博士論文（要約）

On the epsilon factors of ℓ-adic sheaves on varieties

（多様体上のℓ進層のイプシロン因子について）

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Let k be a finite field and X be a smooth variety over k. For an ℓ-adic sheaf $\mathcal{F}$ on X (where ℓ is a prime number invertible in k), Saito [7] constructs the characteristic cycle $CC(\mathcal{F})$ as a $\mathbb{Z}$-linear combination of the irreducible components of the singular support $SS(\mathcal{F})$ defined by Beilinson [1]. The singular support is a closed subset of the cotangent bundle T^*X which controls the universal local acyclicity of morphisms. The cycle $CC(\mathcal{F})$ is characterized by a Milnor-type formula computing the total dimensions of the vanishing cycles complexes. Furthermore, when X is projective, the intersection number with the 0-section $T^*_X X \subset T^*X$ equals to the Euler-Poincaré characteristic:

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC(\mathcal{F}), T^*_X X)_{T^*X}.$$

In this dissertation, we give a refinement, the epsilon cycle, of the characteristic cycle by which we can treat the global epsilon factor

$$\varepsilon(X, \mathcal{F}) = \prod_i \det(-\mathrm{Frob}_k, H^i(X_{\bar{k}}, \mathcal{F}))^{(-1)^{i+1}}$$

modulo roots of unity.

This dissertation consists of three parts (Parts I, II, III). In Part I, we prove a continuity theorem of the local epsilon factors of vanishing cycles in a family, which is a key ingredient of the following parts. In Part II, we construct the epsilon cycles of ℓ-adic sheaves and study various properties of them. In the final part, Part III, we treat the special case of constant sheaves in Part II without taking modulo roots of unity. In this part, we compute the local epsilon factors of the vanishing cycles (with constant coefficient) of isolated singularities in terms of non-degenerate symmetric bilinear forms over k associated with the singularities.

Using the generalizations of local epsilon factors to general perfect field cases due to Yasuda [8] and Guignard [4], the results in Parts I, III are generalized to any perfect base field of positive characteristic. Also, Part II can be generalized to the case where the base field is the perfection of a finitely generated field, but is allowed to be of characteristic 0. For simplicity, in this summary, we only treat the case where the base field is finite.

In Part I, we consider a family of vanishing cycles complexes. To be precise, let S be a scheme of finite type over $\mathbb{F}_p$ and consider the following commutative diagram of S-schemes

(1)
$$\begin{array}{ccccc} Z & \hookrightarrow & X & \xrightarrow{f} & \mathbb{A}^1_S \\ & & & \underset{\pi}{\searrow} \quad S \quad \swarrow & \end{array}$$

and a constructible complex of ℓ-adic sheaves $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$ on which the following conditions are imposed: π is of finite type and universally locally acyclic relatively to $\mathcal{F}$, f is universally locally acyclic relatively to $\mathcal{F}$ outside a closed subscheme $Z \subset X$ which is assumed finite over S.

Let $s \in |S|$ be a closed point. Taking the fibers of the diagram (1) and the sheaf $\mathcal{F}$, we get a morphism $f_s \colon X_s \to \mathbb{A}^1_s$ which is universally locally acyclic relatively to $\mathcal{F}_s$ outside the finite subset Z_s. For each point $z \in Z_s$, let us write $\mathbb{A}^1_{s,(f(z))}$ for the henselization of $\mathbb{A}^1_s$ at the image $f(z)$ and $\mathbb{A}^1_{s,(z)}$ for the unramified extension of $\mathbb{A}^1_{s,(f(z))}$ corresponding to the residue extension $k(z)/k(f(z))$. The vanishing cycles complex $R\Phi_{f_s}(\mathcal{F}_s)_z$ supported at z gives a bounded complex of ℓ-adic representations of the absolute Galois group of the function field of $\mathbb{A}^1_{s,(z)}$. As the completion of the function field is a local field, one can associate the local epsilon factor

$$\varepsilon_0(\mathbb{A}^1_{s,(z)}, R\Phi_{f_s}(\mathcal{F}_s)_z, dt) \in \overline{\mathbb{Q}_\ell}^\times$$

in the sense of Langlands-Deligne [3].

For a family as in (1), we consider the function on $|S|$ defined by

$$\theta_{\mathrm{ep}} \colon |S| \to \overline{\mathbb{Q}_\ell}^\times, \quad s \mapsto \prod_{z \in Z_s} (-1)^{a_z} \varepsilon_0(\mathbb{A}^1_{s,(z)}, R\Phi_{f_s}(\mathcal{F}_s)_z, dt).$$

Here we set $a_z = [k(z)\colon k(s)] \cdot \mathrm{dimtot}_z(R\Phi_{f_s}(\mathcal{F}_s)_z) = [k(z)\colon k(s)] \cdot (\dim(R\Phi_{f_s}(\mathcal{F}_s)_z) + \mathrm{Sw}_z(R\Phi_{f_s}(\mathcal{F}_s)_z))$. The main result in Part I states that this function satisfies the reciprocity law of the class field theory:

Theorem 1. *Suppose that S is connected. Then, there exists a continuous character $\rho_{\mathcal{F},t} \colon \pi_1^{ab}(S) \to \overline{\mathbb{Q}_\ell}^\times$ which makes the following diagram commutative*

$$\begin{array}{ccc} |S| & \xrightarrow{\ \theta_{\mathrm{ep}}\ } & \overline{\mathbb{Q}_\ell}^\times \\ & \searrow \quad \nearrow_{\rho_{\mathcal{F},t}} & \\ & \pi_1^{ab}(S). & \end{array}$$

Here the slant arrow $|S| \to \pi_1^{ab}(S)$ sends closed points of $|S|$ to the geometric Frobeniuses at the points.

Let us explain the proof of the theorem. In the proof, we use oriented products of topoi, by which we can treat vanishing cycles complexes over general base schemes. More precisely, we use the oriented products of the form $S \overleftarrow{\times}_{\mathbb{P}^1_S} \mathbb{A}^1_S$. When $S = \mathrm{Spec}(k)$ is the spectrum of a field k, this topos is canonically equivalent to the étale topos of the generic point of the henselization $\mathbb{P}^1_{k,(\infty)}$ at the infinity. In our context, the topos $S \overleftarrow{\times}_{\mathbb{P}^1_S} \mathbb{A}^1_S$ is treated as a family of such étale topoi parametrized by S. Using the oriented product, we treat a family of the local Fourier transforms of the vanishing cycles and prove that they glue to a complex of smooth ℓ-adic sheaves on $S \overleftarrow{\times}_{\mathbb{P}^1_S} \mathbb{A}^1_S$. Then, taking the determinant, the theorem follows from Laumon's cohomological interpretation of local epsilon factors [6, (3.5.1.1)]. We note that, if one takes the rank of the complex, we obtain an alternative proof of the local constancy of the total dimensions in [7, 2.16] in our setting.

As an application of Theorem 1, we construct epsilon cycles in Part II.

Let X be a smooth variety over a finite field k. For an ℓ-adic sheaf $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$, Beilinson defines the singular support $SS(\mathcal{F})$. This is a closed subset of the cotangent bundle T^*X stable under the $\mathbb{G}_m$-action. If one writes its irreducible decomposition as

$$SS(\mathcal{F}) = \cup_a C_a,$$

each irreducible component C_a has the same dimension as X. The epsilon cycle of $\mathcal{F}$, denoted by $\mathcal{E}(\mathcal{F})$, is a cycle supported on $SS(\mathcal{F})$

$$\mathcal{E}(\mathcal{F}) = \sum_a \xi_a \otimes [C_a] \quad \xi_a \in \overline{\mathbb{Q}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$$

whose coefficients take values in the group $\overline{\mathbb{Q}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. This cycle is characterized by a Milnor-type formula for local epsilon factors. To explain it precisely, take k-morphisms

$$X \xleftarrow{j} U \xrightarrow{f} \mathbb{A}^1_k \tag{2}$$

where j is étale. Suppose that an isolated $SS(\mathcal{F})$-characteristic point $z \in U$ with respect to f is given. Then f is universally locally acyclic relatively to $\mathcal{F}$ outside that point. Therefore the local epsilon factor $\varepsilon_0(\mathbb{A}^1_{k,(z)}, R\Phi_f(\mathcal{F})_z, dt)$ is defined. On the other hand, since the differential $df = f^*dt$ (where t is the standard coordinate of $\mathbb{A}^1$) does not intersect $SS(\mathcal{F})$ away from z, the intersection number $(C_a, df)_z$ at z is defined for each C_a.

Theorem 2. *For a diagram (2) with an isolated $SS(\mathcal{F})$-characteristic point $z \in U$, we have*

$$\varepsilon_0(\mathbb{A}^1_{k,(z)}, R\Phi_f(\mathcal{F})_z, dt)^{-1} = (\mathcal{E}(\mathcal{F}), df)_z^{[k(z):k]} := \prod_a \xi_a^{[k(z):k]\cdot(C_a, df)_z}$$

in the quotient group $\overline{\mathbb{Q}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\overline{\mathbb{Q}_\ell}^\times$.

In Part II, we study and prove various properties of epsilon cycles which are analogous to characteristic cycles. Among them, we give a pull-back formula and a product formula, both of which can be proven by the same methods given for characteristic cycles by Beilinson and T. Saito:

Theorem 3. *Let X be a smooth k-variety purely of dimension n and $\mathcal{F}$ be an ℓ-adic sheaf on it.*

1. *(compatibility with pull-backs) Let $h\colon W \to X$ be a separated morphism from a smooth k-variety W. Suppose that h is properly $SS(\mathcal{F})$-transversal. We also suppose that W is purely of dimension m. Then, we have*

   $$\mathcal{E}(h^*\mathcal{F}) = h^!(\mathcal{E}(\mathcal{F})(\frac{n-m}{2})).$$

   Here $h^!$ denotes the pull-back of cycles defined for properly $SS(\mathcal{F})$-transversal morphisms and $(\frac{n-m}{2})$ denotes the Tate twist.

2. *(product formula) When X is projective, we have the following equality in the group $\overline{\mathbb{Q}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$:*

   $$\prod_i \det(\mathrm{Frob}_k, H^i(X_{\bar{k}}, \mathcal{F}))^{(-1)^i} = (\mathcal{E}(\mathcal{F}), T^*_X X)_{T^*X} := \prod_a \xi_a^{(C_a, T^*_X X)_{T^*X}}$$

Here $T^*_X X \subset T^*X$ in the right-hand side denotes the 0*-section of* T^*X*. The intersection number* $(C_a, T^*_X X)_{T^*X}$ *is defined as both of the* $C_a, T^*_X X \cong X$ *are* n *dimensional and* T^*X *is* $2n$ *dimensional.*

The proof of Theorem 2 goes as the corresponding Milnor-type formula for characteristic cycles. Since the assertion is étale local, we may assume that X is quasi-projective. Embedding X into a projective space, we apply the continuity of local epsilon factors (Theorem 1) to the universal family of pencils constructed from the embedding. By Katz-Lang's finiteness theorem [5], for a normal connected scheme S of finite type over k, the composition of a continuous character $\rho\colon \pi_1^{ab}(S) \to \overline{\mathbb{Q}_\ell}^\times$ and the quotient map $\overline{\mathbb{Q}_\ell}^\times \to \overline{\mathbb{Q}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ factors through the structure map $\pi_1^{ab}(S) \to \pi_1^{ab}(k)$. From this observation, we know that the local epsilon factors behave as if they were locally constant in the group $\overline{\mathbb{Q}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, from which the theorem follows.

In Part III, we consider a refinement of Theorem 2. Restricting ourselves to the case where $\mathcal{F} = \overline{\mathbb{Z}_\ell}$, we compute the local epsilon factor $\varepsilon_0(\mathbb{A}^1_{k,(z)}, R\Phi_f(\overline{\mathbb{Z}_\ell})_z, dt)$ without the ambiguity on roots of unity. The result can be regarded as a refinement of the Milnor formula in positive characteristic to the local epsilon factors.

When $\mathcal{F}$ is the constant sheaf $\overline{\mathbb{Z}_\ell}$, the singular support $SS(\mathcal{F})$ is equal to the 0-section and the intersection number $(T^*_X X, df)_z$ appearing in Theorem 2 is the Milnor number $\mu(f, z)$ of the isolated singularity z. As classically known, one can associate a non-degenerate symmetric bilinear form $(\varphi_f, B_{f,dt})$ with an isolated singularity as a linear-algebraic enhancement of the Milnor number.

First, we explain the construction of the bilinear form $(\varphi_f, B_{f,dt})$. Consider a commutative diagram of schemes as (1). In the sequel, S can be a general scheme (not necessarily over $\mathbb{F}_p$), and we suppose that X is a smooth S-scheme and that Z is the singular locus of f (namely, the closed subscheme defined by the intersection of df and $T^*_X X$) which is assumed finite over S. Let $i\colon Z \hookrightarrow X$ denote the closed immersion and $g\colon Z \to S$, $\pi\colon X \to S$ denote the structure maps respectively. By the compatibility of !-pull-backs for coherent sheaves, we have a canonical isomorphism

$$g^!\mathcal{O}_S \cong i^!\pi^!\mathcal{O}_S.$$

As π is smooth, we have $\pi^!\mathcal{O}_S \cong \omega_{X/S}[n]$ ($\omega_{X/S}$ is the canonical bundle, $n = \dim X$). Since i is a closed immersion, we have a canonical quasi-isomorphism of complexes of $\mathcal{O}_X$-modules

$$i_*g^!\mathcal{O}_S \cong i_*i^!\pi^!\mathcal{O}_S \cong R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \omega_{X/S}[n]). \tag{3}$$

By the assumption that $Z \to S$ is finite, the Koszul complex defined from the section df on the locally free $\mathcal{O}_X$-module $\Omega^1_{X/S}$ gives a locally free $\mathcal{O}_X$-resolution of $i_*\mathcal{O}_Z$. Computing the right-hand side of (3) in terms of this resolution, we have a canonical isomorphism

$$g_*g^!\mathcal{O}_S \cong g_*(i^*\omega_{X/S}^{\otimes 2})$$

of $\mathcal{O}_S$-modules. Set $\varphi_f = g_*i^*\omega_{X/S}$. This is a locally free $\mathcal{O}_S$-module. We define the non-degenerate symmetric bilinear form $B_{f,dt}$ on it to be the composition of

$$\varphi_f \times \varphi_f \to g_*(i^*\omega_{X/S}^{\otimes 2}) \cong g_*g^!\mathcal{O}_S \to \mathcal{O}_S.$$

Here the last arrow $g_*g^!\mathcal{O}_S \to \mathcal{O}_S$ comes from the adjunction of $(g^!, g_*)$.

We explain the main result of Part III when k is of odd characteristic.

Theorem 4. *Let X be an n dimensional smooth k-variety and $f\colon X \to \mathbb{A}^1_k$ be a k-morphism. Let $z \in X$ be an isolated singularity of f. We write* $\mathrm{disc}B_{f,dt} \in k^\times/(k^\times)^2$ *for the discriminant of the bilinear form $(\varphi_f, B_{f,dt})$ defined above. When k is of odd characteristic, we have the following equality:*

$$(-1)^{[k(z):k]\mathrm{dimtot} R\Phi_f(\overline{\mathbb{Z}_\ell})_z} \cdot \varepsilon_0(\mathbb{A}^1_{k,(z)}, R\Phi_f(\overline{\mathbb{Z}_\ell})_z, dt) = \left(\frac{(-2)^{n\mu(f,z)}\mathrm{disc}B_{f,dt}}{k}\right) \cdot \tau_\psi^{(-1)^{n+1}n\mu(f,z)}.$$

Here $\mu(f,z)$ denotes the Milnor number, $(\frac{}{k})$ denotes the Legendre symbol, and $\tau_\psi = -\sum_{a\in k}\psi(a^2)$ denotes the quadratic Gauss sum attached to the non-trivial additive character $\psi\colon k \to \overline{\mathbb{Q}_\ell}^\times$ used for defining the local epsilon factor.

We explain the results in characteristic 2. In characteristic 2, we consider a lift to the Witt vectors. Let us write $W_3(k)$ for the ring of Witt vectors of length 3. We take a $W_3(k)$-morphism

$$\tilde{f}\colon \tilde{X} \to \mathbb{A}^1_{W_3(k)}$$

which is a lift of $f\colon X \to \mathbb{A}^1_k$. Namely, $\tilde{X}$ is a smooth $W_3(k)$-scheme and the reduction $\tilde{f} \otimes_{W_3(k)} k$ is isomorphic to the initial one f. Setting $(f, S) = (\tilde{f}, \mathrm{Spec}(W_3(k)))$ in the diagram (1), we obtain a bilinear form over $W_3(k)$, and taking its discriminant, we get an element $\mathrm{disc}B_{\tilde{f},dt}$ in $W_3(k)^\times/(W_3(k)^\times)^2$. We note that the group $k/\wp(k)$ (where $\wp$ is the map $x \mapsto x^2 - x$) is contained in $W_3(k)^\times/(W_3(k)^\times)^2$ under the map

$$\alpha\colon k/\wp(k) \hookrightarrow W_3(k)^\times/(W_3(k)^\times)^2, \quad x \mapsto 1 + 4[x]. \tag{4}$$

Theorem 5. *Suppose that k is of characteristic 2. Let N be the integer $n\mu(f,z)$. Then, N is even. The signed discriminant*

$$(-1)^{\frac{N}{2}}\mathrm{disc}B_{\tilde{f},dt} \in W_3(k)^\times/(W_3(k)^\times)^2$$

constructed from a $W_3(k)$-lift $\tilde{f}\colon \tilde{X} \to \mathbb{A}^1_{W_3(k)}$ admits the following properties:

1. *$(-1)^{\frac{N}{2}}\mathrm{disc}B_{\tilde{f},dt}$ belongs to the image of α in (4).*
2. *$(-1)^{\frac{N}{2}}\mathrm{disc}B_{\tilde{f},dt}$ is independent of the choice of the lift $\tilde{f}$.*

Accordingly, there exists a unique element $a \in k/\wp(k)$ by which the discriminant $\mathrm{disc}B_{\tilde{f},dt}$ is written as the form $\mathrm{disc}B_{\tilde{f},dt} = (-1)^{\frac{N}{2}}(1 + 4[a])$. We call this constant a the Arf invariant of f and write $\mathrm{Arf}(f,z)$ for it. When the singularity of f is non-degenerate quadratic, this constant coincides with the classical Arf invariant of quadratic forms (cf. [2]).

Using this Arf invariant, we can state the formula of local epsilon factors in characteristic 2 as follows.

Theorem 6. *Let the notations be as in Theorem 4. We assume that the cardinality q of k is a power of 2. Then, the ratio*

$$(-1)^{[k(z):k]\mathrm{dimtot} R\Phi_f(\overline{\mathbb{Z}_\ell})_z} \cdot \varepsilon_0(\mathbb{A}^1_{k,(z)}, R\Phi_f(\overline{\mathbb{Z}_\ell})_z, dt)/q^{\frac{(-1)^{n+1}n\mu(f,z)}{2}}$$

is equal to ± 1. This sign is equal to 1 if and only if the Arf invariant $\mathrm{Arf}(f,z) \in k/\wp(k)$ is trivial.

Theorems 4, 5, 6 are proven by induction on the Milnor number. We construct a family (1) of isolated singularities parametrized by a smooth connected k-scheme S with the following properties: the family generically contains quadratic singularities with Milnor number 1 or 2 and the initial singularity (f, z) appears in the family. By Chebotarev density and the induction hypothesis on the Milnor number, Theorem 1 reduces the proofs to such quadratic singularities, which cases can be verified by direct computations.

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