博士論文

論文題目　Local in time solvability for reaction-diffusion systems with rapidly growing nonlinear terms

（速く増大する非線形項を持つ連立反応拡散方程式の時間局所可解性）

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Abstract

In this dissertation we consider initial value problems (initial boundary value problems) for reaction-diffusion systems with rapidly growing nonlinear terms. The dissertation consists of three chapters. In each chapter the aim is to obtain integrability conditions of initial data which determine the existence/nonexistence of a local in time solution.

In Chapters 1 and 2 we study the reaction-diffusion system

$$(1) \qquad \begin{cases} \partial_t u = \Delta u + f(u,v) & \text{in } \Omega \times (0,T), \\ \partial_t v = \Delta v + g(u,v) & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is the entire $\mathbb{R}^N$ or a bounded domain in $\mathbb{R}^N$ with C^2 boundary, $N \geq 1$, $T > 0$ and the initial functions u_0 and v_0 are nonnegative. If the boundary does not exist, the boundary condition is not imposed in (1).

It is well known that the nonlinear reaction-diffusion equation

$$(2) \qquad \begin{cases} \partial_t u = \Delta u + f(u) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

possesses a local in time solution if $u_0 \in L^\infty(\Omega)$. On the other hand, if $u_0 \notin L^\infty(\Omega)$, it is known that the existence of a solution depends on the balance between the growth rate of the nonlinearity f and the singularity of the initial function u_0. When the nonlinear terms are polynomials such as u^p, the existence and nonexistence of a local in time solution were studied in both scalar problems and systems ([Weissler, Indiana Univ. Math. J. 1980] and [Quittner and Souplet, Proc. Roy. Soc. Edinburgh. 2001]). In [Fujishima and Ioku, J. Math. Pures Appl. 2018] the authors have studied (2) when $\Omega = \mathbb{R}^N$, $u_0 \geq 0$ and f grows rapidly such as $f(u) = e^u$, e^{u^2}. However, when the nonlinear terms are exponential, solutions of systems have not been well studied. In the two chapters we mainly consider (1) in the case where f and g have an exponential or superexponential growth.

In Chapter 1 we study (1) when it becomes a weakly coupled system. Here, a weakly coupled nonlinearity means that $f(u,v)$ and $g(u,v)$ depend on only v and u, respectively. In this chapter the nonlinear terms $f(u,v)$ and $g(u,v)$ are considered as $g(v)$ and $f(u)$, respectively. Let $\Omega = \mathbb{R}^N$ and we assume that $f, g \in C^1([0,\infty))$ satisfy some conditions including

$$\lim_{s\to\infty} f'(s)F(s) = 1 \text{ and } \lim_{s\to\infty} g'(s)G(s) = 1$$

where

$$F(s) := \int_s^\infty \frac{d\sigma}{f(\sigma)} \text{ and } G(s) := \int_s^\infty \frac{d\sigma}{g(\sigma)}.$$

A typical example satisfying all the conditions is $(f(u), g(v)) = (e^{u^p}, e^{v^q})$, $p \geq 1$ and $q \geq 1$. It is known that the above limits are the Hölder conjugate of the growth rate of f and g ([Miyamoto, J. Differential Equations, 2018]). We mainly consider the case where $f(u)$ and $g(v)$ have an exponential or superexponential growth, and deduce existence and nonexistence results.

In Chapter 2 we study (1) with $f(u,v) = e^{p_1 u + p_2 v}$ and $g(u,v) = e^{q_1 u + q_2 v}$, where $p_i \geq 0$ and $q_i \geq 0$, $i = 1, 2$, with $(p_1, p_2) \neq (0, 0)$ and $(q_1, q_2) \neq (0, 0)$. The domain Ω is $\mathbb{R}^N$ or a bounded domain in $\mathbb{R}^N$ with C^2 boundary, $N \geq 1$. In this chapter we obtain integrability conditions of (u_0, v_0) which explicitly determine the existence/nonexistence of a local in time solution in all the cases (p_1, p_2, q_1, q_2). For the existence result, we can take a wider class of initial functions than previously suggested (when a local in time solution is considered in the sense of the definition in this chapter). We also concretely gave an example of initial data which satisfy our integrability conditions but do not belong to the suggested class. Moreover, applying the obtained results, we proved existence and nonexistence results of (1) when $f(u,v)$ and $g(u,v)$ are both $e^{u^2+v^2}$.

In Chapter 3 we study the fractional in time weakly coupled reaction-diffusion system

$$\begin{cases} \partial_t^{\alpha_1} u = \Delta u + f_1(x,t,v) & \text{in } \Omega \times (0,T), \\ \partial_t^{\alpha_2} v = \Delta v + f_2(x,t,u) & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where $0 < \alpha_1 \leq \alpha_2 < 1$ and $T > 0$. The fractional derivative is meant in a generalized Caputo sense. The domain Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with C^2 boundary. We mainly consider the case where the nonlinear terms f_1 and f_2 polynomially grow with respect to v and u, respectively, and may have singularities with respect to x and t. In this chapter we obtain existence and nonexistence results by using methods including [Gal and Warma, HAL, 2017], which studies a scalar case. The integrability is determined by several parameters, which describe the balance between factors including the growth rate of f_1 and f_2 with respect to u or v and the singularity of u_0 and v_0. In particular, when $0 < \alpha_1 = \alpha_2 < 1$, we can explicitly determine the existence/nonexistence of a local in time solution for all the possible combinations of constants, which appear in our assumption. Moreover, when $0 < \alpha_1 = \alpha_2 < 1$, $f_1(x,t,v) = |v|^{p_1-1}v$ and $f_2(x,t,u) = |u|^{p_2-1}u$, our results correspond to the results of [Quittner and Souplet, Proc. Roy. Soc. Edinburgh. 2001], the case where the system has integer in time derivatives, $\alpha_1 = \alpha_2 = 1$.

We mention that Chapter 1, Chapter 2 and Chapter 3 correspond to [1], [2] and [3] below, respectively.

[1] Y. Miyamoto and M. Suzuki, *Weakly coupled reaction-diffusion systems with rapidly growing nonlinearities and singular initial data*, Nonlinear Anal. **189** (2019), 111576.

[2] M. Suzuki, *Local existence and nonexistence for reaction-diffusion systems with coupled exponential nonlinearities*, J. Math. Anal. Appl. **477** (2019), 776–804.

[3] M. Suzuki, *Local existence and nonexistence for fractional in time weakly coupled reaction-diffusion systems*, to appear in SN Partial Differ. Equ. Appl.

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Contents

Chapter 1

Weakly coupled reaction-diffusion systems with rapidly growing nonlinearities and singular initial data

We study existence and nonexistence of a local in time solution for the weakly coupled reaction-diffusion system

$$\begin{cases} \partial_t u = \Delta u + g(v) & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t v = \Delta v + f(u) & \text{in } \mathbb{R}^N \times (0,T), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \mathbb{R}^N, \end{cases}$$

where $f(u)$ and $g(v)$ grow rapidly, u_0 and v_0 are possibly unbounded nonnegative initial functions in $\mathbb{R}^N$ $(N \geq 1)$ and T is a positive constant. A typical example is $(f(u), g(v)) = (e^{u^p}, e^{v^q})$, $p \geq 1$ and $q \geq 1$. We show that if (u_0, v_0) satisfies a certain integrability condition, then the local in time solution exists. Moreover, we show that there exists (u_0, v_0) not satisfying the integrability condition such that the solution does not exist.

Keywords: Existence and nonexistence; Local in time solutions; Weakly coupled parabolic system; Superexponential nonlinearity.

1.1 Introduction

We study existence and nonexistence of a local in time solution for the reaction-diffusion system

$$\begin{cases} \partial_t u = \Delta u + g(v) & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t v = \Delta v + f(u) & \text{in } \mathbb{R}^N \times (0,T), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1.1}$$

where u_0 and v_0 are possibly unbounded nonnegative initial functions in $\mathbb{R}^N$ $(N \geq 1)$, $f, g \in C^1([0,\infty))$ are nonnegative increasing functions and T is a positive constant. In this chapter we mainly consider the case where f and g grow more rapidly than the exponential function. We define $F(s)$ and $G(s)$ by

$$F(s) := \int_s^\infty \frac{d\sigma}{f(\sigma)} \quad \text{and} \quad G(s) := \int_s^\infty \frac{d\sigma}{g(\sigma)}.$$

Throughout the present chapter we assume the following:

$$f(s) > 0,\ g(s) > 0,\ f'(s) > 0,\ g'(s) > 0,\ F(s) < \infty \text{ and } G(s) < \infty \text{ for } s > 0, \tag{1.1.2}$$

$$\lim_{s\to\infty} f'(s)F(s) = 1 \text{ and } \lim_{s\to\infty} g'(s)G(s) = 1. \tag{1.1.3}$$

In some cases we also assume the following:

$$\text{There exists } s_1 > 0 \text{ such that } f'(s)F(s) \le 1 \quad \text{for } s \ge s_1, \tag{1.1.4}$$

$$\text{there exists } s_2 > 0 \text{ such that } g'(s)G(s) \le 1 \quad \text{for } s \ge s_2. \tag{1.1.5}$$

A typical example is

$$f(u) = e^{u^p}\ (p \ge 1) \quad \text{and} \quad g(v) = e^{v^q}\ (q \ge 1). \tag{1.1.6}$$

In this case we can show that (1.1.2)–(1.1.5) hold. See [10, Theorem 5.2]. If $\frac{d^2}{ds^2}\log f(s) \ge 0$, then (1.1.4) holds. Hence, (1.1.4) and (1.1.5) indicate that the nonlinearities of (1.1.1) are exponential or superexponential.

Let us introduce functional spaces and a definition of a solution of (1.1.1). Let $B_\rho(x)$ denote the ball of radius $\rho > 0$ centered at $x \in \mathbb{R}^N$. For $1 \le p < \infty$, define the uniformly local L^p space $L^p_{ul,\rho}(\mathbb{R}^N)$ by

$$L^p_{ul,\rho}(\mathbb{R}^N) := \left\{ u \in L^p_{loc}(\mathbb{R}^N) : \|u\|_{L^p_{ul,\rho}(\mathbb{R}^N)} := \sup_{y\in\mathbb{R}^N} \left(\int_{B_\rho(y)} |u(x)|^p dx \right)^{1/p} < \infty \right\}.$$

The norm of $L^\infty_{ul,\rho}(\mathbb{R}^N)$ is defined by $\|u\|_{L^\infty_{ul,\rho}(\mathbb{R}^N)} := \sup_{y\in\mathbb{R}^N} \|u\|_{L^\infty(B_\rho(y))}$. Since $\|u\|_{L^\infty_{ul,\rho}(\mathbb{R}^N)} = \|u\|_{L^\infty(\mathbb{R}^N)}$, we see that $L^\infty_{ul,\rho}(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$. Let $\mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$ denote the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ in the space $L^p_{ul,\rho}(\mathbb{R}^N)$, i.e.,

$$\mathcal{L}^p_{ul,\rho}(\mathbb{R}^N) := \overline{BUC(\mathbb{R}^N)}^{\|\cdot\|_{L^p_{ul,\rho}(\mathbb{R}^N)}}.$$

We define

$$(e^{t\Delta}\phi)(x) := (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \phi(y) dy$$

for $\phi \in L^1_{ul,\rho}(\mathbb{R}^N)$. Then, $e^{t\Delta}\phi$ gives a solution for the heat equation in $\mathbb{R}^N$ with the initial function ϕ.

Definition 1.1.1 (Classical solution). Let X, Y be Banach spaces. We call a function $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ a classical solution of (1.1.1) in $X \times Y$ if u and v satisfy the equations in the classical sense in $\mathbb{R}^N \times (0,T)$, $\|u(t) - e^{t\Delta}u_0\|_X \to 0$ and $\|v(t) - e^{t\Delta}v_0\|_Y \to 0$ as $t \to 0$.

In Definition 1.1.1 we do not adopt the following: $\|u(t) - u_0\|_X \to 0$ and $\|v(t) - v_0\|_Y \to 0$, since $\|u(t)-u_0\|_{L^\infty(\mathbb{R}^N)}$ and $\|v(t)-v_0\|_{L^\infty(\mathbb{R}^N)}$ do not necessarily converge to zero.

In [17, Definition 15.1] they defined a classical $L^\infty(\mathbb{R}^N)$-solution, i.e., $u\in C^{2,1}(\mathbb{R}^N\times(0,T))\cap C((0,T);L^\infty(\mathbb{R}^N))$ is a classical solution for $t\in(0,T)$ and $\|u(t)-e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N))}\to 0$ as $t\to 0$. Note that $C((0,T);L^\infty(\mathbb{R}^N))\subset L^\infty_{loc}((0,T);L^\infty(\mathbb{R}^N))$. When u is a classical $L^\infty(\mathbb{R}^N)$-solution, u is a solution in the sense of Definition 1.1.1. On the other hand, if (u,v) satisfies

$$
\begin{aligned}
u(t) &= e^{t\Delta}u_0+\int_0^t e^{(t-s)\Delta}g(v(s))ds,\\
v(t) &= e^{t\Delta}v_0+\int_0^t e^{(t-s)\Delta}f(u(s))ds
\end{aligned}
\tag{1.1.7}
$$

for almost every $(x,t)\in\mathbb{R}^N\times(0,T)$, then (u,v) is called an integral solution. See [17, p.78]. We can show that if (u,v) is a solution in the sense of Definition 1.1.1, then (u,v) is an integral solution for small $t>0$.

We are in a position to state our existence result.

Theorem 1.1.2 (Local in time existence). *Let $N\geq 1$ and $\rho>0$, and let $f,g\in C^1([0,\infty))$ be functions such that (1.1.2) and (1.1.3) hold. Let u_0,v_0 be nonnegative measurable initial functions. Suppose that one of the following holds:*

(i) *the functions u_0,v_0 satisfy*

$$F(u_0)^{-r_1}\in L^1_{ul,\rho}(\mathbb{R}^N) \text{ for some } r_1>N/2 \text{ and } G(v_0)^{-r_2}\in L^1_{ul,\rho}(\mathbb{R}^N) \text{ for some } r_2>N/2,$$

(ii) $\lim_{s\to\infty} f(s)F(s)>0$, *(1.1.5) holds and u_0,v_0 satisfy*

$$F(u_0)^{-r_1}\in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N) \text{ for some } r_1>N/2 \text{ and } G(v_0)^{-\frac{N}{2}}\in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N),$$

(iii) $\lim_{s\to\infty} g(s)G(s)>0$, *(1.1.4) holds and u_0,v_0 satisfy*

$$F(u_0)^{-\frac{N}{2}}\in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N) \text{ and } G(v_0)^{-r_2}\in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N) \text{ for some } r_2>N/2,$$

(iv) $\lim_{s\to\infty} f(s)F(s)>0$, $\lim_{s\to\infty} g(s)G(s)>0$, *both (1.1.4) and (1.1.5) hold and u_0,v_0 satisfy*

$$F(u_0)^{-\frac{N}{2}}\in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N) \text{ and } G(v_0)^{-\frac{N}{2}}\in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N).$$

Then, there exist $T>0$ and a local in time nonnegative classical solution $(u,v)\in C^{2,1}(\mathbb{R}^N\times(0,T))\times C^{2,1}(\mathbb{R}^N\times(0,T))$ of (1.1.1) in $L^\infty(\mathbb{R}^N)\times L^\infty(\mathbb{R}^N)$ in the sense of Definition 1.1.1.

Note that even though the norm is considered to be $L^\infty(\mathbb{R}^N)$, the initial data can be singular. Moreover, the solution (u,v) of (1.1.1) satisfies (1.1.7) in $\mathbb{R}^N\times(0,T)$. The convergence in the sense of Definition 1.1.1 means that both the $L^\infty(\mathbb{R}^N)$ norms of the integral terms in (1.1.7) go to zero as $t\to 0$.

In Theorem 1.1.2 (ii) (iii) and (iv) we assume $\lim_{s\to\infty} f(s)F(s)>0$ or $\lim_{s\to\infty} g(s)G(s)>0$. The nonlinearity u^pe^u $(p\geq 1)$ satisfies the condition. However, e^{u^p} $(p>1)$ does not satisfy the condition.

The second result is about the nonexistence of a solution.

Theorem 1.1.3 (Local in time nonexistence). *Let $N \geq 1$ and $\rho > 0$, and let $f, g \in C^1([0,\infty))$ be functions such that (1.1.2)–(1.1.5) hold. Then, for any $0 < r < N/2$, there exists a nonnegative measurable function u_0 satisfying $F(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$ (resp. v_0 satisfying $G(v_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$) such that the following holds: For each fixed $a > 0$, if $v_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ satisfies $v_0 \geq a$ in $\mathbb{R}^N$ (resp. $u_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ satisfies $u_0 \geq a$ in $\mathbb{R}^N$), then (1.1.1) admits no local in time nonnegative classical solution $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ of (1.1.1) in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 1.1.1. Moreover, if $f'(0) > 0$ and $g'(0) > 0$, then the parameter a can be chosen as $a \geq 0$.*

It is worth noting that the simple constant initial condition above may lead to the nonexistence of a solution even in the case where $a = 0$ and $f(0) = 0$.

The two exponents r_1 and r_2 appearing in Theorem 1.1.2 (i) are important. The problem (1.1.1) has a classical solution if $r_1 > N/2$ and $r_2 > N/2$. On the other hand, Theorem 1.1.3 says that if $0 < r_1 < N/2$ or $0 < r_2 < N/2$, there is a nonnegative initial function such that no classical solution exists. If $r_1 = N/2$ or $r_2 = N/2$, we need limit conditions of f and g to apply Theorem 1.1.2 (ii) to (iv). We leave open the case where $r_1 = N/2$ or $r_2 = N/2$, and the limit conditions do not hold.

Example 1.1.4. *(i) If*

$$f(u) = e^{-u+e^u} \quad \text{and} \quad g(v) = e^{-v+e^v},$$

then $F(u) = e^{-e^u}$ and $G(v) = e^{-e^v}$. Hence, (1.1.2)–(1.1.5) are satisfied. Theorem 1.1.2 (i) tells us that the solution exists if both $F(u_0)^{-r}$ and $G(v_0)^{-r}$ are in $L^1_{ul,\rho}(\mathbb{R}^N)$ for some $r > N/2$. However, it follows from Theorem 1.1.3 that, for each $0 < r < N/2$, there is (u_0, v_0) such that $F(u_0)^{-r}$ or $G(v_0)^{-r}$ is in $L^1_{ul,\rho}(\mathbb{R}^N)$ and the solution does not exist.
(ii) If f and g are given by (1.1.6), then (1.1.2)–(1.1.5) are satisfied. Therefore, the same results as (i) hold. Moreover, if $(p,q) = (1,1)$, then $\lim_{s\to\infty} f(s)F(s) = 1$ and $\lim_{s\to\infty} g(s)G(s) = 1$, and hence Theorem 1.1.2 (ii)–(iv) are also applicable.
(iii) If

$$f(u) = \underbrace{\exp(\cdots\exp(u)\cdots)}_{n\text{times}}\ (n \geq 1) \quad \text{and} \quad g(v) = \underbrace{\exp(\cdots\exp(v)\cdots)}_{m\text{times}}\ (m \geq 1),$$

then (1.1.2)–(1.1.5) are satisfied. See [14, Section 2.3]. Therefore, the same results as (i) hold.

Let us mention related results. First we consider the reaction-diffusion equation

$$\begin{cases} \partial_t u = \Delta u + f(u) & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1.8}$$

where we impose the Dirichlet boundary condition if Ω has a boundary. We assume that f is of class C^1. If $u_0 \in L^\infty(\Omega)$, then it is well known that (1.1.8) has a local in time solution for arbitrary nonlinearities. On the other hand, if $u_0 \notin L^\infty(\Omega)$, it is known that the existence of a solution depends on the balance between the growth rate of the nonlinearity f and the singularity of the initial function u_0. Weissler [18] studied the Dirichlet problem when Ω is bounded. In the model case $f(u) = |u|^{\nu-1}u$ ($\nu > 1$) he constructed a solution when $u_0 \in L^r(\Omega)$ for $r > \frac{N}{2}(\nu - 1)$ and $r \geq 1$, or $r = \frac{N}{2}(\nu-1) > 1$. Baras-Pierre [2] extended results in [18]. Peral-Vázquez [15] studied various properties of solutions when $f(u) = e^u$ and Ω is a ball. Laister *et.al.* [12] studied the optimal growth rate of f. Specifically, when $u_0 \in L^r(\Omega)$ and $r > 1$, the solution exists if $\limsup_{s\to\infty} s^{-(1+2r/N)} f(s) < \infty$, and there exists $u_0 \in L^r(\Omega)$ such that the solution does not exist if $\limsup_{s\to\infty} s^{-(1+2r/N)}f(s) = \infty$ and $r \geq 1$. Fujishima-Ioku [10] obtained the optimal integrability condition of the initial data if the growth rate of f is given. Suppose that the limit

$$q := \lim_{s\to\infty} f'(s)F(s) \tag{1.1.9}$$

exists and $q - 1 < N/2$. If $r > N/2$, then under some additional assumptions they constructed a solution with u_0 satisfying $F(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$. If $r < N/2$, then under some additional assumptions they showed that there exists a nonnegative u_0 satisfying $F(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$ such that a solution does not exist. When $f(u) = u^\nu$, we see that $F(u)^{-r} = Cu^{r(\nu-1)}$, and hence it is an extension for nonnegative solutions of [18]. They also obtained existence results for the borderline case $r = N/2$. These results hold for the case $q = 1$. The exponent (1.1.9) is the Hölder conjugate of the growth rate of f, since a formal calculation gives

$$q = \lim_{s\to\infty} \frac{F'(s)}{(1/f'(s))'} = \lim_{s\to\infty} \frac{f'(s)^2}{f(s)f''(s)} = \lim_{s\to\infty} \frac{1/\left(1 - \frac{f(s)f''(s)}{f'(s)^2}\right)}{1/\left(1 - \frac{f(s)f''(s)}{f'(s)^2}\right) - 1} = \frac{\lim\limits_{s\to\infty} \dfrac{(s)'}{(f(s)/f'(s))'}}{\lim\limits_{s\to\infty} \dfrac{(s)'}{(f(s)/f'(s))'} - 1} = \frac{p}{p-1}$$

and $p = \lim_{s\to\infty} sf'(s)/f(s)$ is the growth rate of f. Therefore, if $q = 1$, then f grows faster than any algebraic function. See the assumption (1.1.3). The exponent (1.1.9) was introduced by Dupaigne-Farina [6]. The purpose of the chapter is to extend the result of [10] to the system (1.1.1).

For the proof of the theorems, we introduce the reaction-diffusion system

$$\begin{cases} \partial_t u = \Delta u + \tilde{f}(u,v) & \text{in } \Omega \times (0,T), \\ \partial_t v = \Delta v + \tilde{g}(u,v) & \text{in } \Omega \times (0,T), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \Omega. \end{cases} \tag{1.1.10}$$

The exponential or superexponential system (1.1.1) can be reduced into (1.1.10) with

$$\tilde{f}(u,v) = C_1 u^{1+\varepsilon_1} v^{3+\varepsilon_1} \quad \text{and} \quad \tilde{g}(u,v) = C_2 u^{3+\varepsilon_2} v^{1+\varepsilon_2}, \tag{1.1.11}$$

where $C_i > 0$ and $\varepsilon_i \geq 0$ for $i = 1, 2$. Then a supersolution for (1.1.1) can be constructed by using a solution of the system (1.1.10) with (1.1.11).

We recall previous research. A typical example of (1.1.10) is

$$\tilde{f}(u,v) = v^q \quad \text{and} \quad \tilde{g}(u,v) = u^p. \tag{1.1.12}$$

The problem (1.1.10) with (1.1.12) is called a weakly coupled system, which was studied in [1, 4, 5, 7–9, 11, 16]. See [17] for a survey. In [7, 8] initial functions are assumed to be bounded. Quittner-Souplet [16] studied (1.1.12) with singular initial data when Ω is bounded. Among other things, they proved the existence (resp. nonexistence) of a solution if $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ and

$$\frac{N}{2} \max\left\{\frac{p}{r_1} - \frac{1}{r_2}, \frac{q}{r_2} - \frac{1}{r_1}\right\} \leq 1, \ p, q > 1 \text{ and } r_1, r_2 > 1 \tag{1.1.13}$$

(resp. $\frac{N}{2} \max\left\{\frac{p}{r_1} - \frac{1}{r_2}, \frac{q}{r_2} - \frac{1}{r_1}\right\} > 1$, $p, q > 0$ and $r_1, r_2 \geq 1$). When the domain Ω is possibly unbounded, Ishige *et al.* [11] constructed a solution if $(u_0, v_0) \in L^{r_1,\infty}_{ul,\rho}(\Omega) \times L^{r_2,\infty}_{ul,\rho}(\Omega)$,

$\frac{N}{2}\max\left\{\frac{p}{r_1}-\frac{1}{r_2},\frac{q}{r_2}-\frac{1}{r_1}\right\}\le 1$, $p,q>0$ and $r_1,r_2\ge 1$. Here we omit the definition of $L^{r_1,\infty}_{ul,\rho}(\Omega)$. See [11, p.6087] for details. However, it is known that $L^{r_1}_{ul,\rho}(\Omega)\subset L^{r_1,\infty}_{ul,\rho}(\Omega)$ if $1<r_1<\infty$. Our problem (1.1.1) can be seen as an exponential or superexponential version of (1.1.12).

Let us explain technical details. In the existence part we use change of variables. The problem (1.1.10) with (1.1.11), which is (1.3.21), is called a strongly coupled system. Let (U,V) be a solution of (1.3.21). Then $(F^{-1}(U^{-3}),G^{-1}(V^{-3}))$ becomes a supersolution for (1.1.1), where F^{-1} and G^{-1} are the inverse functions of F and G, respectively. When a supersolution for (1.1.1) exists, a solution can be constructed by induction method. Therefore, we can obtain a solution of (1.1.1) if we have a solution of the system (1.3.21). In constructing a solution of the system we use the method introduced in [11]. Specifically, we prove the existence of a solution u of (1.1.8), where f has a polynomial growth. Then $(u^{1/r_1},u^{1/r_2})$ becomes a supersolution for the system with different (r_1,r_2). See Proposition 1.3.1 for details. By induction method we obtain a solution of the system, and hence we also obtain a solution of (1.1.1).

In the nonexistence part we give an initial function u_0 of an integral system such that the solution does not exist. Since the singularity of u_0 is strong, $\|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)}$ diverges rapidly as $t\to 0$. This contradicts the upper bound of $\|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)}$ given in Lemma 1.4.2 which holds for (u_0,v_0) satisfying the integrability condition in Theorem 1.1.3. Thus we obtain the nonexistence of the solution of the integral system. Using this initial function, we prove the nonexistence of the solution of (1.1.1). In [18] Weissler established the nonexistence result in the scalar case. This method was refined in [2, 19]. We extend the method to the system case. A careful choice of various exponents is needed throughout the present chapter.

This chapter consists of four sections. In Section 1.2 we prove fundamental lemmas which are used in the proof of Theorems 1.1.2 and 1.1.3. In Sections 1.3 and 1.4 we prove Theorems 1.1.2 and 1.1.3, respectively.

1.2 Preliminaries

In this section we recall known results and prove some lemmas which are natural extensions of the scalar case.

For any set X and the mappings $a=a(x)$ and $b=b(x)$ from X to $[0,\infty)$, we say

$$a(x)\lesssim b(x)\quad\text{for all } x\in X$$

if there exists a positive constant C such that $a(x)\le Cb(x)$ for all $x\in X$.

Lemma 1.2.1. ([13, Corollary 3.1]) *Let $N\ge 1$ and $1\le p\le q\le\infty$. Then there exists $C_1=C_1(N,p,q)>0$ such that*

$$\|e^{t\Delta}u\|_{L^q_{ul,\rho}(\mathbb{R}^N)}\le C_1\left(\rho^{-N\left(\frac{1}{p}-\frac{1}{q}\right)}+t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right)\|u\|_{L^p_{ul,\rho}(\mathbb{R}^N)}$$

for $t>0$, $\rho>0$ and $u\in L^p_{ul,\rho}(\mathbb{R}^N)$.

Proposition 1.2.2. *The following (i) and (ii) hold:*
(i) (Subcritical case) Let $N\ge 1$, $p>1$ and $\tilde{f}\in C^1([0,\infty))$. Assume that $\tilde{f}(s)\ge 0$ for $s\ge 0$ and that $\tilde{f}'(s)=O(s^{p-1})$ as $s\to\infty$. If $q\ge 1$ and $q>\frac{N}{2}(p-1)$, then, for a nonnegative function

$w_0 \in L^q_{ul,\rho}(\mathbb{R}^N)$, there exist $T > 0$ and a nonnegative classical solution of

$$
\begin{cases} \partial_t w = \Delta w + \tilde{f}(w) & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = w_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{1.2.1}
$$

in $L^q_{ul,\rho}(\mathbb{R}^N)$ in the sense of Definition 1.1.1 which satisfies

$$
\|w(t) - e^{t\Delta}w_0\|_{L^q_{ul,\rho}(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t \to 0. \tag{1.2.2}
$$

Moreover, $w \in C((0,T); L^q_{ul,\rho}(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T); L^\infty(\mathbb{R}^N)) \cap C^{2,1}(\mathbb{R}^N \times (0,T))$.
(ii) (Critical case) Let $N \geq 1$, $p > 1$ and $\tilde{f} \in C^1([0,\infty))$. Assume that $\tilde{f}$ satisfies the assumptions of (i). If $q := \frac{N}{2}(p-1) > 1$, then, for a nonnegative function $w_0 \in \mathcal{L}^q_{ul,\rho}(\mathbb{R}^N)$, there exist $T > 0$ and a nonnegative classical solution of (1.2.1) in $L^q_{ul,\rho}(\mathbb{R}^N)$ in the sense of Definition 1.1.1 which satisfies (1.2.2). Moreover, $w \in C([0,T); \mathcal{L}^q_{ul,\rho}(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T); L^\infty(\mathbb{R}^N)) \cap C^{2,1}(\mathbb{R}^N \times (0,T))$.

Furthermore, for both of (i) and (ii), the following holds: There exists $C_2 = C_2(N,p,q,\rho) > 0$ such that

$$
\|w(t)\|_{L^r_{ul,\rho}(\mathbb{R}^N)} \leq C_2 t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|w_0\|_{L^q_{ul,\rho}(\mathbb{R}^N)} \tag{1.2.3}
$$

for sufficiently small $t > 0$ and $r \in [q,\infty]$.

The existence result in Proposition 1.2.2 can be proved using the contraction mapping theorem in [3, 18]. The inequality (1.2.3) is obtained in a similar way to [17]. We omit the proof. See also [10, Propositions 2.2 and 2.3].

Remark 1.2.3. ([10, Remark 2.2 (ii)]) *Assume that all the assumptions of Proposition 1.2.2 (ii) hold. Let $\alpha > 0$ and $\sigma := \frac{N}{2}\left(\frac{1}{q} - \frac{1}{\alpha}\right)$. If α satisfies $\max\{p,q\} < \alpha < pq$, then*

$$
\lim_{t\to 0} t^\sigma \|w(t)\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)} = 0.
$$

Lemma 1.2.4. *Let J be a function from $[0,\infty)$ to itself. If J is convex, then, for a nonnegative function $\phi \in L^1_{ul,\rho}(\mathbb{R}^N)$,*

$$
J(e^{t\Delta}\phi(x)) \leq [e^{t\Delta}(J\phi)](x) \quad \text{in} \quad \mathbb{R}^N \times (0,\infty).
$$

Lemma 1.2.4 follows from Jensen's inequality. For the proof in the case of the pure power nonlinearity see [18, Lemma 5.1].

Lemma 1.2.5. *Let $1 \leq p < \infty$. Then the following (i) and (ii) hold:*
(i) $u \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$ if and only if $|u|^p \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$.
(ii) Let $C > 0$. Then, for any $u \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$, $\max\{u, C\} \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$.

Proof. The assertion (i) follows from [10, Lemma 2.2, Remark 2.1].
We prove (ii). Let $u \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$. Then there exists a sequence $\{u_n\} \subset BUC(\mathbb{R}^N)$ such that $u_n \to u$ in $L^p_{ul,\rho}(\mathbb{R}^N)$ as $n \to \infty$. Put $v_n := \max\{u_n, C\}$ $(n \geq 1)$. We see that $\{v_n\} \subset BUC(\mathbb{R}^N)$ and that

$$
|\max\{u,C\} - v_n| \leq |u - u_n| \to 0 \;\text{ in } L^p_{ul,\rho}(\mathbb{R}^N) \text{ as } n \to \infty.
$$

Hence, $\max\{u, C\} \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$. $\square$

Proposition 1.2.6. *Let $\tilde{f}(u,v), \tilde{g}(u,v) \in C^{1,1}([0,\infty)\times[0,\infty))$ satisfy the following:*

$$\tilde{f}(u,v)\ge 0,\ \ \tilde{f}_u(u,v)\ge 0,\ \ \tilde{f}_v(u,v)\ge 0,\ \ \tilde{g}(u,v)\ge 0,\ \ \tilde{g}_u(u,v)\ge 0 \text{ and } \tilde{g}_v(u,v)\ge 0$$

for $u\ge 0$ and $v\ge 0$. Let $u_0, v_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ be nonnegative functions. Suppose that there exist $T>0$ and nonnegative functions $(\bar{u},\bar{v}) \in C^{2,1}(\mathbb{R}^N\times(0,T))\times C^{2,1}(\mathbb{R}^N\times(0,T))$ such that

$$\begin{aligned} \bar{u}(t) &\ge e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\tilde{f}(\bar{u}(s),\bar{v}(s))ds, \\ \bar{v}(t) &\ge e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\tilde{g}(\bar{u}(s),\bar{v}(s))ds \end{aligned} \tag{1.2.4}$$

for almost every $(x,t)\in\mathbb{R}^N\times(0,T)$. Then there exists a solution $(u,v)\in C^{2,1}(\mathbb{R}^N\times(0,T))\times C^{2,1}(\mathbb{R}^N\times(0,T))$ of the integral system

$$\begin{aligned} u(t) &= e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\tilde{f}(u(s),v(s))ds \quad \text{in } \mathbb{R}^N\times(0,T), \\ v(t) &= e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\tilde{g}(u(s),v(s))ds \quad \text{in } \mathbb{R}^N\times(0,T) \end{aligned} \tag{1.2.5}$$

which satisfies

$$\begin{cases} \partial_t u = \Delta u + \tilde{f}(u,v) & \text{in } \mathbb{R}^N\times(0,T), \\ \partial_t v = \Delta v + \tilde{g}(u,v) & \text{in } \mathbb{R}^N\times(0,T). \end{cases}$$

Furthermore, $0\le u(x,t)\le \bar{u}(x,t)$ in $\mathbb{R}^N\times(0,T)$ and $0\le v(x,t)\le\bar{v}(x,t)$ in $\mathbb{R}^N\times(0,T)$.

Proposition 1.2.6 is a slight generalization of [11, Lemma 2.2], and the proof is similar to [11, Lemma 2.2]. However, we show the proof for readers' convenience.

Proof. Let $u_1 := 0$ and $v_1 := 0$. For $n\ge 2$, define the functions u_n and v_n by

$$\begin{aligned} u_n(t) &= e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\tilde{f}(u_{n-1}(s),v_{n-1}(s))ds, \\ v_n(t) &= e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\tilde{g}(u_{n-1}(s),v_{n-1}(s))ds. \end{aligned} \tag{1.2.6}$$

We use an induction method. Suppose that $\bar{u}(x,t)\ge u_{n-1}(x,t)$ and $\bar{v}(x,t)\ge v_{n-1}(x,t)$. We see that $\tilde{f}$ and $\tilde{g}$ are nondecreasing. Then it follows from (1.2.4) and (1.2.6) that

$$\bar{u}(t) \ge e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\tilde{f}(u_{n-1}(s),v_{n-1}(s))ds = u_n(t).$$

In the same way we can see that $\bar{v}(t)\ge v_n(t)$. Since $\bar{u}(x,t)\ge e^{t\Delta}u_0\ge 0 = u_1(x,t)$ and $\bar{v}(x,t)\ge 0 = v_1(x,t)$, it follows from induction that, for $n\in\mathbb{N}$, $\bar{u}(x,t)\ge u_n(x,t)$ and $\bar{v}(x,t)\ge v_n(x,t)$. By the definitions of u_n and v_n with $n=2$ we obtain

$$\begin{aligned} u_2(t) &= e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\tilde{f}(u_1(s),v_1(s))ds \ge 0 = u_1(t), \\ v_2(t) &= e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\tilde{g}(u_1(s),v_1(s))ds \ge 0 = v_1(t). \end{aligned}$$

Since $\tilde{f}$ and $\tilde{g}$ are nondecreasing, we have

$$\begin{aligned} u_3(t) &= e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\tilde{f}(u_2(s), v_2(s))ds \\ &\geq e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\tilde{f}(u_1(s), v_1(s))ds = u_2(t), \\ v_3(t) &= e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\tilde{g}(u_2(s), v_2(s))ds \\ &\geq e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\tilde{g}(u_1(s), v_1(s))ds = v_2(t). \end{aligned}$$

Repeating the above argument, we have

$$\begin{aligned} 0 = u_1 \leq u_2 \leq \cdots \leq u_{n-1} \leq u_n \leq \bar{u} \quad \text{in } \mathbb{R}^N \times (0,T), \\ 0 = v_1 \leq v_2 \leq \cdots \leq v_{n-1} \leq v_n \leq \bar{v} \quad \text{in } \mathbb{R}^N \times (0,T). \end{aligned}$$

Then, we can define the limit functions u and v by

$$u(x,t) := \lim_{n\to\infty} u_n(x,t) \quad \text{and} \quad v(x,t) := \lim_{n\to\infty} v_n(x,t),$$

respectively. Since $\tilde{f}$ and $\tilde{g}$ are nondecreasing, it follows from the monotone convergence theorem that (1.2.5) holds. Applying the standard regularity theory for parabolic equations, we obtain $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$. $\square$

The following lemma is a system version of [10, Lemma 2.3].

Lemma 1.2.7. *Let $\tilde{f}(u,v) \in C^{1,1}([0,\infty) \times [0,\infty))$ satisfy*

$$\tilde{f}(u,v) \geq 0, \;\; \tilde{f}_u(u,v) \geq 0 \;\text{ and }\; \tilde{f}_v(u,v) \geq 0$$

for $u \geq 0$ and $v \geq 0$. Let $T > 0$ and let $\bar{u}, \bar{v} \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N))$ such that $\bar{u} \geq 0$ and $\bar{v} \geq 0$. Then the following hold:
(i) Let $u_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$. Suppose that

$$\partial_t \bar{u} = \Delta \bar{u} + \tilde{f}(\bar{u}, \bar{v}) \quad \text{in } \mathbb{R}^N \times (0,T) \tag{1.2.7}$$

and

$$\lim_{t\to 0} \|\bar{u}(t) - e^{t\Delta}u_0\|_{L^1_{ul,\rho}(\mathbb{R}^N)} = 0. \tag{1.2.8}$$

Then $\bar{u}$ satisfies

$$\bar{u}(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\tilde{f}(\bar{u}(s), \bar{v}(s))ds \quad \text{in } \mathbb{R}^N \times (0,T). \tag{1.2.9}$$

(ii) Let $u_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$. Suppose that

$$\partial_t \bar{u} \geq \Delta \bar{u} + \tilde{f}(\bar{u}, \bar{v}) \quad \text{in } \mathbb{R}^N \times (0,T). \tag{1.2.10}$$

If either (1.2.8) holds or

$$\bar{u}(t) \geq e^{t\Delta}u_0 \quad in \quad \mathbb{R}^N \times (0,T), \tag{1.2.11}$$

then $\bar{u}$ satisfies

$$\bar{u}(t) \geq e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\tilde{f}(\bar{u}(s),\bar{v}(s))ds \quad in \quad \mathbb{R}^N \times (0,T). \tag{1.2.12}$$

Proof. (i) It follows from (1.2.7) that

$$\bar{u}(t) = e^{(t-\tau)\Delta}\bar{u}(\tau) + \int_\tau^t e^{(t-s)\Delta}\tilde{f}(\bar{u}(s),\bar{v}(s))ds \tag{1.2.13}$$

for $0<\tau<t<T$. See [17, p.77] for details. By Lemma 1.2.1 we have

$$\begin{aligned}
\|e^{(t-\tau)\Delta}\bar{u}(\tau) - e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} &= \|e^{(t-\tau)\Delta}(\bar{u}(\tau) - e^{\tau\Delta}u_0)\|_{L^\infty(\mathbb{R}^N)} \\
&\lesssim \left(\rho^{-N} + (t-\tau)^{-\frac{N}{2}}\right)\|\bar{u}(\tau) - e^{\tau\Delta}u_0\|_{L^1_{ul,\rho}(\mathbb{R}^N)} \\
&\to 0 \quad (\tau \to 0).
\end{aligned}$$

Let $\chi_{(\tau,t)}$ be a characteristic function. It follows from the monotone convergence theorem that

$$\lim_{\tau\to 0}\int_\tau^t e^{(t-s)\Delta}f(\overline{u}(s),\overline{v}(s))ds = \lim_{\tau\to 0}\int_0^t \chi_{(\tau,t)}(s)e^{(t-s)\Delta}f(\overline{u}(s),\overline{v}(s))ds = \int_0^t e^{(t-s)\Delta}f(\overline{u}(s),\overline{v}(s))ds$$

for each $(x,t)\in\mathbb{R}^N\times(0,T)$. Taking the limit of (1.2.13) as $\tau\to 0$, we obtain (1.2.9).
(ii) If (1.2.8) holds, then we can prove (1.2.12) in a similar way as above. On the other hand, if (1.2.11) holds, then it follows from (1.2.10) that

$$\bar{u}(t) \geq e^{(t-\tau)\Delta}e^{\tau\Delta}u_0 + \int_\tau^t e^{(t-s)\Delta}\tilde{f}(\bar{u}(s),\bar{v}(s))ds = e^{t\Delta}u_0 + \int_\tau^t e^{(t-s)\Delta}\tilde{f}(\bar{u}(s),\bar{v}(s))ds.$$

Taking the limit as $\tau\to 0$, we obtain (1.2.12). □

1.3 Existence result

In this section we prove Theorem 1.1.2. In the proof we use a solution of the problem

$$\begin{cases}
\partial_t u = \Delta u + C_1 u^{p_1}v^{p_2} & \text{in } \mathbb{R}^N\times(0,T), \\
\partial_t v = \Delta v + C_2 u^{q_1}v^{q_2} & \text{in } \mathbb{R}^N\times(0,T), \\
(u(x,0),v(x,0)) = (u_0(x),v_0(x)) & \text{in } \mathbb{R}^N,
\end{cases} \tag{1.3.1}$$

where C_1 and C_2 are arbitrary positive constants. In the study of (1.3.1) the following two exponents play an important role:

$$A_1 := 1 + \frac{p_1-1}{r_1} + \frac{p_2}{r_2} \quad\text{and}\quad A_2 := 1 + \frac{q_1}{r_1} + \frac{q_2-1}{r_2}, \tag{1.3.2}$$

where r_1 and r_2 are chosen in accordance with Proposition 1.3.1.

In [11, Theorem 4.1] the existence of a solution of (1.3.1) is established when the initial data is in $L^{r_1,\infty}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2,\infty}_{ul,\rho}(\mathbb{R}^N)$ which includes our space $L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$. However, [11, Theorem 4.1] does not claim in which sense the initial condition is satisfied. In Proposition 1.3.1 we establish the existence of a solution in the sense of Definition 1.1.1.

Proposition 1.3.1. *Let $N \geq 1$ and $\rho > 0$. Let $p_1, p_2, q_1, q_2 \geq 1$ and $C_1, C_2 > 0$. The following (i) and (ii) hold:*
(i) Let $1 \leq r_1, r_2 < \infty$. Suppose that

$$\frac{N}{2} \max\{A_1 - 1, A_2 - 1\} < 1. \tag{1.3.3}$$

Then for a nonnegative initial data $(u_0, v_0) \in L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$, there exist $T > 0$ and a local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)) \times C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N))$ of (1.3.1) in $L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$ in the sense of Definition 1.1.1.
(ii) Let $1 < r_1, r_2 < \infty$. Suppose that

$$\frac{N}{2} \max\{A_1 - 1, A_2 - 1\} = 1. \tag{1.3.4}$$

Then for a nonnegative initial data $(u_0, v_0) \in \mathcal{L}^{r_1}_{ul,\rho}(\mathbb{R}^N) \times \mathcal{L}^{r_2}_{ul,\rho}(\mathbb{R}^N)$, there exist $T > 0$ and a local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)) \times C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N))$ of (1.3.1) in $L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$ in the sense of Definition 1.1.1.

The conditions (1.3.3) and (1.3.4) in Proposition 1.3.1 appear in [11, Theorem 4.1] and these conditions are extensions of (1.1.13).

Proof. We prove (i). Let $w_0(x) := \max\{u_0(x)^{r_1} + v_0(x)^{r_2}, 1\}$. We see that $w_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$. We consider the problem

$$\begin{cases} \partial_t w = \Delta w + P(w) & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = w_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3.5}$$

where $P(w) := r_1 C_1 w^{A_1} + r_2 C_2 w^{A_2}$, C_1 and C_2 are positive constants given in (1.3.1) and A_1 and A_2 are given by (1.3.2). We can apply Proposition 1.2.2 (i) with $p := \max\{A_1, A_2\}$ and $q := 1$, because $P'(w) = O(w^{p-1})$ as $w \to \infty$ and the inequality $q > N(p-1)/2$ follows from (1.3.3). Let w be a solution of (1.3.5) given by Proposition 1.2.2 (i). Put $(\bar{u}, \bar{v}) := (w^{1/r_1}, w^{1/r_2})$. By direct calculation we have

$$\begin{cases} \partial_t \bar{u} \geq \Delta \bar{u} + C_1 \bar{u}^{p_1} \bar{v}^{p_2} & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t \bar{v} \geq \Delta \bar{v} + C_2 \bar{u}^{q_1} \bar{v}^{q_2} & \text{in } \mathbb{R}^N \times (0,T). \end{cases} \tag{1.3.6}$$

Since w is a classical solution of (1.3.5), it follows from Lemma 1.2.7 (i) that

$$w(t) = e^{t\Delta} w_0 + \int_0^t e^{(t-s)\Delta} P(w(s)) ds. \tag{1.3.7}$$

Let $J(s) := s^{r_1}$ $(s \geq 0)$. Since J is convex, it follows from Lemma 1.2.4 and (1.3.7) that

$$J(\bar{u}(t)) = w(t) \geq e^{t\Delta} w_0 \geq e^{t\Delta}(J(u_0)) \geq J(e^{t\Delta} u_0) \quad \text{in } \mathbb{R}^N \times (0,T).$$

Since J is strictly increasing, we have

$$\bar{u}(t) \geq e^{t\Delta}u_0. \tag{1.3.8}$$

Because of (1.3.6) and (1.3.8), we can apply Lemma 1.2.7 (ii), and hence

$$\bar{u}(t) \geq e^{t\Delta}u_0 + C_1 \int_0^t e^{(t-s)\Delta}\bar{u}(s)^{p_1}\bar{v}(s)^{p_2}ds. \tag{1.3.9}$$

In the same way we can obtain

$$\bar{v}(t) \geq e^{t\Delta}v_0 + C_2 \int_0^t e^{(t-s)\Delta}\bar{u}(s)^{q_1}\bar{v}(s)^{q_2}ds. \tag{1.3.10}$$

Let $\tilde{f}(u,v) := C_1 u^{p_1}v^{p_2}$ and $\tilde{g}(u,v) := C_2 u^{q_1}v^{q_2}$. By (1.3.9) and (1.3.10) we see that $\tilde{f}$ and $\tilde{g}$ satisfy all the assumptions of Proposition 1.2.6. It follows from Proposition 1.2.6 that (1.2.5) has a solution $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)) \times C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N))$. We prove the following limits:

$$\lim_{t\to 0} \|u(t) - e^{t\Delta}u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} = 0 \text{ and } \lim_{t\to 0} \|v(t) - e^{t\Delta}v_0\|_{L^{r_2}_{ul,\rho}(\mathbb{R}^N)} = 0. \tag{1.3.11}$$

Since $u \leq \bar{u} = w^{1/r_1}$ and $v \leq \bar{v} = w^{1/r_2}$, by (1.2.5) and Lemma 1.2.1 we have

$$\begin{aligned}
\|u(t) - e^{t\Delta}u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} &\leq C_1 \int_0^t \|e^{(t-s)\Delta}w(s)^{\frac{p_1}{r_1}+\frac{p_2}{r_2}}\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)}ds \\
&\lesssim \int_0^t \|w(s)^{\frac{p_1}{r_1}+\frac{p_2}{r_2}}\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)}ds \\
&= \int_0^t \|w(s)\|^{\frac{p_1}{r_1}+\frac{p_2}{r_2}}_{L^{r_1\left(\frac{p_1}{r_1}+\frac{p_2}{r_2}\right)}_{ul,\rho}(\mathbb{R}^N)}ds.
\end{aligned} \tag{1.3.12}$$

By (1.2.3) we have

$$\|w(s)\|_{L^{r_1\left(\frac{p_1}{r_1}+\frac{p_2}{r_2}\right)}_{ul,\rho}(\mathbb{R}^N)} \lesssim s^{-\frac{N}{2}\left\{1-\frac{1}{r_1\left(\frac{p_1}{r_1}+\frac{p_2}{r_2}\right)}\right\}}\|w_0\|_{L^1_{ul,\rho}(\mathbb{R}^N)} \tag{1.3.13}$$

for sufficiently small $s > 0$. It follows from (1.3.12) and (1.3.13) that

$$\|u(t) - e^{t\Delta}u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} \lesssim \int_0^t s^{-\frac{N}{2}\left(\frac{p_1-1}{r_1}+\frac{p_2}{r_2}\right)}ds\|w_0\|^{\frac{p_1}{r_1}+\frac{p_2}{r_2}}_{L^1_{ul,\rho}(\mathbb{R}^N)}, \tag{1.3.14}$$

provided that $t > 0$ is small. By (1.3.2) and (1.3.3) we see that $-\frac{N}{2}\left(\frac{p_1-1}{r_1}+\frac{p_2}{r_2}\right) = -\frac{N}{2}(A_1-1) > -1$. Thus, the right hand side of (1.3.14) converges to 0 as $t \to 0$. Hence, we obtain the first limit of (1.3.11). In the same way we can obtain the second limit of (1.3.11). Thus, (u,v) is a local in time classical solution of (1.3.1) in $L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$ in the sense of Definition 1.1.1. The proof of (i) is complete.

Next, we prove (ii). Since $r_1, r_2 > 1$, there exists a small $\varepsilon > 0$ such that

$$r_1' := \frac{r_1}{1+\varepsilon} > 1,\ r_2' := \frac{r_2}{1+\varepsilon} > 1 \text{ and } \varepsilon < \frac{2}{N}. \tag{1.3.15}$$

We define A_1' and A_2' by

$$A_1' := 1 + \frac{p_1 - 1}{r_1'} + \frac{p_2}{r_2'} \quad \text{and} \quad A_2' := 1 + \frac{q_1}{r_1'} + \frac{q_2 - 1}{r_2'}.$$

Then by (1.3.4) we have

$$\frac{N}{2} \max\left\{A_1' - 1, A_2' - 1\right\} = 1 + \varepsilon.$$

Let $w_0(x) := \max\{u_0(x)^{r_1'} + v_0(x)^{r_2'}, 1\}$. It follows from Lemma 1.2.5 that $w_0 \in \mathcal{L}^{1+\varepsilon}_{ul,\rho}(\mathbb{R}^N)$. By Proposition 1.2.2 (ii) we see that there exist a constant $T > 0$ and a nonnegative classical solution $w \in C([0,T) : \mathcal{L}^{1+\varepsilon}_{ul,\rho}(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)) \cap C^{2,1}(\mathbb{R}^N \times (0,T))$ for the problem

$$\begin{cases} \partial_t w = \Delta w + Q(w) & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = w_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3.16}$$

where $Q(w) := r_1' C_1 w^{A_1'} + r_2' C_2 w^{A_2'}$ and C_1 and C_2 are positive constants given in (1.3.1). Put $(\bar{u}, \bar{v}) := (w^{1/r_1'}, w^{1/r_2'})$. Then $(\bar{u}, \bar{v})$ satisfies (1.3.6). In the same way by Proposition 1.2.6 the integral system (1.2.5) with $\tilde{f}(u,v) = C_1 u^{p_1} v^{p_2}$ and $\tilde{g}(u,v) = C_2 u^{q_1} v^{q_2}$ possesses a solution (u,v) which satisfies (1.3.1). We prove the following limits:

$$\lim_{t\to 0} \|u(t) - e^{t\Delta} u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} = 0 \quad \text{and} \quad \lim_{t\to 0} \|v(t) - e^{t\Delta} v_0\|_{L^{r_2}_{ul,\rho}(\mathbb{R}^N)} = 0.$$

Let $\alpha_0 := 1 + 2(1+\varepsilon)/N$. By α we define $\alpha := \alpha_0 + \delta$, where $\delta > 0$ is small and δ is defined later. We show that Remark 1.2.3 is applicable for w if $p = \alpha_0$ and $q = 1 + \varepsilon$. By (1.3.15) we see that $\max\{p,q\} = \max\{\alpha_0, 1+\varepsilon\} = \alpha_0 < \alpha$. Since $\alpha_0 < \alpha_0(1+\varepsilon) = pq$ and δ is small, we see that $\alpha < pq$. Since $\max\{p,q\} < \alpha < pq$, Remark 1.2.3 can be applied. Let $k := (1+\varepsilon)(2/N + 1/r_1)$. Then,

$$\frac{p_1}{r_1'} + \frac{p_2}{r_2'} \leq \frac{2}{N}(1+\varepsilon) + \frac{1}{r_1'} = k \quad \text{and} \quad \frac{k}{\alpha} > \frac{1}{r_1}. \tag{1.3.17}$$

Note that the second inequality in (1.3.17) holds, since δ is small. Since $u \leq \bar{u} = w^{1/r_1'}$, $v \leq \bar{v} = w^{1/r_2'}$ and $w \geq e^{t\Delta} w_0 \geq 1$, it follows from (1.3.17) and Lemma 1.2.1 that

(1.3.18)

$$\begin{aligned}
\|u(t) - e^{t\Delta} u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} &\leq C_1 \int_0^t \|e^{(t-s)\Delta} w(s)^{\frac{p_1}{r_1'} + \frac{p_2}{r_2'}}\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} ds \\
&\leq C_1 \int_0^t \|e^{(t-s)\Delta} w(s)^k\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} ds \\
&\lesssim \int_0^t \left(\rho^{-N\left(\frac{k}{\alpha} - \frac{1}{r_1}\right)} + (t-s)^{-\frac{N}{2}\left(\frac{k}{\alpha} - \frac{1}{r_1}\right)}\right) \|w(s)^k\|_{L^{\frac{\alpha}{k}}_{ul,\rho}(\mathbb{R}^N)} ds \\
&\lesssim \int_0^t \left(\rho^{-N\left(\frac{k}{\alpha} - \frac{1}{r_1}\right)} + (t-s)^{-\frac{N}{2}\left(\frac{k}{\alpha} - \frac{1}{r_1}\right)}\right) s^{-\sigma k} ds \sup_{0<s<t} s^{\sigma k} \|w(s)\|^k_{L^\alpha_{ul,\rho}(\mathbb{R}^N)},
\end{aligned}$$

where $\sigma := \frac{N}{2}\left(\frac{1}{1+\varepsilon} - \frac{1}{\alpha}\right)$. We easily see that

$$-\frac{N}{2}\left(\frac{k}{\alpha} - \frac{1}{r_1}\right) > -1, \ -\sigma k > -1 \quad \text{and} \quad -\frac{N}{2}\left(\frac{k}{\alpha} - \frac{1}{r_1}\right) - \sigma k + 1 = 0.$$

Thus,

$$\int_0^t \left(\rho^{-N\left(\frac{k}{\alpha}-\frac{1}{r_1}\right)} + (t-s)^{-\frac{N}{2}\left(\frac{k}{\alpha}-\frac{1}{r_1}\right)}\right) s^{-\sigma k} ds < C \tag{1.3.19}$$

uniformly for small $t > 0$. Therefore, by (1.3.18), (1.3.19) and Remark 1.2.3 we see that $\|u(t) - e^{t\Delta}u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} \to 0$ as $t \to 0$. In the same way we can obtain $\lim_{t\to 0} \|v(t) - e^{t\Delta}v_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} = 0$. The proof of (ii) is complete. □

Lemma 1.3.2. ([10, Lemma 3.1]) *Let $f \in C^1([0,\infty))$ satisfy (1.1.2) and (1.1.3). Define $h(s) := f(F^{-1}(s))$. Then for any $\varepsilon > 0$,*

$$h(s) \lesssim s^{-1-\varepsilon}$$

for sufficiently small $s > 0$. Moreover, if f satisfies (1.1.4) for some $s_1 > 0$, then $h(s) \lesssim s^{-1}$ for sufficiently small $s > 0$.

Proof. Since $(F^{-1}(s))' = -f(F^{-1}(s)) = -h(s)$, we have

$$h'(s) = f'(F^{-1}(s))(F^{-1}(s))' = -f'(F^{-1}(s))h(s).$$

Let $\varepsilon > 0$. Since $F^{-1}(s)$ is nonincreasing with respect to s and $F^{-1}(s) \to \infty$ as $s \to 0$, it follows from (1.1.3) that

$$sf'(F^{-1}(s)) = f'(F^{-1}(s))F(F^{-1}(s)) \le 1 + \varepsilon \tag{1.3.20}$$

for sufficiently small $s > 0$. Then we have

$$\frac{h'(s)}{h(s)} \ge -(1+\varepsilon)s^{-1}.$$

Integrating both sides, we have $h(s) \lesssim s^{-1-\varepsilon}$ for sufficiently small $s > 0$. If f satisfies (1.1.4) for some $s_1 > 0$, then (1.3.20) holds with $\varepsilon = 0$. Thus we prove Lemma 1.3.2. □

Proof of Theorem 1.1.2. We consider the initial value problem

$$\begin{cases} \partial_t U = \Delta U + C_1 U^{1+\varepsilon_1} V^{3+\varepsilon_1} & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t V = \Delta V + C_2 U^{3+\varepsilon_2} V^{1+\varepsilon_2} & \text{in } \mathbb{R}^N \times (0,T), \\ (U(x,0), V(x,0)) = (U_0(x), V_0(x)) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3.21}$$

where $U_0 := \max\{F(u_0)^{-1/3}, M_1\}$, $V_0 := \max\{G(v_0)^{-1/3}, M_2\}$. Here, for $i = 1, 2$, $\varepsilon_i \ge 0$, $C_i > 0$ and $M_i > 0$ and we choose them later. It follows from (1.1.3) that there exist constants $m_1 > 0$ and $m_2 > 0$ such that

$$f'(s)F(s) < \frac{4}{3} \ \text{ for } s \ge m_1, \ g'(s)G(s) < \frac{4}{3} \ \text{ for } s \ge m_2. \tag{1.3.22}$$

Let $M_1 > F(m_1)^{-1/3}$ and $M_2 > G(m_2)^{-1/3}$ be sufficiently large.

We prove (i). Since $F(u_0)^{-r_1} \in L^1_{ul,\rho}(\mathbb{R}^N)$ and $G(v_0)^{-r_2} \in L^1_{ul,\rho}(\mathbb{R}^N)$, we have $U_0 \in L^{3r_1}_{ul,\rho}(\mathbb{R}^N)$

and $V_0 \in L^{3r_2}_{ul,\rho}(\mathbb{R}^N)$. Since $r_1 > N/2$ and $r_2 > N/2$, we have

$$(1.3.23) \qquad \frac{N}{2}\left(\frac{\varepsilon_1}{3r_1} + \frac{3+\varepsilon_1}{3r_2}\right) < 1 \quad \text{and} \quad \frac{N}{2}\left(\frac{3+\varepsilon_2}{3r_1} + \frac{\varepsilon_2}{3r_2}\right) < 1$$

by choosing $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ sufficiently small. By (1.3.23) we can apply Proposition 1.3.1 (i) to (1.3.21). There exist $T > 0$ and a local in time classical solution $(U, V) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)) \times C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N))$ of (1.3.21) in $L^{3r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{3r_2}_{ul,\rho}(\mathbb{R}^N)$ which satisfies

(1.3.24)
$$0 \le U(x,t) \le w(x,t)^{1/(3r_1)} \text{ in } \mathbb{R}^N \times (0,T) \text{ and } 0 \le V(x,t) \le w(x,t)^{1/(3r_2)} \text{ in } \mathbb{R}^N \times (0,T).$$

Here, w is a local in time classical solution of (1.3.5) with $(p_1, p_2, q_1, q_2) = (1+\varepsilon_1, 3+\varepsilon_1, 3+\varepsilon_2, 1+\varepsilon_2)$ and r_1 and r_2 replaced by $3r_1$ and $3r_2$, respectively. Moreover, $w_0(x) := \max\{U_0(x)^{3r_1} + V_0(x)^{3r_2}, 1\} \in L^1_{ul,\rho}(\mathbb{R}^N)$. Then we define

$$(1.3.25) \qquad \bar{u} := F^{-1}(U^{-3}) \quad \text{and} \quad \bar{v} := G^{-1}(V^{-3}).$$

Since (U, V) is a local in time classical solution of (1.3.21), we have

$$(1.3.26) \qquad \partial_t \bar{u} = \Delta \bar{u} + \frac{4/3 - f'(\bar{u})F(\bar{u})}{f(\bar{u})F(\bar{u})}|\nabla \bar{u}|^2 + 3C_1 f(\bar{u})F(\bar{u})^{1-\varepsilon_1/3}G(\bar{v})^{-1-\varepsilon_1/3}.$$

We see that $U \ge e^{t\Delta}U_0 \ge F(m_1)^{-1/3}$ which implies that $\bar{u} \ge m_1$. By (1.3.22) we have $4/3 - f'(\bar{u})F(\bar{u}) > 0$. Moreover, it follows from (1.1.3) that $f(s)F(s)^{1-\varepsilon_1/3}$ (resp. $g(s)G(s)^{1+\varepsilon_1/3}$) is increasing (resp. decreasing) for large $s > 0$. Then there is a constant $C_0 > 0$ such that

$$f(\bar{u})F(\bar{u})^{1-\varepsilon_1/3} \ge C_0 \quad \text{and} \quad G(\bar{v})^{-1-\varepsilon_1/3} \ge C_0 g(\bar{v}).$$

Choosing C_1 properly, we obtain

$$(1.3.27) \qquad \partial_t \bar{u} \ge \Delta \bar{u} + g(\bar{v}) \quad \text{in } \mathbb{R}^N \times (0,T).$$

Let $J(s) := F(s)^{-1/3}$ $(s \ge m_1)$. By direct calculation we see that J is convex. It follows from Lemma 1.2.4 that

$$J(\bar{u}(t)) = U(t) \ge e^{t\Delta}U_0 \ge e^{t\Delta}(J(\max\{u_0, m_1\})) \ge J(e^{t\Delta}(\max\{u_0, m_1\})) \quad \text{in } \mathbb{R}^N \times (0,T).$$

Since J is strictly increasing, we have

$$(1.3.28) \qquad \bar{u}(t) \ge e^{t\Delta}u_0.$$

By (1.3.27) and (1.3.28) we can apply Lemma 1.2.7 (ii). Then we have

$$(1.3.29) \qquad \bar{u}(t) \ge e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} g(\bar{v}(s))ds.$$

In the same way we can obtain the inequality

$$(1.3.30) \qquad \bar{v}(t) \ge e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} f(\bar{u}(s))ds.$$

Therefore, it follows from (1.3.29), (1.3.30) and Proposition 1.2.6 that there exists a solution $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ of (1.1.1). We prove the following limits:

$$\lim_{t\to 0} \|u(t) - e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} = 0 \quad \text{and} \quad \lim_{t\to 0} \|v(t) - e^{t\Delta}v_0\|_{L^\infty(\mathbb{R}^N)} = 0.$$

Since $V \geq e^{t\Delta}V_0 \geq M_2$ and M_2 is large, V^{-3} is small. Then, we can apply Lemma 1.3.2. It follows from (1.3.24) and Lemma 1.3.2 that

$$\begin{aligned} u(t) - e^{t\Delta}u_0 &= \int_0^t e^{(t-s)\Delta} g(v(s))ds \leq \int_0^t e^{(t-s)\Delta} g(\bar{v}(s))ds \\ &= \int_0^t e^{(t-s)\Delta} g\left(G^{-1}(V(s)^{-3})\right) ds \lesssim \int_0^t e^{(t-s)\Delta} V(s)^{3+\varepsilon_1} ds \\ &\leq \int_0^t e^{(t-s)\Delta} w(s)^{\frac{3+\varepsilon_1}{3r_2}} ds. \end{aligned} \tag{1.3.31}$$

Let $r > r_2$. By Lemma 1.2.1 and (1.2.3) we have

(1.3.32)
$$\begin{aligned} \int_0^t \|e^{(t-s)\Delta} w(s)^{\frac{3+\varepsilon_1}{3r_2}}\|_{L^\infty(\mathbb{R}^N)} ds &\lesssim \int_0^t \left(\rho^{-\frac{N}{r}} + (t-s)^{-\frac{N}{2r}}\right) \|w(s)^{\frac{3+\varepsilon_1}{3r_2}}\|_{L^r_{ul,\rho}(\mathbb{R}^N)} ds \\ &= \int_0^t \left(\rho^{-\frac{N}{r}} + (t-s)^{-\frac{N}{2r}}\right) \|w(s)\|^{\frac{3+\varepsilon_1}{3r_2}}_{L^{\frac{r(3+\varepsilon_1)}{3r_2}}_{ul,\rho}(\mathbb{R}^N)} ds \\ &\lesssim \int_0^t \left(\rho^{-\frac{N}{r}} + (t-s)^{-\frac{N}{2r}}\right) s^{-\frac{N}{2}\left(\frac{3+\varepsilon_1}{3r_2} - \frac{1}{r}\right)} ds \|w_0\|^{\frac{3+\varepsilon_1}{3r_2}}_{L^1_{ul,\rho}(\mathbb{R}^N)} \end{aligned}$$

for sufficiently small $t > 0$. By (1.3.23) we have

$$-\frac{N}{2r} > -1, \ -\frac{N}{2}\left(\frac{3+\varepsilon_1}{3r_2} - \frac{1}{r}\right) > -1 \text{ and } -\frac{N}{2}\left(\frac{3+\varepsilon_1}{3r_2} - \frac{1}{r}\right) - \frac{N}{2r} + 1 > 0.$$

Therefore,

$$\int_0^t \left(\rho^{-\frac{N}{r}} + (t-s)^{-\frac{N}{2r}}\right) s^{-\frac{N}{2}\left(\frac{3+\varepsilon_1}{3r_2} - \frac{1}{r}\right)} ds \to 0 \quad \text{as} \quad t \to 0.$$

Thus it follows from (1.3.31) and (1.3.32) that $\lim_{t\to 0} \|u(t) - e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} = 0$. In the same way we can obtain $\lim_{t\to 0} \|v(t) - e^{t\Delta}v_0\|_{L^\infty(\mathbb{R}^N)} = 0$. The proof of (i) is complete.

Next, we prove (ii). We consider the initial value problem (1.3.21) with $\varepsilon_1 = 0$. Since $F(u_0)^{-r_1} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$ and $G(v_0)^{-N/2} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$, it follows from Lemma 1.2.5 that $U_0 \in \mathcal{L}^{3r_1}_{ul,\rho}(\mathbb{R}^N)$ and $V_0 \in \mathcal{L}^{3N/2}_{ul,\rho}(\mathbb{R}^N)$. Moreover, since $r_1 > N/2$, we have

$$\frac{N}{2} \max\left\{\frac{1-1}{3r_1} + \frac{2}{N}, \frac{3+\varepsilon_2}{3r_1} + \frac{2\varepsilon_2}{3N}\right\} = 1 \tag{1.3.33}$$

by choosing $\varepsilon_2 > 0$ sufficiently small. By (1.3.33) we see that (1.3.4) holds, where $(p_1, p_2, q_1, q_2) = (1, 3, 3+\varepsilon_2, 1+\varepsilon_2)$ and r_1 and r_2 are replaced by $3r_1$ and $3N/2$, respectively. We can apply Proposition 1.3.1 (ii). Then there exist $T > 0$ and a local in time classical solution $(U,V) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)) \times C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N))$ of (1.3.1) in $L^{3r_1}_{ul,\rho}(\mathbb{R}^N)\times L^{3N/2}_{ul,\rho}(\mathbb{R}^N)$ such that

$$0\le U(x,t)\le w(x,t)^{(1+\varepsilon)/(3r_1)} \text{ in } \mathbb{R}^N\times(0,T) \text{ and } 0\le V(x,t)\le w(x,t)^{2(1+\varepsilon)/(3N)} \text{ in } \mathbb{R}^N\times(0,T), \tag{1.3.34}$$

where w is a local in time classical solution of (1.3.16). The constant $\varepsilon>0$ is chosen as in the proof of Proposition 1.3.1 (ii). Then we define $(\bar u,\bar v)$ by (1.3.25) and obtain (1.3.26) with $\varepsilon_1=0$. It follows from (1.1.5) that $g(s)G(s)$ is decreasing for $s\ge s_2$. Hence, there is $C>0$ such that $G(\bar v)^{-1}\ge Cg(\bar v)$. Moreover, since $\lim_{s\to\infty}f(s)F(s)>0$, we have (1.3.27). Since we can obtain

$$\partial_t\bar v\ge\Delta\bar v+f(\bar u) \text{ in } \mathbb{R}^N\times(0,T)$$

in the same way as in the proof of (i), the inequalities (1.3.29) and (1.3.30) hold. Thus it follows from Proposition 1.2.6 that there exists a solution $(u,v)\in C^{2,1}(\mathbb{R}^N\times(0,T))\times C^{2,1}(\mathbb{R}^N\times(0,T))$ of (1.1.1).

We prove the following limits:

$$\lim_{t\to0}\|u(t)-e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)}=0 \text{ and } \lim_{t\to0}\|v(t)-e^{t\Delta}v_0\|_{L^\infty(\mathbb{R}^N)}=0.$$

By (1.3.34) and Lemma 1.3.2 we have

$$\begin{aligned}u(t)-e^{t\Delta}u_0&=\int_0^t e^{(t-s)\Delta}g(v(s))ds\le\int_0^t e^{(t-s)\Delta}g(\bar v(s))ds\\&=\int_0^t e^{(t-s)\Delta}g\left(G^{-1}(V(s)^{-3})\right)ds\lesssim\int_0^t e^{(t-s)\Delta}V(s)^3ds\\&\le\int_0^t e^{(t-s)\Delta}w(s)^{\frac{2}{N}(1+\varepsilon)}ds.\end{aligned}\tag{1.3.35}$$

Let $\alpha_0:=1+2(1+\varepsilon)/N$. By α we define $\alpha:=\alpha_0+\delta$, where $\delta>0$ is small. When $p:=\alpha_0$ and $q:=1+\varepsilon$, we already see in the proof of Proposition 1.3.1 (ii) that $\max\{p,q\}<\alpha<pq$. We can use Remark 1.2.3. By Lemma 1.2.1 we have

$$\begin{aligned}&\int_0^t\|e^{(t-s)\Delta}w(s)^{\frac{2}{N}(1+\varepsilon)}\|_{L^\infty(\mathbb{R}^N)}ds\\&\lesssim\int_0^t\left(\rho^{-\frac{2(1+\varepsilon)}{\alpha}}+(t-s)^{-\frac{1+\varepsilon}{\alpha}}\right)\|w(s)^{\frac{2(1+\varepsilon)}{N}}\|_{L^{\frac{N\alpha}{2(1+\varepsilon)}}_{ul,\rho}(\mathbb{R}^N)}ds\\&\le\int_0^t\left(\rho^{-\frac{2(1+\varepsilon)}{\alpha}}+(t-s)^{-\frac{1+\varepsilon}{\alpha}}\right)s^{-\frac{2(1+\varepsilon)}{N}\sigma}ds\sup_{0<s<t}s^{\frac{2(1+\varepsilon)}{N}\sigma}\|w(s)\|^{\frac{2(1+\varepsilon)}{N}}_{L^\alpha_{ul,\rho}(\mathbb{R}^N)},\end{aligned}\tag{1.3.36}$$

where $\sigma:=\frac{N}{2}\left(\frac{1}{1+\varepsilon}-\frac{1}{\alpha}\right)$. Since

$$-\frac{1+\varepsilon}{\alpha}>-1,\ -\frac{2(1+\varepsilon)}{N}\sigma>-1 \text{ and } -\frac{1+\varepsilon}{\alpha}-\frac{2(1+\varepsilon)}{N}\sigma+1=0,$$

we have

$$\int_0^t\left(\rho^{-\frac{2(1+\varepsilon)}{\alpha}}+(t-s)^{-\frac{1+\varepsilon}{\alpha}}\right)s^{-\frac{2(1+\varepsilon)}{N}\sigma}ds<C$$

uniformly for small $t>0$. Thus, it follows from (1.3.35), (1.3.36) and Remark 1.2.3 that $\|u(t)-e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)}\to0$ as $t\to0$. In the same way we can obtain $\lim_{t\to0}\|v(t)-e^{t\Delta}v_0\|_{L^\infty(\mathbb{R}^N)}=0$.

The proof of (ii) is complete.

The existence results of (iii) and (iv) can be shown in a similar way. □

1.4 Nonexistence result

Proposition 1.4.1. *Let $N \geq 1$, $\rho > 0$, $0 < r < N/2$ and $s_0 \geq 0$. Let $f(s)$ and $g(s)$ be convex C^2 functions in $[s_0, \infty)$ and let $F_1(s) := \int_s^\infty e^{-f(\sigma)} d\sigma$. Suppose that f and g have the inverse functions and that there exists $C > 0$ such that $f(s) \geq Cs$ for $s \geq s_0$ and $g(s) \geq Cs$ for $s \geq s_0$. Suppose that f and g satisfy*

$$\frac{f''(s)}{f'(s)^2} \to 0 \ (s \to \infty) \quad \text{and} \quad \frac{g''(s)}{g'(s)^2} \to 0 \ (s \to \infty). \tag{1.4.1}$$

Then there exists $u_0 \geq s_0$ such that $F_1(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$ and that for every $T > 0$ and for every $v_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ satisfying $v_0 \geq s_0$ in $\mathbb{R}^N$, there is no nonnegative solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0, T)) \times C^{2,1}(\mathbb{R}^N \times (0, T))$ of the integral system

$$\begin{aligned} u(t) &= e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \left(e^{g(v(s))} - 1 \right) ds \ \ \text{in} \ \ \mathbb{R}^N \times (0, T), \\ v(t) &= e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} \left(e^{f(u(s))} - 1 \right) ds \ \ \text{in} \ \ \mathbb{R}^N \times (0, T). \end{aligned} \tag{1.4.2}$$

In particular, there is no nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0, T)) \times C^{2,1}(\mathbb{R}^N \times (0, T))$ of

$$\begin{cases} \partial_t u = \Delta u + e^{g(v)} - 1 & \text{in } \mathbb{R}^N \times (0, T), \\ \partial_t v = \Delta v + e^{f(u)} - 1 & \text{in } \mathbb{R}^N \times (0, T), \\ (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) & \text{in } \mathbb{R}^N \end{cases} \tag{1.4.3}$$

in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 1.1.1.

For the proof of Proposition 1.4.1 we need the following lemma:

Lemma 1.4.2. *Let $s_0 \geq 0$ and let f and g be convex functions in $[s_0, \infty)$ satisfying the following: f and g have the inverse functions and there exists $C > 0$ such that $f(s) \geq Cs$ for $s \geq s_0$ and $g(s) \geq Cs$ for $s \geq s_0$. Let $u_0, v_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$. If $u_0 \geq s_0$ in $\mathbb{R}^N$, $v_0 \geq s_0$ in $\mathbb{R}^N$ and there exist $T > 0$ and a local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0, T)) \times C^{2,1}(\mathbb{R}^N \times (0, T))$ satisfying (1.4.2), then, for $k \in \{2, 3, \ldots\}$, there exists a constant $C_k \in \mathbb{R}$ such that*

$$\begin{aligned} \|e^{t\Delta} u_0\|_{L^\infty(\mathbb{R}^N)} &\leq f^{-1} \left(\frac{k}{k-1} \log \frac{1}{t} + C_k + e^{-C_k} t^{\frac{k}{k-1}} \right) \quad \text{for } t \in (0, T), \\ \|e^{t\Delta} v_0\|_{L^\infty(\mathbb{R}^N)} &\leq g^{-1} \left(\frac{k}{k-1} \log \frac{1}{t} + C_k + e^{-C_k} t^{\frac{k}{k-1}} \right) \quad \text{for } t \in (0, T). \end{aligned}$$

The following proof is a system version of [18].

Proof. Let $k \in \{2, 3, \ldots\}$ be fixed. We prove the following:

$$\text{(1.4.4)} \qquad \text{If } l \text{ is odd, then} \begin{cases} u(t) \geq \dfrac{C^{a_l - 1}}{\prod_{i=1}^{l} (k! a_i)^{k^{l-i}}} t^{a_l} \left\{ e^{f(e^{t\Delta} u_0)} - 1 \right\}^{k^l} & \text{in } \mathbb{R}^N \times (0, T), \\ v(t) \geq \dfrac{C^{a_l - 1}}{\prod_{i=1}^{l} (k! a_i)^{k^{l-i}}} t^{a_l} \left\{ e^{g(e^{t\Delta} v_0)} - 1 \right\}^{k^l} & \text{in } \mathbb{R}^N \times (0, T), \end{cases}$$

and

$$\text{(1.4.5)} \qquad \text{if } l \text{ is even, then} \begin{cases} u(t) \geq \dfrac{C^{a_l - 1}}{\prod_{i=1}^{l} (k! a_i)^{k^{l-i}}} t^{a_l} \left\{ e^{g(e^{t\Delta} v_0)} - 1 \right\}^{k^l} & \text{in } \mathbb{R}^N \times (0, T), \\ v(t) \geq \dfrac{C^{a_l - 1}}{\prod_{i=1}^{l} (k! a_i)^{k^{l-i}}} t^{a_l} \left\{ e^{f(e^{t\Delta} u_0)} - 1 \right\}^{k^l} & \text{in } \mathbb{R}^N \times (0, T). \end{cases}$$

Here, $a_l := k^l + k^{l-1} + \cdots + k + 1 = (k^{l+1} - 1)/(k - 1)$. We prove the case $l = 1$. Since (u, v) satisfies the integral system (1.4.2), it is easy to show that $u(t) \geq e^{t\Delta} u_0 \geq s_0$ in $\mathbb{R}^N \times (0, T)$. Moreover, since $e^{f(\eta)} - 1$ is convex in $[s_0, \infty)$, it follows from Lemma 1.2.4 that

$$v(t) \geq \int_0^t e^{(t-s)\Delta} \left\{ e^{f\left(e^{s\Delta} u_0\right)} - 1 \right\} ds \geq \int_0^t \left\{ e^{f\left(e^{(t-s)\Delta} e^{s\Delta} u_0\right)} - 1 \right\} ds \geq t \left\{ e^{f\left(e^{t\Delta} u_0\right)} - 1 \right\}.$$

We see that $e^{g(\eta)} - 1 \geq \frac{g(\eta)^k}{k!} \geq \frac{C^k}{k!} \eta^k$ for $\eta \geq s_0$. Then it follows from Lemma 1.2.4 that

$$\begin{aligned} u(t) &\geq \int_0^t e^{(t-s)\Delta} \left(\frac{C^k v(s)^k}{k!} \right) ds \geq \frac{C^k}{k!} \int_0^t e^{(t-s)\Delta} \left\{ s^k \left(e^{f(e^{s\Delta} u_0)} - 1 \right)^k \right\} ds \\ &\geq \frac{C^k}{k!} \int_0^t s^k \left\{ e^{f(e^{t\Delta} u_0)} - 1 \right\}^k ds = \frac{C^k}{(k+1)!} \cdot t^{k+1} \left\{ e^{f(e^{t\Delta} u_0)} - 1 \right\}^k. \end{aligned}$$

We can obtain the inequality corresponding to v. This proves (1.4.4) with $l = 1$.

We suppose that (1.4.4) holds for $l = 2n - 1$. By similar calculation we have

$$\begin{aligned} u(t) &\geq \frac{C^k}{k!} \int_0^t \frac{C^{k(a_l - 1)} s^{k a_l} \left\{ e^{g(e^{t\Delta} v_0)} - 1 \right\}^{k^{l+1}}}{\prod_{i=1}^{l} (k! a_i)^{k^{l+1-i}}} ds \\ &= \frac{C^{k a_l} t^{k a_l + 1} \left\{ e^{g(e^{t\Delta} v_0)} - 1 \right\}^{k^{l+1}}}{k! (k a_l + 1) \prod_{i=1}^{l} (k! a_i)^{k^{l+1-i}}} = \frac{C^{a_{l+1} - 1} t^{a_{l+1}} \left\{ e^{g(e^{t\Delta} v_0)} - 1 \right\}^{k^{l+1}}}{\prod_{i=1}^{l+1} (k! a_i)^{k^{l+1-i}}}. \end{aligned}$$

We can obtain the inequality corresponding to v. This proves (1.4.5) with $l = 2n$. We see that if (1.4.5) holds for $l = 2n$, then (1.4.4) holds for $l = 2n + 1$ in the same way. By induction we see that both (1.4.4) and (1.4.5) hold.

We are ready to prove Lemma 1.4.2. It follows from (1.4.4) and (1.4.5) that, for $l \in \{1, 2, \ldots\}$,

$$\left(u(t)^{\frac{1}{k^l}} + v(t)^{\frac{1}{k^l}} \right) \prod_{i=1}^{l} (k! a_i)^{k^{-i}} \geq C^{\frac{k}{k-1} - \frac{k}{k^l (k-1)}} \cdot t^{\frac{k}{k-1} - \frac{1}{k^l (k-1)}} \left\{ e^{f(e^{t\Delta} u_0)} - 1 + e^{g(e^{t\Delta} v_0)} - 1 \right\}.$$

When $u(x,t) \neq 0$ or $v(x,t) \neq 0$ at $(x,t) \in \mathbb{R}^N \times (0,T)$, by taking $l \to \infty$ we have

$$2\prod_{i=1}^{\infty}(k!a_i)^{k^{-i}} \geq C^{\frac{k}{k-1}} \cdot t^{\frac{k}{k-1}} \left\{ e^{f(e^{t\Delta}u_0)} - 1 + e^{g(e^{t\Delta}v_0)} - 1 \right\}. \tag{1.4.6}$$

If $u(x,t) = v(x,t) = 0$ at some point $(x,t) \in \mathbb{R}^N \times (0,T)$, then by (1.4.4) and (1.4.5) we see that the right hand side of (1.4.6) is 0, and hence (1.4.6) also holds. Remark that the left hand side of (1.4.6) converges. Indeed, we see that

$$\log\left(\prod_{i=1}^{\infty}(k!a_i)^{k^{-i}}\right) \leq \sum_{i=1}^{\infty} \frac{(i+1)\log k + \log(k!)}{k^i} < \infty.$$

Then we have

$$e^{f(e^{t\Delta}u_0)} - 1 \leq \left(2C^{-\frac{k}{k-1}} \prod_{i=1}^{\infty}(k!a_i)^{k^{-i}}\right) t^{-\frac{k}{k-1}} =: e^{C_k} t^{-\frac{k}{k-1}},$$

which implies that

$$e^{t\Delta}u_0 \leq f^{-1}\left(\log\left(e^{C_k}t^{-\frac{k}{k-1}} + 1\right)\right).$$

By the mean value theorem we see that $\log(a+1) - \log a \leq \frac{1}{a}$ for $a > 0$. Thus we have

$$\|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} \leq f^{-1}\left(\frac{k}{k-1}\log\frac{1}{t} + C_k + e^{-C_k}t^{\frac{k}{k-1}}\right) \quad \text{for } t \in (0,T).$$

We can obtain the inequality corresponding to v_0. The proof is complete. □

We are ready to prove Proposition 1.4.1. The following method is almost the same as [10, Theorem 1.2]. However, Proposition 1.4.1 is needed in the system case. We show the proof.

Proof of Proposition 1.4.1. Fix $r < N/2$ and $2 < \alpha < N/r$. Let $0 < \varepsilon < 1$ be a sufficiently small constant such that $(1+\varepsilon)\alpha r < N$. Define

$$u_0(x) := \begin{cases} f^{-1}\left(\alpha \log \dfrac{1}{|x|}\right) & \text{if } |x| < r_0, \\ s_0 & \text{if } |x| \geq r_0, \end{cases} \tag{1.4.7}$$

where $r_0 > 0$ is chosen to satisfy $f^{-1}(\alpha\log(1/r_0)) = s_0$. Then we have $e^{(1+\varepsilon)rf(u_0)} \in L^1_{ul,\rho}(\mathbb{R}^N)$, because $(1+\varepsilon)\alpha r < N$.

We prove that $F_1(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$. By (1.4.1) there exists a large $s_1 > s_0$ such that $f'(s) > 0$ for $s \geq s_1$ and $f''(s) \leq \varepsilon(f'(s))^2$ for $s \geq s_1$. Then we can easily see that

$$f'(s) \leq f'(s_1)e^{\varepsilon(f(s)-f(s_1))} \leq f'(s_1)e^{\varepsilon f(s)} \quad \text{for } s \geq s_1. \tag{1.4.8}$$

On the other hand, since it follows from (1.4.1) that

$$\frac{f''(s)}{(f'(s))^2}e^{-f(s)} \leq \frac{f''(s)}{2(f'(s))^2}e^{-f(s)} + \frac{1}{2}e^{-f(s)} = \frac{1}{2}\frac{d}{ds}\left(-\frac{1}{f'(s)}e^{-f(s)}\right) \quad \text{for } s \geq s_1,$$

we have

$$\int_s^\infty \frac{f''(\sigma)}{(f'(\sigma))^2}e^{-f(\sigma)}d\sigma \leq \frac{1}{2f'(s)}e^{-f(s)} \quad \text{for } s \geq s_1$$

and by (1.4.8) we obtain

$$\begin{aligned} F_1(s) &= \int_s^\infty \frac{d\sigma}{e^{f(\sigma)}} = \frac{1}{f'(s)} e^{-f(s)} - \int_s^\infty \frac{f''(\sigma)}{(f'(\sigma))^2} e^{-f(\sigma)} d\sigma \\ &\geq \frac{1}{2f'(s)} e^{-f(s)} \geq \frac{1}{2f'(s_1)} e^{-(1+\varepsilon)f(s)} \quad \text{for } s \geq s_1. \end{aligned}$$

This together with $e^{(1+\varepsilon)rf(u_0)} \in L^1_{ul,\rho}$ implies that $F_1(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$.

Putting $y = \sqrt{t}z$, we have

$$\begin{aligned} \|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} &\geq (4\pi t)^{-\frac{N}{2}} \int_{|y|\leq r_0} e^{-\frac{|y|^2}{4t}} f^{-1}\left(\alpha \log \frac{1}{|y|}\right) dy \\ &= (4\pi)^{-\frac{N}{2}} \int_{|z|\leq r_0 t^{-1/2}} e^{-\frac{|z|^2}{4}} f^{-1}\left(\frac{\alpha}{2}\log\frac{1}{t} + \alpha \log\frac{1}{|z|}\right) dz. \end{aligned}$$

Let $\delta > 0$ be a sufficiently small constant such that $\alpha/2 - \delta > 1 + 2\delta$. Then we have $t^{-\delta/\alpha} \leq r_0 t^{-1/2}$ for sufficiently small $t > 0$. By calculating the integral we have

$$\begin{aligned} \|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} &\geq (4\pi)^{-\frac{N}{2}} \int_{|z|\leq t^{-\delta/\alpha}} e^{-\frac{|z|^2}{4}} f^{-1}\left(\frac{\alpha}{2}\log\frac{1}{t} + \alpha\log\frac{1}{|z|}\right) dz \\ &\geq f^{-1}\left(\left(\frac{\alpha}{2} - \delta\right)\log\frac{1}{t}\right) \cdot (4\pi)^{-\frac{N}{2}} \int_{|z|\leq t^{-\delta/\alpha}} e^{-\frac{|z|^2}{4}} dz \\ &= f^{-1}\left(\left(\frac{\alpha}{2} - \delta\right)\log\frac{1}{t}\right) \cdot \left(1 + O\left(e^{-\frac{t^{-2\delta/\alpha}}{8}}\right)\right) \end{aligned} \tag{1.4.9}$$

for sufficiently small $t > 0$. We prove that this yields a contradiction. Assume that there exists a classical solution (u, v) of (1.4.2) with $u \geq 0, v \geq 0$. Then, by Lemma 1.4.2 and (1.4.9) we have

$$\begin{aligned} &f^{-1}\left(\left(\frac{\alpha}{2} - \delta\right)\log\frac{1}{t}\right) \cdot \left(1 + O\left(e^{-\frac{t^{-2\delta/\alpha}}{8}}\right)\right) \\ &\leq f^{-1}\left(\frac{k}{k-1}\log\frac{1}{t} + C_k + e^{-C_k t^{\frac{k}{k-1}}}\right) \leq f^{-1}\left(\left(\frac{k}{k-1} + \delta\right)\log\frac{1}{t}\right) \end{aligned} \tag{1.4.10}$$

for small $t > 0$. Here we take a sufficiently large $k \in \{2, 3, \ldots\}$ satisfying $k/(k-1) + \delta < 1 + 2\delta$. Then, since $\alpha/2 - \delta > 1 + 2\delta$, we have $k/(k-1) + \delta < \alpha/2 - \delta$. In the following, we prove that (1.4.10) yields a contradiction. For simplicity, define $a := \alpha/2 - \delta$, $b := k/(k-1) + \delta$, $c := 2\delta/\alpha$ and $\tau := \log(1/t)$. Then $a > b$ and (1.4.10) implies that

$$f^{-1}(a\tau) \cdot \left(1 + O\left(e^{-e^{c\tau}/8}\right)\right) \leq f^{-1}(b\tau) \tag{1.4.11}$$

for large $\tau > 0$. Since $(f^{-1}(s))'' = -f''(f^{-1}(s))/f'(f^{-1}(s))^3 \leq 0$, by the mean value theorem and (1.4.8) we see that

$$f^{-1}(a\tau) \geq f^{-1}(b\tau) + \frac{1}{f'(f^{-1}(a\tau))}(a-b)\tau \geq f^{-1}(b\tau) + \frac{d}{e^{\varepsilon a\tau}}(a-b)\tau,$$

where $d := 1/f'(s_1)$, which implies that

$$f^{-1}(a\tau)\cdot\left(1+O\left(e^{-e^{c\tau}/8}\right)\right) \ge f^{-1}(b\tau)+\frac{d}{e^{\varepsilon a\tau}}(a-b)\tau+O\left(f^{-1}(a\tau)e^{-e^{c\tau}/8}\right) > f^{-1}(b\tau)$$

for large $\tau > 0$. Note that $f^{-1}(a\tau) = O(\tau)$ for large $\tau > 0$, since $f(s) \ge Cs$ for $s \ge s_0$. This contradicts (1.4.11), and hence (1.4.10) yields a contradiction.

We finally prove the latter assertion of Proposition 1.4.1. Assume that there exist a constant $T > 0$ and a classical solution (u, v) of (1.4.3) with $u \ge 0$ and $v \ge 0$ satisfying $\lim_{t\to 0} \|u(t) - e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} = 0$ and $\lim_{t\to 0} \|v(t) - e^{t\Delta}v_0\|_{L^\infty(\mathbb{R}^N)} = 0$. By Lemma 1.2.7 (i) we see that (u, v) satisfies (1.4.2) with T replaced by a sufficiently small $T' > 0$. This is a contradiction. Thus, we complete the proof of Proposition 1.4.1. $\square$

Lemma 1.4.3. *Let $a < b$ and let $f \in C([a, b])$ be a positive function. Then for any (m_1, m_2) satisfying $0 < m_1 < f(b)$ and $m_2 \ge 0$, there exists $g \in C^1([a, b])$ such that the following hold: $0 < g(s) < f(s)$, $g'(s) \ge 0$ for $s \in [a, b]$, $g(b) = m_1$ and $g'(b) = m_2$.*

Proof. Since f is continuous, there exists a sufficiently small constant $\delta > 0$ such that $f(s) > m_1$ for $s \in [b-\delta, b]$. Put $m_0 := \min_{s\in[a,b-\delta]} f(s) > 0$. We can construct $g \in C^1([a, b])$ satisfying $g'(s) \ge 0$ for $s \in [a, b]$, $g(b) = m_1$, $g'(b) = m_2$, $0 < g(s) < m_0$ for $s \in [a, b - \delta]$ and $0 < g(s) \le m_1$ for $s \in [b - \delta, b]$. This g satisfies the desired conditions. $\square$

Proof of Theorem 1.1.3. We prove only the existence of u_0 such that, for any $v_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ satisfying $v_0 \ge a$, (1.1.1) has no nonnegative classical solution. The other case can be proved in a similar way.

Let $a > 0$ and $0 < \varepsilon < 1$. By (1.1.3) and (1.1.4) there exists a large $b > a$ such that

$$\frac{1-\varepsilon}{f(s)F(s)} < \frac{f'(s)}{f(s)+1} \quad \text{and} \quad f'(s)F(s) \le 1 \quad \text{for } s \ge b.$$

Then, $(1 - \varepsilon)/(f(b)F(b)) < f'(b)/(f(b) + 1)$. We apply Lemma 1.4.3 with $(m_1, m_2) = ((1 - \varepsilon)/(f(b)F(b)), (1-\varepsilon)\{1-f'(b)F(b)\}/(f(b)^2F(b)^2))$. Then, there exists $f_1 \in C^1([a, b])$ such that $0 < f_1(s) < f'(s)/(f(s)+1)$ for $a \le s \le b$, $f_1'(s) \ge 0$ for $a \le s \le b$, $f_1(b) = (1-\varepsilon)/(f(b)F(b))$ and $f_1'(b) = (1 - \varepsilon)\{1 - f'(b)F(b)\}/(f(b)^2F(b)^2)$. Note that if $f'(0) > 0$, then $f'(s)/(f(s) + 1) > 0$ for $s \ge 0$ and we can apply Lemma 1.4.3 with $a = 0$. Put

$$\tilde{f}_1(s) := \begin{cases} f_1(s) & \text{if } s \in [a, b], \\ \frac{1-\varepsilon}{f(s)F(s)} & \text{if } s \in (b, \infty). \end{cases}$$

Then $\tilde{f}_1(s) < f'(s)/(f(s) + 1)$ for $s \ge a$. Integrating both sides of this inequality, we have

$$\exp(f_2(s)) := \exp\left(\int_a^s \tilde{f}_1(\sigma)d\sigma + \log(f(a)+1)\right) \le f(s)+1. \tag{1.4.12}$$

We can construct g_2 in the same way. By direct calculation we see that f_2, g_2 are convex C^2 functions in $[a,\infty)$ and that $f_2'(s) > 0$ and $g_2'(s) > 0$ for $s \ge a$. Thus f_2 and g_2 have the inverse functions ${f_2}^{-1}$ and ${g_2}^{-1}$, respectively. Moreover, there exists $C > 0$ such that $f_2(s) \ge Cs$ for $s \ge a$ and $g_2(s) \ge Cs$ for $s \ge a$. Note that if $f'(0) > 0$ and $g'(0) > 0$, we can choose $a = 0$.

Moreover, by (1.1.3) we see that

$$\frac{{f_2}''(s)}{{f_2}'(s)^2} \to 0 \ (s \to \infty) \ \text{ and } \ \frac{{g_2}''(s)}{{g_2}'(s)^2} \to 0 \ (s \to \infty).$$

Let $0 < r < N/2$. We choose $r' \in (r, N/2)$. It follows from Proposition 1.4.1 that there exists $u_0 \geq a$ satisfying

$$F_2(u_0)^{-r'} \in L^1_{ul,\rho}(\mathbb{R}^N) \ \text{ with } \ F_2(s) := \int_s^\infty \frac{d\sigma}{e^{f_2(\sigma)}}$$

such that for every $T > 0$ and for every $v_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ satisfying $v_0 \geq a$ in $\mathbb{R}^N$, there is no nonnegative solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0, T)) \times C^{2,1}(\mathbb{R}^N \times (0, T))$ of (1.4.2) with (f, g) replaced by (f_2, g_2).

We prove that $F(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$. We see that

$$F_2(s) = \int_s^\infty \frac{d\sigma}{e^{f_2(\sigma)}} = \int_s^\infty \frac{1}{e^{f_2(\sigma)}} \cdot \frac{f_2'(\sigma)}{f_2'(\sigma)} d\sigma \leq \frac{1}{f_2'(s)} \int_s^\infty \frac{f_2'(\sigma)}{e^{f_2(\sigma)}} d\sigma = \frac{1}{f_2'(s) e^{f_2(s)}} \quad \text{for } s > b.$$

Here we use $f_2'(s) = \tilde{f}_1(s) = \frac{1-\varepsilon}{f(s)F(s)}$ is nondecreasing. Thus we have

$$F_2(s)^{-r'} \geq f_2'(s)^{r'} e^{r' f_2(s)} \quad \text{for } s > b. \tag{1.4.13}$$

On the other hand, we see that $(\log f(s))' \leq \frac{1}{f(s)F(s)} = \frac{\tilde{f}_1(s)}{1-\varepsilon}$ for $s > b$. Integrating both sides of this inequality, we have

$$\log f(s) - \log f(b) \leq \frac{1}{1-\varepsilon} \int_b^s \tilde{f}_1(\sigma) d\sigma \leq \frac{1}{1-\varepsilon} \int_a^s \tilde{f}_1(\sigma) d\sigma = \frac{1}{1-\varepsilon} \left\{ f_2(s) - \log(f(a) + 1) \right\},$$

which implies that $f(s) \lesssim \exp(\frac{f_2(s)}{1-\varepsilon})$ for $s > b$. It follows that

$$(1-\varepsilon) F(s)^{-1} = f(s) \tilde{f}_1(s) = f(s) f_2'(s) \lesssim f_2'(s) \exp\left(\frac{f_2(s)}{1-\varepsilon}\right) \quad \text{for } s > b.$$

Thus we have

$$F(s)^{-r} \lesssim f_2'(s)^r \exp\left(\frac{r f_2(s)}{1-\varepsilon}\right) \quad \text{for } s > b. \tag{1.4.14}$$

Since $0 < r < r'$ and $f_2'(s) \geq \frac{1-\varepsilon}{f(b)F(b)}$ for $s > b$, we have $f_2'(s)^r \lesssim f_2'(s)^{r'}$ for $s > b$. Choosing $0 < \varepsilon < 1$ such that $r < (1-\varepsilon) r'$, by (1.4.13) and (1.4.14) we see that

$$F(s)^{-r} \lesssim F_2(s)^{-r'} \quad \text{for } s > b.$$

Since $F_2(u_0)^{-r'} \in L^1_{ul,\rho}(\mathbb{R}^N)$, we obtain $F(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$.

We prove the theorem by contradiction. Let $v_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ such that $v_0 \geq a$ in $\mathbb{R}^N$. Suppose that (1.1.1) possesses a local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0, T)) \times C^{2,1}(\mathbb{R}^N \times (0, T))$ in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$. Then by Lemma 1.2.7 (i) and (1.4.12) we obtain the integral inequalities

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}g(v(s))ds \geq e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left(e^{g_2(v(s))} - 1\right)ds \ \text{ in } \mathbb{R}^N \times (0, T'),$$
$$v(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}f(u(s))ds \geq e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\left(e^{f_2(u(s))} - 1\right)ds \ \text{ in } \mathbb{R}^N \times (0, T'),$$

where $T' > 0$ is sufficiently small. By Proposition 1.2.6 there exists a local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0, T')) \times C^{2,1}(\mathbb{R}^N \times (0, T'))$ satisfying (1.4.2) with (f, g) and T replaced by (f_2, g_2) and T', respectively. This contradicts the nonexistence of the solution of (1.4.2). $\square$

References

[1] D. Andreucci, M. Herrero and J. Velázquez, *Liouville theorems and blow up behaviour in semilinear reaction diffusion systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 1–53.

[2] P. Baras and M. Pierre, *Critère d'existence de solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), no. 3, 185–212.

[3] H. Brezis and T. Cazenave, *A nonlinear heat equation with singular initial data*, J. Anal. Math. **68** (1996), 277–304.

[4] F. Dickstein and M. Loayza, *Life span of solutions of a strongly coupled parabolic system*, Mat. Contemp. **32** (2007), 85–106.

[5] F. Dickstein and M. Loayza, *Life span of solutions of a weakly coupled parabolic system*, Z. Angew. Math. Phys. **59** (2008), 1–23.

[6] L. Dupaigne and A. Farina, *Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$*, J. Eur. Math. Soc. **12** (2010), 855–882.

[7] M. Escobedo and M. Herrero, *Boundedness and blow up for a semilinear reaction-diffusion system*, J. Differential Equations **89** (1991), 176–202.

[8] M. Escobedo and M. Herrero, *A semilinear parabolic system in a bounded domain*, Ann. Mat. Pura Appl. **165** (1993), 315–336.

[9] L. Ferreira and E. Mateus, *Self-similarity and uniqueness of solutions for semilinear reaction-diffusion systems*, Adv. Differential Equations **15** (2010), 73–98.

[10] Y. Fujishima and N. Ioku, *Existence and nonexistence of solutions for the heat equation with a superlinear source term*, J. Math. Pures Appl. **118** (2018), 128–158.

[11] K. Ishige, T. Kawakami and M. Sierżęga, *Supersolutions for a class of nonlinear parabolic systems*, J. Differential Equations **260** (2016), 6084–6107.

[12] R. Laister, J. C. Robinson, M. Sierżęga and A. Vidal-López, *A complete characterisation of local existence for semilinear heat equations in Lebesgue spaces*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 6, 1519–1538.

[13] Y. Maekawa and Y. Terasawa, *The Navier-Stokes equations with initial data in uniformly local L^p spaces*, Differential Integral Equations **19** (2006), 369–400.

[14] Y. Miyamoto, *A limit equation and bifurcation diagrams of semilinear elliptic equations with general supercritical growth*, J. Differential Equations **264** (2018), 2684–2707.

[15] I. Peral and J. Vázquez, *On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term*, Arch. Rational Mech. Anal. **129** (1995), 201–224.

[16] P. Quittner and P. Souplet, *Admissible L_p norms for local existence and for continuation in semilinear parabolic systems are not the same*, Proc. Roy. Soc. Edinburgh **131** (2001), 1435–1456.

[17] P. Quittner and P. Souplet, Superlinear Parabolic Problems, Blow–up, Global Existence and Steady States, Birkhäuser Advanced Texts: Basler Lehrbücher Birkhäuser Verlag, Basel, 2007.

[18] F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in L^p*, Indiana Univ. Math. J. **29** (1980), 79–102.

[19] F. B. Weissler, *L^p-energy and blow-up for a semilinear heat equation*, Proc. Symp. Pure Math. **45** (1986), 545–551.

Chapter 2

Local existence and nonexistence for reaction-diffusion systems with coupled exponential nonlinearities

We study the reaction-diffusion system with coupled exponential nonlinearities

$$\begin{cases} \partial_t u = \Delta u + e^{p_1 u + p_2 v} & \text{in } \Omega \times (0,T), \\ \partial_t v = \Delta v + e^{q_1 u + q_2 v} & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where $T > 0$, $p_i \geq 0$ and $q_i \geq 0$ $(i = 1, 2)$ with $(p_1, p_2) \neq (0, 0)$ and $(q_1, q_2) \neq (0, 0)$. The domain Ω is $\mathbb{R}^N$ $(N \geq 1)$ or a bounded domain in $\mathbb{R}^N$ with C^2 boundary and the initial functions u_0 and v_0 are nonnegative and measurable. For each (p_1, p_2, q_1, q_2), we obtain integrability conditions of (u_0, v_0) which explicitly determine the existence/nonexistence of a local in time nonnegative classical solution. For the existence result, we can take a wider class of initial functions than previously suggested. Our analysis can be applied to other nonlinearities including superexponential ones.

Keywords: Reaction-diffusion systems; Existence and nonexistence; Local in time solutions; Singular initial functions; Exponential nonlinearities.

2.1 Introduction

Let Ω be $\mathbb{R}^N$ $(N \geq 1)$ or a bounded domain in $\mathbb{R}^N$ with C^2 boundary. We study existence and nonexistence of a local in time solution for the reaction-diffusion system

$$(2.1.1) \qquad \begin{cases} \partial_t u = \Delta u + e^{p_1 u + p_2 v} & \text{in } \Omega \times (0,T), \\ \partial_t v = \Delta v + e^{q_1 u + q_2 v} & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where $T>0$, $p_i\geq 0$ and $q_i\geq 0$ $(i=1,2)$ with $(p_1,p_2)\neq(0,0)$ and $(q_1,q_2)\neq(0,0)$. We assume that u_0 and v_0 are nonnegative measurable initial functions in Ω.

Let us recall previous studies. We consider the scalar problem

$$\begin{cases}\partial_t u=\Delta u+f(u) & \text{in } \Omega\times(0,T),\\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x) & \text{in } \Omega,\end{cases} \tag{2.1.2}$$

where $f\in C^1$ and Ω is possibly unbounded. It is well known that problem (2.1.2) possesses a local in time classical solution if $u_0\in L^\infty(\Omega)$ (cf. [11,23]). On the other hand, if $u_0\notin L^\infty(\Omega)$, it is known that the existence of solutions heavily depends on the balance between the growth rate of f and the singularity of u_0 (cf. [11]). Weissler [27] studied (2.1.2) when $f(u)=|u|^{p-1}u$ $(p>1)$ and Ω is bounded. He constructed a local in time solution when $u_0\in L^r(\Omega)$ for $r>\frac{N}{2}(p-1)$ and $r\geq 1$, or $r=\frac{N}{2}(p-1)$ and $r>1$. He also proved that if $1\leq r<\frac{N}{2}(p-1)$, then there exists a nonnegative initial function $u_0\in L^r(\Omega)$ such that, for every $T>0$, (2.1.2) admits no nonnegative solution.

We refer to previous results of the reaction-diffusion system

$$\begin{cases}\partial_t u=\Delta u+f(u,v) & \text{in } \Omega\times(0,T),\\ \partial_t v=\Delta v+g(u,v) & \text{in } \Omega\times(0,T),\\ u(x,t)=v(x,t)=0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x), v(x,0)=v_0(x) & \text{in } \Omega.\end{cases} \tag{2.1.3}$$

Quittner-Souplet [21] studied (2.1.3), where $f(u,v)=|v|^{p-1}v$, $g(u,v)=|u|^{q-1}u$ $(p,q>0)$ and Ω is bounded. This is called a weakly coupled system. They proved the existence (resp. nonexistence) of a local in time solution for $(u_0,v_0)\in L^{r_1}(\Omega)\times L^{r_2}(\Omega)$ and

$$\frac{N}{2}\max\left\{\frac{p}{r_2}-\frac{1}{r_1},\frac{q}{r_1}-\frac{1}{r_2}\right\}\leq 1,\ p,q>1 \text{ and } r_1,r_2>1$$

(resp. $\frac{N}{2}\max\left\{\frac{p}{r_2}-\frac{1}{r_1},\frac{q}{r_1}-\frac{1}{r_2}\right\}>1$, $p,q>0$ and $r_1,r_2\geq 1$). In addition, the system (2.1.3) is called a strongly coupled system when $f(u,v)=u^{\alpha_1}v^{\alpha_2}$ and $g(u,v)=u^{\beta_1}v^{\beta_2}$ $(\alpha_i,\beta_i>0$ for $i=1,2)$. See also [1,3–10,17,18,22]. In particular, initial functions were assumed to be possibly unbounded in [4,5,10,17]. They considered various functional spaces which initial data belong to.

We introduce some functional spaces. Let $B(x,\rho)$ denote the ball of radius $\rho>0$ centered at $x\in\mathbb{R}^N$. For $1\leq p<\infty$, define the uniformly local L^p space $L^p_{ul,\rho}(\mathbb{R}^N)$ by

$$L^p_{ul,\rho}(\mathbb{R}^N):=\left\{u\in L^p_{loc}(\mathbb{R}^N):\|u\|_{L^p_{ul,\rho}(\mathbb{R}^N)}:=\sup_{y\in\mathbb{R}^N}\left(\int_{B(y,\rho)}|u(x)|^p dx\right)^{1/p}<\infty\right\}.$$

For simplicity of notation we set $L^\infty_{ul,\rho}(\mathbb{R}^N):=L^\infty(\mathbb{R}^N)$. Let $\mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$ denote the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ in the space $L^p_{ul,\rho}(\mathbb{R}^N)$, i.e.,

$$\mathcal{L}^p_{ul,\rho}(\mathbb{R}^N):=\overline{BUC(\mathbb{R}^N)}^{\|\cdot\|_{L^p_{ul,\rho}(\mathbb{R}^N)}}.$$

Let us go back to (2.1.2). When the nonlinear term f has a rapid growth such as e^u and e^{u^2}, (2.1.2) has not been well studied until recently. Fujishima-Ioku [11] has studied (2.1.2) when $u_0 \geq 0$, $\Omega = \mathbb{R}^N$ and $f \in C^1([0,\infty))$ satisfies the following conditions:

$$f(s) > 0,\ f'(s) > 0 \text{ and } F(s) < \infty \text{ for } s > 0,$$
$$\lim_{s\to\infty} f'(s)F(s) = 1 \text{ and there exists } s_0 > 0 \text{ such that } f'(s)F(s) \leq 1 \text{ for } s \geq s_0.$$

Here $F(s) := \int_s^\infty d\sigma/f(\sigma)$. Typical examples are $f(u) = e^u$, e^{u^2}, e^{e^u}. They obtained the local in time existence (resp. nonexistence) result when $F(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$ for $r > N/2$ or $F(u_0)^{-\frac{N}{2}} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$ (resp. $F(u_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N)$ for $0 < r < N/2$). See also [14–16, 24–26].

When the nonlinear terms are polynomials such as u^p, the existence and nonexistence of a local in time solution were studied in both scalar problems and systems. However, when the nonlinear terms are exponential, solutions of systems have not been well studied. For recent studies, see [13, 20].

We study (2.1.1) in all the cases (p_1, p_2, q_1, q_2), and obtain integrability conditions of initial data which *explicitly* determine the existence/nonexistence of a local in time solution. In the proof we combine and improve methods of [11] and Ishige *et al.* [17]. Our analysis of (2.1.1) enables us to study other systems. For example, we can treat a system with superexponential nonlinear terms by our results and the transformation approach in [11]. See Section 2.5.

We begin to state an existence result in $\mathbb{R}^N$. Define

$$(e^{t\Delta}\phi)(x) := (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \phi(y)dy$$

for $\phi \in L^1_{ul,\rho}(\mathbb{R}^N)$. Then $e^{t\Delta}\phi$ gives the solution for the heat equation in $\mathbb{R}^N$ with the initial function ϕ.

Definition 2.1.1 (Classical solution in $\mathbb{R}^N$). Let X, Y be Banach spaces. We call a function $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ a classical solution of (2.1.1) in $X \times Y$ if (u,v) satisfies the equations in the classical sense in $\mathbb{R}^N \times (0,T)$, $\|u(t) - e^{t\Delta}u_0\|_X \to 0$ $(t \to 0)$ and $\|v(t) - e^{t\Delta}v_0\|_Y \to 0$ $(t \to 0)$.

In Definition 2.1.1 we do not adopt the following conditions: $\|u(t) - u_0\|_X \to 0$ $(t \to 0)$ and $\|v(t) - v_0\|_Y \to 0$ $(t \to 0)$, since we consider the case where $\|u(t) - u_0\|_{L^\infty(\mathbb{R}^N)}$ and $\|v(t) - v_0\|_{L^\infty(\mathbb{R}^N)}$ do not necessarily converge to 0.

Theorem 2.1.2 (Local in time existence in $\mathbb{R}^N$). *Let $N \geq 1$ and $\rho > 0$. Suppose that one of the following holds:*

(i) *there exist $r_1 > N/2$ and $r_2 > N/2$ such that the nonnegative functions $(u_0, v_0) \in L^1_{ul,\rho}(\mathbb{R}^N) \times L^1_{ul,\rho}(\mathbb{R}^N)$ satisfy*

$$e^{r_1(p_1u_0+p_2v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N) \text{ and } e^{r_2(q_1u_0+q_2v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N), \tag{2.1.4}$$

(ii) *the nonnegative functions $(u_0, v_0) \in L^1_{ul,\rho}(\mathbb{R}^N) \times L^1_{ul,\rho}(\mathbb{R}^N)$ satisfy*

$$e^{r_1(p_1u_0+p_2v_0)} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N) \text{ and } e^{r_2(q_1u_0+q_2v_0)} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N) \tag{2.1.5}$$

with $r_1 = N/2$ and $r_2 = N/2$.

Then there exist $T > 0$ and a local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ of (2.1.1) in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 2.1.1.

We deduce the following corollaries from Theorem 2.1.2.

Corollary 2.1.3. *Let $N \geq 1$, $\rho > 0$, $p_1 \geq q_1$ and $p_2 \geq q_2$. Suppose that the nonnegative functions $(u_0, v_0) \in L^1_{ul,\rho}(\mathbb{R}^N) \times L^1_{ul,\rho}(\mathbb{R}^N)$ satisfy*

$$e^{r_1(p_1 u_0 + p_2 v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N) \ \text{ for some } r_1 > N/2 \ \text{ or } \ e^{r_1(p_1 u_0 + p_2 v_0)} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N) \ \text{ with } r_1 = N/2.$$

Then there exist $T > 0$ and a local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ of (2.1.1) in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 2.1.1.

Corollary 2.1.4. *Let $N \geq 1$, $\rho > 0$, $p_1 < q_1$, $p_2 > q_2$ and $u_0, v_0 \geq 0$. Suppose that one of the following holds:*

(i) *$p_1 = 0$ and $q_2 > 0$: (2.1.4) holds for some $(r_1, r_2) \in (0, N/2] \times (Np_2/(2q_2), \infty)$ or (2.1.5) holds for some $(r_1, r_2) \in (0, N/2] \times \{Np_2/(2q_2)\}$,*

(ii) *$p_1 > 0$ and $q_2 = 0$: (2.1.4) holds for some $(r_1, r_2) \in (Nq_1/(2p_1), \infty) \times (0, N/2]$ or (2.1.5) holds for some $(r_1, r_2) \in \{Nq_1/(2p_1)\} \times (0, N/2]$,*

(iii) *$p_1 > 0$ and $q_2 > 0$: (2.1.4) holds for some $(r_1, r_2) \in ((0, N/2] \times (Np_2/(2q_2), \infty)) \cup ((Nq_1/(2p_1), \infty) \times (0, N/2])$ or (2.1.5) holds for some $(r_1, r_2) \in ((0, N/2] \times \{Np_2/(2q_2)\}) \cup (\{Nq_1/(2p_1)\} \times (0, N/2])$.*

Then there exist $T > 0$ and a local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ of (2.1.1) in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 2.1.1.

We mention Corollary 2.1.4. In the case (i) if the former assumption holds, then we see that

$$e^{r_3\left(\frac{p_2 q_1}{q_2} u_0 + p_2 v_0\right)} = e^{r_2(q_1 u_0 + q_2 v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N), \ \text{where } r_3 = r_2 q_2/p_2 > N/2.$$

If the latter assumption holds, then we see that

$$e^{r_3\left(\frac{p_2 q_1}{q_2} u_0 + p_2 v_0\right)} = e^{r_2(q_1 u_0 + q_2 v_0)} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N), \ \text{where } r_3 = r_2 q_2/p_2 = N/2.$$

Thus Theorem 2.1.2 implies that there exists a nonnegative classical solution of (2.1.1) with p_1, q_1 and q_2 replaced by $p_2 q_1/q_2$, $p_2 q_1/q_2$ and p_2, respectively. Since it is a supersolution for (2.1.1), (2.1.1) also possesses a solution. We can obtain the other cases (ii) and (iii), and Corollary 2.1.3 in a similar argument.

Corollaries 2.1.3 and 2.1.4 hold if p_i and q_i ($i = 1, 2$) are exchanged, respectively. For example, let $q_1 \geq p_1$ and $q_2 \geq p_2$. If $e^{r_2(q_1 u_0 + q_2 v_0)}$ is in $L^1_{ul,\rho}(\mathbb{R}^N)$ for some $r_2 > N/2$ or in $\mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$ with $r_2 = N/2$, then (2.1.1) has a nonnegative classical solution.

It is suggested in [17, p.6105, 6106] that the existence of a solution of (2.1.1) can be obtained from using a solution of the strongly coupled system (2.1.3). In this chapter we devise a different transformation and deduce the existence conditions of initial data (2.1.4) and (2.1.5). Compared to the case using the idea suggested in [17], we can take a wider class of initial data where (2.1.1) possesses a solution in the sense of Definition 2.1.1. See Appendix 2.A for an example of initial data. Moreover, we also obtain a nonexistence result, which is not mentioned in [17].

Theorem 2.1.5 (Local in time nonexistence in $\mathbb{R}^N$). *Let $N \geq 1$ and $\rho > 0$. Then the following* (i) *and* (ii) *hold:*

(i) *Let $p_1 \geq q_1$ and $p_2 \geq q_2$. Then for any $0 < r_1 < N/2$, there exist nonnegative functions (u_0, v_0) satisfying $e^{r_1(p_1u_0+p_2v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N)$ such that, for every $T > 0$, problem* (2.1.1) *admits no local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0, T)) \times C^{2,1}(\mathbb{R}^N \times (0, T))$ in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 2.1.1.*

(ii) *Let $p_1 < q_1$ and $p_2 > q_2$. Put*

$$I_1 := (0, Nq_1/(2p_1)) \text{ if } p_1 > 0, \quad I_1 := (0, \infty) \text{ if } p_1 = 0 \text{ and}$$
$$I_2 := (0, Np_2/(2q_2)) \text{ if } q_2 > 0, \quad I_2 := (0, \infty) \text{ if } q_2 = 0.$$

Then for any $(r_1, r_2) \in (I_1 \times (0, N/2)) \cup ((0, N/2) \times I_2)$, there exist nonnegative functions (u_0, v_0) satisfying (2.1.4) *such that, for every $T > 0$, problem* (2.1.1) *admits no local in time nonnegative classical solution $(u, v) \in C^{2,1}(\mathbb{R}^N \times (0, T)) \times C^{2,1}(\mathbb{R}^N \times (0, T))$ in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 2.1.1.*

Theorem 2.1.5 holds if p_i and q_i $(i = 1, 2)$ are exchanged, respectively as well as Corollaries 2.1.3 and 2.1.4.

Remark 2.1.6. *Using Theorems 2.1.2 and 2.1.5 and Corollaries 2.1.3 and 2.1.4, we can determine the existence and nonexistence of a solution in all the cases. We consider only the following two cases* (1) *and* (2)*, since the other cases are obtained from exchanging p_i and q_i $(i = 1, 2)$, respectively. In conclusion, we cover all the cases $(p_1, p_2, q_1, q_2, r_1, r_2)$.*

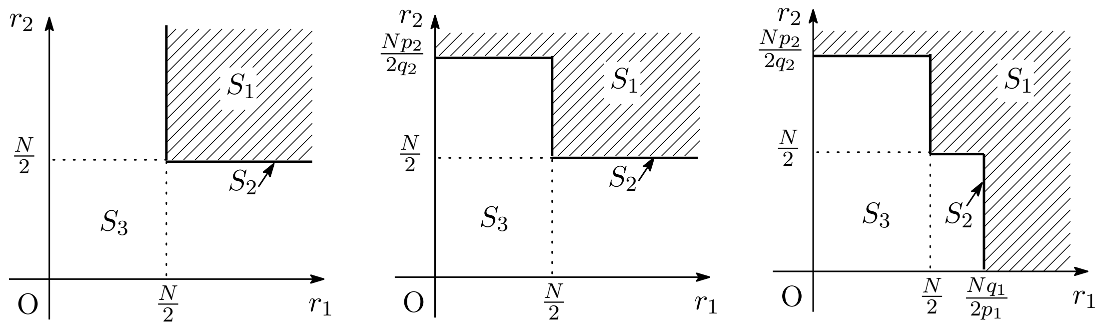
(1) *Let $p_1 \geq q_1$ and $p_2 \geq q_2$. If $e^{r_1(p_1u_0+p_2v_0)}$ is in $L^1_{ul,\rho}(\mathbb{R}^N)$ for some $r_1 > N/2$ or in $\mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$ with $r_1 = N/2$, then Corollary 2.1.3 implies that* (2.1.1) *has a nonnegative classical solution. On the other hand, if $0 < r_1 < N/2$, then Theorem 2.1.5* (i) *implies that there are nonnegative functions (u_0, v_0) such that $e^{r_1(p_1u_0+p_2v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N)$ and the solution does not exist.*

(2) *Let $p_1 < q_1$ and $p_2 > q_2$. We divide into the following three cases. Note that when $p_1 > 0$ and $q_2 = 0$, we consider in a similar way to the case* (ii)*.*

   (i) *$p_1 = 0$ and $q_2 = 0$: Set $S_1 := (N/2, \infty) \times (N/2, \infty)$, $S_2 := \partial S_1$ and $S_3 := ((0, N/2) \times (0, \infty)) \cup ((0, \infty) \times (0, N/2))$. If* (2.1.4) *holds for some $(r_1, r_2) \in S_1$ or* (2.1.5) *holds for some $(r_1, r_2) \in S_2$, then Theorem 2.1.2 claims that* (2.1.1) *has a nonnegative classical solution. On the other hand, by Theorem 2.1.5* (ii) *for each $(r_1, r_2) \in S_3$, there are nonnegative functions (u_0, v_0) such that* (2.1.4) *holds and the solution does not exist. See Fig. 2.1* (i)*.*

   (ii) *$p_1 = 0$ and $q_2 > 0$: Set $S_1 := ((0, N/2] \times (Np_2/(2q_2), \infty)) \cup ((N/2, \infty) \times (N/2, \infty))$, $S_3 := ((0, N/2) \times (0, Np_2/(2q_2))) \cup ((0, \infty) \times (0, N/2))$ and $S_2 := \partial S_1 \cap \partial S_3$. For this case, the same conclusions as the case* (i) *hold using Corollary 2.1.4 and Theorem 2.1.5* (ii)*. See Fig. 2.1* (ii)*.*

   (iii) *$p_1 > 0$ and $q_2 > 0$: Set $S_1 := ((0, N/2] \times (Np_2/(2q_2), \infty)) \cup ((N/2, \infty) \times (N/2, \infty)) \cup ((Nq_1/(2p_1), \infty) \times (0, N/2])$, $S_3 := ((0, N/2) \times (0, Np_2/(2q_2))) \cup ((0, Nq_1/(2p_1)) \times (0, N/2))$ and $S_2 := \partial S_1 \cap \partial S_3$. For this case, the same conclusions as the case* (i) *also hold in the same way as the case* (ii)*. See Fig. 2.1* (iii)*.*



(i) $p_1 = 0$ and $q_2 = 0$ (ii) $p_1 = 0$ and $q_2 > 0$ (iii) $p_1 > 0$ and $q_2 > 0$

Fig. 2.1: When $p_1 < q_1$ and $p_2 > q_2$, the solution exists on $S_1 \cup S_2$ and the nonexistence result holds on S_3.

We consider the case where Ω is a bounded domain with C^2 boundary. Let $1 \le p < \infty$ and let Δ_Ω be the Laplace operator with the domain $D_p(\Delta_\Omega) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. It is known that $e^{t\Delta_\Omega}$ is an analytic semigroup on $L^p(\Omega)$. Moreover, $e^{t\Delta_\Omega}$ has positivity preserving and contractibility on $L^p(\Omega)$ and there exists a positive C^∞-function $G_\Omega : \Omega \times \Omega \times (0,\infty) \to \mathbb{R}$ (Dirichlet heat kernel) such that

$$\left(e^{t\Delta_\Omega}\phi\right)(x) = \int_\Omega G_\Omega(x,y,t)\phi(y)dy$$

for $\phi \in L^p(\Omega)$ (cf. [23]). Then $e^{t\Delta_\Omega}\phi$ gives the solution for the heat equation in Ω with the initial function ϕ. After that, we abbreviate G_Ω as G. See Proposition 2.2.10.

We state existence and nonexistence results in Ω, a bounded domain in $\mathbb{R}^N$ ($N \ge 1$) with C^2 boundary.

Definition 2.1.7 (Classical solution in a bounded domain). Let X, Y be Banach spaces. We call a function $(u,v) \in C^{2,1}(\Omega \times (0,T)) \times C^{2,1}(\Omega \times (0,T))$ a classical solution of (2.1.1) in $X \times Y$ if (u,v) satisfies the boundary conditions, the equations in the classical sense in $\Omega \times (0,T)$, $\|u(t) - e^{t\Delta_\Omega}u_0\|_X \to 0$ $(t \to 0)$ and $\|v(t) - e^{t\Delta_\Omega}v_0\|_Y \to 0$ $(t \to 0)$.

Theorem 2.1.8 (Local in time existence in a bounded domain). *Suppose that the nonnegative functions $(u_0, v_0) \in L^1(\Omega) \times L^1(\Omega)$ satisfy*

$$e^{\frac{N}{2}(p_1u_0+p_2v_0)} \in L^1(\Omega) \text{ and } e^{\frac{N}{2}(q_1u_0+q_2v_0)} \in L^1(\Omega). \tag{2.1.6}$$

Then there exist $T > 0$ and a local in time nonnegative classical solution $(u,v) \in C^{2,1}(\Omega \times (0,T)) \times C^{2,1}(\Omega \times (0,T))$ of (2.1.1) in $L^\infty(\Omega) \times L^\infty(\Omega)$ in the sense of Definition 2.1.7.

The assumption (2.1.6) corresponds to both of the conditions (2.1.4) and (2.1.5) in Theorem 2.1.2. Indeed, if $e^{r_1(p_1u_0+p_2v_0)} \in L^1(\Omega)$ and $e^{r_2(q_1u_0+q_2v_0)} \in L^1(\Omega)$ hold for some $r_1 > N/2$ and $r_2 > N/2$, then u_0 and v_0 satisfy (2.1.6), and (2.1.1) has a local in time nonnegative classical solution.

Theorem 2.1.9 (Local in time nonexistence in a bounded domain). *The following* (i) *and* (ii) *hold:*

(i) *Let $p_1 \ge q_1$ and $p_2 \ge q_2$. Then for any $0 < r_1 < N/2$, there exist nonnegative functions (u_0, v_0) satisfying $e^{r_1(p_1u_0+p_2v_0)} \in L^1(\Omega)$ such that, for every $T > 0$, problem* (2.1.1) *admits* *no local in time nonnegative classical solution* $(u, v) \in C^{2,1}(\Omega \times (0,T)) \times C^{2,1}(\Omega \times (0,T))$ *in* $L^\infty(\Omega) \times L^\infty(\Omega)$ *in the sense of Definition 2.1.7.*

(ii) *Let* $p_1 < q_1$ *and* $p_2 > q_2$. *Put* I_1 *and* I_2 *in the same way as Theorem 2.1.5* (ii). *Then for any* $(r_1, r_2) \in (I_1 \times (0, N/2)) \cup ((0, N/2) \times I_2)$, *there exist nonnegative functions* (u_0, v_0) *satisfying* (2.1.4) *with* $L^1_{ul,\rho}(\mathbb{R}^N)$ *replaced by* $L^1(\Omega)$ *such that, for every* $T > 0$, *problem* (2.1.1) *admits no local in time nonnegative classical solution* $(u, v) \in C^{2,1}(\Omega \times (0,T)) \times C^{2,1}(\Omega \times (0,T))$ *in* $L^\infty(\Omega) \times L^\infty(\Omega)$ *in the sense of Definition 2.1.7.*

Theorem 2.1.9 holds if p_i and q_i $(i = 1, 2)$ are exchanged, respectively.

Remark 2.1.10. *Remark 2.1.6 holds if both of* $L^1_{ul,\rho}(\mathbb{R}^N)$ *and* $\mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$ *are replaced by* $L^1(\Omega)$.

Let us mention a sketch of the proofs. First we mention the existence part. Let (u, v) be a nonnegative solution of (2.1.1). Put $a := p_1 u + p_2 v$ and $b := q_1 u + q_2 v$. Then (a, b) satisfies

$$\begin{cases} \partial_t a = \Delta a + p_1 e^a + p_2 e^b & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t b = \Delta b + q_1 e^a + q_2 e^b & \text{in } \mathbb{R}^N \times (0,T), \\ a(x,0) = a_0(x), b(x,0) = b_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1.7}$$

where $a_0 := p_1 u_0 + p_2 v_0$ and $b_0 := q_1 u_0 + q_2 v_0$. On the other hand, let (U, V) be a nonnegative solution of

$$\begin{cases} \partial_t U = \Delta U + \dfrac{p_1}{\eta} U^{\eta+1} + \dfrac{p_2}{\eta} U V^\eta & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t V = \Delta V + \dfrac{q_1}{\eta} U^\eta V + \dfrac{q_2}{\eta} V^{\eta+1} & \text{in } \mathbb{R}^N \times (0,T), \\ U(x,0) = \exp(a_0(x)/\eta), V(x,0) = \exp(b_0(x)/\eta) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1.8}$$

where η is a large positive constant. Then $(\overline{a}, \overline{b}) := (\eta \log U,\ \eta \log V)$ becomes a supersolution for (2.1.7). By induction method we can construct a local in time nonnegative classical solution of (2.1.7). Thus it suffices to prove the existence of a local in time nonnegative classical solution of (2.1.8).

In [17] they have studied a supersolution for the strongly coupled system (2.1.3). The supersolution is constructed by transforming a solution of the scalar problem $\partial_t w = \Delta w + P(w)$, where P has a polynomial growth. We use this method and prove that (2.1.8) possesses a solution. See Proposition 2.3.1 for details. Finally, we show the following limits by integral estimates: $\lim_{t\to 0} \left\| a(t) - e^{t\Delta} a_0 \right\|_{L^\infty(\mathbb{R}^N)} = 0$ and $\lim_{t\to 0} \left\| b(t) - e^{t\Delta} b_0 \right\|_{L^\infty(\mathbb{R}^N)} = 0$, which lead to the existence of a local in time classical solution of (2.1.1) in the sense of Definition 2.1.1.

Theorem 2.1.8 is obtained from Theorem 2.1.2. Let (u, v) be a local in time classical solution of (2.1.1) in $\mathbb{R}^N$. If Ω is bounded, then $(u|_\Omega, v|_\Omega)$ is a supersolution and (2.1.1) has a local in time classical solution.

We explain the nonexistence part. In the proof of Theorem 2.1.5 it is important to consider a nonexistence result of the scalar integral problem

$$u(t) = e^{t\Delta} u_0 + C \int_0^t e^{(t-s)\Delta} e^{u(s)} ds \quad \text{in } \mathbb{R}^N \times (0,T), \tag{2.1.9}$$

where $C > 0$. This is shown by contradiction. If (2.1.9) has a local in time nonnegative solution, then $\|e^{t\Delta} u_0\|_{L^\infty(\mathbb{R}^N)}$ has an upper bound depending on t. See Lemma 2.4.2 (i). On the other hand, we can construct a nonnegative initial function u_0 satisfying a suitable integrability condition such that $\|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)}$ increases faster than the upper bound as $t \to 0$. Thus this is a contradiction and we obtain the nonexistence result of (2.1.9). We prove Theorem 2.1.5, using u_0.

Theorem 2.1.9 is obtained from the nonexistence result of the scalar integral equation (2.1.9) which replaced Δ by Δ_Ω and $\mathbb{R}^N$ by Ω, respectively. Note that we use an upper bound of $\|e^{t\Delta_\Omega}u_0\|_{L^\infty(\Omega)}$ which is different from the case $\Omega = \mathbb{R}^N$. In particular, when Ω is bounded, we cannot obtain the same upper bound of $\|e^{t\Delta_\Omega}u_0\|_{L^\infty(\Omega)}$ as $\Omega = \mathbb{R}^N$ by applying the method in $\mathbb{R}^N$. We use an inequality of the Dirichlet heat kernel in Ω, integral estimates and an induction method. See Proposition 2.4.1 (ii) and Lemma 2.4.2 (ii) for details.

This chapter is organized as follows. In Section 2.2 we recall some results of the existence of solutions and some properties of a semigroup. In Section 2.3 we prove the existence of a local in time classical solution of (2.1.8). We use this result and prove Theorems 2.1.2 and 2.1.8. In Section 2.4 we show the local in time nonexistence of (2.1.9) and (2.4.1) and prove Theorems 2.1.5 and 2.1.9. In Section 2.5 we apply our results of (2.1.1) to several examples of nonlinear reaction-diffusion systems. In Appendix 2.A we give an example of initial data which satisfy (2.1.4) but do not belong to the class given in [17].

2.2 Preliminaries

For any set X and the mappings $a = a(x)$ and $b = b(x)$ from X to $[0, \infty)$, we say

$$a(x) \lesssim b(x) \quad \text{for all } x \in X$$

if there exists a positive constant C such that $a(x) \le Cb(x)$ for all $x \in X$.

After this section, let $N \ge 1$, $\rho > 0$ and $\mathbb{N} = \{1, 2, \ldots\}$. Moreover, Ω is assumed to be a bounded domain in $\mathbb{R}^N$ with C^2 boundary unless otherwise noted.

Lemma 2.2.1. ([19, Corollary 3.1]) *Let $1 \le p \le q \le \infty$. Then there exists $C = C(N, p, q) > 0$ such that*

$$\|e^{t\Delta}u\|_{L^q_{ul,\rho}(\mathbb{R}^N)} \le C\left(\rho^{-N\left(\frac{1}{p}-\frac{1}{q}\right)} + t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right)\|u\|_{L^p_{ul,\rho}(\mathbb{R}^N)}$$

for $t > 0$, $\rho > 0$ and $u \in L^p_{ul,\rho}(\mathbb{R}^N)$.

Proposition 2.2.2. *Let $p > 1$ and let $f \in C^1([0, \infty))$ be a nonnegative function. Suppose that $f'(s) = O(s^{p-1})$ as $s \to \infty$. Then the following* (i) *and* (ii) *hold:*

(i) (Subcritical case) *If $q \ge 1$ and $q > \frac{N}{2}(p-1)$, then for a nonnegative function $w_0 \in L^q_{ul,\rho}(\mathbb{R}^N)$, there exist $T > 0$ and a nonnegative classical solution $w \in C((0,T) : L^q_{ul,\rho}(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)) \cap C^{2,1}(\mathbb{R}^N \times (0,T))$ such that*

$$(2.2.1) \qquad \begin{cases} \partial_t w = \Delta w + f(w) & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = w_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

and

$$(2.2.2) \qquad \lim_{t \to 0} \|w(t) - e^{t\Delta}w_0\|_{L^q_{ul,\rho}(\mathbb{R}^N)} = 0.$$

(ii) (Critical case) *If* $q = \frac{N}{2}(p-1) > 1$, *then for a nonnegative function* $w_0 \in \mathcal{L}^q_{ul,\rho}(\mathbb{R}^N)$, *there exist* $T > 0$ *and a nonnegative classical solution* $w \in C([0,T) : \mathcal{L}^q_{ul,\rho}(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)) \cap C^{2,1}(\mathbb{R}^N \times (0,T))$ *such that* (2.2.1) *and* (2.2.2) *hold.*

Moreover, for both of (i) *and* (ii), *there exists* $C = C(N,p,q,\rho) > 0$ *such that*

$$\|w(t)\|_{L^r_{ul,\rho}(\mathbb{R}^N)} \le Ct^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|w_0\|_{L^q_{ul,\rho}(\mathbb{R}^N)} \tag{2.2.3}$$

for sufficiently small $t > 0$ *and* $r \in [q,\infty]$.

Since we can prove Proposition 2.2.2 in a similar way to [2, 23, 27], we omit the proof. See also [11, Propositions 2.2 and 2.3].

Remark 2.2.3. ([11, Remark 2.2 (ii)]) *Suppose that all the assumptions of Proposition 2.2.2* (ii) *hold. Let* $\max\{p,q\} < \alpha < pq$ *and* $\sigma := \frac{N}{2}\left(\frac{1}{q}-\frac{1}{\alpha}\right)$. *Then it follows that*

$$\lim_{t\to 0} t^\sigma \|w(t)\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)} = 0.$$

Lemma 2.2.4. *The following* (i) *and* (ii) *hold:*

(i) *Let* J *be a convex function from* $[0,\infty)$ *to itself. Then for a nonnegative function* $\phi \in L^1_{ul,\rho}(\mathbb{R}^N)$,

$$J(e^{t\Delta}\phi(x)) \le [e^{t\Delta}(J\phi)](x) \quad \textit{in } \mathbb{R}^N \times (0,\infty).$$

(ii) *Let* J *be a convex function from* $[0,\infty)$ *to itself. Let* $C_0(\Omega)$ *denote the space of continuous functions on* $\overline{\Omega}$ *vanishing at* $\partial\Omega$. *If* $J(0) = 0$, *then for a nonnegative function* $\phi \in C_0(\Omega)$,

$$J(e^{t\Delta_\Omega}\phi(x)) \le [e^{t\Delta_\Omega}(J\phi)](x) \quad \textit{in } \Omega \times (0,\infty).$$

Since the assertions (i) and (ii) follow from [11, Lemma 2.4] and [27, Lemma 5.1], respectively, we omit the proof.

Lemma 2.2.5. *Let* $1 \le p < \infty$. *Then the following* (i) *and* (ii) *hold:*

(i) $u \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$ *if and only if* $|u|^p \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$.

(ii) *Let* $C > 0$. *If* $u \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$, *then* $\max\{u, C\} \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$.

Proof. The assertion (i) follows from [11, Lemma 2.2 and Remark 2.1].

We prove the assertion (ii). Let $u \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$. Then there exists a sequence $\{u_n\} \subset BUC(\mathbb{R}^N)$ such that $u_n \to u$ in $L^p_{ul,\rho}(\mathbb{R}^N)$ as $n \to \infty$. For $n \in \mathbb{N}$, put $v_n := \max\{u_n, C\}$. We see that $\{v_n\} \subset BUC(\mathbb{R}^N)$ and obtain

$$|\max\{u,C\} - v_n| \le |u - u_n| \to 0 \;\text{ in } L^p_{ul,\rho}(\mathbb{R}^N) \text{ as } n \to \infty.$$

Thus $\max\{u, C\} \in \mathcal{L}^p_{ul,\rho}(\mathbb{R}^N)$. □

Proposition 2.2.6. ([17, Lemma 2.2]) *The following* (i) *and* (ii) *hold:*

(i) *Let* $f(u) \in C^1([0,\infty))$ *satisfy*

$$f(u) \geq 0 \text{ and } f'(u) \geq 0 \text{ for } u \geq 0.$$

Let $u_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ *be a nonnegative function. If there exist* $T > 0$ *and a nonnegative function* $\overline{u} \in C^{2,1}(\mathbb{R}^N \times (0,T))$ *such that*

$$\overline{u}(t) \geq e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(\overline{u}(s))ds$$

for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$*, then there exists a solution* $u \in C^{2,1}(\mathbb{R}^N \times (0,T))$ *of the integral equation*

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u(s))ds \quad \text{in } \mathbb{R}^N \times (0,T)$$

which satisfies

$$\partial_t u = \Delta u + f(u) \quad \text{in } \mathbb{R}^N \times (0,T).$$

Furthermore, $0 \leq u(x,t) \leq \overline{u}(x,t)$ *in* $\mathbb{R}^N \times (0,T)$.

(ii) *Let* $f(u,v), g(u,v) \in C^{1,1}([0,\infty) \times [0,\infty))$ *satisfy*

$$f(u,v) \geq 0,\ f_u(u,v) \geq 0,\ f_v(u,v) \geq 0,\ g(u,v) \geq 0,\ g_u(u,v) \geq 0 \text{ and } g_v(u,v) \geq 0$$

for $u \geq 0$ *and* $v \geq 0$*. Let* $u_0, v_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ *be nonnegative functions. If there exist* $T > 0$ *and nonnegative functions* $(\bar{u}, \bar{v}) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ *such that*

$$\begin{aligned} \bar{u}(t) &\geq e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(\bar{u}(s), \bar{v}(s))ds, \\ \bar{v}(t) &\geq e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} g(\bar{u}(s), \bar{v}(s))ds \end{aligned} \tag{2.2.4}$$

for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$*, then there exists a solution* $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ *of the integral system*

$$\begin{aligned} u(t) &= e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u(s), v(s))ds \quad \text{in } \mathbb{R}^N \times (0,T), \\ v(t) &= e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} g(u(s), v(s))ds \quad \text{in } \mathbb{R}^N \times (0,T) \end{aligned} \tag{2.2.5}$$

which satisfies

$$\begin{cases} \partial_t u = \Delta u + f(u,v) & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t v = \Delta v + g(u,v) & \text{in } \mathbb{R}^N \times (0,T). \end{cases}$$

Furthermore, $0 \leq u(x,t) \leq \bar{u}(x,t)$ *in* $\mathbb{R}^N \times (0,T)$ *and* $0 \leq v(x,t) \leq \bar{v}(x,t)$ *in* $\mathbb{R}^N \times (0,T)$.

Remark 2.2.7. *Proposition 2.2.6* (i) *and* (ii) *hold for a bounded domain. More precisely, both of* (i) *and* (ii) *hold if* $\mathbb{R}^N$, $L^1_{ul,\rho}(\mathbb{R}^N)$ *and* $e^{t\Delta}$ *are replaced by* Ω, $L^1(\Omega)$ *and* $e^{t\Delta_\Omega}$*, respectively.*

Lemma 2.2.8. (cf. [11, Lemma 2.3]) *Let $f(u,v) \in C^{1,1}([0,\infty)\times[0,\infty))$ satisfy*

$$f(u,v) \geq 0,\ f_u(u,v) \geq 0 \ \ \text{and} \ \ f_v(u,v) \geq 0 \text{ for } u \geq 0 \text{ and } v \geq 0.$$

Let $T > 0$ and let $\overline{u}, \overline{v} \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N))$ be nonnegative functions. Then the following (i) *and* (ii) *hold:*

(i) *Let $u_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$. Suppose that*

$$\partial_t \overline{u} = \Delta \overline{u} + f(\overline{u},\overline{v}) \quad \text{in } \mathbb{R}^N \times (0,T) \tag{2.2.6}$$

and that

$$\lim_{t\to 0} \|\overline{u}(t) - e^{t\Delta}u_0\|_{L^1_{ul,\rho}(\mathbb{R}^N)} = 0. \tag{2.2.7}$$

Then $\overline{u}$ satisfies

$$\overline{u}(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(\overline{u}(s),\overline{v}(s))ds \quad \text{in } \mathbb{R}^N \times (0,T). \tag{2.2.8}$$

(ii) *Let $u_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$. Suppose that*

$$\partial_t \overline{u} \geq \Delta \overline{u} + f(\overline{u},\overline{v}) \quad \text{in } \mathbb{R}^N \times (0,T) \tag{2.2.9}$$

and that either (2.2.7) *or the following inequality holds:*

$$\overline{u}(t) \geq e^{t\Delta}u_0 \quad \text{in } \mathbb{R}^N \times (0,T). \tag{2.2.10}$$

Then $\overline{u}$ satisfies

$$\overline{u}(t) \geq e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(\overline{u}(s),\overline{v}(s))ds \quad \text{in } \mathbb{R}^N \times (0,T). \tag{2.2.11}$$

Proof. We prove the assertion (i). It follows from (2.2.6) that

$$\overline{u}(t) = e^{(t-\tau)\Delta}\overline{u}(\tau) + \int_\tau^t e^{(t-s)\Delta} f(\overline{u}(s),\overline{v}(s))ds \tag{2.2.12}$$

for $0 < \tau < t < T$. See [23, p.77] for details. We obtain from Lemma 2.2.1 and (2.2.7) that

$$\begin{aligned}\|e^{(t-\tau)\Delta}\overline{u}(\tau) - e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} &= \|e^{(t-\tau)\Delta}(\overline{u}(\tau) - e^{\tau\Delta}u_0)\|_{L^\infty(\mathbb{R}^N)} \\ &\lesssim \left(\rho^{-N} + (t-\tau)^{-\frac{N}{2}}\right) \|\overline{u}(\tau) - e^{\tau\Delta}u_0\|_{L^1_{ul,\rho}(\mathbb{R}^N)} \\ &\to 0 \ \ (\tau \to 0).\end{aligned}$$

Let $\chi_{(\tau,t)}$ be a characteristic function. By the monotone convergence theorem we have

$$\lim_{\tau\to 0}\int_\tau^t e^{(t-s)\Delta} f(\overline{u}(s),\overline{v}(s))ds = \lim_{\tau\to 0}\int_0^t \chi_{(\tau,t)}(s) e^{(t-s)\Delta} f(\overline{u}(s),\overline{v}(s))ds = \int_0^t e^{(t-s)\Delta} f(\overline{u}(s),\overline{v}(s))ds$$

for each $(x,t) \in \mathbb{R}^N \times (0,T)$. Therefore, we deduce (2.2.8) by taking the limit as $\tau \to 0$ of

(2.2.12).

We prove the assertion (ii). If (2.2.7) holds, then the inequality (2.2.11) can be obtained in the same way as the assertion (i). On the other hand, if (2.2.10) holds, then (2.2.9) implies that

$$\overline{u}(t) \geq e^{(t-\tau)\Delta}\overline{u}(\tau) + \int_\tau^t e^{(t-s)\Delta} f(\overline{u}(s), \overline{v}(s))ds \geq e^{t\Delta}u_0 + \int_\tau^t e^{(t-s)\Delta} f(\overline{u}(s), \overline{v}(s))ds.$$

Taking the limit as $\tau \to 0$, we obtain (2.2.11). □

Lemma 2.2.8 corresponds to [11, Lemma 2.3]. We assume that $\overline{u}, \overline{v} \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N))$, which is needed to obtain (2.2.12). This assumption is not assumed in [11]. However, we do not suppose the limit condition corresponding to [11, (2.1)].

Remark 2.2.9. *Lemma 2.2.8* (i) *and* (ii) *hold for a bounded domain. This is similar to Proposition 2.2.6 and Remark 2.2.7. In this case we impose the Dirichlet boundary condition on $\overline{u}$ and $\overline{v}$.*

Proposition 2.2.10. (cf. [23, Proposition 49.10]) *Let Ω be an arbitrary domain in $\mathbb{R}^N$ $(N \geq 1)$. Let $y \in \Omega$, $\delta(y) := dist(y, \partial\Omega)$, $B := B(y, \delta(y))$ and denote*

$$K_y(x,t) := (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}, \quad x \in B \text{ and } t > 0.$$

Then there exists $m > 0$ depending only on N such that the Dirichlet heat kernel $G(x,y,t)$ in Ω satisfies

$$G(x,y,t) \geq K_y(x,t) - m\delta(y)^{-N} \quad \text{in } B \times (0,\infty).$$

2.3 Existence result

In this section we prove Theorems 2.1.2 and 2.1.8. In the proof of Theorem 2.1.2 we use a solution of the problem

$$\begin{cases} \partial_t u = \Delta u + C_1 u^{p_0} + C_2 u^{p_1} v^{p_2} & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t v = \Delta v + C_3 v^{q_0} + C_4 u^{q_1} v^{q_2} & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.3.1}$$

where $p_0, q_0 > 1$, $p_1, p_2, q_1, q_2 \geq 1$ and C_i $(i = 1,2,3,4)$ are arbitrary nonnegative constants. We assume that (u_0, v_0) is in $L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$ or in $\mathcal{L}^{r_1}_{ul,\rho}(\mathbb{R}^N) \times \mathcal{L}^{r_2}_{ul,\rho}(\mathbb{R}^N)$, where $r_1 \geq 1$ and $r_2 \geq 1$. The following four exponents are key factors to study (2.3.1):

$$\begin{aligned} &A_1 := 1 + \frac{p_0 - 1}{r_1}, \quad A_2 := 1 + \frac{p_1 - 1}{r_1} + \frac{p_2}{r_2}, \\ &A_3 := 1 + \frac{q_0 - 1}{r_2} \quad \text{and} \quad A_4 := 1 + \frac{q_1}{r_1} + \frac{q_2 - 1}{r_2}. \end{aligned} \tag{2.3.2}$$

The existence of a solution of (2.3.1) is established in a similar manner to [17, Theorem 4.1] when $(u_0, v_0) \in L^{r_1,\infty}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2,\infty}_{ul,\rho}(\mathbb{R}^N)$ which includes our space $L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$. However, [17, Theorem 4.1] does not claim in which sense the initial condition is satisfied. In Proposition 2.3.1 we establish the existence of a solution in the sense of Definition 2.1.1.

Proposition 2.3.1. *Let $p_0, q_0 > 1$, $p_1, p_2, q_1, q_2 \geq 1$ and $r_1, r_2 \geq 1$. For $i = 1,2,3,4$, let $C_i \geq 0$ and let A_i be given by* (2.3.2). *Then the following* (i) *and* (ii) *hold:*

(i) *Let $1 \leq r_1, r_2 < \infty$. Suppose that*

$$\frac{N}{2}\max\{A_1 - 1, A_2 - 1, A_3 - 1, A_4 - 1\} < 1. \tag{2.3.3}$$

Then for any nonnegative functions $(u_0, v_0) \in L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$, there exist $T > 0$ and a local in time nonnegative classical solution $(u, v) \in (C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)))^2$ of (2.3.1) *in $L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$ in the sense of Definition 2.1.1.*

(ii) *Let $1 < r_1, r_2 < \infty$. Suppose that*

$$\frac{N}{2}\max\{A_1 - 1, A_2 - 1, A_3 - 1, A_4 - 1\} = 1. \tag{2.3.4}$$

Then for any nonnegative functions $(u_0, v_0) \in \mathcal{L}^{r_1}_{ul,\rho}(\mathbb{R}^N) \times \mathcal{L}^{r_2}_{ul,\rho}(\mathbb{R}^N)$, there exist $T > 0$ and a local in time nonnegative classical solution $(u, v) \in (C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)))^2$ of (2.3.1) *in $L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$ in the sense of Definition 2.1.1.*

Proof. It suffices to prove when $C_i > 0$ for $i = 1,2,3,4$.

We prove the assertion (i). Setting $w_0(x) := \max\{u_0(x)^{r_1} + v_0(x)^{r_2}, 1\}$, we have $w_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$. We consider the problem

$$\begin{cases} \partial_t w = \Delta w + P(w) & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = w_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.3.5}$$

where $P(w) := r_1 C_1 w^{A_1} + r_1 C_2 w^{A_2} + r_2 C_3 w^{A_3} + r_2 C_4 w^{A_4}$. We can apply Proposition 2.2.2 (i) with $p = \max\{A_1, A_2, A_3, A_4\}$ and $q = 1$, since $P'(w) = O(w^{p-1})$ as $w \to \infty$ and the inequality $q > N(p-1)/2$ follows from (2.3.3). Let w be a solution of (2.3.5) given by Proposition 2.2.2 (i). Put $(\bar{u}, \bar{v}) := (w^{1/r_1}, w^{1/r_2})$. Direct calculations show that

$$\begin{cases} \partial_t \bar{u} \geq \Delta u + C_1 \overline{u}^{p_0} + C_2 \overline{u}^{p_1} \overline{v}^{p_2} & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t \bar{v} \geq \Delta v + C_3 \overline{v}^{q_0} + C_4 \overline{u}^{q_1} \overline{v}^{q_2} & \text{in } \mathbb{R}^N \times (0,T). \end{cases} \tag{2.3.6}$$

On the other hand, by (2.2.2) and Lemma 2.2.8 (i) we have

$$w(t) = e^{t\Delta} w_0 + \int_0^t e^{(t-s)\Delta} P(w(s))ds. \tag{2.3.7}$$

Let $J(s) := s^{r_1}$ $(s \geq 0)$. Since J is convex, we deduce from Lemma 2.2.4 (i) and (2.3.7) that

$$J(\overline{u}(t)) = w(t) \geq e^{t\Delta} w_0 \geq e^{t\Delta}(J(u_0)) \geq J(e^{t\Delta} u_0) \quad \text{in } \mathbb{R}^N \times (0,T).$$

Since J is strictly increasing, we have (2.2.10). Using (2.2.10) and (2.3.6), by Lemma 2.2.8 (ii) we have

$$\overline{u}(t) \geq e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \left(C_1 \overline{u}(s)^{p_0} + C_2 \overline{u}(s)^{p_1} \overline{v}(s)^{p_2} \right) ds. \tag{2.3.8}$$

In the same way we can obtain

$$\overline{v}(t) \geq e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\left(C_3\overline{v}(s)^{q_0} + C_4\overline{u}(s)^{q_1}\overline{v}(s)^{q_2}\right) ds. \tag{2.3.9}$$

Put $f(u,v) := C_1u^{p_0} + C_2u^{p_1}v^{p_2}$ and $g(u,v) := C_3v^{q_0} + C_4u^{q_1}v^{q_2}$. We observe from (2.3.8) and (2.3.9) that (f,g) and $(\overline{u},\overline{v})$ satisfy all the assumptions of Proposition 2.2.6 (ii), which implies that (2.2.5) possesses a solution $(u,v) \in (C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)))^2$ satisfying (2.3.1) and

$$0 \leq u(x,t) \leq \overline{u}(x,t) \text{ and } 0 \leq v(x,t) \leq \overline{v}(x,t) \text{ in } \mathbb{R}^N \times (0,T). \tag{2.3.10}$$

It remains to prove the following limits:

$$\lim_{t\to 0}\|u(t) - e^{t\Delta}u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} = 0 \text{ and } \lim_{t\to 0}\|v(t) - e^{t\Delta}v_0\|_{L^{r_2}_{ul,\rho}(\mathbb{R}^N)} = 0. \tag{2.3.11}$$

Since $u \leq \overline{u} = w^{1/r_1}$ and $v \leq \overline{v} = w^{1/r_2}$, it follows from Lemma 2.2.1 and (2.2.5) that

$$\begin{aligned}
&\|u(t) - e^{t\Delta}u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} \\
&\leq C_1\int_0^t \|e^{(t-s)\Delta}w(s)^{\frac{p_0}{r_1}}\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)}ds + C_2\int_0^t \|e^{(t-s)\Delta}w(s)^{\frac{p_1}{r_1}+\frac{p_2}{r_2}}\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)}ds \\
&\lesssim \int_0^t \|w(s)^{\frac{p_0}{r_1}}\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)}ds + \int_0^t \|w(s)^{\frac{p_1}{r_1}+\frac{p_2}{r_2}}\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)}ds \\
&= \int_0^t \|w(s)\|^{\frac{p_0}{r_1}}_{L^{p_0}_{ul,\rho}(\mathbb{R}^N)}ds + \int_0^t \|w(s)\|^{\frac{p_1}{r_1}+\frac{p_2}{r_2}}_{L^{r_1\left(\frac{p_1}{r_1}+\frac{p_2}{r_2}\right)}_{ul,\rho}(\mathbb{R}^N)}ds.
\end{aligned} \tag{2.3.12}$$

On the other hand, by (2.2.3) we have

$$\|w(s)\|_{L^{p_0}_{ul,\rho}(\mathbb{R}^N)} \lesssim s^{-\frac{N}{2}\left(1-\frac{1}{p_0}\right)}\|w_0\|_{L^1_{ul,\rho}(\mathbb{R}^N)} \tag{2.3.13}$$

and

$$\|w(s)\|_{L^{r_1\left(\frac{p_1}{r_1}+\frac{p_2}{r_2}\right)}_{ul,\rho}(\mathbb{R}^N)} \lesssim s^{-\frac{N}{2}\left\{1-\frac{1}{r_1\left(\frac{p_1}{r_1}+\frac{p_2}{r_2}\right)}\right\}}\|w_0\|_{L^1_{ul,\rho}(\mathbb{R}^N)} \tag{2.3.14}$$

for sufficiently small $s > 0$. The inequalities (2.3.12), (2.3.13) and (2.3.14) imply that

$$\|u(t) - e^{t\Delta}u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} \lesssim \int_0^t s^{-\frac{N}{2}\cdot\frac{p_0-1}{r_1}}ds + \int_0^t s^{-\frac{N}{2}\left(\frac{p_1-1}{r_1}+\frac{p_2}{r_2}\right)}ds, \tag{2.3.15}$$

provided that $t > 0$ is small. We observe from (2.3.2) and (2.3.3) that

$$\frac{N}{2}\cdot\frac{p_0-1}{r_1} = \frac{N}{2}(A_1-1) < 1 \text{ and } \frac{N}{2}\left(\frac{p_1-1}{r_1}+\frac{p_2}{r_2}\right) = \frac{N}{2}(A_2-1) < 1.$$

Thus, the right hand side of (2.3.15) converges to 0 as $t \to 0$. Hence, we obtain the first limit of (2.3.11). In the same way we can obtain the second limit of (2.3.11). Thus, (u,v) is a local in time nonnegative classical solution of (2.3.1) in $L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$ in the sense of

Definition 2.1.1. The proof of the assertion (i) is complete.

We prove the assertion (ii). Since $r_1, r_2 > 1$, there exists $\varepsilon > 0$ such that

$$r_1' := \frac{r_1}{1+\varepsilon} > 1,\ r_2' := \frac{r_2}{1+\varepsilon} > 1 \text{ and } \varepsilon < \frac{2}{N}. \tag{2.3.16}$$

We define A_i' $(i = 1, 2, 3, 4)$ by

$$\begin{aligned} A_1' &:= 1 + \frac{p_0 - 1}{r_1'}, \quad A_2' := 1 + \frac{p_1 - 1}{r_1'} + \frac{p_2}{r_2'}, \\ A_3' &:= 1 + \frac{q_0 - 1}{r_2'} \quad \text{and} \quad A_4' := 1 + \frac{q_1}{r_1'} + \frac{q_2 - 1}{r_2'}. \end{aligned}$$

Due to (2.3.4), we have

$$\frac{N}{2} \max\left\{A_1' - 1, A_2' - 1, A_3' - 1, A_4' - 1\right\} = 1 + \varepsilon.$$

Let $w_0(x) := \max\{u_0(x)^{r_1'} + v_0(x)^{r_2'}, 1\}$. Then Lemma 2.2.5 implies that $w_0 \in \mathcal{L}^{1+\varepsilon}_{ul,\rho}(\mathbb{R}^N)$. By Proposition 2.2.2 (ii) there exist $T > 0$ and a nonnegative classical solution $w \in C([0,T) : \mathcal{L}^{1+\varepsilon}_{ul,\rho}(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)) \cap C^{2,1}(\mathbb{R}^N \times (0,T))$ for the problem

$$\begin{cases} \partial_t w = \Delta w + Q(w) & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = w_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.3.17}$$

where $Q(w) := r_1' C_1 w^{A_1'} + r_1' C_2 w^{A_2'} + r_2' C_3 w^{A_3'} + r_2' C_4 w^{A_4'}$. Put $(\bar{u}, \bar{v}) := (w^{1/r_1'}, w^{1/r_2'})$. Then $(\overline{u}, \overline{v})$ satisfies (2.3.6), (2.3.8) and (2.3.9). Thus Proposition 2.2.6 (ii) implies that (2.2.5) with $f(u,v) := C_1 u^{p_0} + C_2 u^{p_1} v^{p_2}$ and $g(u,v) := C_3 v^{q_0} + C_4 u^{q_1} v^{q_2}$ possesses a solution $(u,v) \in (C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)))^2$ which satisfies (2.3.1) and (2.3.10).

We prove (2.3.11). Put $\alpha_0 := 1 + 2(1+\varepsilon)/N$ and $\alpha := \alpha_0 + \delta$, where $\delta > 0$ is a sufficiently small constant chosen later. We show that Remark 2.2.3 can be applied for w with $p = \alpha_0$ and $q = 1 + \varepsilon$. By (2.3.16) we see that $\alpha_0 > 1 + 2/N > 1 + \varepsilon$, and hence $\max\{p, q\} = \alpha_0 < \alpha$. Since $\alpha_0 < \alpha_0(1+\varepsilon) = pq$ and δ is small, we obtain $\alpha < pq$. Thus $\max\{p,q\} < \alpha < pq$ holds and Remark 2.2.3 can be applied. Let $k := (1+\varepsilon)(2/N + 1/r_1)$. By (2.3.4) we have

$$\frac{p_0}{r_1'} \le \frac{2}{N}(1+\varepsilon) + \frac{1}{r_1'} = k \quad \text{and} \quad \frac{p_1}{r_1'} + \frac{p_2}{r_2'} \le k. \tag{2.3.18}$$

Since δ is small, we obtain

$$\frac{k}{\alpha} > \frac{1}{r_1}. \tag{2.3.19}$$

The inequalities $u \le \bar{u} = w^{1/r_1'}$, $v \le \bar{v} = w^{1/r_2'}$, $w \ge e^{t\Delta} w_0 \ge 1$, (2.3.18) and (2.3.19) along with

Lemma 2.2.1 imply that

$$
\begin{aligned}
&\|u(t) - e^{t\Delta}u_0\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} \\
&\le C_1 \int_0^t \|e^{(t-s)\Delta} w(s)^{\frac{p_0}{r_1{}'}}\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} ds + C_2 \int_0^t \|e^{(t-s)\Delta} w(s)^{\frac{p_1}{r_1{}'}+\frac{p_2}{r_2{}'}}\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} ds \\
&\le (C_1 + C_2) \int_0^t \|e^{(t-s)\Delta} w(s)^k\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)} ds \\
&\lesssim \int_0^t \left(\rho^{-N\left(\frac{k}{\alpha}-\frac{1}{r_1}\right)} + (t-s)^{-\frac{N}{2}\left(\frac{k}{\alpha}-\frac{1}{r_1}\right)}\right) \|w(s)^k\|_{L^{\frac{\alpha}{k}}_{ul,\rho}(\mathbb{R}^N)} ds \\
&\le \int_0^t \left(\rho^{-N\left(\frac{k}{\alpha}-\frac{1}{r_1}\right)} + (t-s)^{-\frac{N}{2}\left(\frac{k}{\alpha}-\frac{1}{r_1}\right)}\right) s^{-\sigma k} ds \cdot \sup_{0<s<t} s^{\sigma k} \|w(s)\|^k_{L^{\alpha}_{ul,\rho}(\mathbb{R}^N)},
\end{aligned} \tag{2.3.20}
$$

where $\sigma := \frac{N}{2}\left(\frac{1}{1+\varepsilon} - \frac{1}{\alpha}\right)$. We easily see that

$$
\frac{N}{2}\left(\frac{k}{\alpha} - \frac{1}{r_1}\right) < 1, \ \sigma k < 1 \ \text{ and } \ \frac{N}{2}\left(\frac{k}{\alpha} - \frac{1}{r_1}\right) + \sigma k = 1.
$$

Thus

$$
\int_0^t \left(\rho^{-N\left(\frac{k}{\alpha}-\frac{1}{r_1}\right)} + (t-s)^{-\frac{N}{2}\left(\frac{k}{\alpha}-\frac{1}{r_1}\right)}\right) s^{-\sigma k} ds < C \tag{2.3.21}
$$

uniformly for small $t > 0$. Then due to (2.3.20) and (2.3.21), Remark 2.2.3 implies that $\left\|u(t) - e^{t\Delta}u_0\right\|_{L^{r_1}_{ul,\rho}(\mathbb{R}^N)}$ converges to 0 as $t \to 0$. We can obtain $\lim_{t\to 0} \left\|v(t) - e^{t\Delta}v_0\right\|_{L^{r_2}_{ul,\rho}(\mathbb{R}^N)} = 0$ in the same way. Therefore, the proof of the assertion (ii) is complete. $\square$

Proof of Theorem 2.1.2. Let $a_0 := p_1 u_0 + p_2 v_0$ and $b_0 := q_1 u_0 + q_2 v_0$. We recall that if (u, v) satisfies (2.1.1), $a = p_1 u + p_2 v$ and $b = q_1 u + q_2 v$, then (a, b) satisfies (2.1.7).

We consider the case where $\det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \neq 0$. Assume that (2.1.7) possesses a local in time nonnegative classical solution $(a, b) \in (C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)))^2$ in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 2.1.1. Then $\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$ satisfies (2.1.1) and

$$
\lim_{t\to 0} \left\|u(t) - e^{t\Delta}u_0\right\|_{L^\infty(\mathbb{R}^N)} = 0 \ \text{ and } \ \lim_{t\to 0} \left\|v(t) - e^{t\Delta}v_0\right\|_{L^\infty(\mathbb{R}^N)} = 0. \tag{2.3.22}
$$

Moreover, Lemma 2.2.8 (i) implies that (a, b) satisfies

$$
\begin{aligned}
a(t) &= e^{t\Delta}a_0 + \int_0^t e^{(t-s)\Delta}\left(p_1 e^{a(s)} + p_2 e^{b(s)}\right) ds \ \text{ in } \mathbb{R}^N \times (0,T), \\
b(t) &= e^{t\Delta}b_0 + \int_0^t e^{(t-s)\Delta}\left(q_1 e^{a(s)} + q_2 e^{b(s)}\right) ds \ \text{ in } \mathbb{R}^N \times (0,T).
\end{aligned} \tag{2.3.23}
$$

We obtain from (2.3.23) that (u, v) satisfies (2.2.5) with $f(u, v) := e^{p_1 u + p_2 v}$ and $g(u, v) := e^{q_1 u + q_2 v}$, and hence $u \ge 0$ and $v \ge 0$.

On the other hand, if $\det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} = 0$, then there exists $\lambda > 0$ such that $(q_1, q_2) =$

$(\lambda p_1, \lambda p_2)$. Since (2.1.4) holds for some $r_1 > N/2$ and $r_2 > N/2$, or (2.1.5) holds with $r_1 = N/2$ and $r_2 = N/2$, there exist $T > 0$ and a nonnegative classical solution $\overline{a} \in C^{2,1}(\mathbb{R}^N \times (0,T))$ of

$$\overline{a}(t) = e^{t\Delta}a_0 + (p_1 + p_2)\int_0^t e^{(t-s)\Delta}e^{\max\{\lambda,1\}\overline{a}(s)}ds \quad \text{in } \mathbb{R}^N \times (0,T)$$

which satisfies

$$\lim_{t\to 0}\|\overline{a}(t) - e^{t\Delta}a_0\|_{L^\infty(\mathbb{R}^N)} = 0. \tag{2.3.24}$$

The existence result follows from [11, Theorem 1.1 and Proposition 2.4]. By Proposition 2.2.6 (i) there exists a solution $a \in C^{2,1}(\mathbb{R}^N \times (0,T))$ of

$$a(t) = e^{t\Delta}a_0 + \int_0^t e^{(t-s)\Delta}\left(p_1e^{a(s)} + p_2e^{\lambda a(s)}\right)ds \quad \text{in } \mathbb{R}^N \times (0,T)$$

which satisfies $0 \le a \le \overline{a}$ in $\mathbb{R}^N \times (0,T)$. Putting

$$u(t) := e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}e^{a(s)}ds \quad \text{and} \quad v(t) := e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}e^{\lambda a(s)}ds,$$

we have $a = p_1u + p_2v$, and hence (2.1.1) holds. We observe that (2.3.24) yields (2.3.22). Therefore, it suffices to prove the existence of a local in time nonnegative classical solution $(a,b) \in (C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)))^2$ of (2.1.7) in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 2.1.1 when $\det\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2\end{pmatrix} \neq 0$.

We choose $\eta > \max\{2/N, 1\}$ and consider the problem (2.1.8). Put $U_0 := \exp(a_0/\eta)$ and $V_0 := \exp(b_0/\eta)$.

We prove the assertion (i). Since (2.1.4) holds for some $r_1 > N/2$ and $r_2 > N/2$, we see that $(U_0, V_0) \in L^{\eta r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{\eta r_2}_{ul,\rho}(\mathbb{R}^N)$. By direct calculation we have (2.3.3). Since $\eta > \max\{2/N, 1\}$, we can apply Proposition 2.3.1 (i) to (2.1.8). Then there exist $T > 0$ and a local in time classical solution $(U,V) \in (C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)))^2$ of (2.1.8) in $L^{\eta r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{\eta r_2}_{ul,\rho}(\mathbb{R}^N)$ which satisfies

$$0 \le U(x,t) \le w(x,t)^{1/\eta r_1} \quad \text{and} \quad 0 \le V(x,t) \le w(x,t)^{1/\eta r_2} \quad \text{in } \mathbb{R}^N \times (0,T). \tag{2.3.25}$$

Here w is a local in time classical solution of (2.3.5) with $P(w) = (p_1r_1 + q_1r_2)w^{1+1/r_1} + (p_2r_1 + q_2r_2)w^{1+1/r_2}$ and $w_0(x) := \max\{U_0(x)^{\eta r_1} + V_0(x)^{\eta r_2}, 1\} \in L^1_{ul,\rho}(\mathbb{R}^N)$. Since $U \ge e^{t\Delta}U_0 \ge 1$ and $V \ge e^{t\Delta}V_0 \ge 1$, we can define

$$(\overline{a}, \overline{b}) := (\eta \log U,\ \eta \log V). \tag{2.3.26}$$

Since (U,V) is a local in time classical solution of (2.1.8), we have

$$\begin{cases} \partial_t\overline{a} \ge \Delta\overline{a} + p_1e^{\overline{a}} + p_2e^{\overline{b}} & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t\overline{b} \ge \Delta\overline{b} + q_1e^{\overline{a}} + q_2e^{\overline{b}} & \text{in } \mathbb{R}^N \times (0,T). \end{cases} \tag{2.3.27}$$

Let $J(s) := e^{s/\eta}$ $(s \ge 0)$. We see that J is convex. It follows from Lemma 2.2.4 (i) that

$$J(\overline{a}(t)) = U(t) \ge e^{t\Delta}U_0 = e^{t\Delta}(J(a_0)) \ge J(e^{t\Delta}a_0) \quad \text{in } \mathbb{R}^N \times (0,T).$$

Since J is strictly increasing, we have

$$\overline{a}(t) \geq e^{t\Delta} a_0. \tag{2.3.28}$$

Combining (2.3.27) and (2.3.28), we deduce from Lemma 2.2.8 (ii) that

$$\overline{a}(t) \geq e^{t\Delta} a_0 + \int_0^t e^{(t-s)\Delta} \left(p_1 e^{\overline{a}(s)} + p_2 e^{\overline{b}(s)} \right) ds. \tag{2.3.29}$$

In the same way we can obtain

$$\overline{b}(t) \geq e^{t\Delta} b_0 + \int_0^t e^{(t-s)\Delta} \left(q_1 e^{\overline{a}(s)} + q_2 e^{\overline{b}(s)} \right) ds. \tag{2.3.30}$$

Therefore, using (2.3.29) and (2.3.30), we observe from Proposition 2.2.6 (ii) that there exists a solution $(a, b) \in (C^{2,1}(\mathbb{R}^N \times (0,T)) \cap L^\infty_{loc}((0,T) : L^\infty(\mathbb{R}^N)))^2$ of the integral system (2.3.23) satisfying

$$0 \leq a(x,t) \leq \overline{a}(x,t) \ \text{ and } \ 0 \leq b(x,t) \leq \overline{b}(x,t) \ \text{ in } \mathbb{R}^N \times (0,T). \tag{2.3.31}$$

It remains to prove the following limits:

$$\lim_{t\to 0} \left\| a(t) - e^{t\Delta} a_0 \right\|_{L^\infty(\mathbb{R}^N)} = 0 \ \text{ and } \ \lim_{t\to 0} \left\| b(t) - e^{t\Delta} b_0 \right\|_{L^\infty(\mathbb{R}^N)} = 0. \tag{2.3.32}$$

Combining (2.3.23), (2.3.25), (2.3.26) and (2.3.31), we obtain

$$\begin{aligned} a(t) - e^{t\Delta} a_0 &\leq \int_0^t e^{(t-s)\Delta} \left(p_1 e^{\overline{a}(s)} + p_2 e^{\overline{b}(s)} \right) ds \\ &= \int_0^t e^{(t-s)\Delta} \left(p_1 U(s)^\eta + p_2 V(s)^\eta \right) ds \\ &\leq \int_0^t e^{(t-s)\Delta} \left(p_1 w(s)^{\frac{1}{r_1}} + p_2 w(s)^{\frac{1}{r_2}} \right) ds. \end{aligned} \tag{2.3.33}$$

This along with Lemma 2.2.1 and (2.2.3) yields

(2.3.34)

$$\begin{aligned} &\|a(t) - e^{t\Delta} a_0\|_{L^\infty(\mathbb{R}^N)} \\ &\leq p_1 \int_0^t \|e^{(t-s)\Delta} w(s)^{\frac{1}{r_1}}\|_{L^\infty(\mathbb{R}^N)} ds + p_2 \int_0^t \|e^{(t-s)\Delta} w(s)^{\frac{1}{r_2}}\|_{L^\infty(\mathbb{R}^N)} ds \\ &\lesssim \int_0^t \left(\rho^{-\frac{N}{2r_1}} + (t-s)^{-\frac{N}{4r_1}} \right) \|w(s)\|^{\frac{1}{r_1}}_{L^2_{ul,\rho}(\mathbb{R}^N)} ds + \int_0^t \left(\rho^{-\frac{N}{2r_2}} + (t-s)^{-\frac{N}{4r_2}} \right) \|w(s)\|^{\frac{1}{r_2}}_{L^2_{ul,\rho}(\mathbb{R}^N)} ds \\ &\lesssim \int_0^t \left(\rho^{-\frac{N}{2r_1}} + (t-s)^{-\frac{N}{4r_1}} \right) s^{-\frac{N}{4r_1}} ds \|w_0\|^{\frac{1}{r_1}}_{L^1_{ul,\rho}(\mathbb{R}^N)} + \int_0^t \left(\rho^{-\frac{N}{2r_2}} + (t-s)^{-\frac{N}{4r_2}} \right) s^{-\frac{N}{4r_2}} ds \|w_0\|^{\frac{1}{r_2}}_{L^1_{ul,\rho}(\mathbb{R}^N)} \end{aligned}$$

for sufficiently small $t > 0$. Since $r_1 > N/2$ and $r_2 > N/2$, we have

$$\int_0^t \left(\rho^{-\frac{N}{2r_1}} + (t-s)^{-\frac{N}{4r_1}} \right) s^{-\frac{N}{4r_1}} ds \to 0 \ \text{ and } \ \int_0^t \left(\rho^{-\frac{N}{2r_2}} + (t-s)^{-\frac{N}{4r_2}} \right) s^{-\frac{N}{4r_2}} ds \to 0 \ \text{ as } t \to 0.$$

Thus we deduce from (2.3.33) and (2.3.34) that $\left\| a(t) - e^{t\Delta} a_0 \right\|_{L^\infty(\mathbb{R}^N)}$ converges to 0 as $t \to 0$.

We can obtain $\lim_{t\to 0}\left\|b(t)-e^{t\Delta}b_0\right\|_{L^\infty(\mathbb{R}^N)}=0$ in the same way. The proof of the assertion (i) is complete.

We prove the assertion (ii). We consider (2.1.8) again. Due to (2.1.5) and Lemma 2.2.5 (i), we get $(U_0,V_0)\in\mathcal{L}^{N\eta/2}_{ul,\rho}(\mathbb{R}^N)\times\mathcal{L}^{N\eta/2}_{ul,\rho}(\mathbb{R}^N)$. By direct calculation we have $A_i=1+2/N$ for $i=1,2,3,4$, and hence (2.3.4) holds. We can apply Proposition 2.3.1 (ii). Then there exist $T>0$ and a local in time classical solution $(U,V)\in(C^{2,1}(\mathbb{R}^N\times(0,T))\cap L^\infty_{loc}((0,T):L^\infty(\mathbb{R}^N)))^2$ of (2.1.8) in $L^{N\eta/2}_{ul,\rho}(\mathbb{R}^N)\times L^{N\eta/2}_{ul,\rho}(\mathbb{R}^N)$ such that

$$0\le U(x,t),V(x,t)\le w(x,t)^{2(1+\varepsilon)/(N\eta)}\quad\text{in }\mathbb{R}^N\times(0,T),\tag{2.3.35}$$

where w is a local in time classical solution of (2.3.17). The constant $\varepsilon>0$ is chosen as in the proof of Proposition 2.3.1 (ii). Then we define $(\overline{a},\overline{b})$ in the same way as (2.3.26) and obtain (2.3.27). By (2.3.29), (2.3.30) and Proposition 2.2.6 (ii) there exists a solution $(a,b)\in(C^{2,1}(\mathbb{R}^N\times(0,T))\cap L^\infty_{loc}((0,T):L^\infty(\mathbb{R}^N)))^2$ of (2.3.23) satisfying (2.3.31). We prove (2.3.32). In the same way as (2.3.33) we obtain from (2.3.35) that

$$a(t)-e^{t\Delta}a_0\le(p_1+p_2)\int_0^t e^{(t-s)\Delta}w(s)^{\frac{2(1+\varepsilon)}{N}}ds.\tag{2.3.36}$$

Put $\alpha_0:=1+2(1+\varepsilon)/N$ and $\alpha:=\alpha_0+\delta$, where $\delta>0$ is sufficiently small. When $p=\alpha_0$ and $q=1+\varepsilon$, we already see in the proof of Proposition 2.3.1 (ii) that $\max\{p,q\}<\alpha<pq$. Thus we can apply Remark 2.2.3. It follows from Lemma 2.2.1 that

$$\begin{aligned}&\int_0^t\left\|e^{(t-s)\Delta}w(s)^{\frac{2(1+\varepsilon)}{N}}\right\|_{L^\infty(\mathbb{R}^N)}ds\\&\lesssim\int_0^t\left(\rho^{-\frac{2(1+\varepsilon)}{\alpha}}+(t-s)^{-\frac{1+\varepsilon}{\alpha}}\right)\|w(s)^{\frac{2(1+\varepsilon)}{N}}\|_{L^{\frac{N\alpha}{2(1+\varepsilon)}}_{ul,\rho}(\mathbb{R}^N)}ds\\&\le\int_0^t\left(\rho^{-\frac{2(1+\varepsilon)}{\alpha}}+(t-s)^{-\frac{1+\varepsilon}{\alpha}}\right)s^{-\frac{2(1+\varepsilon)}{N}\sigma}ds\cdot\sup_{0<s<t}s^{\frac{2(1+\varepsilon)}{N}\sigma}\|w(s)\|^{\frac{2(1+\varepsilon)}{N}}_{L^\alpha_{ul,\rho}(\mathbb{R}^N)},\end{aligned}\tag{2.3.37}$$

where $\sigma:=\frac{N}{2}\left(\frac{1}{1+\varepsilon}-\frac{1}{\alpha}\right)$. Since

$$\frac{1+\varepsilon}{\alpha}<1,\ \ \frac{2(1+\varepsilon)}{N}\sigma<1\ \ \text{and}\ \ \frac{1+\varepsilon}{\alpha}+\frac{2(1+\varepsilon)}{N}\sigma=1,$$

we have

$$\int_0^t\left(\rho^{-\frac{2(1+\varepsilon)}{\alpha}}+(t-s)^{-\frac{1+\varepsilon}{\alpha}}\right)s^{-\frac{2(1+\varepsilon)}{N}\sigma}ds<C$$

uniformly for small $t>0$. Then due to (2.3.36) and (2.3.37), Remark 2.2.3 implies that $\|a(t)-e^{t\Delta}a_0\|_{L^\infty(\mathbb{R}^N)}$ converges to 0 as $t\to 0$. We can obtain $\lim_{t\to 0}\|b(t)-e^{t\Delta}b_0\|_{L^\infty(\mathbb{R}^N)}=0$ in the same way. Therefore, the proof of the assertion (ii) is complete. □

Proof of Theorem 2.1.8. Define

$$\tilde{u}_0(x):=\begin{cases}u_0(x)&\text{if }x\in\Omega,\\0&\text{if }x\in\mathbb{R}^N\setminus\Omega\end{cases}\quad\text{and}\quad\tilde{v}_0(x):=\begin{cases}v_0(x)&\text{if }x\in\Omega,\\0&\text{if }x\in\mathbb{R}^N\setminus\Omega.\end{cases}$$

Let $\rho > 0$. We prove $e^{\frac{N}{2}(p_1\tilde{u}_0+p_2\tilde{v}_0)} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$. Since $e^{\frac{N}{2}(p_1u_0+p_2v_0)} - 1 \in L^1(\Omega)$, there exists a sequence $\{f_n\} \subset C_c^\infty(\Omega)$ such that

$$f_n \to e^{\frac{N}{2}(p_1u_0+p_2v_0)} - 1 \ \text{ in } L^1(\Omega) \text{ as } n \to \infty. \tag{2.3.38}$$

Here we call $C_c^\infty(\Omega)$ the space of C^∞-functions with compact support in Ω. For $n \in \mathbb{N}$, define

$$g_n(x) := \begin{cases} f_n(x)+1 & \text{if } x \in \Omega, \\ 1 & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Since $\{f_n\} \subset C_c^\infty(\Omega)$, we see that $\{g_n\} \subset BUC(\mathbb{R}^N)$. Due to (2.3.38), we have

$$g_n \to e^{\frac{N}{2}(p_1\tilde{u}_0+p_2\tilde{v}_0)} \ \text{ in } L^1_{ul,\rho}(\mathbb{R}^N) \text{ as } n \to \infty,$$

hence $e^{\frac{N}{2}(p_1\tilde{u}_0+p_2\tilde{v}_0)} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$. We obtain $e^{\frac{N}{2}(q_1\tilde{u}_0+q_2\tilde{v}_0)} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)$ in the same way. Theorem 2.1.2 (ii) implies that there exist $T > 0$ and a nonnegative local in time classical solution $(U,V) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ of (2.1.1) in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ satisfying

$$\begin{aligned} U(t) &= e^{t\Delta}\tilde{u_0} + \int_0^t e^{(t-s)\Delta}e^{p_1U(s)+p_2V(s)}ds \ \text{ in } \mathbb{R}^N \times (0,T), \\ V(t) &= e^{t\Delta}\tilde{v_0} + \int_0^t e^{(t-s)\Delta}e^{q_1U(s)+q_2V(s)}ds \ \text{ in } \mathbb{R}^N \times (0,T). \end{aligned} \tag{2.3.39}$$

By the maximum principle for the heat equation we have

$$e^{t\Delta}\tilde{u}_0 \geq e^{t\Delta_\Omega}u_0 \ \text{ in } \Omega \times (0,T) \tag{2.3.40}$$

and

$$\int_0^t e^{(t-s)\Delta}e^{p_1U(s)+p_2V(s)}ds \geq \int_0^t e^{(t-s)\Delta_\Omega}e^{p_1U|_\Omega(s)+p_2V|_\Omega(s)}ds \ \text{ in } \Omega \times (0,T). \tag{2.3.41}$$

Combining (2.3.39), (2.3.40) and (2.3.41), we obtain

$$U|_\Omega(t) \geq e^{t\Delta_\Omega}u_0 + \int_0^t e^{(t-s)\Delta_\Omega}e^{p_1U|_\Omega(s)+p_2V|_\Omega(s)}ds \ \text{ in } \Omega \times (0,T).$$

Since we obtain

$$V|_\Omega(t) \geq e^{t\Delta_\Omega}v_0 + \int_0^t e^{(t-s)\Delta_\Omega}e^{q_1U|_\Omega(s)+q_2V|_\Omega(s)}ds \ \text{ in } \Omega \times (0,T)$$

in the same way, Proposition 2.2.6 (ii) and Remark 2.2.7 imply that (2.2.5) with $f(u,v) := e^{p_1u+p_2v}$ and $g(u,v) := e^{q_1u+q_2v}$ possesses a solution $(u,v) \in C^{2,1}(\Omega \times (0,T)) \times C^{2,1}(\Omega \times (0,T))$ satisfying (2.1.1) in $\Omega \times (0,T)$. It remains to prove the following limits:

$$\lim_{t\to 0}\left\|u(t) - e^{t\Delta_\Omega}u_0\right\|_{L^\infty(\Omega)} = 0 \ \text{ and } \ \lim_{t\to 0}\left\|v(t) - e^{t\Delta_\Omega}v_0\right\|_{L^\infty(\Omega)} = 0.$$

Using $0 \leq u(x,t) \leq U|_\Omega(x,t)$ and $0 \leq v(x,t) \leq V|_\Omega(x,t)$ in $\Omega \times (0,T)$, (2.3.39) and (2.3.41), we deduce that

$$u(t) - e^{t\Delta_\Omega}u_0 \le \int_0^t e^{(t-s)\Delta_\Omega} e^{p_1 U|_\Omega(s) + p_2 V|_\Omega(s)} ds \le \int_0^t e^{(t-s)\Delta} e^{p_1 U(s) + p_2 V(s)} ds$$
$$= U(t) - e^{t\Delta}\tilde{u_0} \to 0 \ \text{ in } L^\infty(\mathbb{R}^N) \text{ as } t \to 0,$$

hence $\|u(t) - e^{t\Delta_\Omega}u_0\|_{L^\infty(\Omega)}$ converges to 0 as $t \to 0$. We have $\lim_{t\to 0} \|v(t) - e^{t\Delta_\Omega}v_0\|_{L^\infty(\Omega)} = 0$ in the same way. $\square$

2.4 Nonexistence result

Proposition 2.4.1. *Let $0 < r < N/2$ and $C > 0$. Then the following* (i) *and* (ii) *hold:*

(i) *There exists a nonnegative function u_0 in $\mathbb{R}^N$ satisfying $e^{ru_0} \in L^1_{ul,\rho}(\mathbb{R}^N)$ such that, for every $T > 0$, there is no nonnegative solution $u \in C^{2,1}(\mathbb{R}^N \times (0,T))$ of* (2.1.9).

(ii) *There exists a nonnegative function u_0 in Ω satisfying $e^{ru_0} \in L^1(\Omega)$ such that, for every $T > 0$, there is no nonnegative solution $u \in C^{2,1}(\Omega \times (0,T))$ of*

$$u(t) = e^{t\Delta_\Omega}u_0 + C\int_0^t e^{(t-s)\Delta_\Omega} e^{u(s)} ds \quad \text{in } \Omega \times (0,T). \tag{2.4.1}$$

Proposition 2.4.1 (i) follows from [11, Theorem 1.2]. In this proposition we establish the nonexistence result in a bounded domain. We prove both of the assertions (i) and (ii) for convenience. Let us introduce the following lemma.

Lemma 2.4.2. *Let $C > 0$. Then the following* (i) *and* (ii) *hold:*

(i) *Let $u_0 \in L^1_{ul,\rho}(\mathbb{R}^N)$ be a nonnegative function. Suppose that there exist $T > 0$ and a local in time nonnegative classical solution $u \in C^{2,1}(\mathbb{R}^N \times (0,T))$ of* (2.1.9). *Then it follows that*

$$\|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} \le \log\frac{1}{Ct} \quad \text{for } t \in (0,T).$$

(ii) *Let $u_0 \in L^1(\Omega)$ be a nonnegative function. Suppose that there exist $T > 0$ and a local in time nonnegative classical solution $u \in C^{2,1}(\Omega \times (0,T))$ of* (2.4.1). *Then for $k \in \{2,3,\dots\}$, there exists a constant $C_k \in \mathbb{R}$ such that*

$$\|e^{t\Delta_\Omega}u_0\|_{L^\infty(\Omega)} \le \frac{k}{k-1}\log\frac{1}{t} + C_k + e^{-C_k}t^{\frac{k}{k-1}} \quad \text{for } t \in (0,T).$$

Proof. Since the assertion (i) follows from [11, Lemma 4.1], we omit the proof. To prove the assertion (i), we use the developed argument of Fujita [12, Theorem 2.2] and Weissler [28, Theorem 1]. The proof of the assertion (ii) is based on [27].

We prove the assertion (ii). Let $k \in \{2,3,\dots\}$ be fixed and let $0 < \varepsilon < T$. We show that for each $l \in \mathbb{N}$,

$$u(t) \ge \frac{\{C(t-\varepsilon)\}^{a_l}\left\{\exp(e^{t\Delta_\Omega}u_0) - 1\right\}^{k^l}}{\prod_{i=1}^{l}(k!a_i)^{k^{l-i}}} \quad \text{in } \Omega \times (\varepsilon, T), \tag{2.4.2}$$

where $a_l := k^l + k^{l-1} + \cdots + k + 1 = (k^{l+1} - 1)/(k-1)$.

We prove the case $l = 1$. Since u satisfies (2.4.1) and $e^s - 1$ is convex in $[0, \infty)$, Lemma 2.2.4 (ii) yields

$$\begin{aligned} u(t) &\geq C \int_\varepsilon^t e^{(t-s)\Delta_\Omega} \left\{ \exp(e^{s\Delta_\Omega} u_0) - 1 \right\} ds \\ &\geq C \int_\varepsilon^t \left\{ \exp(e^{(t-s)\Delta_\Omega} e^{s\Delta_\Omega} u_0) - 1 \right\} ds \geq C(t-\varepsilon) \left\{ \exp(e^{t\Delta_\Omega} u_0) - 1 \right\}. \end{aligned}$$

This together with $e^{u(s)} \geq \frac{u(s)^k}{k!}$ in $\Omega \times (\varepsilon, T)$, and Lemma 2.2.4 (ii) imply that

$$\begin{aligned} u(t) &\geq C \int_\varepsilon^t e^{(t-s)\Delta_\Omega} \left(\frac{u(s)^k}{k!} \right) ds \geq \frac{C^{k+1}}{k!} \int_\varepsilon^t e^{(t-s)\Delta_\Omega} \left\{ (s-\varepsilon)^k \left(\exp(e^{s\Delta_\Omega} u_0) - 1 \right)^k \right\} ds \\ &\geq \frac{C^{k+1}}{k!} \int_\varepsilon^t (s-\varepsilon)^k \left\{ \exp(e^{t\Delta_\Omega} u_0) - 1 \right\}^k ds = \frac{C^{k+1}}{(k+1)!} \cdot (t-\varepsilon)^{k+1} \left\{ \exp(e^{t\Delta_\Omega} u_0) - 1 \right\}^k. \end{aligned}$$

This proves (2.4.2) with $l = 1$.

We suppose that (2.4.2) holds for $l = n$. Applying $e^{u(s)} \geq \frac{u(s)^k}{k!}$, (2.4.2) and Lemma 2.2.4 (ii), we have

$$\begin{aligned} u(t) &\geq C \int_\varepsilon^t e^{(t-s)\Delta_\Omega} \left(\frac{u(s)^k}{k!} \right) ds \geq \frac{C^{ka_n+1}}{k!} \int_\varepsilon^t \frac{(s-\varepsilon)^{ka_n} \left\{ \exp\left(e^{t\Delta_\Omega} u_0\right) - 1 \right\}^{k^{n+1}}}{\prod_{i=1}^n (k! a_i)^{k^{n+1-i}}} ds \\ &= \frac{\{C(t-\varepsilon)\}^{ka_n+1} \left\{ \exp\left(e^{t\Delta_\Omega} u_0\right) - 1 \right\}^{k^{n+1}}}{k!(ka_n+1) \prod_{i=1}^n (k! a_i)^{k^{n+1-i}}} = \frac{\{C(t-\varepsilon)\}^{a_{n+1}} \left\{ \exp\left(e^{t\Delta_\Omega} u_0\right) - 1 \right\}^{k^{n+1}}}{\prod_{i=1}^{n+1} (k! a_i)^{k^{n+1-i}}}. \end{aligned}$$

This implies that (2.4.2) holds for $l = n+1$. By induction we see that (2.4.2) holds for each $l \in \mathbb{N}$.

We are ready to prove Lemma 2.4.2 (ii). It follows from (2.4.2) that for $l \in \mathbb{N}$,

$$u(t)^{\frac{1}{k^l}} \prod_{i=1}^l (k! a_i)^{k^{-i}} \geq (Ct)^{\frac{k}{k-1} - \frac{1}{k^l(k-1)}} \left\{ \exp\left(e^{t\Delta_\Omega} u_0\right) - 1 \right\} \quad \text{in } \Omega \times (0, T).$$

Here we take the limit as $\varepsilon \to +0$. If $u(x,t) \neq 0$ at $(x,t) \in \Omega \times (0,T)$, then by taking $l \to \infty$ we obtain

$$\prod_{i=1}^\infty (k! a_i)^{k^{-i}} \geq (Ct)^{\frac{k}{k-1}} \left\{ \exp\left(e^{t\Delta_\Omega} u_0\right) - 1 \right\}. \tag{2.4.3}$$

If $u(x,t) = 0$ at some point $(x,t) \in \Omega \times (0,T)$, then (2.4.2) implies that the right hand side of (2.4.3) is 0, and hence (2.4.3) also holds. Remark that the left hand side of (2.4.3) converges. Indeed, we have

$$\log \left(\prod_{i=1}^\infty (k! a_i)^{k^{-i}} \right) \leq \sum_{i=1}^\infty \frac{(i+1)\log k + \log(k!)}{k^i} < \infty.$$

Then we have

$$\exp\left(e^{t\Delta_\Omega}u_0\right) - 1 \le \left(C^{-\frac{k}{k-1}} \prod_{i=1}^{\infty} (k!a_i)^{k^{-i}}\right) \cdot t^{-\frac{k}{k-1}} =: e^{C_k} t^{-\frac{k}{k-1}},$$

which implies that

$$e^{t\Delta_\Omega}u_0 \le \log\left(e^{C_k} t^{-\frac{k}{k-1}} + 1\right).$$

We observe from the mean value theorem that $\log(a+1) - \log a \le \frac{1}{a}$ for $a > 0$. Thus we obtain

$$\|e^{t\Delta_\Omega}u_0\|_{L^\infty(\Omega)} \le \frac{k}{k-1}\log\frac{1}{t} + C_k + e^{-C_k} t^{\frac{k}{k-1}} \quad \text{for } t \in (0,T).$$

This proves Lemma 2.4.2 (ii). $\square$

Proof of Proposition 2.4.1. The proof is by contradiction. We construct an initial function u_0 and suppose that there exist $T > 0$ and a local in time nonnegative classical solution u satisfying (2.1.9) for the assertion (i), or (2.4.1) for the assertion (ii).

We prove the assertion (i). Choose α such that $2 < \alpha < N/r$ and define

$$u_0(x) := \max\left\{\alpha \log\frac{1}{|x|}, 0\right\}. \tag{2.4.4}$$

Then we have $e^{ru_0} \in L^1_{ul,\rho}(\mathbb{R}^N)$. We choose $\varepsilon > 0$ such that $\alpha/2 - \varepsilon\alpha > 1$. Putting $y = \sqrt{t}z$, we obtain

$$\begin{aligned} \left\|e^{t\Delta}u_0\right\|_{L^\infty(\mathbb{R}^N)} &\ge (4\pi t)^{-\frac{N}{2}} \int_{B(0,1)} e^{-\frac{|y|^2}{4t}} \log\left(|y|^{-\alpha}\right) dy \\ &= (4\pi)^{-\frac{N}{2}} \int_{B(0,t^{-1/2})} e^{-\frac{|z|^2}{4}} \log\left(t^{-\frac{\alpha}{2}}|z|^{-\alpha}\right) dz. \end{aligned} \tag{2.4.5}$$

We observe from the choice of ε that $t^{-\varepsilon} < t^{-1/2}$ for $0 < t < 1$. Then we deduce that

$$\begin{aligned} \left\|e^{t\Delta}u_0\right\|_{L^\infty(\mathbb{R}^N)} &\ge (4\pi)^{-\frac{N}{2}} \int_{B(0,t^{-\varepsilon})} e^{-\frac{|z|^2}{4}} \log\left(t^{-\frac{\alpha}{2}}|z|^{-\alpha}\right) dz \\ &\ge (4\pi)^{-\frac{N}{2}} \log\left(t^{-\frac{\alpha}{2}+\varepsilon\alpha}\right) \int_{B(0,t^{-\varepsilon})} e^{-\frac{|z|^2}{4}} dz \\ &= \left(\frac{\alpha}{2} - \varepsilon\alpha\right) \log\frac{1}{t} \cdot \left(1 + O\left(e^{-t^{-2\varepsilon}/8}\right)\right) \end{aligned} \tag{2.4.6}$$

for sufficiently small $t > 0$. This along with Lemma 2.4.2 (i) yields

$$\log\frac{1}{Ct} \ge \left(\frac{\alpha}{2} - \varepsilon\alpha\right) \log\frac{1}{t} \cdot \left(1 + O\left(e^{-t^{-2\varepsilon}/8}\right)\right). \tag{2.4.7}$$

Let $\tau := \log(1/t)$. Due to (2.4.7), we have

$$1 + \frac{1}{\tau}\log\frac{1}{C} \ge \left(\frac{\alpha}{2} - \varepsilon\alpha\right) \cdot \left(1 + O\left(e^{-e^{2\varepsilon\tau}/8}\right)\right)$$

for sufficiently large $\tau > 0$. Taking the limit as $\tau \to \infty$, we obtain $1 \ge \alpha/2 - \varepsilon\alpha$. This contradicts the choice of ε.

We prove the assertion (ii). Without loss of generality, we suppose that $0 \in \Omega$. Choose ρ and α such that

$$0 < \rho < \delta(0)/3, \ 2 < \alpha < N/r,$$

where $\delta(0) = \mathrm{dist}(0, \partial\Omega)$. Define u_0 by (2.4.4) and we have $e^{ru_0} \in L^1(\Omega)$. Let $y \in B(0,\rho)$. We show that $B(0,\rho) \subset B(y, \delta(y))$. If $z \in B(0,\rho)$, then we have

$$|z-y| \le |z| + |y| < 2\rho < \frac{2}{3}\delta(0).$$

It follows from the triangular inequality that

$$\mathrm{dist}(y, \partial\Omega) \ge \mathrm{dist}(0, \partial\Omega) - |y|,$$

which implies that $\delta(y) > 2\delta(0)/3$. Thus we obtain $|z-y| < \delta(y)$ for $z \in B(0,\rho)$. Hence, we deduce that $B(0,\rho) \subset B(y,\delta(y))$. This together with Proposition 2.2.10 implies that there exists $m > 0$ depending only on N such that for every $y \in B(0,\rho)$,

$$G(x,y,t) \ge K_y(x,t) - m\delta(y)^{-N} \ \text{ in } B(0,\rho) \times (0,\infty).$$

Since $\delta(y) > 2\delta(0)/3$, it follows that for every $y \in B(0,\rho)$,

$$\begin{aligned} G(x,y,t) &> K_y(x,t) - m(2\delta(0)/3)^{-N} \\ &= (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}} - M_1 \quad \text{in } B(0,\rho) \times (0,\infty), \end{aligned} \tag{2.4.8}$$

where $M_1 := m(2\delta(0)/3)^{-N}$. We choose $\varepsilon > 0$ such that $\alpha/2 - \varepsilon\alpha > 1$ again. Let $c := \min\{\rho, 1\}$. In a similar calculation of (2.4.5) and (2.4.6) we obtain from (2.4.8) that for sufficiently small $t > 0$,

$$\begin{aligned} \left\| e^{t\Delta_\Omega} u_0 \right\|_{L^\infty(\Omega)} &\ge \int_{B(0,c)} G(0,y,t) u_0(y) dy \\ &\ge \int_{B(0,c)} \left\{ (4\pi t)^{-\frac{N}{2}} e^{-\frac{|y|^2}{4t}} - M_1 \right\} \log\left(|y|^{-\alpha}\right) dy \\ &\ge \left(\frac{\alpha}{2} - \varepsilon\alpha\right) \log\frac{1}{t} \cdot \left(1 + O\left(e^{-t^{-2\varepsilon}/8}\right)\right) - M_2, \end{aligned} \tag{2.4.9}$$

where $M_2 := M_1 \int_{B(0,c)} \log\left(|y|^{-\alpha}\right) dy < \infty$. Then (2.4.9) together with Lemma 2.4.2 (ii) implies that

$$\frac{k}{k-1}\log\frac{1}{t} + C_k + e^{-C_k} t^{\frac{k}{k-1}} \ge \left(\frac{\alpha}{2} - \varepsilon\alpha\right) \log\frac{1}{t} \cdot \left(1 + O\left(e^{-t^{-2\varepsilon}/8}\right)\right) - M_2 \tag{2.4.10}$$

for sufficiently small $t > 0$. Here we take a large $k \in \{2, 3, \ldots\}$ such that $k/(k-1) < \alpha/2 - \varepsilon\alpha$. Setting $\tau := \log(1/t)$, we observe from (2.4.10) that

$$\frac{k}{k-1} + \left(C_k + e^{-C_k - \frac{k}{k-1}\tau}\right)/\tau \ge \left(\frac{\alpha}{2} - \varepsilon\alpha\right) \cdot \left(1 + O\left(e^{-e^{2\varepsilon\tau}/8}\right)\right) - M_2/\tau$$

for sufficiently large $\tau > 0$. Taking the limit as $\tau \to \infty$, we obtain $k/(k-1) \ge \alpha/2 - \varepsilon\alpha$. This contradicts the choice of k. $\square$

Proofs of Theorems 2.1.5 and 2.1.9. We prove Theorem 2.1.5 (i). We consider the case where $p_1 > 0$. Let $0 < r_1 < N/2$. By Proposition 2.4.1 (i) there exists a nonnegative function a_0 in $\mathbb{R}^N$ satisfying $e^{r_1 a_0} \in L^1_{ul,\rho}(\mathbb{R}^N)$ such that, for every $T > 0$, there is no nonnegative solution $a \in C^{2,1}(\mathbb{R}^N \times (0,T))$ of the scalar integral problem

$$a(t) = e^{t\Delta}a_0 + p_1 \int_0^t e^{(t-s)\Delta}e^{a(s)}ds \quad \text{in } \mathbb{R}^N \times (0,T). \tag{2.4.11}$$

Setting $u_0(x) := a_0(x)/p_1$ and $v_0(x) := 0$, we obtain $e^{r_1(p_1u_0+p_2v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N)$.

The proof is by contradiction. Suppose that problem (2.1.1) possesses a local in time nonnegative classical solution $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$. Putting $\overline{a} := p_1u + p_2v$, we have

$$\partial_t \overline{a} \geq \Delta\overline{a} + p_1 e^{\overline{a}} \quad \text{in } \mathbb{R}^N \times (0,T).$$

Moreover, since (u,v) satisfies $\|u(t) - e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} \to 0$ and $\|v(t) - e^{t\Delta}v_0\|_{L^\infty(\mathbb{R}^N)} \to 0$ $(t \to 0)$, we obtain $\|\overline{a}(t) - e^{t\Delta}a_0\|_{L^\infty(\mathbb{R}^N)} \to 0$ $(t \to 0)$. Thus Lemma 2.2.8 (ii) implies that

$$\overline{a}(t) \geq e^{t\Delta}a_0 + p_1 \int_0^t e^{(t-s)\Delta}e^{\overline{a}(s)}ds \quad \text{in } \mathbb{R}^N \times (0,T'),$$

where $T' > 0$ is sufficiently small. By Proposition 2.2.6 (i) there exists a nonnegative solution $a \in C^{2,1}(\mathbb{R}^N \times (0,T'))$ of (2.4.11) with T replaced by T'. This contradicts the nonexistence result of (2.4.11).

Next we consider the case where $p_1 = 0$. Since $(p_1,p_2) \neq (0,0)$, $p_2 > 0$. Choose α such that $2 < \alpha < N/r_1$ and put $u_0(x) := 0$ and $v_0(x) := \max\left\{\frac{\alpha}{p_2}\log\frac{1}{|x|}, 0\right\}$. Then we obtain $e^{r_1(p_1u_0+p_2v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N)$.

The proof is also by contradiction. Suppose that problem (2.1.1) possesses a local in time nonnegative classical solution $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$. Since (u,v) satisfies $\|u(t) - e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} \to 0$ $(t \to 0)$, Lemma 2.2.8 (i) implies that

$$\left\|\int_0^t e^{(t-s)\Delta}e^{p_2v(s)}ds\right\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad (t \to 0).$$

On the other hand, it follows from Lemma 2.2.4 (i) that

$$\int_0^t e^{(t-s)\Delta}e^{p_2v(s)}ds \geq \int_0^t e^{(t-s)\Delta}e^{p_2e^{s\Delta}v_0}ds \geq \int_0^t e^{p_2\left(e^{(t-s)\Delta}e^{s\Delta}v_0\right)}ds \geq te^{p_2e^{t\Delta}v_0}.$$

Then we have

$$\left\|\int_0^t e^{(t-s)\Delta}e^{p_2v(s)}ds\right\|_{L^\infty(\mathbb{R}^N)} \geq te^{p_2\|e^{t\Delta}v_0\|_{L^\infty(\mathbb{R}^N)}}. \tag{2.4.12}$$

We obtain in a similar manner to (2.4.6) that for sufficiently small $t > 0$,

$$\begin{aligned}
\log\left(te^{p_2\|e^{t\Delta}v_0\|_{L^\infty(\mathbb{R}^N)}}\right) &= p_2\left\|e^{t\Delta}v_0\right\|_{L^\infty(\mathbb{R}^N)} - \log\frac{1}{t} \\
&\geq \left\{\left(\frac{\alpha}{2} - \varepsilon\alpha\right)\cdot(4\pi)^{-\frac{N}{2}}\int_{B(0,t^{-\varepsilon})} e^{-\frac{|z|^2}{4}}dz - 1\right\}\cdot\log\frac{1}{t},
\end{aligned} \tag{2.4.13}$$

where $\varepsilon > 0$ is sufficiently small such that $\alpha/2 - \varepsilon\alpha > 1$. Thus it follows from (2.4.12) and (2.4.13) that

$$\left\|\int_0^t e^{(t-s)\Delta}e^{p_2v(s)}ds\right\|_{L^\infty(\mathbb{R}^N)} \to \infty \quad (t \to 0).$$

This is a contradiction. Therefore, the proof of Theorem 2.1.5 (i) is complete.

We prove Theorem 2.1.5 (ii). It suffices to prove only the case where $(r_1, r_2) \in I_1 \times (0, N/2)$, since one can prove the other case in the same way. We can choose α such that

$$2 < \alpha < N/r_2 \ \ (\text{if } p_1 = 0), \ \ 2 < \alpha < N/r_2 \text{ and } r_1\alpha p_1/q_1 < N \ \ (\text{if } p_1 > 0).$$

Define (u_0, v_0) by $u_0(x) := \max\left\{\frac{\alpha}{q_1}\log\frac{1}{|x|}, 0\right\}$ and $v_0(x) := 0$. Then (2.1.4) holds. When $q_2 > 0$, we can prove the assertion (ii) in a similar argument in the proof of the assertion (i), the case where $p_1 > 0$. When $q_2 = 0$, we can also prove the assertion (ii) by considering [20, Proposition 4.1] with $f(s) = q_1 s$ and $g(s) = p_2 s$. We omit the details. Finally, both of Theorem 2.1.9 (i) and (ii) can be obtained in a similar way using Remark 2.2.9. □

2.5 Applications

In this section we apply our study to the existence and nonexistence results of the reaction-diffusion systems below. We leave the proofs to readers.

We consider the n-component system

$$\begin{cases}
\partial_t u_1 = \Delta u_1 + e^{p_{11}u_1 + p_{12}u_2 + \cdots + p_{1n}u_n} & \text{in } \mathbb{R}^N \times (0, T), \\
\partial_t u_2 = \Delta u_2 + e^{p_{21}u_1 + p_{22}u_2 + \cdots + p_{2n}u_n} & \text{in } \mathbb{R}^N \times (0, T), \\
\qquad \vdots & \\
\partial_t u_n = \Delta u_n + e^{p_{n1}u_1 + p_{n2}u_2 + \cdots + p_{nn}u_n} & \text{in } \mathbb{R}^N \times (0, T), \\
u_1(x, 0) = u_{1,0}(x), \ldots, u_n(x, 0) = u_{n,0}(x) & \text{in } \mathbb{R}^N,
\end{cases} \tag{2.5.1}$$

where $n \geq 3$, $T > 0$ and $p_{ij} \geq 0$ $(1 \leq i, j \leq n)$. We assume that for each $i = 1, 2, \ldots, n$, $(p_{i1}, p_{i2}, \ldots, p_{in}) \neq (0, 0, \ldots, 0)$. The initial functions $u_{i,0}$ $(i = 1, 2, \ldots, n)$ are nonnegative and measurable.

In this case the following existence and nonexistence results can be established:

(i) (Existence) Suppose that one of the following holds:

  (a) there exist $r_1, r_2, \ldots, r_n > N/2$ such that the nonnegative initial functions belonging to $L^1_{ul,\rho}(\mathbb{R}^N)$ satisfy

  (2.5.2) $\exp\{r_i(p_{i1}u_{1,0} + p_{i2}u_{2,0} + \cdots + p_{in}u_{n,0})\} \in L^1_{ul,\rho}(\mathbb{R}^N)$ for $i = 1, 2, \ldots, n$,

(b) the nonnegative initial functions belonging to $L^1_{ul,\rho}(\mathbb{R}^N)$ satisfy

$$\exp\left\{\frac{N}{2}(p_{i1}u_{1,0}+p_{i2}u_{2,0}+\cdots+p_{in}u_{n,0})\right\}\in\mathcal{L}^1_{ul,\rho}(\mathbb{R}^N)\quad\text{for } i=1,2,\dots,n.$$

Then there exist $T>0$ and a local in time nonnegative classical solution $(u_1,u_2,\dots,u_n)\in(C^{2,1}(\mathbb{R}^N\times(0,T)))^n$ of (2.5.1) which satisfies

$$\lim_{t\to 0}\|u_i(t)-e^{t\Delta}u_{i,0}\|_{L^\infty(\mathbb{R}^N)}=0\quad\text{for } i=1,2,\dots,n. \tag{2.5.3}$$

(ii) (Nonexistence) The following (a) and (b) hold:

(a) Suppose that there exists $i\in\{1,2,\dots,n\}$ such that $p_{ij}\ge p_{i'j}$ for each $i'\neq i$ and $j=1,2,\dots,n$. Then for any $0<r_i<N/2$, there exist nonnegative functions $(u_{1,0},u_{2,0},\dots,u_{n,0})$ satisfying $\exp\{r_i(p_{i1}u_{1,0}+p_{i2}u_{2,0}+\cdots+p_{in}u_{n,0})\}\in L^1_{ul,\rho}(\mathbb{R}^N)$ such that, for every $T>0$, problem (2.5.1) admits no local in time nonnegative classical solution $(u_1,u_2,\dots,u_n)\in(C^{2,1}(\mathbb{R}^N\times(0,T)))^n$ of (2.5.1) satisfying (2.5.3).

(b) Suppose that there exists $i\in\{1,2,\dots,n\}$ such that $p_{ii}>p_{i'i}$ for each $i'\neq i$. Then for any $0<r_i<N/2$, there exist nonnegative functions $(u_{1,0},u_{2,0},\dots,u_{n,0})$ satisfying (2.5.2) for some $r_1,\dots,r_{i-1},r_{i+1},\dots,r_n>N/2$ such that, for every $T>0$, problem (2.5.1) admits no local in time nonnegative classical solution $(u_1,u_2,\dots,u_n)\in(C^{2,1}(\mathbb{R}^N\times(0,T)))^n$ of (2.5.1) satisfying (2.5.3).

The assertions (i) and (ii) (a) correspond to Theorem 2.1.2 and Theorem 2.1.5 (i), respectively. On the other hand, the assertion (ii) (b) does not correspond to Theorem 2.1.5 (ii). When $n\ge 3$, it is not easy to give an explicit condition for the existence and nonexistence results.

We consider the system with superexponential nonlinearities

$$\begin{cases}\partial_t u=\Delta u+e^{u^2+v^2} & \text{in } \mathbb{R}^N\times(0,T),\\ \partial_t v=\Delta v+e^{u^2+v^2} & \text{in } \mathbb{R}^N\times(0,T),\\ u(x,0)=u_0(x),\, v(x,0)=v_0(x) & \text{in } \mathbb{R}^N,\end{cases} \tag{2.5.4}$$

where $T>0$ and the initial functions u_0 and v_0 are nonnegative and measurable. In this case also the existence and nonexistence results can be established. Combining our results and the transformation approach in [11], we obtain the existence of a local in time nonnegative classical solution of (2.5.4). Let $F(s):=\int_s^\infty e^{-\sigma^2}d\sigma$ and let (U,V) be a nonnegative classical solution of

$$\begin{cases}\partial_t U=\Delta U+e^{U+V} & \text{in } \mathbb{R}^N\times(0,T),\\ \partial_t V=\Delta V+e^{U+V} & \text{in } \mathbb{R}^N\times(0,T),\\ U(x,0)=U_0(x),\, V(x,0)=V_0(x) & \text{in } \mathbb{R}^N,\end{cases}$$

where $(U_0,V_0):=(-\log F(u_0),-\log F(v_0))$. Then $(\overline{u},\overline{v}):=\left(F^{-1}(e^{-U}),F^{-1}(e^{-V})\right)$ becomes a supersolution for (2.5.4), and hence we can construct a solution. The nonexistence result is obtained from the nonexistence result of the scalar problem $\partial_t u=\Delta u+e^{u^2}$. See [11, Theorems 1.2 and 5.2] for details.

(i) (Existence) Suppose that one of the following holds:

(a) there exists $r > N/2$ such that the nonnegative initial functions satisfy

$$F(u_0)^{-r}F(v_0)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N), \tag{2.5.5}$$

(b) the nonnegative initial functions satisfy

$$F(u_0)^{-\frac{N}{2}}F(v_0)^{-\frac{N}{2}} \in \mathcal{L}^1_{ul,\rho}(\mathbb{R}^N).$$

Then there exist $T > 0$ and a local in time nonnegative classical solution $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ of (2.5.4) in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 2.1.1.

(ii) (Nonexistence) For any $0 < r < N/2$, there exist nonnegative functions (u_0, v_0) satisfying (2.5.5) such that, for every $T > 0$, problem (2.5.4) admits no local in time nonnegative classical solution $(u,v) \in C^{2,1}(\mathbb{R}^N \times (0,T)) \times C^{2,1}(\mathbb{R}^N \times (0,T))$ in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ in the sense of Definition 2.1.1.

Appendix

2.A Initial data and their range for the existence result

In [17] the following idea is suggested. They construct a local in time nonnegative classical solution (U, V) of the system

$$
\begin{cases}
\partial_t U = \Delta U + U^{p_1+1} V^{p_2} & \text{in } \mathbb{R}^N \times (0, T), \\
\partial_t V = \Delta V + U^{q_1} V^{q_2+1} & \text{in } \mathbb{R}^N \times (0, T), \\
U(x, 0) = e^{u_0(x)}, V(x, 0) = e^{v_0(x)} & \text{in } \mathbb{R}^N.
\end{cases} \tag{2.A.1}
$$

Put $\overline{u} := \log U$, $\overline{v} := \log V$. Since $(\overline{u}, \overline{v})$ satisfies

$$
\begin{cases}
\partial_t \overline{u} \geq \Delta \overline{u} + e^{p_1 \overline{u} + p_2 \overline{v}} & \text{in } \mathbb{R}^N \times (0, T), \\
\partial_t \overline{v} \geq \Delta \overline{v} + e^{q_1 \overline{u} + q_2 \overline{v}} & \text{in } \mathbb{R}^N \times (0, T),
\end{cases}
$$

(2.1.1) has a local in time nonnegative classical solution.

Since $(p_1, p_2) \neq (0, 0)$, $(q_1, q_2) \neq (0, 0)$, $U(x, 0) \geq 1$ and $V(x, 0) \geq 1$, we obtain in a similar manner to the proof of Proposition 2.3.1 that if

(2.A.2)

$$(e^{u_0}, e^{v_0}) \in L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N) \ \text{ for some } 1 \leq r_1, r_2 < \infty \text{ satisfying } \frac{N}{2} \max\left\{\frac{p_1}{r_1} + \frac{p_2}{r_2}, \frac{q_1}{r_1} + \frac{q_2}{r_2}\right\} < 1$$

or

(2.A.3)

$$(e^{u_0}, e^{v_0}) \in \mathcal{L}^{r_1}_{ul,\rho}(\mathbb{R}^N) \times \mathcal{L}^{r_2}_{ul,\rho}(\mathbb{R}^N) \ \text{ for some } 1 < r_1, r_2 < \infty \text{ satisfying } \frac{N}{2} \max\left\{\frac{p_1}{r_1} + \frac{p_2}{r_2}, \frac{q_1}{r_1} + \frac{q_2}{r_2}\right\} = 1,$$

then (2.A.1) possesses a local in time nonnegative classical solution. The following proposition refers to the range of initial data for the existence result.

Proposition 2.A.1. *Then the following* (i) *and* (ii) *hold:*

(i) *If $u_0 \geq 0$ and $v_0 \geq 0$ satisfy* (2.A.2) *(resp.* (2.A.3)*), then* (2.1.4) *holds for some $r_1 > N/2$ and $r_2 > N/2$ (resp.* (2.1.5) *holds with $r_1 = r_2 = N/2$).*

(ii) *Let $p_i > 0$ and $q_i > 0$ for $i = 1, 2$. Then there exist nonnegative functions (u_0, v_0) such that* (2.1.4) *holds for some $r_1 > N/2$ and $r_2 > N/2$ but* (2.A.2) *does not hold.*

By Theorem 2.1.2 and the assertion (i) we can take a wider class of initial data where (2.1.1) has a solution in the sense of Definition 2.1.1 comparing to the idea in [17]. On the other hand, by the assertion (ii) we can give an example of (u_0, v_0) such that we can prove that (2.1.1) has a solution in the sense of Definition 2.1.1 due to Theorem 2.1.2 but cannot prove the same result using the idea in [17].

Proof. We prove the assertion (i). Suppose that (u_0, v_0) satisfies (2.A.2). Put $r' := 1/\left(\frac{p_1}{r_1} + \frac{p_2}{r_2}\right)$. By (2.A.2) we have $r' > N/2$. Since we see that

$$1/\left(\frac{r_1}{r'p_1}\right) + 1/\left(\frac{r_2}{r'p_2}\right) = 1, \quad e^{r'p_1u_0} \in L^{r_1/(r'p_1)}_{ul,\rho}(\mathbb{R}^N) \ \text{ and } \ e^{r'p_2v_0} \in L^{r_2/(r'p_2)}_{ul,\rho}(\mathbb{R}^N),$$

it follows from Hölder's inequality that $e^{r'(p_1u_0+p_2v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N)$. We can obtain $e^{r''(q_1u_0+q_2v_0)} \in L^1_{ul,\rho}(\mathbb{R}^N)$, where $r'' := 1/\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right) > N/2$ in the same way. Since we can obtain (2.1.5) from (2.A.3) in a similar way, details are omitted.

We prove the assertion (ii). Let $1 < \alpha < 2$ be chosen later. Put

(2.A.4)
$$u_0(x) := \max\left\{\frac{\alpha}{\max\{p_1, q_1\}} \log \frac{1}{|x|}, 0\right\} \ \text{ and } \ v_0(x) := \max\left\{\frac{\alpha}{\max\{p_2, q_2\}} \log \frac{1}{|x-\xi|}, 0\right\},$$

where $\xi := 1$ (if $N = 1$) and $\xi := (0, 0, \ldots, 0, 1) \in \mathbb{R}^N$ (if $N \geq 2$). By direct calculation (2.1.4) holds for sufficiently small $r_1 > N/2$ and $r_2 > N/2$. On the other hand, we prove that (2.A.2) does not hold when $1 < \alpha < 2$ is sufficiently large. If $(e^{u_0}, e^{v_0}) \in L^{r_1}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2}_{ul,\rho}(\mathbb{R}^N)$, then we have

$$\frac{r_1\alpha}{\max\{p_1, q_1\}} < N \ \text{ and } \ \frac{r_2\alpha}{\max\{p_2, q_2\}} < N, \tag{2.A.5}$$

which imply that

$$\begin{aligned}
&\frac{N}{2}\left(\frac{p_1}{r_1} + \frac{p_2}{r_2}\right) > \frac{\alpha}{2}\left(\frac{p_1}{\max\{p_1, q_1\}} + \frac{p_2}{\max\{p_2, q_2\}}\right) \ \text{ and} \\
&\frac{N}{2}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right) > \frac{\alpha}{2}\left(\frac{q_1}{\max\{p_1, q_1\}} + \frac{q_2}{\max\{p_2, q_2\}}\right).
\end{aligned}$$

If $p_1 \geq q_1$ and $p_2 \geq q_2$, then we see that $\frac{N}{2}\left(\frac{p_1}{r_1} + \frac{p_2}{r_2}\right) > \alpha > 1$, and hence (2.A.2) does not hold. If $p_1 < q_1$ and $p_2 > q_2$, then $\frac{p_1}{\max\{p_1,q_1\}} + \frac{p_2}{\max\{p_2,q_2\}} > 1$ holds. Thus we obtain $\frac{\alpha}{2}\left(\frac{p_1}{\max\{p_1,q_1\}} + \frac{p_2}{\max\{p_2,q_2\}}\right) > 1$ by choosing $1 < \alpha < 2$ sufficiently large. Then (2.A.2) does not hold. The same conclusion can be said for the other cases. □

In [17, Theorem 4.1] they consider strongly couple systems when the initial functions belong to $L^{r_1,\infty}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2,\infty}_{ul,\rho}(\mathbb{R}^N)$ $(1 \leq r_1, r_2 < \infty)$. This theorem corresponds to [17, Theorem 3.1]. Set (u_0, v_0) as (2.A.4). Then [17, Theorem 4.1] cannot be applied. Indeed, if $(e^{u_0}, e^{v_0}) \in L^{r_1,\infty}_{ul,\rho}(\mathbb{R}^N) \times L^{r_2,\infty}_{ul,\rho}(\mathbb{R}^N)$, then (2.A.5) also holds. Since we deduce $\frac{N}{2}\max\left\{\frac{p_1}{r_1} + \frac{p_2}{r_2}, \frac{q_1}{r_1} + \frac{q_2}{r_2}\right\} > 1$ in the same way, the assumptions corresponding to [17, Theorem 3.1 (i) and (ii)] do not hold. Moreover, choosing $1 < \alpha < 2$ so large and using a similar evaluation in [23, p.274], the assumption corresponding to [17, Theorem 3.1 (iii)] does not hold. In conclusion, (2.A.4) is an example of initial data where (2.1.1) has a solution in the sense of Definition 2.1.1 but the existence result in [17] cannot be obtained.

References

[1] D. Andreucci, M. A. Herrero and J. J. L. Velázquez, *Liouville theorems and blow up behaviour in semilinear reaction diffusion systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 1–53.

[2] H. Brezis and T. Cazenave, *A nonlinear heat equation with singular initial data*, J. Anal. Math. **68** (1996), 277–304.

[3] H. Chen, *Global existence and blow-up for a nonlinear reaction diffusion system*, J. Math. Anal. Appl. **212** (1997), 481–492.

[4] F. Dickstein and M. Loayza, *Life span of solutions of a strongly coupled parabolic system*, Mat. Contemp. **32** (2007), 85–106.

[5] F. Dickstein and M. Loayza, *Life span of solutions of a weakly coupled parabolic system*, Z. Angew. Math. Phys. **59** (2008), 1–23.

[6] M. Escobedo and M. A. Herrero, *Boundness and blow up for a semilinear reaction-diffusion system*, J. Differential Equations **89** (1991), 176–202.

[7] M. Escobedo and M. A. Herrero, *A uniqueness result for a semilinear reaction diffusion system*, Proc. Amer. Math. Soc. **112** (1991), 175–185.

[8] M. Escobedo and M. A. Herrero, *A semilinear parabolic system in a bounded domain*, Ann. Mat. Pura Appl. **165** (1993), 315–336.

[9] M. Escobedo and H. A. Levine, *Critical blowup and global existence numbers for a weakly coupled system of reaction-diffusion equations*, Arch. Ration. Mech. Anal. **129** (1995), 47–100.

[10] L. C. F. Ferreira and E. Mateus, *Self-similarity and uniqueness of solutions for semilinear reaction-diffusion systems*, Adv. Differential Equations **15** (2010), 73–98.

[11] Y. Fujishima and N. Ioku, *Existence and nonexistence of solutions for the heat equation with a superlinear source term*, J. Math. Pures Appl. **118** (2018), 128–158.

[12] H. Fujita, *On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations*, Nonlinear Functional Analysis (1970), 105–113.

[13] T-E. Ghoul, V. T. Nguyen and H. Zaag, *Blowup solutions for a reaction-diffusion system with exponential nonlinearities*, J. Differential Equations **264** (2018), 7523–7579.

[14] S. Ibrahim, R. Jrad, M. Majdoub and T. Saanouni, *Local well posedness of a 2D semilinear heat equation*, Bull. Belg. Math. Soc. Simon Stevin **21** (2014), 535–551.

[15] N. Ioku, *The Cauchy problem for heat equations with exponential nonlinearity*, J. Differential Equations **251** (2011), 1172–1194.

[16] N. Ioku, B. Ruf and E. Terraneo, *Existence, non-existence, and uniqueness for a heat equation with exponential nonlinearity in* $\mathbb{R}^2$, Math. Phys. Anal. Geom. **18** (2015), 18:29.

[17] K. Ishige, T. Kawakami and M. Sierżęga, *Supersolutions for a class of nonlinear parabolic systems*, J. Differential Equations **260** (2016), 6084–6107.

[18] G. Lu and B. D. Sleeman, *Subsolutions and supersolutions to systems of parabolic equations with applications to generalized Fujita-type systems*, Math. Methods Appl. Sci. **17** (1994), 1005–1016.

[19] Y. Maekawa and Y. Terasawa, *The Navier-Stokes equations with initial data in uniformly local* L^p *spaces*, Differential Integral Equations **19** (2006), 369–400.

[20] Y. Miyamoto and M. Suzuki, *Weakly coupled reaction-diffusion systems with rapidly growing nonlinearities and singular initial data*, Nonlinear Anal. **189** (2019), 111576.

[21] P. Quittner and P. Souplet, *Admissible* L_p *norms for local existence and for continuation in semilinear parabolic systems are not the same*, Proc. Roy. Soc. Edinburgh **131** (2001), 1435–1456.

[22] P. Quittner and P. Souplet, *Global existence from single-component* L^p *estimates in a semilinear reaction-diffusion system*, Proc. Amer. Math. Soc. **130** (2002), 2719–2724.

[23] P. Quittner and P. Souplet, "Superlinear Parabolic Problems, Blow–up, Global Existence and Steady States", Birkhäuser Advanced Texts: Basler Lehrbücher Birkhäuser Verlag, Basel, 2007.

[24] J. C. Robinson and M. Sierżęga, *Supersolutions for a class of semilinear heat equations*, Rev. Mat. Complut. **26** (2013), 341–360.

[25] J. C. Robinson and M. Sierżęga, *A note on well-posedness of semilinear reaction-diffusion problem with singular initial data*, J. Math. Anal. Appl. **385** (2012), 105–110.

[26] B. Ruf and E. Terraneo, *The Cauchy problem for a semilinear heat equation with singular initial data*, Progr. Nonlinear Differential Equations Appl. **50** (2002), 295–309.

[27] F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in* L^p, Indiana Univ. Math. J. **29** (1980), 79–102.

[28] F. B. Weissler, L^p*-energy and blow-up for a semilinear heat equation*, Nonlinear functional analysis and its applications, Proc. Sympos. Pure Math. **45** (1986), 545–551.

Chapter 3

Local existence and nonexistence for fractional in time weakly coupled reaction-diffusion systems

We study a fractional in time weakly coupled reaction-diffusion system in a bounded domain with the Dirichlet boundary condition. The domain is imbedded in an N-dimensional space and it has C^2 boundary, and fractional derivatives are meant in a generalized Caputo sense. The system can be referred to as a standard reaction-diffusion system in two components with polynomial growth. We obtain integrability conditions on the initial state functions which determine the existence/nonexistence of a local in time mild solution.

Keywords: Local in time solutions; Caputo fractional derivative; Singular initial functions; Uniqueness; Cauchy sequences; Semigroup estimates.

3.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with C^2 boundary. We study existence and nonexistence of a local in time solution of the fractional in time weakly coupled reaction-diffusion system

$$\begin{cases} \partial_t^{\alpha_1} u = \Delta u + f_1(x,t,v) & \text{in } \Omega \times (0,T), \\ \partial_t^{\alpha_2} v = \Delta v + f_2(x,t,u) & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{3.1.1}$$

where $0 < \alpha_1 \leq \alpha_2 < 1$ and $T > 0$. The fractional derivatives $\partial_t^{\alpha_1}$ and $\partial_t^{\alpha_2}$ are meant in a generalized Caputo sense, i.e.,

$$\partial_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(s) ds \quad \text{for } 0 < \alpha < 1,$$

where Γ denotes the usual Gamma function. In the present chapter we suppose the following:

Assumption A. Let $1 < p_1, p_2 < \infty$, $q_1, q_2 \in (1,\infty] \cap \left(\frac{N}{2}, \infty\right]$, $m_1 \in [0, \alpha_1)$ and $m_2 \in [0, \alpha_2)$

be given constants. There exist nonnegative functions $c_1 \in L^{q_1}(\Omega)$ and $c_2 \in L^{q_2}(\Omega)$ such that the following hold:

(F1) for $i = 1, 2,\ f_i(x, t, \cdot) : \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$|f_i(x,t,\xi)| \le c_i(x) \cdot t^{-m_i}(1 + |\xi|)^{p_i} \quad \text{for } \xi \in \mathbb{R}, \text{ a.e. } (x,t) \in \Omega \times (0,\infty),$$

(F2) for $i = 1, 2,\ f_i$ satisfies the local Lipschitz condition

$$|f_i(x,t,\xi) - f_i(x,t,\eta)| \le c_i(x) \cdot t^{-m_i}(1 + |\xi| + |\eta|)^{p_i - 1}|\xi - \eta| \quad \text{for } \xi, \eta \in \mathbb{R}, \\ \text{a.e. } (x,t) \in \Omega \times (0,\infty).$$

Let us start with classical equations, $\alpha = 1$. We consider the scalar problem

$$\text{(3.1.2)} \qquad \begin{cases} \partial_t u = \Delta u + f(u) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $f \in C^1$ and Ω is a (possibly unbounded) smooth domain. It is well known that the problem (3.1.2) possesses a local in time classical solution for a general nonlinear term f if $u_0 \in L^\infty(\Omega)$ (cf. [7,16]). On the other hand, in the case where $u_0 \notin L^\infty(\Omega)$, the existence of solutions heavily depends on the balance between the growth rate of f and the singularity of u_0 (cf. [7]). In Weissler [20], (3.1.2) was studied when $f(u) = |u|^{p-1}u$, $p > 1$, and Ω is bounded. A local in time solution was constructed when $u_0 \in L^r(\Omega)$ for $r > \frac{N}{2}(p-1)$ and $r \ge 1$, or $r = \frac{N}{2}(p-1)$ and $r > 1$. It was also shown that if $1 \le r < \frac{N}{2}(p-1)$, then there exists a nonnegative initial function $u_0 \in L^r(\Omega)$ such that, for every $T > 0$, (3.1.2) admits no nonnegative solution.

Next, we consider the reaction-diffusion system

$$\text{(3.1.3)} \qquad \begin{cases} \partial_t u = \Delta u + f_1(u,v) & \text{in } \Omega \times (0,T), \\ \partial_t v = \Delta v + f_2(u,v) & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega. \end{cases}$$

Quittner-Souplet [15] studied (3.1.3), where $f_1(u,v) = |v|^{p_1-1}v$, $f_2(u,v) = |u|^{p_2-1}u$ ($p_1, p_2 > 0$) and Ω is bounded. This is called a weakly coupled system. The existence (resp. nonexistence) of a local in time solution was proved when $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ and

$$\frac{N}{2} \max\left\{ \frac{p_1}{r_2} - \frac{1}{r_1}, \frac{p_2}{r_1} - \frac{1}{r_2} \right\} \le 1, \ p_1, p_2 > 1 \text{ and } r_1, r_2 > 1$$

(resp. $\frac{N}{2} \max\left\{ \frac{p_1}{r_2} - \frac{1}{r_1}, \frac{p_2}{r_1} - \frac{1}{r_2} \right\} > 1$, $p_1, p_2 > 0$ and $r_1, r_2 \ge 1$). Moreover, weakly and strongly coupled variants of system (3.1.3) were studied in [4–6, 11, 13, 19]. They assumed that initial functions may have singularities.

We obtain integrability conditions of (u_0, v_0) which determine the existence/nonexistence of a local in time solution of (3.1.1). In the proof we combine methods of Gal-Warma [9] and

Quittner-Souplet [15, 16].

To define a mild solution, we recall the Wright type function [10] defined by

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} \quad \text{for } 0 < \alpha < 1 \text{ and } z \in \mathbb{C}.$$

This is also sometimes called the Mainardi function and studied in [14, 18, 22]. It follows from [2, 10] that

$$\Phi_\alpha(t) \geq 0 \ \text{ for } t \geq 0 \ \text{ and } \ \int_0^\infty \Phi_\alpha(t)dt = 1. \tag{3.1.4}$$

We prove our nonexistence result using (3.1.4). It is a key point that the function $\Phi_\alpha(t)$ is nonnegative and integrable. Moreover, it is well known ([10]) that

$$\int_0^\infty t^p \Phi_\alpha(t)dt = \frac{\Gamma(p+1)}{\Gamma(\alpha p + 1)} \ \text{ for } p > -1 \text{ and } 0 < \alpha < 1. \tag{3.1.5}$$

Next, we consider functional spaces. Let $1 < r < \infty$. We denote by $W^{k,r}(\Omega)$ the Sobolev space (resp. the Sobolev-Slobodeckii space) if k is an integer (resp. if k is not an integer). We also denote by $\mathcal{D}(\Omega)$ the space of C^∞-functions with compact support in Ω. Put $X_0(r) := L^r(\Omega)$ and $X_1(r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$, where $W_0^{1,r}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,r}(\Omega)$. Let Δ be the Laplace operator with the domain $D(\Delta) = X_1(r)$. Then Δ generates a C^0 analytic semigroup in X_0 by [16, Examples 51.4 (i)]. Let $X_{-1}(r)$ be the completion of $X_0(r)$ endowed with the norm $|x|_{X_{-1}(r)} := |(\omega + \Delta)^{-1}x|_{X_0(r)}$, where $\omega \in \mathbb{R}$ satisfies that $\omega + \Delta : X_1(r) \to X_0(r)$ is an isomorphism ([16, p.466, 467]). For $0 < \theta < 1$, set $X_\theta(r) := (X_0(r), X_1(r))_\theta$ and $X_{-1+\theta}(r) := (X_{-1}(r), X_0(r))_\theta$, where $(\cdot, \cdot)_\theta$ is the complex interpolation functor if $\theta = \dfrac{1}{2}$ and the real interpolation functor $(\cdot, \cdot)_{\theta,r}$ otherwise. Due to [16, Theorem 51. 1 (i) and Examples 51.4 (i)], we have $X_{\theta_1}(r) \hookrightarrow X_{\theta_2}(r)$ if $-1 \leq \theta_2 \leq \theta_1 \leq 1$,

$$X_\theta(r) \hookrightarrow W^{2\theta,r}(\Omega) \ \text{ if } \theta \geq 0, \ \text{ and } \ X_\theta(r) \doteq (X_{-\theta}(r'))' \ \text{ if } \theta < 0, \tag{3.1.6}$$

where r' is the conjugate exponent of r, i.e., $\dfrac{1}{r} + \dfrac{1}{r'} = 1$. We denote X' by the (topological) dual space if X is a Banach space. We write $X \hookrightarrow Y$ if X is continuously embedded in Y. Moreover, $X \doteq Y$ means that $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

We are ready to introduce some operators related to fractional derivatives. Let $0 \leq \theta \leq 1$ and $1 < r < \infty$. We observe from [16, Theorem 51.1 (iv)] that the operator Δ also generates a C^0 analytic semigroup in $X_\theta(r)$, which is denoted by $S(t)$. For $0 < \alpha < 1$ and $t > 0$, we define

$$S_\alpha(t) : X_\theta(r) \to X_\theta(r), \ P_\alpha(t) : X_\theta(r) \to X_\theta(r)$$

by

$$\begin{cases} S_\alpha(t)w := \displaystyle\int_0^\infty \Phi_\alpha(\tau) S(\tau t^\alpha) w d\tau, \\ P_\alpha(t)w := \alpha t^{\alpha-1} \displaystyle\int_0^\infty \tau \Phi_\alpha(\tau) S(\tau t^\alpha) w d\tau \end{cases} \tag{3.1.7}$$

for $w \in X_\theta(r)$. Note that if $w \in L^r(\Omega)$ for some $1 \leq r < \infty$, then we can also define in the same way as (3.1.7) (cf. [9]). Moreover, by definition the operator S_α is strongly continuous, i.e.,

$$\lim_{t\to 0}\|S_\alpha(t)w - w\|_X = 0 \quad \text{for } w \in X, \tag{3.1.8}$$

where $X = X_\theta(r)$, $0 \le \theta \le 1$ and $1 < r < \infty$, or $X = L^r(\Omega)$, $1 \le r < \infty$.

Definition 3.1.1 (Mild solution). By a mild solution of (3.1.1) on $[0,T)$ we mean that the measurable functions (u,v) have the following properties:

(a) $u(t) = u(\cdot,t) \in L^1(\Omega)$ and $v(t) = v(\cdot,t) \in L^1(\Omega)$ for $t \in (0,T)$,

(b) $f_1(t,v(t)) = f_1(\cdot,t,v(\cdot,t)) \in L^1(\Omega)$ and $f_2(t,u(t)) = f_2(\cdot,t,u(\cdot,t)) \in L^1(\Omega)$ for almost all $t \in (0,T)$,

(c) $\displaystyle\int_0^t \|f_1(s,v(s))\|_{L^1(\Omega)}\,ds < \infty$ and $\displaystyle\int_0^t \|f_2(s,u(s))\|_{L^1(\Omega)}\,ds < \infty$ for $t \in (0,T)$,

(d) the functions (u,v) satisfy

$$u(t) = S_{\alpha_1}(t)u_0 + \int_0^t P_{\alpha_1}(t-s)f_1(s,v(s))ds \quad \text{in } \Omega \times (0,T),$$
$$v(t) = S_{\alpha_2}(t)v_0 + \int_0^t P_{\alpha_2}(t-s)f_2(s,u(s))ds \quad \text{in } \Omega \times (0,T),$$

where the integral terms are absolutely converging Bochner integrals in $L^1(\Omega)$,

(e) the initial functions (u_0,v_0) satisfy

$$\|u(t)-u_0\|_{L^{r_1}(\Omega)} \to 0 \quad (t\to 0) \quad \text{and} \quad \|v(t)-v_0\|_{L^{r_2}(\Omega)} \to 0 \quad (t \to 0)$$

for $(u_0,v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$, if $1 \le r_1, r_2 < \infty$.

It follows from (3.1.8) and [9, Remark 3.1.2] that the property of Definition 3.1.1 (e) holds if and only if

$$\lim_{t\to 0}\|u(t) - S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)} = 0 \quad \text{and} \quad \lim_{t\to 0}\|v(t) - S_{\alpha_2}(t)v_0\|_{L^{r_2}(\Omega)} = 0, \tag{3.1.9}$$

which are equivalent to the convergence in the norms of the integral terms in Definition 3.1.1 (d) to 0.

We are ready to state our main results.

Theorem 3.1.2 (Local in time existence). *Let $N \ge 1$, $0 < \alpha_1 \le \alpha_2 < 1$, $1 < p_1, p_2 < \infty$, $q_1, q_2 \in (1,\infty] \cap \left(\frac{N}{2},\infty\right]$, $1 < r_1, r_2 < \infty$, $m_1 \in [0,\alpha_1)$ and $m_2 \in [0,\alpha_2)$. Suppose that Assumption A holds. Put*

$$\mathcal{P} := \frac{N}{2}\left(\frac{p_1}{r_2} - \frac{1}{r_1} + \frac{1}{q_1}\right) + \frac{m_1}{\alpha_1}, \quad \mathcal{Q} := \frac{N}{2}\left(\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) + \frac{m_2}{\alpha_2} \quad \textit{and}$$
$$\mathcal{R} := \frac{N}{2}\left(\frac{p_1}{r_2} - \frac{1}{r_1} + \frac{1}{q_1}\right) + \frac{m_1}{\alpha_2} + \left(1 - \frac{\alpha_1}{\alpha_2}\right)\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right)\right\}.$$

Suppose that one of the following holds:

(a) $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} < 1$,

(b) $\alpha_1 = \alpha_2$ *and* $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} = 1$,

(c) $\alpha_1 < \alpha_2$ *and* $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} = 1$, *where we also suppose that* $m_1 \in (0, \alpha_1)$ *when* $\mathcal{P} = 1$.

Then for any $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$, *there exist* $T > 0$ *and a unique local in time mild solution* (u, v) *of* (3.1.1) *in the sense of Definition 3.1.1 on the interval* $[0, T)$.

We also obtain the following nonexistence result.

Theorem 3.1.3 (Local in time nonexistence). *Let* $N \geq 1$, $0 < \alpha_1 \leq \alpha_2 < 1$, $0 < p_1, p_2 < \infty$, $q_1, q_2 \in [1, \infty]$, $1 \leq r_1, r_2 < \infty$, $m_1 \in [0, \alpha_1)$ *and* $m_2 \in [0, \alpha_2)$. *Put*

$$\mathcal{Q}_1 := \frac{N\alpha_1}{2\alpha_2}\left(\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) + \frac{m_2}{\alpha_2} \quad \text{and} \quad \mathcal{Q}_2 := \frac{N}{2}\left(\frac{\alpha_1 p_2}{\alpha_2 r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) + \frac{m_2}{\alpha_2}.$$

Put $\mathcal{P}$ *and* $\mathcal{R}$ *in the same way as in Theorem 3.1.2. Suppose that* $\max\{\mathcal{P}, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{R}\} > 1$. *Then there exist nonnegative functions* $(c_1, c_2, u_0, v_0) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega) \times L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ *such that, for every* $T > 0$, *the problem* (3.1.1) *with* $f_1(x, t, v) = c_1(x) \cdot t^{-m_1} v^{p_1}$ *and* $f_2(x, t, u) = c_2(x) \cdot t^{-m_2} u^{p_2}$ *admits no local in time nonnegative mild solution* (u, v) *in the sense of Definition 3.1.1 on the interval* $[0, T)$.

If $1 < p_1, p_2 < \infty$, then the nonlinear terms $f_1(x, t, v) = c_1(x) \cdot t^{-m_1} v^{p_1}$ and $f_2(x, t, u) = c_2(x) \cdot t^{-m_2} u^{p_2}$ mentioned in Theorem 3.1.3 satisfy Assumption A with $\mathbb{R}$ replaced by $[0, \infty)$.

We deduce the following corollary from Theorems 3.1.2 and 3.1.3.

Corollary 3.1.4. *Let* $N \geq 1$ *and* $0 < \alpha_1 = \alpha_2 < 1$. *Then the following are true:*

(i) *Let* $1 < p_1, p_2 < \infty$, $q_1, q_2 \in (1, \infty] \cap \left(\frac{N}{2}, \infty\right]$, $1 < r_1, r_2 < \infty$, $m_1 \in [0, \alpha_1)$ *and* $m_2 \in [0, \alpha_2)$. *Suppose that Assumption A holds. Put* $\mathcal{P}$ *and* $\mathcal{Q}$ *in the same way as in Theorem 3.1.2. If* $\max\{\mathcal{P}, \mathcal{Q}\} \leq 1$, *then for any* $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$, *there exist* $T > 0$ *and a unique local in time mild solution* (u, v) *of* (3.1.1).

(ii) *Let* $0 < p_1, p_2 < \infty$, $q_1, q_2 \in [1, \infty]$, $1 \leq r_1, r_2 < \infty$, $m_1 \in [0, \alpha_1)$ *and* $m_2 \in [0, \alpha_2)$. *If* $\max\{\mathcal{P}, \mathcal{Q}\} > 1$, *then there exist nonnegative functions* $(c_1, c_2, u_0, v_0) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega) \times L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ *such that, for every* $T > 0$, *the problem* (3.1.1) *with* $f_1(x, t, v) = c_1(x) \cdot t^{-m_1} v^{p_1}$ *and* $f_2(x, t, u) = c_2(x) \cdot t^{-m_2} u^{p_2}$ *admits no local in time nonnegative mild solution* (u, v).

Corollary 3.1.4 implies that we can *explicitly* determine the existence/nonexistence of a solution when $\alpha_1 = \alpha_2$. Our conditions cover all the cases $(p_1, p_2, q_1, q_2, m_1, m_2)$ in Assumption A. Moreover, Corollary 3.1.4 leads to the following pure power case result, which corresponds to [15].

Corollary 3.1.5. *Let* $N \geq 1$ *and* $0 < \alpha_1 = \alpha_2 < 1$. *Then the following are true:*

(i) *Let* $1 < p_1, p_2 < \infty$ *and* $1 < r_1, r_2 < \infty$. *Put*

$$\widetilde{\mathcal{P}} := \frac{N}{2}\left(\frac{p_1}{r_2} - \frac{1}{r_1}\right) \quad \text{and} \quad \widetilde{\mathcal{Q}} := \frac{N}{2}\left(\frac{p_2}{r_1} - \frac{1}{r_2}\right).$$

If $\max\{\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}}\} \leq 1$, *then for any* $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$, *there exist* $T > 0$ *and a unique local in time mild solution* (u, v) *of* (3.1.1) *with* $f_1(x, t, v) = |v|^{p_1 - 1} v$ *and* $f_2(x, t, u) = |u|^{p_2 - 1} u$.

(ii) *Let* $0 < p_1, p_2 < \infty$ *and* $1 \le r_1, r_2 < \infty$. *If* $\max\{\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}}\} > 1$, *then there are nonnegative functions* $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ *such that, the problem* (3.1.1) *with* $f_1(x,t,v) = v^{p_1}$ *and* $f_2(x,t,u) = u^{p_2}$ *has no local in time nonnegative mild solution on any time interval.*

Let us recall fundamental properties of scalar problems. Fractional in time parabolic equations with nonlinear terms have not been well studied until recently. Gal-Warma [9] has studied the fractional in time scalar problem

$$\begin{cases} \partial_t^\alpha u = Au + f(x,t,u) & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.1.10}$$

where A is a differential operator which generates a strongly continuous semigroup on $L^2(\Omega)$. Detailed results can be found in [1, 3, 8, 12]. Let $1 \le p < \infty$ and $q_1, q_2 \in [1,\infty]$ be given constants. In [9] the authors assumed that there exists a nonnegative function $c \in L_{q_1,q_2}$ such that the following hold:

(F1') $f(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$|f(x,t,\xi)| \le c(x,t)\ (1 + |\xi|)^p \quad \text{for } \xi \in \mathbb{R}, \text{ a.e. } (x,t) \in \Omega \times (0,\infty),$$

(F2') f satisfies the local Lipschitz condition

$$|f(x,t,\xi) - f(x,t,\eta)| \le c(x,t)\ (1+|\xi|+|\eta|)^{p-1}|\xi-\eta| \quad \text{for } \xi,\eta \in \mathbb{R}, \text{ a.e. } (x,t) \in \Omega \times (0,\infty).$$

Here L_{q_1,q_2} denotes the Banach space defined by

$$L_{q_1,q_2} := \left\{ c : \Omega \times (0,\infty) \to \mathbb{R} \text{ measurable}, \ \ \|c\|_{L_{q_1,q_2}} := \sup_{\substack{t_1,t_2 \in (0,\infty), \\ 0 \le t_2 - t_1 \le 1}} \left(\int_{t_1}^{t_2} \|c(\cdot,s)\|_{L^{q_1}(\Omega)}^{q_2} ds \right)^{\frac{1}{q_2}} < \infty \right\}$$

for $q_1 \in [1,\infty]$ and $q_2 \in [1,\infty)$ with the obvious modifications when $q_2 = \infty$. Proposition 2.2.2 of [9, p.21, 22] states the following existence result. Let $0 < \alpha < 1$, $1 \le p < \infty$, $q_1 \in [1,\infty] \cap (\beta_A, \infty]$ and $q_2 \in \left(\dfrac{1}{\alpha}, \infty\right]$. The constant β_A is related to the L^p-L^q estimate of the semigroup generated by A. Assume **(F1')** and **(F2')**. If

$$\beta_A \left(\frac{p-1}{r} + \frac{1}{q_1} \right) + \frac{1}{\alpha q_2} < 1 \text{ and } 1 \le p, r < \infty, \text{ or}$$
$$\beta_A \left(\frac{p-1}{r} + \frac{1}{q_1} \right) + \frac{1}{\alpha q_2} = 1 \text{ and } 1 < p, r < \infty,$$

then for any $u_0 \in L^r(\Omega)$, there exist $T > 0$ and a unique local in time solution of (3.1.10). Note that if Ω is a bounded domain in $\mathbb{R}^N$, $N \ge 1$, with C^2 boundary, $A = \Delta$ and $f(x,t,u) = |u|^{p-1}u$, $p > 1$, then $\beta_A = \dfrac{N}{2}$, $q_1 = q_2 = \infty$ and hence this result corresponds to the existence part in [20].

In [9, Remark 5.0.2], based on [20, 21], they conjectured the nonexistence of a local in time solution of (3.1.10) in the super-critical case

$$\beta_A \left(\frac{p-1}{r} + \frac{1}{q_1} \right) + \frac{1}{\alpha q_2} > 1. \tag{3.1.11}$$

We give an affirmative answer to the conjecture when $A = \Delta$. Section 3.5 is devoted to this nonexistence result.

Let us explain a sketch of the proofs. The main points of the proofs are Cauchy sequences for the existence part, including

$$u_n(t) = S_{\alpha_1}(t)u_0 + \int_0^t P_{\alpha_1}(t-s)f_1(s, v_{n-1}(s))ds, \tag{3.1.12}$$

$$v_n(t) = S_{\alpha_2}(t)v_0 + \int_0^t P_{\alpha_2}(t-s)f_2(s, u_{n-1}(s))ds \tag{3.1.13}$$

for $n \geq 2$, $u_1 = v_1 = 0$, and the contradiction argument for the nonexistence part.

For the existence part by induction method we can show that if $T > 0$ is sufficiently small, then $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are Cauchy sequences. Then limits u and v of the sequences exist and (u, v) is a mild solution of (3.1.1) in the sense of Definition 3.1.1. In order to show that these are indeed Cauchy sequences, it is crucial to find various exponents including θ_1, θ_2 and θ_3. However, it is not obvious how to find these exponents. Section 3.3 addresses this aspect.

For the nonexistence part we construct initial data (u_0, v_0). Assume that (3.1.1) has a local in time nonnegative mild solution (u, v). Since the singularity of the constructed functions are strong, the norm of at least one integral term in Definition 3.1.1 (d) diverges as $t \to 0$, which follows from estimates of $S_\alpha(t)$ and $P_\alpha(t)$. This is a contradiction. It is known (cf. [16, p.440]) that there exists a positive C^∞-function $G_\Omega : \Omega \times \Omega \times (0, \infty) \to \mathbb{R}$ (Dirichlet heat kernel) such that

$$(S(t)\phi)(x) = \int_\Omega G_\Omega(x, y, t)\phi(y)dy$$

for $\phi \in L^r(\Omega)$, $1 \leq r \leq \infty$. After that, we abbreviate G_Ω as G. Since G has a lower bound with respect to t, we obtain estimates of $S_\alpha(t)$ and $P_\alpha(t)$ from (3.1.7).

This chapter is organized as follows. In Section 3.2 we give and recall some properties of $S_\alpha(t)$, $P_\alpha(t)$ and the Dirichlet heat kernel. In Sections 3.3 and 3.4 we use these properties and prove Theorems 3.1.2 and 3.1.3, respectively. In Section 3.5 we give a nonexistence result for scalar problems. In Section 3.6 we discuss our results and explain possible future problems ensuing from the current analysis.

3.2 Preliminaries

For any set $\mathcal{X}$ and the mappings $a = a(x)$ and $b = b(x)$ from $\mathcal{X}$ to $[0, \infty)$, we say

$$a(x) \lesssim b(x) \quad \text{for all } x \in \mathcal{X}$$

if there exists a positive constant C such that $a(x) \leq Cb(x)$ for all $x \in \mathcal{X}$.

Proposition 3.2.1. *Let $0 < \alpha < 1$, $1 < r < \infty$ and $-1 \leq \theta_1 \leq \theta_2 \leq 1$. Then the following are true:*

(i) *If $\theta_2 - \theta_1 < 1$, then there exists $C > 0$ such that*

$$|S_\alpha(t)w|_{X_{\theta_2}(r)} \leq Ct^{\alpha(\theta_1 - \theta_2)}|w|_{X_{\theta_1}(r)}$$

for $t > 0$ and $w \in X_{\theta_1}(r)$.

(ii) *If $\theta_1 > -1$ or $\theta_2 < 1$, then there exists $C > 0$ such that*

$$|t^{1-\alpha} P_\alpha(t) w|_{X_{\theta_2}(r)} \leq C t^{\alpha(\theta_1-\theta_2)} |w|_{X_{\theta_1}(r)}$$

for $t > 0$ and $w \in X_{\theta_1}(r)$.

Proof. We prove (i) in a similar manner to [9, Proposition 2.2.2]. Using (3.1.5), (3.1.7) and [15, Theorem 51.1 (iv)], we have

$$\begin{aligned}
|S_\alpha(t) w|_{X_{\theta_2}(r)} &\leq \int_0^\infty \Phi_\alpha(\tau) |S(\tau t^\alpha) w|_{X_{\theta_2}(r)} d\tau \\
&\lesssim \int_0^\infty \Phi_\alpha(\tau) \tau^{\theta_1-\theta_2} t^{\alpha(\theta_1-\theta_2)} |w|_{X_{\theta_1}(r)} d\tau \\
&= t^{\alpha(\theta_1-\theta_2)} |w|_{X_{\theta_1}(r)} \int_0^\infty \Phi_\alpha(\tau) \tau^{\theta_1-\theta_2} d\tau \\
&= \frac{\Gamma(1+\theta_1-\theta_2)}{\Gamma(1+\alpha(\theta_1-\theta_2))} t^{\alpha(\theta_1-\theta_2)} |w|_{X_{\theta_1}(r)}.
\end{aligned}$$

We can obtain the assertion (ii) in the same way as the assertion (i). □

Lemma 3.2.2. *Let $0 < \alpha_1 \leq \alpha_2 < 1$ and $0 < T < \infty$. For $i = 1, 2$, let $0 < \theta_i < 1$, $1 < r_i < \infty$ and $\Pi_i \subset X_0(r_i)$ $(= L^{r_i}(\Omega))$. Suppose that for $i = 1, 2$,*

$$\kappa(\Pi_i) := \left\{ u |u|^{-1}_{X_0(r_i)} : u \in \Pi_i,\ u \neq 0 \right\}$$

is precompact in $X_0(r_i)$. Then there exists a continuous and nondecreasing function $g : (0, T) \to (0, \infty)$, depending on α_i, θ_i, r_i and Π_i $(i = 1, 2)$ such that the following are true:

(i) *For $i = 1, 2$, the following is true:*

$$|S_{\alpha_i}(t) u|_{X_{\theta_i}(r_i)} \lesssim g(t) \cdot t^{-\alpha_2 \theta_i} |u|_{X_0(r_i)} \quad \text{for } 0 < t < T \text{ and } u \in \Pi_i.$$

(ii) *We have $\lim_{t \to 0} g(t) = 0$. For $i = 1, 2$, the function $w_i = w_i(t)$ defined by*

$$(w_i(t))^{-\alpha_2 \theta_i} = g(t) \cdot t^{-\alpha_2 \theta_i}$$

has the properties

$$\lim_{t \to 0} w_i(t) = 0 \text{ and } \min\left\{ t, t^{1-\frac{\min\{\theta_1,\theta_2\}}{2\theta_i}} \right\} \leq w_i(t) \leq t^{1-\frac{\min\{\theta_1,\theta_2\}}{2\max\{\theta_1,\theta_2\}}}.$$

Proof. Define

$$\begin{cases}
h_1(t, u) := |S_{\alpha_1}(t) u|_{X_{\theta_1}(r_1)} C_0^{-1} t^{\alpha_1 \theta_1} |u|^{-1}_{X_0(r_1)} & \text{for } (t, u) \in (0, T) \times \Pi_1 \backslash \{0\}, \\
\overline{h_1}(t) := \sup \{ h_1(s, u) : s \in (0, t],\ u \in \Pi_1 \backslash \{0\} \} & \text{for } t \in (0, T),
\end{cases}$$

where $C_0 (= C) > 0$ is the constant from Proposition 3.2.1 (i). We observe from [9, Lemma A.0.2] that $0 \leq \overline{h_1} \leq 1$, $\lim_{t \to 0} \overline{h_1}(t) = 0$ and

$$\begin{aligned}
|S_{\alpha_1}(t) u|_{X_{\theta_1}(r_1)} &\leq C_0 \overline{h_1}(t) \cdot t^{-\alpha_1 \theta_1} |u|_{X_0(r_1)} \\
&\lesssim C_0 \overline{h_1}(t) \cdot t^{-\alpha_2 \theta_1} |u|_{X_0(r_1)} \quad \text{for } 0 < t < T \text{ and } u \in \Pi_1.
\end{aligned}$$

Put $\overline{h_2}$ in the same way. We set

$$g(t) := \max\left\{\overline{h_1}(t), \overline{h_2}(t), t^{\delta\alpha_2\theta_1}, t^{\delta\alpha_2\theta_2}\right\},$$

where $\delta := \dfrac{\min\{\theta_1, \theta_2\}}{2\max\{\theta_1, \theta_2\}}$.

It suffices to prove the estimates of $w_i(t)$ for $i = 1, 2$. Since $g(t) \geq t^{\delta\alpha_2\theta_1}$, we obtain

$$w_1(t) = g(t)^{-\frac{1}{\alpha_2\theta_1}} \cdot t \leq t^{1-\delta} \quad \text{for } t > 0.$$

On the other hand, let $0 < t < \min\{T, 1\}$. Due to $\overline{h_1}, \overline{h_2} \leq 1$, we have $g(t) \leq 1$ and hence $w_1(t) \geq t$. If $t \geq 1$, then $g(t) = t^{\delta\alpha_2 \max\{\theta_1,\theta_2\}}$. Thus it follows that

$$w_1(t) = t^{1-\frac{\delta\max\{\theta_1,\theta_2\}}{\theta_1}} = t^{1-\frac{\min\{\theta_1,\theta_2\}}{2\theta_1}}.$$

Therefore, we deduce the desired estimate of $w_1(t)$. We can obtain the estimate of $w_2(t)$ in the same way. $\square$

Proposition 3.2.3. ([16, Proposition 49.10]) *Let $N \geq 1$ and let Ω be an arbitrary domain in $\mathbb{R}^N$. There exist constants $c_1 > 0$ and $c_2 \geq 2$ depending only on N, such that the Dirichlet heat kernel $G(x, y, t)$ in Ω satisfies*

$$G(x, y, t) \geq c_1 t^{-\frac{N}{2}}$$

for $t > 0$ and $x, y \in \Omega$ such that

$$dist(x, \partial\Omega) \geq c_2\sqrt{t} \quad \text{and} \quad |x - y| \leq \sqrt{t}.$$

3.3 Existence result

Proposition 3.3.1. *Let $N \geq 1$, $0 < \alpha_1 \leq \alpha_2 < 1$, $1 < p_1, p_2 < \infty$, $q_1, q_2 \in (1, \infty] \cap \left(\dfrac{N}{2}, \infty\right]$, $1 < r_1, r_2 < \infty$, $m_1 \in [0, \alpha_1)$ and $m_2 \in [0, \alpha_2)$. Put $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ in the same way as in Theorem 3.1.2. If $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} < 1$, then there exist $(s_1, \theta_2, z_1, z_2)$, $(s_2, \theta_1, z_3, z_4) \in (0, 1) \times (0, 1) \times (1, \infty) \times (1, \infty)$ such that*

$$\frac{1}{z_1} - \frac{2}{N}(1 - s_1) = \frac{1}{r_1}, \ \frac{1}{q_1} + \frac{p_1}{z_2} = \frac{1}{z_1} \ \text{and} \ \frac{1}{r_2} - \frac{2}{N}\theta_2 \leq \frac{1}{z_2}, \tag{3.3.1}$$

$$\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2 - \alpha_1)\theta_1 > 0, \tag{3.3.2}$$

$$m_1 + p_1\alpha_2\theta_2 < 1, \tag{3.3.3}$$

$$\theta_1 < s_1, \tag{3.3.4}$$

$$\frac{1}{z_3} - \frac{2}{N}(1 - s_2) = \frac{1}{r_2}, \ \frac{1}{q_2} + \frac{p_2}{z_4} = \frac{1}{z_3} \ \text{and} \ \frac{1}{r_1} - \frac{2}{N}\theta_1 \leq \frac{1}{z_4}, \tag{3.3.5}$$

$$\alpha_2 s_2 - m_2 - p_2\alpha_2\theta_1 > 0, \tag{3.3.6}$$

$$\theta_2 < s_2. \tag{3.3.7}$$

Moreover, if $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} = 1$, then there exist $(s_1, \theta_2, z_1, z_2)$, $(s_2, \theta_1, z_3, z_4) \in (0, 1) \times (0, 1) \times (1, \infty) \times (1, \infty)$ such that (3.3.1)–(3.3.7) *hold with* (3.3.2) *and* (3.3.6) *replaced by $\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2 - \alpha_1)\theta_1 \geq 0$ and $\alpha_2 s_2 - m_2 - p_2\alpha_2\theta_1 \geq 0$, respectively.*

Proof. Suppose that $\max\{\mathcal{P},\mathcal{Q},\mathcal{R}\}<1$. For $1-\frac{N}{2}\left(1-\frac{1}{r_1}\right)<s<\min\left\{1,1-\frac{N}{2}\left(\frac{1}{q_1}-\frac{1}{r_1}\right)\right\}$, put

$$\theta_2(s):=\frac{N}{2p_1}\left(\frac{p_1}{r_2}-\frac{1}{r_1}+\frac{1}{q_1}\right)-\frac{1}{p_1}(1-s),\quad \frac{1}{z_1(s)}:=\frac{2}{N}(1-s)+\frac{1}{r_1}\ \text{and}$$
$$\frac{1}{z_2(s)}:=\frac{1}{p_1}\left\{\frac{1}{r_1}-\frac{1}{q_1}+\frac{2}{N}(1-s)\right\}.$$

We see that

$$\max\left\{\frac{1}{q_1},\frac{1}{r_1}\right\}<\frac{1}{z_1(s)}<1\ \text{and}\ \max\left\{\frac{1}{p_1}\left(\frac{1}{r_1}-\frac{1}{q_1}\right),0\right\}<\frac{1}{z_2(s)}<\frac{1}{p_1}\left(1-\frac{1}{q_1}\right).$$

Hence, $z_1(s)\in(1,\infty)$ and $z_2(s)\in(1,\infty)$. Let $(s_1,\theta_2,z_1,z_2)=(s,\theta_2(s),z_1(s),z_2(s))$. By direct calculation we have (3.3.1). In the same way we derive (3.3.5) with $(s_2,\theta_1,z_3,z_4)=(\tilde{s},\theta_1(\tilde{s}),z_3(\tilde{s}),z_4(\tilde{s}))$, where

$$\theta_1(\tilde{s}):=\frac{N}{2p_2}\left(\frac{p_2}{r_1}-\frac{1}{r_2}+\frac{1}{q_2}\right)-\frac{1}{p_2}(1-\tilde{s}),\quad \frac{1}{z_3(\tilde{s})}:=\frac{2}{N}(1-\tilde{s})+\frac{1}{r_2}\ \text{and}$$
$$\frac{1}{z_4(\tilde{s})}:=\frac{1}{p_2}\left\{\frac{1}{r_2}-\frac{1}{q_2}+\frac{2}{N}(1-\tilde{s})\right\}$$

for $1-\frac{N}{2}\left(1-\frac{1}{r_2}\right)<\tilde{s}<\min\left\{1,1-\frac{N}{2}\left(\frac{1}{q_2}-\frac{1}{r_2}\right)\right\}$. Moreover, we see that (3.3.3) and (3.3.6) follow from $\mathcal{P}<1$ and $\mathcal{Q}<1$, respectively.

Substituting $(s_1,\theta_2,s_2,\theta_1)=(s,\theta_2(s),\tilde{s},\theta_1(\tilde{s}))$ into (3.3.4) and (3.3.7), we obtain

$$\frac{N}{2p_1}\left(\frac{p_1}{r_2}-\frac{1}{r_1}+\frac{1}{q_1}\right)-\frac{1}{p_1}(1-s)<\tilde{s}, \tag{3.3.8}$$
$$\frac{N}{2p_2}\left(\frac{p_2}{r_1}-\frac{1}{r_2}+\frac{1}{q_2}\right)-\frac{1}{p_2}(1-\tilde{s})<s. \tag{3.3.9}$$

We show that the inequality (3.3.8) holds with $s=\max\left\{1-\frac{N}{2}\left(1-\frac{1}{r_1}\right),\frac{m_1}{\alpha_1}\right\}$ and $\tilde{s}=\min\left\{1,1-\frac{N}{2}\left(\frac{1}{q_2}-\frac{1}{r_2}\right)\right\}$. Since $\mathcal{P}<1$, we have

$$\frac{N}{2p_1}\left(\frac{p_1}{r_2}-1+\frac{1}{q_1}\right)<\frac{N}{2p_1}\left(\frac{p_1}{r_2}-\frac{1}{r_1}+\frac{1}{q_1}\right)<\frac{1}{p_1}\left(1-\frac{m_1}{\alpha_1}\right)<1.$$

Moreover, it follows from $q_1,q_2\in(1,\infty]\cap\left(\frac{N}{2},\infty\right]$ that $\frac{N}{2p_1}\left(\frac{p_1}{r_2}-1+\frac{1}{q_1}\right)<\frac{N}{2r_2}<1-\frac{N}{2}\left(\frac{1}{q_2}-\frac{1}{r_2}\right)$. Thus (3.3.8) holds when $s=1-\frac{N}{2}\left(1-\frac{1}{r_1}\right)$. Since $\mathcal{P}<1$, the left hand side of (3.3.8) is negative when $s=\frac{m_1}{\alpha_1}$. Therefore, (3.3.8) holds with $s=\max\left\{1-\frac{N}{2}\left(1-\frac{1}{r_1}\right),\frac{m_1}{\alpha_1}\right\}$ and $\tilde{s}=\min\left\{1,1-\frac{N}{2}\left(\frac{1}{q_2}-\frac{1}{r_2}\right)\right\}$. In the same way the inequality (3.3.9) holds with $s=\min\left\{1,1-\frac{N}{2}\left(\frac{1}{q_1}-\frac{1}{r_1}\right)\right\}$ and $\tilde{s}=\max\left\{1-\frac{N}{2}\left(1-\frac{1}{r_2}\right),\frac{m_2}{\alpha_2}\right\}$.

We divide the possibilities into two cases: $\alpha_1 = \alpha_2$ and $\alpha_1 < \alpha_2$.

Let $\alpha_1 = \alpha_2$. Since $\mathcal{P} < 1$, we obtain (3.3.2). Put

$$I_1 := \left(\max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right), \frac{m_1}{\alpha_1}\right\}, \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_1} - \frac{1}{r_1}\right)\right\}\right),$$
$$I_2 := \left(\max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_2}\right), \frac{m_2}{\alpha_2}\right\}, \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)\right\}\right) \quad \text{and} \quad R := I_1 \times I_2.$$

The area of $(s, \tilde{s})$ satisfying (3.3.8) and (3.3.9) is the shaded portion on the graphs. See Fig. 3.1. Due to the above calculation results, the position of R is as shown in the left figure, but not as shown in the right one. Thus there exists $(s, \tilde{s}) \in R$ such that (3.3.8) and (3.3.9) hold.

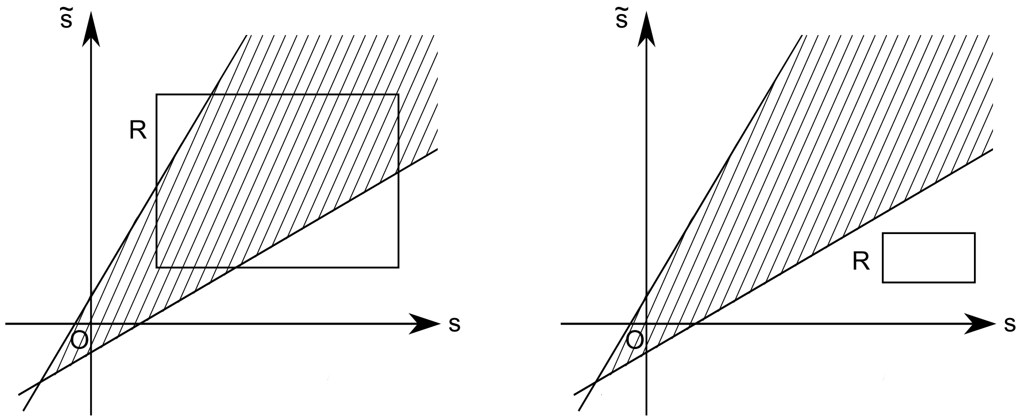


Fig. 3.1: Possible relative positions of the rectangle R and sector defined by (3.3.8)-(3.3.9).

It remains to consider θ_1 and θ_2. We observe from (3.3.8) and (3.3.9) that $\theta_2(s) < 1$ and $\theta_1(\tilde{s}) < 1$. If $\theta_2(s) \le 0$, then we replace $\theta_2 = \theta_2(s)$ with $\theta_2 = \varepsilon > 0$, where ε is sufficiently small such that $\alpha_1 s_1 - m_1 - p_1\alpha_2\varepsilon > 0$ and $\varepsilon < \tilde{s}$. We see that (3.3.1), (3.3.2), (3.3.3) and (3.3.7) hold. If $\theta_1(\tilde{s}) \le 0$, then we can replace $\theta_1(\tilde{s})$ with $\theta_1 \in (0, 1)$ in the same way.

Let $\alpha_1 < \alpha_2$. We recall that

$$(s_1, \theta_2, z_1, z_2) = (s, \theta_2(s), z_1(s), z_2(s)) \quad \text{and} \quad (s_2, \theta_1, z_3, z_4) = (\tilde{s}, \theta_1(\tilde{s}), z_3(\tilde{s}), z_4(\tilde{s})).$$

We divide the possibilities into two cases: (i) $\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2} > 0$ and (ii) $\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2} \le 0$.

We prove (i). It follows that (3.3.1), (3.3.3), (3.3.5) and (3.3.6) hold. The inequality (3.3.2) is equivalent to

$$\frac{N}{2p_2}\left(\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) - \frac{1}{p_2}(1 - \tilde{s}) > s + \frac{\frac{N}{2}\left(\frac{p_1}{r_2} - \frac{1}{r_1} + \frac{1}{q_1}\right) + \frac{m_1}{\alpha_2} - 1}{1 - \frac{\alpha_1}{\alpha_2}}, \tag{3.3.10}$$

which holds with $(s, \tilde{s}) = \left(\max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right), \frac{m_1}{\alpha_1}\right\}, \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)\right\}\right)$. Indeed, the right hand side of (3.3.10) is negative when $s = \frac{m_1}{\alpha_1}$ (resp. when $s = 1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right)$), since $\mathcal{P} < 1$ (resp. since $\mathcal{R} < 1$). On the other hand, since $\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2} > 0$, the left hand side of (3.3.10) is positive.

The area of $(s, \tilde{s})$ satisfying (3.3.8), (3.3.9) and (3.3.10) is the shaded portion on the graphs. See Fig. 3.2. Here, R is defined in the same way as when $\alpha_1 = \alpha_2$. Due to the above calculation results, the position of R is as shown in the left figure, but not as shown in the right one. Thus there exists $(s, \tilde{s}) \in R$ such that (3.3.8), (3.3.9) and (3.3.10) hold.

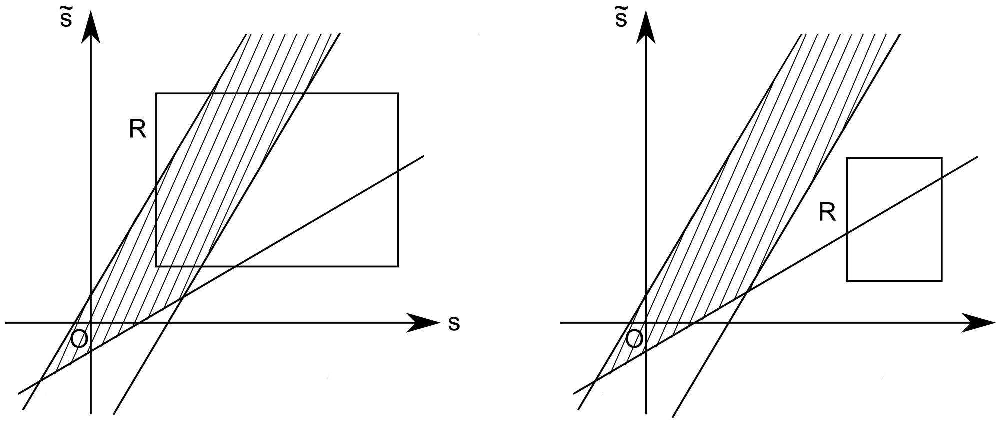


Fig. 3.2: Possible relative positions of the rectangle R and the strip defined by (3.3.8)-(3.3.10).

It remains to consider θ_1 and θ_2. We observe from (3.3.8) and (3.3.9) that $\theta_2(s) < 1$ and $\theta_1(\tilde{s}) < 1$. If $\theta_2(s) \leq 0$ or $\theta_1(\tilde{s}) \leq 0$, then it is sufficient to replace it/them in the same way as when $\alpha_1 = \alpha_2$.

We prove (ii). Let $\varepsilon > 0$ be sufficiently small. Put

$$s := \max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right), \frac{m_1}{\alpha_1}\right\} + \varepsilon \in I_1 \quad \text{and} \quad \tilde{s} := \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)\right\} - \varepsilon \in I_2$$

Due to $\theta_1(\tilde{s}) < 0$ in this case, we replace $\theta_1 = \theta_1(\tilde{s})$ with $\theta_1 = \varepsilon$. It follows that (3.3.1), (3.3.3) and (3.3.5) hold. Since $\theta_1 = \varepsilon$ is sufficiently small, $s \geq \varepsilon > 0$ and $\tilde{s} > \dfrac{m_2}{\alpha_2}$, we obtain (3.3.4) and (3.3.6). The inequality (3.3.2) is equivalent to (3.3.10) with the left hand side replaced by ε. Then by $\max\{\mathcal{P}, \mathcal{R}\} < 1$ we deduce (3.3.2). Moreover, since (3.3.8) holds with $(s, \tilde{s}) = \left(\max\left\{1 - \dfrac{N}{2}\left(1 - \dfrac{1}{r_1}\right), \dfrac{m_1}{\alpha_1}\right\}, \min\left\{1, 1 - \dfrac{N}{2}\left(\dfrac{1}{q_2} - \dfrac{1}{r_2}\right)\right\}\right)$, (3.3.8) also holds with $(s, \tilde{s}) = \left(\max\left\{1 - \dfrac{N}{2}\left(1 - \dfrac{1}{r_1}\right), \dfrac{m_1}{\alpha_1}\right\} + \varepsilon, \min\left\{1, 1 - \dfrac{N}{2}\left(\dfrac{1}{q_2} - \dfrac{1}{r_2}\right)\right\} - \varepsilon\right)$.

It remains to consider θ_2. We observe from (3.3.8) that $\theta_2(s) < 1$. If $\theta_2(s) \leq 0$, then it is sufficient to replace it in the same way as when $\alpha_1 = \alpha_2$.

Since we can prove the case where $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} = 1$ in a similar way, we omit the proof. □

Lemma 3.3.2. *Suppose that $\alpha_1 < \alpha_2$ and $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} < 1$ hold in particular in the assumptions of Proposition 3.3.1. Let $s_1 \in (0, 1)$ be chosen in Proposition 3.3.1. Put*

$$\mathcal{S} := \frac{1 - \frac{N}{2}\left(\frac{p_1}{r_2} - \frac{1}{r_1} + \frac{1}{q_1}\right) - \frac{m_1}{\alpha_2}}{1 - \frac{\alpha_1}{\alpha_2}}.$$

If $s_1 \geq \mathcal{S}$, then the following are true:

(i) *There exists $s_3 \in (0, 1)$ such that $1 - \dfrac{N}{2}\left(1 - \dfrac{1}{r_1}\right) < s_3 < \mathcal{S}$ and that $\theta_3 := \theta_2(s_3) > 0$,* *where $\theta_2(s)$ is defined in the proof of Proposition 3.3.1.*

(ii) $\theta_2(s_1) > 0$.

Proof. By direct calculation we have $p_1\alpha_2\theta_2(\mathcal{S}) = \alpha_1\mathcal{S} - m_1$. It follows from $\mathcal{P} < 1$ that $\mathcal{S} > \dfrac{m_1}{\alpha_1}$. Then $\theta_2(\mathcal{S}) > 0$ holds. Note that $\mathcal{R} < 1$ yields $1 - \dfrac{N}{2}\left(1 - \dfrac{1}{r_1}\right) < \mathcal{S}$. Choosing $\max\left\{1 - \dfrac{N}{2}\left(1 - \dfrac{1}{r_1}\right), \dfrac{m_1}{\alpha_1}\right\} < s_3 < \mathcal{S}$ sufficiently large, we obtain $\theta_2(s_3) > 0$. Moreover, we see that

$$\max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right), \frac{m_1}{\alpha_1}\right\} < s_3 < \mathcal{S} \le s_1 < 1.$$

Thus $s_3 \in (0,1)$ holds. Since $\theta_2(s)$ is increasing with respect to s, we have $\theta_2(s_1) > \theta_2(s_3) > 0$. The proof is complete. $\square$

Proof of Theorem 3.1.2. Set $u_1 := 0$ and $v_1 := 0$. For $n \ge 2$, define the functions u_n and v_n by (3.1.12) and (3.1.13), respectively. We introduce the Banach space defined by

$$Y_{\theta,r,T} := \left\{u \in L^\infty_{loc}((0,T], X_\theta(r)) : \|u\|_{Y_{\theta,r,T}} := \sup_{t\in(0,T)} t^{\alpha_2\theta}|u(t)|_{X_\theta(r)} < \infty\right\}$$

for $0 < \theta < 1$, $1 < r < \infty$ and $T > 0$.

We prove the case (a). Choose $(s_1, \theta_2, z_1, z_2)$ and $(s_2, \theta_1, z_3, z_4)$ as in Proposition 3.3.1.

We consider the existence part. Let $T > 0$. Assume that $(u_{n-1}, v_{n-1}) \in Y_{\theta_1,r_1,T} \times Y_{\theta_2,r_2,T}$. We obtain from Proposition 3.2.1 that for $0 < t < T$,

$$\begin{aligned} t^{\alpha_2\theta_1}|u_n|_{X_{\theta_1}(r_1)} &\le t^{\alpha_2\theta_1}|S_{\alpha_1}(t)u_0|_{X_{\theta_1}(r_1)} + t^{\alpha_2\theta_1}\int_0^t |P_{\alpha_1}(t-s)f_1(s, v_{n-1}(s))|_{X_{\theta_1}(r_1)}ds \\ &\lesssim |u_0|_{X_0(r_1)} + t^{\alpha_2\theta_1}\int_0^t (t-s)^{\alpha_1(s_1-\theta_1)-1}|f_1(s, v_{n-1}(s))|_{X_{s_1-1}(r_1)}ds. \end{aligned} \tag{3.3.11}$$

It follows from (3.1.6) and the first equality of (3.3.1) that $X_{1-s_1}(r_1') \hookrightarrow L^{z_1'}(\Omega)$, which implies that $L^{z_1}(\Omega) \hookrightarrow X_{s_1-1}(r_1)$. By the last inequality of (3.3.1) we have $X_{\theta_2}(r_2) \hookrightarrow L^{z_2}(\Omega)$. Then we can deduce from Assumption A and (3.3.1) that for $0 < s < t$,

$$\begin{aligned} |f_1(s, v_{n-1}(s))|_{X_{s_1-1}(r_1)} &\lesssim \|f_1(s, v_{n-1}(s))\|_{L^{z_1}(\Omega)} \le \|c_1 \cdot s^{-m_1}(1 + |v_{n-1}(s)|)^{p_1}\|_{L^{z_1}(\Omega)} \\ &\le s^{-m_1} \cdot \|c_1\|_{L^{q_1}(\Omega)}\|(1 + |v_{n-1}(s)|)^{p_1}\|_{L^{\frac{z_2}{p_1}}(\Omega)} \\ &= s^{-m_1} \cdot \|c_1\|_{L^{q_1}(\Omega)}\|1 + |v_{n-1}(s)|\|^{p_1}_{L^{z_2}(\Omega)} \\ &\lesssim s^{-m_1} \cdot \|c_1\|_{L^{q_1}(\Omega)}|1 + |v_{n-1}(s)||^{p_1}_{X_{\theta_2}(r_2)} \\ &\le s^{-m_1-p_1\alpha_2\theta_2} \cdot \|c_1\|_{L^{q_1}(\Omega)}\|1 + |v_{n-1}|\|^{p_1}_{Y_{\theta_2,r_2,T}}, \end{aligned} \tag{3.3.12}$$

which yields

$$
\begin{aligned}
&t^{\alpha_2\theta_1}\int_0^t (t-s)^{\alpha_1(s_1-\theta_1)-1}|f_1(s,v_{n-1}(s))|_{X_{s_1-1}(r_1)}ds\\
&\lesssim t^{\alpha_2\theta_1}\int_0^t (t-s)^{\alpha_1(s_1-\theta_1)-1}s^{-m_1-p_1\alpha_2\theta_2}\cdot\|c_1\|_{L^{q_1}(\Omega)}\|1+|v_{n-1}|\|^{p_1}_{Y_{\theta_2,r_2,T}}ds\\
&\le \|c_1\|_{L^{q_1}(\Omega)}\|1+|v_{n-1}|\|^{p_1}_{Y_{\theta_2,r_2,T}}\cdot t^{\alpha_1 s_1-m_1-p_1\alpha_2\theta_2+(\alpha_2-\alpha_1)\theta_1}\\
&\qquad\times\int_0^1(1-s)^{\alpha_1(s_1-\theta_1)-1}s^{-m_1-p_1\alpha_2\theta_2}ds
\end{aligned}
\tag{3.3.13}
$$

for $0<t<T$. Here we use $\alpha_1(s_1-\theta_1)-1>-1$ and $0<m_1+p_1\alpha_2\theta_2<1$, which follow from (3.3.4) and (3.3.3), respectively. Combining (3.3.2), (3.3.11) and (3.3.13), we have $u_n\in Y_{\theta_1,r_1,T}$. We can obtain $v_n\in Y_{\theta_2,r_2,T}$ in the same way. By induction we derive $(u_n,v_n)\in Y_{\theta_1,r_1,T}\times Y_{\theta_2,r_2,T}$ for $n\ge 1$.

We divide the possibilities into two cases: (i) $\alpha_1=\alpha_2$, or $\alpha_1<\alpha_2$ and $s_1<\mathcal{S}$, and (ii) $\alpha_1<\alpha_2$ and $s_1\ge\mathcal{S}$. In the case (ii) we also choose s_3 and put θ_3 as in Lemma 3.3.2. Moreover, let us introduce the Banach space defined by

$$
Y_{\theta_2,\theta_3,r_2,T}:=\left\{u\in L^\infty_{loc}((0,T],X_{\theta_2}(r_2)\cap X_{\theta_3}(r_2)):\|u\|_{Y_{\theta_2,\theta_3,r_2,T}}:=\max\left\{\|u\|_{Y_{\theta_2,r_2,T}},\|u\|_{Y_{\theta_3,r_2,T}}\right\}<\infty\right\}.
$$

In a similar way to (3.3.11), (3.3.12) and (3.3.13) we obtain

(3.3.14)

$$
\begin{aligned}
&t^{\alpha_2\theta_3}|v_n|_{X_{\theta_3}(r_2)}\\
&\le t^{\alpha_2\theta_3}|S_{\alpha_2}(t)v_0|_{X_{\theta_3}(r_2)}+t^{\alpha_2\theta_3}\int_0^t|P_{\alpha_2}(t-s)f_2(s,u_{n-1}(s))|_{X_{\theta_3}(r_2)}ds\\
&\lesssim |v_0|_{X_0(r_2)}+t^{\alpha_2\theta_3}\int_0^t(t-s)^{\alpha_2(s_2-\theta_3)-1}|f_2(s,u_{n-1}(s))|_{X_{s_2-1}(r_2)}ds\\
&\lesssim |v_0|_{X_0(r_2)}+t^{\alpha_2\theta_3}\int_0^t(t-s)^{\alpha_2(s_2-\theta_3)-1}s^{-m_2-p_2\alpha_2\theta_1}\cdot\|c_2\|_{L^{q_2}(\Omega)}\|1+|u_{n-1}|\|^{p_2}_{Y_{\theta_1,r_1,T}}ds\\
&\le |v_0|_{X_0(r_2)}+\|c_2\|_{L^{q_2}(\Omega)}\|1+|u_{n-1}|\|^{p_2}_{Y_{\theta_1,r_1,T}}\cdot t^{\alpha_2 s_2-m_2-p_2\alpha_2\theta_1}\\
&\qquad\times\int_0^1(1-s)^{\alpha_2(s_2-\theta_3)-1}s^{-m_2-p_2\alpha_2\theta_1}ds.
\end{aligned}
$$

Here we apply $s_2>\theta_2=\theta_2(s_1)>\theta_2(s_3)=\theta_3$, which follows from (3.3.7) and Lemma 3.3.2. Then $u_{n-1}\in Y_{\theta_1,r_1,T}$ leads not only to $v_n\in Y_{\theta_2,r_2,T}$ but also to $v_n\in Y_{\theta_3,r_2,T}$. Thus $v_n\in Y_{\theta_2,\theta_3,r_2,T}$ for $n\ge 1$.

We consider the rest of the existence part only in the case (i). In the case (ii) it can be proved by replacing $Y_{\theta_2,r_2,T}$ and Y_{θ_2,r_2,T_*} with $Y_{\theta_2,\theta_3,r_2,T}$ and $Y_{\theta_2,\theta_3,r_2,T_*}$, respectively.

In a similar way there exist $C_1>0$ and $C_2>0$ such that for $n\ge 2$,

(3.3.15)

$$
\begin{aligned}
&\|u_{n+1}-u_n\|_{Y_{\theta_1,r_1,T}}\\
&\qquad\le C_1T^{\alpha_1 s_1-m_1-p_1\alpha_2\theta_2+(\alpha_2-\alpha_1)\theta_1}\|c_1\|_{L^{q_1}(\Omega)}\|1+|v_n|+|v_{n-1}|\|^{p_1-1}_{Y_{\theta_2,r_2,T}}\|v_n-v_{n-1}\|_{Y_{\theta_2,r_2,T}},\\
&\|v_{n+1}-v_n\|_{Y_{\theta_2,r_2,T}}\le C_2T^{\alpha_2 s_2-m_2-p_2\alpha_2\theta_1}\|c_2\|_{L^{q_2}(\Omega)}\|1+|u_n|+|u_{n-1}|\|^{p_2-1}_{Y_{\theta_1,r_1,T}}\|u_n-u_{n-1}\|_{Y_{\theta_1,r_1,T}}.
\end{aligned}
$$

Put $U := 2\max\left\{\|u_2\|_{Y_{\theta_1,r_1,T}}, \|v_2\|_{Y_{\theta_2,r_2,T}}\right\}$ and choose a sufficiently small time $T_* > 0$ such that

$$\begin{aligned} C_1 {T_*}^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2-\alpha_1)\theta_1} \|c_1\|_{L^{q_1}(\Omega)}(1+2U)^{p_1-1} &\le \frac{1}{2}, \\ C_2 {T_*}^{\alpha_2 s_2 - m_2 - p_2\alpha_2\theta_1} \|c_2\|_{L^{q_2}(\Omega)}(1+2U)^{p_2-1} &\le \frac{1}{2}. \end{aligned} \tag{3.3.16}$$

Note that T_* is independent of n. By induction together with (3.3.15) and (3.3.16) we have

$$\begin{cases} \|u_n\|_{Y_{\theta_1,r_1,T_*}} \le U & \text{for } n \ge 1, \\ \|v_n\|_{Y_{\theta_2,r_2,T_*}} \le U & \text{for } n \ge 1, \\ \|u_{n+1} - u_n\|_{Y_{\theta_1,r_1,T_*}} \le \dfrac{1}{2}\|v_n - v_{n-1}\|_{Y_{\theta_2,r_2,T_*}} & \text{for } n \ge 2, \\ \|v_{n+1} - v_n\|_{Y_{\theta_2,r_2,T_*}} \le \dfrac{1}{2}\|u_n - u_{n-1}\|_{Y_{\theta_1,r_1,T_*}} & \text{for } n \ge 2. \end{cases} \tag{3.3.17}$$

By iteration in (3.3.17) the sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are Cauchy in the Banach spaces Y_{θ_1,r_1,T_*} and Y_{θ_2,r_2,T_*}, respectively. Consequently, there exist limits $u \in Y_{\theta_1,r_1,T_*}$ and $v \in Y_{\theta_2,r_2,T_*}$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{Y_{\theta_1,r_1,T_*}} = 0 \text{ and } \lim_{n\to\infty} \|v_n - v\|_{Y_{\theta_2,r_2,T_*}} = 0. \tag{3.3.18}$$

We show that the limits u and v have all the properties of Definition 3.1.1 on $[0, T_*)$. Property (a) immediately follows from $u \in Y_{\theta_1,r_1,T_*}$ and $v \in Y_{\theta_2,r_2,T_*}$. We obtain in the same way as we deduce (3.3.12) that

$$\begin{aligned} \int_0^{T_*} \|f_1(s,v(s))\|_{L^1(\Omega)} ds &\lesssim \int_0^{T_*} \|f_1(s,v(s))\|_{L^{z_1}(\Omega)} ds \\ &\lesssim \int_0^{T_*} s^{-m_1-p_1\alpha\theta_2} \cdot \|c_1\|_{L^{q_1}(\Omega)} \|1+|v|\|^{p_1}_{Y_{\theta_2,r_2,T_*}} ds \\ &= \frac{(T_*)^{1-m_1-p_1\alpha_2\theta_2}}{1-m_1-p_1\alpha_2\theta_2} \cdot \|c_1\|_{L^{q_1}(\Omega)} \|1+|v|\|^{p_1}_{Y_{\theta_2,r_2,T_*}}. \end{aligned}$$

Here we use $m_1 + p_1\alpha_2\theta_2 < 1$ by (3.3.3). Since the same is true for f_2, (u, v) satisfies the properties (b) and (c).

We mention the properties (d) and (e). We divide the possibilities into the same two cases as in the existence part.

In the case (i) it follows that

$$\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 > 0. \tag{3.3.19}$$

Indeed, when $\theta_2 = \theta_2(s_1)$, (3.3.19) follows from $\mathcal{P} < 1$ (resp. $s_1 < \mathcal{S}$) if $\alpha_1 = \alpha_2$ (resp. if $\alpha_1 < \alpha_2$). On the other hand, when $\theta_2 = \varepsilon$, i.e., $\theta_2(s_1) \le 0$, (3.3.19) follows from the choice of ε in the proof of Proposition 3.3.1. Then in the same way as (3.3.15) with θ_1 replaced by 0 we can evaluate

$$\begin{aligned} &\left\| \int_0^t P_{\alpha_1}(t-s)\left(f_1(s,v_n(s)) - f_1(s,v(s))\right) ds \right\|_{L^{r_1}(\Omega)} \\ &\le C_1 (T_*)^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2} \|c_1\|_{L^{q_1}(\Omega)} \|1+|v_n|+|v|\|^{p_1-1}_{Y_{\theta_2,r_2,T_*}} \|v_n - v\|_{Y_{\theta_2,r_2,T_*}}, \end{aligned} \tag{3.3.20}$$

which converges to zero as $n \to \infty$, by (3.3.18).

In the case (ii) it follows from Lemma 3.3.2 (ii) that $\theta_2 = \theta_2(s_1)$. Since $s_3 < \mathcal{S}$, we have

$$\alpha_1 s_3 - m_1 - p_1\alpha_2\theta_3 = \alpha_1 s_3 - m_1 - p_1\alpha_2\theta_2(s_3) > 0. \tag{3.3.21}$$

Then we obtain in the same way as (3.3.11), (3.3.12) and (3.3.13) with θ_1 and $(s_1, \theta_2, z_1, z_2) = (s_1, \theta_2(s_1), z_1(s_1), z_2(s_1))$ replaced by 0 and $(s_3, \theta_3, z_1(s_3), z_2(s_3))$, respectively that

$$\begin{aligned}
&|u_{n+1} - u_n|_{X_{\theta_1}(r_1)} \\
&\quad \le C_1(T_*)^{\alpha_1 s_3 - m_1 - p_1\alpha_2\theta_3} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_n| + |v_{n-1}|\|^{p_1-1}_{Y_{\theta_3,r_2,T_*}} \|v_n - v_{n-1}\|_{Y_{\theta_3,r_2,T_*}} \\
&\quad \le C_1(T_*)^{\alpha_1 s_3 - m_1 - p_1\alpha_2\theta_3} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_n| + |v_{n-1}|\|^{p_1-1}_{Y_{\theta_2,\theta_3,r_2,T_*}} \|v_n - v_{n-1}\|_{Y_{\theta_2,\theta_3,r_2,T_*}},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left\| \int_0^t P_{\alpha_1}(t-s)\,(f_1(s, v_n(s)) - f_1(s, v(s)))\,ds \right\|_{L^{r_1}(\Omega)} \\
&\qquad \le C_1(T_*)^{\alpha_1 s_3 - m_1 - p_1\alpha_2\theta_3} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_n| + |v|\|^{p_1-1}_{Y_{\theta_2,\theta_3,r_2,T_*}} \|v_n - v\|_{Y_{\theta_2,\theta_3,r_2,T_*}}.
\end{aligned} \tag{3.3.22}$$

Both (3.3.18) and (3.3.20) (or (3.3.22)) enable us to take the limit of (3.1.12) as $n \to \infty$ in $L^1(\Omega)$. Since we can take the limit of (3.1.13) in a similar way, we deduce the integral system in Definition 3.1.1 (d). For the last property (e), it suffices to prove (3.1.9). In the case (i) it follows from (3.3.13) with θ_1 and v_{n-1} replaced by 0 and v, respectively and (3.3.19) that

$$\begin{aligned}
&\|u(t) - S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)} \\
&\quad \lesssim \int_0^t (t-s)^{\alpha_1 s_1 - 1} |f_1(s, v(s))|_{X_{s_1-1}(r_1)} ds \\
&\quad \lesssim \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v|\|^{p_1}_{Y_{\theta_2,r_2,T_*}} \cdot t^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2} \int_0^1 (1-s)^{\alpha_1 s_1 - 1} s^{-m_1 - p_1\alpha_2\theta_2} ds \\
&\quad \to 0 \ \ (t \to 0).
\end{aligned} \tag{3.3.23}$$

In the case (ii) we observe from considering (3.3.13) and (3.3.19) that (3.3.23) also holds with s_1 and θ_2 replaced by s_3 and θ_3, respectively. Moreover, in both cases (i) and (ii) since we can obtain $\lim_{t\to 0} \|v(t) - S_{\alpha_2}(t)v_0\|_{L^{r_2}(\Omega)} = 0$ in a similar way, (3.1.9) holds. Thus (u, v) possesses the desired properties of Definition 3.1.1.

We consider the uniqueness of the mild solution only in the case (i). In the case (ii) it can be proved by replacing $Y_{\theta_2,r_2,T}$ with $Y_{\theta_2,\theta_3,r_2,T}$. We can derive the uniqueness from calculations similar to (3.3.15). Indeed, let $T \in (0, T_*]$ and let $(u, v), (\tilde{u}, \tilde{v}) \in Y_{\theta_1,r_1,T} \times Y_{\theta_2,r_2,T}$ be any two mild solutions of (3.1.1) with the same initial data (u_0, v_0). In a similar way to deduce (3.3.15) it follows that

(3.3.24)

$$\begin{aligned}
&\|u - \tilde{u}\|_{Y_{\theta_1,r_1,T}} \le C_1 T^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2 - \alpha_1)\theta_1} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v| + |\tilde{v}|\|^{p_1-1}_{Y_{\theta_2,r_2,T}} \|v - \tilde{v}\|_{Y_{\theta_2,r_2,T}}, \\
&\|v - \tilde{v}\|_{Y_{\theta_2,r_2,T}} \le C_2 T^{\alpha_2 s_2 - m_2 - p_2\alpha_2\theta_1} \|c_2\|_{L^{q_2}(\Omega)} \|1 + |u| + |\tilde{u}|\|^{p_2-1}_{Y_{\theta_1,r_1,T}} \|u - \tilde{u}\|_{Y_{\theta_1,r_1,T}}.
\end{aligned}$$

We see that there exists a sufficiently small time $\widehat{T} \in (0, T]$ such that

$$C_1 \widehat{T}^{\alpha_1 s_1 - m_1 - p_1 \alpha_2 \theta_2 + (\alpha_2 - \alpha_1)\theta_1} \|c_1\|_{L^{q_1}(\Omega)} \| 1 + |v| + |\tilde{v}| \|^{p_1 - 1}_{Y_{\theta_2, r_2, T}} \le \frac{1}{2},$$
$$C_2 \widehat{T}^{\alpha_2 s_2 - m_2 - p_2 \alpha_2 \theta_1} \|c_2\|_{L^{q_2}(\Omega)} \| 1 + |u| + |\tilde{u}| \|^{p_2 - 1}_{Y_{\theta_1, r_1, T}} \le \frac{1}{2}.$$

By this together with (3.3.24) we have $u(\cdot, t) \equiv \tilde{u}(\cdot, t)$ and $v(\cdot, t) \equiv \tilde{v}(\cdot, t)$ for $t \in [0, \widehat{T}]$. We obtain from a standard continuation argument (cf. [17]) that the uniqueness over the whole interval $[0, T_*]$ holds.

We prove the cases (b) and (c). Choose $(s_1, \theta_2, z_1, z_2)$ and $(s_2, \theta_1, z_3, z_4)$ as in Proposition 3.3.1. We divide the possibilities into two cases: (i) $\alpha_1 = \alpha_2$, or $\alpha_1 < \alpha_2$ and $s_1 \le \mathcal{S}$ and (ii) $\alpha_1 < \alpha_2$ and $s_1 > \mathcal{S}$.

We consider the existence part in the case (i). Set $\Pi_1 := \{u_0\} \subset L^{r_1}(\Omega)$ and $\Pi_2 := \{v_0\} \subset L^{r_2}(\Omega)$. Let $0 < T < \infty$. We can apply Lemma 3.2.2 and consider the constructed functions g and w_i $(i = 1, 2)$. For $i = 1, 2$, put

(3.3.25)
$$Y_{w_i, \theta_i, r_i, T} := \left\{ u \in L^\infty_{loc}((0, T], X_{\theta_i}(r_i)) : \|u\|_{Y_{w_i, \theta_i, r_i, T}} := \sup_{t \in (0, T)} (w_i(t))^{\alpha_2 \theta_i} |u(t)|_{X_{\theta_i}(r_i)} < \infty \right\}.$$

Lemma 3.2.2 (ii) implies that the spaces $Y_{w_i, \theta_i, r_i, T}$, $i = 1, 2$, are Banach spaces. Note that for $i = 1, 2$, $Y_{w_i, \theta_i, r_i, T} \subset Y_{\theta_i, r_i, T}$ follows if $0 < T < \infty$. Define the functions (u_n, v_n) in the same way as in the case (a). Assume that $(u_{n-1}, v_{n-1}) \in Y_{w_1, \theta_1, r_1, T} \times Y_{w_2, \theta_2, r_2, T}$. In the same way as (3.3.11) and (3.3.13) we have for $0 < t < T$,

$$\begin{aligned}
&(w_1(t))^{\alpha_2 \theta_1} |u_n|_{X_{\theta_1}(r_1)} \\
&\quad \le (w_1(t))^{\alpha_2 \theta_1} |S_{\alpha_1}(t) u_0|_{X_{\theta_1}(r_1)} + (w_1(t))^{\alpha_2 \theta_1} \int_0^t |P_{\alpha_1}(t - s) f_1(s, v_{n-1}(s))|_{X_{\theta_1}(r_1)} ds \\
&\quad \lesssim |u_0|_{X_0(r_1)} + (w_1(t))^{\alpha_2 \theta_1} \int_0^t (t - s)^{\alpha_1 (s_1 - \theta_1) - 1} |f_1(s, v_{n-1}(s))|_{X_{s_1 - 1}(r_1)} ds
\end{aligned}$$

and

(3.3.26)
$$\begin{aligned}
&(w_1(t))^{\alpha_2 \theta_1} \int_0^t (t - s)^{\alpha_1 (s_1 - \theta_1) - 1} |f_1(s, v_{n-1}(s))|_{X_{s_1 - 1}(r_1)} ds \\
&\quad \lesssim (w_1(t))^{\alpha_2 \theta_1} \int_0^t (t - s)^{\alpha_1 (s_1 - \theta_1) - 1} s^{-m_1} (w_2(s))^{-p_1 \alpha_2 \theta_2} \cdot \|c_1\|_{L^{q_1}(\Omega)} \| 1 + |v_{n-1}| \|^{p_1}_{Y_{w_2, \theta_2, r_2, T}} ds \\
&\quad \le \|c_1\|_{L^{q_1}(\Omega)} \| 1 + |v_{n-1}| \|^{p_1}_{Y_{w_2, \theta_2, r_2, T}} \cdot (g(t))^{p_1 - 1} \cdot t^{\alpha_2 \theta_1} \int_0^t (t - s)^{\alpha_1 (s_1 - \theta_1) - 1} s^{-m_1 - p_1 \alpha_2 \theta_2} ds \\
&\quad = \|c_1\|_{L^{q_1}(\Omega)} \| 1 + |v_{n-1}| \|^{p_1}_{Y_{w_2, \theta_2, r_2, T}} \\
&\qquad\qquad \cdot (g(t))^{p_1 - 1} \cdot t^{\alpha_1 s_1 - m_1 - p_1 \alpha_2 \theta_2 + (\alpha_2 - \alpha_1)\theta_1} \int_0^1 (1 - s)^{\alpha_1 (s_1 - \theta_1) - 1} s^{-m_1 - p_1 \alpha_2 \theta_2} ds.
\end{aligned}$$

We recall that $\lim_{t \to 0} g(t) = 0$ and $\alpha_1 s_1 - m_1 - p_1 \alpha_2 \theta_2 + (\alpha_2 - \alpha_1)\theta_1 \ge 0$. Hence, $u_n \in Y_{w_1, \theta_1, r_1, T}$. We can obtain $v_n \in Y_{w_2, \theta_2, r_2, T}$ in the same way. By induction we derive $(u_n, v_n) \in Y_{w_1, \theta_1, r_1, T} \times$

$Y_{w_2,\theta_2,r_2,T}$ for $n \geq 1$. In a similar way there exist $C_3 > 0$ and $C_4 > 0$ such that for $n \geq 2$,

$$
\begin{aligned}
\|u_{n+1} - u_n\|_{Y_{w_1,\theta_1,r_1,T}} &\leq C_3(g(T))^{p_1-1} T^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2-\alpha_1)\theta_1} \\
&\qquad \cdot \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_n| + |v_{n-1}|\|^{p_1-1}_{Y_{w_2,\theta_2,r_2,T}} \|v_n - v_{n-1}\|_{Y_{w_2,\theta_2,r_2,T}}, \\
\|v_{n+1} - v_n\|_{Y_{w_2,\theta_2,r_2,T}} &\leq C_4(g(T))^{p_2-1} T^{\alpha_2 s_2 - m_2 - p_2\alpha_2\theta_1} \\
&\qquad \cdot \|c_2\|_{L^{q_2}(\Omega)} \|1 + |u_n| + |u_{n-1}|\|^{p_2-1}_{Y_{w_1,\theta_1,r_1,T}} \|u_n - u_{n-1}\|_{Y_{w_1,\theta_1,r_1,T}}.
\end{aligned}
$$

Put $V := 2\max\left\{\|u_2\|_{Y_{w_1,\theta_1,r_1,T}}, \|v_2\|_{Y_{w_2,\theta_2,r_2,T}}\right\}$ and choose a sufficiently small time $T_{**} > 0$ such that

$$
\begin{aligned}
C_3(g(T_{**}))^{p_1-1} {T_{**}}^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2-\alpha_1)\theta_1} \|c_1\|_{L^{q_1}(\Omega)} (1+2V)^{p_1-1} &\leq \frac{1}{2}, \\
C_4(g(T_{**}))^{p_2-1} {T_{**}}^{\alpha_2 s_2 - m_2 - p_2\alpha_2\theta_1} \|c_2\|_{L^{q_2}(\Omega)} (1+2V)^{p_2-1} &\leq \frac{1}{2}.
\end{aligned}
$$

We deduce the analogue of (3.3.17) in $Y_{w_1,\theta_1,r_1,T_{**}} \times Y_{w_2,\theta_2,r_2,T_{**}}$ instead of $Y_{\theta_1,r_1,T_*} \times Y_{\theta_2,r_2,T_*}$. Therefore, there exist limits $u \in Y_{w_1,\theta_1,r_1,T_{**}}$ and $v \in Y_{w_2,\theta_2,r_2,T_{**}}$ such that

$$
\lim_{n\to\infty} \|u_n - u\|_{Y_{w_1,\theta_1,r_1,T_{**}}} = 0 \text{ and } \lim_{n\to\infty} \|v_n - v\|_{Y_{w_2,\theta_2,r_2,T_{**}}} = 0.
$$

We consider the case (ii). Let $s_3 = \mathcal{S}$ and $\theta_3 = \theta_2(s_3)$, where $\theta_2(s)$ is defined in the proof of Proposition 3.3.1. It follows from $\mathcal{P} < 1$, or $\mathcal{P} = 1$ and $m_1 > 0$ that $s_3 > 0$ and $\theta_3 \geq 0$ hold. By this together with $\mathcal{S} < s_1 < 1$ and $\theta(s_1) < 1$ we have $s_3 \in (0,1)$ and $\theta_3 \in [0,1)$. Let $0 < T < \infty$. In a similar way to Lemma 3.2.2 we can construct a continuous and nondecreasing function $g : (0,T) \to (0,\infty)$ such that

$$
\begin{aligned}
|S_{\alpha_1}(t)u_0|_{X_{\theta_1}(r_1)} \lesssim g(t)\cdot t^{-\alpha_2\theta_1}|u_0|_{X_0(r_1)},\ & |S_{\alpha_2}(t)v_0|_{X_{\theta_2}(r_2)} \lesssim g(t)\cdot t^{-\alpha_2\theta_2}|v_0|_{X_0(r_2)} \text{ and} \\
& |S_{\alpha_3}(t)v_0|_{X_{\theta_3}(r_2)} \lesssim g(t)\cdot t^{-\alpha_2\theta_3}|v_0|_{X_0(r_2)} \text{ for } 0 < t < T,
\end{aligned}
$$

$\lim_{t\to 0} g(t) = 0$, and $\lim_{t\to 0} w_i(t) = 0$ for $i = 1,2,3$, where $(w_i(t))^{-\alpha_2\theta_i} = g(t)\cdot t^{-\alpha_2\theta_i}$. Put $Y_{w_1,\theta_1,r_1,T}$, $Y_{w_2,\theta_2,r_2,T}$ and $Y_{w_3,\theta_3,r_2,T}$ in the same way as (3.3.25). Then it follows that $(u_n, v_n) \in Y_{w_1,\theta_1,r_1,T} \times Y_{w_2,\theta_2,r_2,T}$ for $n \geq 1$. Let us introduce the Banach space defined by

$$
\begin{aligned}
& Y_{w_2,\theta_2,w_3,\theta_3,r_2,T} \\
& := \left\{u \in L^\infty_{loc}((0,T], X_{\theta_2}(r_2) \cap X_{\theta_3}(r_2)) : \|u\|_{Y_{w_2,\theta_2,w_3,\theta_3,r_2,T}} := \max\left\{\|u\|_{Y_{w_2,\theta_2,r_2,T}}, \|u\|_{Y_{w_3,\theta_3,r_2,T}}\right\} < \infty\right\}.
\end{aligned}
$$

We observe from a similar manner to (3.3.14) that $u_{n-1} \in Y_{w_1,\theta_1,r_1,T}$ leads not only to $v_n \in Y_{w_2,\theta_2,r_2,T}$ but also to $v_n \in Y_{w_3,\theta_3,r_2,T}$. Thus $v_n \in Y_{w_2,\theta_2,w_3,\theta_3,r_2,T}$ for $n \geq 1$. We omit the rest of the existence part, since it can be proved by replacing $Y_{w_2,\theta_2,r_2,T}$ and Y_{w_2,θ_2,r_2,T_*} with $Y_{w_2,\theta_2,w_3,\theta_3,r_2,T}$ and $Y_{w_2,\theta_2,w_3,\theta_3,r_2,T_*}$, respectively.

We can obtain in a similar way to the case (a) that u and v have the properties (a)-(d) of Definition 3.1.1 on $[0, T_{**})$. In the case (i) it follows that $\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 \geq 0$. Then we deduce from (3.3.26) by replacing θ_1 and v_{n-1} with 0 and v, respectively that

(3.3.27)
$$\begin{aligned}
&\|u(\cdot,t)-S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)} \\
&\quad\lesssim \int_0^t (t-s)^{\alpha_1 s_1-1}|f_1(s,v(s))|_{X_{s_1-1}(r_1)}ds \\
&\quad\lesssim \|c_1\|_{L^{q_1}(\Omega)}\|1+|v|\|^{p_1}_{Y_{w_2,\theta_2,r_2,T_{**}}}\cdot(g(t))^{p_1-1}\cdot t^{\alpha_1 s_1-m_1-p_1\alpha_2\theta_2}\int_0^1(1-s)^{\alpha_1 s_1-1}s^{-m_1-p_1\alpha_2\theta_2}ds \\
&\quad\to 0 \quad (t\to 0).
\end{aligned}$$

In the same way it follows that $\lim_{t\to 0}\|v(t)-S_{\alpha_2}(t)v_0\|_{L^{r_2}(\Omega)}=0$. In the case (ii) we observe from considering (3.3.26) that (3.3.27) also holds with s_1, θ_2 and $Y_{w_2,\theta_2,r_2,T_{**}}$ replaced by s_3, θ_3 and $Y_{w_3,\theta_3,r_2,T_{**}}$, respectively. Thus (u,v) is a mild solution of (3.1.1) in the sense of Definition 3.1.1 on $[0,T_{**})$.

Note that in the case of (ii) how to derive the limit corresponding to (3.3.27) differs between when $\mathcal{R}<1$ and when $\mathcal{R}=1$. When $\mathcal{R}<1$, by considering (3.3.11) and (3.3.12) we can evaluate

$$\begin{aligned}
\|u(\cdot,t)-S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)} &\lesssim \int_0^t(t-s)^{\alpha_1 s_3-1}|f_1(s,v(s))|_{X_{s_3-1}(r_1)}ds \\
&\lesssim \int_0^t(t-s)^{\alpha_1 s_3-1}\|f_1(s,v(s))\|_{L^{z_1}(\Omega)}ds,
\end{aligned}$$

where $z_1=z_1(s_3)\in(1,\infty)$. On the other hand, when $\mathcal{R}=1$, since $z_1=z_1(s_3)=1$, the inequality $|f_1(s,v(s))|_{X_{s_3-1}(r_1)}\lesssim\|f_1(s,v(s))\|_{L^{z_1}(\Omega)}$ does not hold. Then we use the L^p-L^q estimate of the heat semigroup ([16, Proposition 48.4 (c)-(e)]) and deduce that

$$\begin{aligned}
&\|P_{\alpha_1}(t-s)f_1(s,v(s))\|_{L^{r_1}(\Omega)} \\
&\quad\le \alpha_1(t-s)^{\alpha_1-1}\int_0^\infty \tau\Phi_{\alpha_1}(\tau)\|S(\tau(t-s)^{\alpha_1})f_1(s,v(s))\|_{L^{r_1}(\Omega)}d\tau \\
&\quad\lesssim \alpha_1(t-s)^{\alpha_1-1}\int_0^\infty \tau\Phi_{\alpha_1}(\tau)\cdot\tau^{-\frac{N}{2}\left(1-\frac{1}{r_1}\right)}(t-s)^{-\frac{N}{2}\alpha_1\left(1-\frac{1}{r_1}\right)}\|f_1(s,v(s))\|_{L^1(\Omega)}d\tau \\
&\quad= \alpha_1(t-s)^{\alpha_1 s_3-1}\|f_1(s,v(s))\|_{L^1(\Omega)}\int_0^\infty\tau^{s_3}\Phi_{\alpha_1}(\tau)d\tau \\
&\quad= \frac{\alpha_1\Gamma(s_3+1)}{\Gamma(\alpha_1 s_3+1)}(t-s)^{\alpha_1 s_3-1}\|f_1(s,v(s))\|_{L^1(\Omega)}.
\end{aligned}$$

Here we apply (3.1.5) and $\mathcal{S}=1-\dfrac{N}{2}\left(1-\dfrac{1}{r_1}\right)$, which follows from $\mathcal{R}=1$. Thus we have

$$\|u(\cdot,t)-S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)}\lesssim\int_0^t(t-s)^{\alpha_1 s_3-1}\|f_1(s,v(s))\|_{L^1(\Omega)}ds.$$

We also obtain the uniqueness of the mild solution in the space $Y_{w_1,\theta_1,r_1,T_{**}}\times Y_{w_2,\theta_2,r_2,T_{**}}$ through an argument similar to (3.3.24). We omit the details. □

3.4 Nonexistence result

Proof of Theorem 3.1.3. Without loss of generality, we suppose that $0 \in \Omega$. Choose $\rho > 0$ such that $B(0,\rho) \subset \Omega$, where $B(0,\rho)$ denotes the ball of radius $\rho > 0$ centered at 0.

We prove the case where $\max\{\mathcal{P},\mathcal{R}\} > 1$. Let $0 < l < \dfrac{N}{q_1}$ if $1 \le q_1 < \infty$, $l = 0$ if $q_1 = \infty$ and $0 < k < \dfrac{N}{r_2}$. Note that k is chosen to be sufficiently large and the same is valid for l when $1 \le q_1 < \infty$. Put

$$c_1(x) := |x|^{-l},\ c_2(x) := 1,\ u_0(x) := 0 \text{ and } v_0(x) := |x|^{-k}\chi_{B(0,\rho)}(x),$$

where $\chi_{B(0,\rho)}$ is a characteristic function. Then $(c_1, c_2, u_0, v_0) \in L^{q_1}(\Omega)\times L^{q_2}(\Omega)\times L^{r_1}(\Omega)\times L^{r_2}(\Omega)$ holds.

The proof is by contradiction. Assume that there exists $T > 0$ such that the problem (3.1.1) with $f_1(x,t,v) = c_1(x)\cdot t^{-m_1}v^{p_1}$ and $f_2(x,t,u) = c_2(x)\cdot t^{-m_2}u^{p_2}$ possesses a local in time nonnegative mild solution (u,v) in the sense of Definition 3.1.1 on $[0,T)$. Let $1 < \tau < 2$ and let $s > 0$ be sufficiently small. Let $|x| \le \dfrac{\sqrt{s^{\tilde{\alpha}}}}{2}$, where $\tilde{\alpha} = \alpha_1$ or $\tilde{\alpha} = \alpha_2$. Then we can apply Proposition 3.2.3 and obtain

$$\begin{aligned}
\left(S(\tau s^{\alpha_2})v_0\right)(x) &= \int_{\{|y|<\rho\}} G(x,y,\tau s^{\alpha_2})|y|^{-k}dy \ge \int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} G(x,y,\tau s^{\alpha_2})|y|^{-k}dy \\
&\ge c_1 \int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} (\tau s^{\alpha_2})^{-\frac{N}{2}}|y|^{-k}dy \gtrsim s^{-\frac{N}{2}\alpha_2}\int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} |y|^{-k}dy \\
&\ge s^{-\frac{N}{2}\alpha_2}\left(\sqrt{\tau s^{\alpha_2}} + \frac{\sqrt{s^{\tilde{\alpha}}}}{2}\right)^{-k}\int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} dy \gtrsim s^{-\frac{k}{2}\tilde{\alpha}}.
\end{aligned}$$

Due to (3.1.4) and (3.1.7), we have

$$\left(S_{\alpha_2}(s)v_0\right)(x) \ge \int_1^2 \Phi_{\alpha_2}(\tau)\left(S(\tau s^{\alpha_2})v_0\right)(x)d\tau \gtrsim \int_1^2 \Phi_{\alpha_2}(\tau)d\tau\cdot s^{-\frac{k}{2}\tilde{\alpha}}$$

and hence

$$S_{\alpha_2}(s)v_0 \gtrsim s^{-\frac{k}{2}\tilde{\alpha}}\chi_{B\left(0,\frac{\sqrt{s^{\tilde{\alpha}}}}{2}\right)} \tag{3.4.1}$$

for sufficiently small $s > 0$.

Let $t > 0$ be sufficiently small and let $\dfrac{t}{3} \le s \le \dfrac{t}{2}$, $|x| < \dfrac{\sqrt{s^{\alpha_1}}}{2}$ and $1 < \tau < 2$. Then for $|y| < \dfrac{\sqrt{s^{\tilde{\alpha}}}}{2}$,

$$|x-y| < \sqrt{s^{\alpha_1}} \le \sqrt{(t-s)^{\alpha_1}} < \sqrt{\tau(t-s)^{\alpha_1}}.$$

Using Proposition 3.2.3, we obtain

$$G(x,y,\tau(t-s)^{\alpha_1}) \gtrsim \tau^{-\frac{N}{2}}(t-s)^{-\frac{N}{2}\alpha_1} \gtrsim s^{-\frac{N}{2}\alpha_1} \tag{3.4.2}$$

for $|y| < \dfrac{\sqrt{s^{\tilde{\alpha}}}}{2}$. It follows from (3.4.1) and (3.4.2) that

$$\begin{aligned}
\left(S(\tau(t-s)^{\alpha_1})f_1(s, S_{\alpha_2}(s)v_0)\right)(x) &= \int_\Omega G(x,y,\tau(t-s)^{\alpha_1})|y|^{-l}s^{-m_1}\left((S_{\alpha_2}(s)v_0)(y)\right)^{p_1}dy \\
&\gtrsim s^{-m_1-\frac{k}{2}p_1\tilde{\alpha}} \int_{\left\{|y|<\frac{\sqrt{s^{\tilde{\alpha}}}}{2}\right\}} G(x,y,\tau(t-s)^{\alpha_1})|y|^{-l}dy \\
&\gtrsim s^{-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1} \int_{\left\{|y|<\frac{\sqrt{s^{\tilde{\alpha}}}}{2}\right\}} dy \\
&\gtrsim s^{-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}},
\end{aligned}$$

which yields

$$\begin{aligned}
\left(P_{\alpha_1}(t-s)f_1(s, S_{\alpha_2}(s)v_0)\right)(x) &\geq \alpha_1(t-s)^{\alpha_1-1}\int_1^2 \tau\Phi_{\alpha_1}(\tau)\left(S(\tau(t-s)^{\alpha_1})f_1(s,S_{\alpha_2}(s)v_0)\right)(x)d\tau \\
&\gtrsim (t-s)^{\alpha_1-1}s^{-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}} \\
&\gtrsim (t-s)^{\alpha_1-1}t^{-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}}.
\end{aligned}$$

Here we use $s \leq \dfrac{t}{2}$. By direct calculation we have

$$\begin{aligned}
\int_0^t \left(P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0)\right)(x)ds &\geq \int_{\frac{t}{3}}^{\frac{t}{2}}\left(P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0)\right)(x)ds \\
&\gtrsim t^{\alpha_1-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
\left\|\int_0^t P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0)ds\right\|_{L^{r_1}(\Omega)}^{r_1} &\gtrsim \int_{\left\{|x|<\frac{\sqrt{s^{\alpha_1}}}{2}\right\}} t^{r_1\left(\alpha_1-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}\right)}dx \\
&\gtrsim t^{\frac{N}{2}\alpha_1+r_1\left(\alpha_1-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}\right)}.
\end{aligned}$$

Taking the limit as $l \to \dfrac{N}{q_1}$ (if $1 \leq q_1 < \infty$) and $k \to \dfrac{N}{r_2}$, we deduce that

$$\frac{N}{2}\alpha_1 + r_1\left(\alpha_1 - \frac{l}{2}\tilde{\alpha} - m_1 - \frac{k}{2}p_1\tilde{\alpha} - \frac{N}{2}\alpha_1 + \frac{N}{2}\tilde{\alpha}\right) \to \begin{cases} \alpha_1 r_1(1-\mathcal{P}) & \text{when } \tilde{\alpha} = \alpha_1, \\ \alpha_2 r_1(1-\mathcal{R}) & \text{when } \tilde{\alpha} = \alpha_2. \end{cases}$$

Thus we obtain $\left\|\int_0^t P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0)ds\right\|_{L^{r_1}(\Omega)}^{r_1} \to \infty$ as $t \to 0$. By Definition 3.1.1 (d) we have $v(t) \geq S_{\alpha_2}(t)v_0$ in $\Omega \times (0,T)$, which yields

$$\left\|\int_0^t P_{\alpha_1}(t-s)f_1(s,v(s))ds\right\|_{L^{r_1}(\Omega)} \geq \left\|\int_0^t P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0)ds\right\|_{L^{r_1}(\Omega)} \to \infty \quad (t \to 0).$$

This contradicts the property of Definition 3.1.1 (e).

We prove the case where $\max\{\mathcal{Q}_1, \mathcal{Q}_2\} > 1$. Let $0 < \tilde{l} < \dfrac{N}{q_2}$ if $1 \leq q_2 < \infty$, $\tilde{l} = 0$ if $q_2 = \infty$

and $0 < \tilde{k} < \dfrac{N}{r_1}$. Note that $\tilde{k}$ is chosen to be sufficiently large and the same is valid for $\tilde{l}$ when $1 \le q_2 < \infty$. Put

$$c_1(x) := 1,\ c_2(x) := |x|^{-\tilde{l}},\ u_0(x) := |x|^{-\tilde{k}}\chi_{B(0,\rho)}(x) \text{ and } v_0(x) := 0.$$

Then $(c_1, c_2, u_0, v_0) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega) \times L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ holds.

The proof is also by contradiction. As in the case where $\max\{\mathcal{P}, \mathcal{R}\} > 1$, we assume that there exists a local in time nonnegative mild solution (u, v) in the sense of Definition 3.1.1. We obtain in the same way as we deduce (3.4.1) that

$$S_{\alpha_1}(s)u_0 \gtrsim s^{-\frac{\tilde{k}}{2}\alpha_1}\chi_{B\left(0,\frac{\sqrt{s^{\alpha_1}}}{2}\right)} \tag{3.4.3}$$

for sufficiently small $s > 0$.

We prove the case where $\mathcal{Q}_1 > 1$. Let $t > 0$ be sufficiently small and let $\dfrac{t}{3} \le s \le \dfrac{t}{2}$, $|x| < \dfrac{\sqrt{s^{\alpha_1}}}{3}$ and $1 < \tau < 2$. It follows from Proposition 3.2.3 and (3.4.3) that

$$\begin{aligned}
\left(S(\tau(t-s)^{\alpha_2})f_2(s, S_{\alpha_1}(s)u_0)\right)(x) &= \int_\Omega G(x, y, \tau(t-s)^{\alpha_2})|y|^{-\tilde{l}}s^{-m_2}\left((S_{\alpha_1}(s)u_0)(y)\right)^{p_2} dy \\
&\gtrsim s^{-\frac{N}{2}\alpha_2}\int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} |y|^{-\tilde{l}}s^{-m_2}\left((S_{\alpha_1}(s)u_0)(y)\right)^{p_2} dy \\
&\gtrsim s^{-\frac{N}{2}\alpha_2-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1}\int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} dy \\
&\gtrsim s^{-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1}.
\end{aligned}$$

Note that we apply $|y| \le |y-x| + |x| < \sqrt{\tau s^{\alpha_2}} + \dfrac{\sqrt{s^{\alpha_1}}}{3} \le \dfrac{\sqrt{s^{\alpha_1}}}{2}$, since s is sufficiently small. Then by $s \le \dfrac{t}{2}$ we have

$$\begin{aligned}
\left(P_{\alpha_2}(t-s)f_2(s, S_{\alpha_1}(s)u_0)\right)(x) &\ge \alpha_2(t-s)^{\alpha_2-1}\int_1^2 \tau\Phi_{\alpha_2}(\tau)\left(S(\tau(t-s)^{\alpha_2})f_1(s, S_{\alpha_1}(s)u_0)\right)(x)d\tau \\
&\gtrsim (t-s)^{\alpha_2-1}s^{-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1} \\
&\gtrsim (t-s)^{\alpha_2-1}t^{-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1}.
\end{aligned}$$

By direct calculation we get

$$\int_0^t \left(P_{\alpha_2}(t-s)f_2(s, S_{\alpha_1}(s)u_0)\right)(x)ds \ge \int_{\frac{t}{3}}^{\frac{t}{2}} \left(P_{\alpha_2}(t-s)f_2(s, S_{\alpha_1}(s)u_0)\right)(x)ds \gtrsim t^{\alpha_2-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1}$$

and

$$\begin{aligned}
\left\|\int_0^t P_{\alpha_2}(t-s)f_2(s, S_{\alpha_1}(s)u_0)ds\right\|_{L^{r_2}(\Omega)}^{r_2} &\gtrsim \int_{\left\{|x|<\frac{\sqrt{s^{\alpha_1}}}{3}\right\}} t^{r_2\left(\alpha_2-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1\right)} dx \\
&\gtrsim t^{\frac{N}{2}\alpha_1+r_2\left(\alpha_2-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1\right)}.
\end{aligned}$$

Taking the limit as $\tilde{l} \to \dfrac{N}{q_2}$ (if $1 \le q_2 < \infty$) and $\tilde{k} \to \dfrac{N}{r_1}$, we deduce that

$$\frac{N}{2}\alpha_1 + r_2\left(\alpha_2 - \frac{\tilde{l}}{2}\alpha_1 - m_2 - \frac{\tilde{k}}{2}p_2\alpha_1\right) \to \alpha_2 r_2(1-\mathcal{Q}_1) < 0.$$

Thus we obtain $\left\|\int_0^t P_{\alpha_2}(t-s)f_2(s,S_{\alpha_1}(s)u_0)ds\right\|_{L^{r_2}(\Omega)}^{r_2} \to \infty$ as $t \to 0$. This contradicts the property of Definition 3.1.1 (e) in a similar way to the case where $\max\{\mathcal{P},\mathcal{R}\} > 1$.

We prove the case where $\mathcal{Q}_2 > 1$. Let $t > 0$ be sufficiently small and let $\dfrac{t}{3} \le s \le \dfrac{t}{2}$, $|x| < \dfrac{\sqrt{s^{\alpha_2}}}{2}$ and $1 < \tau < 2$. Then for $|y| < \dfrac{\sqrt{s^{\alpha_2}}}{2}$,

$$|x-y| < \sqrt{s^{\alpha_2}} \le \sqrt{(t-s)^{\alpha_2}} < \sqrt{\tau(t-s)^{\alpha_2}}.$$

Due to Proposition 3.2.3, we have

$$G(x,y,\tau(t-s)^{\alpha_2}) \gtrsim \tau^{-\frac{N}{2}}(t-s)^{-\frac{N}{2}\alpha_2} \gtrsim s^{-\frac{N}{2}\alpha_2} \tag{3.4.4}$$

for $|y| < \dfrac{\sqrt{s^{\alpha_2}}}{2}$. Considering in the same way as in the case where $\max\{\mathcal{P},\mathcal{R}\} > 1$, we obtain from (3.4.3) and (3.4.4) that

$$\left(S(\tau(t-s)^{\alpha_2})f_2(s,S_{\alpha_1}(s)u_0)\right)(x) \gtrsim s^{-\frac{\tilde{l}}{2}\alpha_2 - m_2 - \frac{\tilde{k}}{2}p_2\alpha_1},$$

which yields

$$\left(P_{\alpha_2}(t-s)f_2(s,S_{\alpha_1}(s)u_0)\right)(x) \gtrsim (t-s)^{\alpha_2-1}t^{-\frac{\tilde{l}}{2}\alpha_2 - m_2 - \frac{\tilde{k}}{2}p_2\alpha_1}.$$

Moreover, this leads to

$$\left\|\int_0^t P_{\alpha_2}(t-s)f_2(s,S_{\alpha_1}(s)u_0)ds\right\|_{L^{r_2}(\Omega)}^{r_2} \gtrsim t^{\frac{N}{2}\alpha_2 + r_2\left(\alpha_2 - \frac{\tilde{l}}{2}\alpha_2 - m_2 - \frac{\tilde{k}}{2}p_2\alpha_1\right)}.$$

Taking the limit as $\tilde{l} \to \dfrac{N}{q_2}$ (if $1 \le q_2 < \infty$) and $\tilde{k} \to \dfrac{N}{r_1}$, we deduce that

$$\frac{N}{2}\alpha_2 + r_2\left(\alpha_2 - \frac{\tilde{l}}{2}\alpha_2 - m_2 - \frac{\tilde{k}}{2}p_2\alpha_1\right) \to \alpha_2 r_2(1-\mathcal{Q}_2) < 0.$$

Thus we obtain $\left\|\int_0^t P_{\alpha_2}(t-s)f_2(s,S_{\alpha_1}(s)u_0)ds\right\|_{L^{r_2}(\Omega)}^{r_2} \to \infty$ as $t \to 0$. This contradicts the property of Definition 3.1.1 (e). Therefore, the proof is complete. □

3.5 Nonexistence result for scalar problems

In this section we apply our study to the nonexistence of a local in time solution of the scalar problem

$$
\begin{cases}
\partial_t^\alpha u = \Delta u + f(x,t,u) & \text{in } \Omega \times (0,T), \\
u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\
u(x,0) = u_0(x) & \text{in } \Omega.
\end{cases} \tag{3.5.1}
$$

We obtain the following nonexistence theorem.

Theorem 3.5.1. *Let $N \geq 1$, $0 < \alpha < 1$, $0 < p < \infty$, $q_1 \in [1,\infty]$, $q_2 \in \left(\frac{1}{\alpha}, \infty\right]$ and $1 \leq r < \infty$. Suppose that* (3.1.11) *holds with $\beta_A = \frac{N}{2}$. Then there exist nonnegative functions $c(x,t) \in L_{q_1,q_2}$ and $u_0 \in L^r(\Omega)$ such that, for every $T > 0$, the problem* (3.5.1) *with $f(x,t,u) = c(x,t) \cdot u^p$ admits no local in time nonnegative mild solution u in the sense of Definition 3.1.1 (more precisely, in the sense of [9, Definition 3.1.1]) on the interval* $[0,T)$.

Since we can prove in the same way as in the proof of Theorem 3.1.3, we leave the proof to readers. Note that we only solve the nonexistence conjecture in [9] when the nonlinear term f is separable with respect to x, t and u.

Remark that if $1 \leq r < \frac{N}{2}(p-1)$, then there exists a nonnegative initial function $u_0 \in L^r(\Omega)$ such that, the problem (3.5.1) with $f(x,t,u) = u^p$ has no local in time nonnegative mild solution on any time interval. Hence, our nonexistence result corresponds to [20]. In conclusion, for scalar problems and systems with pure power nonlinear terms, the existence/nonexistence results correspond to [20] and [15], respectively.

3.6 Discussion

In this chapter we consider a local in time solution of a time fractional weakly coupled reaction-diffusion system in two components with possibly distinct fractional orders. In Theorems 3.1.2 and 3.1.3 we derive the integrability conditions on the initial state functions for the local in time existence and nonexistence results. The parameters $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{R}$ describe the balance between the factors: the growth rates (resp. the singularities) of the nonlinear terms with respect to u or v (resp. x and t), the singularities of the initial data, and the fractional exponents. For instance, the larger the growth rates p_1 and p_2 become, the larger these five parameters become. Then Theorems 3.1.2 and 3.1.3 imply that the existence result is less likely to hold, and that the nonexistence result is more likely to hold. The integrability is determined by $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\}$ and $\max\{\mathcal{P}, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{R}\}$ in the existence and nonexistence part, respectively.

When $\alpha_1 = \alpha_2$, the equalities $\mathcal{P} = \mathcal{R}$ and $\mathcal{Q} = \mathcal{Q}_1 = \mathcal{Q}_2$ hold. Therefore, as seen in Corollary 3.1.4, we can explicitly determine the existence/nonexistence of a solution. The threshold integrability condition on initial data, which is a pair (r_1, r_2), is defined by $\max\{\mathcal{P}, \mathcal{Q}\} = 1$. When $m_1 > 0$ and $m_2 > 0$, the larger α_1 and α_2 become, the wider the space of initial data for the existence result becomes. On the other hand, when $\alpha_1 < \alpha_2$, the inequalities $\max\{\mathcal{Q}_1, \mathcal{Q}_2\} \leq 1 < \mathcal{Q}$ and $\max\{\mathcal{P}, \mathcal{R}\} \leq 1$ can occur. In this case since Theorems 3.1.2 and 3.1.3 cannot be applied, we cannot determine whether the problem (3.1.1) possesses a local in time solution or not. We mention the following points:

1. In Theorem 3.1.3 even if we assume $\mathcal{Q} > 1$, we cannot obtain the nonexistence result, since the inequality (3.4.3) does not hold with α_1 replaced by α_2 on the right hand side. If this is true, we can evaluate $\left(S(\tau(t-s)^{\alpha_2})f_2(s, S_{\alpha_1}(s)u_0)\right)(x)$ in the same way as in the case where $\max\{\mathcal{P}, \mathcal{R}\} > 1$. Hence, we can get the nonexistence result even if $\mathcal{Q} > 1$.

2. If we assume $\max\{\mathcal{Q}_1, \mathcal{Q}_2\} < 1$ instead of $\mathcal{Q} < 1$, then Proposition 3.3.1 does not hold. In particular, the inequality $\max\{\mathcal{Q}_1, \mathcal{Q}_2\} < 1$ does not lead to (3.3.6) and (3.3.9) with $(s, \tilde{s}) = \left(\min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_1} - \frac{1}{r_1}\right)\right\}, \max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_2}\right), \frac{m_2}{\alpha_2}\right\}\right)$. Thus we cannot obtain the existence result under this assumption.

The author conjectures that the nonexistence result does not hold in the above case, since the former point is a greater reason. In the existence result it seems that the solvability of the problem (3.1.1) in the above case may hold by using a functional space different from the Banach spaces introduced in the proof of Theorem 3.1.2.

Possible future problems ensuing from the current analysis are as follows:

1. What are the consequences of the problem (3.1.1) with a different boundary condition or situation, e.g. the Neumann boundary condition, the boundary condition where u and v are non-zero positive bounded functions, and the situation where the boundary is broken into parts with a condition of a different type set on each?

2. What happens to a local in time solution of (3.1.1) as $\alpha \to 1^-$? Given that (3.1.3) is the limit of (3.1.1) as $\alpha \to 1^-$, is it possible to show that the estimates obtained by this chapter approach those known for (3.1.3)? If not, which is more conservative and why?

3. For the problem (3.1.1), what is the solvability when one equation has an integer in time derivative and the other has a fractional in time derivative?

References

[1] E. Alvarez, C. G. Gal, V. Keyantuo and M. Warma, *Well-posedness results for a class of semi-linear super-diffusive equations*, Nonlinear Anal. **181** (2019), 24–61.

[2] E. Bazhlekova, *Fractional evolution equations in Banach spaces*, Ph.D. Thesis, Eindhoven University of Technology, 2001.

[3] M. D'Abbicco, M. R. Ebert and T. Picon, *Global Existence of Small Data Solutions to the Semilinear Fractional Wave Equation*, in: New trends in analysis and interdisciplinary applications, in: Trends Math. Res. Perspect., Birkhäuser/Springer, Cham (2017), 465–471.

[4] F. Dickstein and M. Loayza, *Life span of solutions of a strongly coupled parabolic system*, Mat. Contemp. **32** (2007), 85–106.

[5] F. Dickstein and M. Loayza, *Life span of solutions of a weakly coupled parabolic system*, Z. Angew. Math. Phys. **59** (2008), 1–23.

[6] L. C. F. Ferreira and E. Mateus, *Self-similarity and uniqueness of solutions for semilinear reaction-diffusion systems*, Adv. Differential Equations **15** (2010), 73–98.

[7] Y. Fujishima and N. Ioku, *Existence and nonexistence of solutions for the heat equation with a superlinear source term*, J. Math. Pures Appl. **118** (2018), 128–158.

[8] C. G. Gal and M. Warma, *Nonlocal transmission problems with fractional diffusion and boundary conditions on non-smooth interfaces*, Comm. Partial Differential Equations **42** (2017), 579–625.

[9] C. G. Gal and M. Warma, *Fractional in time semilinear parabolic equations and applications*, HAL Id: hal-01578788, 2017.

[10] R. Gorenflo, Y. Luchko and F. Mainardi, *Analytical properties and applications of the Wright function*, Fract. Calc. Appl. Anal. **2** (1999), 383–414.

[11] K. Ishige, T. Kawakami and M. Sierżęga, *Supersolutions for a class of nonlinear parabolic systems*, J. Differential Equations **260** (2016), 6084–6107.

[12] Y. Kian and M. Yamamoto, *On existence and uniqueness of solutions for semilinear fractional wave equations*, Fract. Calc. Appl. Anal. **20** (2017), 117-138.

[13] Y. Miyamoto and M. Suzuki, *Weakly coupled reaction-diffusion systems with rapidly growing nonlinearities and singular initial data*, Nonlinear Anal. **189** (2019), 111576.

[14] I. Podlubny, "Fractional Differential Equations", **198** Academic Press, San Diego, California, USA, 1999.

[15] P. Quittner and P. Souplet, *Admissible L_p norms for local existence and for continuation in semilinear parabolic systems are not the same*, Proc. Roy. Soc. Edinburgh **131** (2001), 1435–1456.

[16] P. Quittner and P. Souplet, "Superlinear Parabolic Problems, Blow–up, Global Existence and Steady States", Birkhäuser Advanced Texts: Basler Lehrbücher Birkhäuser Verlag, Basel, 2007.

[17] F. Rothe, "Global Solutions of Reaction-diffusion Systems", Lecture Notes in Math. 1072: Springer-Verlag, Berlin, New York, 1984.

[18] S. Samko, A. A. Kilbas and O. I. Marichev, "Fractional Integrals and Derivatives", Theory and Applications, Gordon and Beach Science Publishers, Yverdon, 1993.

[19] M. Suzuki, *Local existence and nonexistence for reaction-diffusion systems with coupled exponential nonlinearities*, J. Math. Anal. Appl. **477** (2019), 776–804.

[20] F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in L^p*, Indiana Univ. Math. J. **29** (1980), 79–102.

[21] F. B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math. **38** (1981), 29–40.

[22] G. M. Wright, *The generalized Bessel function of order greater than one*, Quart. J. Math., Oxford Ser. **11** (1940), 36–48.