博士論文

論文題目: Feigin-Semikhatov conjecture
and
its applications
(Feigin-Semikhatov 予想とその応用)

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Abstract

In this thesis, we investigate relationships between the subregular $\mathcal{W}$-algebra for $\mathfrak{sl}_n$ and the principal $\mathcal{W}$-superalgebra for $\mathfrak{sl}_{1|n}$ in terms of their algebraic structure and representation theory.

Firstly, we show that the Heisenberg cosets of these two (super)algebras are isomorphic if the levels satisfy a certain"duality relation" and a "non-degenerate condition". This isomorphism for generic levels has been conjectured originally by Feigin and Semikhatov [Nuclear Phys., 2004], where another construction of the subregular $\mathcal{W}$-algebra, which is denoted by $\mathcal{W}_n^{(2)}$, was defined in terms of screening operators.

Secondly, we enhance this duality: we prove a reconstruction type theorem, which asserts that the Heisenberg coset of the tensor product of one of the above $\mathcal{W}$-(super)algebras and a certain lattice vertex superalgebra is isomorphic to the other $\mathcal{W}$-(super)algebra. In the case $n = 2$, this reconstruction theorem states that the $\mathcal{N} = 2$ superconformal algebra is constructed from the affine vertex algebra for $\mathfrak{sl}_2$ and a certain lattice vertex superalgebra, and vice versa. This coincides with the Kazama–Suzuki coset construction [Nuclear Phys. B, 1989] of the $\mathcal{N} = 2$ superconformal algebra and its inverse construction due to Feigin, Semikhatov and Tipunin [J. Math. Phys., 1998].

Thirdly, by using this reconstruction theorem, we describe the simple principal $\mathcal{W}$-superalgebras for $\mathfrak{sl}_{1|n}$ at certain levels as simple current extensions of tensor products of simple principal $\mathcal{W}$-algebras and lattice vertex superalgebras, based on the corresponding result for the subregular $\mathcal{W}$-algebras obtained by Creutzig and Linshaw [arXiv:2005.10234].

Finally, we use this extension description to show that the simple principal $\mathcal{W}$-superalgebra for $\mathfrak{sl}_{1|n}$ at those levels are rational and C_2-cofinite, and then to derive the classification of their irreducible modules and determine their fusion rules. In the case of $n = 2$, this coincides with the unitary minimal series representations of the $\mathcal{N} = 2$ superconformal algebra.

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Contents

1. Introduction

This thesis consists of three parts. In the first part (§2), we reconstruct the screening operators for the $\mathcal{W}$-algebras [G1] from the (dual of) generalized Bernstein-Gel'fand-Gel'fand resolutions of finite dimensional simple Lie algebras, and discuss their relationship with quantum groups. This part is based on [N1]. The second part (§3-5) deals with the duality of the subregular $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ for $\mathfrak{sl}_n$ and the principal $\mathcal{W}$-superalgebra $\mathcal{W}^\ell(\mathfrak{sl}_{1|n})$ for $\mathfrak{sl}_{1|n}$. This part is based on [CGN]. The third part (§6-7) deals with the representation theory of $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ and $\mathcal{W}^\ell(\mathfrak{sl}_{1|n})$. This part is based on [CGNS]. In the below, we explain the detail.

1.1. **Background.** Let $\mathfrak{g}$ be a basic classical simple Lie superalgebra, k a complex number and f an even nilpotent element in $\mathfrak{g}$. Then one associates with the universal affine vertex superalgebra $V^k(\mathfrak{g})$ at level k the universal $\mathcal{W}$-superalgebra $\mathcal{W}^k(\mathfrak{g}, f)$ via the quantum Hamiltonian redcution [KRW, FFr2]. Especially, the $\mathcal{W}$-superalgebras associated with a principal nilpotent element f_{prin} (which is unique up to conjugations), are conventionally denoted by $\mathcal{W}^k(\mathfrak{g})$. Affine vertex superalgebras and their $\mathcal{W}$-superalgebras are most important families of vertex superalgebras due to their essential role in various aspects of representation theory, geometry, and physics. This traces back to constructions of knot invariants [W1], extensions of the Virasoro symmetry in the 2 dimensional chiral conformal field theories [FL1, FaZa, Za], and quantizations of the symmetries in soliton equations [DS, FL2, FFr2, FFr7, GD]. Nowadays, their significance becomes larger, ranging from the (quantum) geometric Langlands program [F2, FrGaits, AFO] and invariants of low dimensional manifolds [CCFGH, FeGu] to invariants of 3 and 4 dimensional supersymmetric quantum field theories [AGT, BMR, CG, FrGaio, GR] or symmetries of 6 dimensional conformal field theories [BRvR].

The celebrated Feigin–Frenkel duality [FFr4] of the principal $\mathcal{W}$-algebras associated with simple Lie algebras asserts that for all non-critical levels k one has an isomorphism

$$\mathcal{W}^k(\mathfrak{g}) \simeq \mathcal{W}^\ell({}^L\mathfrak{g}), \tag{1.1}$$

where ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra of $\mathfrak{g}$ and the dual level ℓ is defined by

$$r(k+h^\vee)(\ell + {}^Lh^\vee) = 1 \tag{1.2}$$

with r the lacity of $\mathfrak{g}$ and $h^\vee$, (resp. ${}^Lh^\vee$), the dual Coxeter numbers of $\mathfrak{g}$, (resp. ${}^L\mathfrak{g}$). However, dualities for the general $\mathcal{W}$-(super)algebras have been mysterious for many years.

One of the differences between the principal $\mathcal{W}$-algebras and the other types of $\mathcal{W}$-(super)algebras is that the former do not have affine vertex subalgebras, but the latter do. Based on the study of S-dualities among 4 dimensional $\mathcal{N}=4$ super Yang–Mills theories, a remarkable conjecture was proposed by Gaiotto and Rapčák [GR], which asserts that *the coset vertex algebras of $\mathcal{W}$-superalgebras by the affine vertex subalgebras admit dualities.* According to [GR], even if we consider the affine coset of a $\mathcal{W}$-algebra associated with a simple Lie algebra $\mathfrak{g}$, the affine coset on the other side in the duality is not of a $\mathcal{W}$-algebra associated with the Langlands dual of $\mathfrak{g}$, but of a $\mathcal{W}$-*super*algebra associated with a certain Lie *super*algebra. Therefore, we obtain a conjectural relationship of $\mathcal{W}$-algebras associated with Lie *algebras* and those associated with Lie *superalgebras*.

Actually, the most fundamental example of these new dualities has been well-known in both mathematics and physics literature: the duality between the Heisenberg cosets of $V^k(\mathfrak{sl}_2)$ and $\mathcal{W}^\ell(\mathfrak{sl}_{1|2})$, (also known as the $\mathcal{N}=2$ superconformal algebra, $\mathcal{N}=2$ SCA). In this case, the duality is an immediate consequence of the famous *Kazama-Suzuki coset construction* [KaSu] of the $\mathcal{N}=2$ SCA. A generalization of this particular duality has already been suggested by an impressive work of Feigin and Semikhatov [FS], which is the main objective of this thesis.

1.2. **Feigin–Semikhatov Conjecture.** In [FS], Feigin and Semikhatov introduced a family of vertex algebras, called the $\mathcal{W}^{(2)}_n$-algebras, which are proven to be isomorphic to the subregular $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ [G1]. They constructed $\mathcal{W}^{(2)}_n$ as subalgebras of the joint kernel of a set of screening operators acting on some Heisenberg vertex algebras. These screening operators are associated not to the set of simple positive roots of $\mathfrak{sl}_n$, but to the set of simple positive roots of $\mathfrak{sl}_{1|n}$. Note that in contrast to the case of simple Lie algebras, the Borel subalgebras of $\mathfrak{sl}_{1|n}$ are *not* unique up to conjugations, and neither are the sets of simple positive roots. In [FS], the authors constructed a set of screening operators for each set of simple positive roots.

Recall that by the Wakimoto realization [FFr1, Wak1], the affine vertex algebra $V^k(\mathfrak{g})$ is also embedded into a Heisenberg vertex algebra whose image for a generic level coincides with the joint kernel of certain screening operators. By [FFr7], these screening operators can be seen as generators of the quantum group $U_q(\mathfrak{n}_+)$ where $\mathfrak{n}_+ \subset \mathfrak{g}$ is the nilpotent Lie subalgebra in the positive part. In this sense, $U_q(\mathfrak{n}_+)$ and $V^k(\mathfrak{g})$ form a commuting pair in the Heisenberg vertex algebra where the screening operators act. This relation holds for the general $\mathcal{W}$-algebras and their screening operators, which was first observed by Feigin and Frenkel [FFr7] for the principal $\mathcal{W}$-algebras, see Remark 2.8 for the general $\mathcal{W}$-algebras. Therefore, the construction of $\mathcal{W}^{(2)}_n$ implies a hidden connection between $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ and an affine vertex superalgebra $V^\ell(\mathfrak{sl}_{1|n})$ at a certan level ℓ or its $\mathcal{W}$-superalgebras. They suggested an isomorphism between the Heisenberg cosets of $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ and $\mathcal{W}^\ell(\mathfrak{sl}_{1|n})$ whose levels (k,ℓ) are related in a similar way as (1.2), see [FS, eqn 1.6].

To show that the $\mathcal{W}^{(2)}_n$-algebras constructed from different screening operators are all isomorphic, the authors related the Heiseberg cosets of $\mathcal{W}^{(2)}_n$ to the same coset vertex algebra of $V^{\ell'}(\mathfrak{sl}_{1|n})$ at another level ℓ' by the affine vertex subalgebra $V^{\ell'}(\mathfrak{gl}_n)$. Many impressive operator product computations then suggest that the Heisenberg cosets of the $\mathcal{W}^{(2)}_n$-algebras associated with different screening operators are actually isomorphic to the latter coset algebras.

Conjecture. (Feigin and Semikhatov [FS]) *Let π_{H_1}, (resp. π_{H_2}), be the Heisenberg subalgebra of $\mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}})$, (resp. $\mathcal{W}^{k_2}(\mathfrak{sl}_{1|n})$). Then we have the following.*

(1) *For generic levels $(k_1,k_2) \in \mathbb{C}^2$ satisfying $(k_1+n)(k_2+n-1)=1$,*
$$\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n})\right).$$

(2) *For generic levels $(k_1,k_2) \in \mathbb{C}^2$ satisfying $\frac{1}{k_1+n} + \frac{1}{k_3+n} = 1$,*
$$\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(V^{k_3}(\mathfrak{gl}_n), V^{k_3}(\mathfrak{sl}_{1|n})\right).$$

Several special cases have already proven in the literature by using explicit OPE formulas. When $n=2$, $\mathcal{W}^{k_1}(\mathfrak{sl}_2, f_{\mathrm{sub}}) = V^{k_1}(\mathfrak{sl}_2)$ and $\mathcal{W}^{k_2}(\mathfrak{sl}_{1|2})$ is the $\mathcal{N}=2$ superconformal algebra. Then Conjecture (1) is essentially the well-known Kazama–Suzuki coset construction of the $\mathcal{N}=2$ superconformal algebra [KaSu] and Conjecture (2) follows from a some relation between $V^k(\mathfrak{sl}_{1|2})$ and $L_1(\mathfrak{d}(2,1;-(k+1)))$, [BFST, CG]. When $n=3$, Conjecture (2) is proven in [ACL1, Theorem 6.2]. This approach might not be applied in the higher rank cases since it is difficult to determine explicit OPE formulas in general.

In this thesis, we prove Conjecture (1) in general by the same way as the Feigin–Frenkel duality (1.1) was proved [FFr4]. We embed the Heisenberg cosets into certain Heisenberg vertex algebras and characterize them at generic levels as joint kernels of screening operators acting on the Heisenberg vertex algebras. This is based on the description of the $\mathcal{W}$-superalgebras as joint kernel of certain screening operator, which has been developed by Genra in terms of cohomological method [G1]-[G2]. In §2, the screening operators for the $\mathcal{W}$-algebras associated with $\mathfrak{g}$ are reconstructed in terms of the differentials of the (dual of) generalized Bernstein -Gel'fand-Gel'fand resolution of the trivial representation $\mathbb{C}$ of $\mathfrak{g}$, which clarifies their meaning.

This presentation of the Heisenberg cosets reduces the proof to a much simpler problem, that is, identifying the Heisenberg vertex algebras and screening operators. The isomorphisms thus obtained at generic levels extend naturally to all levels except for some "critical levels" when the Heisenberg subalgebras of the $\mathcal{W}$-superalgebras degenerate. In application to the representation theory of the $\mathcal{W}$-superalgebras $\mathcal{W}^k(\mathfrak{g}, f)$, the isomorphism in Conjecture (1) for their simple quotients, which we denote by $\mathcal{W}_k(\mathfrak{g}, f)$ will be important. Motivated by this, we prove the commutativity of taking coset and taking simple quotient (Corollary 5.6), which gives the isomorphism between their simple quotients as well. To summarize, we prove the following:

Theorem A. (Theorem 4.6 (i), Corollary 5.7 (i), [CGN]) *For* $(k_1, k_2) \in \mathbb{C}^2$ *satisfying*

$$(k_1, k_2) \neq (-n, -n+1), \left(-n + \frac{n}{n-1}, \frac{(n-1)^2}{n}\right), \quad (k_1+n)(k_2+n-1) = 1,$$

we have isomorphisms of vertex algebras

(1) $\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n})\right)$,

(2) $\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}_{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}_{k_2}(\mathfrak{sl}_{1|n})\right)$.

We note that Theorem A is proven independently by Creutzig and Linshaw [CL4] where a large part of the conjecture of Gaiotto and Rapčák [GR] is proven. Their proof relies on an analysis of strong generators of the affine cosets of the $\mathcal{W}$-superalgebras (of type A). This enables them to describe these coset algebras as quotients of the universal two-parameter $\mathcal{W}_\infty$-algebra $\mathcal{W}[c, \lambda]$ constructed by Linshaw [Lin], and to obtain isomorphisms between them. Their approach is powerful in the *regular* cases. In this case, they also identified the coset algebra in Theorem A (2) with a certain principal $\mathcal{W}$-algebra, which will be used in this thesis.

1.3. **Relation to Y_{r_1,r_2,r_3} and $\mathcal{W}_{r_1,r_2,r_3}$-algebras.** In [GR], the family of vertex algebras, called the vertex algebras at the corner or the Y_{r_1,r_2,r_3}-algebras $(r_1, r_2, r_3) \in \mathbb{Z}^3_{\geq 0}$ are introduced and conjectured to enjoy a triality

$$Y_{r_1,r_2,r_3} \simeq Y_{r_2,r_3,r_1} \simeq Y_{r_3,r_1,r_2}.$$

The Feigin–Semikhatov conjecture is equivalent to the case $(r_1, r_2, r_3) = (1, n, 0)$ since $Y_{0,n,1} = \mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right)$, $Y_{1,0,n} = \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n})\right)$, and $Y_{n,1,0} = \mathrm{Com}\left(V^{k_3}(\mathfrak{gl}_n), V^{k_3}(\mathfrak{sl}_{1|n})\right)$, respectively. Therefore, Theorem A is equivalent to the isomorphism $Y_{0,n,1} \simeq Y_{1,0,n}$. The proof in this thesis relates the $Y_{0,n,1}$-algebra to the $\mathcal{W}_{r_1,r_2,r_3}$-algebra introduced in [BFM] with $(r_1, r_2, r_3) = (0, n, 1)$, since the latter algebra is defined as the joint kernel of the same screening operators that we use for the $Y_{0,n,1}$-algebra. Therefore, as a byproduct, we have proved an isomorphism $\mathcal{W}_{r_1,r_2,r_3} \simeq Y_{r_1,r_2,r_3}$ for $(r_1, r_2, r_3) = (0, n, 1)$, which is conjectured by Procházka and Rapčák [PR] in general. We expect that we can prove by the same argument more general cases $\mathcal{W}_{0,r_1,r_2} \simeq Y_{0,r_1,r_2}$, which are the coset vertex algebra of certain $\mathcal{W}$-algebra by the affine vertex subalgebra. This will be interesting since the $\mathcal{W}_{r_1,r_2,r_3}$-algebra is shown to act on the equivariant cohomology of the moduli spaces of spiked instantons of Nekrasov [RSYZ], which generalizes a result of Schiffmann and Vasserot [SV] for the case $\mathcal{W}_{n,0,0}$, which is isomorphic to the principal $\mathcal{W}$-algebra for $\mathfrak{gl}_n$.

1.4. **Kazama–Suzuki cosets.** The Kazama–Suzuki coset construction [KaSu], appeared in the 1980's as building blocks of sigma models in string theory. Mathematically, it gives a family of vertex superalgeras which have the $\mathcal{N}=2$ superaonformal algebra ($\mathcal{N}=2$ SCA) as vertex subalgebras. This construction is very beneficial since it avoids the difficulty of extending by hand the Virasoro symmetry (which arbitrary vertex operator superalgebras possess) to the $\mathcal{N}=2$ SCA symmetry. The idea lies in the same line as the celebrated Goddard–Kent–Olive construction of the unitary minimal series representations of the Virasoro or $\mathcal{N}=1$ super-Virasoro algebra [GKO1, GKO2].

For simplicity, let us consider the type A case. Then the idea of Kazama and Suzuki is to use the tensor product of $V^k(\mathfrak{sl}_{n+1})$ and n pairs of free fermions G_n. Since the latter algebra carries an action of $L_1(\mathfrak{gl}_n)$, $V^{k+1}(\mathfrak{gl}_n)$ acts diagonally on the tensor product, whose coset vertex superalgebra automatically has odd fields in conformal weight $\frac{3}{2}$ weakly generating the $\mathcal{N}=2$ SCA. The coset is conjecturally isomorphic to the principal $\mathcal{W}$-superalgebras of $\mathfrak{sl}_{n+1|n}$. The case $n=1$ is the relation between the $\mathcal{N}=2$ SCA and $V^k(\mathfrak{sl}_2)$. We note that this relation has been first used in [DVPYZ] to construct explicitly the unitary minimal representations of the $\mathcal{N}=2$ SCA, which is followed later by Adamović [Ad1] in the mathematics literature. However, in [Ad1, DVPYZ], the authors only used an embedding of the $\mathcal{N}=2$ SCA into the coset and the honest isomorphism (surjectivity) seems to be established quite recently [CL3]. See [GL] for the case $n=2$ and [ACL2, Corollary 14.1] for the regularity of the type A cases in general.

The idea of Kazama and Suzuki is expected to be efficient in general to construct regular vertex operator superalgebras. The regularity is equivalent to the C_2-cofiniteness of the superalgebra, which ensures the finiteness of the number of the inequivalent simple modules, and the rationality, i.e., the semisimplicity of the module category [ABD]. Let $\mathcal{A}^k$ be a vertex operator algebra (VOA) equipped with a $V^k(\mathfrak{g})$-action where $\mathfrak{g}$ is a reductive Lie algebra. Then take a tensor product $\mathcal{A}^k \otimes G_*$ of $\mathcal{A}^k$ and several pairs of free fermions G_* generated by the natural representation of $\mathfrak{g}$ so that $\mathcal{A}^k \otimes G_*$ has a diagonal $V^{k+1}(\mathfrak{g})$-action. The coset vertex superalgebra $\mathrm{Com}(V^{k+1}(\mathfrak{g}), \mathcal{A}^k \otimes G_*)$ by the diagonal action is our new variant of Kazama–Suzuki coset, see [Sa2] for a recent work. We expect that the regularity is preserved under this construction.

In this thesis, we treat the case $\dim \mathfrak{g} = 1$, when $V^k(\mathfrak{g})$ is nothing but a rank one Heisenberg vertex algebra. A remarkable observation was made by Feigin, Semikhatov, and Tipunin [FST]: the coset construction of the $\mathcal{N}=2$ SCA from the $V^k(\mathfrak{sl}_2)$ can be reversed, i.e., $V^k(\mathfrak{sl}_2)$ can be also obtained from the $\mathcal{N}=2$ SCA by a coset construction. These two constructions serve as a dictionary between the representation theories of the $\mathcal{N}=2$ SCA and $V^k(\mathfrak{sl}_2)$. This was used efficiently by Adamović to determine the *fusion rules* of the unitary minimal series representation of the $\mathcal{N}=2$ SCA, and furthermore, to *classify* the irreducible modules of the $\mathcal{N}=2$ SCA *out of unitary minimal series* [Ad2]. See [Sa1, KoSa, CLRW] for recent developments in this direction and also [CR1, CR3, AP] for a similar relation between the $\beta\gamma$-system and $V^k(\mathfrak{gl}_{1|1})$. By using lattice vertex superalgebras $V_{\mathbb{Z}}$ and $V_{\mathbb{Z}\sqrt{-1}}$, our second main theorem can be stated as the following Kazama–Suzuki type coset theorem and its Feigin–Semikhatov–Tipunin type inverse:

Theorem B. (Theorem 4.6 (ii), (iii), Corollary 5.7 (ii), (iii), [CGN]) *For $(k_1,k_2)\in\mathbb{C}^2$ satisfying $(k_1+n)(k_2+n-1)=1$, we have isomorphisms of vertex superalgebras*

(1) $\mathcal{W}^{k_2}(\mathfrak{sl}_{1|n}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right)$,

(2) $\mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n}) \otimes V_{\mathbb{Z}\sqrt{-1}}\right)$,

(3) $\mathcal{W}_{k_2}(\mathfrak{sl}_{1|n}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_1}, \mathcal{W}_{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right)$,

(4) $\mathcal{W}_{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}_{k_2}(\mathfrak{sl}_{1|n}) \otimes V_{\mathbb{Z}\sqrt{-1}}\right)$,

where $\pi_{\widetilde{H}_i}$, $(i=1,2)$, is a certain Heisenberg vertex subalgebra.

This result enables us to determine the levels $k\in\mathbb{C}$ when the $\mathcal{W}$-superalgebra $\mathcal{W}_k(\mathfrak{sl}_{1|n})$ is regular. Moreover, we classify the irreducible modules and derive their fusion rules explicitly at these levels.

1.5. **Regularity of $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ and $\mathcal{W}_\ell(\mathfrak{sl}_{1|n})$.** Despite of its importance, the representation theory of the $\mathcal{W}$-superalgebras has not yet been understood very much in contrast to that of the affine vertex algebras $V^k(\mathfrak{g})$ or its simple quotient $L_k(\mathfrak{g})$ [Kac2, DFMS]. The difference is that the $\mathcal{W}$-superalgebras are defined as cohomologies and the defining operator product expansions among the strong generators are not known explicitly, and even worse, are non-linear in general. The non-linearity implies that the $\mathcal{W}$-superalgebras are not induced from some infinite dimensional Lie superalgebras, which makes it difficult even to determine the irreducible ordinary (i.e., highest weight) representations. On the other hand, when the $\mathcal{W}$-superalgebras are induced from some infinite dimensional Lie superalgebras, the representation theory has been developed. The prominent example is the case of the Virasoro vertex algebra, that is, $\mathcal{W}^k(\mathfrak{sl}_2)$, known as the Feigin-Fuchs theory [FFu, IK2].

The case of the principal $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g})$, which are non-linear, has been studied intensively by Arakawa and the C_2-cofiniteness and rationality of the simple quotient $\mathcal{W}_k(\mathfrak{g})$ at non-degenerate admissible levels are established in [Ar6] and [Ar5], respectively. Beyond the principal cases, the C_2-cofininteness is established for exceptional $\mathcal{W}$-algebras including $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ [Ar6] and the rationality in those levels has been proven under a certain assumption [AvE1], see also [Ar4, CL2] for earlier results and [Kaw, AM] for non-exceptional cases.

In contrast to this, the regularity of the principal $\mathcal{W}$-superalgebras and $\mathcal{W}$-superalgebras in general is an open problem except for the rationality of $\mathcal{W}_k(\mathfrak{g})$ with $\mathfrak{g}=\mathfrak{osp}_{1|2}$, $\mathfrak{sl}_{1|2}$ [Ad1, Ad3]. We note that they are actually induced by infinite dimensional Lie superalgebras called the $\mathcal{N}=1$ and $\mathcal{N}=2$ super-Virasoro algebras, respectively.

Our third result is the regularity of the simple $\mathcal{W}$-superalgebra $\mathcal{W}_\ell(\mathfrak{sl}_{1|n})$ at certain levels. By Theorem B (3), this follows from the corresponding result of the simple subregular $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ [AvE2, CL4]. In [CL4], Creutzig and Linshaw proved that the coset algebra of $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n) := \mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}})$, $(r\geq 3)$ by π_{H_1} is isomorphic to the principal $\mathcal{W}$-algebra $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) := \mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r)$, generalizing the level-rank duality for the parafermion algebra [ALY] for the case $n=2$ and [ACL1] for the case $n=3$. Moreover, they proved that the coset $\mathrm{Com}(\mathcal{W}_{(n,1)}(\mathfrak{sl}_r), \mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n))$ is isomorphic to a lattice vertex algebra $V_{\sqrt{nr}\mathbb{Z}}$ and

$$\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n) \simeq \bigoplus_{i\in\mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{\frac{ni}{\sqrt{nr}}+\sqrt{nr}\mathbb{Z}}, \tag{1.3}$$

as $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)\otimes V_{\sqrt{nr}\mathbb{Z}}$-modules, see §7.2-7.3 for details. The modules of $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)$ and $V_{\sqrt{nr}\mathbb{Z}}$ appearing in this decomposition are very special, called *simple currents* (see §1.7 below) and the regularity of the subalgebra $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)\otimes V_{\sqrt{nr}\mathbb{Z}}$ is equivalent to that of $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$. By using Theorem B (4), we can derive a parallel statement for the principal $\mathcal{W}$-superalgebra $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n}) := \mathcal{W}_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n})$.

Theorem C. (Theorem 7.4, [CGNS]) *For $r\geq 3$ and $n\geq 2$, there exists an isomorphism of vertex superalgebras*

$$\mathrm{Com}\left(\mathcal{W}_{(n,1)}(\mathfrak{sl}_r),\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})\right)\simeq V_{\sqrt{(n+r)r}\mathbb{Z}}.$$

In this case, we have

$$\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})\simeq\bigoplus_{i\in\mathbb{Z}_r}\mathbf{L}_{\mathcal{W}}(n\varpi_i)\otimes V_{\frac{(n+r)i}{\sqrt{(n+r)r}}+\sqrt{(n+r)r}\mathbb{Z}} \tag{1.4}$$

as $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)\otimes V_{\sqrt{(n+r)r}\mathbb{Z}}$-modules. In particular, $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$ is a simple current extension of $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)\otimes V_{\sqrt{(n+r)r}\mathbb{Z}}$ and thus is regular.

We note that the conformal vector of $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$ used for the regularity in Theorem C is not the standard one in [KRW], but taken so that (1.4) gives a conformal embedding of $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)\otimes V_{\sqrt{(n+r)r}\mathbb{Z}}$.

1.6. **Simple Currents.** Let V be a simple (self-dual) regular VOA. The theory of the tensor product on the category V-mod of (ordinary) V-modules has been developed by Huang and Lepowsky in a series of works [HL1]-[HL3] and subsequent works of Huang [H2]-[H7]. In particular, V-mod has a natural structure of modular tensor category. In this thesis, we call the tensor product of V-modules M and N the *fusion product* of M and N and write $M\boxtimes N$ in order to distinguish it from $\otimes_{\mathbb{C}}$. A V-module is called a *simple current* if the irreducibility of an arbitrary V-module is preserved under the fusion product with M. A simple current extension (SCE) of V is a VOA extension of V of the form

$$\mathcal{E}=\bigoplus_{g\in G}S_g \tag{1.5}$$

where $\{S_g\}_{G\in g}$ is a set of simple current V-modules parametrized by a finite abelian group G in the sense that the unit $e\in G$ corresponds to $S_e=V$ and $S_g\boxtimes S_h\simeq S_{gh}$. Such extensions provide a realistic way of constructing new VOAs from known ones [Ca, DLM1, DM2, La, Li1, Y], especially holomorphic VOAs [LS] or *irrational C_2-cofinite* VOAs [CKL]. A notable example is the moonshine VOA $V^\natural$ (whose automorphism group is the Monster group $\mathbb{M}$) [H1, FLM]. Simple current extensions can be used to show the *regularity of the $\mathcal{W}$-superalgebras*, for example, the $\mathcal{W}$-algebras related to Deligne exceptional series [Kaw], the subregular $\mathcal{W}$-algebras for $\mathfrak{sl}_n$ [ACL1, CL4] and the principal $\mathcal{W}$-superalgebras in Theorem C in this thesis.

By [HKL, CKL, CKM1, KO], given a VOA V, VOA extensions or more generally vertex operator *super*algebra (VOsA) extensions are essentially commutative *super*algebra objects in the category V-mod which admits a structure of braid tensor category by [HLZ1]-[HLZ8]. As in Theorem B, we naturally encounter with VOsA extensions of VOsAs. Unfortunately, to the best of our knowledge, there is no reference in this setting, and even for the category V-mod of a vertex operator superalgebra V. In the super setting, the category V-mod is no longer a braided tensor category, but is a $\mathbb{C}$-linear braided monoidal supercategory whose underlying category $\underline{V\text{-mod}}$ (whose morphisms are *even*) is abelian, see §6.1 for details. This is due to the fact that parity-inhomogeneous morphisms usually do not admit kernel or cokernel objects. Therefore, we begin with proving some basics like Schur's lemma (Lemma B.2) in this abstract super settng. (In the literature, it is proven for finite dimensional simple $\mathbb{C}$-superalgebras [CW].) This implies that the center of a simple vertex superalgebra is trivial (Lemma 5.1), which is also well-known for a vertex algebra [LLi].

Then we study in Appendix B, simple currents in a $\mathbb{C}$-linear braided monoidal supercategory $\mathcal{C}$ whose underlying category $\underline{\mathcal{C}}$ is abelian. In this generality, simple currents are *invertible objects* with respect to the tensor product. Each simple current generates an abelian group by the tensor product, (more precisely a groupoid), and gives rise to some filtration and decomposition of $\mathcal{C}$ by tensor product and monodromy. This point of view is efficiently used to analyze the (super)category $\mathrm{Rep}(\mathcal{E})$ of (super)module objects for a simple current extension $\mathcal{E}$. For example, see Proposition B.19 for the equivalence of the semisimplicity of $\mathcal{C}$ and $\mathrm{Rep}(\mathcal{E})$ and Corollary B.20 and B.21 for the classification of simple objects in $\mathrm{Rep}(\mathcal{E})$ and the description of the Grothendieck ring $\mathcal{K}(\mathrm{Rep}(\mathcal{E}))$ of $\mathrm{Rep}(\mathcal{E})$, respectively. We note that these results are already obtained within the theory of simple regular VOAs. The above equivalence of the semisimplicity of two categories is due to Carnahan and Miyamoto [CaMi]. The objects in $\mathrm{Rep}(\mathcal{E})$ are called twisted modules and simple objects in $\mathrm{Rep}(\mathcal{E})$ are constructed by Yamauchi [Y]. The description of the Grothendieck ring $\mathcal{K}(\mathrm{Rep}(\mathcal{E}))$ in general is due to Creutzig, Kanade and McRae [CKM1].

1.7. **Fusion Rules of Simple Current Extensions by Lattice Theory.** Let V be a simple C_2-cofinite $\frac{1}{2}\mathbb{Z}$-graded vertex superalgebra of CFT type and $\mathcal{E}$ a simple current extension of V. Then the representation theory of $\mathcal{E}$ is controlled by that of V. We can not expect the converse in general. However, if we replace V with a tensor product $V \otimes V_L$ where V_L is the lattice vertex superalgebra associated with a positive-definite integral lattice L, then we may recover the representation theory of $\mathcal{E}$ from that of V. More precisely, take a sublattice N of the dual lattice $L' = \{a \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid (a, L) \subset \mathbb{Z}\}$ containing L. This defines a group of simple currents $\{V_{a+L}\}_{a \in N/L}$ in V_L-mod. Take a group of simple currents $\{S_a\}_{a \in N/L}$ of V-modules satisfying $S_a \boxtimes S_b \simeq S_{a+b}$. Let $\mathcal{E}$ be a simple C_2-cofinite vertex operator superalgebra of CFT type of the form

$$\mathcal{E} = \bigoplus_{a \in N/L} S_a \otimes V_{a+L}.$$

Then the set of simple $\mathcal{E}$-modules $\mathrm{Irr}(\mathcal{E})$ and the fusion ring $\mathcal{K}(\mathcal{E})$ are determined as follows:

Theorem D. (Theorem 6.3, [CGNS])

(1) *The set of simple $\mathcal{E}$-modules is in one-to-one correspondence*
$$\mathrm{Irr}(\mathcal{E}) \simeq \{(M, a) \in \mathrm{Irr}(V) \times (L'/L) \mid \phi_M \phi_{V_{a+L}} = 1\}\big/(N/L)$$
by $(M, a) \mapsto \mathcal{F}(M \otimes V_{a+L}) = \mathcal{E} \boxtimes_{V \otimes V_L} (M \otimes V_{a+L})$. *In particular,*
$$|\,\mathrm{Irr}(\mathcal{E})| = \frac{|\,\mathrm{Irr}(V)||N'/L|}{|N/L|}$$
and
$$\mathrm{Pic}(\mathcal{E}) \simeq \{(M, a) \in \mathrm{Pic}(V) \times (L'/L) \mid \phi_M \phi_{V_{a+L}} = 1\}\big/(N/L).$$

(2) *Suppose that the fusion products* $\boxtimes$ *on* V-mod *and* $\mathrm{Rep}(\mathcal{E})$ *are exact. Then we have an isomorphism of rings*
$$\mathcal{K}(\mathcal{E}) \simeq \left(\mathcal{K}(V) \bigotimes_{\mathbb{Z}[N/L]} \mathbb{Z}[L'/L]\right)^{N/L} \tag{1.6}$$

where the tensor product over $\mathbb{Z}[N/L]$ *is given by* $[M \boxtimes S_a] \otimes b = [M] \otimes (-a+b)$ *for* $a \in N/L$

Conversely, we may recover the fusion data of V-mod in terms of $\mathcal{E}$-mod as follows:

Theorem E. (Theorem 6.5, [CGNS])

(1) *The set of simple* V*-modules* $\mathrm{Irr}(V)$ *is in one-to-one correspondence to*

$$\{(M,a) \in \mathrm{Irr}(\mathcal{E}) \times (L'/L) \mid \mathcal{M}_{V_{b+L}, M\boxtimes_{\mathcal{E}} V_{a+L}} = \mathrm{id},\ (\forall b \in N'/L)\}/(N'/L)$$

by $(M,a) \mapsto N$ *so that* $\mathcal{F}(N \otimes V_L) \simeq M \boxtimes_{\mathcal{E}} V_{a+L}$ *holds. In particular,*

$$|\operatorname{Irr}(V)| = \frac{|\operatorname{Irr}(\mathcal{E})||N/L|}{|N'/L|}$$

and the group $\mathrm{Pic}(V)$ *is naturally isomorphic to*

$$\{(M,a) \in \mathrm{Pic}(\mathcal{E}) \times (L'/L) \mid \mathcal{M}_{V_{b+L}, M\boxtimes_{\mathcal{E}} V_{a+L}} = \mathrm{id},\ (\forall b \in N'/L)\}/(N'/L)$$

(2) *Suppose that the fusion products* $\boxtimes$ *on* V-mod *and* $\mathrm{Rep}(\mathcal{E})$ *are exact. Then we have an isomorphism of rings*

$$\mathcal{K}(V) \simeq \left(\mathcal{K}(\mathcal{E}) \bigotimes_{\mathbb{Z}[N'/L]} \mathbb{Z}[L'/L]\right)^{N'/L} \tag{1.7}$$

where the tensor product over $\mathbb{Z}[N'/L]$ *is given by* $[M \boxtimes_{\mathcal{E}} V_{a+L}] \otimes b = [M] \otimes (a+b)$ *for* $a \in N'/L$.

These theorems are established by Yamada and Yamauchi [YY] in the regular, non-super cases and also essentially by Creutzig, Kanade, and McRae [CKM1] in the regular, super cases. But the description of the fusion rings in (2) of this form seems new. (We note that the duality between $\mathcal{K}(V)$ and $\mathcal{K}(\mathcal{E})$ is also discussed in [YY].) The exactness of the fusion product follows not only from the semisimplicity of the the module category, but also the rigidity, i.e., the property that every module has a left and right dual. By [CMY], if V is self-dual and every irreducible V-module is rigid, then V-mod is rigid. In this case, $\mathrm{Rep}(\mathcal{E})$ and $\mathcal{E}$-mod are also rigid, and thus the assumption of (2) in Theorem D and E is satisfied.

The later theorem can be applied to the parafermion vertex algebra $K_k(\mathfrak{g})$, which is the coset vertex algebra of the simple affine VOA $L_k(\mathfrak{g})$ by a Heisenberg vertex algebra (or a certain lattice vertex algebra V_L) and $L_k(\mathfrak{g})$ is a simple current extension of $K_k(\mathfrak{g}) \otimes V_L$. Then we recover the fusion rules of $K_k(\mathfrak{g})$, see [ADJR, DR, DW].

We note that the description of the whole fusion rings in Theorem D and E in this form are very useful to comparing fusion rings of other VOsAs. We will see its efficiency in the next subsection.

1.8. **Fusion rules of** $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$ **and** $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$**.** By [Ar2], the set of (inequivalent) simple $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)$-modules is in one-to-one correspondence with that of $L_n(\mathfrak{sl}_r)$, that is, $\mathrm{Irr}(L_n(\lambda)) = \{L_n(\lambda) \mid \lambda \in P^n_+(r)\}$. Here $P^n_+(r)$ is the set of dominant integral weights of $\mathfrak{sl}_r$ at level n. Let $\mathbf{L}_{\mathcal{W}}(\lambda)$ denote the simple $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)$-module corresponding to $\lambda \in P^n_+(r)$. Then by [AvE1, Cr, FKW], it satisfies the same fusion rules as that of $L_n(\mathfrak{sl}_r)$ i.e.,

$$\mathbf{L}_{\mathcal{W}}(\lambda) \boxtimes \mathbf{L}_{\mathcal{W}}(\mu) \simeq \bigoplus_{\nu \in P^n_+(r)} N^{\nu}_{\lambda,\mu}(\widehat{\mathfrak{sl}}_{r,n}) \mathbf{L}_{\mathcal{W}}(\nu),$$

where the coefficient $N^{\nu}_{\lambda,\mu}(\widehat{\mathfrak{sl}}_{r,n})$ is given by

$$\mathbf{L}_n(\lambda) \boxtimes \mathbf{L}_n(\mu) \simeq \bigoplus_{\nu \in P^n_+(r)} N^{\nu}_{\lambda,\mu}(\widehat{\mathfrak{sl}}_{r,n}) \mathbf{L}_n(\nu).$$

Now, we can apply Theorem D to the SCEs in (1.3) and (1.4), respectively, and obtain the classification of irreducible modules and the fusion rings:

Theorem F. (Theorem 7.2, Theorem 7.5, [CGNS]) *Let $r \geq 3$ and $n \geq 2$.*

(1) *There exists a one-to-one correspondence*

$$\mathrm{Irr}(\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)) \simeq \left\{(\lambda, a) \in P^n_+(r) \times \mathbb{Z}_{nr} \mid \pi_{P/Q}(\lambda) = a \in \mathbb{Z}_r\right\} \Big/ \mathbb{Z}_r,$$

where $\mathbb{Z}_r$ acts on $P^n_+(r) \times \mathbb{Z}_{nr}$ by $m \cdot (\lambda, a) = (\sigma^m(\lambda), a + mn)$, ($m \in \mathbb{Z}_r$). The $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$-module $\mathbf{L}_{sub}(\lambda, a)$ corresponding to (λ, a) decomposes into

$$\mathbf{L}_{sub}(\lambda, a) \simeq \bigoplus_{i \in \mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(\sigma^i(\lambda)) \otimes V_{\frac{a+ni}{\sqrt{nr}} + \sqrt{nr}\mathbb{Z}} \tag{1.8}$$

as a $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{nr}\mathbb{Z}}$-module and satisfies the fusion rules

$$\mathbf{L}_{sub}(\lambda, a) \boxtimes \mathbf{L}_{sub}(\mu, b) \simeq \bigoplus_{\nu \in P^n_+(r)} N^{\nu}_{\lambda,\mu}(\widehat{\mathfrak{sl}}_{r,n}) \mathbf{L}_{sub}(\nu, a + b).$$

The fusion ring is isomorphic to

$$\mathcal{K}(\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)) \simeq \left(\mathcal{K}(L_n(\mathfrak{sl}_r)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_r]} \mathbb{Z}[\mathbb{Z}_{nr}]\right)^{\mathbb{Z}_r}.$$

(2) *There exists a one-to-one correspondence*

$$\mathrm{Irr}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})) \simeq \left\{(\lambda, a) \in P^n_+(r) \times \mathbb{Z}_{(n+r)r} \mid \pi_{P/Q}(\lambda) = a \in \mathbb{Z}_r\right\} \Big/ \mathbb{Z}_r,$$

where $\mathbb{Z}_r$ acts on $P^n_+(r) \times \mathbb{Z}_{(n+r)r}$ by $m \cdot (\lambda, a) = \big(\sigma^m(\lambda), a + m(n+r)\big)$, ($m \in \mathbb{Z}_r$). The $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$-module $\mathbf{L}_{sp}(\lambda, a)$ corresponding to (λ, a) decomposes into

$$\mathbf{L}_{sp}(\lambda, a) \simeq \bigoplus_{i \in \mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(\sigma^i(\lambda)) \otimes V_{\frac{a+(n+r)i}{\sqrt{(n+r)r}} + \sqrt{(n+r)r}\mathbb{Z}} \tag{1.9}$$

as a $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{(n+r)r}\mathbb{Z}}$-module and satisfies the fusion rules

$$\mathbf{L}_{sp}(\lambda, a) \boxtimes \mathbf{L}_{sp}(\mu, b) \simeq \bigoplus_{\nu \in P^n_+(r)} N^{\nu}_{\lambda,\mu}(\widehat{\mathfrak{sl}}_{r,n}) \mathbf{L}_{sp}(\nu, a + b).$$

The fusion ring is isomorphic to

$$\mathcal{K}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})) \simeq \left(\mathcal{K}(L_n(\mathfrak{sl}_r)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_r]} \mathbb{Z}[\mathbb{Z}_{(n+r)r}]\right)^{\mathbb{Z}_r}.$$

See §7 for details. In the literature, the classification of the irreducible $\mathcal{N} = 2$ SCA-modules and their fusion rules are already determined in [Ad2] and [Ad3], respectively. The classification of the irreducible $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$-modules for $r \geq 0$ with n even is established in [AvE2] based on Zhu's theory [Zh] and their fusion rules are determined for n even via the Verlinde formula [H5]. In [AvE2], the fusion ring is determined to be

$$\mathcal{K}(\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)) \simeq \mathcal{K}(L_r(\mathfrak{sl}_n)),$$

which is a priori different from Theorem F (1). It turns out that the compatibility of these two expressions can be explained by the *level-rank duality* between affine vertex algebras [Fr, OS]. Namely, we have

$$\mathcal{K}(L_m(\mathfrak{sl}_n)) \simeq \left(\mathcal{K}(L_n(\mathfrak{sl}_m)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_m]} \mathbb{Z}[\mathbb{Z}_{nm}] \right)^{\mathbb{Z}_m}, \quad (n, m \geq 2), \tag{1.10}$$

see Proposition 7.1. We note that the isomorphism (1.10) restricted to certain subalgebras $\mathcal{K}^0(L_m(\mathfrak{sl}_n)) \simeq \mathcal{K}^0(L_n(\mathfrak{sl}_m))$ where "0" means the grading $\pi_{P/Q} = 0$ part is established in [Fr, OS]. But the results in [Fr, OS] together with Theorem D improve their results to (1.10). Then Theorem F (1) implies

Corollary G. (Corollary 7.3, [CGNS]) *For $r \geq 3$, there exists an isomorphism*

$$\mathcal{K}(\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)) \simeq \mathcal{K}(L_r(\mathfrak{sl}_n)).$$

Now, we can use this isomorphism to obtain a similar description for the principal $\mathcal{W}$-superalgebras by Theorem B (4) and Theorem F (2):

Corollary H. (Theorem 7.7, [CGNS]) *For $r \geq 3$, there exists an isomorphism*

$$\mathcal{K}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})) \simeq \left(\mathcal{K}(L_r(\mathfrak{sl}_n)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_n]} \mathbb{Z}[\mathbb{Z}_{n(n+r)}] \right)^{\mathbb{Z}_n}. \tag{1.11}$$

This formula gives a correct formulation of Wakimoto's observation in his unpublished manuscript [Wak2] that the fusion rules of the $\mathcal{N} = 2$ SCA may somehow resemble that of the affine algebra $\widehat{\mathfrak{sl}}_2$ twisted by certain cyclic groups.

Interestingly, the fusion rings $\mathcal{K}(\mathcal{W}_k(\mathfrak{g}, f))$ for rational and C_2-cofinite $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{g}, f)$ are often described by the fusion rings $\mathcal{K}(L_\ell(\mathfrak{g}))$ of the affine vertex algebra of the same $\mathfrak{g}$ at nother levels ℓ. This is the case for the principal $\mathcal{W}$-algebras [Cr] and the subregular $\mathcal{W}$-algebras ([AvE2], Corollary G).

We expect that the right hand side in the isomorphism (1.11) is also related to some fusion ring of $L_r(\mathfrak{sl}_{1|n})$ at some level. (Note that the even part of $\mathfrak{sl}_{1|n}$ is exactly $\mathfrak{sl}_n \oplus \mathbb{C}$, which is compatible of the right-hand side of (1.11).) But it is known that the representation theory of $L_r(\mathfrak{sl}_{1|n})$ at positive integer level $r \in \mathbb{Z}_{\geq 0}$ is very different from the non-super cases. See [ERF, GS, KW1] for the classification of suitable irreducible modules of $L_r(\mathfrak{sl}_{1|n})$ and [GK], [KW3]-[KW5] for character formulas. We hope to come back to this point in our future work.

1.9. **Future perspective.** There are several problems left for future study.

First of all, normalized q-characters of irreducible $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$-modules should be interesting. In the case of $n = 2$, it is written in terms of theta functions [KW4, Mat] and we expect it to hold in general. Beyond the levels considered in this thesis, the classification of the (ordinary) irreducible $\mathcal{W}_k(\mathfrak{sl}_{1|n})$-modules and their q-characters when k is a principal admissible number for $\mathfrak{sl}_{1|n}$ will be very interesting. The case $n = 2$, the classification is established in [Ad2] and there are uncountably many inequivalent irreducible modules. Moreover, their q-characters are described by the Appel-Lerch sums, which are special kind of mock modular forms [Sa2]. Therefore, the usual Verlinde formula does not apply, but a variant of it is expected to hold, see [AC, Sa2]. It is interesting to consider its relation to the modular completion of the q-characters [KW5]. It is natural to explore the higher rank cases, which will be difficult in general at this moment. We may study the case $n = 3$ via the Kazama–Suzuki coset construction Theorem B since the subregular $\mathcal{W}$-algebra in this case is *minimal*, which has very special properties [Ar1] and whose relaxed highest weight modules are classified very recently [AKR, FKR].

A certain interesting non-regular case can be found in the context of 4d-2d correspondence which is under intensive investigation in the last decade. For the subregular $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sl}_{p-1}, f_{\mathrm{sub}})$, this is the case when $k = 2 - p - p^{-1}$. The subregular $\mathcal{W}$-algebra at this level, also known as the $\mathcal{B}^{(p)}$-algebra [CRW] turns out to be the chiral algebra for the Argyres–Douglas theory of type (A_1, A_{2p-3}) [ACGY] and the representation theory has been studied [ACKR]. One can use these results to also study the representation theory of $\mathcal{W}_{3-p+(p-1)^{-1}}(\mathfrak{sl}_{p-1|1})$, which will be a good example of the representation theory of non C_2-cofinite simple vertex operator superalgebras.

Secondly, we would like to point out that the representation theory of the $\mathcal{W}$-(super)algebras beyond the principal cases has a new feature from the beginning. In contrast to the representation theory of well-studied vertex operator algebras like the affine vertex algebras, the Virasoro vertex algebra, and their conformal extensions including lattice vertex algebras, it does not have a natural choice of conformal structure but a family of conformal structures parametrized by good gradings. It leads us to study the effect of the choice of conformal structures on the representation theory of VOAs. The classification of conformal vectors for $\mathbb{Z}$-graded simple self-dual VOA of (strong) CFT type is established in [Mo], see also [MN] for an earlier result. The choice of conformal structures changes the category of ordinary modules and the braided tensor category structure on it in the sense of Huang–Lepowsky. We expect that in the C_2-cofinite setting, for conformal structures whose difference are derivations of Cartan subalgebra, the braided tensor categories are equivalent. We expect this to happen for simple C_2-cofinite $\mathcal{W}$-algebras. In this case, the most natural choice of good gradings should be the Dynking grading, which makes the $\mathcal{W}$-algebra self-dual, a necessary property of the rigidity of the module category.

Thirdly, the dualities among the $\mathcal{W}$-superlagebras conjectured by Gaiotto–Rapčák [GR] give a clue to understand the representation theory of the $\mathcal{W}$-superalgebras, which is widely open. As is exemplified in this work, the duality itself is not sufficient for this purpose, but reconstruction theorems like Kazama–Suzuki coset construction play a significant role. Since the Kazama–Suzuki coset construction works only for Heisenberg cosets, we would like to establish a reconstruction theorem for vertex superalgebras V_i $(i = 1, 2)$ whose coset by affine vertex subalgebras are isomorphic. For the $\mathcal{W}$-superalgebras, we expect that the (quantum) geometric Langlands kernel algebras [CG] or some variants play a role of "adhesive material". More precisely, the kernel algebra is a vertex (super)algebra, which is a conformal extension of the affine vertex algebras used in the coset and the relative semi-infinite cohomology of the diagonal action on the tensor product of V_1 and the kernel algebra gives the other algebra V_2 and vise versa. We have checked this construction for the pair $(\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}}), \mathcal{W}^\ell(\mathfrak{sl}_{1|n}))$ studied in this thesis ([CGNS]). This construction will give a certain equivalence of full subcategories of V_1-modules and V_2-modules even beyond the rational cases. This equivalence will give us a large enough class of modules to have resolutions of objects by universal objects, like Verma modules. See [KoSa] for the case of the $\mathcal{N} = 2$ SCA. These universal objects are never modules of simple quotients of V_i, we would like to derive dualities for the *universal* $\mathcal{W}$-superlagebras. For this purpose, the screening kernel description of the $\mathcal{W}$-superalgebras, including affine vertex superalgebras (at generic levels) will play an important role since some universal objects called generalized Wakimoto representations [G2] are constructed similarly, see §2. The case of basic classical affine vertex superalgebras is already achieved and in preparation by a joint work of the author and Genra, see [IK1, R, IMP1, IMP2] for earlier results. Such a description is obtained as a direct generalization of the work [FFr8] and interestingly, the classical limit (to a Poisson vertex superalgebra) gives a realization of an affine Poisson vertex superalgebra as a coordinate ring of an open cell of a thick affine (super)Grassmanian, see also [FFr7, FFr8, BZF, N3]. This implies a possible geometric realization of the $\mathcal{W}$-superalgebras associated with thick affine (super)Grassmanians.

Fourthly, the screening realization of the $\mathcal{W}$-superalgebras does fail at some levels in the sense that the screening kernels get larger than the $\mathcal{W}$-superalgebras themselves. It already happens for the Virasoro vertex algebra and gives a construction of the *singlet algebras* [Ad4], which is known as a building block of the famous triplet $\mathcal{W}$-algebra [TW], see also [Su]. The singlet algebra has several important features. It is a simple, non C_2-cofinite vertex algebra which admits a natural module category equipped with a rigid braided tensor category structure in the sense of Huang–Lepowsky [CMY]. Moreover, it has a class of modules whose q-characters are expressed by false theta functions. This gives a clue for the study of the relationship of a certain modularity of q-characters (more precisely, conformal blocks) and the fusion rules beyond the case of the regular VOAs [Zh, H5], see [CrMi] and references therein. Surprisingly, $\mathbb{Z}$-linear combinations of the false theta functions used in the q-characters appear in the so-called *homological blocks* in the study of invariants of plumbed 3 manifolds, which are refinements of the Witten–Reshetikhin–Turaev invariants [CCFGH]. We expect that we may associate at least with Brieskorn spheres certain modules of order 4 conformal extensions of the singlet algebras whose reduced characters coincides with homological blocks.

Finally, returning to the context of the trialities of Gaiotto–Rapčák, another related direction worth further investigation is to show these type of trialities for the corresponding Zhu's algebras and associated varieties. Since these algebras and varieties are well-known for the universal $\mathcal{W}$-superalgebras (finite $\mathcal{W}$-superalgeras and Slodowy slices, respectively), the problems reduce to the relationship of taking coset vertex algebras by affine vertex subalgebras and taking Zhu's algebras (or associated varieties). We expect that these two procedures commute under a suitable assumption. A nice corollary of this expectation is that C_2-cofiniteness will be inherited under taking Heisenberg cosets. To the best of our knowledge, it is established when the Heisenberg vertex algebra extends to a lattice vertex algebra (for example, when the vertex algebra is strongly rational and self-contragradient [Mas]) and thus the cyclic orbifold theory may apply [Mi2].

1.10. **Outline.** In §2, we review realizations of the affine vertex algebras and $\mathcal{W}$-superalgebras at generic levels as joint kernels of screening operators and then in the non-super case, we show that these constructions are induced by the generalized BGG resolutions of relevant finite dimensional simple Lie algebras, and thus prove their compatibility. In §3, we derive a free field realization of the subregular $\mathcal{W}$-algebras and the principal $\mathcal{W}$-superalgebras with a screening kernel description at generic levels from the result in §2 and a free field realization of the affine vertex superalgebra $V^{\kappa}(\mathfrak{gl}_{1|1})$ developed in Appendix A. In §4, we relate the screening operators of the principal $\mathcal{W}$-superalgebra with those of the subregular $\mathcal{W}$-algebra by the Feigin–Frenkel duality for the Virasoro vertex algebra and prove Theorem A and B at generic levels and then at all levels except for some prohibited levels by an argument of continuity. In §5, we prove the commutativity of taking simple quotient and taking Heisenberg cosets, and as an application,show the assertions in Theorem A and B for the simple quotients. Based on a detailed analysis of basic properties of simple currents in a $\mathbb{C}$-linear braided monoidal supercategories in Appendix B, we show Theorem D and E in §6. Thereafter, by using a result of Creutzig and Linshaw, we prove Theorem C, E and F and then Corollary G and H in the last section.

2. Screening operators for $\mathcal{W}$-superalgebras

Following [Kac3], throughout this paper, given a vertex superalgebra V, $|0\rangle$ denotes the vacuum vector, ∂ the translation operator, $Y(a,z)=\sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$ is the field corresponding to an element $a\in V$.

2.1. **$\mathcal{W}$-superalgebras.** We review the definition of the (affine) $\mathcal{W}$-algebras, following [KRW]. Let $\mathfrak{g}=\mathfrak{g}^{\bar 0}\oplus\mathfrak{g}^{\bar 1}$ be a finite dimensional basic classical simple Lie superalgebra over $\mathbb{C}$ with a non-degenerate even supersymmetric invariant bilinear form $(\cdot|\cdot)$. $f\in\mathfrak{g}^{\bar 0}$ a nilpotent element, and a good grading of $\mathfrak{g}$ for f

$$\Gamma\colon \mathfrak{g}=\bigoplus_{j\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_j, \tag{2.1}$$

i.e., a $\frac{1}{2}-\mathbb{Z}$-grading such that $[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$, $(i,j\in\frac{1}{2}\mathbb{Z})$, $f\in\mathfrak{g}_{-1}$ and $\operatorname{ad}f\colon\mathfrak{g}_j\to\mathfrak{g}_{j-1}$ is injective (resp. surjective) for $j\geq\frac{1}{2}$, (resp. $j\leq\frac{1}{2}$). Then there exists a semisimple element $h\in\mathfrak{g}$ such that the grading Γ of $\mathfrak{g}$ is the eigenspace decomposition of $\operatorname{ad}(\frac{1}{2}h)$. By Jacobson–Morozov Theorem, we may extend $\{h,f\}$ to an $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{g}^{\bar 0}$. Choose a Cartan subalgebra $\mathfrak{h}$ containing h so that $\mathfrak{h}\subset\mathfrak{g}_0$. Let Δ be the set of roots, Δ_+ the set of positive roots such that $\bigoplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha\subset\mathfrak{g}_{\geq 0}$, where $\mathfrak{g}_\alpha$ is the root space of $\alpha\in\Delta$. Let I be the set of simple roots, $\Delta_j=\{\alpha\in\Delta\mid\mathfrak{g}_\alpha\subset\mathfrak{g}_j\}$ and $I_j=I\cap\Delta_j$, $(j\in\frac{1}{2}\mathbb{Z})$. Set $\Delta_0^+=\Delta_0\cap\Delta_+$. Then

$$\Delta=\bigsqcup_{j\in\frac{1}{2}\mathbb{Z}}\Delta_j,\quad \Delta_+=\Delta_0^+\sqcup\bigsqcup_{j>0}\Delta_j,\quad I=I_0\sqcup I_{\frac{1}{2}}\sqcup I_1,$$

see [EK]. Denote by $\deg_\Gamma\alpha=j$ if $\alpha\in\Delta_j$. Fix a basis $\{e_i\}_{i\in I}$ of $\mathfrak{h}$, a root vector $e_\alpha\in\mathfrak{g}$, $(\alpha\in\Delta)$ so that $\{e_\alpha\}_{\alpha\in I\sqcup\Delta}$ forms a (parity homogeneous) basis of $\mathfrak{g}$. We also set $h_\alpha:=[e_\alpha,e_{-\alpha}]$ for $\alpha\in\Delta$. We denote by $\bar\alpha$ the parity of e_α, $c_{\alpha,\beta}^\gamma$ the structure constants $[e_\alpha,e_\beta]=\sum_{\gamma\in I\sqcup\Delta}c_{\alpha,\beta}^\gamma e_\gamma$ extended to $[u,v]=\sum_\gamma c_{u,v}^\gamma e_\gamma$, $(u,v\in\mathfrak{g})$, and normalize $(\cdot|\cdot)$ so that $(\theta|\theta)=2$ for the highest root θ of $\mathfrak{g}^{\bar 0}$. Let $\chi\colon\mathfrak{g}\to\mathbb{C}$ be a linear map defined by $\chi(u)=(f|u)$ for $u\in\mathfrak{g}$. Denote by $\mathfrak{n}_\pm=\bigoplus_{\alpha\in\Delta_\pm}\mathfrak{g}_\alpha$ and $\mathfrak{b}_\pm=\mathfrak{h}\oplus\mathfrak{n}_\pm$.

Let $V^k(\mathfrak{g})$ be the universal affine vertex superalgebra associated with $\mathfrak{g}$ at level $k\in\mathbb{C}$ which is generated by the fields $u(z)$, ($u\in\mathfrak{g}$), satisfying the OPEs

$$u(z)v(w)\sim\frac{[u,v](w)}{z-w}+\frac{k(u|v)}{(z-w)^2},\quad (u,v\in\mathfrak{g}),$$

$F_{\mathrm{ch}}(\mathfrak{g}_{>0})$ the charged fermion vertex superalgebra associated with $\mathfrak{g}_{>0}$, which is generated by the fields $\varphi_\alpha(z),\varphi^\alpha(z)$, $(\alpha\in\Delta_{>0})$ of parity reversed to e_α, satisfying the OPEs

$$\varphi_\alpha(z)\varphi^\beta(w)\sim\frac{\delta_{\alpha,\beta}}{z-w},\quad \varphi_\alpha(z)\varphi_\beta(w)\sim 0\sim\varphi^\alpha(z)\varphi^\beta(w),\quad (\alpha,\beta\in\Delta_{>0}).$$

By setting the degree $\deg_{\mathrm{ch}}(\varphi_\alpha(z))=-1$ and $\deg_{\mathrm{ch}}(\varphi^\alpha(z))=1$, $(\alpha\in\Delta_{>0})$, we obtain a degree decomposition $F_{\mathrm{ch}}(\mathfrak{g}_{>0})=\bigoplus_{n\in\mathbb{Z}}F_{\mathrm{ch}}^n$, where $F_{\mathrm{ch}}^n=\{A\in F_{\mathrm{ch}}(\mathfrak{g}_{>0})\mid\deg_{\mathrm{ch}}(A)=n\}$. Let $\Phi(\mathfrak{g}_{\frac{1}{2}})$ be the neutral free superfermion vertex superalgebra associated with $\mathfrak{g}_{\frac{1}{2}}$ which is generated by the fields $\Phi_\alpha(z)$, $(\alpha\in\Delta_{\frac{1}{2}})$ satisfying the OPEs

$$\Phi_\alpha(z)\Phi_\beta(w)\sim\frac{\chi([e_\alpha,e_\beta])}{z-w},\quad (\alpha,\beta\in\Delta_{\frac{1}{2}}).$$

The following is proven straightforwardly:

Lemma 2.1. *The map $f_\Phi\colon e_\alpha(z)\mapsto\delta_{\alpha\in\Delta_{1/2}}\Phi_\alpha(z)+\chi(e_\alpha)$, $(\alpha\in\Delta_{>0})$, defines an action of the loop algera $L\mathfrak{g}_{>0}:=\mathfrak{g}_{>0}((t))$ on $\Phi(\mathfrak{g}_{\frac{1}{2}})$.*

Given a $\mathfrak{g}_{>0}[[t]]$-integrable $\mathfrak{g}_{>0}((t))$-module M, the semi-infinite cohomology of $L\mathfrak{g}_{>0}$ with coefficients in M is defined by

$$H^{\frac{\infty}{2}+\bullet}(L\mathfrak{g}_{>0}; M) := H^{\bullet}(M \otimes F_{\mathrm{ch}}(\mathfrak{g}_{>0}), d_{\mathrm{st}})$$

where $d_{\mathrm{st}} = \int d_{\mathrm{st}}(z)dz$ with

$$d_{\mathrm{st}}(z) = \sum_{\alpha \in \Delta_{>0}} (-1)^{\bar{\alpha}} : e_{\alpha}(z)\varphi^{\alpha}(z) : -\frac{1}{2} \sum_{\alpha,\beta,\gamma \in \Delta_{>0}} (-1)^{\bar{\alpha}\bar{\gamma}} c_{\alpha,\beta}^{\gamma} : \varphi_{\gamma}(z)\varphi^{\alpha}(z)\varphi^{\beta}(z) :,$$

see [Fei]. Then the Drinfeld–Sokolov reduction cohomology functor associated with f is defined by

$$H_f^{\bullet}(M) := H^{\frac{\infty}{2}+\bullet}(L\mathfrak{g}_{>0}; M \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}))$$

for a $V^k(\mathfrak{g})$-module M where $L\mathfrak{g}_{>0}$ acts on $M \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$ diagonally. In particular, $\mathcal{W}^k(\mathfrak{g}, f; \Gamma) := H_f^{\bullet}(V^k(\mathfrak{g}))$ is naturally a vertex superalgebra and called the $\mathcal{W}$-superalgebra $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ associated with $(\mathfrak{g}, f, \Gamma)$ at level k. Equivalently, $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ is the vertex superalgebra defined as the cohomology the d.g. vertex superalgebra, called the BRST complex associated with $(\mathfrak{g}, f, \Gamma)$ at level k:

$$C_k^{\bullet}(\mathfrak{g}, f; \Gamma) := V^k(\mathfrak{g}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \otimes F_{\mathrm{ch}}(\mathfrak{g}_{>0}) = \bigoplus_{n \in \mathbb{Z}} V^k(\mathfrak{g}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \otimes F_{\mathrm{ch}}^n$$

with differential $d = \int d(z)\ dz$, where $d(z) = d_{\mathrm{st}}(z) + d_{\mathrm{ne}}(z) + d_{\chi}(z)$ and

$$d_{\mathrm{ne}}(z) = \sum_{\alpha \in \Delta_{1/2}} : \varphi^{\alpha}(z)\Phi_{\alpha}(z) :, \quad d_{\chi}(z) = \sum_{\alpha \in \Delta_1} \chi(e_{\alpha})\varphi^{\alpha}(z).$$

By [KW2], $\mathcal{W}^k(\mathfrak{g}, f; \Gamma) = H^0(C_k^{\bullet}(\mathfrak{g}, f; \Gamma), d)$ and when $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$, the vertex algebra structure on $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ does not depend on Γ, which determines only the conformal structure [BG, AKM]. For an arbitrary $V^k(\mathfrak{g})$-module M, $H_f^n(M)$ is naturally a $\mathcal{W}^k(\mathfrak{g}, f, \Gamma)$-module. This defines a $\mathbb{C}$-linear functor between the module categories

$$H_f^n(?) \colon V^k(\mathfrak{g})\text{-mod} \to \mathcal{W}^k(\mathfrak{g}, f; \Gamma)\text{-mod}.$$

2.2. **Miura maps and Screening operators.** The calculation of the cohomology $H^{\bullet}(C_k(\mathfrak{g}, f; \Gamma), d)$ in [KW2] via a certain spectral sequence implies an embedding $\mathcal{W}^k(\mathfrak{g}, f; \Gamma) \hookrightarrow V^{\tau_k}(\mathfrak{g}_{\leq 0}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$ where the level τ_k of the affine vertex superalgebra $V^{\tau_k}(\mathfrak{g}_{\leq 0})$ is given by

$$\tau_k(u|v) = k(u|v) + \frac{1}{2}\kappa_{\mathfrak{g}}(u|v) - \frac{1}{2}\kappa_{\mathfrak{g}_0}(u|v), \quad u, v \in \mathfrak{g}_0. \tag{2.2}$$

By composing it with the natural projection $V^{\tau_k}(\mathfrak{g}_{\leq 0}) \twoheadrightarrow V^{\tau_k}(\mathfrak{g}_0)$, we obtain a vertex superalgebra homomorphism

$$\mu_k^{\Gamma} \colon \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \to V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}), \tag{2.3}$$

called *the Miura map*, which is injective by [Ar7, DKV, N2]. Following [F1, G1], we describe the image of (2.3) at generic levels $k \in \mathbb{C}$ as the joint kernel of certain screening operators. By [KW2], the complex $C_k(\mathfrak{g}, f; \Gamma)$ is quasi-isomorphic to a subcomplex C_k^{+} generated as a vertex subalgebra by the fields

$$J^{(u)}(z) = u(z) + \sum_{\alpha,\beta \in \Delta_{>0}} c_{u,\beta}^{\alpha} : \varphi_{\alpha}(z)\varphi^{\beta}(z) :, \quad (u \in \mathfrak{g}_{\leq 0}),$$

$$\varphi^{\alpha}(z),\ (\alpha \in \Delta_{>0}), \quad \Phi_{\beta}(z),\ (\beta \in \Delta_{\frac{1}{2}}).$$

Note that the fields $J^{(u)}(z)$, $(u \in \mathfrak{g}_{\leq 0})$ generate an affine vertex superalgebra $V^{\tau_k}(\mathfrak{g}_{\leq 0})$ and the fields $\Phi_{\beta}(z)$, $(\beta \in \Delta_{\frac{1}{2}})$, generate $\Phi(\mathfrak{g}_{\frac{1}{2}})$. Thus $C_k^{+} = V^{\tau_k}(\mathfrak{g}_{\leq 0}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \otimes F_{\mathrm{ch},+}$ where $F_{\mathrm{ch},+} \subset F_{\mathrm{ch}}(\mathfrak{g}_{>0})$ denotes the subalgebra generated by the fields $\varphi^{\alpha}(z)$, $(\alpha \in \Delta_{>0})$. Note that the complex $C_k^{+} = \oplus_{n \geq 0} C_k^{+,n}$ has non-negative cohomological degrees and thus the zero-th cohomology $\mathcal{W}^k(\mathfrak{g},f;\Gamma)\simeq H^0(C_k^+,d)$ is a vertex subalgebra of C_k^+. Define a weight decomposition $C_k^+=\oplus_{n\in\mathbb{Z}}C_{k,n}^+$ by setting

$$\mathrm{wt}(J^{(u)})=-2\mathrm{deg}(u),\quad \mathrm{wt}(\Phi_\alpha)=0,\quad \mathrm{wt}(\varphi^\alpha)=2\mathrm{deg}(\alpha) \tag{2.4}$$

and $\mathrm{wt}(ab)=\mathrm{wt}(a)+\mathrm{wt}(b)$, $\mathrm{wt}(\partial a)=\mathrm{wt}(a)$ for $a,b\in C_k^+$. Then the differential d has the following weights:

$$\mathrm{wt}(d_{\mathrm{st}})=0,\quad \mathrm{wt}(d_{\mathrm{ne}})=1,\quad \mathrm{wt}(d_\chi)=2.$$

Thus d preserves the decreasing filtration $\{F_nC_k^+=\oplus_{j\geq n}C_{k,j}^+\}_{n\in\mathbb{Z}}$ and a convergent associated spectral sequence $\{E_r^\bullet,d_r\}_{r=1}^\infty\Rightarrow \mathrm{gr}_F H^\bullet(C_k^+,d)$ is obtained. For $\alpha\in I_{>0}$, set $[\alpha]=\{\beta\in\Delta\mid\beta-\alpha\in\mathbb{Z}\Delta_0\}$ and define an irreducible $\mathfrak{g}_0$-module $\mathbb{C}^{[\alpha]}$ by

$$\mathbb{C}^{[\alpha]}=\bigoplus_{\beta\in[\alpha]}\mathbb{C}v_\beta,\quad u\cdot v_\beta=\sum_\gamma c_{\gamma,u}^\beta v_\gamma,\ (u\in\mathfrak{g}_0), \tag{2.5}$$

and an induced module M_α of the affine Lie algebra $\hat{\mathfrak{g}}_{0,\tau_k}$ at level τ_k

$$M_\alpha:=U(\hat{\mathfrak{g}}_{0,\tau_k})\otimes_{U(\mathfrak{g}_0[t]\oplus\mathbb{C}K)}\mathbb{C}^{[\alpha]}. \tag{2.6}$$

Then $E_1^\bullet$ contains

$$E_1^0\simeq V^{\tau_k}(\mathfrak{g}_0)\otimes\Phi(\mathfrak{g}_{\frac12}),\quad E_1^1\simeq\bigoplus_{\alpha\in I_{>0}}M_{[\alpha]}\otimes\Phi(\mathfrak{g}_{\frac12})$$

as $V^{\tau_k}(\mathfrak{g}_0)\otimes\Phi(\mathfrak{g}_{\frac12})$-modules. For generic $k\in\mathbb{C}$, the Miura map μ_k^Γ appears as the inclusion $\mathcal{W}^k(\mathfrak{g},f;\Gamma)\hookrightarrow E_1^k$ and moreover, a linear map $\tilde{d}:=d_{\mathrm{ne}}+d_\chi$ induces a differential $[\tilde{d}]\colon E_1^0\to E_1^1$, which gives screening operators for $\mathcal{W}^k(\mathfrak{g},f;\Gamma)$. More precisely, introduce an element $V_\alpha\in M_\alpha\otimes\Phi(\mathfrak{g}_{\frac12})$ by

$$V_\alpha:=\sum_{\beta\in[\alpha]}v_\beta\otimes f_\Phi(e_\beta) \tag{2.7}$$

Then the structure morphism of module $Y(?,z)\colon(V^{\tau_k}(\mathfrak{g}_0)\otimes\Phi(\mathfrak{g}_{\frac12}))\otimes(M_\alpha\otimes\Phi(\mathfrak{g}_{\frac12}))\to(M_\alpha\otimes\Phi(\mathfrak{g}_{\frac12}))((z))$ induces an intertwining operator

$$Y(V_\alpha,z)\colon V^{\tau_k}(\mathfrak{g}_0)\otimes\Phi(\mathfrak{g}_{\frac12})\to(M_\alpha\otimes\Phi(\mathfrak{g}_{\frac12}))((z)),$$

for each V_α by the skew-symmetry e.g. [X, Theorem 3.4.10] and thus a linear map

$$\mathbf{S}_\alpha^\Gamma:=\int Y(V_\alpha,z)dz\colon V^{\tau_k}(\mathfrak{g}_0)\otimes\Phi(\mathfrak{g}_{\frac12})\to M_\alpha\otimes\Phi(\mathfrak{g}_{\frac12}). \tag{2.8}$$

For generic $k\in\mathbb{C}$, we have $[\tilde{d}]=\oplus_{\alpha\in I_{>0}}\mathbf{S}_\alpha^\Gamma$ and the injective homomorphism (2.3) is extended to a short exact sequence

$$0\to\mathcal{W}^k(\mathfrak{g},f;\Gamma)\xrightarrow{\mu_k^\Gamma}V^{\tau_k}(\mathfrak{g}_0)\otimes\Phi(\mathfrak{g}_{\frac12})\xrightarrow{\oplus_\alpha\mathbf{S}_\alpha^\Gamma}\bigoplus_{\alpha\in I_{>0}}M_\alpha\otimes\Phi(\mathfrak{g}_{\frac12}). \tag{2.9}$$

and in particular an isomorphism of vertex superalgebras at generic $k\in\mathbb{C}$

$$\mathcal{W}^k(\mathfrak{g},f;\Gamma)\simeq\bigcap_{\alpha\in I_{>0}}\mathrm{Ker}\left(\mathbf{S}_\alpha^\Gamma\colon V^{\tau_k}(\mathfrak{g}_0)\otimes\Phi(\mathfrak{g}_{\frac12})\to M_\alpha\otimes\Phi(\mathfrak{g}_{\frac12})\right). \tag{2.10}$$

2.3. **Generalized Wakimoto representations for $V^k(\mathfrak{g})$.** In the case when $\mathfrak{g}$ is purely even, we introduce the generalized Wakimoto resolution $D_{\Gamma,\bullet}^{W,k}$ for an affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g},f;\Gamma)$ obtained in [N1]. For this purpose, we first review generalized Wakimoto representations of an affine vertex algebra $V^k(\mathfrak{g})$ at generic level $k\in\mathbb{C}$.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with a good grading (2.1) and G a connected algebraic group with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. By the correspondence of Lie subalgebra of $\mathfrak{g}$ and closed subgroups of G, there exist unique closed subgroups N_+, H, N_-, B_- corresponding to the Lie subalgebras $\mathfrak{n}_+$, $\mathfrak{h}$, $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$, respectively. Since $\mathfrak{n}_- \subset \mathfrak{b}_-$ is an ideal, we have the decomposition $B_- = N_-.H$ and thus the principal H-bundle $\pi \colon \mathcal{P} := N_-\backslash G \twoheadrightarrow B_-\backslash G$. Note that $\mathcal{P}$ is G-equivariant with respect to the natural right G-action. The infinitesimal $\mathfrak{g}$-action on the open subset $N_+ \subset B_-\backslash G$ gives rise to an algebra homomorphism

$$\rho_{\mathrm{fin}} \colon U(\mathfrak{g}) \to \mathcal{D}_{N_+} \otimes U(\mathfrak{h}).$$

Remark 2.2. *We remark that in the literature ρ_{fin} is derived from the right coset construction $G/N_- \twoheadrightarrow G/B_-$, cf.,* [ACL2, F2, G2]. *In this approach, the naive homomorphism ρ_{fin} is anti-algebra homomorphism since left group actions induce right representations of groups on the coordinate rings, and a certain automorphism of $U(\mathfrak{g})$ is used implicitly to obtain an algebra homomorphism. Therefore, it is more natural to use the left coset constructions, which induce algebra homomorphisms directly. Thus, our notation for left/right is switched from those in the literature.*

Here $\mathcal{D}_{N_+}$ is the algebra of differential operators and $U(\mathfrak{h})$ is regarded as the left H-invariant differential operators on the fiber $\pi^{-1}(e) = H$. The map ρ_{fin} is injective by [F2, §7.1.1]. We may describe ρ_{fin} more explicitly as follows. Since N_+ is unipotent, the exponential map $\exp \colon \mathfrak{n}_+ \simeq N_+$ gives an isomorphism of affine varieties. By taking the coordinates $\{x_\alpha\}_{\alpha\in\Delta_+}$ of $\mathfrak{n}_+$ with respect to the basis $\{e_\alpha\}_{\alpha\in\Delta_+}$, we have $\mathbb{C}[N_+] \simeq \mathbb{C}[\mathfrak{n}_+] = \mathbb{C}[x_\alpha | \alpha \in \Delta_+]$. Then ρ_{fin} satisfies

$$\begin{aligned} \rho_{\mathrm{fin}}(e_\alpha) &= \sum_{\beta\in\Delta_+} P^\beta_\alpha(x)\partial_\beta, \\ \rho_{\mathrm{fin}}(e_{-\alpha}) &= \sum_{\beta\in\Delta_+} Q^\beta_\alpha(x)\partial_\beta + x_\alpha h_\alpha, \\ \rho_{\mathrm{fin}}(h_\alpha) &= -\sum_{\beta\in\Delta_+} \beta(h_\alpha) x_\beta \partial_\beta + h_\alpha, \end{aligned} \tag{2.11}$$

for certain polynomials $P^\beta_\alpha, Q^\beta_\alpha \in \mathbb{C}[N_+]$, $(\alpha \in \Delta_+)$, [F2, §5.2.5]. The map ρ_{fin} admits the following vertex algebra analogue. Let $\mathcal{A}_{\Delta_+}$ denote the $\beta\gamma$-system generated by the fields $a_\alpha(z)$, $a^*_\alpha(z)$, $(\alpha \in \Delta_+)$, satisfying the OPEs

$$a_\alpha(z)a^*_\beta(w) \sim \frac{\delta_{\alpha,\beta}}{z-w}, \quad a_\alpha(z)a_\beta(w) \sim 0 \sim a^*_\alpha(z)a^*_\beta(w)$$

and $\pi^{k+h^\vee}_{\mathfrak{h}}$ the Heisenberg vertex algebra generated by the fields $u(z)$, $(u \in \mathfrak{h})$, satisfying the OPEs

$$u(z)v(w) \sim \frac{(k+h^\vee)(u|v)}{(z-w)^2}, \quad (u, v \in \mathfrak{h}).$$

Lemma 2.3 ([FFr1, F1])**.** *There exists an unique injective vertex algebra homomorphism $\hat{\rho}_k \colon V^k(\mathfrak{g}) \to \mathcal{A}_{\Delta_+} \otimes \pi^{k+h^\vee}_{\mathfrak{h}}$ such that*

$$\begin{aligned} \hat{\rho}_k(e_\alpha(z)) &= \sum_{\beta\in\Delta_+} :P^\beta_\alpha(a^*)(z)a_\beta(z):, \\ \hat{\rho}_k(e_{-\alpha}(z)) &= \sum_{\beta\in\Delta_+} :Q^\beta_\alpha(a^*)(z)a_\beta(z): + :h_\alpha(z)a^*_\alpha(z): + ((e_\alpha|f_\alpha)k + c_\alpha)\partial a^*_\alpha(z), \\ \hat{\rho}_k(h_\alpha(z)) &= -\sum_{\beta\in\Delta_+} \beta(h_\alpha) :a^*_\beta(z)a_\beta(z): + h_\alpha(z) \end{aligned}$$

for some constant $c_\alpha \in \mathbb{C}$, $(\alpha \in \Pi)$.

The realization $\hat{\rho}_k$ is called the *Wakimoto realization* of $V^k(\mathfrak{g})$. Let $\pi^{k+h^\vee}_{\mathfrak{h},\beta}$, $(\beta \in \mathfrak{h}^*)$, denote the Fock module of $\pi^{k+h^\vee}_{\mathfrak{h}}$ generated by a vector $|\beta\rangle$ satisfying

$$u_{(n)}|\beta\rangle = \delta_{n,0}\beta(u)|\alpha\rangle, \quad (u \in \mathfrak{h}, n \geq 0).$$

Then the $\mathcal{A}_{\Delta_+} \otimes \pi^{k+h^\vee}_{\mathfrak{h}}$-module $W^k(\beta) := \mathcal{A}_{\Delta_+} \otimes \pi^{k+h^\vee}_{\mathfrak{h},\beta}$ admits a $V^k(\mathfrak{g})$-module structure through $\hat{\rho}_k$. It is called the *Wakimoto module* of highest weight β at level $k \in \mathbb{C}$. For generic $k \in \mathbb{C}$, the injective homomorphism $\hat{\rho}_k$ is extended to an exact sequence

$$0 \to V^k(\mathfrak{g}) \xrightarrow{\hat{\rho}_k} W^k(0) \xrightarrow{\bigoplus \mathbf{Q}_\alpha} \bigoplus_{\alpha \in I} W^k(-\alpha) \tag{2.12}$$

where

$$\mathbf{Q}_\alpha = \int Y(\hat{\rho}^L(e_\alpha)|-\alpha\rangle, z)dz, \quad (\alpha \in I). \tag{2.13}$$

Here we have used a vertex superalgebra homomorphism

$$\hat{\rho}^L\colon V^0(\mathfrak{n}_+) \to \mathcal{A}_{\Delta_+}, \quad e_\alpha(z) \mapsto \sum_{\beta \in \Delta_+} :P^{\beta,L}_\alpha(a^*)(z)a_\beta(z): \tag{2.14}$$

induced from the left N_+-action on the open dense subset $N_+ \subset B_-\backslash G$

$$N_+ \times N_+ \to N_+, \quad (g,h) \mapsto g^{-1}h,$$

which gives rise to an algebra homomorphism $U(\mathfrak{n}_+) \to \mathcal{D}_{N_+}$.

The above construction is generalized to an arbitrary parabolic subalgebra of $\mathfrak{g}$. In particular, we consider the case $\mathfrak{g}_{\geq 0} \subset \mathfrak{g}$ with respect to the good grading (2.1). In this case, let $G>0$, G_0, $G_{\leq 0}$, $G_{<0}$ denote the closed subgroups of G corresponding to $\mathfrak{g}_{>0}$, $\mathfrak{g}_0$, $\mathfrak{g}_{\leq 0}$, $\mathfrak{g}_{<0}$, respectively. Then we consider the principal G_0-bundle $\pi_\Gamma\colon \mathcal{P}_\Gamma = G_{<0}\backslash G \twoheadrightarrow G_{\leq 0}\backslash G$, which induces an algebra homomorphism

$$\rho_{\mathrm{fin},\Gamma}\colon U(\mathfrak{g}) \to \mathcal{D}_{G_{>0}} \otimes U(\mathfrak{g}_0).$$

Following [G2], we embed $G_{>0} \hookrightarrow N_+$ by the decomposition $N_+ = N_{+,0}.G_{>0}$ where $N_{+,0} \subset G$ is the closed subgroup with $\mathrm{Lie}(N_{+,0}) = \mathfrak{n}_{+,0} := \mathfrak{g}_0 \cap \mathfrak{n}_+$ and consider the isomorphism $\mathfrak{n}_+ = \mathfrak{n}_{+,0} \oplus \mathfrak{g}_{>0} \simeq N_{+,0}.G_{>0} = N_+$, $((a,b) \mapsto \exp(a)\exp(b))$. Then in the above notation, we have $\mathbb{C}[N_{+,0}] \simeq \mathbb{C}[\mathfrak{n}_{+,0}] = \mathbb{C}[x_\alpha|\alpha \in \Delta^0_+]$, $\mathbb{C}[G_{>0}] \simeq \mathbb{C}[\mathfrak{g}_{>0}] = \mathbb{C}[x_\alpha|\alpha \in \Delta_{>0}]$, and $\mathbb{C}[N_+] \simeq \mathbb{C}[N_{+,0}] \otimes \mathbb{C}[G_{>0}]$. Moreover, the formulas (2.11) and thus Lemma 2.3 does not change under this modification. By [F1], $\rho_{\mathrm{fin},\Gamma}$ satisfies

$$\begin{aligned} \rho_{\mathrm{fin},\Gamma}(e_\alpha) &= \sum_{\beta \in \Delta_{>0}} P^\beta_{\Gamma,\alpha}(x)\partial_\beta, \quad (\alpha \in \Delta_{>0}), \\ \rho_{\mathrm{fin},\Gamma}(u) &= -\sum_{\beta,\gamma \in \Delta_{>0}} c^\alpha_{u,\beta} x_\beta \partial_\alpha + u, \quad (u \in \mathfrak{g}_0). \end{aligned} \tag{2.15}$$

The projection π_Γ factors through

$$\begin{tikzcd} G_{<0}\backslash G \arrow[r, two heads] \arrow[d, "\pi_\Gamma"] & N_-\backslash G \arrow[d, "\pi"] \\ G_{\leq 0}\backslash G & B_-\backslash G \arrow[l, two heads] \end{tikzcd}$$

in a right G-equivariant way and the the projection $B_-\backslash G \twoheadrightarrow G_{\leq 0}\backslash G$ is a principal $B_-\backslash G_{\leq 0}(\simeq B_{-,0}\backslash G_0)$-bundle. Here $B_{-,0} = B_- \cap G_0$ is a Borel subgroup of G_0.

Then it follows that ρ_{fin} factors through

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[r, "\rho_{\mathrm{fin},\Gamma}"] \arrow[dr, "\rho_{\mathrm{fin}}"'] & \mathcal{D}_{G_{>0}} \otimes U(\mathfrak{g}_0) \arrow[d, "\rho_{\mathrm{fin},0}"] \\ & \mathcal{D}_{N_+} \otimes U(\mathfrak{h}) \end{tikzcd} .$$

Here $\rho_{\mathrm{fin},0} \colon U(\mathfrak{g}_0) \to \mathcal{D}_{N_{+,0}} \otimes U(\mathfrak{h}) \subset \mathcal{D}_{N_+} \otimes U(\mathfrak{h})$ is an algebra homomorphism defined in the same way as ρ_{fin} with $\mathfrak{g}$ replaced by $\mathfrak{g}_0$. The map $\rho_{\mathrm{fin},\Gamma}$ admits the following vertex algebra analogue.

Lemma 2.4 ([F1])**.** *There exists an unique injective vertex algebra homomorphism $\hat{\rho}_{k,\Gamma} \colon V^k(\mathfrak{g}) \to \mathcal{A}_{\Delta_{>0}} \otimes V^{\tau_k}(\mathfrak{g}_0)$ such that the composition*

$$\hat{\rho}_{\tau_k,0} \circ \hat{\rho}_{k,\Gamma} \colon V^k(\mathfrak{g}) \to \mathcal{A}_{\Delta_{>0}} \otimes V^{\tau_k}(\mathfrak{g}_0) \to \mathcal{A}_{\Delta_+} \otimes \pi_{\mathfrak{h}}^{k+h^\vee}$$

coincides with $\hat{\rho}_k$ where $\hat{\rho}_{\tau_k,0} \colon V^{\tau_k}(\mathfrak{g}_0) \to \mathcal{A}_{\Delta_+^0} \otimes \pi_{\mathfrak{h}}^{k+h^\vee}$ is the Wakimoto realization of $V^{\tau_k}(\mathfrak{g}_0)$. Moreover, $\hat{\rho}_{k,\Gamma}$ satisfies

$$\begin{aligned} \hat{\rho}_{k,\Gamma}(e_\alpha(z)) &= \sum_{\beta \in \Delta_{>0}} : P^\beta_{\Gamma,\alpha}(a^*)(z) a_\beta(z) :, \quad (\alpha \in \Delta_{>0}), \\ \hat{\rho}_{k,\Gamma}(u(z)) &= -\sum_{\alpha,\beta \in \Delta_{>0}} c^\alpha_{u,\beta} : a^*_\beta(z) a_\alpha(z) : + u(z), \quad (u \in \mathfrak{g}_0). \end{aligned} \tag{2.16}$$

The realization $\hat{\rho}_{k,\Gamma}$ is called a *generalized Wakimoto realization* of $V^k(\mathfrak{g})$. Let $P_{0,+} = \{\beta \in \mathfrak{h}^* \mid (\beta|\alpha_i^\vee) \in \mathbb{Z}_{\geq 0}\ (\forall i \in I)\}$. For $\beta \in P_{0,+}$, let $L_0(\beta)$ denote the simple $\mathfrak{g}_0$-module with highest weight β and $\mathbb{V}_0^{\tau_k}(\beta)$ the Weyl module of $\hat{\mathfrak{g}}_0$ at level τ_k

$$\mathbb{V}_0^{\tau_k}(\beta) = U(\hat{\mathfrak{g}}_{0,\tau_k}) \underset{U(\mathfrak{g}_0[t] \oplus \mathbb{C}K)}{\otimes} L_0(\beta),$$

where $\mathfrak{g}_0[t]$ acts on $L_0(\beta)$ through the projection $\mathfrak{g}_0[t] \twoheadrightarrow \mathfrak{g}_0$ and K by k. Then $\mathbb{V}_0^{\tau_k}(\beta)$ is a $V^\tau(\mathfrak{g}_0)$-module and thus $W^k_\Gamma(\beta) := \mathcal{A}_{\Delta_{>0}} \otimes \mathbb{V}_0^{\tau_k}(\beta)$ admits a $V^k(\mathfrak{g})$-module structure through $\hat{\rho}_{k,\Gamma}$. It is called the *generalized Wakimoto representation* of $V^k(\mathfrak{g})$ with highest weight β at level $k \in \mathbb{C}$.

2.4. **Resolutions for $V^k(\mathfrak{g})$.** Here we extend the injective homomorphisms $\hat{\rho}_k$ and $\hat{\rho}_{k,\Gamma}$ to exact sequences, which give resolutions of $V^k(\mathfrak{g})$ by generalized Wakimoto representations at generic level $k \in \mathbb{C}$. Let W (resp. W_0) denote the Weyl group of $\mathfrak{g}$ (resp. $\mathfrak{g}_0$), $\ell(w)$ the length of $w \in W$, $w_\circ$ the longest element in W, $\lambda \mapsto w \circ \lambda$, $(\lambda \in \mathfrak{h}^*)$, the dot action of $w \in W$. We denote by $M(\lambda)$ (resp. $M_0(\lambda)$) the Verma module of $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) with highest weight $\lambda \in \mathfrak{h}^*$ and by $L(\lambda)$ (resp. $L_0(\lambda)$) be the simple quotient of $M(\lambda)$ (resp. $M_0(\lambda)$). The generalized Verma module $M_\Gamma(\lambda)$ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ for the parabolic subalgebra $\mathfrak{g}_{\geq 0}$ is defined by

$$M_\Gamma(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} L_0(\lambda),$$

where $\mathfrak{g}_{\geq 0}$ acts on $L_0(\lambda)$ through the projection $\mathfrak{g}_{\geq 0} \twoheadrightarrow \mathfrak{g}_0$. Then we have canonical projections $M(\lambda) \twoheadrightarrow M_\Gamma(\lambda) \twoheadrightarrow L(\lambda)$ of $\mathfrak{g}$-modules and thus canonical embeddings of their dual $\mathfrak{g}$-modules $L(\lambda)^\vee \hookrightarrow M_\Gamma(\lambda)^\vee \hookrightarrow M(\lambda)^\vee$. By [Ku, Theorem 9.2.18], we have the (horizontally exact) commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] & C_0^\vee \arrow[r] & C_1^\vee \arrow[r] & \cdots \arrow[r] & C^\vee_{\ell(w'_\circ)} \arrow[r] & \cdots \arrow[r] & C^\vee_{\ell(w_\circ)} \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[u, equal] & C^\vee_{\Gamma,0} \arrow[r] \arrow[u, hook] & C^\vee_{\Gamma,1} \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & C^\vee_{\Gamma,\ell(w'_\circ)} \arrow[r] \arrow[u, hook] & 0, \end{tikzcd} \tag{2.17}$$

where

$$C_i^\vee = \bigoplus_{\substack{w \in W \\ \ell(w)=i}} M(w^{-1}\circ 0)^\vee, \quad C_{\Gamma,i}^\vee = \bigoplus_{\substack{w \in W_0' \\ \ell(w)=i}} M_\Gamma(w^{-1}\circ 0)^\vee,$$

$$W_0' = \{w \in W \mid \forall u \in W_0,\ \ell(wu) \geq \ell(w)\},$$

with $w_\circ' \in W_0$ denotes the longest element. Here the first row is the dual of the BGG resolution of the trivial $\mathfrak{g}$-module $\mathbb{C}$ and the second row is the dual of the parabolic BGG resolution of $\mathbb{C}$ associated with a Levi subalgebra $\mathfrak{g}_0$ embedded into the first row by canonical embeddings.

Next, we apply Fiebig's equivalence [Fie] to (2.17) in order to deduce the corresponding resolution for $V^k(\mathfrak{g})$ at generic level $k \in \mathbb{C}$. For generic $k \in \mathbb{C}$, the affine vertex algebra $V^k(\mathfrak{g})$ admits a conformal vector

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \frac{1}{2(k+h^\vee)} \sum_{i\alpha \in I \sqcup \Delta} :e_\alpha(z) e^\alpha(z):,$$

by Sugawara construction where $\{e^\alpha\}_\alpha \subset \mathfrak{g}$ is the dual basis of $\{e_\alpha\}_\alpha$ by $(\cdot|\cdot)$. Let $\widehat{\mathcal{O}}_k$ denote the full subcategory of the category of $V^k(\mathfrak{g})$-modules consisting of objects satisfying the following properties: (1) $\mathfrak{h}$ and L_0 acts on M semisimply, (2) $\dim U(\hat{\mathfrak{n}}_+)m < \infty$ for all $m \in M$ where $\hat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus \mathfrak{g}[t]t$. For $M \in \widehat{\mathcal{O}}_k$, let $M = \oplus_{\Delta\in\mathbb{C}} M_\Delta$ be the L_0-grading decomposition and $M^{\mathrm{top}} \subset M$ be the $\mathfrak{g}$-submodule defined by the direct sum of M_r such that $M_{r+n} = 0$ for all $n \in \mathbb{Z}_{<0}$. By Fiebig's equivalence [Fie], the functor

$$(?)^{\mathrm{top}} : \widehat{\mathcal{O}}_k \to \mathcal{O}_{\mathrm{fin}}, \quad M \mapsto M^{\mathrm{top}},$$

gives an equivalence of categories of $\widehat{\mathcal{O}}_k$ and the BGG category $\mathcal{O}_{\mathrm{fin}}$ of $\mathfrak{g}$-modules (see e.g. [Hum]) and the inverse is give by the induction functor

$$\mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}_k}(?) \colon \mathcal{O}_{\mathrm{fin}} \to \widehat{\mathcal{O}}_k, \quad N \mapsto U(\hat{\mathfrak{g}}) \underset{U(\mathfrak{g}[t]\oplus\mathbb{C}K)}{\otimes} N,$$

where $\mathfrak{g}[t] \oplus \mathbb{C}K$ acts on N through $\mathfrak{g}[t] \oplus \mathbb{C}K \twoheadrightarrow \mathfrak{g} \oplus \mathbb{C}K$ and K acts by $k \in \mathbb{C}$.

For $M \in \widehat{\mathcal{O}}_k$, let $M = \oplus_{\lambda\in\mathfrak{h}^*} M^\lambda$ be the $\mathfrak{h}$-weight decomposition, $M_\Delta^\lambda = M_\Delta \cap M^\lambda$ and $M^\vee = \bigoplus_{\lambda,\Delta} \mathrm{Hom}_F\left(M_\Delta^\lambda, F\right)$ be the contragredient dual of M. Since $((\mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}^F}(N))^\vee)^{\mathrm{top}} = N^\vee$, we have $(\mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}_k}(N))^\vee \simeq \mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}^F}(N^\vee)$ for $N \in \mathcal{O}_{\mathrm{fin}}$. Let $\mathbb{M}^k(\lambda) = \mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}_k}(M(\lambda))$ be the Verma module of $\hat{\mathfrak{g}}$ with highest weight $\lambda \in \mathfrak{h}^*$ at level $k \in \mathbb{C}$, and $\mathbb{M}_\Gamma^k(\lambda) = \mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}_k}(M_\Gamma(\lambda))$. Then $\mathbb{M}^k(\lambda), \mathbb{M}_\Gamma^k(\lambda) \in \widehat{\mathcal{O}}_k$ and their contragradient duals are generalized Wakimoto representations

$$\mathbb{M}^k(\lambda)^\vee \simeq W^k(\lambda),\ (\lambda \in \mathfrak{h}^*), \quad \mathbb{M}_\Gamma^k(\lambda)^\vee \simeq W_\Gamma^k(\lambda),\ (\lambda \in P_{+,0}). \tag{2.18}$$

by [ACL2, Proposition 3.3] and [G2, Lemma A.2] respectively, see also [FFr3]. Note that under (2.18), the canonical embedding $M_\Gamma(\lambda)^\vee \hookrightarrow M(\lambda)^\vee$ induces a canonical embedding

$$\hat{\rho}_{\tau_k,0,\lambda} \colon W_\Gamma^k(\lambda) \hookrightarrow W^k(\lambda), \tag{2.19}$$

which generalizes the Wakimoto realization $\hat{\rho}_{\tau_k,0,} \colon \mathcal{A}_{\Delta_{>0}} \otimes V^{\tau_k}(\mathfrak{g}_0) \hookrightarrow \mathcal{A}_{\Delta_+} \otimes \pi_{\mathfrak{h}}^{k+^\vee}$ corresponding to the case $\lambda = 0$. W apply $\mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}^F}$ to (2.17) and obtain a horizontally exact commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & V^k(\mathfrak{g}) \arrow[r, "\epsilon"] \arrow[d, equal] & D_0^k \arrow[r, "d_0"] & D_1^k \arrow[r] & \cdots \arrow[r] & D^k_{\ell(w'_\circ)} \arrow[r] & \cdots \arrow[r] & D^k_{\ell(w_\circ)} \arrow[r] & 0 \\
0 \arrow[r] & V^k(\mathfrak{g}) \arrow[r, "\epsilon_\Gamma"] & D^k_{\Gamma,0} \arrow[r, "d_{\Gamma,0}"] \arrow[u, hook, "\iota_0"] & D^k_{\Gamma,1} \arrow[r] \arrow[u, hook, "\iota_1"] & \cdots \arrow[r] & D^k_{\Gamma,\ell(w'_0)} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}
\tag{2.20}
$$

where

$$D_i^k = \bigoplus_{\substack{w\in W \\ \ell(w)=i}} W^k(w^{-1}\circ 0), \quad D^k_{\Gamma,i} = \bigoplus_{\substack{w\in W'_0 \\ \ell(w)=i}} W^k_\Gamma\left(w^{-1}\circ 0\right),$$

and in particular, $D_0^k = W^k(0)$ and $D^k_{\Gamma,0} = W^k_\Gamma(0)$. Then by Fiebig's equivalence,

$$\dim \operatorname{Hom}_{\hat{\mathcal{O}}_k}(V^k(\mathfrak{g}), D_0^k) = \dim \operatorname{Hom}_{\mathcal{O}_{\mathrm{fin}}}(\mathbb{C}, C_0^\vee) = 1.$$

It follows that $\epsilon = \hat{\rho}_k$ and $\epsilon_\Gamma = \hat{\rho}_{k,\Gamma}$ by looking at the images of the vacuum vector $|0\rangle$. Similarly, we have $d_0 = \bigoplus_{\alpha\in I}\mathbf{Q}_\alpha$, see (2.13). We note that the higher differentials $d_i\colon D_i^k \to D^k_{i+1}$ with $i \geq 1$ are unique up to scalar by Fiebig's equivalence and [Ku, Lemma 9.2.16] and thus the exact sequence in the first row in (2.20) coincides with the one considered in [FFr8] up to scalar for the choices of higher differentials. Next, consider the column arrows $\iota_i\colon D^k_{\Gamma,i} \to D_i^k$, $(i \geq 0)$. Again by Fiebig's equivalence,

$$\dim_{\hat{\mathcal{O}}_k}\left(W^k_\Gamma(\lambda), W^k(\lambda)\right) = \dim_{\mathcal{O}_{\mathrm{fin}}}\left(M_\Gamma(\lambda)^\vee, M(\lambda)^\vee\right) = 1, \quad \lambda \in P_{0,+}.$$

Thus, we have $\iota_0 = \hat{\rho}_{\tau_k,0}$, $\iota_1 = \bigoplus_{\alpha\in\Gamma}\hat{\rho}_{\tau_k,0,-\alpha}$ and

$$\iota_i = \bigoplus_{\substack{w\in W'_0 \\ \ell(w)=i}} c_w \hat{\rho}_{\tau_k,0,w^{-1}\circ 0}, \quad (i \geq 2),$$

for some invertibles $c_w \in \mathbb{C}$. Finally, consider the second row. We already know $\epsilon_\Gamma = \hat{\rho}_{k,\Gamma}$. Since $d_{\Gamma,0}$ is the restriction of $d_0 = \oplus_{\alpha\in I}\mathbf{Q}_\alpha$ to $W^k_\Gamma(0)$ through ι_0, we have the decomposition $d_{\Gamma,0} = \oplus_{\alpha\in I_{>0}}\mathbf{Q}^\Gamma_\alpha$ so that the diagram

$$
\begin{tikzcd}
W^k(0) \arrow[r, "\mathbf{Q}_\alpha"] & W^k(-\alpha) \\
W^k_\Gamma(0) \arrow[r, "\mathbf{Q}^\Gamma_\alpha"] \arrow[u, hook, "\iota_0"] & W^k_\Gamma(-\alpha) \arrow[u, hook, "\iota_1"]
\end{tikzcd}
\tag{2.21}
$$

commutes for $\alpha \in I_{>0}$. Since $W^k(\lambda)^{\mathrm{top}} = \mathbb{C}[a^*_\beta|\beta\in\Delta_+]\otimes\mathbb{C}|\lambda\rangle$ and $W^k_\Gamma(\lambda)^{\mathrm{top}} = \mathbb{C}[a^*_\beta|\beta\in\Delta_{>0}]\otimes L_0(\lambda)$, the top component $(?)^{\mathrm{top}}$ of (2.21) is of the shape

$$
\begin{tikzcd}
\mathbb{C}[a^*_\beta|\beta\in\Delta_+] \arrow[r, "\mathbf{Q}^{\mathrm{top}}_\alpha"] & \mathbb{C}[a^*_\beta|\beta\in\Delta_+]\otimes\mathbb{C}|-\alpha\rangle \\
\mathbb{C}[a^*_\beta|\beta\in\Delta_{>0}] \arrow[r, "\mathbf{Q}^{\Gamma,\mathrm{top}}_\alpha"] \arrow[u, hook, "\iota_0"] & \mathbb{C}[a^*_\beta|\beta\in\Delta_{>0}]\otimes L_0(-\alpha) \arrow[u, hook, "\iota_1"]
\end{tikzcd}
$$

and by (2.13),

$$\mathbf{Q}^{\mathrm{top}}_\alpha = \sum_{\beta\in\Delta_+}\mathrm{mult}(\bar{v}_\beta)\partial_{a^*_\beta}, \quad \bar{v}_\beta = P^{\beta,L}_\alpha(a^*)|-\alpha\rangle. \tag{2.22}$$

For $\alpha \in I_{>0}$ and $\beta \in [\alpha]$, it follows from $a^*_\beta \in W^k_\Gamma(\lambda)^{\mathrm{top}}$ that $\bar{v}_\beta = \mathbf{Q}^{\mathrm{top}}_\alpha(a^*_\beta) \in W^k_\Gamma(-\alpha)^{\mathrm{top}}$. Note that $L_0(-\alpha)$ is isomorphism to $\mathbb{C}^{[\alpha]}$ in (2.5) by

$$\eta_\alpha \colon L_0(-\alpha) \simeq \mathbb{C}^{[\alpha]} = \bigoplus_{\beta\in[\alpha]} \mathbb{C}v_\beta, \quad |-\alpha\rangle \mapsto v_\alpha. \tag{2.23}$$

Lemma 2.5. *For $\beta \in [\alpha]$, we have $\bar{v}_\beta \in L_0(-\alpha)$ and, moreover, $I_\alpha(\bar{v}_\beta) = v_\beta$.*

Proof. The first assertion is immediate from the weight consideration. For the second assertion, note that the $(-\beta)$-weight space of $\mathbb{C}^{[\alpha]}$ is $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_\beta, \mathbb{C})$ and thus 1 dimensional. Since $I_\alpha(\bar{v}_\beta)$, v_β are of weight $-\beta$, it follows that $I_\alpha(\bar{v}_\beta) = c_{\alpha,\beta}v_\beta$ for some element $c_{\alpha,\beta} \in \mathbb{C}$. We have $c_{\alpha,\alpha} = 1$ since $I_\alpha(\bar{v}_\alpha) = I_\alpha(|-\alpha\rangle) = v_\alpha$. Since $\mathbf{Q}_\alpha$ is a residue of an $V^k(\mathfrak{g})$-intertwining operator, it follows that $\mathbf{Q}_\alpha$ is a $\mathfrak{g}_0$-homomorphism. Thus $[u, \mathbf{Q}^{\mathrm{top}}_\alpha] = 0$ for $u \in \mathfrak{g}_0$. By (2.16), $\mathfrak{g}_0$ acts on $\mathbb{C}[a^*_\beta | \beta \in \Delta_{>0}]$ by

$$u \mapsto -\sum_{p,q\in\Delta_{>0}} c^q_{u,p} a^*_q \partial_{a^*_p}, \quad (u \in \mathfrak{g}_0)$$

Hence it follows from $[u, \mathbf{Q}^{\mathrm{top}}_\alpha] = 0$ that

$$\sum_{\beta\in\Delta_{>0}} u(\bar{v}_\beta)\partial_{a^*_\beta} = -\sum_{\beta\in\Delta_{>0}} \bar{v}_\beta \left[u, \partial_{a^*_\beta}\right] = -\sum_{\beta,\gamma\in\Delta_{>0}} c^\beta_{u,\gamma}\bar{v}_\beta \partial_{a^*_\gamma},$$

and thus $u(\bar{v}_\beta) = \sum_{\gamma\in\Delta_{>0}} c^\gamma_{\beta,u}\bar{v}_\gamma$. Therefore, by the simplicity of $L_0(-\alpha)$, $c_{\alpha,\beta} = c_{\alpha,\alpha} = 1$. This completes the proof. $\square$

Thus, by using a vertex superalgebra homomorphism $\hat{\rho}^L_{\mathfrak{g}_{>0}} \colon V^0(\mathfrak{g}_{>0}) \to \mathcal{A}_{\Delta_{>0}}$ defined in the same way as (2.14), we have a decomposition $\mathbf{Q}^\Gamma_\alpha = \mathbf{Q}^{\Gamma,(1)}_\alpha + \mathbf{Q}^{\Gamma,(2)}_\alpha$ where

$$\begin{aligned} \mathbf{Q}^{\Gamma,(1)}_\alpha &= \sum_{\beta\in[\alpha]} \int Y(\hat{\rho}^L_{\mathfrak{g}_{>0}}(e_\beta)v_\beta, z)\ dz \colon W^k_\Gamma \to W^k_\Gamma(-\alpha), \\ \mathbf{Q}^{\Gamma,(2)}_\alpha &= \sum_{\beta\in\Delta_{>0}\backslash[\alpha]} \int Y(a_\beta\bar{v}_\beta, z)\ dz \colon W^k_\Gamma \to W^k_\Gamma(-\alpha). \end{aligned} \tag{2.24}$$

In summary, (2.20) is of the shape

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & V^k(\mathfrak{g}) & \xrightarrow{\hat{\rho}_k} & D^k_0 & \xrightarrow{\oplus\mathbf{Q}_\alpha} & D^k_1 & \longrightarrow \cdots \longrightarrow & D^k_{\ell(w'_\circ)} & \longrightarrow & \cdots \\ & & \| & & \big\uparrow{\scriptstyle\hat{\rho}_{\tau_k,0}} & & \big\uparrow{\scriptstyle\oplus\hat{\rho}_{\tau_k,0,-\alpha}} & & \big\uparrow & & \\ 0 & \longrightarrow & V^k(\mathfrak{g}) & \xrightarrow{\hat{\rho}_{k,\Gamma}} & D^k_{\Gamma,0} & \xrightarrow{\oplus\mathbf{Q}^\Gamma_\alpha} & D^k_{\Gamma,1} & \longrightarrow \cdots \longrightarrow & D^k_{\Gamma,\ell(w'_0)} & \longrightarrow & 0. \end{array} \tag{2.25}$$

2.5. **Resolutions for $\mathcal{W}$-algebras.** We apply $H^0_f(?)$ to (2.25).

Proposition 2.6.
(i) ([FFr5, ACL2, G2]*) There exists an isomorphism $H^0_f(W^k(0)) \simeq \mathcal{A}_{\Delta^+_0} \otimes \pi^{k+h^\vee}_{\mathfrak{h}} \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$ of vertex superalgebras. For $\lambda \in \mathfrak{h}^*$, we have an isomorphism*

$$H^n_f(W^k(\lambda)) \simeq \delta_{n,0}\mathcal{A}_{\Delta^+_0} \otimes \pi^{k+h^\vee}_{\mathfrak{h}} \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$$

as $\mathcal{A}_{\Delta^+_0} \otimes \pi^{k+h^\vee}_{\mathfrak{h}} \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$-modules.
(ii) There exists an isomorphism $H^0_f(W^k_\Gamma(0)) \simeq V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$ of vertex superalgebras. For $\lambda \in P_{+,0}$, we have an isomorphism

$$H^n_f(W^k_\Gamma(\lambda)) \simeq \delta_{n,0}\mathbb{V}^{\tau_k}_0(\lambda) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$$

as $V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$-modules.

Proof. The proof of (i) in [ACL2, Lemma 3.1, Lemma 5.2] can also be applied for (ii), but we include the proof of (ii) for the completeness of the paper. Recall that the complex for $H_f^\bullet(W_\Gamma^k(0))$ is $\mathcal{C} := \mathcal{A}_{\Delta_{>0}} \otimes V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \otimes F_{\mathrm{ch}}(\mathfrak{g}_{>0})$. Since the $L\mathfrak{g}_{>0}$-action on $V^{\tau_k}(\mathfrak{g}_0)$ is trivial, $H_f^\bullet(W_\Gamma^k(0)) \simeq H_f^\bullet(\mathcal{A}_{\Delta_{>0}}) \otimes V^{\tau_k}(\mathfrak{g}_0)$ as vertex superalgebras. Since $\mathcal{A}_{\Delta_{>0}}$ is a semi-regular bimodule of $L\mathfrak{g}_{>0}$ by the actions $\hat{\rho}_k^R$, $\hat{\rho}_{\mathfrak{g}_{>0}}^L$ [V1, V2], we have a vertex superalgebra isomorphism

$$\Psi \colon \mathcal{A}_{\Delta_{>0}} \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \xrightarrow{\sim} \mathcal{A}_{\Delta_{>0}} \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$$

such that

$$\begin{aligned} &\Psi \circ \left(\hat{\rho}_k^R(u(z)) \otimes 1 + 1 \otimes f_\Phi(u(z))\right) = \hat{\rho}_k^R(u(z)) \circ \Psi, \\ &\Psi \circ (\hat{\rho}_{\mathfrak{g}_{>0}}^L(u(z)) \otimes 1) = \left(\hat{\rho}_{\mathfrak{g}_{>0}}^L(u(z)) \otimes 1 + 1 \otimes f_\Phi(u(z))\right) \circ \Psi, \end{aligned} \tag{2.26}$$

where f_Ψ denotes the action in Lemma 2.1 by [ACL2, Proposition 4.5]. Extending Ψ to $\mathcal{C}$ by identity on $V^{\tau_k}(\mathfrak{g}_0) \otimes F_{\mathrm{ch}}(\mathfrak{g}_{>0})$, we obtain

$$\begin{aligned} H_f^n(W_\Gamma^k(0)) &\xrightarrow[{[\Psi]}]{\sim} H^{\frac{\infty}{2}+n}(W_\Gamma^k(0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})) \\ &= H^{\frac{\infty}{2}+n}(\mathcal{A}_{\Delta_{>0}}) \otimes V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \simeq \delta_{n,0} V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \end{aligned}$$

as vertex superalgebras since $H^{\frac{\infty}{2}+n}(\mathcal{A}_{\Delta_{>0}}) \simeq \delta_{n,0}\mathbb{C}$ by [Fei]. For the second assertion, note that we may take an isomorphism $V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \simeq H_f^0(W_\Gamma^k)$ as

$$\begin{array}{ccl} V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) & \simeq & H_f^0(W_\Gamma^k) \\ u(z) \otimes 1 & \mapsto & [u(z) \otimes 1] \\ 1 \otimes \Psi_\alpha(z) & \mapsto & \left[\Phi_\alpha(z) - \sum_{\beta \in \Delta_{1/2}} \chi([e_\beta, e_\alpha]) a_\beta^*(z)\right], \end{array} \tag{2.27}$$

by the proof of [G2, Proposition 4.5]. Since the $L\mathfrak{g}_{>0}$-action on $\mathbb{V}_0^{\tau_k}(\lambda)$ is trivial, we have

$$[\Psi] \colon H_f^\bullet(W_\Gamma^k(\lambda)) \simeq H_f^\bullet(\mathcal{A}_{\Delta_{>0}}) \otimes \mathbb{V}_0^{\tau_k}(\lambda) \simeq \Phi(\mathfrak{g}_{\frac{1}{2}}) \otimes \mathbb{V}_0^{\tau_k}(\lambda) \tag{2.28}$$

as $V^{\tau_k}(\mathfrak{g}_0)$-modules where $V^{\tau_k}(\mathfrak{g}_0)$ acts only on $\mathbb{V}_0^{\tau_k}(\lambda)$. Now, the assertion is obvious by (2.27). □

By Proposition 2.6, we obtain from (2.29) a horizontally exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \arrow[r, "{[\hat{\rho}_k]}"] \arrow[d, equal] & D_0^{\mathcal{W},k} \arrow[r, "{\oplus[\mathbf{Q}_\alpha]}"] & D_1^{\mathcal{W},k} \arrow[r] & \cdots \arrow[r] & D_{\ell(w_\circ')}^{\mathcal{W},k} \arrow[r] & \cdots \\ 0 \arrow[r] & \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \arrow[r, "{[\hat{\rho}_{k,\Gamma}]}"] & D_{\Gamma,0}^{\mathcal{W},k} \arrow[r, "{\oplus[\mathbf{Q}_\alpha^\Gamma]}"] \arrow[u, hook, "\hat{\rho}_{\tau_k,0}"'] & D_{\Gamma,1}^{\mathcal{W},k} \arrow[r] \arrow[u, hook, "\oplus\hat{\rho}_{\tau_k,0,-\alpha}"'] & \cdots \arrow[r] & D_{\Gamma,\ell(w_\circ')}^{\mathcal{W},k} \arrow[r] \arrow[u, hook] & 0, \end{tikzcd} \tag{2.29}$$

where

$$D_i^{\mathcal{W},k} = \bigoplus_{\substack{w \in W \\ \ell(w)=i}} \mathcal{A}_{\Delta_+^0} \otimes \pi_{\mathfrak{h}}^{k+h^\vee} \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}), \quad D_{\Gamma,i}^{\mathcal{W},k} = \bigoplus_{\substack{w \in W_0' \\ \ell(w)=i}} \mathbb{V}_0^{\tau_k}(w^{-1} \circ 0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}).$$

By construction, the resolution $D_\bullet^{\mathcal{W},k}$ of $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ in the first row coincides with the Wakimoto resolution [G2, §4.3]. We call the resolution $D_{\Gamma,\bullet}^{\mathcal{W},k}$ in the second raw the *generalized Wakimoto resolution* of $W^k(\mathfrak{g}, f, \Gamma)$. Note that the isomorphism (2.23) induces an isomorphism

$$\hat{\eta}_\alpha \colon \mathbb{V}_0^{\tau_k}(-\alpha) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \simeq M_\alpha \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}), \quad (\alpha \in I_{>0}),$$

as $V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$-modules, see (2.6).

Theorem 2.7. *Under the isomorphisms $\hat{\eta}_\alpha$, $(\alpha \in I_{>0})$, the exact sequence*

$$0 \to \mathcal{W}^k(\mathfrak{g}, f, \Gamma) \xrightarrow{[\hat{\rho}_{k,\Gamma}]} D^{\mathcal{W},k}_{\Gamma,0} \xrightarrow{\oplus[\mathbf{Q}^\Gamma_\alpha]} D^{\mathcal{W},k}_{\Gamma,1}$$

coincides with (2.9).

The rest of this subsection is devoted to prove the theorem. Since $D^{\mathcal{W},k}_{\Gamma,0} = V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$ and $\oplus\hat{\eta}_\alpha \colon D^{\mathcal{W},k}_{\Gamma,1} \simeq \bigoplus_{\alpha \in I_{>0}} M_\alpha \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$, it suffices to show (i) $[\hat{\rho}_{k,\Gamma}] = \mu^\Gamma_k$ and (ii) $[\mathbf{Q}^\Gamma_\alpha] = \mathbf{S}^\Gamma_\alpha$ for $\alpha \in I_{>0}$ under $\hat{\eta}_\alpha$.

Proof of (ii). Here, we show $[\mathbf{Q}^\Gamma_\alpha] = \mathbf{S}^\Gamma_\alpha$, $(\alpha \in I_{>0})$. By weight consideration, we have $[\mathbf{Q}^\Gamma_\alpha] = [\mathbf{Q}^{\Gamma,(1)}_\alpha + \mathbf{Q}^{\Gamma,(2)}_\alpha] = [\mathbf{Q}^{\Gamma,(1)}_\alpha]$. By (2.25) and (2.26), we have under the isomorphism (2.28)

$$\left[\sum_{\beta \in [\alpha]} \hat{\rho}^L_{\mathfrak{g}_{>0}}(e_\beta) v_\beta\right] = \left[\sum_{\beta \in [\alpha]} (\hat{\rho}^L_{\mathfrak{g}_{>0}}(e_\beta) + f_\Phi(e_\beta)) v_\beta\right] = \left[\sum_{\beta \in [\alpha]} f_\Phi(e_\beta) v_\beta\right]. \quad (2.30)$$

We note that in the second equality $\hat{\rho}^L_{\mathfrak{g}_{>0}}(e_\beta)$ goes to zero by $H^{\frac{\infty}{2}+0}(\mathcal{A}_{\Delta_{>0}}) \simeq \mathbb{C}$ due to weight consideration. Now, we obtain the equality $[\hat{\rho}_{k,\Gamma}] = \mu^\Gamma_k$ since $\mathbf{Q}^{\Gamma,(1)}_\alpha$ is the residue of the intertwining operator associated with the left-hand side in (2.30) and μ^Γ_k is the one associated with the right-hand side. □

To show (i), recall from §2.2 that the BRST complex $C_k(\mathfrak{g}, f; \Gamma)$ is quasi-isomorphic to the subcomplex $C^+_k = V^{\tau_k}(\mathfrak{g}_{\leq 0}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \otimes F^+_{\mathrm{ch}} \subset C_k(\mathfrak{g}, f; \Gamma)$. Thus the map $[\hat{\rho}^R_{k,\Gamma}] \colon \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \to V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$ is identical to the restriction $[\hat{\rho}^R_{k,\Gamma}] \colon H^0(C^+_k, d) \to H^0(\widetilde{C}^+_k, d)$ where $\widetilde{C}^+_k = \mathcal{A}_{\Delta_{>0}} \otimes V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \otimes F_{\mathrm{ch}}(\mathfrak{g}_{>0})$. Recall that μ^Γ_k is constructed via a spectral sequence associated with a filtration of $F_\bullet C^+_k$ induced by weights (2.4). To relate $[\hat{\rho}^R_{k,\Gamma}]$ to μ^Γ_k, we introduce a weight decomposition $\widetilde{C}^+_k = \oplus_{n \in \mathbb{Z}} \widetilde{C}^+_{k,n}$ by setting

$$\mathrm{wt}(\varphi_\alpha) = -2\mathrm{deg}(\alpha), \quad \mathrm{wt}(a^*_\alpha) = 2\mathrm{deg}(\alpha) = -\mathrm{wt}(a_\alpha)$$

and (2.4). Then the map $\hat{\rho}_{k,\Gamma} \colon C^+_k \to \widetilde{C}^+_k$ preserves the weights and the differential d preserves the decreasing filtration $\{F_n \widetilde{C}^+_k = \bigoplus_{j \geq n} \widetilde{C}^+_{k,j}\}_{n \in \mathbb{Z}}$ as is the case of C^+_k. Note that the spectral sequence $\{\widetilde{E}^\bullet_q\}^\infty_{q=1}$ associated with the filtration $F_\bullet \widetilde{C}^+_k$ collapses at $r = 1$ since

$$\widetilde{E}^n_1 = H^{\frac{\infty}{2}+n}(L\mathfrak{g}_{>0}; \mathcal{A}_{\Delta_{>0}} \otimes V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})) \simeq \delta_{n,0} V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}).$$

Proof of (ii). The map $\hat{\rho}_{k,\Gamma} \colon C^+_k \to C^W_k$ induces a map $[\hat{\rho}_{k,\Gamma}]_1 \colon E^0_1 \to \widetilde{E}^0_1$, which is

$$[\hat{\rho}_{k,\Gamma}]_1 \colon V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \to V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}).$$

By (2.9) and the collapsing of the spectral sequences, it suffices to show $[\hat{\rho}_{k,\Gamma}]_1 = \mathrm{id}$. By definition, $\hat{\rho}_{k,\Gamma}$ on $\Phi(\mathfrak{g}_{\frac{1}{2}})$ is the identity, so is $[\hat{\rho}_{k,\Gamma}]_1$ on $\Phi(\mathfrak{g}_{\frac{1}{2}})$. By (2.16), we have

$$\begin{aligned}
[\hat{\rho}_{k,\Gamma}]_1([u(z)]) &= \left[u(z) - \sum_{\alpha,\beta \in \Delta_{>0}} c^\alpha_{u,\beta}(:a^*_\beta(z) a_\alpha(z): - :\varphi_\alpha(z)\varphi^\beta(z):)\right] \\
&= [u(z)] - \left[\sum_{\alpha,\beta \in \Delta_{>0}} c^\alpha_{u,\beta}(:a^*_\beta(z) a_\alpha(z): - :\varphi_\alpha(z)\varphi^\beta(z):)\right]
\end{aligned}$$

for $u \in \mathfrak{g}_0$. By weight consideration, the second term is equal to 0 by $H^{\frac{\infty}{2}+0}(\mathcal{A}_{\Delta_{>0}}) = \mathbb{C}$. Thus $[\hat{\rho}_{k,\Gamma}]_1 = \mathrm{id}$ on $V^{\tau_k}(\mathfrak{g}_0)$. This completes the proof. □

Remark 2.8. *Recall that the first differentials in the first row in* (2.17) *are associated with the simple root vectors* $\{e_i\}_{i\in I}$. *By* [Kos], $\{e_i\}_{i\in I}$ *is interpreted geometrically as the infinitesimal action of the nilpotent Lie subalgerba* $\mathfrak{n}_+$ *on the space of functions* $C_0^\vee \simeq \mathbb{C}[N_+]$. *Similarly, Feigin and Frenkel interpreted that the screening operators* $\{\mathbf{Q}_{\alpha_i}\}_{i\in I}$ *in* (2.25) *are generators of the quantum group* $U_q(\mathfrak{n}_+)$ *with* $q = e^{\frac{\pi\sqrt{-1}}{r(k+h^\vee)}}$, *which is deformation of the enveloping algebra* $U(\mathfrak{n}_+)$ [FFr6, FFr7, FFr8]. *In this sense,* $V^k(\mathfrak{g})$ *and* $U_q(\mathfrak{n}_+)$ *form a commuting pair in the free field algebra* $D_0^k = \mathcal{A}_{\Delta_+} \otimes \pi_{\mathfrak{h}}^{k+^\vee}$. *After the Drinfeld-Sokolov reduction, this implies that* $\mathcal{W}^k(\mathfrak{g}, f;\Gamma)$ *and* $U_q(\mathfrak{n}_+)$ *form a commuting pair in the free field algebra* $D_0^{\mathcal{W},k} = \mathcal{A}_{\Delta_+^0} \otimes \pi_{\mathfrak{h}}^{k+^\vee} \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$. *In the same way, recalling that the first differentials in the second row in* (2.17) *are associated with the simple root vectors* $\{e_i\}_{i\in I_{>0}}$, *we find that* $\mathcal{W}^k(\mathfrak{g}, f;\Gamma)$ *and* $U_q(\mathfrak{g}_{>0})$ *form a commuting pair in the vertex algebra* $D_{\Gamma,0}^{\mathcal{W},k} = V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$.

3. Free field realization of $\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$ and $\mathcal{W}^k(\mathfrak{sl}_{1|n+1}, f_{\mathrm{prin}})$

Here we derive a free field realization of the subregular $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$ and the principal $\mathcal{W}$-superalgebra $\mathcal{W}^k(\mathfrak{sl}_{1|n+1}, f_{\mathrm{prin}})$.

3.1. **Subregular $\mathcal{W}$-algebras.**

We describe the isomorphism (2.10) more explicitly in the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ with a subregular nilpotent element f_{sub}. By using the natural representation $\mathfrak{sl}_{n+1} \hookrightarrow \mathrm{End}(\mathbb{C}^{n+1})$ and the elementary matrices $e_{i,j} \in \mathrm{End}(\mathbb{C}^{n+1})$, the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_{n+1}$ is spanned by $h_i = e_{i,i} - e_{i+1,i+1}$, $(i = 1,\dots,n)$. The normalized invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{sl}_{n+1}$ is given by $(u|v) = \mathrm{tr}(uv)$, which induces an isomorphism $\nu\colon \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$. Then the set of simple roots of $\mathfrak{sl}_{n+1}$ is $I = \{\alpha_i\}_{i=1}^n$ where $\alpha_i = \nu(h_i) \in \mathfrak{h}^*$. Set

$$f_{\mathrm{sub}} = \sum_{i=2}^n e_{i+1,i}, \quad x = \frac{1}{2(n+1)} \sum_{i=1}^n (n-i+1)(in+i-2)h_i.$$

The element f_{sub} is a subregular nilpotent element (which is unique up to conjugation) and the adjoint action ad_x of x gives a good $\mathbb{Z}$-grading $\Gamma_{\mathrm{sub}}\colon \mathfrak{sl}_{n+1} = \oplus_{d=-n+1}^{n-1} \mathfrak{sl}_{n+1,d}$ whose weighted Dynkin diagram is

$$\begin{array}{ccccccccccc} 0 & & 1 & & & & 1 & & 1 & & 1 \\ \bigcirc & \text{---} & \bigcirc & \text{---} & \cdots & \text{---} & \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc\,. \\ \alpha_1 & & \alpha_2 & & & & \alpha_{n-2} & & \alpha_{n-1} & & \alpha_n \end{array}$$

The associated $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) := \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}};\Gamma_{\mathrm{sub}})$ is called the subregular $\mathcal{W}$-algebra of type $\mathfrak{sl}_{n+1}$.

The Lie subalgebra $\mathfrak{sl}_{n+1,0}$ decomposes as

$$\mathfrak{sl}_{n+1,0} = \mathfrak{sl}_{n+1,0}^{\mathrm{red}} \oplus \mathfrak{z},\ \mathfrak{sl}_{n+1,0}^{\mathrm{red}} := \mathrm{Span}\{e_1, h_1, f_1\},\ \mathfrak{z} := \mathrm{Span}\{\widetilde{h}_2, h_3, \dots, h_n\},$$

where $\widetilde{h}_2 = h_2 + \frac{1}{2}h_1$ and $e_1 = e_{1,2}$, $f_1 = e_{2,1}$. The Lie subalgebra $\mathfrak{z}$ is commutative and $\mathfrak{sl}_{n+1,0}^{\mathrm{red}}$ is isomorphic to $\mathfrak{sl}_2$ by

$$\mathfrak{sl}_2 \xrightarrow{\sim} \mathfrak{sl}_{n+1,0}^{\mathrm{red}}, \quad e,h,f \mapsto e_1, h_1, f_1.$$

The restriction of τ_k (2.2) on $\mathfrak{sl}_{n+1,0}$ is

$$\tau_k(u|v) = \begin{cases} (k+n+1)(u|v), & u,v \in \mathfrak{z}, \\ (k+n-1)(u|v), & u,v \in \mathfrak{sl}_{n+1,0}^{\mathrm{red}}, \\ 0, & u \in \mathfrak{z},\ v \in \mathfrak{sl}_{n+1,0}^{\mathrm{red}}, \end{cases}$$

since the dual Coxeter number of $\mathfrak{sl}_{n+1}$ is $h^\vee = n+1$. Therefore, the affine vertex algebra $V^{\tau_k}(\mathfrak{sl}_{n+1,0})$ decomposes as

$$V^{\tau_k}(\mathfrak{sl}_{n+1,0}) \simeq V^{k+n-1}(\mathfrak{sl}_2) \otimes \pi_{\mathfrak{z}}^{k+n+1}, \tag{3.1}$$

where $\pi_{\mathfrak{z}}^{k+n+1}$ is the Heisenberg vertex algebra associated with $\mathfrak{z}$ at level $k+n+1$, see §A.1. Since

$$[\alpha_2] = \{\alpha_2, \alpha_1 + \alpha_2\}, \quad [\alpha_i] = \{\alpha_i\}, \quad i = 3, \dots, n, \tag{3.2}$$

the screening operators (2.8) for $\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$ are

$$\int Y(v_{\alpha_i}, z) dz \colon V^{\tau_k}(\mathfrak{sl}_{n+1,0}) \to M_{\alpha_i}, \quad i = 2, \dots, n, \tag{3.3}$$

First, consider (3.3) in the case $i = 3, \dots, n$. The orthogonal decomposition

$$\mathfrak{z} = \mathfrak{z}_i^\perp \oplus \mathfrak{z}_i, \quad \mathfrak{z}_i = \mathbb{C}h_i, \quad \mathfrak{z}_i^\perp = \{h \in \mathfrak{z} \mid \alpha_i(h) = 0\}.$$

induces the decomposition of the Heisenberg vertex algebra

$$\pi_{\mathfrak{z}}^{k+n+1} \simeq \pi_{\mathfrak{z}_i^\perp}^{k+n+1} \otimes \pi_{\mathfrak{z}_i}^{k+n+1}.$$

By (3.1) and (3.2), we have an isomorphism of $V^{\tau_k}(\mathfrak{sl}_{n+1,0})$-modules

$$\begin{array}{ccc} M_{\alpha_i} & \simeq & \widetilde{M}_{\alpha_i} := V^{k+n-1}(\mathfrak{sl}_2) \otimes \pi_{\mathfrak{z}_i^\perp}^{k+n+1} \otimes \pi_{\mathfrak{z}_i, -\alpha_i}^{k+n+1} \\ v_{\alpha_i} & \mapsto & |0\rangle \otimes |0\rangle \otimes |-\alpha_i\rangle. \end{array}$$

and thus

$$\int Y(v_{\alpha_i}, z)\ dz = \int \ :\mathrm{e}^{-\frac{1}{k+n+1}\int \alpha_i(z)}:\ dz. \tag{3.4}$$

Next, consider (3.3) in the case $\alpha = \alpha_2$. By Lemma 2.3, we have an embedding

$$\begin{aligned} &\rho_{\mathfrak{sl}_2} \colon V^{k+n-1}(\mathfrak{sl}_2) \hookrightarrow M_{\mathfrak{sl}_2} \otimes \pi_a^{k+n+1}, \\ &e(z) \mapsto \beta(z), \quad h(z) \mapsto -2 :\gamma(z)\beta(z): + a(z), \\ &f(z) \mapsto -:\gamma(z)^2\beta(z): + (k+n-1)\partial\gamma(z) + \gamma(z)a(z), \end{aligned}$$

which gives an isomorphism for generic k

$$V^{k+n-1}(\mathfrak{sl}_2) \simeq \mathrm{Ker}\left(\int :\beta(z)\mathrm{e}^{-\frac{1}{k+n+1}\int a(z)}:\ dz : M_{\mathfrak{sl}_2} \otimes \pi_a^{k+n+1} \to M_{\mathfrak{sl}_2} \otimes \pi_{a,-a}^{k+n+1} \right).$$

by (2.12). Here $M_{\mathfrak{sl}_2}$ is the $\beta\gamma$-system vertex algebra generated by the even fields $\beta(z), \gamma(z)$ with OPEs

$$\beta(z)\gamma(w) \sim \frac{1}{z-w}, \quad \beta(z)\beta(w) \sim 0 \sim \gamma(z)\gamma(w),$$

and π_a^{k+n+1} is the Heisenberg vertex algebra generated by an even field $a(z)$ with an OPE

$$a(z)a(w) \sim \frac{2(k+n+1)}{(z-w)^2}.$$

Then it follows from (3.1) and the isomorphism of vertex algebras

$$\begin{aligned} &\pi_{\mathfrak{h}}^{k+n+1} \xrightarrow{\sim} \pi_a^{k+n+1} \otimes \pi_{\mathfrak{z}}^{k+n+1} \\ &\alpha_1(z) \mapsto a(z), \quad \alpha_2(z) \mapsto \widetilde{h}_2(z) - \frac{1}{2}a(z), \quad \alpha_i(z) \mapsto h_i(z),\ (i = 3, \dots, n), \end{aligned}$$

that we have a vertex algebra embedding

$$\rho_{\mathfrak{sl}_{n+1,0}} \colon V^{\tau_k}(\mathfrak{sl}_{n+1,0}) \hookrightarrow M_{\mathfrak{sl}_2} \otimes \pi_{\mathfrak{h}}^{k+n+1}, \tag{3.5}$$

which gives an isomorphism for generic k

$$V^{\tau_k}(\mathfrak{sl}_{n+1,0}) \simeq \operatorname{Ker}\left(\int : \beta(z) \mathrm{e}^{-\frac{1}{k+n+1}\int a(z)} : \; dz : \; M_{\mathfrak{sl}_2} \otimes \pi_{\mathfrak{h}}^{k+n+1} \to M_{\mathfrak{sl}_2} \otimes \pi_{\mathfrak{h},-\alpha_1}^{k+n+1}\right). \tag{3.6}$$

It gives a $V^{\tau_k}(\mathfrak{sl}_{n+1,0})$-module structure on $M_{\mathfrak{sl}_2} \otimes \pi_{\mathfrak{h},-\alpha_2}^{k+n+1}$. Let $\widetilde{M}_{\alpha_2}$ be a $V^{\tau_k}(\mathfrak{sl}_{n+1,0})$-submodule generated by the subspace

$$\widetilde{\mathbb{C}}^{[\alpha_2]} = \mathbb{C}|-\alpha_2\rangle \oplus \mathbb{C}\gamma_{(-1)}|-\alpha_2\rangle.$$

Lemma 3.1. *For generic k, $M_{\alpha_2} \simeq \widetilde{M}_{\alpha_2}$ as $V^{\tau_k}(\mathfrak{sl}_{n+1,0})$-modules.*

Proof. The linear map

$$\mathbb{C}^{[\alpha_2]} \xrightarrow{\sim} \widetilde{\mathbb{C}}^{[\alpha_2]}, \quad v_{\alpha_2} \mapsto |-\alpha_2\rangle, \quad v_{\alpha_1+\alpha_2} \mapsto -\gamma_{(-1)}|-\alpha_2\rangle$$

gives an isomorphism as $(\mathfrak{sl}_{n+1,0}$-modules. By the universality of the induced modules, it induces a surjective $V^{\tau_k}(\mathfrak{sl}_{n+1,0})$-module homomorphism $M_{\alpha_2} \twoheadrightarrow \widetilde{M}_{\alpha_2}$. It is an isomorphism for generic k since $\mathbb{C}^{[\alpha_2]}$ is simple as a $\mathfrak{sl}_{n+1,0}$-module. □

Under the realization (3.6), Lemma 3.1 implies the identification

$$\int Y(v_{\alpha_2}, z)\, dz = \int : \mathrm{e}^{-\frac{1}{k+n+1}\int \alpha_2(z)} : \; dz. \tag{3.7}$$

By (2.10), (3.4) and (3.7), we conclude

$$\begin{aligned}
\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) &\simeq \bigcap_{i=2}^{n} \operatorname{Ker}\Big(\int Y(v_{\alpha_i}, z) dz \colon V^{\tau_k}(\mathfrak{sl}_{n+1,0}) \to M_{\alpha_i}\Big) \\
&\simeq \bigcap_{i=2}^{n} \operatorname{Ker}\Big(\int : \mathrm{e}^{-\frac{1}{k+n+1}\int \alpha_i(z)} : \; dz \quad : \operatorname{Im}(\rho_{\mathfrak{sl}_{n+1,0}}) \to \widetilde{M}_{\alpha_i}\Big).
\end{aligned}$$

Therefore, the composition Υ_1 of (2.10) and (3.5) gives the following free field realization of the subregular $\mathcal{W}$-algebra:

Theorem 3.2. *We have an embedding $\Upsilon_1 : \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) \hookrightarrow M_{\mathfrak{sl}_2} \otimes \pi_{\mathfrak{h}}^{k+h^\vee}$ of vertex algebras for an arbitrary $k \in \mathbb{C}$ and moreover, for generic k the image coincides with*

$$\operatorname{Im}(\Upsilon_1) = \bigcap_{i=1}^{n} \operatorname{Ker} \int Q_i(z) dz$$

where

$$Q_1(z) =: \beta(z) \mathrm{e}^{-\frac{1}{k+n+1}\int \alpha_1(z)} :, \quad Q_i(z) =: \mathrm{e}^{-\frac{1}{k+n+1}\int \alpha_i(z)} :, \ (i = 2, \dots, n).$$

3.2. **Principal $\mathcal{W}$-superalgebras.** We describe the isomorphism (2.10) more explicitly in the case $\mathfrak{g} = \mathfrak{sl}_{1|n+1}$ with a principal nilpotent element. We use the natural representation $\mathfrak{sl}_{1|n+1} \hookrightarrow \operatorname{End}(\mathbb{C}^{1|n+1})$, (e.g. [Kac1, §2]). Let $\{e_i\}_{i \in J}$ denote the standard basis of $\mathbb{C}^{1|n+1}$ with index sets

$$J = J_{\bar{0}} \sqcup J_{\bar{1}}, \quad J_{\bar{0}} = \{0\}, \quad J_{\bar{1}} = \{1, \dots, n+1\},$$

where e_i is even, (resp. odd), if $i \in J_{\bar{0}}$, (resp. $i \in J_{\bar{1}}$), and let $e_{i,j} \in \mathfrak{gl}(\mathbb{C}^{1|n+1})$ be the elementary matrix. Then the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_{1|n+1}$ is spanned by

$$h_0 = -e_{0,0} - e_{1,1}, \quad h_i = e_{i,i} - e_{i+1,i+1}, \ (i = 1, \dots, n).$$

The normalized invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{sl}_{1|n+1}$ is given by the supertrace $(u|v) = -\operatorname{str}(uv)$. It induces an isomorphism $\nu \colon \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$. Then the set of the simple roots is $I=\{\alpha_i\}_{i=0}^n$ where $\alpha_i=\nu(h_i)\in\mathfrak{h}^*$. Set

$$f_{\mathrm{prin}}=\sum_{i=1}^{n}e_{i+1,i},\quad x=\sum_{i=0}^{n+1}\left(\frac{n+1}{2}-i+1\right)e_{i,i}-e_{0,0}.$$

The element f_{prin} is an even principal nilpotent element (which is unique upt to conjugation) and ad_x gives a good $\mathbb{Z}$-grading $\Gamma_{\mathrm{prin}}\colon\mathfrak{sl}_{1|n+1}=\bigoplus_{d=n}^{n}\mathfrak{sl}_{1|n+1,d}$ whose weighted Dynkin diagram is

$$\begin{array}{ccccccccc} 0 & & 1 & & & & 1 & & 1 & & 1 \\ \otimes & — & \bigcirc & — & \cdots & — & \bigcirc & — & \bigcirc & — & \bigcirc \\ \alpha_0 & & \alpha_1 & & & & \alpha_{n-2} & & \alpha_{n-1} & & \alpha_n \end{array}.$$

The associated $\mathcal{W}$-superalgebra $\mathcal{W}^k(\mathfrak{sl}_{1|n+1}):=\mathcal{W}^k(\mathfrak{sl}_{1|n+1},f_{\mathrm{prin}};\Gamma_{\mathrm{prin}})$ is called the principal $\mathcal{W}$-superalgebra of type $\mathfrak{sl}_{1|n+1}$.

The Lie subalgebra $\mathfrak{sl}_{1|n+1,0}$ decomposes as

$$\mathfrak{sl}_{1|n+1,0}=\mathfrak{sl}_{1|n+1,0}^{\mathrm{red}}\oplus\mathfrak{z},\ \mathfrak{sl}_{1|n+1,0}^{\mathrm{red}}:=\mathrm{Span}\{e_0,h_0,h_1,f_0\},\ \mathfrak{z}:=\mathrm{Span}\{\widetilde{h}_2,h_3,\dots,h_n\},$$

where $e_0=e_{0,1}$, $f_0=e_{1,0}$, and $\widetilde{h}_2=h_2-h_0$. The Lie subalgebra $\mathfrak{z}$ is commutative and $\mathfrak{sl}_{1|n+1,0}^{\mathrm{red}}$ is isomorphic to the Lie superalgebra $\mathfrak{gl}_{1|1}$ by

$$\begin{array}{cccc}\iota: & \mathfrak{gl}_{1|1} & \xrightarrow{\sim} & \mathfrak{sl}_{1|n+1,0}^{\mathrm{red}}\\ & E_{1,1},E_{2,2},E_{1,2},E_{2,1} & \mapsto & -(h_0+h_1),rh_1,e_0,f_0.\end{array}$$

The restriction of τ_k on $\mathfrak{sl}_{1|n+1,0}$ is

$$\tau_k(u|v)=\begin{cases}(k+n)(u|v), & u,v\in\mathfrak{z},\\ (k_1\kappa_1+k_2\kappa_2)(\iota^{-1}(u)|\iota^{-1}(v)), & u,v\in\mathfrak{sl}_{1|n+1,0}^{\mathrm{red}},\\ 0, & u\in\mathfrak{z},\ v\in\mathfrak{sl}_{1|n+1,0}^{\mathrm{red}},\end{cases}$$

where $k_1=-(k+n)$, $k_2=(k+n)+1$ since the dual Coxeter number of $\mathfrak{sl}_{1|n+1}$ is $h^\vee=n$. Then the affine vertex superalgebra $V^{\tau_k}(\mathfrak{sl}_{1|n+1,0})$ decomposes as

$$V^{\tau_k}(\mathfrak{sl}_{1|n+1,0})\simeq V^{\kappa}(\mathfrak{gl}_{1|1})\otimes\pi_{\mathfrak{z}}^{k+n},\tag{3.8}$$

where $\kappa=k_1\kappa_1+k_2\kappa_2$ and $\pi_{\mathfrak{z}}^{k+n}$ is the Heisenberg vertex algebra associated with $\mathfrak{z}$ at level $k+n$, see also §A.2. Since

$$[\alpha_1]=\{\alpha_1,\alpha_0+\alpha_1\},\quad[\alpha_i]=\{\alpha_i\},\quad i=2,\dots,n,\tag{3.9}$$

the screening operators (2.8) for $\mathcal{W}^k(\mathfrak{sl}_{1|n+1})$ are

$$\int Y(v_{\alpha_i},z)dz\colon V^{\tau_k}(\mathfrak{sl}_{1|n+1,0})\to M_{\alpha_i},\quad i=1,\dots,n.\tag{3.10}$$

First, consider (3.10) in the case $i=2,\dots,n$. The orthogonal decomposition

$$\mathfrak{sl}_{1|n+1,0}=\mathfrak{h}_i^{\perp}\oplus\mathfrak{h}_i,\quad\mathfrak{h}_i=\mathbb{C}h_i,\quad\mathfrak{h}_i^{\perp}=\mathrm{Span}\{e_0,f_0,h\mid h\in\mathfrak{h},\alpha_i(h)=0\}$$

induces the decomposition of the affine vertex superalgebra $V^{\tau_k}(\mathfrak{sl}_{1|n+1,0})$

$$V^{\tau_k}(\mathfrak{sl}_{1|n+1,0})\simeq V^{\tau_k}(\mathfrak{h}_i^{\perp})\otimes\pi_{\mathfrak{h}_i}^{k+n}.$$

By (3.8) and (3.9), we have an isomorphism of $V^{\tau_\kappa}(\mathfrak{sl}_{1|n+1,0})$-modules

$$\begin{array}{ccc}M_{\alpha_i} & \simeq & \widetilde{M}_{\alpha_i}:=V^{\tau_k}(\mathfrak{h}_i^{\perp})\otimes\pi_{\mathfrak{h}_i,-\alpha_i}^{k+n}\\ v_{\alpha_i} & \mapsto & |0\rangle\otimes|-\alpha_i\rangle.\end{array}$$

and thus

$$\int Y(v_{\alpha_i},z)\ dz=\int\ :\mathrm{e}^{-\frac{1}{k+n}\int\alpha_i(z)}:\ dz\quad:V^{\tau_k}(\mathfrak{h}_i^{\perp})\otimes\pi_{\mathfrak{h}_i}^{k+n}\to\widetilde{M}_{\alpha_i}.\tag{3.11}$$

Next, consider (3.10) in the case $n=1$. By Proposition A.1, we have a homomorphism

$$\rho_{\mathfrak{gl}_{1|1}}\colon V^{\kappa}(\mathfrak{gl}_{1|1}) \hookrightarrow M_{\mathfrak{gl}_{1|1}} \otimes \pi_{\chi}^{\kappa-\kappa_2},$$

which is injective by Lemma A.2 and gives an isomorphism

$$\begin{aligned} &V^{\kappa}(\mathfrak{gl}_{1|1}) \\ &\simeq \mathrm{Ker}\left(\int :b(z)\mathrm{e}^{\frac{1}{k+n}\int(\chi_1+\chi_2)(z)}:dz \ \ : M_{\mathfrak{gl}_{1|1}} \otimes \pi_{\chi}^{\kappa-\kappa_2} \to M_{\mathfrak{gl}_{1|1}} \otimes \pi_{\chi,-(\chi_1+\chi_2)}^{\kappa-\kappa_2}\right). \end{aligned} \tag{3.12}$$

for $k \neq -n$ by Proposition A.5. Here $M_{\mathfrak{gl}_{1|1}}$ is the bc-system vertex superalgebra and $\pi_{\chi}^{\kappa-\kappa_2}$ is the Heisenberg vertex algebra generated by even fields $\chi_1(z)$, $\chi_2(z)$ with OPEs (A.1). Then it follows from (3.8) and the isomorphism of vertex algebras

$$\begin{aligned} &\pi_{\mathfrak{h}}^{k+n} \xrightarrow{\simeq} \pi_{\chi}^{\kappa-\kappa_2} \otimes \pi_{\mathfrak{z}}^{k+n} \\ &\alpha_0(z) \mapsto -(\chi_1+\chi_2)(z),\ \alpha_1(z) \mapsto \chi_2(z),\ \alpha_2(z) \mapsto \widetilde{h}_2(z) + (\chi_1+\chi_2)(z) \\ &\alpha_i(z) \mapsto h_i(z), \quad (i=3,\dots,n), \end{aligned}$$

that we have a vertex superalgebra embedding

$$\rho_{\mathfrak{sl}_{1|n+1,0}}\colon V^{\tau_k}(\mathfrak{sl}_{1|n+1,0}) \hookrightarrow M_{\mathfrak{gl}_{1|1}} \otimes \pi_{\mathfrak{h}}^{k+n}, \tag{3.13}$$

which gives an isomorphism for $k \neq -n$

$$\begin{aligned} &V^{\tau_k}(\mathfrak{sl}_{1|n+1,0}) \\ &\simeq \mathrm{Ker}\left(\int :b(z)\mathrm{e}^{-\frac{1}{k+n}\int\alpha_0(z)}:\ dz \quad : M_{\mathfrak{gl}_{1|1}} \otimes \pi_{\mathfrak{h}}^{k+n} \to M_{\mathfrak{gl}_{1|1}} \otimes \pi_{\mathfrak{h},-\alpha_0}^{k+n}\right). \end{aligned} \tag{3.14}$$

Then it gives a $V^{\tau_k}(\mathfrak{sl}_{1|n+1,0})$-module structure on $\widetilde{M}_{\alpha_1} := M_{\mathfrak{gl}_{1|1}} \otimes \pi_{\mathfrak{h},-\alpha_1}^{k+n}$. We have an isomorphism

$$\widetilde{M}_{\alpha_1} \simeq \widehat{V}_{\frac{3}{2},-1}^{+,\kappa} \otimes \pi_{\mathfrak{z}}^{k+n} \tag{3.15}$$

as $V^{\tau_k}(\mathfrak{sl}_{1|n+1,0}) \simeq V^{\kappa}(\mathfrak{gl}_{1|1}) \otimes \pi_{\mathfrak{z}}^{k+n}$-module for $k \notin \mathbb{Q}$ by Lemma A.3, (1) and Proposition A.4. Note that as a $V^{\tau_k}(\mathfrak{sl}_{1|n+1,0})$-module, $\widetilde{M}_{\alpha_1}$ is generated by a $\mathfrak{sl}_{1|n+1,0}$-submodule

$$\widetilde{\mathbb{C}}^{[\alpha_1]} = \mathbb{C}|-\alpha_1\rangle \oplus \mathbb{C}c_{(-1)}|-\alpha_1\rangle.$$

Lemma 3.3. *For $k \notin \mathbb{Q}$, we have $M_{\alpha_1} \simeq \widetilde{M}_{\alpha_1}$ as $V^{\tau_k}(\mathfrak{sl}_{1|n+1,0})$-modules.*

Proof. It follows from (3.15) and the $\mathfrak{sl}_{1|n+1,0}$-module isomorphism

$$\mathbb{C}^{[\alpha_1]} \xrightarrow{\simeq} \widetilde{\mathbb{C}}^{[\alpha_1]}, \quad v_{\alpha_1} \mapsto |-\alpha_1\rangle, \quad v_{\alpha_0+\alpha_1} \mapsto -c_{(-1)}|-\alpha_1\rangle.$$

□

Under the realization (3.14), Lemma 3.3 implies the identification

$$\int Y(v_{\alpha_1},z)dz = \int :\mathrm{e}^{-\frac{1}{k+n}\int\alpha_1(z)}:dz. \tag{3.16}$$

By (2.10), (3.11) and (3.16), we conclude

$$\begin{aligned} \mathcal{W}^k(\mathfrak{sl}_{1|n+1}) &\simeq \bigcap_{i=1}^{n} \mathrm{Ker}\left(\int Y(v_{\alpha_i},z)\ dz\colon V^{\tau_k}(\mathfrak{sl}_{1|n+1,0}) \to M_{\alpha_i}\right) \\ &\simeq \bigcap_{i=1}^{n} \mathrm{Ker}\left(\int :\mathrm{e}^{-\frac{1}{k+n}\int\alpha_i(z)}:\ dz \quad : \mathrm{Im}(\rho_{\mathfrak{sl}_{1|n+1,0}}) \to \widetilde{M}_{\alpha_i}\right). \end{aligned}$$

The composition Ψ_1 of (2.10) and (3.13) gives the following free field realization of the principal $\mathcal{W}$-superalgebra:

Theorem 3.4. *We have an embedding $\Psi_1 : \mathcal{W}^k(\mathfrak{sl}_{1|n+1}) \hookrightarrow M_{\mathfrak{gl}_{1|1}} \otimes \pi_{\mathfrak{h}}^{k+n}$ of vertex superalgebras for an arbitrary $k \in \mathbb{C}$ and moreover, for generic k the image coincides with*

$$\mathrm{Im}(\Psi_1) = \bigcap_{i=0}^{n} \mathrm{Ker} \int Q_i(z) dz$$

where

$$Q_0(z) =: b(z) \mathrm{e}^{-\frac{1}{k+n}\int \alpha_0(z)} :, \quad Q_i(z) =: \mathrm{e}^{-\frac{1}{k+n}\int \alpha_i(z)} :, \ (i = 1, \dots, n).$$

4. Dualities in coset vertex algebras

4.1. **Coset vertex algebras.** Given a vertex superalgebra V and a subalgebra $W \subset V$, the subspace

$$\mathrm{Com}(W, V) := \{a \in V \mid \forall b \in W,\ Y(b,z)Y(a,w) \sim 0\},$$

is a vertex subalgebra, called the coset vertex superalgebra of the pair (V, W).

We consider the coset vertex algebra of the pair $(\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}), \pi_{H_1})$ where π_{H_1} is a Heisenberg vertex subalgebra generated by the field

$$H_1(z) = \omega_1^\vee(z) - :\beta(z)\gamma(z):, \quad \omega_1^\vee = \frac{1}{n+1}\sum_{i=1}^{n}(n-i+1)\alpha_i. \tag{4.1}$$

on $M_{\mathfrak{sl}_2} \otimes \pi_{\mathfrak{h}}^{k+h^\vee}$. Note that $\omega_1^\vee$ represents the first fundamental coweight of $\mathfrak{sl}_{n+1}$. It defines a field on $\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$ by Theorem 3.2 since $H_1(z)$ lies in the kernels of the screening operators $\int Q_i(z) dz$, $(1 \le i \le n)$.

We describe the coset vertex algebra $\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})\right)$ in terms of screening operators for generic k. Let $L_1 = \mathbb{Z}x \oplus \mathbb{Z}y$ be a $\mathbb{Z}$-lattice equipped with a bilinear form $(\cdot|\cdot)$ given by $(ax+by|cx+dy) = ac - bd$, π_{L_1} the Heisenberg vertex algebra associate with the commutative Lie algebra $L_1 \otimes_{\mathbb{Z}} \mathbb{C}$, (§A.1), and

$$V_{L_1} := \bigoplus_{(m,n)\in\mathbb{Z}^2} \pi_{L_1, mx+ny}$$

the lattice vertex superalgebra associated with L_1 and the vertex subalgera

$$V_{x+y} := \bigoplus_{n \in \mathbb{Z}} \pi_{L_1, n(x+y)}.$$

The Friedan–Martinec–Shenker bosonization [F2, Chapter 7] gives a vertex algebra embedding

$$\begin{aligned} \Upsilon_2 \colon M_{\mathfrak{sl}_2} &\hookrightarrow V_{x+y} \\ \beta(z) &\mapsto : \mathrm{e}^{\int (x+y)(z)} :, \\ \gamma(z) &\mapsto - : x(z) \mathrm{e}^{-\int (x+y)(z)} :, \end{aligned}$$

whose image is equal to the kernel of the screening operator $\int : \mathrm{e}^{\int x(z)} : dz$

$$\mathrm{Im}(\Upsilon_2) = \mathrm{Ker}\left(\int : \mathrm{e}^{\int x(z)} :\ dz \quad : V_{x+y} \to \bigoplus_{n \in \mathbb{Z}} \pi_{L_1, (n+1)x+ny} \right).$$

By composing it with Υ_1 in Theorem 3.2, we obtain a vertex algebra embedding

$$\Upsilon := \Upsilon_2 \circ \Upsilon_1 \colon \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) \hookrightarrow V_{x+y} \otimes \pi_{\mathfrak{h}}^{k+n+1}, \tag{4.2}$$

whose image for generic k coincides with

$$\begin{aligned}\mathrm{Im}(\Upsilon) = &\mathrm{Ker} \int :\mathrm{e}^{\int x(z)}: \ dz \ \cap \ \mathrm{Ker} \int :\mathrm{e}^{-\frac{1}{k+n+1}\int (\alpha_1-(k+n+1)(x+y))(z)}: dz \\ &\cap \bigcap_{i=2}^{n} \mathrm{Ker} \int :\mathrm{e}^{-\frac{1}{k+n+1}\int \alpha_i(z)}: dz\end{aligned} \tag{4.3}$$

Let $\pi^{k+n+1}_{\widetilde{\alpha}} \subset \pi_{L_1} \otimes \pi^{k+n+1}_{\mathfrak{h}}$ be the Heisenberg vertex subalgebra generated by

$$\begin{aligned}&\widetilde{\alpha}_0(z) = x(z), \quad \widetilde{\alpha}_1(z) = (\alpha_1 - (k+n+1)(x+y))(z),\\ &\widetilde{\alpha}_i(z) = \alpha_i(z), \quad (i = 2, \dots, n).\end{aligned}$$

Since Υ_2 induces an isomorphism

$$\begin{array}{ccl}\pi_{H_1} \otimes \pi^{k+n+1}_{\widetilde{\alpha}} & \xrightarrow{\sim} & \pi_{L_1} \otimes \pi^{k+n+1}_{\mathfrak{h}} \\ 1 \otimes \widetilde{\alpha}_i(z) & \mapsto & 1 \otimes \widetilde{\alpha}_i(z), \quad (i = 0, \dots, n), \\ H_1(z) \otimes 1 & \mapsto & \omega_1^\vee(z) - y(z),\end{array}$$

we have $\mathrm{Com}\left(\pi_{H_1}, V_{x+y} \otimes \pi^{k+n+1}_{\mathfrak{h}}\right) = \pi^{k+n+1}_{\widetilde{\alpha}}$. Therefore, Υ restricts to

$$\Upsilon\colon \mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})\right) \hookrightarrow \pi^{k+n+1}_{\widetilde{\alpha}}, \tag{4.4}$$

and thus we obtain the following proposition.

Proposition 4.1. *For generic $k \in \mathbb{C}$, we have an isomorphism of vertex algebras*

$$\begin{aligned}&\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})\right) \\ &\simeq \mathrm{Ker}_{\pi^{k+n+1}_{\widetilde{\alpha}}} \int :\mathrm{e}^{\int \widetilde{\alpha}_0(z)}: \ dz \cap \bigcap_{i=1}^{n} \mathrm{Ker}_{\pi^{k+n+1}_{\widetilde{\alpha}}} \int :\mathrm{e}^{-\frac{1}{k+n+1}\int \widetilde{\alpha}_i(z)}: dz.\end{aligned}$$

Similarly, we consider the coset vertex (super)algebra of the pair $(\mathcal{W}^k(\mathfrak{sl}_{1|n+1}), \pi_{H_2})$ where π_{H_2} is a Heisenberg vertex subalgebra generated by the field

$$H_2(z) = \omega_0^\vee(z) + :b(z)c(z):, \quad \omega_0^\vee = -\frac{1}{n}\sum_{i=0}^{n}(n-i+1)\alpha_i. \tag{4.5}$$

on $M_{\mathfrak{gl}_{1|1}} \otimes \pi^{k+h^\vee}_{\mathfrak{h}}$. Note that $\omega_0^\vee$ represents the 0-th fundamental coweight of $\mathfrak{sl}_{1|n+1}$. It defines a field on $\mathcal{W}^k(\mathfrak{sl}_{1|n+1})$ by Theorem 3.4 since $H_2(z)$ lies in the kernels of the screening operators $\int Q_i(z)dz$, $(0 \leq i \leq n)$.

We describe the coset vertex (super)algebra $\mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^k(\mathfrak{sl}_{1|n+1})\right)$ in terms of screening operators for generic k. Let $\mathbb{Z} = \mathbb{Z}\phi$ a $\mathbb{Z}$-lattice equipped with a bilinear form $(m\phi|n\phi) = mn$, $\pi_{\mathbb{Z}}$ the Heisenberg vertex algebra associate with the commutative Lie algebra $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{C}$, (§A.1), and

$$V_{\mathbb{Z}} := \bigoplus_{m \in \mathbb{Z}} \pi_{\mathbb{Z}, m\phi}$$

the lattice vertex superalgebra associated with $\mathbb{Z}$. By the boson-fermion correspondence, e.g. [FBZ, Chapter 5], we have an isomorphism

$$\begin{array}{rcc}\Psi_2\colon M_{\mathfrak{gl}_{1|1}} & \xrightarrow{\sim} & V_{\mathbb{Z}} \\ b(z) & \mapsto & :\mathrm{e}^{\int \phi(z)}:, \\ c(z) & \mapsto & :\mathrm{e}^{\int -\phi(z)}:.\end{array}$$

Composing it with Υ_1 in Theorem 3.4, we obtain a vertex algebra embedding

$$\Psi := \Psi_2 \circ \Psi_1\colon \mathcal{W}^k(\mathfrak{sl}_{1|n+1}) \hookrightarrow V_{\mathbb{Z}} \otimes \pi^{k+n}_{\mathfrak{h}}, \tag{4.6}$$

whose image for generic k coincides with

$$\mathrm{Im}(\Psi) = \mathrm{Ker} \int : \mathrm{e}^{-\frac{1}{k+n}\int (\alpha_0-(k+n)\phi)(z)} : dz \ \cap \ \bigcap_{i=1}^{n} \mathrm{Ker} \int : \mathrm{e}^{-\frac{1}{k+n}\int \alpha_i(z)} : dz. \tag{4.7}$$

Let $\pi_{\widetilde{\beta}}^{k+n} \subset V_{\mathbb{Z}} \otimes \pi_{\mathfrak{h}}^{k+n}$ be the Heisenberg vertex subalgebra generated by

$$\widetilde{\beta}_0(z) = -\frac{1}{k+n}\alpha_0(z) + \phi(z), \quad \widetilde{\beta}_i(z) = -\frac{1}{k+n}\alpha_i(z), \ (i = 1, \dots, n).$$

Since Ψ_2 induces an isomorphism

$$\begin{array}{ccc} \pi_{H_2} \otimes \pi_{\widetilde{\beta}}^{k+n} & \xrightarrow{\simeq} & \pi_{\mathbb{Z}} \otimes \pi_{\mathfrak{h}}^{k+n} \\ 1 \otimes \widetilde{\beta}_i(z) & \mapsto & \widetilde{\beta}_i(z), \quad (i = 0, \dots, n) \\ H_2(z) \otimes 1 & \mapsto & \omega_0^{\vee}(z) + \phi(z), \end{array}$$

we have $\mathrm{Com}\left(\pi_{H_2}, V_{\mathbb{Z}} \otimes \pi_{\mathfrak{h}}^{k+n}\right) = \pi_{\widetilde{\beta}}^{k+n}$. Therefore, Ψ restricts to

$$\Psi \colon \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^k(\mathfrak{sl}_{1|n+1})\right) \hookrightarrow \pi_{\widetilde{\beta}}^{k+n}, \tag{4.8}$$

and thus we obtain the following proposition.

Proposition 4.2. *For generic k, we have an isomorphism*

$$\mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^k(\mathfrak{sl}_{1|n+1})\right) \simeq \bigcap_{i=0}^{n} \mathrm{Ker}_{\pi_{\widetilde{\beta}}^{k+n}} \int : \mathrm{e}^{\int \widetilde{\beta}_i(z)} : dz.$$

4.2. **A conjecture of Feigin and Semikhatov.**

Theorem 4.3. *Let $k_1, k_2 \in \mathbb{C}$ be generic levels satisfying the relation*

$$(k_1 + n + 1)(k_2 + n) = 1. \tag{4.9}$$

Then we have an isomorphism

$$\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1})\right).$$

Proof. For simplicity, set $(h_1^{\vee}, h_2^{\vee}) = (n+1, n)$. For $k_1, k_2 \in \mathbb{C}$ satisfying (4.9), we have an isomorphism

$$\begin{array}{ccc} \pi_{\widetilde{\alpha}}^{k_1+h_1^{\vee}} & \to & \pi_{\widetilde{\beta}}^{k_2+h_2^{\vee}} \\ \widetilde{\alpha}_i(z) & \mapsto & \widetilde{\beta}_i(z), \quad (i = 0, \dots, n), \end{array}$$

since both of the Gram matrices for $\{\widetilde{\alpha}_i\}_{i=0}^{n}$ and $\{\widetilde{\beta}_i\}_{i=0}^{n}$ are

$$\begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & n-1 & n \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{array} & \begin{pmatrix} 1 & -\kappa & 0 & \cdots & 0 & 0 \\ -\kappa & 2\kappa & -\kappa & \cdots & 0 & 0 \\ 0 & -\kappa & 2\kappa & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\kappa & -\kappa \\ 0 & 0 & 0 & \cdots & -\kappa & 2\kappa \end{pmatrix} \end{array},$$

where $\kappa = k_1 + h_1^{\vee}$. By applying the Feigin–Frenkel duality for the Virasoro vertex algebras (cf. [FBZ, Chapter 15]), we have

$$\mathrm{Ker}_{\pi_{\widetilde{\alpha}}^{k_1+h_1^{\vee}}} \int : \mathrm{e}^{-\frac{1}{k_1+h_1^{\vee}}\int \widetilde{\alpha}_i(z)} : dz \ = \ \mathrm{Ker}_{\pi_{\widetilde{\alpha}}^{k_1+h_1^{\vee}}} \int : \mathrm{e}^{\int \widetilde{\alpha}_i(z)} : dz, \quad (i = 1, \dots, n-1),$$

$$\mathrm{Ker}_{\pi_{\widetilde{\alpha}}^{k_1+h_1^{\vee}}} \int : \mathrm{e}^{-\frac{1}{r(k_1+h_1^{\vee})}\int \widetilde{\alpha}_n(z)} : dz \ = \ \mathrm{Ker}_{\pi_{\widetilde{\alpha}}^{k_1+h_1^{\vee}}} \int : \mathrm{e}^{\int \widetilde{\alpha}_n(z)} : dz,$$

for generic k_1. Hence, for generic $k_1, k_2 \in \mathbb{C}$ satisfying (4.9),

$$\operatorname{Com}\left(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})\right) \simeq \bigcap_{i=0}^{n} \operatorname{Ker}_{\pi_{\widetilde{\alpha}}^{k_1+h_1^\vee}} \int :\mathrm{e}^{\int \widetilde{\alpha}_i(z)}: dz$$

$$\simeq \bigcap_{i=0}^{n} \operatorname{Ker}_{\pi_{\widetilde{\beta}}^{k_2+h_2^\vee}} \int :\mathrm{e}^{\int \widetilde{\beta}_i(z)}: dz \simeq \operatorname{Com}\left(\pi_{H_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1})\right)$$

by Proposition 4.1 and Proposition 4.2. □

4.3. **Reconstruction theorem.** Let $\pi_{\widetilde{H}_1} \subset \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}$ be the Heisenberg vertex subalgebra generated by the field

$$\widetilde{H}_1(z) := \phi(z) - H_1(z) = -\omega_1^\vee(z) + y(z) + \phi(z), \tag{4.10}$$

see also (4.1). Next, consider the lattice $\mathbb{Z}\sqrt{-1} \subset \mathbb{C}$, i.e., the lattice $\mathbb{Z}\psi$, spanned by ψ, equipped with a bilinear form $(\cdot|\cdot)$, which satisfies $(m\psi|n\psi) = -mn$. Let $\pi_{\sqrt{-1}\mathbb{Z}}$ be the Heisenberg vertex algebra associated with the abelian Lie algebra $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}\sqrt{-1}$ and

$$V_{\mathbb{Z}\sqrt{-1}} := \bigoplus_{n\in\mathbb{Z}} \pi_{\mathbb{Z}\sqrt{-1}, n\psi}.$$

the lattice vertex superalgebra associated with $\mathbb{Z}\sqrt{-1}$. Let $\pi_{\widetilde{H}_2} \subset \mathcal{W}^k(\mathfrak{sl}_{1|n+1}) \otimes V_{\mathbb{Z}\sqrt{-1}}$, be the Heisenberg vertex subalgebra generated by the field

$$\widetilde{H}_2(z) := \psi(z) + H_2(z) = \omega_0^\vee(z) + \phi(z) + \psi(z), \tag{4.11}$$

see also (4.5).

Theorem 4.4. *Let $k_1, k_2 \in \mathbb{C}$ be generic levels satisfying* (4.9).
(i) $\mathcal{W}^{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) \simeq \operatorname{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1}) \otimes V_{\mathbb{Z}\sqrt{-1}}\right)$,
(ii) $\mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1}) \simeq \operatorname{Com}\left(\pi_{\widetilde{H}_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right)$.

Proof. For $\mathfrak{sl}_{n+1}$, (resp. $\mathfrak{sl}_{1|n+1}$), let $\mathfrak{h}_1$ (resp. $\mathfrak{h}_2$), denote the Cartan subalgebra, $h_1^\vee$, (resp. $h_2^\vee$), its dual Coxeter number, $\{\alpha_i\}_{i=1}^n$, (resp. $\{\beta_i\}_{i=0}^n$), the set of simple roots and $\alpha_i(z)$ the corresponding fields on $\pi_{\mathfrak{h}_1}^{k_1+h_1^\vee}$, (resp. $\beta_i(z)$ on $\pi_{\mathfrak{h}_2}^{k_2+h_2^\vee}$).

First, we show (1). By (4.6), we have a vertex superalgebra embedding

$$\Psi \otimes \mathrm{id}\colon \quad \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1}) \otimes V_{\mathbb{Z}\sqrt{-1}} \hookrightarrow V_{\mathbb{Z}} \otimes \pi_{\mathfrak{h}_2}^{k_2+h_2^\vee} \otimes V_{\mathbb{Z}\sqrt{-1}}. \tag{4.12}$$

Let $V_{\mathbb{Z}(\phi+\psi)} \subset V_{\mathbb{Z}} \otimes V_{\mathbb{Z}\sqrt{-1}}$ be the lattice vertex subalgebra corresponding to the sublattice $\mathbb{Z}(\phi+\psi) \subset \mathbb{Z} + \mathbb{Z}\sqrt{-1}$ and

$$V_{X+Y} \subset V_{\mathbb{Z}} \otimes \pi_{\mathfrak{h}_2}^{k_2+h_2^\vee} \otimes V_{\mathbb{Z}\sqrt{-1}}$$

the vertex subalgebra generated by $V_{\mathbb{Z}(\phi+\psi)}$ and the Heisenberg vertex subalgebra $\pi_{X,Y}$ generated by the fields

$$X(z) = -\frac{1}{k_2 + h_2^\vee}\beta_0(z) + \phi(z), \quad Y(z) = \frac{1}{k_2 + h_2^\vee}\beta_0(z) + \psi(z).$$

Let $\pi_A \subset V_{\mathbb{Z}} \otimes \pi_{\mathfrak{h}_2}^{k_2+h_2^\vee} \otimes V_{\mathbb{Z}\sqrt{-1}}$ be the Heisenberg vertex subalgebra generated by the fields

$$A_1(z) = \beta_1(z) - \phi(z) - \psi(z), \quad A_i(z) = \beta_i(z), \ (i = 2, \dots, n).$$

It follows from

$$X(z)A_i(w) \sim 0 \sim Y(z)A_i(w), \quad i = 1, \dots, n,$$

that $V_{X+Y}\otimes\pi_A\subset V_{\mathbb{Z}}\otimes\pi_{\mathfrak{h}_2}^{k_2+h_2^\vee}\otimes V_{\mathbb{Z}\sqrt{-1}}$. By (4.11), we have

$$\mathrm{Com}\left(\pi_{\widetilde{H}_2}, V_{\mathbb{Z}}\otimes\pi_{\mathfrak{h}_2}^{k_2+h_2^\vee}\otimes V_{\mathbb{Z}\sqrt{-1}}\right)=V_{X+Y}\otimes\pi_A,$$

and thus (4.12) implies

$$\mathrm{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1})\otimes V_{\mathbb{Z}\sqrt{-1}}\right)\hookrightarrow V_{X+Y}\otimes\pi_A,$$

whose image for generic k_2 coincides with

$$\begin{aligned}&\Psi\otimes\mathrm{id}\left(\mathrm{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1})\otimes V_{\mathbb{Z}\sqrt{-1}}\right)\right)\\ &=\mathrm{Ker}\int :\mathrm{e}^{\int X(z)}:dz\cap\ \mathrm{Ker}\int :\mathrm{e}^{-\frac{1}{k_2+h_2^\vee}\int(A_1+X+Y)(z)}:dz\\ &\qquad\qquad\cap\bigcap_{i=2}^{n}\mathrm{Ker}\int :\mathrm{e}^{-\frac{1}{k_2+h_2^\vee}\int A_i(z)}:dz\end{aligned}$$

by (4.7). Since $(A_1+X+Y)_{(0)}|n(\phi+\psi)\rangle=0$, we have

$$\begin{aligned}&\mathrm{Ker}\int :\mathrm{e}^{-\frac{1}{k_2+h_2^\vee}\int(A_1+X+Y)(z)}:dz\\ &=\bigoplus_{n\in\mathbb{Z}}\left(\mathrm{Ker}_{\pi_{X,Y}}\int :\mathrm{e}^{-\frac{1}{k_2+h_2^\vee}\int(A_1+X+Y)(z)}:dz\right)_{(-1)}|n(\phi+\psi)\rangle\\ &=\bigoplus_{n\in\mathbb{Z}}\left(\mathrm{Ker}_{\pi_{X,Y}}\int :\mathrm{e}^{\int(A_1+X+Y)(z)}:dz\right)_{(-1)}|n(\phi+\psi)\rangle\\ &=\mathrm{Ker}\int :\mathrm{e}^{\int(A_1+X+Y)(z)}:dz.\end{aligned}$$

by the Feigin–Frenkel duality for the Virasoro vertex algebras, (cf. [FBZ, Chapter 15]). Similarly, we have

$$\mathrm{Ker}\int :\mathrm{e}^{-\frac{1}{k_2+h_2^\vee}\int A_i(z)}:dz=\mathrm{Ker}\int :\mathrm{e}^{\int A_i(z)}:dz,\quad i=1,\dots,n.$$

Therefore, we have

$$\begin{aligned}&(\Psi\otimes\mathrm{id})\left(\mathrm{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1})\otimes V_{\mathbb{Z}\sqrt{-1}}\right)\right)\\ &=\mathrm{Ker}\int :\mathrm{e}^{\int X(z)}:dz\ \cap\ \mathrm{Ker}\int :\mathrm{e}^{\int(A_1+X+Y)(z)}:dz\ \cap\ \bigcap_{i=2}^{n}\mathrm{Ker}\int :\mathrm{e}^{\int A_i(z)}:dz.\end{aligned}$$

Now (i) follows from the above equality, (4.3), and the isomorphism

$$\begin{aligned}&V_{X+Y}\otimes\pi_A\xrightarrow{\sim}V_{x+y}\otimes\pi_{\mathfrak{h}_1}^{k_1+h_1^\vee}\\ &X(z)\mapsto x(z),\quad Y(z)\mapsto y(z),\quad A_i(z)\mapsto-\frac{1}{k_1+h_1^\vee}\alpha_i(z),\ (i=1,\dots,n).\end{aligned}$$

Next, we will show (ii) in the same way as the proof of (1). By (4.2), we have a vertex superalgebra embedding

$$\Upsilon\otimes\mathrm{id}:\mathcal{W}^{k_1}(\mathfrak{sl}_{n+1},f_{\mathrm{sub}})\otimes V_{\mathbb{Z}}\hookrightarrow V_{x+y}\otimes\pi_{\mathfrak{h}_1}^{k_1+h_1^\vee}\otimes V_{\mathbb{Z}}.$$

Let $V_{\widetilde{\mathbb{Z}}}$ be the vertex superalgebra generated by the fields $:\mathrm{e}^{\int\pm\widetilde{\phi}(z)}:$ where

$$\widetilde{\phi}(z)=x(z)+y(z)+\phi(z),$$

and π_B the Heisenberg vertex subalgebra generated by the fields

$$B_i(z) = \begin{cases} -y(z) - \phi(z), & i = 0, \\ \alpha_1(z) - (k_1 + h_1^\vee)(x+y)(z), & i = 1, \\ \alpha_i(z), & i = 2, \dots, n. \end{cases}$$

Then we have $\mathrm{Com}\left(\pi_{\widetilde{H}_1}, V_{x+y} \otimes \pi_{\mathfrak{h}_1}^{k_1+h_1^\vee} \otimes V_{\mathbb{Z}}\right) = V_{\widetilde{\mathbb{Z}}} \otimes \pi_B$ and thus

$$\Upsilon \otimes \mathrm{id} : \mathrm{Com}\left(\pi_{\widetilde{H}_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_{1|n+1}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right) \hookrightarrow V_{\widetilde{\mathbb{Z}}} \otimes \pi_B,$$

whose image for generic k_1 coincides with

$$\begin{aligned} &\mathrm{Ker} \int : \mathrm{e}^{\int (B_0 + \widetilde{\phi})(z)} : \, dz \ \cap \ \bigcap_{i=1}^{n} \mathrm{Ker} \int : \mathrm{e}^{-\frac{1}{k_1+h_1^\vee} \int B_i(z)} : dz \\ &= \mathrm{Ker} \int : \mathrm{e}^{\int (B_0 + \widetilde{\phi})(z)} : \, dz \ \cap \ \bigcap_{i=1}^{n} \mathrm{Ker} \int : \mathrm{e}^{\int B_i(z)} : dz \end{aligned}$$

by (4.3) and the Feigin–Frenkel duality for the Virasoro vertex algebras. Now (ii) follows from the above equation, (4.7) and the isomorphism

$$\begin{aligned} &V_{\widetilde{\mathbb{Z}}} \otimes \pi_B \xrightarrow{\sim} V_{\mathbb{Z}} \otimes \pi_{\mathfrak{h}_2}^{k_2+h_2^\vee} \\ &: \mathrm{e}^{\int \pm \widetilde{\phi}(z)} : \mapsto : \mathrm{e}^{\int \pm \phi(z)} :, \quad B_i(z) \mapsto -\frac{1}{k_2 + h_2^\vee} \beta_i(z), \ (i = 0, \dots, n). \end{aligned}$$

□

4.4. **Dualities for Non-critical levels.** Let V be a finite dimensional vector space over $\mathbb{C}$. A family of vector subspaces $\{W^\alpha\}_{\alpha \in \mathbb{C}}$ is called continuous if they are of the same dimension $d \in \mathbb{Z}_{\geq 0}$ and the induced map $\mathbb{C} \to \mathrm{Gr}(d, V)$ to the Grassmannian manifold is continuous [T]. For a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ such that $\mathrm{dim} V_n < \infty$, $(n \in \mathbb{Z})$, a family of graded vector subspaces $\{W^\alpha\}_{\alpha \in \mathbb{C}}$, $(W^\alpha = \bigoplus_{n \in \mathbb{Z}} W^\alpha_n)$, is called continuous if the homogeneous subspaces $\{W^\alpha_n\}_{\alpha \in \mathbb{C}}$, $(n \in \mathbb{Z})$, are continuous families. The following principle is obvious, but useful to generalize a result at generic levels to all levels, see [AFO].

Lemma 4.5. *Let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be a $\mathbb{Z}$-graded vertex superalgebra with $\mathrm{dim} V_n < \infty$, $(n \in \mathbb{Z})$. Let $\{W^1_\alpha\}_{\alpha \in \mathbb{C}}$, $\{W^2_\alpha\}_{\alpha \in \mathbb{C}}$ be $\mathbb{Z}$-graded vertex (super)subalgebras which form continuous families as vector spaces. If $W^1_\alpha = W^2_\alpha$ on some open dense subset $U \subset \mathbb{C}$, then $W^1_\alpha = W^2_\alpha$ for all $\alpha \in \mathbb{C}$ as vertex superalgebras.*

Define $x_i \in \mathbb{Q}$, $(i = 1, 2)$, by

$$(x_1, x_2) = \left(-n + \frac{1}{n}, -\frac{n^2}{n+1}\right) \tag{4.13}$$

and, set $K_i := \{-h_i^\vee\}$, $S_i := \{-h_i^\vee, x_i\}$. Then we have the following.

Corollary 4.6. *Let $k_1, k_2 \in \mathbb{C}$ satisfy* (4.9)
(i) For $k_1 \in \mathbb{C} \setminus S_1$ and $k_2 \in \mathbb{C} \setminus S_2$,

$$\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1})\right),$$

(ii) For $k_1 \in \mathbb{C} \setminus K_1$ and $k_2 \in \mathbb{C} \setminus K_2$,

$$\mathcal{W}^{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1}) \otimes V_{\mathbb{Z}\sqrt{-1}}\right),$$

(iii) For $k_1 \in \mathbb{C} \setminus K_1$ and $k_2 \in \mathbb{C} \setminus K_2$,

$$\mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right).$$

Proof. We show (i). First, note that the pair (x_1, x_2) in (4.13) satisfies (4.9). Recall that non-degenerate Heisenberg vertex algebras are all isomorphic if and only if their ranks are equal and that they have the conformal gradings by the Segal-Sugawara conformal vector with finite dimension homogeneous subspaces. Next, note that the excluded level $(k_1, k_2) = (x_1, x_2)$ is exactly when the Heisenberg vertex algebras π_{H_1} π_{H_2} degenerate. Therefore, $\{\mathrm{Com}(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}))\}_{k_1 \in \mathbb{C}\setminus S_1}$ is a continuous family of vertex algebras inside a non-degenerate Heisenberg vertex algebra of rank $n+1$ by (4.4) and so is $\{\mathrm{Com}(\pi_{H_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n+1})\}_{k_2 \in \mathbb{C}\setminus S_2}$ by (4.8). They are isomorphic if a generic level (k_1, k_2) with (4.9), i.e., all values for $k_1 \in \mathbb{C}\setminus S_1$, (equivalently $k_2 \in \mathbb{C}\setminus S_2$ by (4.9)), except for countably many values. Thus we may apply Lemma 4.5 with $\mathbb{C}$ replaced by $\mathbb{C}\setminus S_1$. This completes the proof. (ii) and (iii) are proved in the same way. □

5. Duality for the simple quotients

We show that the simplicity is inherited under taking coset vertex algebras. We apply it to show the commutativity of taking Heisenberg cosets and taking simple quotients, see [CKLR, Li2] for earlier literature. In this section, a vertex superalgebra is always defined over $\mathbb{C}$ and of countable dimension.

We begin with Schur's lemma for vertex superalgebras, see e.g. [LLi, Proposition 4.5.5] for the purely even case.

Lemma 5.1. *For a simple vertex superalgebra V, the center is trivial, i.e.,*

$$\mathrm{Com}(V, V) = \mathbb{C}|0\rangle.$$

Proof. Take a nonzero parity homogeneous element $a \in \mathrm{Com}(V, V)$. Since $a_{(n)} \in \mathrm{End}_V(V)$, we have $a_{(n)} \in \mathbb{C}\,\mathrm{id}_V$ if a is even and $a_{(n)} \in \mathbb{C}\Pi$ if a is odd by Lemma B.2. It follows from the identity $a_{(-1)}|0\rangle = a$, $a \in \mathbb{C}|0\rangle$ if a is even. Suppose that a is odd. By the same identity, $a = a_{(-1)}|0\rangle \in \mathbb{C}\Pi|0\rangle$ and thus we may assume $a_{(-1)} = \Pi$ without loss of generality. Since $a \in \mathrm{Com}(V, V)$, $|0\rangle = \Pi^2|0\rangle = \frac{1}{2}[a_{(-1)}, a_{(-1)}] = 0$, a contradiction. Therefore, $\mathrm{Com}(V, V)_{\bar{1}} = 0$. This completes the proof. □

The following observation is straightforward to show, but very useful.

Proposition 5.2. *(cf.* [CL3, Theorem 8.1, Remark 8.3]*) Let A be a vertex superalgebra, $B \subset A$ a simple vertex subalgebra and $C \subset A$, an arbitrary vertex subalgebra such that $B \otimes C \subset A$. Suppose that A decomposes as*

$$A \simeq \bigoplus_{\lambda \in \Lambda} B_\lambda \otimes C_\lambda$$

as $B \otimes C$-modules where Λ is a set with $0 \in \Lambda$ labeling a set of inequivalent *simple B-modules $\{B_\lambda\}_{\lambda\in\Lambda}$ with $B_0 = B$ and a set of C-modules $\{C_\lambda\}_{\lambda\in\Lambda}$ with $C_0 = C$. Let A_s be an arbitrary quotient A_s as vertex superalgebras.Then $W \subset A_s$ and $D := \mathrm{Com}(B, A_s)$ is a quotient of C. Moreover, A_s decomposes as*

$$A_s \simeq \bigoplus_{\lambda \in I} B_\lambda \otimes D_\lambda$$

as $B \otimes D$-modules where D_λ, $(\lambda \in \Lambda)$, is a quotient of C_λ as a C-module which is naturally a D-module.

Next, we consider a criterion for the simplicity of the coset vertex superalgebra of a pair of simple vertex superalgebras.

Lemma 5.3. *Let V be a vertex superalgebra and M a V-module. Let $m \in M$ be a vacuum-like vector, i.e., satisfies $a_{(n)}m = 0$ for all $a \in V$, $n \geq 0$. Then the $\mathbb{C}$-linear map*

$$F_m : \ V \to M, \quad a \mapsto a_{(-1)}m$$

is a V-module homomorphism.

Proof. We need to show $(a_{(n)}b)_{(-1)}m = a_{(n)}(b_{(-1)}m)$ for all $a, b \in V$ and $n \in \mathbb{Z}$, which are special cases of [LLi, Proposition 4.5.6]. $\square$

Let $W \subset V$ be simple vertex superalgebras such that V is semisimple as a W-module:

$$V = \bigoplus_{\lambda \in \Lambda} \widehat{W}_\lambda,$$

where Λ is an index set that labels the inequivalent simple W-modules W_λ appearing in V, and $\widehat{W}_\lambda$ is the W-submodule spanned by all the simple W-submodules isomorphic to W_λ. We may assume $0 \in \Lambda$ and $W_0 = W$.

Lemma 5.4. *The subspace $\widehat{W}_0$ is isomorphic to $W \otimes \mathrm{Com}(W, V)$ as a W-module. In particular, $\widehat{W}_0$ has the structure of a vertex superalgebra.*

Proof. Since $\widehat{W}_0$ is semi-simple as a W-module, we have $\widehat{W}_0 = \bigoplus_{\alpha \in I} M_\alpha$ for some W-submodules M_α, $(\alpha \in \Lambda)$ all isomorphic to W as W-modules. By Lemma 5.1, $\mathrm{Com}(W, W) = \mathbb{C}|0\rangle$ and thus the space of vacuum-like vectors in M_α is one dimensional. Note that the vacuum-like vectors in V with respect to W is nothing but $\mathrm{Com}(W, V)$. Therefore, the linear map

$$W \otimes \mathrm{Com}(W, V) \to \widehat{W}_0, \quad w \otimes u \mapsto w_{(-1)}u$$

is an isomorphism of W-modules by Lemma 5.3. $\square$

Now we have a criterion for the simplicity of $\mathrm{Com}(W, V)$.

Proposition 5.5. *Let $W \subset V$ be simple vertex superalgebras. Suppose that V is semisimple as a W-module and $W \otimes \mathrm{Com}(W, V)$ has a conformal vector ω which is also a conformal vector of V. Then $\mathrm{Com}(W, V)$ is also simple.*

Proof. Take a nonzero ideal $\mathcal{I} \subset \mathrm{Com}(W, V)$ and let $\widehat{\mathcal{I}} \subset V$ denote the ideal of V generated by $\mathcal{I}$. We show

$$\widehat{\mathcal{I}} \cap (W \otimes \mathrm{Com}(W, V)) = W \otimes \mathcal{I}. \tag{5.1}$$

Indeed, by [LLi, Proposition 4.5.6], we have

$$\widehat{\mathcal{I}} = \mathrm{Span}_{\mathbb{C}}\{a_{(n)}u \mid a \in V,\ u \in \mathcal{I},\ n \in \mathbb{Z}\}. \tag{5.2}$$

Then by using the conformal field $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ and the skew-symmetry

$$Y(a, z)u = -(-1)^{\bar{a}\bar{u}} e^{zL_{-1}} Y(u, -z)a,$$

we have

$$\widehat{\mathcal{I}} = \mathrm{Span}_{\mathbb{C}}\{u_{(n)}a \mid u \in \mathcal{I},\ a \in V,\ n \in \mathbb{Z}\} = \sum_{\lambda \in \Lambda} \sum_{n \in \mathbb{Z}} \mathcal{I}_{(n)} \widehat{W}_\lambda.$$

Note that for $u \in \mathrm{Com}(W, V)$, the linear map $a_{(n)}$ restrict to a W-homomorphism $W_\lambda \to V$. By Lemma B.2, we have $\mathcal{I}_{(n)} \widehat{W}_\lambda \subset (W \otimes \mathrm{Com}(W, V))_{(n)} \widehat{W}_\lambda \subset \widehat{W}_\lambda$ and thus

$$\begin{aligned} \widehat{\mathcal{I}} \cap (W \otimes \mathrm{Com}(W, V)) &= \mathrm{Span}_{\mathbb{C}}\{u_{(n)}a \mid u \in \mathcal{I},\ a \in W \otimes \mathrm{Com}(W, V),\ n \in \mathbb{Z}\} \\ &= W \otimes \mathcal{I}. \end{aligned}$$

Since V is simple, $\widehat{\mathcal{I}} = V$ and thus $\mathcal{I} = \mathrm{Com}(W, V)$. This completes the proof. $\square$

Corollary 5.6 ([CKLR, Proposition 3.2, Theorem 2.9])**.** *Let $V = \oplus_{n \in \mathbb{Z}} V_n$ be a conformal vertex operator superalgebra and $\pi \subset V$ a simple Heisenberg vertex subalgebra generated by a subspace $\mathcal{H} \subset V_1$ of primary fields. Suppose the following conditions:*

(1) $V = \oplus_{\lambda \in \mathcal{H}^*} \Omega_\lambda(V)$, $\quad \Omega_\lambda(V) := \{a \in V \mid \forall h \in \mathcal{H},\ h_{(0)}a = \lambda(h)a\}$.

(2) *For each $\lambda \in \mathcal{H}^*$, the conformal degrees of $\Omega_\lambda(V)$ are bounded from below.*

Then, we have
(i) π acts on V semisimply:

$$V \simeq \bigoplus_{\lambda \in \mathcal{H}^*} \pi_\lambda \otimes C_\lambda, \quad C_\lambda := \left\{a \in V \mid h_{(n)}a = \delta_{n,0}\lambda(h)a,\ \forall h \in \mathcal{H}, n \geq 0\right\} \tag{5.3}$$

(ii) For an arbitrary simple quotient $V \twoheadrightarrow V_s$, $\pi \subset V_s$ and π acts on V_s semisimply

$$V_s \simeq \bigoplus_{\lambda \in \mathcal{H}^*} \pi_\lambda \otimes C^s_\lambda, \quad C^s_\lambda := \left\{a \in V_s \mid h_{(n)}a = \delta_{n,0}\lambda(h)a,\ \forall h \in \mathcal{H}, n \geq 0\right\}. \tag{5.4}$$

Moreover, each C^s_λ is a simple quotient of C_λ as a $C_0 = \mathrm{Com}(\pi, V)$-module, which is naturally a $C^s_0 = \mathrm{Com}(\pi, V_s)$-module.

Proof. The proof of (i) in [FLM, Theorem 1.7.3] when V is purely even applies in this general setting. We show (ii). Let $\omega \in V$ denote the conformal vector of V and $\omega_\pi \in \pi$ the conformal vector of π by the Segal-Sugawara construction. Since $\mathcal{H}$ consist of primary vector with respect to ω, ω^ω_π lies in $\mathrm{Com}(\pi, V)$ and is a conformal vector. Therefore, we may apply Proposition 5.2. Then it remains to show that C^0_λ is a simple $\mathrm{Com}(\pi, V)$-module. By Proposition 5.5, C^s_0 is simple. The same proof applies for C^s_λ, $\lambda \in \mathcal{H}^*$. □

Let $\mathcal{W}_k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$, (resp. $\mathcal{W}_k(\mathfrak{sl}_{1|n+1})$), denote the (unique) simple quotient of $\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$, (resp. $\mathcal{W}^k(\mathfrak{sl}_{1|n+1})$). By [KRW, KW2], $\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$ (resp. $\mathcal{W}^k(\mathfrak{sl}_{1|n+1})$) is semisimple as a π_{H_1}-module (resp. π_{H_2}-module), so is the simple quotient $\mathcal{W}_k(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$, (resp. $\mathcal{W}_k(\mathfrak{sl}_{1|n+1}) \otimes V_{\mathbb{Z}\sqrt{-1}}$). The following is immediate from Corollary 4.6 and Corollary 5.6.

Corollary 5.7. *Let $k_1, k_2 \in \mathbb{C}$ satisfy* (4.9).
(i) For $k_1 \in \mathbb{C} \setminus S_1$ and $k_2 \in \mathbb{C} \setminus S_2$,

$$\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}_{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}_{k_2}(\mathfrak{sl}_{1|n+1})\right)$$

(ii) For $k_1 \in \mathbb{C} \setminus K_1$ and $k_2 \in \mathbb{C} \setminus K_2$,

$$\mathcal{W}_{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}_{k_2}(\mathfrak{sl}_{1|n+1}) \otimes V_{\mathbb{Z}\sqrt{-1}}\right),$$

(iii) For $k_1 \in \mathbb{C} \setminus K_1$ and $k_2 \in \mathbb{C} \setminus K_2$,

$$\mathcal{W}_{k_2}(\mathfrak{sl}_{1|n+1}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_1}, \mathcal{W}_{k_1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right).$$

6. Fusion rules of lattice cosets

6.1. **Vertex superalgebras and their modules.** In this section, we consider a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator superalgebra of CFT type, i.e., a vertex superalgebra with a Virasoro field $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ such that L_0 acts on V semisimply and gives the $\frac{1}{2}\mathbb{Z}_{\geq 0}$-grading

$$V = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_{\geq 0}} V_\Delta, \quad V_{\bar{i}} = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_{\geq 0}} V_{\bar{i},\Delta}, \quad (V_{\bar{i},\Delta} = V_{\bar{i}} \cap V_\Delta),$$

for $\bar{i} \in \mathbb{Z}_2 (= \mathbb{Z}/2\mathbb{Z})$ such that $\dim(V_\Delta) < \infty$ for all Δ and $V_0 = \mathbb{C}|0\rangle$. Note that the decomposition

$$V = \bigoplus_{\Delta \in \mathbb{Z}} V_\Delta \oplus \bigoplus_{\Delta \in \frac{1}{2} + \mathbb{Z}} V_\Delta \tag{6.1}$$

does not necessarily give the parity decomposition. If the decomposition (6.1) agrees with the parity decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$, then we call V of correct statistics. Otherwise, we call V of wrong statistics.

From now on, we always assume the $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator superalgebra V to be simple and C_2-cofinite. We start with clarifying what we mean by a category of V-modules and what properties it has in general, following the works of Huang [H2]-[H7], Huang–Lepowsky [HL1]-[HL3], Huang–Lepowsky–Zhang [HLZ1]-[HLZ8], [HKL], and [CKM1]. For this purpose, we view V as a superalgebra object (see Remark B.14) of a suitable module category of the subalgebra $V_{\bar{0},\mathbb{Z}} := \bigoplus_{\Delta\in\mathbb{Z}} V_{\bar{0},\Delta}$. Therefore, we make a short digression on $V_{\bar{0},\mathbb{Z}}$.

Note that $V_{\bar{0},\mathbb{Z}}$ is a $\mathbb{Z}_{\geq 0}$-graded vertex operator algebra of CFT type, characterized as the fixed-point subalgebra of V with respect to the finite abelian group generated by the twist $\theta_V := e^{2\pi i L_0} \in \mathrm{End}_{\mathbb{C}}(V)_{\bar{0}}$ and the parity involution $P_V = \mathrm{id}_{V_{\bar{0}}} - \mathrm{id}_{V_{\bar{1}}} \in \mathrm{End}_{\mathbb{C}}(V)_{\bar{0}}$. Hence, $V_{\bar{0},\mathbb{Z}}$ is simple and C_2-cofinite by [DM1] and [Mi2], respectively. Let $V_{\bar{0},\mathbb{Z}}$-mod denote the category of grading-restricted generalized $V_{\bar{0},\mathbb{Z}}$-modules, see [CKM1, Definition 3.1], ([HLZ1] for the non-super case). By [H7, Theorem 3.24, Proposition 4.1], $V_{\bar{0},\mathbb{Z}}$-mod is a $\mathbb{C}$-linear finite abelian category. In particular, $V_{\bar{0},\mathbb{Z}}$-mod satisfies Assumption 1 and 4 in Appendix B. Moreover, $V_{\bar{0},\mathbb{Z}}$-mod has a structure of a braided tensor category by [H7, Theorem 4.13], see also [CKM1, §3] for a detailed review. By [HLZ2, Proposition 4.26], the fusion product $\boxtimes$ on $V_{\bar{0},\mathbb{Z}}$-mod is right exact. Let $\mathcal{S}(V_{\bar{0},\mathbb{Z}}\text{-mod})$ denote the superization of $V_{\bar{0},\mathbb{Z}}$-mod, see Remark B.14. Then $\mathcal{S}(V_{\bar{0},\mathbb{Z}}\text{-mod})$ is a braided tensor supercategory satisfying Assumption 1,2 and 4 and that the fusion product $\boxtimes$ is right exact. Therefore, the vertex superalgebra V is an algebra object of $\mathcal{S}(V_{\bar{0},\mathbb{Z}}\text{-mod})$, see also Remark B.13. Then, by [CKM1, Theorem 3.65], the supercategory $\mathrm{Rep}^0(V)$ consisting of (categorical) V-module objects in $\mathcal{S}(V_{\bar{0},\mathbb{Z}}\text{-mod})$ coincides with the supercategory V-mod of grading-restricted generalized V-modules as $\mathbb{C}$-linear additive supercategories. Moreover, the braided monoidal supercategory structure thus induced on V-mod coincides with the one in the sense of Huang–Lepowsky–Zhang, whose existence is not a priori guaranteed by [H7]. Now, the following is clear (see also Appendix B).

Lemma 6.1. *The supercategory V-mod is $\mathbb{C}$-linear monoidal supercategory whose underlying category is an abelian category satisfying Assumption 1,2, and 4 in Appendix B and whose fusion product $\boxtimes$ is right exact. Moreover, if the supercategory V-mod is rigid, then $\boxtimes$ is exact.*

Remark 6.2. *The exactness of $\boxtimes$ holds without rigidity if V-mod is semisimple. Even if V-mod is semisimple, the rigidity of V-mod is important in the theory of vertex algebras since it is necessary for V-mod to be a modular tensor category. In the literature, the rigidity of V-mod is established for a $\mathbb{Z}_{\geq 0}$-graded simple vertex operator algebra of CFT type equipped with a non-degenerate invariant bilinear form by Huang* [H6, Theorem 3.8].

6.2. **Simple current extensions by Lattice.** Here we study simple current extensions of tensor products of vertex superalgebras and lattice vertex superalgebras under suitable conditions.

Let V_L be the lattice vertex superalgebra associated with a positive-definite integral lattice L of finite rank with a bilinear form $(\cdot|\cdot)\colon L \times L \to \mathbb{Z}$. It is a simple $\frac{1}{2}$-graded vertex operator superalgebra of CFT type. It is well-known that V_L is rational and C_2-cofinite and that the set of irreducible V_L-modules is $\mathrm{Irr}(V_L\text{-mod}) = \{V_{a+L} \mid a \in L'/L\}$ with fusion product $V_{a+L} \boxtimes V_{b+L} \simeq V_{a+b+L}$, where $L' = \{a \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid (a|L) \subset \mathbb{Z}\}$, (see e.g. [DLM2]). In particular, we have $\mathrm{Irr}(V_L\text{-mod}) = \mathrm{Pic}(V_L\text{-mod}) \simeq L'/L$ (as groups) and thus V_L-mod is rigid. The monodromy among them is given by $\mathcal{M}_{V_{a+L},V_{b+L}} = e^{2\pi i(a|b)}$, (see e.g. [CKL]). By abuse of notation, an element $a \in L'$ also denotes the corresponding element in quotients L'/N by subgroups $N \subset L'$

Let V be a simple, C_2-cofinite, $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator superalgebra, then so is the tensor product $V \otimes V_L$. The monoidal supercategory $V \otimes V_L$-mod is naturally equivalent to the Deligne product $(V\text{-mod})\widehat{\otimes}(V_L\text{-mod})$. Indeed, we have a natural superfunctor $(V\text{-mod})\widehat{\otimes}(V_L\text{-mod}) \to V \otimes V_L$-mod such that $M\widehat{\otimes}N \mapsto M \otimes N (= M \otimes_{\mathbb{C}} N)$. Since every weak V_L-module is complete reducible, it admits a quasi-inverse

$$\begin{array}{ccc} (V \otimes V_L)\text{-mod} & \simeq & (V\text{-mod})\widehat{\otimes}(V_L\text{-mod}) \\ M & \mapsto & \bigoplus_{a \in L'/L} \Omega_a(M)\widehat{\otimes}V_{a+L} \end{array} \tag{6.2}$$

where

$$\Omega_a(M) := \{m \in M \mid h_{(n)}m = \delta_{n,0}(a|h)m, (\forall h \in \mathbb{C} \otimes_{\mathbb{Z}} L)\} \subset M.$$

(In what follows, we will write $\otimes$ also for $\widehat{\otimes}$ by abuse of notation.) In particular, $\mathrm{Pic}(V)$ decomposes into the product

$$\mathrm{Pic}(V) \times \mathrm{Pic}(V_L) \simeq \mathrm{Pic}(V \otimes V_L), \quad (M, V_{a+L}) \mapsto M \otimes V_{a+L}. \tag{6.3}$$

Let N be a sublattice of L' containing L and consider a categorical simple current extension $(\mathcal{E}, \mu_{\mathcal{E}})$ of $V \otimes V_L$ of the form

$$\mathcal{E} = \bigoplus_{a \in N/L} \mathcal{E}_a = \bigoplus_{a \in N/L} S_a \otimes V_{a+L} \tag{6.4}$$

for some finite subgroup $\{S_a\}_{a\in N/L}$ of $\mathrm{Pic}(V)$ satisfying (S1) in §B.5 together with $\theta_{\mathcal{E}}^2 = \mathrm{id}$ and $\theta_{\mathcal{E}_a}\theta_{\mathcal{E}_b} = \theta_{\mathcal{E}_{a+b}}$. Note that (S2) in §B.5 is automatically satisfied since $\mathrm{Irr}(V_L) = \mathrm{Pic}(V_L)$ forms a group. In this case, the categorical simple current extension $\mathcal{E}$ is a simple, C_2-cofinite, $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator superalgebra by [CKM1, Theorem 3.42], and $\mathrm{Rep}^0(\mathcal{E})$ is equivalent to $\mathcal{E}$-mod as $\mathbb{C}$-linear braided monoidal supercategories, (see [Ca, HKL] for the purely even cases). We note that the larger monoidal supercategory $\mathrm{Rep}(\mathcal{E})$ is equivalent to a supercategory of certain twisted $\mathcal{E}$-modules, see [YY]. We apply a general theory in §B.5. Recall that the simple currents $\{S_a\}_{a\in N/L}$ in V-mod give a monodromy decomposition

$$V\text{-mod} = \bigoplus_{\phi \in (N/L)^\vee} V\text{-mod}_\phi,$$

where $(N/L)^\vee = \mathrm{Hom}_{\mathrm{Grp}}(N/L, \mathbb{C}^*)$ and $V\text{-mod}_\phi$ is the full subcategory of V-mod consisting of objects M such that $\mathcal{M}_{S_a,M} = \phi(a)\,\mathrm{id}_{S_a \boxtimes M}$, $(a \in N/L)$. Let ϕ_M denote the character $\phi \in (N/L)^\vee$ associated with M. Similarly, the simple currents $\{V_{a+L}\}_{a\in N/L}$ in V_L-mod give a monodromy decomposition

$$V_L\text{-mod} = \bigoplus_{\phi \in (N/L)^\vee} V_L\text{-mod}_\phi.$$

In this case, $V_L\text{-mod}_\phi$ is semisimple and

$$\mathrm{Irr}(V_L\text{-mod}_\phi) = \left\{ V_{b+L} \,\middle|\, b \in L'/L,\ e^{2\pi\sqrt{-1}(a|b)} = \phi(a),\ (\forall a \in N/L) \right\}.$$

The group homomorphism $L' \to (N/L)^\vee$ $(a \mapsto e^{2\pi\sqrt{-1}(a|\bullet)})$ induces an isomorphism $L'/N' \simeq (N/L)^\vee$. Then we have $\mathrm{Irr}(V_L\text{-mod}_\phi) = \{V_{b+L} \mid b \in L'/L, b = \phi \in L'/N'\}$. By (6.2), the supercategory of $\mathcal{E}$-local $V \otimes V_L$-modules is

$$(V \otimes V_L)\text{-mod}^0 \simeq \bigoplus_{\phi \in (N/L)^\vee} (V\text{-mod})_\phi \otimes (V_L\text{-mod})_{\phi^{-1}}. \tag{6.5}$$

Theorem 6.3. (cf. [CKM1, YY])
(i) The set of simple $\mathcal{E}$-modules (in $\mathrm{Rep}^0(\mathcal{E})$*) is in one-to-one correspondence*

$$\mathrm{Irr}(\mathcal{E}) \simeq \{(M,a) \in \mathrm{Irr}(V) \times (L'/L) \mid \phi_M \phi_{V_{a+L}} = 1\} \big/ (N/L)$$

by $(M,a) \mapsto \mathcal{F}(M \otimes V_{a+L}) = \mathcal{E} \boxtimes_{V \otimes V_L} (M \otimes V_{a+L})$*. In particular,*

$$|\mathrm{Irr}(\mathcal{E})| = \frac{|\mathrm{Irr}(V)||N'/L|}{|N/L|}$$

and

$$\mathrm{Pic}(\mathcal{E}) \simeq \{(M,a) \in \mathrm{Pic}(V) \times (L'/L) \mid \phi_M \phi_{V_{a+L}} = 1\} \big/ (N/L).$$

(ii) Suppose that the fusion product $\boxtimes$ on V-mod *and* $\mathrm{Rep}(\mathcal{E})$ *are exact. Then we have an isomorphism of rings*

$$\mathcal{K}(\mathcal{E}) \simeq \left(\mathcal{K}(V) \bigotimes_{\mathbb{Z}[N/L]} \mathbb{Z}[L'/L] \right)^{N/L} \tag{6.6}$$

where the tensor product over $\mathbb{Z}[N/L]$ is given by $[M \boxtimes S_a] \otimes b = [M] \otimes (-a+b)$ for $a \in N/L$

Proof. (i) follows from Corollary B.20. For (ii), note that $\mathcal{K}(V_L) \simeq \mathbb{Z}[L'/L]$, ($[V_{a+L}] \mapsto a$). Since $\mathcal{K}(\mathcal{E})$ and $\mathcal{K}(V_L)$ are $(N/L)^\vee$-graded $\mathbb{Z}[N/L]$-algebras by Theroem B.12, the fusion algebra $\mathcal{K}\left((V \otimes V_L)\text{-mod}^0\right)$ is a diagonal N/L-invariant subalgebra

$$\begin{aligned} \mathcal{K}\left((V \otimes V_L)\text{-mod}^0\right) &\simeq \left(\mathcal{K}(V) \otimes \mathcal{K}(V_L)\right)^{N/L} \\ &\simeq \left(\mathcal{K}(V) \otimes \mathbb{Z}[L'/L]\right)^{N/L}. \end{aligned}$$

Hence, by Corollary B.21 the induction functor $\mathcal{E} \boxtimes_{V \otimes V_L} \bullet$ induces an isomorphism

$$\mathcal{K}\mathcal{E}) \simeq \mathcal{K}\left((V \otimes V_L)\text{-mod}^0\right)/\mathcal{J}$$

where $\mathcal{J}$ is generated by

$$[M] \otimes b - [M \boxtimes S_a] \otimes (a+b)$$

for $M \in Ob(V\text{-mod})$, $a \in N/L$, $b \in L'/L$. This implies (6.6). □

Therefore, we obtain a concrete description of the fusion data of $\mathcal{E}$-modules in terms of V. Let us consider a converse description. Note that by (S2), the monoidal superfunctor $\mathcal{F}\colon (V\text{-mod}) \otimes (V_L\text{-mod}) \to \mathrm{Rep}(\mathcal{E})$ gives embeddings

$$\begin{aligned} V\text{-mod} &\to \mathrm{Rep}(\mathcal{E}), \quad N \mapsto \mathcal{F}(N \otimes V_L), \\ V_L\text{-mod} &\to \mathrm{Rep}(\mathcal{E}), \quad N \mapsto \mathcal{F}(V \otimes N), \end{aligned}$$

as $\mathbb{C}$-linear monoidal supercategories and thus, we may consider V-mod and V_L-mod as subcategories of $\mathrm{Rep}(\mathcal{E})$. We use the abbreviation, e,g., $M \boxtimes_{\mathcal{E}} V_{a+L} (= M \boxtimes_{\mathcal{E}} \mathcal{F}(V \boxtimes V_{a+L}))$ for $M \in Ob(\mathrm{Rep}(\mathcal{E}))$ and $a \in L'/L$.

Lemma 6.4. *Every simple object N in* $\mathrm{Rep}(\mathcal{E})$ *is isomorphic to $M \boxtimes_{\mathcal{E}} V_{a+L}$ for some $M \in \mathrm{Irr}(\mathcal{E})$ and $a \in L'/L$. Moreover, the set of such pairs (M,a) forms an N'/L-torsor by*

$$b.(M,a) = (M \boxtimes_{\mathcal{E}} V_{b+L}, a-b), \quad (b \in N'/L).$$

Proof. By Proposition B.19 (ii), we have $N \simeq \mathcal{F}(M \otimes V_{a+L})$ for some $M \in \mathrm{Irr}(V)$ and $a \in L'/L$. Since $\mathcal{F}$ is mononoidal, we may decompose

$$\mathcal{F}(M \otimes V_{a+L}) \simeq \mathcal{F}(M \otimes V_{b+L}) \boxtimes_{\mathcal{E}} V_{a-b+L} \tag{6.7}$$

for any $b \in L'/L$. By using the isomorphism $L'/N' \simeq (N/L)^\vee$, we may take $b \in L'/L$ so that $M \otimes V_{b+L}$ is $\mathcal{E}$-local and thus$\mathcal{F}(M \otimes V_{b+L}) \in \mathrm{Irr}(\mathcal{E})$. This proves the first part of the statement. The element $b \in L'/L$ in th decomposition (6.7) such that $M \otimes V_{b+L}$ is $\mathcal{E}$-local are uniquely determined up to N'/L. This implies the second part of the statement. $\square$

Since objects of $\mathrm{Rep}(\mathcal{E})$ are pairs (μ_M, M) of $M \in Ob(V\text{-mod} \otimes V_L\text{-mod})$ and a morphism $\mu_M : \mathcal{E} \boxtimes_{V \otimes V_L} M \to M$ of V-mod $\otimes$ V_L-mod, we have two family of monodromy actions

$$\mathcal{M}_{N,\bullet} := \mathcal{M}_{N \otimes V_L, \bullet}, \ (N \in Ob(V\text{-mod})), \quad \mathcal{M}_{V_{a+L}, \bullet} := \mathcal{M}_{V \otimes V_{a+L}, \bullet}, \ (a \in L'/L).$$

Clearly, any object N in the subcategory V-mod $\subset \mathrm{Rep}(\mathcal{E})$ has trivial monodromy with V_{a+L}, $(a \in N'/L)$, i.e., $\mathcal{M}_{V_{a+L}, \mathcal{F}(N \otimes V_L)} = \mathrm{id}_{(V \otimes V_{a+L}) \boxtimes_{V \otimes V_L} (N \otimes V_L)}$. Conversely, any simple object in $\mathrm{Rep}(\mathcal{E})$ satisfying this monodromy-free property lies in V-mod. This implies the following theorem.

Theorem 6.5. (cf. [CKM1, YY])
(i) The set of simple V-modules $\mathrm{Irr}(V)$ is in one-to-one correspondence to

$$\{(M, a) \in \mathrm{Irr}(\mathcal{E}) \times (L'/L) \mid \mathcal{M}_{V_{b+L}, M \boxtimes_{\mathcal{E}} V_{a+L}} = \mathrm{id}, \ (\forall b \in N'/L)\}/(N'/L)$$

by $(M, a) \mapsto N$ so that $\mathcal{F}(N \otimes V_L) \simeq M \boxtimes_{\mathcal{E}} V_{a+L}$ holds. In particular,

$$|\mathrm{Irr}(V)| = \frac{|\mathrm{Irr}(\mathcal{E})||N/L|}{|N'/L|}$$

and the group $\mathrm{Pic}(V)$ is naturally isomorphic to

$$\{(M, a) \in \mathrm{Pic}(\mathcal{E}) \times (L'/L) \mid \mathcal{M}_{V_{b+L}, M \boxtimes_{\mathcal{E}} V_{a+L}} = \mathrm{id}, \ (\forall b \in N'/L)\}/(N'/L)$$

(ii) Suppose that the fusion product $\boxtimes$ on V-mod and $\mathrm{Rep}(\mathcal{E})$ are exact. Then we have an isomorphism of rings

$$\mathcal{K}(V) \simeq \left(\mathcal{K}(\mathcal{E}) \bigotimes_{\mathbb{Z}[N'/L]} \mathbb{Z}[L'/L] \right)^{N'/L} \tag{6.8}$$

where the tensor product over $\mathbb{Z}[N'/L]$ is given by $[M \boxtimes_{\mathcal{E}} V_{a+L}] \otimes b = [M] \otimes (a+b)$ for $a \in N'/L$.

Proof. (i) is immediate from Lemma 6.4. We show (ii). By Lemma 6.1, every object in $\mathrm{Rep}(\mathcal{E})$, (resp. $\mathcal{E}$-mod), has finite length. Thus, we may take a basis of $\mathcal{K}(\mathrm{Rep}(\mathcal{E}))$, (resp. $\mathcal{E}$-mod), by $\mathrm{Irr}(\mathrm{Rep}(\mathcal{E})$, (resp. $\mathrm{Irr}(\mathcal{E})$). Then by Lemma 6.4, we have a natural isomorphism

$$\mathcal{K}(\mathrm{Rep}(\mathcal{E})) \simeq \mathcal{K}(\mathcal{E}) \bigotimes_{\mathbb{Z}[L'/N]} \mathcal{K}(V_L) \simeq \mathcal{K}(\mathcal{E}) \bigotimes_{\mathbb{Z}[L'/N]} \mathbb{Z}[L'/L].$$

Note that $\mathcal{K}(\mathrm{Rep}(\mathcal{E}))$ is an $(N'/L)^\vee$-graded ring by the monodromy action of $\{V_{a+L}\}_{a \in N'/L}$ whose trivial grading part is spanned by $\mathrm{Irr}(V)$, we obtain the assertion. $\square$

We end up this subsection by giving two sufficient conditions for the exactness of the fusion products of V-mod and $\mathrm{Rep}(\mathcal{E})$.

Lemma 6.6. *The monoidal superfunctor $\boxtimes$ on $\mathrm{Rep}(\mathcal{E})$ is exact if one of the following conditions holds:*
(i) V-mod is semisimple.
(ii) V is self-dual and V-mod is rigid.

Proof. (i) is obvious since $\mathrm{Rep}(\mathcal{E})$ is also semisimple by Proposition B.19. We show (ii). By assumption V-mod is rigid and thus so is $V \otimes V_L$-mod. Since the induction functor $\mathcal{F}$ maps rigid objects to rigid objects, simple objects in $\mathrm{Rep}(\mathcal{E})$ (resp. $\mathrm{Rep}^0(\mathcal{E})$) is rigid by (i). Since V is self-dual, $V \otimes V_L$ is self-dual. It implies that $\mathcal{E}$ is also self-dual since $S_a \otimes V_{a+L}$ has a contragradient module $S_{-a} \otimes V_{-a+L}$ inside $\mathcal{E}$. The assertion follows from [CMY, Theorem 4.4.1] since every object $\mathrm{Rep}(\mathcal{E})$ (resp. $\mathrm{Rep}^0(\mathcal{E})$) has finite length. $\square$

7. Fusion rules of $\mathcal{W}$-algebras at rational levels

Here we study the fusion rules of the subregular $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ and the principal $\mathcal{W}$-superalgebra $\mathcal{W}_k(\mathfrak{sl}_{1|n})$ in the rational cases.

7.1. **Fusion rules of $L_m(\mathfrak{sl}_n)$ and level-rank duality.** In this subsection, we review the fusion rules of the simple affine vertex algebra associated with $\mathfrak{sl}_n$ in the rational cases. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{sl}_n$, $\{\alpha_i^\vee\}_{i=1}^{n-1}$ the set of simple coroots, $\{\alpha_i\}_{i=1}^{n-1}$ the set of simple roots, ϖ_i the i-th fundamental weight, and $\rho = \sum_{i=1}^{n-1} \varpi_i$ the Weyl vector of $\mathfrak{sl}_n$. Then $\{\alpha_i\}_{i=1}^{n-1}$ (resp. $\{\alpha_i^\vee\}_{i=1}^{n-1}$) forms a basis of $\mathfrak{h}^*$ (resp. $\mathfrak{h}$), which are naturally identified by the normalized invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ (see also §3.1).

Let $Q = Q(A_{n-1}) = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$ denote the root lattice of $\mathfrak{sl}_n$, $P = P(A_{n-1}) = \bigoplus_{i=1}^{n-1} \mathbb{Z}\varpi_i$ the weight lattice, $P_+ = P_+(A_{n-1}) = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\varpi_i$ the set of integral dominant weights. Then by setting $\varpi_0 = 0$, $\{\varpi_i\}_{i\in\mathbb{Z}_n}$ forms a set of representatives of P/Q in P and $P/Q \xrightarrow{\simeq} \mathbb{Z}_n$, $(\varpi_i \mapsto i)$, gives an isomorphism of abelian groups.

Let $L_m(\mathfrak{sl}_n)$ denote the simple affine vertex algebra of $\mathfrak{sl}_n$ at level $m \in \mathbb{C}$, which is the unique simple quotient of the universal affine vertex algebra $V^k(\mathfrak{sl}_n)$. If $m \in \mathbb{Z}_{\geq 0}$, then $L_m(\mathfrak{sl}_n)$ is regular [DLM2], equivalent to say, rational and C_2-cofinite [ABD]. (See also [FrZh] for the rationality). By [FrZh], the set of irreducible modules is one-to-one correspondence with the set

$$P_+^m(n) := \left\{ \lambda = \sum_{i=1}^{n-1} a_i\varpi_i \in P_+(A_{n-1}) \,\middle|\, \sum_{i=1}^{n-1} a_i \leq m \right\}$$

by

$$P_+^m(n) \simeq \mathrm{Irr}(L_m(\mathfrak{sl}_n)), \quad \lambda \mapsto L_k(\lambda)$$

where $L_k(\lambda)$ is the (unique) simple quotient of the Verma module $\mathbb{M}^m(\lambda)$ of $V^k(\mathfrak{sl}_n)$ with highest weight λ at level m, see §2.4. Note that the group homomorphism

$$\pi_{P/Q} \colon P \twoheadrightarrow P/Q \simeq \mathbb{Z}_n, \quad \Lambda_i \mapsto i, \tag{7.1}$$

induces the following decomposition

$$P_+^m(n) = \bigsqcup_{i\in\mathbb{Z}_n} P_+^m(n)_i, \quad P_+^m(n)_i := \pi_{P/Q}^{-1}(i).$$

Let $N_{\lambda,\mu}^\nu(\widehat{\mathfrak{sl}}_{n,m})$ denote the fusion rule of $L_m(\mathfrak{sl}_n)$, i.e., the non-negative integer given by

$$L_m(\lambda) \boxtimes L_m(\mu) \simeq \bigoplus_{\nu\in P_+^m(n)} N_{\lambda,\mu}^\nu(\widehat{\mathfrak{sl}}_{n,m}) L_m(\nu). \tag{7.2}$$

The explicit formula for $N_{\lambda,\mu}^\nu(\widehat{\mathfrak{sl}}_{n,m})$ is given by the Kac–Walton formula [DFMS, §16.2], see also [Gep, W2] for the relations to Schubert calculus and quantum cohomology ring of Grassmannians. By [Fu], the group of simple currents is $\mathrm{Pic}(L_m(\mathfrak{sl}_n)) = \{L_m(n\varpi_i)\}_{i\in\mathbb{Z}_n}$ and is isomorphic to $\mathbb{Z}_n$ by $L_m(n\varpi_i) \mapsto i$, which induces a $\mathbb{Z}_n$-action on $\mathrm{Irr}(L_m(\mathfrak{sl}_n))$ by fusion product. More explicitly,

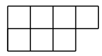
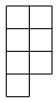
$$L_m(n\varpi_i) \boxtimes L_m(\lambda) \simeq L_m(\sigma^i(\lambda)), \quad \lambda \in P_+^m(n), \tag{7.3}$$

where σ is the cyclic permutation $\sigma(\varpi_i) = \varpi_{i+1}$ for $i \in \mathbb{Z}_n$

For later purpose, we recall the level-rank duality between $L_m(\mathfrak{sl}_n)$ and $L_n(\mathfrak{sl}_m)$ for $n, m \geq 0$, [Fr, OS]. The isomorphism $\mathbb{C}^{nm} \simeq \mathbb{C}^n \otimes \mathbb{C}^m$ induces an embedding of Lie algebras $\mathfrak{sl}_n \oplus \mathfrak{sl}_m \hookrightarrow \mathfrak{sl}_{nm}$ and thus $L_m(\mathfrak{sl}_n) \otimes L_n(\mathfrak{sl}_m) \hookrightarrow L_1(\mathfrak{sl}_{nm})$. It is a conformal embedding and gives a finite decomposition

$$L_1(\mathfrak{sl}_{nm}) \simeq \bigoplus_{\lambda \in P_+^m(n)_0} L_m(\lambda) \otimes L_n(\lambda^t)$$

as $L_m(\mathfrak{sl}_n) \otimes L_n(\mathfrak{sl}_m)$-modules where $\lambda \to \lambda^t$ denotes the transpose. More precisely, let $C_{n,m}$ denote the set of Young digrams lying in the $n \times m$ rectangle. Then we have an embedding $P_+^m(n) \hookrightarrow C_{n,m}$, $(\lambda = \sum_{i \in \mathbb{Z}_n} a_i \Lambda_i \mapsto \sqcup_{i=1}^{n-1} a_i R_i)$ where R_i denote the column of boxes of height i. (R_n is identified with the empty set.) Then the transpose $\lambda \mapsto \lambda^t$ is just the transpose of Young diagrams, e.g.,

$P_+^5(3) \ni \Lambda_0 + \Lambda_1 + 3\Lambda_2 =$  $\xrightarrow{t}$  $= \Lambda_0 + \Lambda_3 + \Lambda_4 \in P_+^3(5)$.

Note that $\pi_{P/Q}(\lambda) = \ell(\lambda) \in \mathbb{Z}_n$ where $\ell(\lambda)$ denotes the number of boxes of the Young diagram of λ. By the Frenkel–Kac construction, we have an embedding $L_1(\mathfrak{sl}_{nm}) \simeq V_{Q(A_{nm-1})} \hookrightarrow V_{\mathbb{Z}^{nm}}$. Then $\mathrm{Com}(L_1(\mathfrak{sl}_{nm}), V_{\mathbb{Z}^{nm}}) \simeq V_{\sqrt{nm}\mathbb{Z}}$ with $\sqrt{nm}\mathbb{Z} \hookrightarrow \mathbb{Z}^{nm}$, $(a\sqrt{nm} \mapsto (a, a, \cdots, a))$ and we have

$$V_{\mathbb{Z}^{nm}} \simeq \bigoplus_{a \in \mathbb{Z}_{nm}} L_1(\Lambda_a) \otimes V_{\frac{a}{\sqrt{nm}} + \sqrt{nm}\mathbb{Z}}$$

as $L_1(\mathfrak{sl}_{nm}) \otimes V_{\sqrt{nm}\mathbb{Z}}$-modules. Now, by the branching law of $L_1(\Lambda_a)$ as an $L_m(\mathfrak{sl}_n) \otimes L_n(\mathfrak{sl}_m)$-module [OS, Theorem 4.1], we obtain

$$V_{\mathbb{Z}^{nm}} \simeq \bigoplus_{a \in \mathbb{Z}_{nm}} \left(\bigoplus_{\lambda \in P_+^m(n)_a} L_m(\lambda) \otimes L_n(\sigma^{\frac{a-\ell(\lambda)}{n}}(\lambda^t)) \right) \otimes V_{\frac{a}{\sqrt{nm}} + \sqrt{nm}\mathbb{Z}} \qquad (7.4)$$

as $L_m(\mathfrak{sl}_n) \otimes L_n(\mathfrak{sl}_m) \otimes V_{\sqrt{nm}\mathbb{Z}}$-modules. Thus, $\mathcal{E}_{m,n} := \mathrm{Com}(L_m(\mathfrak{sl}_n), V_{\mathbb{Z}^{nm}})$ is a simple current extension of $L_n(\mathfrak{sl}_m) \otimes V_{\sqrt{nm}\mathbb{Z}}$

$$\mathrm{Com}(L_m(\mathfrak{sl}_n), V_{\mathbb{Z}^{nm}}) \simeq \bigoplus_{a \in \mathbb{Z}_m} L_n(n\Lambda_a) \otimes V_{\frac{an}{\sqrt{nm}} + \sqrt{nm}\mathbb{Z}}$$

of order m. Therefore, by Theorem 6.3, we have

$$\mathrm{Irr}(\mathcal{E}_{m,n}) \simeq \left\{ (\lambda, a) \in (P_+^n(m) \times \mathbb{Z}_{nm}) \mid \pi_{P/Q}(\lambda) = a \in \mathbb{Z}_m \right\} / \mathbb{Z}_m,$$

where $\mathbb{Z}_m$ acts on $P_+^n(m) \times \mathbb{Z}_{nm}$ by $r \cdot (\lambda, a) = (\sigma^r(\lambda), a + rn)$, $(r \in \mathbb{Z}_m)$. Let $\mathbf{M}(\lambda, a)$ denote the simple $\mathcal{E}_{m,n}$-module corresponding to (λ, a). Then we have an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{K}(\mathcal{E}_{m,n}) \simeq (\mathcal{K}(L_n(\mathfrak{sl}_m)) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} \mathbb{Z}[\mathbb{Z}_{nm}])^{\mathbb{Z}_m}, \quad \mathbf{M}(\lambda, a) \mapsto L_n(\lambda) \otimes [a].$$

Now, (7.4) implies the decomposition

$$V_{\mathbb{Z}^{nm}} \simeq \bigoplus_{\lambda \in P_+^m(n)} L_m(\lambda) \otimes \mathbf{M}(\lambda^t, \ell(\lambda))$$

as $L_m(\mathfrak{sl}_n) \otimes \mathcal{E}_{m,n}$-modules. This gives an one-to-one correspondence

$$\mathrm{Irr}(L_m(\mathfrak{sl}_n)) \to \mathrm{Irr}(\mathcal{E}_{m,n}), \quad L_m(\lambda) \mapsto \mathbf{M}(\lambda^t, \ell(\lambda))$$

between irreducible modules, which implies a braided-reverse equivalence of braided tensor categories between $L_m(\mathfrak{sl}_n)$-mod and $\mathcal{E}_{m,n}$-mod by [CKM2]. Thus we obtain the following isomorphism between the fusion algebras.

Proposition 7.1. *We have an isomorphism of rings*

$$\mathcal{K}(L_m(\mathfrak{sl}_n)) \simeq \left(\mathcal{K}(L_n(\mathfrak{sl}_m)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_m]} \mathbb{Z}[\mathbb{Z}_{nm}]\right)^{\mathbb{Z}_m}, \quad L_m(\lambda) \mapsto L_n(\lambda^t) \otimes [\ell(\lambda)].$$

7.2. **Fusion rules of principal $\mathcal{W}$-algebras.** The (simple) principal $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sl}_r)$ is C_2-cofinite if the level $k \in \mathbb{C}$ is a non-degenerate admissible number, i.e., of the form

$$k_{p,q} = -r + \frac{r+p}{r+q}, \quad p, q \in \mathbb{Z}_{\geq 0},\ (r+p, r+q) = 1,$$

by [Ar4]. In this case, $\mathcal{W}_k(\mathfrak{sl}_r)$ is also denoted by $\mathcal{W}_{(p,q)}(\mathfrak{sl}_r)$ and is rational by [Ar5]. The set $\mathrm{Irr}(\mathcal{W}_{(p,q)}(\mathfrak{sl}_r))$ of irreducible $\mathcal{W}_{(p,q)}(\mathfrak{sl}_r)$ is determined by [Ar2] and the fusion rules are determined by [AvE1, Cr, FKW]. In the case $(p,q) = (n,1)$, $\mathrm{Irr}(\mathcal{W}_{(n,1)}(\mathfrak{sl}_r))$ is in one-to-one correspondence to $P_+^n(r)$ by

$$P_+^n(r) \simeq \mathrm{Irr}(\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)), \quad \lambda \mapsto \mathbf{L}_{\mathcal{W}}(\lambda),$$

where $\mathbf{L}_{\mathcal{W}}(\lambda) = H^0_-\left(L_{k_{n,1}}(\lambda - (k_{n,1}+r)\rho)\right)$ and $H^0_-(?)$ denotes the "$-$"-reduction functor introduced in [FKW]. The conformal dimension of $\mathbf{L}_{\mathcal{W}}(\lambda)$ is given by the formula

$$h_\lambda^{\mathcal{W}} := \frac{(\lambda|\lambda+2\rho)}{2(k_{n,1}+r)} - (\lambda|\rho). \tag{7.5}$$

The fusion rules are given by

$$\mathbf{L}_{\mathcal{W}}(\lambda) \boxtimes \mathbf{L}_{\mathcal{W}}(\mu) \simeq \bigoplus_{\nu \in P_+^n(r)} N_{\lambda,\mu}^{\nu}(\widehat{\mathfrak{sl}}_{r,n}) \mathbf{L}_{\mathcal{W}}(\nu),$$

Therefore, we have an isomorphism of fusion algebras

$$\mathcal{K}(L_n(\mathfrak{sl}_r)) \xrightarrow{\simeq} \mathcal{K}(\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)), \quad L_n(\lambda) \mapsto \mathbf{L}_{\mathcal{W}}(\lambda). \tag{7.6}$$

and the group of simple currents is

$$\mathrm{Pic}(\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)) = \{\mathbf{L}_{\mathcal{W}}(n\varpi_i)\}_{i \in \mathbb{Z}_r} \simeq \mathbb{Z}_r, \quad \mathbf{L}_{\mathcal{W}}(n\varpi_i) \mapsto i.$$

7.3. **Fusion rules of subregular $\mathcal{W}$-algebras.** The (simple) subregular $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ is C_2-cofinite if the level $k \in \mathbb{C}$ is of the form

$$k_r := -n + \frac{n+r}{n-1}, \quad r \in \mathbb{Z}_{\geq 0},\ (n+r, n-1) = 1.$$

by [Ar4]. In this case, we denote $\mathcal{W}_{k_r}(\mathfrak{sl}_n, f_{\mathrm{sub}})$ also by $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$ for simplicity. In the rest of this subsection, we assume $r \geq 3$. In this case, by [CL1, Theorem 9.4], we have an isomorphism of vertex algebras

$$\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)\right) \simeq \mathcal{W}_{(n,1)}(\mathfrak{sl}_r). \tag{7.7}$$

and moreover,

$$\mathrm{Com}\left(\mathcal{W}_{(n,1)}(\mathfrak{sl}_r), \mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)\right) \simeq V_{\sqrt{nr}\mathbb{Z}}, \tag{7.8}$$

$$\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n) \simeq \bigoplus_{i \in \mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{\frac{ni}{\sqrt{nr}} + \sqrt{nr}\mathbb{Z}}. \tag{7.9}$$

Since $\mathbf{L}_{\mathcal{W}}(n\varpi_i)$, $(i \in \mathbb{Z}_r)$, is a simple current of $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)$, (7.9) shows that $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$ is a simple current extension of $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{nr}\mathbb{Z}}$ of order r. Since $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)$ and $V_{\sqrt{nr}\mathbb{Z}}$ are rational, so is $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$ by Proposition B.19. It is immediate that every simple ordinary module of a simple C_2-cofinite vertex operator superalgebra of CFT type is also C_2-cofinite. Since $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r)$ and $V_{\sqrt{nr}\mathbb{Z}}$ are C_2-cofinite, so is $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$ as a $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{nr}\mathbb{Z}}$-module and thus is C_2-cofinite as a vertex algebra. We note that the rationality for $r \geq 0$ is proven independently in [AvE2] by the analysis of Zhu's algebra.

By the results in §6.2, the decomposition (7.9) gives the following description of the set of irreducible $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$-modules.

Theorem 7.2. *For $r \geq 3$, there exists a one-to-one correspondence*

$$\mathrm{Irr}(\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)) \simeq \left\{(\lambda, a) \in P^n_+(r) \times \mathbb{Z}_{nr} \mid \pi_{P/Q}(\lambda) = a \in \mathbb{Z}_r\right\} \Big/ \mathbb{Z}_r,$$

where $\mathbb{Z}_r$ acts on $P^n_+(r) \times \mathbb{Z}_{nr}$ by $m \cdot (\lambda, a) = (\sigma^m(\lambda), a + mn)$, ($m \in \mathbb{Z}_r$). The $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)$-module $\mathbf{L}_{\mathit{sub}}(\lambda, a)$ corresponding to (λ, a) decomposes into

$$\mathbf{L}_{\mathit{sub}}(\lambda, a) \simeq \bigoplus_{i \in \mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(\sigma^i(\lambda)) \otimes V_{\frac{a+ni}{\sqrt{nr}} + \sqrt{nr}\,\mathbb{Z}} \tag{7.10}$$

as a $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{nr}\,\mathbb{Z}}$-module and satisfies the fusion rules

$$\mathbf{L}_{\mathit{sub}}(\lambda, a) \boxtimes \mathbf{L}_{\mathit{sub}}(\mu, b) \simeq \bigoplus_{\nu \in P^n_+(r)} N^{\nu}_{\lambda,\mu}(\widehat{\mathfrak{sl}}_{r,n}) \mathbf{L}_{\mathit{sub}}(\nu, a+b).$$

Proof. By (7.5), we have $h^{\mathcal{W}}_{n\Lambda_i} = in(r-i)/2r$. Hence the monodromy $\mathcal{M}_{\mathbf{L}_{\mathcal{W}}(n\varpi_i), \mathbf{L}_{\mathcal{W}}(\lambda)}$ is

$$\mathcal{M}_{\mathbf{L}_{\mathcal{W}}(n\varpi_i), \mathbf{L}_{\mathcal{W}}(\lambda)} = \zeta_r^{-i\pi_{P/Q}(\lambda)}, \quad \zeta_r = \mathrm{e}^{\frac{2\pi\sqrt{-1}}{r}}.$$

Thus $\mathbf{L}_{\mathcal{W}}(\lambda) \otimes V_{\frac{an}{rn} + \sqrt{nr}\,\mathbb{Z}}$ is local for the simple currents $\mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{\frac{ni}{nr} + \sqrt{nr}\,\mathbb{Z}}$ ($i \in \mathbb{Z}_r$) if and only if

$$\pi_{P/Q}(\lambda) = a \in \mathbb{Z}_r$$

where $a \in \mathbb{Z}_{nr}$ is seen as an element of $\mathbb{Z}_r$ by the natural projection $\mathbb{Z}_{nr} \twoheadrightarrow \mathbb{Z}_r$. Therefore, the assertions follow from Theorem 6.3. $\square$

By Proposition 7.1, Theorem 7.2 implies the following, cf. [AvE2].

Corollary 7.3. *The one-to-one correspondence*

$$\mathrm{Irr}(L_r(\mathfrak{sl}_n)) \simeq \mathrm{Irr}(\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)), \quad L_m(\lambda) \mapsto \mathbf{L}_{\mathit{sub}}(\lambda^t, \ell(\lambda)),$$

gives an isomorphism of fusion rings

$$\mathcal{K}(L_r(\mathfrak{sl}_n)) \simeq \mathcal{K}(\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)).$$

7.4. **Principal $\mathcal{W}$-superalgebras.** By the Kazama–Suzuki coset construction Corollary 5.7 for the simple quotients, the decomposition (7.9) implies a description of principal $\mathcal{W}$-superalgebras as a simple current extension of the tensor product of a principal $\mathcal{W}$-algebra and a lattice vertex superalgera. For simplicity, we write

$$\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n}) = \mathcal{W}_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n}), \quad r \in \mathbb{Z}_{\geq 0}, \ (n+r, n-1) = 1.$$

Theorem 7.4. *For $r \geq 3$, there exists an isomorphism of vertex superalgebras*

$$\mathrm{Com}\left(\mathcal{W}_{(n,1)}(\mathfrak{sl}_r), \mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})\right) \simeq V_{\sqrt{(n+r)r}\,\mathbb{Z}}.$$

In this case, we have as isomorphism

$$\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n}) \simeq \bigoplus_{i \in \mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{\frac{(n+r)i}{\sqrt{(n+r)r}} + \sqrt{(n+r)r}\,\mathbb{Z}} \tag{7.11}$$

as $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{(n+r)r}\,\mathbb{Z}}$-modules. In particular, $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$ is a simple current extension of $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{(n+r)r}\,\mathbb{Z}}$ and thus is C_2-cofinite and rational.

Proof. Note that the element nH_1 is identified with the element $\sqrt{nr}$, a generator of $\sqrt{nr}\,\mathbb{Z}$. Then we may write (7.9) as

$$\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n) \simeq \bigoplus_{i \in \mathbb{Z}_r} \bigoplus_{a \in \mathbb{Z}} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes \pi^{H_1}_{ar+i}.$$

where $\pi_a^{H_1}$ is the Fock module such that $H_{1,(0)}$ acts by $a \in \mathbb{C}$. Let ϕ denote the Heisenberg field in $\pi_{\mathbb{Z}} \subset V_{\mathbb{Z}}$ corresponding to $1 \in \mathbb{Z}$. Then the Heienberg field $\psi(z) = \frac{1}{(n+r)}(nH_1(z) + r\phi(z))$ is orthogonal to $H_1(z)$. We have isomorphisms

$$\begin{aligned}\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n) \otimes V_{\mathbb{Z}} &\simeq \bigoplus_{i\in\mathbb{Z}_r}\bigoplus_{a,b\in\mathbb{Z}} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes \pi^{H_1}_{ar+i} \otimes \pi^{\phi}_{b} \\ &\simeq \bigoplus_{i\in\mathbb{Z}_r}\bigoplus_{a,b\in\mathbb{Z}} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes \pi^{\widetilde{H}_1}_{b-ar-i} \otimes \pi^{(n+r)\psi}_{nra+in+rb}\end{aligned} \tag{7.12}$$

as $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes \pi^{\widetilde{H}_1} \otimes \pi^{(n+r)\psi}$-modules. Thus, by Corollary 5.7,

$$\begin{aligned}\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n}) &\simeq \mathrm{Com}\left(\pi^{\widetilde{H}_1}, \mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n) \otimes V_{\mathbb{Z}}\right) \\ &\simeq \bigoplus_{i\in\mathbb{Z}_r}\bigoplus_{a\in\mathbb{Z}} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes \pi^{(n+r)\psi}_{(n+r)(ra+i)}.\end{aligned} \tag{7.13}$$

This implies the first assertion since $\mathrm{Com}(\mathcal{W}_{(n,1)}(\mathfrak{sl}_r), \mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})) \simeq \bigoplus_{\lambda\in\mathbb{Z}} \pi^{(n+r)\psi}_{(n+r)r\lambda}$ is a lattice vertex superalgebra $V_{\sqrt{(n+r)r}\mathbb{Z}}$. Now the second assertion follows from rewriting the decomposition (7.13) as $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes \pi^{(n+r)\psi}$-modules into the one as $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{(n+r)r}\mathbb{Z}}$-modules. The third assertion follows from the same argument as in the case of the subregular $\mathcal{W}$-algebra, see the argument just after (7.9). □

We note that the $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{(n+r)r}\mathbb{Z}}$-module

$$\bigoplus_{i\in\mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{-\frac{(n+r)i}{\sqrt{(n+r)r}} + \sqrt{(n+r)r}\mathbb{Z}},$$

which looks like a "reversed gluing" has also a natural structure of vertex superalgebra. Indeed, it is also isomorphic to $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$. In the case $n = 2$, the isomorphism of these two extensions coincides with the famous automorphism of the $\mathcal{N} = 2$ SCA, which maps the strong generators to

$$L(z) \mapsto L(z), \quad J(z) \mapsto -J(z), \quad G^{\pm}(z) \mapsto G^{\mp}(z).$$

Here $L(z)$ is the Virasoro field, $J(z)$ the Heisenberg field, and $G^{\pm}(z)$ are primary odd fields of conformal weight $\frac{3}{2}$ weakly generating the $\mathcal{N} = 2$ SCA, see e.g. [Sa1]. This isomorphism for $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$ follows from a general theorem of Shimakura [Shi] describing isomorphisms between simple current extensions of a vertex algebra and automorphisms of lattice vertex algebra $V_{\sqrt{(n+r)r}\mathbb{Z}}$ [DN] appearing in the decomposition.

By the results in §6.2, Theorem 7.11 implies the following description of the set of irreducible $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$-modules.

Theorem 7.5. *There exists a one-to-one correspondence*

$$\mathrm{Irr}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})) \simeq \left\{(\lambda, a) \in P^n_+(r) \times \mathbb{Z}_{(n+r)r} \mid \pi_{P/Q}(\lambda) = a \in \mathbb{Z}_r\right\} \Big/ \mathbb{Z}_r,$$

where $\mathbb{Z}_r$ acts on $P^n_+(r) \times \mathbb{Z}_{(n+r)r}$ by $m \cdot (\lambda, a) = \big(\sigma^m(\lambda), a + m(n+r)\big)$, ($m \in \mathbb{Z}_r$). The $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$-module $\mathbf{L}_{sp}(\lambda, a)$ corresponding to (λ, a) has a decomposition

$$\mathbf{L}_{sp}(\lambda, a) \simeq \bigoplus_{i\in\mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(\sigma^i(\lambda)) \otimes V_{\frac{a+(n+r)i}{\sqrt{(n+r)r}} + \sqrt{(n+r)r}\mathbb{Z}} \tag{7.14}$$

as a $\mathcal{W}_{(n,1)}(\mathfrak{sl}_r) \otimes V_{\sqrt{(n+r)r}\mathbb{Z}}$-module and satisfies the fusion rules

$$\mathbf{L}_{sp}(\lambda, a) \boxtimes \mathbf{L}_{sp}(\mu, b) \simeq \bigoplus_{\nu\in P^n_+(r)} N^{\nu}_{\lambda\mu}(\widehat{\mathfrak{sl}}_{r,n}) \mathbf{L}_{sp}(\nu, a+b)$$

The proof is very similar to that of Theorem 7.2 and thus we omit it.

Corollary 7.6.
(i) The fusion ring of $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$ *is isomorphic to*

$$\mathcal{K}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})) \simeq \left(\mathcal{K}(L_n(\mathfrak{sl}_r)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_r]} \mathbb{Z}[\mathbb{Z}_{(n+r)r}]\right)^{\mathbb{Z}_r}.$$

(ii) The number of inequivalent simple $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$*-modules is* $\binom{n+r}{n}$.
(iii) The group of simple currents of $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$*-modules is*

$$\mathrm{Pic}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})) = \{\mathbf{L}_{sp}(0, ar)\}_{a\in\mathbb{Z}_{n+r}} \simeq \mathbb{Z}_{n+r}, \quad \mathbf{L}_{sp}(0, ar) \mapsto a.$$

We give another description of $\mathrm{Irr}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n}))$ in terms of $P^r_+(n)$. It follows from (7.11) and (7.12) that

$$\mathrm{Com}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n}), \mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n) \otimes V_{\mathbb{Z}}) \simeq \bigoplus_{a\in\mathbb{Z}} \pi^{\widetilde{H}_1}_{(n+r)a} \simeq V_{\sqrt{(n+r)n}\,\mathbb{Z}}$$

as vertex algebras. Hence, we have

$$\begin{aligned}\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n) \otimes V_{\mathbb{Z}} &\simeq \bigoplus_{i\in\mathbb{Z}_r}\bigoplus_{a\in\mathbb{Z}_{n+r}} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{\frac{i(n+r)+ra}{\sqrt{r(n+r)}}+\sqrt{r(n+r)}\,\mathbb{Z}} \otimes V_{\frac{na}{\sqrt{n(n+r)}}+\sqrt{n(n+r)}\,\mathbb{Z}}\\ &\simeq \bigoplus_{a\in\mathbb{Z}_{n+r}} \mathbf{L}_{\mathsf{sp}}(0, ar) \otimes V_{\frac{na}{\sqrt{n(n+r)}}+\sqrt{n(n+r)}\,\mathbb{Z}}\end{aligned}$$

as $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n}) \otimes V_{\sqrt{n(n+r)}\,\mathbb{Z}}$-modules. Thus, $\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n) \otimes V_{\mathbb{Z}}$ is an order $n+r$ simple current extension of $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n}) \otimes V_{\sqrt{n(n+r)}\,\mathbb{Z}}$. Since $V_{\mathbb{Z}}$ is a holomorphic vertex operator superalgebra,

$$\mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)\text{-mod} \simeq \mathcal{W}^{\mathrm{sub}}_{(r)}(\mathfrak{sl}_n)\text{-mod} \otimes V_{\mathbb{Z}}\text{-mod}, \quad M \mapsto M \otimes V_{\mathbb{Z}}$$

is an equivalence of braided tensor categories. Then Theorem 6.5 implies the following.

Theorem 7.7. *There exists a one-to-one correspondence*

$$\mathrm{Irr}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})) \simeq \left\{(\lambda, a) \in P^r_+(n) \times \mathbb{Z}_{(n+r)n} \mid \pi_{P/Q}(\lambda) = -a \in \mathbb{Z}_n\right\}/\mathbb{Z}_n$$

where $\mathbb{Z}_n$ *acts on* $P^r_+(n)\times\mathbb{Z}_{(n+r)n}$ *by* $m\cdot(\lambda, a) = \big(\sigma^m(\lambda), a - m(n+r)\big)$, $(m \in \mathbb{Z}_n)$. *Let* $\mathbf{L}_{\mathcal{SW}}(\lambda, a)$ *denote the* $\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})$*-module corresponding to* (λ, a)*. Then we have an isomorphism of rings*

$$\mathcal{K}(\mathcal{W}_{(r)}(\mathfrak{sl}_{1|n})) \simeq \left(\mathcal{K}(L_r(\mathfrak{sl}_n)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_n]} \mathbb{Z}[\mathbb{Z}_{n(n+r)}]\right)^{\mathbb{Z}_n}, \quad \mathbf{L}_{\mathcal{SW}}(\lambda, a) \mapsto L_r(\lambda) \otimes [a].$$

Appendix A. Free field realization for $V^{\kappa}(\mathfrak{gl}_{1|1})$

We study a free field realization of the universal affine vertex superalgebra $V^{\kappa}(\mathfrak{gl}_{1|1})$. We also realize it as the kernel of a certain screening operator.

A.1. **Heisenberg vertex algebra.** We use the language of the λ-bracket for vertex superalgebras, cf. [DK]. For a finite dimensional commutative Lie algebra $\mathfrak{h}$ over $\mathbb{C}$ with a symmetric bilinear form κ, the Heisenberg vertex algebra $\pi^{\kappa}_{\mathfrak{h}}$ associated with $\mathfrak{h}$ at level κ is defined by the vertex algebra generated by the fields $u(z) = \sum_{n\in\mathbb{Z}} u_{(n)} z^{-n-1}$, $(u \in \mathfrak{h})$, satisfying the OPEs

$$u(z)v(w) \sim \frac{\kappa(u, v)}{(z-w)^2}, \quad u, v \in \mathfrak{h}.$$

If κ is non-degenerate, $\pi^{\kappa}_{\mathfrak{h}}$ is simple, called a non-degenerate Heisenberg vertex algebra. The dimension $\dim\mathfrak{h}$ of $\mathfrak{h}$ is equal to that of subspace of $\pi^{\kappa}_{\mathfrak{h}}$ with conformal degree 1, called the rank of $\pi^\kappa_{\mathfrak{h}}$. For $\mu \in \mathfrak{h}^*$, let $\pi^\kappa_{\mathfrak{h},\mu}$ denote the Fock module of $\pi_{\mathfrak{h}}$ with highest weight μ, generated by a highest vector $|\mu\rangle$ satisfying

$$u_{(n)}|\mu\rangle = \delta_{n,0}\mu(u)|\mu\rangle, \quad n \geq 0, \ u \in \mathfrak{h}.$$

If we have a non-degenerate bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$, then $\mathfrak{h}$ is identified with $\mathfrak{h}^*$ by $h \mapsto (\nu(h) \colon h' \mapsto (h|h'))$. For $\kappa = k(\cdot|\cdot)$, we write $\pi^k_{\mathfrak{h}}$, (resp. $\pi^k_{\mathfrak{h},\mu}$) instead of $\pi^\kappa_{\mathfrak{h}}$, (resp. $\pi^\kappa_{\mathfrak{h},\mu}$), and denote by $\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_{(n)} z^{-n-1} := \nu^{-1}(\alpha)(z)$ for $\alpha \in \mathfrak{h}^*$. We call $\pi^k_{\mathfrak{h}}$ the Heisenberg vertex algebra associated with $\mathfrak{h}$ at level k.

A.2. **Wakimoto representations of $\widehat{\mathfrak{gl}}(1|1)_\kappa$.** Let $\mathfrak{gl}_{1|1}$ denote the Lie superalgebra $\mathrm{End}(\mathbb{C}^{1|1})$ with Lie superbracket

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx, \quad x, y \in \mathrm{End}(\mathbb{C}^{1|1}),$$

where $\bar{x} \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ denotes the parity of $x \in \mathrm{End}(\mathbb{C}^{1|1})$. Let $\{E_{i,j}\}_{1\leq i,j\leq 2}$ denote the elementary matrices of $\mathfrak{gl}_{1|1} = \mathrm{End}(\mathbb{C}^{1|1})$, $\mathfrak{h} = \mathbb{C}E_{1,1} \oplus \mathbb{C}E_{2,2}$, $\mathfrak{n}_+ = \mathbb{C}E_{1,2}$ and $\mathfrak{n}_- = \mathbb{C}E_{2,1}$. Note that the parity of $E_{i,j}$ is $\overline{i+j}$. For an even supersymmetric invariant bilinear form κ on $\mathfrak{gl}_{1|1}$, there exist unique $k_1, k_2 \in \mathbb{C}$ such that

$$\kappa = k_1\kappa_1 + k_2\kappa_2,$$

where $\kappa_1(x|y) = \mathrm{str}_{\mathbb{C}^{1|1}}(xy)$ and $\kappa_2(x|y) = -\frac{1}{2}\mathrm{str}_{\mathfrak{gl}_{1|1}}(\mathrm{ad}(x)\,\mathrm{ad}(y))$, and $\mathrm{str}_V(?)$ denotes the supertrace over a vector superspace V. The non-zero parings of κ are:

$$\kappa(E_{1,1}|E_{1,1}) = k_1 + k_2, \quad \kappa(E_{2,2}|E_{2,2}) = -k_1 + k_2,$$
$$\kappa(E_{1,1}|E_{2,2}) = -k_2, \quad \kappa(E_{1,2}|E_{2,1}) = k_1.$$

Define $\chi_i \in \mathfrak{h}^*$ by

$$\chi_i(E_{j,j}) = (-1)^{i+1}\delta_{i,j}, \quad i, j \in \{1, 2\}.$$

We identify $\mathfrak{h}$ with $\mathfrak{h}^*$ by κ_1, under which $E_{i,i}$ is identified with χ_i, $(i = 1, 2)$, since $\kappa_1(E_{i,i}|E_{j,j}) = \chi_i(E_{j,j})$. Let $\pi^{\kappa-\kappa_2} := \pi^{\kappa-\kappa_2}_{\mathfrak{h}}$ be the Heisenberg vertex algebra associated with $\mathfrak{h}$ at level $\kappa-\kappa_2$, which is freely generated by fields $\chi_i(z)$, $(i = 1, 2)$, with OPEs

$$\chi_i(z)\chi_j(w) \sim \frac{(\kappa - \kappa_2)(E_{i,i}|E_{j,j})}{(z - w)^2}, \quad i, j = 1, 2. \tag{A.1}$$

Let $V^\kappa(\mathfrak{gl}_{1|1})$ denote the universal affine vertex superalgebra associated with $\mathfrak{sl}_{1|1}$ at level κ, which is freely generated by the fields $x(z) = \sum_{n\in\mathbb{Z}} x_{(n)} z^{-n-1}$, $(x \in \mathfrak{gl}_{1|1})$ with OPEs

$$x(z)y(w) \sim \frac{[x, y]}{z - w} + \frac{\kappa(x, y)}{(z - w)^2}, \quad x, y \in \mathfrak{gl}_{1|1}.$$

If $k_1 \neq 0$, then it admits the Segal-Sugawara conformal field

$$\begin{aligned} T(z) := \frac{1}{2k_1}\Big(&\frac{1 - k_2}{k_1} :(E_{1,1}(z) + E_{2,2}(z))^2: + :E_{1,1}(z)^2: - :E_{2,2}(z)^2:) \\ &- :E_{1,2}(z)E_{2,1}(z): + :E_{2,1}(z)E_{1,2}(z):\Big), \end{aligned} \tag{A.2}$$

whose central charge is 0, i.e., the superdimension of $\mathfrak{gl}_{1|1}$.

Let $M_{\mathfrak{gl}_{1|1}}$ be the bc-system vertex superalgebra, which is generated by odd fields $b(z), c(z)$ satisfying the OPEs

$$b(z)c(w) \sim \frac{1}{z - w}, \quad b(z)b(w) \sim 0 \sim c(z)c(w).$$

The following proposition follows from direct calculations.

Proposition A.1. *There exists a homomorphism of vertex superalgebras* $\rho\colon V^{\kappa}\left(\mathfrak{gl}_{1|1}\right) \to M_{\mathfrak{gl}_{1|1}} \otimes \pi^{\kappa-\kappa_2}$*, which satisfies*

$$\begin{aligned} &E_{1,2}(z) \mapsto b(z), \quad E_{2,1}(z) \mapsto :c(z)\left(\chi_1(z)+\chi_2(z)\right): + k_1 \partial c(z), \\ &E_{1,1}(z) \mapsto -:c(z)b(z): + \chi_1(z), \quad E_{2,2}(z) \mapsto :c(z)b(z): + \chi_2(z). \end{aligned} \tag{A.3}$$

We denote a $V^{\kappa}\left(\mathfrak{gl}_{1|1}\right)$-module $M_{\mathfrak{gl}_{1|1}} \otimes \pi^{\kappa-\kappa_2}$ by W^{κ}. Note that $T(z)$ maps to

$$:\partial c(z)\ b(z): + \frac{1-k_2}{2k_1^2} :(\chi_1+\chi_2)(z)^2: + \frac{1}{2k_1}\left(:\chi_1(z)^2: - :\chi_2(z)^2: + \partial(\chi_1+\chi_2)(z)\right).$$

Lemma A.2. *ρ is injective for all κ.*

Proof. Define conformal gradings Δ on $V^{\kappa}\left(\mathfrak{gl}_{1|1}\right)$ and W^{κ} by setting

$$\begin{aligned} &\Delta(|0\rangle) = \Delta(c_{(-1)}|0\rangle) = 0, \\ &\Delta(x_{(-1)}|0\rangle) = \Delta(b_{(-1)}|0\rangle) = \Delta(\chi_{i(-1)}|0\rangle) = 1, \quad x \in \mathfrak{gl}_{1|1}, \quad i = 1,2 \end{aligned} \tag{A.4}$$

and $\Delta(A_{(n)}B) = \Delta(A) + \Delta(B) - n - 1$. We choose the set $\mathcal{A}$ of homogeneous strong generators to be $\{X_{(-1)}|0\rangle\}_{X \in \mathfrak{gl}_{1|1}}$ for $V^{\kappa}(\mathfrak{gl}_{1|1})$ and $\{b_{(-1)}|0\rangle, c_{(-1)}|0\rangle, \chi_{1(-1)}|0\rangle, \chi_{2(-1)}|0\rangle\}$ for W^{κ}. Then the associated standard filtrations are

$$F_n V := \mathrm{Span}\left\{ a^{i_1}_{(-n_1)} \cdots a^{i_r}_{(-n_r)}|0\rangle; a^{i_j} \in \mathcal{A}, \sum_j \Delta(a^{i_j}) \leq n, r \geq 0, n_i \geq 0 \right\}$$

for $V = V^{\kappa}\left(\mathfrak{gl}_{1|1}\right)$, W^{κ} respectively and their associated graded superspaces

$$\mathrm{gr}_F V := \bigoplus_{n=0}^{\infty} \frac{F_n V}{F_{n-1} V}$$

admit a structure of Poisson vertex superalgebra [Ar3, Li3]. Since ρ preserves the gradings Δ by (A.3), ρ induces a homomorphism of Poisson vertex superalgebras

$$\mathrm{gr}_F \rho\colon \mathrm{gr}_F V^{\kappa}\left(\mathfrak{gl}_{1|1}\right) \to \mathrm{gr}_F W^{\kappa}.$$

We have

$$\begin{aligned} &\mathrm{gr}_F V^{\kappa}\left(\mathfrak{gl}_{1|1}\right) = \mathbb{C}[\partial^n E_{i,j} \mid 1 \leq i,j \leq 2,\ n \in \mathbb{Z}_{\geq 0}], \\ &\mathrm{gr}_F W^{\kappa} = \mathbb{C}[\partial^n b, \partial^n c, \partial^n \chi_i \mid i = 1,2,\ n \in \mathbb{Z}_{\geq 0}], \end{aligned}$$

where $\partial^n A$ is the image of $\frac{1}{n!} A_{(-n-1)}|0\rangle \in F_{\Delta(A)+n} V$ in $F_{\Delta(A)+n} V / F_{\Delta(A)+n-1} V$. Next, define weight gradings wt on $\mathrm{gr}_F V^{\kappa}\left(\mathfrak{gl}_{1|1}\right)$ and $\mathrm{gr}_F W^{\kappa}$ by setting

$$\begin{aligned} &\mathrm{wt}(\partial^n E_{1,2}) = \mathrm{wt}(\partial^n b) = \mathrm{wt}(\partial^n c) = 0, \\ &\mathrm{wt}(\partial^n E_{i,i}) = \mathrm{wt}(\partial^n E_{2,1}) = \mathrm{wt}(\partial^n \chi_i) = 1, \quad i = 1,2 \end{aligned}$$

and $\mathrm{wt}(AB) = \mathrm{wt}(A) + \mathrm{wt}(B)$. They yield filtrations $G_n \overline{V} = \mathrm{Span}\{A \in \overline{V}; \mathrm{wt}(A) \leq n\}$ on $\mathrm{gr}_F \bar{V}$ for $\bar{V} = \mathrm{gr}_F V^{\kappa}\left(\mathfrak{gl}_{1|1}\right)$, $\mathrm{gr}_F W^{\kappa}$. Since $\{G_m \overline{V}_{\lambda} G_n \overline{V}\} \subset G_{m+n} \overline{V}[\lambda]$, the associated graded superspace

$$\mathrm{gr}_G \overline{V} := \bigoplus_{n=0}^{\infty} \frac{G_n \overline{V}}{G_{n-1} \overline{V}}$$

also has a structure of Poisson vertex superalgebra. Since $\mathrm{gr}_F \rho$ preserves the weight gradings by (A.3), $\mathrm{gr}_F \rho$ induces a homomorphism of Poisson vertex superalgebras

$$\begin{aligned} \mathrm{gr}_G \mathrm{gr}_F \rho\colon \mathrm{gr}_G \mathrm{gr}_F V^{\kappa}\left(\mathfrak{gl}_{1|1}\right) &\to \mathrm{gr}_G \mathrm{gr}_F W^{\kappa}, \\ E_{1,2} \mapsto b, \quad & E_{2,1} \mapsto c(\chi_1+\chi_2), \\ E_{i,i} \mapsto \chi_i, \quad & i = 1,2, \end{aligned}$$

where A denotes the image of $A \in G_{\mathrm{wt}(A)}\overline{V}$ in $G_{\mathrm{wt}(A)}\overline{V}/G_{\mathrm{wt}(A)-1}\overline{V}$ by abuse of notation. Since $\mathrm{gr}_G\mathrm{gr}_F\rho$ is injective, so is ρ. $\square$

By using ρ in Proposition A.1, the W^κ-module

$$W^\kappa_\mu := M_{\mathfrak{gl}_{1|1}} \otimes \pi^{\kappa-\kappa_2}_\mu, \quad \mu \in \mathfrak{h}^*$$

becomes a $V^\kappa\left(\mathfrak{gl}_{1|1}\right)$-module, which we call the Wakimoto representation of $\widehat{\mathfrak{gl}}(1|1)_\kappa$ with highest weight μ, (cf. [F1]).

A.3. **Wakimoto representations at generic level.** Here we study the Wakimoto representations W^κ_μ for generic level κ. Let

$$\mu_i := \mu(E_{i,i}), \quad i = 1, 2.$$

Consider the highest, (resp. lowest), Verma module of $\mathfrak{gl}_{1|1}$

$$V^\pm_{n,e} := U\left(\mathfrak{gl}_{1|1}\right) \otimes_{U(\mathfrak{b}_\pm)} \mathbb{C}v_{n,e}, \quad n, e \in \mathbb{C},$$

where $\mathfrak{b}_+ := \mathrm{Span}\{E_{1,2}, N, E\}$, (resp. $\mathfrak{b}_- := \mathrm{Span}\{E_{2,1}, N, E\}$), with

$$N := \frac{1}{2}(E_{1,1} - E_{2,2}), \quad E := E_{1,1} + E_{2,2}$$

and $\mathbb{C}v_{n,e}$ is the one dimensional $\mathfrak{b}_\pm$-module such that $E_{1,2}$, (resp. $E_{2,1}$), acts by 0, N by n, and E by e. If $e \neq 0$, then they are irreducible and we have an isomorphism $V^+_{n,e} \simeq V^-_{n-1,e}$. If $e = 0$, then they are only indecomposable and we have the following short exact sequences

$$0 \to A_{n\pm1} \to V^\pm_{n,0} \to A_n \to 0 \tag{A.5}$$

where $A_q = \mathbb{C}w_q$, ($q \in \mathbb{C}$), is the one dimensional $\mathfrak{gl}_{1|1}$-module such that $E_{1,2}$, $E_{2,1}$, E acts by 0, and N acts by q.

Define the $V^\kappa(\mathfrak{gl}_{1|1})$-modules

$$\hat{V}^{\pm,\kappa}_{n,e} := U(\widehat{\mathfrak{gl}}(1|1)_\kappa) \otimes_{U(\mathfrak{gl}_{1|1,\kappa,+})} V^\pm_{n,e}, \quad \hat{A}^\kappa_n := U(\widehat{\mathfrak{gl}}_{1|1,\kappa}) \otimes_{U(\mathfrak{gl}_{1|1,\kappa,+})} A_n.$$

Lemma A.3 ([CR2])**.** *(i) If $e \neq 0$, then $\hat{V}^{\pm,\kappa}_{n,e}$ is irreducible for $\frac{e}{k_1} \notin \mathbb{Q}$, and there exists an isomorpshim*

$$\hat{V}^{+,\kappa}_{n,e} \simeq \hat{V}^{-,\kappa}_{n-1,e}.$$

(ii) If $e = 0$, then $\hat{V}^{\pm,\kappa}_{n,e}$ admits the following short exact sequence for $k_1 \neq 0$

$$0 \to \hat{A}^\kappa_{n\pm1} \to \hat{V}^{\pm,\kappa}_{n,0} \to \hat{A}^\kappa_n \to 0. \tag{A.6}$$

(iii) $\hat{A}^\kappa_n$ is irreducible for $k_1 \neq 0$.

Proof. We include the proof for the completeness of the paper, following the argument in [CR2, Sec. 3.2]. We may assume $k_1 \neq 0$. For (i), the isomorphism $V^+_{n,e} \simeq V^-_{n-1,e}$ induces an isomorphism $\hat{V}^{+,\kappa}_{n,e} \simeq \hat{V}^{-,\kappa}_{n-1,e}$. The conformal dimension of $1 \otimes v_{n,e} \in \hat{V}^{\pm,\kappa}_{n,e}$ with respect to (A.2) is

$$\Delta_{n,e} := \frac{1}{2k_1}\left(\frac{1-k_2}{k_1}e^2 + 2en \mp 1\right).$$

Suppose that $\hat{V}^{-,\kappa}_{n,e}$ is reducible. Then we have a nontrivial $V^\kappa(\mathfrak{gl}_{1|1})$-module homomorphism

$$\hat{V}^{-,\kappa}_{n',e} \to \hat{V}^{-,\kappa}_{n,e}, \quad \exists n' \in n + \mathbb{Z},$$

and thus $\Delta_{n',e} = \Delta_{n,e} + j$ for some $j \in \mathbb{Z}_{>0}$, that is, $\frac{e}{k_1}(n' - n) = j$. Since $n' - n \in \mathbb{Z}$, it is impossible when $\frac{e}{k_1} \notin \mathbb{Q}$, in which case $\hat{V}^{-,\kappa}_{n,e}$ is irreducible. This completes the proof of (i). For (iii), note that the conformal dimension of $1 \otimes w_n \in \hat{A}^\kappa_n$ with respect to (A.2) is always 0. Suppose that $\hat{A}^\kappa_n$ is reducible. Then we have a nontrivial $V^\kappa(\mathfrak{gl}_{1|1})$-module homomorphism

$$\hat{V}^{+,\kappa}_{n',0} \to \hat{A}^\kappa_n \quad \text{or} \quad \hat{V}^{-,\kappa}_{n',0} \to \hat{A}^\kappa_n, \quad \exists n' \in \mathbb{Z}$$

such that the image of $1 \otimes v_{n,0}$ is not contained in $\mathbb{C}w_n$. It is impossible since the conformal dimension of $1 \otimes v_{n,0} \in \hat{V}^{\pm,\kappa}_{n,0}$ is 0. Thus $\hat{A}^\kappa_n$ is irreducible for $k_1 \neq 0$. For (ii), (A.5) induces an exact sequence of $V^\kappa(\mathfrak{gl}_{1|1})$-modules

$$\hat{A}^\kappa_{n\pm1} \to \hat{V}^{\pm,\kappa}_{n,0} \to \hat{A}^\kappa_n \to 0$$

by the right exactness of the tensor functor $U(\widehat{\mathfrak{gl}}(1|1)_\kappa) \otimes_{U(\mathfrak{gl}_{1|1,\kappa,+})}$ (?). The map $\hat{A}^\kappa_{n\pm1} \to \hat{V}^{\pm,\kappa}_{n,0}$ is also injective by (iii) for $k_1 \neq 0$. This completes the proof of (ii). □

Proposition A.4. *We have an isomorphism*

$$W^\kappa_\mu \simeq \hat{V}^{-,\kappa}_{n(\mu),e(\mu)}$$

of $V^\kappa(\mathfrak{gl}_{1|1})$-modules where $n(\mu) = \frac{\mu_1+\mu_2}{2} - 1$, $e(\mu) = \mu_1 - \mu_2$ for $\frac{e(\mu)}{k_1} \notin \mathbb{Q}$ if $e(\mu) \neq 0$ and for $k_1 \neq 0$ if $e(\mu) = 0$.

Proof. Define a conformal grading Δ on W^κ_μ by $\Delta(|\mu\rangle) = 0$ and (A.4):

$$W^\kappa_\mu = \bigoplus_{d\geq0} W^\kappa_{\mu,d}. \tag{A.7}$$

Then the subspace $W^\kappa_{\mu,0} = \mathbb{C}|\mu\rangle \oplus \mathbb{C}c_{(-1)}|\mu\rangle$ is a $\hat{\mathfrak{gl}}(1|1)_{\kappa,+}$-module, which is isomorphic to $V^-_{n(\mu),e(\mu)}$ by

$$\begin{aligned} &V^-_{n(\mu),e(\mu)} \to W^\kappa_{\mu,0}, \\ &v_{n(\mu),e(\mu)} \mapsto c_{(-1)}|\mu\rangle, \ E_{1,2}v_{n(\mu),e(\mu)} \mapsto |\mu\rangle. \end{aligned}$$

Thus, by the universality of the induced modules, we have a $V^\kappa(\mathfrak{gl}_{1|1})$-module homomorphism

$$\hat{V}^{-,\kappa}_{n(\mu),e(\mu)} \to W^\kappa_{-n\alpha}. \tag{A.8}$$

If $\mu_1 \neq \mu_2$, then the map (A.8) is injective by Lemma A.3 (1). If $\mu_1 = \mu_2$, then $(n(\mu), e(\mu)) = (-n-1, 0)$. It follows from (A.6) that the singular vectors of $\hat{V}^{-,\kappa}_{-n-1,0}$ for $k_1 \neq 0$ belong to the subspace $A_{-n} \subset \hat{A}^\kappa_{-n}$, which is clearly embedded by (A.8). Thus the map (A.8) is injective by Lemma A.3 (2) also in this case. Therefore, the map (A.8) is injective for generic κ for any highest weight $\mu \in \mathfrak{h}^*$.

Define a conformal degree Δ of $\hat{V}^{-,\kappa}_{n(\mu),e(\mu)}$ by $\Delta(V^-_{n(\mu),e(\mu)}) = 0$, $\Delta(X_{(-1)}) = 1$, $(X \in \mathfrak{gl}_{1|1})$, and $\Delta(X_{(-n-1)}Y) = \Delta(X) + \Delta(Y) + n + 1$. Then the map (A.8) preserves the conformal gradings by Proposition A.1. Now its surjectivity follows from the equality of the characters:

$$\mathrm{ch}\left[\widehat{V}^-_{n(\mu),e(\mu)}\right] = 2\prod_{n=1}^\infty (1+q^n)^2(1+q^n)^{-2} = \mathrm{ch}[W^\kappa_\mu].$$

Thus (A.8) is an isomorphism. □

A.4. **Resolution.** For $\alpha = \chi_1 + \chi_2 \in \mathfrak{h}^*$, define an intertwining operator $S(z)$: $W^\kappa_{-n\alpha} \xrightarrow{S} W^\kappa_{-(n+1)\alpha}((z))$, $(k_1 \neq 0)$, by

$$S(z) = :b(z)\mathrm{e}^{-\frac{1}{k_1}\int \alpha(z)}:,$$

where

$$:\mathrm{e}^{-\frac{1}{k_1}\int\alpha(z)}:=T_{-\alpha}\mathrm{exp}\left(\frac{1}{k_1}\sum_{n<0}\frac{\alpha_{(n)}}{n}z^{-n}\right)\mathrm{exp}\left(\frac{1}{k_1}\sum_{n>0}\frac{\alpha_{(n)}}{n}z^{-n}\right),\tag{A.9}$$

and $T_{-\alpha}$ is the translation operator $\pi^\kappa_{-n\alpha}\to\pi^\kappa_{-(n+1)\alpha}$ sending the highest weight vector to the highest weight vector and commuting with all $\chi_{i(n)}$, $i=1,2$, $n\neq 0$. By direct calculation, one can show that the residue

$$S:=\int S(z)dz$$

satisfies $S(u_{(-1)}|0\rangle)=0$, $u\in\mathfrak{gl}_{1|1}$. It follows that S is a $V^\kappa(\mathfrak{gl}_{1|1})$-module homomorphism from $W^\kappa_{-n\alpha}$ to $W^\kappa_{-(n+1)\alpha}$ (cf. [F1]).

Using Wakimoto representations, we can extend the injective morphism ρ to a long exact sequence as in the following proposition.

Proposition A.5. *The sequence*

$$0\to V^\kappa(\mathfrak{gl}_{1|1})\xrightarrow{\rho}W^\kappa_0\xrightarrow{S}W^\kappa_{-\alpha}\to\cdots\to W^\kappa_{-n\alpha}\xrightarrow{S}W^\kappa_{-(n+1)\alpha}\to\cdots.\tag{A.10}$$

is a complex of $V^\kappa(\mathfrak{gl}_{1|1})$-modules and exact for $k_1\neq 0$.

Proof. To show that (A.10) is a complex, we have to show (1) $\mathrm{Im}(\rho)\subset\mathrm{Ker}(S)$ and (2) $S\circ S=0$. (1) follows from $S(x_{(-1)}|0\rangle)=0$, $x\in\mathfrak{gl}_{1|1}$. To show (2), notice that we can extend the vertex superalgebra W^κ to the direct sum of W^κ-modules $\bigoplus_{n\geq 0}W^\kappa_{-n\alpha}$ by setting

$$Y(|-n\alpha\rangle,z)=:\mathrm{e}^{-\frac{n}{k_1}\int\alpha(z)}:,$$

where $|-n\alpha\rangle$ is a fixed non-zero highest weight vector of $\pi^\kappa_{n\alpha}$ (see (A.9)). This is due to $\kappa(n\alpha,m\alpha)=0$ for $n,m\geq 0$ (cf. [FBZ, Chap. 5]). Then S is the 0-th mode $Q_{(0)}$ of the field corresponding to $Q=b_{(-1)}|-\alpha\rangle$. By the Jacobi identity of the λ-bracket, we have

$$S\circ S(a)=[Q_\lambda[Q_\mu a]]_{\lambda=\mu=0}=\frac{1}{2}[[Q_\lambda Q]_{\lambda+\mu}a]]_{\lambda=\mu=0}=0,\quad a\in W^\kappa_{-n\alpha}.$$

Thus $S\circ S=0$ follows.

It is clear that the complex (A.10) preserves the conformal gradings (A.7). In particular, the subcomplex of conformal grading 0 is isomorphic to

$$0\to A_0\to V^-_{-1,0}\to V^-_{-2,0}\to\cdots\to V^-_{-n,0}\to V^-_{-n-1,0}\to\cdots,$$

which is a complex of $\mathfrak{gl}_{1|1}$-modules. Since $S(|-n\alpha\rangle)=0$ and $S(c_{(-1)}|-n\alpha\rangle)=|-(n+1)\alpha\rangle$, it is a successive composition of (A.5), and thus exact. Now it follows that the complex (A.10) for $k_1\neq 0$ is a successive composition of (A.6) since $\hat{A}^\kappa_{-n}$, $(n\geq 0)$, are irreducible for $k_1\neq 0$. Thus (A.10) is exact. □

Appendix B. Categorical aspects of simple currents

We study the theory of simple currents of vertex operator algebras purely in a categorical manner, following [CKL, CKM1].

B.1. **Preliminaries.** A supercategory $\mathcal{C}$ is a category enriched by the category of $\mathbb{Z}_2$-graded sets , [BE]. By definition, for objects M,N,L in $\mathcal{C}$, the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(M,N)$ carries a $\mathbb{Z}_2$-grading

$$\mathrm{Hom}_{\mathcal{C}}(M,N)=\bigoplus_{i\in\mathbb{Z}_2}\mathrm{Hom}_{\mathcal{C}}(M,N)_i$$

and the composition of morphisms preserves the $\mathbb{Z}_2$-grading by addition

$$\circ\colon\ \mathrm{Hom}_{\mathcal{C}}(M,N)_i\times\mathrm{Hom}_{\mathcal{C}}(L,M)_j\to\mathrm{Hom}_{\mathcal{C}}(L,N)_{i+j},\quad(f,g)\mapsto f\circ g.$$

The underlying category $\underline{\mathcal{C}}$ of $\mathcal{C}$ is defined as the category whose objects are the same as $\mathcal{C}$ and whose morphisms are the *even* morphisms of $\mathcal{C}$. For an additive supercategory $\mathcal{C}$ such that $\underline{\mathcal{C}}$ is an *abelian category*, by a subquotient object in $\mathcal{C}$ we mean one in the underlying category $\underline{\mathcal{C}}$. We use the notion of simplicity as well. We stress that we use this terminology even if $\mathcal{C}$ is an *abelian supercategory*, that is, an additive supercategory such that every (parity-inhomogeneous) morphism in $\mathcal{C}$ admits a kernel and cokernel.

Let $\mathcal{C}$ be an essentially small, $\mathbb{C}$-linear, monoidal supercategory whose underlying category is abelian. We denote by

$$\boxtimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad (M, N) \mapsto M \boxtimes N$$

the monoidal product superfunctor with unit object $1_{\mathcal{C}}$, by

$$l_\bullet \colon 1_{\mathcal{C}} \boxtimes \bullet \xrightarrow{\simeq} \bullet, \quad (l_M \colon 1_{\mathcal{C}} \boxtimes M \xrightarrow{\simeq} M),$$

$$r_\bullet \colon \bullet \boxtimes 1_{\mathcal{C}} \xrightarrow{\simeq} \bullet, \quad (r_M \colon M \boxtimes 1_{\mathcal{C}} \xrightarrow{\simeq} M),$$

$$\mathcal{A}_{\bullet,\bullet,\bullet} \colon \bullet \boxtimes (\bullet \boxtimes \bullet) \xrightarrow{\simeq} (\bullet \boxtimes \bullet) \boxtimes \bullet, \quad \Big(\mathcal{A}_{M,N,L} \colon M \boxtimes (N \boxtimes L) \xrightarrow{\simeq} (M \boxtimes N) \boxtimes L\Big),$$

the structual natural isomorphisms of superfunctors satisfying the pentagon and triangle axioms, see [BK, EGNO]. The last one is called the associativity isomorphism. Here we use the convention that the parity of a parity-homogeneous morphism f is denoted by $\bar{f}$ and the composition of two parity-homogeneous morphisms (f_1, f_2), (g_1, g_2) in $\mathcal{C} \times \mathcal{C}$ is given by $(-1)^{\bar{f}_2 \bar{g}_1}(f_1 g_1, f_2 g_2)$.

For an object $M \in Ob(\mathcal{C})$, a right dual of M is a triple (M^*, e^R_M, i^R_M) of $M^* \in Ob(\mathcal{C})$ and even morphisms

$$e^R_M \colon M^* \boxtimes M \to 1_{\mathcal{C}}, \quad i^R_M \colon 1_{\mathcal{C}} \to M \boxtimes M^*, \tag{B.1}$$

satisfying the rigidity axioms. A left dual of M is similarly defined to be a triple $({}^*M, e^L_M, i^L_M)$ consisting of ${}^*M \in Ob(\mathcal{C})$ and even morphisms

$$e^L_M \colon M \boxtimes {}^*M \to 1_{\mathcal{C}}, \quad i^L_M \colon 1_{\mathcal{C}} \to {}^*M \boxtimes M. \tag{B.2}$$

By [EGNO, Proposition 2.10.15], right (left) duals are unique up to even isomorphisms if they exist. The category $\mathcal{C}$ is called rigid if every object in $\mathcal{C}$ has a right and left dual. We call an object M of $\mathcal{C}$ an *invertible object* if it admits a left and right dual such that the morphisms in (B.1) and (B.2) are isomorphisms.

Lemma B.1. *For an invertible object S, the superfunctors*

$$S \boxtimes \bullet \colon \mathcal{C} \to \mathcal{C}, \quad M \mapsto S \boxtimes M,$$
$$\bullet \boxtimes S \colon \mathcal{C} \to \mathcal{C}, \quad M \mapsto M \boxtimes S,$$

are exact.

Proof. Since the proofs for $S \boxtimes \bullet$ and $\bullet \boxtimes S$ are similar, we prove only for $S \boxtimes \bullet$. Note that any object lies in the image of $S \boxtimes \bullet$ since for any object $M \in Ob(\mathcal{C})$,

$$S \boxtimes (S^* \boxtimes M) \simeq (S \boxtimes S^*) \boxtimes M \simeq 1_{\mathcal{C}} \boxtimes M \simeq M.$$

Now, take a short exact sequence in $\mathcal{C}$

$$0 \to M \to N \to L \to 0.$$

We show that the complex

$$0 \to S \boxtimes M \to S \boxtimes N \to S \boxtimes L \to 0 \tag{B.3}$$

is exact. For the left exactness of (B.3), it suffices to show that for any object $A \in Ob(\mathcal{C})$, the induced complex

$$0 \to \mathrm{Hom}_{\mathcal{C}}(A, S \boxtimes M) \to \mathrm{Hom}_{\mathcal{C}}(A, S \boxtimes N) \to \mathrm{Hom}_{\mathcal{C}}(A, S \boxtimes L) \tag{B.4}$$

is exact. Indeed, by the left exactness of $\mathrm{Hom}_{\mathcal{C}}(A,\bullet)$, we have the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(A,M) \to \mathrm{Hom}_{\mathcal{C}}(A,N) \to \mathrm{Hom}_{\mathcal{C}}(A,L). \tag{B.5}$$

Then, by the functoriality of the following isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(A,M) &\simeq \mathrm{Hom}_{\mathcal{C}}(A,1_{\mathcal{C}}\boxtimes M) \simeq \mathrm{Hom}_{\mathcal{C}}(A,({}^*S\boxtimes S)\boxtimes M)\\ &\simeq \mathrm{Hom}_{\mathcal{C}}(A,{}^*S\boxtimes (S\boxtimes M)) \simeq \mathrm{Hom}_{\mathcal{C}}(S\boxtimes A,S\boxtimes M),\end{aligned} \tag{B.6}$$

(see [EGNO, Proposition 2.10.8]), the exactness of (B.5) implies that of

$$0 \to \mathrm{Hom}_{\mathcal{C}}(S\boxtimes A,S\boxtimes M) \to \mathrm{Hom}_{\mathcal{C}}(S\boxtimes A,S\boxtimes N) \to \mathrm{Hom}_{\mathcal{C}}(S\boxtimes A,S\boxtimes L).$$

Replacing A by $S^*\boxtimes A$ in $S\boxtimes A$, we conclude that (B.4) is exact. We can prove the right exactness of (B.3) in a similar way by showing the exactness of

$$0 \to \mathrm{Hom}_{\mathcal{C}}(S\boxtimes L,A) \to \mathrm{Hom}_{\mathcal{C}}(S\boxtimes N,A) \to \mathrm{Hom}_{\mathcal{C}}(S\boxtimes M,A)$$

for any object $A\in Ob(\mathcal{C})$ and thus we omit it. This completes the proof. $\square$

We call a simple invertible object a *simple current* following the terminology of the theory of vertex algebra. By Lemma B.1, a simple current exists if and only if the unit object $1_{\mathcal{C}}$ is simple.

Assumption 1. *The unit object $1_{\mathcal{C}}$ is simple.*

A braided monodial supercategory is a monoidal supercategory $\mathcal{C}$ equipped with a natural isomorphism of superfunctors, called the *braiding*,

$$\mathcal{R}_{\bullet,\bullet}\colon (\bullet\boxtimes\bullet) \xrightarrow{\simeq} (\bullet\boxtimes\bullet)\circ\sigma, \quad (\mathcal{R}_{M,N}\colon M\boxtimes N \xrightarrow{\simeq} N\boxtimes M),$$

where σ is the superfunctor

$$\sigma\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}\times\mathcal{C}, \quad (M,N)\mapsto (N,M),$$

under which every parity-homogeneous morphism (f,g) maps to $(-1)^{\bar{f}\bar{g}}(g,f)$. The natural isomorphism $\mathcal{R}_{\bullet,\bullet}$ is required to satisfy the hexagon identity. The natural isomorphism

$$\mathcal{M}_{\bullet,\bullet} := \mathcal{R}^2_{\bullet,\bullet}\colon (\bullet\boxtimes\bullet) \xrightarrow{\simeq} (\bullet\boxtimes\bullet), \quad (\mathcal{M}_{M,N}\colon M\boxtimes N \xrightarrow{\simeq} M\boxtimes N)$$

is called the *monodromy*. It is straightforward to check that, in a braided monoidal supercategory, the existence of a right dual implies that of a left dual and vice versa. Indeed, given a right dual (M^*, e^R_M, i^R_M) of an object M, the triple $(M^*, e^R_M\circ\mathcal{R}^{-1}_{M,M^*}, \mathcal{R}_{M,M^*}\circ i^R_M)$ defines a left dual of M and conversely, given a left dual $({}^*M, e^L_M, i^L_M)$, a triple $({}^*M, e^L_M\circ\mathcal{R}_{M,{}^*M}, \mathcal{R}^{-1}_{M,{}^*M}\circ i^L_M)$ defines a right dual.

One of the simplest examples of braided monoidal supercategory is the supercategory $\mathrm{SVect}_{\mathbb{C}}$ of vector superspaces over $\mathbb{C}$ of at most countable dimension equipped with the tensor product $M\boxtimes N = M\otimes_{\mathbb{C}} N$ and the braiding

$$R_{M,N}\colon M\otimes_{\mathbb{C}} N\to N\otimes_{\mathbb{C}} M, \quad m\otimes n\mapsto (-1)^{\bar{n}\bar{m}} n\otimes m.$$

Obviously, the supercategory $\mathcal{C}$ is $\mathbb{C}$-linear and its underlying category $\underline{\mathcal{C}}$ is abelian. Note that the supercategory $\mathrm{SVect}_{\mathbb{C}}$ admits a natural parity reversing endofunctor Π, which exchanges the parity of objects.

In this paper, we make the following assumption on our monoidal supercategory $\mathcal{C}$ in consideration:

Assumption 2. *Every object in $\mathcal{C}$ has a structure of a $\mathbb{C}$-vector superspace of at most countable dimension, the forgetful functor $\mathcal{C}\to\mathrm{SVect}_{\mathbb{C}}$ is a $\mathbb{C}$-linear exact faithful superfunctor, and there exists an involutive autofunctor $\Pi_{\mathcal{C}}$ of $\mathcal{C}$ which coincides with the parity reversing autofunctor Π of $\mathrm{SVect}_{\mathbb{C}}$ through the forgetful functor. In addition, each $\mathbb{C}$-vector superspace of morphisms in $\mathcal{C}$ has a finite dimension.*

Such a situation naturally appears when we consider module categories of suitable superalgebras over $\mathbb{C}$ like vertex operator superalgebras or Hopf superalgebras consisting of modules of at most countable dimension. The existence of the forgetful functor implies Diximir's lemma (or Schur's lemma) for $\mathcal{C}$.

Lemma B.2.
(i) For non-isomorphic simple objects $M, N \in Ob(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(M,N) = 0$.
(ii) For a simple object $M \in Ob(\mathcal{C})$, $\mathrm{End}_{\mathcal{C}}(M)_{\bar{0}} = \mathbb{C}\,\mathrm{id}_M$ and $\mathrm{End}_{\mathcal{C}}(M)_{\bar{1}} = 0$ or $\mathbb{C}\Pi$ for some odd isomorphism Π such that $\Pi^2 = \mathrm{id}_M$.

Proof. It is easy to see that (i) and the first assertion in (ii) follow from the argument in the purely even case, see e.g. [Wal, §1.2]. For the second one in (ii), we suppose $\mathrm{End}_{\mathcal{C}}(M)_{\bar{1}} \neq 0$ and take a nonzero morphism $f \in \mathrm{End}_{\mathcal{C}}(M)_{\bar{1}}$. We prove $\mathrm{End}_{\mathcal{C}}(M)_{\bar{1}} = \mathbb{C}f$. Since f^2 is an even isomorphism, we may assume that $f^2 = \mathrm{id}_{\mathcal{C}}$ by rescalling. Then it suffices to show $g = \pm f$ for $g \in \mathrm{End}_{\mathcal{C}}(M)_{\bar{1}}$ such that $g^2 = \mathrm{id}_{\mathcal{C}}$. Let $\alpha \in \mathbb{C}$ be the scalar such that $fg = \alpha \mathrm{id}_{\mathcal{C}}$. Since $\mathrm{id}_{\mathcal{C}} = f(fg)g = \alpha fg = \alpha^2\,\mathrm{id}_{\mathcal{C}}$, we obtain $\alpha = \pm 1$. Then, by $fg = \pm\mathrm{id}_{\mathcal{C}}$ and $(fg)(gf) = \mathrm{id}_{\mathcal{C}}$, we have $fg = gf$. Now, we have $(f+g)(f-g) = f^2 - g^2 = 0$, which implies that $f + g = 0$ or $f - g = 0$. This completes the proof. $\square$

In practice, the case $\dim_{\mathbb{C}} \mathrm{End}_{\mathcal{C}}(M)_{\bar{1}} = 1$ occurs when $\mathcal{C}$ is a module category of a $\mathbb{C}$-superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and an object $M = M_{\bar{0}} \oplus M_{\bar{1}}$ satisfies $f\colon M_{\bar{0}} \simeq M_{\bar{1}}$ as $A_{\bar{0}}$-modules. In this case, an isomorphism $(f, f^{-1})\colon M = M_{\bar{0}} \oplus M_{\bar{1}} \simeq M_{\bar{1}} \oplus M_{\bar{0}} = M$ as $A_{\bar{0}}$-modules has odd parity and it might define an isomorphism as A-modules, see [CW, §3.1].

B.2. **Block decomposition and Monodromy filtration by Simple currents.** Let $\mathcal{C}$ be a monoidal supercategory as in §2.1 equipped with a braided monoidal supercategory structure. Here we study decompositions and filtrations on $\mathcal{C}$ induced by simple currents. Take a simple current $S \in Ob(\mathcal{C})$. We set

$$S^n := \begin{cases} S \boxtimes (S \boxtimes (\cdots (S \boxtimes S)\cdots), & (n \geq 1), \\ 1_{\mathcal{C}}, & (n = 0), \\ S^* \boxtimes (S^* \boxtimes (\cdots (S^* \boxtimes S^*)\cdots), & (n \leq -1). \end{cases}$$

Since simple currents are closed under taking left, (right), dual and monoidal product by [EGNO, Proposition 2.11.3], S^n, $(n \in \mathbb{Z})$, are simple currents and satisfy

$$S^n \boxtimes S^m \simeq S^{n+m}, \quad (S^n)^* \simeq S^{-n}.$$

Then by (B.6), we have

$$\mathrm{End}_{\mathcal{C}}(M) \simeq \mathrm{End}_{\mathcal{C}}(S^n \boxtimes M), \quad (M \in Ob(\mathcal{C})).$$

Lemma B.3. *The map*

$$\mathrm{End}_{\mathcal{C}}(M) \to \mathrm{End}_{\mathcal{C}}(S^n \boxtimes M), \quad f \mapsto \mathrm{id}_{S^n} \boxtimes f \tag{B.7}$$

is an isomorphism of $\mathbb{C}$-superalgebras.

Proof. It is straightforward to show that the map

$$\mathrm{End}_{\mathcal{C}}(S^n \boxtimes M) \to \mathrm{End}_{\mathcal{C}}\left(S^{-n} \boxtimes (S^n \boxtimes M)\right), \quad f \mapsto \mathrm{id}_{S^{-n}} \boxtimes (\mathrm{id}_{S^n} \boxtimes f)$$

is an inverse of (B.7) under the natural isomorphisms

$$\begin{aligned}\mathrm{End}_{\mathcal{C}}\left(S^{-n} \boxtimes (S^n \boxtimes M)\right) &\simeq \mathrm{End}_{\mathcal{C}}\left((S^{-n} \boxtimes S^n) \boxtimes M\right) \\ &\simeq \mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}} \boxtimes M) \simeq \mathrm{End}_{\mathcal{C}}(M).\end{aligned}$$

$\square$

By Lemma B.3, the monodromy

$$\mathcal{M}_{S^n,M} = \mathcal{R}_{M,S^n} \circ \mathcal{R}_{S^n,M} \in \mathrm{End}_{\mathcal{C}}(S^n \boxtimes M)$$

defines a unique element $m_S(n) \in \mathrm{End}_{\mathcal{C}}(M)^{\bar{0}}$ satisfying

$$\mathcal{M}_{S^n,M} = \mathrm{id}_{S^n} \boxtimes m_S(n).$$

Proposition B.4. *The endomorphism $m_S(n)$ is invertible and moreover*

$$m_S(n) = m_S(1)^n \quad (n \in \mathbb{Z}).$$

Proof. We show the assertion for $n \geq 1$ by induction on n. By the hexagon identity, the following diagram commutes.

$$\begin{tikzcd}
(S \boxtimes S^{n-1}) \boxtimes M \arrow[r, "\mathcal{R}_{S^n,M}"] \arrow[d, "\simeq"'] & M \boxtimes (S \boxtimes S^{n-1}) \arrow[r, "\mathcal{R}_{M,S^n}"] \arrow[ddd, bend left, "\simeq"] & (S \boxtimes S^{n-1}) \boxtimes M \\
S \boxtimes (S^{n-1} \boxtimes M) \arrow[d, "\mathrm{id} \boxtimes \mathcal{R}"'] & & S \boxtimes (S^{n-1} \boxtimes M) \arrow[u, "\simeq"'] \\
S \boxtimes (M \boxtimes S^{n-1}) \arrow[d, "\simeq"'] & & S \boxtimes (M \boxtimes S^{n-1}) \arrow[u, "\mathrm{id} \boxtimes \mathcal{R}"'] \\
(S \boxtimes M) \boxtimes S^{n-1} \arrow[r, "\mathcal{R} \boxtimes \mathrm{id}"'] & (M \boxtimes S) \boxtimes S^{n-1} \arrow[r, "\mathcal{R} \boxtimes \mathrm{id}"'] \arrow[uuu, bend left, "\simeq"] & (S \boxtimes M) \boxtimes S^{n-1}. \arrow[u, "\simeq"']
\end{tikzcd}$$

Here all the isomorphisms without symbols are associativity isomorphisms. Then the morphisms in the bottom are composed as

$$\begin{aligned}
(\mathcal{R}_{M,S} \boxtimes \mathrm{id}_{S^{n-1}}) \circ (\mathcal{R}_{S,M} \boxtimes \mathrm{id}_{S^{n-1}}) &= \mathcal{M}_{S,M} \boxtimes \mathrm{id}_{S^{n-1}} \\
&= (\mathrm{id}_S \boxtimes m_S(1)) \boxtimes \mathrm{id}_{S^{n-1}} .
\end{aligned}$$

Now, we may use the naturality of the associativity $\mathcal{A}_{\bullet,\bullet,\bullet}$ and the braiding $\mathcal{R}_{\bullet,\bullet}$ to conclude $\mathcal{M}_{S^n,M} = \mathrm{id}_{S^n} \boxtimes m_S(1)^n$ since

$$\begin{aligned}
\mathcal{M}_{S^n,M} &= \mathcal{R}_{M,S^n} \circ \mathcal{R}_{S^n,M} \\
&= \mathcal{A}_{S,S^{n-1},M} \circ (\mathrm{id}_S \boxtimes \mathcal{R}_{S^{n-1},M}) \circ \mathcal{A}^{-1}_{S,M,S^{n-1}} \\
&\qquad \circ (\mathcal{M}_{S,M} \boxtimes \mathrm{id}_{S^{n-1}}) \circ \mathcal{A}_{S,M,S^{n-1}} \circ (\mathrm{id}_S \boxtimes \mathcal{R}_{S^{n-1},M}) \circ \mathcal{A}^{-1}_{S,S^{n-1},M} \\
&= (\mathrm{id}_{S^n} \boxtimes m_S(1)) \circ \mathcal{A}_{S,S^{n-1},M} \circ (\mathrm{id}_S \boxtimes \mathcal{R}_{S^{n-1},M}) \circ \mathcal{A}^{-1}_{S,M,S^{n-1}} \\
&\qquad \circ \mathcal{A}_{S,M,S^{n-1}} \circ (\mathrm{id}_S \boxtimes \mathcal{R}_{S^{n-1},M}) \circ \mathcal{A}^{-1}_{S,S^{n-1},M} \\
&= (\mathrm{id}_{S^n} \boxtimes m_S(1)) \circ \mathcal{A}_{S,S^{n-1},M} \circ (\mathrm{id}_S \boxtimes \mathcal{M}_{S^{n-1},M}) \circ \mathcal{A}^{-1}_{S,S^{n-1},M} \\
&= (\mathrm{id}_{S^n} \boxtimes m_S(1)) \circ \mathcal{A}_{S,S^{n-1},M} \circ (\mathrm{id}_S \boxtimes (\mathrm{id}_{S^{n-1}} \boxtimes m_S(1)^{n-1})) \circ \mathcal{A}^{-1}_{S,S^{n-1},M} \\
&= \mathrm{id}_{S^n} \boxtimes m_S(1)^n.
\end{aligned}$$

Similarly, we can show $m_S(-n) = m_S(-1)^n$ for $n \geq 1$. Thus it remains to prove

$$m_S(1) m_S(-1) = \mathrm{id}_M .$$

For this, note that by using the same argument as above, we obtain that

$$\mathcal{M}_{S \boxtimes S^*,M} = \mathrm{id}_{S \boxtimes S^*} \boxtimes (m_S(1) m_S(-1)).$$

Then $S \boxtimes S^* \simeq 1_{\mathcal{C}}$ and $\mathcal{M}_{1_{\mathcal{C}},M} = \mathrm{id}_{1_{\mathcal{C}} \boxtimes M}$ (see [Kas, Proposition XIII. 1.2]) implies the assertion. This completes the proof. $\square$

Proposition B.5. *An object $M \in Ob(\mathcal{C})$ admits the generalized eigenspace decomposition of $m_S(1)$*

$$M = \bigoplus_{\alpha \in \mathbb{C}^*} M_\alpha,$$

where
$$M_\alpha = \bigcup_{n\in\mathbb{Z}_{\geq 0}} M_\alpha[n], \quad M_\alpha[n] := \mathrm{Ker}(m_S(1)-\alpha)^n.$$

Proof. It is immediate that we have
$$M \supset \sum_{\alpha\in\mathbb{C}^*} M_\alpha = \bigoplus_{\alpha\in\mathbb{C}^*} M_\alpha \tag{B.8}$$
and that $\{M_\alpha[n]\}_{n\in\mathbb{Z}_{\geq 0}}$ defines a filtration $M_\alpha[p] \subset M_\alpha[q]$, $(p<q)$. Thus it remains to prove the equality of (B.8). For this, we use the multiplication of $m_S(1)$ on $\mathrm{End}_{\mathcal{C}}(M)$. Since the $\mathbb{C}$-vector superspace $\mathrm{End}_{\mathcal{C}}(M)$ is finite dimensional by Assumption 2, the multiplication of $m_S(1)$ gives a generalized eigenspace decomposition
$$\begin{aligned}&\mathrm{End}_{\mathcal{C}}(M) = \bigoplus_{\alpha\in\mathbb{C}^*} \mathrm{End}_{\mathcal{C}}(M)_\alpha,\\ &\mathrm{End}_{\mathcal{C}}(M)_\alpha := \{f \in \mathrm{End}_{\mathcal{C}}(M)_\alpha \mid (m_S(1)-\alpha)^n f = 0, \quad (\forall n \gg 0)\}.\end{aligned}$$
This gives the decomposition $\mathrm{id}_M = \sum_\alpha \pi_\alpha$. Thus we obtain
$$M = \mathrm{id}_M M = \sum_{\alpha\in\mathbb{C}^*} \pi_\alpha M \subset \sum_{\alpha\in\mathbb{C}^*} M_\alpha.$$
This completes the proof. □

We introduce full subcategories $\mathcal{C}_\alpha[n] \subset \mathcal{C}$, $(\alpha\in\mathbb{C}^*, n\in\mathbb{Z}_{\geq 0})$ defined by
$$\mathcal{C}_\alpha[n] := \{M \in Ob(\mathcal{C}) \mid (m_S(1)-\alpha)^{n+1}\,\mathrm{id}_M = 0\}.$$
For a fixed $\alpha\in\mathbb{C}^*$, they give a filtration
$$\mathcal{C}_\alpha[0] \subset \mathcal{C}_\alpha[1] \subset \cdots \subset \mathcal{C}_\alpha := \bigcup_{n\geq 0} \mathcal{C}_\alpha[n]. \tag{B.9}$$
Then by Proposition B.5, we have the following block decomposition of $\mathcal{C}$:
$$\mathcal{C} = \bigoplus_{\alpha\in\mathbb{C}^*} \mathcal{C}_\alpha. \tag{B.10}$$
We call it the *monodromy decomposition* of $\mathcal{C}$ by S and (B.9) the *monodromy filtration* of $\mathcal{C}$ by S.

Proposition B.6. *(i) Any object M in $\mathcal{C}_\alpha[m]$ is an extension*
$$0 \to M_1 \to M \to M_2 \to 0$$
for some $M_1 \in Ob(\mathcal{C}_\alpha[0])$ and $M_2 \in Ob(\mathcal{C}_\alpha[m-1])$.
(ii) We have
$$\boxtimes\colon \mathcal{C}_\alpha[m] \times \mathcal{C}_\beta[n] \to \mathcal{C}_{\alpha\beta}[m+n].$$

Proof. (i) is immediate from the definition of $\mathcal{C}_\alpha[m]$. We prove (ii). Take $M \in Ob(\mathcal{C}_\alpha[m])$ and $N \in Ob(\mathcal{C}_\beta[n])$. Let us write the monodromy operator $m_S(1)$ for M as $m_{S,M}(1)$ for clarity. By using the same diagram in the proof of Proposition B.4, we obtain
$$\mathcal{M}_{S,M\boxtimes N} = \mathrm{id}_S \boxtimes (m_{S,M}(1) \boxtimes m_{S,N}(1)),$$

and thus $m_{S,M\boxtimes N}(1) = m_{S,M}(1) \boxtimes m_{S,N}(1)$. Now the assertion holds since

$$\begin{aligned}
&(m_{S,M\boxtimes N}(1) - \alpha\beta)^{m+n+1}\\
&= (m_{S,M}(1) \boxtimes m_{S,N}(1) - \alpha\beta)^{m+n+1}\\
&= ((m_{S,M}(1) - \alpha) \boxtimes m_{S,N}(1) + \alpha \operatorname{id}_M \boxtimes (m_{S,N}(1) - \beta))^{m+n+1}\\
&= \sum_{k=0}^{m+n+1} \binom{m+n+1}{k} \alpha^k (m_{S,M}(1) - \alpha)^{m+n+1-k} \boxtimes m_{S,N}(1)^{m+n+1-k}(m_{S,N}(1) - \beta)^k\\
&= 0.
\end{aligned}$$

Corollary B.7. *The full subcategory*

$$\mathcal{C}[0] := \bigoplus_{\alpha\in\mathbb{C}^*} \mathcal{C}_\alpha[0]$$

is a braided monoidal supercategory.

Remark B.8. *If the simple current S is of finite order $S^n \simeq 1_{\mathcal{C}}$, then the decomposition* (B.10) *holds without Assumption 2. Indeed, by the complete reducibility of representations of finite abelian groups, $m_S(1)^n = 1$ implies the block decomposition*

$$\mathcal{C} = \mathcal{C}[0] = \bigoplus_{\alpha\in\mathbb{Z}/n\mathbb{Z}} \mathcal{C}_\alpha[0],$$

where $\mathbb{Z}_n \hookrightarrow \mathbb{C}^*$, $[1] \mapsto e^{2\pi i/n}$ (cf. [CKL, Lemma 3.17]*).*

Next, we generalize (B.10) to a simultaneous decomposition by simple currents $\{S_g\}_{g\in G}$ parametrized by a group G, i.e., $S_g \boxtimes S_h \simeq S_{gh}$, $(g, h \in G)$. Since $\mathcal{C}$ is braided, we have

$$S_{gh} \simeq S_g \boxtimes S_h \xrightarrow[\mathcal{R}_{S_g,S_h}]{\simeq} S_h \boxtimes S_g \simeq S_{hg},$$

and G is necessarily abelian. As the functoriality of $\mathcal{M}_{\bullet,\bullet}$ implies that each monodromy operator $m_{S_g}(1)$ on $M \in Ob(\mathcal{C})$ lies in the center of $\mathrm{End}_{\mathcal{C}}(M)$, the abelian group G acts on M by $g \mapsto m_{S_g}(1)$, $(g \in G)$. We denote by $G^\vee := \mathrm{Hom}_{\mathrm{Grp}}(G, \mathbb{C}^*)$ the dual of G and introduce full subcategories $\mathcal{C}_\phi \subset \mathcal{C}$, $(\phi \in G^\vee)$, by

$$\mathcal{C}_\phi := \{M \in Ob(M) \mid \forall g \in G, (m_{S_g}(1) - \phi(g))^N = 0, (\forall N \gg 0)\}.$$

Then Proposition B.5 and the proof of Proposition B.6 implies the following immediately.

Theorem B.9. *The supercategory $\mathcal{C}$ admits a decomposition*

$$\mathcal{C} = \bigoplus_{\phi\in G^\vee} \mathcal{C}_\phi$$

as additive supercategories and the monoidal product respects the decomposition, i.e., $\boxtimes\colon \mathcal{C}_\phi \times \mathcal{C}_\psi \to \mathcal{C}_{\phi\psi}$.

Finally, we consider the case that G is finitely generated. In this case, by the fundamental theorem of finitely generated abelian group, we have $G \simeq G_{\mathrm{fin}} \times \mathbb{Z}^n$ for some finite abelian group G_{fin} and non-negative integer $n \geq 0$. Then the dual group $G^\vee$ is $G^\vee \simeq G_{\mathrm{fin}}^\vee \times (\mathbb{C}^*)^n$. Let $\mathcal{C}_{\phi,\alpha}[p] \subset \mathcal{C}$, $(\phi \in G_{\mathrm{fin}}^\vee$, $\alpha \in (\mathbb{C}^*)^n$, $p \in \mathbb{Z}_{\geq 0}^n)$, denote the full subcategory whose objects consist of $M \in Ob(\mathcal{C})$ such that

$$m_{S_g}(1) = \phi(g),\ (\forall g \in G_{\mathrm{fin}}),\quad (m_{S,1_i} - \alpha_i)^{p_i}|_M = 0,\ (1 \leq \forall i \leq n),$$

where 1_i, $(1 \leq i \leq n)$, denotes the generator of the i-th component of $\mathbb{Z}^n \subset G$. For each $(\phi, \alpha) \in G^\vee$, we have $\mathcal{C}_{\phi,\alpha}[p] \subset \mathcal{C}_{\phi,\alpha}[q]$ if $q - p \in \mathbb{Z}_{\geq 0}^n$ and define

$$\mathcal{C}_{\phi,\alpha} := \bigcup_{p\in\mathbb{Z}_{\geq 0}^n} \mathcal{C}_{\phi,\alpha}[p].$$

Then we have the following:

Theorem B.10. *Assume that the group G is finitely generated. Then,*
(i) the supercategory $\mathcal{C}$ admits a decomposition

$$\mathcal{C} = \bigoplus_{(\phi,\alpha)\in G^\vee} \mathcal{C}_{\phi,\alpha}. \tag{B.11}$$

and the monoidal product respects the filtration, i.e.,

$$\boxtimes\colon \mathcal{C}_{\phi,\alpha}[p] \times \mathcal{C}_{\psi,\beta}[q] \to \mathcal{C}_{\phi\psi,\alpha\beta}[p+q],$$

(ii) the full subcategory

$$\mathcal{C}[0] := \bigoplus_{(\phi,\alpha)\in G^\vee} \mathcal{C}_{\phi,\alpha}[0]$$

is naturally a braided monoidal subsupercategory,
(iii) every object in $\mathcal{C}_{\phi,\alpha}[p]$ is expressed as an extension of certain objects in $\mathcal{C}_{\phi,\alpha}[0]$ and objects in $\mathcal{C}_{\phi,\alpha}[(p_1,\cdots,p_j-1,\cdots,p_n)]$ for some j.

We call the decomposition (B.11) the *monodromy decompositions* by G and the filtration $\{\mathcal{C}_{\phi,\alpha}[p]\}$ the *monodromy filtration* by G.

Remark B.11. *By Theorem B.10 (iii), every simple object lies in $\mathcal{C}[0]$. Thus $\mathcal{C} = \mathcal{C}[0]$ holds if $\mathcal{C}$ is semisimple. Equivalently, $\mathcal{C} \neq \mathcal{C}[0]$ implies that the supercategory $\mathcal{C}$ is not semisimple.*

B.3. **Fusion rings.** Let $\mathcal{C}$ be a braided monoidal supercategory as in §2.2 with Assumption 1 and 2. Let $\mathcal{K}(\mathcal{C})$ denote its Grothendieck group, which is generated over $\mathbb{Z}$ by isomorphism classes $[M]$ of objects M in $\mathcal{C}$. Note that, if every object in $\mathcal{C}$ of finite length, then the set $\operatorname{Irr}\mathcal{C}$ of simple objects of $\mathcal{C}$ gives a $\mathbb{Z}$-basis of $\mathcal{K}(\mathcal{C})$. From now on, we assume the following condition:

Assumption 3. *The bifunctor $\boxtimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is biexact.*

Then $\mathcal{K}(\mathcal{C})$ is a commutative ring by

$$\mathcal{K}(\mathcal{C})\times\mathcal{K}(\mathcal{C})\to\mathcal{K}(\mathcal{C}),\quad ([M],[N])\to[M\boxtimes N]$$

and is called the fusion ring of $\mathcal{C}$ We remark that $\mathcal{K}(\mathcal{C})$ does not have a natural $\mathbb{Z}_2$-graded structure. The set of simple currents in $\mathcal{C}$, denoted by $\operatorname{Pic}\mathcal{C}$, is naturally an abelian group by $\boxtimes$ and we regard the group ring $\mathbb{Z}[\operatorname{Pic}\mathcal{C}]$ as a subring of $\mathcal{K}(\mathcal{C})$.

Suppose that we have a set of simple currents $\{S_g\}_{g\in G}$ parametrized by a finitely generated abelian group G as in §B.2. By Theorem B.10 (i), (ii), the fusion ring is $G^\vee$-graded

$$\mathcal{K}(\mathcal{C}) = \bigoplus_{\xi\in G^\vee}\mathcal{K}(\mathcal{C}_\xi).$$

By Theorem B.10 (iii), the embedding $\mathcal{C}[0]\subset\mathcal{C}$ induces an isomorphism $\mathcal{K}(\mathcal{C}[0])\simeq\mathcal{K}(\mathcal{C})$. Finally, we note that $\mathcal{K}(\mathcal{C})$ is a $\mathbb{Z}[G]$-algebra, where $\mathbb{Z}[G]$ denotes the group ring of G, by

$$\mathbb{Z}[G]\times\mathcal{K}(\mathcal{C})\to\mathcal{K}(\mathcal{C}),\quad (g,[M])\to[S_g\boxtimes M].$$

Before we remark a criterion for the $\mathbb{Z}[G]$-freeness of $\mathcal{K}(\mathcal{C})$, we recall that Lemma B.1 implies that G acts on $\operatorname{Irr}\mathcal{C}$ by

$$G\times\operatorname{Irr}\mathcal{C}\to\operatorname{Irr}\mathcal{C},\quad (g,M)\to S_g\boxtimes M.$$

Then it is clear that if every object in $\mathcal{C}$ is of finite length, then $\mathcal{K}(\mathcal{C})$ is free over $\mathbb{Z}[G]$ if and only if the G-action on $\operatorname{Irr}\mathcal{C}$ is free. Thus we have proved the following proposition:

Proposition B.12. *For a set of simple currents $\{S_g\}_{g\in G}$ in $\mathcal{C}$ parametrized a finitely generated abelian group G, the fusion rings $\mathcal{K}(\mathcal{C}[0])$ and $\mathcal{K}(\mathcal{C})$ are $G^\vee$-graded*

$\mathbb{Z}[G]$-algebras. Moreover, the embedding $\mathcal{C}[0] \subset \mathcal{C}$ gives an isomorphism $\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}(\mathcal{C}[0])$ as $G^\vee$-graded $\mathbb{Z}[G]$-algebras. If every object in $\mathcal{C}$ is of finite length, then $\mathcal{K}(\mathcal{C})$ is free over $\mathbb{Z}[G]$ if and only if the G-action on $\operatorname{Irr}\mathcal{C}$ is free.

B.4. **Algebra objects and Induction functor.** Following [CKM1], we review the notion of (unital, associative, commutative) algebra objects and their module objects in a braided monoidal supercategory, originally introduced by [KO] in the purely even case. We note that the authors of [CKM1] deal with superizations of non-super categories (see Remark B.14), but the proofs apply to our setting.

Let $\mathcal{C}$ be a braided monoidal supercategory as in §B.3.

Remark B.13. *In this subsection, we may replace Assumption 3 by a weaker one, that is, the bifunctor $\boxtimes$ is right exact.*

An algebra object in $\mathcal{C}$ is a triple $(\mathcal{E}, \mu_{\mathcal{E}}, \iota)$, (or $\mathcal{E}$ for simplicity), consisting of $\mathcal{E} \in Ob(\mathcal{C})$ and even morphisms $\mu_{\mathcal{E}} \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{E})_{\bar{0}}$ and $\iota \in \operatorname{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, \mathcal{E})_{\bar{0}}$ satisfying the following commutative diagrams:

(A0) The morphism $\iota\colon 1_{\mathcal{C}} \to \mathcal{E}$ is injective.

(A1) Unity

$$\begin{tikzcd} 1_{\mathcal{C}} \boxtimes \mathcal{E} \arrow[r, "\iota\boxtimes\mathrm{id}_{\mathcal{E}}"] \arrow[dr, "l_{\mathcal{E}}"'] & \mathcal{E} \boxtimes \mathcal{E} \arrow[d, "\mu_{\mathcal{E}}"] \\ & \mathcal{E}. \end{tikzcd}$$

(A2) Associativity

$$\begin{tikzcd} \mathcal{E} \boxtimes (\mathcal{E} \boxtimes \mathcal{E}) \arrow[r, "\mathrm{id}_{\mathcal{E}}\boxtimes\mu_{\mathcal{E}}"] \arrow[d, "\mathcal{A}_{\mathcal{E},\mathcal{E},\mathcal{E}}"'] & \mathcal{E} \boxtimes \mathcal{E} \arrow[r, "\mu_{\mathcal{E}}"] & \mathcal{E} \\ (\mathcal{E} \boxtimes \mathcal{E}) \boxtimes \mathcal{E} \arrow[r, "\mu_{\mathcal{E}}\boxtimes\mathrm{id}_{\mathcal{E}}"] & \mathcal{E} \boxtimes \mathcal{E}. \arrow[ur, "\mu_{\mathcal{E}}"'] & \end{tikzcd}$$

(A3) Commutativity

$$\begin{tikzcd} \mathcal{E} \boxtimes \mathcal{E} \arrow[r, "\mathcal{R}_{\mathcal{E},\mathcal{E}}"] \arrow[dr, "\mu_{\mathcal{E}}"'] & \mathcal{E} \boxtimes \mathcal{E} \arrow[d, "\mu_{\mathcal{E}}"] \\ & \mathcal{E}. \end{tikzcd}$$

Remark B.14.
(1) A typical example of our braided monoidal supercategory is the superization $\mathcal{SC}$ of a braided abelian monoidal category $\mathcal{C}$, whose objects are pairs $M = (M_{\bar{0}}, M_{\bar{1}})$ of $M_{\bar{i}} \in Ob(\mathcal{C})$. This induces a $\mathbb{Z}_2$-graded structure on the set of morphisms. In this case, an algebra object in $\mathcal{SC}$ is called a superalgebra object in $\mathcal{C}$, see [CKM1].
(2) For an application to extensions of vertex superalgebras, the condition

$$\operatorname{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, \mathcal{E}) \simeq \operatorname{End}_{\mathcal{C}}(1_{\mathcal{C}})$$

is often assumed so that the extended vertex superalgebra is of CFT type.

An $\mathcal{E}$-module is a pair (M, μ_M), (or M for simplicity), consisting of $M \in Ob(\mathcal{C})$ and an even morphism $\mu_M \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{E} \boxtimes M, M)_{\bar{0}}$ satisfying the following commutative diagrams:

(M1) Unity

$$\begin{tikzcd} 1_{\mathcal{C}} \boxtimes M \arrow[r, "\iota\boxtimes\mathrm{id}_M"] \arrow[dr, "l_M"'] & \mathcal{E} \boxtimes M \arrow[d, "\mu_M"] \\ & M. \end{tikzcd}$$

(M2) Associativity

$$\begin{tikzcd}
\mathcal{E} \boxtimes (\mathcal{E} \boxtimes M) \arrow[r, "\mathrm{id}_{\mathcal{E}} \boxtimes \mu_M"] \arrow[d, "\mathcal{A}_{\mathcal{E},\mathcal{E},M}"'] & \mathcal{E} \boxtimes M \arrow[r, "\mu_M"] & M \\
(\mathcal{E} \boxtimes \mathcal{E}) \boxtimes M \arrow[r, "\mu_{\mathcal{E}} \boxtimes \mathrm{id}_M"] & \mathcal{E} \boxtimes M. \arrow[ur, "\mu_M"'] &
\end{tikzcd}$$

An $\mathcal{E}$-module (M, μ_M) is called *local* if it further satisfies the following commutative diagram:

(M3) Locality

$$\begin{tikzcd}
\mathcal{E} \boxtimes M \arrow[r, "\mathcal{M}_{\mathcal{E},M}"] \arrow[dr, "\mu_M"'] & \mathcal{E} \boxtimes M \arrow[d, "\mu_M"] \\
& M.
\end{tikzcd}$$

A morphism of $\mathcal{E}$-modules from (M, μ_M) to (N, μ_N) is a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ satisfying the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E} \boxtimes M \arrow[r, "\mathrm{id}_{\mathcal{E}} \boxtimes f"] \arrow[d, "\mu_M"'] & \mathcal{E} \boxtimes N \arrow[d, "\mu_N"] \\
M \arrow[r, "f"] & N.
\end{tikzcd}$$

Let $\mathrm{Rep}(\mathcal{E})$ denote the supercategory of $\mathcal{E}$-modules with morphisms of $\mathcal{E}$-modules, and $\mathrm{Rep}^0(\mathcal{E})$ the full subcategory of $\mathrm{Rep}(\mathcal{E})$ consisting of local $\mathcal{E}$-modules.

Although $\mathrm{Rep}(\mathcal{E})$ is just a $\mathbb{C}$-linear additive supercategory, the underlying category $\underline{\mathrm{Rep}(\mathcal{E})}$ is an abelian category. Furthermore, by the existence of the involutive autofunctor $\Pi_{\mathcal{C}}$, every parity-homogeneous morphism in $\mathrm{Rep}(\mathcal{E})$ admits kernel and cokernel objects. The same is true for $\mathrm{Rep}^0(\mathcal{E})$. Each $\mathrm{Rep}(\mathcal{E})$ and $\mathrm{Rep}^0(\mathcal{E})$ admits a natural monoidal structure in the following way [CKM1, §2]. Consider the two compositions

$$\xi_1 \colon \mathcal{E} \boxtimes (M \boxtimes N) \xrightarrow{\mathcal{A}_{\mathcal{E},\mathcal{E},M}} (\mathcal{E} \boxtimes M) \boxtimes N \xrightarrow{\mu_M \boxtimes \mathrm{id}_N} M \boxtimes N,$$

$$\xi_2 \colon \mathcal{E} \boxtimes (M \boxtimes N) \xrightarrow{\mathcal{A}_{\mathcal{E},M,N}} (\mathcal{E} \boxtimes M) \boxtimes N \xrightarrow{\mathcal{R}_{\mathcal{E},M} \boxtimes \mathrm{id}_N} (M \boxtimes \mathcal{E}) \boxtimes N$$
$$\xrightarrow{\mathcal{A}^{-1}_{M,\mathcal{E},N}} M \boxtimes (\mathcal{E} \boxtimes N) \xrightarrow{\mathrm{id}_M \boxtimes \mu_N} M \boxtimes N.$$

Then $M \boxtimes_{\mathcal{E}} N$ is defined by $M \boxtimes_{\mathcal{E}} N := \mathrm{Coker}(\xi_1 - \xi_2)$, which is an object of $\mathcal{C}$. Let $\eta_{M,N}$ denote the canonical surjection $M \boxtimes N \to M \boxtimes_{\mathcal{E}} N$. The $\mathcal{E}$-module structure $\mu_{M \boxtimes_{\mathcal{E}} N} \colon \mathcal{E} \boxtimes (M \boxtimes_{\mathcal{E}} N) \to M \boxtimes_{\mathcal{E}} N$ is the unique even morphism, which makes the diagram

$$\begin{tikzcd}
\mathcal{E} \boxtimes (M \otimes N) \arrow[r, "\xi_i"] \arrow[d, "\mathrm{id}_{\mathcal{E}} \boxtimes \eta_{M,N}"'] & M \boxtimes N \arrow[d, "\eta_{M,N}"] \\
\mathcal{E} \boxtimes (M \boxtimes_{\mathcal{E}} N) \arrow[r, "\mu_{M \boxtimes N}"] & M \boxtimes_{\mathcal{E}} N.
\end{tikzcd}$$

commutes for $i = 1, 2$. The associativity $\mathcal{A}^{\mathcal{E}}_{\bullet,\bullet,\bullet} \colon \bullet \boxtimes_{\mathcal{E}} (\bullet \boxtimes_{\mathcal{E}} \bullet) \simeq (\bullet \boxtimes_{\mathcal{E}} \bullet) \boxtimes_{\mathcal{E}} \bullet$ is given by the family of unique even morphisms $\mathcal{A}^{\mathcal{E}}_{M,N,L} \colon M \boxtimes_{\mathcal{E}} (N \boxtimes_{\mathcal{E}} L) \simeq (M \boxtimes_{\mathcal{E}} N) \boxtimes_{\mathcal{E}} L$

for $M, N, L \in Ob(\mathrm{Rep}(\mathcal{E}))$, which make the following diagram commute:

$$\begin{tikzcd}
M \boxtimes (N \otimes L) \arrow[r, "\mathcal{A}_{M,N,L}"] \arrow[d, "\mathrm{id}_M \boxtimes \eta_{N,L}"'] & (M \boxtimes N) \boxtimes L \arrow[d, "\eta_{M,N}"] \\
M \boxtimes (N \boxtimes_{\mathcal{E}} L) \arrow[d, "\eta_{M,N\boxtimes_{\mathcal{E}} L}"'] & (M \boxtimes_{\mathcal{E}} N) \boxtimes L \arrow[d, "\eta_{M\boxtimes_{\mathcal{E}} N,L}"] \\
M \boxtimes_{\mathcal{E}} (N \boxtimes_{\mathcal{E}} L) \arrow[r, "\mathcal{A}^{\mathcal{E}}_{M,N,L}"] & (M \boxtimes_{\mathcal{E}} N) \boxtimes_{\mathcal{E}} L.
\end{tikzcd}$$

The unit object of $\mathrm{Rep}(\mathcal{E})$ and $\mathrm{Rep}^0(\mathcal{E})$ is $\mathcal{E}$ equipped with even natural morphisms

$$l^{\mathcal{E}}_{\bullet} \colon \mathcal{E} \boxtimes_{\mathcal{E}} \bullet \simeq \bullet, \quad r^{\mathcal{E}}_{\bullet} \colon \bullet \boxtimes_{\mathcal{E}} \mathcal{E} \simeq \bullet$$

given by the family of unique even morphisms $l^{\mathcal{E}}_M \colon \mathcal{E} \boxtimes_{\mathcal{E}} M \simeq M$ and $r^{\mathcal{E}}_M \colon M \boxtimes_{\mathcal{E}} \mathcal{E} \simeq M$, which make the following diagrams commute:

$$\begin{tikzcd}
\mathcal{E} \boxtimes M \arrow[r, "\mu_M"] \arrow[d, "\eta_{\mathcal{E},N}"'] & M \arrow[d, equal] \\
\mathcal{E} \boxtimes_{\mathcal{E}} M \arrow[r, "l^{\mathcal{E}}_M"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \boxtimes \mathcal{E} \arrow[r, "\mathcal{R}^{-1}_{M,\mathcal{E}}"] \arrow[d, "\eta_{M,\mathcal{E}}"'] & \mathcal{E} \boxtimes M \arrow[r, "\mu_M"] & M \arrow[d, equal] \\
M \boxtimes_{\mathcal{E}} \mathcal{E} \arrow[rr, "r^{\mathcal{E}}_M"] & & M.
\end{tikzcd}$$

The braiding $\mathcal{R}_{\bullet,\bullet}$ on $\mathcal{C}$ induces a braiding $\mathcal{R}^{\mathcal{E}}_{\bullet,\bullet}$ on $\mathrm{Rep}^0(\mathcal{E})$, which is a family of unique even morphisms $\mathcal{R}^{\mathcal{E}}_{M,N} \colon M \boxtimes_{\mathcal{E}} N \simeq N \boxtimes_{\mathcal{E}} M$, which make the diagram

$$\begin{tikzcd}
M \boxtimes N \arrow[r, "\mathcal{R}_{M,N}"] \arrow[d, "\eta_{M,N}"'] & N \boxtimes M \arrow[d, "\eta_{N,M}"] \\
M \boxtimes_{\mathcal{E}} N \arrow[r, "\mathcal{R}^{\mathcal{E}}_{M,N}"] & N \boxtimes_{\mathcal{E}} M
\end{tikzcd}$$

commute. To summarize, we obtain the following.

Theorem B.15 ([CKM1, Theorem 2.53])**.** *The supercatregory* $\mathrm{Rep}(\mathcal{E})$ *(resp.* $\mathrm{Rep}^0(\mathcal{E})$*) is naturally a* $\mathbb{C}$*-linear additive (resp. braided) monoidal supercategory such that the underlying categoty* $\underline{\mathrm{Rep}(\mathcal{E})}$*, (resp.* $\underline{\mathrm{Rep}^0(\mathcal{E})}$*) is an abelian category.*

For $M \in Ob(\mathcal{C})$, we define an $\mathcal{E}$-module $(\mathcal{F}(M), \mu_{\mathcal{F}(M)})$ by

$$\mathcal{F}(M) := \mathcal{E} \boxtimes M,$$

$$\mu_{\mathcal{F}(M)} \colon \mathcal{E} \boxtimes (\mathcal{E} \boxtimes M) \simeq (\mathcal{E} \boxtimes \mathcal{E}) \boxtimes M \xrightarrow{\mu_{\mathcal{E}} \boxtimes \mathrm{id}_M} \mathcal{E} \boxtimes M.$$

We also define a superfunctor $\mathcal{F} \colon \mathcal{C} \to \mathrm{Rep}(\mathcal{E})$ by $M \mapsto (\mathcal{F}(M), \mu_{\mathcal{F}(M)})$ and $\mathcal{F}(f) := \mathrm{id}_{\mathcal{E}} \boxtimes f \in \mathrm{Hom}_{\mathrm{Rep}(\mathcal{E})}(\mathcal{F}(M), \mathcal{F}(N))$ for $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$. The superfunctor $\mathcal{F}$ is called the induction functor and enjoys the following property.

Theorem B.16 ([CKM1, Theorem 2.59])**.** *The induction functor* $\mathcal{F} \colon \mathcal{C} \to \mathrm{Rep}(\mathcal{E})$ *is a* $\mathbb{C}$*-linear, additive, strong monoidal superfunctor.*

Let $\mathcal{C}^0$ denote the full subcategory of $\mathcal{C}$ consisting of objects $M \in Ob(\mathcal{C})$ such that $\mathcal{M}_{\mathcal{E},M} = \mathrm{id}_{\mathcal{E},M}$. We call $\mathcal{C}^0$ the category of $\mathcal{E}$-local objects.

Theorem B.17 ([CKM1])**.** *We have the following.*

(1) *The supercategory* $\mathcal{C}^0$ *is a braided monoidal subsupercategory of* $\mathcal{C}$*.*
(2) *For* $M \in Ob(\mathcal{C})$*, the object* $\mathcal{F}(M)$ *lies in* $\mathrm{Rep}^0(\mathcal{E})$ *if and only if* $M \in Ob(\mathcal{C}^0)$*.*
(3) *The restriction of* $\mathcal{F}$ *to* $\mathcal{C}^0$ *gives a braided monoidal superfunctor*

$$\mathcal{F} \colon \mathcal{C}^0 \to \mathrm{Rep}^0(\mathcal{E}).$$

Proof. By [CKM1, Theorem 2.67], $\mathcal{C}^0$ is a $\mathbb{C}$-linear additive braided monoidal supercategory, Thus to show (1), it remains to show that $\mathcal{C}^0$ is closed under kernel and cokernel, which immediately follows from the exactness of $\mathcal{E} \boxtimes \bullet$ in Assumption 3. (2) is [CKM1, Proposition 2.65] and (3) is [CKM1, Theorem 2.67]. □

At last, the induction functor $\mathcal{F}$ is related to the forgetful functor

$$\mathcal{G}\colon \mathrm{Rep}(\mathcal{E}) \to \mathcal{C}, \quad (M, \mu_M) \mapsto M.$$

by the Frobenius reciprocity:

Proposition B.18 ([CKM1, Lemma 2.61])**.** *The superfunctor $\mathcal{G}\colon \mathrm{Rep}(\mathcal{E}) \to \mathcal{C}$ is right adjoint to the superfunctor $\mathcal{F}\colon \mathcal{C} \to \mathrm{Rep}(\mathcal{E})$, that is, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Rep}(\mathcal{E})}(\mathcal{F}(N), M) \simeq \mathrm{Hom}_{\mathcal{C}}(N, \mathcal{G}(M)),$$

for $M \in Ob(\mathrm{Rep}(\mathcal{E}))$ and $N \in Ob(\mathcal{C})$. More explicitly, for $f \in \mathrm{Hom}_{\mathcal{C}}(N, \mathcal{G}(M))$, the corresponding morphism of $\mathcal{E}$-modules is given by

$$\mathcal{F}(N) = \mathcal{E} \boxtimes N \to M, \quad a \boxtimes m \mapsto \mu_M(a \boxtimes f(m)).$$

B.5. **Categorical simple current extensions.** Let $\mathcal{C}$ be a braided monoidal supercategory as in §B.2, satisfying Assumption 1–3 and the following assumption.

Assumption 4. *Every object in $\mathcal{C}$ has finite length.*

Let $\mathcal{E}$ be an algebra object in $\mathcal{C}$ of the form

$$\mathcal{E} = \bigoplus_{g \in G} S_g,$$

where $\{S_g\}_{g \in G} \subset \mathrm{Pic}\,\mathcal{C}$ is a set of simple currents parametrized by a finite abelian gorup G with $S_e = 1_{\mathcal{C}}$. Here $e \in G$ denotes the unit of G. We call $\mathcal{E}$ a categorical simple current extension of $1_{\mathcal{C}}$. In the rest of this subsection, we assume

- (S1) the product $\mu_{\mathcal{E}}$ restricts to a non-zero morphism $S_g \boxtimes S_h \to S_{gh}$ for $g, h \in G$,
- (S2) the action of G on $\mathrm{Irr}\,\mathcal{C}$ is fixed-point free.

Note that these assumptions imply $S_g \simeq S_h$ if and only if $g = h$.

Proposition B.19. *Suppose (S1) and (S2).*

1. *If M is a simple object in $\mathcal{C}$, so is $\mathcal{F}(M)$ in $\mathrm{Rep}(\mathcal{E})$.*
2. *For a simple object M in $\mathrm{Rep}(\mathcal{E})$, there exists a simple object N in $\mathcal{C}$ such that $M \simeq \mathcal{F}(N)$ in $\mathrm{Rep}(\mathcal{E})$.*
3. *For simple objects M and N in $\mathcal{C}$, we have $\mathcal{F}(M) \simeq \mathcal{F}(N)$ in $\mathrm{Rep}(\mathcal{E})$ if and only if we have $M \simeq S_g \boxtimes N$ for some $g \in G$.*
4. *The supercategory $\mathcal{C}$ is semisimple if and only if $\mathrm{Rep}(\mathcal{E})$ is semisimple. In this case, $\mathcal{C}^0 \subset \mathcal{C}$ and $\mathrm{Rep}^0(\mathcal{E}) \subset \mathrm{Rep}(\mathcal{E})$ are also semisimple.*

Proof. Although these statements are well-known in the theory of vertex algebras, (see e.g., [CKM1, Proposition 4.5]), we include a proof for the completeness of the paper. First we prove (1). Let M be a simple object in $\mathcal{C}$. Then $\mathcal{F}(M) = \oplus_{g \in G} S_g \boxtimes M$ and all summands are pairwise non-isomorphic by (S2). Let $\mathcal{N}$ be a nonzero subobject of $\mathcal{F}(M)$ in $\mathrm{Rep}(\mathcal{E})$. Since $\mathcal{N}$ is a semisimple object in $\mathcal{C}$, it has a simple subobject which is isomorphic to $S_g \boxtimes M$ for some $g \in G$. By (S1), the structure morpihsm $\mu_{\mathcal{F}(M)}$ restricts to an isomorphism $S_h \boxtimes (S_g \boxtimes M) \xrightarrow{\simeq} S_{hg} \boxtimes M$ for any $h \in G$. Since $\mathcal{N}$ is closed under the $\mathcal{E}$-action, it contains $\sum_{h \in G} \mu_{\mathcal{F}(M)}(S_h \boxtimes (S_g \boxtimes M)) = \oplus_{h \in G} S_{hg} \boxtimes M = \mathcal{F}(M)$. This proves (1). Then (2) and (3) follow from (1) and Proposition B.18. Finally, we show (4). Assume that $\mathrm{Rep}(\mathcal{E})$ is semisimple and take $N \in Ob(\mathcal{C})$. Since $\mathcal{F}(N)$ is semisimple, we have

$$\mathcal{F}(N) \simeq \bigoplus_{i \in I} \mathcal{N}_i$$

for some simple objects $\mathcal{N}_i$ in $\mathrm{Rep}(\mathcal{E})$ indexed by a finite set I. Then by (2), we may replace $\mathcal{N}_i$ by $\mathcal{F}(N_i)$ for some simple objects N_i in $\mathcal{C}$. Thus,

$$M \subset \mathcal{F}(N) \simeq \bigoplus_{i\in I} \mathcal{F}(N_i) = \bigoplus_{\substack{i\in I\\ g\in G}} S_g \boxtimes N_i.$$

Since $S_g \boxtimes N_i$ are all simple objects in $\mathcal{C}$, N is semisimple. To prove the converse, assume that $\mathcal{C}$ is semisimple. Since every object in $\mathcal{C}$ has finite length, so does every object in $\mathrm{Rep}(\mathcal{E})$. Thus to show that $\mathrm{Rep}(\mathcal{E})$ is semisimple, it suffices to show the splitting of any short exact sequence

$$0 \to \mathcal{N}_1 \to \mathcal{N} \to \mathcal{N}_2 \to 0 \tag{B.12}$$

in $\mathrm{Rep}(\mathcal{E})$ where $\mathcal{N}_1$, $\mathcal{N}_2$ are simple. By (2), we may assume $\mathcal{N}_i = \mathcal{F}(N_i)$ for some simple object $N_i \in Ob(\mathcal{C})$. If $\mathcal{F}(N_1) \not\simeq \mathcal{F}(N_2)$, then $S_g \boxtimes N_1 \not\simeq S_h \boxtimes N_2$ for all $g, h \in G$ in $\mathcal{C}$. This implies the splitting of (B.12) in $\mathrm{Rep}(\mathcal{E})$. Thus we may assume $\mathcal{F}(N_1) \simeq \mathcal{F}(N_2)$ and moreover $N := N_1 = N_2$ from the beginning. Since G is a finite abelian group, it is isomorphic to some direct product of cyclic groups $\prod_i \mathbb{Z}_{n_i}$. Then it suffices to show the splitting in the case of $G = \mathbb{Z}_n$ for some $n \in \mathbb{Z}_{>0}$. Let S_p denote the simple current corresponding to $p \in \mathbb{Z}_n$. Since the space of intertwining operators $I\binom{S_{p+1}\boxtimes N}{S_1\ \ S_p\boxtimes N}$ is one dimensional, we may take its basis by the intertwining operator

$$S_1 \boxtimes (S_p \boxtimes N) \simeq (S_1 \boxtimes S_p) \boxtimes N \simeq S_{p+1} \boxtimes N$$

used in the definition of $\mathcal{F}(N)$. On the other hand, the restriction of the $\mathcal{E}$-module structure of $\mathcal{N}$ gives an intertwining operator $S_1 \boxtimes \mathcal{N} \to \mathcal{N}$. Along with the decomposition $\mathcal{N} \simeq \mathcal{F}(N) \oplus \mathcal{F}(N)$ in $\mathcal{C}$, the intertwining operator is expressed by the matrix:

$$K := \left(\begin{array}{c|c} E & A \\ \hline 0 & E \end{array}\right)$$

where $E = \sum_{i\in\mathbb{Z}_n} E_{i+1,i}$ and $A = \sum_{i\in\mathbb{Z}_n} a_i E_{i+1,i}$ for some $a_i \in \mathbb{C}$. Since $S_1^n \simeq 1_{\mathcal{C}}$, we have $\sum_i a_i = 0$. Then it is straightforward to check that K is conjugate to the matrix K with $a_i = 0$ for all i. This implies that we may take a decomposition $\mathcal{N} = \mathcal{F}(N) \oplus \mathcal{F}(N)$ in $\mathcal{C}$ which is preserved by the action of $S_1 \subset \mathcal{E}$. Since the other action of $S_p \subset \mathcal{E}$ is obtained from the action of S_1 via iteration, the above decomposition of $\mathcal{N}$ is actually the decomposition as an $\mathcal{E}$-module. The remaining statements in (4) are now obvious. This completes the proof. $\square$

In particular, we have the following.

Corollary B.20. *The induction functor* $\mathcal{F}\colon \mathcal{C} \to \mathrm{Rep}(\mathcal{E})$ *induces the following natural identifications:*

(1) $\mathrm{Irr}(\mathrm{Rep}(\mathcal{E})) \simeq \mathrm{Irr}(\mathcal{C})/G$ *and* $\mathrm{Pic}(\mathrm{Rep}(\mathcal{E})) \simeq \mathrm{Pic}(\mathcal{C})/G$*;*
(2) $\mathrm{Irr}(\mathrm{Rep}^0(\mathcal{E})) \simeq \mathrm{Irr}(\mathcal{C}^0)/G$ *and* $\mathrm{Pic}(\mathrm{Rep}^0(\mathcal{E})) \simeq \mathrm{Pic}(\mathcal{C}^0)/G$*.*

Since the induction functor is a monoidal superfunctor, we may write the fusion ring of $\mathrm{Rep}(\mathcal{E})$ in terms of $\mathcal{C}$.

Corollary B.21. *Suppose (S1) and (S2). If the superfunctor* $\boxtimes_{\mathcal{E}}$ *is exact, then the induction functor* $\mathcal{F}\colon \mathcal{C} \to \mathrm{Rep}(\mathcal{E})$ *induces the following isomorphisms of rings:*

(1) $\mathcal{K}(\mathrm{Rep}(\mathcal{E})) \simeq \mathcal{K}(\mathcal{C})/\mathcal{I}$ *where* $\mathcal{I} = \langle [M] - [S_g \boxtimes M] \mid g \in G, M \in Ob(\mathcal{C})\rangle$*;*
(2) $\mathcal{K}(\mathrm{Rep}^0(\mathcal{E})) \simeq \mathcal{K}(\mathcal{C}^0)/\mathcal{I}^0$ *where* $\mathcal{I}^0 = \langle [M] - [S_g \boxtimes M] \mid g \in G, M \in Ob(\mathcal{C}^0)\rangle$*.*

Proof. (1) follow from Proposition B.19. (2) follows from (1) and Theorem B.17. $\square$

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