}/4N\mathbb{Z}$. Thus, an order 2 element in $(\mathbb{Z}/4N\mathbb{Z})^\times$ corresponds to the outer automorphism of the modular tensor category.*

6. Mass formula: application to chiral conformal field theory

A current-current deformation may produce new vertex algebras from a vertex algebra. In this section, we gives a formula to count the number of algebras constructed from a current-current deformation.

6.1. **Genus and mass of lattices.** An *integral lattice* of rank $n \in \mathbb{N}$ is a rank n free abelian group L equipped with a $\mathbb{Z}$-valued symmetric bilinear form

$$(\ , \ ) : L \times L \to \mathbb{Z}.$$

A lattice L is said to be *even* if

$$(\alpha, \alpha) \in 2\mathbb{Z} \text{ for any } \alpha \in L,$$

and *positive-definite* if

$$(\alpha, \alpha) > 0 \text{ for any } \alpha \in L \setminus \{0\}.$$

For an integral lattice L and a unital commutative ring R, we can extend the bilinear form $(\ , \ )$ bilinearly to $L \otimes_{\mathbb{Z}} R$ and L is said to be *non-degenerate* if the bilinear form on $L \otimes_{\mathbb{Z}} \mathbb{R}$ is non-degenerate. The dual of L is the set

$$L^{\vee} = \{\alpha \in L \otimes_{\mathbb{Z}} \mathbb{R} \mid (\alpha, L) \subset \mathbb{Z}\}.$$

The lattice L is said to be unimodular if $L = L^{\vee}$.

Two integral lattices L and M are said to be equivalent or *in the same genus* if

$$L \otimes_{\mathbb{Z}} \mathbb{R} \simeq M \otimes_{\mathbb{Z}} \mathbb{R}, \quad L \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq M \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

for all the prime integers p, where $\mathbb{Z}_p$ is the ring of p-adic integers. Denote by genus(L) the genus of lattices which contains L. If L is positive-definite, then a mass of its genus mass$(L) \in \mathbb{Q}$ is defined by

$$\mathrm{mass}(L) = \sum_{L' \in \mathrm{genus}(L)} \frac{1}{|\mathrm{Aut}\, L'|}, \tag{6.1}$$

where $\mathrm{Aut}\, L'$ is the automorphism group of the lattice L'.

Lattices over $\mathbb{R}$ are completely determined by the signature. Similarly, lattices over $\mathbb{Z}_p$ are determined by some invariant, called p-adic signatures (If $p = 2$, we have to consider another invariant, called an oddity). The Smith-Minkowski-Siegel's mass formula is a formula which computes mass(L) by using those invariants (see [Si, Mi, CS, Kitao]).

Consider the unique even unimodular lattice $II_{1,1}$ of signature $(1, 1)$. The proof of the following lemma can be found in [KP]:

Lemma 6.1. *The lattices L_1 and L_2 are in the same genus if and only if*

$$L_1 \otimes II_{1,1} \simeq L_2 \otimes II_{1,1}$$

as lattices.

6.2. **Genus of vertex algebra and current-current deformation.** In the previous section, we recall the notion of a genus of lattices, which is an equivalence relation of lattices and important to classify lattices. By using Lemma 6.1, we generalize it and define a genus of $\mathcal{H}$-vertex algebra.

Let us consider the lattice vertex algebra $V_{II_{1,1}}$ associated with the rank 2 lattice $II_{1,1}$ (see Section 5.3) and let (V, H) be an $\mathcal{H}$-vertex algebra. Then, by Proposition 3.12, $V \otimes C_s$ is a full $\mathcal{H}$-vertex algebra and $V \otimes V_{II_{1,1}}$ is an $\mathcal{H}$-vertex algebra.

$\mathcal{H}$-vertex algebras (V, H) and (V', H') are said to be equivalent (or in the same genus) if $(V \otimes V_{II_{1,1}}, H \oplus H_{II_{1,1}})$ and $(V' \otimes V_{II_{1,1}}, H' \oplus H_{II_{1,1}})$ are isomorphic as $\mathcal{H}$-vertex algebras, which defines an equivalent relation on $\mathcal{H}$-vertex algebras. An equivalent class is called a *genus* of $\mathcal{H}$-vertex algebras. The equivalent classes of an $\mathcal{H}$-vertex algebra (V, H) is denoted by genus(V, H) or genus(V) for short.

Theorem 6.2. *Let (V, H) and (V', H') be $\mathcal{H}$-vertex algebras. Then, the following conditions are equivalent:*

(1) $\mathcal{H}$-vertex algebras (V, H) and (V', H') are in the same genus;
(2) There exits a current-current deformation between the full $\mathcal{H}$-vertex algebras $V \otimes C_s$ and $V' \otimes C_s$;
(3) Generalized full vertex algebras $(\Omega_{V,H} \otimes \mathbb{C}[\widehat{II_{1,1}}], H \oplus H_{II_{1,1}})$ and $(\Omega_{V',H'} \otimes \mathbb{C}[\widehat{II_{1,1}}], H \oplus H_{II_{1,1}})$ are isomorphic as generalized full vertex algebras.

proof of Theorem 6.2. Since the vacuum spaces of $V \otimes C_s$ and $V \otimes V_{II_{1,1}}$ are isomorphic to $\Omega_{V,H} \otimes \mathbb{C}[\widehat{II_{1,1}}]$, (1) or (2) implies (3). Assume that (3) holds. Since all fields in $V \otimes V_{II_{1,1}}$ and $V' \otimes V_{II_{1,1}}$ are holomorphic, they are isomorphic to $F_{\Omega_{V,H} \otimes \mathbb{C}[\widehat{II_{1,1}}], H \oplus H_{II_{1,1}}, \mathrm{id}}$, where $\mathrm{id} \in P(H \oplus H_{II_{1,1}})$ is the identity map. Similarly, by Lemma 5.1, the projections which define $V \otimes C_s$ and $V' \otimes C_s$ is in the same orbit of $O(H \oplus H_{II_{1,1}}; \mathbb{R})$ since the signature of the anti-holomorphic part $\ker p$ must be $(0, 1)$. Hence, (3) implies (1) and (2). □

6.3. **From vertex algebra to lattice.** In this section, we construct an even H-lattice from an $\mathcal{H}$-vertex algebra (V, H). Let (V, H) be an $\mathcal{H}$-vertex algebra and $(\Omega_{V,H}, H)$ the generalized vertex algebra constructed in Proposition 4.17 and $A_{\Omega_{V,H}}$ the even AH pair constructed in Proposition 4.12. The $\mathcal{H}$-vertex algebra (V, H) is good if $A_{\Omega_{V,H}}$ is good. By the following lemmas, almost all natural $\mathcal{H}$-vertex algebras are good:

Lemma 6.3. *If V is a simple vertex algebra and $V_0^0 = \mathbb{C}\mathbf{1}$, then $A_{\Omega_{V,H}}$ is a good AH pair.*

Proof. (GAH1) follows from $V_0^0 = \mathbb{C}\mathbf{1}$. Let $a \in A_{\Omega_{V,H}}^{\alpha}$ and $b \in A_{\Omega_{V,H}}^{\beta}$ be non-zero vectors for some $\alpha, \beta \in H$. Then, $ab \neq 0$ if and only if $\hat{Y}(a, z)b \neq 0$. By the definition of $\hat{Y}(-, z)$, $ab \neq 0$ if and only if $Y(a, z)b \neq 0$. Thus, by Lemma 2.11, (GAH2) holds. □

Lemma 6.4 (Lemma 3.13 in [Mo1])**.** *Let (V, H) be an $\mathcal{H}$-vertex algebra. Then, $A_{\Omega_{V \otimes V_{II_{1,1}}, H \oplus H_{II_{1,1}}}}$ is isomorphic to $A_{\Omega_{V,H}} \otimes \mathbb{C}[\widehat{II_{1,1}}]$ as an even AH pair. In particular, $A_{\Omega_{V \otimes V_{II_{1,1}}, H \oplus H_{II_{1,1}}}}$ is good if and only if $A_{\Omega_{V,H}}$ is good.*

Let (V, H) be a good $\mathcal{H}$-vertex algebra. Then, by Proposition 4.15, we have the lattice pair $(A_{\Omega_{V,H}}^{\mathrm{lat}}, H)$ and the even H-lattice $L_{\Omega_{V,H}, H}$. Set $L_{V,H} = L_{\Omega_{V,H}, H}$. By Proposition 4.14, $A_{\Omega_{V,H}}^{\mathrm{lat}}$ is isomorphic to the twisted group algebra $\mathbb{C}[\widehat{L_{V,H}}]$. Since $\mathbb{C}[\widehat{L_{V,H}}]$ is a subalgebra of the even AH pair $A_{\Omega_{V,H}}$, by the equivalence of categories the lattice vertex algebra $V_{L_{V,H}}$ is a subalgebra of V as an $\mathcal{H}$-vertex algebra. This lattice subalgebra has the following universal property:

Proposition 6.5. *For any even H-lattice $M \subset H$ and an $\mathcal{H}$-vertex algebra homomorphism $\phi : V_M \to V$,*

$$\begin{array}{ccc} V_M & \xrightarrow{\ \phi\ } & V \ . \\ {\scriptstyle \exists !}\downarrow & \nearrow{\scriptstyle i} & \\ V_{L_{V,H}} & & \end{array}$$

Proof. By using adjoint functors, we have

$$\begin{aligned}
\mathrm{Hom}_{\underline{\mathcal{H}\text{-VA}}}(V_M, V) &\cong \mathrm{Hom}_{\underline{\text{G-VA}}}(\mathbb{C}[\hat{M}], \Omega_{V,H}) \\
&\cong \mathrm{Hom}_{\underline{\text{even AH pair}}}(\mathbb{C}[\hat{M}], A_{\Omega_{V,H}}) \\
&\cong \mathrm{Hom}_{\underline{\text{good AH pair}}}(\mathbb{C}[\hat{M}], A_{\Omega_{V,H}}) \\
&\cong \mathrm{Hom}_{\underline{\text{Lattice pair}}}(\mathbb{C}[\hat{M}], A^{\text{lat}}_{\Omega_{V,H}}) \\
&\cong \mathrm{Hom}_{\underline{\text{Lattice pair}}}(\mathbb{C}[\hat{M}], \mathbb{C}[\hat{L_{V,H}}]).
\end{aligned}$$

□

Let $\mathrm{Aut}\,(V,H)$ the $\mathcal{H}$-vertex algebra automorphism group of (V,H), that is,

$$\mathrm{Aut}\,(V,H) = \{f \in \mathrm{Aut}\,(V) \mid f(H) = H\}.$$

Then, similarly to Section 4.2, there is a group homomorphism $\mathrm{Aut}\,(V,H) \to O(H;\mathbb{R})$. Then, by the equivalence of categories, we have:

Lemma 6.6. *For an $\mathcal{H}$-vertex algebra (V,H), $\mathrm{Aut}\,(V,H)$ is isomorphic to the automorphism group of the generalized vertex algebra $(\Omega_{V,H}, H)$.*

By construction, the group $\mathrm{Aut}\,(V,H)$ acts on the lattice pair $A^{\text{lat}}_{\Omega_{V,H}}$. Thus, we have a group homomorphism $\mathrm{Aut}\,(V,H) \to \mathrm{Aut}\,(L_{V,H})$, where $\mathrm{Aut}\,(L_{V,H})$ is the lattice automorphism group. The image of $\mathrm{Aut}\,(V,H)$ in $\mathrm{Aut}\,(L_{V,H})$ is denoted by $G_{V,H}$. The following lemma is clear from the definition:

Lemma 6.7. *If $L_{V,H}$ is a free abelian group of rank equal to $\dim_{\mathbb{C}} H$, then $G_{V,H}$ is equal to the duality group $D_{V,H}$ in $\mathrm{O}(H;\mathbb{R})$.*

6.4. **Mass formula.** In this section, we recall the mass formula [Mo1]. A $\mathcal{H}$-vertex algebra (V,H) is called positive if $(H,(-,-))$ is positive-definite. We note that since H is positive-definite, $L_{V,H}$ is a positive-definite lattice and $\mathrm{Aut}\,(L_{V,H})$ and $G_{V,H}$ is a finite group.

Let (V,H) be a good positive-definite $\mathcal{H}$-vertex algebra.

By Lemma 6.4, all $\mathcal{H}$-vertex algebra in the genus $\mathrm{mass}(V,H)$ are good and positive-definite. The mass of the genus $\mathrm{genus}(V,H)$ is a rational number defined by

$$\mathrm{mass}(V,H) = \sum_{(W,H_W)\in \mathrm{genus}(V,H)} \frac{1}{\# G_{W,H_W}}.$$

In [Mo1], we prove the following result:

Theorem 6.8. *Let (V,H) be a simple positive-definite $\mathcal{H}$-vertex algebra with $V_0^0 = \mathbb{C}\mathbf{1}$. If the index of the groups $[\mathrm{Aut}\,(L_{V,H} \oplus II_{1,1}) : G_{V\otimes V_{II_{1,1}}, H\oplus H_{II_{1,1}}}]$ is finite, then $\frac{mass(V,H)}{mass(L_{V,H})} = [\mathrm{Aut}\,(L_{V,H} \oplus II_{1,1}) : G_{V\otimes V_{II_{1,1}}, H\oplus H_{II_{1,1}}}]$.*

Thus, all the isomorphism classes of simple positive-definite $\mathcal{H}$-vertex algebras produced by the current-current deformation can be counted by the mass formula.

6.5. **Example.** As an application, we consider a current-current deformation of a vertex operator algebra constructed in [LS]. In [LS], Lam and Shimakura constructed a vertex operator algebra of central charge 24 as an extension of the vertex operator algebra $V_{\mathrm{E}_{8,2}} \otimes V_{\mathrm{B}_{8,1}}$, where $V_{\mathrm{E}_{8,2}}$ and $V_{\mathrm{B}_{8,1}}$ are affine vertex algebras associated with simple Lie algebras E_8 and B_8 at level 2 and 1, respectively. We denote it by $V^{\text{hol}}_{E_{8,2}B_{8,1}}$. A Cartan subalgebra $H_{E_8\oplus B_8}$ of $E_8 \oplus B_8$ defines an $\mathcal{H}$-vertex algebra structure on $V^{\mathrm{hol}}_{E_{8,2}B_{8,1}}$. In [Mo1, Proposition 5.7], we determine the maximal lattice:

$$L_{V^{\mathrm{hol}}_{E_{8,2}B_{8,1}},H_{E_8\oplus B_8}} = \sqrt{2}E_8 \oplus D_8.$$

Thus, by Lemma 6.7, the duality group is equal to $G_{V^{\mathrm{hol}}_{E_{8,2}B_{8,1}},H_{E_8\oplus B_8}}$ and by [Mo1, Proposition 5.7],

$$D_{V^{\mathrm{hol}}_{E_{8,2}B_{8,1}},H_{E_8\oplus B_8}} = \mathrm{Aut}\,(\sqrt{2}E_8 \oplus D_8).$$

Set

$$II_{17,1}(2_{II}^{+10}) = II_{1,1} \oplus \sqrt{2}E_8 \oplus D_8,$$

which is an even lattice of signature $(17, 1)$. By [Mo1, Lemma 5.14 and Proposition 5.7], we have:

Proposition 6.9. *The duality group $D_{V^{\mathrm{hol}}_{E_{8,2}B_{8,1}}\otimes V_{II_{1,1}},H_{E_8\oplus B_8}\oplus H_{II_{1,1}}}$ is isomorphic to the automorphism group of the lattice $\mathrm{Aut}\,II_{17,1}(2_{II}^{+10})$ and the genus of the $\mathcal{H}$-vertex algebra $V^{\mathrm{hol}}_{E_{8,2}B_{8,1}}$ contains exactly 17 non-isomorphic $\mathcal{H}$-vertex algebras.*

Thus, we have:

Proposition 6.10. *The current-current deformation of the full $\mathcal{H}$-vertex algebra $V^{\mathrm{hol}}_{E_{8,2}B_{8,1}} \otimes C_s$ is parametrized by*

$$\mathrm{Aut}\,II_{17,1}(2_{II}^{+10})\backslash O(17,1;\mathbb{R})/O(17;\mathbb{R}) \times O(1;\mathbb{R})$$

and there are exactly 17 non-isomorphic $\mathcal{H}$-vertex algebras V such that $V \otimes C_s$ is contained in this family.

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