博士論文

論文題目: Finite element analysis for radially symmetric solutions of nonlinear heat equations
(非線形熱方程式の球対称解に対する有限要素解析)

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Contents

Preface

This thesis studies the finite element method (FEM) applied to a semilinear parabolic equation with a singular convection term,

$$u_t = u_{xx} + \frac{N-1}{x}u_x + f(u), \qquad x \in I = (0,1),\ t > 0, \tag{1a}$$

$$u_x(0,t) = u(1,t) = 0, \qquad t > 0, \tag{1b}$$

$$u(x,0) = u^0(x), \qquad x \in I. \tag{1c}$$

Therein, $u = u(x,t)$, $x \in \overline{I} = [0,1], t \geq 0$, denotes the function to be find, f a given locally Lipschitz continuous function and u^0 a given continuous function. Throughout this paper, we assume that

$$N \text{ is an integer} \geq 2. \tag{2}$$

We first clarify the motivation of this study. In many engineering problems, the space dimension of a mathematical model is at most three. Solving partial differential equations (PDEs) in more than three spatial dimension is usually motivated by mathematical interests. Mathematicians understand that solving the problem in a general setting can reveal hidden natures of PDEs. One successful result is the discovery of Fujita's blow-up exponent for the semilinear heat equation of $U = U(\boldsymbol{x}, t)$ given as

$$U_t = \Delta U + f(U) \qquad (\boldsymbol{x} \in \mathbb{R}^N,\ t > 0), \tag{3}$$

where N and $f(U)$ are defined above. Assuming $f(U) = U|U|^\alpha$ with $\alpha > 0$, Fujita showed that any positive solution blows up in finite time if $1+\alpha < 1+2/N$, but a solution remains smooth at any time if the initial value is small and $1+\alpha > 1+2/N$. The quantity $p_c = 1+2/N$ is known as Fujita's critical exponent, and Eq.(3) is called Fujita's equation.

Therefore, we found it is interesting to study the numerical methods for computing the solution of nonlinear partial differential equations in an N-dimensional space. Non-stationary problems in four dimensional space are difficult to solve by numerical methods, even on modern computers. Therefore, the present paper investigates radially symmetric solutions of Eq.(3). Assuming radial symmetry of the solution and the given data, the N-dimensional equation reduces to a one-dimensional equation. More specifically, considering (3) in an N-dimensional unit ball $\mathrm{B} = \{\boldsymbol{x} \in \mathbb{R}^N \mid |\boldsymbol{x}|_{\mathbb{R}^N} < 1\}$ with the homogeneous Dirichlet boundary condition on the boundary and assuming U is expressed as $u(x) = U(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathrm{B}$ and $x = |\boldsymbol{x}|_{\mathbb{R}^N}$, we came to consider the problem (1).

The finite difference method was already studied in [8] and [15]. In particular, the error estimates were established. Because their finite difference schemes use special approximations around the origin to maintain some analytical properties of the solution, they should be performed on a uniform spatial mesh. Conversely, when seeking the blow up solution, non-uniform partitions of the space variable are useful for examining highly concentrated solutions at the origin. For this purpose, we developed the FEM scheme.

The solution of (1) maybe blow up in finite time. It is interesting to study the relationship between the blow-up and the space dimension N. To this end, we try to apply the Nakagawa's time-increment control strategy (see [32]) which is a powerful technique for approximating the blow-up time.

In Chapter 1, the standard finite element methods are considered. FEM analyses of the linear case, in which $f(u)$ in Eq.(1) is replaced by a given function $f(x,t)$, are not new. Eriksson and Thomée [18] and Thomée [42] studied the convergence property of the elliptic equation, and proposed two schemes: the symmetric scheme, in which the optimal-order error is estimated in the weighted L^2 norm, and the nonsymmetric scheme, in which the L^∞ error is estimated. The main purpose of this chapter is to derive various optimal order error estimates for the symmetric and nonsymmetric schemes of [18, 42] applied to (1). These schemes are described below as (Sym) and (Non-Sym). In Section 1.2, we introduce standard symmetric scheme (Sym) and standard nonsymmetric scheme (Non-Sym).

First, we derive two alternate weak formulations of (1). Letting $\chi \in \dot{H}^1 = \{v \in H^1(I) \mid v(1) = 0\}$ be arbitrary, then multiplying both sides of (1a) by $x^{N-1}\chi$ and using integration by parts over I, we obtain

$$\int_I x^{N-1} u_t \chi \, dx + \int_I x^{N-1} u_x \chi_x \, dx = \int_I x^{N-1} f(u) \chi \, dx. \tag{4}$$

Otherwise, if we multiply both sides of (1a) by $x\chi$ instead of $x^{N-1}\chi$ and integrate it over I, then we have

$$\int_I x u_t \chi \, dx + \int_I [x u_x \chi_x + (2-N) u_x \chi] \, dx = \int_I x f(u) \chi \, dx. \tag{5}$$

We designate (4) the *symmetric* weak form because of the symmetric bilinear form associated with the differential operator $u_{xx} + \frac{N-1}{x} u_x$. In contrast, (5) is the *nonsymmetric* weak form. Both forms are identical at $N = 2$.

We now establish the finite element schemes based on these identities. For a positive integer m, we introduce node points

$$0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < \cdots < x_{m-1} < x_m = 1,$$

and set $I_j = (x_{j-1}, x_j)$ and $h_j = x_j - x_{j-1}$, where $j = 1, \ldots, m$. The granularity parameter is defined as $h = \max_{1 \le j \le m} h_j$. Let $\mathcal{P}_k(J)$ be the set of all polynomials in an interval J of degree $\le k$. We define the P1 finite element space as

$$S_h = \{v \in H^1(I) \mid v \in \mathcal{P}_1(I_j) \ (j = 1, \cdots, m), \ v(1) = 0\}. \tag{6}$$

Its standard basis function ϕ_j $(j = 0, 1, \cdots, m-1)$ is defined as

$$\phi_j(x_i) = \delta_{ij},$$

where δ_{ij} denotes Kronecker's delta.

For time discretization, we introduce non-uniform partitions

$$t_0 = 0, \quad t_n = \sum_{j=0}^{n-1} \tau_j \quad (n \ge 1),$$

where $\tau_j > 0$ denotes the time increment.

Generally, we write $\partial_{\tau_n} u_h^{n+1} = (u_h^{n+1} - u_h^n)/\tau_n$.
We are now in a position to state the finite element schemes to be considered in Chapter 1.

(Sym) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\int_I x^{N-1} \partial_{\tau_n} u_h^{n+1} \chi \, dx + \int_I x^{N-1} (u_h^{n+1})_x \chi_x \, dx = \int_I x^{N-1} f(u_h^n) \chi \, dx \quad (\chi \in S_h), \tag{7}$$

where $u_h^0 \in S_h$ is assumed to be given.

(Non-Sym) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\int_I x \partial_{\tau_n} u_h^{n+1} \chi \, dx + \int_I x (u_h^{n+1})_x \chi_x \, dx + (2-N) \int_I (u_h^{n+1})_x \chi \, dx = \int_I x f(u_h^n) \chi \, dx \quad (\chi \in S_h). \tag{8}$$

In Section 1.3, we show the well-posedness for (Sym) and (Non-Sym), and positivity preserving for (Sym) with a restriction of time increment.
In Section 1.4, we discuss the convergence property for (Sym) and (Non-Sym).

Given $T > 0$ and setting $Q_T = [0, 1] \times [0, T]$, we assume that u is sufficiently smooth such that

$$\kappa_\nu(u) = \sum_{j=0}^{2} \|\partial_x^j u\|_{L^\infty(Q_T)} + \sum_{l=1}^{2+\nu} \|\partial_t^l u\|_{L^\infty(Q_T)} + \sum_{k=1}^{1+\nu} \|\partial_t^k \partial_x^2 u\|_{L^\infty(Q_T)} < \infty, \tag{9}$$

where ν is either 0 or 1.

The partition $\{x_i\}_{j=0}^{m}$ of $\bar{I} = [0,1]$ is assumed to be quasi-uniform, with a positive constant β independent of h such that

$$h \le \beta \min_{1\le j\le m} h_j. \tag{10}$$

Finally, the approximate initial value u_h^0 is chosen as

$$\|u_h^0 - u^0\|_{L^\infty(I)} \le C_0 h^2 \tag{11}$$

for a positive constant C_0.

If f is *locally* Lipschitz continuous and $N \le 3$, then there exists an $h_1 = h_1(T, \kappa_0(u), C_0, N, \beta)$ such that, for any $h \le h_1$, we have (see Theorem 1.4.3 in Chapter 1)

$$\sup_{0\le t_n\le T} \|x^{\frac{N-1}{2}}(u_h^n - u(\cdot,t_n))\|_{L^2(I)} \le C_1(h^2+\tau), \tag{12}$$

where $C_1 = C_1(T, \kappa_0(u), C_0, N, \beta)$ and u_h^n is the solution of (Sym).

Letting $f(s) = s|s|^\alpha$ for $s \in \mathbb{R}$, where $\alpha \ge 1$, then given $T > 0$, we assume (9) with $\nu = 1$ and uniform time increment. Then, there exists an $h_2 = h_2(T, \kappa_1(u), C_0, N, \beta)$ such that, for any $h \le h_2$, we have (see Theorem 1.4.6 in Chapter 1)

$$\sup_{0\le t_n\le T} \|u_h^n - u(\cdot,t_n)\|_{L^\infty(I)} \le C_2 \left(\log\frac{1}{h}\right)^{\frac{1}{2}} (h^2+\tau), \tag{13}$$

where $C_2 = C_2(T, \kappa_1(u), C_0, N, \beta)$ and u_h^n is the solution of (Non-Sym).

In Section 1.5, we report some numerical examples to validate our theoretical results. Section 1.6 proves some inequalities for the proof of L^∞ error estimate of (Non-Sym), and Section 1.7 proves the decreasing property of energy functional for (Sym) that plays an important role in blow-up analysis.

In Chapter 1, we examined the standard finite element method. However, there are some obstacles to apply their finite element schemes to the semilinear heat equation (1). As the non-symmetric scheme seems to be incompatible with Nakagwa's time-increment control strategy, we pose the following question: Can the restriction $N \le 3$ be removed from the symmetric scheme? In fact, this restriction is imposed by the inverse inequality Lemma 1.4.8 in Chapter 1 and the necessity of finding the boundedness of the finite element solution (see the proof of Theorem 1.4.3 in Chapter 1). To surmount this difficulty, the L^∞ estimates for the FEM can be directly derived using the discrete maximum principle (DMP). As the DMP is based largely on the nonnegativity of the finite element solution, the time derivative term should be approximated by the mass-lumping approximation. Unfortunately, we tried but failed to prove the convergence property of the finite element solution by this approximation (see (14) below).
Therefore, we propose a special mass-lumping approximation (15) in Chapter 2. Using the special approximation, we prove the DMP and the convergence property of the finite element solution, and perform the blow up analysis for any $N \ge 2$.

Section 2.2 presents our finite element schemes and the convergence theorems. Under the notation of Chapter 1, the mass-lumping approximation of the weighted L^2 inner product can be naturally defined as

$$\int_I x^{N-1} wv\, dx \approx w(x_0)v(x_0)\int_0^{x_{1/2}} x^{N-1}\, dx + \sum_{i=1}^{m-1} w(x_i)v(x_i)\int_{x_{i-1/2}}^{x_{i+1/2}} x^{N-1}\, dx, \tag{14}$$

where $x_{i-1/2} = (x_i + x_{i-1})/2$. As mentioned above, this standard formulation is useless for our purpose. Instead, we define

$$\int_I x^{N-1} wv\, dx \approx \sum_{i=0}^{m-1} w(x_i)v(x_i)\int_I x^{N-1}\phi_i\, dx = \int_I x^{N-1}\Pi_h(wv)\, dx. \tag{15}$$

The Lagrange interpolation operator Π_h of $\dot{H}^1 \to S_h$ is defined as $\Pi_h w = \sum_{j=0}^{m-1} w(x_j)\phi_j$ for $w \in \dot{H}^1$.

The mass lumping finite element schemes are then stated as follows.

(ML–1) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\int_I x^{N-1}\Pi_h(\partial_{\tau_n} u_h^{n+1}\chi)\ dx + \int_I x^{N-1}(u_h^{n+1})_x\chi_x\ dx = \int_I x^{N-1} f(u_h^n)\chi\ dx \quad (\chi \in S_h), \tag{16}$$

where $u_h^0 \in S_h$ is assumed to be given.

(ML–2) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\int_I x^{N-1}\Pi_h(\partial_{\tau_n} u_h^{n+1}\chi)\ dx + \int_I x^{N-1}(u_h^{n})_x\chi_x\ dx = \int_I x^{N-1}\Pi_h(f(u_h^n)\chi)\ dx \quad (\chi \in S_h). \tag{17}$$

For (ML-1) and (ML-2), we can get the following positivity preserving properties.

In addition to the basic assumption on f, assume that f is a non-decreasing function with $f(0) \geq 0$. If $u_h^n \geq 0$, then the solution u_h^{n+1} of (ML–1) satisfies $u_h^{n+1} \geq 0$. Under the assumptions above, further assume that

$$\tau_n \leq \frac{\beta^2}{N+1} h^2. \tag{18}$$

Then the solution u_h^{n+1} of (ML–2) satisfies $u_h^{n+1} \geq 0$.

We assume

$$\kappa(u) = \sum_{k=0}^{2} \|\partial_x^k u\|_{L^\infty(Q_T)} + \sum_{l=1}^{2} \|\partial_t^l u\|_{L^\infty(Q_T)} + \sum_{k=1}^{2} \|\partial_t \partial_x^k u\|_{L^\infty(Q_T)} < \infty, \tag{19}$$

where $T > 0$ and $Q_T = [0,1] \times [0,T]$. Assume that (10) and (11) are satisfied. Then, for sufficiently small h and τ, we have (see Theorem 2.2.4 and Theorem 2.2.5 in Chapter 2)

$$\sup_{0 \leq t_n \leq T} \|x^{\frac{N-1}{2}}(u_h^n - u(\cdot, t_n))\|_{L^2(I)} \leq C_3(h^2 + \tau), \tag{20}$$

$$\sup_{0 \leq t_n \leq T} \|u_h^n - u(\cdot, t_n)\|_{L^\infty(I)} \leq C_3\left(h + \tau\right), \tag{21}$$

where $C_3 = C_3(T, f, \kappa(u), C_0, N, \beta)$ and u_h^n is the solution of (ML–1).

For (ML-2), we get the following. For sufficiently small h and τ, we have (see Theorem 2.2.6 in Chapter 2)

$$\sup_{0 \leq t_n \leq T} \|u_h^n - u(\cdot, t_n)\|_{L^\infty(I)} \leq C_4\left(h + \tau\right), \tag{22}$$

where $C_4 = C_4(T, f, \kappa(u), C_0, N, \beta)$ and u_h^n is the solution of (ML–2).

After having described some preliminary results in Section 2.3, we prove the convergence theorems in Section 2.4. Blow-up analysis is reported in Section 2.5. We employ the finite element version of the eigenvalue problem:

$$\int_I x^{N-1}(\hat{\psi}_h)_x\chi_x\ dx = \hat{\mu}_h \int_I x^{N-1}\Pi_h(\hat{\psi}_h\chi)\ dx \quad (\chi \in S_h). \tag{23}$$

Let $\hat{\psi}_h \in S_h$ be the eigenfunction associated with the smallest eigenvalue $\hat{\mu}_h > 0$ of (23). For the eigenvalue problem (23), we can obtain the following result (see Proposition 2.5.5 in Chapter 2).

(i) $\hat{\mu}_h \to \mu$ as $h \to 0$.

(ii) The first eigenfunction $\hat{\psi}_h$ of (23) does not change sign.

(iii) $\|x^{\frac{N-1}{2}}(\hat{\psi}_h - \psi)_x\|_{L^2(I)} \to 0$ as $h \to 0$.

Here $\psi \in \dot{H}^1$ denotes the eigenfunction associated with the first eigenvalue $\mu > 0$ of the eigenvalue problem

$$\int_I x^{N-1}\psi_x\chi_x\ dx = \mu \int_I x^{N-1}\psi\chi\ dx \quad (\chi \in \dot{H}^1). \tag{24}$$

Therefore, without loss of generality, we can assume that $\hat{\psi}_h \geq 0$ in I and $\int_I x^{N-1}\hat{\psi}_h(x)\ dx = 1$.

For $v \in \dot{H}^1$, we set

$$K(v) = \frac{1}{2}\|x^{\frac{N-1}{2}} v_x\|_{L^2(I)}^2 - \frac{1}{\alpha+2}\int_I x^{N-1}|v(x)|^{\alpha+2}\, dx,$$
$$I(v) = \int_I x^{N-1} v(x)\psi(x)\, dx.$$

For $v \in S_h$, we set

$$K_h(v) = \frac{1}{2}\|x^{\frac{N-1}{2}} v_x\|_{L^2(I)}^2 - \frac{1}{\alpha+2}\sum_{i=0}^{m}|v(x_i)|^{\alpha+2}\int_I x^{N-1}\phi_i\, dx,$$
$$I_h(v) = \int_I x^{N-1}\Pi_h(v\hat{\psi}_h)(x)\, dx.$$

We introduce the approximate blow-up time $\hat{T}_\infty(h)$ by setting

$$\hat{T}_\infty(h) = \lim_{n\to\infty} t_n = \lim_{n\to\infty}\sum_{j=0}^{n-1}\tau_j. \tag{25}$$

Suppose that the solution u of (1) blows up at finite time T_∞ in the sense that

$$\|u(\cdot,t)\|_{L^\infty(I)} \to \infty \quad \text{and} \quad \|x^{\frac{N-1}{2}} u(\cdot,t)\|_{L^2(I)} \to \infty \quad (t \to T_\infty - 0). \tag{26}$$

Assume that for any $T < T_\infty$, u is sufficiently smooth that (19) holds. Assuming also that (10) is satisfied, we set

$$\tau = \delta\frac{\beta^2}{N+1}h^2 \tag{27}$$

for some $\delta \in (0,1]$. The time increment τ_n is iteratively defined as

$$\tau_n = \tau_n(h) = \tau\min\left\{1, \frac{1}{\left\{\int_I x^{N-1}\Pi_h((u_h^n)^2)\, dx\right\}^{\frac{\alpha}{2}}}\right\}, \tag{28}$$

where we have used the solution u_h^n of (ML–2) with (11). Moreover, assume that (18) is satisfied and that

$$\forall T < T_\infty, \quad \lim_{h\to 0}\sup_{0\le t_n\le T}|K(u(\cdot,t_n)) - K_h(u_h^n)| = 0. \tag{29}$$

We then have (see Theorem 2.5.6 in Chapter 2)

$$\lim_{h\to 0}\hat{T}_\infty(h) = T_\infty. \tag{30}$$

Suppose that the solution u of (1) blows up at finite time T_∞ in the sense that

$$I(u(\cdot,t)) \to \infty \quad \text{and} \quad \|u(\cdot,t)\|_{L^\infty(I)} \to \infty \quad (t \to T_\infty - 0). \tag{31}$$

Assume that, for any $T < T_\infty$, u is sufficiently smooth that (19) holds. Assuming also that (10) is satisfied, we set τ by (27) with some $\delta \in (0,1]$. The time increment τ_n is iteratively defined as

$$\tau_n = \tau_n(h) = \tau\min\left\{1, \frac{1}{I_h(u_h^n)^\alpha}\right\}, \tag{32}$$

where we have used the solution u_h^n of (ML–2) with (11). We then obtain (30) (see Theorem 2.5.7 in Chapter 2).

The above theorems differ in that the first theorem requires the convergence property (29) of the discrete energy functional $K_h(u_h^n)$, whereas no convergence property of I_h is necessary in the second theorem.

Section 2.6 presents some numerical examples that validate our theoretical results. In Section 2.7, we mention the proof of some auxiliary results on the eigenvalue problems.

In Chapter 3, we examine the time-increment control methods proposed by Cho-Okamoto [15], Chen [8] and Groisman [23]. In particular, we study the numbers of the blow-up points and the blow-up rates of the finite element solutions.

Chapter 4 is devoted to an application to the Keller-Segel system which describes the aggregation of slime molds resulting from their chemotactic features. We consider the radially symmetric solutions for the parabolic-parabolic and parabolic-elliptic systems and offer the finite element schemes that preserve positivity and mass-conservation properties. The validity is verified by numerical examples.

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Chapter 1

The standard finite element method

1.1 Introduction

This chapter was conducted to investigate the convergence property of finite element method (FEM) applied to a parabolic equation with singular coefficients for the function $u = u(x,t)$, $x \in \overline{I} = [0,1]$, and $t \geq 0$, as expressed in

$$
\begin{aligned}
& u_t = u_{xx} + \frac{N-1}{x} u_x + f(u), && x \in I = (0,1),\ t > 0, && \text{(1.1a)}\\
& u_x(0,t) = u(1,t) = 0, && t > 0, && \text{(1.1b)}\\
& u(x,0) = u^0(x), && x \in I, && \text{(1.1c)}
\end{aligned}
$$

where f is a given locally Lipschitz continuous function, u^0 is a given continuous function, and

$$N \geq 2 \quad \text{integer} \tag{1.2}$$

is a given parameter.

In the study of an N-dimensional semilinear heat equation, the following problem arises as

$$
\begin{aligned}
& U_t = \Delta U + f(U), && \boldsymbol{x} \in \Omega,\ t > 0 && \text{(1.3a)}\\
& U = 0, && \boldsymbol{x} \in \partial\Omega,\ t > 0, && \text{(1.3b)}\\
& U(0,\boldsymbol{x}) = U^0(\boldsymbol{x}), && \boldsymbol{x} \in \Omega, && \text{(1.3c)}
\end{aligned}
$$

where Ω represents a bounded domain in $\mathbb{R}^N$. If one is concerned with the *radially symmetric solution* $u(|\boldsymbol{x}|) = U(\boldsymbol{x})$ in the N-dimensional ball $\Omega = \{\boldsymbol{x} \in \mathbb{R}^N \mid |\boldsymbol{x}| = |\boldsymbol{x}|_{\mathbb{R}^N} < 1\}$, then (1.3) implies (1.1), where $x = |\boldsymbol{x}|$ and $u^0(x) = U_0(\boldsymbol{x})$.

For a linear case in which $f(u) = 0$ is replaced by a given function $f(x,t)$, the works [18, 42] studied the convergence property of the FEM to (1.1) along with the corresponding steady-state problem, and two proposed schemes: the symmetric scheme, wherein they established the optimal order error estimate in the weighted L^2 norm; and the nonsymmetric scheme, wherein they proved the L^∞ error estimate. In this chapter, both schemes are applied to the semilinear heat equation (1.1) to derive various error estimates. Moreover, this chapter includes a discussion of discrete positivity conservation properties, which earlier studies [18, 42] failed to embrace, but which are actually important in the study of diffusion-type equations.

Our emphasis is on FEM because we are able to use non-uniform partitions of the space variable. Therefore, the method is deemed useful for examining highly concentrated solutions at the origin. On this connection, we present our motivation for this chapter. The critical phenomenon appearing in the semilinear heat equation of the form

$$U_t = \Delta U + U^{1+\alpha}, \quad \alpha > 0$$

in a multidimensional space has attracted considerable attention since the pioneering work of Fujita [22]. According to him, the equation is in the whole N dimensional space. Any positive solution blows up in a finite time if $\alpha \leq 2/N$, whereas a solution is smooth at any time for a small initial value if $\alpha > 2/N$. Therefore, expression $p_c = 1 + 2/N$ is known as Fujita's critical exponent ([31, 16] provides some critical exponents of other equations). Generally, similar critical exponents can be found for an initial-boundary value problem for the semilinear heat equation. Some examples are given in reports of earlier studies [27, 31, 16]. However, the concrete values of those critical conditions are apparently unknown. Therefore, we found it interesting to study the numerical methods for computing the solutions of nonlinear partial differential equations in an N-dimensional space. However, computing the non-stationary four-space dimensional problem is difficult, even for modern computers. We consider the FEM to solve the one space dimensional equation (1.1). However, we face another difficulty in dealing with the singular coefficient $(N-1)/x$, which the FEM reasonably simplified, as explained later.

As described above, the main purpose of this chapter is to derive various optimal order error estimates for the symmetric and nonsymmetric schemes of [18, 42] applied to (1.1). These schemes are described below as (Sym) and (Non-Sym). To this end, we address mostly the general nonlinearity $f(u)$. Moreover, we study discrete positivity conservation properties. We summarize our typical results here.

- The solution of (Sym) is positive if f and the discretization parameters satisfy some conditions, as shown by Theorem 1.3.2.
- If f is a *globally* Lipschitz continuous function, then the solution of (Sym) converges to the solution of (1.1) in the weighted L^2 norm for the space and in the L^∞ norm for time. Moreover, the convergence is at the optimal order, as shown by Theorem 1.4.1.
- If f is a *locally* Lipschitz continuous function and $N \leq 3$, then the solution of (Sym) converges to the solution of (1.1) in the weighted L^2 norm for the space and in the L^∞ norm for time. The convergence is at the optimal order, as shown by Theorem 1.4.3.
- If $f(u) = u|u|^\alpha$ with $\alpha \geq 1$ and if the time partition is uniform, then the solution of (Non-Sym) converges to the solution of (1.1) in the $L^\infty(0, T; L^\infty(I))$ norm. The convergence is at the optimal order up to the logarithm factor, as shown by Theorem 1.4.6.

However, we do not proceed to applications of our schemes to the blow-up computation in this work. In fact, from the main results presented in this chapter, we infer that the standard schemes of [18, 42] do not fit for the blow-up computation for large N. For the symmetric scheme, the restriction $N \leq 3$ reduces interest in considering radially symmetric problems. Moreover, for the nonsymmetric scheme, the use of uniform time-partitions makes it difficult to apply Nakagawa's time-partitions control strategy: a powerful technique for computing the approximate blow-up time, as described in earlier reports [8, 32, 39, 14, 9, 40, 7]. Nevertheless, we believe that our results are of interest to researchers in this and related fields. In fact, the validity issue of the symmetric scheme only for $N \leq 3$ was pointed out earlier in [2] for a nonlinear Schrödinger equation with no mathematical evidence. The analysis reported herein reveals weak points of the two standard schemes. As a sequel to this chapter, we propose a new finite element scheme for (1.1). The scheme, which uses a nonstandard mass-lumping approximation, is shown to be positivity-preserving and convergent for any $N \geq 2$. Details will be reported in the next chapter.

It is noteworthy that the finite difference method for (1.1) has been studied and that its optimal order convergence was proved in an earlier report [8]. Its finite difference scheme uses a special approximation around the origin to assume a uniform spatial mesh.

This chapter comprises seven sections. Section 1.2 presents our finite element schemes. Well-posedness and positivity conservation are examined in Section 1.3. Section 1.4 presents the error estimates and their proofs and Section 1.5 presents some numerical examples that validate our theoretical results. Section 1.6 proves the inequalities for the proof of L^∞ error estimate of (Non-Sym), and Section 1.7 proves the decreasing property of energy functional for (Sym) that plays important role in blow-up analysis.

1.2 Finite element method

First, we derive two alternate weak formulations of (1.1). Unless otherwise stated explicitly, we assume that f is a locally Lipschitz continuous function such that

$$\forall \mu > 0,\ \exists M_\mu > 0 :\ |f(s) - f(s')| \leq M_\mu |s - s'| \quad (s, s' \in \mathbb{R}, |s|, |s'| \leq \mu). \tag{f1}$$

Letting $\chi \in \dot{H}^1 = \{v \in H^1(I) \mid v(1) = 0\}$ be arbitrary, then multiplying both sides of (1.1a) by $x^{N-1}\chi$ and using integration by parts over I, we obtain

$$\int_I x^{N-1} u_t \chi \, dx + \int_I x^{N-1} u_x \chi_x \, dx = \int_I x^{N-1} f(u) \chi \, dx. \tag{1.4}$$

Otherwise, if we multiply both sides of (1.1a) by $x\chi$ instead of $x^{N-1}\chi$ and integrate it over I, then we have

$$\int_I x u_t \chi \, dx + \int_I [x u_x \chi_x + (2-N) u_x \chi] \, dx = \int_I x f(u) \chi \, dx. \tag{1.5}$$

We designate (1.4) the *symmetric* weak form because of the symmetric bilinear form associated with the differential operator $u_{xx} + \frac{N-1}{x} u_x$. In contrast, (1.5) is the *nonsymmetric* weak form. Both forms are identical at $N = 2$.

We now establish the finite element schemes based on these identities. For a positive integer m, we introduce node points

$$0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < \cdots < x_{m-1} < x_m = 1,$$

and set $I_j = (x_{j-1}, x_j)$ and $h_j = x_j - x_{j-1}$, where $j = 1, \ldots, m$. The granularity parameter is defined as $h = \max_{1 \le j \le m} h_j$. Let $\mathcal{P}_k(J)$ be the set of all polynomials in an interval J of degree $\le k$. We define the P1 finite element space as

$$S_h = \{v \in H^1(I) \mid v \in \mathcal{P}_1(I_j) \ (j = 1, \cdots, m), \ v(1) = 0\}. \tag{1.6}$$

Its standard basis function ϕ_j, $j = 0, 1, \cdots, m-1$, is defined as

$$\phi_j(x_i) = \delta_{ij},$$

where δ_{ij} denotes Kronecker's delta.

For time discretization, we introduce non-uniform partitions

$$t_0 = 0, \quad t_n = \sum_{j=0}^{n-1} \tau_j \quad (n \ge 1),$$

where $\tau_j > 0$ denotes the time increment. Furthermore, we set

$$\tau = \sup_{j \ge 0} \tau_j.$$

Generally, we write $\partial_{\tau_n} u_h^{n+1} = (u_h^{n+1} - u_h^n)/\tau_n$.

We are now in a position to state the finite element schemes to be considered.

(Sym) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\left(\partial_{\tau_n} u_h^{n+1}, \chi\right) + A(u_h^{n+1}, \chi) = (f(u_h^n), \chi) \quad (\chi \in S_h, \ n = 0, 1, \ldots), \tag{1.7}$$

where $u_h^0 \in S_h$ is assumed to be given. Hereinafter, we set

$$(w, v) = \int_I x^{N-1} w v \, dx, \qquad \|w\|^2 = (w, w) = \int_I x^{N-1} w^2 \, dx, \tag{1.8a}$$

$$A(w, v) = \int_I x^{N-1} w_x v_x \, dx. \tag{1.8b}$$

(Non-Sym) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\left\langle \partial_{\tau_n} u_h^{n+1}, \chi \right\rangle + B(u_h^{n+1}, \chi) = \langle f(u_h^n), \chi \rangle \quad (\chi \in S_h, \ n = 0, 1, \ldots), \tag{1.9}$$

where

$$\langle w, v \rangle = \int_I x w v \, dx, \qquad |||w|||^2 = \langle w, w \rangle = \int_I x w^2 \, dx, \tag{1.10a}$$

$$B(w, v) = \int_I x w_x v_x \, dx + (2-N) \int_I w_x v \, dx. \tag{1.10b}$$

It is noteworthy that $B(\cdot, \cdot)$ is coercive in $\dot{H}^1$ such that

$$B(w, w) = \langle w_x, w_x \rangle + (2-N) \int_I w_x w dx = |||w_x|||^2 + \frac{N-2}{2} w(0)^2 \ge |||w_x|||^2. \tag{1.11}$$

1.3 Well-posedness and positivity conservation

In this section, we prove the following theorems.

Theorem 1.3.1 (Well-posedness of (Sym))**.** For a given $u_h^n \in S_h$ with $n \geq 0$, the scheme (Sym) admits a unique solution $u_h^{n+1} \in S_h$.

Theorem 1.3.2 (Positivity of (Sym))**.** In addition to the basic assumption (f1), assume that

$$f \text{ is a non-decreasing function with } f(0) \geq 0. \tag{f2}$$

Letting $n \geq 0$ and $u_h^n \geq 0$, and assuming that

$$\tau_n \geq \frac{1}{4}h^2, \tag{1.12}$$

then the solution u_h^{n+1} of (Sym) satisfies $u_h^{n+1} \geq 0$.

Theorem 1.3.3 (Comparison principle for (Sym))**.** We let $n \geq 0$ and assume that $u_h^n,\ \tilde{u}_h^n \in S_h$ satisfy $u_h^n \leq \tilde{u}_h^n$ in I. Furthermore, we assume that (f1) and (f2) are satisfied. Similarly, we let $u_h^{n+1}, \tilde{u}_h^{n+1} \in S_h$ be the solutions of (Sym) with $u_h^n, \tilde{u}_h^n$, respectively, using the same time increment τ_n. Moreover, we assume that (1.12) is satisfied. Consequently, we obtain $u_h^{n+1} \leq \tilde{u}_h^{n+1}$ in I. The equality holds true if and only if $u_h^n = \tilde{u}_h^n$ in I.

Theorem 1.3.4 (Well-posedness of (Non-Sym))**.** For a given $u_h^n \in S_h$ with $n \geq 0$, the scheme (Non-Sym) admits a unique solution $u_h^{n+1} \in S_h$.

To prove these theorems, we conveniently rewrite (1.7) into a matrix form. That is, we introduce

$$\begin{aligned}
&\mathcal{M} = (\mu_{i,j})_{0\leq i,j\leq m-1} \in \mathbb{R}^{m\times m}, && \mu_{i,j} = (\phi_j, \phi_i),\\
&\mathcal{A} = (a_{i,j})_{0\leq i,j\leq m-1} \in \mathbb{R}^{m\times m}, && a_{i,j} = A(\phi_j, \phi_i),\\
&\boldsymbol{u}^n = (u_j^n)_{0\leq j\leq m-1} \in \mathbb{R}^{m}, && u_j^n = u_h^n(x_j),\\
&\boldsymbol{F}^n = (F_j^n)_{0\leq j\leq m-1} \in \mathbb{R}^{m}, && F_j^n = (f(u_h^n), \phi_j),
\end{aligned}$$

and express (1.7) as

$$(\mathcal{M} + \tau_n\mathcal{A})\boldsymbol{u}^{n+1} = \mathcal{M}\boldsymbol{u}^n + \tau_n\boldsymbol{F}^n \quad (n = 0, 1, \ldots), \tag{1.13}$$

where $u_m^n = u_h^n(x_m)$ is understood as $u_m^n = 0$.

Lemma 1.3.5. $\mathcal{M}$ and $\mathcal{A}$ are both tri-diagonal and positive-definite matrices.

Theorem 1.3.1 is a direct consequence of this lemma. We proceed to proofs of other theorems.

Proof of Theorem 1.3.2. We use the representative matrix (1.13) instead of (1.7) and set

$$\mathcal{C} = (c_{i,j})_{0\leq i,j\leq m-1} = \mathcal{M} + \tau_n\mathcal{A}, \quad c_{i,j} = \mu_{i,j} + \tau_n a_{i,j}.$$

If $\mathcal{C}^{-1} \geq O$, then we obtain

$$\boldsymbol{u}^{n+1} = \mathcal{C}^{-1}\left(\mathcal{M}\boldsymbol{u}^n + \tau_n\boldsymbol{F}^n\right) \geq \boldsymbol{0},$$

because $\mathcal{M} \geq O$ and $\boldsymbol{F}^n \geq \boldsymbol{0}$ in view of (f2). The proof that $\mathcal{C}^{-1} \geq O$ is true under (1.12) is divided into three steps, each described as presented below.

Step 1. We show that

$$\sum_{j=0}^{m-1} c_{i,j} > 0 \qquad (0 \leq i \leq m-1). \tag{1.14}$$

Letting $1 \leq i \leq m-2$, we calculate

$$\begin{aligned}
\sum_{j=0}^{m-1} c_{i,j} &= \sum_{j=i-1}^{i+1} \mu_{i,j} + \tau_n \sum_{j=i-1}^{i+1} a_{i,j}\\
&= \sum_{j=i-1}^{i+1} \mu_{i,j} + \tau_n \int_{x_{i-1}}^{x_{i+1}} x^{N-1}(\phi_{i-1} + \phi_i + \phi_{i+1})_x(\phi_i)_x\ dx\\
&= \sum_{j=i-1}^{i+1} \mu_{i,j} > 0,
\end{aligned}$$

because $\phi_{i-1}+\phi_i+\phi_{i+1}\equiv 1$ in (x_{i-1},x_{i+1}). Cases $i=0$ and $i=m-1$ are verified similarly.
Step 2. We show that, if

$$\tau_n \geq -\frac{\mu_{i,i+1}}{a_{i,i+1}}, -\frac{\mu_{i,i-1}}{a_{i,i-1}} \qquad (i=0,1,\cdots,m-1), \tag{1.15}$$

then $\mathcal{C}^{-1}\geq O$. First, (1.15) implies that $c_{i,i-1},c_{i,i+1}\leq 0$ for $0\leq i\leq m-1$ because $a_{i,i-1},a_{i,i+1}<0$. Matrix $\mathcal{C}$ is decomposed as $\mathcal{C}=\mathcal{D}(\mathcal{I}-\mathcal{E})$, where $\mathcal{D}=(d_{i,j})_{0\leq i,j\leq m-1}$ and $\mathcal{E}=(e_{i,j})_{0\leq i,j\leq m-1}$ are defined as

$$d_{i,j}=\begin{cases} c_{i,i} & (i=j)\\ 0 & (i\neq j)\end{cases}, \qquad e_{i,j}=\begin{cases} 0 & (i=j)\\ -\frac{c_{i,j}}{c_{i,i}} & (i\neq j),\end{cases}$$

and where I is the identity matrix. Apparently, $\mathcal{I}-\mathcal{E}$ is non-singular and $\mathcal{D}\geq O$. Using (1.14), we deduce

$$\|\mathcal{E}\|_\infty = \max_{0\leq i\leq m-1}\left(-\frac{c_{i,i-1}}{c_{i,i}}-\frac{c_{i,i+1}}{c_{i,i}}\right)<1.$$

Therefore, matrix $\mathcal{I}-\mathcal{E}$ is non-singular and $(\mathcal{I}-\mathcal{E})^{-1}=\sum_{k=0}^{\infty}\mathcal{E}^k\geq O$. Consequently, we have $\mathcal{C}^{-1}=(\mathcal{I}-\mathcal{E})^{-1}\mathcal{D}^{-1}\geq O$.
Step 3. Finally, we demonstrate that (1.12) implies (1.15). We calculate

$$\begin{aligned}\mu_{i,i+1} &= \int_{x_i}^{x_{i+1}} x^{N-1}\frac{1}{h_{i+1}^2}(x-x_i)(x_{i+1}-x)\,dx \leq \frac{1}{4}h_{i+1}^2\int_{x_i}^{x_{i+1}}\frac{1}{h_{i+1}^2}x^{N-1}\,dx,\\ -a_{i,i+1} &= \int_{x_i}^{x_{i+1}} x^{N-1}\frac{1}{h_{i+1}^2}\,dx.\end{aligned}$$

Therefore, we deduce $-\frac{\mu_{i,i+1}}{a_{i,i+1}}\leq\frac{1}{4}h^2$. □

Proof of Theorem 1.3.3. Because $f(\tilde{u}_h^n)-f(u_h^n)\geq 0$ in I, the proof follows exactly the same pattern as that of the proof of Proposition 1.3.2. □

We proceed to the result for (Non-Sym):

$$\begin{aligned}
&\mathcal{M}'=(\mu'_{i,j})_{0\leq i,j\leq m-1}\in\mathbb{R}^{m\times m}, && \mu'_{i,j}=\langle\phi_j,\phi_i\rangle,\\
&\mathcal{B}=(b_{i,j})_{0\leq i,j\leq m-1}\in\mathbb{R}^{m\times m}, && b_{i,j}=B(\phi_j,\phi_i),\\
&\boldsymbol{G}^n=(G_j^n)_{0\leq j\leq m-1}\in\mathbb{R}^m, && G_j^n=\langle f(u_h^n),\phi_j\rangle,
\end{aligned}$$

and express (1.9) as

$$(\mathcal{M}'+\tau_n\mathcal{B})\boldsymbol{u}^{n+1}=\mathcal{M}'\boldsymbol{u}^n+\tau_n\boldsymbol{G}^n \quad (n=0,1,\ldots). \tag{1.16}$$

In view of (1.11), $\mathcal{M}'$ and $\mathcal{B}$ are both tri-diagonal and positive-definite matrices. Therefore, the proof is completed.

1.4 Convergence and error analysis

1.4.1 Results

Our convergence results for (Sym) and (Non-Sym) are stated under a smoothness assumption of the solution u of (1.1): given $T>0$ and setting $Q_T=[0,1]\times[0,T]$, we assume that u is sufficiently smooth such that

$$\kappa_\nu(u)=\sum_{j=0}^{2}\|\partial_x^j u\|_{L^\infty(Q_T)}+\sum_{l=1}^{2+\nu}\|\partial_t^l u\|_{L^\infty(Q_T)}+\sum_{k=1}^{1+\nu}\|\partial_t^k\partial_x^2 u\|_{L^\infty(Q_T)}<\infty, \tag{1.17}$$

where ν is either 0 or 1.

The partition $\{x_i\}_{j=0}^{m}$ of $\bar{I}=[0,1]$ is assumed to be quasi-uniform, with a positive constant β independent of h such that

$$h \le \beta \min_{1\le j\le m} h_j. \tag{1.18}$$

Finally, the approximate initial value u_h^0 is chosen as

$$\|u_h^0 - u^0\| \le C_0 h^2 \tag{1.19}$$

for a positive constant C_0.

Moreover, for $k=1,2,\ldots$, we express the positive constants $C_k = C_k(\gamma_1,\gamma_2,\ldots)$ and $h_k = h_k(\gamma_1,\gamma_2,\ldots)$ according to the parameters $\gamma_1,\gamma_2,\ldots$. Particularly, C_k and h_k are independent of h and τ.

Next we state the following theorems.

Theorem 1.4.1 (Convergence for (Sym) in $\|\cdot\|$, I)**.** Assume that f is a globally Lipschitz continuous function; assume (f1) and

$$M = \sup_{\mu>0} M_\mu < \infty. \tag{f3}$$

Assume that, for $T>0$, solution u of (1.1) is sufficiently smooth that (1.17) for $\nu=0$ holds true. Moreover, assume that (1.18) and (1.19) are satisfied. Then, there exists an $h_1 = h_1(N,\beta)$ such that, for any $h\le h_1$, we have

$$\sup_{0\le t_n\le T} \|u_h^n - u(\cdot,t_n)\| \le C_1(h^2+\tau),$$

where $C_1 = C_1(T,M,\kappa_0(u),C_0,N,\beta)$ and u_h^n is the solution of (Sym).

For L^∞ error estimates, we must further assume that u_h^0 is chosen as

$$A(u_h^0 - u^0, v_h) = 0 \quad (v_h\in S_h). \tag{1.20}$$

Theorem 1.4.2 (Convergence for (Sym) in $\|\cdot\|_{L^\infty(\sigma,1)}$, I)**.** In addition to the assumption of Theorem 1.4.1, assume that (1.20) is satisfied. Furthermore, let $\sigma\in(0,1)$ be arbitrary. Then, there exists an $h_2 = h_2(N,\beta)$ such that, for any $h\le h_2$, we have

$$\sup_{0\le t_n\le T} \|u_h^n - u(\cdot,t_n)\|_{L^\infty(\sigma,1)} \le C_2\left(h^2\log\frac{1}{h}+\tau\right),$$

where $C_2 = C_2(T,M,\kappa_0(u),C_0,N,\beta,\sigma)$ and u_h^n is the solution of (Sym).

The restriction that f is a globally Lipschitz continuous function with (f3) can be removed in the following manner.

Theorem 1.4.3 (Convergence of (Sym) in $\|\cdot\|$, II)**.** Given that $T>0$ and that only (f1) is satisfied, we assume that (1.17) with $\nu=0$, (1.18), and (1.19) are satisfied. Furthermore, assume that $N\le 3$ and that there exist positive constants c_1 and σ such that

$$\tau h^{-N/2} \le c_1 h^\sigma. \tag{1.21}$$

Then there exists an $h_3 = h_3(T,\kappa_0(u),C_0,N,\beta)$ such that, for any $h\le h_3$, we have

$$\sup_{0\le t_n\le T} \|u_h^n - u(\cdot,t_n)\| \le C_2(h^2+\tau),$$

where $C_3 = C_3(T,\kappa_0(u),C_0,N,\beta)$ and u_h^n is the solution of (Sym).

Theorem 1.4.4 (Convergence for (Sym) in $\|\cdot\|_{L^\infty(\sigma,1)}$, II)**.** Given that $T>0$ and that (f1) is satisfied, we assume that (1.17) with $\nu=0$, (1.18), (1.19), (1.20) and (1.21) are satisfied. Consequently, there exists an $h_4 = h_4(T,\kappa_0(u),C_0,N,\beta)$ such that, for any $h\le h_4$, we have

$$\sup_{0\le t_n\le T} \|u_h^n - u(\cdot,t_n)\|_{L^\infty(\sigma,1)} \le C_4\left(h^2\log\frac{1}{h}+\tau\right),$$

where $C_4 = C_4(T,\kappa_0(u),C_0,N,\beta)$ and u_h^n is the solution of (Sym).

Subsequently, let us proceed to error estimates for (Non-Sym). For the approximate initial value u_h^0, we choose

$$B(u_h^0 - u^0, v_h) = 0 \qquad (v_h \in S_h). \tag{1.22}$$

Quasi-uniformity is also required for the time partition. Therefore, there exists a positive constant $\gamma > 0$ such that

$$\tau \leq \gamma \tau_{\min}, \tag{1.23}$$

where $\tau_{\min} = \min_{n \geq 0} \tau_n$. Moreover, we set

$$\delta = \sup_{t_{k+1} \in [0,T]} |\tau_k - \tau_{k+1}|. \tag{1.24}$$

Theorem 1.4.5 (Convergence for (Non-Sym), I)**.** Let f be a C^1 function satisfying

$$M_1 = \sup_{s \in \mathbb{R}} |f'(s)| < \infty, \quad M_2 = \sup_{s \neq s' \in \mathbb{R}} \frac{|f'(s) - f'(s')|}{|s - s'|} < \infty. \tag{f4}$$

Given $T > 0$, we assume that the solution u of (1.1) is sufficiently smooth that (1.17) for $\nu = 1$ holds true. Furthermore, we assume that (1.18), (1.22) and (1.23) are satisfied. Then, there exists an $h_5 = h_5(T, \kappa_1(u), M_1, M_2, \gamma, N, \beta)$ such that, for any $h \leq h_5$, we have

$$\sup_{0 \leq t_n \leq T} \|u_h^n - u(\cdot, t_n)\|_{L^\infty(I)} \leq C_5 \left(\log \frac{1}{h}\right)^{\frac{1}{2}} \left(h^2 + \tau + \frac{\delta}{\tau_{\min}}\right),$$

where $C_5 = C_5(T, \kappa_1(u), M_1, M_2, \gamma, N, \beta) > 0$ and u_h^n is the solution of (Non-Sym).

Finally, we state the error estimates for non-globally Lipschitz continuous function f. To avoid unnecessary complexity, we deal only with the power nonlinearity $f(s) = s|s|^\alpha$.

Theorem 1.4.6 (Convergence for (Non-Sym), II)**.** Letting $f(s) = s|s|^\alpha$ for $s \in \mathbb{R}$, where $\alpha \geq 1$, then given $T > 0$, we assume that (1.17) with $\nu = 1$, (1.18) and (1.22) are satisfied. We assume the uniform time increment $\gamma = 1$. Then, there exists an $h_6 = h_6(T, \kappa_1(u), N, \beta)$ such that, for any $h \leq h_6$, we have

$$\sup_{0 \leq t_n \leq T} \|u_h^n - u(\cdot, t_n)\|_{L^\infty(I)} \leq C_6 \left(\log \frac{1}{h}\right)^{\frac{1}{2}} (h^2 + \tau),$$

where $C_6 = C_6(T, \kappa_1(u), N, \beta)$ and u_h^n is the solution of (Non-Sym).

1.4.2 Proof of Theorems 1.4.1 and 1.4.2

We use the projection operator P_A of $\dot{H}^1 \to S_h$ associated with $A(\cdot, \cdot)$, defined for $w \in \dot{H}^1$ as

$$P_A w \in S_h, \quad A(P_A w - w, \chi) = 0 \qquad (\chi \in S_h). \tag{1.25}$$

In [18] and [29], the following error estimates are proved.

Lemma 1.4.7. Letting $w \in C^2(\bar{I}) \cap \dot{H}^1$, and (1.18) be satisfied, then for $h \leq h_7 = h_7(N, \beta)$, we obtain

$$\|P_A w - w\| \leq Ch^2 \|w_{xx}\|, \tag{1.26}$$

$$\|P_A w - w\|_{L^\infty(I)} \leq C \left(\log \frac{1}{h}\right) h^2 \|w_{xx}\|_{L^\infty(I)}, \tag{1.27}$$

where C is a positive constant depending only on N and β.

Proof of Theorem 1.4.1. Using $P_A u$, we distribute the error in the form shown below.

$$u_h^n - u(t_n) = \underbrace{(u_h^n - P_A u(t_n))}_{=\theta^n} + \underbrace{(P_A u(t_n) - u(t_n))}_{=\rho^n}$$

From (1.26), it is known that

$$\|\rho^n\| \le Ch^2\|u_{xx}(t_n)\| \le Ch^2\|u_{xx}\|_{L^\infty(Q_T)}. \tag{1.28}$$

Next we derive an estimate for θ^n. By considering the symmetric weak form (1.4) at $t = t_{n+1}$, we obtain

$$\begin{aligned}(\partial_{\tau_n} u(t_{n+1}), \chi) + A(P_A u(t_{n+1}), \chi) = (f(u(t_n)), \chi)& \\ + (f(u(t_{n+1})) - f(u(t_n)), \chi) + (\partial_{\tau_n} u(t_{n+1}) - u_t(t_{n+1}), \chi)&\end{aligned}$$

which, together with (1.7), implies that

$$\begin{aligned}\left(\partial_{\tau_n}\theta^{n+1}, \chi\right) + A(\theta^{n+1}, \chi) = (f(u_h^n) - f(u(t_n)), \chi)& \\ - (f(u(t_{n+1})) - f(u(t_n)), \chi) - (\partial_{\tau_n} u(t_{n+1}) - u_t(t_{n+1}), \chi) - \left(\partial_{\tau_n}\rho^{n+1}, \chi\right).&\end{aligned} \tag{1.29}$$

Substituting this expression for $\chi = \theta^{n+1}$ yields the following:

$$\begin{aligned}\frac{1}{\tau_n}\left\{\|\theta^{n+1}\|^2 - \|\theta^n\| \cdot \|\theta^{n+1}\|\right\} \le M\|\theta^n + \rho^n\| \cdot \|\theta^{n+1}\|& \\ + M\tau_n\|u_t\|_{L^\infty(Q_T)} \cdot \|\theta^{n+1}\| + C\tau_n\|u_{tt}\|_{L^\infty(Q_T)}\|\theta^{n+1}\| + \left\|\partial_{\tau_n}\rho^{n+1}\right\| \cdot \|\theta^{n+1}\|.&\end{aligned}$$

Correspondingly, because

$$\partial_{\tau_n}\rho^{n+1} = P_A\left(\frac{u(t_{n+1}) - u(t_n)}{\tau_n}\right) - \frac{u(t_{n+1}) - u(t_n)}{\tau_n},$$

we provide an estimate

$$\left\|\partial_{\tau_n}\rho^{n+1}\right\| \le Ch^2\left\|\frac{u_{xx}(t_{n+1}) - u_{xx}(t_n)}{\tau_n}\right\| \le Ch^2\|u_{xxt}\|_{L^\infty(Q_T)}. \tag{1.30}$$

To sum up, we obtain

$$\|\theta^{n+1}\| - \|\theta^n\| \le \tau_n M\|\theta^n\| + Ch^2M\tau_n + CM\tau_n^2 + C\tau_n^2 + Ch^2\tau_n.$$

Therefore,

$$\begin{aligned}\|\theta^n\| &\le e^{MT}\|u_h^0 - P_A u^0\| + C\frac{e^{MT} - 1}{M}(\tau + h^2) \\ &\le e^{MT}(\|u_h^0 - u^0\| + \|u^0 - P_A u^0\|) + C\frac{e^{MT} - 1}{M}(\tau + h^2) \\ &\le C'(\tau + h^2),\end{aligned} \tag{1.31}$$

where $C' = C'(T, \kappa_0(u), M, N, \beta, C_0) > 0$. By combining this expression with (1.28), one can deduce the desired error estimate. □

Proof of Theorem 1.4.2. We use the same error decomposition process as that used in the previous proof where $u_h^n - u(t_n) = \theta^n + \rho^n$. Also, we apply (1.27) to estimate $\|\rho^n\|_{L^\infty(I)}$. Because

$$\|\theta^n\|_{L^\infty(\sigma,1)} \le \|\theta_x^n\|_{L^1(\sigma,1)} \le C(\sigma, N)\|\theta_x^n\|, \tag{1.32}$$

we perform an estimation for $\|\theta_x^n\|$.

Substituting (1.29) for $\chi = \partial_{\tau_n}\theta^{n+1}$, we obtain the following.

$$\begin{aligned}\left\|\partial_{\tau_n}\theta^{n+1}\right\|^2 + A(\theta^{n+1}, \partial_{\tau_n}\theta^{n+1}) \le M\|\theta^n\| \cdot \left\|\partial_{\tau_n}\theta^{n+1}\right\|& \\ + M\|\rho^n\| \cdot \left\|\partial_{\tau_n}\theta^{n+1}\right\| + M\tau_n\|u_t\|_{L^\infty(Q_T)} \cdot \left\|\partial_{\tau_n}\theta^{n+1}\right\|& \\ + \|u_{tt}\|_{L^\infty(Q_T)}\tau_n\left\|\partial_{\tau_n}\theta^{n+1}\right\| + \left\|\partial_{\tau_n}\rho^{n+1}\right\| \cdot \left\|\partial_{\tau_n}\theta^{n+1}\right\|&\end{aligned}$$

Correspondingly, we apply the elementary identity shown below

$$\begin{aligned} A\left(\theta^{n+1}, \partial_{\tau_n}\theta^{n+1}\right) &= \frac{1}{2} A\left(\theta^{n+1} - \theta^n + \theta^{n+1} + \theta^n, \partial_{\tau_n}\theta^{n+1}\right) \\ &\geq \frac{1}{2\tau_n}\left[A\left(\theta^{n+1}, \theta^{n+1}\right) - A\left(\theta^n, \theta^n\right)\right] \end{aligned}$$

along with Young's inequality to obtain

$$\begin{aligned} \frac{1}{2\tau_n}\left[A(\theta^{n+1}, \theta^{n+1}) - A(\theta^n, \theta^n)\right] &\leq \frac{1}{2}\frac{M^2}{\delta_0^2}\|\theta^n\|^2 + \frac{1}{2}\delta_0^2\left\|\partial_{\tau_n}\theta^{n+1}\right\|^2 \\ &+ \frac{1}{2}\frac{M^2}{\delta_1^2}\|\rho^n\|^2 + \frac{1}{2}\delta_1^2\left\|\partial_{\tau_n}\theta^{n+1}\right\|^2 + \frac{1}{2}\frac{C^2}{\delta_2^2}\tau_n^2 + \frac{1}{2}\delta_2^2\left\|\partial_{\tau_n}\theta^{n+1}\right\|^2 \\ &+ \frac{1}{2}\left\|\partial_{\tau_n}\rho^{n+1}\right\|^2 + \frac{1}{2}\left\|\partial_{\tau_n}\theta^{n+1}\right\|^2 - \left\|\partial_{\tau_n}\theta^{n+1}\right\|^2, \end{aligned}$$

where $\delta_0, \delta_1, \delta_2 > 0$ are constants. After setting $\delta_0^2 + \delta_1^2 + \delta_2^2 = 1$, we obtain

$$A(\theta^{n+1}, \theta^{n+1}) - A(\theta^n, \theta^n) \leq \tau_n\left[\frac{C^2}{\delta_0^2}\|\theta^n\|^2 + \frac{C^2}{\delta_1^2}\|\rho^n\|^2 + \left\|\partial_{\tau_n}\rho^{n+1}\right\|^2 + \frac{C^2}{\delta_2^2}\tau^2\right].$$

Therefore,

$$A(\theta^n, \theta^n) \leq A(\theta^0, \theta^0) + C^2 t_n \sup_{1\leq k\leq n}\left[\|\theta^{k-1}\|^2 + \|\rho^{k-1}\|^2 + \left\|\partial_{\tau_{k-1}}\rho^k\right\|^2 + \tau^2\right].$$

Consequently, using (1.20), (1.30), and (1.31), we deduce

$$\|\theta_x^n\| \leq C t_n^{\frac{1}{2}}\left(\tau + h^2\right).$$

This, together with (1.27) and (1.32), implies the desired estimate. □

1.4.3 Proof of Theorems 1.4.3 and 1.4.4

For the proof, we use the inverse inequality that follows.

Lemma 1.4.8 (Inverse inequality)**.** Under condition (1.18),

$$\|v_h\|_{L^\infty(I)} \leq C_\star h^{-\frac{N}{2}}\|v_h\| \qquad (v_h \in S_h),$$

where $C_\star$ is a positive constant depending only on N and β.

Proof. Let $v_h \in S_h$ be arbitrary. From the norm equivalence in $\mathbb{R}^2$, we know that

$$\begin{aligned} &\|v_h\|_{L^\infty(I_1)} \leq C_{\star\star} h_1^{-1/2}\|v_h\|_{L^2(\frac{h_1}{2}, h_1)}, \\ &\|v_h\|_{L^\infty(I_j)} \leq C_{\star\star} h_j^{-1/2}\|v_h\|_{L^2(I_j)} \quad (j = 2, \ldots, m), \end{aligned}$$

where $C_{\star\star}$ denotes the positive constant. Given that $\|v_h\|_{L^\infty(I)} = \|v_h\|_{L^\infty(I_1)}$, the expression is calculable as

$$\begin{aligned} \|v_h\|_{L^\infty(I_1)}^2 &\leq C_{\star\star}^2 h_1^{-1}\int_{h_1/2}^{h_1} x^{-(N-1)} x^{N-1} v_h^2 \, dx \\ &\leq C_{\star\star}^2 h_1^{-1}\left(\frac{h_1}{2}\right)^{-(N-1)}\int_{h_1/2}^{h_1} x^{N-1} v_h^2 \, dx \\ &\leq C_{\star\star}^2 2^{N-1} h^{-N}\left(\frac{h_1}{h}\right)^{-N}\int_{h_1/2}^{h_1} x^{N-1} v_h^2 \, dx \\ &\leq C_\star^2 h^{-N}\|v_h\|^2. \end{aligned}$$

The case $\|v_h\|_{L^\infty(I)} = \|v_h\|_{L^\infty(I_j)}$ with $j = 2, \ldots, m$ is examined similarly. □

Proof of Theorem 1.4.3. Consider (1.1) and (Sym) with replacement $f(s)$ in

$$\tilde{f}(s) = \begin{cases} f(\mu) & (s \geq \mu) \\ f(s) & (-\mu \leq s \leq \mu) \\ f(-\mu) & (s \leq -\mu), \end{cases}$$

where $\mu > 0$ is determined later. Then, $\tilde{f}$ satisfies condition (f3) in Theorem 1.4.1 such that

$$\sup_{s,s' \in \mathbb{R}, s \neq s'} \frac{|\tilde{f}(s) - \tilde{f}(s')|}{|s - s'|} \leq M \equiv \sup_{|\lambda| \leq \mu} M_\lambda < \infty.$$

Let $\tilde{u}$ and $\tilde{u}_h^n$ be the solutions of (1.1) and (Sym) with $\tilde{f}$, respectively, such that

$$\|\tilde{u}_h^n\|_{L^\infty(I)} \leq \|\theta^n\|_{L^\infty(I)} + \|P_A \tilde{u}(t_n)\|_{L^\infty(I)},$$

where $\theta^n = \tilde{u}_h^n - P_A \tilde{u}(t_n)$ and $\rho^n = P_A \tilde{u}(t_n) - \tilde{u}(t_n)$. Applying Theorem 1.4.1 to $\tilde{u}$ and $\tilde{u}_h^n$, one obtains

$$\sup_{0 \leq t_n \leq T} \|\tilde{u}_h^n - \tilde{u}(\cdot, t_n)\| \leq C_2(h^2 + \tau), \tag{1.33}$$

where $C_2 = C_2(T, \kappa_0(\tilde{u}), \mu, C_0, N, \beta)$. Moreover, an estimate (1.31) for θ^n is available. In view of Lemmas 1.4.7 and 1.4.8, we determine those estimates as

$$\|\theta^n\|_{L^\infty(I)} \leq C_\star h^{-\frac{N}{2}} \|\theta^n\| \leq C_3 h^{-\frac{N}{2}}(h^2 + \tau),$$
$$\|\rho^n\|_{L^\infty(I)} \leq C_4 \left(h^2 \log \frac{1}{h}\right) \|\tilde{u}_{xx}(t_n)\|_{L^\infty(I)},$$

where $C_3 = C_3(T, \kappa_0(\tilde{u}), \mu, C_0, N, \beta)$ and $C_4 = C_4(N, \beta)$. Therefore, we have

$$\|P_A \tilde{u}(t_n)\|_{L^\infty(I)} \leq \|\tilde{u}(t_n)\|_{L^\infty(I)} + C_4 \left(h^2 \log \frac{1}{h}\right) \|\tilde{u}_{xx}(t_n)\|_{L^\infty(I)}$$

and

$$\|\tilde{u}_h^n\|_{L^\infty(I)} \leq C_3(h^{2-\frac{N}{2}} + h^{-\frac{N}{2}}\tau) + \|\tilde{u}(t_n)\|_{L^\infty(I)} + C_4 \left(h^2 \log \frac{1}{h}\right) \|\tilde{u}_{xx}(t_n)\|_{L^\infty(I)}.$$

At this stage, we set $\mu = 1 + \|u\|_{L^\infty(Q_T)}$ to obtain $u = \tilde{u}$ in Q_T by uniqueness. Moreover, because $N < 4$, we can take a very small h such that

$$C_3(h^{2-\frac{N}{2}} + h^{-\frac{N}{2}}\tau) \leq \frac{1}{2}, \quad C_4 \left(h^2 \log \frac{1}{h}\right) \|u_{xx}(t_n)\|_{L^\infty(I)} \leq \frac{1}{2}.$$

Consequently, $\|\tilde{u}_h^n\|_{L^\infty(I)} \leq \mu$. Also, by uniqueness $u_h^n = \tilde{u}_h^n$. Therefore, (1.33) implies the desired conclusion. □

Proof of Theorem 1.4.4. The proof follows the same pattern as that for Theorem 1.4.3, but using Theorem 1.4.2 instead of Theorem 1.4.1. □

1.4.4 Proof of Theorems 1.4.5 and 1.4.6

We use the projection operator P_B of $\dot{H}^1 \to S_h$ associated with $B(\cdot, \cdot)$:

$$B(P_B w - w, \chi) = 0 \qquad (\chi \in S_h). \tag{1.34}$$

In [18], the following error estimates are proved.

Lemma 1.4.9. Letting $w \in C^2(\bar{I}) \cap \dot{H}^1$ and (1.18) be satisfied, then for $h \leq h_7 = h_7(N, \beta)$ we obtain

$$\|P_B w - w\|_{L^\infty(I)} \leq C_7 h^2 \|w_{xx}\|_{L^\infty(I)}, \tag{1.35}$$

where $C_7 = C_7(N, \beta)$.

We also use a version of Poincaré's inequality (see [42, Lemma 18.1]).

Lemma 1.4.10. We have

$$|||w||| \leq |||w_x||| \qquad (w \in \dot{H}(I)). \tag{1.36}$$

We can now state the proof that follows.

Proof of Theorem 1.4.5. Using $P_B u(t) \in S_h$, we decompose the error into

$$u_h^n - u(t_n) = \underbrace{(u_h^n - P_B u(t_n))}_{=\theta^n} + \underbrace{(P_B u(t_n) - u(t_n))}_{=\rho^n}.$$

We know from (1.35) that

$$|||\rho^n||| \leq \|\rho^n\|_{L^\infty(I)} \leq Ch^2 \|u_{xx}\|_{L^\infty(Q_T)}, \tag{1.37a}$$

$$|||\partial_{\tau_n}\rho^{n+1}||| \leq \|\partial_{\tau_n}\rho^{n+1}\|_{L^\infty(I)} \leq Ch^2 \|u_{xxt}\|_{L^\infty(Q_T)}. \tag{1.37b}$$

Therefore, we will specifically examine estimation of $|||\theta_x^n|||$ because we are aware that

$$\|\chi\|_{L^\infty(I)} \leq \|\chi_x\|_{L^1(I)} \leq C\left(\log \frac{1}{h}\right)^{\frac{1}{2}} |||\chi_x||| \qquad (\chi \in S_h).$$

Furthermore, (1.5) and (1.9) give

$$\begin{aligned}\left\langle \partial_{\tau_n}\theta^{n+1} + \partial_{\tau_n}\rho^{n+1}, \chi\right\rangle + B(\theta^{n+1}, \chi) &= \langle f(u_h^n) - f(u(t_n)), \chi\rangle \\ &- \langle f(u(t_{n+1})) - f(u(t_n)), \chi\rangle - \langle \partial_{\tau_n} u(t_{n+1}) - u_t(t_{n+1}), \chi\rangle\end{aligned} \tag{1.38}$$

for $\chi \in S_h$. Substituting this for $\chi = \theta^{n+1}$, we have

$$\begin{aligned}&\left\langle \partial_{\tau_n}\theta^{n+1}, \theta^{n+1}\right\rangle + B(\theta^{n+1}, \theta^{n+1}) \\ &\quad = \left\langle f(u_h^n) - f(u(t_n)), \theta^{n+1}\right\rangle - \left\langle f(u(t_{n+1})) - f(u(t_n)), \theta^{n+1}\right\rangle \\ &\qquad - \left\langle \partial_{\tau_n} u(t_{n+1}) - u_t(t_{n+1}), \theta^{n+1}\right\rangle - \left\langle \partial_{\tau_n}\rho^{n+1}, \theta^{n+1}\right\rangle.\end{aligned} \tag{1.39}$$

This, together with (1.11), implies that

$$\begin{aligned}|||\theta_x^{n+1}|||^2 &\leq M|||u_h^n - u(t_n)||| \cdot |||\theta^{n+1}||| \\ &+ M|||u(t_{n+1}) - u(t_n)||| \cdot |||\theta^{n+1}||| + |||\partial_{\tau_n} u(t_{n+1}) - u_t(t_{n+1})||| \cdot |||\theta^{n+1}||| \\ &+ |||\partial_{\tau_n}\rho^{n+1}||| \cdot |||\theta^{n+1}||| + |||\partial_{\tau_n}\theta^{n+1}||| \cdot |||\theta^{n+1}|||.\end{aligned}$$

Therefore, using (1.36), we deduce that

$$\begin{aligned}|||\theta_x^{n+1}||| &\leq M|||u_h^n - u(t_n)||| + M|||u(t_{n+1}) - u(t_n)||| \\ &\quad + |||\partial_{\tau_n} u(t_{n+1}) - u_t(t_{n+1})||| + |||\partial_{\tau_n}\rho^{n+1}||| + |||\partial_{\tau_n}\theta^{n+1}||| \\ &\leq M(|||\theta^n||| + |||\rho^n|||) + M\tau_n \|u_t\|_{L^\infty(Q_T)} \\ &\quad + \tau_n \|u_{tt}\|_{L^\infty(Q_T)} + |||\partial_{\tau_n}\rho^{n+1}||| + |||\partial_{\tau_n}\theta^{n+1}|||.\end{aligned} \tag{1.40}$$

These estimates actually hold. Nevertheless, their proof is postponed for Section 1.6:

$$|||\theta^n||| \leq C(h^2 + \tau), \tag{1.41a}$$

$$|||\partial_{\tau_n}\theta^{n+1}||| \leq C\left(h^2 + \tau + \frac{\delta}{\tau}\right). \tag{1.41b}$$

Using (1.37a), (1.37b), (1.41a), and (1.41b), we deduce

$$|||\theta_x^{n+1}||| \leq C\left(h^2 + \tau + \frac{\delta}{\tau}\right),$$

which completes the proof of Theorem 1.4.5. ☐

Finally, we state the following proof.

Proof of Theorem 1.4.6. Consider problems (1.1) and (1.9) with replacement $f(s) = s|s|^\alpha$ by

$$\tilde{f}(s) = \begin{cases} s|s|^\alpha & (|s| \le \mu) \\ [(1+\alpha)\mu^\alpha s - \alpha\mu^{1+\alpha}]\,\mathrm{sgn}(s) & (|s| \ge \mu), \end{cases}$$

where $\mu > 0$ is determined later. Then, $\tilde{f}$ is a C^1 function and the corresponding values of $\tilde{M}_1$ and $\tilde{M}_2$ in (f4) are expressed as $\tilde{M}_1 = (1+\alpha)\mu^\alpha$ and $\tilde{M}_2 = (1+\alpha)\alpha\mu^{\alpha-1}$.

Let $\tilde{u}$ and $\tilde{u}_h^n$ respectively represent the solutions of (1.1) and (1.9) with $\tilde{f}$. If $\mu \ge \kappa_1(u)$, then $u = \tilde{u}$ holds true by uniqueness. Consequently, we can apply Theorem 1.4.5 to obtain

$$\|\tilde{u}_h^n - u(t_n)\|_{L^\infty(I)} \le C \left(\log \frac{1}{h}\right)^{\frac{1}{2}} (h^2 + \tau), \tag{1.42}$$

where $C = C(T, \kappa_1(u), \gamma, N, \beta)$. At this juncture, we apply small h and τ such that $C\left(\log \frac{1}{h}\right)^{\frac{1}{2}}(h^2+\tau) < 1$, and set $\mu = \kappa_1(u) + 1$. As $\|\tilde{u}_h^n\|_{L^\infty(I)} \le \kappa_1(u) + 1 = \mu$, we obtain $\tilde{u}_h^n = u_h^n$ by the uniqueness theorem. Therefore, (1.42) implies the desired estimate. □

1.5 Numerical examples

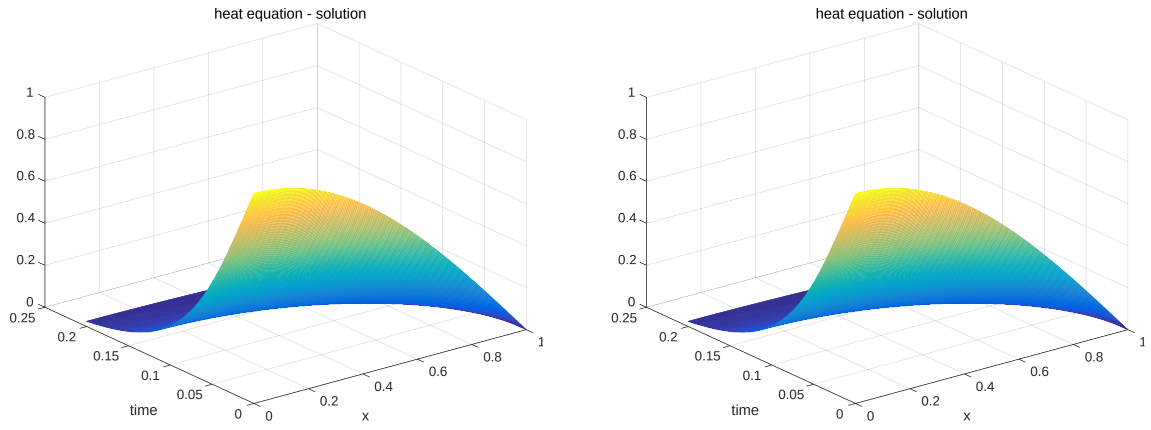
This section presents some numerical examples to validate our theoretical results. For this purpose, throughout this section, we set

$$f(s) = s|s|^\alpha, \ \alpha > 0.$$

If this were the case, then the solution of (1) might blow up in the finite time. Therefore, one must devote particular attention to setting of the time increment τ_n. Particularly, following Nakagawa [32] (see also Chen [8] and Cho–Hamada–Okamoto [14]), we use the time-increment control

$$\tau_n = \tau \cdot \min\left\{1, \ \frac{1}{\|u_h^n\|_2^\alpha}\right\} \qquad \left(\|u_h^n\|_2^2 = \sum_{j=0}^{m-1} h x_{j+1}^{N-1} u_h^n(x_j)^2\right), \tag{1.43}$$

where $\tau = \lambda h^2$ and $\lambda = 1/2$.

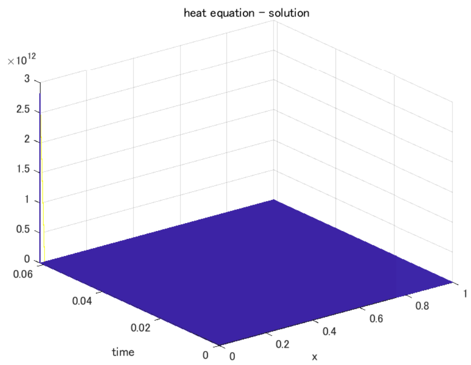


(a) (Sym) (b) (Non-Sym)

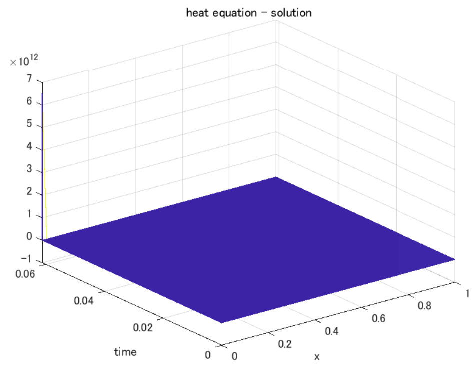
Figure 1.1: $N = 5$, $\alpha = \frac{4}{3}$ and $u(0,x) = \cos\frac{\pi}{2}x$.

First, we compared the shapes of both solutions of (Sym) and (Non-Sym), as shown in Fig. 1.1 for $N = 5$, $\alpha = \frac{4}{3}$ and $u(0,x) = \cos\frac{\pi}{2}x$. We used the uniform space mesh $x_j = jh$ $(j = 0, \ldots, m)$ and $h = 1/m$ with $m = 50$. For the choice of u_h^0, we used the linear interpolation of $u(0,x)$.
We computed them continuously until $t_n = T = 0.2$ or $\|u_h^n\|_2^{-1} < \epsilon = 10^{-8}$, wherein both solutions exist globally in time and approach 0 uniformly in $\overline{I}$ as $t \to \infty$. No marked differences were observed in

Figs. 1.1(a) and 1.1(b). Subsequently, we took Fig. 1.2 for the case in which the initial value was $u(0,x) = 13\cos\frac{\pi}{2}x$. The rest of the parameters are the same. At this point, the solutions of (Sym) and (Non-Sym) blew up after $x = 0.06$ with the distinct observation that the solution of the former blew up earlier than that of the latter. Furthermore, the solution of (Non-Sym) had negative values whereas that of (Sym) was always positive.



(a) (Sym)



(b) (Non-Sym)

Figure 1.2: $N = 5$, $\alpha = \frac{4}{3}$ and $u(0,x) = 13\cos\frac{\pi}{2}x$.

We examined the error estimates of the solutions for the same uniform space mesh $x_j = jh$ ($j = 0,\ldots,m$) and $h = 1/m$. Also, we regarded the numerical solution with $h' = 1/480$ as the exact solution. The following quantities were compared:

$$\begin{aligned}
&L^1\text{err} && \|u_{h'}^n - u_h^n\|_{L^1(I)};\\
&L^2\text{err} && \|u_{h'}^n - u_h^n\| = \left\|x^{\frac{N-1}{2}}(u_{h'}^n - u_h^n)\right\|_{L^2(I)};\\
&L^\infty\text{err} && \|u_{h'}^n - u_h^n\|_{L^\infty(I)}.
\end{aligned}$$

Fig. 1.3 presents results for $N = 3$, $\alpha = \frac{4}{3}$ and $u(0,x) = \cos\frac{\pi}{2}x$. We used the uniform time increment $\tau_n = \tau = \lambda h^2$ ($n = 0,1,\ldots$) with $\lambda = 1/2$ and computed until $t \le T = 0.005$. We took the linear interpolation of $u(0,x)$ as $u_h^0(x)$ in (Sym), and we took $P_B u(0,x)$ as $u_h^0(x)$ in (Non-Sym). For (Sym), we observed the theoretical convergence rate $h^2 + \tau$ in the $\|\cdot\|$ norm (see Theorem 1.4.3), whereas the rate in the L^∞ norm deteriorated slightly. For (Non-Sym), we observed second-order convergence in the L^∞ norm, which supports the results presented in Theorem 1.4.5.

Moreover, we considered the case for $N = 4$, which is not supported in Theorem 1.4.3 for (Sym). Also, we chose $\alpha = 4$ and $u(0,x) = 3\cos\frac{\pi}{2}x$ for this case. Fig. 1.4(d) displays the shape of the solution, which blew up at approximately $T = 0.0035$. Furthermore, we computed errors until $T = 0.0011, 0.0022$, and 0.0033 using the uniform meshes x_j and τ_n with $\lambda = 0.11$. From Fig. 1.4, we observed the second-order convergence in the $\|\cdot\|$ norm, suggesting the possibility of removing assumption $N \le 3$.

Finally, we observed the non-increasing property of the energy functional. The energy functional associated with (1) is given as

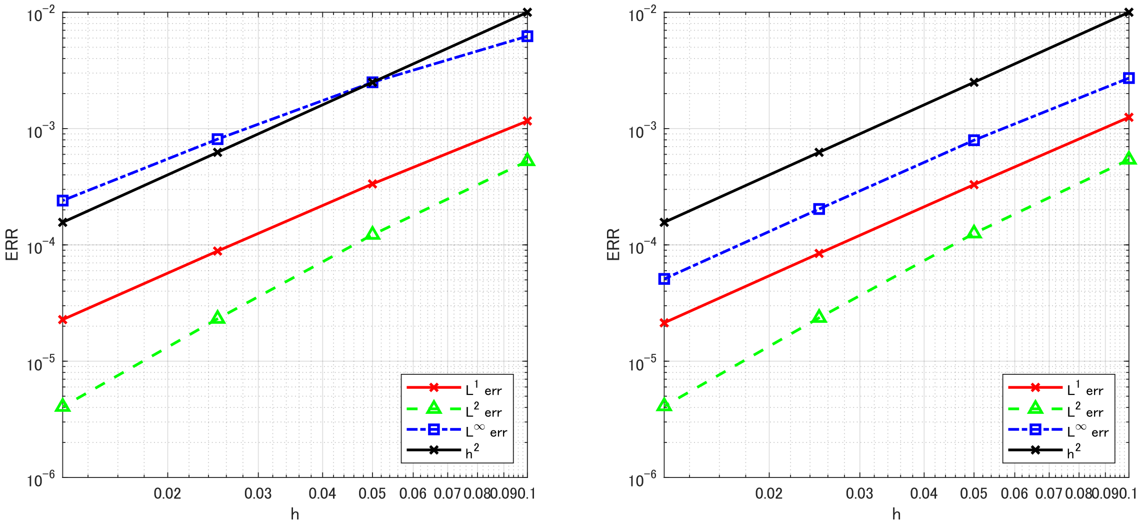
$$J(t) = \frac{1}{2}\|u_x\|^2 - \frac{1}{\alpha+2}\int_I x^{N-1}|u|^{\alpha+2}\,dx.$$

We can use the standard method to prove that $J(t)$ is non-increasing in t.

This non-increasing property plays an important role in the blow-up analysis of the solution of (1), as presented by Nakagawa [32]. Therefore, it is of interest whether a discrete version of this non-increasing property holds true. Actually, introducing the discrete energy functional associated with (Sym) as

$$J_h(n) = \frac{1}{2}\|(u_h^n)_x\|^2 - \frac{1}{\alpha+2}\int_I x^{N-1}|u_h^n|^{\alpha+2}\,dx,$$

we prove the following. Section 1.7 presents the proof.



(a) (Sym) (b) (Non-Sym)

Figure 1.3: Errors. $N=3$, $\alpha=\frac{4}{3}$ and $u(0,x)=\cos\frac{\pi}{2}x$.

Proposition 1.5.1. *$J_h(n)$ is a non-increase sequence of n.*

Now let $N=3$, $\alpha=\frac{4}{3}$, and $u(0,x)=\cos\frac{\pi}{2}x,\ 13\cos\frac{\pi}{2}x$. We determined the time increment τ_n through (1.43) for the uniform space mesh $x_j=jh$ with $h=1/m$ and $m=50$. We took the linear interpolation of $u(0,x)$ as u_h^0. Fig. 1.5 presents the results, which support that of Proposition 1.5.1.

1.6 Proofs of (1.41a) and (1.41b)

Proofs of (1.41a) and (1.41b) are stated in this section using the same notation as that used in Section 1.4.

Proof of (1.41a). By application of (1.39), (1.37a), and (1.37b), we derived the expression

$$\begin{aligned}\frac{1}{\tau_n}\left(|||\theta^{n+1}|||^2-|||\theta^{n+1}|||\cdot|||\theta^n|||\right)\le M(|||\theta^n|||+Ch^2\|u_{xx}\|_{L^\infty(Q_T)})\cdot|||\theta^{n+1}|||\\ +M\tau_n\|u_t\|_{L^\infty(Q_T)}\cdot|||\theta^{n+1}|||+\tau_n\|u_{tt}\|_{L^\infty(Q_T)}\cdot|||\theta^{n+1}|||\\ +Ch^2\|u_{xxt}\|_{L^\infty(Q_T)}\cdot|||\theta^{n+1}|||.\end{aligned}$$

Consequently, we have

$$|||\theta^{n+1}|||\le(1+\tau_nM)|||\theta^n|||+C\tau_n(h^2+\tau_n).$$

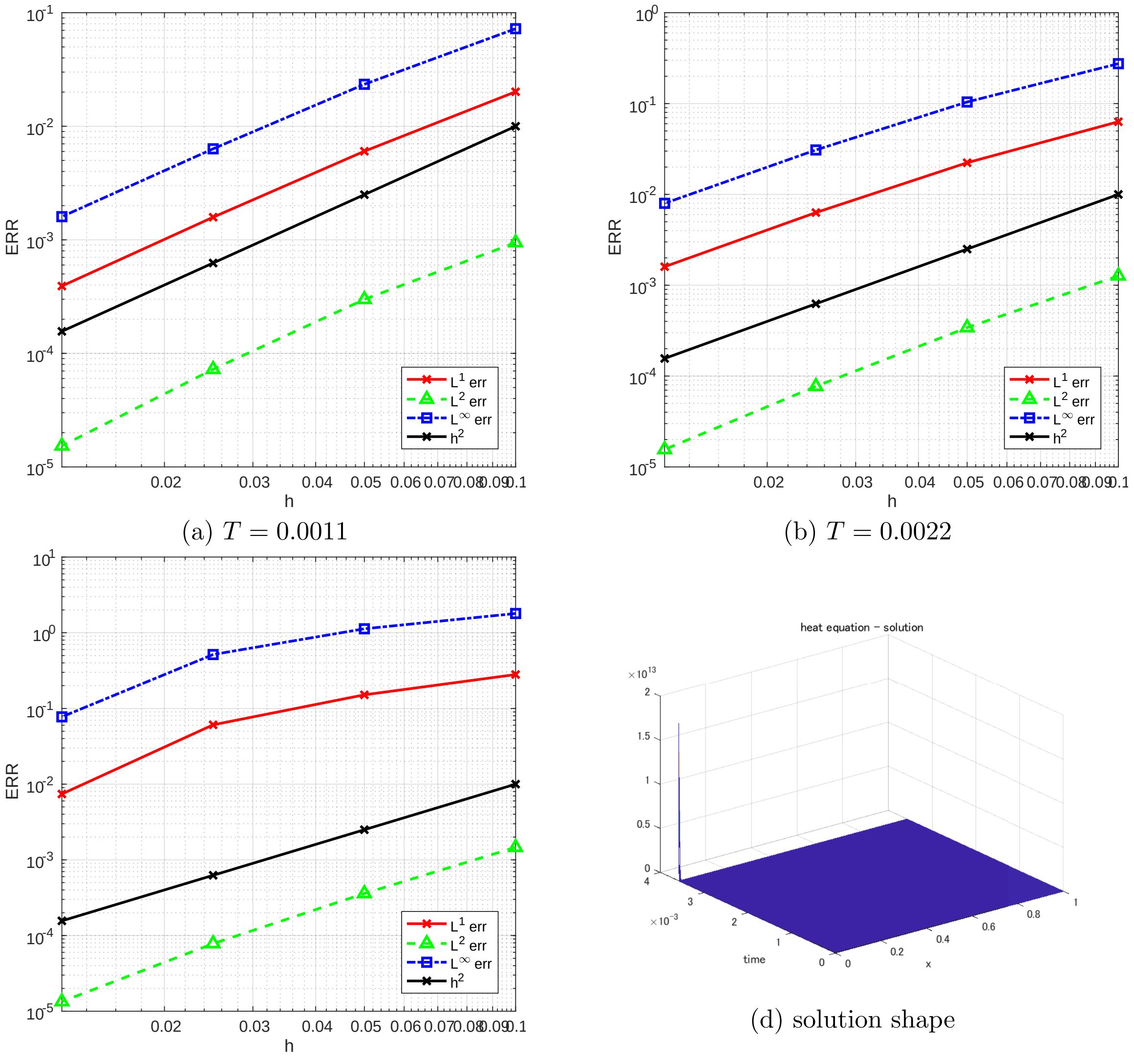
Therefore, similarly to the derivation of (1.31), we obtain from (1.22) the expression of

$$|||\theta^n|||\le C\left(h^2+\tau\right)$$

to complete the proof. □

Proof of (1.41b). First, we prove the case of $n=0$. Substituting (1.38) for $n=0$ and $\chi=\theta^1$, we obtain

$$\begin{aligned}\left\langle\frac{\theta^1-\theta^0}{\tau_0},\theta^1\right\rangle+B(\theta^1,\theta^1)\le\left\langle f(u_h^0)-f(u^0),\theta^1\right\rangle\\ -\left\langle f(u(t_1))-f(u^0),\theta^1\right\rangle-\left\langle\partial_{\tau_0}u(t_1)-u_t(t_1),\theta^1\right\rangle-\left\langle\frac{\rho^1-\rho^0}{\tau_0},\theta^1\right\rangle.\end{aligned}$$



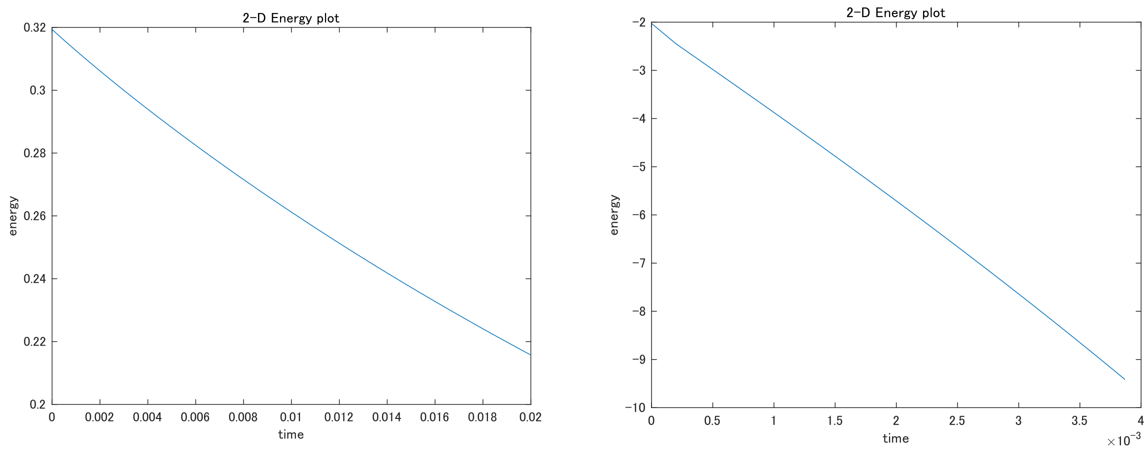
(a) $T = 0.0011$

(b) $T = 0.0022$

(c) $T = 0.0033$

(d) solution shape

Figure 1.4: Errors. $N = 4$, $\alpha = 4$ and $u(0, x) = 3\cos\frac{\pi}{2}x$.



(Sym) & $u(0, x) = \cos\frac{\pi}{2}x$

(Sym) & $u(0, x) = 13\cos\frac{\pi}{2}x$

Figure 1.5: Energy functional.

Because $\theta^0 = 0$, we apply (1.37b) to get

$$\begin{aligned}\frac{1}{\tau_0}|||\theta^1|||^2 &\le M|||\rho^0|||\cdot|||\theta^1||| + M\tau_0\|u_t\|_{L^\infty(Q_T)}|||\theta^1||| \\ &\quad + \tau_0\|u_{tt}\|_{L^\infty(Q_T)}|||\theta^1||| + |||\partial_{\tau_0}\rho^1|||\cdot|||\theta^1||| \\ &\le C(\tau_0 + h^2)|||\theta^1|||.\end{aligned}$$

Repeatedly using $\theta^0 = 0$, we obtain

$$|||\partial_{\tau_0}\theta^1||| \le C(\tau_0 + h^2). \tag{1.44}$$

Next we assume $n \ge 0$ and $t_{n+2} \le T$. Consequently, from (1.38), we derive

$$\begin{aligned}&\langle \partial_{\tau_{n+1}}\theta^{n+2} - \partial_{\tau_n}\theta^{n+1}, \chi\rangle + B(\theta^{n+2} - \theta^{n+1}, \chi) \\ &= \langle \underbrace{f(u_h^{n+1}) - f(u(t_{n+1})) - f(u_h^n) + f(u(t_n))}_{=J_1}, \chi\rangle \\ &\quad - \langle \underbrace{f(u(t_{n+2})) - f(u(t_{n+1})) - f(u(t_{n+1})) + f(u(t_n))}_{=J_2}, \chi\rangle \\ &\quad - \langle \underbrace{\partial_{\tau_{n+1}}u(t_{n+2}) - u_t(t_{n+2}) - \partial_{\tau_n}u(t_{n+1}) + u_t(t_{n+1})}_{=J_3}, \chi\rangle \\ &\quad - \langle \underbrace{\partial_{\tau_{n+1}}\rho^{n+2} - \partial_{\tau_n}\rho^{n+1}}_{=J_4}, \chi\rangle\end{aligned} \tag{1.45}$$

for any $\chi \in S_h$. Substituting this expression for $\chi = \partial_{\tau_{n+1}}\theta^{n+2}$, we obtain

$$|||\partial_{\tau_{n+1}}\theta^{n+2}|||^2 - |||\partial_{\tau_n}\theta^{n+1}|||\cdot|||\partial_{\tau_{n+1}}\theta^{n+2}||| \le |||\partial_{\tau_{n+1}}\theta^{n+2}|||\sum_{j=1}^{4}|||J_j|||.$$

Here, we accept the following estimates:

$$|||J_1||| \le C\tau_n(1+\tau_n)|||\partial_{\tau_n}\theta^{n+1}||| + C\tau_n(h^2 + \tau_n + \tau_n h^2), \tag{1.46a}$$

$$|||J_2|||, |||J_3||| \le C\tau_{n+1}(\tau_{n+1} + \tau_n) + C|\tau_{n+1} - \tau_n|, \tag{1.46b}$$

$$|||J_4||| \le C(\tau_{n+1} + \tau_n)h^2. \tag{1.46c}$$

In view of the quasi-uniformity of time partition (1.23), we have

$$\tau_{n+1} = \tau_n \frac{\tau_{n+1}}{\tau_n} \le \gamma\tau_n.$$

Summing up, we deduce

$$b_{n+1} - b_n \le C\tau_n b_n + C\tau_n\left(h^2 + \tau + \frac{\delta}{\tau_{\min}}\right), \tag{1.47}$$

where $b_n = |||\partial_{\tau_n}\theta^{n+1}|||$. Therefore,

$$b_n \le e^{CT}b_0 + C(e^{CT} - 1)\left(h^2 + \tau + \frac{\delta}{\tau_{\min}}\right),$$

which, together with (1.44), implies the desired inequality (1.41b).

We now prove (1.46a)–(1.46c).

Estimation for J_1. We apply Taylor's theorem to obtain

$$\begin{aligned}J_1 &= f'(s_1)(u_h^{n+1} - u_h^n) - f'(s_2)(u(t_{n+1}) - u(t_n)) \\ &= f'(s_1)[(\theta^{n+1} + \rho^{n+1}) - (\theta^n + \rho^n)] + \frac{f'(s_1) - f'(s_2)}{s_1 - s_2}(s_1 - s_2)(u(t_{n+1}) - u(t_n)),\end{aligned}$$

where $s_1 = u_h^{n+1} - \mu_1(u_h^{n+1} - u_h^n)$ and $s_2 = u(t_{n+1}) - \mu_2[u(t_{n+1}) - u(t_n)]$ for some $\mu_1, \mu_2 \in [0,1]$. In view of (1.37a), (1.37b), and (1.41a), we find the following estimates

$$\begin{aligned}
|||J_1||| &\le \tau_n M |||\partial_{\tau_n}\theta^{n+1}||| + \tau_n M |||\partial_{\tau_n}\rho^{n+1}||| + \left|\left|\left| \frac{f'(s_1) - f'(s_2)}{s_1 - s_2}(s_1 - s_2) \right|\right|\right| \cdot \tau_n \|u_t\|_{L^\infty(Q_T)}, \\
&\le \tau_n M |||\partial_{\tau_n}\theta^{n+1}||| + C\tau_n M h^2 \|u_{txx}\|_{L^\infty(Q_T)} + \left|\left|\left| \frac{f'(s_1) - f'(s_2)}{s_1 - s_2}(s_1 - s_2) \right|\right|\right| \cdot \tau_n \|u_t\|_{L^\infty(Q_T)},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left|\left| \frac{f'(s_1) - f'(s_2)}{s_1 - s_2}(s_1 - s_2) \right|\right|\right| \\
\le& M_2 |||\theta^{n+1} + \rho^{n+1} - \mu_1(\theta^{n+1} + \rho^{n+1} - \theta^n - \rho^n) + (\mu_2 - \mu_1)(u(t_{n+1}) - u(t_n))||| \\
\le& M_2\{|||\theta^{n+1}||| + |||\rho^{n+1}||| + \tau_n |||\partial_{\tau_n}\theta^{n+1}||| + \tau_n |||\partial_{\tau_n}\rho^{n+1}||| + \tau_n \|u_t\|_{L^\infty(Q_T)}\} \\
\le& M_2\{C(h^2 + \tau) + Ch^2\|u_{xx}\|_{L^\infty(Q_T)} + \tau_n |||\partial_{\tau_n}\theta^{n+1}||| + C\tau_n h^2 \|u_{txx}\|_{L^\infty(Q_T)} + \tau_n \|u_t\|_{L^\infty(Q_T)}\}.
\end{aligned}$$

Estimation for J_2. We begin with

$$\begin{aligned}
J_2 &= f'(s_3)(u(t_{n+2}) - u(t_{n+1})) - f'(s_4)(u(t_{n+1}) - u(t_n)) \\
&= \frac{f'(s_3) - f'(s_4)}{s_3 - s_4}(s_3 - s_4)\tau_{n+1}u_t(\eta_1) + f'(s_4)(\tau_{n+1}u_t(\eta_1) - \tau_n u_t(\eta_2)) \\
&= \frac{f'(s_3) - f'(s_4)}{s_3 - s_4}(s_3 - s_4)\tau_{n+1}u_t(\eta_1) \\
&\quad + f'(s_4)\tau_{n+1}(u_t(\eta_1) - u_t(\eta_2)) + f'(s_4)(\tau_{n+1} - \tau_n)u_t(\eta_2),
\end{aligned}$$

where $s_3 = u(t_{n+1}) + \mu_3(u(t_{n+2}) - u(t_{n+1}))$ and $s_4 = u(t_{n+1}) + \mu_4(u(t_n) - u(t_{n+1}))$ for some $\mu_3,\ \mu_4 \in [0,1]$, $\eta_1 \in [t_{n+1}, t_{n+2}]$, and $\eta_2 \in [t_n, t_{n+1}]$. Next, we obtain the following estimate:

$$\begin{aligned}
|||J_2||| \le& \tau_{n+1} \left|\left|\left| \frac{f'(s_3) - f'(s_4)}{s_3 - s_4}(s_3 - s_4) \right|\right|\right| \cdot \|u_t\|_{L^\infty(Q_T)} \\
&+ M_1 \tau_{n+1}(\tau_{n+1} + \tau_n)\|u_{tt}\|_{L^\infty(Q_T)} + M_1|\tau_{n+1} - \tau_n| \cdot \|u_t\|_{L^\infty(Q_T)};
\end{aligned}$$

$$\left|\left|\left| \frac{f'(s_3) - f'(s_4)}{s_3 - s_4}(s_3 - s_4) \right|\right|\right| \le CM_2(\tau_{n+1} + \tau_n)\|u_t\|_{L^\infty(Q_T)}.$$

Estimation for J_3. We express J_3 as

$$\begin{aligned}
J_3 &= \frac{\tau_{n+1}u_t(t_{n+2}) - \frac{1}{2}\tau_{n+1}^2 u_{tt}(s_5)}{\tau_{n+1}} - u_t(t_{n+2}) \\
&\quad - \left(\frac{\tau_n u_t(t_{n+1}) - \frac{1}{2}\tau_n^2 u_{tt}(s_6)}{\tau_n} - u_t(t_{n+1}) \right) \\
&= -\frac{1}{2}\tau_{n+1}u_{tt}(s_5) + \frac{1}{2}\tau_n u_{tt}(s_6) \\
&= \frac{1}{2}\tau_{n+1}(u_{tt}(s_6) - u_{tt}(s_5)) - \frac{1}{2}(\tau_{n+1} - \tau_n)u_{tt}(s_6) \\
&= \frac{1}{2}\tau_{n+1}u_{ttt}(s_7)(s_5 - s_6) - \frac{1}{2}(\tau_{n+1} - \tau_n)u_{tt}(s_6)
\end{aligned}$$

for some $s_5 \in [t_{n+1}, t_{n+2}]$, $s_6 \in [t_n, t_{n+1}]$ and $s_7 \in [s_6, s_5] \subset [t_n, t_{n+2}]$. Therefore,

$$|||J_3||| \le \frac{1}{2}\tau_{n+1}(\tau_{n+1} + \tau_n)\|u_{ttt}\|_{L^\infty(Q_T)} + \frac{1}{2}|\tau_{n+1} - \tau_n| \cdot \|u_{tt}\|_{L^\infty(Q_T)}.$$

Estimation for J_4. For some $s_8 \in [t_{n+1}, t_{n+2}]$, $s_9 \in [t_n, t_{n+1}]$, and $s_{10} \in [s_9, s_8]$, we obtain the expression

$$\frac{\rho^{n+2} - \rho^{n+1}}{\tau_{n+1}} - \frac{\rho^{n+1} - \rho^n}{\tau_n} = \rho_t(s_8) - \rho_t(s_9) = (s_8 - s_9)\rho_{tt}(s_{10})$$

Therefore, using (1.35),

$$|||J_4||| \le C(\tau_{n+1} + \tau_n)h^2 \|u_{ttxx}\|_{L^\infty(Q_T)}.$$

□

1.7 Proof of Proposition 1.5.1

Proof. Substituting $\chi = \partial_{\tau_n} u_h^{n+1}$ for (1.7), we have

$$\|\partial_{\tau_n} u_h^{n+1}\|^2 = -\left((u_h^{n+1})_x, \frac{(u_h^{n+1})_x - (u_h^n)_x}{\tau_n}\right) + \left(u_h^n|u_h^n|^\alpha, \frac{u_h^{n+1} - u_h^n}{\tau_n}\right).$$

Therefore, for the conditions

$$\left((u_h^{n+1})_x, \frac{(u_h^{n+1})_x - (u_h^n)_x}{\tau_n}\right) \ge \frac{1}{2}\left((u_h^n)_x + (u_h^{n+1})_x, \frac{(u_h^{n+1})_x - (u_h^n)_x}{\tau_n}\right), \tag{1.48a}$$

$$\left(u_h^n|u_h^n|^\alpha, \frac{u_h^{n+1} - u_h^n}{\tau_n}\right) \le \frac{1}{\tau_n(\alpha+2)}\left[\int_I x^{N-1}(|u_h^{n+1}|^{\alpha+2} - |u_h^n|^{\alpha+2})\ dx\right], \tag{1.48b}$$

we obtain

$$\left\|\partial_{\tau_n} u_h^{n+1}\right\|^2 \le -\frac{1}{\tau_n}(J_h(n+1) - J_h(n)), \tag{1.49}$$

which implies that $J_h(n+1) \le J_h(n)$.

We can validate (1.48a) and (1.48b). Also, (1.48a) is derived readily. To prove (1.48b), we set $g(s) = \frac{1}{\alpha+2}|s|^{\alpha+2}$, and apply the mean value theorem to deduce

$$g(u_h^{n+1}) - g(u_h^n) = w|w|^\alpha(u_h^{n+1} - u_h^n),$$

where $w = w(x) = u_h^n + \sigma(u_h^{n+1} - u_h^n)$ and $\sigma = \sigma(x) \in (0,1)$. Consequently,

$$\begin{aligned} J \equiv \frac{1}{\tau_n(\alpha+2)}\left[\int_I x^{N-1}(|u_h^{n+1}|^{\alpha+2} - |u_h^n|^{\alpha+2})\ dx\right] - \int_I x^{N-1}u_h^n|u_h^n|^\alpha \frac{u_h^{n+1} - u_h^n}{\tau_n}\ dx \\ = \frac{1}{\tau_n}\int_I x^{N-1}\left[w|w|^\alpha - u_h^n|u_h^n|^\alpha\right](u_h^{n+1} - u_h^n)\ dx. \end{aligned}$$

Then we repeat the mean value theorem to resolve

$$w|w|^\alpha - u_h^n|u_h^n|^\alpha = (\alpha+1)|\tilde{w}|^\alpha(w - u_h^n) = (\alpha+1)|\tilde{w}|^\alpha\tilde{\sigma}(u_h^{n+1} - u_h^n),$$

where $\tilde{w} = u_h^n + \tilde{\sigma}(w - u_h^n)$ and $\tilde{\sigma} = \tilde{\sigma}(x) \in (0,1)$. Therefore,

$$J = \frac{1}{\tau_n}\int_I x^{N-1}(\alpha+1)|\tilde{w}|^\alpha\tilde{\sigma}(u_h^{n+1} - u_h^n)^2\ dx \ge 0,$$

which gives (1.48b). □

1.8 Note

This chapter was taken directly from our previous paper, Nakanishi-Saito [34].

Chapter 2

A mass-lumping finite element method

2.1 Introduction

This chapter applies the finite element method (FEM) to a semilinear parabolic equation with a singular convection term:

$$u_t = u_{xx} + \frac{N-1}{x}u_x + f(u), \qquad x \in I = (0,1),\ t > 0, \tag{2.1a}$$

$$u_x(0,t) = u(1,t) = 0, \qquad t > 0, \tag{2.1b}$$

$$u(x,0) = u^0(x), \qquad x \in I. \tag{2.1c}$$

Here, $u = u(x,t)$, $x \in \overline{I} = [0,1], t \geq 0$ denotes the function to be found, f is a given locally Lipschitz continuous function, and u^0 is a given continuous function. Throughout this chapter, we assume that

$$N \text{ is an integer} \geq 2. \tag{2.2}$$

To compute the blow-up solution of (2.1), we apply Nakagawa's time-increment control strategy (see [32] and Section 2.6 of the present chapter), a powerful technique for approximating blow-up times. As recalled below, the standard finite element approximation is unuseful for achieving this purpose. We thus propose a special mass-lumping finite element approximation, prove its convergence, and apply it to a blow-up analysis.

We first clarify the motivation of this chapter. In many engineering problems, the spatial dimension of a mathematical model is at most three. Solving partial differential equations (PDEs) in more than three spatial dimensions is usually motivated by mathematical interests. Mathematicians understand that solving problems in a general setting can reveal the hidden natures of PDEs. One successful result is the discovery of Fujita's blow-up exponent for the semilinear heat equation of $U = U(\boldsymbol{x}, t)$ given as

$$U_t = \Delta U + f(U) \qquad (\boldsymbol{x} \in \mathbb{R}^N,\ t > 0), \tag{2.3}$$

where N and $f(U)$ are defined above. Assuming $f(U) = U|U|^\alpha$ with $\alpha > 0$, Fujita showed that any positive solution blows up in finite time if $1 + \alpha < 1 + 2/N$, but a solution remains smooth at any time if the initial value is small and $1 + \alpha > 1 + 2/N$. The quantity $p_c = 1 + 2/N$ is known as Fujita's critical exponent, and Eq.(2.3) is called Fujita's equation. Since Fujita's work, a huge number of studies have been devoted to critical phenomena in nonlinear PDEs of several kinds (see [16, 31, 38] for details). The knowledge gained by these studies has been applied to problems with spatial dimensions of three or fewer. However, many problems related to stochastic analysis are formulated as higher-dimensional PDEs. These problems have attracted much interest, but are beyond the scope of the present chapter.

Non-stationary problems in four dimensional space are difficult to solve by numerical methods, even on modern computers. Consequently, numerical analyses of the blow-up solutions of nonlinear PDEs have been restricted to two-dimensional space (see for example [1, 4, 5, 6, 10, 12, 13, 19, 23, 25, 24, 26, 28, 33, 36]). Although Nakagawa's time-increment control strategy is applied to various nonlinear PDEs including the nonlinear heat, wave and Schrödinger equations, these equations are considered only in the one-space dimension; see [7, 9, 10, 14, 26, 39, 40]. We know two notably exceptions; the one is [33] where the finite element method to a semilinear heat equation in a two dimensional polygonal domain was considered, and the other is [8] where the finite difference method to the radially symmetric solution of the semilinear heat equation in an N dimensional ball was studied.

Following [8], the present chapter investigates radially symmetric solutions to Eq.(2.3). Assuming radial symmetry of the solution and the given data, the N-dimensional equation reduces to a one-dimensional equation. More specifically, considering (2.3) in an N-dimensional unit ball $\mathrm{B} = \{\boldsymbol{x} \in \mathbb{R}^N \mid |\boldsymbol{x}|_{\mathbb{R}^N} < 1\}$ with the homogeneous Dirichlet boundary condition on the boundary and assuming U is expressed as $u(x) = U(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathrm{B}$ and $x = |\boldsymbol{x}|_{\mathbb{R}^N}$, we came to consider the problem (2.1).

After completing the present chapter, we learned that Cho and Okamoto [15] extended the work in [8]. The time dimension was discretized by the semi-implicit Euler method in [8], but Cho and Okamoto [15] explored the explicit scheme, then proved optimal-order convergence with Nakagawa's strategy. Because their schemes use special approximations around the origin to maintain some analytical properties of the solution, they should be performed on a uniform spatial mesh. Conversely, when seeking the blow up solution, non-uniform partitions of the space variable are useful for examining highly concentrated solutions at the origin. For this purpose, we developed the FEM scheme.

FEM analyses of the linear case, in which $f(u) = 0$ in Eq.(2.1) is replaced by a given function $f(x,t)$, are not new. Eriksson and Thomée [18] and Thomée [42] studied the convergence property of the elliptic equation, and proposed two schemes: the symmetric scheme, in which the optimal-order error is estimated in the weighted L^2 norm, and the nonsymmetric scheme, in which the L^∞ error is estimated. However, their finite element schemes are not easily adaptable to the semilinear heat equation (2.1), as reported in our earlier study [34]. Our earlier results are briefly summarized below:

- If f is *globally* Lipschitz continuous, the solution of the symmetric scheme converges to the solution of (2.1) in the weighted L^2 norm in space and in the L^∞ norm in time. Moreover, the convergence is at the optimal order (see Theorem 4.1 in [34]).
- If f is *locally* Lipschitz continuous and $N \leq 3$, the same conclusion holds (see Theorem 4.3 in [34]). However, if $N \geq 4$, the convergence properties are not guaranteed. For this reason, interest in radially symmetric problems has diminished.
- If $f(u) = u|u|^\alpha$ with $\alpha \geq 1$ and the time partition is uniform, the solution of the non-symmetric scheme converges to the solution of (2.1) in the $L^\infty(0,T;L^\infty(I))$ norm. Optimal-order convergence holds up to the logarithmic factor (see Theorem 4.6 in [34]). Nakagawa's time-increment control strategy is difficult to apply in such cases.

As the non-symmetric scheme seems to be incompatible with Nakagwa's time-increment control strategy, we pose the following question: Can the restriction $N \leq 3$ be removed from the symmetric scheme? In fact, this restriction is imposed by the inverse inequality Lemma 4.8 in [34] and the necessity of finding the boundedness of the finite element solution (see the proof of Theorem 4.3 in [34]). To surmount this difficulty, the L^∞ estimates for the FEM can be directly derived using the discrete maximum principle (DMP). As the DMP is based largely on the nonnegativity of the finite element solution, the time derivative term should be approximated by the mass-lumping approximation. Unfortunately, we tried but failed to prove the convergence property of the finite element solution by this approximation (see (2.8) below). Therefore, we propose a special mass-lumping approximation (2.9) in this chapter. Using the special approximation, we prove the DMP and the convergence property of the finite element solution, and perform the blow up analysis for any $N \geq 2$.

Our typical results are summarized below. Here, our schemes are denoted as (ML–1) and (ML–2).

- The solution of (ML–1) is non-negative if f and u^0 satisfy some conditions (Theorem 2.2.2). Furthermore, if the time increment satisfies condition (2.12), the solution of (ML–2) is also non-negative (Theorem 2.2.2). Theorem 2.2.3 gives a useful sufficient condition of (2.12).
- The solution of (ML–1) converges to the solution of (2.1) in the weighted L^2 norm in space and in the L^∞ norm in time. Moreover, the convergence is at the optimal order (Theorem 2.2.4). The proof is based on a sub-optimal estimate in the $L^\infty(0,T;L^\infty(I))$ norm (Theorem 2.2.5).
- If condition (2.12) is satisfied, then the solution of (ML–2) converges to the solution of (2.1) in the $L^\infty(0,T;L^\infty(I))$ norm (Theorem 2.2.6). Unfortunately, the order of the convergence is sub-optimal.

- The solution of (ML–2) reproduces the blow up property of the solution of (2.1) (Theorems 2.5.6 and 2.5.7).

This chapter comprises seven sections. Section 2.2 presents our finite element schemes and the convergence theorems (Theorems 2.2.4–2.2.6). After describing our preliminary results in Section 2.3, we prove our convergence theorems in Section 2.4. Section 2.5 reports the results of our blow-up analysis, and Section 2.6 validates our theoretical results with numerical examples. Section 2.7 presents the proofs of some auxiliary results on the eigenvalue problems.

2.2 The schemes and their convergence results

Throughout this chapter, f is assumed as a locally Lipschitz continuous function of $\mathbb{R} \to \mathbb{R}$.

For some arbitrary $\chi \in \dot{H}^1 = \{v \in H^1(I) \mid v(1) = 0\}$, we multiply both sides of (2.1a) by $x^{N-1}\chi$ and integrate by parts over I. We thus obtain

$$\int_I x^{N-1} u_t \chi \, dx + \int_I x^{N-1} u_x \chi_x \, dx = \int_I x^{N-1} f(u) \chi \, dx. \tag{2.4}$$

Therefore a weak formulation of (2.1) is stated as follows. For $t > 0$, find $u(\cdot, t) \in \dot{H}^1$ such that

$$(u_t, \chi) + A(u, \chi) = (f(u), \chi) \quad (\forall \chi \in \dot{H}^1), \tag{2.5}$$

where

$$(w, v) = \int_I x^{N-1} wv \, dx, \quad A(w, v) = \int_I x^{N-1} w_x v_x \, dx. \tag{2.6}$$

We now introduce the finite element method. For a positive integer m, we introduce node points

$$0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < \cdots < x_{m-1} < x_m = 1,$$

and set $I_j = (x_{j-1}, x_j)$ and $h_j = x_j - x_{j-1}$, where $j = 1, \ldots, m$. The granularity parameter is defined as $h = \max_{1 \le j \le m} h_j$. Let $\mathcal{P}_k(J)$ be the set of all polynomials in an interval J of degree $\le k$. The $\mathcal{P}_1$ finite element space is defined as

$$S_h = \{v \in H^1(I) \mid v \in \mathcal{P}_1(I_j) \ (j = 1, \cdots, m), \ v(1) = 0\}. \tag{2.7}$$

The standard basis function $\phi_j \in S_h$, $j = 0, 1, \cdots, m-1$ is defined as

$$\phi_j(x_i) = \delta_{ij},$$

where δ_{ij} denotes Kronecker's delta. We note that $S_h \subset \dot{H}^1$ and that any function of $\dot{H}^1$ is identified with a continuous function. The Lagrange interpolation operator Π_h of $\dot{H}^1 \to S_h$ is defined as $\Pi_h w = \sum_{j=0}^{m-1} w(x_j)\phi_j$ for $w \in \dot{H}^1$.

The mass-lumping approximation of the weighted L^2 norm $(\cdot, \cdot)$ can be naturally defined as

$$(w, v) \approx w(x_0)v(x_0) \int_0^{x_{1/2}} x^{N-1} \, dx + \sum_{i=1}^{m-1} w(x_i)v(x_i) \int_{x_{i-1/2}}^{x_{i+1/2}} x^{N-1} \, dx, \tag{2.8}$$

where $x_{i-1/2} = (x_i + x_{i-1})/2$. As mentioned in the Introduction, this standard formulation is useless for our purpose. Instead, we define

$$\langle w, v \rangle = \sum_{i=0}^{m-1} w(x_i)v(x_i)(1, \phi_i) \qquad (w, v \in \dot{H}^1). \tag{2.9}$$

This definition leads to the following result, which can be verified by direct calculation.

Lemma 2.2.1. We have $\langle 1, w \rangle = (1, \Pi_h w)$ for any $w \in \dot{H}^1$.

In [37], the formulation similar to (2.9) was introduced. [37] considered two-spatial dimensional parabolic problems with degenerate coefficients.

The associated norms with $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are respectively given by

$$\|v\| = (v,v)^{1/2} \quad \text{and} \quad |||v||| = \langle v,v\rangle^{1/2}.$$

These norms are equivalent in S_h as mentioned in Lemma 2.3.4.

The time discretization is non-uniformly partitioned as

$$t_0 = 0, \qquad t_n = \sum_{j=0}^{n-1} \tau_j \quad (n \geq 1),$$

where $\tau_j > 0$ denotes the time increment. Furthermore, we set

$$\tau = \sup_{j\geq 0} \tau_j.$$

In general, we write $\partial_{\tau_n} u_h^{n+1} = (u_h^{n+1} - u_h^n)/\tau_n$.

The finite element schemes are then stated as follows.

(ML–1) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\left\langle \partial_{\tau_n} u_h^{n+1}, \chi \right\rangle + A(u_h^{n+1}, \chi) = (f(u_h^n), \chi) \quad (\chi \in S_h), \tag{2.10}$$

where $u_h^0 \in S_h$ is assumed to be given.

(ML–2) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\left\langle \partial_{\tau_n} u_h^{n+1}, \chi \right\rangle + A(u_h^n, \chi) = \langle f(u_h^n), \chi \rangle \quad (\chi \in S_h). \tag{2.11}$$

Below, we will show the optimal order error estimate in the weighted L^2 norm for the solution of (ML–1). On the other hand, we are able to show only a sub-optimal order error estimate in the L^∞ norm for the solution of (ML–2). Nevertheless, we consider (ML–2) because it is suitable for the blow-up analysis (see Section 2.5).

We also summarize the well-posedness of our schemes. The proof is omitted because it is identical to Theorems 3.1 and 3.2 in [34].

Theorem 2.2.2. Suppose that $n \geq 0$ and $u_h^n \in S_h$ are given.

(i) Schemes (ML–1) and (ML–2) admit unique solutions $u_h^{n+1} \in S_h$.

(ii) In addition to the basic assumption on f, assume that f is a non-decreasing function with $f(0) \geq 0$. If $u_h^n \geq 0$, then the solution u_h^{n+1} of (ML–1) satisfies $u_h^{n+1} \geq 0$.

(iii) Under the assumptions of (ii) above, further assume that

$$\tau \leq \min_{0\leq i\leq m-1} \frac{(1,\phi_i)}{A(\phi_i,\phi_i)}. \tag{2.12}$$

Then the solution u_h^{n+1} of (ML–2) satisfies $u_h^{n+1} \geq 0$.

To provide a useful sufficient condition under which (2.12) holds, we assume that the partition $\{x_i\}_{j=0}^m$ of $\bar{I} = [0,1]$ is quasi-uniform, that is,

$$h \leq \beta \min_{1\leq j\leq m} h_j, \tag{2.13}$$

where β is a positive constant independent of h.

Theorem 2.2.3. Inequality (2.12) holds if

$$\tau \leq \frac{\beta^2}{N+1} h^2. \tag{2.14}$$

Proof. A direct calculation gives

$$\frac{(1,\phi_i)}{A(\phi_i,\phi_i)} \geq \frac{1}{N+1}h_i^2 \geq \frac{\beta^2}{N+1}h^2$$

for any i. Therefore, (2.14) implies (2.12). □

We now proceed to the convergence analysis. Our results for (ML–1) and (ML–2) assume a smooth solution u of (2.1): given $T>0$ and setting $Q_T=[0,1]\times[0,T]$, we assume that u is sufficiently smooth such that

$$\kappa(u)=\sum_{k=0}^{2}\|\partial_x^k u\|_{L^\infty(Q_T)}+\sum_{l=1}^{2}\|\partial_t^l u\|_{L^\infty(Q_T)}+\sum_{k=1}^{2}\|\partial_t\partial_x^k u\|_{L^\infty(Q_T)}<\infty. \tag{2.15}$$

Here, we have used the conventional $\|v\|_{L^\infty(\omega)}=\max_{\overline{\omega}}|v|$ for a continuous function v defined in a bounded set ω in $\mathbb{R}^p$, $p\geq 1$.

Moreover, the approximate initial value u_h^0 is chosen as

$$\|u_h^0-u^0\|_{L^\infty(I)}\leq C_0h^2 \tag{2.16}$$

for a positive constant C_0.

We now express positive constants $C=C(\gamma_1,\gamma_2,\ldots)$ depending only on the parameters $\gamma_1,\gamma_2,\ldots$. Particularly, C is independent of h and τ.

Theorem 2.2.4 (Optimal L^2 error estimate for (ML-1))**.** Assume that, for $T>0$, the solution u of (2.1) is sufficiently smooth that (2.15) holds. Moreover, assume that (2.13) and (2.16) are satisfied. Then, for sufficiently small h and τ, we have

$$\sup_{0\leq t_n\leq T}\|u_h^n-u(\cdot,t_n)\|\leq C(h^2+\tau), \tag{2.17}$$

where $C=C(T,f,\kappa(u),C_0,N,\beta)$ and u_h^n is the solution of (ML–1).

The following result, which is worth a separate mention, gives only a sub-optimal error estimate but is useful for proving Theorem 2.2.4.

Theorem 2.2.5 (Sub-optimal L^∞ error estimate for (ML–1))**.** Under the assumptions of Theorem 2.2.4 and for sufficiently small h and τ, we have

$$\sup_{0\leq t_n\leq T}\|u_h^n-u(\cdot,t_n)\|_{L^\infty(I)}\leq C\left(h+\tau\right), \tag{2.18}$$

where $C=C(T,f,\kappa(u),C_0,N,\beta)$ and u_h^n is the solution of (ML–1).

Theorem 2.2.6 (Sub-optimal L^∞ error estimate for (ML–2))**.** Also under the assumptions of Theorem 2.2.4, assume that (2.12) is satisfied. Then, for sufficiently small h and τ, we have

$$\sup_{0\leq t_n\leq T}\|u_h^n-u(\cdot,t_n)\|_{L^\infty(I)}\leq C\left(h+\tau\right), \tag{2.19}$$

where $C=C(T,f,\kappa(u),C_0,N,\beta)$ and u_h^n is the solution of (ML–2).

Remark 2.2.7. Other schemes based on the mass-lumping $\langle\cdot,\cdot\rangle$ are possible. For example, the scheme

$$\left\langle\partial_{\tau_n}u_h^{n+1},\chi\right\rangle+A(u_h^{n+1},\chi)=\langle f(u_h^n),\chi\rangle\quad(\chi\in S_h) \tag{2.20}$$

and

$$\left\langle\partial_{\tau_n}u_h^{n+1},\chi\right\rangle+A(u_h^{n},\chi)=(f(u_h^n),\chi)\quad(\chi\in S_h) \tag{2.21}$$

have very similar properties to those of (ML–1) and (ML–2). We omit the details because the modifications are easily performed.

2.3 Preliminaries

This section gives some preliminary results of the theorem proofs. The quasi-uniformity condition (2.13) is always assumed.

For some $w \in \dot{H}^1$, the projection operator P_A of $\dot{H}^1 \to S_h$ associated with $A(\cdot,\cdot)$ is defined as

$$P_A w \in S_h, \quad A(P_A w - w, \chi) = 0 \qquad (\chi \in S_h). \tag{2.22}$$

The following error estimates are proved in [18] and [29].

Lemma 2.3.1. Letting $w \in C^2(\bar{I}) \cap \dot{H}^1$, and h be sufficient small, we obtain

$$\|P_A w - w\| \le Ch^2 \|w_{xx}\|, \tag{2.23a}$$

$$\|P_A w - w\|_{L^\infty(I)} \le C\left(\log \frac{1}{h}\right) h^2 \|w_{xx}\|_{L^\infty(I)}, \tag{2.23b}$$

where $C = (\beta, N) > 0$.

Lemma 2.3.2 (Inverse estimate)**.** There exists a constant $C = C(\beta, N) > 0$ such that

$$\|w_x\| \le Ch^{-1}\|w\| \qquad (w \in S_h).$$

Proof. The proof is identical to that of the standard inverse estimate. □

Lemma 2.3.3. Let $w \in C(\bar{I})$ be a piecewise quadratic function in I, that is, $w|_{I_j} \in \mathcal{P}_2(I_j)$ $(j = 1, \cdots, m)$. Then, we have

$$\int_I x^{N-1}|\Pi_h w - w| \; dx \le Ch^2 \|x^{N-1} w_{xx}\|_{L^1(I)}, \tag{2.24}$$

where $C = C(\beta, N) > 0$.

Proof. To prove (2.24), it suffices to replace $I = I_j$ with $j = 1, \dots, m$. First, let $j \ge 2$. By Taylor's theorem, we can write

$$|\Pi_h w(x) - w(x)| \le Ch_j \int_{I_j} |w_{xx}(\xi)| \; d\xi \quad (x \in I_j). \tag{2.25}$$

Referring to (2.13), we see that

$$\frac{x}{x_{j-1}} = 1 + \frac{x - x_{j-1}}{x_{j-1}} \le 1 + \frac{h_j}{h_{j-1}} \le 1 + \beta \quad (x \in \bar{I}_j). \tag{2.26}$$

Combining (2.25) and (2.26), we deduce that

$$x^{N-1}|\Pi_h w(x) - w(x)| \le C(1+\beta)^{N-1} h_j \int_{I_j} x^{N-1}|w_{xx}(\xi)| \; d\xi \quad (x \in I_j).$$

Integrating both sides, we obtain (2.24) for $I = I_j$ and $j \ge 2$. We now proceed to the case $m = 1$. Setting $w(x) = ax^2 + bx + c$ for $x \in I_1$, where a, b and c are constants, we express $\Pi_h w(x) - w(x) = ax(h_1 - x)$ for $x \in I_1$. Therefore, we directly obtain

$$\int_{I_1} x^{N-1}|\Pi_h w(x) - w(x)| \; dx = \frac{|a|}{(N+1)(N+2)} h_1^{N+2},$$
$$h_1^2 \|x^{N-1} w_{xx}\|_{L^1(I_1)} = \frac{2}{N}|a| h_1^{N+2},$$

which implies (2.24) for $I = I_1$. □

Lemma 2.3.4. There exist constants $C = C(\beta, N)$ and $C' = C'(\beta, N)$ such that

$$C' |||w||| \le \|w\| \le C |||w||| \qquad (w \in S_h).$$

Proof. Let $w \in S_h$. To prove the first inequality, we note that w^2 is a piecewise quadratic function and $(w^2)_{xx} = 2(w_x)^2$. By Lemmas 2.3.3 and 2.3.2, we get

$$\begin{aligned} |\langle w, w\rangle - (w,w)| &\le Ch^2 \|x^{N-1}(w^2)_{xx}\|_{L^1(I)} \\ &\le Ch^2\|w_x\|^2 \\ &\le C\|w\|^2, \end{aligned}$$

which implies the first inequality.

To prove the second inequality, we estimate $\|w\|$ as

$$\begin{aligned} \|w\|^2 &\le \sum_{j=1}^{m} \left[w(x_{j-1})^2 \int_{x_{j-1}}^{x_j} x^{N-1} dx + w(x_j)^2 \int_{x_{j-1}}^{x_j} x^{N-1} dx \right] \\ &= w(x_0)^2 \int_{x_0}^{x_1} x^{N-1} dx + \sum_{j=1}^{m-1} w(x_j)^2 \int_{x_{j-1}}^{x_{j+1}} x^{N-1} dx. \end{aligned}$$

We also express $|||w|||$ as

$$|||w|||^2 = w(x_0)^2 \int_{x_0}^{x_1} x^{N-1}\phi_0(x)\ dx + \sum_{j=1}^{m-1} w(x_j)^2 \int_{x_{j-1}}^{x_{j+1}} x^{N-1}\phi_j(x)\ dx.$$

Therefore, it suffices to show that

$$\int_{x_0}^{x_1} x^{N-1} dx \le C_1 \int_{x_0}^{x_1} x^{N-1}\phi_0\ dx, \tag{2.27a}$$

$$\int_{x_{j-1}}^{x_{j+1}} x^{N-1} dx \le C_2 \int_{x_{j-1}}^{x_{j+1}} x^{N-1}\phi_j\ dx \quad (j = 1, \ldots, m-1), \tag{2.27b}$$

where $C_1 = C_1(N) > 0$ and $C_2 = C_2(N) > 0$.

Equation (2.27b) is directly verified using (2.26). Equation (2.27a) is obtained by the change-of-variables technique, setting $\xi = x/h_1$. □

We here introduce two auxiliary problems. Given $n \ge 0$, $g_h^n \in S_h$ and $u_h^n \in S_h$, we seek $u_h^{n+1} \in S_h$ such that

$$\left\langle \partial_{\tau_n} u_h^{n+1}, \chi \right\rangle + A(u_h^{n+1}, \chi) = \langle g_h^n, \chi\rangle \qquad (\chi \in S_h), \tag{2.28}$$

and

$$\left\langle \partial_{\tau_n} u_h^{n+1}, \chi \right\rangle + A(u_h^{n}, \chi) = \langle g_h^n, \chi\rangle \qquad (\chi \in S_h). \tag{2.29}$$

Lemma 2.3.5. Suppose that $n \ge 0$ and $u_h^n,\ g_h^n \in S_h$ are given. Then, problem (2.28) admits a unique solution $u_h^{n+1} \in S_h$ and it satisfies

$$\|u_h^{n+1}\|_{L^\infty(I)} \le \|u_h^n\|_{L^\infty(I)} + \tau_n \|g_h^n\|_{L^\infty(I)}. \tag{2.30}$$

Problem (2.29) also admits a unique solution $u_h^{n+1} \in S_h$ that satisfies (2.30) under condition (2.12).

Proof. The unique existence of the solution of (2.28) can be verified by a standard approach (see Theorems 3.1, 3.2 in [34]). Substituting $\chi = \phi_i$, $i = 0, \ldots, m-1$, in (2.28), we have

$$\frac{\tau_n a_{i,i-1}}{m_i} u_{i-1}^{n+1} + \left(1 + \frac{\tau_n a_{i,i}}{m_i}\right) u_i^{n+1} + \frac{\tau_n a_{i,i+1}}{m_i} u_{i+1}^{n+1} = u_i^n + \tau_n g_i^n,$$

where $u_i^n = u_h^n(x_i)$, $g_i^n = g_h^n(x_i)$, $a_{i,j} = A(\phi_j, \phi_i)$ and $m_i = (1, \phi_i)$. Therein, we should understand that $a_{0,-1} = 0$, $m_0 = 1$ and $u_{-1}^{n+1} = 1$. Moreover, substituting $\chi = \phi_i$ in (2.29), we get

$$u_i^{n+1} = -\frac{\tau_n a_{i,i-1}}{m_i} u_{i-1}^n + \left(1 - \frac{\tau_n a_{i,i}}{m_i}\right) u_i^n - \frac{\tau_n a_{i,i+1}}{m_i} u_{i+1}^n + \tau_n g_i^n.$$

From these expressions, (2.30) is deduced by a standard argument. □

2.4 Proofs of Theorems 2.2.4, 2.2.5 and 2.2.6

Proof of Theorem 2.2.4 using Theorem 2.2.5. This proof is divided into the following two steps:
Step 1. We prove Theorem 2.2.4 under an additional assumption: f is a globally Lipschitz function. That is, we assume

$$M = \sup_{\substack{s,s'\in\mathbb{R}\\ s\neq s'}} \frac{|f(s)-f(s')|}{|s-s'|} < \infty. \tag{2.31}$$

Using $P_A u$, we divide the error into the form

$$u_h^n - u(\cdot,t_n) = \underbrace{(u_h^n - P_A u(\cdot,t_n))}_{=v_h^n} + \underbrace{(P_A u(\cdot,t_n) - u(\cdot,t_n))}_{=w^n}. \tag{2.32}$$

From (2.23a), we know that

$$\|w^n\| \le Ch^2\|u_{xx}(t_n)\| \le Ch^2\|u_{xx}\|_{L^\infty(Q_T)} \tag{2.33}$$

and that $\partial_{\tau_n} P_A v = P_A \partial_{\tau_n} v$ for $v \in C(\overline{I})$.

We now estimate v_h^n. Using the weak form (2.5) at $t = t_{n+1}$, scheme (ML–1), and the property of P_A, we obtain

$$\left\langle \partial_{\tau_n} v_h^{n+1}, \chi\right\rangle + A(v_h^{n+1},\chi) = (\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V})(\chi) \quad (\chi \in S_h), \tag{2.34}$$

where

$$\begin{aligned}
\mathrm{I}(\chi) &= (f(u_h^n),\chi) - (f(u(\cdot,t_n)),\chi),\\
\mathrm{II}(\chi) &= (u_t(t_{n+1}),\chi) - (\partial_{\tau_n} u(\cdot,t_{n+1}),\chi),\\
\mathrm{III}(\chi) &= (f(u(\cdot,t_n)),\chi) - (f(u(\cdot,t_{n+1})),\chi),\\
\mathrm{IV}(\chi) &= (\partial_{\tau_n} u(\cdot,t_{n+1}),\chi) - (P_A\partial_{\tau_n} u(\cdot,t_{n+1}),\chi) = (\partial_{\tau_n} w^{n+1},\chi),\\
\mathrm{V}(\chi) &= (\partial_{\tau_n} P_A u(\cdot,t_{n+1}),\chi) - \langle \partial_{\tau_n} P_A u(\cdot,t_{n+1}),\chi\rangle .
\end{aligned}$$

The estimations of I–IV are straightforward. That is, we have

$$\begin{aligned}
|\mathrm{I}(\chi)| &\le M(\|w^n\| + \|v_h^n\|)\cdot\|\chi\|,\\
|\mathrm{II}(\chi)| &\le \tau_n\|u_{tt}\|_{L^\infty(Q_T)}\|\chi\|,\\
|\mathrm{III}(\chi)| &\le \tau_n M\|u_t\|_{L^\infty(Q_T)}\|\chi\|,\\
|\mathrm{IV}(\chi)| &\le Ch^2\|u_{txx}\|_{L^\infty(Q_T)}\|\chi\|.
\end{aligned}$$

To estimate V, we use Lemmas 2.2.1 and 2.3.3. Lemma 2.3.3 is applicable because $\partial_{\tau_n} P_A u(\cdot,t_{n+1})\chi$ is a piecewise quadratic function. That is,

$$\begin{aligned}
|\mathrm{V}(\chi)| &= (1, \Pi_h(\partial_{\tau_n} P_A u(\cdot,t_{n+1})\chi) - \partial_{\tau_n} P_A u(\cdot,t_{n+1})\chi)\\
&\le Ch^2\|x^{N-1}\{\partial_{\tau_n} P_A u(\cdot,t_{n+1})\chi\}_{xx}\|_{L^1(I)}\\
&\le Ch^2\|x^{N-1}(P_A\partial_{\tau_n} u(\cdot,t_{n+1}))_x\cdot\chi_x\|_{L^1(I)}\\
&\le Ch^2\|(P_A\partial_{\tau_n} u(\cdot,t_{n+1}))_x\|\cdot\|\chi_x\|\\
&\le Ch^2\|(\partial_{\tau_n} u(\cdot,t_{n+1}))_x\|\cdot\|\chi_x\|\\
&\le Ch^2\|u_{tx}\|_{L^\infty(Q_T)}\|\chi_x\|.
\end{aligned}$$

Substituting $\chi = v_h^{n+1}$ in (2.34) gives

$$\frac{1}{2\tau_n}\left(|||v_h^{n+1}|||^2 - |||v_h^n|||^2\right) + \|(v_h^{n+1})_x\|^2 \le C|||v_h^n|||\cdot\|(v_h^{n+1})_x\| + C(h^2+\tau_n)\kappa(u)\|(v_h^{n+1})_x\|.$$

Herein, we have used Lemma 2.3.4 and the Poincaré inequality (Lemma 18.1 in [42]). By Young's inequality, we then deduce that

$$\frac{1}{\tau_n}\left(|||v_h^{n+1}|||^2 - |||v_h^n|||^2\right) \le C|||v_h^n|||^2 + C(h^2+\tau_n)^2\kappa(u)^2.$$

Therefore,

$$|||v_h^n|||^2 \le e^{CT}|||v_h^0|||^2 + C(e^{CT}-1)(h^2+\tau_n)^2\kappa(u)^2,$$

which completes the proof.

Step 2. Let $r = 1 + \|u\|_{L^\infty(Q_T)}$. Consider (2.1) and (ML–1) with replacement $f(s)$ in

$$\tilde{f}(s) = \begin{cases} f(r) & (s \ge r) \\ f(s) & (|s| \le r) \\ f(-r) & (s \le -r). \end{cases}$$

The function $\tilde{f}$ is a locally Lipschitz function satisfying

$$M = \sup_{\substack{s,s'\in\mathbb{R} \\ s\neq s'}} \frac{|\tilde{f}(s)-\tilde{f}(s')|}{|s-s'|} = \sup_{\substack{|s|,|s'|\le r \\ s\neq s'}} \frac{|f(s)-f(s')|}{|s-s'|}.$$

Let $\tilde{u}$ and $\tilde{u}_h^n$ be the solutions of (2.1) and (ML-1) with $\tilde{f}$, respectively. Applying Step 1 and Theorem 2.2.5 to $\tilde{u}$ and $\tilde{u}_h^n$, we obtain

$$\sup_{0\le t_n\le T} \|\tilde{u}_h^n - \tilde{u}(\cdot,t_n)\| \le C(h^2+\tau), \tag{2.35a}$$

$$\sup_{0\le t_n\le T} \|\tilde{u}_h^n - \tilde{u}(\cdot,t_n)\|_{L^\infty(I)} \le C\left(h + h^2\log\frac{1}{h} + \tau\right). \tag{2.35b}$$

By the definition of r and the uniqueness of the solution of (2.1), we know that $u = \tilde{u}$ in Q_T. For sufficiently small h and τ, we have

$$C\left(h + h^2\log\frac{1}{h} + \tau\right) \le 1.$$

Consequently, $\|\tilde{u}_h^n\|_{L^\infty(I)} \le r$ for $0 \le t_n \le T$ and, by the uniqueness of the solution of (ML–1), we have $u_h^n = \tilde{u}_h^n$. Therefore, (2.35a) implies the desired result. □

We now proceed to the proof of Theorem 2.2.5.

Proof of Theorem 2.2.5. The notation is that of the previous proof. It suffices to prove Theorem 2.2.5 under assumption (2.31), which is generalizable to an arbitrary f as demonstrated in the previous proof. By (2.23b), we have $\|v_h^0\|_{L^\infty(I)} \le Ch^2\log(1/h)\kappa(u)$ and $\|w^n\|_{L^\infty(I)} \le Ch^2\log(1/h)\kappa(u)$ for $0 \le t_n \le T$. Therefore, it remains to estimate v_h^n when $0 < t_n \le T$. Setting

$$G_h^n = \sum_{i=0}^{m-1} G_i^n\phi_i, \quad G_i^n = \frac{(\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V})(\phi_i)}{(1,\phi_i)},$$

we rewrite (2.34) as

$$\left\langle \partial_{\tau_n} v_h^{n+1}, \chi\right\rangle + A(v_h^{n+1},\chi) = \langle G_h^n, \chi\rangle \quad (\chi \in S_h).$$

Showing that

$$\|G_h^n\|_{L^\infty(I)} \le M\|v_h^n\|_{L^\infty(I)} + C\,(h+\tau)\,\kappa(u), \tag{2.36}$$

we can apply Lemma 2.3.5 to obtain

$$\|v_h^{n+1}\|_{L^\infty(I)} \le (1+M\tau_n)\|v_h^n\|_{L^\infty(I)} + \tau_n \cdot C(h+\tau)\kappa(u),$$

and consequently

$$\|v_h^n\|_{L^\infty(I)} \le e^{Mt_n}\|v_h^0\|_{L^\infty(I)} + \frac{e^{Mt_n}-1}{M}C(h+\tau)\kappa(u).$$

Thereby, we deduce the desired estimate.

Below we prove the truth of (2.36). Recall that we assumed global Lipschitz continuity (2.31) on f.

I(ϕ_i)–IV(ϕ_i) are straightforwardly estimated as follows:

$$\begin{aligned}
|\mathrm{I}(\phi_i)| &\le M[\|v_h^n\|_{L^\infty(I)} + Ch^2\log(1/h)\kappa(u)]\cdot(1,\phi_i),\\
|\mathrm{II}(\phi_i)| &\le \tau_n\kappa(u)(1,\phi_i),\\
|\mathrm{III}(\phi_i)| &\le M\tau_n\kappa(u)(1,\phi_i),\\
|\mathrm{IV}(\phi_i)| &\le Ch^2\log(1/h)\kappa(u)(1,\phi_i).
\end{aligned}$$

To estimate V(ϕ_i), we write

$$\mathrm{V}(\phi_i) = \mathrm{V}_1(\phi_i) + \mathrm{V}_2(\phi_i) + \mathrm{V}_3(\phi_i),$$

where

$$\begin{aligned}
\mathrm{V}_1(\phi_i) &= (\partial_{\tau_n}P_A u(\cdot,t_{n+1}),\phi_i) - (\partial_{\tau_n}u(\cdot,t_{n+1}),\phi_i),\\
\mathrm{V}_2(\phi_i) &= (\partial_{\tau_n}u(\cdot,t_{n+1}),\phi_i) - \langle\partial_{\tau_n}u(\cdot,t_{n+1}),\phi_i\rangle\,,\\
\mathrm{V}_3(\phi_i) &= \langle\partial_{\tau_n}u(\cdot,t_{n+1}),\phi_i\rangle - \langle\partial_{\tau_n}P_A u(\cdot,t_{n+1}),\phi_i\rangle\,.
\end{aligned}$$

The above terms are respectively estimated as:

$$\begin{aligned}
|\mathrm{V}_1(\phi_i)| &\le \|P_A(\partial_{\tau_n}u(\cdot,t_{n+1})) - \partial_{\tau_n}u(\cdot,t_{n+1})\|_\infty(1,\phi_i)\\
&\le Ch^2\log(1/h)\kappa(u)(1,\phi_i);\\
|\mathrm{V}_2(\phi_i)| &\le \int_I x^{N-1}\left|\partial_{\tau_n}u(x,t_{n+1}) - \partial_{\tau_n}u(x_i,t_{n+1})\right|\phi_i(x)\ dx\\
&\le Ch\kappa(u)(1,\phi_i);\\
|\mathrm{V}_3(\phi_i)| &\le |\partial_{\tau_n}u(x_i,t_{n+1}) - P_A\partial_{\tau_n}u(x_i,t_{n+1})|(1,\phi_i)\\
&\le Ch^2\log(1/h)\kappa(u)(1,\phi_i).
\end{aligned}$$

We thereby deduce that

$$\|G_h^n\|_{L^\infty(I)} \le M\|v_h\|_{L^\infty(I)} + C\left(h + h^2\log(1/h) + \tau\right)\kappa(u),$$

which implies (2.36). This step completes the proof. □

Proof of Theorem 2.2.6. The proof is identical to that of Theorem 2.2.5. □

2.5 Blow-up analysis

2.5.1 Results

This section considers the special nonlinearlity

$$f(s) = s|s|^\alpha,\quad \alpha > 0.$$

As we are interested in non-negative solutions, we assume that

$$u^0 \ge 0, \not\equiv 0,\quad u_h^0 \ge 0, \not\equiv 0. \tag{2.37}$$

Therefore, the solution u of (2.1) is non-negative and the solution u_h^n of (ML–2) is also non-negative under condition (2.12). Generally, the solution of (2.1) blows up when the initial data u_0 are sufficiently large, and the blow up is controlled by the energy functional associated with (2.1). Herein, we study whether or not the numerical solution behaves similarly by initially defining some properties of the solution u of (2.1). In particular, we see that (ML–2) is suitable for this purpose.

The energy functionals associated with (2.1) are defined as

$$\begin{aligned}
K(v) &= \frac{1}{2}\|v_x\|^2 - \frac{1}{\alpha+2}\int_I x^{N-1}|v(x)|^{\alpha+2}\ dx,\\
I(v) &= \int_I x^{N-1}v(x)\psi(x)\ dx,
\end{aligned}$$

where $\psi \in \dot{H}^1$ denotes the eigenfunction associated with the first eigenvalue $\mu > 0$ of the eigenvalue problem

$$A(\psi, \chi) = \mu(\psi, \chi) \quad (\chi \in \dot{H}^1). \tag{2.38}$$

Without loss of generality, we assume that $\psi \geq 0$ in I and $\int_I x^{N-1}\psi(x)\ dx = 1$.

The following propositions 2.5.1, 2.5.2, and 2.5.3 are often applied to the semilinear heat equation in a bounded domain. They are easily extended to the radially symmetric case.

Proposition 2.5.1. $K(u(t))$ is a non-increasing function of t, where u is the solution of (2.1).

Proposition 2.5.2. Suppose that $u^0 \geq 0, \not\equiv 0$ and u is the solution of (2.1). Then, the following statements are equivalent:

(i) There exists $T_\infty > 0$ such that u blows up at $t = T_\infty$ in the sense that $\lim_{t \to T_\infty} \|u(\cdot, t)\| = \infty$.

(ii) There exists $t_0 \geq 0$ such that $K(u(\cdot, t_0)) < 0$.

Proposition 2.5.3. Suppose that $u^0 \geq 0, \not\equiv 0$ and that u is the solution of (2.1). Then, the following statements are equivalent:

(i) There exists $T_\infty > 0$ such that u blows up at $t = T_\infty$ in the sense that $\lim_{t \to T_\infty} I(u(\cdot, t)) = \infty$.

(ii) There exists $t_0 \geq 0$ such that $I(u(\cdot, t_0)) > \mu^{\frac{1}{\alpha}}$.

Remark 2.5.4. In Propositions 2.5.2 and 2.5.3, the blow up time T_∞ is estimated respectively as

$$T_\infty \leq t_0 + \frac{\alpha + 2}{\alpha^2} N^{-\frac{\alpha}{2}} \|u(\cdot, t_0)\|^{-\alpha},$$

and

$$T_\infty \leq t_0 + \int_{I(u(\cdot, t_0))}^{\infty} \frac{ds}{-\mu s + s^{1+\alpha}}.$$

We now proceed to the discrete energy functionals. To this end, we employ the finite element version of the eigenvalue problem:

$$A(\hat{\psi}_h, \chi_h) = \hat{\mu}_h \left\langle \hat{\psi}_h, \chi_h \right\rangle \quad (\chi_h \in S_h). \tag{2.39}$$

Let $\hat{\psi}_h \in S_h$ be the eigenfunction associated with the smallest eigenvalue $\hat{\mu}_h > 0$ of (2.39). For the eigenvalue problem (2.39), we state the following result, postponing the proof to Section 2.7.

Proposition 2.5.5. If the partition $\{x_j\}_{j=0}^m$ is quasi-uniform, that is, satisfies (2.13), we have the following:

(i) $\hat{\mu}_h \to \mu$ as $h \to 0$.

(ii) The first eigenfunction $\hat{\psi}_h$ of (2.39) does not change sign.

(iii) $\|(\hat{\psi}_h - \psi)_x\| \to 0$ as $h \to 0$.

Therefore, without loss of generality, we can assume that $\hat{\psi}_h \geq 0$ and $\int_I x^{N-1}\hat{\psi}_h(x)\ dx = 1$.

For $v \in S_h$, we set

$$K_h(v) = \frac{1}{2}\|v_x\|^2 - \frac{1}{\alpha + 2}\sum_{i=0}^{m} |v(x_i)|^{\alpha+2}(1, \phi_i),$$

$$I_h(v) = \left\langle v, \hat{\psi}_h \right\rangle = \int_I x^{N-1}\Pi_h(v\hat{\psi}_h)(x)\ dx.$$

We introduce the approximate blow-up time $\hat{T}_\infty(h)$ by setting

$$\hat{T}_\infty(h) = \lim_{n \to \infty} t_n = \lim_{n \to \infty} \sum_{j=0}^{n-1} \tau_j. \tag{2.40}$$

We are now in a position to mention the main theorems in this section:

Theorem 2.5.6. Let (2.37) be satisfied. Suppose that the solution u of (2.1) blows up at finite time T_∞ in the sense that

$$\|u(\cdot,t)\|_{L^\infty(I)} \to \infty \quad \text{and} \quad \|u(\cdot,t)\| \to \infty \quad (t \to T_\infty - 0). \tag{2.41}$$

Assume that for any $T < T_\infty$, u is sufficiently smooth that (2.15) holds. Assuming also that (2.13) is satisfied, we set

$$\tau = \delta \frac{\beta^2}{N+1} h^2 \tag{2.42}$$

for some $\delta \in (0,1]$. The time increment τ_n is iteratively defined as

$$\tau_n = \tau_n(h) = \tau \min\left\{1, \frac{1}{|||u_h^n|||^\alpha}\right\}, \tag{2.43}$$

where we have used the solution u_h^n of (ML–2). Moreover, assume that (2.12) is satisfied and that

$$\forall T < T_\infty, \quad \lim_{h\to 0} \sup_{0 \le t_n \le T} |K(u(\cdot,t_n)) - K_h(u_h^n)| = 0. \tag{2.44}$$

We then have

$$\lim_{h\to 0} \hat{T}_\infty(h) = T_\infty. \tag{2.45}$$

Theorem 2.5.7. Let (2.37) be satisfied. Suppose that the solution u of (2.1) blows up at finite time T_∞ in the sense that

$$I(u(\cdot,t)) \to \infty \quad \text{and} \quad \|u(\cdot,t)\|_{L^\infty(I)} \to \infty \quad (t \to T_\infty - 0). \tag{2.46}$$

Assume that, for any $T < T_\infty$, u is sufficiently smooth that (2.15) holds. Assuming also that (2.13) is satisfied, we set τ by (2.42) with some $\delta \in (0,1]$. The time increment τ_n is iteratively defined as

$$\tau_n = \tau_n(h) = \tau \min\left\{1, \frac{1}{I_h(u_h^n)^\alpha}\right\}, \tag{2.47}$$

where we have used the solution u_h^n of (ML–2) with (2.16). We then obtain (2.45).

Remark 2.5.8. The above theorems differ in that Theorem 2.5.6 requires the convergence property (2.44) of the discrete energy functional $K_h(u_h^n)$, whereas no convergence property of I_h is necessary in Theorem 2.5.7.

Remark 2.5.9. Unfortunately, we could not prove Theorems 2.5.6 and 2.5.7 using the solution of (ML–1). In particular, the proof of the difference inequalities (2.48) and (2.51) failed in scheme (ML–1).

2.5.2 Proof of Theorem 2.5.6

To prove Theorem 2.5.6, we follow Nakagawa's blow-up analysis [32]. For this purpose, we must derive the difference inequality (2.48) and the boundedness (2.49) of $\hat{T}_\infty$ (see Lemmas 2.5.11 and 2.5.12). The original proof in [32] immediately follows from these results; see also [7], [8], and [14]. Therefore, we concentrate our efforts on proving Lemmas 2.5.11 and 2.5.12.

Throughout this subsection, we take the same assumptions of Theorem 2.5.6; in particular, the time-increment control (2.43). Note that condition (2.12) is satisfied by the definition of τ_n. Consequently, the solution u of (2.1) and the solution u_h^n of (ML–2) are non-negative.

Lemma 2.5.10. $K_h(u_h^n)$ is a non-increasing sequence of n.

Proof. Fixing some $n \ge 0$, we write $w = u_h^{n+1}$, $u = u_h^n$, $w_j = w(x_j)$, and $u_j = u(x_j)$. To show that $K_h(w) - K_h(u) \le 0$, we perform the following division:

$$K_h(w) - K_h(u) = \mathrm{X} + \mathrm{Y},$$

where

$$\mathrm{X} = \frac{1}{2}\|w_x\|^2 - \frac{1}{2}\|u_x\|^2,$$
$$\mathrm{Y} = -\frac{1}{\alpha+2}\sum_{j=0}^{m-1} w_j^{\alpha+2}(1,\phi_j) + \frac{1}{\alpha+2}\sum_{j=0}^{m-1} u_j^{\alpha+2}(1,\phi_j).$$

X is expressed as

$$\mathrm{X} = A(u, w-u) + \frac{1}{2}A(w-u, w-u).$$

By the mean value theorem, there exists $\theta_j \in [0,1]$ such that

$$w_j^{\alpha+2} - u_j^{\alpha+2} = (\alpha+2)\tilde{u}_j^{\alpha+1}(w_j - u_j),$$

where $\tilde{u}_j = u + \theta_j(w-u)$. Therefore,

$$\begin{aligned}
\mathrm{Y} &= -\sum_{j=0}^{m-1} \tilde{u}_j^{\alpha+1}(1,\phi_j)(w_j-u_j) \\
&= -\sum_{j=0}^{m-1} [\tilde{u}_j^{\alpha+1} - u^{\alpha+1}](1,\phi_j)(w_j-u_j) - \sum_{j=0}^{m-1} u_j^{\alpha+1}(1,\phi_j)(w_j-u_j) = \mathrm{Y}_1 + \mathrm{Y}_2.
\end{aligned}$$

We calculate

$$A(u, w-u) + \mathrm{Y}_2 = -\left\langle \frac{w-u}{\tau_n}, w-u \right\rangle = -\frac{1}{\tau}|||w-u|||^2$$

and

$$\mathrm{Y}_1 = -\sum_{j=0}^{m-1} (\alpha+1)\theta_j(1,\phi_j)\hat{u}_j^{\alpha}(w_j-u_j)^2 \leq 0,$$

where $\hat{u}_j = u + \hat{\theta}_j(\tilde{u}_j - u_j)$ with some $\hat{\theta}_j \in [0,1]$.

Meanwhile, for $v_h \in S_h(I)$, we write

$$\begin{aligned}
A(v_h, v_h) &\leq 2\sum_{j=1}^{m} \int_{I_j} x^{N-1} \cdot \frac{1}{h_j^2}\ dx \cdot (v_j^2 + v_{j-1}^2) \\
&= 2\sum_{j=0}^{m-1} a_{jj} v_j^2.
\end{aligned}$$

Using (2.12), we have

$$A(v_h, v_h) \leq 2\sum_{j=0}^{m-1} \frac{(1,\phi_j)}{\tau_n} v_j^2 = \frac{2}{\tau_n}\langle v_h, v_h\rangle = \frac{2}{\tau_n}|||v_h|||^2 \quad (v_h \in S_h).$$

We thereby deduce that

$$\mathrm{X} + \mathrm{Y} = -\frac{1}{\tau_n}|||w-u|||^2 + \frac{1}{2}A(w-u, w-u) + \mathrm{Y}_1 \leq 0,$$

which implies that $K_h(u_h^n)$ is non-increasing in n. □

Lemma 2.5.11. If there exists a non-negative integer n' such that $K_h(u_h^n) \leq 0$ for all $n \geq n'$, we have

$$\frac{1}{2}\partial_{\tau_n}|||u_h^{n+1}|||^2 \geq \frac{\alpha}{\alpha+2} N^{\frac{\alpha}{2}} |||u_h^n|||^{\alpha+2} \quad (n \geq n'). \tag{2.48}$$

Proof. Substituting $\chi_h = u_h^n$ in (ML–2), we obtain

$$\left\langle \frac{u_h^{n+1} - u_h^n}{\tau_n}, u_h^n \right\rangle + A(u_h^n, u_h^n) = \langle u_h^n (u_h^n)^{\alpha}, u_h^n \rangle .$$

We note that

$$\langle u_h^{n+1} - u_h^n, u_h^n \rangle \leq \left\langle u_h^{n+1} - u_h^n, \frac{1}{2}(u_h^{n+1} + u_h^n) \right\rangle = \frac{1}{2}(|||u_h^{n+1}|||^2 - |||u_h^n|||^2).$$

By the decreasing property of $K_h(u_h^n)$, we have

$$\|(u_h^n)_x\|^2 \le \frac{2}{\alpha+2}\left\langle (u_h^n)^{\alpha+2}, 1\right\rangle \quad (n \ge n').$$

Combining these results, we get

$$\frac{1}{2}\cdot\frac{1}{\tau_n}(|||u_h^{n+1}|||^2 - |||u_h^n|||^2) \ge \frac{\alpha}{\alpha+2}\left\langle (u_h^n)^{\alpha+2}, 1\right\rangle.$$

Using Hölder's inequality, we calculate

$$|||u_h^n|||^2 \le (1/N)^{\frac{\alpha}{\alpha+2}} \cdot \left\langle (u_h^n)^{\alpha+2}, 1\right\rangle^{\frac{2}{\alpha+2}}.$$

We thereby deduce (2.48). □

Lemma 2.5.12. If $K_h(u_h^{n_0}) \le 0$ and $|||u_h^{n_0}||| \ge 1$ for some integer $n_0 \ge 0$, then we have

$$\hat{T}_\infty(h) \le t_{n_0} + \left\{\frac{\alpha+2}{\alpha^2}N^{-\frac{\alpha}{2}} + \tau\left(1+\frac{2}{\alpha}\right)\right\}|||u_h^{n_0}|||^{-\alpha}. \tag{2.49}$$

Proof. From Lemma 2.5.11,

$$|||u_h^{n+1}|||^2 \ge (1+2\tau_n C|||u_h^n|||^\alpha)|||u_h^n|||^2 = (1+2\tau C)|||u_h^n|||^2,$$

where $C = \frac{\alpha}{\alpha+2}N^{\frac{\alpha}{2}}$. Therefore,

$$\lim_{n\to\infty}|||u_h^n||| = \infty$$

and, for $n \ge n_0$,

$$t_n = t_{n_0} + \sum_{m=n_0}^{n-1}\tau_m = t_{n_0} + \sum_{m=n_0}^{n-1}\frac{\tau}{|||u_h^m|||^\alpha}.$$

The remainder is identical to that of Corollary 2.1 in [14], so the details are omitted here. □

2.5.3 Proof of Theorem 2.5.7

To prove Theorem 2.5.7, we apply abstract theory by (Propositions 4.2 and 4.3) in [40]. In this subsection, we take the same assumptions of Theorem 2.5.7, in particular, the time-increment control (2.47).

Lemma 2.5.13. We have $T_\infty \le \liminf_{h\to 0}\hat{T}_\infty(h)$.

Proof. The proof is shown by contradiction. Setting $S_\infty = \liminf_{h\to 0}\hat{T}_\infty(h)$, we assume that $S_\infty < T_\infty$. Then, there exists $h_0 > 0$ such that $\inf_{h'\le h}\hat{T}_\infty(h') < M$ for all $h \le h_0$, where $M = \frac{T_\infty+S_\infty}{2} < T_\infty$. That is, for some fixed $h \le h_0$, we have $t_n \le \hat{T}_\infty(h) < M$ and $I_h(u_h^n) \to \infty$ as $n \to \infty$. This implies that for some $0 \le j(n) \le m-1$, we have $u_h^n(x_{j(n)}) \to \infty$, and consequently $\|u_h^n\|_{L^\infty(I)} \to \infty$. However, from Theorem 2.2.6, we observe that

$$\lim_{h\to 0}\sup_{0\le t_n\le M}\|u_h^n - u(\cdot,t_n)\|_{L^\infty(I)} = 0.$$

If this expression is true, then T_∞ cannot be the blow-up time of the solution u of (2.1). This contradiction completes the proof. □

Lemma 2.5.14. For any $T < T_\infty$, we have

$$\lim_{h\to 0}\sup_{0\le t_n\le T}|I_h(u_h^n) - I(u(\cdot,t_n))| = 0.$$

Proof. We derive separate estimations for $|I_h(u_h^n) - \tilde{I}_h(u_h^n)|$ and $|\tilde{I}_h(u_h^n) - I(u(\cdot, t_n))|$, where $\tilde{I}_h(v)$ denotes the auxiliary functional

$$\tilde{I}_h(v) = \int_I x^{N-1} v(x) \hat{\psi}_h(x)\ dx.$$

From Theorem 2.2.6, Lemma 2.3.3 and Proposition 2.5.5 (iii), we first derive

$$\begin{aligned}
|I_h(u_h^n) - \tilde{I}_h(u_h^n)| &\le Ch^2 \|x^{N-1}(u_h^n \hat{\psi}_h)_{xx}\|_{L^1(I)} \\
&\le Ch^2 \|(u_h^n)_x\| \cdot \|\hat{\psi}_h'\| \\
&\le Ch \|u_h^n\|_{L^\infty(I)} \|\hat{\psi}_h'\| \\
&\le Ch(\|u(\cdot, t_n)\|_{L^\infty(I)} + 1)(\|\psi'\| + 1),
\end{aligned}$$

where we have used the elemental inequality $\|\psi_x\| \le Ch^{-1}\|\psi\|_{L^\infty(I)}$ for $\psi \in S_h$. This implies that $|I_h(u_h^n) - \tilde{I}_h(u_h^n)| \to 0$ as $h \to 0$.

On the other hand, as $h \to 0$, we have

$$\begin{aligned}
|\tilde{I}_h(u_h^n) - I(u(\cdot, t_n))| &= \left| \int_I x^{N-1}(u_h^n - u(\cdot, t_n))\hat{\psi}_h\ dx \right| + \left| \int_I x^{N-1} u(\cdot, t_n)(\hat{\psi}_h - \psi)\ dx \right| \\
&\le \|u_h^n - u(\cdot, t_n)\| \cdot \|\hat{\psi}_h\| + \|u(\cdot, t_n)\| \cdot \|\hat{\psi}_h - \psi\| \to 0.
\end{aligned}$$

This expression concludes the proof. □

The following is a readily obtainable consequence of Lemma 2.5.14.

Lemma 2.5.15. For any s_0, there exists a nonnegative integer $n_0 = n_0(h)$ such that $I_h(u_h^{n_0}) > s_0$.

The following lemma is elementary and was originally stated as Lemma 3.1 in [14].

Lemma 2.5.16. There exists $s_0 > 1$ satisfying

$$\frac{1}{2} f(s) + (1 + \mu)s \le f(s) \quad (s \ge s_0),$$

where $f(s) = s^{1+\alpha}$.

We can now prove Theorem 2.5.7.

Proof of Theorem 2.5.7. It remains to verify that

$$T_\infty \ge \limsup_{h \to 0}\ \hat{T}_\infty(h). \tag{2.50}$$

To this end, we apply abstract theory (Propositions 4.2 and 4.3) in [40]. Adopting the notation of [40], we respectively set X, X_h, G, H, J and J_h in [40] as

$$\begin{aligned}
&X = \dot{H}^1, \quad \|v\|_X = \|v\|_{L^\infty(I)}, \quad X_h = S_h, \\
&G(s) = \frac{1}{2} f(s) = \frac{1}{2} s^{1+\alpha}, \quad H(s) = s^\alpha \quad (s \ge 0), \\
&J(t, v) = \int_I x^{N-1} v(x) \cdot \psi(x)\ dx, \quad (t, v) \in (0, \infty) \times X, \\
&J_h(t, v_h) = I_h(v_h) = \sum_{j=0}^{m-1} v_j \hat{\psi}_h(x_j)(1, \phi_j), \quad (t, v_h) \in (0, \infty) \times X_h, \qquad v_j = v_h(x_j).
\end{aligned}$$

Here, G and H are functions of class (G) and class (H), respectively.

To avoid unnecessary complexity, we assume that $n_0 = 0$ (see Lemma 2.5.15). Our problem setting matches problem settings (I)–(VIII) in §4 of [40].

It is readily verified that conditions (H1), (H3), (H4) and (H5) in §4 of [40] hold. We need only check that condition (H2)

$$\partial_{\tau_n} I_h(u_h^{n+1}) \ge \frac{1}{2} f(I_h(u_h^n)) \quad (n \ge 0) \tag{2.51}$$

also holds.

Substituting $\chi_h = \hat{\psi}_h \in S_h$ in (ML–2) and using the relation (2.39), we have

$$\partial_{\tau_n} I_h(v_h^{n+1}) + \hat{\mu}_h I_h(v_h^{n+1}) = \left\langle f(v_h^n), \hat{\psi}_h \right\rangle.$$

From Proposition 2.5.5 (i), we know that $\hat{\mu}_h < \mu + 1$. Moreover, because $\left\langle 1, \hat{\psi}_h \right\rangle = 1$, we can apply Jensen's inequality to get

$$\partial_{\tau_n} I_h(u_h^{n+1}) \geq -(\mu+1) \cdot I_h(u_h^n) + f(I_h(u_h^n)).$$

By Lemma 2.5.16, $-(\mu+1)s + f(s) \geq \frac{1}{2} f(s)$ for $s \geq s_0$. Because $I_h(u_h^0) > s_0$ by Lemma 2.5.15, we deduce that

$$\partial_{\tau_{n_0}} I_h(u_h^1) \geq \frac{1}{2} f(I_h(u_h^0)).$$

We thus obtain $I_h(v_h^1) > s_0$. By this process, we finally obtain

$$\partial_{\tau_n} I_h(u_h^{n+1}) \geq \frac{1}{2} f(I_h(u_h^n)), \quad I_h(u_h^n) > s_0 \quad (n \geq 0).$$

□

Remark 2.5.17. Theorem 2.5.7 remains true after replacing (2.47) by

$$\tau_n = \tau_n(h) = \tau \min \left\{ 1, \frac{1}{\|u_h^n\|_{L^\infty(I)}^\alpha} \right\}.$$

However, this definition increases the computational cost over that based on (2.47). Therefore, in the following numerical evaluation, we adopt (2.47).

2.6 Numerical examples

This section validates our theoretical results with numerical examples.

We first examine the error estimates of the solutions on a uniform spatial mesh $x_j = jh$ $(j = 0, \ldots, m)$ with $h = 1/m$, regarding the numerical solution with $h' = 1/480$ as the exact solution. The following quantities were compared:

$$\begin{aligned}
\mathcal{E}_1(h) &= \|u_{h'}^n - u_h^n\|_{L^1(I)}, \\
\mathcal{E}_2(h) &= \|u_{h'}^n - u_h^n\| = \left\| x^{\frac{N-1}{2}} (u_{h'}^n - u_h^n) \right\|_{L^2(I)}, \\
\mathcal{E}_\infty(h) &= \|u_{h'}^n - u_h^n\|_{L^\infty(I)}.
\end{aligned}$$

Fig. 2.1 shows the results for $N = 3$, $\alpha = 4$ and $u(0,x) = \cos \frac{\pi}{2} x$. The time increment was uniform ($\tau_n = \tau = \lambda h^2$, $n = 0, 1, \ldots$, $\lambda = 1/2$) and the iterations were continued until $t \leq T = 0.005$. Hereinafter, we set $u_h^0 = \Pi_h u^0$. In scheme (ML-1), the numerical convergence rate was $h^2 + \tau$ in the $\|\cdot\|$ norm (see Theorem 2.2.4), but was slightly deteriorated in the L^∞ norm.

In the case $N = 4$, which is not supported in the convergence property of the standard symmetric FEM [34], we chose $\alpha = 3$ and $u(0,x) = 3 \cos \frac{\pi}{2} x$. The errors were computed up to $T = 0.0033$ on the non-uniform meshes with $x_i = \sin \frac{(i-1)\pi}{2m}$ and τ_n with $\lambda = 0.11$. As shown in Fig. 2.2, the $\|\cdot\|$ norm showed second-order convergence in both (ML-1) and standard FEM, but the $\|\cdot\|_{L^\infty(I)}$ norm showed first-order convergence in the standard FEM.

Secondly, we confirmed the non-increasing property of the energy functional $K_h(u_h^n)$ in scheme (ML–2) with $N = 5$, $\alpha = \frac{4}{3}$, and $u(0,x) = \cos \frac{\pi}{2} x,\ 13 \cos \frac{\pi}{2} x$. The time increment τ_n was determined through Theorem 5.6 with $\beta = 1$ and $\delta = 1$. Simulations were performed on a uniform spatial mesh $x_j = jh$ with $h = 1/m$ and $m = 50$. The results (see Fig. 2.3) support Lemma 5.9. As shown in Fig. 2.4, the energy functional $I_h(u_h^n)$ increased exponentially after $t = 0.04$ when the initial data was large, but vanishes when the initial data was small. For $I_h(u_h^n)$, we used the time increment in Theorem 5.7 with $\delta = 1$.

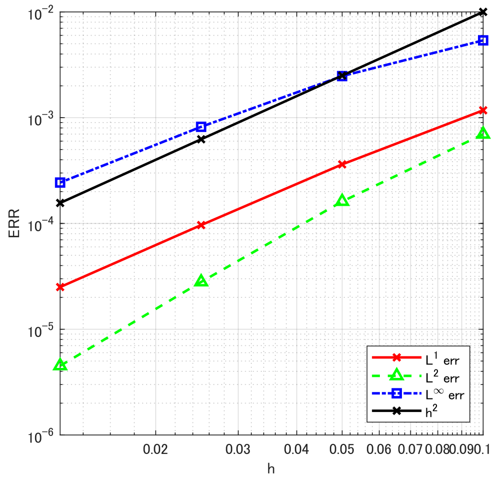
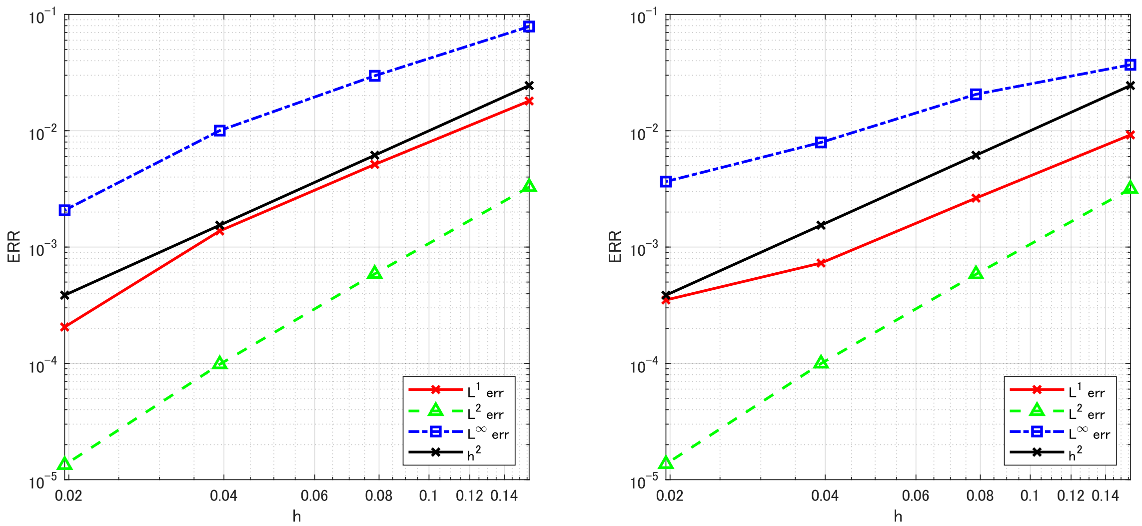
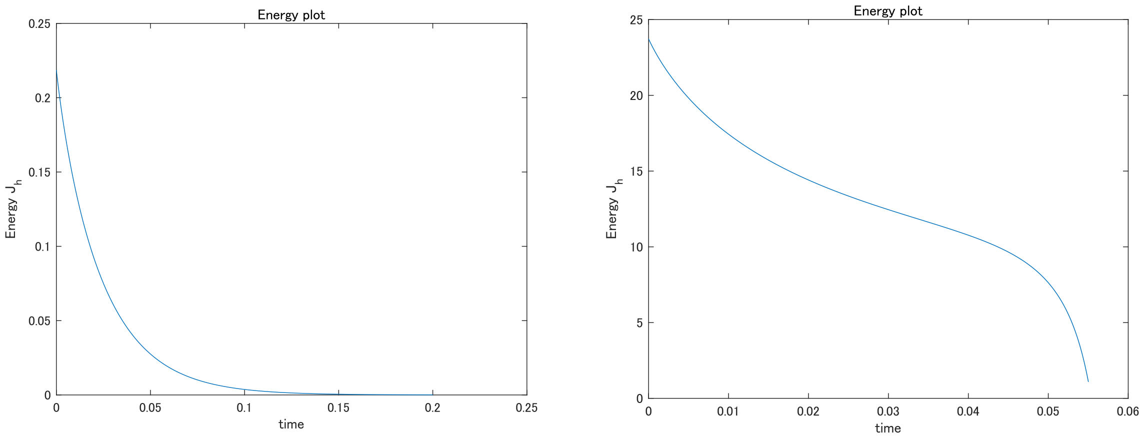


Figure 2.1: Error convergences versus granularity: $N = 3$, $\alpha = 4$ and $u(0, x) = \cos \frac{\pi}{2} x$



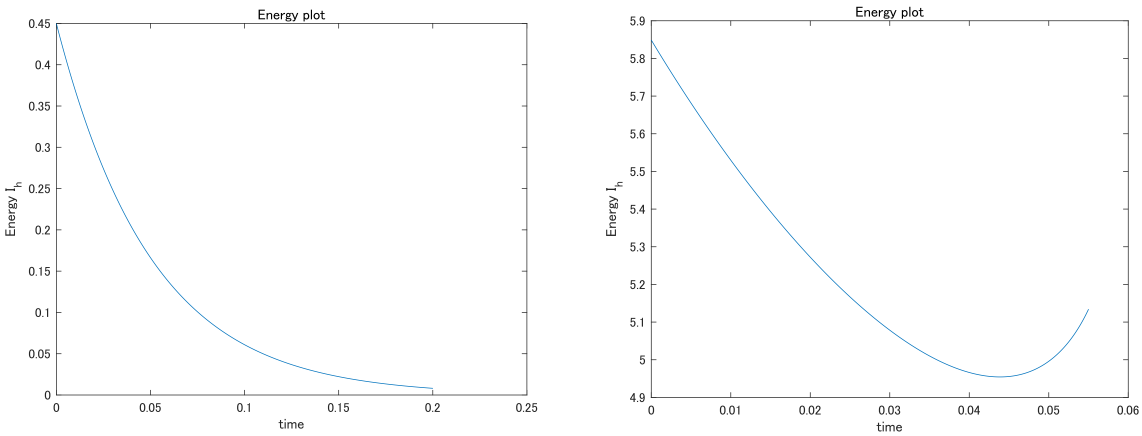
(a) (ML-1) (b) Standard FEM

Figure 2.2: Error convergences in the (ML–1) schemes (left) and the standard finite element method (right): $N = 4$, $\alpha = 3$ and $u(0, x) = 3 \cos \frac{\pi}{2} x$



(ML–2) & $u(0,x) = \cos\frac{\pi}{2}x$ (ML–2) & $u(0,x) = 13\cos\frac{\pi}{2}x$

Figure 2.3: Temporal dynamics of the energy functional $K_h(u_h^n)$ in scheme (ML–2) with different initial conditions



(ML–2) & $u(0,x) = \cos\frac{\pi}{2}x$ (ML–2) & $u(0,x) = 13\cos\frac{\pi}{2}x$

Figure 2.4: Temporal dynamics of the energy functional $I_h(u_h^n)$ in scheme (ML–2) with small (left) and large (right) data inputs

Finally, we calculated the numerical blow-up time in scheme (ML–2). Here, we set $h = 1/m$ ($m = 16, 32, 64$). The time increments were defined as

$$\text{(K) } \tau_n = \frac{1}{N+1} \min\left\{1, \frac{1}{|||u_h^n|||^\alpha}\right\} \quad \text{(see Theorem 2.5.6)},$$
$$\text{(I) } \tau_n = \frac{1}{N+1} \min\left\{1, \frac{1}{I_h(u_h^n)^\alpha}\right\} \quad \text{(see Theorem 2.5.7)}.$$

For a comparison evaluation, we executed the FDM of Chen [8] and the FDM of Cho and Okamoto [15]. Specifically, set $\tau_n = \frac{1}{2}h^2 \cdot \min\left\{1, \frac{1}{\|u_h^n\|_2^\alpha}\right\}$ in Chen's FDM and $\tau_n = \frac{1}{3N}h^2 \cdot \min\left\{1, \frac{1}{\|u_h^n\|_2^\alpha}\right\}$, $\sigma = \frac{3}{2N}$ in Cho and Okamoto's FDM.

We then introduced the *truncated numerical blow-up time* $\hat{T}_M(h)$:

$$\hat{T}_M(h) = \min\left\{t_n \mid \|u_h^n\|_\infty > M = 10^8\right\}.$$

Evaluations were performed with three parameter sets:

Case 1 $N = 5$, $\alpha = 0.39$, and $u(0, x) = 8000 \cos \frac{\pi}{2} x$;

Case 2 $N = 4$, $\alpha = 0.49$, and $u(0, x) = 800(1 - x^2)$;

Case 3 $N = 3$, $\alpha = 0.66$, and $u(0, x) = 1000(e^{-x^2} - e^{-1})$.

Note that if $1 > \frac{N\alpha}{2}$ holds true and $u^0 \geq 0$ is decreasing in x, then $I(u(t))$, $\|u(t)\|$ and $\|u(t)\|_{L^\infty(I)}$ blow up simultaneously; see [21]. We chose these settings so that the assumptions in Theorem 2.5.6 and Theorem 2.5.7 hold true.

Fig. 2.5 compares the truncated numerical blow-up times $\hat{T}_M(h)$ as functions of h in the four schemes. The solution of Chen's FDM blew up later than the other schemes, whereas that of (ML–2) with $I_h(u_h^n)$ blew up sooner than the other schemes.

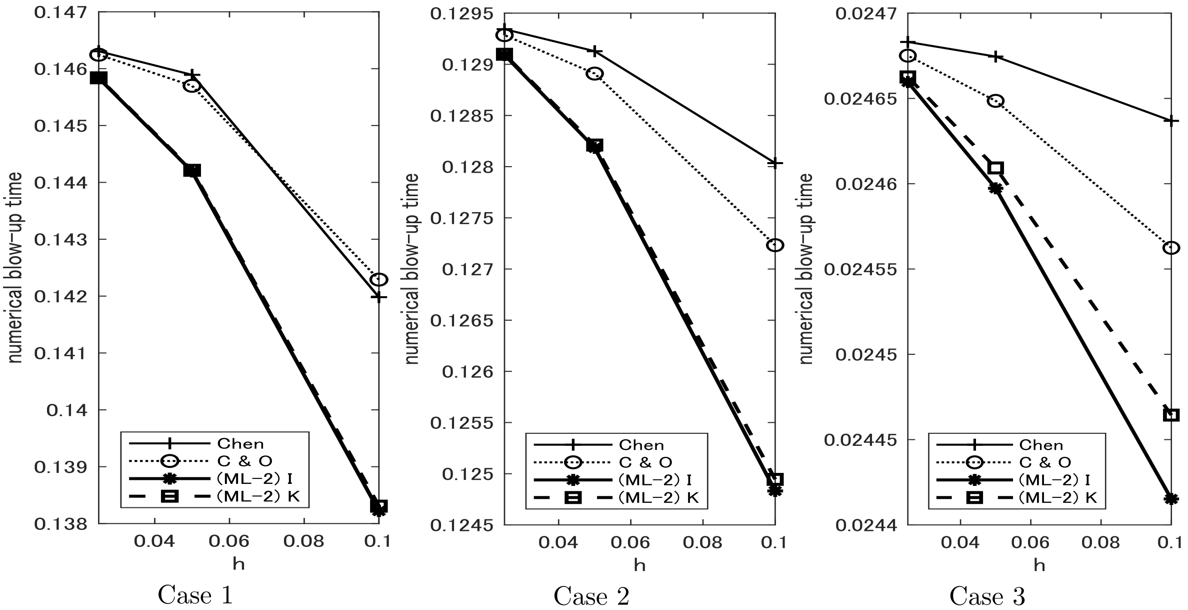


Figure 2.5: Truncated numerical blow-up times $\hat{T}_M(h)$ in the four schemes with three parameter settings

2.7 Approximate eigenvalue problems

This chapter establishes the proof of Proposition 2.5.5. Recall that μ and $\hat{\mu}_h$ are the smallest eigenvalues of (2.38) and (2.39), respectively. Functions ψ and $\hat{\psi}_h$ are the eigenfunctions associated with μ and $\hat{\mu}_h$, respectively.

We introduce the following linear operators $T : \dot{H}^1 \to \dot{H}^1$, $T_h : S_h \to S_h$ and $\hat{T}_h : S_h \to S_h$ by

$$\begin{aligned} &A(Tv, \chi) = (v, \chi) && (\chi \in \dot{H}^1, v \in \dot{H}^1), \\ &A(T_h v_h, \chi_h) = (v_h, \chi_h) && (\chi_h \in S_h, v_h \in S_h), \\ &A(\hat{T}_h v_h, \chi_h) = \langle v_h, \chi_h \rangle && (\chi_h \in S_h, v_h \in S_h). \end{aligned}$$

We also write $v' = v_x$ for some function $v = v(x)$.

Lemma 2.7.1 (Lemma 2.2 in [41], Theorems 13 and 14 in [20])**.** For any $f_h \in S_h \subset \dot{H}^1$,

$$Tf_h \in H^2(I), \ \|(Tf_h)''\| \leq C\|f_h\|.$$

For a linear operator $B : X \subset \dot{H} \to \dot{H}$, we define

$$\|B\|_{1,X} = \sup_{v \in X, \ v \neq 0} \frac{\|(Bv)'\|}{\|v'\|}.$$

Lemma 2.7.2 (Lemma 3.3 in [3])**.** $\|T - \hat{T}_h\|_{1,S_h} \to 0$ as $h \to 0$.

Remark 2.7.3. Lemma 2.7.2 does not exactly agree with Lemma 3.3 in [3], but its proof is identical to that of Lemma 3.3 in [3].

The resolvent operator $R_z(\hat{T}_h)$ for $z \in \mathbb{C}$ is defined as

$$R_z(\hat{T}_h) = (zI - \hat{T}_h)^{-1} : S_h \to S_h,$$

where $\hat{T}_h : S_h \to S_h$ and $z \in \rho(\hat{T}_h)$. Let I be the identity operator and $\rho(B)$ be the resolvent set of a linear operator B.

Lemma 2.7.4 ([17, Lemma 1])**.** For any closed set $F \subset \rho(T)$, there exists $h_0 > 0$ such that for any $h \leq h_0$, $R_z(\hat{T}_h)$ exists and

$$\|R_z(\hat{T}_h)\|_{1,S_h} \leq C \quad (\forall z \in F),$$

where C is independent of h.

We now define spectral projections of T and $\hat{T}_h$. Let $\Gamma \subset \mathbb{C}$ be a circle centered at $\frac{1}{\mu}$ enclosing no other points of $\sigma(T)$ which stands for the spectral set of T. Let $E = E(\frac{1}{\mu}) : \dot{H}^1 \to \dot{H}^1$ and $\hat{E}_h = \hat{E}_h(\frac{1}{\mu}) : S_h \to S_h$ be the spectral projection operators associated with T and $\hat{T}_h$ and the parts of the corresponding spectrum enclosed by Γ, respectively:

$$E = \frac{1}{2\pi i} \int_\Gamma R_z(T) \, dz, \quad \hat{E}_h = \frac{1}{2\pi i} \int_\Gamma R_z(\hat{T}_h) \, dz.$$

Remark 2.7.5. By Lemma 2.7.4, when h is sufficiently small, $\Gamma \subset \rho(\hat{T}_h)$ holds and $\|R_z(\hat{T}_z)\|_{1,S_h}$ is bounded for all $z \in \Gamma$. Thus the integral of $\hat{E}_h$ exists.

To examine the convergence property of $\hat{E}_h$, we use the following lemma.

Lemma 2.7.6 (Lemma 2 in [17])**.** $\|E - \hat{E}_h\|_{1,S_h} \to 0$ as $h \to 0$.

We use the following symbols.

- $\mathrm{dist}(w, A) = \inf_{y \in A} \|w' - y'\| \quad (w \in \dot{H}^1, \, A \subset \dot{H}^1)$,
- $\delta(\hat{E}_h(S_h), E(\dot{H}^1)) = \sup_{v_h \in \hat{E}_h(S_h), \ \|v_h'\|=1} \mathrm{dist}(v_h, E(\dot{H}^1))$,
- $\delta(E(\dot{H}^1), \hat{E}_h(S_h)) = \sup_{v \in E(\dot{H}^1), \ \|v'\|=1} \mathrm{dist}(v, \hat{E}_h(S_h))$,
- $\hat{\delta}(E(\dot{H}^1), \hat{E}_h(S_h)) = \max\{\delta(\hat{E}_h(S_h), E(\dot{H}^1)), \ \delta(E(\dot{H}^1), \hat{E}_h(S_h))\}$.

The next result follows from the property of the spectral projection operator.

Corollary 2.7.7. $\delta(\hat{E}_h(S_h), E(\dot{H}^1)) \to 0$ as $h \to 0$.

Lemma 2.7.8 (Theorem 2 in [17])**.**

$$\lim_{h\to 0} \inf_{\chi_h \in S_h} \|(u-\chi_h)'\| = 0 \quad (u \in \dot{H}^1).$$

Corollary 2.7.9 (Theorem 3 in [17])**.** $\delta(E(\dot{H}^1), \hat{E}_h(S_h)) \to 0$ as $h \to 0$.

Lemma 2.7.10 (Corollary 2.6 in [30])**.** If $\hat{\delta}(E(\dot{H}^1), \hat{E}_h(S_h)) < 1$, then $\dim E(\dot{H}^1) = \dim \hat{E}_h(S_h)$.

For sufficiently small h, we observe that $\hat{\delta}(E(\dot{H}^1), \hat{E}_h(S_h)) < 1$, that is, $\dim E(\dot{H}^1) = \dim \hat{E}_h(S_h)$.

Therefore, $\dim E(\dot{H}^1) = 1$, because $E(\dot{H}^1)$ is the eigenspace of the smallest eigenvalue of (2.38). Then, the unique eigenvalue of $\hat{T}_h$ (denoted by $\frac{1}{\hat{\nu}_h}$) is located inside Γ. Then, there exists $\hat{\xi}_h(\neq 0) \in S_h$ such that

$$A(\hat{\xi}_h, \chi_h) = \hat{\nu}_h \left\langle \hat{\xi}_h, \chi_h \right\rangle, \ \chi_h \in S_h.$$

Corollary 2.7.11. $\hat{\nu}_h \to \mu$ as $h \to 0$.

Proof. For some arbitrary $\epsilon > 0$, we set $\Gamma_\epsilon = B_{\frac{1}{\mu}}(\epsilon) = \{z \in \mathbb{C} \mid |z - \frac{1}{\mu}| = \epsilon\}$. As stated above, there exists $h_\epsilon > 0$ such that the eigenvalue $\frac{1}{\hat{\nu}_h}$ of $\hat{T}_h$ is inside Γ_ϵ for all $h < h_\epsilon$. Therefore,

$$\left| \frac{1}{\hat{\nu}_h} - \frac{1}{\mu} \right| \le \epsilon.$$

Because μ is positive, the proof is complete. $\square$

Remark 2.7.12. Similarly, we find that a unique eigenvalue $\frac{1}{\nu_h}$ of T_h exists inside Γ, and that $\nu_h \to \mu$ as $h \to 0$.

Let $\mu_h > 0$ be the smallest eigenvalue of

$$A(\psi_h, \chi_h) = \mu_h(\psi_h, \chi_h), \ \chi \in S_h, \tag{2.52}$$

where $\psi_h \in S_h$.

Lemma 2.7.13. For sufficiently small $h > 0$, we have $\nu_h = \mu_h$. In particular, $\mu_h \to \mu$ as $h \to 0$.

Proof. We know that $\dim E(\dot{H}^1) = 1$. By variational characterization, we obtain

$$\mu = \inf_{v \in \dot{H}^1, v \neq 0} \frac{\|v'\|^2}{\|v\|^2} \le \inf_{v_h \in S_h, v_h \neq 0} \frac{\|v_h'\|^2}{\|v_h\|^2}.$$

Here μ_h is the smallest eigenvalue of (2.52), that is,

$$\mu_h = \inf_{v_h \in S_h, v_h \neq 0} \frac{\|v_h'\|^2}{\|v_h\|^2}.$$

All eigenvalues of (2.52) are greater than or equal to μ. As the eigenvalue of T_h enclosed by Γ is unique, we obtain $\nu_h = \mu_h$ for sufficiently small $h > 0$, and $\mu_h \to \mu$ as $h \to 0$. $\square$

We now state the following proof.

Proof of Proposition 2.5.5 (i). By variational characterization, we get

$$\hat{\mu}_h = \inf_{v_h \in S_h, v_h \neq 0} \frac{\|v_h'\|^2}{|||v_h|||^2} = \left(\sup_{v_h \in S_h, v_h \neq 0} \frac{A(\hat{T}_h v_h, v_h)}{\|v_h'\|^2} \right)^{-1},$$

$$\mu_h = \inf_{v_h \in S_h, v_h \neq 0} \frac{\|v_h'\|^2}{\|v_h\|^2} = \left(\sup_{v_h \in S_h, v_h \neq 0} \frac{A(T_h v_h, v_h)}{\|v_h'\|^2} \right)^{-1}.$$

Then,

$$\sup_{v_h \in S_h, v_h \neq 0} \frac{A(\hat{T}_h v_h, v_h)}{\|v_h'\|^2} = \sup_{v_h \in S_h, v_h \neq 0} \left(\frac{A(T_h v_h, v_h)}{\|v_h'\|^2} + \frac{A((\hat{T}_h - T_h) v_h, v_h)}{\|v_h'\|^2} \right)$$
$$\leq \sup_{v_h \in S_h, v_h \neq 0} \frac{A(T_h v_h, v_h)}{\|v_h'\|^2} + \|\hat{T}_h - T_h\|_{1,S_h}.$$

Similarly,

$$\sup_{v_h \in S_h, v_h \neq 0} \frac{A(T_h v_h, v_h)}{\|v_h'\|^2} \leq \sup_{v_h \in S_h, v_h \neq 0} \frac{A(\hat{T}_h v_h, v_h)}{\|v_h'\|^2} + \|\hat{T}_h - T_h\|_{1,S_h}.$$

Applying Lemma 2.7.2, we obtain $\hat{\mu}_h = \mu$ as $h \to 0$. $\square$

Remark 2.7.14. As the eigenvalue of $\hat{T}_h$ enclosed by Γ is unique, we conclude that $\hat{\nu}_h = \hat{\mu}_h$ for sufficiently small $h > 0$.

Proof of Proposition 2.5.5 (ii). We write (2.39) in matrix form:

$$\mathcal{A}\boldsymbol{\psi} = \hat{\mu}_h \mathcal{M} \boldsymbol{\psi},$$

where $\mathcal{M} = \mathrm{diag}(\mu_i)_{0 \leq i \leq m-1}$ and $\mathcal{A} = (a_{i,j})_{0 \leq i,j \leq m-1}$ are defined as $m_i = (1, \phi_i)$ and $a_{i,j} = A(\phi_j, \phi_i)$, respectively. Moreover, $\boldsymbol{\psi} = (\psi_i)_{0 \leq i \leq m-1} \in \mathbb{R}^m$, $\psi_i = \hat{\psi}_h(x_i)$. We thus express (2.39) as

$$\mathcal{M}^{-1}\mathcal{A}\boldsymbol{\psi} = \hat{\mu}_h \boldsymbol{\psi}.$$

Because $\mathcal{M}$ is a diagonal matrix, the diagonal components $a_{i,i}/\mu_{i,i}$ of $\mathcal{M}^{-1}\mathcal{A}$ are all positive and the non-diagonal components are all non-positive.

Writing

$$\left(-\mathcal{M}^{-1}\mathcal{A} + \max_{0 \leq i \leq m-1} \frac{a_{i,i}}{m_{i,i}} \mathcal{I} \right) \boldsymbol{\psi} = \left(-\hat{\mu}_h + \max_{0 \leq i \leq m-1} \frac{a_{i,i}}{m_{i,i}} \right) \boldsymbol{\psi}, \tag{2.53}$$

where $\mathcal{I} \in \mathbb{R}^{m \times m}$ is the identity matrix, then we can see that all components of the matrix on the left-hand side are non-negative. We now consider the following eigenvalue problem:

$$\left(-\mathcal{M}^{-1}\mathcal{A} + \max_{0 \leq i \leq m-1} \frac{a_{i,i}}{m_{i,i}} \mathcal{I} \right) \vec{x} = \tilde{\mu}_h \vec{x}. \tag{2.54}$$

By the Perron–Frobenius theorem, we can take a positive eigenvector for the largest eigenvalue of (2.54). $-\hat{\mu}_h + \max_{0 \leq i \leq m-1} \frac{a_{i,i}}{m_{i,i}}$ is the largest eigenvalue of (2.54), because $\hat{\mu}_h$ is the smallest eigenvalue of (2.39).

Consequently, the sign of the first eigenfunction of (2.39) is unchanged. $\square$

Proof of Proposition 2.5.5 (iii). We assume that

$$\hat{\psi}_h \geq 0 \quad \text{and} \quad \int_I x^{N-1} \hat{\psi}_h(x)\, dx = 1.$$

Setting $\phi = \psi / \|\psi'\|$ and $\hat{\phi}_h = \hat{\psi}_h / \|\hat{\psi}_h'\|$, applying Corollary 2.7.7 and Proposition 2.5.5 (i), and setting $v_h = \hat{\phi}_h$, we obtain

$$\mathrm{dist}(\hat{\phi}_h, E(\dot{H}^1)) \to 0 \quad \text{as} \quad h \to 0.$$

Because $\dim E(\dot{H}^1) = 1$ and $E(\dot{H}^1)$ is a closed subspace in $\dot{H}^1$, we find that

$$E(\dot{H}^1) = \{ z\phi \in \dot{H}^1 \mid z \in \mathbb{C} \}$$

and

$$\mathrm{dist}(\hat{\phi}_h, E(\dot{H}^1)) = \|\hat{\phi}_h' - c_h \phi'\|,$$

where $c_h \in \mathbb{C}$.

Therefore, $|c_h| \to 1$ as $h \to 0$. Using $\hat{\phi}_h,\ \phi \geq 0$ and $\|\hat{\phi}_h - c_h\phi\| \leq \|\hat{\phi}'_h - c_h\phi'\|$, we find that $c_h \to 1$ as $h \to 0$. That is, as $h \to 0$,

$$\begin{aligned}\|\hat{\phi}'_h - \phi'\| &\leq \|\hat{\phi}'_h - c_h\phi'\| + \|c_h\phi' - \phi'\| \\ &= \|\hat{\phi}'_h - c_h\phi'\| + |c_h - 1| \cdot \|\phi'\| \to 0.\end{aligned}$$

On the other hand,

$$\|(\psi - \hat{\psi}_h)'\| \leq \frac{1}{\int_I x^{N-1}\phi(x)\ dx}\|\phi' - \hat{\phi}'_h\| + \left|\frac{1}{\int_I x^{N-1}\phi(x)\ dx} - \frac{1}{\int_I x^{N-1}\hat{\phi}_h(x)\ dx}\right|.$$

This, together with

$$\left|\int_I x^{N-1}\phi(x)\ dx - \int_I x^{N-1}\hat{\phi}_h(x)\ dx\right| \leq \left(\frac{1}{N}\right)^{\frac{1}{2}} \cdot \|\phi - \hat{\phi}_h\| \leq \left(\frac{1}{N}\right)^{\frac{1}{2}} \cdot \|\phi' - \hat{\phi}'_h\| \to 0 \quad \text{as} \quad h \to 0$$

implies that $\|(\psi - \hat{\psi}_h)'\| \to 0$ as $h \to 0$, which completes the proof. □

2.8 Note

This chapter was taken directly from Nakanishi-Saito [35].

Chapter 3

Asymptotic behavior of the finite element blow-up solutions

3.1 The schemes

This chapter is a supplement to Chapter 2; we follow the same notation and consider the same problem as Chapter 2. Then, we further introduce the following schemes.

(ML-2) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\left\langle \partial_{\tau_n} u_h^{n+1}, \chi \right\rangle + A(u_h^n, \chi) = \langle f(u_h^n), \chi \rangle \quad (\chi \in S_h). \tag{3.1}$$

(ML-3) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\left\langle \partial_{\tau_n} u_h^{n+1}, \chi \right\rangle + A(u_h^{n+1}, \chi) = \langle f(u_h^n), \chi \rangle \quad (\chi \in S_h). \tag{3.2}$$

For (ML-3), we obtain the following positivity preserving.

Theorem 3.1.1 (Well-posedness and positivity preserving property)**.** Suppose that $n \geq 0$ and $u_h^n \in S_h$ are given.

1. The scheme (ML-3) admits a unique solution $u_h^{n+1} \in S_h$.
2. In addition to the basic assumption on f, assume that f is a non-decreasing function with $f(0) \geq 0$. If $u_h^n \geq 0$, then the solution u_h^{n+1} of (ML-3) satisfies $u_h^{n+1} \geq 0$.

Since we can prove this similarly as Theorem 1.3.1 and Theorem 1.3.2 in Chapter 1, we omit the proof. For (ML-2), see Theorem 2.2.2 in Chapter 2.

For the monotonicity of the numerical solution, we get the following results.

Theorem 3.1.2 (Order preserving property)**.** Suppose that $n \geq 0$ and $u_h^n \in S_h$ are given. In addition to the basic assumption on f, assume that f is a non-decreasing function with $f(0) \geq 0$.

1. Assume that

$$\tau_n \leq \min \left\{ \min_{0 \leq i \leq m-1} \frac{(1, \phi_i)}{A(\phi_i, \phi_i)}, \min_{0 \leq i \leq m-2} \left(\frac{-A(\phi_i, \phi_{i+1})}{(1, \phi_i)} + \frac{-A(\phi_{i+1}, \phi_i)}{(1, \phi_{i+1})} \right)^{-1} \right\}. \tag{3.3}$$

   If u_h^n is a positive and decreasing function, then u_h^{n+1} of (ML-2) is decreasing in x.

2. Assume that

$$\tau_n < \min_{1 \leq i \leq m-2} \left\{ \frac{-(1, \phi_1)}{A(\phi_2, \phi_1)}, \left(\frac{-A(\phi_{i-1}, \phi_i)}{(1, \phi_i)} + \frac{-A(\phi_{i+2}, \phi_{i+1})}{(1, \phi_{i+1})} \right)^{-1} \right\}. \tag{3.4}$$

   If u_h^n is a positive and decreasing function, then u_h^{n+1} of (ML-3) is decreasing in x.

Their proofs are postponed for Section 3.6.

Remark 3.1.3. From a direct calculation, the sufficient condition for (3.3) and (3.4) is

$$\tau_n \leq \frac{\beta^2}{2(N+1)} h^2,$$

where β is defined by (2.13).

We will prove this in Section 3.6.

We proceed to the convergence analysis for (ML-3).

Theorem 3.1.4 (Convergence for (ML-3) in $\|\cdot\|_{L^\infty(I)}$)**.** We assume the same condition as Theorem 2.2.5. For sufficient small h and τ, we get

$$\sup_{0\leq t_n \leq T} \|u_h^n - u(\cdot, t_n)\|_{L^\infty(I)} \leq C' \left(h + h^2 \log \frac{1}{h} + \tau \right),$$

where $C' = C'(T, f, \kappa(u), C_0, N, \beta)$ and u_h^n is the solution of (ML-3).

Since we can prove this similarly as Theorem 2.2.5 in Chapter 2, we omit the proof. For (ML-2), see Theorem 2.2.6 in Chapter 2.

3.2 Adaptive time step control (Chen and Cho-Okamoto)

We set $f(s) = s^{1+\alpha}$. For blow-up analysis, we now apply [8, 15] to (ML-2) and (ML-3).

We state a sufficient condition for the blow-up of (ML-2).

Assumption 3.2.1 (Assumption (H) in [15])**.** There exists a positive constant a such that

$$W_j^0 = \delta^2 u_j^0 + (1-a)(u_j^0)^{1+\alpha} \geq 0, \ j = 0, 1, \cdots, m-1, \tag{3.5}$$

where

$$\begin{aligned} \delta^2 u_j^n &= -\frac{1}{(1, \phi_j)} \sum_{k=0}^{m-1} A(\phi_j, \phi_k) u_k^n, \\ u_j^n &= u_h^n(x_j), \end{aligned}$$

and u_h^0 is a strictly decreasing function in x.

Remark 3.2.2. We assume $u^0 \geq 0$, $u_x^0 \leq 0$ in I and $u^0(1) = 0$. It is well-known that the solution u of (2.1) blows up in finite time if $(u^0)_{xx} + \frac{N-1}{x}(u^0)_x + u^0|u^0|^\alpha \geq 0$ in I (see [21]). Assumption 3.2.1 is the discrete version of this condition.

Lemma 3.2.3 (Sufficient condition for blow-up, Lemma 4.3 in [15])**.** We assume Assumption 3.2.1 and (3.3). Then the solution of (ML-2) satisfies

$$W_j^n \geq 0, \ \text{for } j = 0, 1, \cdots, m-1, \ \text{and } n \geq 0,$$

where $W_j^n = \delta^2 u_j^n + (1-a)(u_j^n)^{1+\alpha}$. In particular,

$$\partial_{\tau_n} \|u_h^{n+1}\|_{L^\infty(I)} = \partial_{\tau_n} u_0^{n+1} \geq a (u_0^n)^{1+\alpha}, \ n \geq 0.$$

We will prove this in Section 3.6.

Theorem 3.2.4 (Convergence for numerical blow-up time, Theorem 4.4 in [15])**.** In addition to the assumption of Theorem 2.2.6, we assume Assumption 3.2.1 and $\tau = \delta \frac{\beta^2}{2(N+1)} h^2$, where $\delta \in (0, 1]$. We adopt the time step control

$$\tau_n = \tau \cdot \min \left\{ 1, \frac{1}{\|u_h^n\|_{L^\infty(I)}^\gamma} \right\}, \tag{3.6}$$

where $0 < \gamma < 1 + \alpha + \tau^{-1}$. Then for the solution of (ML-2) we get

$$T_h = \sum_{n=0}^{\infty} \tau_n < \infty, \quad \lim_{h \to 0} T_h = T_\infty,$$

where T_∞ denotes the blow-up time for L^∞ norm of (2.1).

For the proof, see Section 3.6.

Remark 3.2.5. Theorem 3.2.4 does not impose the assumption that $\|u(t)\|_{L^\infty(I)}$, $\|u(t)\|$ and $I(u(t))$ blow up simultaneously, but assumes Assumption 3.2.1. Below we set $\gamma = \alpha$.

We now study the asymptotic behavior of the finite element solutions.

Theorem 3.2.6 (Asymptotic behavior for (ML-2), I, Theorem 4.9 in [15] and Theorem 4.1 in [14])**.** We assume Assumption 3.2.1 and $\tau \le \frac{\beta^2}{2(N+1)} h^2$. We adopt the time increment (3.6) with $\gamma = \alpha$. For the solution of (ML-2), $u_h^n(x_1)$ is bounded if $\alpha > 1$. On the other hand, if $\alpha \le 1$, then $\lim_{n \to \infty} u_h^n(x_1) = \infty$.

For the proof, see Section 3.6.

Theorem 3.2.7 (Asymptotic behavior of (ML-2), II, Theorem 4.12 in [15] and Theorem 4.2 in [14])**.** Under the same assumption and time increment as Theorem 3.2.6, we let a number $k \in \mathbb{N}$ be

$$\frac{1}{k+1} < \alpha \le \frac{1}{k}.$$

Then the solution of (ML-2) blows up exactly at $k+1$ points and it is bounded at all other points. Namely

$$\lim_{n \to \infty} u_h^n(x_j) = \infty \text{ if and only if } 0 \le j \le k.$$

For the proof, see Section 3.6.

Theorem 3.2.8 (Asymptotic behavior of (ML-3), Theorem 3.2 in [8])**.** We assume that u_h^0 is positive and decreasing in x, and $\tau \le \frac{\beta^2}{2(N+1)} h^2$. We adopt the time increment (3.6) with $\gamma = \alpha$. For the blow-up solution of (ML-3), we assume that $\lim_{n \to \infty} u_h^n(x_0)^{\alpha-1} u_h^n(x_1) = \infty$ for $\alpha > 1$. Then $u_h^n(x_1)$ is bounded if $\alpha > 1$. On the other hand, if $\alpha \le 1$, then $\lim_{n \to \infty} u_h^n(x_1) = \infty$. If $\alpha = 1$, then $u_h^n(x_2)$ is bounded.

For the proof, see Section 3.6.

Remark 3.2.9. It is well-known that if u^0 is decreasing in x, then the blow-up solution u of (2.1) blows up only at the origin (see [21]). Thus if $\alpha > 1$, then (ML-2) reproduces one-point blow-up.

3.3 Uniform time step control (Cho-Okamoto)

We shall apply the uniform time increment to (ML-2). Recall that we are considering $f(s) = s^{1+\alpha}$.

Theorem 3.3.1 (Convergence of the numerical blow-up time, Theorem 4.13 in [15] and Theorem 3.4 in [11])**.** We define a strictly increasing function $H : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{s \to \infty} H(s) = \infty, \quad \lim_{\tau \to 0} \tau \log\left(H^{-1}\left(\frac{1}{\tau}\right)\right) = 0.$$

In addition to the assumption Theorem 2.2.6, we assume Assumption 3.2.1 and $\tau = \delta \frac{\beta^2}{2(N+1)} h^2$ where $\delta \in (0, 1]$. We adopt the uniform time step control

$$\tau_n = \tau. \tag{3.7}$$

We then define the numerical blow-up time T_h by

$$T_h = n_h \tau,$$

where $n_h \in \mathbb{N}$ satisfies

$$\tau H(\|u_h^{n_h-1}\|_\infty) < 1, \ \tau H(\|u_h^{n_h}\|_\infty) \geq 1.$$

Then for the solution of (ML-2) we get

$$\lim_{h\to 0} T_h = T_\infty,$$

where T_∞ denotes the exact blow-up time for L^∞ norm of (2.1).

Since we can prove this similarly as Theorem 3.2 in [11], we omit the proof.

Remark 3.3.2. Theorem 3.2.4 does not impose the assumption that $\|u(t)\|_{L^\infty(I)}$, $\|u(t)\|$ and $I(u(t))$ blow up simultaneously, but assumes Assumption 3.2.1.

By uniform time increment, we can find the boundedness of L^p norm of the exact solution.

Theorem 3.3.3 (Boundedness of L^p norm, Theorem 4.16 in [15])**.** For $1 \leq p < \infty$, we assume that

$$\limsup_{h\to 0} \max_{0\leq n\leq n_h} \|u_h^n\|_p = M < \infty,$$

where $\tau,\ n_h$ are defined in Theorem 3.3.1 and

$$\|v\|_p = \left\{ \int_I x^{N-1}|v(x)|^p \ dx \right\}^{\frac{1}{p}}.$$

Under the assumption of Theorem 3.3.1, we get

$$\limsup_{t\to T_\infty} \|u(t,\cdot)\|_p < \infty,$$

where T_∞ denotes the blow-up time for L^∞ norm of (2.1).

Since we can prove this similarly as Theorem 4.16 in [15], we omit the proof.

Remark 3.3.4. It is known that

$$\liminf_{t\to T_\infty} \|u(t,\cdot)\|_p = \infty, \quad p > \frac{N\alpha}{2},$$
$$\limsup_{t\to T_\infty} \|u(t,\cdot)\|_p < \infty, \quad p < \frac{N\alpha}{2},$$

(see [21]). For the case of $p = \frac{N\alpha}{2}$, see [43].
Without the knowledge, we can predict the boundedness of L^p norm of the exact solution from numerical experiments.

In numerical experiments, we set $H(s) = s^\alpha$.

3.4 Adaptive time step control (Groisman)

We still assume $f(s) = s^{1+\alpha}$. We shall adopt the time step control in Groisman [23]. Setting $m_k = \langle \phi_k, \phi_k \rangle$ and $a_{j,k} = A(\phi_j, \phi_k)$ in Groisman's explicit scheme (cf. (1.4) in [23]), we can obtain the following results.

Assumption 3.4.1 (Assumption (H1)-(H2) in [23])**.** We consider the semi-discrete problem for (2.1). Find $u_h(t) \in S_h,\ t > 0$ such that

$$\langle u_h'(t), \chi\rangle + A(u_h(t), \chi) = \langle f(u_h(t)), \chi\rangle, \ \chi \in S_h,$$

where $u_h(0) \in S_h$ is given. We assume that for any $0 \leq T < T_\infty$,

$$\|u - u_h\|_{L^\infty(I\times(0,T))} \to 0 \ (h \to 0) \text{ and } \|(u - u_h)_x\|(T) \to 0 \ (h \to 0).$$

Theorem 3.4.2 (Convergence of blow-up time, Corollary 2.1 in [23])**.** Under the assumption of Theorem 2.2.6 and Assumption 3.4.1, we suppose that the solution u of (2.1) blows up at finite time T_∞ in the sense that

$$\|u(\cdot,t)\|_{L^\infty(I)} \to \infty \quad \text{and} \quad \|u(\cdot,t)\| \to \infty \quad (t \to T_\infty - 0). \tag{3.8}$$

The time increment τ_n is iteratively defined as

$$\tau_n = \tau_n(h,\tau) = \tau \min\left\{1, \frac{1}{\langle 1, u_h^n\rangle^{1+\alpha}}\right\}, \tag{3.9}$$

where we have used the solution u_h^n of (ML–2). Moreover, assume that

$$\tau_n \leq \min_{0\leq i\leq m-1} \frac{m_i}{a_{i,i}} \tag{3.10}$$

and

$$\forall T < T_\infty, \quad \lim_{h\to 0}\lim_{\tau\to 0}\sup_{0\leq t_n\leq T} |K(u(\cdot,t_n)) - K_h(u_h^n)| = 0. \tag{3.11}$$

For small h and τ, we then have

$$T_{h,\tau} = \sum_{n=0}^{\infty} \tau_n < \infty, \;\; \lim_{h\to 0}\lim_{\tau\to 0} T_{h,\tau} = T_\infty. \tag{3.12}$$

Remark 3.4.3. Theorem 3.4.2 means the convergence of an iterated limit.

Remark 3.4.4. For the same time increment and the solution u_h^n of (ML–2), Lemma 2.3 and Lemma 2.4 of [23] show that

$$\|u_h^n\|_{L^\infty(I)} \sim \langle 1, u_h^n\rangle \sim n, \text{ if } \|u_h^n\|_{L^\infty(I)} > \kappa,$$

where $f \sim g$ is defined by

$$cg \leq f \leq Cg, \text{ (c and C are independent of f and g)}$$

and

$$\kappa = \frac{\left(2\sum_{i=1}^{m-1}\sum_{k=1}^{m-1} a_{i,k}\right)^{\frac{1}{\alpha}} \left(\min_{0\leq k\leq m-1} m_k\right)^{\frac{p-2}{p-1}}}{\left(\sum_{k=1}^{m-1} m_k\right)^{p-1}}.$$

Theorem 3.4.5 (Blow-up rate for numerical solution, Theorem 2.2 in [23])**.** Under the same increment τ_n as (3.9) and (3.10), we can get

$$\lim_{n\to\infty} \|u_h^n\|_{L^\infty(I)} (T_{h,\tau} - t_n)^{1/\alpha} = C_\alpha = (1/\alpha)^{1/\alpha},$$

where we have used the blow-up solution u_h^n of (ML–2).

Remark 3.4.6. We assume $u^0 \geq 0$, $u_x^0 \leq 0$ in I and $u^0(1) = 0$. It is well-known that

$$\lim_{t\to T_\infty - 0} u(0,t)(T_\infty - t)^{1/\alpha} = C_\alpha,$$

if $(u^0)_{xx} + \frac{N-1}{x}(u^0)_x + u^0|u^0|^\alpha \geq 0$ in I and $N = 1,\ 2$ or $\alpha + 1 \leq \frac{N+2}{N-2}$ when $N \geq 3$ (see [21]).

Theorem 3.4.7 (Numerical blow up set, Theorem 2.3 in [23])**.** For the time increment (3.9)–(3.10) and the blow-up solution of (ML-2), we define the set B by

$$B = \left\{ j : \lim_{n\to\infty} u_j^n (T_{h,\tau} - t_n)^{1/\alpha} = C_\alpha \right\}.$$

We assume that if $d(j') < d(j)$, then

$$u_h^n(x_j) < u_h^n(x_{j'}) \text{ for any } n > 0,$$

where

$$d(j) = \min_{i \in B} |i - j|.$$

Then $u_h^n(x_j) \to \infty$ $(n \to \infty)$ holds true if and only if $d(j) \le K$ holds, where

$$K = \left\lfloor \frac{1}{\alpha} \right\rfloor \text{ and } \lfloor \cdot \rfloor \text{ denotes the floor function.}$$

If $d(k) \le K$, then we can get

$$u_h^n(x_k) \sim \begin{cases} (T_{h,\tau} - t_n)^{-\frac{1}{\alpha} + d(k)}, & d(k) \ne \frac{1}{\alpha}, \\ \log(T_{h,\tau} - t_n), & d(k) = \frac{1}{\alpha}. \end{cases}$$

For (ML-3), see §3 in [23].

3.5 Numerical examples

In this section, we introduce some numerical examples about the asymptotic behavior and numerical blow-up time of (ML-2).

First, we observe asymptotic behaviors of (ML-2) with adaptive time increment (3.6). We calculate the following quantities:

$$\text{ratio\_i} = \frac{u_{i-1}^{n+1} - u_{i-1}^n}{u_{i-1}^n}, \; i = 1, \cdots, 4.$$

with uniform spatial mesh $m = 20$, $\tau = \frac{1}{2(N+1)} h^2$, $u_h^0 = \Pi_h u^0$, and $\gamma = \alpha$.

Evaluations were performed with three parameter sets:

Case 1 $N = 3$, $\alpha = 2$, and $u(0, x) = 6 \cos \frac{\pi}{2} x$;

Case 2 $N = 4$, $\alpha = 1$, and $u(0, x) = 100(1 - x^2)$;

Case 3 $N = 5$, $\alpha = \frac{1}{2}$, and $u(0, x) = 8000(e^{-x^2} - e^{-1})$.

We compute them until $\|u_h^n\|_{L^\infty(I)} > 10^6$. Note that by Theorem 3.2.6-3.2.7, Case 1 blows up only at the origin, Case 2 blows up at two points, and Case 3 blows up at three points. From Fig. 3.1, we see that ratio_1 converges to a positive constant in Case 1 and Case 2, which means that u_0^n increases exponentially. On the other hand, ratio_2–ratio_4 converges to 0 in Case 1 and Case 2, which means that rates of increase in $u_1^n, \cdots, u_3^n$ are decreasing. In Case 3, convergences of ratio_1–ratio_4 are slow.

Secondly, we compute the numerical blow-up time of (ML-2) with uniform time increment in Theorem 3.3.1. We adopt the increasing function $H(s) = s^\alpha$ in Theorem 3.3.1, $\tau = \frac{1}{2(N+1)} h^2$ with uniform spatial mesh, and $u_h^0 = \Pi_h u^0$. Here, we set $h = 1/m$ $(m = 16, 32, 64, 128)$. For a comparison evaluation, we executed FDM of Chen [8] and the FDM of Cho and Okamoto [15].

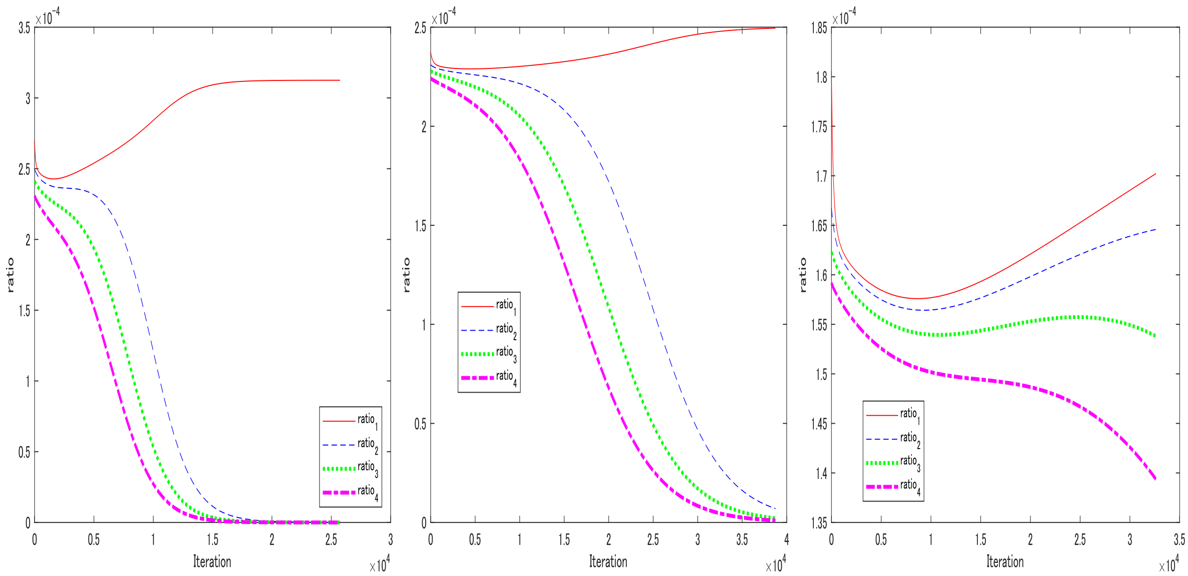
Evaluations were performed with three parameter sets:

Case 4 $N = 5$, $\alpha = \frac{4}{3}$, and $u(0, x) = 13 \cos \frac{\pi}{2} x$;

Case 5 $N = 4$, $\alpha = 3$, and $u(0, x) = 3(1 - x^2)$;

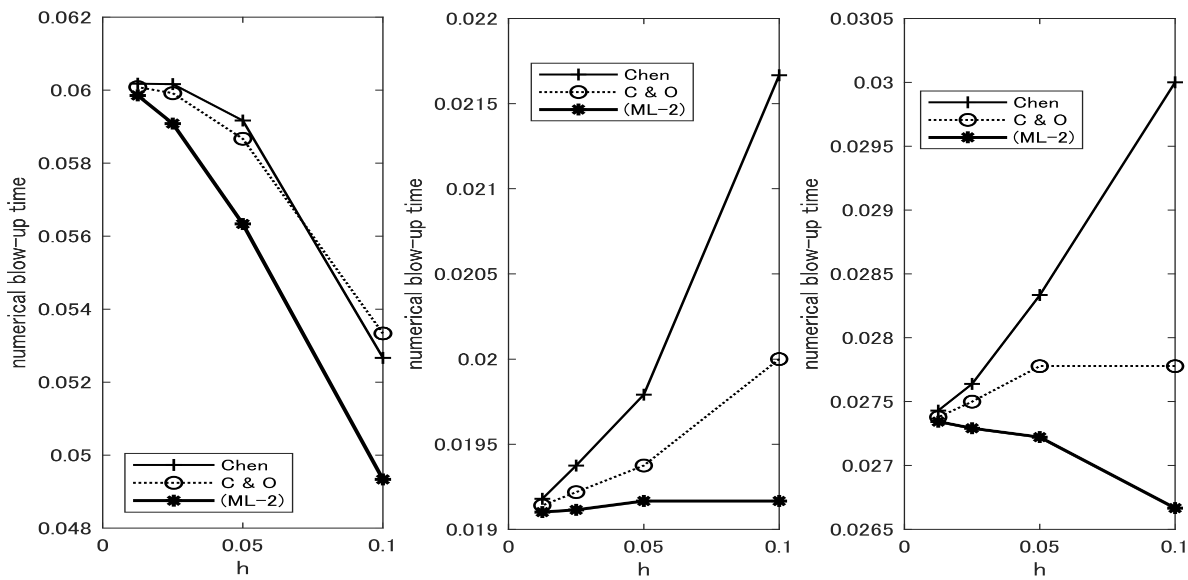
Case 6 $N = 3$, $\alpha = 3$, and $u(0, x) = \frac{9}{2}(e^{-x^2} - e^{-1})$.

These settings correspond to critical, super-critical, and sub-critical cases, respectively; see [27] for example. The solution of Chen's FDM blew up later than the other schemes, whereas that of (ML–2) blew up sooner than the other schemes (see Fig. 3.2).



Case 1 Case 2 Case 3

Figure 3.1: Asymptotic behaviors of (ML-2) with three parameter settings



Case 4 Case 5 Case 6

Figure 3.2: Numerical blow-up times with three parameter settings

3.6 Proof of Theorems

We introduce a lemma to prove boundedness of sequence.

Lemma 3.6.1 (p.461–462 in [8], Lemma 4.7 in [14] and Lemma 4.10 in [15])**.** We define a sequence $\{x_n\}$ of non-negative numbers by

$$x_{n+1} \le A_n x_n + B_n,\ x_0 \text{ is given.}$$

If $A_n \ge 0$, $B_n \ge 0$, $\displaystyle\prod_{n=1}^{\infty} A_n < \infty$ and $\displaystyle\sum_{n=0}^{\infty} B_n < \infty$, then $\{x_n\}$ is bounded sequence.

First we show Theorem 3.1.2.

Proof of Theorem 3.1.2. We rewrite (ML-2) into

$$\frac{u_0^{n+1}-u_0^n}{\tau_n}(1,\phi_0) + u_0^n A(\phi_0,\phi_0) + u_1^n A(\phi_1,\phi_0) = f(u_0^n)\cdot(1,\phi_0),$$
$$\frac{u_i^{n+1}-u_i^n}{\tau_n}(1,\phi_i) + u_{i-1}^n A(\phi_{i-1},\phi_i) + u_i^n A(\phi_i,\phi_i) + u_{i+1}^n A(\phi_{i+1},\phi_i) = f(u_i^n)\cdot(1,\phi_i),\ 1\le i\le m-1,$$

where $u_i^n = u_h^n(x_i)$. We set $m_i = (1,\phi_i)$ and $a_{i,j} = A(\phi_i,\phi_j)$. Then we can obtain

$$u_0^{n+1} = \left(1-\tau_n\frac{a_{0,0}}{m_0}\right)u_0^n - \tau_n\frac{a_{1,0}}{m_0}u_1^n + \tau_n f(u_0^n).$$

For $1\le i\le m-1$, we can get

$$u_i^{n+1} = -\tau_n\frac{a_{i-1,i}}{m_i}u_{i-1}^n + \left(1-\tau_n\frac{a_{i,i}}{m_i}\right)u_i^n - \tau_n\frac{a_{i+1,i}}{m_i}u_{i+1}^n + \tau_n f(u_i^n).$$

We estimate $u_0^{n+1}-u_1^{n+1}$.

$$\begin{aligned}
u_0^{n+1}-u_1^{n+1} &= \left(1-\tau_n\frac{a_{0,0}}{m_0}\right)u_0^n - \tau_n\frac{a_{1,0}}{m_0}u_1^n + \tau_n f(u_0^n)\\
&\quad + \tau_n\frac{a_{0,1}}{m_1}u_0^n - \left(1-\tau_n\frac{a_{1,1}}{m_1}\right)u_1^n + \tau_n\frac{a_{2,1}}{m_1}u_2^n - \tau_n f(u_1^n)\\
&\ge \left(1-\tau_n\frac{a_{0,0}}{m_0}+\tau_n\frac{a_{0,1}}{m_1}\right)u_0^n - \left(1-\tau_n\frac{a_{1,1}}{m_1}+\tau_n\frac{a_{1,0}}{m_0}-\tau_n\frac{a_{2,1}}{m_1}\right)u_1^n + \tau_n\left\{f(u_0^n)-f(u_1^n)\right\}
\end{aligned}$$

We note that f is increasing function and

$$\begin{aligned}
1-\tau_n\frac{a_{1,1}}{m_1}+\tau_n\frac{a_{1,0}}{m_0}-\tau_n\frac{a_{2,1}}{m_1} &= 1+\tau_n\frac{a_{1,0}}{m_0}+\tau_n\frac{a_{0,1}}{m_1},\\
1-\tau_n\frac{a_{0,0}}{m_0}+\tau_n\frac{a_{0,1}}{m_1} &= 1+\tau_n\frac{a_{1,0}}{m_0}+\tau_n\frac{a_{0,1}}{m_1}.
\end{aligned}$$

Therefore

$$u_0^{n+1}-u_1^{n+1} \ge \left(1+\tau_n\frac{a_{1,0}}{m_0}+\tau_n\frac{a_{0,1}}{m_1}\right)(u_0^n-u_1^n) \ge 0.$$

For $1\le i\le m-2$, we calculate $u_i^{n+1}-u_{i+1}^{n+1}$.

$$\begin{aligned}
u_i^{n+1}-u_{i+1}^{n+1} &= -\tau_n\frac{a_{i-1,i}}{m_i}u_{i-1}^n + \left(1-\tau_n\frac{a_{i,i}}{m_i}\right)u_i^n - \tau_n\frac{a_{i+1,i}}{m_i}u_{i+1}^n + \tau_n f(u_i^n)\\
&\quad + \tau_n\frac{a_{i,i+1}}{m_{i+1}}u_i^n - \left(1-\tau_n\frac{a_{i+1,i+1}}{m_{i+1}}\right)u_{i+1}^n + \tau_n\frac{a_{i+2,i+1}}{m_{i+1}}u_{i+2}^n - \tau_n f(u_{i+1}^n)\\
&\ge \left(-\tau_n\frac{a_{i-1,i}}{m_i}+1-\tau_n\frac{a_{i,i}}{m_i}+\tau_n\frac{a_{i,i+1}}{m_{i+1}}\right)u_i^n\\
&\quad - \left(\tau_n\frac{a_{i+1,i}}{m_i}+1-\tau_n\frac{a_{i+1,i+1}}{m_{i+1}}-\tau_n\frac{a_{i+2,i+1}}{m_{i+1}}\right)u_{i+1}^n + \tau_n\left\{f(u_i^n)-f(u_{i+1}^n)\right\}
\end{aligned}$$

We note that f is increasing function and

$$\begin{aligned}
-\tau_n\frac{a_{i-1,i}}{m_i}+1-\tau_n\frac{a_{i,i}}{m_i}+\tau_n\frac{a_{i,i+1}}{m_{i+1}}&=1+\tau_n\frac{a_{i+1,i}}{m_i}+\tau_n\frac{a_{i,i+1}}{m_{i+1}},\\
\tau_n\frac{a_{i+1,i}}{m_i}+1-\tau_n\frac{a_{i+1,i+1}}{m_{i+1}}-\tau_n\frac{a_{i+2,i+1}}{m_{i+1}}&=1+\tau_n\frac{a_{i+1,i}}{m_i}+\tau_n\frac{a_{i,i+1}}{m_{i+1}}.
\end{aligned}$$

Therefore

$$u_i^{n+1}-u_{i+1}^{n+1}\geq\left(1+\tau_n\frac{a_{i+1,i}}{m_i}+\tau_n\frac{a_{i,i+1}}{m_{i+1}}\right)(u_i^n-u_{i+1}^n)\geq 0.$$

For $i=m-1$, we can obtain $u_{m-1}^{n+1}\geq u_m^{n+1}=0$ by positivity preserving.
Secondly we consider (ML-3). We can rewrite into

$$\begin{aligned}
&\frac{u_0^{n+1}-u_0^n}{\tau_n}(1,\phi_0)+u_0^{n+1}A(\phi_0,\phi_0)+u_1^{n+1}A(\phi_1,\phi_0)=f(u_0^n)(1,\phi_0),\\
&\frac{u_i^{n+1}-u_i^n}{\tau_n}(1,\phi_i)+u_{i-1}^{n+1}A(\phi_{i-1},\phi_i)+u_i^{n+1}A(\phi_i,\phi_i)+u_{i+1}^{n+1}A(\phi_{i+1},\phi_i)=f(u_i^n)(1,\phi_i),\ 1\leq i\leq m-1.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{u_0^{n+1}-u_0^n}{\tau_n}-\frac{A(\phi_1,\phi_0)}{(1,\phi_0)}(u_0^{n+1}-u_1^{n+1})=f(u_0^n),\\
&\frac{u_1^{n+1}-u_1^n}{\tau_n}+\frac{A(\phi_0,\phi_1)}{(1,\phi_1)}(u_0^{n+1}-u_1^{n+1})-\frac{A(\phi_2,\phi_1)}{(1,\phi_1)}(u_1^{n+1}-u_2^{n+1})=f(u_1^n).
\end{aligned}$$

For $1\leq i\leq m-1$,

$$\begin{aligned}
\frac{u_i^{n+1}-u_i^n}{\tau_n}+\frac{A(\phi_{i-1},\phi_i)}{(1,\phi_i)}(u_{i-1}^{n+1}-u_i^{n+1})-\frac{A(\phi_{i+1},\phi_i)}{(1,\phi_i)}(u_i^{n+1}-u_{i+1}^{n+1})&=f(u_i^n),\\
\frac{u_{i+1}^{n+1}-u_{i+1}^n}{\tau_n}+\frac{A(\phi_i,\phi_{i+1})}{(1,\phi_{i+1})}(u_i^{n+1}-u_{i+1}^{n+1})-\frac{A(\phi_{i+2},\phi_{i+1})}{(1,\phi_{i+1})}(u_{i+1}^{n+1}-u_{i+2}^{n+1})&=f(u_{i+1}^n).
\end{aligned}$$

We set $w_i^n=u_i^n-u_{i+1}^n$ $(0\leq i\leq m-1)$. From positivity preserving, $w_{m-1}^{n+1}\geq 0$. Then

$$\frac{w_0^{n+1}-w_0^n}{\tau_n}-\left(\frac{A(\phi_1,\phi_0)}{(1,\phi_0)}+\frac{A(\phi_0,\phi_1)}{(1,\phi_1)}\right)w_0^{n+1}+\frac{A(\phi_2,\phi_1)}{(1,\phi_1)}w_1^{n+1}=f(u_0^n)-f(u_1^n).$$

For $1\leq i\leq m-2$,

$$\frac{w_i^{n+1}-w_i^n}{\tau_n}+\frac{A(\phi_{i-1},\phi_i)}{(1,\phi_i)}w_{i-1}^{n+1}-\left(\frac{A(\phi_{i+1},\phi_i)}{(1,\phi_i)}+\frac{A(\phi_i,\phi_{i+1})}{(1,\phi_{i+1})}\right)w_i^{n+1}+\frac{A(\phi_{i+2},\phi_{i+1})}{(1,\phi_{i+1})}w_{i+1}^{n+1}=f(u_i^n)-f(u_{i+1}^n).$$

We assume that $w_j^{n+1}=\min_{0\leq i\leq m-2}w_i^{n+1}$. We divide into two cases; the case of $j=0$ and the case of $1\leq j\leq m-2$.
(i) The case of $j=0$.
We can obtain

$$\frac{w_0^{n+1}-w_0^n}{\tau_n}-\left(\frac{A(\phi_1,\phi_0)}{(1,\phi_0)}+\frac{A(\phi_0,\phi_1)}{(1,\phi_1)}\right)w_0^{n+1}+\frac{A(\phi_2,\phi_1)}{(1,\phi_1)}w_0^{n+1}\geq 0.$$

By $w_0^n\geq 0$,

$$\left\{1-\left(\frac{A(\phi_1,\phi_0)}{(1,\phi_0)}+\frac{A(\phi_0,\phi_1)}{(1,\phi_1)}-\frac{A(\phi_2,\phi_1)}{(1,\phi_1)}\right)\tau_n\right\}w_0^{n+1}\geq 0.$$

Thus if $\tau_n<-\frac{(1,\phi_1)}{A(\phi_2,\phi_1)}$, then $w_0^{n+1}\geq 0$.
(ii) The case of $1\leq j\leq m-2$.
We can obtain

$$\frac{w_j^{n+1}-w_j^n}{\tau_n}+\frac{A(\phi_{j-1},\phi_j)}{(1,\phi_j)}w_j^{n+1}-\left(\frac{A(\phi_{j+1},\phi_j)}{(1,\phi_j)}+\frac{A(\phi_j,\phi_{j+1})}{(1,\phi_{j+1})}\right)w_j^{n+1}+\frac{A(\phi_{j+2},\phi_{j+1})}{(1,\phi_{j+1})}w_j^{n+1}\geq 0.$$

By $w_j^n \geq 0$,

$$\left\{1 - \left(-\frac{A(\phi_{j-1}, \phi_j)}{(1, \phi_j)} + \frac{A(\phi_{j+1}, \phi_j)}{(1, \phi_j)} + \frac{A(\phi_j, \phi_{j+1})}{(1, \phi_{j+1})} - \frac{A(\phi_{j+2}, \phi_{j+1})}{(1, \phi_{j+1})}\right)\tau_n\right\} w_j^{n+1} \geq 0.$$

Thus if $\tau_n < \left(-\frac{A(\phi_{j-1},\phi_j)}{(1,\phi_j)} - \frac{A(\phi_{j+2},\phi_{j+1})}{(1,\phi_{j+1})}\right)^{-1}$, then $w_j^{n+1} \geq 0$.

From (i) and (ii), we can get $w_i^{n+1} \geq 0$ $(0 \leq i \leq m-1)$. □

We shall show Remark 3.6.

Proof of Remark 3.6. First we show that if (3.6) holds, then (3.3) holds. We know that

$$\min_{0 \leq i \leq m-1} \frac{(1, \phi_i)}{A(\phi_i, \phi_i)} \geq \frac{\beta^2}{N+1} h^2 > \frac{\beta^2}{2(N+1)} h^2.$$

For $0 \leq i \leq m-2$, we get

$$\left(-\frac{A(\phi_i, \phi_{i+1})}{(1, \phi_{i+1})} - \frac{A(\phi_{i+1}, \phi_i)}{(1, \phi_i)}\right)^{-1} > \left(\frac{\int_{x_i}^{x_{i+1}} x^{N-1} \frac{1}{h_{i+1}^2}\, dx}{\int_{x_i}^{x_{i+1}} x^{N-1} \phi_i(x)\, dx} + \frac{\int_{x_i}^{x_{i+1}} x^{N-1} \frac{1}{h_{i+1}^2}\, dx}{\int_{x_i}^{x_{i+1}} x^{N-1} \phi_{i+1}(x)\, dx}\right)^{-1}.$$

From direct calculation, we can obtain

$$\frac{\int_{x_{i-1}}^{x_i} x^{N-1} \phi_i(x)\, dx}{\int_{x_{i-1}}^{x_i} x^{N-1}\, dx} \geq \frac{1}{2}, \ (1 \leq i \leq m-1) \tag{3.13}$$

$$\frac{\int_{x_i}^{x_{i+1}} x^{N-1} \phi_i(x)\, dx}{\int_{x_i}^{x_{i+1}} x^{N-1}\, dx} > \frac{1}{N+1}, \ (1 \leq i \leq m-1). \tag{3.14}$$

From (3.13),

$$\frac{\int_{x_i}^{x_{i+1}} x^{N-1}\, dx}{\int_{x_i}^{x_{i+1}} x^{N-1} \phi_i(x)\, dx} \frac{1}{h_{i+1}^2} < (N+1) \frac{1}{\beta^2 h^2}, \ (1 \leq i \leq m-1).$$

For $i = 0$, we can get

$$\frac{\int_{x_0}^{x_1} x^{N-1}\, dx}{\int_{x_0}^{x_i} x^{N-1} \phi_0\, dx} \frac{1}{h_1^2} = (N+1) \frac{1}{h_1^2} \leq (N+1) \frac{1}{\beta^2 h^2}.$$

Combining the above inequalities, we can get

$$\frac{\int_{x_i}^{x_{i+1}} x^{N-1}\, dx}{\int_{x_i}^{x_{i+1}} x^{N-1} \phi_i(x)\, dx} \frac{1}{h_{i+1}^2} \leq \frac{N+1}{\beta^2 h^2}, \ 0 \leq i \leq m-2.$$

From (3.14),

$$\frac{\int_{x_i}^{x_{i+1}} x^{N-1}\, dx}{\int_{x_i}^{x_{i+1}} x^{N-1} \phi_{i+1}(x)\, dx} \frac{1}{h_{i+1}^2} \leq \frac{2}{\beta^2 h^2} \leq \frac{N+1}{\beta^2 h^2}, \ 0 \leq i \leq m-2.$$

Therefore

$$\left(-\frac{A(\phi_i, \phi_{i+1})}{(1, \phi_{i+1})} - \frac{A(\phi_{i+1}, \phi_i)}{(1, \phi_i)}\right)^{-1} > \frac{\beta^2 h^2}{2(N+1)}.$$

We show that if (3.6) holds, then (3.4) holds.

$$-\frac{(1, \phi_1)}{A(\phi_2, \phi_1)} = \frac{\int_{x_0}^{x_2} x^{N-1} \phi_1(x)\, dx}{\int_{x_1}^{x_2} x^{N-1} \frac{1}{h_2^2}\, dx} > \frac{\int_{x_1}^{x_2} x^{N-1} \phi_1(x)\, dx}{\int_{x_1}^{x_2} x^{N-1}\, dx} h_2^2.$$

By (3.14), we can obtain

$$\frac{\int_{x_1}^{x_2} x^{N-1} \phi_1(x)\, dx}{\int_{x_1}^{x_2} x^{N-1}\, dx} > \frac{1}{N+1}.$$

Thus

$$-\frac{(1,\phi_1)}{A(\phi_2,\phi_1)} > \frac{1}{N+1}\beta^2 h^2 > \frac{1}{2(N+1)}\beta^2 h^2.$$

For $1 \le i \le m-2$,

$$\left(-\frac{A(\phi_{i-1},\phi_i)}{(1,\phi_i)} - \frac{A(\phi_{i+2},\phi_{i+1})}{(1,\phi_{i+1})}\right)^{-1} > \left(\frac{\int_{x_{i-1}}^{x_i} x^{N-1}\frac{1}{h_i^2\ dx}}{\int_{x_{i-1}}^{x_i} x^{N-1}\phi_i(x)\ dx} + \frac{\int_{x_{i+1}}^{x_{i+2}} x^{N-1}\frac{1}{h_{i+2}^2\ dx}}{\int_{x_{i+1}}^{x_{i+2}} x^{N-1}\phi_{i+1}(x)\ dx}\right)^{-1}.$$

From (3.13) and (3.14),

$$\frac{\int_{x_{i-1}}^{x_i} x^{N-1}\ dx}{\int_{x_{i-1}}^{x_i} x^{N-1}\phi_i(x)\ dx} \le 2 < N+1,\ 1 \le i \le m-2,$$

$$\frac{\int_{x_{i+1}}^{x_{i+2}} x^{N-1}\ dx}{\int_{x_{i+1}}^{x_{i+2}} x^{N-1}\phi_{i+1}(x)\ dx} < N+1,\ 1 \le i \le m-2.$$

Thus

$$\left(-\frac{A(\phi_{i-1},\phi_i)}{(1,\phi_i)} - \frac{A(\phi_{i+2},\phi_{i+1})}{(1,\phi_{i+1})}\right)^{-1} > \frac{1}{2(N+1)}\beta^2 h^2,\ 1 \le i \le m-2.$$

□

We shall show Lemma 3.2.3.

Proof of Lemma 3.2.3. We show $W_j^n \ge 0$ $(0 \le j \le m-1,\ n \ge 0)$ by induction. We assume that $W_j^n \ge 0$ $(0 \le j \le m-1)$ holds true. Then we show $W_j^{n+1} \ge 0$ $(0 \le j \le m-1)$. From definition of W_j^n, we get

$$\frac{W_j^{n+1} - W_j^n}{\tau_n} = \frac{\delta^2 u_j^{n+1} - \delta^2 u_j^n}{\tau_n} + (1-a)\frac{(u_j^{n+1})^{1+\alpha} - (u_j^n)^{1+\alpha}}{\tau_n}\ (0 \le j \le m-1).$$

We set $V_j^n = \frac{u_j^{n+1} - u_j^n}{\tau_n}$. Then

$$\frac{W_j^{n+1} - W_j^n}{\tau_n} = \delta^2 V_j^n + (1-a)\frac{(u_j^{n+1})^{1+\alpha} - (u_j^n)^{1+\alpha}}{\tau_n}\ (0 \le j \le m-1).$$

From (ML-2),

$$\frac{u_j^{n+1} - u_j^n}{\tau_n} = \delta^2 u_j^n + (u_j^n)^{1+\alpha}\ (0 \le j \le m-1).$$

Thus $W_j^n = V_j^n - a(u_j^n)^{1+\alpha}$ $(0 \le j \le m-1)$.

We can obtain

$$\begin{aligned}\frac{W_j^{n+1} - W_j^n}{\tau_n} - \delta^2 W_j^n &= \delta^2 V_j^n + (1-a)\frac{(u_j^{n+1})^{1+\alpha} - (u_j^n)^{1+\alpha}}{\tau_n} - \delta^2\left(V_j^n - a(u_j^n)^{1+\alpha}\right)\\ &= (1-a)\frac{(u_j^{n+1})^{1+\alpha} - (u_j^n)^{1+\alpha}}{\tau_n} + a\delta^2\{(u_j^n)^{1+\alpha}\}.\end{aligned}$$

First, we estimate $\frac{(u_j^{n+1})^{1+\alpha} - (u_j^n)^{1+\alpha}}{\tau_n}$. From mean value theorem and $W_j^n \ge 0$ for $0 \le j \le m-1$, we get

$$\frac{(u_j^{n+1})^{1+\alpha} - (u_j^n)^{1+\alpha}}{\tau_n} \ge (1+\alpha)(u_j^n)^\alpha V_j^n.$$

Secondly, we estimate $\delta^2\{(u_j^n)^{1+\alpha}\}$. We set $m_j = (1,\phi_j)$, $a_{i,j} = A(\phi_i,\phi_j)$. For the case of $1 \le j \le m-1$,

$$\begin{aligned}\delta^2\{(u_j^n)^{1+\alpha}\} &= -\frac{1}{m_j}a_{j-1,j}(u_{j-1}^n)^{1+\alpha} - \frac{1}{m_j}a_{j,j}(u_j^n)^{1+\alpha} - \frac{1}{m_j}a_{j+1,j}(u_{j+1}^n)^{1+\alpha}\\ &= \frac{1}{m_j}\left\{-a_{j+1,j}((u_{j+1}^n)^{1+\alpha} - (u_j^n)^{1+\alpha}) + a_{j-1,j}((u_j^n)^{1+\alpha} - (u_{j-1}^n)^{1+\alpha})\right\}\\ &= \frac{1}{m_j}\left\{-a_{j+1,j}(1+\alpha)(\tilde{u}_{j,1}^n)^\alpha(u_{j+1}^n - u_j^n) + a_{j-1,j}(1+\alpha)(\tilde{u}_{j,2}^n)^\alpha(u_j^n - u_{j-1}^n)\right\},\end{aligned}$$

where $\tilde{u}^n_{j,1}$ and $\tilde{u}^n_{j,2}$ satisfy

$$u^n_{j+1} \le \tilde{u}^n_{j,1} \le u^n_j \le \tilde{u}^n_{j,2} \le u^n_{j-1}.$$

Thus

$$\begin{aligned}\delta^2\{(u^n_j)^{1+\alpha}\} &\ge \frac{1}{m_j}\left\{-a_{j+1,j}(1+\alpha)(u^n_j)^\alpha(u^n_{j+1}-u^n_j) + a_{j-1,j}(1+\alpha)(u^n_j)^\alpha(u^n_j - u^n_{j-1})\right\}\\ &= (1+\alpha)(u^n_j)^\alpha\delta^2(u^n_j).\end{aligned}$$

For the case of $j=0$,

$$\begin{aligned}\delta^2\{(u^n_0)^{1+\alpha}\} &= -\frac{1}{m_0}a_{0,0}(u^n_0)^{1+\alpha} - \frac{1}{m_0}a_{1,0}(u^n_1)^{1+\alpha}\\ &= -\frac{1}{m_0}a_{0,0}\left\{(u^n_0)^{1+\alpha} - (u^n_1)^{1+\alpha}\right\}\\ &= -\frac{1}{m_0}a_{0,0}(1+\alpha)(\tilde{u}^n_{0,1})^\alpha(u^n_0 - u^n_1),\end{aligned}$$

where $\tilde{u}^n_{0,1}$ satisfies

$$u^n_1 \le \tilde{u}^n_{0,1} \le u^n_0.$$

Thus

$$\begin{aligned}\delta^2\{(u^n_0)^{1+\alpha}\} &\ge -\frac{1}{m_0}a_{0,0}(1+\alpha)(u^n_0)^\alpha(u^n_0 - u^n_1)\\ &= (1+\alpha)(u^n_0)^\alpha\delta^2(u^n_0).\end{aligned}$$

From the above estimations,

$$\begin{aligned}\frac{W^{n+1}_j - W^n_j}{\tau_n} - \delta^2 W^n_j &\ge (1-a)(1+\alpha)(u^n_j)^\alpha V^n_j + a(1+\alpha)(u^n_j)^\alpha\delta^2(u^n_j)\\ &= (1+\alpha)(u^n_j)^\alpha\{(1-a)V^n_j + a\delta^2(u^n_j)\}.\end{aligned}$$

Here, we can see that

$$\begin{aligned}(1-a)V^n_j + a\delta^2(u^n_j) &= (1-a)\{\delta^2(u^n_j) + (u^n_j)^{1+\alpha}\} + a\delta^2(u^n_j)\\ &= \delta^2(u^n_j) + (1-a)(u^n_j)^{1+\alpha}\\ &= W^n_j.\end{aligned}$$

Therefore

$$\frac{W^{n+1}_j - W^n_j}{\tau_n} - \delta^2 W^n_j \ge (1+\alpha)(u^n_j)^\alpha W^n_j.$$

We can obtain

$$\frac{W^{n+1}_0 - W^n_0}{\tau_n} + \frac{a_{0,0}}{m_0}W^n_0 + \frac{a_{1,0}}{m_0}W^n_1 \ge (1+\alpha)(u^n_0)^\alpha W^n_0.$$

Thus

$$\begin{aligned}W^{n+1}_0 &\ge W^n_0 - \frac{a_{0,0}}{m_0}\tau_n W^n_0 + (1+\alpha)(u^n_0)^\alpha W^n_0\\ &\ge (1 - \frac{a_{0,0}}{m_0}\tau_n)W^n_0.\end{aligned}$$

For $1 \le j \le m-1$,

$$\frac{W^{n+1}_j - W^n_j}{\tau_n} + \frac{a_{j-1,j}}{m_j}W^n_{j-1} + \frac{a_{j,j}}{m_j}W^n_j + \frac{a_{j+1,j}}{m_j}W^n_{j+1} \ge (1+\alpha)(u^n_j)^\alpha W^n_j.$$

Thus

$$\begin{aligned} W_j^{n+1} &\geq W_j^n - \frac{a_{j,j}}{m_j}\tau_n W_j^n + (1+\alpha)(u_j^n)^\alpha W_j^n \\ &\geq \left(1 - \frac{a_{j,j}}{m_j}\tau_n\right) W_j^n. \end{aligned}$$

From the time step condition, we can get $W_j^{n+1} \geq 0$ $(0 \leq j \leq m-1)$. Since u_h^{n+1} is a decreasing function, we get $u_0^{n+1} = \|u_h^{n+1}\|_\infty$, where $\|\cdot\|_\infty$ denotes the L^∞ norm. Thus

$$\frac{\|u_h^{n+1}\|_\infty - \|u_h^n\|_\infty}{\tau_n} = \frac{u_0^{n+1} - u_0^n}{\tau_n} \geq a(u_0^n)^{1+\alpha}.$$

□

We will prove Lemma 3.2.4.

Proof of Lemma 3.2.4. First, we show $T_\infty \leq \liminf_{h\to 0} T_h = T_*$ by contradiction. We assume that $T_\infty > T_*$. Then we can take a sequence $\{h_i\}_{i=1}^\infty$ such that $h_i \to 0$ $(i \to \infty)$, and

$$T_{h_i} \leq T_* + \delta < T_\infty,$$

where $\delta = (T_\infty - T_*)/2$. Thus $\|u_{h_i}^n\|_{L^\infty(I)} \to \infty$ $(n \to \infty)$. Since T_∞ is the blow-up time of $u(t,x)$, we can see

$$\max_{0\leq t\leq T_*+\delta} \|u(t)\|_\infty < \infty,$$

where $\|v\|_\infty = \|v\|_{L^\infty(I)}$. However $u_{h_i}^n$ satisfies

$$\lim_{n\to\infty} \|u_{h_i}^n\|_\infty = \infty.$$

This contradicts the convergence property of (ML-2).

Secondly, we take $\tilde{t} \in (0, T_\infty)$ such that $\|u(\tilde{t})\|_\infty > 2$. Then there exists $h_1 > 0$ such that $\tilde{t} < T_h$ for all $h < h_1$. From the convergence property of (ML-2), there exists $n_0(h) \in \mathbb{N}$ such that

$$\begin{aligned} &\|u_h^{n_0(h)}\|_\infty \geq 1,\ 0 < t_{n_0(h)} \leq \tilde{t}, \\ &|\tilde{t} - t_{n_0(h)}| < \tau. \end{aligned}$$

We set $v_h^j = u_h^{n_0(h)+j}$, $v(t) = u(t+\tilde{t})$. Then we define the blow-up time T_∞^v of v by

$$T_\infty^v = T_\infty - \tilde{t}.$$

Similarly we can define the numerical blow-up time T_h^v of v_h^n by

$$T_h^v = T_h - t_{n_0(h)}.$$

Then we prove

$$\lim_{h\to 0} T_h^v = T_\infty^v. \tag{3.15}$$

If we can show (3.15), then

$$\begin{aligned} |T_h - T_\infty| &= |(T_h - t_{n_0(h)}) - (T_\infty - \tilde{t}) + (t_{n_0(h)} - \tilde{t})| \\ &\leq |T_h^v - T_\infty^v| + |t_{n_0(h)} - \tilde{t}| \\ &\leq |T_h^v - T_\infty^v| + \tau \to 0\ (h \to 0). \end{aligned}$$

Therefore we check 14 conditions of Saito-Sasaki's paper to prove (3.15). In their paper, we set $X = L^\infty(I)$, $J(t,v) = \|v\|_{L^\infty(I)}$, and $\alpha = 1$. Moreover, we define $G(s) = as^{1+\alpha}$, $H(s) = s^\gamma$ and $f(s) = s + \tau\frac{G(s)}{H(s)}$. Then we can see that $f(s)$ is increasing function in $s \in [1,\infty)$. In fact,

$$\begin{aligned} f(s) &= s + a\tau s^{1+\alpha-\gamma}, \\ f'(s) &= 1 + a\tau(1+\alpha-\gamma)s^{\alpha-\gamma}. \end{aligned}$$

If $\alpha \geq \gamma$, then $f'(s) \geq 0$. If $\alpha < \gamma$, then we can see that $f'(s) \geq 1 + a\tau(-\tau^{-1})1 = 1 - a > 0$ since $-\tau^{-1} < 1 + \alpha - \gamma < 1 + \alpha$. In both cases, f is a increasing function. Thus we complete the proof. □

We prove Theorem 3.2.6. We use $\|\cdot\|_\infty$ as L^∞ norm.

Proof of Theorem 3.2.6. (Step 1) First, we can see that $\lim_{n\to\infty}\|u_h^n\|_\infty = \lim_{n\to\infty} u_0^n = \infty$ from Lemma 3.2.3. Actually, from Lemma 3.2.3, we get

$$\frac{\|u_h^{n+1}\|_\infty - \|u_h^n\|_\infty}{\tau_n} \geq a\|u_h^n\|_\infty^{1+\alpha}.$$

Thus $\|u_h^n\|_\infty$ is increasing sequence. If $\|u_h^n\|_\infty \geq 1$, then $\tau_n = \tau$ and $\|u_h^{n+1}\|_\infty \geq (1+a\tau)\|u_h^n\|_\infty$. Thus $\lim_{n\to\infty}\|u_h^n\|_\infty = \infty$. If $\|u_h^n\|_\infty < 1$, then

$$\begin{aligned}\|u_h^{n+1}\|_\infty - \|u_h^n\|_\infty \geq a\tau\|u_h^n\|_\infty^{1+\alpha} \geq a\tau\|u_h^0\|_\infty^\alpha\|u_h^n\|_\infty,\\ \|u_h^{n+1}\|_\infty \geq (1+a\tau\|u_h^0\|_\infty^\alpha)\|u_h^n\|_\infty.\end{aligned}$$

Thus there exists $N \in \mathbb{N}$ such that $\|u_h^N\|_\infty > 1$. In both cases, we can see $\lim_{n\to\infty}\|u_h^n\|_\infty = \infty$.

(Step 2) Secondly, we can see

$$\lim_{n\to\infty}\frac{\|u_h^{n+1}\|_\infty}{\|u_h^n\|_\infty} = 1+\tau. \tag{3.16}$$

In fact, from (ML-2), Lemma 3.1.2,

$$\begin{aligned}\frac{\|u_h^{n+1}\|_\infty}{\|u_h^n\|_\infty} = \frac{u_0^{n+1}}{u_0^n} &= \left(1-\tau_n\frac{a_{0,0}}{m_0}\right) - \tau_n\frac{a_{1,0}}{m_0}\frac{u_1^n}{u_0^n} + \tau\\ &\to 1+\tau \ (n\to\infty).\end{aligned}$$

(Step 3) Next, we prove $\lim_{n\to\infty}\dfrac{u_1^n}{u_0^n} = 0$. From (ML-2),

$$\frac{u_1^{n+1}}{u_0^{n+1}} = \frac{-\tau_n\frac{a_{0,1}}{m_1}u_0^n + (1-\tau_n\frac{a_{1,1}}{m_1})u_1^n - \tau_n\frac{a_{2,1}}{m_1}u_2^n + \tau_n(u_1^n)^{1+\alpha}}{(1-\tau_n\frac{a_{0,0}}{m_0})u_0^n - \tau_n\frac{a_{1,0}}{m_0}u_1^n + \tau_n(u_0^n)^{1+\alpha}}.$$

We set $a_n = \frac{u_1^n}{u_0^n}$. Then

$$a_{n+1} = \frac{u_1^{n+1}}{u_0^{n+1}} = \frac{-\tau_n\frac{a_{0,1}}{m_1} + (1-\tau_n\frac{a_{1,1}}{m_1})a_n - \tau_n\frac{a_{2,1}}{m_1}\frac{u_2^n}{u_0^n} + a_n^{1+\alpha}\tau_n(u_0^n)^\alpha}{(1-\tau_n\frac{a_{0,0}}{m_0}) - \tau_n\frac{a_{1,0}}{m_0}a_n + \tau_n(u_0^n)^\alpha}.$$

Since $\frac{u_2^n}{u_0^n} \leq \frac{u_1^n}{u_0^n} = a_n$,

$$a_{n+1} \leq \frac{(1-\tau_n\frac{a_{1,1}}{m_1} - \tau_n\frac{a_{2,1}}{m_1})a_n - \tau_n\frac{a_{0,1}}{m_1} + a_n^{1+\alpha}\tau_n(u_0^n)^\alpha}{(1-\tau_n\frac{a_{0,0}}{m_0}) - \tau_n\frac{a_{1,0}}{m_0}a_n + \tau_n(u_0^n)^\alpha}.$$

We assume that there does not exist $\lim_{n\to\infty} a_n$. Then from Lemma 3.1.2, we get

$$0 \leq \underline{a} := \liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n =: \overline{a} \leq 1.$$

For any $\kappa \in (\underline{a}, \overline{a})$, there exists a subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that

$$a_{n_k} < \kappa,\ a_{n_k+1} \geq \kappa.$$

Then

$$\begin{aligned}\kappa &\leq \limsup_{k\to\infty} a_{n_k+1} \leq \limsup_{k\to\infty}\frac{(1-\tau_{n_k}\frac{a_{1,1}}{m_1} - \tau_{n_k}\frac{a_{2,1}}{m_1})a_{n_k} - \tau_{n_k}\frac{a_{0,1}}{m_1} + a_{n_k}^{1+\alpha}\tau_{n_k}(u_0^{n_k})^\alpha}{(1-\tau_{n_k}\frac{a_{0,0}}{m_0}) - \tau_{n_k}\frac{a_{1,0}}{m_0}a_{n_k} + \tau_{n_k}(u_0^{n_k})^\alpha}\\ &\leq \frac{\kappa+\kappa^{1+\alpha}\tau}{1+\tau} = \frac{1+\kappa^\alpha\tau}{1+\tau}\kappa < \kappa.\end{aligned}$$

Thus we can see that there exists $a = \lim_{n\to\infty} a_n \in [0,1]$. We can get the similar inequality for a:

$$a \leq \frac{a + a^{1+\alpha}\tau}{1+\tau} = \frac{1+\tau a^{\alpha}}{1+\tau} a.$$

Thus $a = 0$ or 1. We will prove $a \neq 1$.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &\leq \frac{(1+\tau_n \frac{a_{0,1}}{m_1})a_n - \tau_n \frac{a_{0,1}}{m_1} + a_n^{1+\alpha}\tau_n (u_0^n)^{\alpha}}{a_n\{(1+\tau_n\frac{a_{1,0}}{m_0}) - \tau_n \frac{a_{1,0}}{m_0} a_n + \tau_n (u_0^n)^{\alpha}\}} \\
&= \frac{a_n - \tau_n \frac{a_{0,1}}{m_1}(1-a_n) + a_n^{1+\alpha}\tau_n (u_0^n)^{\alpha}}{a_n + \tau_n \frac{a_{1,0}}{m_0} a_n(1-a_n) + \tau_n a_n (u_0^n)^{\alpha}} = \frac{C_n}{D_n},
\end{aligned}$$

where

$$\begin{aligned}
C_n &:= a_n - \tau_n \frac{a_{0,1}}{m_1}(1-a_n) + a_n^{1+\alpha}\tau_n(u_0^n)^{\alpha}, \\
D_n &:= a_n + \tau_n \frac{a_{1,0}}{m_0} a_n(1-a_n) + a_n\tau_n(u_0^n)^{\alpha}.
\end{aligned}$$

Then

$$\begin{aligned}
C_n - D_n &= (a_n - \tau_n \frac{a_{0,1}}{m_1}(1-a_n) + a_n^{1+\alpha}\tau_n(u_0^n)^{\alpha}) - (a_n + \tau_n \frac{a_{1,0}}{m_0} a_n(1-a_n) + a_n\tau_n(u_0^n)^{\alpha}) \\
&= (1-a_n)\tau_n \left\{ -\frac{a_{0,1}}{m_1} - \frac{a_{1,0}}{m_0}a_n - \frac{a_n - a_n^{1+\alpha}}{1-a_n}(u_0^n)^{\alpha} \right\}.
\end{aligned}$$

We divide two cases: the case of $0 < \alpha < 1$ and the case of $\alpha \geq 1$.
(i) The case of $0 < \alpha < 1$
For $x \in [0,1]$, we can see that

$$\begin{aligned}
1 - x &\leq \frac{1}{\alpha}(1-x^{\alpha}), \\
\alpha &\leq \frac{1-x^{\alpha}}{1-x}.
\end{aligned}$$

Thus

$$\begin{aligned}
C_n - D_n &= (1-a_n)\tau_n \left\{ -\frac{a_{0,1}}{m_1} - \frac{a_{1,0}}{m_0}a_n - a_n\frac{1-a_n^{\alpha}}{1-a_n}(u_0^n)^{\alpha} \right\} \\
&\leq (1-a_n)\tau_n \left\{ -\frac{a_{0,1}}{m_1} - \frac{a_{1,0}}{m_0} - \alpha a_n (u_0^n)^{\alpha} \right\}.
\end{aligned}$$

If $a = 1$, then $a_n(u_0^n)^{\alpha} \to \infty$ as $n \to \infty$. Thus $\{a_n\}$ is a decreasing sequence. This contradicts $a = 1$.

(ii) The case of $\alpha \geq 1$
From $1 - a_n \leq 1 - a_n^{\alpha}$,

$$C_n - D_n \leq (1-a_n)\tau_n \left\{ -\frac{a_{0,1}}{m_1} - \frac{a_{1,0}}{m_0} - a_n(u_0^n)^{\alpha} \right\}.$$

Similarly we can see that $\{a_n\}$ is decreasing sequence if we assume that $a = 1$. In both cases, we can obtain $a = 0$.

(Step 4) Next we prove

$$\lim_{n\to\infty} u_1^n = \infty \text{ when } 0 < \alpha \leq 1. \tag{3.17}$$

From the positivity preserving, we can get

$$\begin{aligned}
u_1^{n+1} &= -\tau_n \frac{a_{0,1}}{m_1} u_0^n + \left(1 - \tau_n \frac{a_{1,1}}{m_1}\right) u_1^n - \tau_n \frac{a_{2,1}}{m_1} u_2^n + \tau_n (u_1^n)^{1+\alpha} \\
&\geq -\tau_n \frac{a_{0,1}}{m_1} u_0^n + \left(1 - \tau_n \frac{a_{1,1}}{m_1}\right) u_1^n.
\end{aligned}$$

Thus

$$u_1^{n+1} - u_1^n \geq -\tau_n \left(\frac{a_{0,1}}{m_1} u_0^n + \frac{a_{1,1}}{m_1} u_1^n \right).$$

Here since $\displaystyle\lim_{n\to\infty} \frac{u_1^n}{u_0^n} = 0$, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\frac{u_1^n}{u_0^n} < \epsilon$, for $n \geq N$. Thus $u_1^n < \epsilon u_0^n$, for $n > N$. In this case, we set $\epsilon < -\frac{a_{0,1}}{a_{1,1}}$. Then

$$u_1^{n+1} - u_1^n \geq -\tau_n \left(\frac{a_{0,1}}{m_1} + \epsilon \frac{a_{1,1}}{m_1} \right) u_0^n.$$

Since $\alpha \leq 1$, we can get $\tau_n u_0^n = \frac{\tau}{(u_0^n)^\alpha} u_0^n \geq \tau$ for sufficient large n. Thus

$$u_1^{n+1} - u_1^n \geq \tau \left(-\frac{a_{0,1}}{m_1} - \epsilon \frac{a_{1,1}}{m_1} \right).$$

We can obtain $\displaystyle\lim_{n\to\infty} u_1^n = \infty$.

(Step 5) We shall show $\displaystyle\lim_{n\to\infty} \frac{u_1^n}{u_1^{n+1}} = (1+\tau)^{\alpha-1}$ for $0 < \alpha < 1$. We set $w_n = \frac{u_1^n}{u_1^{n+1}}$. Then we can see that $0 < w_n < 1$ for sufficient large n. We assume that $\displaystyle\liminf_{n\to\infty} w_n < \limsup_{n\to\infty} w_n$. Then there exists $r \in [0,1]$ such that $\displaystyle\liminf_{n\to\infty} w_n < r < \limsup_{n\to\infty} w_n$. Thus there exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ such that

$$w_{n_k} \leq r, \ r < w_{n_k+1}.$$

We can get

$$u_1^{n_k} \leq r u_1^{n_k+1}, \ r u_1^{n_k+2} < u_1^{n_k+1}.$$

We shall estimate $u_1^{n_k+1} - \left\{ 1 - \tau_{n_k} \frac{a_{1,1}}{m_1} + \tau_{n_k} (u_1^{n_k})^\alpha \right\} u_1^{n_k}$. Then

$$\begin{aligned} u_1^{n_k+1} - \left\{ 1 - \tau_{n_k} \frac{a_{1,1}}{m_1} + \tau_{n_k} (u_1^{n_k})^\alpha \right\} u_1^{n_k} &\geq u_1^{n_k+1} - r \left\{ 1 - \tau_{n_k} \frac{a_{1,1}}{m_1} + \tau_{n_k} (u_1^{n_k})^\alpha \right\} u_1^{n_k+1} \\ &= \left(1 - r + r\tau_{n_k} \frac{a_{1,1}}{m_1} - r\tau_{n_k} (u_1^{n_k})^\alpha \right) u_1^{n_k+1}. \end{aligned}$$

On the other hand, we will estimate $u_1^{n_k+1} - \left\{ 1 - \tau_{n_k} \frac{a_{1,1}}{m_1} + \tau_{n_k} (u_1^n)^\alpha \right\} u_1^{n_k}$. Then

$$\begin{aligned} &u_1^{n_k+2} - \left\{ 1 - \tau_{n_k+1} \frac{a_{1,1}}{m_1} + \tau_{n_k+1} (u_1^{n_k+1})^\alpha \right\} u_1^{n_k+1} \\ &< u_1^{n_k+2} - \left(1 - \tau_{n_k+1} \frac{a_{1,1}}{m_1} \right) r u_1^{n_k+2} - \tau_{n_k+1} (u_1^{n_k+1})^{\alpha+1} \\ &< \left(1 - r + r\tau_{n_k+1} \frac{a_{1,1}}{m_1} \right) u_1^{n_k+2}. \end{aligned}$$

Therefore

$$\frac{u_1^{n_k+1} - \{1 - \tau_{n_k} \frac{a_{1,1}}{m_1} + \tau_{n_k} (u_1^{n_k})^\alpha\} u_1^{n_k}}{u_1^{n_k+2} - \{1 - \tau_{n_k+1} \frac{a_{1,1}}{m_1} + \tau_{n_k+1} (u_1^{n_k+1})^\alpha\} u_1^{n_k+1}} > \frac{\{1 - r + r\tau_{n_k} \frac{a_{1,1}}{m_1} - r\tau_{n_k} (u_1^{n_k})^\alpha\} u_1^{n_k+1}}{(1 - r + r\tau_{n_k+1} \frac{a_{1,1}}{m_1}) u_1^{n_k+2}}.$$

We define

$$A := \lim_{k\to\infty} \frac{u_1^{n_k+1} - \{1 - \tau_{n_k} \frac{a_{1,1}}{m_1} + \tau_{n_k} (u_1^{n_k})^\alpha\} u_1^{n_k}}{u_1^{n_k+2} - \{1 - \tau_{n_k+1} \frac{a_{1,1}}{m_1} + \tau_{n_k+1} (u_1^{n_k+1})^\alpha\} u_1^{n_k+1}}.$$

Then

$$\begin{aligned} A &= \lim_{k\to\infty} \frac{-\tau_{n_k} \frac{a_{0,1}}{m_1} u_0^{n_k} - \tau_{n_k} \frac{a_{2,1}}{m_1} u_2^{n_k}}{-\tau_{n_k+1} \frac{a_{0,1}}{m_1} u_0^{n_k+1} - \tau_{n_k+1} \frac{a_{2,1}}{m_1} u_2^{n_k+1}} \\ &= \lim_{k\to\infty} \frac{\tau_{n_k}}{\tau_{n_k+1}} \frac{-\frac{a_{0,1}}{m_1} \frac{u_0^{n_k}}{u_0^{n_k+1}} - \frac{a_{2,1}}{m_1} \frac{u_2^{n_k}}{u_0^{n_k+1}}}{-\frac{a_{0,1}}{m_1} - \frac{a_{2,1}}{m_1} \frac{u_2^{n_k+1}}{u_0^{n_k+1}}}. \end{aligned}$$

Since $\lim_{k\to\infty} \dfrac{u_0^{n_k}}{u_0^{n_k+1}} = (1+\tau)^{-1}$, we can get

$$\frac{\tau_{n_k}}{\tau_{n_k+1}} = \frac{(u_0^{n_k+1})^\alpha}{(u_0^{n_k})^\alpha} \to (1+\tau)^\alpha,\ k \to \infty.$$

Since

$$0 \le \frac{u_2^{n_k}}{u_0^{n_k+1}} \le \frac{u_1^{n_k}}{u_0^{n_k}} \to 0\ (k \to \infty),$$

we can get

$$\lim_{k\to\infty} \frac{u_2^{n_k}}{u_0^{n_k+1}} = 0.$$

Similarly since

$$0 \le \frac{u_2^{n_k+1}}{u_0^{n_k+1}} \le \frac{u_1^{n_k+1}}{u_0^{n_k+1}} \to 0\ (k \to \infty),$$

we can get

$$\lim_{k\to\infty} \frac{u_2^{n_k+1}}{u_0^{n_k+1}} = 0.$$

Thus

$$A = (1+\tau)^{\alpha-1}.$$

We can get

$$\begin{aligned}(1+\tau)^{\alpha-1} &\ge \limsup_{k\to\infty} \frac{\{1-r+r\tau_{n_k}\frac{a_{1,1}}{m_1} - r\tau_{n_k}(u_1^{n_k})^\alpha\}u_1^{n_k+1}}{(1-r+r\tau_{n_k+1}\frac{a_{1,1}}{m_1})u_1^{n_k+2}} \\ &= \limsup_{k\to\infty} w_{n_k+1} \ge r.\end{aligned}$$

On the other hand, there exists a subsequence $\{w_{\overline{n}_j}\} \subset \{w_n\}$ such that

$$w_{\overline{n}_j} > r,\ w_{\overline{n}_j+1} \le r.$$

Similarly we can get $r \le (1+\tau)^{\alpha-1}$. Thus $r = (1+\tau)^{\alpha-1}$. This contradicts the arbitrariness of r. Thus there exists $\lim_{n\to\infty} w_n \in [0,1]$.

We can obtain

$$\lim_{n\to\infty} \frac{u_1^{n+1} - (1-\tau_n\frac{a_{1,1}}{m_1} + \tau_n(u_1^n)^\alpha)u_1^n}{u_1^{n+2} - (1-\tau_{n+1}\frac{a_{1,1}}{m_1} + \tau_{n+1}(u_1^{n+1})^\alpha)u_1^{n+1}} = (1+\tau)^{\alpha-1}.$$

By setting $\gamma_n = -\tau_n\frac{a_{1,1}}{m_1} + \tau_n(u_1^n)^\alpha$, we can get

$$\begin{aligned}\frac{u_1^{n+1} - (1+\gamma_n)u_1^n}{u_1^{n+2} - (1+\gamma_{n+1})u_1^{n+1}} &= \frac{u_1^{n+1}}{u_1^{n+2}} \frac{1-(1+\gamma_n)\frac{u_1^n}{u_1^{n+1}}}{1-(1+\gamma_{n+1})\frac{u_1^{n+1}}{u_1^{n+2}}} \\ &= w_{n+1}\frac{1-(1+\gamma_n)w_n}{1-(1+\gamma_{n+1})w_{n+1}}.\end{aligned}$$

We set $w = \lim_{n\to\infty} w_n$. From $\lim_{n\to\infty}\gamma_n = 0$, if $w \ne 1$, then $\lim_{n\to\infty} w_n = (1+\tau)^{\alpha-1}$. We will show $w < 1$ by contradiction. We assume that $w = 1$. Then

$$\lim_{n\to\infty} \frac{\xi_n}{\xi_{n+1}} = (1+\tau)^{\alpha-1} < 1,$$

where $\xi_n = 1-(1+\gamma_n)w_n$. There exists $\rho < 1$ such that

$$\frac{\xi_n}{\xi_{n+1}} < \rho,\ \text{that is, } \rho^{-1}\xi_n < \xi_{n+1},\ \text{for sufficiently large } n.$$

On the other hand, from $w=1$, we get $\lim_{n\to\infty}\xi_n = 0$. This contradicts $\rho^{-1}\xi_n < \xi_{n+1}$. Thus we can get

$$\lim_{n\to\infty}\frac{u_1^n}{u_1^{n+1}} = (1+\tau)^{\alpha-1}\ (0<\alpha<1).$$

(Step 6) For $\alpha>1$, we will prove $\lim_{n\to\infty}\frac{u_1^n}{u_1^{n+1}} = 1$. From (ML-2), we can get

$$1 = -\tau_n\frac{a_{0,1}}{m_1}\frac{u_0^n}{u_1^{n+1}} + \left(1-\tau_n\frac{a_{1,1}}{m_1}+\tau a_n^{\alpha}\right)\frac{u_1^n}{u_1^{n+1}} - \tau_n\frac{a_{2,1}}{m_1}\frac{u_2^n}{u_1^{n+1}}.$$

We estimate the right hand side of the above equation.

$$\begin{aligned}-\tau_n\frac{a_{0,1}}{m_1}\frac{u_0^n}{u_1^{n+1}} &= -\frac{a_{0,1}}{m_1}\frac{\tau}{(u_0^n)^{\alpha}}\frac{u_0^n}{u_1^{n+1}}\\ &= \frac{-a_{0,1}\tau}{m_1}\frac{1}{(u_0^n)^{\alpha-1}}\frac{1}{u_1^{n+1}}.\end{aligned}$$

Here we can get $\inf_{n\in\mathbb{N}} u_1^n > 0$. In fact,

$$\begin{aligned}u_1^{n+1}-u_1^n &= -\tau_n\frac{a_{0,1}}{m_1}u_0^n - \tau_n\frac{a_{1,1}}{m_1}u_1^n - \tau_n\frac{a_{2,1}}{m_1}u_2^n + \tau_n(u_1^n)^{1+\alpha}\\ &\geq -\tau_n\left(\frac{a_{0,1}}{m_1}u_0^n + \frac{a_{1,1}}{m_1}u_1^n\right).\end{aligned}$$

From $\lim_{n\to\infty}\frac{u_1^n}{u_0^n} = 0$, for any $\epsilon>0$, there exists $N\in\mathbb{N}$ such that $\frac{u_1^n}{u_0^n}<\epsilon$ for $n\geq N$, that is, $u_1^n < \epsilon u_0^n$. Thus

$$u_1^{n+1}-u_1^n \geq \tau_n\left(-\frac{a_{0,1}}{m_1} - \epsilon\frac{a_{1,1}}{m_1}\right)u_0^n.$$

Since we can take $\epsilon < -\frac{a_{0,1}}{a_{1,1}}$,

$$u_1^n \geq \tau_N\left(-\frac{a_{0,1}}{m_1} - \epsilon\frac{a_{1,1}}{m_1}\right)u_0^N$$

for $n\geq N$. In particular, $\inf_{n\in\mathbb{N}} u_1^n > 0$. Therefore

$$-\tau_n\frac{a_{0,1}}{m_1}\frac{u_0^n}{u_1^{n+1}} \leq \frac{-a_{0,1}\tau}{m_1}\frac{1}{(u_0^n)^{\alpha-1}}\frac{1}{\inf_{n\in\mathbb{N}} u_1^n} \to 0\ (n\to\infty).$$

Since $u_2^n \leq u_0^n$, we can obtain

$$\lim_{n\to\infty}\left(-\tau_n\frac{a_{2,1}}{m_1}\frac{u_2^n}{u_1^{n+1}}\right) = 0.$$

Therefore $\lim_{n\to\infty}\frac{u_1^n}{u_1^{n+1}} = 1$.

(Step 7) Next we prove $\lim_{n\to\infty}\frac{u_1^n}{u_1^{n+1}} = 1$ for $\alpha=1$. Since

$$1 = -\tau_n\frac{a_{0,1}}{m_1}\frac{u_0^n}{u_1^{n+1}} + \left(1-\tau_n\frac{a_{1,1}}{m_1}+\tau a_n^{\alpha}\right)\frac{u_1^n}{u_1^{n+1}} - \tau_n\frac{a_{2,1}}{m_1}\frac{u_2^n}{u_1^{n+1}},$$

we will estimate $-\tau_n\frac{a_{0,1}}{m_1}\frac{u_0^n}{u_1^{n+1}}$.

$$-\tau_n\frac{a_{0,1}}{m_1}\frac{u_0^n}{u_1^{n+1}} = -\frac{a_{0,1}}{m_1}\frac{\tau}{u_1^{n+1}} \to 0\ (n\to\infty).$$

Here we used $\lim_{n\to\infty} u_1^n = \infty$ for $0 < \alpha \leq 1$. From order preserving property,

$$-\tau_n \frac{a_{2,1}}{m_1} \frac{u_2^n}{u_1^{n+1}} \to 0 \ (n \to \infty).$$

Therefore $\lim_{n\to\infty} \frac{u_1^n}{u_1^{n+1}} = 1$.

We shall summarize the above results: For $\alpha > 0$,

$$\lim_{n\to\infty} \frac{u_1^n}{u_1^{n+1}} = (1+\tau)^{\min(1,\alpha)-1}, \tag{3.18}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{u_1^{n+1}}{u_0^{n+1}} \frac{u_0^n}{u_1^n} \\
&= \lim_{n\to\infty} \frac{u_1^{n+1}}{u_1^n} \frac{u_0^n}{u_0^{n+1}} \\
&= (1+\tau)^{-\min(1,\alpha)}.
\end{aligned} \tag{3.19}$$

(Step 8) Finally we prove the boundedness of $\{u_1^n\}$ for $\alpha > 1$. We can get

$$\begin{aligned}
u_1^{n+1} &= -\tau_n \frac{a_{0,1}}{m_1} u_0^n + \left(1 - \tau_n \frac{a_{1,1}}{m_1} + \tau a_n^\alpha\right) u_1^n - \tau_n \frac{a_{2,1}}{m_1} u_2^n \\
&\leq -\tau_n \frac{a_{0,1}}{m_1} u_0^n - \tau_n \frac{a_{2,1}}{m_1} u_0^n + (1 + \tau a_n^\alpha) u_1^n \\
&= \tau \frac{a_{1,1}}{m_1} (u_0^n)^{1-\alpha} + (1 + \tau a_n^\alpha) u_1^n =: A_n u_1^n + B_n,
\end{aligned}$$

where $A_n = 1 + \tau a_n^\alpha$ and $B_n = \tau \frac{a_{1,1}}{m_1} (u_0^n)^{1-\alpha}$. If we can prove $\prod_{n=0}^{\infty} A_n < \infty$ and $\sum_{n=0}^{\infty} B_n < \infty$, then we can see that $\{u_1^n\}$ is bounded from Lemma 3.6.1. Thus we shall show $\prod_{n=0}^{\infty} A_n < \infty$ and $\sum_{n=0}^{\infty} B_n < \infty$. Since

$$\lim_{n\to\infty} \frac{(u_0^{n+1})^{1-\alpha}}{(u_0^n)^{1-\alpha}} = (1+\tau)^{1-\alpha} < 1,$$

we can get $\sum_{n=0}^{\infty} B_n < \infty$ from d'Alembert's ratio test. On the other hand, since

$$\log\left(\prod_{n=0}^{\infty} A_n\right) = \log\left(\prod_{n=0}^{\infty} (1 + \tau a_n^\alpha)\right) = \sum_{n=0}^{\infty} \log(1 + \tau a_n^\alpha) \leq \sum_{n=0}^{\infty} \tau a_n^\alpha,$$

it suffices to show $\sum_{n=0}^{\infty} \tau a_n^\alpha < \infty$. Since

$$\lim_{n\to\infty} \frac{a_{n+1}^\alpha}{a_n^\alpha} = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)^\alpha = (1+\tau)^{-\alpha \min(1,\alpha)} < 1,$$

we can get $\sum_{n=0}^{\infty} \tau a_n^\alpha < \infty$ from d'Alembert's ratio test. Thus we can obtain the boundedness of $\{u_1^n\}$. □

We shall show Theorem 3.2.7.

Proof. (Step 1) First we show for $0 < \alpha < 1$,

$$\lim_{n\to\infty} \frac{(u_0^n)^{1-\alpha}}{u_1^n} = \frac{(1+\tau)^{-\alpha+1} - 1}{-\tau \frac{a_{0,1}}{m_1}} \ (< \infty). \tag{3.20}$$

From (ML-2),

$$\frac{u_1^{n+1}}{u_1^n} = -\tau\frac{a_{0,1}}{m_1}\frac{(u_0^n)^{1-\alpha}}{u_1^n} + \left(1-\tau_n\frac{a_{1,1}}{m_1}\right) - \tau_n\frac{a_{2,1}}{m_1}\frac{u_2^n}{u_1^n} + \tau a_n^\alpha.$$

Since

$$\lim_{n\to\infty}\frac{u_1^{n+1}}{u_1^n} = (1+\tau)^{-\alpha+1},$$

$$\lim_{n\to\infty}\left(-\tau_n\frac{a_{2,1}}{m_1}\frac{u_2^n}{u_1^n}\right) = \lim_{n\to\infty}\left(-\frac{a_{2,1}}{m_1}\right)\tau\frac{1}{(u_0^n)^\alpha}\frac{u_2^n}{u_1^n} = 0,\ (\because u_2^n \le u_1^n.)$$

we can obtain

$$(1+\tau)^{-\alpha+1} = -\tau\frac{a_{0,1}}{m_1}\lim_{n\to\infty}\frac{(u_0^n)^{1-\alpha}}{u_1^n} + 1.$$

For $0<\alpha<1$,

$$\lim_{n\to\infty}\frac{(u_0^n)^{1-\alpha}}{u_1^n} = \frac{(1+\tau)^{-\alpha+1}-1}{-\tau\frac{a_{0,1}}{m_1}}\ (<\infty).$$

(Step 2) Secondly, for $0<\alpha<1$, we show

$$\lim_{n\to\infty}\frac{u_2^n}{u_1^n} = 0. \tag{3.21}$$

We set $c_n = \frac{u_2^n}{u_1^n}$. Then

$$\begin{aligned} c_{n+1} &\le -\tau_n\frac{a_{1,2}}{m_2}\frac{u_1^n}{u_1^{n+1}} + \left(1+\tau_n\frac{a_{1,2}}{m_2}\right)\frac{u_2^n}{u_1^{n+1}} + \tau_n(u_2^n)^\alpha\frac{u_2^n}{u_1^{n+1}} \\ &\le -\tau_n\frac{a_{1,2}}{m_2}\frac{u_1^n}{u_1^{n+1}} + (1+\tau_n(u_2^n)^\alpha)\frac{u_1^n}{u_1^{n+1}}c_n. \end{aligned}$$

Since $\tau_n\to 0$, $\frac{u_1^n}{u_1^{n+1}} \to (1+\tau)^{\alpha-1}$ $(n\to\infty)$ and $0\le\tau_n(u_2^n)^\alpha \le \left(\frac{u_1^n}{u_0^n}\right)^\alpha \to 0$ $(n\to\infty)$, we get

$$\limsup_{n\to\infty} c_{n+1} \le 0 + (1+\tau)^{\alpha-1}\limsup_{n\to\infty} c_n.$$

Therefore $\lim\limits_{n\to\infty} c_n = 0$.

(Step 3) Next we shall show the following four equations by induction: Letting $\alpha \le \frac{1}{k}$ and $k\in\mathbb{N}$,

- for $j=1,\cdots,k$,

$$\lim_{n\to\infty} u_j^n = \infty, \tag{3.22}$$

$$\lim_{n\to\infty}\frac{u_j^n}{u_j^{n+1}} = (1+\tau)^{\min(1,j\alpha)-1}, \tag{3.23}$$

- for $\alpha<\frac{1}{k}$ and $j=k$,

$$\lim_{n\to\infty}\frac{(u_j^n)^{\frac{1-(j-1)\alpha}{1-j\alpha}}}{u_{j-1}^n} \in (0,\infty), \tag{3.24}$$

$$\lim_{n\to\infty}\frac{u_{j+1}^n}{u_j^n} = 0. \tag{3.25}$$

Before we prove these properties, we show that if (3.22)–(3.25) hold true, then Theorem 3.2.7 holds true. We assume that (3.22)–(3.25) hold true. Then for $\frac{1}{k+1}<\alpha\le\frac{1}{k}$, if $j\le k$, then $\lim\limits_{n\to\infty} u_j^n = \infty$ by (3.22). On the other hand, for $j=k+1$,

$$\begin{aligned} u_{k+1}^{n+1} &\le -\tau_n\frac{a_{k,k+1}}{m_{k+1}}u_k^n + \left(1+\tau_n\frac{a_{k,k+1}}{m_{k+1}}\right)u_{k+1}^n + \tau\left(\frac{u_{k+1}^n}{u_0^n}\right)^\alpha u_{k+1}^n \\ &\le -\tau_n\frac{a_{k,k+1}}{m_{k+1}}u_k^n + \left(1+\tau_n\frac{a_{k,k+1}}{m_{k+1}}\right)u_{k+1}^n + \tau a_n^\alpha u_{k+1}^n, \end{aligned}$$

where $a_n = \frac{u_1^n}{u_0^n}$. Thus

$$u_{k+1}^{n+1} \leq -\tau_n \frac{a_{k,k+1}}{m_{k+1}} u_k^n + (1+\tau a_n^\alpha) u_{k+1}^n.$$

To show the boundedness of u_{k+1}^{n+1}, we show

$$\sum_{n=0}^{\infty} \left(-\tau_n \frac{a_{k,k+1}}{m_{k+1}} u_k^n \right) < \infty, \quad \prod_{n=0}^{\infty} (1+\tau a_n^\alpha) < \infty.$$

Since

$$\begin{aligned}
\lim_{n\to\infty} \frac{\tau_{n+1} u_k^{n+1}}{\tau_n u_k^n} &= \lim_{n\to\infty} \left(\frac{u_0^n}{u_0^{n+1}} \right)^\alpha \frac{u_k^{n+1}}{u_k^n} \\
&= (1+\tau)^{-\alpha} (1+\tau)^{-\min(1,k\alpha)+1} \ (\because (3.16) \text{ and } (3.23).) \\
&= (1+\tau)^{-(k+1)\alpha} < 1,
\end{aligned}$$

we can obtain $\sum_{n=0}^{\infty} \left(-\tau_n \frac{a_{k,k+1}}{m_{k+1}} u_k^n \right) < \infty$ by d'Alembert's ratio test.

Since

$$\log \prod_{n=0}^{\infty} (1+\tau a_n^\alpha) = \sum_{n=0}^{\infty} \log(1+\tau a_n^\alpha) \leq \sum_{n=0}^{\infty} \tau a_n^\alpha,$$

it suffices to show $\sum_{n=0}^{\infty} \tau a_n^\alpha < \infty$. From (3.19), we can get

$$\lim_{n\to\infty} \frac{a_{n+1}^\alpha}{a_n^\alpha} = (1+\tau)^{-\min(1,\alpha)\alpha} < 1.$$

Thus we can get $\prod_{n=0}^{\infty} (1+\tau a_n^\alpha) < \infty$. From the boundedness of $\{u_{k+1}^n\}$, we can see that if $\lim_{n\to\infty} u_j^n = \infty$, then $j \leq k$. Hence if (3.22)–(3.25) holds true for all $k \in \mathbb{N}$, then Theorem 3.2.7 holds true.

(Step 4) Hereinafter we prove (3.22)–(3.25).
First, we see that (3.22)–(3.25) hold true for $k = 1$.
For $0 < \alpha \leq 1$,

$$\lim_{n\to\infty} u_1^n = \infty, \ (\because (3.17))$$

$$\lim_{n\to\infty} \frac{u_1^n}{u_1^{n+1}} = (1+\tau)^{\min(1,\alpha)-1}. \ (\because (3.18))$$

Thus (3.22)–(3.23) hold true for $k = 1$.
For $0 < \alpha < 1$ and $j = 1$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{(u_1^n)^{\frac{1}{1-\alpha}}}{u_0^n} &= \lim_{n\to\infty} \left(\frac{u_1^n}{(u_0^n)^{1-\alpha}} \right)^{\frac{1}{1-\alpha}} \\
&= \left(\frac{-\tau \frac{a_{0,1}}{m_1}}{(1+\tau)^{-\alpha+1} - 1} \right)^{\frac{1}{1-\alpha}} < \infty \ (\because (3.20)).
\end{aligned}$$

Thus (3.24) holds true for $k = 1$. Moreover, from (3.21), we can get (3.25) for $k = 1$.

Secondly we assume that (3.22)–(3.25) hold true for all number less than $k+1$. Then we show (3.22)–(3.25) hold true for $k+1$. We show $\lim_{n\to\infty} u_{k+1}^n = \infty$ for $\alpha \leq \frac{1}{k+1}$. From (ML-2),

$$\begin{aligned}
u_{k+1}^{n+1} &\geq -\tau_n \frac{a_{k,k+1}}{m_{k+1}} u_k^n + \left(1 - \tau_n \frac{a_{k+1,k+1}}{m_{k+1}}\right) u_{k+1}^n \ (\because \text{ Positivity preserving.}) \\
&= -\tau \frac{a_{k,k+1}}{m_{k+1}} \left\{ \frac{(u_1^n)^{\frac{1}{1-\alpha}}}{u_0^n} \right\}^\alpha \left\{ \frac{(u_2^n)^{\frac{1-\alpha}{1-2\alpha}}}{u_1^n} \right\}^{\frac{\alpha}{1-\alpha}} \cdots \left\{ \frac{(u_k^n)^{\frac{1-(k-1)\alpha}{1-k\alpha}}}{u_{k-1}^n} \right\}^{\frac{\alpha}{1-(k-1)\alpha}} (u_k^n)^{\frac{1-(k+1)\alpha}{1-k\alpha}} \\
&\quad + \left(1 - \tau_n \frac{a_{k+1,k+1}}{m_{k+1}}\right) u_{k+1}^n.
\end{aligned}$$

Then from (3.24),

$$\lim_{n\to\infty} -\tau\frac{a_{k,k+1}}{m_{k+1}}\left\{\frac{(u_1^n)^{\frac{1}{1-\alpha}}}{u_0^n}\right\}^{\alpha}\left\{\frac{(u_2^n)^{\frac{1-\alpha}{1-2\alpha}}}{u_1^n}\right\}^{\frac{\alpha}{1-\alpha}}\cdots\left\{\frac{(u_k^n)^{\frac{1-(k-1)\alpha}{1-k\alpha}}}{u_{k-1}^n}\right\}^{\frac{\alpha}{1-(k-1)\alpha}} (=: M) < \infty.$$

Moreover by (3.22),

$$\lim_{n\to\infty}(u_k^n)^{\frac{1-(k+1)\alpha}{1-k\alpha}} = \infty.$$

Since $\liminf_{n\to\infty} u_{k+1}^{n+1} \geq M + \liminf_{n\to\infty} u_{k+1}^n$, we can get $\liminf_{n\to\infty} u_{k+1}^n = \infty$. Thus (3.22) holds true for $k+1$.

Secondly, we show (3.23) for $k+1$.

$$\frac{u_{k+1}^{n+1} - \left\{\left(1-\tau_n\frac{a_{k+1,k+1}}{m_{k+1}}\right)u_{k+1}^n + \tau_n(u_k^n)^\alpha\right\}u_k^n}{u_{k+1}^{n+2} - \left\{\left(1-\tau_{n+1}\frac{a_{k+1,k+1}}{m_{k+1}}\right)u_{k+1}^{n+1} + \tau_{n+1}(u_k^{n+1})^\alpha\right\}u_k^{n+1}} = \frac{\tau_n\left(-\frac{a_{k,k+1}}{m_{k+1}}u_k^n - \frac{a_{k+2,k+1}}{m_{k+1}}u_{k+2}^n\right)}{\tau_{n+1}\left(-\frac{a_{k,k+1}}{m_{k+1}}u_k^{n+1} - \frac{a_{k+2,k+1}}{m_{k+1}}u_{k+2}^{n+1}\right)}$$
$$\to (1+\tau)^{(k+1)\alpha-1} \ (n\to\infty).$$

Here we used the following equations:

$$\lim_{n\to\infty}\frac{\tau_n}{\tau_{n+1}} = \lim_{n\to\infty}\left(\frac{u_0^{n+1}}{u_0^n}\right)^\alpha = (1+\tau)^\alpha,\ (\because (3.16))$$
$$\lim_{n\to\infty}\frac{u_k^n}{u_k^{n+1}} = (1+\tau)^{k\alpha-1}.\ (\because (3.23))$$

Moreover,

$$0 \leq \limsup_{n\to\infty}\frac{u_{k+2}^n}{u_k^{n+1}} \leq \limsup_{n\to\infty}\frac{u_{k+1}^n}{u_k^{n+1}}$$
$$= \lim_{n\to\infty}\frac{u_{k+1}^n}{u_k^n}\frac{u_k^n}{u_k^{n+1}} = 0.\ (\because (3.23) \text{ and } (3.25))$$

Thus we have $\lim_{n\to\infty}\frac{u_{k+2}^n}{u_k^{n+1}} = 0$. Similarly, we can get $\lim_{n\to\infty}\frac{u_{k+2}^{n+1}}{u_k^{n+1}} = 0$.

Now, we set $w_n = \frac{u_{k+1}^n}{u_{k+1}^{n+1}}$. Then we shall prove the existence of $\lim_{n\to\infty} w_n$. From (ML-2), we can get

$$u_{k+1}^{n+1} - u_{k+1}^n \geq -\tau_n\frac{a_{k,k+1}}{m_{k+1}}u_k^n - \tau_n\frac{a_{k+1,k+1}}{m_{k+1}}u_{k+1}^n.$$

From $\lim_{n\to\infty}\frac{u_{k+1}^n}{u_k^n} = 0$, for sufficiently large n,

$$-\tau_n\frac{a_{k,k+1}}{m_{k+1}}u_k^n - \tau_n\frac{a_{k+1,k+1}}{m_{k+1}}u_{k+1}^n > 0.$$

Thus $0 \leq w_n < 1$ for sufficiently large n. To show the existence of $\lim_{n\to\infty} w_n$, we assume that

$$\liminf_{n\to\infty} w_n < \limsup_{n\to\infty} w_n.$$

Then there exists $r \in (0,1)$ such that $\liminf_{n\to\infty} w_n < r < \limsup_{n\to\infty} w_n$. There exists a subsequence $\{w_{n_j}\} \subset \{w_n\}$ such that

$$w_{n_j} \leq r,\ r < w_{n_j+1}.$$

Thus

$$u_{k+1}^{n_j} \leq r u_{k+1}^{n_j+1},\ r u_{k+1}^{n_j+2} < u_{k+1}^{n_j+1}.$$

We will estimate $u_{k+1}^{n_j+1} - \left(1 - \tau_{n_j}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j}(u_{k+1}^{n_j})^\alpha\right)u_{k+1}^{n_j}$.

$$
\begin{aligned}
u_{k+1}^{n_j+1} - \left(1 - \tau_{n_j}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j}(u_{k+1}^{n_j})^\alpha\right)u_{k+1}^{n_j} &\geq u_{k+1}^{n_j+1} - r\left(1 - \tau_{n_j}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j}(u_{k+1}^{n_j})^\alpha\right)u_{k+1}^{n_{j+1}} \\
&= \left\{1 - r\left(1 - \tau_{n_j}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j}(u_{k+1}^{n_j})^\alpha\right)\right\}u_{k+1}^{n_j+1}.
\end{aligned}
$$

On the other hand, we estimate $u_{k+1}^{n_j+2} - \left(1 - \tau_{n_j+1}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j+1}(u_{k+1}^{n_j+1})^\alpha\right)u_{k+1}^{n_j+1}$.

$$
\begin{aligned}
&u_{k+1}^{n_j+2} - \left(1 - \tau_{n_j+1}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j+1}(u_{k+1}^{n_j+1})^\alpha\right)u_{k+1}^{n_j+1} \\
&< \left\{1 - r\left(1 - \tau_{n_j+1}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j+1}(u_{k+1}^{n_j+1})^\alpha\right)\right\}u_{k+1}^{n_j+2}.
\end{aligned}
$$

Here we can see that

$$
\begin{aligned}
A &:= \lim_{j\to\infty}\frac{u_{k+1}^{n_j+1} - \left(1 - \tau_{n_j}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j}(u_{k+1}^{n_j})^\alpha\right)u_{k+1}^{n_j}}{u_{k+1}^{n_j+2} - \left(1 - \tau_{n_j+1}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j+1}(u_{k+1}^{n_j+1})^\alpha\right)u_{k+1}^{n_j+1}} \\
&= \lim_{j\to\infty}\frac{-\tau_{n_j}\frac{a_{k,k+1}}{m_{k+1}}u_k^{n_j} - \tau_{n_j}\frac{a_{k+2,k+1}}{m_{k+1}}u_{k+2}^{n_j}}{-\tau_{n_j+1}\frac{a_{k,k+1}}{m_{k+1}}u_k^{n_j+1} - \tau_{n_j+1}\frac{a_{k+2,k+1}}{m_{k+1}}u_{k+2}^{n_j+1}}.
\end{aligned}
$$

We can obtain

$$
\begin{aligned}
&\lim_{j\to\infty}\frac{u_k^{n_j}}{u_k^{n_j+1}} = (1+\tau)^{k\alpha-1},\ (\because (3.23)) \\
&\lim_{j\to\infty}\frac{\tau_{n_j}}{\tau_{n_j+1}} = \lim_{j\to\infty}\left(\frac{u_0^{n_j+1}}{u_0^{n_j}}\right)^\alpha = (1+\tau)^\alpha\ (\because (3.16)).
\end{aligned}
$$

Since

$$
\frac{u_{k+2}^{n_j}}{u_k^{n_j+1}} \leq \frac{u_{k+1}^{n_j}}{u_k^{n_j}}\frac{u_k^{n_j}}{u_k^{n_j+1}} \to 0\ (j\to\infty),\ (\because (3.23) \text{ and } (3.25))
$$

we can get

$$
\lim_{j\to\infty}\frac{u_{k+2}^{n_j}}{u_k^{n_j+1}} = 0.
$$

Similarly, we can get

$$
\lim_{j\to\infty}\frac{u_{k+2}^{n_j+1}}{u_k^{n_j+1}} = 0.
$$

Thus

$$
A = (1+\tau)^{k\alpha-1}(1+\tau)^\alpha = (1+\tau)^{(k+1)\alpha-1}.
$$

$$
\begin{aligned}
(1+\tau)^{(k+1)\alpha-1} &\geq \limsup_{j\to\infty}\frac{\left\{1 - r\left(1 - \tau_{n_j}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j}(u_{k+1}^{n_j})^\alpha\right)\right\}u_{k+1}^{n_j+1}}{\left\{1 - r\left(1 - \tau_{n_j+1}\frac{a_{k+1,k+1}}{m_{k+1}} + \tau_{n_j+1}(u_{k+1}^{n_j+1})^\alpha\right)\right\}u_{k+1}^{n_j+2}} \\
&= \limsup_{j\to\infty} w_{n_j+1} = r.
\end{aligned}
$$

Similarly, we can take a subsequence $\{w_{\overline{n}_j}\} \subset \{w_n\}$ such that

$$
w_{\overline{n}_j} > r,\ w_{\overline{n}_j+1} \leq r.
$$

Using this subsequence, we can obtain

$$
(1+\tau)^{(k+1)\alpha-1} \leq r.
$$

Therefore

$$(1+\tau)^{(k+1)\alpha-1}=r.$$

This contradicts the arbitrariness of r. Thus we can see the existence of $\lim_{n\to\infty} w_n \in [0,1]$.

Next we prove

$$\lim_{n\to\infty} w_n = (1+\tau)^{(k+1)\alpha-1}.$$

From above argument, we can get

$$\lim_{n\to\infty}\frac{u_{k+1}^{n+1}-\left(1-\tau_n\frac{a_{k+1,k+1}}{m_{k+1}}-\tau_n(u_{k+1}^n)^\alpha\right)u_{k+1}^n}{u_{k+1}^{n+2}-\left(1-\tau_{n+1}\frac{a_{k+1,k+1}}{m_{k+1}}-\tau_{n+1}(u_{k+1}^{n+1})^\alpha\right)u_{k+1}^{n+1}}=(1+\tau)^{(k+1)\alpha-1}.$$

We set $\gamma_n=-\tau_n\frac{a_{k+1,k+1}}{m_{k+1}}-\tau_n(u_{k+1}^n)^\alpha$. Then we can get

$$\frac{u_{k+1}^{n+1}-(1+\gamma_n)u_{k+1}^n}{u_{k+1}^{n+2}-(1+\gamma_{n+1})u_{k+1}^{n+1}}=w_{n+1}\frac{1-(1+\gamma_n)w_n}{1-(1+\gamma_{n+1})w_{n+1}}.$$

We set $w=\lim_{n\to\infty} w_n$. From $\lim_{n\to\infty}\gamma_n=0$, we can see that if $w\neq 1$, then $w=(1+\tau)^{(k+1)\alpha-1}$. If $w=1$, then

$$\lim_{n\to\infty}\frac{\xi_n}{\xi_{n+1}}=(1+\tau)^{(k+1)\alpha-1},$$

where $\xi_n=1-(1+\gamma_n)w_n$. We assume that $\alpha<\frac{1}{k+1}$. Then

$$(1+\tau)^{(k+1)\alpha-1}<1.$$

There exists $\rho<1$ such that

$$\frac{\xi_n}{\xi_{n+1}}<\rho,\ \text{for sufficiently large } n.$$

Hence

$$\rho^{-1}\xi_n<\xi_{n+1}.$$

This contradicts $\lim_{n\to\infty}\xi_n=0$. Thus $\lim_{n\to\infty} w_n\neq 1$. For $\alpha<\frac{1}{k+1}$, we can get

$$\lim_{n\to\infty} w_n=(1+\tau)^{(k+1)\alpha-1}.$$

On the other hand, we assume that $\alpha=\frac{1}{k+1}$. Then from (ML-2),

$$1=-\tau_n\frac{a_{k,k+1}}{m_{k+1}}\frac{u_k^n}{u_{k+1}^{n+1}}+\left(1-\tau_n\frac{a_{k+1,k+1}}{m_{k+1}}+\tau_n(u_{k+1}^n)^\alpha\right)\frac{u_{k+1}^n}{u_{k+1}^{n+1}}-\tau_n\frac{a_{k+2,k+1}}{m_{k+1}}\frac{u_{k+2}^n}{u_{k+1}^{n+1}}.$$

Since

$$\tau_n\frac{u_k^n}{u_{k+1}^{n+1}}=\frac{\tau}{(u_0^n)^\alpha}\frac{u_k^n}{u_{k+1}^{n+1}}=\frac{\tau}{(u_0^n)^{\frac{1}{k+1}}}\frac{u_k^n}{u_{k+1}^{n+1}},$$

we consider the limit of $\frac{u_k^n}{(u_0^n)^{\frac{1}{k+1}}}$. We can obtain

$$\frac{u_k^n}{(u_0^n)^{\frac{1}{k+1}}}=\frac{u_k^n}{(u_{k-1}^n)^{\frac12}}\frac{(u_{k-1}^n)^{\frac12}}{(u_{k-2}^n)^{\frac13}}\cdots\frac{(u_1^n)^{\frac1k}}{(u_0^n)^{\frac{1}{k+1}}},$$

and by (3.24), there exists $\lim_{n\to\infty}\frac{(u_j^n)^{\frac{1-(j-1)\alpha}{1-j\alpha}}}{u_{j-1}^n}\in(0,\infty)$ for $j=1,\cdots,k$. Since $\alpha=\frac{1}{k+1}$, we can get

$$\frac{(u_j^n)^{\frac{1-(j-1)\frac{1}{k+1-j}}{1-j\frac{1}{k+1}}}}{u_{j-1}^n}=\left(\frac{(u_j^n)^{\frac{1}{k+1-j}}}{(u_{j-1}^n)^{\frac{1}{k+1-(j-1)}}}\right)^{k+1-(j-1)}.$$

Thus there exists $\lim_{n\to\infty} \frac{(u_j^n)^{\frac{1}{k+1-j}}}{(u_{j-1}^n)^{\frac{1}{k+1-(j-1)}}} \in (0,\infty),\ j=1,\cdots,k$. Since

$$\lim_{n\to\infty} \frac{u_k^n}{(u_0^n)^{\frac{1}{k+1}}} \in (0,\infty),$$

we can get

$$\lim_{n\to\infty} \frac{\tau}{(u_0^n)^{\frac{1}{k+1}}} \frac{u_{k+2}^n}{u_{k+1}^{n+1}} = 0.$$

Thus if $\alpha = \frac{1}{k+1}$, then

$$\lim_{n\to\infty} \frac{u_{k+1}^n}{u_{k+1}^{n+1}} = 1.$$

For $\alpha \le \frac{1}{k+1}$, we can get

$$\lim_{n\to\infty} \frac{u_{k+1}^n}{u_{k+1}^{n+1}} = (1+\tau)^{(k+1)\alpha-1}.$$

Thus (3.23) holds true. Next we prove (3.24) for $\alpha < \frac{1}{k+1}$. From (ML-2),

$$1 = -\tau_n \frac{a_{k,k+1}}{m_{k+1}} \frac{u_k^n}{u_{k+1}^{n+1}} + \left(1 - \tau_n \frac{a_{k+1,k+1}}{m_{k+1}} + \tau_n (u_{k+1}^n)^\alpha\right) \frac{u_{k+1}^n}{u_{k+1}^{n+1}} - \tau_n \frac{a_{k+2,k+1}}{m_{k+1}} \frac{u_{k+2}^n}{u_{k+1}^{n+1}}.$$

Here we can get

$$\begin{aligned}
\tau_n \frac{u_{k+2}^n}{u_{k+1}^{n+1}} &= \tau \frac{1}{(u_0^n)^\alpha} \frac{u_{k+2}^n}{u_{k+1}^{n+1}} \le \tau \frac{1}{(u_0^n)^\alpha} \frac{u_{k+1}^n}{u_{k+1}^{n+1}} \to 0\ (n\to\infty),\\
\tau_n (u_{k+1}^n)^\alpha &= \tau \left(\frac{u_{k+1}^n}{u_0^n}\right)^\alpha \le \tau \left(\frac{u_{k+1}^n}{u_k^n}\right)^\alpha \to 0\ (n\to\infty).
\end{aligned}$$

Combining these equations, we can get

$$\lim_{n\to\infty} \tau_n \left(-\frac{a_{k,k+1}}{m_{k+1}}\right) \frac{u_k^n}{u_{k+1}^{n+1}} = 1 - (1+\tau)^{(k+1)\alpha-1}.$$

We can get

$$\begin{aligned}
\tau_n \frac{u_k^n}{u_{k+1}^{n+1}} &= \tau \frac{1}{(u_0^\alpha)^\alpha} \frac{u_k^n}{u_{k+1}^{n+1}}\\
&= \tau \left\{\frac{(u_1^n)^{\frac{1}{1-\alpha}}}{u_0^n}\right\}^\alpha \left\{\frac{(u_2^n)^{\frac{1-\alpha}{1-2\alpha}}}{u_1^n}\right\}^{\frac{\alpha}{1-\alpha}} \cdots \left\{\frac{(u_k^n)^{\frac{1-(k-1)\alpha}{1-k\alpha}}}{u_{k-1}^n}\right\}^{\frac{\alpha}{1-(k-1)\alpha}} \frac{(u_k^n)^{\frac{1-(k+1)\alpha}{1-k\alpha}}}{u_{k+1}^n} \frac{u_{k+1}^n}{u_{k+1}^{n+1}}.
\end{aligned}$$

There exists

$$\lim_{n\to\infty} \frac{(u_k^n)^{\frac{1-(k+1)\alpha}{1-k\alpha}}}{u_{k+1}^n} \in (0,\infty).$$

Thus (3.24) holds true.

Finally, we prove (3.25) for $\alpha < \frac{1}{k+1}$.

$$\begin{aligned}
\frac{u_{k+2}^{n+1}}{u_{k+1}^{n+1}} &= -\tau_n \frac{a_{k+1,k+2}}{m_{k+2}} \frac{u_{k+1}^n}{u_{k+1}^{n+1}} + \left(1 - \tau_n \frac{a_{k+2,k+2}}{m_{k+2}} + \tau_n (u_{k+2}^n)^\alpha\right) \frac{u_{k+2}^n}{u_{k+1}^{n+1}} - \tau_n \frac{a_{k+3,k+2}}{m_{k+2}} \frac{u_{k+3}^n}{u_{k+1}^{n+1}}\\
&\le \left\{-\tau_n \frac{a_{k+1,k+2}}{m_{k+2}} + \left(1 + \tau_n (u_{k+2}^n)^\alpha\right) \frac{u_{k+2}^n}{u_{k+1}^n}\right\} \frac{u_{k+1}^n}{u_{k+1}^{n+1}}.
\end{aligned}$$

Thus

$$\limsup_{n\to\infty} \frac{u_{k+2}^{n+1}}{u_{k+1}^{n+1}} \le (1+\tau)^{(k+1)\alpha-1} \limsup_{n\to\infty} \frac{u_{k+2}^n}{u_{k+1}^n}.$$

Since $\alpha < \frac{1}{k+1}$, we can see $(1+\tau)^{(k+1)\alpha-1} < 1$. We can get

$$\limsup_{n\to\infty} \frac{u_{k+2}^n}{u_{k+1}^n} = 0.$$

Thus (3.25) holds true. Therefore we can obtain (3.22)–(3.25) by induction.

□

We prove Theorem 3.2.8.

Proof of Theorem 3.2.8. (Step 1) First we show

$$\lim_{n\to\infty} \frac{u_0^{n+1}}{u_0^n} = 1+\tau.$$

From (ML-3), we can get

$$\begin{aligned}
\left(1+\tau_n \frac{a_{0,0}}{m_0}\right) u_0^{n+1} + \tau_n \frac{a_{1,0}}{m_0} u_1^{n+1} &= u_0^n + \tau_n (u_0^n)^{1+\alpha}, \\
\left(1+\tau_n \frac{a_{0,0}}{m_0}\right) \frac{u_0^{n+1}}{u_0^n} + \tau_n \frac{a_{1,0}}{m_0} \frac{u_1^{n+1}}{u_0^n} &= 1+\tau, \text{ for large } n, \\
\left(1+\tau_n \frac{a_{0,0}}{m_0} + \tau_n \frac{a_{1,0}}{m_0} \frac{u_1^{n+1}}{u_0^{n+1}}\right) \frac{u_0^{n+1}}{u_0^n} &= 1+\tau.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{u_0^{n+1}}{u_0^n} = 1+\tau.$$

(Step 2) Next we will prove

$$\lim_{n\to\infty} u_1^n = \infty,$$

for $0 < \alpha \le 1$. We shall estimate u_0^{n+1} and u_1^{n+1}.

$$\begin{aligned}
u_0^{n+1} &= \frac{-\tau_n \frac{a_{1,0}}{m_0} u_1^n + (1+\tau_n (u_0^n)^\alpha) u_0^n}{1+\tau_n \frac{a_{0,0}}{m_0}} \\
&\ge \frac{(1+\tau_n (u_0^n)^\alpha) u_0^n}{1+\tau_n \frac{a_{0,0}}{m_0}}.
\end{aligned}$$

From (ML-3),

$$\tau_n \frac{a_{0,1}}{m_1} u_0^{n+1} + \left(1+\tau_n \frac{a_{1,1}}{m_1}\right) u_1^{n+1} + \tau_n \frac{a_{2,1}}{m_1} u_2^{n+1} = u_1^n + \tau_n (u_1^n)^{1+\alpha}.$$

Thus

$$\begin{aligned}
u_1^{n+1} &= \frac{-\tau_n \frac{a_{0,1}}{m_1} u_0^{n+1} - \tau_n \frac{a_{2,1}}{m_1} u_2^{n+1} + (1+\tau_n (u_1^n)^\alpha) u_1^n}{1+\tau_n \frac{a_{1,1}}{m_1}} \\
&\ge \frac{-\tau_n \frac{a_{0,1}}{m_1} u_0^{n+1} + (1+\tau_n (u_1^n)^\alpha) u_1^n}{1+\tau_n \frac{a_{1,1}}{m_1}} \\
&\ge \frac{-\tau_n \frac{a_{0,1}}{m_1} (1+\tau_n (u_0^n)^\alpha) u_0^n + (1+\tau_n \frac{a_{0,0}}{m_0})(1+\tau_n (u_1^n)^\alpha) u_1^n}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)} \\
&\ge \frac{-\tau_n \frac{a_{0,1}}{m_1} u_0^n + u_1^n}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)},
\end{aligned}$$

for $n \ge 0$.

Thus

$$\liminf_{n\to\infty} u_1^{n+1} \geq -\frac{a_{0,1}}{m_1} \liminf_{n\to\infty} \frac{\tau(u_0^n)^{1-\alpha}}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)} + \liminf_{n\to\infty} u_1^n.$$

Here,

$$\liminf_{n\to\infty} \frac{\tau(u_0^n)^{1-\alpha}}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)} = \begin{cases} \infty, & 0<\alpha<1 \\ \tau, & \alpha = 1. \end{cases}$$

Therefore

$$\lim_{n\to\infty} u_1^n = \infty,\ 0<\alpha\leq 1.$$

(Step 3) Next we will show

$$\lim_{n\to\infty} a_n = 0,\ a_n = \frac{u_1^n}{u_0^n}.$$

We can see that

$$\begin{aligned} a_{n+1} = \frac{u_1^{n+1}}{u_0^{n+1}} &= \frac{u_1^{n+1}(1+\tau_n \frac{a_{0,0}}{m_0})}{-\tau_n \frac{a_{1,0}}{m_0} u_1^{n+1} + (1+\tau_n (u_0^n)^\alpha) u_0^n} \\ &= \frac{1+\tau_n \frac{a_{0,0}}{m_0}}{-\tau_n \frac{a_{1,0}}{m_0} + (1+\tau_n (u_0^n)^\alpha) \frac{u_0^n}{u_1^{n+1}}}. \end{aligned}$$

To get lower and upper bound of a_{n+1}, we calculate lower and upper bound of u_1^{n+1}.

$$\begin{aligned} u_1^{n+1} &= \frac{-\tau_n \frac{a_{0,1}}{m_1} u_0^{n+1} - \tau_n \frac{a_{2,1}}{m_1} u_2^{n+1} + (1+\tau_n (u_1^n)^\alpha) u_1^n}{1+\tau_n \frac{a_{1,1}}{m_1}} \\ &\leq \frac{-\tau_n \frac{a_{0,1}}{m_1} u_0^{n+1} - \tau_n \frac{a_{2,1}}{m_1} u_1^{n+1} + (1+\tau_n (u_1^n)^\alpha) u_1^n}{1+\tau_n \frac{a_{1,1}}{m_1}} \\ &\leq \frac{-\tau_n \frac{a_{0,1}}{m_1} (-\tau_n \frac{a_{1,0}}{m_0} u_1^{n+1} + (1+\tau_n (u_0^n)^\alpha) u_0^n)}{\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)} \\ &\quad + \frac{-\tau_n \frac{a_{2,1}}{m_1} u_1^{n+1} + (1+\tau_n (u_1^n)^\alpha) u_1^n}{1+\tau_n \frac{a_{1,1}}{m_1}}. \end{aligned}$$

Thus

$$\begin{aligned} &\left(1 - \frac{\tau_n^2 \frac{a_{0,1}}{m_1} \frac{a_{1,0}}{m_0}}{\left(1+\frac{a_{0,0}}{m_0}\right)\left(1+\frac{a_{1,1}}{m_1}\right)} + \frac{\tau_n \frac{a_{2,1}}{m_1}}{1+\tau_n \frac{a_{1,1}}{m_1}}\right) u_1^{n+1} \\ &\qquad \leq \frac{-\tau_n \frac{a_{0,1}}{m_1} (1+\tau_n (u_0^n)^\alpha) u_0^n + \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau_n (u_1^n)^\alpha) u_1^n}{\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)}. \end{aligned}$$

We compute left-hand side.

$$\begin{aligned} (LHS) &= \frac{1+\tau_n \frac{a_{0,0}}{m_0} + \tau_n \frac{a_{1,1}}{m_1} + \tau_n^2 \frac{a_{0,0}}{m_0} \frac{a_{1,1}}{m_1} - \tau_n^2 \frac{a_{0,1}}{m_1} \frac{a_{1,0}}{m_0} + \tau_n \frac{a_{2,1}}{m_1} + \tau_n^2 \frac{a_{0,0}}{m_0} \frac{a_{2,1}}{m_1}}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)} u_1^{n+1} \\ &= \frac{1+\tau_n \frac{a_{0,0}}{m_0} - \tau_n \frac{a_{0,1}}{m_1} + \tau_n^2 \frac{a_{0,0}}{m_0} \frac{a_{1,1}}{m_1} + \tau_n^2 \frac{a_{0,1}}{m_1} \frac{a_{0,0}}{m_0} + \tau_n^2 \frac{a_{0,0}}{m_0} \frac{a_{2,1}}{m_1}}{\left(1+\tau_n \frac{a_{1,1}}{m_1} \frac{a_{0,0}}{m_0}\right)} u_1^{n+1} \\ &= \frac{1+\tau_n \frac{a_{0,0}}{m_0} - \tau_n \frac{a_{0,1}}{m_1} + \tau_n^2 \frac{a_{0,0}}{m_0} \left(a_{1,1}+a_{0,1}+a_{2,1}\right) \frac{1}{m_1}}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)} u_1^{n+1} \\ &= \frac{1+\tau_n \frac{a_{0,0}}{m_0} - \tau_n \frac{a_{0,1}}{m_1}}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)} u_1^{n+1}. \end{aligned}$$

Thus we can get

$$u_1^{n+1} \le \frac{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+\tau_n\frac{a_{0,0}}{m_0} - \tau_n\frac{a_{0,1}}{m_1}}. \tag{3.26}$$

For the lower bound of u_1^{n+1}, we can get

$$\begin{aligned} u_1^{n+1} &\ge \frac{-\tau_n\frac{a_{0,1}}{m_1}u_0^{n+1} + (1+\tau_n(u_1^n)^\alpha)u_1^n}{1+\tau_n\frac{a_{1,1}}{m_1}} \\ &= \frac{-\tau_n\frac{a_{0,1}}{m_1}\left(-\tau_n\frac{a_{1,0}}{m_0}u_1^{n+1} + (1+\tau_n(u_0^n)^\alpha)u_0^n\right)}{\left(1+\tau_n\frac{a_{1,1}}{m_1}\right)\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)} + \frac{(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+\tau_n\frac{a_{1,1}}{m_1}}. \end{aligned}$$

Thus

$$\begin{aligned} &\left\{1 - \frac{\tau_n^2\frac{a_{0,1}}{m_1}\left(-\tau_n\frac{a_{1,0}}{m_0}u_1^{n+1} + (1+\tau_n(u_0^n)^\alpha)u_0^n\right)}{\left(1+\tau_n\frac{a_{1,1}}{m_1}\right)\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)}\right\}u_1^{n+1} \\ &\qquad \ge \frac{-\tau_n\frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n + (1+\tau_n\frac{a_{0,0}}{m_0})(1+\tau_n(u_1^n)^\alpha)u_1^n}{\left(1+\tau_n\frac{a_{1,1}}{m_1}\right)\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)}. \end{aligned}$$

Here we calculate the left-hand side.

$$\begin{aligned} (LHS) &= \frac{1+\tau_n\frac{a_{1,1}}{m_1} + \tau_n\frac{a_{0,0}}{m_0} + \tau_n^2\frac{a_{1,1}}{m_1}\frac{a_{0,0}}{m_0} - \tau_n^2\frac{a_{0,1}}{m_1}\frac{a_{1,0}}{m_0}}{\left(1+\tau_n\frac{a_{1,1}}{m_1}\right)\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)}u_1^{n+1} \\ &= \frac{1+\tau_n\frac{a_{1,1}}{m_1} + \tau_n\frac{a_{0,0}}{m_0} + \tau_n^2\frac{a_{1,1}}{m_1}\frac{a_{0,0}}{m_0} + \tau_n^2\frac{a_{0,1}}{m_1}\frac{a_{0,0}}{m_0}}{\left(1+\tau_n\frac{a_{1,1}}{m_1}\right)\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)}u_1^{n+1} \\ &= \frac{1+\tau_n\frac{a_{1,1}}{m_1} + \tau_n\frac{a_{0,0}}{m_0} - \tau_n^2\frac{a_{0,0}}{m_0}\frac{a_{2,1}}{m_1}}{\left(1+\tau_n\frac{a_{1,1}}{m_1}\right)\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)}u_1^{n+1}. \end{aligned}$$

Therefore

$$u_1^{n+1} \ge \frac{-\tau_n\frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n + (1+\tau_n\frac{a_{0,0}}{m_0})(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+\tau_n\frac{a_{1,1}}{m_1} + \tau_n\frac{a_{0,0}}{m_0} - \tau_n^2\frac{a_{0,0}}{m_0}\frac{a_{2,1}}{m_1}}. \tag{3.27}$$

From (3.26), we can get

$$\begin{aligned} &a_{n+1} = \frac{\left(1+\frac{a_{0,0}}{m_0}\right)u_1^{n+1}}{-\tau_n\frac{a_{1,0}}{m_0}u_1^{n+1} + (1+\tau_n(u_0^n)^\alpha)u_0^n} \\ &\le \frac{\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)\left\{-\tau_n\frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n\right\}}{-\tau_n\frac{a_{1,0}}{m_0}\left\{-\tau_n\frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n\right\} + (1+\tau_n(u_0^n)^\alpha)u_0^n(1+\tau_n\frac{a_{0,0}}{m_0} - \frac{a_{0,1}}{m_1})} \\ &= \frac{\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)\left\{-\tau_n\frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n\right\}}{\left(1+\tau_n\frac{a_{0,0}}{m_0} - \tau_n\frac{a_{0,1}}{m_1} + \tau_n^2\frac{a_{1,0}}{m_0}\frac{a_{0,1}}{m_1}\right)(1+\tau_n(u_0^n)^\alpha)u_0^n - \tau_n\frac{a_{1,0}}{m_0}\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n}. \end{aligned}$$

Since

$$\begin{aligned} 1+\tau_n\frac{a_{0,0}}{m_0} - \tau_n\frac{a_{0,1}}{m_1} + \tau_n^2\frac{a_{1,0}}{m_0}\frac{a_{0,1}}{m_1} &= 1+\tau_n\frac{a_{0,0}}{m_0} - \tau_n\frac{a_{0,1}}{m_1} - \tau_n^2\frac{a_{0,0}}{m_0}\frac{a_{0,1}}{m_1} \\ &= \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)\left(1+\tau_n\frac{a_{1,1}}{m_1}\right), \end{aligned}$$

we can obtain

$$a_{n+1} = \frac{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau_n (u_0^n)^\alpha)u_0^n + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n}{\left(1-\tau_n\frac{a_{0,1}}{m_1}\right)(1+\tau_n(u_0^n)^\alpha)u_0^n - \tau_n\frac{a_{1,0}}{m_0}(1+\tau_n(u_1^n)^\alpha)u_1^n}$$
$$= \frac{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau)u_0^n + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau a_n^\alpha)u_1^n}{\left(1-\tau_n\frac{a_{0,1}}{m_1}\right)(1+\tau)u_0^n - \tau_n\frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)u_1^n}, \text{ for large } n.$$

Therefore we can obtain the upper bound of a_{n+1},

$$a_{n+1} \le \frac{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau) + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau a_n^\alpha)a_n}{\left(1-\tau_n\frac{a_{0,1}}{m_1}\right)(1+\tau) - \tau_n\frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)a_n}. \tag{3.28}$$

For the lower bound of a_{n+1},

$$a_{n+1} = \frac{u_1^{n+1}}{u_0^{n+1}} = \frac{\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)u_1^{n+1}}{-\tau_n\frac{a_{1,0}}{m_0}u_1^{n+1} + (1+\tau_n(u_0^n)^\alpha)u_0^n}$$
$$= \frac{1+\tau_n\frac{a_{0,0}}{m_0}}{-\tau_n\frac{a_{1,0}}{m_0} + (1+\tau_n(u_0^n)^\alpha)\frac{u_0^n}{u_1^{n+1}}} \ge \frac{X}{Y} \ (\because (3.26)),$$

where

$$X := \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)\left\{-\tau_n\frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n\right\},$$
$$Y := -\tau_n\frac{a_{1,0}}{m_0}\left\{-\tau_n\frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n\right\}$$
$$+ (1+\tau_n(u_0^n)^\alpha)u_0^n\left(1+\tau_n\frac{a_{1,1}}{m_1}+\tau_n\frac{a_{0,0}}{m_0}-\tau_n^2\frac{a_{0,0}}{m_0}\frac{a_{2,1}}{m_1}\right).$$

We calculate X.

$$X = \tau_n^2\frac{a_{1,0}}{m_0}\frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n - \tau_n\frac{a_{1,0}}{m_0}\left(1+\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n$$
$$+ (1+\tau_n\frac{a_{1,1}}{m_1}+\tau_n\frac{a_{0,0}}{m_0}-\tau_n^2\frac{a_{0,0}}{m_0}\frac{a_{2,1}}{m_1})(1+\tau_n(u_0^n)^\alpha)u_0^n.$$

Here, we can see that

$$\begin{aligned}
&1+\tau_n\frac{a_{1,1}}{m_1}+\tau_n\frac{a_{0,0}}{m_0}-\tau_n^2\frac{a_{0,0}}{m_0}\frac{a_{2,1}}{m_1}+\tau_n^2\frac{a_{1,0}}{m_0}\frac{a_{0,1}}{m_1}\\
=&1+\tau_n\frac{a_{1,1}}{m_1}+\tau_n\frac{a_{0,0}}{m_0}-\tau_n^2\frac{a_{0,0}}{m_0}\frac{a_{2,1}}{m_1}-\tau_n^2\frac{a_{0,0}}{m_0}\frac{a_{0,1}}{m_1}\\
=&1+\tau_n\frac{a_{1,1}}{m_1}+\tau_n\frac{a_{0,0}}{m_0}+\tau_n^2\frac{a_{0,0}}{m_0}\frac{a_{1,1}}{m_1}\\
=&\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)\left(1+\tau_n\frac{a_{1,1}}{m_1}\right).
\end{aligned}$$

Thus

$$X = \left(1+\tau_n\frac{a_{1,1}}{m_1}\right)\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_0^n)^\alpha)u_0^n - \tau_n\frac{a_{1,0}}{m_0}\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n.$$

Therefore

$$
\begin{aligned}
a_{n+1} &\geq \frac{\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)\left\{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau_n (u_0^n)^\alpha)u_0^n + \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau_n (u_1^n)^\alpha)u_1^n\right\}}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau_n (u_0^n)^\alpha)u_0^n - \tau_n \frac{a_{1,0}}{m_0}\left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau_n (u_1^n)^\alpha)u_1^n} \\
&= \frac{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau_n (u_0^n)^\alpha)u_0^n + \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau_n (u_1^n)^\alpha)u_1^n}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)(1+\tau_n (u_0^n)^\alpha)u_0^n - \tau_n \frac{a_{1,0}}{m_0}(1+\tau_n (u_1^n)^\alpha)u_1^n} \\
&= \frac{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau)u_0^n + \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau a_n^\alpha)u_1^n}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)(1+\tau)u_0^n - \tau_n \frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)u_1^n} \text{ for large } n.
\end{aligned}
$$

Thus we get the lower bound for a_{n+1},

$$
a_{n+1} \geq \frac{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau) + \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau a_n^\alpha)a_n}{\left(1+\tau_n \frac{a_{1,1}}{m_1}\right)(1+\tau) - \tau_n \frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)a_n}. \tag{3.29}
$$

Next we check the decreasing property of a_n.

$$
\begin{aligned}
a_{n+1} - a_n &\leq \frac{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau) + (1+\tau_n \frac{a_{0,0}}{m_0})(1+\tau a_n^\alpha)a_n}{\left(1-\tau_n \frac{a_{0,1}}{m_1}\right)(1+\tau) - \tau_n \frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)a_n} - a_n \\
&= \frac{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau) + (1+\tau_n \frac{a_{0,0}}{m_0})(1+\tau a_n^\alpha)a_n - a_n\left\{\left(1-\tau_n \frac{a_{0,1}}{m_1}\right)(1+\tau) - \tau_n \frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)a_n\right\}}{\left(1-\tau_n \frac{a_{0,1}}{m_1}\right)(1+\tau) - \tau_n \frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)a_n}.
\end{aligned}
$$

Then we calculate the numerator of right-hand side, where we define it by Z.

$$
\begin{aligned}
Z &= -\tau_n \frac{a_{0,1}}{m_1}(1+\tau) + (1+\tau_n \frac{a_{0,0}}{m_0})(1+\tau a_n^\alpha)a_n - \left(1-\tau_n \frac{a_{0,1}}{m_1}\right)(1+\tau)a_n + \tau_n \frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)a_n^2 \\
&= \left\{-\tau_n \frac{a_{0,1}}{m_1} - \left(1-\tau_n \frac{a_{0,1}}{m_1}\right)a_n\right\}(1+\tau) + \left\{(1+\tau_n a_{0,0} m_0) - \tau_n \frac{a_{0,0}}{m_0}\right\}(1+\tau a_n^\alpha)a_n \\
&= -a_n(1+\tau) - \tau_n \frac{a_{0,1}}{m_1}(1-a_n)(1+\tau) + (1+\tau a_n^\alpha)a_n + a_n \tau_n \frac{a_{0,0}}{m_0}(1-a_n)(1+\tau a_n^\alpha) \\
&= -\tau_n \frac{a_{0,1}}{m_1}(1-a_n)(1+\tau) + a_n \tau_n \frac{a_{0,0}}{m_0}(1-a_n)(1+\tau a_n^\alpha) + \tau a_n(a_n^\alpha - 1).
\end{aligned}
$$

Then we can get

$$
a_{n+1} - a_n \leq \frac{-\tau_n \frac{a_{0,1}}{m_1}(1-a_n)(1+\tau) + a_n \tau_n \frac{a_{0,0}}{m_0}(1-a_n)(1+\tau a_n^\alpha) + \tau a_n(a_n^\alpha - 1)}{\left(1-\tau_n \frac{a_{0,1}}{m_1}\right)(1+\tau) - \tau_n \frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)a_n}.
$$

From $0 \leq a_n \leq 1$, for large n,

$$
a_{n+1} - a_n \leq \frac{\tau_n(1-a_n)\left\{-\frac{a_{0,1}}{m_1} + \frac{a_{0,0}}{m_0}\right\}(1+\tau) + \tau_n (u_0^n)^\alpha (a_n^\alpha - 1)a_n}{\left(1-\tau_n \frac{a_{0,1}}{m_1}\right)(1+\tau) - \tau_n \frac{a_{1,0}}{m_0}(1+\tau a_n^\alpha)a_n}.
$$

Here we can see that

$$
0 < a_n^\alpha \leq a_n < 1,
$$

for $\alpha \geq 1$, and

$$
0 < a_n \leq a_n^\alpha < 1,
$$

for $0<\alpha<1$. We set $K=\lfloor \frac{1}{\alpha}\rfloor$, where $\lfloor\cdot\rfloor$ denotes the floor function. For $0<\alpha<1$,

$$\begin{aligned}
1-a_n &\le 1-a_n+a_n^{1-K\alpha}\left(1-a_n^{(K+1)\alpha-1}\right)\\
&=1+\sum_{i=1}^{K}a_n^{1-i\alpha}-a_n^{\alpha}-\sum_{j=1}^{K}a_n^{1-(j-1)\alpha}\\
&=(1+\sum_{i=1}^{K}a_n^{1-i\alpha})(1-a_n^{\alpha})\\
&\le (K+1)(1-a_n^{\alpha}).
\end{aligned}$$

Thus

$$1-a_n\le (K+1)(1-a_n^{\alpha}),\ 0<\alpha<1.$$

For $\alpha\ge 1$, we can see that

$$a_{n+1}-a_n\le \frac{\tau_n(1-a_n)\left\{\left(-\frac{a_{0,1}}{m_1}+\frac{a_{0,0}}{m_0}\right)(1+\tau)-(u_0^n)^{\alpha}a_n\right\}}{\left(1-\tau_n\frac{a_{0,1}}{m_1}\right)(1+\tau)-\tau_n\frac{a_{1,0}}{m_0}(1+\tau a_n^{\alpha})a_n}.$$

Since $(u_0^n)^{\alpha}a_n=(u_0^n)^{\alpha-1}u_1^n$, if $\alpha=1$, then from $\lim_{n\to\infty}u_1^n=\infty$, we can get $a_{n+1}-a_n\le 0$. If $\alpha>1$, then from the assumption $\lim_{n\to\infty}(u_0^n)^{\alpha-1}u_1^n=\infty$, we can get $a_{n+1}-a_n\le 0$. Thus for $\alpha\ge 1$, from $0<a_n<1$ and the decreasing property of $\{a_n\}$, we can see that

$$a=\lim_{n\to\infty}a_n\in[0,1).$$

From (3.28), we can get

$$a\le \frac{(1+a^{\alpha}\tau)a}{1+\tau}.$$

On the other hand, from (3.29), we can see that

$$a\ge \frac{(1+a^{\alpha}\tau)a}{1+\tau}.$$

Thus $a=a^{1+\alpha}$, so we get $a=0$ since $a\in[0,1)$. Therefore we can obtain $\lim_{n\to\infty}a_n=0$ for $\alpha\ge 1$.

Next for $0<\alpha<1$, we shall prove $\lim_{n\to\infty}a_n=0$. For $0<\alpha<1$, we can see that

$$\begin{aligned}
a_{n+1}-a_n &\le \frac{\tau(1-a_n^{\alpha})\left\{\left(-\frac{a_{0,1}}{m_1}+\frac{a_{0,0}}{m_0}\right)(1+\tau)\frac{1-a_n}{1-a_n^{\alpha}}-(u_0^n)^{\alpha}a_n\right\}}{\left(1-\tau_n\frac{a_{0,1}}{m_1}\right)(1+\tau)-\tau_n\frac{a_{1,0}}{m_0}(1+\tau a_n^{\alpha})a_n}\\
&\le \frac{\tau(1-a_n^{\alpha})\left\{\left(-\frac{a_{0,1}}{m_1}+\frac{a_{0,0}}{m_0}\right)(K+1)(1+\tau)-(u_0^n)^{\alpha}a_n\right\}}{\left(1-\tau_n\frac{a_{0,1}}{m_1}\right)(1+\tau)-\tau_n\frac{a_{1,0}}{m_0}(1+\tau a_n^{\alpha})a_n}.
\end{aligned}\tag{3.30}$$

If $\{a_n\}$ is a convergent sequence, then from the same argument as the case of $\alpha\ge 1$, we can get

$$\lim_{n\to\infty}(u_0^n)^{\alpha}a_n=\infty.$$

Thus

$$\lim_{n\to\infty}a_n=0.$$

If $\{a_n\}$ is not a convergent sequence, there exist $a_*=\liminf_{n\to\infty}a_n$ and $a^*=\limsup_{n\to\infty}a_n$ such that

$$0\le a_*<a^*\le 1.$$

Then we fix $\gamma \in (a_*, a^*)$. We define subsequences A, B, and $\tilde{A}$ such that

$$\begin{aligned}A &= \{a_n | a_n \leq \gamma\},\\ B &= \{a_n | a_n > \gamma\},\\ \tilde{A} &= \{a_{n_i} | a_{n_i} \in A, a_{n_i+1} \in B\}.\end{aligned}$$

We can take a subsequence $\{a_{n_i}\} \in \tilde{A}$. Then from (3.28),

$$\begin{aligned}a_{n_i+1} &\leq \frac{-\tau_{n_i}\frac{a_{0,1}}{m_1}(1+\tau) + (1+\tau_{n_i}\frac{a_{0,0}}{m_0})(1+\tau a_{n_i}^{\alpha})a_{n_i}}{(1-\tau_{n_i}\frac{a_{0,1}}{m_1})(1+\tau) - \tau_{n_i}\frac{a_{1,0}}{m_0}(1+\tau a_{n_i}^{\alpha})a_{n_i}}\\ &\leq \frac{-\tau_{n_i}\frac{a_{0,1}}{m_1}(1+\tau) + (1+\tau_{n_i}\frac{a_{0,0}}{m_0})(1+\tau \gamma^{\alpha})\gamma}{(1-\tau_{n_i}\frac{a_{0,1}}{m_1})(1+\tau) - \tau_{n_i}\frac{a_{1,0}}{m_0}(1+\tau a_{n_i}^{\alpha})a_{n_i}}.\end{aligned}$$

Thus from $\gamma \in (0,1)$,

$$\limsup_{i\to\infty} a_{n_i+1} \leq \frac{(1+\tau\gamma^{\alpha})\gamma}{1+\tau} < \gamma.$$

This contradicts $a_{n_i+1} \in B$. Thus we can get

$$\lim_{n\to\infty} a_n = 0,\ 1.$$

If $\lim_{n\to\infty} a_n = 1$ holds true, then

$$\lim_{n\to\infty} a_n (u_0^n)^{\alpha} = \infty.$$

However, from (3.30), $a_{n+1} - a_n < 0$ for large n. Therefore

$$a = \lim_{n\to\infty} a_n < 1.$$

This contradicts $a = 1$. Thus we can obtain $\lim_{n\to\infty} a_n = 0$ for $0 < \alpha < 1$.

(Step 4) Next for $\alpha \geq 1$, we shall show

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{1+\tau}.$$

We can see that

$$\lim_{n\to\infty} (u_0^n)^{1-\alpha}(u_1^n)^{-1} = 0.$$

Thus from (3.29),

$$\begin{aligned}\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} &\geq \liminf_{n\to\infty} \frac{1}{a_n} \frac{-\tau_n\frac{a_{0,1}}{m_1}(1+\tau) + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau a_n^{\alpha})a_n}{\left(1+\tau_n\frac{a_{1,1}}{m_1}\right)(1+\tau) - \tau_n\frac{a_{1,0}}{m_0}(1+\tau a_n^{\alpha})a_n}\\ &= \lim_{n\to\infty} \frac{-\frac{\tau_n}{a_n}\frac{a_{0,1}}{m_1}(1+\tau) + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau a_n^{\alpha})}{\left(1+\tau_n\frac{a_{1,1}}{m_1}\right)(1+\tau) - \tau_n\frac{a_{1,0}}{m_0}(1+\tau a_n^{\alpha})a_n}\\ &= \frac{1}{1+\tau}.\end{aligned}$$

On the other hand, from (3.28),

$$\begin{aligned}\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} &\leq \limsup_{n\to\infty} \frac{1}{a_n} \frac{-\tau_n\frac{a_{0,1}}{m_1}(1+\tau) + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau a_n^{\alpha})a_n}{\left(1-\tau_n\frac{a_{0,1}}{m_1}\right)(1+\tau) - \tau_n\frac{a_{1,0}}{m_0}(1+\tau a_n^{\alpha})a_n}\\ &= \lim_{n\to\infty} \frac{-\frac{\tau_n}{a_n}\frac{a_{0,1}}{m_1}(1+\tau) + \left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau a_n^{\alpha})}{\left(1-\tau_n\frac{a_{0,1}}{m_1}\right)(1+\tau) - \tau_n\frac{a_{1,0}}{m_0}(1+\tau a_n^{\alpha})a_n}\\ &= \frac{1}{1+\tau}.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{1+\tau}.$$

(Step 5) Next we shall prove that $\{u_1^n\}$ is bounded above. From (3.26),

$$\begin{aligned}
u_1^{n+1} &\le -\tau_n\frac{a_{0,1}}{m_1}(1+\tau_n(u_0^n)^\alpha)u_0^n+\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n\\
&\le -\tau_n\frac{a_{0,1}}{m_1}(1+\tau)^\alpha u_0^n+\left(1+\tau_n\frac{a_{0,0}}{m_0}\right)(1+\tau_n(u_1^n)^\alpha)u_1^n.
\end{aligned}$$

Thus

$$u_1^{n+1}-u_1^n\le -\tau_n\frac{a_{0,1}}{m_1}(1+\tau)^\alpha u_0^n+\left(\tau_n(u_1^n)^\alpha+\tau_n\frac{a_{0,0}}{m_0}+\tau_n^2\frac{a_{0,0}}{m_0}(u_1^n)^\alpha\right)u_1^n.$$

Hence

$$\begin{aligned}
u_1^n &= \sum_{k=0}^{n-1}(u_1^k-u_1^{k-1})+u_1^0\\
&\le \sum_{k=0}^{\infty}\left\{-\tau_k\frac{a_{0,1}}{m_1}(1+\tau)u_0^k+\left(\tau_k(u_1^k)^\alpha+\tau_k\frac{a_{0,0}}{m_0}+\tau_k^2\frac{a_{0,0}}{m_0}(u_1^k)^\alpha\right)u_1^k\right\}+u_1^0.
\end{aligned}$$

Here since

$$\lim_{n\to\infty}\frac{u_0^{n+1}}{u_0^n}=1+\tau,\ \ \alpha>0,$$

we can get

$$\lim_{k\to\infty}\frac{\tau_{k+1}u_0^{k+1}}{\tau_k u_0^k}=\lim_{k\to\infty}\frac{(u_0^{k+1})^{1-\alpha}}{(u_0^k)^{1-\alpha}}=(1+\tau)^{1-\alpha}<1.$$

Also, from $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\dfrac{1}{1+\tau}$ for $\alpha\ge 1$, we can get

$$\begin{aligned}
\lim_{k\to\infty}\frac{\tau_{k+1}(u_1^{k+1})^{1+\alpha}}{\tau_k(u_1^k)^{1+\alpha}} &= \lim_{k\to\infty}\frac{a_{k+1}^{\alpha+1}u_0^{k+1}}{a_k^{\alpha+1}u_0^k}\\
&=(1+\tau)^{-\alpha-1}(1+\tau)\\
&=(1+\tau)^{-\alpha}<1,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{k\to\infty}\frac{\tau_{k+1}u_1^{k+1}}{\tau_k u_1^k} &= \lim_{k\to\infty}\frac{(u_0^{k+1})^{-\alpha}u_1^{k+1}}{(u_0^k)^{-\alpha}u_1^k}\\
&=\lim_{k\to\infty}\frac{(u_0^{k+1})^{-\alpha+1}a_{k+1}}{(u_0^k)^{-\alpha+1}a_k}\\
&=(1+\tau)^{-\alpha}<1.
\end{aligned}$$

We can see that

$$\sum_{k=0}^{\infty}\tau_k^2(u_1^k)^{1+\alpha}\le\sum_{k=0}^{\infty}\tau_k(u_1^k)^{1+\alpha}.$$

From the above inequalities and d'Alembert's ratio test, we can get

$$u_1^n<\infty.$$

(Step 6) Finally, we show $\{u_2^n\}$ is a bounded for $\alpha=1$. From (ML-3), we can get

$$\frac{a_{1,2}}{m_2}\tau_n u_1^{n+1}+\left(1+\tau_n\frac{a_{2,2}}{m_2}\right)u_2^{n+1}+\frac{a_{3,2}}{m_2}\tau_n u_3^{n+1}=u_2^n+\tau_n(u_2^n)^{1+\alpha}.$$

Thus

$$\begin{aligned}u_2^{n+1} &= \frac{-\frac{a_{1,2}}{m_2}\tau_n u_1^{n+1} - \frac{a_{3,2}}{m_2}\tau_n u_3^{n+1} + (1+\tau_n (u_2^n)^\alpha)u_2^n}{1+\tau_n \frac{a_{2,2}}{m_2}} \\ &\le \frac{-\frac{a_{1,2}}{m_2}\tau_n u_1^{n+1} - \frac{a_{3,2}}{m_2}\tau_n u_2^{n+1} + (1+\tau_n (u_2^n)^\alpha)u_2^n}{1+\tau_n \frac{a_{2,2}}{m_2}}.\end{aligned}$$

Solving for u_2^{n+1}, we can see that

$$\begin{aligned}&\left(1+\tau_n \frac{a_{2,2}}{m_2}\right) u_2^{n+1} + \frac{a_{3,2}}{m_2}\tau_n u_2^{n+1} \le -\frac{a_{1,2}}{m_2}\tau_n u_1^{n+1} + (1+\tau_n (u_2^n)^\alpha)u_2^n, \\ &\left(1-\tau_n \frac{a_{1,2}}{m_2}\right) u_2^{n+1} \le -\frac{a_{1,2}}{m_2}\tau_n u_1^{n+1} + (1+\tau_n (u_2^n)^\alpha)u_2^n.\end{aligned}$$

Thus

$$\begin{aligned}u_2^{n+1} &\le \frac{-\frac{a_{1,2}}{m_2}\tau_n u_1^{n+1} + (1+\tau_n (u_2^n)^\alpha)u_2^n}{1-\tau_n \frac{a_{1,2}}{m_2}} \\ &\le -\frac{a_{1,2}}{m_2}\tau_n u_1^{n+1} + (1+\tau_n (u_2^n)^\alpha)u_2^n.\end{aligned}$$

Moreover, from (3.26),

$$u_1^{n+1} \le -\tau_n \frac{a_{0,1}}{m_1}(1+\tau_n (u_0^n)^\alpha)u_0^n + \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau_n (u_1^n)^\alpha)u_1^n.$$

Hence

$$\begin{aligned}u_2^{n+1} &\le -\frac{a_{1,2}}{m_2}\tau_n \left\{-\tau_n \frac{a_{0,1}}{m_1}(1+\tau_n (u_0^n)^\alpha)u_0^n + \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau_n (u_1^n)^\alpha)u_1^n\right\} + (1+\tau_n (u_2^n)^\alpha)u_2^n \\ &= \frac{a_{1,2}}{m_2}\frac{a_{0,1}}{m_1}\tau_n^2(1+\tau_n (u_0^n)^\alpha)u_0^n - \frac{a_{1,2}}{m_2}\tau_n \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau_n (u_1^n)^\alpha)u_1^n + (1+\tau_n (u_2^n)^\alpha)u_2^n \\ &\le \frac{a_{1,2}}{m_2}\frac{a_{0,1}}{m_1}\tau_n^2(1+\tau)u_0^n - \frac{a_{1,2}}{m_2}\tau_n \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau a_n^\alpha)u_1^n + (1+\tau a_n^\alpha)u_2^n.\end{aligned}$$

We set

$$\begin{aligned}A_n &= 1+\tau a_n^\alpha, \\ B_n &= \frac{a_{1,2}}{m_2}\frac{a_{0,1}}{m_1}\tau_n^2(1+\tau)u_0^n - \frac{a_{1,2}}{m_2}\tau_n \left(1+\tau_n \frac{a_{0,0}}{m_0}\right)(1+\tau a_n^\alpha)u_1^n.\end{aligned}$$

Then we can get $u_2^{n+1} \le A_n u_2^n + B_n$. Here

$$\log\left(\prod_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \log A_n = \sum_{n=0}^{\infty} \log(1+\tau a_n^\alpha) \le \sum_{n=0}^{\infty} \tau a_n^\alpha.$$

Since

$$\lim_{n\to\infty} \frac{a_{n+1}^\alpha}{a_n^\alpha} = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^\alpha = (1+\tau)^{-\alpha} < 1,\ \alpha \ge 1,$$

we can get

$$\prod_{n=0}^{\infty} A_n < \infty.$$

Also, we can obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{\tau_{n+1}^2 u_0^{n+1}}{\tau_n^2 u_0^n} &= \lim_{n\to\infty} \frac{(u_0^n)^{2\alpha} u_0^{n+1}}{(u_0^{n+1})^{2\alpha} u_0^n} \\ &= \lim_{n\to\infty} \frac{(u_0^{n+1})^{1-2\alpha}}{(u_0^n)^{1-2\alpha}} \\ &= (1+\tau)^{1-2\alpha} < 1,\end{aligned}$$

and

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\tau_{n+1}(1+\tau_{n+1}(u_1^{n+1})^\alpha)u_1^{n+1}}{\tau_{n+1}(1+\tau_n(u_1^n)^\alpha)u_1^n} &= \lim_{n\to\infty} \frac{(1+a_{n+1}^\alpha)u_1^{n+1}(u_0^n)^\alpha}{(1+a_n^\alpha)u_1^n(u_0^{n+1})^\alpha} \\
&= \lim_{n\to\infty} \frac{(1+a_{n+1}^\alpha)a_{n+1}u_0^{n+1}(u_0^n)^\alpha}{(1+a_n^\alpha)a_n u_0^n(u_0^{n+1})^\alpha} \\
&= \lim_{n\to\infty} \frac{1+a_{n+1}^\alpha}{1+a_n^\alpha}\frac{a_{n+1}}{a_n}\left(\frac{u_0^n}{u_0^{n+1}}\right)^{\alpha-1} \\
&= (1+\tau)^{-\alpha} < 1.
\end{aligned}
$$

Therefore we can get

$$\sum_{n=0}^{\infty} B_n < \infty.$$

From Lemma 3.6.1, we can get

$$u_2^n < \infty,\ n \geq 0.$$

Thus we complete the proof. $\square$

Chapter 4

Application to the Keller-Segel systems

4.1 The schemes

The purpose of this chapter is to study the finite element method (FEM) applied to the parabolic-elliptic system,

$$
\begin{aligned}
&u_t = x^{1-N}(x^{N-1}(u_x - uv_x))_x, && x \in I = (0,1),\ t > 0, && (4.1a)\\
&0 = x^{1-N}(x^{N-1}v_x)_x - v + u, && x \in I,\ t > 0, && (4.1b)\\
&u_x(0,t) = u_x(1,t) = v_x(0,t) = v_x(1,t) = 0, && t > 0, && (4.1c)\\
&u(x,0) = u^0(x), && x \in I, && (4.1d)
\end{aligned}
$$

and the parabolic-parabolic system

$$
\begin{aligned}
&u_t = x^{1-N}(x^{N-1}(u_x - uv_x))_x, && x \in I = (0,1),\ t > 0, && (4.2a)\\
&v_t = x^{1-N}(x^{N-1}v_x)_x - v + u, && x \in I,\ t > 0, && (4.2b)\\
&u_x(0,t) = u_x(1,t) = v_x(0,t) = v_x(1,t) = 0, && t > 0, && (4.2c)\\
&u(x,0) = u^0(x),\ v(x,0) = v^0(x), && x \in I. && (4.2d)
\end{aligned}
$$

Therein, $u = u(x,t), v = v(x,t)$, $x \in \overline{I} = [0,1], t \geq 0$, denote the functions to be find and $u^0, v^0 \geq 0$ $(\not\equiv 0)$ given continuous functions. In [44], they consider the finite volume method for (4.1).

We should recall parabolic-elliptic and parabolic-parabolic Keller-Segel systems:

$$
\begin{aligned}
&U_t = \nabla \cdot (\nabla U - U \cdot \nabla V), && \vec{x} \in \Omega,\ t > 0, && (4.3a)\\
&-\Delta V + V = U, && \vec{x} \in \Omega,\ t > 0, && (4.3b)\\
&\frac{\partial U}{\partial \nu}(\vec{y},t) = \frac{\partial V}{\partial \nu}(\vec{y},t) = 0, && \vec{y} \in \Gamma = \partial\Omega,\ t > 0, && (4.3c)\\
&U(\vec{x},0) = U^0(\vec{x}), && \vec{x} \in \Omega, && (4.3d)
\end{aligned}
$$

and

$$
\begin{aligned}
&U_t = \nabla \cdot (\nabla U - U \cdot \nabla V), && \vec{x} \in \Omega,\ t > 0, && (4.4a)\\
&V_t = \Delta V - V + U, && \vec{x} \in \Omega,\ t > 0, && (4.4b)\\
&\frac{\partial U}{\partial \nu}(\vec{y},t) = \frac{\partial V}{\partial \nu}(\vec{y},t) = 0, && \vec{y} \in \Gamma = \partial\Omega,\ t > 0, && (4.4c)\\
&U(\vec{x},0) = U^0(x),\ V(\vec{x},0) = V^0(\vec{x}), && \vec{x} \in \Omega, && (4.4d)
\end{aligned}
$$

where $U = U(\vec{x},t), V = V(\vec{x},t)$, $\vec{x} \in \overline{\Omega}$, $t \geq 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with the boundary Γ, ν is the outer unit normal vector to Γ, $\frac{\partial U}{\partial \nu} = \nabla U \cdot \nu$, and $U^0, V^0 \geq 0$ $(\not\equiv 0)$ are given continuous functions. In particular, the systems (4.3) and (4.4) denote the aggregation of slime molds resulting from their chemotactic features. Then U and V express the density of the cellular slime molds and the concentration of the chemical substance.

We can see that the solutions of (4.1) and (4.2) correspond to the radially symmetric solutions of (4.3) and (4.4). It is well-known that the solutions of (4.3) and (4.4) satisfy conservation of positivity, conservation of mass and conservation of L^1 norm:

$$U(\vec{x},t) \geq 0,\ \vec{x} \in \overline{\Omega},\ t \geq 0, \tag{4.5}$$

$$\int_\Omega U(\vec{x},t)\ d\vec{x} = \int_\Omega U^0(\vec{x})\ d\vec{x},\ t \geq 0, \tag{4.6}$$

$$\|U(\cdot,t)\|_{L^1(\Omega)} = \|U^0\|_{L^1(\Omega)},\ t \geq 0. \tag{4.7}$$

We first mention the weak formulation of (4.1). By multiplying $x^{N-1}\chi$ and integrating in I, we can get

$$\begin{aligned} \int_I x^{N-1} u_t \chi\ dx &= -\int_I x^{N-1}(u_x - u v_x)\chi'\ dx, \\ 0 &= -\int_I x^{N-1} v_x \chi'\ dx + \int_I x^{N-1}(-v+u)\chi\ dx\ (\chi \in H^1(I)). \end{aligned}$$

We set

$$(u,v) = \int_I x^{N-1} uv\ dx,\ A(u,v) = \int_I x^{N-1} u'v'\ dx,\ \text{and } B(u,v,w) = -\int_I x^{N-1} uv'w'\ dx.$$

Therefore we can rewrite this to

$$\begin{aligned} (u_t,\chi) + A(u,\chi) + B(u,v,\chi) &= 0, \\ A(u,\chi) - (u-v,\chi) &= 0\ (\chi \in H^1(I)). \end{aligned}$$

Similarly, we can get the weak formulation of (4.2):

$$\begin{aligned} (u_t,\chi) + A(u,\chi) + B(u,v,\chi) &= 0, \\ (v_t,\chi) + A(u,\chi) - (u-v,\chi) &= 0\ (\chi \in H^1(I)). \end{aligned}$$

For a positive integer m, we introduce node points

$$0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < \cdots < x_{m-1} < x_m = 1,$$

and set $I_j = (x_{j-1}, x_j)$ and $h_j = x_j - x_{j-1}$, where $j = 1, \ldots, m$. The granularity parameter is defined as $h = \max_{1\leq j \leq m} h_j$. Let $\mathcal{P}_k(J)$ be the set of all polynomials in an interval J of degree $\leq k$. We define the P1 finite element space as

$$S_h = \{v \in H^1(I) \mid v \in \mathcal{P}_1(I_j)\ (j = 1, \cdots, m)\}. \tag{4.8}$$

Its standard basis function ϕ_j, $j = 0, 1, \cdots, m$, is defined as

$$\phi_j(x_i) = \delta_{ij},$$

where δ_{ij} denotes Kronecker's delta.

For time discretization, we introduce non-uniform partitions

$$t_0 = 0, \quad t_n = \sum_{j=0}^{n-1} \tau_j \quad (n \geq 1),$$

where $\tau_j > 0$ denotes the time increment.

Generally, we write $\partial_{\tau_n} u_h^{n+1} = (u_h^{n+1} - u_h^n)/\tau_n$.

We define

$$\langle w, v\rangle = \sum_{i=0}^{m} w(x_i)v(x_i)(1,\phi_i) \qquad (w, v \in H^1(I)). \tag{4.9}$$

We introduce the finite element schemes (KS-1) for (4.1) and (KS-2) for (4.2).

(KS-1) Find $u_h^{n+1},\ v_h^n \in S_h$, $n = 0, 1, \ldots$, such that

$$\left\langle \partial_{\tau_n} u_h^{n+1}, \chi \right\rangle + A(u_h^{n+1}, \chi) + B_h(u_h^{n+1}, v_h^n, \chi) = 0, \tag{4.10}$$

$$A(v_h^n, \chi) - \langle u_h^n - v_h^n, \chi \rangle = 0,\ (\chi \in S_h) \tag{4.11}$$

where $u_h^0 \in S_h$ is assumed to be given.

(KS-2) Find $u_h^{n+1},\ v_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that

$$\left\langle \partial_{\tau_n} u_h^{n+1}, \chi \right\rangle + A(u_h^{n+1}, \chi) + B_h(u_h^{n+1}, v_h^n, \chi) = 0, \tag{4.12}$$

$$\left\langle \partial_{\tau_n} v_h^{n+1}, \chi \right\rangle + A(v_h^n, \chi) - \langle u_h^n - v_h^n, \chi \rangle = 0,\ (\chi \in S_h) \tag{4.13}$$

where $u_h^0,\ v_h^0 \in S_h$ are assumed to be given.

Here,

$$B_h(u, v, w) = \sum_{i=0}^{m} w(x_i) \sum_{j \in \Lambda_i} \left\{ u(x_i)\beta_{i,j}^+(v) - u(x_j)\beta_{i,j}^-(v) \right\},$$

where

$$\Lambda_i = \begin{cases} \{1\} & i = 0 \\ \{i-1, i+1\} & i \neq 0, m \\ \{m-1\} & i = m, \end{cases}$$

and

$$\beta_{i,j}^{\pm}(v) = \left| \int_{x_i}^{x_j} x^{N-1} \max\left\{ 0, \pm \frac{v(x_j) - v(x_i)}{|x_j - x_i|} \right\} (\phi_i)_x \, dx \right|.$$

The solutions of (KS-1) and (KS-2) reproduce (4.5), (4.6), as stated below.

Proposition 4.1.1 (Positivity preserving and conservation of the mass, (KS-1))**.** For (KS-1), if $u_h^0 \geq 0$ in I, then the solution (u_h^n, v_h^n) satisfies positivity preserving and conservation law:

$$u_h^n,\ v_h^n \geq 0 \text{ in } I,\ n \geq 0,$$

$$\langle u_h^n, 1 \rangle = \left\langle u_h^0, 1 \right\rangle = \left\langle v_h^0, 1 \right\rangle = \langle v_h^n, 1 \rangle.$$

For the proof, see Section 4.3.

Proposition 4.1.2 (Positivity preserving and conservation of the mass, (KS-2))**.** For (KS-2), if $u_h^0,\ v_h^0 \geq 0$ in I, then the solution (u_h^n, v_h^n) satisfies positivity preserving and conservation law:

$$u_h^n,\ v_h^n \geq 0 \text{ in } I,\ n \geq 0,$$

$$\langle u_h^n, 1 \rangle = \left\langle u_h^0, 1 \right\rangle.$$

For the proof, see Section 4.3.

4.2 Numerical examples

We calculate the radially symmetric solutions of Keller-Segel systems (4.1)–(4.2). We set $m = 200,\ T = 0.2,\ \tau_n = \tau = \frac{1}{50}h,\ u_h^0 = \Pi_h u^0,\ v_h^0 = \Pi_h v^0$,

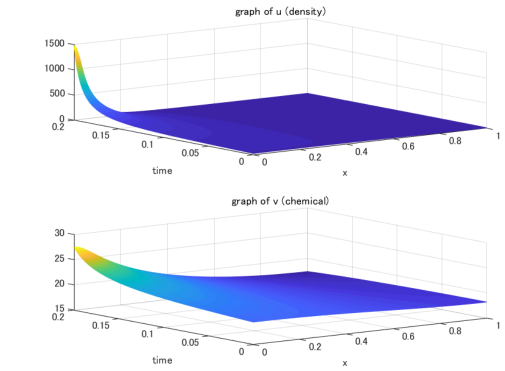
$$u^0(x) = \mu \left\{ 10e^{-2x^2} + 20e^{-2(x-0.3)^2} \right\}, \text{ and } v^0(x) = \mu \cos \frac{\pi}{2} x,$$

where $\Pi_h v$ denotes the linear interpolation for $v \in H^1(I)$. We adopt $N = 2$ for (KS-1), and $N = 3$ for (KS-2). Fig. 4.1 shows the graphs of (KS-1) and the amounts of change of the following quantities:

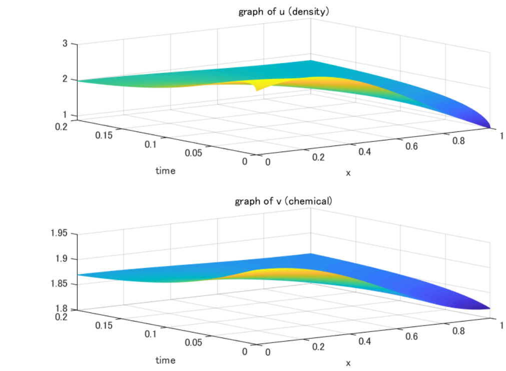
$$\text{mass} = \int_I x^{N-1} u_h^n \, dx, \tag{4.14}$$

$$\text{chemical} = \int_I x^{N-1} v_h^n \, dx. \tag{4.15}$$

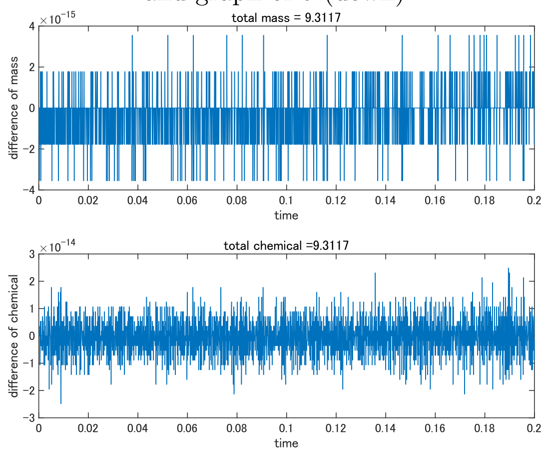
Fig. 4.2 shows the graphs of (KS-2) and the amounts of change of (4.14) and (4.15). We observe that Fig. 4.1 (a) and Fig. 4.2 concentrate on the origin and Fig. 4.1 (b) distributes uniformly. Fig. 4.1 (c), (d) and Fig. 4.2 show that (KS-1) conserves both (4.14) and (4.15) numerically, while (KS-2) conserves (4.14) numerically.



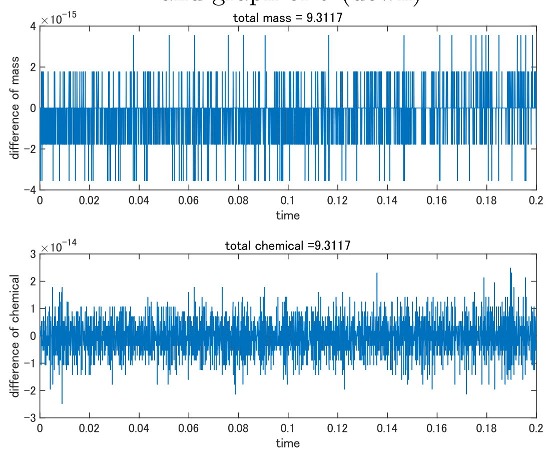
(a) (KS-1), $\mu = 1$, graph of u (up) and graph of v (down)



(b) (KS-1), $\mu = \frac{1}{10}$, graph of u (up) and graph of v (down)



(c) (KS-1), $\mu = 1$, change of mass (up) and change of chemical (down)



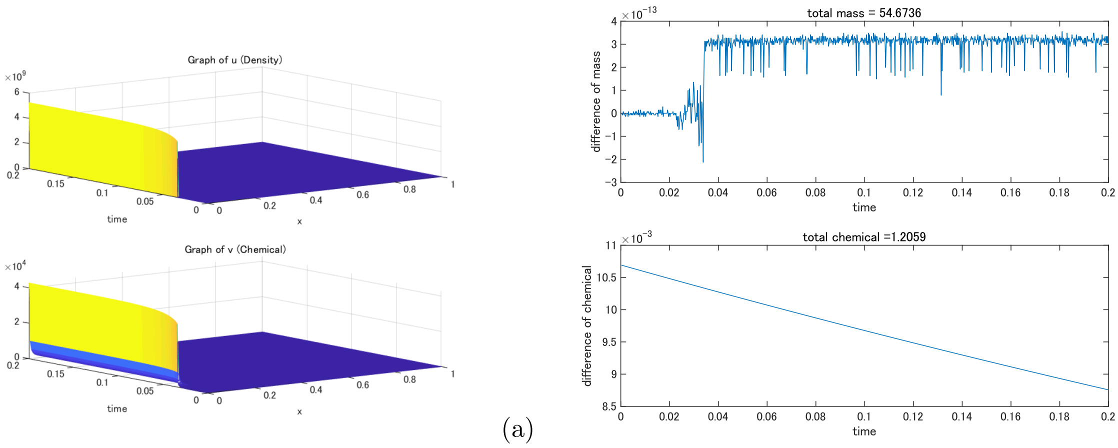
(d) (KS-1), $\mu = \frac{1}{10}$, change of mass (up) and change of chemical (down)

Figure 4.1: Graphs and mass conservation law, (KS-1)

4.3 Proof of Propositions

We shall prove Proposition 4.1.1.

Proof of Proposition 4.1.1. We shall prove the positivity preserving. We set $u_i^n = u_h^n(x_i)$, $v_i^n = v_h^n(x_i)$.



(a) graph of u (up) and graph of v (down)

(b) change of mass (up) and change of chemical (down)

Figure 4.2: Graphs and mass conservation law, (KS-2) with $N = 3$ and $\mu = 10$

Substituting $\chi = \phi_i$ in (4.10), we get

$$\begin{aligned}
&\frac{u_i^{n+1} - u_i^n}{\tau_n} + u_{i-1}^{n+1} A(\phi_{i-1}, \phi_i) + u_i^n A(\phi_i, \phi_i) + u_{i+1}^n A(\phi_{i+1}, \phi_i) \\
&+ u_i^{n+1} \beta_{i,i-1}^+(v_h^n) - u_{i-1}^{n+1} \beta_{i,i-1}^-(v_h^n) + u_i^{n+1} \beta_{i,i+1}^+(v_h^n) - u_{i+1}^{n+1} \beta_{i,i+1}^-(v_h^n) = 0,
\end{aligned}$$

where we understand that $u_{-1}^n = u_{m+1}^n = A(\phi_{-1}, \phi_0) = A(\phi_{m+1}, \phi_m) = \beta_{0,-1}^\pm(v_h^n) = \beta_{m,m+1}^\pm(v_h^n) = 0$. Then we set $m_i = (1, \phi_i)$, $a_{i,j} = A(\phi_i, \phi_j)$, and $\beta_{i,j}^\pm = \beta_{i,j}^\pm(v_h^n)$. Thus

$$\begin{aligned}
&\frac{1}{\tau_n} m_i (u_i^{n+1} - u_i^n) + a_{i-1,i} u_{i-1}^{n+1} + a_{i,i} u_i^{n+1} + a_{i+1,i} u_{i+1}^{n+1} \\
&+ u_i^{n+1} \beta_{i,i-1}^+ - u_{i-1}^{n+1} \beta_{i,i-1}^- + u_i^{n+1} \beta_{i,i+1}^+ - u_{i+1}^{n+1} \beta_{i,i+1}^- = 0, \\
&(a_{i-1,i} - \beta_{i,i-1}^-) u_{i-1}^{n+1} + \left(\frac{1}{\tau} + a_{i,i} + \beta_{i,i-1}^+ + \beta_{i,i+1}^+ \right) u_i^{n+1} + (a_{i+1,i} - \beta_{i,i+1}^-) u_{i+1}^{n+1} \\
&= \frac{1}{\tau_n} m_i u_i^n.
\end{aligned}$$

Here we set

$$\begin{aligned}
A_{i,i-1}^1 &= a_{i-1,i} - \beta_{i,i-1}^-, \\
A_{i,i}^1 &= \frac{1}{\tau_n} m_i + a_{i,i} + \beta_{i,i-1}^+ + \beta_{i,i+1}^+, \\
A_{i,i+1}^1 &= a_{i+1,i} - \beta_{i,i+1}^-.
\end{aligned}$$

We define

$$\begin{aligned}
A_1 &= (A_{i,j}^1)_{0 \le i,j \le m} \in \mathbb{R}^{m+1,m+1}, \\
M_1 &= \operatorname{diag}\left(\frac{m_i}{\tau_n} \right) \in \mathbb{R}^{m+1,m+1}, \\
\vec{u}^{(n)} &= (u_i^n)_{0 \le i \le m} \in \mathbb{R}^{m+1},
\end{aligned}$$

where $A_{i,j}^1 = 0$ ($|i - j| > 1$) and $\operatorname{diag}(x_i)$ denotes the diagonal matrix whose (i,i)th entry is x_i.

Substituting $\chi = \phi_i$ in (4.11), we get

$$a_{i-1,i} v_{i-1}^n + (a_{i,i} + m_i) v_i^n + a_{i+1,i} v_{i+1}^n = m_i u_i^n.$$

We set

$$A_{i,i-1}^2 = a_{i-1,i}, \quad A_{i,i}^2 = a_{i,i} + m_i, \quad A_{i,i+1}^2 = a_{i+1,i}.$$

We define

$$\begin{aligned} A_2 &= (A^2_{ij})_{0\le i,j\le m} \in \mathbb{R}^{m+1,m+1},\\ M_2 &= \mathrm{diag}(m_i) \in \mathbb{R}^{m+1},\\ \vec{v}^n &= (v_i^n)_{0\le i\le m} \in \mathbb{R}^{m+1}, \end{aligned}$$

where $A^2_{i,j} = 0$ $(|i-j|>1)$.

We can rewrite (KS-1) as follows:

$$A_1\vec{u}^{n+1} = M_1\vec{u}^n, \tag{4.16}$$
$$A_2\vec{v}^n = M_2\vec{u}^n. \tag{4.17}$$

It suffices to prove that $A_1^{-1}, A_2^{-1} \ge 0$, that is, all components in A_1^{-1} and A_2^{-1} are nonnegative. We can get

$$A^1_{i,i} > 0,\ A^1_{i,j} \le 0,\ A^2_{i,i} > 0,\ A^2_{i,j} \le 0,\ (i \ne j). \tag{4.18}$$

By a direct calculation, we can see that

$$\begin{aligned} &a_{i,i-1} + a_{i,i} + a_{i,i+1} = 0,\\ &\beta^+_{i,i-1} - \beta^-_{i-1,i} = 0,\ \beta^+_{i,i+1} - \beta^-_{i+1,i} = 0. \end{aligned}$$

Therefore

$$\begin{aligned} A^1_{i-1,i} + A^1_{i,i} + A^1_{i+1,i} &= \frac{m_i}{\tau_n} > 0,\\ A^2_{i-1,i} + A^2_{i,i} + A^2_{i+1,i} &= m_i > 0. \end{aligned}$$

Thus

$$\begin{aligned} |A^1_{i,i}| = A^1_{i,i} &= \frac{m_i}{\tau_n} - A^1_{i-1,i} - A^1_{i+1,i}\\ &> \sum_{0\le j\le m,\ j\ne i} |A^1_{j,i}|,\\ |A^2_{i,i}| = A^2_{i,i} &= m_i - A^2_{i-1,i} - A^2_{i+1,i}\\ &> \sum_{0\le j\le m,\ j\ne i} |A^2_{j,i}|. \end{aligned}$$

We then see that A_1^T and A_2^T are diagonally dominant. From (4.18) and the above results, we can get

$$(A_1^T)^{-1},\ (A_2^T)^{-1} \ge 0.$$

Thus we can see the positivity of u_h^n and v_h^n.

Secondly we prove the conservation law. Summing all components of (4.16) and (4.17), we get

$$\begin{aligned} \sum_{j=0}^m \frac{m_j}{\tau_n} u_j^{n+1} &= \sum_{j=0}^m \frac{m_j}{\tau_n} u_j^n,\\ \sum_{i=0}^m m_j v_j^n &= \sum_{i=0}^m m_j u_j^n. \end{aligned}$$

Thus

$$\left\langle u_h^{n+1}, 1\right\rangle = \left\langle u_h^n, 1\right\rangle,\ \ \left\langle v_h^n, 1\right\rangle = \left\langle u_h^n, 1\right\rangle.$$

We complete the proof. □

Finally we prove Proposition 4.1.2.

Proof of Proposition 4.1.2. We use the same notations as the previous proof. First we show the positivity. Substituting $\chi = \phi_i$ in (4.13), we can get

$$a_{i-1,i}v_{i-1}^{n+1} + \left(\frac{m_i}{\tau_n} + a_{i,i} + m_i\right) v_i^{n+1} + a_{i+1,i}v_{i+1}^{n+1} = \frac{m_i}{\tau_n}v_i^n + m_i u_i^n.$$

We define

$$A_{i,i-1}^3 = a_{i-1,i},\ A_{i,i}^3 = \frac{m_i}{\tau_n} + a_{i,i} + m_i,\ A_{i,i+1}^3 = a_{i+1,i}.$$

We can rewrite (KS-2) into

$$A_1\vec{u}^{n+1} = M_1\vec{u}^n, \tag{4.19}$$

$$A_3\vec{v}^{n+1} = M_1\vec{v}^n + M_2\vec{u}^{n+1}. \tag{4.20}$$

From the proof of Proposition 4.1.1, we can see $A_1^{-1} \geq 0$. Thus it suffices to show that $A_3^{-1} \geq 0$. Similarly as Proposition 4.1.1, we can see that

$$A_{i,i}^3 > 0,\ A_{i,j}^3 \leq 0\ (i \neq j),$$

and

$$A_{i-1,i}^3 + A_{i,i}^3 + A_{i+1,i}^3 = \frac{m_i}{\tau_n} + m_i.$$

Thus

$$\begin{aligned} |A_{i,i}^3| = A_{i,i}^3 &= \frac{m_i}{\tau_n} + m_i - A_{i-1,i}^3 - A_{i+1,i}^3 \\ &> \sum_{0\leq j\leq m,\ j\neq i} |A_{i,j}^3|. \end{aligned}$$

By making the same argument as Proposition 4.1.1, we can obtain

$$A_3^{-1} \geq 0.$$

Secondly, we show the conservation law. Summing all components in (4.19)–(4.20), we can see that

$$\begin{aligned} \sum_{j=0}^{m} \frac{m_j}{\tau_n}u_j^{n+1} &= \sum_{j=0}^{m} \frac{m_j}{\tau_n}u_j^n, \\ \sum_{i=0}^{m} \left(\frac{m_j}{\tau_n} + m_j\right) v_j^{n+1} &= \sum_{j=0}^{m} \frac{m_j}{\tau_n}v_j^n + \sum_{j=0}^{m} m_j u_j^{n+1}. \end{aligned}$$

Thus

$$\left\langle u_h^{n+1}, 1\right\rangle = \left\langle u_h^n, 1\right\rangle, \tag{4.21}$$

$$\left(\frac{1}{\tau_n} + 1\right)\left\langle v_h^{n+1}, 1\right\rangle = \frac{1}{\tau_n}\left\langle v_h^n, 1\right\rangle + \left\langle u_h^n, 1\right\rangle. \tag{4.22}$$

Thus we complete the proof. □

Remark 4.3.1. If we choose the uniform time increment $\tau_n = \tau$, then from (4.22) we get

$$\left\langle v_h^n, 1\right\rangle = \left\langle u_h^0, 1\right\rangle + \left(\frac{1}{1+\tau}\right)^n \left\langle v_h^0 - u_h^0, 1\right\rangle.$$

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