博士論文

論文題目　Spectral analysis on complete anti-de Sitter 3-manifolds
（完備な3次元反ド・ジッター多様体上のスペクトル解析）

氏　　名　甘中 一輝

Contents

1 Introduction

Let Γ be a discontinuous group for the three-dimensional *anti-de Sitter space* $\mathrm{AdS}^3 := \mathrm{SO}_0(2,2)/\mathrm{SO}_0(2,1)$. In this thesis, we study

- a growth rate of the counting of Γ-orbits at infinity;
- the discrete spectrum of the Laplacian of the complete anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$.

The anti-de Sitter space AdS^3 is a Lorentzian manifold with constant sectional curvature -1 of which the identity component of the isometry group is the Lie group $\mathrm{SO}_0(2,2)$. Discontinuous groups for AdS^3 and their deformation theory have been developed by renowned mathematicians, William Goldman, Toshiyuki Kobayashi, and Fanny Kassel, among others.

1.1 Relationship between the sharpness of the Γ-action and a growth rate of its counting at infinity

Traditionaly, the terminology "discontinuous groups" was used to denote the same meaning of discrete subgroups. Indeed, the action of a discrete group of isometries is automatically properly discontinuous in the Riemannian setting. Kobayashi [17] advocated to distinguish two terminologies: discontinuous groups for the property of *actions*, and discrete subgroups for the property of *groups*, in his study of the action of discrete groups beyond the Riemannian setting. Following this principle, we call a discrete subgroup Γ of a Lie group G is a *discontinuous group for* a homogeneous manifold G/H if the natural Γ-action on G/H from the left is properly discontinuous and free ([17, Def. 1.3]). Then the Γ-orbit meets a compact subset of G/H in at most finitely many points, and thus we may consider the cardinality of the intersection points. Kassel-Kobayashi [11] introduced a compact subset $B(R)$ called a pseudo-ball of radius $R > 0$ in AdS^3, more generally in any semisimple symmetric space G/H, and studied a growth rate of the *counting*

$$N_\Gamma(x, R) := \#(\Gamma x \cap B(R))$$

of the Γ-orbit through $x \in G/H$ as $R \to \infty$.

When the metric tensor is indefinite such as the anti-de Sitter space AdS^3, an isotropy subgroup of the isometry group is not necessarily compact and an orbit of a discrete subgroup Γ of isometries may have accumulation points. In particular, Γ may not act on G/H properly discontinuously. Generalizing a pioneering work of Kobayashi [14] on the properness criterion by means of the Cartan projection for homogeneous manifolds of reductive type, Kobayashi [15] and Benoist [1] established a criterion for a general discrete subgroup Γ of a reductive Lie group G to act properly discontinuously on G/H. As a slightly stronger condition than this criterion, Kassel-Kobayashi [11] introduced the notion of (c, C)-*sharpness* ($c > 0$, $C \geq 0$) of a discontinuous group which quantifies proper discontinuity. Loosely speaking, the parameter $c > 0$ indicates that the "degree of proper discontinuity" of the Γ-action is weaker if c approaches to 0. Then they gave an upper estimate of the counting for (c, C)-sharp discontinuous groups for AdS^3 (more generally, any semisimple symmetric space G/H) by means of the two constants c and C, and proved that the counting $N_\Gamma(x,R)$ is of exponential growth uniformly with respect to $x \in G/H$ as $R \to \infty$:

Fact 1.1 (Kassel-Kobayashi [11, Lem. 4.6 (4)])**.** *Let $c > 0$ and $C \geq 0$. There exists $A_0 > 0$ independent of c and C such that for any torsion-free (c,C)-sharp discontinuous group Γ for AdS^3, one has*

$$\forall x \in \mathrm{AdS}^3,\ \forall R > 0,\ N_\Gamma(x,R) \leq A_0 \exp(\frac{4(R+C)}{c}).$$

On the other hand, there has been no existing literature about the counting for a non-sharp discontinuous group (the case $c = 0$) to the best knowledge of the author. In Chapter 2, generalizing a non-sharp example of Guéritaud-Kassel [7], we construct a family of infinitely generated subgroups of $\mathrm{SO}_0(2,2)$. Our subgroup has four sequences $(a_-(k), a_+(k), r(k), R(k))_{k\in\mathbb{N}}$ as parameters. We find a properness criterion and a sharpness criterion for the actions of our subgroups on AdS^3 as conditions of these parameters (Propositions 2.22 and 2.31, respectively). Then we investigate phenomena that may happen for the counting of orbits of non-sharp discontinuous groups obtained by using these criteria. In particular, we give constructive proofs of the following:

Theorem A (Theorem 2.1)**.** *There exists a non-sharp discontinuous group for AdS^3 such that*

$$\forall x \in \mathrm{AdS}^3,\ \forall R > 0,\ N_\Gamma(x,R) \leq 4^R.$$

Theorem B (Theorem 2.2)**.** *For any monotone increasing function $f\colon \mathbb{R} \to \mathbb{R}_{>0}$ and any $x \in \mathrm{AdS}^3$, there exists a discontinuous group $\Gamma \equiv \Gamma_{f,x}$ for AdS^3 satisfying*

$$\lim_{R\to\infty} \frac{N_\Gamma(x,R)}{f(R)} = \infty.$$

For example, applying Theorem B to $f(R) = \exp(e^R)$, we can construct a discontinuous group Γ satisfying

$$\lim_{R\to\infty} \frac{\#(\Gamma x \cap B(R))}{\mathrm{vol}(B(R))} = \infty.$$

1.2 Discrete spectrum of non-sharp anti-de Sitter manifolds

In the second half of Chapter 2, we consider the discrete spectrum of the Laplacian of the noncompact anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ for a non-sharp discontinuous group Γ.

Let us recall some basic notions. A *pseudo-Riemannian manifold* is a C^∞-manifold equipped with a smooth non-degenerate symmetric bilinear tensor of signature (p,q). It is called *Riemannian* if $q = 0$ and *Lorentzian* if $q = 1$. As in the Riemannian case, $\square = \mathrm{div} \circ \mathrm{grad}$ defines a second order differential operator (the *Laplacian*) on a pseudo-Riemannian manifold. In contrast to the Riemannian setting, the Laplacian on a Lorentzian manifold is not an elliptic differential operator but a hyperbolic differential operator, and its eigenfunction is not analytic in general.

We write $L^2(M)$ for the Hilbert space of square integrable functions with respect to the Radon measure induced by the pseudo-Riemannian structure of M, and set

$$L^2_\lambda(M) := \{f \in L^2(M) \mid \Box_M f = \lambda f \text{ in the weak sense}\}$$

for $\lambda \in \mathbb{C}$. Then the set of L^2-eigenvalues

$$\mathrm{Spec}_d(\Box_M) := \{\lambda \in \mathbb{C} \mid L^2_\lambda(M) \neq 0\}$$

is called the *discrete spectrum* of the Laplacian of M.

Let us recall a known result by applying to AdS^3 the theory of Kassel-Kobayashi [11] on the discrete spectrum of "intrinsic" differential operators on locally semisimple symmetric spaces. Let Γ be a discontinuous group for AdS^3. Then the quotient space $\Gamma\backslash\mathrm{AdS}^3$ is a C^∞-manifold and the quotient map $\mathrm{AdS}^3 \to \Gamma\backslash\mathrm{AdS}^3$ is a covering map of C^∞-class. The quotient manifold $\Gamma\backslash\mathrm{AdS}^3$ admits a Lorentzian structure with constant sectional curvature -1 via this covering map. Kassel-Kobayashi [11] initiated the study of the discrete spectrum $\mathrm{Spec}_d(\Box)$ of the hyperbolic Laplacian $\Box$ on the anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$.

They introduced "the Γ-averages of non-periodic eigenfunctions" as a generalization of Poincaré series to construct L^2-eigenvalues. If an eigenfunction φ of the Laplacian on AdS^3 is integrable, then the *generalized Poincaré series*

$$\varphi^\Gamma(\Gamma x) := \sum_{\gamma \in \Gamma} \varphi(\gamma^{-1}x)$$

defines an integrable function on the anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$, and is an eigenfunction of the Laplacian with same eigenvalue. It is known that there exists an L^2-eigenfunction of the Laplacian on AdS^3 with eigenvalue

$$\lambda_m := 4m(m-1) \quad (m \in \mathbb{Z} \text{ and } m \geq 2).$$

As an application of an upper estimate of the counting as in Fact 1.1, they proved L^2-convergence and non-vanishing of generalized Poincaré series of eigenfunctions with sufficiently large eigenvalue λ_m, and obtained the following theorem:

Fact 1.2 ([11])**.** *For any sharp discontinuous group Γ for AdS^3, there exists a constant $m_0(\Gamma) > 0$ such that*

$$\mathrm{Spec}_d(\Box_{\Gamma\backslash\mathrm{AdS}^3}) \supset \{\lambda_m \mid m \in \mathbb{Z},\ m > m_0(\Gamma)\}.$$

A natural question would be whether the Laplacian on the anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ still has an L^2-eigenvalue if the discontinuous group Γ is non-sharp. As an application of an upper estimate of the counting as Theorem A, we prove the following by applying the machinery developed in [11]:

Theorem C (Theorem 2.5)**.** *There exist a non-sharp discontinuous group Γ for AdS^3 and a constant $m_0'(\Gamma) > 0$ such that*

$$\mathrm{Spec}_d(\Box_{\Gamma\backslash\mathrm{AdS}^3}) \supset \{\lambda_m \mid m \in \mathbb{Z},\ m > m_0'(\Gamma)\}.$$

1.3 Multiplicity of the discrete spectrum

In Chapter 3, we study the multiplicity of the L^2-eigenvalue λ_m of the Laplacian of an anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ constructed by generalized Poincaré series. Here, for a pseudo-Riemannian manifold M,

$$\mathcal{N}_M(\lambda) := \dim_{\mathbb{C}} L^2_\lambda(M) \in \mathbb{N} \cup \{\infty\}$$

is called the multiplicity of an L^2-eigenvalue λ. The Laplacian on a Riemannian manifold is an elliptic differential operator and the multiplicity of an L^2-eigenvalue is always finite if M is compact. However, in the Lorentzian setting, the multiplicity may be finite or may not even if M is compact.

Kassel-Kobayashi [12] proved that $\mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) = \infty$ for sufficiently large $m \in \mathbb{N}$ if a discontinuous group Γ for AdS^3 is torsion-free and standard ([11, Def. 1.4]). On the other hand, there exists a non-standard discontinuous group Γ, for example a finitely generated discontinuous group Γ which is Zariski-dense in the Lie group $\mathrm{SO}(2,2)$ ([16], [13]). However, it is not known whether the multiplicity of the Laplacian is finite in this case. In this thesis, we prove that the multiplicities of the Laplacian on the anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ are unbounded for such Γ:

Theorem D (Theorem 3.1)**.** *For any finitely generated discontinuous group* Γ *for* AdS^3*,*

$$\lim_{m\to\infty} \mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) = \infty.$$

In the proof, we find an explicit constant $m_\Gamma(k) \in \mathbb{R}$ for any $k \in \mathbb{N}$ such that $\mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) \geq k$ if $m > m_\Gamma(k)$. To be more precise, we use $\mathrm{SO}(2) \times \mathrm{SO}(2)$-finite L^2-eigenfunctions of the Laplacian on AdS^3 with eigenvalue λ_m vanishing at the origin. We note that such eigenfunctions decay more rapidly at infinity than at the origin with respect to geodesic parameters. We choose an L^2-eigenfunction with eigenvalue λ_m for each $j = 0, 1, \ldots, k-1$ which decays at the origin as rapidly as R^{3^j} when a "pseudo-distance" R from the origin tends to zero, and show the linear independence of their generalized Poincaré series when $m \geq m_\Gamma(k)$ (Proposition 3.15).

In the second half of Chapter 3, we study how the multiplicities of L^2-eigenvalues behave under a small deformation of a discrete subgroup. The study of local rigidity and stability of discontinuous groups for non-Riemannian homogeneous manifolds was initiated by Kobayashi [16] and Kobayashi-Nasrin [19]. In our AdS^3 setting, any cocompact discontinuous group is not locally rigid and its proper discontinuity is stable under any small deformation ([16], [13]). Moreover, Kassel-Kobayashi [11] constructed infinitely many *stable L^2-eigenvalues* of the Laplacian of any compact anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ under any small deformation of Γ. More specifically, for sufficiently large $m \in \mathbb{N}$, one has

$$\lambda_m \in \bigcap_{\Gamma'} \mathrm{Spec}_d(\Box_{\Gamma'\backslash\mathrm{AdS}^3}),$$

where Γ' runs over a sufficiently small neighborhood of Γ in the compact-open topology ([11, Cor. 9.10]). In this thesis, we introduce in Definition 3.3 the multiplicities of stable eigenvalues denoted by

$$\widetilde{\mathcal{N}}_{\Gamma\backslash\mathrm{AdS}^3} \colon \mathbb{C} \to \mathbb{N} \cup \{\infty\}.$$

This function has the following properties:

- $\widetilde{\mathcal{N}}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda) \neq 0$ if and only if λ is a stable L^2-eigenvalue of $\Box_{\Gamma\backslash\mathrm{AdS}^3}$;
- $\mathcal{N}_{\Gamma'\backslash\mathrm{AdS}^3}(\lambda) \geq \widetilde{\mathcal{N}}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda)$ for any Γ' sufficiently close to Γ.

Moreover, we prove the following:

Theorem E (Theorem 3.4)**.** *For any cocompact discontinuous group* Γ *for* AdS^3,

$$\lim_{m\to\infty} \widetilde{\mathcal{N}}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) = \infty.$$

The explicit constant $m_\Gamma(k)$ also plays a crucial role in the proof of Theorem E. Here, recall

$$\mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) \geq k \text{ for any integer } m > m_\Gamma(k).$$

The constant $m_\Gamma(k)$ is defined in the proof of Theorem D by using

- a growth rate of the counting $N_\Gamma(x, R)$ as $R \to \infty$;
- the "injective radius" of the anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$.

We control these two quantities simultaneously using Lipschitz constants associated to Γ introduced by Guéritaud-Kassel [7], and show that $m_\Gamma(k)$ depends "continuously" on a small deformation of Γ. We prove that the larger $m \in \mathbb{N}$ is, the more L^2-eigenfunctions of the Laplacian of the compact anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ can be constructed and that their construction is stable under any small deformation of Γ.

Acknowledgements

Before anything else, I would like to show my greatest appreciation to my supervisor Toshiyuki Kobayashi. He suggested many new and inspiring ideas to me. I could not start my work without his theory on global analysis on non-Riemannian locally symmetric spaces.

I would also like to thank all of the members of our seminar. I have greatly benefited from conversations about proper actions with Taro Yoshino, Takayuki Okuda, Yosuke Morita, Koichi Tojo, and Keiichi Maeta. In particular, Yosuke Morita taught me various topics related to geometry. I also received helpful comments on representation theory and related topics from Hideko Sekiguchi, Toshihisa Kubo, Yoshiki Oshima, Yuichiro Tanaka, Masatoshi Kitagawa, Ryosuke Nakahama, Leontiev Oleksii, Naoya Shimamoto, Taito Tauchi, Yohei Ito, Hiroyoshi Tamori, and Takashi Satomi.

I would also like to express my gratitude to Satoshi Wakatsuki and Kentaro Ito for their interest in my work.

Special thanks to all my friends who studied and discussed together. Thanks to them, I was able to have a fun and fruitful university life.

Lastly, I am deeply grateful to my family for their continuous support.

This work was supported by JSPS KAKENHI Grant Number 18J20157 and the Program for Leading Graduate Schools, MEXT, Japan.

2 Counting orbits of certain infinitely generated non-sharp discontinuous groups for the anti-de Sitter space

2.1 Introduction

2.1.1 Construction of $\Gamma_\nu(a_-, a_+, r, R)$ and the counting

In this paper, we construct a family of discrete groups Γ of isometries of the 3-dimensional anti-de Sitter space AdS^3 such that

- Γ act properly discontinuously on AdS^3;
- the counting has an arbitrary growth rate at infinity,

generalizing an example of Guéritaud-Kassel [7]. By counting, we mean the cardinality of a Γ-orbit contained in a compact set called a pseudo-ball $B(R)$ of radius $R > 0$.

In contrast to the Riemannian case, a discrete group of isometries of a pseudo-Riemannian manifold such as AdS^3 may act with non-closed orbits. We recall some basic notions and facts. A *pseudo-Riemannian manifold* is a smooth manifold X equipped with a smooth non-degenerate symmetric tensor of signature (p, q). It is *Riemannian* if $q = 0$ and *Lorentzian* if $q = 1$. A discrete group Γ of isometries of a pseudo-Riemannian manifold X is called a *discontinuous group for* X if Γ acts on X properly discontinuously and freely (we include freeness in the definition as in Kobayashi [17, Def. 1.3]). Then there are at most finite elements in any orbit of a discontinuous group Γ contained in any compact subset of X, hence we may think of its cardinality. A semisimple symmetric space $X = G/H$ is a typical example of a pseudo-Riemannian manifold, of which the isometry group is "large". Kassel and Kobayashi proved in [11] for a discontinuous group $\Gamma(\subset G)$ for an arbitrary semisimple symmetric space G/H that the counting is at most of exponential growth if Γ is *sharp* (a notion for "strong" proper discontinuity, see [11, Def. 4.2]).

The 3-dimensional anti-de Sitter space AdS^3 is the simplest example of a Lorentzian semisimple symmetric space that admits infinite discontinuous groups. Let us recall the counting result of Kassel-Kobayashi [11] in this specific setting where $X = \mathrm{AdS}^3$ and $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. They considered a compact subset $B(R)$ of X called a *pseudo-ball* of radius $R > 0$, of which the volume is of exponential growth as $R \to \infty$, see Section 2.2.1. They proved that if a discontinuous group $\Gamma \subset G$ is sharp, then the *counting*

$$N_\Gamma(x, R) := \#(\Gamma x \cap B(R)) \quad \text{for } x \in X \text{ and } R > 0$$

has an exponential growth uniformly on $x \in X$ ([11, Lem. 4.6 (4)]):

$$\exists A > 0,\ \exists a > 0,\ \forall x \in X,\ \forall R > 0,\ N_\Gamma(x, R) \leq Ae^{aR}. \tag{2.1}$$

In particular, one has

$$\exists a > 0,\ \forall x \in \mathrm{AdS}^3,\ \limsup_{R\to\infty} \frac{N_\Gamma(x, R)}{e^{aR}} < \infty. \tag{2.2}$$

Any finitely generated discontinuous group for AdS^3 is sharp by the results of Kassel [8] and Guéritaud-Kassel [7], hence its counting always satisfies the exponential growth condition (2.1).

On the other hand, the counting for a *non-sharp* discontinuous group has not been well-understood. In this paper, we investigate what can happen about the asymptotic behavior for the counting $N_\Gamma(x, R)$ when Γ is non-sharp. For this, we construct a family of subgroups $\Gamma_\nu \equiv \Gamma_\nu(a_-, a_+, r, R)$ of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ for sufficiently large $\nu \in \mathbb{N}$ associated to quadruples (a_-, a_+, r, R) of real-valued sequences in Section 2.3, and study how the properties of Γ_ν depend on the data (a_-, a_+, r, R). For instance, we find a necessary and sufficient condition for the quadruple (a_-, a_+, r, R) that Γ_ν is a discontinuous group for AdS^3 in Proposition 2.22. Moreover we determine when the Γ_ν-action on AdS^3 is sharp in Proposition 2.31. With these criteria, we present various non-sharp discontinuous groups for which different phenomena happen about the counting by choosing appropriate data (a_-, a_+, r, R):

Theorem 2.1. *There exists a non-sharp discontinuous group* Γ *for* AdS^3 *satisfying*

$$\forall x \in \mathrm{AdS}^3,\ \forall R > 0,\ N_\Gamma(x, R) \leq 4^R.$$

Theorem 2.2. *Let* $x \in \mathrm{AdS}^3$. *For any increasing function* $f \colon \mathbb{R} \to \mathbb{R}_{>0}$, *there exists a discontinuous group* $\Gamma \equiv \Gamma_{f,x}$ *for* AdS^3 *satisfying*

$$\lim_{R\to\infty} \frac{N_\Gamma(x, R)}{f(R)} = \infty.$$

Remark 2.3. *Theorem 2.2 applied to the function* $f(R) = \exp(\exp(R))$ *shows*

$$\lim_{R\to\infty} \frac{N_\Gamma(x, R)}{\mathrm{vol}(B(R + c))} = \infty$$

for any $c > 0$ *since the volume* $\mathrm{vol}(B(R))$ *is of exponential growth as* $R \to \infty$. *Thus an analogue of the Riemannian case (2.3) below does not hold.*

The above theorems deal with the setting where the metric tensor of X is indefinite and Γ is a discontinuous group for X. Let us compare them with some known results in the following different settings:

- Γ is a discontinuous group for X, but X is Riemannian (the metric tensor of X is positive definite);
- the metric tensor of X is indefinite, but Γ is not a discontinuous group for X (e.g. Γ is a lattice of the isometry group of X).

Suppose that X is a complete Riemannian manifold, and that Γ is a discrete group of isometries of X. We write $B(R)$ for the ball of radius R centered at a fixed point in X. Then,

$$\forall x \in X,\ \exists c > 0,\ \limsup_{R\to\infty} \frac{\#(\Gamma x \cap B(R))}{\mathrm{vol}(B(R + c))} < \infty. \tag{2.3}$$

The estimate (2.3) does not require that Γ is finitely generated, but the Riemannian assumption is crucial as shown in Remark 2.3.

A semisimple symmetric space $X = G/H$ admits a G-invariant pseudo-Riemannian structure. Eskin-Mcmullen [4] studied the counting of an orbit of a lattice Γ of G in $X = G/H$. Their result [4, Thm. 1.4] applied to the specific case where $X = \mathrm{AdS}^3$ and $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ tells us that if $\Gamma \cap H$ is a lattice in H, then at the base point $o = eH \in X$

$$\lim_{R\to\infty} \frac{N_\Gamma(o,R)}{\mathrm{vol}(B(R))} = \frac{\mathrm{vol}(\Gamma\backslash G)}{\mathrm{vol}((\Gamma\cap H)\backslash H)}. \tag{2.4}$$

In the right-hand side, the Haar measures of G and H (and therefore, the induced measures of $\Gamma\backslash G$ and $\Gamma \cap H\backslash H$) are normalized such that the Fubini theorem for the fibration $H \to G \to X = G/H$ is given by $dg = dxdh$, where dx is the volume element of the anti-de Sitter space $X = \mathrm{AdS}^3$ (see Section 2.2). We note that their setting is different from ours: Γ in [4] is a lattice of G, hence does not act properly discontinuously on X.

We summarize these results about the asymptotic behaviors of $N_\Gamma(x,R)$ in each setting in Table 2.1.1 below:

Table 2.1: The asymptotic behaviors of $N_\Gamma(x,R)$

	Γ	$x \in \mathrm{AdS}^3$	$N_\Gamma(x,R)$
Eskin-Mcmullen [4]	$\forall$ lattice in G	special x	$\sim Ae^R$
Kassel-Kobayashi [11]	$\forall$ sharp discont. gp.	general x	$\leq Ae^{aR}$
Theorem 2.1	$\exists$ non-sharp discont. gp.	general x	$\leq Ae^{aR}$
Theorem 2.2	$\exists$ non-sharp discont. gp.	general x	$\gg \exp(e^R)$

Remark 2.4. *Kassel-Kobayashi [11] gave a uniform estimate of $N_\Gamma(x,R)$ with respect to $x \in \mathrm{AdS}^3$. We prove such a uniform estimate for Theorem 2.1, but not for Theorem 2.2.*

2.1.2 Spectrum of the Laplacian on $\Gamma_\nu\backslash\mathrm{AdS}^3$

Let Γ be a discontinuous group for the anti-de Sitter space $X := \mathrm{AdS}^3$. Then the quotient space $X_\Gamma := \Gamma\backslash X$ is a C^∞-manifold and the quotient map $X \to X_\Gamma$ is a smooth covering. Thus X_Γ inherits an anti-de Sitter structure from X, and in particular, is a Lorentzian manifold. As in the Riemannian case, one defines the Laplacian $\Box_{X_\Gamma} := \mathrm{div} \circ \mathrm{grad}$, a second-order differential operator on X_Γ.

Kassel-Kobayashi [11] initiated the study of global analysis on the anti-de Sitter manifold X_Γ (actually in a much more general setting). They studied the *discrete spectrum*, namely the set of L^2-eigenvalues of the Laplacian $\Box_{X_\Gamma}$ on X_Γ, denoted by

$$\mathrm{Spec}_d(\Box_{X_\Gamma}) := \{\lambda \in \mathbb{C} \mid \exists f \in L^2(X_\Gamma) \setminus \{0\},\ \Box_{X_\Gamma} f = \lambda f \text{ in the weak sense}\}.$$

Here $L^2(X_\Gamma)$ is the Hilbert space of square integrable functions on X_Γ with respect to the Radon measure induced by the Lorentzian structure. We note that in contrast to the Riemannian case where the Laplacian is an elliptic differential operator, the Laplacian for the Lorentzian manifold X_Γ is a hyperbolic operator, and thus eigenfunctions may and may not be smooth functions by the failure of the elliptic regularity theorem (see [12, Sect. 3.1] for example).

Kassel-Kobayashi [11] proved the following: if Γ is sharp, then there exists $m_0 = m_0(\Gamma) > 0$ such that

$$\mathrm{Spec}_d(\Box_{X_\Gamma}) \supset \{4m(m-1) \mid m \in \mathbb{Z} \text{ and } m > m_0\}.$$

In particular, they proved that the discrete spectrum $\mathrm{Spec}_d(\Box_{X_\Gamma})$ is infinite in the setting where Γ is sharp.

A natural question would be whether the Laplacian $\Box_{X_\Gamma}$ still has an L^2-eigenvalue if the discontinuous group Γ is non-sharp. As an application of the sharpness criterion (Proposition 2.31) and an upper estimate of the counting as in Theorem 2.1, we can apply the machinery developed in [11] also to the non-sharp setting, and prove:

Theorem 2.5 (see Theorem 2.39 and Example 2.40)**.** *There exist a non-sharp discontinuous group* Γ *for* AdS^3 *and* $m_0 = m_0(\Gamma) > 0$ *such that*

$$\mathrm{Spec}_d(\Box_{X_\Gamma}) \supset \{4m(m-1) \mid m \in \mathbb{Z} \textit{ and } m > m_0\}.$$

2.1.3 Organization of the paper

In Section 2.2, we give preliminary results including a pseudo-ball $B(R)$ and the Kobayashi-Benoist properness criterion applied to our AdS^3 setting. In Section 2.3, we construct a family of infinitely generated Schottky-like discontinuous groups $\Gamma_\nu \equiv \Gamma_\nu(a_-, a_+, r, R)$ for AdS^3 associated to quadruples (a_-, a_+, r, R) of real-valued sequences satisfying some conditions for sufficiently large $\nu \in \mathbb{N}$. Moreover, we recall the notion of sharpness for discontinuous groups, and find a necessary and sufficient condition on the quadruple (a_-, a_+, r, R) such that Γ_ν is sharp. In Section 2.4, we find a lower bound for the counting $N_{\Gamma_\nu}(x, R)$, and prove Theorem 2.2. In Section 2.5, we find a sufficient condition on the quadruple (a_-, a_+, r, R) such that the counting $N_{\Gamma_\nu}(x, R)$ is at most of exponential growth, and complete the proof of Theorem 2.1 with the sharpness criterion given in Section 2.3. The proof of Theorem 2.5 is then given by applying the method established by Kassel-Kobayashi [11].

Notation. $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$.

2.2 Preliminary results about AdS^3

In this section, we collect some preliminary results about AdS^3 that will be needed for later sections.

Let V be a four-dimensional real vector space equipped with a quadratic form Q of signature $(2, 2)$ on V, and X the hypersurface given by $X = \{v \in V \mid Q(v) = 1\}$. The tangent space T_vX at $v \in X$ is identified with the orthogonal complement $(\mathbb{R}v)^\perp$ in V with respect to Q. The restriction of $-Q$ to the hyperplane $(\mathbb{R}v)^\perp$ is a quadratic form of signature $(2, 1)$, which induces a Lorentzian structure on X with constant sectional curvature -1. The resulting Lorentzian manifold is called the 3-dimensional anti-de Sitter space AdS^3.

2.2.1 Pseudo-balls in AdS^3

In this subsection, we consider pseudo-balls $B(R)$ on the Lorentzian manifold AdS^3. We work with coordinates on AdS^3 by choosing $V = \mathrm{M}(2, \mathbb{R})$ and $Q =$

det. Then AdS^3 is identified with $\mathrm{SL}(2,\mathbb{R})$. The direct product group $\mathrm{SL}(2,\mathbb{R})\times \mathrm{SL}(2,\mathbb{R})$ acts on $V = \mathrm{M}(2,\mathbb{R})$ by left and right multiplication, which induces an isometric and transitive action on AdS^3. Thus

$$\mathrm{AdS}^3 \cong (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))/\mathrm{diag}(\mathrm{SL}(2,\mathbb{R})).$$

Let o be the base point in AdS^3 corresponding to the identity matrix in $\mathrm{SL}(2,\mathbb{R})$. The *pseudo-distance* $\|g\|(\geq 0)$ of $g \in \mathrm{SL}(2,\mathbb{R})$ from the base point o is defined by the formula

$$2\cosh\|g\| = \mathrm{Tr}({}^t gg). \tag{2.5}$$

We give two equivalent definitions of the pseudo-distance $\|g\|$ as below.

First, for $\theta \in [0,2\pi]$ and $t \geq 0$, we set $k(\theta) := \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ and $a(t) := \begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix}$. Any element $g \in \mathrm{SL}(2,\mathbb{R})$ can be expressed by the Cartan decomposition $g = k(\theta_1)a(t)k(\theta_2)$ with $\theta_1,\theta_2 \in [0,2\pi]$ and unique $t \geq 0$. Then (2.5) implies

$$\|g\| = 2t. \tag{2.6}$$

This interpretation shows readily that the map $\|\cdot\|\colon \mathrm{SL}(2,\mathbb{R}) \to [0,\infty)$ is proper and that for any $R > 0$,

$$B(R) := \{g \in \mathrm{SL}(2,\mathbb{R}) \mid \|g\| \leq R\} \tag{2.7}$$

is a compact subset of $\mathrm{SL}(2,\mathbb{R})$, to which we refer as the *pseudo-ball* of radius R. The family $\{B(R)\}_{R>0}$ is well-rounded (Eskin-Mcmullen [4, Thm. 6.1]).

Second, we realize the hyperbolic space $\mathbb{H}^2$ as the upper-half plane $\{x + \sqrt{-1}y \in \mathbb{C} \mid y > 0\}$ endowed with the metric tensor $ds^2 = y^{-2}(dx^2 + dy^2)$. We write $d_{\mathbb{H}^2}$ for the hyperbolic distance of $\mathbb{H}^2$. The group $\mathrm{SL}(2,\mathbb{R})$ acts isometrically on $\mathbb{H}^2$ by linear fractional transformations. In this model, the pseudo-distance $\|g\|$ is computed by (2.6) as follows:

Lemma 2.6 (see e.g. [7, (A.1) and (A.2)])**.** *For any $g \in \mathrm{SL}(2,\mathbb{R})$,*

$$\|g\| = d_{\mathbb{H}^2}(g\sqrt{-1}, \sqrt{-1}).$$

In particular, for any point $x + \sqrt{-1}y \in \mathbb{H}^2$,

$$2\cosh d_{\mathbb{H}^2}(x + \sqrt{-1}y, \sqrt{-1}) = \frac{x^2 + y^2 + 1}{y}.$$

The following properties of the pseudo-distance follow from Lemma 2.6:

Lemma 2.7. *For $g, g' \in \mathrm{SL}(2,\mathbb{R})$,*

(1) $\|g^{-1}\| = \|g\|$.

(2) $|\|g\| - \|g'\|| \leq \|gg'\| \leq \|g\| + \|g'\|$.

The Jacobian of the Cartan decomposition $(0,2\pi)\times(0,\infty)\times(0,2\pi)\to \mathrm{SL}(2,\mathbb{R})$ defined by $(\theta_1,t,\theta_2)\mapsto k(\theta_1)a(t)k(\theta_2)$ equals $\sinh(2t)$ with respect to this Lorentzian structure on $\mathrm{SL}(2,\mathbb{R})\cong \mathrm{AdS}^3$ and the standard metrics of the intervals $(0,2\pi)$ and $(0,\infty)$. Hence the following integral formula holds:

$$\int_{\mathrm{SL}(2,\mathbb{R})} f(g)dg = \int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi} f(k(\theta_1)a(t)k(\theta_2))\sinh(2t)d\theta_1 dt d\theta_2. \tag{2.8}$$

Therefore, the volume $\mathrm{vol}(B(R))$ equals $2\pi^2(\cosh(R)-1)$ since $k(\theta_1)a(t)k(\theta_2)\in B(R)$ if and only if $2t\le R$.

2.2.2 Discontinuous groups for AdS^3

Let G be a Lie group, H a closed subgroup of G, and Γ a discrete subgroup of G, which acts naturally on $X := G/H$ from the left. In this subsection, we explain the Kobayashi-Benoist criterion for the proper discontinuity of the Γ-action on X applied to our specific setting where $G=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$, $H=\mathrm{diag}(\mathrm{SL}(2,\mathbb{R}))$, and $X=\mathrm{AdS}^3$.

Throughout this paper, we mean by a discontinuous group for X a discrete subgroup Γ of G acting properly discontinuously and freely on X (Kobayashi [17, Def. 1.3]). For torsion-free Γ, it is a discontinuous group for X if and only if Γ acts properly discontinuously on X. Proper discontinuity is a serious condition when the isotropy subgroup of G on X is noncompact. Geometrically, one should note that not every discrete subgroup of isometries can act properly discontinuously on a pseudo-Riemannian manifold X. Kobayashi [15] and Benoist [1] established a properness criterion for reductive G generalizing the original properness criterion of Kobayashi [14].

Applying the Kobayashi-Benoist properness criterion to our specific setting, we can determine whether the Γ-action on AdS^3 is properly discontinuous in terms of the pseudo-distance defined in Section 2.2.1 as follows:

Fact 2.8 (Kobayashi [15] and Benoist [1])**.** *Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. The following are equivalent:*

(i) *The action of Γ on AdS^3 is properly discontinuous.*

(ii) *For any $C>0$, the set $\{(\alpha,\beta)\in\Gamma \mid |\|\alpha\|-\|\beta\||<C\}$ is finite.*

2.3 Discontinuous groups $\Gamma_\nu(a_-,a_+,r,R)$ for AdS^3

In this section, we introduce a family of Schottky-like subgroups $\Gamma_\nu(a_-,a_+,r,R)$ of $G=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ in Definition 2.15 associated to the following data:

- $\nu\in\mathbb{N}$;
- $a_-,a_+\colon\mathbb{N}\to\mathbb{R}$ and $r,R\colon\mathbb{N}\to\mathbb{R}_{>0}$ satisfying Assumptions 2.9–2.11 below.

We find a properness criterion and a sharpness criterion for the action of $\Gamma_\nu(a_-,a_+,r,R)$ on AdS^3. In particular, we use three constants

$$\nu_{dis}\le\nu_{pro}\le\nu_{sha}$$

with $\nu_{sha}\equiv\nu_{sha}(c)$ depending on $c>0$ for sufficiency of discreteness, properness, and sharpness of $\Gamma_\nu(a_-,a_+,r,R)$ as follows:

- if $\nu \geq \nu_{dis}$, then $\Gamma_\nu(a_-, a_+, r, R)$ is an infinitely generated, free discrete subgroup of G (Proposition 2.18);
- if $\nu \geq \nu_{pro}$, then the action of $\Gamma_\nu(a_-, a_+, r, R)$ on AdS^3 is properly discontinuous (Proposition 2.21);
- if $\nu \geq \nu_{sha}(c)$, then $\Gamma_\nu(a_-, a_+, r, R)$ is $(c, 0)$-sharp for AdS^3 in the sense of Kassel-Kobayashi [11] (Proposition 2.31).

2.3.1 Construction of discrete subgroups $\Gamma_\nu(a_-, a_+, r, R)$

In this subsection, we construct $\Gamma_\nu(a_-, a_+, r, R)$ and see that it is an infinitely generated, free discrete subgroup of G for any integer $\nu \geq \nu_{dis}$.

We assume that $a_-, a_+ \colon \mathbb{N} \to \mathbb{R}$ and $r, R \colon \mathbb{N} \to \mathbb{R}_{>0}$ satisfy the following:

Assumption 2.9. *There exists $\nu_{dis} \in \mathbb{R}$ such that for any integer $k \geq \nu_{dis}$,*

$$\begin{aligned} r(k) &< R(k), \\ a_-(k) + R(k) &< a_+(k) - R(k), \\ a_+(k) + R(k) &< a_-(k+1) - R(k+1). \end{aligned}$$

Assumption 2.10.

$$\lim_{k\to\infty} a_+(k) = \lim_{k\to\infty} a_-(k) = \infty.$$

For $\nu \in \mathbb{N}$, we set

$$\eta(\nu) := \sup\left\{ \left| \frac{R(k)}{a_\delta(k) - a_\epsilon(\ell)} \right| \,\middle|\, \delta, \epsilon \in \{+, -\},\ k, \ell \geq \nu, \text{ and } k \neq \ell \text{ if } \delta = \epsilon \right\}. \tag{2.9}$$

Assumption 2.11.

$$\lim_{\nu\to\infty} \eta(\nu) = 0.$$

Such quadruples (a_-, a_+, r, R) may be obtained as follows:

Lemma 2.12. *Given a real-valued monotone increasing C^2-function $p(x)$ defined for sufficiently large x, say $x \geq \nu_0$, such that $\lim_{x\to\infty} p(x) = \infty$ and that the second derivative $p''(x)$ is nowhere vanishing, we set $a_-(k) := p(k)$ and $a_+(k) := p(k + \frac{1}{2})$ for any integer $k \geq \nu_0$. Take any sequence $R(k) > 0$ such that the positive-valued sequence*

$$p_R(k) := \frac{R(k)}{\min\{p'(k-1), p'(k+1)\}}$$

converges to 0 as k tends to infinity, and choose any sequence $r(k) > 0$ such that $r(k) < R(k)$. Take $\nu_{dis} \in \mathbb{R}$ with $\nu_{dis} \geq \nu_0$ such that $p_R(k) < \frac{1}{4}$ for any integer $k \geq \nu_{dis}$. Then the quadruple (a_-, a_+, r, R) satisfies Assumptions 2.9–2.11. Moreover,

$$\eta(\nu) \leq 2 \max_{k\geq\nu} p_R(k) \text{ for } \nu \geq \nu_{dis}. \tag{2.10}$$

Remark 2.13. *The definition of $p_R(k)$ cannot be replaced by a simpler one such as $p_R(k) := R(k)/p'(k)$, as the third condition of Assumption 2.9 does not necessarily follow.*

Proof of Lemma 2.12. Assumption 2.10 is obviously satisfied. We note that the derivative $p'(x)$ on any bounded interval attains its minimum at one of the ends of the interval because the second derivative $p''(x)$ is nowhere vanishing. By the mean value theorem, we have

$$p(y) - p(x) \geq \frac{(y-x)R(k)}{p_R(k)} \qquad \text{for } k-1 \leq x < y \leq k+1, \tag{2.11}$$
$$p(t) - p(s) \geq \frac{(t-s)R(k+1)}{p_R(k+1)} \qquad \text{for } k \leq s < t \leq k+1.$$

Hence, for any integer $k \geq \nu_{dis}$, we have

$$a_+(k) - a_-(k) = p(k+\frac{1}{2}) - p(k) \geq \frac{R(k)}{2p_R(k)} > 2R(k),$$

$$a_-(k+1) - a_+(k) = p(k+1) - p(k+\frac{1}{2}) \geq \frac{R(k)}{4p_R(k)} + \frac{R(k+1)}{4p_R(k+1)} > R(k) + R(k+1).$$

Thus Assumption 2.9 is verified.

Take any $k, \ell \geq \nu_{dis}$ and $\delta, \epsilon \in \{+, -\}$. We assume $k \neq \ell$ if $\delta = \epsilon$. Since the function $p(x)$ is monotone increasing,

$$|a_\delta(k) - a_\epsilon(\ell)| \geq \min\left\{p(k) - p(k-\frac{1}{2}), p(k+\frac{1}{2}) - p(k), p(k+1) - p(k+\frac{1}{2})\right\}$$
$$\geq \frac{R(k)}{2p_R(k)},$$

where the second inequality follows from (2.11). Hence we get (2.10) and thus Assumption 2.11 is verified since $\lim_{k\to\infty} p_R(k) = 0$. $\square$

Example 2.14. *The quadruples (a_-, a_+, r, R) in (1), (2), and (3) of Table 2.2 are obtained by applying Lemma 2.12 to $p(x) = \exp(e^x)$, x^2, and $\log x$, respectively. For the reader's convenience, we list in Table 2.2 also the asymptotic behaviors of the counting $N_\Gamma(x, R)$ as R tends to infinity where Γ are the discontinuous groups $\Gamma_\nu(a_-, a_+, r, R)$ associated to the quadruples (a_-, a_+, r, R) in (1) and (3). We refer to Example 2.40 (2) and Example 2.38 below for details about the counting.*

Table 2.2: Examples of (a_-, a_+, r, R) satisfying Assumptions 2.9-2.11

	$a_-(k)$	$a_+(k)$	$r(k)$	$R(k)$	$N_\Gamma(x,R)$
(1)	$\exp(e^k)$	$\exp(e^{k+\frac{1}{2}})$	1	e^k	$\leq 4^R$
(2)	k^2	$k^2+k+\frac{1}{4}$	1	$\log k$	
(3)	$\log k$	$\log(k+\frac{1}{2})$	$(k^2 \log k)^{-1}$	k^{-2}	$\geq \exp(e^{\frac{R}{4}}) - \nu$

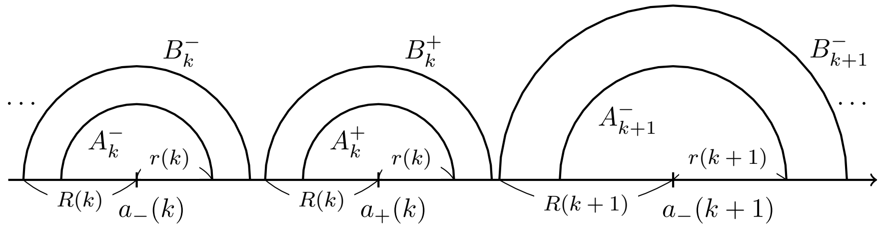

Figure 2.1: $A_k^\pm$ and $B_k^\pm$ in $\mathbb{H}^2$

We introduce a coordinate map $\tau\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}_{>0}\to \mathrm{SL}(2,\mathbb{R})$ by

$$\tau = \tau(x_-, x_+, u) := \frac{1}{u}\begin{pmatrix} x_+ & -(x_-x_+ + u^2) \\ 1 & -x_- \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{2.12}$$

Then the quadruple (a_-, a_+, r, R) of functions defines a sequence of elements $(\alpha_k, \beta_k) \in G$ by

$$\alpha_k := \tau(a_-(k), a_+(k), r(k)),\ \beta_k := \tau(a_-(k), a_+(k), R(k)) \in \mathrm{SL}(2,\mathbb{R}). \tag{2.13}$$

Definition 2.15. *Let (a_-, a_+, r, R) be a quadruple of sequences satisfying Assumptions 2.9–2.11. For $\nu \in \mathbb{N}$, we define $\Gamma_\nu(a_-, a_+, r, R)$ as the subgroup of G generated by $\{(\alpha_k, \beta_k) \mid k = \nu, \nu+1, \ldots\}$.*

Notation 2.16. *Let F^∞ denote the free group generated by countably many elements $\{\gamma_k\}_{k\in\mathbb{N}}$. Let $(\alpha_k, \beta_k) \in G$ be a sequence of elements associated to a quadruple (a_-, a_+, r, R) satisfying Assumptions 2.9–2.11 by (2.13). Then there exist group homomorphisms $j\colon F^\infty \to \mathrm{SL}(2,\mathbb{R})$ and $\rho\colon F^\infty \to \mathrm{SL}(2,\mathbb{R})$ such that $j(\gamma_k) = \alpha_k$ and $\rho(\gamma_k) = \beta_k$ for all $k \in \mathbb{N}$. For $\nu \in \mathbb{N}$, let F_ν^∞ be the subgroup of generated by $\{\gamma_k\}_{k=\nu}^\infty$.*

Then, by Definition 2.15,

$$\Gamma_\nu(a_-, a_+, r, R) = \{(j(\gamma), \rho(\gamma)) \mid \gamma \in F_\nu^\infty\}.$$

Example 2.17. *The subgroup $\Gamma_\nu(a_-, a_+, r, R)$ for $(a_-(k), a_+(k), r(k), R(k)) = (k^2, k^2+k, 1, \log k)$ coincides with $\Gamma_\nu^{j,\rho}$ in Guéritaud-Kassel [7, Sect. 10.1].*

Proposition 2.18. *Let (a_-, a_+, r, R) be a quadruple of sequences satisfying Assumptions 2.9–2.11, and $\nu_{dis} \in \mathbb{R}$ as in Assumption 2.9. Then the subgroup $\Gamma_\nu(a_-, a_+, r, R)$ of G is discrete and free for any integer $\nu \geq \nu_{dis}$.*

The proof of Proposition 2.18 is based on a ping-pong lemma. For this, we need some setups. Let $|\cdot|$ denote the Euclidean norm in the upper-half plane $\mathbb{H}^2$. Associated to the quadruple (a_-, a_+, r, R), we set

$$A_k^\epsilon := \{z \in \mathbb{H}^2 \mid |z - a_\epsilon(k)| \leq r(k)\},\ B_k^\epsilon := \{z \in \mathbb{H}^2 \mid |z - a_\epsilon(k)| \leq R(k)\}. \tag{2.14}$$

for $k \in \mathbb{N}$ and $\epsilon \in \{+,-\}$, see Figure 2.1. Take any integer $\nu \geq \nu_{dis}$. Then we claim:

- $A_\nu^-, A_\nu^+, A_{\nu+1}^-, A_{\nu+1}^+, \ldots$ are disjoint;

- $\alpha_k(\mathbb{H}^2 \setminus A_k^-) \subset A_k^+$ for $k \geq \nu$;
- $\bigcup_{k\geq\nu}(A_k^- \cup A_k^+)$ is a proper closed subset of $\mathbb{H}^2$.

The first claim is immediate from Assumption 2.9 and thus the third claim is obvious from Assumption 2.10. To see the second claim, we use the following key property of the map $\tau = \tau(x_-, x_+, u)$ in (2.12):

$$|z - x_-| > u \text{ if and only if } |\tau(z) - x_+| < u \text{ for } z \in \mathbb{H}^2, \tag{2.15}$$

which is readily seen from the identity

$$\tau(z) - x_+ = -u^2(z - x_-)^{-1}. \tag{2.16}$$

Proposition 2.18 holds because the subgroup of $\mathrm{SL}(2,\mathbb{R})$ generated by $\{\alpha_k \mid k = \nu, \nu+1, \ldots\}$ is free and discrete by a standard ping-pong argument, namely, by applying Lemma 2.19 below to $H = \mathrm{SL}(2,\mathbb{R})$, $Y = \mathbb{H}^2$, and $Y_k^{\pm} = A_{k+\nu}^{\pm}$.

Lemma 2.19 (Ping-pong lemma). *Let H be a topological group acting continuously on a topological space Y, and Γ the subgroup generated by $h_0, h_1, \ldots \in H$. Suppose that there exist disjoint closed subsets $Y_0^-, Y_0^+, Y_1^-, Y_1^+, \ldots$ of Y satisfying the following:*

(i) *$h_k(Y \setminus Y_k^-) \subset Y_k^+$ for any $k \in \mathbb{N}$.*

(ii) *$\bigcup_{k\in\mathbb{N}}(Y_k^- \cup Y_k^+)$ is a proper closed subset of Y.*

Then, Γ is a free discrete subgroup of H.

Although the proof of Lemma 2.19 is standard, we give a proof for the sake of completeness.

Proof. The conditions (i) and (ii) may be restated as $h_k^{-1}(Y \setminus Y_k^+) \subset Y_k^-$ for any $k \in \mathbb{N}$ and $U := Y \setminus \bigcup_{i\in\mathbb{N}}(Y_i^- \cup Y_i^+)$ is a non-empty open subset of Y, respectively. Take any $h = h_{i_0}^{s_0} \cdots h_{i_n}^{s_n} \in \Gamma$. Suppose that this is a reduced expression, namely, $s_0, \ldots, s_n \in \{1, -1\}$ and $s_k = s_{k+1}$ whenever $i_k = i_{k+1}$ for $0 \leq k < n$. Then we have $h(U) \subset Y_{i_0}^{s_0}$ and thus $h(U) \cap U = \emptyset$. Hence $h \neq e$ and $\{h_0, h_1, \ldots\}$ is a free generator of Γ. Take a neighborhood V of e in H and a non-empty open subset W of U such that $V \cdot W \subset U$. Then $\Gamma \cap V = \{e\}$ and thus Γ is discrete in H. $\square$

2.3.2 Proper discontinuity of the action of $\Gamma_\nu(a_-, a_+, r, R)$

In this subsection, we find a necessary and sufficient condition for $\Gamma_\nu(a_-, a_+, r, R)$ to act properly discontinuously on AdS^3.

We introduce a constant $\varepsilon(\nu)$ for this. Let (a_-, a_+, r, R) be a quadruple satisfying Assumptions 2.9–2.11, and $\eta(\nu)$ as in (2.9). Let $(\alpha_k, \beta_k) \in G$ be the sequence of the elements associated to the quadruple (a_-, a_+, r, R) by (2.13). For $\nu \in \mathbb{N}$, we set

$$\varepsilon(\nu) := \max_{k\geq\nu} \left\{ \frac{24R(k)}{a_-(k)},\ \frac{6(R(k)^2+1)}{(a_-(k) - R(k))^2},\ 8\eta(k) \right\}. \tag{2.17}$$

The sequence $\varepsilon(\nu)$ is monotone decreasing. We claim $\lim_{\nu\to\infty}\varepsilon(\nu)=0$. To see this, we note

$$\lim_{k\to\infty}\frac{R(k)}{a_+(k)}=0,\ \lim_{k\to\infty}\frac{R(k)}{a_-(k)}=0. \tag{2.18}$$

In fact, for sufficiently large $k\in\mathbb{N}$, the sequences $a_-(k), a_+(k)$ are monotone increasing by Assumption 2.9, and $a_-(k), a_+(k)>0$ by Assumption 2.10. Hence we get (2.18) by Assumption 2.11. Therefore, $\lim_{\nu\to\infty}\varepsilon(\nu)=0$ follows again from Assumptions 2.9–2.11.

It is convenient to note the following:

Lemma 2.20. *Let (a_-, a_+, r, R) be a quadruple of sequences satisfying Assumptions 2.9–2.11, $\nu_{dis}\in\mathbb{R}$ as in Assumption 2.9, $\eta(\nu)$ as in (2.9), and $B_k^{\pm}$ as in (2.14). Take any integer $\nu\geq\nu_{dis}$ satisfying $\varepsilon(\nu)<1$. Then,*

$$\begin{aligned}\varepsilon(\nu)\geq \max_{\substack{k\geq\nu,\\ \delta\in\{1,-1\}}}\Bigg\{&12\left|\log\left(1+\frac{\delta R(k)}{a_-(k)}\right)\right|,\ 6\log\left(1+e^{-2d_{\mathbb{H}^2}(B_k^\delta,\sqrt{-1})}\right),\\ &6\log\left(1+\frac{R(k)^2+1}{(a_-(k)-R(k))^2}\right),\ 4\left|\log(1+\delta\eta(k))\right|\Bigg\}.\end{aligned}$$

Here we have used the convention that $B_k^{\pm1}=B_k^{\pm}$.

Proof. For $\delta\in\{1,-1\}$, we claim

$$e^{-2d_{\mathbb{H}^2}(B_k^\delta,\sqrt{-1})}<\frac{R(k)}{a_-(k)}. \tag{2.19}$$

Then the assertion follows from the inequalities $t\geq\log(1+t)$ for $t\geq0$ and $2s\geq|\log(1-s)|$ for $0\leq s\leq\frac{1}{2}$ since $\varepsilon(\nu)<1$.

We now prove (2.19). Note $3R(k)(<24R(k))<a_-(k)$ and $a_-(k)-R(k)(>\sqrt{6})>2$ for any integer $k\geq\nu$ since $\varepsilon(\nu)<1$. Thus

$$(a_-(k)-R(k))^2>2(a_-(k)-R(k))>a_-(k)+R(k). \tag{2.20}$$

We write $x_\delta(k)+\sqrt{-1}y_\delta(k)\in\mathbb{H}^2$ for the closest point of B_k^δ to $\sqrt{-1}$ with respect to the hyperbolic distance. Obviously, we have

$$x_\delta(k)\geq a_\delta(k)-R(k),\ y_\delta(k)\leq R(k).$$

Thus, by Lemma 2.6,

$$2\cosh d_{\mathbb{H}^2}(B_k^\delta,\sqrt{-1})>\frac{x_\delta(k)^2}{y_\delta(k)}\geq\frac{(a_\delta(k)-R(k))^2}{R(k)}\geq\frac{(a_-(k)-R(k))^2}{R(k)},$$

where the last inequality follows from Assumption 2.9. Hence, since $e^{-d_{\mathbb{H}^2}(B_k^\delta,\sqrt{-1})}\leq 1$, we have

$$e^{d_{\mathbb{H}^2}(B_k^\delta,\sqrt{-1})}>\frac{(a_-(k)-R(k))^2}{R(k)}-1>\frac{a_-(k)}{R(k)}$$

where the second inequality follows from (2.20). Therefore,

$$e^{-2d_{\mathbb{H}^2}(B_k^\delta,\sqrt{-1})}\leq e^{-d_{\mathbb{H}^2}(B_k^\delta,\sqrt{-1})}<\frac{R(k)}{a_-(k)}.$$

This proves (2.19) and thus the lemma holds. □

The action of the Schottky-like discrete group $\Gamma_\nu(a_-,a_+,r,R)$ on AdS^3 is not always properly discontinuous. We give a necessary and sufficient condition for this action to be properly discontinuous:

Proposition 2.21. *Let (a_-,a_+,r,R) be a quadruple of sequences satisfying Assumptions 2.9–2.11, and $\nu_{dis}\in\mathbb{R}$ as in Assumption 2.9. The action of $\Gamma_\nu(a_-,a_+,r,R)$ on AdS^3 is properly discontinuous for sufficiently large $\nu\in\mathbb{N}$ if and only if*

$$\lim_{k\to\infty}\frac{R(k)}{r(k)}=\infty. \tag{2.21}$$

In this case, take $\nu_{pro}\in\mathbb{R}$ with $\nu_{pro}\geq\nu_{dis}$ such that $\varepsilon(k)<1$ and $\log(R(k)/r(k))>1$ for any integer $k\geq\nu_{pro}$. Then the action of $\Gamma_\nu(a_-,a_+,r,R)$ is properly discontinuous for any $\nu\geq\nu_{pro}$.

Postponing the proof of Proposition 2.21, we state its immediate consequences in Proposition 2.22 and Lemma 2.24 as below. First, since the group $\Gamma_\nu(a_-,a_+,r,R)$ is torsion-free, any properly discontinuous action is free, hence we obtain:

Proposition 2.22. *In the setting of Proposition 2.21, assume the condition (2.21). Then $\Gamma_\nu(a_-,a_+,r,R)$ is a discontinuous group for AdS^3 if $\nu\geq\nu_{pro}$.*

Example 2.23. *All (a_-,a_+,r,R) in Table 2.2 apply to Proposition 2.22.*

Second, we formulate the following lemma which is actually a special case of Proposition 2.22. We will use Lemma 2.24 in Section 2.4 for a lower bound of the counting $N_\Gamma(o,R)$ and in Section 2.5 for the proof of Theorem 2.1:

Lemma 2.24. *Let $p(t)$ be a monotone increasing C^2-function defined for sufficiently large t, say $t\geq\nu_0$, such that the second derivative $p''(t)$ is nowhere vanishing and $\lim_{t\to\infty}p(t)=\infty$, and $q(t)$ a continuous function satisfying $\lim_{t\to\infty}q(t)=\infty$. We set $\delta:=\mathrm{sgn}(p''(t))\in\{\pm1\}$, which is independent of $t\geq\nu_0$. Let $\Gamma_\nu(p,q)$ denote the group $\Gamma_\nu(a_-,a_+.r,R)$ associated to the sequences*

$$a_-(k):=p(k),a_+(k):=p(k+\frac{1}{2}),r(k):=\frac{p'(k-\delta)}{p(k)q(k)},R(k):=\frac{p'(k-\delta)}{q(k)} \tag{2.22}$$

for $k\geq\nu_0$. Then $\Gamma_\nu(p,q)$ is a discontinuous group for AdS^3 if $\nu\gg0$.

Proof of Lemma 2.24. The quadruple (a_-,a_+,r,R) applies to Lemma 2.12 and satisfies the condition (2.21). Hence the assertion follows from Proposition 2.22. □

Remark 2.25. *We shall study some basic properties of the action of the group $\Gamma_\nu(p,q)$ on AdS^3 in later sections: for instance, see Lemma 2.34 for a necessary and sufficient condition on the pair (p,q) such that the discontinuous group $\Gamma_\nu(p,q)$ is sharp in the sense of Kassel-Kobayashi [11], and see Proposition 2.37 for the counting of the $\Gamma_\nu(p,q)$-orbit through the base point o. We then provide an example of a non-sharp discontinuous group for which the counting is at most of exponential growth by taking $(p(t),q(t))=(e^t,\exp(e^t))$ in Example 2.40 (1).*

The following lemma for Proposition 2.21 will be used also to prove the counting result (Theorem 2.39) in Section 2.5.

Lemma 2.26. *Given a quadruple (a_-, a_+, r, R) satisfying Assumptions 2.9–2.11, let j, ρ, and F_ν^∞ be as in Notation 2.16. Suppose $\nu \geq \nu_{dis}$ and $\varepsilon(\nu) < 1$. Let $\gamma \neq e$ be an arbitrary element of F_ν^∞ and $m := \ell(\gamma)$ the word length of γ. We write $\gamma = \gamma_{k_1}^{s_1} \dots \gamma_{k_m}^{s_m}$ for the reduced expression where $s_1, \dots, s_m \in \{1, -1\}$ and $k_1, \dots, k_m \geq \nu$. Then,*

$$\left| \|j(\gamma)\| - \|\rho(\gamma)\| - 2\sum_{i=1}^{\ell(\gamma)} \log \frac{R(k_i)}{r(k_i)} \right| \leq \ell(\gamma)\varepsilon(\nu). \tag{2.23}$$

We postpone the proof of Lemma 2.26 until the end of this subsection, and first prove Proposition 2.21.

Proof of Proposition 2.21. Recall $\Gamma_\nu(a_-, a_+, r, R) = \{(j(\gamma), \rho(\gamma)) \mid \gamma \in F_\nu^\infty\}$. The Kobayashi-Benoist properness criterion (Fact 2.8) tells us that $\Gamma_\nu(a_-, a_+, r, R)$ acts properly discontinuously on AdS^3 if and only if

$$\forall C > 0,\ \#\{\gamma \in F_\nu^\infty \mid |\|j(\gamma)\| - \|\rho(\gamma)\|| < C\} < \infty.$$

We suppose that the action of $\Gamma_\nu(a_-, a_+, r, R)$ on AdS^3 is properly discontinuous for any sufficiently large $\nu \in \mathbb{N}$. Then $\lim_{k\to\infty} |\|j(\gamma_k)\| - \|\rho(\gamma_k)\|| = \infty$ by Fact 2.8. By Lemma 2.26,

$$2\log \frac{R(k)}{r(k)} \geq |\|j(\gamma_k)\| - \|\rho(\gamma_k)\|| - \varepsilon(\nu)$$

for any $k \geq \nu$ and thus $\lim_{k\to\infty} R(k)/r(k) = \infty$.

Conversely, we suppose $\lim_{k\to\infty} R(k)/r(k) = \infty$. Take any integer $\nu \geq \nu_{pro}$. Then $\log(R(k)/r(k)) > 1$ for any $k \geq \nu$ and $\varepsilon(\nu) < 1$. Hence note $\|j(\gamma)\| - \|\rho(\gamma)\| \geq 0$ by Lemma 2.26 for any $\gamma \in F_\nu^\infty$. Assume that $\gamma \in F_\nu^\infty \setminus \{e\}$ satisfies $|\|j(\gamma)\| - \|\rho(\gamma)\|| < C$. Let $m := \ell(\gamma)$ be the word length of γ and we write $\gamma = \gamma_{k_1}^{s_1} \cdots \gamma_{k_m}^{s_m}$ for the reduced expression where $s_1, \dots, s_m \in \{1, -1\}$ and $k_1, \dots, k_m \geq \nu$. By Lemma 2.26,

$$\ell(\gamma) > \ell(\gamma)\varepsilon(\nu) \geq 2\sum_{i=1}^{\ell(\gamma)} \log \frac{R(k_i)}{r(k_i)} - (\|j(\gamma)\| - \|\rho(\gamma)\|) > 2\ell(\gamma) - C.$$

Hence $\ell(\gamma) < C$. Again by Lemma 2.26, we get

$$\sum_{i=1}^{\ell(\gamma)} \log \frac{R(k_i)}{r(k_i)} < \frac{1}{2}((\|j(\gamma)\| - \|\rho(\gamma)\|) + \ell(\gamma)\varepsilon(\nu)) < C$$

and there are only finitely many γ satisfying this inequality. By Fact 2.8, the action of $\Gamma_\nu(a_-, a_+, r, R)$ on AdS^3 is properly discontinuous. $\square$

Guéritaud-Kassel [7, Sect. 10.1] gave an upper bound of

$$\left| \|j(\gamma)\| - \|\rho(\gamma)\| - 2\sum_{i=1}^{\ell(\gamma)} \log \frac{R(k_i)}{r(k_i)} \right|$$

for the quadruple $(a_-(k), a_+(k), r(k), R(k)) = (k^2, k^2 + k, 1, \log k)$, see Table 2.2 (2). However, since the explanation given there was not clear to the author, we take an alternative approach to prove the inequality (2.23) in our general setting where a quadruple (a_-, a_+, r, R) is arbitrary subject to Assumptions 2.9–2.11.

The rest of this subsection is devoted to the proof of Lemma 2.26. We need:

Lemma 2.27. *In the setting of Lemma 2.26, the following hold for both $\varphi = j$ and $\varphi = \rho$:*

(1) $\varphi(\gamma)\sqrt{-1} \in B_{k_1}^{s_1}$.

(2) $\|\varphi(\gamma)\| \geq d_{\mathbb{H}^2}(B_{k_1}^{s_1}, \sqrt{-1})$.

(3) $\mathrm{Re}(\varphi(\gamma)\sqrt{-1}) \geq a_-(k_1) - R(k_1)$.

(4) $\mathrm{Im}(\varphi(\gamma)\sqrt{-1}) \leq R(k_1)$.

(5) $\left|2\log\dfrac{\mathrm{Re}(j(\gamma)\sqrt{-1})}{\mathrm{Re}(\rho(\gamma)\sqrt{-1})}\right| \leq \dfrac{1}{3}\varepsilon(\nu)$.

(6) $\left|-\log\dfrac{\mathrm{Im}(j(\gamma)\sqrt{-1})}{\mathrm{Im}(\rho(\gamma)\sqrt{-1})} - 2\displaystyle\sum_{i=1}^{m}\log\dfrac{R(k_i)}{r(k_i)}\right| \leq (\ell(\gamma)-1)\varepsilon(\nu)$.

Here $\ell(\gamma)$ is the word length of γ, and we have used the convention that $B_k^{\pm 1} = B_k^{\pm}$ and $a_{\pm 1} = a_{\pm}$.

Proof. The assertion (1) follows from Assumption 2.9 by a standard ping-pong argument (see the proofs of Proposition 2.18 and Lemma 2.19), hence we get (3) and (4). By Lemma 2.6, (2) follows from (1).

By (1), we get $\left|\mathrm{Re}(\varphi(\gamma)\sqrt{-1}) - a_{s_1}(k_1)\right| \leq R(k_1)$. Thus noting $a_{s_1}(k_1) > R(k_1)$ by $\varepsilon(\nu) < 1$, we have

$$\left|\log\frac{\mathrm{Re}(\varphi(\gamma)\sqrt{-1})}{a_{s_1}(k_1)}\right| \leq \max_{\delta=\pm1}\left\{\left|\log\left(1+\frac{\delta R(k_1)}{a_{s_1}(k_1)}\right)\right|\right\} \leq \frac{1}{12}\varepsilon(\nu) \tag{2.24}$$

by Lemma 2.20. Hence (5) follows.

In the following, we set for $k \in \mathbb{N}$

$$r_\varphi(k) := \begin{cases} r(k) & \text{for } \varphi = j, \\ R(k) & \text{for } \varphi = \rho. \end{cases}$$

Then, by (2.16),

$$\mathrm{Im}(\varphi(\gamma_k^s)z) = \frac{r_\varphi(k)^2\mathrm{Im}z}{|z - a_{-s}(k)|^2} \text{ for } z \in \mathbb{H}^2,\ k \in \mathbb{N}, \text{ and } s = \pm1. \tag{2.25}$$

Let us prove (6). Define $\sigma_0, \ldots, \sigma_m \in F_\nu^\infty$ by $\sigma_i = \gamma_{k_{i+1}}^{s_{i+1}} \ldots \gamma_{k_m}^{s_m}$ for $0 \leq i < m$ and $\sigma_m = 1$. We note $\sigma_0 = \gamma$. For $0 \leq i \leq m$, we set

$$Q(\sigma_i) := \frac{\mathrm{Im}(j(\sigma_i)\sqrt{-1})}{\mathrm{Im}(\rho(\sigma_i)\sqrt{-1})},\ D(\varphi(\sigma_i)) := \left|\varphi(\sigma_i)\sqrt{-1} - a_{-s_i}(k_i)\right|.$$

We claim:

$$\left|\log\frac{Q(\sigma_i)}{Q(\sigma_{i+1})} + 2\log\frac{R(k_{i+1})}{r(k_{i+1})}\right| = \left|2\log\frac{D(j(\sigma_{i+1}))}{D(\rho(\sigma_{i+1}))}\right| \text{ for } 0 \leq i < m, \tag{Q_i}$$

$$\left|2\log\frac{D(j(\sigma_i))}{D(\rho(\sigma_i))}\right| \le \begin{cases}\varepsilon(\nu) & \text{if } 0<i<m,\\ 0 & \text{if } i=m.\end{cases} \tag{D$_i$}$$

Then, we get (6) by summing up (Q$_i$) and (D$_i$) for all i because $\log Q(\sigma_m)=0$.

It remains to verify (Q$_i$) and (D$_i$). Because $\sigma_i=\gamma_{k_{i+1}}^{s_{i+1}}\sigma_{i+1}$, we have

$$\operatorname{Im}(\varphi(\sigma_i)\sqrt{-1})=\frac{r_\varphi(k_{i+1})^2\operatorname{Im}(\varphi(\sigma_{i+1})\sqrt{-1})}{D(\varphi(\sigma_{i+1}))^2} \tag{I$_i$}$$

for $0\le i<m$ by (2.25). Thus (Q$_i$) follows from the formula

$$\frac{Q(\sigma_i)}{Q(\sigma_{i+1})}=\left(\frac{D(\rho(\sigma_{i+1}))}{D(j(\sigma_{i+1}))}\right)^2\left(\frac{r(k_{i+1})}{R(k_{i+1})}\right)^2.$$

We observe that (D$_i$) for $i=m$ is obvious because $D(\varphi(\sigma_m))=|\sqrt{-1}-a_{-s_m}(k_m)|$ is independent of φ. For $0<i<m$, by the triangle inequality, we have

$$\left|D(\varphi(\sigma_i))-\left|a_{s_{i+1}}(k_{i+1})-a_{-s_i}(k_i)\right|\right|\le\left|\varphi(\sigma_i)\sqrt{-1}-a_{s_{i+1}}(k_{i+1})\right|\le R(k_{i+1})$$

since $\varphi(\sigma_i)\sqrt{-1}\in B_{k_{i+1}}^{s_{i+1}}$ by (1). Hence noting

$$|a_{s_{i+1}}(k_{i+1})-a_{-s_i}(k_i)|\ge\eta(\nu)^{-1}R(k_{i+1})>R(k_{i+1})$$

by $\varepsilon(\nu)<1$, we obtain

$$\begin{aligned}\left|\log\frac{D(\varphi(\sigma_i))}{\left|a_{s_{i+1}}(k_{i+1})-a_{-s_i}(k_i)\right|}\right| &\le \max_{\delta=\pm1}\left\{\left|\log\left(1+\frac{\delta R(k_{i+1})}{\left|a_{s_{i+1}}(k_{i+1})-a_{-s_i}(k_i)\right|}\right)\right|\right\}\\ &\le\max_{\delta=\pm1}\{|\log(1+\delta\eta(\nu))|\}\le\frac{1}{4}\varepsilon(\nu).\end{aligned} \tag{D$'_i$}$$

The second and third inequalities follow from the definition (2.9) of $\eta(\nu)$ and Lemma 2.20, respectively. Hence $\left|\log\frac{D(j(\sigma_i))}{D(\rho(\sigma_i))}\right|\le\frac{\varepsilon(\nu)}{2}$. Thus (D$_i$) holds for all i and the proof of (6) is completed. □

We are ready to prove Lemma 2.26.

Proof of Lemma 2.26. We have

$$\begin{aligned}|\|\varphi(\gamma)\|-\log(2\cosh\|\varphi(\gamma)\|)| &= \log\left(1+e^{-2\|\varphi(\gamma)\|}\right)\\ &\le\log\left(1+e^{-2d_{\mathbb{H}^2}(B_{k_1}^{s_1},\sqrt{-1})}\right)\le\frac{1}{6}\varepsilon(\nu),\end{aligned} \tag{2.26}$$

where the second and third inequalities follow from Lemmas 2.27 (2) and 2.20, respectively. Hence we get

$$\left|\|j(\gamma)\|-\|\rho(\gamma)\|-\log\frac{\cosh\|j(\gamma)\|}{\cosh\|\rho(\gamma)\|}\right|\le\frac{1}{3}\varepsilon(\nu). \tag{2.27}$$

By Lemma 2.6,

$$2\cosh\|\varphi(\gamma)\|=\frac{\operatorname{Re}(\varphi(\gamma)\sqrt{-1})^2+\operatorname{Im}(\varphi(\gamma)\sqrt{-1})^2+1}{\operatorname{Im}(\varphi(\gamma)\sqrt{-1})}.$$

Therefore, we have

$$\begin{aligned}&\left|\log(2\cosh\|\varphi(\gamma)\|) - 2\log\operatorname{Re}(\varphi(\gamma)\sqrt{-1}) + \log\operatorname{Im}(\varphi(\gamma)\sqrt{-1})\right|\\ &= \log\left(1+\frac{\operatorname{Im}(\varphi(\gamma)\sqrt{-1})^2+1}{\operatorname{Re}(\varphi(\gamma)\sqrt{-1})^2}\right) \le \log\left(1+\frac{R(k)^2+1}{(a_-(k)-R(k))^2}\right) \le \frac{1}{6}\varepsilon(\nu).\end{aligned} \tag{2.28}$$

The second and third inequalities follow from Lemma 2.27 (3), (4) and Lemma 2.20, respectively. Hence

$$\left|\log\frac{\cosh\|j(\gamma)\|}{\cosh\|\rho(\gamma)\|} - 2\log\frac{\operatorname{Re}(j(\gamma)\sqrt{-1})}{\operatorname{Re}(\rho(\gamma)\sqrt{-1})} + \log\frac{\operatorname{Im}(j(\gamma)\sqrt{-1})}{\operatorname{Im}(\rho(\gamma)\sqrt{-1})}\right| \le \frac{1}{3}\varepsilon(\nu). \tag{2.29}$$

Summing up Lemma 2.27 (5), (6), (2.27), and (2.29), we obtain (2.23). □

We shall use the following estimate of $\|j(\gamma)\|$ in the next subsection, which is obtained as a by-product of the proofs of Lemmas 2.26 and 2.27:

Lemma 2.28. *We assume the same setting as Lemma 2.26. That is to say, given a quadruple (a_-, a_+, r, R) satisfying Assumptions 2.9–2.11, let j, ρ, and F_ν^∞ be as in Notation 2.16. Suppose $\nu \ge \nu_{dis}$ and $\varepsilon(\nu) < 1$. Let $\gamma \neq e$ be an arbitrary element of F_ν^∞ and $m := \ell(\gamma)$ the word length of γ. We write $\gamma = \gamma_{k_1}^{s_1} \dots \gamma_{k_m}^{s_m}$ for the reduced expression where $s_1, \dots, s_m \in \{1, -1\}$ and $k_1, \dots, k_m \ge \nu$. Then,*

$$\|j(\gamma)\| \le \left(\sum_{i=1}^m 2\log\frac{a_+(k_i)a_-(k_i)}{r(k_i)}\right) + m\varepsilon(\nu).$$

Proof. We have

$$\|j(\gamma)\| \le 2\log\operatorname{Re}(j(\gamma)\sqrt{-1}) - \log\operatorname{Im}(j(\gamma)\sqrt{-1}) + \frac{1}{3}\varepsilon(\nu) \tag{2.30}$$

by summing up (2.26) and (2.28) for $\varphi = j$. By (2.24), we have

$$2\log\operatorname{Re}(j(\gamma)\sqrt{-1}) \le 2\log a_{s_1}(k_1) + \frac{1}{6}\varepsilon(\nu). \tag{2.31}$$

We claim:

$$-\log\operatorname{Im}(j(\gamma)\sqrt{-1}) + 2\log a_{s_1}(k_1) \le \left(\sum_{i=1}^m 2\log\frac{a_+(k_i)a_-(k_i)}{r(k_i)}\right) + \frac{m}{2}\varepsilon(\nu). \tag{2.32}$$

Then Lemma 2.28 is proved by summing up (2.30), (2.31), and (2.32) since $m = \ell(\gamma) \ge 1$.

It remains to verify (2.32). As in the proof of Lemma 2.27, $\gamma = \gamma_{k_1}^{s_1} \cdots \gamma_{k_m}^{s_m} \in F_\nu^\infty$ defines a sequence of elements $\sigma_0, \dots, \sigma_m \in F_\nu^\infty$ by $\sigma_i = \gamma_{k_{i+1}}^{s_{i+1}} \dots \gamma_{k_m}^{s_m}$ for $0 \le i < m$ with $\sigma_0 = \gamma$ and $\sigma_m = 1$. We set $D(j(\sigma_i)) := \left|j(\sigma_i)\sqrt{-1} - a_{-s_i}(k_i)\right|$. Multiplying all (I_i) for $0 \le i < m$, with $\varphi = j$, we get $\operatorname{Im}(j(\gamma)\sqrt{-1}) = \prod_{i=1}^m (r(k_i)/D(j(\sigma_i)))^2$. Hence

$$-\log\operatorname{Im}(j(\gamma)\sqrt{-1}) = 2\sum_{i=1}^m \left(\log D(j(\sigma_i)) - \log r(k_i)\right). \tag{2.33}$$

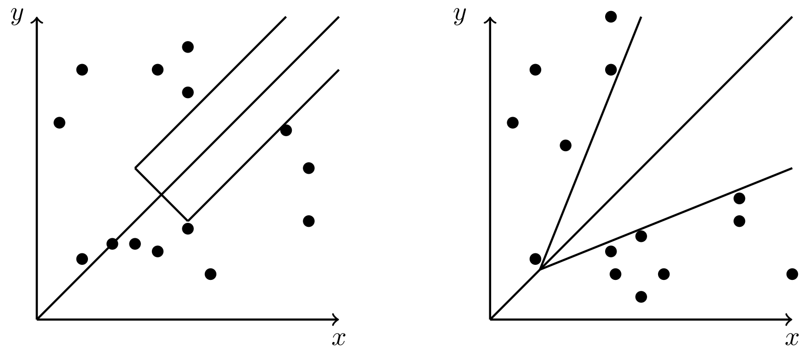

Figure 2.2: Properly discontinuous actions and sharp actions

Since $\sigma_m = 1$, we have

$$\begin{aligned} 2\left(\log D(j(\sigma_m)) - \log a_{-s_m}(k_m)\right) &= \log\left(1 + \frac{1}{a_{-s_m}(k_m)^2}\right) \leq \frac{1}{a_{-s_m}(k_m)^2} \\ &\leq \frac{3(R(k_m)^2+1)}{(a_-(k_m) - R(k_m))^2} \leq \frac{1}{2}\varepsilon(\nu), \end{aligned} \tag{2.34}$$

where the third and fourth inequalities follow from Assumption 2.9 and the definition (2.17) of $\varepsilon(\nu)$, respectively.

Note $|s-t| \leq st$ for $s, t > 1$ and $a_+(k) > a_-(k) > 1$ for all $k \geq \nu$ by Assumption 2.9 and $\varepsilon(\nu) < 1$. Thus, by (D'_i) for $1 \leq i < m$, we have

$$\begin{aligned} 2\log D(j(\sigma_i)) &\leq 2\log|a_{s_{i+1}}(k_{i+1}) - a_{-s_i}(k_i)| + \frac{1}{2}\varepsilon(\nu) \\ &\leq 2(\log a_{s_{i+1}}(k_{i+1}) + \log a_{-s_i}(k_i)) + \frac{1}{2}\varepsilon(\nu). \end{aligned} \tag{2.35}$$

Now the inequality (2.32) follows from (2.33), (2.34), and (2.35). Thus the proof of Lemma 2.28 is completed. □

2.3.3 Sharpness of the $\Gamma_\nu(a_-, a_+, r, R)$-action

The notion of sharpness was introduced in Kassel-Kobayashi [11], although the idea was already implicit in [16]. It is defined for a general homogeneous space of reductive type, however, in this section, we explain it only for AdS^3. Moreover, we find a necessary and sufficient condition that the discontinuous group $\Gamma_\nu(a_-, a_+, r, R)$ for AdS^3 is sharp.

The Cartan projection μ for the direct product group $G = \mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ is given by $\mu(g) = (\|\alpha\|, \|\beta\|)$ for $g = (\alpha, \beta) \in G$, where we recall that $\|\cdot\|$ is the pseudo-distance in $\mathrm{SL}(2,\mathbb{R})$. By Fact 2.8, a discrete subgroup Γ of G acts properly discontinuously on AdS^3 if and only if $\mu(\Gamma)$ "goes away from the line $x = y$ at infinity". The condition of sharpness is stronger than the condition of proper discontinuity as Definition 2.29 below, and Γ is sharp for AdS^3 if $\mu(\Gamma)$ "goes away from the line $x = y$ at infinity" with a speed that is at least linear ([11, p. 152]) as in Figure 2.2. See also [11, Ch. 4, Fig. 1] for the illustration of sharp actions in the general setting.

Definition 2.29 (Kassel-Kobayashi [11, Def. 4.2])**.** *Let $c \in (0,1]$ and $C \geq 0$. A discrete subgroup $\Gamma \subset G$ is called (c,C)-sharp for* AdS^3 *if for any $\gamma = (\alpha, \beta) \in \Gamma$,*

$$||\alpha\| - \|\beta\|| \geq c\sqrt{\|\alpha\|^2 + \|\beta\|^2} - C. \tag{2.36}$$

If Γ is (c,C)-sharp for some c and C, then Γ is called sharp for AdS^3.

Remark 2.30. *Our definition of (c,C)-sharpness is a little different from the original definition of (c,C)-sharpness. In fact, the inequality (4.7) in [11, Def. 4.2] for (c,C)-sharpness ($c \in (0,1]$, $C \geq 0$) is equivalent to the following inequality by the pseudo-distance $\|\cdot\|$ in our* AdS^3 *setting:*

$$||\alpha\| - \|\beta\|| \geq \sqrt{2}(c\sqrt{\|\alpha\|^2 + \|\beta\|^2} - C).$$

We have replaced $(\sqrt{2}c, \sqrt{2}C)$ with (c,C) in this paper. However, we still suppose $c \in (0,1]$ in Definition 2.29 since $||\alpha\| - \|\beta||/\sqrt{\|\alpha\|^2 + \|\beta\|^2} \leq 1$.

Kassel [8] and Guériataud-Kassel [7] proved that any finitely generated discontinuous group $\Gamma \subset G$ for AdS^3 is sharp. However, our discontinuous group $\Gamma_\nu(a_-, a_+, r, R)$ is infinitely generated, and actually, $\Gamma_\nu(a_-, a_+, r, R)$ may and may not be sharp for AdS^3. The next proposition gives a necessary and sufficient condition on the quadruple (a_-, a_+, r, R) such that the action of $\Gamma_\nu(a_-, a_+, r, R)$ on AdS^3 is sharp:

Proposition 2.31. *Let (a_-, a_+, r, R) be a quadruple of sequences satisfying Assumptions 2.9–2.11 and the condition (2.21), and $\nu_{pro} \in \mathbb{R}$ as in Proposition 2.21. We set*

$$A \equiv A(a_-, a_+, r, R) := \liminf_{k \to \infty} \log\left(\frac{R(k)}{r(k)}\right)\left(\log\frac{a_-(k)a_+(k)}{r(k)}\right)^{-1}. \tag{2.37}$$

Then $0 \leq A \leq 1$. Moreover, take any integer $\nu \geq \nu_{pro}$. Then the following hold for the discontinuous group $\Gamma_\nu \equiv \Gamma_\nu(a_-, a_+, r, R)$ for AdS^3*:*

(1) *if $c > A/\sqrt{1 + (1-A)^2}$, then Γ_ν is not (c,C)-sharp for any $C \geq 0$;*

(2) *if $0 < c < A/\sqrt{1 + (1-A)^2}$, then Γ_ν is $(c,0)$-sharp for any $\nu \geq \nu_{sha}(c)$. Here $\nu_{sha}(c)$ is a real number $\geq \nu_{pro}$ such that for all $k \geq \nu_{sha}(c)$,*

$$\frac{B+1}{A-B} < \log\frac{a_-(k)a_+(k)}{r(k)} < \frac{2}{A+B}\log\frac{R(k)}{r(k)},$$

where $B \in (0, A)$ is defined by $c = B/\sqrt{1 + (1-B)^2}$.

In particular, Γ_ν is sharp for $\nu \gg 0$ if and only if $A \neq 0$.

Proof. Take an arbitrary integer $\nu \geq \nu_{pro} (\geq \nu_{dis})$. Then $R(k) > r(k)$ and $a_+(k) > a_-(k)$ for any integer $k \geq \nu$ by Assumption 2.9 since $k \geq \nu_{dis}$. Moreover, $a_-(k) > \max\{R(k), 1\}$ since $\varepsilon(\nu) < 1$. Hence $a_-(k)a_+(k) > R(k) > r(k)$ and thus $0 \leq A \leq 1$.

(1) Recall that Γ_ν is generated by $\{(\alpha_k, \beta_k) \mid k = \nu, \nu+1, \ldots\}$. We claim:

$$\liminf_{k \to \infty} \frac{||\alpha_k\| - \|\beta_k||}{\sqrt{\|\alpha_k\|^2 + \|\beta_k\|^2}} \leq \frac{A}{\sqrt{1 + (1-A)^2}}. \tag{2.38}$$

Then (1) is obvious. Note $\lim_{k\to\infty}\sqrt{\|\alpha_k\|^2+\|\beta_k\|^2}=\infty$ since the Cartan projection $\mu\colon G\to\mathbb{R}^2$ is proper and since $\{(\alpha_k,\beta_k)\mid k=\nu,\nu+1,\ldots\}$ is an infinite discrete subset of G.

We now prove (2.38). By Lemma 2.26, for $k\geq\nu$,

$$|\|\alpha_k\|-\|\beta_k\||\leq 2\log\left(\frac{R(k)}{r(k)}\right)+\varepsilon(\nu). \tag{2.39}$$

Since $\alpha_k=r(k)^{-1}\begin{pmatrix}a_+(k) & -(a_-(k)a_+(k)+r(k)^2)\\ 1 & -a_-(k)\end{pmatrix}$ by the definition (2.13),

$$\|\alpha_k\|\geq\log(2\cosh\|\alpha_k\|)-\frac{1}{6}\varepsilon(\nu)\geq 2\log\left(\frac{a_-(k)a_+(k)}{r(k)}\right)-\frac{1}{6}\varepsilon(\nu), \tag{2.40}$$

where the first and second inequalities follow from (2.26) and the definition (2.5), respectively. Similarly,

$$\begin{aligned}\|\beta_k\|&\geq 2\log\left(\frac{a_-(k)a_+(k)}{R(k)}\right)-\frac{1}{6}\varepsilon(\nu)\\ &=2\left(\log\left(\frac{a_-(k)a_+(k)}{r(k)}\right)-\log\left(\frac{R(k)}{r(k)}\right)\right)-\frac{1}{6}\varepsilon(\nu),\end{aligned} \tag{2.41}$$

The inequality (2.38) follows from (2.39), (2.40), and (2.41) since the function $x/\sqrt{1+(1-x)^2}$ is monotone increasing on the interval $[0,1]$. Hence (1) holds.

(2) Suppose $0<c<A/\sqrt{1+(1-A)^2}$ and $\nu\geq\nu_{sha}(c)(\geq\nu_{pro})$. Then $\varepsilon(\nu)<1$. Setting $2\xi:=A-B(>0)$, for any integer $k\geq\nu$, we have

$$(B+\xi)\log\frac{a_-(k)a_+(k)}{r(k)}<\log\frac{R(k)}{r(k)}, \tag{2.42}$$

$$(B+1)\varepsilon(\nu)<2\xi\log\frac{a_-(k)a_+(k)}{r(k)}. \tag{2.43}$$

Let $\gamma\neq e$ be an arbitrary element of F_ν^∞, and $m:=\ell(\gamma)$ the word length of γ. We write $\gamma=\gamma_{k_1}^{s_1}\cdots\gamma_{k_m}^{s_m}$ for the reduced expression where $s_1,\ldots,s_m\in\{1,-1\}$ and $k_1,\ldots,k_m\geq\nu$. By Lemma 2.28, we have

$$\begin{aligned}B\|j(\gamma)\|&\leq\sum_{i=1}^m\left(2B\log\left(\frac{a_-(k_i)a_+(k_i)}{r(k_i)}\right)+B\varepsilon(\nu)\right)\\ &<\sum_{i=1}^m\left(2\log\left(\frac{R(k_i)}{r(k_i)}\right)-\left(2\xi\log\left(\frac{a_-(k)a_+(k)}{r(k)}\right)-B\varepsilon(\nu)\right)\right)\\ &<\sum_{i=1}^m\left(2\log\left(\frac{R(k_i)}{r(k_i)}\right)-\varepsilon(\nu)\right)\leq\|j(\gamma)\|-\|\rho(\gamma)\|.\end{aligned}$$

Here the second, third, and fourth inequalities follow from (2.42), (2.43), and Lemma 2.26, respectively. Then $\|\rho(\gamma)\|<(1-B)\|j(\gamma)\|$ and thus

$$\frac{\|j(\gamma)\|-\|\rho(\gamma)\|}{\sqrt{\|j(\gamma)\|^2+\|\rho(\gamma)\|}}>\frac{B}{\sqrt{1+(1-B)^2}}=c.$$

Hence Γ_ν is $(c,0)$-sharp, which proves (2).

□

Remark 2.32. *In Proposition 2.31, we did not treat the case that c takes the critical value $c(a_-, a_+, r, R) := A(1 + (1 - A)^2)^{-\frac{1}{2}}$. At this critical value, the discontinuous group $\Gamma_\nu(a_-, a_+, r, R)$ may be (c, C)-sharp for all $C \geq 0$, and may not be (c, C)-sharp for all $C \geq 0$. We give such examples as below.*

Let $\delta = \pm 1$ and $b \geq 0$. We define the following quadruple (a_-, a_+, r, R_δ):

$$a_-(k) := e^{\frac{b}{2}(k - \frac{1}{4}) + 1},\ a_+(k) := e^{\frac{b}{2}(k + \frac{1}{4}) + 1},\ r(k) := e^{-k},\ R_\delta(k) := (\log k)^\delta$$

for $k \geq 2$. This quadruple applies to Lemma 2.12 and satisfies the condition (2.21). Then the critical value $c \equiv c(a_-, a_+, r, R_\delta)$ amounts to $(b^2 + (b+1)^2)^{-\frac{1}{2}}$ because

$$\log \frac{R_\delta(k)}{r(k)} = k + \delta \log \log k,\ \log \frac{a_-(k) a_+(k)}{r(k)} = (b+1)k + 2.$$

Suppose $\nu \geq \nu_{pro}$. Let $\Gamma_{\nu,\delta}$ be the discontinuous group $\Gamma_\nu(a_-, a_+, r, R_\delta)$ for AdS^3 *(see Proposition 2.22). Then, the following hold:*

(1) *if $\delta = 1$, then $\Gamma_{\nu,\delta}$ is $(c, 0)$-sharp for $\nu \gg 0$, and thus (c, C)-sharp for all $C \geq 0$;*

(2) *if $\delta = -1$, then $\Gamma_{\nu,\delta}$ is not (c, C)-sharp for all $C \geq 0$.*

Indeed, suppose $\delta = 1$. Then we may and do take $\nu \gg 0$ such that for all $k \geq \nu$,

$$\frac{1}{b+1}\left(2 \log \left(\frac{a_-(k) a_+(k)}{r(k)}\right) + \varepsilon(\nu)\right) \leq 2 \log \left(\frac{R_\delta(k)}{r(k)}\right) - \varepsilon(\nu).$$

Then the inequality $(b+1)^{-1}\|j(\gamma)\| \leq \|j(\gamma)\| - \|\rho(\gamma)\|$ for any $\gamma \in F_\nu^\infty$ follows from Lemmas 2.26 and 2.28 by an argument similar to the proof of Proposition 2.31 (2). Hence $\|\rho(\gamma)\| \leq b(1+b)^{-1}\|j(\gamma)\|$ and thus (1) holds since then

$$\frac{\|j(\gamma)\| - \|\rho(\gamma)\|}{\sqrt{\|j(\gamma)\|^2 + \|\rho(\gamma)\|}} \geq \frac{1}{\sqrt{b^2 + (1+b)^2}} = c.$$

On the other hand, suppose $\delta = -1$. Since $\varepsilon(\nu) < 1$ by $\nu \geq \nu_{pro}$, we have $|\|\alpha_k\| - \|\beta_k\|| \leq 2(k - \log \log k) + 1$, $\|\alpha_k\| \geq 2(b+1)k$, and $\|\beta_k\| \geq 2bk$ for any integer $k \geq \nu$ by (2.39), (2.40), and (2.41), respectively. Thus (2) follows readily from

$$\liminf_{k \to \infty} \left(c\sqrt{\|\alpha_k\|^2 + \|\beta_k\|^2} - |\|\alpha_k\| - \|\beta_k\||\right) \geq \liminf_{k \to \infty} (2 \log \log k - 1) = \infty.$$

Example 2.33. *The discontinuous groups $\Gamma_\nu(a_-, a_+, r, R)$ associated to the quadruples (a_-, a_+, r, R) in Table 2.2 are all non-sharp by Proposition 2.31.*

As an application of Proposition 2.31, we obtain the following lemma, which will be used for the proof of the counting result (Theorem 2.1) in Section 2.5:

Lemma 2.34. *Suppose $\Gamma_\nu(p, q)$ is the discontinuous group for* AdS^3 *associated to a pair of functions $(p(t), q(t))$ as in Lemma 2.24. We set $\delta := \mathrm{sgn}(p''(t)) \in \{\pm 1\}$, which is independent of $t \gg 0$. Then $\Gamma_\nu(p, q)$ is sharp for $\nu \gg 0$ if and only if the following conditions hold:*

case (i) $\delta = 1$: $\limsup_{k\to\infty} \dfrac{\log p(k+\frac{1}{2})}{\log p(k)} < \infty$ *and* $\limsup_{k\to\infty} \dfrac{\log q(k)}{\log p(k)} < \infty$;

case (ii) $\delta = -1$: $\limsup_{k\to\infty} \dfrac{-\log p'(k+1)}{\log p(k)} < \infty$ *and* $\limsup_{k\to\infty} \dfrac{\log q(k)}{\log p(k)} < \infty$.

Proof. Note that

$$\log \frac{a_-(k)a_+(k)}{r(k)} = 2\log p(k) + \log p(k+\frac{1}{2}) + \log q(k) - \log p'(k-\delta), \tag{2.44}$$

$$\log \frac{R(k)}{r(k)} = \log p(k). \tag{2.45}$$

Proposition 2.31 tells us that the discontinuous group $\Gamma_\nu(p,q)$ for AdS^3 is sharp for $\nu \gg 0$ if and only if $\limsup_{k\to\infty}(2.44)/(2.45) < \infty$. Therefore the assertion follows from elementary properties of C^2-functions $p(t)$ of one-variable, summarized in Lemma 2.35 below. $\square$

Lemma 2.35. *Let $p(t)$ be a C^2-function defined in $t \gg 0$ such that the second derivative p'' is nowhere vanishing and $\lim_{t\to\infty} p(t) = \infty$. We set $\delta := \mathrm{sgn}(p''(t)) \in \{\pm 1\}$, which is independent of $t \gg 0$. Then the following hold:*

(1) *if $\delta = 1$ (resp. -1), then $-\log p'(t)$ is bounded from the above (resp. below);*

(2) *if $\delta = 1$, then $\log p(t) - \log p'(t-1)$ is bounded from the below;*

(3) *if $\delta = -1$, then $p(t+\frac{1}{2}) < 2p(t)$ for $t \gg 0$.*

Proof. (1) If $\delta = 1$ (resp. -1), then $p'(t)$ is monotone increasing (resp. decreasing). Hence (1) is obvious.

(2) Suppose $\delta = 1$. Since then p is convex, we have $p(t) \geq p(t-1) + p'(t-1) > p'(t-1)$ for $t \gg 0$ by $\lim_{t\to\infty} p(t) = \infty$. Hence (2) holds.

(3) Suppose $\delta = -1$. Then $p'(t)$ is monotone decreasing. Hence we have $p'(t) < 2p(t)$ for $t \gg 0$ by $\lim_{t\to\infty} p(t) = \infty$. Hence, since p is concave, we have $p(t+\frac{1}{2}) \leq p(t) + \frac{1}{2}p'(t) < 2p(t)$, which proves (3).

$\square$

2.4 Construction of Γ with large counting

In this section, we explain a construction of discontinuous groups Γ for which the asymptotic growth of the counting $N_\Gamma(x,R)$ is as rapid as we wish, and thus complete the proof of Theorem 2.2.

We begin with a lemma which reduces the estimate of $N_\Gamma(x,R)$ to the case $x = o$.

Lemma 2.36. *Let Γ be a discontinuous group for AdS^3. Set $\|g\| := \|\alpha\| + \|\beta\|$ for $g = (\alpha,\beta) \in G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Then, for $x \in \mathrm{AdS}^3$ and $R > 0$,*

$$N_\Gamma(gx, R - \|g\|) \leq N_{g^{-1}\Gamma g}(x,R) \leq N_\Gamma(gx, R + \|g\|).$$

Proof. By Lemma 2.7, we have $\|gx\| = \|\alpha x \beta^{-1}\| \leq \|x\| + \|g\|$ and thus $gB(R) \subset B(R + \|g\|)$. Similarly, $g^{-1}B(R - \|g\|) \subset B(R)$ and thus

$$B(R - \|g\|) \subset gB(R) \subset B(R + \|g\|).$$

The assertion follows immediately from these inclusion relations. $\square$

The next proposition gives a lower bound of the counting $N_\Gamma(o, R)$ for the discontinuous group $\Gamma = \Gamma_\nu(p, q)$ introduced in Lemma 2.24.

Proposition 2.37. *Let $\Gamma_\nu \equiv \Gamma_\nu(p, q)$ be the discontinuous group for* AdS^3 *associated to a pair of functions $(p(t), q(t))$ as in Lemma 2.24. Here we have supposed $\nu \geq \nu_{pro}$ as in Proposition 2.21. Then*

$$N_{\Gamma_\nu}(o, 4 \log p(t)) \geq t - \nu \text{ for all } t \geq \nu.$$

Proof. We note $p(k) \geq 2$ for any $k \geq \nu$ since $\log p(k) = \log(R(k)/r(k)) > 1$ holds from $k \geq \nu_{pro}$. We also recall from Definition 2.15 that the group Γ_ν is generated by $\{(\alpha_k, \beta_k) \mid k = \nu, \nu + 1, \ldots\}$.

By the definitions (2.13) and (2.22), for $k \geq \nu$, we have

$$\alpha_k^{-1}\beta_k = \begin{pmatrix} p(k)^{-1} & p(k)^2 - 1 \\ 0 & p(k) \end{pmatrix},$$

whence the pseudo-distance of $\alpha_k^{-1}\beta_k \in \mathrm{SL}(2, \mathbb{R})$ is computed by (2.5):

$$2 \cosh \|\alpha_k^{-1}\beta_k\| = p(k)^4 - p(k)^2 + 1 + p(k)^{-2} < p(k)^4.$$

We then observe for any $R > 0$:

$$\Gamma_\nu o \cap B(R) \supset \{\alpha_k^{-1}\beta_k \mid \|\alpha_k^{-1}\beta_k\| \leq R\} \supset \{\alpha_k^{-1}\beta_k \mid p(k)^4 \leq e^R\}$$

since $2\cosh R > e^R$. Since Γ_ν acts freely on AdS^3, we deduce

$$N_{\Gamma_\nu}(o, R) \geq \#\{k \in \mathbb{N} \mid \nu \leq k \text{ and } 4 \log p(k) \leq R\}.$$

Recall that $p(t)$ is monotone increasing. Hence we conclude $N_{\Gamma_\nu}(o, 4\log p(t)) \geq \#\{k \in \mathbb{N} \mid \nu \leq k \leq t\} > t - \nu$ for all $t \geq \nu$. $\square$

We are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Since the G-action on AdS^3 is transitive, we may and do assume $x = o$ by Lemma 2.36. Moreover, it suffices to consider the case where $f(t)$ is a monotone increasing C^2-function such that $f''(t) > 0$ for any t and $\lim_{t\to\infty} f(t) = \infty$.

Let $p(t)$ be the inverse function of $sf(s)$ defined for $t \gg 0$. Then $p(t)$ is a monotone increasing C^2-function such that $p''(t) < 0$ for $t \gg 0$ and that $\lim_{t\to\infty} p(t) = \infty$. Take an arbitrary continuous function $q(t)$ satisfying $\lim_{t\to\infty} q(t) = \infty$. Then $\Gamma_\nu(p, q)$ is a discontinuous group for AdS^3 if $\nu \gg 0$ by Lemma 2.24. Suppose $\nu \geq \nu_{pro}$ as in Proposition 2.21 and set $\Gamma \equiv \Gamma_{f,o} := \Gamma_\nu(p, q)$. By Proposition 2.37, $N_\Gamma(o, R) \geq e^{R/4} f(e^{R/4}) - \nu$ for all $R \geq 4 \log p(\nu)$ since $R = 4 \log p(t)$ if and only if $e^{R/4} f(e^{R/4}) = t$. Thus $\lim_{R\to\infty} N_\Gamma(o, R)/f(R) = \infty$ and the proof of Theorem 2.2 is completed. $\square$

Example 2.38. *Let $\Gamma_\nu \equiv \Gamma_\nu(a_-, a_+, r, R)$ be the discontinuous group for* AdS^3 *associated to the quadruple (a_-, a_+, r, R) in Table 2.2 (3). Here we have supposed $\nu \geq \nu_{pro}$ as in Proposition 2.21. By an argument similar to the proof of Proposition 2.37, we have*

$$N_{\Gamma_\nu}(o, R) \geq \exp(e^{\frac{R}{4}}) - \nu \textit{ for all } R \geq 4 \log\log \nu.$$

2.5 Application to the spectral analysis

In this section, we complete the proofs of Theorems 2.1 and 2.5.

Associated to a quadruple (a_-, a_+, r, R) of sequences, we have defined in Section 2.3 the subgroup $\Gamma_\nu \equiv \Gamma_\nu(a_-, a_+, r, R)$ of $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, and proved that Γ_ν is a discontinuous group for $X = \mathrm{AdS}^3$ when $\nu \geq \nu_{pro}$ if Assumptions 2.9–2.11 and the condition (2.21) are satisfied. Then the quotient space $X_{\Gamma_\nu} := \Gamma_\nu \backslash X$ admits an anti-de Sitter structure via the covering $X \to X_{\Gamma_\nu}$. Let $\Box_{X_{\Gamma_\nu}}$ be the Laplacian on X_{Γ_ν}, which is not an elliptic operator, but a hyperbolic operator because X_{Γ_ν} is a Lorentzian manifold. Our interest here is the discrete spectrum $\mathrm{Spec}_d(\Box_{X_{\Gamma_\nu}})$ of the Laplacian $\Box_{X_{\Gamma_\nu}}$.

In this section, we find a sufficient condition on (a_-, a_+, r, R) such that the counting $N_{\Gamma_\nu}(x, R)$ grows at most 4^R, in particular, satisfies the exponential growth condition (2.1). By the criterion of sharpness in Section 2.3.3, we give non-sharp Γ_ν such that the counting $N_{\Gamma_\nu}(x, R)$ grows at most 4^R. Moreover, we use this counting result to construct an infinite subset of $\mathrm{Spec}_d(\Box_{X_{\Gamma_\nu}})$ for such Γ_ν. We show:

Theorem 2.39. *Let (a_-, a_+, r, R) be a quadruple of sequences satisfying Assumptions 2.9-2.11, and $\nu_{pro} \in \mathbb{N}$ as in Proposition 2.21. Moreover, take an arbitrary integer $\nu \geq \max\{\nu_{pro}, 2\}$. If $R(k)/r(k) \geq e^k$ for any integer $k \geq \nu$, then*

(1) *$\Gamma_\nu \equiv \Gamma_\nu(a_-, a_+, r, R)$ is a discontinuous group for X;*

(2) *$N_{\Gamma_\nu}(x, R) \leq 4^R$ for any $x \in X$ and any $R > 0$;*

(3) *there exists $m_0 = m_0(\Gamma_\nu) > 0$ such that*

$$\mathrm{Spec}_d(\Box_{X_{\Gamma_\nu}}) \supset \{4m(m-1) \mid m \in \mathbb{Z} \textit{ and } m > m_0\}.$$

Example 2.40. *The following quadruples (a_-, a_+, r, R) apply to Theorem 2.39:*

(1) $a_-(k) = e^k,\ a_+(k) = e^{k+\frac{1}{2}},\ r(k) = \dfrac{e^{k-1}}{\exp(k + e^k)},\ R(k) = \dfrac{e^{k-1}}{\exp(e^k)}$. *This quadruple is obtained by Lemma 2.24 with $p(t) = e^t$ and $q(t) = \exp(e^t)$.*

(2) $a_1(k) = \exp(e^k),\ a_2(k) = \exp(e^{k+\frac{1}{2}}),\ r(k) = 1,\ R(k) = e^k$ *(Table 2.2 (1)).*

Postponing the proof of Theorem 2.39, we give proofs of Theorems 2.1 and 2.5.

Proof of Theorems 2.1 and 2.5. The discontinuous groups $\Gamma_\nu(a_-, a_+, r, R)$ associated to (1) and (2) of Example 2.40 are non-sharp by Lemma 2.34 and Proposition 2.31, respectively. Applying (2) and (3) of Theorem 2.39, we get Theorems 2.1 and 2.5, respectively. $\Box$

To prove Theorem 2.39 (3), namely, to construct an infinite subset of the discrete spectrum, we use the following fact established by Kassel-Kobayashi [11]:

Fact 2.41 ([11]). *Let Γ be a discontinuous group for $X = \mathrm{AdS}^3$ satisfying the exponential growth condition (2.1). Then there exists $m_0(\Gamma) > 0$ such that*

$$\mathrm{Spec}_d(\Box_{X_\Gamma}) \supset \{4m(m-1) \mid m \in \mathbb{Z} \text{ and } m > m_0(\Gamma)\}.$$

Remark 2.42. *Kassel-Kobayashi constructed in [11, Cor. 9.10] an infinite subset of the discrete spectrum which is stable under small deformations of Γ in G.*

We prepare some lemmas needed for the proof of Theorem 2.39 (2). The following lemma was proved in [11] in the general setting where X is a reductive symmetric space. Since it plays a crucial role in proving Theorem 2.39, we give an elementary proof for $X = \mathrm{AdS}^3$ for the convenience of the reader.

Lemma 2.43 ([11, Lem. 4.4 and 4.17]). *For $(\alpha, \beta) \in G$ and $x \in X = \mathrm{AdS}^3$,*

$$\|(\alpha, \beta)x\| + \|x\| \geq |\|\alpha\| - \|\beta\|| \,.$$

Proof. By Lemma 2.7, we have

$$\|(\alpha, \beta)x\| = \|\alpha x \beta^{-1}\| \geq |\|\alpha x\| - \|\beta\|| \,, \tag{2.46}$$

$$\|x\| = \|\alpha^{-1}\alpha x\| \geq |\|\alpha\| - \|\alpha x\||. \tag{2.47}$$

Summing up (2.46) and (2.47), we have $\|(\alpha, \beta)x\| + \|x\| \geq |\|\alpha\| - \|\beta\||$. □

Fact 2.44 ([11, Def–Lem. 4.20]). *Let Γ be a discontinuous group for $X = \mathrm{AdS}^3$. Then, the set*

$$\mathcal{D}_{X_\Gamma} := \{x \in X \mid \forall \gamma \in \Gamma,\ \|\gamma x\| \geq \|x\|\}.$$

is a fundamental domain of X for the action of Γ. In particular, $\Gamma \mathcal{D}_{X_\Gamma} = X$.

Therefore, we may assume $x \in \mathcal{D}_{X_\Gamma}$ to study $N_\Gamma(x, R)$.

Lemma 2.45 (cf. [11, Lem. 4.21]). *For any $x \in \mathcal{D}_{X_\Gamma}$ and any $\gamma = (\alpha, \beta) \in \Gamma$,*

$$\|\gamma x\| \geq \frac{1}{2} |\|\alpha\| - \|\beta\|| \,.$$

Proof. Let $\gamma = (\alpha, \beta) \in \Gamma$ and $x \in \mathcal{D}_{X_\Gamma}$. Then we have

$$2\|\gamma x\| \geq \|\gamma x\| + \|x\| \geq |\|\alpha\| - \|\beta\||$$

by the definition of $\mathcal{D}_{X_\Gamma}$ and Lemma 2.43, and thus Lemma 2.45 holds. □

Lemma 2.46. *We set $S(R) := \bigcup_{m=1}^{\infty} \left\{ (k_1, \ldots, k_m) \in \mathbb{N}_+^m \,\middle|\, \sum_{i=1}^{m} k_i \leq R \right\}$ for any $R \in \mathbb{N}$. Then the cardinality of $S(R)$ equals $2^R - 1$.*

Proof. For $(k_1, \ldots, k_m) \in S(R)$, we define the binary number $1\underbrace{0\ldots0}_{k_1-1}\ldots1\underbrace{0\ldots0}_{k_m-1}$, which induces a bijection $S(R) \xrightarrow{\sim} \mathbb{Z} \cap [1, 2^R - 1]$. Hence $\#S(R) = 2^R - 1$. □

We are ready to prove Theorem 2.39. In the following, let $\lfloor R \rfloor$ denote the largest integer less than or equal to R for $R \in \mathbb{R}$.

Proof of Theorem 2.39. The assertion (1) is verified by Proposition 2.22. We now prove (2). Let $\Gamma_\nu \equiv \Gamma_\nu(a_-, a_+, r, R)$. Take any integer $\nu \geq \max\{\nu_{pro}, 2\}$. Then recall $\varepsilon(\nu)(< 1) \leq 2$. Moreover, take $\gamma \in F_\nu^\infty \setminus \{e\}$. Let $m := \ell(\gamma)$ be the word length of γ, and we write $\gamma = \gamma_{k_1}^{s_1} \cdots \gamma_{k_m}^{s_m}$ for the reduced expression where $s_1, \ldots, s_m \in \{1, -1\}$ and $k_1, \ldots, k_m \geq \nu (\geq 2)$. By Lemma 2.26,

$$\frac{1}{2}(\|j(\gamma)\| - \|\rho(\gamma)\|) \geq \sum_{i=1}^{m} \left(\log \left(\frac{R(k_i)}{r(k_i)} \right) - \frac{1}{2}\varepsilon(\nu) \right) \geq \sum_{i=1}^{m} (k_i - 1). \tag{2.48}$$

To prove (2), we may and do assume $x \in \mathcal{D}_{X_{\Gamma_\nu}}$ by Fact 2.44. Suppose $(j(\gamma), \rho(\gamma))x \in B(R)$. Then $|\|j(\gamma)\| - \|\rho(\gamma)\||/2 \leq R$ by Lemma 2.45. Hence $(k_1 - 1, \ldots, k_m - 1) \in S(\lfloor R \rfloor)$ by the inequality (2.48) and in particular, $m \leq R$. The number of such $(k_1, \ldots, k_m)$ is at most $2^R - 1$ by Lemma 2.46 and thus the number of such $\gamma = \gamma_{k_1}^{s_1} \cdots \gamma_{k_m}^{s_m} (\neq e)$ with $s_1, \ldots, s_m \in \{1, -1\}$ is at most $(2^R - 1)2^R$. Hence we obtain $N_{\Gamma_\nu}(x, R) \leq (2^R - 1)2^R + 1 \leq 4^R$, which proves (2). The assertion (3) follows from (2) and Fact 2.41. $\square$

3 Linear independence of generalized Poincaré series for anti-de Sitter 3-manifolds

3.1 Introduction

A pseudo-Riemannian manifold is a smooth manifold M equipped with a smooth non-degenerate symmetric bilinear tensor g of signature (p, q) on M. It is called Riemannian if $q = 0$, and Lorentzian if $q = 1$. As in the Riemannian case, the Laplacian $\square_M := \mathrm{div}_M \circ \mathrm{grad}_M$ is defined as a second-order differential operator on M. We note that it is a hyperbolic differential operator if M is Lorentzian. We write $L^2(M)$ for the Hilbert space of square-integrable functions on M with respect to the Radon measure induced by the pseudo-Riemannian structure. For $\lambda \in \mathbb{C}$, we denote by

$$L^2_\lambda(M) := \{f \in L^2(M) \mid \square_M f = \lambda f \text{ in the weak sense}\}.$$

The set of L^2-eigenvalues $\mathrm{Spec}_d(\square_M) := \{\lambda \in \mathbb{C} \mid L^2_\lambda(M) \neq 0\}$ is called *discrete spectrum* of $\square_M$.

Our interest is the multiplicity of L^2-eigenvalues λ of $\square_M$, denoted by

$$\mathcal{N}_M(\lambda) := \dim_{\mathbb{C}} L^2_\lambda(M) \in \mathbb{N} \cup \{\infty\}.$$

In the Riemannian case, the Laplacian is an elliptic differential operator and the distribution of its discrete spectrum has been investigated extensively, such as the Weyl law for compact Riemannian manifolds. However, it is not the case for non-Riemannian manifolds. Kobayashi [18], and later Fox-Strichartz [5], investigated the distribution of discrete spectrum of the Laplacian $\square_M$ of some pseudo-Riemannian manifolds, i.e., when M is the flat pseudo-Riemannian manifold $\mathbb{R}^{p,q}/\mathbb{Z}^{p+q}$ and is the Lorentzian manifold $S^1 \times S^q$, respectively.

Let us recall some basic notions. A *discontinuous group* for a homogeneous manifold $X = G/H$ is a discrete subgroup Γ of G acting properly discontinuously and freely on X (Kobayashi [17, Def. 1.3]). In this case, the quotient space $X_\Gamma := \Gamma\backslash X$ carries a C^∞-manifold structure such that the quotient map $p_\Gamma \colon X \to X_\Gamma$ is a covering of C^∞ class, hence X_Γ has a (G, X)-structure induced by p_Γ. If we drop the assumption of freeness, X_Γ is not always a manifold but carries a nice structure called an orbifold or V-manifold. Proper discontinuity is a more serious assumption which assures X_Γ to be Hausdorff in the quotient topology. We remark that the Γ-action on X may fail to be properly discontinuous when H is noncompact. In order to overcome this difficulty, Kobayashi [15] and Benoist [1] established the properness criterion for reductive G generalizing the original criterion by Kobayashi [14]. Whereas discontinuous groups for the de Sitter space $\mathrm{dS}^n := \mathrm{SO}_0(n, 1)/\mathrm{SO}_0(n-1, 1)$ are always finite groups (the Calabi-Markus phenomenon, see [3], [14]), there are a rich family of discontinuous groups for the anti-de Sitter space, see e.g. [6], [16], [22]. We treat, in this article, the three-dimensional anti-de Sitter space $\mathrm{AdS}^3 := \mathrm{SO}_0(2, 2)/(\{\pm 1\} \times \mathrm{SO}_0(2, 1))$.

For $m \in \mathbb{N}$, we set

$$\lambda_m := 4m(m-1). \tag{3.1}$$

We prove:

Theorem 3.1. *For any finitely generated discontinuous group* Γ *for* AdS^3*,*

$$\lim_{m\to\infty} \mathcal{N}_{\Gamma\backslash\,\mathrm{AdS}^3}(\lambda_m) = \infty.$$

Remark 3.2. *(1) If the discontinuous group* Γ *is "standard" and torsion-free, a stronger result holds:* $\mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) = \infty$ *for sufficiently large* $m \in \mathbb{N}$ *(Kassel-Kobayashi [10]), which is derived from the results in Kassel-Kobayashi [12]. On the other hand, Theorem 3.1 is also applicable to "non-standard"* Γ*, for example, in the case where* Γ *is Zariski dense in* $\mathrm{SO}(2,2)$*.*

(2) The assumption that Γ *is finitely generated could be relaxed. In fact, the exponential growth condition (see (3.9)) for* Γ*-orbits is essential in the proof of Theorem 3.1, and there exist infinitely generated discontinuous groups* Γ *satisfying (3.9) and the conclusion of Theorem 3.1 holds for such* Γ *(see Theorem 3.14 which is proved without finitely generated assumption).*

(3) An analogous statement to Theorem 3.1 also holds when $\Gamma\backslash\mathrm{AdS}^3$ *is an orbifold. See Section 3.2.3 for the argument when we drop the assumption that the* Γ*-action is free.*

Now we consider a small deformation of a discrete subgroup. The study of *stability* for properness was intiated by Kobayashi [16] and Kobayashi-Nasrin [19] and has been developed by Kassel [9] and others. Moreover, Kassel-Kobayashi [11] proved the existence of infinite *stable* L^2*-eigenvalues* under any small deformation of discontinuous groups. In this article, we also consider the multiplicities of stable L^2-eigenvalues (Definition 3.3) and prove that they are unbounded.

To be precise, let X_n be the n-fold covering of $X_1 := \mathrm{AdS}^3$ for $1 \le n \le \infty$, and G_n the Lie group of its isometries. Every compact anti-de Sitter 3-manifold M is of the form $M \cong \Gamma\backslash X_n$ for some *finite* n where $\Gamma(\subset G_n)$ is a discontinuous group for X_n by Kulkarni-Raymond [20, Thm. 7.2] and Klingler [13]. We take n to be the smallest integer of this property.

Let $\mathrm{Hom}(\Gamma, G_n)$ be the set of group homomorphisms with the compact-open topology, and $\mathcal{U}_\Gamma$ the set of neighborhoods W in $\mathrm{Hom}(\Gamma, G_n)$ of the natural inclusion $\Gamma \subset G_n$ such that for any $\varphi \in W$, the map φ is injective and $\varphi(\Gamma)$ acts properly discontinuously on X_n. One knows $\mathcal{U}_\Gamma \neq \emptyset$ ([16], [13]). By definition, λ is a stable L^2-eigenvalue if $\min_{\varphi\in W} \mathcal{N}_{\varphi(\Gamma)\backslash X_n}(\lambda) \neq 0$ for some $W \in \mathcal{U}_\Gamma$. Moreover, for any $\lambda \in \mathbb{C}$ and any inclusion $W' \subset W$ in $\mathcal{U}_\Gamma$, we have an obvious inequality

$$\min_{\varphi\in W'} \mathcal{N}_{\varphi(\Gamma)\backslash X_n}(\lambda) \ge \min_{\varphi\in W} \mathcal{N}_{\varphi(\Gamma)\backslash X_n}(\lambda).$$

Definition 3.3. *For a compact anti-de Sitter* 3*-manifold* M*, we say that*

$$\widetilde{\mathcal{N}}_M(\lambda) := \sup_{W\in\mathcal{U}_\Gamma} \min_{\varphi\in W} \mathcal{N}_{\varphi(\Gamma)\backslash X_n}(\lambda)$$

is the multiplicity of a stable L^2*-eigenvalue* λ*.*

There exist infinitely many $m \in \mathbb{N}$ such that $\widetilde{\mathcal{N}}_M(\lambda_m) \ge 1$, namely λ_m is a stable L^2-eigenvalue for sufficiently large m ([11, Cor. 9.10]). However, to the best knowledge of the author, it is not known whether $\widetilde{\mathcal{N}}_M(\lambda)$ is finite. We prove:

Theorem 3.4. *For any compact anti-de Sitter* 3*-manifold* M*,*

$$\lim_{m\to\infty} \widetilde{\mathcal{N}}_M(\lambda_m) = \infty.$$

The organization of this article is as follows. A key step to our proof is to find a family of L^2-eigenfunctions of $\square_{\mathrm{AdS}^3}$ with eigenvalue λ_m on AdS^3 for which the corresponding "generalized Poincaré series" are linearly independent, see Proposition 3.15. In Section 3.2, we recall some facts about L^2-eigenfunctions of $\square_{\mathrm{AdS}^3}$ and their generalized Poincaré series which were introduced in [11] as the Γ-average of these eigenfunctions. We then give a uniform estimate of the "pseudo-distance" between the origin and the second closest point of each Γ-orbit (see Section 3.2.4). In Section 3.3, we complete a proof of Proposition 3.15. In Section 3.4, we prove a generalization of Theorem 3.4 to the case of convex cocompact groups (Definition 3.21).

3.2 Preliminaries about the anti-de Sitter space

In this section, we collect some preliminary results about AdS^3. We refer to [11, Sect. 9] where they illustrate their general theory for reductive symmetric spaces $X = G/H$ in details in the special setting where $X = \mathrm{AdS}^3$.

Let Q be a quadratic form on $\mathbb{R}^4$ defined by $Q(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2$ for $x = (x_1, x_2, x_3, x_4)$ and we set

$$\mathbb{H}^{2,1} := \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid Q(x) = 1\} \cong \mathrm{SO}_0(2,2)/\mathrm{SO}_0(2,1).$$

The tangent space $T_x(\mathbb{H}^{2,1})$ at $x \in \mathbb{H}^{2,1}$ is isomorphic to the orthogonal complement $(\mathbb{R}x)^\perp$ with respect to Q. Then $-Q|_{(\mathbb{R}x)^\perp}$ is a quadratic form of signature $(2,1)$ on $T_x(\mathbb{H}^{2,1}) \cong (\mathbb{R}x)^\perp$ and thus $-Q$ induces a Lorentzian structure on $\mathbb{H}^{2,1}$ with constant sectional curvature -1. The 3-dimensional anti-de Sitter space

$$\mathrm{AdS}^3 := \mathbb{H}^{2,1}/\{\pm 1\} \cong \mathrm{SO}_0(2,2)/(\{\pm 1\} \times \mathrm{SO}_0(2,1)),$$

inherits a Lorentzian structure through the double covering $\pi \colon \mathbb{H}^{2,1} \to \mathrm{AdS}^3$.

3.2.1 Some coordinates and "pseudo-balls"

In this subsection, we work with coordinates on $\mathbb{H}^{2,1}$ and consider "pseudo-balls" in AdS^3. We identify $\mathbb{H}^{2,1}$ with $\mathrm{SL}(2,\mathbb{R})$ using the isomorphism

$$\begin{array}{ccc} \mathbb{H}^{2,1} & \xrightarrow{\cong} & \mathrm{SL}(2,\mathbb{R}) \\ x = (x_1, x_2, x_3, x_4) & \longmapsto & \begin{pmatrix} x_1 + x_4 & -x_2 + x_3 \\ x_2 + x_3 & x_1 - x_4 \end{pmatrix}. \end{array} \tag{3.2}$$

For $t \geq 0$ and $\theta \in \mathbb{R}$, we use the notations

$$k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \ a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{3.3}$$

We embed $\mathbb{H}^{2,1}$ into $\mathbb{C}^2$ by

$$x \mapsto (z_1, z_2) = (x_1 + \sqrt{-1}x_2, x_3 + \sqrt{-1}x_4). \tag{3.4}$$

We note that $z_1 \neq 0$ if $x \in \mathbb{H}^{2,1}$. Via the identification (3.2), we have

$$(z_1, z_2) = ((\cosh t)e^{\sqrt{-1}(\theta_1+\theta_2)}, (\sinh t)e^{\sqrt{-1}(\theta_1-\theta_2)}), \tag{3.5}$$

if $x = k(\theta_1)a(t)k(\theta_2) \in \mathrm{SL}(2,\mathbb{R})$ (a "polar coordinate"). In particular, we have

$$\cosh 2t = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Next, we consider pseudo-balls on AdS^3, as a special case of Kassel-Kobayashi [11] for reductive symmetric spaces.

Definition 3.5. *For $x = (x_1, x_2, x_3, x_4) \in \mathbb{H}^{2,1}$, $\|x\| \in \mathbb{R}_{\geq 0}$ is defined by*

$$\cosh \|x\| := x_1^2 + x_2^2 + x_3^2 + x_4^2 (= \cosh(2t)).$$

This function is invariant under $x \mapsto -x$, hence defines a function on AdS^3, to be also denoted by $\|\cdot\|$ (a "pseudo-distance" from the origin). The compact set

$$B(R) := \{y \in \mathrm{AdS}^3 \mid \|y\| \leq R\}$$

is called a pseudo-ball of radius R.

3.2.2 Square-integrable eigenfunctions of the Laplacian on the anti-de Sitter space

In this subsection, we consider the square-integrable eigenfunctions $\psi_{m,k}$ of $\Box_{\mathrm{AdS}^3}$ with eigenvalues $\lambda_m = 4m(m-1)$. We use the decomposition of the open subset $\{Q > 0\}$ of the flat pseudo-Riemannian manifold $\mathbb{R}^{2,2} = (\mathbb{R}^4, Q(dx))$

$$\begin{array}{ccc} \{Q > 0\} & \xrightarrow{\cong} & \mathbb{R}_{>0} \times \mathbb{H}^{2,1} \\ x & \longmapsto & (\sqrt{Q(x)}, x/\sqrt{Q(x)}). \end{array}$$

Let $r = \sqrt{Q(x)}$. As in [11, p. 215],

$$-r^2 \Box_{\mathbb{R}^{2,2}} = -\left(r\frac{\partial}{\partial r}\right)^2 - 2r\frac{\partial}{\partial r} + \Box_{\mathbb{H}^{2,1}}.$$

Let m be a positive integer and k be a non-negative integer. In the coordinates (3.4), the function $z_1^{-(k+2m)} z_2^k$ is invariant under $(z_1, z_2) \mapsto (-z_1, -z_2)$, hence defines a real analytic function on AdS^3, to be denoted by $\psi_{m,k}$. Then we have

$$\psi_{m,k} \in L^2(\mathrm{AdS}^3),\ \Box_{\mathrm{AdS}^3} \psi_{m,k} = \lambda_m \psi_{m,k}.$$

Discrete spectrum $\mathrm{Spec}_d(\Box_{\mathrm{AdS}^3})$ coincides with $\{\lambda_m \mid m \in \mathbb{N}\}$ and $L^2_{\lambda_m}(\mathrm{AdS}^3)$ is generated by $\psi_{m,0}$ and its complex conjugate $\overline{\psi_{m,0}}$ as a representation of $\mathrm{SO}_0(2,2)$. By (3.5), for $x = k(\theta_1)a(t)k(\theta_2) \in \mathbb{H}^{2,1}$, we have

$$\psi_{m,k}(\pi(x)) = e^{-2\sqrt{-1}(m\theta_1 + (m+k)\theta_2)} \tanh^k t \cosh^{-2m} t. \tag{3.6}$$

We refer to $\psi_{m,k}$ as a spherical function of type $(-m, m+k)$ in accordance with the action of $\mathrm{SO}(2) \times \mathrm{SO}(2)$.

3.2.3 Convergence of generalized Poincaré series

In this subsection, we explain the fact about discrete spectrum of locally symmetric spaces by Kassel-Kobayashi [11] in our AdS^3 setting. We use the following notation.

Notation 3.6.
- *Let* $`G = \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ *and* $G = `G \times `G$.
- *Let* $`K = \mathrm{PSO}(2) = \mathrm{SO}(2)/\{\pm 1\}$ *and* $K = `K \times `K$.
- *Let E and $`E$ be respectively the identity elements of $G$ and $`G$.*

Remark 3.7. *The double covering* $\mathrm{SO}_0(2,2) \to G$ *induces an isomorphism* $\mathrm{AdS}^3 \cong G/\mathrm{diag}`G (\cong `G)$. *From now on, we consider only discontinuous groups* Γ *for* AdS^3 *which are discrete subgroups of G. This is enough for our purpose.*

In order to study $\mathrm{Spec}_d(\Box_{\Gamma\backslash\mathrm{AdS}^3})$, Kassel-Kobayashi [11] considered the convergence and non-vanishing of generalized Poincaré series

$$\varphi^\Gamma(\Gamma x) := \sum_{\gamma\in\Gamma} \varphi(\gamma^{-1}x) \tag{3.7}$$

for K-finite square-integrable eigenfunctions φ of $\Box_{\mathrm{AdS}^3}$. For this, they used an analytic estimate of φ and a geometric estimate of the number of Γ-orbits

$$N_\Gamma(x,R) := \#\{\gamma \in \Gamma \mid \gamma x \in B(R)\} \tag{3.8}$$

in the pseudo-ball $B(R)$ for $R > 0$. Since the Γ-action is properly discontinuous and $B(R)$ is compact, we have $N_\Gamma(x,R) < \infty$.

The convergence of generalized Poincaré series is proved by [11] as follows. For $g \in G$ and a function f on AdS^3, $\ell_g^* f$ is defined by $\ell_g^* f(x) = f(g^{-1}x)$.

Fact 3.8 (Kassel-Kobayashi [11])**.** *Let* $\Gamma \subset G$ *be a discontinuous group for* AdS^3 *satisfying the exponential growth condition*

$$\exists A, a > 0, \forall x \in \mathrm{AdS}^3, \forall R > 0, N_\Gamma(x,R) < Ae^{aR}. \tag{3.9}$$

Then, for any K-finite eigenfunction φ of $\Box_{\mathrm{AdS}^3}$ *with eigenvalue* λ_m *and any* $g \in G$*, if* $m > a$*, then* $(\ell_g^*\varphi)^\Gamma$ *(see (3.7)) is continuous and square-integrable on* $\Gamma\backslash\mathrm{AdS}^3$ *and an eigenfunction of* $\Box_{\Gamma\backslash\mathrm{AdS}^3}$ *with eigenvalue* λ_m.

Remark 3.9. *(1) Fact 3.8 does not assert the non-vanishing of the series* $(\ell_g^*\varphi)^\Gamma$ *which is more involved. Kassel-Kobayashi [11] proved that there exists* $g \in G$ *such that* $(\ell_g^*\psi_{m,0})^\Gamma \neq 0$ *for sufficiently large* $m \in \mathbb{N}$.

(2) By [11, Lem. 4.6.4], if a discontinuous group Γ is sharp in the sense of [11, Def. 4.2], then Γ satisfies the exponential growth condition (3.9). Moreover, Kassel [8] and Guériataud-Kassel [7] proved that finitely generated discontinuous groups for AdS^3 *are always sharp (see Fact 2.21 below).*

The conclusion of Fact 3.8 still holds if we drop the assumption that Γ acts freely on $X = \mathrm{AdS}^3$. In this case, the quotient space $X_\Gamma = \Gamma\backslash X$ is an orbifold. To formulate more precisely in the orbifold case, we observe that the quotient space X_Γ is Hausdorff, and carries a natural Radon measure (see e.g. [2, Chap. VII, §2, No. 2, Prop. 4]). A continuous function g on X_Γ is *smooth* if the pull-back $p_\Gamma^* g$ is a smooth function on X where $p_\Gamma \colon X \to X_\Gamma$ is the natural quotient map. We write $C_c^\infty(X_\Gamma)$ for the set of smooth functions on X_Γ with compact support. For $g \in C_c^\infty(X_\Gamma)$, we define $\Box_{X_\Gamma} g \in C_c^\infty(X_\Gamma)$ by identifying it with the Γ-invariant function $\Box_X(p_\Gamma^* g)$. For $\lambda \in \mathbb{C}$, we define

$$L^2_\lambda(X_\Gamma) := \{f \in L^2(X_\Gamma) \mid \forall g \in C_c^\infty(X_\Gamma), \langle f, \Box_{X_\Gamma} g\rangle_{X_\Gamma} = \lambda \langle f, g\rangle_{X_\Gamma}\}.$$

Discrete spectrum $\mathrm{Spec}_d(\Box_{X_\Gamma})$ and its multiplicity $\mathcal{N}_{X_\Gamma}$ are defined similarly to the case where Γ acts also freely.

3.2.4 "Injectivity radii" of anti-de Sitter 3-manifolds

Let Γ be a discontinuous group for AdS^3. In this subsection, we give a uniform estimate of the pseudo-distance between the origin and the second closest point of each Γ-orbit.

We recall that $\Gamma(\subset `G \times `G)$ acts isometrically on $\mathrm{AdS}^3 (\cong `G)$ by $(\gamma_1, \gamma_2)x = \gamma_1 x \gamma_2^{-1}$ for $(\gamma_1, \gamma_2) \in \Gamma$ and $x \in `G$. We set

$$\varepsilon_\Gamma := \inf_{(\gamma_1,\gamma_2)\in\Gamma\setminus\{E\}} \frac{1}{3} \left| \|\gamma_1\| - \|\gamma_2\| \right|. \tag{3.10}$$

By the inequality (see Lemma 2.43)

$$\|(g_1, g_2)x\| \geq \left| \|g_1\| - \|g_2\| \right| - \|x\| \text{ for } (g_1, g_2) \in G \text{ and } x \in \mathrm{AdS}^3,$$

we get:

Lemma 3.10. *If $\varepsilon_\Gamma > 0$, then $\gamma B(\varepsilon_\Gamma) \cap B(\varepsilon_\Gamma) = \emptyset$ for all $\gamma \in \Gamma \setminus \{E\}$.*

Proposition 3.11. *Let Γ be a discrete subgroup of G acting properly discontinuously on AdS^3. Then there exists $g \in G$ satisfying $\varepsilon_{g^{-1}\Gamma g} > 0$.*

Remark 3.12. *One sees in the proof below that the set of such g is dense in G.*

Proposition 3.11 follows obviously from the proper discontinuity of the Γ-action and the following lemma:

Lemma 3.13. *For any countable subset Γ of G, there exists $g \in G$ such that $\|\gamma_1\| \neq \|\gamma_2\|$ for all $\gamma = (\gamma_1, \gamma_2) \in g^{-1}\Gamma g \setminus \{E\}$.*

Proof of Lemma 3.13. For $\gamma \in \Gamma$, the map $f_\gamma \colon G \to G$ defined by $g \mapsto g^{-1}\gamma g$ is real analytic. For the analytic subset $F = \{(g_1, g_2) \in G \mid \|g_1\| = \|g_2\|\}$ of G, the set $f_\gamma^{-1}(F)$ is a proper subset of G if $\gamma \neq E$. Indeed, if $\|\gamma_1\| \neq \|\gamma_2\|$, then obviously $E \notin f_\gamma^{-1}(F)$. If $\|\gamma_1\| = \|\gamma_2\|$, then $(\gamma_1, \gamma_2) \neq (`E, `E)$. Without loss of generality, we may assume $\gamma_1 \neq `E$. Then there exists $g_1 \in `G$ satisfying $\|g_1^{-1}\gamma_1 g_1\| \neq \|\gamma_1\|$ as one can find g_1 depending on the three cases where γ_1 is hyperbolic, parabolic, or elliptic. Hence $(g_1, `E) \notin f_\gamma^{-1}(F)$.

Therefore there is no interior point in the analytic set $f_\gamma^{-1}(F)$, so is the countable union $\bigcup_{\gamma\in\Gamma\setminus\{E\}} f_\gamma^{-1}(F)$ by the Baire category theorem (see e.g. [21, Thm. 2.2]). Hence there exists an element g of $G \setminus \bigcup_{\gamma\in\Gamma\setminus\{E\}} f_\gamma^{-1}(F)$ and we have $\|\gamma_1\| \neq \|\gamma_2\|$ for all $\gamma = (\gamma_1, \gamma_2) \in g^{-1}\Gamma g \setminus \{E\}$. $\Box$

3.3 Proof of Theorem 3.1

In this section, we prove Theorem 3.1.

More generally, without finitely generated assumption of Γ, we study linear independence of the generalized Poincaré series of the spherical functions $\psi_{m,k}$ of type $(-m, m+k)$ defined in Section 3.2.2. By choosing $k = 3^j$ $(j = 0, 1, 2, \ldots)$, we prove:

Theorem 3.14. *If Γ is a discontinuous group for AdS^3 satisfying the exponential growth condition (3.9), then*

$$\lim_{m\to\infty} \mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) = \infty.$$

Theorem 3.1 is a direct consequence of Theorem 3.14 by Remark 3.9 (2).

Proposition 3.15. *Let Γ be a discrete subgroup of G acting properly discontinuously on AdS^3 and satisfying the exponential growth condition (3.9). If $\varepsilon_\Gamma > 0$, then there exists a real number $m_\Gamma(k)$ (given explicitly by (3.13)) for $k \in \mathbb{N}$ such that $\{(\mathrm{Re}(\psi_{m,3^j}))^\Gamma\}_{j=0}^{k-1} \subset L^2_{\lambda_m}(\Gamma\backslash\mathrm{AdS}^3)$ are linearly independent for all integers $m > m_\Gamma(k)$.*

Proof of Theorem 3.14. We have an obvious equality of the multiplicity of L^2-eigenvalues, $\mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3} = \mathcal{N}_{(g^{-1}\Gamma g)\backslash\mathrm{AdS}^3}$ for any $g \in G$ through the natural isomorphism $\Gamma\backslash\mathrm{AdS}^3 \cong (g^{-1}\Gamma g)\backslash\mathrm{AdS}^3$ as Lorentzian manifolds. By replacing Γ with $g^{-1}\Gamma g$ if necessary, we may and do assume $\varepsilon_\Gamma > 0$ by Proposition 3.11. Then Proposition 3.15 implies that $L^2_{\lambda_m}(\Gamma\backslash\mathrm{AdS}^3)$ contains at least k linearly independent elements if $m > m_\Gamma(k)$ for any fixed $k \in \mathbb{N}$, which means $\dim_{\mathbb{C}} L^2_{\lambda_m}(\Gamma\backslash\mathrm{AdS}^3) \geq k$. Hence Theorem 3.14 follows. □

Kassel-Kobayashi [11] proved the non-vanishing of the generalized Poincaré series $(\psi_{m,0})^\Gamma$ for sufficiently large $m \in \mathbb{N}$ by showing that the first term in the generalized Poincaré series is larger at the origin than the sum of the remaining terms. For this, they utilized the fact that $\psi_{m,0}(`E) = 1$. Our strategy for the proof of Proposition 3.15 is along the same line, however, there are some technical difficulties since $\psi_{m,k}$ for $k \geq 1$ vanishes at the origin. We then make use of an observation that $\psi_{m,k}$ decays more slowly at the origin than at infinity, to be precise, by the following formula, see (3.6):

$$|\psi_{m,k}(x)| = \cosh^{-2m}(\|x\|/2)\tanh^k(\|x\|/2).$$

Actually, we use an analytic lemma (Lemma 3.16) to prove that the first term in the generalized Poincaré series $(\psi_{m,k})^\Gamma$ is larger at points sufficiently close to the origin than the sum of the remaining terms if $m \gg 0$. Moreover, we use a combinatorial lemma (Lemma 3.17) to find points at which leading terms of $(\mathrm{Re}(\psi_{m,k}))^\Gamma$ do not cancel each other for any linear combination.

For $C, a, \varepsilon > 0$ and $s \in \mathbb{N}$, we set

$$m(C, a, \varepsilon, s) := \frac{(\log 2)s + 2a\varepsilon + \log(1 + 2^s C e^{6a\varepsilon})}{\log\cosh\varepsilon} \tag{3.11}$$

and

$$\tilde{m}(C, a, \delta, s) := \inf_{0<\varepsilon<\delta} m(C, a, \varepsilon, s).$$

Note that $\tilde{m}(C, a, \delta, s) = O(\delta^{-2})$ as $\delta \to 0$ and $= O(1)$ as $\delta \to \infty$.

Lemma 3.16. *For any integer $m > m(C, a, \varepsilon, s)$ and any one-variable polynomial f of degree $\leq s$ with non-negative coefficients,*

$$C \sum_{n=1}^{\infty} e^{4a(n+1)\varepsilon} (\cosh 2n\varepsilon)^{-m} f(\tanh 2(n+1)\varepsilon) < (\cosh \varepsilon)^{-m} f(\tanh \varepsilon).$$

Proof. We may assume that $f(x) = x^j$ for $j = 0, 1, \ldots, s$. Since

$$1 \leq \frac{\tanh nx}{\tanh x} \leq n, \ (\cosh x)^n \leq \cosh nx$$

for $x \in \mathbb{R}$, we have

$$\begin{aligned}(\mathrm{LHS})/(\mathrm{RHS}) &= C \sum_{n=1}^{\infty} e^{4a(n+1)\varepsilon} \left(\frac{\cosh 2n\varepsilon}{\cosh \varepsilon} \right)^{-m} \left(\frac{\tanh 2(n+1)\varepsilon}{\tanh \varepsilon} \right)^{j} \\ &\leq C e^{6a\varepsilon} \sum_{n=1}^{\infty} (e^{2a\varepsilon} (\cosh \varepsilon)^{-m})^{2n-1} (2(n+1))^s.\end{aligned}$$

We set $d := e^{2a\varepsilon}(\cosh \varepsilon)^{-m}$. Then $d < 1$ by $m > m(C, a, \varepsilon, s)$. Since $n+1 \leq 2^n$ for all $n \in \mathbb{N}$, we have

$$(\mathrm{LHS})/(\mathrm{RHS}) \leq 2^s C e^{6a\varepsilon} \sum_{n=1}^{\infty} (2^s d)^n = 2^s C e^{6a\varepsilon} \frac{2^s d}{1 - 2^s d}.$$

Again by $m > m(C, a, \varepsilon, s)$, we have $2^s d < (1 + 2^s C e^{6a\varepsilon})^{-1}$. Therefore we obtain $(\mathrm{LHS})/(\mathrm{RHS}) < 1$. □

Let $\chi \colon \{\pm 1\} \to \{0, 1\}$ be the map defined by $\chi(1) = 0$ and $\chi(-1) = 1$. For $a = (a_j)_{j=0}^{k-1} \in \{\pm 1\}^k$ and an odd integer $N \geq 3$, we set

$$\theta_{a,N} := \pi \sum_{i=0}^{k-1} (\chi(a_i) - \chi(a_{i-1})) N^{-i}. \tag{3.12}$$

Here we use the convention $a_{-1} = 1$.

Lemma 3.17. *For any $a = (a_0, \ldots, a_{k-1}) \in \{\pm 1\}^k$ and any odd integer N, we have*

$$a_j \cos(N^j \theta_{a,N}) > 0 \quad \text{for } j = 0, 1, \ldots, k-1.$$

Proof. Since $N^{k-1}\theta_{a,N} \equiv \pi\chi(a_{k-1}) \pmod{2\pi}$, we have $\cos(N^{k-1}\theta_{a,N}) = a_{k-1}$. It is easy to check that $|N^j \theta_{a,N} - N^j \theta_{(a_0, \cdots, a_j), N}| < \pi/2$ for $j = 0, 1, \ldots, k-1$, hence the signature of $\cos(N^j \theta_{a,N})$ is equal to that of $\cos(N^j \theta_{(a_0, \cdots, a_j), N}) = a_j$. □

Remark 3.18. *We have used the geometric progression $(N^j)_{j=0}^{k-1}$ in Lemma 3.17. On the other hand, an analogous statement does not hold if we use arithmetic progressions. For example, there does not exist $\theta \in \mathbb{R}$ satisfying $a_j \cos m_j \theta > 0$ for all $j = 0, 1, 2, 3, 4$ if we choose $(a_j)_{j=0}^4 = (1, 1, 1, -1, 1)$ and an arithmetic progression $(m_j)_{j=0}^4$.*

For a discontinuous group Γ and $k \in \mathbb{N}$, one can take $m_\Gamma(k)$ in Proposition 3.13 by

$$m_\Gamma(k) = \inf_{(A,a)\in \mathrm{C}_{\exp}(\Gamma)} \max\{\tilde{m}(3^{k-1}A, a, \varepsilon_\Gamma/4, 3^{k-1})/2, a\}, \tag{3.13}$$

where $\mathrm{C}_{\exp}(\Gamma) := \{(A,a) \in \mathbb{R}^2 \mid \forall x \in \mathrm{AdS}^3, \forall R > 0, N_\Gamma(x,R) < Ae^{aR}\}$. Here, we adopt the convention that $\inf_\emptyset f = \infty$ for a real-valued function f. In particular, $m_\Gamma(k) = \infty$ when $\mathrm{C}_{\exp}(\Gamma) = \emptyset$ or $\varepsilon_\Gamma = 0$.

Proof of Proposition 3.15. By the exponential growth condition (3.9), $\mathrm{C}_{\exp}(\Gamma) \neq \emptyset$ and thus $m_\Gamma(k) < \infty$. We take an integer $m > m_\Gamma(k)$. Then there exist ε with $0 < \varepsilon < \varepsilon_\Gamma/4$ and $(A,a) \in \mathrm{C}_{\exp}(\Gamma)$ satisfying the inequality $m > \max\{m(3^{k-1}A, a, \varepsilon, 3^{k-1})/2, a\}$.

To see $\mathbb{C}$-linear independence of the real-valued functions $\{(\mathrm{Re}(\psi_{m,3^j}))^\Gamma\}_{j=0}^{k-1}$, it is enough to prove the non-vanishing of the real part $\mathrm{Re}(\psi_{m,b}^\Gamma) = (\mathrm{Re}(\psi_{m,b}))^\Gamma$ of the generalized Poincaré series of a linear combination

$$\psi_{m,b} := \sum_{j=0}^{k-1} b_j \psi_{m,3^j}$$

for any $b = (b_0, b_1, \ldots, b_{k-1}) \in \mathbb{R}^k \setminus \{0\}$. By Lemma 3.10, for $x \in B(4\varepsilon)$, we have

$$\psi_{m,b}^\Gamma(\Gamma x) = \psi_{m,b}(x) + \sum_{\substack{\gamma\in\Gamma \\ \|\gamma^{-1}x\|>4\varepsilon}} \psi_{m,b}(\gamma^{-1}x). \tag{3.14}$$

By (3.6), for any $y \in \mathrm{AdS}^3$, we get

$$|\psi_{m,b}(y)| \leq \left(\cosh\frac{\|y\|}{2}\right)^{-2m} \sum_{j=0}^{k-1} |b_j| \left(\tanh\frac{\|y\|}{2}\right)^{3^j}.$$

We define $a = (a_j)_{j=0}^{k-1}$ by $a_j = 1$ for $b_j \geq 0$ and $a_j = -1$ for $b_j < 0$, and set

$$f_b(u) := \sum_{j=0}^{k-1} b_j \cos(3^j\theta_{a,3}) u^{3^j}.$$

We note that all the coefficients of f_b are non-negative by Lemma 3.17. Moreover, we get $\left|\cos(3^j\theta_{a,3})\right|^{-1} \leq 3^{k-1}$ for all $j = 0, 1, \ldots, k-1$ by using the inequality $\sin(\pi x/2) \geq x$ for $0 \leq x \leq 1$. Thus

$$|\psi_{m,b}(y)| \leq 3^{k-1} \left(\cosh\frac{\|y\|}{2}\right)^{-2m} f_b\left(\tanh\frac{\|y\|}{2}\right)$$

and, for any $x \in B(4\varepsilon)$, we have

$$
\begin{aligned}
|\sum_{\substack{\gamma\in\Gamma \\ \|\gamma^{-1}x\|>4\varepsilon}} \operatorname{Re}(\psi_{m,b}(\gamma^{-1}x))| &\leq \sum_{n=1}^{\infty} \sum_{\substack{\gamma\in\Gamma \\ 4\varepsilon n<\|\gamma^{-1}x\|\leq 4\varepsilon(n+1)}} |\psi_{m,b}(\gamma^{-1}x)| \\
&\leq 3^{k-1} \sum_{n=1}^{\infty} N_{\Gamma}(x, 4\varepsilon(n+1)) \left(\cosh 2\varepsilon n\right)^{-2m} f_b\left(\tanh 2\varepsilon(n+1)\right) \\
&\leq 3^{k-1} A \sum_{n=1}^{\infty} e^{4a\varepsilon(n+1)} \left(\cosh 2\varepsilon n\right)^{-2m} f_b\left(\tanh 2\varepsilon(n+1)\right) \\
&< \left(\cosh \varepsilon\right)^{-2m} f_b\left(\tanh \varepsilon\right). \qquad (3.15)
\end{aligned}
$$

The third and forth inequalities respectively follow from the exponential growth condition (3.9) and Lemma 3.16. On the other hand, we set

$$x_{a,\varepsilon} := k(\frac{\theta_{a,3}}{2})a(\varepsilon)k(\frac{\theta_{a,3}}{2})^{-1} \in B(4\varepsilon).$$

Then it follows from (3.6) that

$$\operatorname{Re} \psi_{m,b}(x_{a,\varepsilon}) = \left(\cosh \varepsilon\right)^{-2m} f_b\left(\tanh \varepsilon\right). \qquad (3.16)$$

By (3.14), (3.15), and (3.16), we obtain $(\operatorname{Re}(\psi_{m,b}))^{\Gamma}(\Gamma x_{a,\varepsilon}) \neq 0$. Hence we complete the proof by the continuity of $\psi^{\Gamma}_{m,b}$ (Fact 3.8). $\square$

3.4 Proof of Theorem 3.4

In this section, we prove Theorem 3.4 by applying Proposition 3.15. We work in the following setting. We allow Δ to have torsion.

Setting 3.19.

- *Δ is a discrete subgroup of $\grave{G} = \mathrm{PSL}(2, \mathbb{R})$.*
- *$j, \rho \colon \Delta \to \grave{G}$ are two group homomorphisms with j injective and discrete.*
- *$\Delta^{j,\rho}$ is a discrete subgroup of $G = \grave{G} \times \grave{G}$ given by $\{(j(\gamma), \rho(\gamma)) \mid \gamma \in \Delta\}$.*

We use the following structural results of discontinuous groups for the proof of Theorem 3.4.

Fact 3.20 ([11, Lem. 9.2])**.** *Let Γ be a finitely generated discrete subgroup of G acting properly discontinuously on AdS^3. Then Γ is of either type (i) or (ii) as follows:*

type (i) *Γ is of the form $\Delta^{j,\rho}$ up to switching the two factors.*

type (ii) *Γ is contained in a conjugate of $\grave{G} \times \grave{K}$ or $\grave{K} \times \grave{G}$.*

A non-elementary discrete subgroup Γ of a connected linear real reductive Lie group L of real rank 1 is called *convex cocompact* if Γ acts cocompactly on the convex hull of its limit set in the Riemannian symmetric space associated to L. For example, cocompact lattices and Schottky groups are convex cocompact. More generally, one may think of the notion of convex cocompactness of discontinuous groups for AdS^3:

Definition 3.21 ([11, Def. 9.1])**.** *A discontinuous group Γ for AdS^3 is called convex cocompact if Γ is of the form $\Delta^{j,\rho}$ up to finite index and switching the two factors, where Δ is torsion-free and $j(\Delta)$ is convex cocompact in ${}^{\backprime}G$.*

We note that a discontinuous group $\Delta^{j,\rho}$ acts cocompactly on AdS^3 if and only if $j(\Delta)$ is cocompact in ${}^{\backprime}G$ because $\Delta^{j,\rho}$ is isomorphic to $j(\Delta)$ as abstract groups. By Fact 3.20, discontinuous groups acting cocompactly on AdS^3 are convex cocompact.

3.4.1 Proof of Theorem 3.4 for Γ of type (i)

In this subsection, we prove Theorem 3.4 for Γ of type (i). For this, we use the constant $C_{Lip}(j,\rho)$ introduced by Kassel [8] and Guéritaud-Kassel [7], which quantifies the properness of the action of $\Delta^{j,\rho}$ on AdS^3.

Definition 3.22. *Let $d_{\mathbb{H}^2}$ be the hyperbolic distance of the 2-dimensional hyperbolic space $\mathbb{H}^2(\cong {}^{\backprime}G/{}^{\backprime}K)$. In Setting 3.19, we denote by $C_{Lip}(j,\rho)$ the infimum of Lipschitz constants*

$$\mathrm{Lip}(f) = \sup_{y\neq y'} \frac{d_{\mathbb{H}^2}(f(y), f(y'))}{d_{\mathbb{H}^2}(y,y')}$$

of maps $f\colon \mathbb{H}^2 \to \mathbb{H}^2$ that are (j,ρ)-equivariant.

The map $(j,\rho)\mapsto C_{Lip}(j,\rho)$ is continuous over the set of $(j,\rho)\in \mathrm{Hom}(\Delta, {}^{\backprime}G)^2$ such that j is injective and $j(\Delta)$ is convex cocompact in ${}^{\backprime}G$ ([7, Prop. 1.5]).

Fact 3.23 ([8],[7])**.** *Assume that Δ is finitely generated. Then the action of $\Delta^{j,\rho}$ on AdS^3 is properly discontinuous if and only if $\min\{C_{Lip}(j,\rho), C_{Lip}(\rho,j)\} < 1$.*

Remark 3.24. *In the setting of Fact 3.23, if $C_{Lip}(\rho,j) < 1$, then ρ is injective and discrete. Moreover, if $j(\Delta)$ is convex cocompact, then so is $\rho(\Delta)$.*

Therefore, Theorem 3.4 for Γ of type (i) reduces to the following:

Theorem 3.25. *In Setting 3.19, we assume that Δ is finitely generated and that $C_{Lip}(j,\rho) < 1$. Then there exists a constant $\mu_1 > 0$ independent of j,ρ and Δ such that for any $m,k\in\mathbb{N}$ with $m > 3^k\mu_1(1-C_{Lip}(j,\rho))^{-2}$,*

$$\mathcal{N}_{\Delta^{j,\rho}\backslash \mathrm{AdS}^3}(\lambda_m) \geq k.$$

For the proof of Theorem 3.25, we need two results from Kassel-Kobayashi [11] applied to our setting $G = {}^{\backprime}G \times {}^{\backprime}G$. If a discontinuous group Γ satisfies the assumption of Fact 3.26 below, then it is $((1-\alpha)/2, 0)$-sharp in the sense of [11, Def. 4.2]. Hence we get the following by applying [11, Lem. 4.6.4]:

Fact 3.26 ([11])**.** *Let $\Gamma\subset G$ be a discontinuous group for AdS^3. We assume that there exists $0\leq\alpha<1$ such that $\|\gamma_2\|\leq\alpha\|\gamma_1\|$ or $\|\gamma_1\|\leq\alpha\|\gamma_2\|$ for any $(\gamma_1,\gamma_2)\in\Gamma$. Then there exists $c>0$ independent of α and Γ such that for any $x\in\mathrm{AdS}^3$ and any $R>0$,*

$$N_\Gamma(x,R) \leq \#(\Gamma\cap K)ce^{8R(1-\alpha)^{-1}}.$$

The following theorem traces back to the Kazhdan-Margulis theorem for discrete subgroups of semisimple groups.

Fact 3.27 ([11, Prop. 8.14]). *There exists a constant $r > 0$ satisfying the following property: for any discrete subgroup ${}^{\backprime}\Gamma$ of ${}^{\backprime}G$, there exists ${}^{\backprime}g \in {}^{\backprime}G$ such that $\|{}^{\backprime}\gamma\| \geq r$ for all ${}^{\backprime}\gamma \in {}^{\backprime}g^{-1}\,{}^{\backprime}\Gamma\,{}^{\backprime}g \setminus \{{}^{\backprime}E\}$.*

In the following, we use the upper half plane model $\{z = x + \sqrt{-1}y \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ equipped with the metric tensor $ds^2 = (dx^2 + dy^2)/y^2$ for the hyperbolic space $\mathbb{H}^2$. Then $\|{}^{\backprime}g\|$ is equal to the hyperbolic distance $d_{\mathbb{H}^2}({}^{\backprime}g\sqrt{-1}, \sqrt{-1})$ for ${}^{\backprime}g \in \mathrm{AdS}^3 \cong {}^{\backprime}G$ (see e.g. [7, (A.1)]).

Proof of Theorem 3.25. The idea of the proof is similar to [11, Thm. 9.9], however, we give a proof for the sake of completeness. By Fact 3.27, replacing j by some conjugate under ${}^{\backprime}G$, we may assume $\|j(\gamma)\| \geq r$ for any $\gamma \in \Delta \setminus \{{}^{\backprime}E\}$. In particular, $\Gamma \cap K = \{E\}$ for such j and for any ρ. We fix $\delta > 0$ such that

$$\alpha := C_{Lip}(j, \rho) + \delta < 1.$$

Then, replacing ρ by some conjugate under ${}^{\backprime}G$, we may assume

$$\|\rho(\gamma)\| \leq \alpha\|j(\gamma)\| \text{ for any } \gamma \in \Delta. \tag{3.17}$$

Indeed, by Definition 3.22, there exists a (j, ρ)-equivariant map $f_\delta \colon \mathbb{H}^2 \to \mathbb{H}^2$ satisfying $\mathrm{Lip}(f_\delta) < \alpha$. We take $g_\delta \in {}^{\backprime}G$ such that $g_\delta\sqrt{-1} = f_\delta(\sqrt{-1})$. Then, for any $\gamma \in \Delta$, we have

$$\|g_\delta^{-1}\rho(\gamma)g_\delta\| = d_{\mathbb{H}^2}(f_\delta(\sqrt{-1}), \rho(\gamma)f_\delta(\sqrt{-1})) < \alpha d_{\mathbb{H}^2}(\sqrt{-1}, j(\gamma)\sqrt{-1}) = \alpha\|j(\gamma)\|.$$

Hence (3.17) holds by replacing ρ with $g_\delta^{-1}\rho(\cdot)g_\delta$, and therefore we get

$$N_\Gamma(x, R) \leq ce^{8R(1-(C_{Lip}(j,\rho)+\delta))^{-1}}$$

by Fact 3.26. Then the constant ε_Γ in (3.10) has the following lower bound:

$$3\varepsilon_\Gamma = \inf_{\gamma \in \Delta\setminus\{{}^{\backprime}E\}} |\|j(\gamma)\| - \|\rho(\gamma)\|| \geq \inf_{\gamma \in \Delta\setminus\{{}^{\backprime}E\}} (1 - \alpha)\|j(\gamma)\| \geq r(1 - \alpha).$$

Note that $\log\cosh t = O(t^2)$ as $t \to 0$. By the explicit description (3.13) of $m_\Gamma(k)$, Theorem 3.25 follows from Proposition 3.15. ☐

3.4.2 Proof of Theorem 3.4 for Γ of type (ii)

In this subsection, we prove Theorem 3.4 for the case where Γ is standard. For this, we use the following fact by Kobayashi [16] and Kassel [9] applied to our AdS^3 setting, which gives the stability for properness under any small deformation of standard convex cocompact discontinuous groups.

Fact 3.28 ([9, Thm. 1.4]). *Let Γ be a convex cocompact discrete subgroup of ${}^{\backprime}G \times {}^{\backprime}K$. Then for any $\alpha, \beta > 0$, there exists a neighborhood $W \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion $\Gamma \subset G$ such that for any $\varphi \in W$,*

$$|\mu(\varphi(\gamma)) - \mu(\gamma)| \leq \begin{cases} \alpha\,|\mu(\gamma)| & \text{if } \gamma \in \Gamma \setminus K, \\ \beta & \text{if } \gamma \in \Gamma \cap K, \end{cases}$$

where $\mu(g_1, g_2) := (\|g_1\|, \|g_2\|) \in \mathbb{R}^2$ for $(g_1, g_2) \in G$, $\|\cdot\|$ is given in Definition 3.5, and $|(x_1, x_2)| := \sqrt{x_1^2 + x_2^2}$ for $(x_1, x_2) \in \mathbb{R}^2$.

We introduce the following terminology for the estimate of discrete spectrum since Γ is not necessarily torsion-free. Let $\mathrm{pr}_j \colon G = {}^\backprime G \times {}^\backprime G \to {}^\backprime G$ be the j-th projection ($j = 1, 2$).

Definition 3.29. *A discrete subgroup Γ of G is said to be standard of class n if $\mathrm{pr}_2(\Gamma)$ is bounded and the cyclic group $\Gamma_1 := \ker(\mathrm{pr}_1|_\Gamma)$ is of order n.*

Remark 3.30. *(1) If Γ is torsion-free, then it is of class 1.*

(2) If $\mathrm{pr}_2(\Gamma)$ is bounded for a discrete subgroup Γ of G, then the group $\mathrm{pr}_1(\Gamma)$ is discrete in ${}^\backprime G$. Moreover, if Γ is of class 1, then it is of the form $\Delta^{j,\rho}$ such that $\Delta = \mathrm{pr}_1(\Gamma)$ and $C_{Lip}(j,\rho) = 0$.

Let $r > 0$ be the constant in Fact 3.27. For an integer $n \geq 2$, we define a positive number η_n by

$$\cosh \eta_n := 1 + 2(\sinh \frac{r}{4} \sin \frac{\pi}{n})^2.$$

We get the following by easy computations:

Lemma 3.31. *By an abuse of notation, we regard $k(\theta)$, $a(t)$ in (3.3) as elements of ${}^\backprime G = \mathrm{PSL}(2,\mathbb{R})$. Then*

$$\|a(\frac{r}{8})^{-1} k(\frac{j\pi}{n}) a(\frac{r}{8})\| \geq \eta_n \quad \text{for } j = 1, \ldots, n-1.$$

We give a uniform estimate of ε_Γ in (3.10) and $N_\Gamma(x, R)$ in (3.8) for standard discrete subgroups Γ of class n after taking a conjugation of Γ.

Lemma 3.32. *Let Γ be a standard discrete subgroup of class $n \geq 2$. There exists $g \in G$ such that $\varepsilon_{g^{-1}\Gamma g} \geq \min\{\eta_n/3, r/6\}$ and $N_{g^{-1}\Gamma g}(x, R) < ce^{16R}$ for any $x \in \mathrm{AdS}^3$ and any $R > 0$.*

Proof. Let $\Gamma_1 = \ker(\mathrm{pr}_1|_\Gamma)$ as in Definition 3.29. Since Γ is of class n, the group $\mathrm{pr}_2(\Gamma_1)$ is generated by $k(\pi/n) \in {}^\backprime G = \mathrm{PSL}(2,\mathbb{R})$. We take ${}^\backprime g \in {}^\backprime G$ in Fact 3.27 applied to ${}^\backprime\Gamma = \mathrm{pr}_1(\Gamma)$ and set $g := ({}^\backprime g, a(r/8)) \in G$. Replacing Γ by $g^{-1}\Gamma g$, we get $\|\gamma_1\| \geq r$ for $(\gamma_1, \gamma_2) \in \Gamma \setminus \Gamma_1$ by Fact 3.27 and $\|\gamma_2\| \geq \eta_n$ for $(\gamma_1, \gamma_2) \in \Gamma_1 \setminus \{E\}$ by Lemma 3.31. Moreover, if $(\gamma_1, \gamma_2) \in \Gamma$, then $\|\gamma_2\| = \|a(r/8)^{-1} k a(r/8)\|$ for some $k \in {}^\backprime K$, hence $\|\gamma_2\| \leq r/2$ because $\|g_1 g_2\| \leq \|g_1\| + \|g_2\|$ for $g_1, g_2 \in {}^\backprime G$ and since $\|a(t)\| = 2t$ for $t \geq 0$ and $\|k\| = 0$ for $k \in {}^\backprime K$. To summarize,

$$\begin{cases} \|\gamma_2\| \leq \frac{r}{2} \leq \frac{\|\gamma_1\|}{2} & \text{if } (\gamma_1, \gamma_2) \in \Gamma \setminus \Gamma_1, \\ \|\gamma_2\| \geq \eta_n & \text{if } (\gamma_1, \gamma_2) \in \Gamma_1 \setminus \{E\}. \end{cases}$$

Then $\varepsilon_\Gamma \geq \min\{\eta_n/3, r/6\}$ and $\Gamma \cap K = \{E\}$. Moreover, $\|\gamma_1\| \leq \|\gamma_2\|/2$ or $\|\gamma_2\| \leq \|\gamma_1\|/2$ for any $(\gamma_1, \gamma_2) \in \Gamma$ and thus $N_\Gamma(x, R) < ce^{16R}$ for any $x \in \mathrm{AdS}^3$ and any $R > 0$ by Fact 3.26. $\square$

Theorem 3.33. *There exists a constant $\mu_n > 0$ depending only on n such that for any convex cocompact standard discrete subgroup Γ of class n and any $m, k \in \mathbb{N}$ with $m > 3^k \mu_n$,*

$$\widetilde{\mathcal{N}}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) \geq k.$$

Proof. If $n = 1$, then this follows from Theorem 3.25 since convex cocompact discontinuous groups are finitely generated, hence we assume that $n \geq 2$. In this case, we shall prove that Γ and its small deformation are standard of class n. When $n \geq 2$, the group $\Gamma_1 = \ker(\mathrm{pr}_1|_\Gamma)$ is a cyclic group of order n. By Fact 3.27, replacing Γ by some conjugate under ${}^{\backprime}G \times \{{}^{\backprime}E\}$, we may and do assume $\|\gamma_1\| \geq r$ for any $(\gamma_1, \gamma_2) \in \Gamma \setminus \Gamma_1$. By Fact 3.28, there exists a neighborhood W of the natural inclusion $\Gamma \subset G$ such that for any $\varphi \in W$, the restriction of φ to the finite subgroup Γ_1 is injective and the inequalities

$$\begin{cases} \|\varphi_1(\gamma)\| \geq \frac{1}{2}r, \ \|\varphi_2(\gamma)\| \leq \frac{1}{2}\|\varphi_1(\gamma)\| & \text{if } \gamma \in \Gamma \setminus \Gamma_1, \\ |\mu(\varphi(\gamma))| < \frac{1}{2}r & \text{if } \gamma \in \Gamma_1 \end{cases} \tag{3.18}$$

hold where $\varphi_i = \mathrm{pr}_i \circ \varphi$ for $i = 1, 2$. Then φ is injective and discrete.

We claim $\varphi_1(\Gamma_1)$ is trivial. Indeed, if there exists $\gamma \in \Gamma_1 \setminus \{E\}$ such that $\varphi_1(\gamma) \neq {}^{\backprime}E$, then the normalizer of $\varphi(\Gamma_1)$ in G is contained in ${}^{\backprime}K_1 \times {}^{\backprime}G$ where ${}^{\backprime}K_1$ is the maximal compact subgroup of ${}^{\backprime}G$ containing $\varphi_1(\Gamma_1)$. Hence $\varphi(\Gamma) \subset {}^{\backprime}K_1 \times {}^{\backprime}G$. By the inequalities (3.18), $\varphi(\Gamma)$ is finite, hence Γ is also finite. This contradicts the assumption that Γ is non-elementary. Thus $\varphi_1(\Gamma_1)$ is trivial and $\varphi_2(\Gamma_1)$ is non-trivial. Hence the normalizer of $\varphi(\Gamma_1)$ in G is contained in ${}^{\backprime}G \times {}^{\backprime}K_2$, where ${}^{\backprime}K_2$ is the maximal compact subgroup of ${}^{\backprime}G$ containing $\varphi_2(\Gamma_1)$. Therefore $\mathrm{pr}_2(\varphi(\Gamma))$ is bounded. Moreover $\varphi(\Gamma)_1 = \varphi(\Gamma_1)$ by the inequalities (3.18), hence the discrete subgroup $\varphi(\Gamma)$ is standard of class n. By the explicit description (3.13) of $m_\Gamma(k)$ and Lemma 3.32, Theorem 3.33 follows from Proposition 3.15. $\square$

Remark 3.34. *In the above proof, we have shown that a convex cocompact standard discrete subgroup Γ of class $n \geq 2$ and its small deformation are standard of class n. Therefore we obtain a stronger result that*

$$\widetilde{\mathcal{N}}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda_m) = \infty \tag{3.19}$$

for any convex cocompact standard discrete subgroup Γ of class $n \geq 2$ and any integer $m > 3\mu_n$ if the following statement holds: $\mathcal{N}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda_m) = \infty$ for any standard discrete subgroup Γ and any $m \in \mathbb{N}$ such that $\mathcal{N}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda_m) \geq 1$. The latter statement is discussed in [10] by using discretely decomposable blanching laws of unitary representations (cf. [12]).

Thus the proof of Theorem 3.4 is completed.

References

[1] Y. Benoist. Actions propres sur les espaces homogènes réductifs. *Ann. of Math. (2)*, 144(2):315–347, 1996.

[2] N. Bourbaki. *Integration. II. Chapters 7–9.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004. Translated from the 1963 and 1969 French originals by Sterling K. Berberian.

[3] E. Calabi and L. Markus. Relativistic space forms. *Ann. of Math. (2)*, 75:63–76, 1962.

[4] A. Eskin and C. McMullen. Mixing, counting, and equidistribution in Lie groups. *Duke Math. J.*, 71(1):181–209, 1993.

[5] J. Fox and R. S. Strichartz. Unexpected spectral asymptotics for wave equations on certain compact spacetimes. *J. Anal. Math.*, 136(1):209–251, 2018.

[6] W. M. Goldman. Nonstandard Lorentz space forms. *J. Differential Geom.*, 21(2):301–308, 1985.

[7] F. Guéritaud and F. Kassel. Maximally stretched laminations on geometrically finite hyperbolic manifolds. *Geom. Topol.*, 21(2):693–840, 2017.

[8] F. Kassel. Quotients compacts d'espaces homogènes réels ou p-adiques. PhD thesis, Université Paris-Sud, 2009.

[9] F. Kassel. Deformation of proper actions on reductive homogeneous spaces. *Math. Ann.*, 353(2):599–632, 2012.

[10] F. Kassel and T. Kobayashi. Analyticity of Poincaré series on standard non-Riemannian locally symmetric spaces. In preparation.

[11] F. Kassel and T. Kobayashi. Poincaré series for non-Riemannian locally symmetric spaces. *Adv. Math.*, 287:123–236, 2016.

[12] F. Kassel and T. Kobayashi. Spectral analysis on standard locally homogeneous spaces. arXiv:1912.12601v1, preprint.

[13] B. Klingler. Complétude des variétés lorentziennes à courbure constante. *Math. Ann.*, 306(2):353–370, 1996.

[14] T. Kobayashi. Proper action on a homogeneous space of reductive type. *Math. Ann.*, 285(2):249–263, 1989.

[15] T. Kobayashi. Criterion for proper actions on homogeneous spaces of reductive groups. *J. Lie Theory*, 6(2):147–163, 1996.

[16] T. Kobayashi. Deformation of compact Clifford-Klein forms of indefinite-Riemannian homogeneous manifolds. *Math. Ann.*, 310(3):395–409, 1998.

[17] T. Kobayashi. Discontinuous groups for non-Riemannian homogeneous spaces. In *Mathematics unlimited—2001 and beyond*, pages 723–747. Springer, Berlin, 2001.

[18] T. Kobayashi. Intrinsic sound of anti–de Sitter manifolds. In *Lie theory and its applications in physics*, volume 191 of *Springer Proc. Math. Stat.*, pages 83–99. Springer, Singapore, 2016.

[19] T. Kobayashi and S. Nasrin. Deformation of properly discontinuous actions of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$. *Internat. J. Math.*, 17(10):1175–1193, 2006.

[20] R. S. Kulkarni and F. Raymond. 3-dimensional Lorentz space-forms and Seifert fiber spaces. *J. Differential Geom.*, 21(2):231–268, 1985.

[21] W. Rudin. *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[22] F. Salein. Variétés anti-de Sitter de dimension 3 exotiques. *Ann. Inst. Fourier (Grenoble)*, 50(1):257–284, 2000.