博士論文

論文題目 Ray-Singer torsion and the Laplacians of the Rumin complex on lens spaces

(レンズ空間上の Ray-Singer 捩率と Rumin 複体のラプラシアン)

氏　　名 北岡　旦

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Preface

Let (M,H) be a compact contact manifold of dimension $2n+1$ and E be a flat vector bundle with a unimodular holonomy on M. Rumin [**14**] introduced a complex $(\mathcal{E}^\bullet(M,E), d_{\mathrm{R}}^\bullet)$, which is a subquotient of the de Rham complex of E. A specific feature of the complex is that the operator $D = d_{\mathrm{R}}^n \colon \mathcal{E}^n(M,E) \to \mathcal{E}^{n+1}(M,E)$ in 'middle degree' is second-order, while $d_{\mathrm{R}}^k \colon \mathcal{E}^k(M,E) \to \mathcal{E}^{k+1}(M,E)$ for $k \neq n$ are first order which are induced by the exterior derivatives. Let $a_k = 1/\max\{1, \sqrt{|n-k|}\}$. Then, $(\mathcal{E}^\bullet(M,E), d_{\mathcal{E}}^\bullet)$, where $d_{\mathcal{E}}^k = a_k d_{\mathrm{R}}^k$, is also a complex. We call $(\mathcal{E}^\bullet(M,E), d_{\mathcal{E}}^\bullet)$ the *Rumin complex*. In virtue of the rescaling, $d_{\mathcal{E}}^\bullet$ satisfies Kähler-type identities on Sasakian manifolds [**16**], which include the case of lens spaces.

The Rumin complex has two aspects. First, it is the Bernstein-Gelfand-Gelfand complex (BGG complex) of the twisted de Rham complex of a flat vector bundle with respect to contact manifolds (e.g. [**17**, §5.3], [**4**, §4]). The BGG complex is defined for parabolic geometry [**3**] and on filtrated Riemannian manifolds with some assumptions [**4**]. As a typical theorem, the cohomology of the BGG complex coincides with the cohomology of the de Rham complex of a flat vector bundle [**15**, Theorem 1], [**3**, Theorem 4.13], [**4**, Corollary 4.20]. This claim is a generalization of the result of the Rumin complex [**14**].

Second, the Rumin complex arises when we take the sub-Riemannian limit. One natural approach to sub-Riemannian geometry lies in the study of the behavior of Riemannian objects in the sub-Riemannian limit. On fibrations of compact manifolds Mazzeo and Melrose [**11**], and on Riemann foliations Forman [**6**] studied spectral sequence using Hodge theoretic techniques and they showed that a part of the spectral sequence can be written in terms of the BGG complex. On contact manifolds, Rumin pointed out [**16**] that the Rumin complex can be derived from a spectral sequence induced by Heisenberg dilations.

Rumin and Seshadri defined the analytic torsion associated with the Rumin complex d_{R}, which we call the Rumin-Seshadri torsion [**18**]. They showed that Rumin-Seshadri torsion agrees with the Ray-Singer torsion for flat bundles with unimodular holonomy on 3-dimensional Sasakian manifolds with S^1-action. With this coincidence, they found a relation between the Ray-Singer torsion and holonomy. It is natural to ask whether such a relation holds for higher dimensions.

In this thesis, we extend this coincidence to lens spaces of arbitrary dimension. First, we determine explicitly eigenvalues of the Laplacian $\Delta_{\mathcal{E}}$ of the Rumin complex with parameters on the trivial bundle over the standard CR spheres $S^{2n+1} \subset \mathbb{C}^{n+1}$, Theorem 2.1.1. In particular, we show that the eigenvalues of $\Delta_{\mathcal{E}}$ are determined by the highest weight of $\mathrm{U}(n+1)$ which acts on S^{2n+1}. This phenomenon also appears in the case of the Hodge-de Rham Laplacian Δ_{dR} on symmetric spaces G/K. Ikeda and Taniguchi [**7**] showed that on the subspaces of k-forms of G/K corresponding to the irreducible component which has the highest weight λ, the eigenvalue of Δ_{dR} is determined by λ [**7**]. Also, this phenomenon does not appear in the Laplacians of the Rumin complex with $a_k = 1$, §4.2.

Next, on flat vector bundles with a unimodular holonomy over lens spaces, we express explicitly the analytic torsion functions associated with the Rumin complex in terms of the Hurwitz zeta function, Theorem 3.1.1. In particular, we determine the analytic torsions. Moreover, we give a formula between this torsion and the Ray-Singer torsion. These were written in the papers [**9**, **10**].

Finally, on 3 and 5 dimensional CR spheres we calculate the analytic torsion associated with the Rumin complex with arbitrary parameters $\{a_k\}$. Here, let g_{std} be the standard metric on the spheres. Weng and You [**20**] showed that on the trivial bundle on the spheres $(S^{2n+1}, 4g_{\mathrm{std}})$ the Ray-Singer torsion is $(4\pi)^{n+1}/n!$. With the result of Rumin and Seshadri in dimensions, we expected that the Rumin-Seshadri torsion agrees with this values on spheres of higher dimensions. However, we showed that on the 5-dimensional CR standard sphere the Rumin-Seshadri torsion is $(4\pi)^3 2^{-5/4+\pi^2/18}$. The precise statement for general $\{a_k\}$ on S^3 and S^5 are given in Theorem 4.1.1.

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CHAPTER 1

The Rumin complex

We call (M, H) an orientable contact manifold of dimension $2n+1$ if H is a subbundle of TM of codimension 1 and there exists a 1-form θ, called a contact form, such that $\operatorname{Ker}(\theta\colon TM \to \mathbb{R}) = H$ and $\theta \wedge (d\theta)^n \neq 0$. The Reeb vector field of θ is the unique vector field T satisfying $\theta(T) = 1$ and $\operatorname{Int}_T d\theta = 0$, where Int_T is the interior product with respect to T.

For H and θ, we call $J \in \operatorname{End}(TM)$ an almost complex structure associated with θ if $J^2 = -\operatorname{Id}$ on H, $JT = 0$, and the Levi form $d\theta(-, J-)$ is positive definite on H. Given θ and J, we define a Riemannian metric $g_{\theta,J}$ on TM by

$$g_{\theta,J}(X,Y) := d\theta(X, JY) + \theta(X)\theta(Y) \qquad \text{for } X, Y \in TM.$$

Let $*$ be the Hodge star operator on $\wedge^\bullet T^*M$ with respect to $g_{\theta,J}$.

Let $\widetilde{M}$ be the universal cover of M and $\pi_1(M)$ be the fundamental group. For a unitary representation $\alpha\colon \pi_1(M) \to \mathrm{U}(r)$, we denote the flat vector bundle associated with α by

$$E_\alpha := \widetilde{M} \times_\alpha \mathbb{C}^r \to M.$$

Let ∇_α be the flat connection on E_α induced from the trivial connection on $\widetilde{M} \times \mathbb{C}^r$, and d^{∇_α} be the exterior covariant derivative of ∇_α.

The Rumin complex [**14**] is defined on contact manifolds as follows. We set $L := d\theta\wedge$ and $\Lambda := *^{-1}L*$, which is the adjoint operator of L with respect to the metric $g_{\theta,J}$ at each point. We set

$$\begin{aligned}
\textstyle\bigwedge^k_{\mathrm{prim}} H^* &:= \left\{ v \in \textstyle\bigwedge^k H^* \;\middle|\; \Lambda v = 0 \right\}, \\
\textstyle\bigwedge^k_L H^* &:= \left\{ v \in \textstyle\bigwedge^k H^* \;\middle|\; L v = 0 \right\}, \\
\mathcal{E}^k(M, E_\alpha) &:= \begin{cases} C^\infty\left(M, \textstyle\bigwedge^k_{\mathrm{prim}} H^* \otimes E_\alpha\right), & k \le n, \\ C^\infty\left(M, \theta \wedge \textstyle\bigwedge^{k-1}_L H^* \otimes E_\alpha\right), & k \ge n+1. \end{cases} \\
\mathcal{E}^k(M) &:= \mathcal{E}^k(M, \underline{\mathbb{C}}),
\end{aligned}$$

where $\underline{\mathbb{C}}$ is the trivial vector bundle. We embed H^* into T^*M as the subbundle $\{\phi \in T^*M \mid \phi(T) = 0\}$ so that we can regard

$$\Omega^k_H(M, E_\alpha) := C^\infty\left(M, \wedge^k H^* \otimes E_\alpha\right)$$

as a subspace of $\Omega^k(M, E_\alpha)$, the space of k-forms. We define $d_b\colon \Omega^k_H(M, E_\alpha) \to \Omega^{k+1}_H(M, E_\alpha)$ by

$$d_b\phi := d^{\nabla_\alpha}\phi - \theta \wedge (\operatorname{Int}_T d^{\nabla_\alpha}\phi),$$

and then $D\colon \mathcal{E}^n(M,E_\alpha)\to\mathcal{E}^{n+1}(M,E_\alpha)$ by

$$D=\theta\wedge(\mathcal{L}_T+d_bL^{-1}d_b), \tag{1.0.1}$$

where $\mathcal{L}_T$ is the Lie derivative with respect to T, and L^{-1} is the inverse of the isomorphism $L\colon \wedge^{n-1}H^*\to\wedge^{n+1}H^*$.

Let $P\colon \wedge^k H^*\to\wedge^k_{\mathrm{prim}}H^*$ be the fiberwise orthogonal projection with respect to $g_{\theta,J}$, which also defines a projection $P\colon \Omega^k(M,E_\alpha)\to\mathcal{E}^k(M,E_\alpha)$. We set

$$d_{\mathrm{R}}^k:=\begin{cases} P\circ d^{\nabla_\alpha} & \text{on } \mathcal{E}^k(M,E_\alpha),\quad k\le n-1,\\ D & \text{on } \mathcal{E}^n(M,E_\alpha),\\ d^{\nabla_\alpha} & \text{on } \mathcal{E}^k(M,E_\alpha),\quad k\ge n+1.\end{cases}$$

Then $(\mathcal{E}^\bullet(M,E_\alpha),d_{\mathrm{R}}^\bullet)$ is a complex. Let $d_{\mathcal{E}}^k=a_kd_{\mathrm{R}}^k$, where $a_k=1/\sqrt{|n-k|}$ for $k\neq n$ and $a_n=1$. We call $(\mathcal{E}^\bullet(M,E_\alpha),d_{\mathcal{E}}^\bullet)$ the Rumin complex.

We define the L^2-inner product on $\Omega^k(M,E_\alpha)$ by

$$(\phi,\psi):=\int_M g_{\theta,J}(\phi,\psi)d\,\mathrm{vol}_g$$

and the L^2-norm on $\Omega^k(M,E_\alpha)$ by $\|\phi\|:=\sqrt{(\phi,\phi)}$. Since the Hodge star operator $*$ induces a bundle isomorphism from $\wedge^k_{\mathrm{prim}}H^*$ to $\theta\wedge\wedge^{2n-k}_L H^*$, it also induces a map $\mathcal{E}^k(M,E_\alpha)\to\mathcal{E}^{2n+1-k}(M,E_\alpha)$. We note that

$$\mathcal{E}^k(M,E_\alpha)=\left\{\phi\in\mathcal{E}^k(\widetilde{M},\mathbb{C}^r)\,\middle|\, t_*\phi=\alpha(t)^{-1}\phi \text{ for } t\in\pi_1(M)\right\}$$

Let $d_{\mathcal{E}}^\#$ and $D^\#$ denote the formal adjoint of $d_{\mathcal{E}}$ and D, respectively, for the L^2-inner product. We define the forth-order Laplacian $\Delta_{\mathcal{E}}$ on $\mathcal{E}^k(M,E_\alpha)$ by

$$\Delta_{\mathcal{E}}^k:=\begin{cases}(d_{\mathcal{E}}^{k-1}d_{\mathcal{E}}^{k-1\#})^2+(d_{\mathcal{E}}^{k\#}d_{\mathcal{E}}^k)^2, & k\neq n,n+1,\\ (d_{\mathcal{E}}^{n-1}d_{\mathcal{E}}^{n-1\#})^2+D^\#D, & k=n,\\ DD^\#+(d_{\mathcal{E}}^{n+1\#}d_{\mathcal{E}}^{n+1})^2, & k=n+1.\end{cases}$$

We call it the Rumin Laplacian [**14**]. Rumin showed that $\Delta_{\mathcal{E}}$ have discrete eigenvalues with finite multiplicities. Since $*$ and $\Delta_{\mathcal{E}}$ commute, to determine the eigenvalue on $\mathcal{E}^\bullet(M,E_\alpha)$, it is sufficient to compute them on $\mathcal{E}^k(M,E_\alpha)$ for $k\le n$.

CHAPTER 2

The eigenvalue of the Rumin Laplacian on the standard CR sphere

2.1. Introduction

In this chapter, we determine the eigenvalues of $\Delta_{\mathcal{E}}$ on the trivial bundle $\underline{\mathbb{C}}$ over the standard CR spheres $S^{2n+1} \subset \mathbb{C}^{n+1}$. Here the *standard CR sphere* is the triple (S^{2n+1}, θ, J), where θ is given the contact form by $\theta = \sqrt{-1}(\bar{\partial} - \partial)|z|^2$ and J is an almost complex structure J induced from the complex structure of $\mathbb{C}^{n+1}$. To state our result we need to introduce notation for highest weight representations of the unitary group $\mathrm{U}(n+1)$ which acts on S^{2n+1}. The irreducible representations of $\mathrm{U}(n+1)$ are classified by the highest weights $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$; the corresponding representation will be denoted by $V(\lambda)$. Julg and Kasparov [**8**] showed that the complexification of $\mathcal{E}^k(S^{2n+1})$, as a $\mathrm{U}(n+1)$-module, is decomposed into the irreducible of the form

$$\Psi_{(q,j,i,p)} := V(q, \underbrace{1, \ldots, 1}_{j \text{ times}}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{i \text{ times}}, -p).$$

See Proposition 2.2.1 below for the relations between k and (q, i, j, p). Since $\Delta_{\mathcal{E}}$ commutes with the $\mathrm{U}(n+1)$-action, it acts as a scalar on each $\Psi_{(q,j,i,p)}$.

THEOREM 2.1.1. *Let S^{2n+1} be the standard CR sphere with the contact from $\theta = \sqrt{-1}(\bar{\partial} - \partial)|z|^2$. Then, on the subspaces of the complexification of $\mathcal{E}^\bullet$ corresponding to the representations $\Psi_{(q,j,i,p)}$, the eigenvalue of $\Delta_{\mathcal{E}}$ is*

$$\frac{((p+i)(q+n-i) + (q+j)(p+n-j))^2}{4(n-i-j)^2}.$$

This theorem claims that the eigenvalues of $\Delta_{\mathcal{E}}$ are determined by the highest weight. This phenomenon also appears in the case of the Hodge-de Rham Laplacian Δ_{dR} on symmetric spaces G/K. Ikeda and Taniguchi [**7**] showed that on the subspaces of k-forms of G/K corresponding to $V(\lambda)$, the eigenvalue of Δ_{dR} is determined by λ. It is a natural question to ask whether the eigenvalues of $\Delta_{\mathcal{E}}$ on a contact homogeneous space G/K are determined by the highest weight of G.

Theorem 2.1.1 unifies the following results on the eigenvalues of Rumin Laplacians on the spheres. Julg and Kasparov [**8**] determined the eigenvalues of $D^{\#}D$. Folland [**5**] calculated the eigenvalue of the sub-Laplacian Δ_b, which agrees with $\Delta_{\mathcal{E}}$ on $\mathcal{E}^0$. Seshadri [**19**] determined the eigenvalues of $d_{\mathcal{E}} d_{\mathcal{E}}^{\#}$ on $\mathcal{E}^1$ in the case S^3. Ørsted and Zhang [**12**] determined eigenvalues of the Laplacian of the holomorphic and anti-holomorphic part of d_{R} except for the ones containing D.

Note that Ørsted and Zhang used d_{R} in place of $d_{\mathcal{E}}$. As a result, the eigenvalues of the Laplacian in their paper are not determined by the highest weights. This also explains the importance the scaling factor a_k.

The chapter is organized as follows. In §2.2, we recall properties of the Rumin complex on S^{2n+1}. In §2.3.1, we construct highest weight vectors, and compute the actions of d_{R} and the Lie derivative $\mathcal{L}_T$ with respect to the Reeb vector field T on these vectors. In §2.3.2, we calculate the L^2-norm of them. Then, in §2.3.3, we compute the eigenvalues of $\Delta_{\mathcal{E}}$ for each irreducible component.

2.2. The Rumin complex on the CR spheres

Let $S := \{z \in \mathbb{C}^{n+1} \mid |z|^2 = 1\}$ and $\theta := \sqrt{-1}(\bar\partial - \partial)|z|^2$. (We will omit the dimension from S^{2n+1} for the simplicity of the notation.) Let g_{std} be the standard metric on S. Then, $g_{\theta,J}$ coincides with $4g_{\mathrm{std}}$. The Reeb vector filed of θ is

$$T = \frac{\sqrt{-1}}{2}\sum_{l=1}^{n+1}\left(z_l\frac{\partial}{\partial z_l} - \bar z_l\frac{\partial}{\partial \bar z_l}\right).$$

With respect to the standard almost complex structure J, we decompose the bundles defined in the previous subsection as follows:

$$\begin{aligned} H^{*1,0} &:= \{v \in \mathbb{C}H^* \mid Jv = \sqrt{-1}v\},\\ H^{*0,1} &:= \{v \in \mathbb{C}H^* \mid Jv = -\sqrt{-1}v\},\\ \textstyle\bigwedge^{i,j} H^* &:= \textstyle\bigwedge^{i} H^{*1,0} \otimes \bigwedge^{j} H^{*0,1},\\ \textstyle\bigwedge^{i,j}_{\mathrm{prim}} H^* &:= \left\{\phi \in \textstyle\bigwedge^{i,j} H^* \,\middle|\, \Lambda\phi = 0\right\},\\ \mathcal{E}^{i,j} &:= C^\infty\left(S, \textstyle\bigwedge^{i,j}_{\mathrm{prim}} H^*\right). \end{aligned}$$

Then $d_b\Omega_H^{i,j} \subset \Omega_H^{i+1,j} \oplus \Omega_H^{i,j+1}$. We define $\partial_b\colon \Omega_H^{i,j} \to \Omega_H^{i+1,j}$ and $\bar\partial_b\colon \Omega_H^{i,j} \to \Omega_H^{i,j+1}$ by

$$d_b = \partial_b + \bar\partial_b.$$

Similarly, we decompose

$$d_{\mathrm{R}} = \partial_{\mathrm{R}} + \bar\partial_{\mathrm{R}}, \quad d_{\mathcal{E}} = \partial_{\mathcal{E}} + \bar\partial_{\mathcal{E}}.$$

In view of the Lefschetz primitive decomposition, we may rewrite (1.0.1) as

$$D = \theta \wedge \left(\mathcal{L}_T - \sqrt{-1}(\partial_b + \bar\partial_b)(\partial_b^{\#} - \bar\partial_b^{\#})\right) \tag{2.2.1}$$

by using $\partial_b^{\#} = \sqrt{-1}[\Lambda, \bar\partial_b]$ and $\bar\partial_b^{\#} = -\sqrt{-1}[\Lambda, \partial_b]$. Note that this equation holds on Sasakian manifolds.

We decompose $\mathcal{E}^{i,j}$ into a direct sum of irreducible representations of the unitary group $\mathrm{U}(n+1)$. Recall that irreducible representations of $\mathrm{U}(m)$ are parametrized by the highest weight $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m$; the representation corresponding to λ will be denoted by $V(\lambda)$. To simplify the notation, we introduce the following abbreviation: for $a_1, \dots, a_l \in \mathbb{Z}$ and $k_1, \dots, k_l \in \mathbb{Z}$, $(\underline{a_1}_{k_1}, \cdots, \underline{a_l}_{k_l})$ denotes the $k_1 + \cdots + k_l$-tuple whose first k_1 entries are a_1, whose next k_2 entries are a_2, etc. For example,

$$(\underline{1}_3, \underline{0}_2, \underline{-1}_2) = (1, 1, 1, 0, 0, -1, -1).$$

We note that $\underline{a}_1$ is a and $\underline{a}_0$ is the zero tuple.

In [**8**], it is shown that the multiplicity of $V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p)$ in $\mathcal{E}^{s,t}$ is at most one. Thus we may set

$$\Psi^{(s,t)}_{(q,j,i,p)} := \mathcal{E}^{s,t} \cap V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p).$$

Proposition 2.2.1. ([**8**, Section 4(b)]) *The irreducible decomposition of the* $\mathrm{U}(n+1)$*-module* $\mathcal{E}^{i,j}$ *is given as follows:*
Case I :

$$\mathcal{E}^{0,0} = \bigoplus_{q\geq 0, p\geq 0} \Psi^{(0,0)}_{(q,0,0,p)}$$

Case II : *For* $i+j \leq n-1$ *with* $i, j > 0$,

$$\mathcal{E}^{i,j} = \bigoplus_{q\geq 1, p\geq 1} \left(\Psi^{(i,j)}_{(q,j,i,p)} \oplus \Psi^{(i,j)}_{(q,j,i-1,p)} \oplus \Psi^{(i,j)}_{(q,j-1,i,p)} \oplus \Psi^{(i,j)}_{(q,j-1,i-1,p)}\right).$$

Case III : *For* $1 \leq i \leq n-1$,

$$\mathcal{E}^{i,0} = \bigoplus_{q\geq 0, p\geq 1} \left(\Psi^{(i,0)}_{(q,0,i,p)} \oplus \Psi^{(i,0)}_{(q,0,i-1,p)}\right).$$

Case IV : *For* $1 \leq j \leq n-1$,

$$\mathcal{E}^{0,j} = \bigoplus_{q\geq 1, p\geq 0} \left(\Psi^{(0,j)}_{(q,j,0,p)} \oplus \Psi^{(0,j)}_{(q,j-1,0,p)}\right).$$

Case V : *For* $i+j = n$ *with* $i, j > 0$,

$$\mathcal{E}^{i,j} = \bigoplus_{q\geq 1, p\geq 1} \left(\Psi^{(i,j)}_{(q,j,i-1,p)} \oplus \Psi^{(i,j)}_{(q,j-1,i,p)} \oplus \Psi^{(i,j)}_{(q,j-1,i-1,p)}\right).$$

Case VI :

$$\mathcal{E}^{n,0} = \bigoplus_{q\geq -1, p\geq 1} \Psi^{(n,0)}_{(q,0,n-1,p)}.$$

Case VII :

$$\mathcal{E}^{0,n} = \bigoplus_{q\geq 1, p\geq -1} \Psi^{(0,n)}_{(q,n-1,0,p)}.$$

2.3. The eigenvalues of the Rumin Laplacian

2.3.1. The action of d_{R} and the Reeb vector field. Setting $\omega_i := dz_i - z_i\partial|z|^2$ and $\overline{\omega}_i := d\bar{z}_i - \bar{z}_i\bar{\partial}|z|^2$, we define differential forms

$$\begin{aligned}
\alpha^{(0,0)}_{(j,0)} &:= \sum_{\nu=1}^{j+1} (-1)^{\nu-1} \bar{z}_\nu \overline{\omega}_1 \wedge \cdots \widehat{\overline{\omega}_\nu} \cdots \wedge \overline{\omega}_{j+1},\\
\alpha^{(0,1)}_{(j,0)} &:= \overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_{j+1},\\
\alpha^{(0,0)}_{(0,i)} &:= \sum_{\mu=n-i+1}^{n+1} (-1)^{\mu-(n-i+1)} z_\mu \omega_{n-i+1} \wedge \cdots \widehat{\omega_\mu} \cdots \wedge \omega_{n+1},\\
\alpha^{(1,0)}_{(0,i)} &:= \omega_{n-i+1} \wedge \cdots \wedge \omega_{n+1}.
\end{aligned}$$

Following [**12**], we see that $\Psi^{(s,t)}_{(q,j,i,p)}$ contains the following element $\psi^{(s,t)}_{(q,j,i,p)}$: for $p, q \geq 1$, $i, j \geq 0$, $i+j \leq n-1$, $a, b \geq 0$ and $a+b \leq 1$,

$$\begin{aligned}
\psi^{(0,0)}_{(0,0,0,0)} &:= 1,\\
\psi^{(i+a,j+b)}_{(q,j,i,p)} &:= \bar{z}_1^{q-1} z_{n+1}^{p-1} \alpha^{(a,0)}_{(0,i)} \wedge \alpha^{(0,b)}_{(j,0)} / \sqrt{2\pi^{n+1}},\\
\psi^{(i+1,j+1)}_{(q,j,i,p)} &:= P\tilde{\psi}^{(i+1,j+1)}_{(q,j,i,p)},
\end{aligned}$$

where $\tilde{\psi}^{(i+1,j+1)}_{(q,j,i,p)} := \overline{z}_1^{q-1} z_{n+1}^{p-1} \alpha^{(1,0)}_{(0,i)} \wedge \alpha^{(0,1)}_{(j,0)} / \sqrt{2\pi^{n+1}}$,

$$\psi^{(0,j+b)}_{(q,j,0,0)} := \overline{z}_1^{q-1} \alpha^{(0,b)}_{(j,0)} / \sqrt{2\pi^{n+1}},$$
$$\psi^{(i+a,0)}_{(0,0,i,p)} := z_{n+1}^{p-1} \alpha^{(a,0)}_{(0,i)} / \sqrt{2\pi^{n+1}},$$
$$\psi^{(0,n)}_{(q,n-1,0,-1)} := \overline{z}_1^{q-1} \alpha^{(0,0)}_{(n,0)} / \sqrt{2\pi^{n+1}},$$
$$\psi^{(n,0)}_{(-1,0,n-1,p)} := z_{n+1}^{p-1} \alpha^{(0,0)}_{(0,n)} / \sqrt{2\pi^{n+1}}.$$

We have used the projection P in the definition of $\psi^{(i+1,j+1)}_{(q,j,i,p)}$. Let us calculate P explicitly (see also Remark 2.3.5 below). Since

$$2\Lambda(\omega_\mu \wedge \overline{\omega}_\nu) = -\sqrt{-1} z_\mu \overline{z}_\nu \quad \text{for } \mu \neq \nu,$$

we have

$$2\Lambda\left(\alpha^{(1,0)}_{(0,i)} \wedge \alpha^{(0,1)}_{(j,0)}\right) = \sqrt{-1}(-1)^{i+1} \alpha^{(0,0)}_{(0,i)} \wedge \alpha^{(0,0)}_{(j,0)}.$$

Thus

$$2\Lambda^2 \tilde{\psi}^{(i+1,j+1)}_{(q,j,i,p)} = \sqrt{-1}(-1)^{i+1} \Lambda \psi^{(i,j)}_{(q,j,i,p)} = 0.$$

By using the Lefschetz primitive decomposition, we get

$$P|_{\Psi^{(i+1,j+1)}_{(q,j,i,p)} \oplus L\Psi^{(i,j)}_{(q,j,i,p)}} = 1 - \frac{1}{n-i-j} L\Lambda. \tag{2.3.1}$$

Proof of (2.3.1). For $k+2 \leq n$, let $\phi = \phi_0 + L\phi_1 \in \Omega^{k+2}(M)$, where $\phi_0 \in C^\infty(M, \wedge^{k+2}_{\mathrm{prim}} H^*)$ and $\phi_1 \in C^\infty(M, \wedge^{k}_{\mathrm{prim}} H^*)$. Then we have

$$\Lambda\phi = \Lambda L \phi_1.$$

Since $[L, \Lambda] = k - n$ on $\Omega^k_H(M)$, we obtain

$$\Lambda\phi = -[L,\Lambda]\phi_1 = -(k-n)\phi_1,$$

and hence

$$L\Lambda\phi = (n-k) L\phi_1.$$

It means that $L\Lambda/(n-k)$ is the projection from
$C^\infty(M, \wedge^{k+2}_{\mathrm{prim}} H^*) \oplus L \cdot C^\infty(M, \wedge^{k}_{\mathrm{prim}} H^*)$ to the second component. Therefore, we obtain (2.3.1). □

Lemma 2.3.1. *If "$i+j \leq n-1$ and $p, q \geq 1$" or "$i \leq n-1$, $j = 0$, $p \geq 1$ and $q = 0$",*

$$\begin{cases} \partial_{\mathrm{R}} \psi^{(i,j)}_{(q,j,i,p)} &= (p+i) \psi^{(i+1,j)}_{(q,j,i,p)}, \\ \bar{\partial}_{\mathrm{R}} \psi^{(j,i)}_{(p,i,j,q)} &= (-1)^j (p+i) \psi^{(j,i+1)}_{(p,i,j,q)}. \end{cases} \tag{2.3.2}$$

If $i+j \leq n-2$, $p, q \geq 1$,

$$\begin{cases} \partial_{\mathrm{R}} \psi^{(i,j+1)}_{(q,j,i,p)} &= (p+i) \psi^{(i+1,j+1)}_{(q,j,i,p)}, \\ \bar{\partial}_{\mathrm{R}} \psi^{(j+1,i)}_{(p,i,j,q)} &= (-1)^{j+1} (p+i) \psi^{(j+1,i+1)}_{(p,i,j,q)}. \end{cases} \tag{2.3.3}$$

Otherwise, $\partial_{\mathrm{R}} \psi^{(s,t)}_{(q,j,i,p)} = 0$ and $\bar{\partial}_{\mathrm{R}} \psi^{(s,t)}_{(p,i,j,q)} = 0$.

Remark 2.3.2. *Since $\Lambda \partial_b \psi^{(i,j)}_{(q,j,i,p)} = 0$ and $\Lambda \bar{\partial}_b \psi^{(i,j)}_{(q,j,i,p)} = 0$, the operators ∂_{R} and $\bar{\partial}_{\mathrm{R}}$ in (2.3.2) coincide with ∂_b and $\bar{\partial}_b$. But, since $\Lambda \partial_b \psi^{(i,j+1)}_{(q,j,i,p)} \neq 0$ and $\Lambda \bar{\partial}_b \psi^{(i+1,j)}_{(q,j,i,p)} \neq 0$, this is not the case for (2.3.3).*

The action of $\mathcal{L}_T$ on $\psi^{(s,t)}_{(q,j,i,p)}$ is also easy to compute. Since

$$2\mathcal{L}_T z_i = \sqrt{-1} z_i, \quad 2\mathcal{L}_T \omega_i = \sqrt{-1}\omega_i,$$

we obtain

$$2\mathcal{L}_T \psi^{(s,t)}_{(q,j,i,p)} = \sqrt{-1}(p+i-j-q)\psi^{(s,t)}_{(q,j,i,p)}. \tag{2.3.4}$$

2.3.2. L^2-norms of highest weight vectors.

LEMMA 2.3.3. ([**12**, Lemma 3.2]) *Let $p, q \geq 1$ and set*

$$C(q,p) = 2^{n+1}\pi^{n+1}(q-1)!(p-1)!/(q+p+n)!,$$
$$D(q) = 2^{n+1}\pi^{n+1}(q-1)!/(q+n)!.$$

If $i+j \leq n-1$,

$$\left\|\psi^{(i,j)}_{(q,j,i,p)}\right\|^2 = \frac{C(q,p)}{2^{j+i}}(q+j)(p+i). \tag{2.3.5}$$

If $j > 0$ and $i+j \leq n-1$,

$$\left\|\psi^{(i+1,j)}_{(q,j,i,p)}\right\|^2 = \left\|\psi^{(j,i+1)}_{(p,i,j,q)}\right\|^2 = \frac{C(q,p)}{2^{j+i+1}}(q+j)(q+n-i). \tag{2.3.6}$$

If $i, j > 0$ and $i+j \leq n-2$,

$$\left\|\psi^{(i+1,j+1)}_{(q,j,i,p)}\right\|^2 = \frac{C(q,p)}{2^{j+i+2}}\frac{(q+n-i)(p+n-j)(n-1-i-j)}{n-i-j}. \tag{2.3.7}$$

If $0 \leq j \leq n-1$,

$$\left\|\psi^{(0,j)}_{(q,j,0,0)}\right\|^2 = \left\|\psi^{(j,0)}_{(0,0,j,q)}\right\|^2 = \frac{D(q)}{2^j}(q+j), \tag{2.3.8}$$

$$\left\|\psi^{(0,j+1)}_{(q,j,0,0)}\right\|^2 = \left\|\psi^{(j+1,0)}_{(0,0,j,q)}\right\|^2 = \frac{D(q)}{2^{j+1}}(n-j). \tag{2.3.9}$$

REMARK 2.3.4. *These formulas are different from those in* [**12**] *by factors in powers of* 2 *due to the choice of the metric g.*

PROOF. We only prove (2.3.7) because others were proved in Lemma 3.2 in [**12**]; see also Remark 2.3.5. Since P is the orthogonal projection and $\psi^{(i+1,j+1)}_{(q,j,i,p)} = P\tilde{\psi}^{(i+1,j+1)}_{(q,j,i,p)}$, the formula (2.3.1) gives

$$\left\|\psi^{(i+1,j+1)}_{(q,j,i,p)}\right\|^2 = \left\|\tilde{\psi}^{(i+1,j+1)}_{(q,j,i,p)}\right\|^2 - \left\|(n-i-j)^{-1}L\Lambda\tilde{\psi}^{(i+1,j+1)}_{(q,j,i,p)}\right\|^2.$$

The first term of the right-hand side can be calculated by using the following facts: the squared norm of $\alpha^{(1,0)}_{(0,i)}$ in g (see [**5**, Lemma 5]) is $\sum_{\mu=1}^{n-i}|z_\mu|^2/2^{i+1}$ and

$$\int_S |z^\alpha|^2 d\,\mathrm{vol}_g = \frac{2^{n+1}\pi^{n+1}\alpha!}{(|\alpha|+n)!}.$$

For the second term, we can use

$$\Lambda\tilde{\psi}^{(i+1,j+1)}_{(q,j,i,p)} = \frac{\sqrt{-1}}{2}(-1)^{i+1}\psi^{(i,j)}_{(q,j,i,p)}$$

and

$$\|Lf\|^2 = (n-i-j)\|f\|^2, \quad f \in \mathcal{E}^{i,j}$$

to reduce it to

$$\frac{1}{4(n-i-j)}\left\|\psi^{(i,j)}_{(q,j,i,p)}\right\|^2.$$

This is given by (2.3.5). $\square$

REMARK 2.3.5. *In* [**12**], *the formula of the projection P, corresponding to our* (2.3.1)*, is not correct. This result in errors in the evaluation of the norm corresponding to our* (2.3.7) *and the computations of the eigenvalues of the Laplacians using that formula.*

2.3.3. Calculation of eigenvalues. Given (q, j, i, p), we list up all (s, t) such that $\Psi^{(s,t)}_{(q,j,i,p)} \neq \{0\}$ and calculate the eigenvalues of $\Delta_{\mathcal{E}}$ on them. In this subsection, we omit the subscripts from $\psi^{(s,t)}_{(q,j,i,p)}$, $\Psi^{(s,t)}_{(q,j,i,p)}$ and write $\psi^{(s,t)}$, $\Psi^{(s,t)}$.

Case I: $i = j = 0$ and $p = q = 0$

The space is $\Psi^{(0,0)}$, and we have $\Delta_{\mathcal{E}} \Psi^{(0,0)} = 0$.

Case II: $i + j \leq n - 2$, $p \geq 1$ and $q \geq 1$

The spaces are $\Psi^{(i,j)}$, $\Psi^{(i+1,j)}$, $\Psi^{(i,j+1)}$ and $\Psi^{(i+1,j+1)}$. Let $\|\partial_{\mathcal{E}}\|$ and $\|\bar{\partial}_{\mathcal{E}}\|$ be the norm of bounded linear operators of $\partial_{\mathcal{E}}$ and $\bar{\partial}_{\mathcal{E}}$. By using Propositions 2.3.1 and 2.3.3, we have

$$\begin{aligned}
\|\partial_{\mathcal{E}}|_{\Psi^{(i,j)}}\|^2 &= \frac{(p+i)^2}{n-i-j} \frac{\|\psi^{(i+1,j)}\|^2}{\|\psi^{(i,j)}\|^2} &&= \frac{(p+i)(q+n-i)}{2(n-i-j)}, \\
\|\bar{\partial}_{\mathcal{E}}|_{\Psi^{(i,j)}}\|^2 &= \frac{(q+j)^2}{n-i-j} \frac{\|\psi^{(i,j+1)}\|^2}{\|\psi^{(i,j)}\|^2} &&= \frac{(q+j)(p+n-j)}{2(n-i-j)}, \\
\|\partial_{\mathcal{E}}|_{\Psi^{(i,j+1)}}\|^2 &= \frac{(p+i)^2}{n-i-j-1} \frac{\|\psi^{(i+1,j+1)}\|^2}{\|\psi^{(i,j+1)}\|^2} &&= \frac{(p+i)(q+n-i)}{2(n-i-j)}, \\
\|\bar{\partial}_{\mathcal{E}}|_{\Psi^{(i+1,j)}}\|^2 &= \frac{(q+j)^2}{n-i-j-1} \frac{\|\psi^{(i+1,j+1)}\|^2}{\|\psi^{(i+1,j)}\|^2} &&= \frac{(q+j)(p+n-j)}{2(n-i-j)}.
\end{aligned}$$

Therefore we can calculate the eigenvalue of $\Delta_{\mathcal{E}}$ on $\Psi^{(i,j)}$ and $\Psi^{(i+1,j+1)}$ are

$$\begin{aligned}
\Delta_{\mathcal{E}}|_{\Psi^{(i,j)}} &= \left(\partial_{\mathcal{E}}{}^{\#}|_{\Psi^{(i+1,j)}} \partial_{\mathcal{E}}|_{\Psi^{(i,j)}} + \bar{\partial}_{\mathcal{E}}{}^{\#}|_{\Psi^{(i,j+1)}} \bar{\partial}_{\mathcal{E}}|_{\Psi^{(i,j)}}\right)^2 \\
&= \frac{((p+i)(q+n-i) + (q+j)(p+n-j))^2}{4(n-i-j)^2}, \\
\Delta_{\mathcal{E}}|_{\Psi^{(i+1,j+1)}} &= \left(\partial_{\mathcal{E}}|_{\Psi^{(i,j+1)}} \partial_{\mathcal{E}}{}^{\#}|_{\Psi^{(i+1,j+1)}} + \bar{\partial}_{\mathcal{E}}|_{\Psi^{(i+1,j)}} \bar{\partial}_{\mathcal{E}}{}^{\#}|_{\Psi^{(i+1,j+1)}}\right)^2 \\
&= \frac{((p+i)(q+n-i) + (q+j)(p+n-j))^2}{4(n-i-j)^2}.
\end{aligned}$$

We consider $(i+j+1)$-form. Since $\operatorname{Im} d_{\mathcal{E}}$ and $\operatorname{Im} d_{\mathcal{E}}{}^{\#}$ are orthogonal, $\Psi^{(i+1,j)} \oplus \Psi^{(i,j+1)} = d_{\mathcal{E}} \Psi^{(i,j)} \oplus d_{\mathcal{E}}{}^{\#} \Psi^{(i+1,j+1)}$. Since $\Delta_{\mathcal{E}} d_{\mathcal{E}} = d_{\mathcal{E}} \Delta_{\mathcal{E}}$ and $\Delta_{\mathcal{E}} d^{\#}_{\mathcal{E}} = d^{\#}_{\mathcal{E}} \Delta_{\mathcal{E}}$, the eigenvalue of $\Delta_{\mathcal{E}}$ on $\Psi^{(i+1,j)} \oplus \Psi^{(i,j+1)}$ is

$$\frac{((p+i)(q+n-i) + (q+j)(p+n-j))^2}{4(n-i-j)^2}.$$

Case III: $i \leq n - 1$, $j = 0$, $p \geq 1$ and $q = 0$

The spaces are $\Psi^{(i,0)}$ and $\Psi^{(i+1,0)}$. We have

$$\|\partial_{\mathcal{E}}|_{\Psi^{(i,0)}}\|^2 = (p+i)/2.$$

In the same way on Case II, the eigenvalue of $\Delta_{\mathcal{E}}$ is

$$(p+i)^2/4.$$

Case IV: $i=0$, $j \leq n-1$, $p=0$ and $q \geq 1$

The spaces are $\Psi^{(0,j)}$ and $\Psi^{(0,j+1)}$. Taking the conjugate of Case III, the eigenvalue of $\Delta_{\mathcal{E}}$ is

$$(q+j)^2/4.$$

Case V: $i+j=n-1$, $p \geq 1$ and $q \geq 1$

The spaces are $\Psi^{(i,j)}$, $\Psi^{(i+1,j)}$ and $\Psi^{(i,j+1)}$. We have

$$\|\partial_{\mathcal{E}}|_{\Psi^{(i,j)}}\|^2 = (p+i)(q+n-i)/2,$$
$$\left\|\bar{\partial}_{\mathcal{E}}|_{\Psi^{(i,j)}}\right\|^2 = (q+j)(p+n-j)/2.$$

Therefore, on $\Psi^{(i,j)}$, the eigenvalue of $\Delta_{\mathcal{E}}$ is

$$((p+i)(q+n-i)+(q+j)(p+n-j))^2/4.$$

Next we consider $W = \Psi^{(i+1,j)} \oplus \Psi^{(i,j+1)}$. We set

$$\underline{\psi}^{(s,t)} = \psi^{(s,t)}/\|\psi^{(s,t)}\|.$$

Let $A = \|\partial_{\mathcal{E}}|_{\Psi^{(i,j)}}\|$ and $B = \left\|\bar{\partial}_{\mathcal{E}}|_{\Psi^{(i,j)}}\right\|$. Then, we have

$$d_{\mathcal{E}}\underline{\psi}^{(i,j)} = A\underline{\psi}^{(i+1,j)} + B\underline{\psi}^{(i,j+1)} \in \operatorname{Im} d_{\mathcal{E}},$$

and

$$\begin{aligned} d_{\mathcal{E}}d_{\mathcal{E}}^{\#}(A\underline{\psi}^{(i+1,j)} + B\underline{\psi}^{(i,j+1)}) &= d_{\mathcal{E}}(A^2+B^2)\underline{\psi}^{(i,j)} \\ &= (A^2+B^2)(A\underline{\psi}^{(i+1,j)} + B\underline{\psi}^{(i,j+1)}). \end{aligned}$$

Therefore, eigenvalue of $\Delta_{\mathcal{E}}$ on $\operatorname{Im} d_{\mathcal{E}}\Psi^{(i,j)}$ is

$$(A^2+B^2)^2 = ((p+i)(q+n-i)+(q+j)(p+n-j))^2/4.$$

Let us find the eigenvalue on $d_{\mathcal{E}}\Psi^{(i,j)\perp}$, which is the orthogonal complement in W. We note that

$$B\underline{\psi}^{(i+1,j)} - A\underline{\psi}^{(i,j+1)} \in d_{\mathcal{E}}\Psi^{(i,j)\perp}.$$

Let $C = (p+i-j-q)/2$, $A' = C - 2A^2$ and $B' = C + 2B^2$. By (2.2.1) and (2.3.4),

$$D(B\underline{\psi}^{(i+1,j)} - A\underline{\psi}^{(i,j+1)}) = \sqrt{-1}\theta \wedge (A'B\underline{\psi}^{(i+1,j)} - B'A\underline{\psi}^{(i,j+1)}).$$

Since $D(A\underline{\psi}^{(i+1,j)} + B\underline{\psi}^{(i,j+1)}) = 0$, we have

$$\begin{aligned} &D^{\#}D(B\underline{\psi}^{(i+1,j)} - A\underline{\psi}^{(i,j+1)}) \\ &= \frac{(A'B)^2+(B'A)^2}{A^2+B^2}(B\underline{\psi}^{(i+1,j)} - A\underline{\psi}^{(i,j+1)}). \end{aligned}$$

We note that

$$\begin{aligned} &\frac{(A'B)^2+(B'A)^2}{A^2+B^2} \\ &= \frac{1}{4}(q+j-i-p)^2 + (p+i)(q+n-i)(q+j)(p+n-j). \end{aligned}$$

Under the condition $i + j = n - 1$, it agrees with

$$((p+i)(q+n-i)+(q+j)(p+n-j))^2/4.$$

Therefore, we see that the eigenvalue on $\operatorname{Im} d_{\mathcal{E}} \Psi^{(i,j)^{\perp}}$ is

$$((p+i)(q+n-i)+(q+j)(p+n-j))^2/4.$$

Case VI: $i = n-1$, $j = 0$, $p \geq 1$ and $q = -1$

The space is $\Psi^{(n,0)}$. Since there is no subspaces of $\mathcal{E}^{n-1}(S)$ corresponding to the $V(\underline{-1}_n, -p)$, we conclude $\partial_b{}^{\#}\Psi^{(n,0)} = \bar{\partial}_b{}^{\#}\Psi^{(n,0)} = \{0\}$. By (2.2.1), we have

$$D\psi^{(n,0)} = \theta \wedge \mathcal{L}_T \psi^{(n,0)}.$$

Therefore, we have

$$\Delta_{\mathcal{E}}\psi^{(n,0)} = (d_{\mathcal{E}} d_{\mathcal{E}}{}^{\#})^2\psi^{(n,0)} + D^{\#}D\psi^{(n,0)} = \mathcal{L}_T{}^{\#}\mathcal{L}_T\psi^{(n,0)},$$

where $\mathcal{L}_T{}^{\#}$ is the formal adjoint of $\mathcal{L}_T$ for the L^2-inner product. By (2.3.4), we see that the eigenvalue of $\Delta_{\mathcal{E}}$ is

$$(p+n)^2/4.$$

Case VII: $i = 0$, $j = n-1$, $p = -1$ and $q \geq 1$

The space is $\Psi^{(0,n)}$. Taking the conjugate of Case VI, the eigenvalue of $\Delta_{\mathcal{E}}$ is

$$(q+n)^2/4.$$

REMARK 2.3.6. *In Cases V-VII, the eigenvalues of $D^{\#}D$ were determined by* [**8**]. *Their choice of highest weight vectors in* $\operatorname{Ker} D$ *and* $\operatorname{Im} D^{\#}$ *are different from ours.*

CHAPTER 3

Ray-Singer Torsion and the Rumin Laplacian on lens spaces

3.1. Introduction

We next introduce the analytic torsion and metric of the Rumin complex $(\mathcal{E}^\bullet(M,E), d_{\mathcal{E}}^\bullet)$ by following [**2**, **18**]. We define the *contact analytic torsion function* associated with $(\mathcal{E}^\bullet(M,E), d_{\mathcal{E}}^\bullet)$ by

$$\kappa_{\mathcal{E}}(M,E,g_{\theta,J})(s) := \sum_{k=0}^{n}(-1)^{k+1}(n+1-k)\zeta(\Delta_{\mathcal{E}}^k)(s), \tag{3.1.1}$$

where $\zeta(\Delta_{\mathcal{E}}^k)(s)$ is the spectral zeta function of $\Delta_{\mathcal{E}}^k$, and the *contact analytic torsion* $T_{\mathcal{E}}$ by

$$2\log T_{\mathcal{E}}(M,E,g_{\theta,J}) = \kappa_{\mathcal{E}}(M,E,g_{\theta,J})'(0).$$

Let $H^\bullet(\mathcal{E}^\bullet, d_{\mathcal{E}}^\bullet)$ be the cohomology of the Rumin complex. We define the contact metric on $\det H^\bullet(\mathcal{E}^\bullet, d_{\mathcal{E}}^\bullet)$ by

$$\| \quad \|_{\mathcal{E}}(M,E,g_{\theta,J}) = T_{\mathcal{E}}^{-1}(M,E,g_{\theta,J})| \quad |_{L^2(\mathcal{E}^\bullet)},$$

where the metric $| \quad |_{L^2(\mathcal{E}^\bullet)}$ is induced by L^2 metric on $\mathcal{E}^\bullet(M,E)$ via identification of the cohomology classes by the harmonic forms on $\mathcal{E}^\bullet(M,E)$.

Rumin and Seshadri [**18**] defined another analytic torsion function κ_{R} from $(\mathcal{E}^\bullet(M,E), d_{\mathrm{R}}^\bullet)$, which is different from $\kappa_{\mathcal{E}}$ except in dimension 3. In dimension 3, they showed that $\kappa_{\mathrm{R}}(M,E,g_{\theta,J})(0)$ is a contact invariant, that is, independent of the metric $g_{\theta,J}$. Moreover, on 3-dimensional Sasakian manifolds with S^1-action, $\kappa_{\mathrm{R}}(M,E,g_{\theta,J})(0) = 0$. Furthermore they showed that the this analytic torsion and the Ray-Singer torsion $T_{\mathrm{dR}}(M,E,g_{\theta,J})$ equal for flat bundles with unimodular holonomy on 3-dimensional Sasakian manifolds with S^1 action. With this coincidence, they found a relation between the Ray-Singer torsion and holonomy.

To extend the coincidence, with $d_{\mathcal{E}}$ instead of d_{R}, the author [**9**] showed that $T_{\mathcal{E}}(S^{2n+1}, \underline{\mathbb{C}}, g_{\theta,J}) = n! T_{\mathrm{dR}}(S^{2n+1}, \underline{\mathbb{C}}, g_{\theta,J})$ on the standard CR spheres $S^{2n+1}(\subset \mathbb{C}^{n+1})$. Bisides, Albin and Quan [**1**] showed the difference between the Ray-Singer torsion and the contact analytic torsion is given by some integrals of universal polynomials in the local invariants of the metric on contact manifolds.

In this chapter, we extend this coincidence on lens spaces and determine explicitly the analytic torsion functions associated with the Rumin complex in terms of the Hurwitz zeta function. Let g_{std} be the standard metric on S^{2n+1} and we note that $g_{\theta,J} = 4g_{\mathrm{std}}$. Let $\mu, \nu_1, \dots, \nu_{n+1}$ be integers such that the ν_j are coprime to μ. Let Γ be the subgroup of $(S^1)^{n+1}$ generated by

$$\gamma = (\gamma_1, \dots, \gamma_{n+1}) := \left(\exp(2\pi\sqrt{-1}\nu_1/\mu), \cdots, \exp(2\pi\sqrt{-1}\nu_{n+1}/\mu)\right).$$

We denote the lens space by

$$K := S^{2n+1}/\Gamma.$$

Let $\underline{\mathbb{C}}$ be the trivial line bundle on K. Fix $u \in \{1, \dots, \mu\}$ and consider the unitary representation $\alpha = \alpha_u \colon \pi_1(K) = \Gamma \to \mathrm{U}(1)$, defined by

$$\alpha_u(\gamma) = \exp\left(2\pi\sqrt{-1}u/\mu\right).$$

Let E_u be the flat vector bundle associated with α_u, which is induced from the trivial bundle on S^{2n+1}.

Our main result is

THEOREM 3.1.1. *Let K be the lens space of which contact form and almost complex structure is induced by the action Γ on the standard CR sphere S^{2n+1}.*

(1) *The contact analytic torsion function of $(K, \underline{\mathbb{C}})$ is given by*

$$\kappa_{\mathcal{E}}(K, \underline{\mathbb{C}}, g_{\theta,J})(s) = -(n+1)\big(1 + 2^{2s+1}\mu^{-2s}\zeta(2s)\big), \tag{3.1.2}$$

where ζ is the Riemann zeta function. In particular, we have

$$\kappa_{\mathcal{E}}(K, \underline{\mathbb{C}}, g_{\theta,J})(0) = 0, \tag{3.1.3}$$

$$T_{\mathcal{E}}(K, \underline{\mathbb{C}}, g_{\theta,J}) = \left(\frac{4\pi}{\mu}\right)^{n+1}. \tag{3.1.4}$$

(2) *The contact analytic torsion function of (K, E_u) for $u \in \{1, \dots, \mu - 1\}$ is given by*

$$\kappa_{\mathcal{E}}(K, E_u, g_{\theta,J})(s) = -\frac{2^{2s}}{\mu^{2s}} \sum_{j=1}^{n+1} \Big(\zeta\big(2s, A_\mu(u\tau_j)/\mu\big) + \zeta\big(2s, A_\mu(-u\tau_j)/\mu\big)\Big), \tag{3.1.5}$$

where $A_\mu(w)$ and τ_j are the integers between 1 and μ such that $A_\mu(w) \equiv w \mod \mu$ and $\tau_j\nu_j \equiv 1$. In particular, we have

$$\kappa_{\mathcal{E}}(K, E_u, g_{\theta,J})(0) = 0, \tag{3.1.6}$$

$$T_{\mathcal{E}}(K, E_u, g_{\theta,J}) = \prod_{j=1}^{n+1} \left| e^{2\pi\sqrt{-1}u\tau_j/\mu} - 1 \right|. \tag{3.1.7}$$

The equations (3.1.2) and (3.1.5) extend the following results of $\kappa_{\mathcal{E}}$ on the spheres to the lens spaces. Rumin and Seshadri [**18**, Theorem 5.4] showed (3.1.2) in the case $(S^3, \underline{\mathbb{C}})$. The author [**9**] showed (3.1.2) in the case $(S^{2n+1}, \underline{\mathbb{C}})$ for arbitrary n.

From (3.1.3) and (3.1.6), we see that the metric $\|\ \ \|_{\mathcal{E}}$ on $(K, E_u, g_{\theta,J})$ is invariant under the constant rescaling $\theta \mapsto C\theta$. The argument is exactly same as the one in [**18**].

The fact that the representations determines the eigenvalues of $\Delta_{\mathcal{E}}$ causes several cancellations in the linear combination (3.1.1), which greatly simplifies the computation of $\kappa_{\mathcal{E}}(s)$. We cannot get such a simple formula for the contact analytic torsion function κ_{R} of $(\mathcal{E}^\bullet, d_{\mathrm{R}}^\bullet)$ for dimensions higher than 3, see chapter 4.

Let us compare the contact analytic torsion with the Ray-Singer torsion on lens spaces. Ray [**13**] showed that for $u \in \{1, \dots, \mu - 1\}$

$$T_{\mathrm{dR}}(K, E_u, 4g_{\mathrm{std}}) = \prod_{j=1}^{n+1} \left| e^{2\pi\sqrt{-1}u\tau_j/\mu} - 1 \right|. \tag{3.1.8}$$

Weng and You [**20**] calculated the Ray-Singer torsion on spheres. We extend their results to the case of the trivial bundle on lens spaces.

Proposition 3.1.2. *In the setting of Theorem 3.1.1, we have*

$$T_{\mathrm{dR}}(K, \underline{\mathbb{C}}, 4g_{\mathrm{std}}) = \frac{(4\pi)^{n+1}}{n!\mu^{n+1}}.$$

The metric $4g_{\mathrm{std}}$ agrees with the metric $g_{\theta,J}$ defined from the contact from $\theta = \sqrt{-1}(\bar{\partial} - \partial)|z|^2$. Since the cohomology of $(\mathcal{E}^\bullet(M,E), d_{\mathcal{E}}^\bullet)$ coincides with that of $(\Omega^\bullet(M,E), d)$ (e.g. [**17**, §5.3], [**4**, §4]), there is a natural isomorphism $\det H^\bullet(\mathcal{E}^\bullet(M,E), d_{\mathcal{E}}^\bullet) \cong \det H^\bullet(\Omega^\bullet(M,E), d)$, which turns out to be isometric for the L^2 metrics. Therefore (3.1.4) and (3.1.7) give

Corollary 3.1.3. *In the setting of Theorem 3.1.1, for a unitary holonomy* $\alpha\colon \pi_1(K) \to \mathrm{U}(r)$, *we have*

$$T_{\mathcal{E}}(K, E_\alpha, g_{\theta,J}) = n!^{\dim H^0(K,E_\alpha)} T_{\mathrm{dR}}(K, E_\alpha, g_{\theta,J}),$$
$$\|\ \ \|_{\mathcal{E}}(K, E_\alpha, g_{\theta,J}) = n!^{-\dim H^0(K,E_\alpha)} \|\ \ \|_{\mathrm{dR}}(K, E_\alpha, g_{\theta,J}).$$

The chapter is organized as follows. In §3.2, we calculate the contact analytic torsion function $\kappa_{\mathcal{E}}$ of flat vector bundles on lens spaces. In §3.3, we compute the Ray-Singer torsion T_{dR} of the trivial vector bundle. In §3.4, we compare the Ray-Singer torsion and the contact analytic torsion.

3.2. Contact analytic torsion of flat vector bundles

Let $\mu, \nu_1, \dots, \nu_{n+1}$ be integers such that the ν_j are coprime to μ. Let Γ be the subgroup of $(S^1)^{n+1}$ generated by

$$\gamma = (\gamma_1, \dots, \gamma_{n+1}) := \left(\exp(2\pi\sqrt{-1}\nu_1/\mu), \cdots, \exp(2\pi\sqrt{-1}\nu_{n+1}/\mu)\right).$$

We denote the lens space by

$$K := S^{2n+1}/\Gamma.$$

Fix $u \in \{1, \dots, \mu\}$ and consider the unitary representation $\alpha = \alpha_u\colon \pi_1(K) = \Gamma \to \mathrm{U}(1)$, defined by

$$\alpha_u(\gamma) = \exp\left(2\pi\sqrt{-1}u/\mu\right).$$

Let E_u be the flat vector bundle associated with α_u, which is induced from the trivial bundle on S^{2n+1}.

Let χ_V be the character of the representation (V, ρ) of $\mathrm{U}(n+1)$. For α_u, we define V^{α_u} by

$$V^{\alpha_u} = \left\{ v \in V \,\middle|\, \alpha_u(\gamma)^{-1} v = \chi_V(\gamma) v \right\}.$$

We note that for (V, ρ),

$$\dim V^{\alpha_u} = \sum_{t \in \Gamma} \chi_V(t) \alpha_u(t) / \#\Gamma.$$

From Theorem 2.1.1, we see that the terms of $\kappa_{\mathcal{E}}(K, E_u, g_{\theta,J})(s)$ in Cases II and V in Proposition 2.2.1 cancel each other. Thus we get

$$\kappa_{\mathcal{E}}(K, E_u, g_{\theta,J})(s) = \kappa_1(K, E_u, g_{\theta,J})(s) + \kappa_2(K, E_u, g_{\theta,J})(s) + \kappa_3(K, E_u, g_{\theta,J})(s), \tag{3.2.1}$$

where

$$\begin{aligned}\kappa_1(K,E_u,g_{\theta,J})(s) &= \sum_{k=0}^{n}(-1)^{k+1}(n+1-k)\dim H^k(K,E_u)\\ &= -(n+1)\dim H^0(K,E_u),\\ &= \begin{cases}-(n+1), & u=0,\\ 0, & u\neq 0,\end{cases}\end{aligned} \tag{3.2.2}$$

which is the sum of the terms of $\kappa_{\mathcal{E}}(K,E_u,g_{\theta,J})(s)$ in Case I, and

$$\begin{aligned}&\kappa_2(K,E_u,g_{\theta,J})(s)\\ &= \sum_{j=0}^{n}(-1)^{j+1}(n+1-j)\sum_{q\geq 1}\left(\frac{\dim V^{\alpha_u}(q,\underline{1}_j,\underline{0}_{n-j})}{\big((q+j)/2\big)^{2s}}+\frac{\dim V^{\alpha_u}(q,\underline{1}_{j-1},\underline{0}_{n-j+1})}{\big((q+j-1)/2\big)^{2s}}\right)\\ &= \sum_{j=0}^{n}(-1)^{j+1}(n+1-j)\sum_{q\geq 1}\frac{\dim V^{\alpha_u}(q,\underline{1}_j,\underline{0}_{n-j})+\dim V^{\alpha_u}(q+1,\underline{1}_{j-1},\underline{0}_{n-j+1})}{\big((q+j)/2\big)^{2s}}\\ &\quad+\sum_{j=1}^{n}(-1)^{j+1}(n+1-j)\frac{\dim V^{\alpha_u}(\underline{1}_j,\underline{0}_{n-j+1})}{\big(j/2\big)^{2s}},\end{aligned} \tag{3.2.3}$$

which is the sum of the terms of $\kappa_{\mathcal{E}}(K,E_u,g_{\theta,J})(s)$ in Cases III and VI, and

$$\kappa_3(K,E_u,g_{\theta,J})(s) = \kappa_2(K,E_{-u},g_{\theta,J})(s), \tag{3.2.4}$$

which is the sum of the terms of $\kappa_{\mathcal{E}}(K,E_u,g_{\theta,J})(s)$ in Cases IV and VII.

By Richardson-Littlewood's rule, we have

$$\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}\chi_{V(q,\underline{0}_n)} = \chi_{V(q,\underline{1}_j,\underline{0}_{n-j})}+\chi_{V(q+1,\underline{1}_{j-1},\underline{0}_{n-j+1})}. \tag{3.2.5}$$

From (3.2.3) and (3.2.5),

$$
\begin{aligned}
&\kappa_2(K, E_u, g_{\theta,J})(s) \\
&= \frac{1}{\mu}\sum_{j=0}^{n}(-1)^{j+1}(n+1-j)\sum_{q\geq 1}\sum_{l=0}^{\mu-1}\frac{\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(\gamma^l)\chi_{V(q,\underline{0}_n)}(\gamma^l)\alpha_u(\gamma^l)}{\big((q+j)/2\big)^{2s}} \\
&\quad + \frac{1}{\mu}\sum_{j=1}^{n}(-1)^{j+1}(n+1-j)\sum_{l=0}^{\mu-1}\frac{\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(\gamma^l)\alpha_u(\gamma^l)}{\big(j/2\big)^{2s}} \\
&= \frac{2^{2s}}{\mu\Gamma(2s)}\sum_{j=0}^{n}(-1)^{j+1}(n+1-j) \\
&\qquad \sum_{q\geq 1}\sum_{l=0}^{\mu-1}\int_0^\infty \chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(\gamma^l)\chi_{V(q,\underline{0}_n)}(\gamma^l)\alpha_u(\gamma^l)e^{-(j+q)x}x^{2s-1}dx \\
&\quad + \frac{2^{2s}}{\mu\Gamma(2s)}\sum_{j=1}^{n}(-1)^{j+1}(n+1-j) \\
&\qquad \sum_{l=0}^{\mu-1}\int_0^\infty \chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(\gamma^l)\alpha_u(\gamma^l)e^{-jx}x^{2s-1}dx \\
&= \frac{2^{2s}}{\mu\Gamma(2s)}\sum_{l=0}^{\mu-1}\int_0^\infty\Bigg(\sum_{j=0}^{n}(-1)^{j+1}(n+1-j)\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(\gamma^l)e^{-jx} \\
&\qquad\qquad \sum_{q\geq 1}\chi_{V(q,\underline{0}_n)}(\gamma^l)e^{-qx} \\
&\quad + \sum_{j=1}^{n}(-1)^{j+1}(n+1-j)\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(\gamma^l)e^{-jx}\Bigg)\alpha_u(\gamma^l)x^{2s-1}dx. \qquad (3.2.6)
\end{aligned}
$$

We consider contents of integral for the last equation. It is known that for $t = (t_1, \cdots, t_{n+1}) \in (S^1)^{n+1}$,

$$
\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(t) = \sum_{\substack{\beta_1+\cdots+\beta_{n+1}=j \\ 0\leq\beta_1,\ldots,\beta_{n+1}\leq 1}} t_1^{\beta_1}\cdots t_{n+1}^{\beta_{n+1}}, \qquad (3.2.7)
$$

$$
\chi_{V(q,\underline{0}_n)}(t) = \sum_{\substack{\alpha_1+\cdots+\alpha_{n+1}=q \\ \alpha_1,\ldots,\alpha_{n+1}\geq 0}} t_1^{\alpha_1}\cdots t_{n+1}^{\alpha_{n+1}}. \qquad (3.2.8)
$$

We set $X := e^{-x}$ and

$$
F_1(t, X) := \sum_{j=0}^{n+1}(-1)^j\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(t)X^j.
$$

Then (3.2.7) gives

$$F_1 = \prod_{j=0}^{n+1}(1 - t_j X), \tag{3.2.9}$$

$$X\frac{\partial F_1}{\partial X} = \sum_{j=0}^{n+1}(-1)^j j\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(t)X^j = -\sum_{i=1}^{n+1} t_i X \prod_{j=1, j\neq i}^{n+1}(1 - t_j X). \tag{3.2.10}$$

From (3.2.9) and (3.2.10), we have

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{j+1}(n+1-j)\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(t)X^j\\ &= (n+1)F_1 - X\frac{\partial F_1}{\partial X}.\end{aligned} \tag{3.2.11}$$

We set

$$F_2(t, X) := \sum_{q\geq 1}\chi_{V(q,\underline{0}_n)}(t)X^q.$$

From (3.2.8), we can rewrite F_2 as

$$F_2 = \prod_{j=1}^{n+1}\frac{1}{1 - t_j X} - 1 = \frac{1}{F_1(t, X)} - 1. \tag{3.2.12}$$

From (3.2.10), (3.2.11) and (3.2.12), we can deduce that

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{j+1}(n+1-j)\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(t)X^j\sum_{q\geq 1}\chi_{V(q,\underline{0}_n)}(t)X^q\\ &+\sum_{j=1}^{n}(-1)^{j+1}(n+1-j)\chi_{V(\underline{1}_j,\underline{0}_{n-j+1})}(t)X^j\\ &= -\left((n+1)F_1 - X\frac{\partial F_1}{\partial X}\right)\left(\frac{1}{F_1} - 1\right)\\ &\quad -\left((n+1)(F_1 - 1) - X\frac{\partial F_1}{\partial X}\right)\\ &= \frac{X\frac{\partial F_1}{\partial X}(t, X)}{F_1} = -\sum_{j=1}^{n+1}\frac{t_j X}{1 - t_j X}.\end{aligned} \tag{3.2.13}$$

From (3.2.6) and (3.2.13), we see

$$\begin{aligned}&\kappa_2(K, E_u, g_{\theta,J})(s)\\ &= -\frac{2^{2s}}{\mu\Gamma(2s)}\sum_{l=0}^{\mu-1}\sum_{j=1}^{n+1}\int_0^\infty \frac{\gamma_j^l e^{-x}}{1 - \gamma_j^l e^{-x}} e^{2\pi\sqrt{-1}ul/\mu}x^{2s-1}dx\\ &= -\sum_{j=1}^{n+1}\frac{2^{2s}}{\mu\Gamma(2s)}\sum_{l=0}^{\mu-1}\int_0^\infty\sum_{q=1}^{\infty} e^{2\pi\sqrt{-1}(q\nu_j+u)l/\mu}e^{-qx}x^{2s-1}dx\\ &= -2^{2s}\sum_{j=1}^{n+1}\sum_{l=0}^{\mu-1}\sum_{q=1}^{\infty}\frac{e^{2\pi\sqrt{-1}(q\nu_j+u)l/\mu}}{\mu}q^{-2s}.\end{aligned} \tag{3.2.14}$$

Let τ_j be the integers in $\{1, \dots, \mu\}$ such that $\tau_j \nu_j \equiv 1 \mod \mu$. Since the multiplication of $\nu_j \in (\mathbb{Z}/\mu\mathbb{Z})^{\times}$ induced the bijective map from $\mathbb{Z}/\mu\mathbb{Z}$ to $\mathbb{Z}/\mu\mathbb{Z}$, we have

$$\begin{aligned}\sum_{l=0}^{\mu-1} \exp\left(2\pi\sqrt{-1}(q\nu_j + u)l/\mu\right) &= \sum_{l=0}^{\mu-1} \exp\left(2\pi\sqrt{-1}(q + u\tau_j)\nu_j l/\mu\right) \\ &= \sum_{l=0}^{\mu-1} \exp\left(2\pi\sqrt{-1}(q + u\tau_j)l/\mu\right) \\ &= \begin{cases} 0, & q \not\equiv -u\tau_j \mod \mu, \\ \mu, & q \equiv -u\tau_j \mod \mu. \end{cases}\end{aligned}$$

For $w \in \mathbb{Z}$, let $A_\mu(w)$ be the integer between 1 and μ which is congruent to w modulo μ, then from (3.2.14), we can rewrite κ_2 as

$$\begin{aligned}\kappa_2(K, E_u, g_{\theta,J})(s) &= -2^{2s} \sum_{j=1}^{n+1} \sum_{\substack{q>0, \\ q \equiv -u\tau_j \mod \mu}}^{\infty} q^{-2s} \\ &= -2^{2s} \sum_{j=1}^{n+1} \sum_{q=0}^{\infty} \left(q\mu + A_\mu(-u\tau_j)\right)^{-2s} \\ &= -2^{2s}\mu^{-2s} \sum_{j=1}^{n+1} \zeta\left(2s, A_\mu(-u\tau_j)/\mu\right), \end{aligned} \tag{3.2.15}$$

where for $0 < a \leq 1$, $\zeta(s, a) := \sum_{q=0}^{\infty}(q + a)^{-s}$ is the Hurwitz zeta function.

Next, we calculate κ_3. As in the case of κ_2, from (3.2.4), we can rewrite κ_3 as

$$\kappa_3(K, E_u, g_{\theta,J})(s) = -2^{2s}\mu^{-2s} \sum_{j=1}^{n+1} \zeta\left(2s, A_\mu(u\tau_j)/\mu\right). \tag{3.2.16}$$

From (3.2.1), (3.2.2), (3.2.15) and (3.2.16), we have

$$\kappa_{\mathcal{E}}(K, E_u, g_{\theta,J})(s) = \begin{cases} -(n+1)\left(1 + 2^{2s+1}\mu^{-2s}\zeta(2s)\right), & \\ & u = 0. \\ -2^{2s}\mu^{-2s} \displaystyle\sum_{j=1}^{n+1} \Big(\zeta\left(2s, A_\mu(u\tau_j)/\mu\right) + \zeta\left(2s, 1 - A_\mu(u\tau_j)/\mu\right)\Big), & \\ & u \neq 0. \end{cases}$$

It is known that $\zeta(0) = -1/2$ and $\zeta'(0) = -\log(2\pi)/2$ and for $0 < a < 1$,

$$\begin{aligned}\zeta(0, a) + \zeta(0, 1 - a) &= 0, \\ \zeta'(0, a) + \zeta'(0, 1 - a) &= -\log|e^{2\pi\sqrt{-1}a} - 1|.\end{aligned}$$

Using the above equations, we conclude for $u = 0$

$$\begin{aligned}\kappa_{\mathcal{E}}(K, \underline{\mathbb{C}}, g_{\theta,J})(0) &= -(n+1)\left(1 + 2\zeta(0)\right) = 0, \\ \kappa_{\mathcal{E}}(K, \underline{\mathbb{C}}, g_{\theta,J})'(0) &= 2(n+1)\log\frac{4\pi}{\mu},\end{aligned}$$

and for $u \neq 0$

$$\kappa_{\mathcal{E}}(K, E_u, g_{\theta,J})(0) = 0,$$
$$\kappa_{\mathcal{E}}(K, E_u, g_{\theta,J})'(0) = 2\sum_{j=1}^{n+1} \log\left|e^{2\pi\sqrt{-1}u\tau_j/\mu} - 1\right|$$

as claimed.

Remark 3.2.1. *In the case S^3, Rumin-Seshadri* [**18**, Theorem 5.4] *gives*

$$\kappa_{\mathcal{E}}(s) = -2(1 + 2\zeta(2s)).$$

This is different from (3.1.2) *and* (3.1.5) *by the factor of 2^{2s}, which is caused by the different choice of contact forms. With our θ in Theorems 2.1.1 and 3.1.1, their contact form can be written as $\theta/2$, whose Reeb vector field generates a flow of period 2π.*

Remark 3.2.2. *For any other choice of $\{a_k\}$, the analytic torsion function κ_a associated with $\{a_k d^k_{\mathrm{R}}\}$ also vanishes at 0. This is clear from the observation that κ_a is, up to constant integers, given by the alternating sum of $\zeta(d^{k\,\#}_{\mathcal{E}} d^k_{\mathcal{E}})$, whose value at 0 is independent of $\{a_k\}$.*

3.3. Ray-Singer torsion of the trivial bundle

We compute $T_{\mathrm{dR}}(K, \underline{\mathbb{C}}, g_{\mathrm{std}})$. Following the derivation of [**13**, (3)], we have

$$2\log T_{\mathrm{dR}}(K, \underline{\mathbb{C}}, g_{\mathrm{std}}) = \frac{1}{\mu}\sum_{l=0}^{\mu-1}\sum_{j=0}^{2n}(-1)^{j+1}\zeta'(0; j, \gamma^l),$$

where λ_m is the m-th eigenvalue of $d^{\#}d$ and

$$\zeta(s; j, \gamma^l) = \sum_{m=0}^{\infty} \lambda_m^{-s}\,\mathrm{Tr}(\gamma^l|_{X_{j,m}}),$$
$$X_{j,m} := \{\phi \in \Omega^j(K) \mid d^{\#}d\phi = \lambda_m\phi\}.$$

We recall from [**13**, page 123]

$$\begin{aligned}&\sum_{j=0}^{2n}(-1)^{j+1}\zeta(s; j, \gamma^l)\\ &= (\Gamma(s))^{-2}\int_0^\infty t^{2s-1}\int_0^1 \big(u(1-u)\big)^{s-1}\left(f_0(t,u) + f_1(t, l\nu/\mu)\right)du dt + \mathcal{O}(s^2),\end{aligned}$$

where for $\sigma \in \mathbb{R}^{n+1}$,

$$f_0(t,u) = \sum_{j=0}^{n+1}(-1)^j\binom{n}{j}\left(\frac{2\sinh tu \sinh t(1-u)}{\sinh t}\right)^j - \frac{e^{-(2n+1)tu}}{2\sinh tu} - \frac{e^{-(2n+1)t(1-u)}}{2\sinh t(1-u)},$$
$$f_1(t,\sigma) = \sum_{k}^{n+1}\left(1 - \frac{\sinh t}{\cosh t - \cos 2\pi\sigma_k}\right).$$

We set

$$h_0(s) := (\Gamma(s))^{-2} \int_0^\infty t^{2s-1} \int_0^1 \big(u(1-u)\big)^{s-1} f_0(t,u) du dt,$$

$$h_1(s,\gamma^l) := (\Gamma(s))^{-2} \int_0^\infty t^{2s-1} \int_0^1 \big(u(1-u)\big)^{s-1} f_1(t,l\nu/\mu) du dt.$$

From [**13**, page 125], it is seen that

$$h_1(s,\gamma^l) = -\frac{1}{2}\sum_{j=0}^{\mu-1} \mathrm{Tr}(\gamma^{jl})\big(\zeta(2s,j/\mu)+\zeta(2s,1-j/\mu)\big) - 2(n+1)\mu^{-2s}\zeta(2s),$$

where

$$\mathrm{Tr}(\gamma^l) = \sum_{j=1}^{n+1}\left(e^{2\pi\sqrt{-1}l\nu_j/\mu} + e^{-2\pi\sqrt{-1}l\nu_j/\mu}\right).$$

Taking the average of $h_1(s,\gamma^l)$, we have

$$\frac{1}{\mu}\sum_{l=0}^{\mu-1} h_1(s,\gamma^l) = -2(n+1)\mu^{-2s}\zeta(2s).$$

Using $\zeta'(0) = -\log(2\pi)/2$, we get

$$\frac{1}{\mu}\sum_{l=0}^{\mu-1} h_1'(0,\gamma^l) = 2(n+1)\log\left(\frac{2\pi}{\mu}\right). \tag{3.3.1}$$

We recall the Ray-Singer torsion on spheres.

PROPOSITION 3.3.1. ([**20**])

$$T_{\mathrm{dR}}(S,\underline{\mathbb{C}},g_{\mathrm{std}}) = \frac{2\pi^{n+1}}{n!}.$$

Let $\mu = 1$, $\nu = (1,\ldots,1)$. It follows from (3.3.1) that

$$\begin{aligned} h_0'(0) &= 2\log T_{\mathrm{dR}}(S,\underline{\mathbb{C}},g_{\mathrm{std}}) - \frac{1}{\mu}\sum_{l=0}^{\mu-1} h_1'(0,\gamma^l) \\ &= 2\log\left(\frac{2\pi^{n+1}}{n!}\right) - 2(n+1)\log(2\pi) = 2\log\left(\frac{2^{-n}}{n!}\right). \end{aligned} \tag{3.3.2}$$

By (3.3.1) and (3.3.2), we conclude

$$\begin{aligned} 2\log T_{\mathrm{dR}}(K,\underline{\mathbb{C}},g_{\mathrm{std}}) &= \frac{1}{\mu}\sum_{l=0}^{\mu-1} h_0'(0) + \frac{1}{\mu}\sum_{l=0}^{\mu-1} h_1'(0,\gamma^l) \\ &= 2\log\left(\frac{2^{-n}}{n!}\right) + 2(n+1)\log\left(\frac{2\pi}{\mu}\right) = 2\log\left(\frac{2\pi^{n+1}}{n!\mu^{n+1}}\right). \end{aligned}$$

3.4. Ray-Singer torsion and the contact torsion

Since $\alpha(\gamma) \in \mathrm{U}(r)$ is diagonalizable by a unitary matrix, we have

$$E_\alpha = E_{u_1} \oplus \cdots \oplus E_{u_r}.$$

From (3.1.8), Theorem 3.1.1, and Proposition 3.1.2, we conclude

$$\begin{aligned} T_{\mathrm{dR}}(K, E_\alpha, g_{\theta,J}) &= \prod_{j=1}^{r} T_{\mathrm{dR}}(K, E_{u_j}, g_{\theta,J}) \\ &= \prod_{j=1}^{r} n!^{-\dim H^0(K, E_{u_j})} T_{\mathcal{E}}(K, E_{u_j}, g_{\theta,J}) \\ &= n!^{-\dim H^0(K, E_\alpha)} T_{\mathcal{E}}(K, E_\alpha, g_{\theta,J}). \end{aligned}$$

CHAPTER 4

Analytic torsions associated with the Rumin complex with parameters

4.1. Introduction

Let (M, H) be a compact contact manifold of dimension $2n+1$. Let θ be a contact form of H and J be an almost complex structure on H. Then we may define a Riemann metric $g_{\theta,J}$ on TM by extending the Levi metric $d\theta(-, J-)$ on H (see chapter 1). Let $\{a_k\}$ be positive real numbers. Then, $(\mathcal{E}^\bullet(M), a_k d_{\mathrm{R}}^\bullet)$ is also a complex. We define the Laplacians $\Delta_{\mathrm{R},\{a_k\}}$ associated with $(\mathcal{E}^\bullet(M), a_k d_{\mathrm{R}}^\bullet)$ and the metric $g_{\theta,J}$ by

$$\Delta_{\mathrm{R},\{a_k\}}^k := \begin{cases} a_{k-1}^4 (d_{\mathrm{R}} d_{\mathrm{R}}{}^\#)^2 + a_k^4 (d_{\mathrm{R}}{}^\# d_{\mathrm{R}})^2, & (k \neq n, n+1), \\ a_{n-1}^4 (d_{\mathrm{R}} d_{\mathrm{R}}{}^\#)^2 + a_n D^\# D, & (k = n), \\ a_n^2 D D^\# + a_{n+1}^4 (d_{\mathrm{R}}{}^\# d_{\mathrm{R}})^2, & (k = n+1). \end{cases}$$

We follow [**2, 18**] to formulate the analytic torsions and metrics of the Rumin complex $(\mathcal{E}^\bullet(M), a_k d_{\mathrm{R}}^\bullet)$. We define the contact analytic torsion function by

$$\kappa_{\mathrm{R},\{a_k\}}(s) = \frac{1}{2} \sum_{k=0}^{2n+1} (-1)^k w(k) \zeta(\Delta_{\mathrm{R},\{a_k\}}^k)(s),$$

where $\zeta(\Delta_{\mathrm{R},\{a_k\}}^k)(s)$ is the spectral zeta function of $\Delta_{\mathrm{R},\{a_k\}}^k$ and $w(k) = k$ for $k \leq n$ and $w(k) = k+1$ for $k \geq n+1$. We define the contact analytic torsion $T_{\mathrm{R},\{a_k\}}$ by

$$2 \log T_{\mathrm{R},\{a_k\}} = \kappa'_{\mathrm{R},\{a_k\}}(0).$$

To normalize parameters, we assume that $a_n = 1$.

Theorem 4.1.1. *On the 3 dimensional standard CR sphere S^3 with the contact from $\theta = \sqrt{-1}(\bar{\partial} - \partial)|z|^2$, the contact analytic torsion associated with $(\mathcal{E}^\bullet(M), a_k d_{\mathrm{R}}^\bullet)$ is given by*

$$T_{\mathrm{R},\{a_k\}} = (4\pi)^2 (a_0 a_2)^{\frac{\pi^2}{32} - 1}.$$

On the 5 dimensional standard CR sphere S^5 with the contact from $\theta = \sqrt{-1}(\bar{\partial} - \partial)|z|^2$, the contact analytic torsion associated with $(\mathcal{E}^\bullet(M), a_k d_{\mathrm{R}}^\bullet)$ is given by

$$T_{\mathrm{R},\{a_k\}} = (4\pi)^3 (2a_0 a_4)^{-\frac{5}{4} + \frac{\pi^2}{18}} (a_1 a_3)^{\frac{113}{144} + \frac{25\pi^2}{2304}}.$$

Recall that Weng-You calculated $T_{\mathrm{dR}}(S^{2n+1}, \underline{\mathbb{C}}, g_{\theta,J}) = (4\pi)^{n+1}/n!$. If we asuume the symmeetry $a_j = a_{2n-j}$, the contact torsion coincides with the Ray-Singer torsion on the standard CR sphere if and only if when $n = 1$,

$$a_0 = 1,$$

when $n = 2$,

$$2^{-\frac{1}{4}+\frac{\pi^2}{18}} a_0^{-\frac{5}{2}+\frac{\pi^2}{9}} a_1^{\frac{113}{72}+\frac{25\pi^2}{1152}} = 1.$$

Theorem 4.1.1 claims that if $a_k = 1$, on 5 dimensional standard CR sphere, the contact analytic torsion does not coincide with the Ray-Singer torsion.

This chapter is organized as follows. In §4.2, we calculate the eigenvalues of the Rumin Laplacians $\Delta_{\mathrm{R},\{a_k\}}$ associated with $(\mathcal{E}^\bullet(S^{2n+1}), a_k d_{\mathrm{R}}^\bullet)$ for each irreducible component. In §4.3, we prepare to calculate the analytic torsion associated with the Rumin complex with parameters. In §4.4 and §4.5, we compute the analytic torsion on the 3 and 5 dimensional CR standard spheres, respectively.

4.2. The Rumin complex with parameters

All irreducible components are calculated by Proposition 2.2.1. It is shown that the multiplicity of $V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p)$ in $\mathcal{E}^{s,t}$ is at most one. In the same way as Theorem 2.1.1 in [**9**], we calculate the eigenvalues of $\Delta^k_{\mathrm{R},\{a_k\}}$.

Case I: On irreducible component $V(q, \underline{0}_n, -p)$ of $\mathcal{E}^0(S^{2n+1})$ such that $q \geq 0$ and $p \geq 0$, the eigenvalue of $\Delta_{\mathrm{R},\{a_k\}}$ is

$$\frac{1}{4}(p(q+n) + q(p+n))^2 a_0^4.$$

Case II: Let $1 \leq k \leq n-1$. On $V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p)$ of $\mathcal{E}^k(S^{2n+1})$ such that $q \geq 1$, $p \geq 1$ and $i + j = k$, the eigenvalue of $\Delta_{\mathrm{R},\{a_k\}}$ is

$$\frac{1}{4}((p+i)(q+n-i) + (q+j)(p+n-j))^2 a_{i+j}^4.$$

On $V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p)$ of $\mathcal{E}^k(S^{2n+1})$ such that $q \geq 1$, $p \geq 1$ and $i+j = k-1$ is multiplicity 2 and on each irreducible components the eigenvalues of $\Delta_{\mathrm{R},\{a_k\}}$ are

$$\frac{1}{4}((p+i)(q+n-i) + (q+j)(p+n-j))^2 a_{i+j}^2,$$

$$\frac{1}{4}\left(\frac{n-1-i-j}{n-i-j}\right)^2 ((p+i)(q+n-i) + (q+j)(p+n-j))^2 a_{i+j+1}^4.$$

If $k \geq 2$, for $V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p)$ of $\mathcal{E}^k(S^{2n+1})$ such that $q \geq 1$, $p \geq 1$ and $i + j = k-2$ the eigenvalue of $\Delta_{\mathrm{R},\{a_k\}}$ is

$$\frac{1}{4}\left(\frac{n-1-i-j}{n-i-j}\right)^2 ((p+i)(q+n-i) + (q+j)(p+n-j))^2 a_{i+j+2}^4.$$

On $V(\underline{0}_{n-i}, \underline{-1}_i, -p)$ and $V(p, \underline{1}_i, \underline{0}_{n-i})$ of $\mathcal{E}^k(S^{2n+1})$ such that $p \geq 1$ and $i = k - k_1$, the eigenvalues of $\Delta_{\mathrm{R},\{a_k\}}$ is

$$\frac{1}{4}((p+i)(n-i+k_1))^2 a_i^4.$$

Case III:

If $n \geq 2$, on $V(q, \underline{1}_j, 0, \underline{-1}_i, -p)$ of $\mathcal{E}^n(S^{2n+1})$ such that $q \geq 1$, $p \geq 1$ and $i + j = n-2$, the eigenvalue of $\Delta_{\mathrm{R},\{a_k\}}$ is

$$\frac{1}{4}((p+i)(q+n-i) + (q+j)(p+n-j))^2 a_{i+j+1}^4.$$

On $V(0,\underline{-1}_{n-1},-p)$ and $V(p,\underline{1}_{n-1},0)$ of $\mathcal{E}^n(S^{2n+1})$ such that $p\geq 1$, the eigenvalues of $\Delta_{\mathrm{R},\{a_k\}}$ is

$$\frac{1}{4}(p+n-1)^2a_{n-1}^4.$$

On $V(q,\underline{1}_j,\underline{-1}_i,-p)$ of $\mathcal{E}^n(S^{2n+1})$ such that $q\geq 1$, $p\geq 1$ and $i+j=n-1$ is multiplicity 2, and on each irreducible component, the eigenvalues of $\Delta_{\mathrm{R},\{a_k\}}$ are for $l=0,1$

$$\frac{1}{4}((p+i)(q+n-i)+(q+j)(p+n-j))^2a_{i+j+l}^4.$$

On $V(\underline{-1}_n,p)$ and $V(p,\underline{1}_n)$ of $\mathcal{E}^n(S^{2n+1})$ such that $p\geq 1$, the eigenvalues of $\Delta_{\mathrm{R},\{a_k\}}$ is

$$\frac{1}{4}(p+n)^2a_n^4.$$

4.3. Analytic torsion function associated with the Rumin complex with parameters

In this section, we assume that $i,j\geq 0$ under additional restriction given in the formulas. To calculate $\kappa_{\mathrm{R},\{a_k\}}(s)$, we set

$$2\kappa_{\mathrm{R},\{a_k\}}(s)=\kappa_{\mathrm{I}}(s)+4^s\Bigg(\sum_{\substack{0\leq k\leq n-2\\ n+3\leq k\leq 2n+1}}\kappa_{\mathrm{II},k}(s)+2\sum_{\substack{0\leq k\leq n-1\\ n+2\leq k\leq 2n+1}}\kappa_{\mathrm{III},k}(s)$$
$$+\kappa_{\mathrm{V},n-1}(s)+\kappa_{\mathrm{V},n+2}(s)+2\kappa_{\mathrm{VI},n}(s)+2\kappa_{\mathrm{VI},n+1}(s)\Bigg),$$

where

$$\kappa_{\mathrm{I}}(s):=(-1)^{2n+1}(2n+2)\dim V(\underline{0}_{n+1})=-(2n+2),$$

$$\kappa_{\mathrm{II},k}(s) := \begin{cases} (-1)^{k+1} \sum_{\substack{i+j=k \\ p,q\geq 1}} \left(a_k^{-4s} - a_{k+1}^{-4s} \left(\frac{n-1-k}{n-k} \right)^{-2s} \right) \\ \qquad \frac{\dim V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p)}{((p+i)(q+n-i)+(q+j)(p+n-j))^{2s}} \\ \qquad (\text{ for } 0 \leq k \leq n-2), \\ (-1)^{k} \sum_{\substack{i+j=2n+1-k \\ p,q\geq 1}} \left(a_{k-1}^{-4s} - a_{k-2}^{-4s} \left(\frac{k-n-2}{k-n-1} \right)^{-2s} \right) \\ \qquad \frac{\dim V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p)}{((p+i)(q+n-i)+(q+j)(p+n-j))^{2s}} \\ \qquad (\text{ for } n+3 \leq k \leq 2n+1), \end{cases}$$

$$\kappa_{\mathrm{III},k}(s) := \begin{cases} (-1)^{k+1} a_k^{-4s} \sum_{p\geq 1} \frac{\dim V(\underline{0}_{n-k}, \underline{-1}_k, -p)}{((p+k)(n-k))^{2s}}, \\ \qquad (\text{ for } 0 \leq k \leq n-1), \\ (-1)^{k} a_{k-1}^{-4s} \sum_{p\geq 1} \frac{\dim V(\underline{0}_{k-n-1}, \underline{-1}_{2n+1-k}, -p)}{((p+2n+1-k)(k-n-1))^{2s}}, \\ \qquad (\text{ for } n+2 \leq k \leq 2n+1), \end{cases}$$

$$\kappa_{\mathrm{V},n-1}(s) := (-1)^n \sum_{\substack{i+j=n-1 \\ p,q\geq 1}} \left(a_{n-1}^{-4s} + n a_n^{-4s} \right) \frac{\dim V(q, \underline{1}_j, \underline{-1}_i, -p)}{((p+i)(q+n-i)+(q+j)(p+n-j))^{2s}},$$

$$\kappa_{\mathrm{V},n+2}(s) := (-1)^{n+1} \sum_{\substack{i+j=n-1 \\ p,q\geq 1}} \left(-a_{n+1}^{-4s} + (n+2) a_n^{-4s} \right) \frac{\dim V(q, \underline{1}_j, \underline{-1}_i, -p)}{((p+i)(q+n-i)+(q+j)(p+n-j))^{2s}},$$

$$\kappa_{\mathrm{VI},n}(s) := (-1)^n n a_n^{-4s} \sum_{p\geq 1} \frac{\dim V(\underline{-1}_n, -p)}{(p+n)^{2s}},$$

$$\kappa_{\mathrm{VI},n+1}(s) := (-1)^{n+1} (n+2) a_n^{-4s} \sum_{p\geq 1} \frac{\dim V(\underline{-1}_n, -p)}{(p+n)^{2s}}.$$

We define for $a \geq 0$ the following holomorphic function:

$$\widetilde{\kappa}_a^n(s) := \sum_{i+j=a} \sum_{\substack{p\geq 1 \\ q\geq 1}} \frac{\dim V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p)}{(2(p+i)(q+n-i)+2(q+j)(p+n-j))^{2s}}.$$

Then, we have the following Proposition:

PROPOSITION 4.3.1. *We have*

$$\widetilde{\kappa}_a^n(s) = \sum_{i+j=a} \binom{n}{j}\binom{n}{i} \sum_{l=0}^{\infty} \binom{-2s}{l} (-1)^l (n-a)^{2l+1}$$

$$\cdot \left(\frac{1}{n!2^{n-1}} \sum_{k_1=0}^{n-1} e_{n-1-k_1}^{n-1}(2n-(2i+n-a), \cdots, \widehat{n-a}, \cdots, \right.$$

$$-\widehat{(n-a)}, \cdots, -(2i+n-a))$$

$$\left. \bigl(\zeta(2, A; 2s+l-1-k_1) - (n-a)^{-2s-l+1+k_1}\bigr) \right)$$

$$\cdot \left(\frac{1}{n!2^{n-1}} \sum_{k_2=0}^{n-1} e_{n-1-k_2}^{n-1}(2n-(2j+n-a), \cdots, \widehat{n-a}, \cdots, \right.$$

$$-\widehat{(n-a)}, \cdots, -(2j+n-a))$$

$$\left. \bigl(\zeta(2, A; 2s+l-k_2) - (n-a)^{-2s-l+k_2}\bigr) \right),$$

where $e_l(X_0, \ldots, X_n)$ *are the elementary symmetric polynomials of* $n-1$ *variables of degree* l*, for* $\alpha > 0$ *and* $\beta \in \mathbb{R}$

$$\zeta(\alpha, \beta; s) := \sum_{p=1}^{\infty} (\alpha p + \beta)^{-s},$$

and

$$A = \begin{cases} -1 & (\textit{if } n-a \textit{ is odd}), \\ 0 & (\textit{if } n-a \textit{ is even}). \end{cases}$$

PROOF OF PROPOSITION 4.3.1. First, in the same way as the calculation of the Hurwitz zeta function at the origin, we have

$$\begin{aligned} &\frac{1}{(2(p+i)(q+n-i)+2(q+j)(p+n-j))^{2s}} \\ &= \frac{1}{((2p+n+i-j)(2q+n-i+j)-(n-i-j)^2)^{2s}} \\ &= \frac{1}{((2p+n+i-j)(2q+n-i+j))^{2s}\left(1-\frac{(n-i-j)^2}{(2p+n+i-j)(2q+n-i+j)}\right)^{2s}} \\ &= \sum_{l=0}^{\infty} \binom{-2s}{l} \frac{(-1)^l (n-i-j)^{2l}}{((2p+n+i-j)(2q+n-i+j))^{2s+l}}. \end{aligned}$$

By Weyl's dimensional formula, we obtain

$$\begin{aligned} &\dim V(q, \underline{1}_j, \underline{0}_{n-1-i-j}, \underline{-1}_i, -p) \\ &= \frac{(n-i-j)pq((2p+n+i-j)+(2q+n-i+j))}{2(p+i)(q+j)(p+n-j)(q+n-i)} \binom{n}{i}\binom{n}{j}\binom{p+n}{n}\binom{q+n}{n}. \end{aligned}$$

Therefore we see

$$
\begin{aligned}
&\widetilde{\kappa}_a^n(s)\\
&= \sum_{\substack{i+j=a\\ p,q\geq 1}} \frac{(n-i-j)qp(2p+n+i-j)+(2q+n-i+j))}{2(q+j)(p+i)(q+n-i)(p+n-j)}\\
&\quad \binom{n}{j}\binom{n}{i}\binom{q+n}{n}\binom{p+n}{n}\sum_{l=0}^{\infty}\binom{-2s}{l}\frac{(-1)^l(n-i-j)^{2l}}{((2p+n+i-j)(2q+n-i+j))^{2s+l}}\\
&= \sum_{l=0}^{\infty}\binom{-2s}{l}(-1)^l(n-a)^{2l+1}\\
&\quad \sum_{\substack{i+j=a\\ p,q\geq 1}} \frac{pq(2p+n+i-j)}{(p+i)(q+j)(p+n-j)(q+n-i)}\\
&\quad \cdot\binom{n}{i}\binom{n}{j}\binom{p+n}{n}\binom{q+n}{n}\frac{1}{((2p+n+i-j)(2q+n-i+j))^{2s+l}}\\
&= \sum_{i+j=a}\binom{n}{i}\binom{n}{j}\sum_{l=0}^{\infty}\binom{-2s}{l}(-1)^l(n-a)^{2l+1}\\
&\quad \cdot\Bigg(\frac{1}{n!2^{n-1}}\sum_{k_1=0}^{n-1}e_{n-1-k_1}^{n-1}(2n-(2i+n-a),\cdots,\widehat{n-a},\cdots,\\
&\qquad -\widehat{(n-a)},\cdots,-(2i+n-a))\zeta(2,n+i-j;2s+l-1-k_1)\Bigg)\\
&\quad \cdot\Bigg(\frac{1}{n!2^{n-1}}\sum_{k_2=0}^{n-1}e_{n-1-k_2}^{n-1}(2n-(2j+n-a),\cdots,\widehat{n-a},\cdots,\\
&\qquad -\widehat{(n-a)},\cdots,-(2j+n-a))\zeta(2,n-i+j;2s+l-k_2)\Bigg).
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{k_2=0}^{n-1}e_{n-1-k_2}^{n-1}&(2n-(2j+n-a),\cdots,\widehat{n-a},\cdots,\\
&-\widehat{(n-a)},\cdots,-(2j+n-a))(2b+1)^{k_2}=0
\end{aligned}
$$

for $1\leq 2b+1\leq n-i+j$ except for $2b+1=n-a$, we obtain the claim. □

We note that for $l\geq 1$ and $\beta\in\mathbb{R}$,

$$
\binom{-2s}{l}=\frac{2(-1)^l}{l}s+\mathcal{O}(s^1),\qquad \operatorname{Res}_{s=1}\zeta(2,\beta;s)=\frac{1}{2}.
$$

If $n=1$ and $a=0$, then

$$
\widetilde{\kappa}_0^1(s)=\sum_{l=0}^{\infty}\binom{-2s}{l}(-1)^l\left(\zeta(2,-1;2s+l-1)-1\right)\left(\zeta(2,-1;2s+l)-1\right).
$$

Substituting $s = 0$ gives

$$\begin{aligned}\widetilde{\kappa}_0^1(0) &= (\zeta(2,-1;-1)-1)(\zeta(2,-1;0)-1) \\ &\quad + \frac{2}{1}(\zeta(2,-1;0)-1)\frac{1}{2}\operatorname{Res}_{t=1}(\zeta(2,-1;t)) \\ &\quad + \frac{2}{2}\frac{1}{2}\operatorname{Res}_{t=1}(\zeta(2,-1;t))(\zeta(2,-1;2)-1) \\ &= \frac{1}{6}+\frac{\pi^2}{32}. \end{aligned} \tag{4.3.1}$$

If $n = 2$ and $a = 0$, then

$$\widetilde{\kappa}_0^2(s) = \frac{1}{16}\sum_{l=0}^{\infty}\binom{-2s}{l}(-1)^l 2^{2l+1} \left(\zeta(2,0;2s+l-2)-2^{-2s-l+2}\right)\left(\zeta(2,0;2s+l-1)-2^{-2s-l+1}\right).$$

Substituting $s = 0$ gives

$$\widetilde{\kappa}_0^2(0) = \frac{\pi^2}{18}. \tag{4.3.2}$$

If $n = 2$ and $a = 1$, then

$$\widetilde{\kappa}_1^2(s) = \frac{1}{4}\sum_{l=0}^{\infty}\binom{-2s}{l}(-1)^l \sum_{k_1=0}^{1} e^1_{1-k_1}(3)e^1_{1-k_1}(-3) (\zeta(2,-1;2s+l-1-k_1)-1)(\zeta(2,-1;2s+l-k_1)-1)$$

Substituting $s = 0$ gives

$$\widetilde{\kappa}_1^2(0) = -\frac{31}{144}+\frac{17\pi^2}{256}. \tag{4.3.3}$$

4.4. The analytic torsion on S^3

We consider the case $n = 1$. We have

$$2\kappa_{\mathrm{R},\{a_k\}}(s) = \kappa_{\mathrm{I}}(s) + 4^s\Bigg(2(\kappa_{\mathrm{III},0}(s)+\kappa_{\mathrm{III},3}(s)) + \kappa_{\mathrm{V},0}(s)+\kappa_{\mathrm{V},3}(s) + 2(\kappa_{\mathrm{VI},2}(s)+\kappa_{\mathrm{VI},3}(s))\Bigg).$$

Let us calculate $\kappa_{\mathrm{III},0}(s)+\kappa_{\mathrm{III},3}(s)$. We have

$$\begin{aligned}\kappa_{\mathrm{III},0}(s)+\kappa_{\mathrm{III},3}(s) &= -(a_0^{-4s}+a_2^{-4s})\sum_{p=1}^{\infty}\frac{\dim V(0,-p)}{p^{2s}} \\ &= -(a_0^{-4s}+a_2^{-4s})\sum_{p=1}^{\infty}\frac{p+1}{p^{2s}} \\ &= -(a_0^{-4s}+a_2^{-4s})(\zeta(2s-1)+\zeta(2s)). \end{aligned} \tag{4.4.1}$$

Let us calculate $\kappa_{\mathrm{VI},1}(s)+\kappa_{\mathrm{VI},2}(s)$. We have

$$\begin{aligned}\kappa_{\mathrm{VI},1}(s)+\kappa_{\mathrm{VI},2}(s) &= 2a_1^{-4s}\sum_{p=1}^{\infty}\frac{\dim V(-1,-p)}{(p+1)^{2s}}\\ &= 2a_1^{-4s}\sum_{p=1}^{\infty}\frac{p}{(p+1)^{2s}}\\ &= 2a_1^{-4s}(\zeta(2s-1)-\zeta(2s)).\end{aligned}\tag{4.4.2}$$

From (4.4.1) and (4.4.2), it follows that

$$\begin{aligned}&\kappa_{\mathrm{III},0}(s)+\kappa_{\mathrm{III},3}(s)+\kappa_{\mathrm{VI},1}(s)+\kappa_{\mathrm{VI},2}(s)\\ &= -(a_0^{-4s}+a_2^{-4s})(\zeta(2s-1)+\zeta(2s))+2a_1^{-4s}(\zeta(2s-1)-\zeta(2s))\\ &= (-a_0^{-4s}+2a_1^{-4s}-a_2^{-4s})\zeta(2s-1)+(-a_0^{-4s}-2a_1^{-4s}-a_2^{-4s})\zeta(2s).\end{aligned}$$

We differentiate the above function at the origin:

$$\begin{aligned}&\kappa'_{\mathrm{III},0}(0)+\kappa'_{\mathrm{III},3}(0)+\kappa'_{\mathrm{VI},1}(0)+\kappa'_{\mathrm{VI},2}(0)\\ &=4\left(\log\frac{a_0a_2}{a_1^2}\right)\zeta(-1)+4\left(\log a_0a_1^2a_2\right)\zeta(0)-8\zeta'(0).\end{aligned}\tag{4.4.3}$$

Let us calculate $\kappa_{\mathrm{V},0}(s)+\kappa_{\mathrm{V},3}(s)$. We have

$$\kappa_{\mathrm{V},0}(s)+\kappa_{\mathrm{V},3}(s)=4^s(-a_0^{-4s}+2a_1^{-4s}-a_2^{-4s})\tilde{\kappa}_0^1(s).$$

Derivating the above function at the origin and (4.3.1) give

$$\begin{aligned}\kappa'_{\mathrm{V},0}(0)+\kappa'_{\mathrm{V},3}(0) &= 4(\log a_0-2\log a_1+\log a_2)\tilde{\kappa}_0^1(0)\\ &= 4(\log a_0-2\log a_1+\log a_2)\left(\frac{1}{6}+\frac{\pi^2}{32}\right).\end{aligned}\tag{4.4.4}$$

From (4.4.3) and (4.4.4), we obtain the value of the contact torsion on S^3.

4.5. The analytic torsion on S^5

We consider the case $n=2$. We have

$$\begin{aligned}2\kappa_{\mathrm{R},\{a_k\}}(s)=\kappa_{\mathrm{I}}(s)+4^s\Bigg(&\kappa_{\mathrm{II},0}(s)+\kappa_{\mathrm{II},5}(s)+2\sum_{0\le k\le 1}(\kappa_{\mathrm{III},k}(s)\kappa_{\mathrm{III},5-k}(s))\\ &+\kappa_{\mathrm{V},1}(s)+\kappa_{\mathrm{V},4}(s)+2(\kappa_{\mathrm{VI},2}(s)\kappa_{\mathrm{VI},3}(s))\Bigg).\end{aligned}$$

Let us calculate $\kappa_{\mathrm{III},k}(s)+\kappa_{\mathrm{III},5-k}(s)$. We see

$$\kappa_{\mathrm{III},k}(s)+\kappa_{\mathrm{III},5-k}(s)=(-1)^{k+1}(a_k^{-4s}+a_{4-k}^{-4s})\sum_{p=1}^{\infty}\frac{\dim V(0,\underline{-1}_k,-p)}{((p+k)(2-k))^{2s}},$$

$$\kappa_{\mathrm{VI},2}(s)+\kappa_{\mathrm{VI},3}(s)=-2a_2^{-4s}\sum_{p=1}^{\infty}\frac{\dim V(\underline{-1}_2,-p)}{(p+k)^{2s}},$$

where

$$\tilde{\kappa}_{\mathrm{III},k}(s):=\sum_{p=1}^{\infty}\frac{\dim V(\underline{0}_{2-k},\underline{-1}_k,-p)}{(p+k)^{2s}}.$$

Thus

$$\sum_{k=0,1} \left(\kappa_{\mathrm{III},k}(s) + \kappa_{\mathrm{III},5-k}(s)\right) + \kappa_{\mathrm{VI},2}(s) + \kappa_{\mathrm{VI},3}(s)$$
$$= \sum_{k=0,1} (-1)^{k+1} \left(\left(\sqrt{2-k}a_k\right)^{-4s} + \left(\sqrt{2-k}a_{4-k}\right)^{-4s} \right) \tilde{\kappa}_{\mathrm{III},k}(s)$$
$$- 2a_2^{-4s}\tilde{\kappa}_{\mathrm{III},2}(s).$$

Noting that

$$(p+2)(p+1)p = \sum_{l=0}^{3} e_{3-l}(2-k, 1-k, -k)(p+k)^l,$$

we have

$$\tilde{\kappa}_{\mathrm{III},k}(s) = \sum_{p=1}^{\infty} \frac{1}{2}\binom{2}{k} \frac{(p+2)(p+1)p}{p+k} \frac{1}{(p+k)^{2s}}$$
$$= \frac{1}{2}\binom{2}{k} \sum_{l=0}^{2} \sum_{p=1}^{\infty} \frac{e_l(2-k, 1-k, -k)}{(p+k)^{2s+1-l}}$$
$$= \frac{1}{2}\binom{2}{k} \sum_{l=1}^{3} e_{3-l}(2-k, 1-k, -k)\zeta(2s+1-l).$$

Then we get

$$\tilde{\kappa}_{\mathrm{III},0}(0) = -\frac{5}{8}, \qquad \tilde{\kappa}_{\mathrm{III},1}(0) = \frac{1}{2}, \qquad \tilde{\kappa}_{\mathrm{III},2}(0) = -\frac{3}{8}.$$

Hence

$$\sum_{k=0,1} \left(\kappa'_{\mathrm{III},k}(0) + \kappa'_{\mathrm{III},5-k}(0)\right) + \kappa'_{\mathrm{VI},2}(0) + \kappa'_{\mathrm{VI},3}(0)$$
$$= \sum_{k=0}^{2} (-1)^{k+1} 4\tilde{\kappa}'_{\mathrm{III},k}(0)$$
$$- 4 \sum_{k=0,1} (-1)^{k+1} \left(\log(2-k)a_k a_{4-k}\right) \tilde{\kappa}_{\mathrm{III},k}(0)$$
$$+ 8\log a_2 \cdot \tilde{\kappa}_{\mathrm{III},2}(0)$$
$$= 6\log 2\pi - \frac{5}{2}\log 2a_0a_4 + 2\log a_1a_3 - 3\log a_2. \tag{4.5.1}$$

Let us calculate $\kappa_{\mathrm{II},0}(s) + \kappa_{\mathrm{II},5}(s)$. We have

$$\kappa_{\mathrm{II},0}(s) + \kappa_{\mathrm{II},5}(s)$$
$$= 4^s \left(-a_0^{-4s} + (\sqrt{2}a_1)^{-4s} - a_4^{-4s} + (\sqrt{2}a_3)^{-4s}\right) \widetilde{\kappa}_0^2(s).$$

From (4.3.2), it follows that

$$\kappa'_{\mathrm{II},0}(0) + \kappa'_{\mathrm{II},5}(0) = 4 + \left(\log \frac{2a_0a_4}{a_1a_3}\right) \frac{\pi^2}{18}. \tag{4.5.2}$$

Let us calculate $\kappa_{\mathrm{V},1}(s) + \kappa_{\mathrm{V},4}(s)$. We have

$$\kappa_{\mathrm{V},1}(s) + \kappa_{\mathrm{V},4}(s) = 4^s \left(a_1^{-4s} - 2a_2^{-4s} + a_3^{-4s}\right) \widetilde{\kappa}_1^2(s).$$

From (4.3.3), we see

$$\kappa'_{\mathrm{V},1}(0) + \kappa'_{\mathrm{V},4}(0) = -4 \left(\log \frac{a_1 a_3}{a_2^2} \right) \left(-\frac{31}{144} + \frac{17\pi^2}{256} \right). \tag{4.5.3}$$

From (4.5.1), (4.5.2) and (4.5.3), we obtain the value of the contact torsion on S^5.

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