Abstract

In this thesis we are concerned with asymptotic analysis for solutions to semilinear heat equations. A nonlinear parabolic equation

$$u_t = \Delta u + u^p \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \quad p > 1, \ N \geq 1, \tag{F}$$

is called Fujita's equation and appears in various mathematical models, in particular, in combustion theory as solid fuel ignition processes. The superlinear term u^p arises from the Arrhenius equation which is a formula for the temperature dependence of reaction rates. Due to the balance between the diffusion term and the nonlinear term, the equation (F) has rich mathematical structure and solutions exhibit various properties. One of characteristic properties of the equation (F) is that the solution does not necessarily exist globally in time. This phenomenon is peculiar to nonlinear problems and called the blow-up of the solution. The equation (F) has been widely studied by many researchers since the pioneering work by Fujita '66. One of the major topics is to obtain the rate of blow-up. Let $T \in (0, \infty)$ be the maximal existence time of a solution u to the equation (F). Then the solution u satisfies

$$\|u(\,\cdot\,, t)\|_{L^\infty(\mathbf{R}^N)} \geq C(T-t)^{-1/(p-1)}, \quad 0 < t < T,$$

for some positive constant C. The blow-up of u is said to be of type I if there exists a positive constant K such that

$$\|u(\,\cdot\,, t)\|_{L^\infty(\mathbf{R}^N)} \leq K(T-t)^{-1/(p-1)}, \quad 0 < t < T.$$

Otherwise the blow-up is said to be of type II. For the Sobolev subcritical case, that is,

$$p < p_{\mathrm{S}} := \begin{cases} \infty, & N = 1, 2, \\ 1 + \dfrac{4}{N-2}, & N \geq 3, \end{cases}$$

Gigs–Kohn '87 and Giga–Matsui–Sasayama '04 proved that every blow-up for the equation (F) is of type I. Furthermore, Matano–Merle '04,'09 and Mizoguchi '11 showed that only type I blow-up occurs for positive radial solutions in the Sobolev supercritical and Joseph–Lundgren subcritical case. On the other hand, due to the well-known result by Herrero–Velázquez '94, we see that if p is greater than Joseph–Lundgren exponent, that is,

$$p > p_{\mathrm{JL}} := \begin{cases} \infty, & N \leq 10, \\ 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}}, & N \geq 11, \end{cases}$$

then there exist radial blow-up solutions $u_{\ell,\mathrm{HV}}(x,t)$, such that

$$\|u_{\ell,\mathrm{HV}}(\,\cdot\,, t)\|_\infty \asymp (T-t)^{-(1+2\omega_\ell)/(p-1)} \quad \text{as} \quad t \to T$$

where ω_ℓ is a positive constant related to eigenvalues for a linearized equation of the equation (F). Mizoguchi '04 improved the proof by Herrero–Velázquez '94 under suitable additional assumptions. In addition, Matano '07 and Mizoguchi '11 assumed that every eigenvalue for linearized operator of the equation (F) is not 0 and proved that actual blow-up rate of a type II blow-up radial solution coincides with one of the blow-up rates of type II blow-up solutions constructed by Herrero–Veázquez '94. Thereafter Seki '18,'19 constructed type II blow-up solutions in the Joseph–Lundgren critical case and the Lepin critical case, respectively. The Lepin critical case corresponds to the case where an eigenvalue for linearized operator of the equation (F) is 0.

Chapter 1 is concerned with a heat equation with space-dependent nonlinearity :

$$u_t = \Delta u + |x|^{2a} u^p \quad \text{in} \quad \big(\mathbf{R}^N \setminus \{0\}\big) \times \mathbf{R}_+, \quad a > -1, \tag{PF}$$

and it is the main ingredient of this thesis. A similar argument to that of the equation (F) implies that type I blow-up rate of a solution to the equation (PF) blowing up at the origin should be $(T-t)^{(1+a)/(p-1)}$. However, in comparison to the case $a = 0$, there are not enough results for type I blow-up solutions to the equation (PF) with $a \neq 0$. In addition, there are no results for type II blow-up solutions.

We follow [1] to find type II blow-up solutions for the equation (PF) in the Joseph–Lundgren supercritical case, that is, $N > 10 + 8a$ and

$$p > p_{\mathrm{JL}}(a) := 1 + \frac{4(1+a)}{N - 2a - 4 - 2\sqrt{(N+a-1)(a+1)}}.$$

Our construction of type II blow-up solutions gives basic and important informations to the analysis of the equation (PF). In particular, the asymptotic behavior we obtained is new even if $a = 0$ because we bring the asymptotic behavior of solutions near the origin in detail. As a corollary of our asymptotic analysis, we prove that the profiles of our solutions at blow-up time is a singular stationary solution near the origin.

In Chapter 1, based on the idea of Herrero–Velázquez '94 and Seki '18, we apply the matched asymptotic expansions to the semilinear parabolic equations in backward similarity variables and obtain the precise description of the asymptotic behavior of type II blowing up solution in a neighborhood of the origin. One of difficulties in our analysis is that unstable modes appears in a Fourier expansion of solutions for a linearized equation. In order to overcome the difficulty, we require to choose the parameter associated with the family of the initial functions by using the degree of mappings.

In Chapter 2, based on [2], we consider the asymptotic behavior of a solution to the heat equation with a inverse square potential :

$$u_t = \Delta u - V(|x|)u \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \tag{P}$$

where V satisfies

$$V(r) \sim \begin{cases} \lambda_1 r^{-2} & \text{as} \quad r \to 0, \\ \lambda_2 r^{-2} & \text{as} \quad r \to \infty, \end{cases} \qquad \text{with} \quad \lambda_1, \lambda_2 \geq \lambda_* := -\frac{(N-2)^2}{4}.$$

Here $-\lambda_*$ is the best constant of Hardy's inequality. The equation (P) often arises in a linearized analysis for nonlinear diffusion equations such as equations (F) and (PF). Let $L := -\Delta + V(|x|)$ be a nonnegative Schrödinger operator on $L^2(\mathbf{R}^N)$. The asymptotic behavior of a solution to the equation (P) depends on a criticality for the operator L. The criticality is classified as subcritical, null-critical, positive-critical, and supercritical. Ishige–Kabeya–Ouhabaz '17 obtained the Gaussian estimate of the fundamental solution and the asymptotic behavior of a positive harmonic function in the subcritical case and the null-critical case. The purpose of this chapter is to establish a method for obtaining the precise description of the large time behavior of solutions to the equation (P) in the subcritical case and the null-critical case.

In Chapter 3, based on [3], we investigate the large time behavior of the hot spots

$$H(u(t)) := \left\{ x \in \mathbf{R}^N \,;\, u(x,t) = \sup_{y \in \mathbf{R}^N} u(y,t) \right\}$$

as an application of the precise description of the large time behavior of solutions. The behavior of the hot spots for parabolic equations in unbounded domains has been studied since the pioneering work by Chavel–Karp '90. In particular, for the heat equation on $\mathbf{R}^N$ with nonnegative initial data $\varphi \in L^\infty_c(\mathbf{R}^N)$, they proved:

1. $H(e^{t\Delta}\varphi)$ is a subset of the closed convex hull of the support of the initial function φ;
2. There exists $T > 0$ such that $H(e^{t\Delta}\varphi)$ consists of only one point and moves along a smooth curve for any $t \geq T$;
3. $\displaystyle\lim_{t\to\infty} H(e^{t\Delta}\varphi) = \int_{\mathbf{R}^N} x\varphi(x)\,dx \Big/ \int_{\mathbf{R}^N} \varphi(x)\,dx.$

Applying the arguments in Chapter 2, we study the following subjects when the hots spots tend to the space infinity as $t \to \infty$:

a. The rate and the direction for the hot spots to tend to the space infinity as $t \to \infty$;

b. The number of the hot spots for sufficiently large t.

On the other hand, when the hots spots accumlate to a point x_*, we characterize the limit point x_* by the positive harmonic function. Furthermore, we give a sufficient condition for the hot spots to consist of only one point and to move along a smooth curve.

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Contents

Chapter 0

Notations

Throughout this thesis, we employ standard notations in asymptotic analysis. For positive functions f and g defined in $(0, R)$ for some $R > 0$, we write

$$f(r) \sim g(r) \quad \text{as} \quad r \to 0 \quad \text{if} \quad \lim_{r\to 0} \frac{f(r)}{g(r)} = 1,$$

$$f(r) \ll g(r) \quad \text{as} \quad r \to 0 \quad \text{if} \quad \lim_{r\to 0} \frac{f(r)}{g(r)} = 0.$$

Similarly, for positive functions f and g defined in (R, ∞) for some $R > 0$, we write

$$f(r) \sim g(r) \quad \text{as} \quad r \to \infty \quad \text{if} \quad \lim_{r\to \infty} \frac{f(r)}{g(r)} = 1,$$

$$f(r) \ll g(r) \quad \text{as} \quad r \to \infty \quad \text{if} \quad \lim_{r\to \infty} \frac{f(r)}{g(r)} = 0.$$

In additional, we write

$$f(r) \asymp g(r) \quad \text{if there exists } c > 0 \text{ such that} \quad \frac{1}{c} \le \frac{f(r)}{g(r)} \le c.$$

for sufficiently small or large $r > 0$. By the letter C, we denote generic positive constants and they may have different values also within the same line.

Chapter 1

Refined construction of type II blow-up solutions for semilinear heat equations with Joseph–Lundgren exponent

1.1 Introduction and main results

In the present article we discuss blow-up behavior for a semilinear heat equation:

$$u_t = \Delta u + |u|^{p-1}u \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \tag{1.1.1}$$

and its variant

$$u_t = \Delta u + |x|^{2a}u^p \quad \text{in} \quad \big(\mathbf{R}^N \setminus \{0\}\big) \times \mathbf{R}_+, \tag{1.1.2}$$

where Δ denotes the Laplacian in $\mathbf{R}^N$, $\mathbf{R}_+ := \{t > 0\}$, $p > 1$ and $a > -1$ are constants. Given an initial datum $u|_{t=0} = u_0 \in L^\infty(\mathbf{R}^N)$, we may uniquely obtain a local-in-time classical solution of (1.1.1) (resp., (1.1.2)). They are "C_B-mild solutions" on $\mathbf{R}^N \times [0,T)$, where T stands for the maximal existence time, that is, bounded and continuous up to $x = 0$ and satisfy the integral equation corresponding to (1.1.2). See, for instance, [41, 48].

1.1.1 Study of equation $u_t = \Delta u + u^p$

The simple equation (1.1.1) has been widely studied by many researchers since the pioneering work [12] by H. Fujita. In particular, describing possible blow-up behavior at the blow-up time has attracted considerable attention in the past decades. We say that a solution u of (1.1.1) blows up in a finite time T if

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = \infty. \tag{1.1.3}$$

A number of sufficient conditions finite time blow-up has been obtained by many reseachers. For example, if the nonnegative initial data u_0 satisfies

$$u_0(x) \geq \lambda U_\kappa(|x|), \quad x \in \mathbf{R}^N \tag{1.1.4}$$

for some constants $\lambda > 1$ and $\kappa > 1$, then the solution of (1.1.1) blows up in finite time, where $U_\alpha(|x|)$ denotes a regular stationary solution of (1.1.1) (see (1.1.9) below). In this article, we study the blow-up rate of $\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)}$ as t approaches the blow-up time T. Local theory implies that there is a constant $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \geq C(T-t)^{-1/(p-1)}, \quad 0 < t < T$$

if the maximal time of existence $T = T(u_0)$ is finite (cf. [41, Chapter II]). On the other hand, it is far from obvious whether the corresponding upper estimate holds. A blow-up is said to be of **type I** if there exists a positive constant K such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \leq K(T-t)^{-1/(p-1)}, \quad 0 < t < T\,; \tag{1.1.5}$$

whereas the blow-up is said to be of **type II** otherwise. In the Sobolev subcritical case $p < p_{\mathrm{S}}$, where

$$p_{\mathrm{S}} := \begin{cases} \infty, & N = 1, 2, \\ 1 + \dfrac{4}{N-2}, & N \geq 3, \end{cases} \tag{1.1.6}$$

every blow-up for (1.1.1) is of type I even for non-radial or sign-changing solutions [13, 14] (see also [7] for a related parabolic system). In the Sobolev supercitical case, the situation drastically changes according to whether or not p is less or greater than the Joseph–Lundgren exponent

$$p_{\mathrm{JL}} := \begin{cases} \infty, & N \leq 10, \\ 1 + \dfrac{4}{N-4-2\sqrt{N-1}}, & N \geq 11. \end{cases} \tag{1.1.7}$$

Indeed, if $p_{\mathrm{S}} < p < p_{\mathrm{JL}}$, only type I blow-up occurs for radial solutions under mild assumptions on initial data [28, 29, 35], whereas type II blow-up does occur for $p > p_{\mathrm{JL}}$ as we are going to recall below. To this end, let us write

$$\beta := \frac{1}{p-1}, \tag{1.1.8a}$$

$$\gamma := \frac{N-2-\sqrt{D}}{2}, \tag{1.1.8b}$$

$$D := 16\beta^2 - 8(N-4)\beta + (N-2)(N-10). \tag{1.1.8c}$$

All the radial regular stationary solutions, denoted by $U_\kappa(r)$, are parametrized by their values at the origin, i.e., $\kappa = U_\kappa(0) \in \mathbf{R}$. It is known (cf. Proposition 1.2.1 below) that, when $p > p_{\mathrm{JL}}$,

$$U_\kappa(|x|) = U_\infty(|x|) - h_\kappa |x|^{-\gamma} + o(|x|^{-\gamma}) \quad \text{as} \quad |x| \to \infty, \tag{1.1.9}$$

where $h_\kappa > 0$ is a constant depending on κ and $U_\infty(r)$ is the singular stationary solution :

$$U_\infty(|x|) := c_*|x|^{-2\beta} \quad \text{with} \quad c_*^{p-1} := 2\beta(N-2-2\beta). \tag{1.1.10}$$

Herrero and Velázquez [23, 24] proved that, as long as $N \geq 11$ and $p_{\rm JL} < p$, type II blow-up actually occurs. They constructed radial blow-up solutions $\{u_{\ell,\,\rm HV}\}_{\ell\in\mathcal{L}}$, $\mathcal{L} \subset \mathbf{N}$, (which we call *HV solutions*) satisfying $\|u_{\ell,\,\rm HV}(\cdot,t)\|_\infty = u_{\ell,\,\rm HV}(0,t)$ and

$$C_1(T-t)^{-\beta-2\beta\omega_\ell} \leq u_{\ell,\,\rm HV}(0,t) \leq C_2(T-t)^{-\beta-2\beta\omega_\ell} \tag{1.1.11a}$$

$$\text{with} \quad \omega_\ell := \frac{\lambda_\ell}{\gamma-2\beta} > 0 \quad \text{and} \quad \lambda_\ell := \beta - \frac{\gamma}{2} + \ell \tag{1.1.11b}$$

for some constants $C_1, C_2 > 0$. The proof requires a long argument. Though the main article [23] containing the full proof remains unpublished yet, the result as well as the idea of the proof is well explained in [24] without arguing the technical detail. A slightly shorter proof was given by [32] under the additional assumption that ℓ is even. These blow-up rates appear also for some non-radial solutions [4, 6]. The method of [23, 24] has become one of the standard tools in the study of type II singularity. Indeed, it has been applied to several nonlinear parabolic problems (cf. for instance, [1, 19–21, 43, 49]). Based on the results of [23, 24], Matano [27] and Mizoguchi [34] independently proved that, if $\lambda_n \neq 0$ for every $n \in \mathbf{N}$ and if a radial solution blows up in finite time with type II regime, then its actual blow-up rate coincides with (1.1.11) for some $\ell \in \mathcal{L}$, where λ_ℓ is as in (1.1.11b). As to this direction, an earlier result [35] includes the same conclusion for $p > p_{\rm L}$ (so that $\lambda_0 < \lambda_1 < 0 < \lambda_2 < \dots$), where p_L stands for the Lepin exponent :

$$p_{\rm L} := \begin{cases} \infty, & N \leq 10, \\ 1 + \dfrac{6}{N-10}, & N \geq 11. \end{cases} \tag{1.1.12}$$

This was first found by [25] in the study of self-similar solutions. See [37, 40] for recent results on this topic.

For $p = p_{\rm L}$, it was proved in [46] that there exist type II blow-up solutions with exact rates much different from (1.1.11a) (see also [1, 2] for related results). Whether or not type II blow-up occurs for $p = p_{\rm JL}$ had been long remained open until it was affirmatively solved in [44]. The analysis in [44, 46] is much delicate than that of [23, 24]. Our principal goal is, using the techniques developed in [44, 46] and elaborate estimates on the heat semigroup in backward similarity variables, to construct refined solutions whose blow-up mechanism is driven by a stable eigenvalue such as HV solutions. As we have already pointed out, the method originated from [23, 24] has been applied to several nonlinear parabolic problems. We expect that the refined technique developed in this article would apply to other nonlinear parabolic problems, thus obtaining completely new results or considerable improvements of the previous results.

As for the case $p = p_{\rm S}$, the existence of type II blow-up solutions have been obtained in [42] for $N = 4$ and in [8, 22] for $N = 5$. An earlier result due to [10] formally indicates that type II blow-up can occur for $N = 3, 4, 5, 6$. It was proved in [5] that type II blow-up solutions do not exist in some class of function spaces for $N \geq 7$.

1.1.2 Study of equation $u_t = \Delta u + |x|^{2a}u^p$

In the early stage of the research on (1.1.2), one of the main topics was to investigate the influence of decay rate of initial data at infinity for global-in-time existence of solutions. For instance, Pinsky [39] showed that the critical exponent for existence of global solutions depends on the behavior of weighted term $|x|^{2a}u^p$ as $|x| \to \infty$. Wang [48] studied sufficient conditions on initial data for global-in-time existence and the asymptotic behavior as $t \to \infty$. A comprehensive survey can be found in the introduction of [47]. In the case $a > 0$, on the other hand, the weighted term can disturb blowing up at the origin. Some recent articles discuss whether the zero point in the nonlinearity (i.e., $x = 0$) can be a blow-up point when blow-up takes place. Several conditions which ensure non-blow-up at the zero point were obtained in [15, 16, 18]. Examples of solutions blowing up at $x = 0$, in contrast, were found in [11, 17]. Filippas and Tertikas [11] constructed self-similar solutions that blow up (in finite time) at $x = 0$ in the cases $p < p_{\rm S}(a)$ or $p_{\rm S}(a) < p < p_{\rm JL}(a)$, where

$$p_{\rm S}(a) := \begin{cases} \infty, & N = 1, 2, \\ 1 + \dfrac{4(1+a)}{N-2}, & N \geq 3 \end{cases} \tag{1.1.13}$$

and

$$p_{\rm JL}(a) := \begin{cases} \infty, & N \leq 10 + 8a, \\ 1 + \dfrac{4(1+a)}{N - 2a - 4 - 2\sqrt{(N+a-1)(a+1)}}, & N > 10 + 8a. \end{cases} \tag{1.1.14}$$

As a matter of fact, they agree with the previous notations (1.1.6), (1.1.7) when $a = 0$. Apart from the explicit examples in [11], Guo and Shimojo [17] proved the existence of a solution that blows up at the origin for $N = 3$ and $p > p_{\rm S}(a)$. The proof of [17] is due to an argument by contradiction. To the best of the authors' knowledge, no other example of such a blow up solution has not been obtained. Our method naturally extends to the case $a > 0$ and $p > p_{\rm JL}(a)$ with $N > 10 + 8a$ (cf. §§1.1.3), thereby giving a new example of solutions that blow up at the zero point. We note that our proof in fact works for $a > -1$ and thus covers the three-dimensional case. The proof is totally different from the indirect construction due to [17]. In addition, our blow-up solutions satisfy

$$\lim_{t \nearrow T} (T-t)^{(1+a)/(p-1)} u(0,t) = +\infty. \tag{1.1.15}$$

Phan [38] has recently established a Liouville-type theorem for (1.1.2) and applied it to show the blow-up rate estimates of the form (in our notation):

$$\|u(t)\|_{L^\infty(\mathbf{R}^N)} \leq C(T-t)^{-(1+a)/(p-1)}, \quad 0 < t < T$$

for $a > 0$, $p < p_{\rm S}(a)$ or for $-1 < a < 0$, $N \geq 2$, $p < p_{\rm S}(a)$ and radially nonincreasing initial data. This estimate is in contrast to (1.1.15).

1.1.3 The statement of the main results

Given a number $a > -1$, we re-define the constants β and D as follows:

$$\beta = \beta(a) := \frac{1+a}{p-1}, \tag{1.1.16}$$

$$D = D(a) := 16\beta^2 - 8(N-4-2a)\beta + (N-2)(N-10-8a), \tag{1.1.17}$$

We keep the notations γ and c_* as (1.1.8b) and (1.1.8c), and (1.1.10) with β replaced by the one above. In the following, let us abbreviate $p_{\rm JL}(a)$ to $p_{\rm JL}$. The family of regular stationary solutions $U_{a,\kappa}(r)$, $\kappa > 0$, of (1.1.2) has the same structure as in the case $a = 0$. In particular, the stationary solution $U_{a,1}$ satisfies

$$U_{a,1}(|x|) = U_\infty(|x|) - h|x|^{-\gamma} + o(|x|^{-\gamma}) \quad \text{as} \quad r \to \infty \tag{1.1.18}$$

for some constant $h > 0$ as long as $p > p_{\rm JL}$, $N > 10 + 8a$. See Proposition 1.2.1 below.

Theorem 1.1.1. *Assume that $a > -1$, $p > p_{\rm JL}$, and $N > 10 + 8a$, hold. Let ℓ be a positive integer such that λ_ℓ in* (1.1.11b) *is positive and set $\omega_\ell := \lambda_\ell/(\gamma - 2\beta)$. Then for every $T > 0$ and $\varrho > 0$, there exists a positive radially decreasing solution u_ℓ of* (1.1.1)*, which blows up at $t = T$, $x = 0$, with the following properties:*

(i) (Exact blow-up rate)

$$\lim_{t \nearrow T} (T-t)^{\beta + 2\omega_\ell \beta} u_\ell(0,t) = K_T^{2\beta} \tag{1.1.19}$$

with $K_T := (T/T_0)^{\omega_\ell}$, where $T_0 \in (0,1)$ is a fixed small constant depending only on N, p, a, ℓ, and ϱ;

(ii) (Estimates in a neighborhood of the inner layer) *There holds*

$$\begin{aligned} &\left| u_\ell(x,t) - \left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}} \right)^{2\beta} U_{a,1}\left(\frac{K_T|x|}{(T-t)^{1/2+\omega_\ell}} \right) \right| \\ &< K_T^{-\theta}(T-t)^{\theta\omega_\ell} \left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}} \right)^{2\beta} \Psi\left(\frac{K_T|x|}{(T-t)^{1/2+\omega_\ell}} \right) \end{aligned} \tag{1.1.20}$$

for $|x| \le K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell}$, $0 < t < T$, where K_T is the constant as in (i)*, $\theta \in (0,1)$ is a constant, and $\Psi(\xi)$ is a positive C^∞-function satisfying*

$$\Psi(\xi) = \begin{cases} O(1) & \text{as} \quad \xi \to 0, \\ O(\xi^{-\gamma}) & \text{as} \quad \xi \to \infty; \end{cases} \tag{1.1.21}$$

(iii) (Estimates in bounded regions) *There holds*

$$\begin{aligned} &\left| u_\ell(x,t) - U_\infty(|x|) + K_T^{-(\gamma-2\beta)} C_*(T-t)^\ell |x|^{-\gamma} L_\ell^{(\sqrt{D}/2)}\left(\frac{|x|^2}{4(T-t)} \right) \right| \\ &< \varrho K_T^{-(\gamma-2\beta)}(T-t)^\ell |x|^{-\gamma} \left(1 + \frac{|x|^2}{T-t} \right)^\ell \end{aligned} \tag{1.1.22}$$

with

$$C_* := \frac{h\Gamma(\sqrt{D}/2+1)\ell!}{\Gamma(\sqrt{D}/2+\ell+1)}$$

for $K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell} < |x| \le T^{1/2}/T_0^{1/2-\omega}$, $0 < t < T$, *where* $\omega \in (0,1/2)$, Γ *is the standard Gamma function,* $L_\ell^{(\varsigma)}(z)$ *denotes the associated Laguerre polynomial of degree* ℓ, *and* T_0, θ *are the constants as in* (i), (ii);

(iv) (Number of intersections) *There exist exactly* ℓ *simple zeros* $\{r_n(t)\}_{n=1}^{\ell}$ *of* $u_\ell(\cdot,t) - U_\infty$ *for every* $t \in (0,T)$, *which satisfy* $r_n(t) = O(\sqrt{T-t})$ *as* $t \nearrow T$ *for* $n = 1, ..., \ell$.

Remark 1.1.1. The constant T_0 is related to ϱ as $T_0 \le \varrho^{1/2\theta\omega_\ell}$. In fact, estimate (1.1.22) is further improved in the intersections with parabolic regions $\{|x| \le R\sqrt{T-t}\}$ with $R > 1$ being an arbitrary constant, so that constant ϱ may be replaced by function $K_T^{-2\theta}(T-t)^{2\theta\omega_\ell}$ there. This can be checked by slight modifications of the proofs of Lemmas 1.4.9–1.4.11.

As the blow-up rate estimate (1.1.19) shows, the solution u_ℓ above is of essentially the same class as of $u_{\ell,\mathrm{HV}}$ obtained by [23, 24]. Theorem 1.1.1 includes, however, more information about local-in-space estimates both near and away from the singularity even for $a = 0$. Indeed, the proof of [23, 24] ensures an estimate of the form

$$C_1 U_{\kappa_1}\left(\frac{|x|}{(T-t)^{1/2+\omega_\ell}}\right) \le (T-t)^{\beta+2\beta\omega_\ell} u_{\ell,\mathrm{HV}}(x,t) \le C_2 U_{\kappa_2}\left(\frac{|x|}{(T-t)^{1/2+\omega_\ell}}\right)$$

with $\kappa_1 < 1 < \kappa_2$ for $|x| = O\big((T-t)^{1/2+\omega_\ell}\big)$. The statement (ii) of Theorem 1.1.1 shows that the leading term of u_ℓ in the region $|x| \le K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell}$ is precisely determined as

$$u_\ell(x,t) \sim \left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}}\right)^{2\beta} U_{a,1}\left(\frac{K_T|x|}{(T-t)^{1/2+\omega_\ell}}\right) \quad \text{as} \quad t \nearrow T$$

as well as the estimates of error terms. The counterparts for their derivatives are given in Corollary 1.1.2 below. Another novelty of Theorem 1.1.1 consists in the estimate (1.1.22) for bounded regions, $|x| \asymp 1$, which extends the region $|x| \le (T-t)^{1/2-\sigma}$, $\sigma \in (0,1/2)$, of validity of the estimate guaranteed for $u_{\ell,\mathrm{HV}}$. Since

$$(T-t)^\ell |x|^{2\beta-\gamma}\left(1+\frac{|x|^2}{T-t}\right)^\ell = |x|^{2\lambda_\ell}\left(\frac{T-t}{|x|^2}+1\right)^\ell$$

for $K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell} < |x| \le T^{1/2}/T_0^{1/2-\omega}$, we deduce from (1.1.22) that

$$C'|x|^{2\lambda_\ell} \le \left|\frac{u_\ell(x,T)}{U_\infty(|x|)} - 1\right| \le C|x|^{2\lambda_\ell} \tag{1.1.23}$$

for every $0 < |x|$ small enough, where $u_\ell(x,T) := \lim_{t\nearrow T} u_\ell(x,t)$ denotes the blow-up profile defined outside the blow-up set. In particular, we have,

$$\lim_{|x|\to 0} \frac{u_\ell(x,T)}{U_\infty(|x|)} = 1. \tag{1.1.24}$$

This was established in [29, Theorem 4.1] as one of the properties characterizing (possibly sign-changing) type II blow-up (with the RHS of (1.1.24)replaced by ± 1) for $p > p_{\rm S}$, but no concrete example directly verifying (1.1.24) has been obtained so far. Our particular solutions do imply (1.1.24) and estimate (1.1.23) includes further information on the convergence. In particular, it shows optimal estimates of the error depending on each eigenvalue.

Arguing as in [44, 46], we obtain further properties on the solution.

Corollary 1.1.2. *Assume the same hypothesis as in Theorem 1.1.1 and $a \geq 0$. Let $u = u_\ell$ be the type II blow-up solution as in Theorem 1.1.1. Then the diffusion term $-\Delta u(x,t)$ exhibits the same growth rate as of the superlinear term $|x|^{2a}u^p(x,t)$:*

$$\begin{aligned} -\Delta u(x,t) &= \left(K_T^{-1}(T-t)^{1/2+\omega_\ell}\right)^{-2(\beta+1)} \\ &\quad\times\left(\left(\frac{|x|}{\varepsilon(\tau)\sqrt{T-t}}\right)^{2a} U_1\left(\frac{|x|}{\varepsilon(\tau)\sqrt{T-t}}\right)^{p} + o(1)\right), \end{aligned} \tag{1.1.25}$$

$$u_t(x,t) = o\big((T-t)^{-2(\beta+1)(1/2+\omega_\ell)}\big), \tag{1.1.26}$$

as $t \nearrow T$ for every $(x,t) \in \mathbf{R}^N \times (0,T)$ with $|x| \leq K_T^{-1}(T-t)^{1/2+\omega_\ell}$.

Remark 1.1.2. Set $m(t) = \|u(\cdot,t)\|_\infty$. The following characterization of blow-up rates for any blow-up solutions of (1.1.1) was proved in [28, Appendix B]:

$$\begin{aligned} &\text{Type I}: \quad m'(t) = O(m(t)^p) \quad \text{as} \quad t \nearrow T, \\ &\text{Type II}: \quad m'(t_n) = o(m(t_n)^p) \quad \text{for some sequence } t_n \nearrow T. \end{aligned} \tag{1.1.27}$$

In particular, (1.1.27) represents the *slow nature* of type II blow-up. Corollary 1.1.2 shows the quantitative information about these amounts (without choosing a particular time-sequence) for the solutions. Thereby they become a prime example of this fact.

Corollary 1.1.3. *Assume the same hypothesis as in Theorem 1.1.1. Let $u = u_\ell$ be the type II blow-up solution as in Theorem 1.1.1. Then for every $q > q_c := N(p-1)/2(1+a)$, there exist constants $C_1, C_2 > 0$ such that*

$$C_1(T-t)^{-(1/2+\omega_\ell)(2\beta-N/q)} \leq \|u(\,\cdot\,,t)\|_{L^q(\mathbf{R}^N)} \leq C_2(T-t)^{-(1/2+\omega_\ell)(2\beta-N/q)} \tag{1.1.28}$$

for $0 < t < T$. More precisely,

$$\int_{\left\{|x|\leq K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell}\right\}} u(x,t)^q\,dx = D_1\left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}}\right)^{2\beta q-N}\big(1+o(1)\big), \tag{1.1.29}$$

$$\int_{\left\{K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell}\leq|x|\right\}} u(x,t)^q\,dx = O\left((T-t)^{-(1/2+\theta\omega_\ell)(2\beta q-N)}\right), \tag{1.1.30}$$

as $t \nearrow T$, *where*

$$D_1 := \int_0^\infty U_{a,1}(r)^q r^{N-1}\, dr < \infty.$$

Corollary 1.1.4. *Assume the same hypothesis as in Theorem 1.1.1. Let* σ *be a constant with* $\sigma > 2\beta$. *Then there exists an initial data* u_0 *satisfying*

$$u_0(x) \le C(1+|x|)^{-\sigma} \quad \text{in} \quad \mathbf{R}^N \tag{1.1.31}$$

for some constant $C > 0$ *such that the corresponding solution* $u = u_\ell \in C\left([0,T); L^{q_c}(\mathbf{R}^N)\right)$ *satisfies the same estimates as in* (i)–(iv) *of Theorem 1.1.1 and*

$$C_3|\log(T-t)| \le \|u(\cdot,t)\|_{L^{q_c}(\mathbf{R}^N)} \le C_4|\log(T-t)| \tag{1.1.32}$$

for $0 < t < T$, *where* $C_3, C_4 > 0$ *are some constants.*

Remark 1.1.3. A recent result [36] shows that the critical L^q norm blow-up does occur for possibly non-radial solutions of (1.1.1) if the blow-up is of type I. The solution u_ℓ as in Theorem 1.1.1 exhibits type II blow-up. Nevertheless, corollary 1.1.4 shows that the critical norm $\|u_\ell(\cdot,t)\|_{L^{q_c}}$ blows up as well and that, moreover, the rate is logarithmic.

The solutions as in Theorem 1.1.1 certainly converge to the singular self-similar solution U_∞, in the self-similar variables (cf. (1.2.1)), locally uniformly in $\mathbf{R}^N \setminus \{0\}$. In the Sobolev subcritical case, the convergence holds with the U_∞ replaced by the positive constant β^β with $a = 0$ [13]. In this case, a small perturbation of some initial data yields the same blow-up mechanism (see, for instance, [3,31]). To the best of the authors' knowledge, no reasonable statement on such stability results of Type II blow-up for equation (1.1.1) or (1.1.1) was known even in the radially symmetric case.

Before closing this introduction, we just comment on relation to some previous results on (1.1.1). As we have already pointed out, several methods to analyze type II singularity have been recently developed. The approach of [42] relies on so called energy method coming from dispersive equations and does not on tools particular for parabolic equations, such as maximum principle. Our approach, on the other hand, does not require energetic structure but uses thoroughly explicit formulas of a semigroup (cf. (1.4.18) below). At this stage, it concerns parabolic problems (in the radial case) only, but can describe subtle local behavior, especially in the region of order one $|x| \asymp 1$ (cf. (1.1.23)) in addition to parabolic regions, where more detailed estimates than [42] are obtained. The approaches of [8,22] seems to have close relation to our matched asymptotics. The authors of [8] developed elegant linear theory for a certain linearized problem applicable to determining the leading order profile in blow-up regions. The construction of our sub- and supersolutions relies on "layer structure" of stationary solutions $U_\kappa(|x|)$, i.e., monotonicity with respect to κ (their values at $x = 0$), which is available only for $p \ge p_{\rm JL}$. We believe, however, that it directly justifies the asymptotic series expansions in the formal construction. Except for this, we dispense with inessential comparison techniques in [23,35]. We hope to develop new ideas without relying on the layer structure, so that our approach will be further extended to analyze various nonlinear parabolic problems. As for the non-radial situation, the authors of [5, 8] independently obtained very interesting blow-up solutions on $\mathbf{R}^{N+1}$ by extending radial solutions on $\mathbf{R}^N$. The author are unaware if their analysis could be carried out for (1.1.2) with $a \neq 0$. Discussing the detail of related results for all type of semilinear equations is beyond the scope of this Chapter. The readers are referred to the above-mentioned articles and references cited therein.

The rest of this article is organized as follows. In §1.2 we first summarize some basic properties of stationary solutions and the linearized operator around U_∞ in the backward similarity variables. By means of matched asymptotic expansions, we then formally describe the leading terms and investigate how large the error terms can be. The last argument leads to a formulation of finite-dimensional reduction for the rigorous construction in §1.3. Theorem 1.1.1 and Corollaries 1.1.2–1.1.4 are proved therein under the assumption that a key *a priori* estimate holds. §1.4–1.6 are devoted to proving the *a priori* estimate.

1.2 Preliminaries

In this section, we review some known facts essentially due to [23] and discuss the formal construction. Introducing the backward similarity variables

$$\Phi(y,\tau) := (T-t)^{\beta} u(x,t) \tag{1.2.1a}$$

$$\text{with} \quad y := \frac{x}{\sqrt{T-t}} \quad \text{and} \quad \tau := -\log(T-t), \tag{1.2.1b}$$

we convert equation (1.1.1) to the rescaled equation:

$$\Phi_\tau = \Delta_y \Phi - \frac{y \cdot \nabla_y \Phi}{2} - \beta\Phi + |y|^{2a}\Phi^p \quad \text{in} \quad \big(\mathbf{R}^N \setminus \{0\}\big) \times (-\log T, \infty), \tag{1.2.2}$$

where ∇_y and Δ_y are the gradient and the Laplacian with respect to y, respectively. Notice that $U_\infty(r)$ as in (1.1.10) with $r = |y|$ is also an unbounded stationary solution of (1.2.2). We shall henceforth abuse notations as well such as $\Phi(r,\tau) = \Phi(y,\tau)$ for simplicity.

1.2.1 Formal asymptotics in the inner region

Suppose that an inner layer near the origin appears in our sought-for solution $\Phi(r,\tau)$ of (1.2.2), where sharp changes in Φ arise when $\tau \to \infty$. Let $\varepsilon(\tau)$ denote the size of the inner layer, which is a priori unknown. We assume

$$\varepsilon(\tau),\, \dot{\varepsilon}(\tau) \ll 1 \quad \text{as} \quad \tau \to \infty. \tag{1.2.3}$$

To see the dynamics near the origin, we introduce inner variables $(U(\xi,\tau),\xi)$ as follows:

$$U(\xi,\tau) := \varepsilon(\tau)^{2\beta}\Phi(y,\tau) \quad \text{with} \quad \xi := \frac{y}{\varepsilon(\tau)}. \tag{1.2.4}$$

A direct computation then shows that

$$\varepsilon(\tau)^2 U_\tau = \Delta_\xi U + |\xi|^{2a} U^p - \big(\varepsilon(\tau)^2 - 2\varepsilon(\tau)\dot{\varepsilon}(\tau)\big)\left(\frac{\xi \cdot \nabla_\xi U}{2} + \beta U\right). \tag{1.2.5}$$

In view of (1.2.3), we infer that the leading term of U as $\tau \to \infty$ would be given by a bounded stationary solution of (1.1.2) for $\xi = o(1/\varepsilon(\tau))$, which amounts to $|y| \ll 1$. The structure of stationary solutions $U_{a,\kappa}$ (abbreviated in the sequel to U_κ for simplicity) of (1.1.2) is well understood, which we just recall here.

Proposition 1.2.1. ([26, Lemma 4.3]) *For any $\alpha > 0$, there exists a unique solution U_κ of*

$$\begin{cases} \dfrac{d^2U}{dr^2} + \dfrac{N-1}{r}\dfrac{dU}{dr} + r^{2a}U^p = 0 \quad \text{in} \quad \mathbf{R}_+, \\ U(0) = \kappa, \quad U'(0) = 0. \end{cases} \tag{1.2.6}$$

If $p > p_{\mathrm{JL}}$ and $N > 10 + 8a$, the family of the solutions $\{U_\kappa\}_{\kappa>0}$ has the ordered structure:

$$\kappa_1 < \kappa_2 \Longrightarrow U_{\kappa_1}(r) < U_{\kappa_2}(r) \quad \text{for all} \quad r > 0. \tag{1.2.7}$$

Moreover,

$$U_1(r) = U_\infty(r) - hr^{-\gamma} + R(r), \tag{1.2.8a}$$

$$U_1'(r) = U_\infty'(r) + h\gamma r^{-\gamma-1} + R(r)O(r^{-1}), \tag{1.2.8b}$$

as $r \to \infty$, *where $h > 0$ is a constant and $R(r) = o(r^{-\gamma})$. More precisely, there holds*

$$R(r) = \begin{cases} O\big(r^{-\gamma-\min\{\gamma-2\beta,\sqrt{D}\}}\big) & \text{if} \quad \sqrt{D} \neq \gamma - 2\beta, \\ O\big(r^{-\gamma-\sqrt{D}}\log r\big) & \text{if} \quad \sqrt{D} = \gamma - 2\beta. \end{cases}$$

Due to (1.2.4) and (1.2.5), it is natural to construct a solution of the form:

$$\Phi_{\mathrm{inn}}(r,\tau) := \varepsilon(\tau)^{-2\beta}U_1\left(\frac{r}{\varepsilon(\tau)}\right) \quad \text{with} \quad r = |y|, \tag{1.2.9}$$

which describes the dynamics in the inner region $r = O(\varepsilon(\tau))$. The asymptotic behavior (1.2.8a) of U_κ then implies

$$\Phi_{\mathrm{inn}}(r,\tau) \sim U_\infty(r) - h\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma} \quad \text{for} \quad \varepsilon(\tau) \ll r \ll 1. \tag{1.2.10}$$

Hence our sought-for solution $\Phi(r,\tau)$ should behave, to the leading term, to $U_\infty(r)$ in the regions where $\varepsilon(\tau) \ll r$ as $\tau \to \infty$. It is therefore natural to linearize equation (1.2.2) around $U_\infty(r)$.

1.2.2 Formal asymptotics in the intermediate region

Let us set

$$v(r,\tau) := \Phi(r,\tau) - U_\infty(r). \tag{1.2.11}$$

and $\rho(r) := \exp\{-r^2/4\}$. It is readily seen that v solves equation

$$v_\tau = -\mathcal{A}v + f(v) \tag{1.2.12}$$

with

$$
\begin{aligned}
-\mathcal{A}v &:= \frac{1}{r^{N-1}\rho(r)}\frac{\partial}{\partial r}\left(r^{N-1}\rho(r)\frac{\partial v}{\partial r}\right) - \beta v + \frac{pc_*^{p-1}}{r^2}v && \text{(1.2.13a)}\\
f(v) &:= r^{2a}\big[(U_\infty + v)^p - U_\infty^p - pU_\infty^{p-1}v\big]. && \text{(1.2.13b)}
\end{aligned}
$$

Let us write

$$
\begin{aligned}
L^2_{\rho,\,\mathrm{rad}}(\mathbf{R}^N) &:= \left\{v \in L^2_{\mathrm{loc}}([0,\infty))\,;\, \|v\|^2 \equiv \|v\|^2_{L^2_{\rho,\,\mathrm{rad}}(\mathbf{R}^N)} := \int_0^\infty v(\eta)^2\eta^{N+3}\rho(\eta)\,d\eta < +\infty\right\},\\
H^1_{\rho,\,\mathrm{rad}}(\mathbf{R}^N) &:= \left\{v \in H^1_{\mathrm{loc}}([0,\infty))\,;\, \|v\|^2_{H^1_{\rho,\,\mathrm{rad}}(\mathbf{R}^N)} := \|v\|^2 + \|v'\|^2 < +\infty\right\}.
\end{aligned}
$$

The linearized operator $Av \equiv \mathcal{A}v$ with $v \in \mathcal{D}(A) = C_0^\infty(\mathbf{R}_+)$ is realized as a symmetric operator in $L^2_{\rho,\,\mathrm{rad}}(\mathbf{R}^N)$. A version of Hardy type inequality as well as integration by parts implies that, if v is smooth,

$$
\begin{aligned}
\langle Av, v\rangle_N &:= \langle Av, v\rangle_{L^2_{\rho,\,\mathrm{rad}}(\mathbf{R}^N)}\\
&= \int_0^\infty \left(\frac{\partial v}{\partial r}\right)^2 r^{N-1}\rho(r)\,dr + \beta\int_0^\infty v^2r^{N-1}\rho(r)\,dr - pc_*^{p-1}\int_0^\infty v^2r^{N-3}\rho(r)\,dr\\
&\geq \left(1 - \frac{4pc_*^{p-1}}{(N-2)^2}\right)\left\|\frac{\partial v}{\partial r}\right\|^2 - \left(\frac{pc_*^{p-1}}{N-2} - \beta\right)\|v\|^2.
\end{aligned}
$$

Consequently, if $p \geq p_{\mathrm{JL}}$, the operator A is lower bounded, i.e., $\langle A\phi, \phi\rangle \geq -C\|\phi\|^2$ for every functions $\phi \in \mathcal{D}(A)$. We still denote by A its Friedrichs extension. The following spectral result is proved by essentially the same argument as in [23, Lemma 2.3], [44, Proposition 2.2].

Proposition 1.2.2. *Assume that $p \geq p_{\mathrm{JL}}$ and $N > 10 + 8a$ be in force. Then the spectrum of A consists only of simple eigenvalues $\{\lambda_n\}_{n=0}^\infty$,*

$$
\lambda_n := \beta - \frac{\gamma}{2} + n \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{1.2.14}
$$

Eigenfunctions of A associated with eigenvalues λ_n are given by

$$
\begin{aligned}
\phi_n(r) &:= c_nr^{-\gamma}\mathcal{M}\left(-n, -\gamma + \frac{N}{2}\,;\, \frac{r^2}{4}\right) \quad \text{for} \quad n = 0, 1, 2, \ldots\,; && \text{(1.2.15a)}\\
\mathcal{M}(a, b\,; z) &:= 1 + \sum_{j=1}^\infty \frac{(a)_j}{j!(b)_j}z^j \quad \text{with} \quad (a)_m = \prod_{j=0}^{m-1}(a+j), && \text{(1.2.15b)}
\end{aligned}
$$

where $c_n > 0$ are constants such that $\|\phi_n\| = 1$. Moreover, the eigenfunctions satisfy

$$
\begin{aligned}
\phi_n(r) &= c_nr^{-\gamma}\big(1 + O(r^2)\big) && \text{as} \quad r \to 0\,; && \text{(1.2.16a)}\\
\phi_n(r) &= \tilde{c}_nr^{-\gamma+2n}\big(1 + O(r^{-2})\big) && \text{as} \quad r \to \infty, && \text{(1.2.16b)}
\end{aligned}
$$

where $\tilde{c}_n \in \mathbf{R}$ are constants such that $(-1)^n \tilde{c}_n > 0$ for $n = 0, 1, 2, \dots$ Furthermore, the constants c_n and $\tilde{c}_n$ in (1.2.16) are represented as

$$c_n := \sqrt{\frac{\Gamma(-\gamma + N/2 + n)}{2^{-2\gamma+N-1} n! \Gamma(-\gamma + N/2)^2}}, \tag{1.2.17a}$$

$$\tilde{c}_n := \frac{(-1)^n}{2^{2n}(-\gamma + N/2)_n} \sqrt{\frac{\Gamma(-\gamma + N/2 + n)}{2^{-2\gamma+N-1} n! \Gamma(-\gamma + N/2)^2}}, \tag{1.2.17b}$$

respectively, where Γ stands for the standard Gamma function.

Remark 1.2.1. By classical results on orthogonal polynomials, the eigenfunctions are expressed by associated Laguerre polynomials $L_n^{(\varsigma)}(z) = (n!)^{-1} e^z z^{-\varsigma} (d^n/dz^n)(e^{-z} z^{n+\varsigma})$:

$$\psi_n(r) := r^\gamma \phi_n(r) = \frac{c_n n! \Gamma(\varsigma + 1)}{\Gamma(\varsigma + n + 1)} L_n^{(\varsigma)} \left(\frac{r^2}{4} \right) \quad \text{with} \quad \varsigma = \frac{\sqrt{D}}{2} = -\gamma + \frac{N}{2} - 1. \tag{1.2.18}$$

Applying Stirling's formula $\Gamma(z) \sim \sqrt{2\pi/z}(z/e)^z$ as $z \to \infty$ to (1.2.17), we have

$$c_n^2 \sim \frac{n^\varsigma}{2^{2\varsigma+1}\Gamma(\varsigma+1)^2} \quad \text{and} \quad \tilde{c}_n^2 \sim \frac{1}{2^{4n}(1+\varsigma)_n^2} \frac{n^\varsigma}{2^{2\varsigma+1}\Gamma(\varsigma+1)^2}, \quad \text{as} \quad n \to \infty. \tag{1.2.19}$$

We recall the well-known estimate [45]

$$\left| L_n^{(\varsigma)}(z) \right| \le \frac{\Gamma(\varsigma + n + 1)}{\Gamma(\varsigma + 1) n!} e^{z/2}, \quad z \ge 0,$$

whence:

$$|\psi_n(r)| \le c_n e^{r^2/8}. \tag{1.2.20}$$

In particular, the polynomials $\psi_n(r)/c_n$ are uniformly bounded in every compact set of $[0, \infty)$.

We shall recall the idea of [23, 24] and then refine their argument. Due to Proposition 1.2.2, the solution $v \in L^2_{\rho,\mathrm{rad}}(\mathbf{R}^N)$ of (1.2.13) may be expanded to a Fourier series: $v(r,\tau) = \sum_{n=0}^\infty a_n(\tau)\phi_n(r)$, where the Fourier coefficients $a_n(\tau) = \langle v(\tau), \phi_n \rangle_N$ satisfy

$$\dot{a}_n(\tau) = -\lambda_n a_n(\tau) + \langle f(v(\tau)), \phi_n \rangle_N. \tag{1.2.21}$$

Consider the situation where a stable mode eventually dominates:

$$v(r,\tau) \sim a_\ell(\tau)\phi_\ell(r) \quad \text{as} \quad \tau \to \infty, \tag{1.2.22}$$

where ℓ is an integer such that $\lambda_\ell > 0$. Suppose that the term $\langle f(v(\tau)), \phi_\ell \rangle_N$ in (1.2.21) would play no role to the leading order. We then expect that the leading term of $a_\ell(\tau)$ would be determined by the homogeneous term of (1.2.21). Hence, as $\tau \to \infty$,

$$U_\infty(r) + v(r,\tau) \sim U_\infty(r) - d_\ell e^{-\lambda_\ell \tau} \phi_\ell(r) =: \Phi_{\mathrm{med}}(r,\tau) \tag{1.2.23}$$

with some constant $d_\ell > 0$. The outer expansion as $r \to 0$ then follows from (1.2.16a):

$$\Phi_{\rm med}(r,\tau) \sim U_\infty(r) - c_\ell d_\ell e^{-\lambda_\ell \tau} r^{-\gamma}. \tag{1.2.24}$$

Matching the inner expansions (1.2.10) with the outer ones (1.2.24) in the intermediate region $\{\varepsilon(\tau) \ll |y| \ll 1\}$ where both expansions make sense, we obtain

$$\varepsilon(\tau)^{\gamma-2\beta} \sim C_\ell e^{-\lambda_\ell \tau} \quad \text{with} \quad C_\ell := \frac{c_\ell d_\ell}{h}. \tag{1.2.25}$$

Substituting (1.2.25) into (1.2.10) and returning to the original variables, we formally obtain the asymptotic expansions of the HV solution $\{u_{\ell,\rm HV}\}$.

While the above argument simply tells us what determines the leading terms of the outer expansions, it does not imply the possible effect of the nonlinear term $f(v)$ to $a_\ell(\tau)$ nor how large the next order corrections can be. We shall derive this result as well as expected error estimates by more careful argument.

Hypothesis 1.2.3. *The blow-up is driven by the stable eigenvalue $\lambda_\ell > 0$:*

$$|a_n(\tau)| \ll |a_\ell(\tau)| \quad \text{as} \quad \tau \to \infty \tag{1.2.26}$$

for $n = 0, 1, \ldots, \ell-1$ and (1.2.22) holds. Moreover, the controlling factor of $a_\ell(\tau)$ is $e^{-\lambda_\ell \tau}$ and the other factors are polynomially bounded as $\tau \to \infty$ in the sense that

$$\frac{C_1}{\tau^k} \le \left| e^{\lambda_\ell \tau} a_\ell(\tau) \right| \le C_2 \tau^k \tag{1.2.27}$$

for some constants $C_1, C_2 > 0$ and $k > 0$.

The rationale behind this hypothesis is the occurrence of possible behavior of $a_\ell(\tau)$, such as $a_\ell(\tau) = Ce^{-\lambda_\ell \tau}\tau^\nu$ with some $C > 0$ and $\nu \neq 0$, which actually arises in the critical case $p = p_{\rm JL}$ [44] or when $\lambda_\ell\,(>0)$ is replaced by a neutral eigenvalue [46]. We will show that such behaviors cannot arise in our situation. In order $\Phi_{\rm med}$ to be matched with the inner expansions (1.2.10) in the intermediate region $\{\varepsilon(\tau) \ll |y| \ll 1\}$, we must have

$$a_\ell(\tau) = -\frac{h}{c_\ell}\varepsilon(\tau)^{\gamma-2\beta} + o(\varepsilon(\tau)^{\gamma-2\beta}). \tag{1.2.28}$$

It then follows from (1.2.27) and (1.2.28) that

$$\varepsilon(\tau)^{\gamma-2\beta} = O\left(e^{-\lambda_\ell \tau}\tau^k\right) \quad \text{as} \quad \tau \to \infty. \tag{1.2.29}$$

For the ease of presentation, we consider only the case where N is not too large so that

$$\chi := \int_0^\infty \xi^{2a-\gamma+N-1}\left[U_1(\xi)^p - U_\infty(\xi)^p - \frac{pc_*^{p-1}}{\xi^{2a+2}}(U_1(\xi) - U_\infty(\xi))\right] d\xi < \infty.$$

Then, arguing as in §2.3 of [46], we obtain

$$\langle f(v(\tau)), \phi_n \rangle_N = \chi c_n \varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}} + o\left(\varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}}\right) \quad \text{as} \quad \tau \to \infty. \tag{1.2.30}$$

We now integrate the ODE (1.2.21) over $[\tau, \infty)$. Since $\int_1^\infty e^{\lambda_\ell s} |\langle f(v(s)), \phi_n \rangle_N| \, ds < \infty$ due to (1.2.29) and (1.2.30), it then turns out that a finite limit $A_n := \lim_{\tau_1 \to \infty} e^{\lambda_n \tau_1} a_n(\tau_1)$ exists,

$$a_n(\tau) = A_n e^{-\lambda_n \tau} - \int_\tau^\infty e^{\lambda_n (s-\tau)} \langle f(v(s)), \phi_n \rangle_N \, ds, \tag{1.2.31}$$

$$\int_\tau^\infty e^{\lambda_n (s-\tau)} |\langle f(v(s)), \phi_n \rangle_N| \, ds = o(e^{-\lambda_\ell \tau}) \quad \text{as} \quad \tau \to \infty \tag{1.2.32}$$

for $n = 0, 1, \ldots, \ell$. Notice that

$$A_n = 0 \quad \text{for} \quad n = 0, 1, \ldots, \ell - 1\,; \tag{1.2.33a}$$

$$A_\ell \neq 0. \tag{1.2.33b}$$

Indeed, (1.2.33a) is a simple consequence of (1.2.26), (1.2.31), and (1.2.32). If (1.2.33b) is false, we deduce from (1.2.29)–(1.2.31) that the controlling factor of $a_\ell(\tau)$ is not $e^{-\lambda_\ell \tau}$, a contradiction. Arguing again as above, we obtain $a_n(\tau) = O(\varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}})$ for $n = 0, 1, \ldots, \ell - 1$ and

$$a_\ell(\tau) - A_\ell e^{-\lambda_\ell \tau} = -\int_\tau^\infty e^{\lambda_\ell (s-\tau)} \langle f(v(s)), \phi_\ell \rangle_N \, ds = O\left(\varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}}\right) \quad \text{as} \quad \tau \to \infty.$$

Then (1.2.25) follows from (1.2.28). In addition, we see that A_ℓ is negative due to the matching condition (1.2.25). The matching condition (1.2.25) suggests that

$$\frac{d}{d\tau}(\varepsilon(\tau)^{\gamma-2\beta}) = -\lambda_\ell \varepsilon(\tau)^{\gamma-2\beta}(1 + o(1)) \quad \text{as} \quad \tau \to \infty. \tag{1.2.34}$$

For $n \geq \ell + 1$, we integrate the ODE (1.2.21) over $[\tau_0, \tau]$. By (1.2.30) and (1.2.34), we get

$$e^{\lambda_n \tau} a_n(\tau) - \widetilde{A}_n = \int_{\tau_0}^\tau e^{\lambda_n s} \langle f(v(s)), \phi_n \rangle_N \, ds \sim \frac{\chi c_n}{\lambda_n - (1+\kappa)\lambda_\ell} e^{\lambda_n \tau} \varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}},$$

where $\widetilde{A}_n := e^{\lambda_n \tau_0} a_n(\tau_0)$ and $\kappa := \sqrt{D}/(\gamma - 2\beta) > 0$. Due to this, we obtain the asymptotoics of $a_n(\tau)$ for $n = \ell + 1, \ell + 2, \ldots$ as $\tau \to \infty$. It follows that

$$R(r, \tau) := v(r, \tau) - \sum_{n=0}^{\ell} a_n(\tau) \phi_n(r) \sim \chi \varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}} F_\ell(r)$$

$$\text{with} \quad F_\ell(r) := \sum_{n=\ell+1}^{\infty} \frac{c_n d_n}{n} \phi_n(r) \quad \text{and} \quad d_n := \frac{1}{1 - (\ell + \kappa\lambda_\ell)/n}, \tag{1.2.35}$$

where the convergence is understood in an appropriate weak sense (cf. [44,46]). We will show that $F_\ell(r) \sim w(r) := r^{-\gamma-\sqrt{D}}/2(\sqrt{D}+1)$ as $r \to 0$. Recall the identity $-2\gamma-\sqrt{D}+N-1=1$ and the exact formula (1.2.18) of $\phi_n(r)$. Then we have

$$(\varsigma+1)\int_0^\infty w(r)\phi_n(r)r^{N-1}e^{-r^2/4}\,dr = \frac{c_n n!\Gamma(\varsigma+1)}{\Gamma(\varsigma+n+1)}\int_0^\infty L_n^{(\varsigma)}(z)e^{-z}\,dz,$$

where $\varsigma = \sqrt{D}/2$ and where the change of variable $z = r^2/4$ has been used as well. Since $n!\int_0^\infty L_n^{(\varsigma)}(z)e^{-z}\,dz = \varsigma(\varsigma+1)\dots(\varsigma+n-1)$, it turns out that

$$\int_0^\infty w(r)\phi_n(r)r^{N-1}e^{-r^2/4}\,dr = \frac{c_n}{\varsigma+n} \quad \text{for} \quad n=0,1,2\dots,$$

$$w(r) = \sum_{n=0}^{\ell}\frac{c_n}{\varsigma+n}\phi_n(r) + \sum_{n=\ell+1}^{\infty}\frac{c_n e_n}{n}\phi_n(r) \quad \text{with} \quad e_n := \frac{1}{1+\varsigma/n}. \tag{1.2.36}$$

Comparing (1.2.35) with (1.2.36) and performing similar computations several times, we have

$$F_\ell(r) = \frac{1}{2(\varsigma+1)}r^{-\gamma-\sqrt{D}} + o\Big(r^{-\gamma-\sqrt{D}}\Big),$$

whence: $R(r,\tau) \sim \chi\varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}}r^{-\gamma-\sqrt{D}}/2(\varsigma+1)$ as $r \to 0$. This is indeed much smaller than $a_\ell(\tau)\phi_\ell(r)$ as long as $\varepsilon(\tau) \ll r \ll 1$, $\tau \to \infty$.

It is also possible to refine inner expansions by computing the next order correction to (1.2.9). To this end, we set

$$U(\xi,\tau) := U_1(\xi) + \mathcal{E}(\tau)H_1(|\xi|) + \dots,$$

where $\mathcal{E}(\tau) := \varepsilon(\tau)^2 - 2\varepsilon(\tau)\dot{\varepsilon}(\tau)$ and $\xi := y/\varepsilon(\tau)$. A standard argument then reveals that $H_1(s) = H_1(|\xi|)$ is a solution of the inhomogeneous linear ODE:

$$H'' + \frac{N-1}{s}H' + pU_1(s)^{p-1}H = \frac{sU_1'(s)}{2} + \beta U_1(s) \quad \text{in} \quad \mathbf{R}_+,$$

satisfying $H(0) = H'(0) = 0$. The solution is expressed by means of variation of constants-formula. Due to Proposition 1.2.1 and L'Hôpital rule, we obtain

$$H_1(s) = C_1 s^{-\gamma+2} + o(s^{-\gamma+2}) \quad \text{with} \quad C_1 = \frac{h(\gamma-2\beta)}{4(2+\sqrt{D})}$$

as $s \to \infty$. Consequently, the two-term expansion for $U(\xi,\tau)$ has been obtained. In terms of the self-similar variables, this expansion reads

$$\Phi_{\text{inn}}(r,\tau) = U_\infty(r) - h\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma} + \dots + C_1\mu(\tau)\varepsilon(\tau)^{\gamma-2\beta-2}r^{-\gamma+2} + \dots, \tag{1.2.37}$$

which is valid in the intermediate region $\{\varepsilon(\tau) \ll r \ll 1\}$.

We can observe the asymptotic matching of the outer and inner expansions even in the higher order computed above if we carefully check their coefficients in detail, but they yield no contribution to the leading terms. Hence we use them only to obtain information about a guide for rigorous construction and estimate the error to the leading terms.

1.2.3 Discussions toward the full construction

We have derived condition (1.2.33) from Hypothesis 1.2.3. The full proof proceeds to the opposite direction. Namely, we will find a suitable small perturbation of initial data such that (1.2.33a) holds and then show that Hypothesis 1.2.3 is true. Following [24], we shall solve this finite-dimensional problem by a topological fixed-point theorem based on mapping degree theory. To this end, we have to set an appropriate functional framework and to show *a priori* estimates for $\Phi(y,\tau)$ ensuring (1.2.10) and (1.2.23). We just mention the region where (1.2.23) is expected to hold. Since $v(r,\tau)=\Phi(r,\tau)-U_\infty(r)$ and

$$e^{-\lambda_\ell\tau}\phi_\ell(r)\asymp e^{-\lambda_\ell\tau}r^{-\gamma+2\ell}=e^{-\lambda_\ell\tau}r^{2\lambda_\ell}r^{-2\beta}\quad\text{as}\quad r\to\infty,$$

the maximal region of the quadratic approximation of $f(v)$ holds is, in principle, $(\varepsilon(\tau)\ll)$ $|y|=O(e^{\tau/2})$ as $\tau\to\infty$. This last amount is not a technical upper bound, since $e^{\tau/2}$ is a characteristic curve for the hyperbolic part of the differential operator $v_s+\mathcal{A}v\asymp v_s+y\cdot\nabla_y v/2+\beta v$ for $|y|\gg 1$. Nonetheless, the authors of [24] had to restrict their *a priori* estimates to $\{|y|\le e^{\sigma\tau}\}$ with $\sigma<1/2$. In view of the original coordinate, the set corresponds to a shrinking domain $|x|\le(T-t)^{(1/2)-\sigma}$, $t<T$. In the following sections, we show that it is possible to set a better functional framework than that of [24] and to prove an *a priori* estimate of the form:

$$|\Phi(r,\tau)-U_\infty(r)-e^{-\lambda_\ell\tau}\phi_\ell(r)|<\nu e^{-\lambda_\ell\tau}r^{-\gamma+2\ell}\quad\text{for}\quad 1\le r\le e^{\tau/2}$$

for every $\nu>0$. Consequently our solution $u(x,t)$ has good estimates in the ball $\{|x|<1\}$ uniformly in $(0,T)$.

1.3 Setting of initial data and functional framework

Let us set

$$\varepsilon_0(\tau):=e^{-\omega_\ell\tau}\quad\text{with}\quad\omega_\ell:=\frac{\lambda_\ell}{\gamma-2\beta},\qquad a_\ell^*(\tau):=-\frac{h}{c_\ell}\varepsilon_0(\tau)^{2|\lambda_0|}=-\frac{h}{c_\ell}e^{-\lambda_\ell\tau}.$$

Let $\omega\in(0,1/2)$, and $\theta\in(0,1)$ be constants such that

$$0<\theta<\frac{\min\{2|\lambda_0|,\sqrt{D}\}}{16(2|\lambda_0|+\sqrt{D})},\tag{1.3.1}$$

and let τ_0, τ_1 be numbers such that $\tau_0\le\tau_1<\infty$. Let us write

$$\widetilde{\phi}_\ell(r):=\begin{cases}\dfrac{1}{a_\ell^*(\tau_0)}\left(\varepsilon_0(\tau_0)^{-2\beta}\left(U_1\left(\dfrac{r}{\varepsilon_0(\tau_0)}\right)-U_\infty\left(\dfrac{r}{\varepsilon_0(\tau_0)}\right)\right)-\displaystyle\sum_{n=0}^{\ell-1}\alpha_n\phi_n(r)\right)\\ \qquad\qquad\qquad\qquad\qquad\qquad\text{for}\quad r\le\varepsilon_0(\tau_0)^{\widetilde{\theta}},\\ \phi_\ell(r)\qquad\qquad\text{for}\quad\varepsilon_0(\tau_0)^{\widetilde{\theta}}<r\le e^{(1/2-\widetilde{\omega})\tau_0},\\ \dfrac{1}{a_\ell^*(\tau_0)}\left(U_\infty(r)G(r;\tau_0)-\displaystyle\sum_{n=0}^{\ell-1}\alpha_n\phi_n(r)\right)\quad\text{for}\quad e^{(1/2-\widetilde{\omega})\tau_0}<r,\end{cases}$$

where $\widetilde{\theta} \in (2\theta, 1)$, $\widetilde{\omega} \in (0, \omega)$, and $G(r;\tau_0)$ is a continuous function satisfying

$$G(r;\tau_0) = O(r^{-\kappa}) \quad \text{as} \quad r \to \infty \tag{1.3.2}$$

where $\kappa > 0$. We set the initial data Φ_0 as

$$\Phi_0(r;\alpha) := U_\infty(r) + a_\ell^*(\tau_0)\widetilde{\phi}_\ell(r) + \sum_{n=0}^{\ell-1} \alpha_n \phi_n(r),$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{\ell-1}) \in \mathbf{R}^\ell$ is a tuple of parameters, so that

$$\Phi_0(r;\alpha) = \begin{cases} \varepsilon_0(\tau_0)^{-2\beta} U_1\left(\dfrac{r}{\varepsilon_0(\tau_0)}\right) & \text{for} \quad r \le \varepsilon_0(\tau_0)^{\widetilde{\theta}}, \\ U_\infty(r) + a_\ell^*(\tau_0)\phi_\ell(r) + \displaystyle\sum_{n=0}^{\ell-1} \alpha_n \phi_n(r) & \text{for} \quad \varepsilon_0(\tau_0)^{\widetilde{\theta}} < r \le e^{(1/2-\widetilde{\omega})\tau_0}, \\ U_\infty(r)\big(1 + G(r;\tau_0)\big), & \text{for} \quad e^{(1/2-\widetilde{\omega})\tau_0} < r. \end{cases} \tag{1.3.3}$$

Concerning the parameter α, we impose

$$|\alpha| < \varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta} \tag{1.3.4}$$

(cf. (1.3.9)). In fact, we will convert our problem to a finite dimensional one which amounts to finding a suitable $\alpha \in \mathbf{R}^\ell$ satisfying (1.3.4) such that the corresponding initial data $\Phi_0(r;\alpha)$ yields a solution $\Phi(r,\tau;\alpha)$ with required estimates. To clarify the estimates, we define a functional framework for Φ in the next subsection.

1.3.1 The functional framework

Let $\tau_0 \le \tau_* \le \tau_1$. Assume $\Theta \in (0,1)$ is sufficiently small constant and Θ is positive constant. We say that a continuous function $\Phi : \mathbf{R}_+ \times [\tau_0, \tau_*] \to \mathbf{R}$ belongs to $\mathcal{A}^\nu_{\tau_0,\tau_*}$ with $\nu \in (0,1]$ if Φ fulfills the following conditions (I), $(\mathrm{II})_\mathrm{A}$, $(\mathrm{II})_\mathrm{B}$ and (Ⅲ):

(I) For $\tau_0 \le \tau \le \tau_*$ and $r \le \varepsilon_0(\tau)^\theta$,

$$\left|\Phi(r,\tau) - \varepsilon_0(\tau)^{-2\beta} U_1\left(\frac{r}{\varepsilon_0(\tau)}\right)\right| < \nu \varepsilon_0(\tau)^{-2\beta+\theta}\left(1 + \frac{r}{\varepsilon_0(\tau)}\right)^{-\gamma}; \tag{1.3.5}$$

$(\mathrm{II})_\mathrm{A}$ For $\tau_0 \le \tau \le \tau_*$ and $\varepsilon_0(\tau)^\theta < r \le 1$,

$$\big|\Phi(r,\tau) - U_\infty(r) - a_\ell^*(\tau)\phi_\ell(r)\big| < \nu \varepsilon_0(\tau)^{\gamma-2\beta+2\theta} r^{-\gamma}; \tag{1.3.6}$$

$(\mathrm{II})_\mathrm{B}$ For $\tau_0 \le \tau \le \tau_*$ and $1 < r \le e^{-\omega\tau_0} e^{\tau/2}$,

$$\big|\Phi(r,\tau) - U_\infty(r) - a_\ell^*(\tau)\phi_\ell(r)\big| < \nu \varepsilon_0(\tau_0)^{2\theta} \varepsilon_0(\tau)^{\gamma-2\beta} r^{-\gamma+2\ell}; \tag{1.3.7}$$

(Ⅲ) For $\tau_0 \le \tau \le \tau_*$ and $e^{-\omega\tau_0}e^{\tau/2} < r$,

$$\big|\Phi(r,\tau) - U_\infty(r)\big| < \nu e^{-\Theta\tau_0} r^{-2\beta}. \tag{1.3.8}$$

We see that $\Phi_0 \in \mathcal{A}^{1/2}_{\tau_0,\tau_0}$ for $|\alpha| < \varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}$ (see §§1.3.1). We now define a subset $\mathcal{U}_{\tau_0,\tau_*} \subset \mathbf{R}^\ell$ as

$$\mathcal{U}_{\tau_0,\tau_*} := \big\{\alpha \in \mathbf{R}^\ell\,;\, \Phi(r,\tau;\alpha) \in \mathcal{A}^1_{\tau_0,\tau_*},\ |\alpha| < \varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}\big\}, \tag{1.3.9}$$

where $\Phi(r,\tau;\alpha)$, $r = |y|$, denotes the solution of (1.2.2) in $\big(\mathbf{R}^N \setminus \{0\}\big) \times (\tau_0, \tau_1]$ with initial data $\Phi_0(r;\alpha)$ at $\tau = \tau_0$. A standard continuous dependence on initial data implies that $\mathcal{U}_{\tau_0,\tau}$ is open with respect to the standard topology of $\mathbf{R}^\ell$. For $\tau \ge \tau_0$, we define a map $Q_\tau : \mathbf{R}^\ell \to \mathbf{R}^\ell$ with domain $\overline{\mathcal{U}_{\tau_0,\tau_1}}$ of definition as

$$\begin{aligned} &Q_\tau : \alpha \mapsto (q_0(\tau;\alpha), \ldots, q_{\ell-1}(\tau;\alpha)) \\ &\text{with}\quad q_k(\tau;\alpha) := \langle v(\,\cdot\,,\tau;\alpha), \phi_k\rangle_N \quad \text{for} \quad k = 0, 1, \ldots, \ell-1, \end{aligned}$$

where $v(r,\tau;\alpha) = \Phi(r,\tau;\alpha) - U_\infty(r)$.

Lemma 1.3.1. *Assume that $Q_{\tau_1}(\alpha) = 0$ for some $\alpha \in \overline{\mathcal{U}_{\tau_0,\tau_1}}$. Then:*

$$|\alpha| < \frac{1}{2}\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta} \quad \text{and} \quad \Phi \in \mathcal{A}^{1/2}_{\tau_0,\tau_1}.$$

The proof of this lemma is postponed to §1.4.

Estimates of the initial data

Due to (1.3.3), the initial data $\Phi_0(r) = \Phi_0(r;\alpha)$ satisfies following estimates.

Lemma 1.3.2. *Let $\theta' \in (0, 1-\widetilde{\theta})$. We then have*

$$\begin{aligned} &r^\gamma\big|\Phi_0(r) - U_\infty(r) - a^*_\ell(\tau_0)\phi_\ell(r)\big| \\ &\le \begin{cases} C\varepsilon_0(\tau_0)^{2|\lambda_0|} + Cr^{2|\lambda_0|}, & r \le \varepsilon_0(\tau_0)^{1-\theta'}, \\ C\varepsilon_0(\tau_0)^{2|\lambda_0|+\mu}r^{-\mu} + C\varepsilon_0(\tau_0)^{2|\lambda_0|}r^2, & \varepsilon_0(\tau_0)^{1-\theta'} < r \le \varepsilon_0(\tau_0)^{\widetilde{\theta}}, \\ C\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}(1+r^{2\ell}), & \varepsilon_0(\tau_0)^{\widetilde{\theta}} < r \le e^{(1/2-\widetilde{\omega})\tau_0}, \\ Cr^{2|\lambda_0|}G(r;\tau_0) + C\varepsilon_0(\tau_0)^{2|\lambda_0|}r^{2\ell}, & e^{(1/2-\widetilde{\omega})\tau_0} < r, \end{cases} \end{aligned} \tag{1.3.10}$$

for sufficiently large τ_0, where $\widetilde{\theta} \in (2\theta, 1)$, $\widetilde{\omega} \in (0,\omega)$, $\mu > 0$ and $\kappa > 0$ are as in (1.4.7) *and* (1.3.2), *respectively. Moreover, the initial datum Φ_0 belongs to $\mathcal{A}^{1/2}_{\tau_0,\tau_0}$.*

Proof. **Estimate for the region $\{r \le \varepsilon_0(\tau_0)^{\widetilde{\theta}}\}$.** We see from (1.2.15) and (1.3.3) that

$$
\begin{aligned}
&\left|\Phi_0(r) - U_\infty(r) - a_\ell^*(\tau_0)\phi_\ell(r)\right| \\
&= \left|\varepsilon(\tau_0)^{-2\beta}\left[U_1\left(\frac{r}{\varepsilon(\tau_0)}\right) - U_\infty\left(\frac{r}{\varepsilon(\tau_0)}\right)\right] + h\varepsilon(\tau_0)^{2|\lambda_0|}r^{-\gamma}\left(1 + \sum_{j=1}^{\ell} h_j r^{2j}\right)\right| \\
&\le \varepsilon(\tau_0)^{-2\beta}\left|U_1\left(\frac{r}{\varepsilon(\tau_0)}\right) - U_\infty\left(\frac{r}{\varepsilon(\tau_0)}\right) + h\left(\frac{r}{\varepsilon(\tau_0)}\right)^{-\gamma}\right| + h\varepsilon(\tau_0)^{2|\lambda_0|}r^{-\gamma}\sum_{j=1}^{\ell}|h_j|r^{2j} \\
&\le \begin{cases} C\varepsilon_0(\tau_0)^{-2\beta}(\eta^{-2\beta} + \eta^{-\gamma}), & \eta \le \varepsilon_0(\tau_0)^{-\theta'}, \\ C\varepsilon_0(\tau_0)^{-2\beta}\eta^{-\gamma-\mu} + C\varepsilon_0(\tau_0)^{-2\beta+2}\eta^{-\gamma+2}, & \varepsilon_0(\tau_0)^{-\theta'} \le \eta \le \varepsilon_0(\tau_0)^{-1+\widetilde{\theta}}, \end{cases}
\end{aligned}
\tag{1.3.11}
$$

where $\eta := r/\varepsilon_0(\tau)$ and $h_j := (-\ell)_j/2^{2j}j!(-\gamma + N/2)_j$ which is appeared as the coefficients of $\phi_\ell(r)$.

Estimate for the region $\{\varepsilon_0(\tau)^{\widetilde{\theta}} < r \le e^{(1/2-\widetilde{\omega})\tau_0}\}$. Due to (1.3.3) and (1.3.4), we have

$$
\left|\Phi_0(r) - U_\infty(r) - a_\ell^*(\tau_0)\phi_\ell(r)\right| \le \sum_{n=0}^{\ell-1}|\alpha_n||\phi_n(r)| \le C\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}r^{-\gamma}(1 + r^{2\ell}). \tag{1.3.12}
$$

Estimate for the region $\{r > e^{(1/2-\widetilde{\omega})\tau_0}\}$. It readily follows from (1.3.2) and (1.3.3) that

$$
\begin{aligned}
\left|\Phi_0(r) - U_\infty(r) - a_\ell^*(\tau_0)\phi_\ell(r)\right| &\le U_\infty(r)G(r;\tau_0) + \frac{h\tilde{c}_\ell}{c_\ell}e^{-\lambda_\ell\tau_0}r^{-\gamma+2\ell} \\
&\le Cr^{-2\beta}G(r;\tau_0) + C\varepsilon_0(\tau_0)^{2|\lambda_0|}r^{-\gamma+2\ell}
\end{aligned}
\tag{1.3.13}
$$

Putting (1.3.11)–(1.3.13) together, we obtain (1.3.10).

We then verify that Φ_0 belongs to $\mathcal{A}^{1/2}_{\tau_0,\tau_0}$. A similar argument to (1.3.11) with (1.3.1) and (1.3.4) shows that

$$
\begin{aligned}
&\left|\Phi_0(r) - \varepsilon_0(\tau_0)^{-2\beta}U_1(\eta)\right| \\
&\le \left|\varepsilon_0(\tau_0)^{-2\beta}\left[U_1\left(\frac{r}{\varepsilon(\tau_0)}\right) - U_\infty\left(\frac{r}{\varepsilon(\tau_0)}\right)\right] - a_\ell^*(\tau_0)\phi_\ell(r)\right| + \sum_{n=0}^{\ell-1}|\alpha_n||\phi_n(r)| \\
&\le C\varepsilon_0(\tau_0)^{2|\lambda_0|+\mu}r^{-\gamma-\mu} + C\varepsilon_0(\tau_0)^{2|\lambda_0|}r^{-\gamma+2} + C\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}r^{-\gamma} \le C\varepsilon_0(\tau_0)^{2|\lambda_0|+2\theta}r^{-\gamma}
\end{aligned}
$$

for $\varepsilon_0(\tau_0)^{\widetilde{\theta}} \le r < \varepsilon_0(\tau_0)^{\theta}$ and sufficiently large τ_0. In addition, it follows from a similar argument to (1.3.13), (1.3.2), and (1.3.4) that

$$
\left|\Phi_0(r) - U_\infty(r)\right| = U_\infty(r)G(r;\tau_0) \ll r^{-2\beta}
$$

for $e^{(1/2-\omega)\tau_0} < r \le e^{(1/2-\widetilde{\omega})\tau_0}$ if τ_0 is sufficiently large. These together with (1.3.12) give $\Phi_0 \in \mathcal{A}^{1/2}_{\tau_0,\tau_0}$. The proof is complete. □

1.3.2 Proofs of Theorem 1.1.1 and Corollaries 1.1.2–1.1.4

Once proving the key *a priori* estimate given in Lemma 1.3.1, we may conclude the proof of Theorem 1.1.1 by the topological argument by means of mapping degree as in [23] (see also [19, 32, 43, 44, 46]). Since the argument is purely topological and independent of particular functional framework, we only write main points without discussing the detail.

Proof of Theorem 1.1.1. Lemma 1.3.1 guarantees that any root of Q_{τ_1} in $\mathcal{U}_{\tau_0,\tau_1}$ is contained in the interior of $\mathcal{U}_{\tau_0,\tau_1}$. The mapping degree of Q_τ is then preserved for $\tau_0 \le \tau \le \tau_1$ by homotopy invariance. Hence there exists $\alpha \in \mathcal{U}_{\tau_0,\tau_1}$ such that $Q_\tau(\alpha) = 0$ as long as $\mathcal{U}_{\tau_0,\tau_1} \neq \emptyset$. This last assumption is guaranteed for $|\tau_1 - \tau_0|$ small enough by standard continuous dependence results. Then, by the method of continuity, we have

$$\sup\{\tau_1 > \tau_0\,;\, \mathcal{U}_{\tau_0,\tau_1} \neq \emptyset\} = \infty. \tag{1.3.14}$$

Let $\{\tau_j\} \subset (\tau_0, \infty)$ be a sequence such that $\tau_0 < \tau_1 < \cdots < \tau_j \nearrow \infty$. Due to (1.3.14), there exists $\alpha_j \in \mathcal{U}_{\tau_0,\tau_j}$ such that $Q_{\tau_j}(\alpha_j) = 0$. Lemma 1.3.1 then implies $\Phi(r, \tau; \alpha_j) \in \mathcal{A}^{1/2}_{\tau_0,\tau_j}$. By taking a subsequence, we may assume that $\{\alpha_j\}$ converges to some $\alpha^* \in \mathbf{R}^\ell$, which completely determines the initial data $\Phi_0(r; \alpha^*)$. The function $u(x, t)$ obtained by scaling back from $\Phi(y, \tau; \alpha^*)$ via (1.2.1) is the desired solution of (1.1.2). The pointwise estimates stated in the theorem are obtained by those for $\Phi(y, \tau; \alpha^*)$ guaranteed by its membership to $\mathcal{A}^1_{\tau_0,\infty} := \cap_{\tau_1 \in (\tau_0,\infty)} \mathcal{A}^1_{\tau_0,\tau_1}$ with $\tau_0 = -\log T_0$. The result for arbitrary blow-up time $T > 0$ is obtained by rescaling, i.e., $u_\lambda(x, t) = \lambda^{2\beta} u(\lambda x, \lambda^2 t)$ with $\lambda = \sqrt{T/T_0}$.

The statement (iv) is proved by standard zero number arguments. Indeed, as a function of $r = |x|$, $u(r, t) - U_\infty(r)$ has ℓ-zeros in $\mathbf{R}_+$ at $t = 0$ due to our choice of initial data at the beginning of §1.3. As statements (ii) and (iii) show, every zero of function $|y| \mapsto \Phi(|y|, \tau) - U_\infty(|y|)$, whose total number is the same as of $u(\cdot, t) - U_\infty(\cdot)$, is located in small neighborhoods of the ℓ-zeros of Laguerre polynomial $L_\ell^{(\sqrt{D}/2)}$ for every $\tau \ge \tau_0$. Since the number of the zeros of function $r \mapsto u(r, t) - U_\infty(r)$ is non-increasing in t due to zero-number theory (cf. [41, §§52.8]), the zeros $\{r_k(t)\}_{k=1}^\ell$ are simple and lie in a parabolic region, i.e., there exist $R > 0$ and $t_1 \in (0, T)$ such that $r_k(t) \le R\sqrt{T - t}$ for all $t_1 < t < T$, whence the claim. The proof is now complete. □

Proof of Corollary 1.1.2. Let us write $V(\xi, \tau) := U(\xi, \tau) - U_1(\xi)$ and $U(\xi, \tau) := \varepsilon(\tau)^{2\beta}\Phi(y, \tau)$ with $\xi := y/\varepsilon(\tau)$ (cf. (1.5.4) below). Introducing new variables

$$s := \int_{\tau_0}^{\tau} \frac{1}{\varepsilon(\tau')^2}\, d\tau' \quad \text{and} \quad W(\xi, s) := V(\xi, \tau),$$

we obtain

$$W_s = \Delta_\xi W + pU_1(\xi)^{p-1}W - \widetilde{\mathcal{E}}(s)\left(\frac{\xi \cdot \nabla_\xi W}{2} + \beta W\right) + \widetilde{f}(\xi, s) \tag{1.3.15}$$

with

$$\widetilde{f}(\xi,s) := |\xi|^{2a}\big[(U_1(\xi)+W(\xi,s))^p - U_1(\xi)^p - pU_1(\xi)^{p-1}W(\xi,s)\big] \\ - \widetilde{\mathcal{E}}(s)\left(\frac{\xi\cdot\nabla_\xi U_1(\xi)}{2} + \beta U_1(\xi)\right),$$

$$\widetilde{\mathcal{E}}(s) := \varepsilon(\tau)^2 - 2\varepsilon(\tau)\dot{\varepsilon}(\tau).$$

Notice that the function $\widetilde{f}(\xi,s)$ is Hölder continuous since $a \geq 0$ by assumption.

We apply standard parabolic estimates for equation (1.3.15) in a space-time region $\Omega := B_R \times (s_1+\delta',\infty)$, where $B_R := \{\xi\,;\,|\xi| < R\}$ and $\delta' > 0$ is arbitrary. Due to (1.3.5) (with $\tau_1 = \infty$), it is readily seen that $\big\|\widetilde{f}\big\|_{L^{\alpha'}(\Omega)} \leq C\varepsilon_0(\tau^*)^{2\theta}$ holds for every $\alpha' > 1$, where τ^* is the time corresponding to $s_1+\delta'$ and $C > 0$ is a constant independent of s_1, δ'. Let $\Omega'' \Subset \Omega' \Subset \Omega$ be sub-cylinders and let $W^{2,1}_{\alpha'}(\Omega)$ denote the Sobolev space based on $L^{\alpha'}(\Omega)$ and defined by parabolic distance. Due to classical L^p estimate for parabolic equations, we obtain an estimate of the form $\|W\|_{W^{2,1}_{\alpha'}(\Omega)} \leq C\varepsilon_0(\tau^*)^{\theta}$. We now choose α' large enough so that $W^{2,1}_{\alpha'}(\Omega')$ is embedded in the Hölder space $C^{\nu',\nu'/2}(\overline{\Omega'})$ of order ν' in $\overline{\Omega'}$ (with respect to parabolic distance) for some $\nu' \in (0,1)$. Notice that the embedding constant does not depend on s_1, δ'. Re-selecting a smaller $\nu' > 0$, if necessary, we apply Schauder's interior estimate for (1.3.15), to get

$$\|W\|_{C^{2+\nu',\,1+\nu'/2}(\overline{\Omega''})} \leq K\left(\|W\|_{L^\infty(\Omega')} + \|\widetilde{f}\|_{C^{\nu',\,\nu'/2}(\overline{\Omega'})}\right) \leq C'\varepsilon(\tau^*)^{\theta}$$

for some constant $K > 0$. Since τ^* is arbitrary, the last estimate implies

$$\sup_{\xi\in B_R}\left|\frac{\partial W}{\partial s}(\xi,s)\right| \leq C\varepsilon(\tau)^{\theta},$$

Since $\partial_s W(\xi,s) = \varepsilon(\tau)^2\partial_\tau\big(\varepsilon(\tau)^{2\beta}\Phi(y,\tau)\big)$ and $|\dot{\varepsilon}(\tau)| \leq C\varepsilon(\tau)$, we have

$$|\Phi_\tau(y,\tau)| \leq C\varepsilon(\tau)^{-2\beta}U_1\left(\frac{|y|}{\varepsilon(\tau)}\right) + C\varepsilon(\tau)^{-2\beta-2}\left|\frac{\partial W}{\partial s}(\xi,s)\right| \leq C\varepsilon(\tau)^{-2\beta-2+\theta}$$

for $|y| \leq R\varepsilon(\tau)$, $\tau \geq \tau_0$. Returning to the original variables, we get the estimate (1.1.26) on u_t. Estimate (1.1.25) is easily obtained by (1.1.26), equation (1.1.2), and (1.1.20). The proof is now complete. □

Proofs of Corollaries 1.1.3 and 1.1.4. We estimate the L^q norm by splitting the region of integration defining $\|u(\cdot,t)\|_{L^q(\mathbf{R}^N)}$. Consider first the case of $q > q_c$. The local L^q norm in $\big\{|x| \leq K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell}\big\}$ may be readily estimated by (1.1.20) and the change of variable

$\eta = K_T|x|/(T-t)^{1/2+\omega_\ell}$. Indeed, we have

$$\begin{aligned}
&\int_{\left\{|x|\le K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell}\right\}} \left| u(x,t)^q - \left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}}\right)^{2\beta q} U_1\left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}}\right)^q \right| dx \\
&\le \frac{(T-t)^{\theta\omega_\ell}}{K_T^\theta}\left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}}\right)^{2\beta q-N} \left|\mathbf{S}^{N-1}\right| \int_0^{K_T^{1-\theta}(T-t)^{-(1-\theta)\omega_\ell}} U_1(\eta)^{q-1}\Psi(\eta)\eta^{N-1}\,d\eta \\
&\le C(T-t)^{-(2\beta q-N)(1/2+\omega_\ell)+\theta\omega_\ell}\int_0^\infty (1+\eta)^{-2\beta(q-1)-\gamma}\eta^{N-1}\,d\eta \le C(T-t)^{-(2\beta q-N)(1/2+\omega_\ell)+\theta\omega_\ell}
\end{aligned}$$

and

$$\int_{\left\{|x|\le K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell}\right\}} \left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}}\right)^{2\beta q} U_1\left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}}\right)^q dx \sim D_1\left(\frac{K_T}{(T-t)^{1/2+\omega_\ell}}\right)^{2\beta q-N}$$

which resulted in (1.1.29). On the other hand, (1.1.22) can be used in $\left\{K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell} \le |x|\right\}$ to get

$$\begin{aligned}
&\int_{\left\{K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell}\le|x|\right\}} u(x,t)^q\,dx \\
&\le (c_*+M)^q\int_{K_T^{-\theta}(T-t)^{1/2+\theta\omega_\ell}}^\infty r^{-2\beta q+N-1}\,dr \le C\left(\frac{K_T^\theta}{(T-t)^{1/2+\theta\omega_\ell}}\right)^{2\beta q-N}.
\end{aligned}$$

Since $2\beta q_c = N$, the last integral is finite for $q > q_c$. As for $|x| \ge 1$, we use simply the decay estimate $u \le C|x|^{-2\beta}$. Hence the corresponding integral may be estimated as above, whence (1.1.30). When $q = q_c$, we need a faster decay $u \le C|x|^{-d}$ for some $d > 2\beta$. Under the condition (1.1.31), the last estimate is guaranteed by [30, Proposition C.3] for $a = 0$, whose proof essentially works for $a > -1$. The detail is left to the reader. □

1.4 A priori estimates in the intermediate region

In this section we prove Lemma 1.3.1. This task is done by showing several *a priori* estimates, which we are going to establish in the following subsections. Let us write

$$v(\,\cdot\,,\tau) = \sum_{n=0}^\infty a_n(\tau)\phi_n \quad \text{in} \quad L^2_{\rho,\,\mathrm{rad}}(\mathbf{R}^N), \tag{1.4.1}$$

where $a_n(\tau) = \langle v(\,\cdot\,,\tau), \phi_n\rangle_N$. We first estimates $a_n(\tau)$ for $n = 0, 1, \dots, \ell-1$ in (1.4.1). This is accomplished in §§1.4.1 under the assumption $Q_{\tau_1}(\alpha) = 0$. The reminder term $R(y,\tau) = v(y,\tau) - \sum_{n=0}^\ell a_n(\tau)\phi_n(y)$ yields smaller contribution than the leading mode $a_\ell(\tau)\phi_\ell(y)$ in the outer region $\{\varepsilon_0(\tau)^\theta < |y| \le e^{-\omega\tau_0}e^{\tau/2}\}$. Those estimates of $a_n(\tau)$ and $E(y,\tau)$ lead to the estimates (II) and (III) with $\nu \ll 1$ in the requirement for $\mathcal{A}^{1/2}_{\tau_0,\tau_1}$. The last §1.5 is devoted to showing the estimate (I) in the inner region. In the following, we denote by C a generic positive constant that may change from line to line.

1.4.1 Estimates of Fourier coefficients

Lemma 1.4.1. *Assume that* $\Phi \in \mathcal{A}^1_{\tau_0,\tau_1}$*. Then:*

$$|f(v(r,\tau))| \le \begin{cases} Cr^{-2\beta-2}, & r \le \varepsilon_0(\tau)^{\widehat{\theta}}, \\ C\varepsilon_0(\tau)^{4|\lambda_0|}r^{-\gamma-2|\lambda_0|-2}(1+r^{4\ell}), & \varepsilon_0(\tau)^{\widehat{\theta}} < r \le e^{\widehat{\omega}\tau}, \\ Cr^{-2\beta-2}, & e^{\widehat{\omega}\tau} < r, \end{cases} \tag{1.4.2}$$

for $\tau_0 \le \tau \le \tau_1$*, where* $\widehat{\theta} \in (\theta, 1)$*, and* $\widehat{\omega} \in (0, 1/2)$*.*

Proof. Let $\Phi_{\mathrm{inn}}(r,\tau)$ be the function as in (1.2.9) and set $v_{\mathrm{inn}}(r,\tau) := \Phi_{\mathrm{inn}}(r,\tau) - U_\infty(r)$,

$$|f(v(r,\tau))| \le |f(v(r,\tau)) - f(v_{\mathrm{inn}}(r,\tau))| + |f(v_{\mathrm{inn}}(r,\tau))| =: r^{2a}(F_1 + F_2).$$

By the condition $\Phi \in \mathcal{A}^1_{\tau_0,\tau_1}$, for $r \le \varepsilon_0(\tau)^\theta$, we know that

$$|\Phi(r,\tau) - \Phi_{\mathrm{inn}}(r,\tau)| \le \varepsilon_0(\tau)^\theta \Phi_{\mathrm{inn}}(r,\tau) \ll \Phi_{\mathrm{inn}}(r,\tau).$$

Let $\eta := r/\varepsilon_0(\tau)$. For $r \le \varepsilon_0(\tau)^\theta$, we then obtain that

$$\begin{aligned} F_1 &\le \left|\Phi(r,\tau)^p - \Phi_{\mathrm{inn}}(r,\tau)^p - p\Phi_{\mathrm{inn}}(r,\tau)^{p-1}(\Phi(r,\tau) - \Phi_{\mathrm{inn}}(r,\tau))\right| \\ &\quad + \left|p(\Phi_{\mathrm{inn}}(r,\tau)^{p-1} - U_\infty(r)^{p-1})(\Phi(r,\tau) - \Phi_{\mathrm{inn}}(r,\tau))\right| \\ &\le C|\Phi_{\mathrm{inn}}(r,\tau)|^{p-2}\left|\Phi(r,\tau) - \Phi_{\mathrm{inn}}(r,\tau)\right|^2 \\ &\quad + C\left|\Phi_{\mathrm{inn}}(r,\tau)^{p-1} - U_\infty(r)^{p-1}\right|\left|\Phi(r,\tau) - \Phi_{\mathrm{inn}}(r,\tau)\right| \\ &\le C\varepsilon_0(\tau)^{-2\beta-2-2a+2\theta}U_1(\eta)^{p-2}(1+\eta)^{-2\gamma} \\ &\quad + C\varepsilon_0(\tau)^{-2\beta-2-2a+\theta}\left|U_1(\eta)^{p-1} - U_\infty(\eta)^{p-1}\right|(1+\eta)^{-\gamma} \\ &\le \begin{cases} C\varepsilon_0(\tau)^{-2\beta-2-2a+\theta}U_\infty(\eta)^p, & \eta \le \varepsilon_0(\tau)^{\widehat{\theta}-1}, \\ C\varepsilon_0(\tau)^{-2\beta-2-2a+\theta}\eta^{-2\gamma}U_\infty(\eta)^{p-2}, & \varepsilon_0(\tau)^{\widehat{\theta}-1} < \eta \le \varepsilon_0(\tau)^{\theta-1}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} F_2 &= \left|(U_\infty(r) + v_{\mathrm{inn}}(r,\tau))^p - U_\infty(r)^p - pU_\infty(r)^{p-1}v_{\mathrm{inn}}(r,\tau)\right| \\ &= \varepsilon_0(\tau)^{-2\beta-2-2a}\left|U_1(\eta)^p - U_\infty(\eta)^p - pU_\infty(\eta)^{p-1}(U_1(\eta) - U_\infty(\eta))\right| \\ &\le \begin{cases} C\varepsilon_0(\tau)^{-2\beta-2-2a}U_\infty(\eta)^p, & \eta \le \varepsilon_0(\tau)^{\widehat{\theta}-1}, \\ C\varepsilon_0(\tau)^{-2\beta-2-2a}\eta^{-2\gamma}U_\infty(\eta)^{p-2}, & \varepsilon_0(\tau)^{\widehat{\theta}-1} < \eta \le \varepsilon_0(\tau)^{\theta-1}, \end{cases} \end{aligned}$$

whence:

$$|f(v(r,\tau))| \le \begin{cases} Cr^{-2\beta-2}, & r \le \varepsilon_0(\tau)^{\widehat{\theta}}, \\ C\varepsilon_0(\tau)^{4|\lambda_0|}r^{-\gamma-2|\lambda_0|-2}, & \varepsilon_0(\tau)^{\widehat{\theta}} < r \le \varepsilon_0(\tau)^\theta. \end{cases} \tag{1.4.3}$$

Since $\Phi \in \mathcal{A}^1_{\tau_0,\tau_1}$, we obtain

$$\begin{aligned} |v(r,\tau)| &\le \left|v(r,\tau) - a^*_\ell(\tau)\phi_\ell(r)\right| + |a^*_\ell(\tau)\phi_\ell(r)| \\ &\le C\varepsilon_0(\tau)^{2|\lambda_0|}r^{-\gamma}(1+r^{2\ell}) \le CU_\infty(r)(\varepsilon_0(\tau)^{2|\lambda_0|}r^{-2|\lambda_0|} + e^{-\lambda_\ell\tau}r^{2\lambda_\ell}) \ll U_\infty(r) \end{aligned}$$

for $\varepsilon_0(\tau)^\theta < r \le e^{\widehat{\omega}\tau}$ and

$$\begin{aligned}|v(r,\tau)| &\le \left|v(r,\tau) - e^{-\beta(\tau-\tau_0)}G(re^{-(\tau-\tau_0)/2};\tau_0)\right| + \left|e^{-\beta(\tau-\tau_0)}G(re^{-(\tau-\tau_0)/2};\tau_0)\right| \\ &\le Ce^{-\beta(\tau-\tau_0)}(re^{-\tau/2})^{2\lambda_\ell}(re^{-(\tau-\tau_0)/2})^{-2\beta} \le C(re^{-\tau/2})^{2\lambda_\ell}r^{-2\beta} \ll U_\infty(r)\end{aligned}$$

for $e^{-\omega\tau_0}e^{\tau/2} < r \le e^{\widehat{\omega}\tau}$. It then follows that

$$\begin{aligned}|f(v(r,\tau))| &\le Cr^{2a}v(r,\tau)^2U_\infty(r)^{p-2} \\ &\le Cr^{2a}\varepsilon_0(\tau)^{4|\lambda_0|}(r^{-4|\lambda_0|} + r^{4\lambda_\ell})U_\infty(r)^p \le C\varepsilon_0(\tau)^{4|\lambda_0|}r^{2\beta-2\gamma-2}(1+r^{4\ell})\end{aligned} \tag{1.4.4}$$

for $\varepsilon_0(\tau)^\theta < r \le e^{\widehat{\omega}\tau}$. Moreover, $|v(r,\tau)| \le CU_\infty(r)$ for $r > e^{\widehat{\omega}\tau}$ due to $(\mathrm{II})_\mathrm{B}$ and (Ⅲ) with $\nu = 1$. We then have

$$|f(v(r,\tau))| \le Cr^{2a}U_\infty(r)^p \le Cr^{-2\beta-2}. \tag{1.4.5}$$

Due to (1.4.3)–(1.4.5), we obtain (1.4.2) and the proof is complete. □

Lemma 1.4.2. *Assume that* $\Phi \in \mathcal{A}^1_{\tau_0,\tau_1}$. *Then:*

$$\left|\langle f(v(\tau)), \phi_n\rangle_N\right| \le Cc_n\varepsilon_0(\tau)^{2|\lambda_0|+\mu} \tag{1.4.6}$$

for $n = 0, 1, \ldots$ *and* $\tau_0 \le \tau \le \tau_1$, *where* $c_n > 0$ *is as in* (1.2.17) *and* μ *is positive constant such that*

$$\mu := \frac{1}{4}\min\left\{2|\lambda_0|, \sqrt{D}, \frac{2|\lambda_0|}{\lambda_\ell}\right\} > 0, \tag{1.4.7}$$

Proof. We set

$$\sum_{k=1}^4 H_{k;n} := \left(\int_0^{\varepsilon_0(\tau)^{\widehat{\theta}}} + \int_{\varepsilon_0(\tau)^{\widehat{\theta}}}^1 + \int_1^{e^{\widehat{\omega}\tau}} + \int_{e^{\widehat{\omega}\tau}}^\infty\right)|f(v(r,s))||\phi_n(r)|r^{N-1}\rho(r)\,dr.$$

We shall use (1.4.2) with $\widehat{\theta} = 1-\theta \in (\theta,1)$ and $N - 2\gamma - 2 = \sqrt{D}$ (cf. (1.1.8b), (1.1.16)) to estimate $|f(v(r,\tau))|$ in each subinterval.

Estimate for $H_{1;n}$. Since $\gamma - 2\beta = 2|\lambda_0|$, we have

$$\begin{aligned}H_{1;n} &\le Cc_n\int_0^{\varepsilon_0(\tau)^{\widehat{\theta}}} r^{-\gamma-2\beta+N-3}\,dr \le Cc_n\int_0^{\varepsilon_0(\tau)^{1-\theta}} r^{2|\lambda_0|+\sqrt{D}-1}\,dr \\ &\le Cc_n\varepsilon_0(\tau)^{(1-\theta)(2|\lambda_0|+\sqrt{D})} \le Cc_n\varepsilon_0(\tau)^{2|\lambda_0|+\sqrt{D}-\theta(2|\lambda_0|+\sqrt{D})};\end{aligned} \tag{1.4.8}$$

Estimate for $H_{2;n}$. We see from $N - 2\gamma - 2 = \sqrt{D}$ that

$$\begin{aligned}H_{2;n} &\le Cc_n\varepsilon_0(\tau)^{4|\lambda_0|}\int_{\varepsilon_0(\tau)^{1-\theta}}^1 r^{-2\gamma-2|\lambda_0|+N-3}\,dr \\ &\le Cc_n\varepsilon_0(\tau)^{4|\lambda_0|}\int_{\varepsilon_0(\tau)^{1-\theta}}^1 r^{-2|\lambda_0|+\sqrt{D}-1}\,dr \\ &\le \begin{cases} Cc_n\varepsilon_0(\tau)^{4|\lambda_0|} & \text{if} \quad 2|\lambda_0| < \sqrt{D}, \\ Cc_n\varepsilon_0(\tau)^{4|\lambda_0|}\left|\log\varepsilon_0(\tau)\right| & \text{if} \quad 2|\lambda_0| = \sqrt{D}, \\ Cc_n\varepsilon_0(\tau)^{4|\lambda_0|+(1-\theta)(-2|\lambda_0|+\sqrt{D})} & \text{if} \quad 2|\lambda_0| > \sqrt{D} \end{cases} \\ &\le C\varepsilon_0(\tau)^{3|\lambda_0|} + C\varepsilon_0(\tau)^{2|\lambda_0|+\sqrt{D}};\end{aligned}$$

Estimate for $H_{3;n}$. Due to $|\tilde{c}_n| \le c_n$ and $-|\lambda_0| + \ell = \lambda_\ell > 0$, we obtain

$$\begin{aligned} H_{3;n} &\le C|\tilde{c}_n|\varepsilon_0(\tau)^{4|\lambda_0|} \int_1^{e^{\widehat{\omega}\tau}} r^{-2\gamma-2|\lambda_0|+2\ell+N-3} e^{-r^2/4}\, dr \\ &\le Cc_n\varepsilon_0(\tau)^{4|\lambda_0|} \int_1^{\infty} r^{2\lambda_\ell+\sqrt{D}-1} e^{-r^2/4}\, dr \le Cc_n\varepsilon_0(\tau)^{4|\lambda_0|}\,; \end{aligned} \tag{1.4.9}$$

Estimate for $H_{4;n}$. It follows from $|\tilde{c}_n| \le c_n$ that

$$\begin{aligned} H_{4;n} &\le C|\tilde{c}_n| \int_{e^{\widehat{\omega}\tau}}^{\infty} \left(\frac{r}{e^{\widehat{\omega}\tau}}\right)^{2\lambda_\ell/\widehat{\omega}} r^{-\gamma+2\ell-2\beta+N-3} e^{-r^2/4}\, dr \\ &\le Cc_n e^{-2\lambda_\ell\tau} \int_1^{\infty} r^{2\lambda_\ell/\widehat{\omega}+2|\lambda_0|+2\ell+\sqrt{D}-1} e^{-r^2/4}\, d \le Cc_n\varepsilon_0(\tau)^{4|\lambda_0|}. \end{aligned} \tag{1.4.10}$$

The condition (1.3.1) and the estimates (1.4.8)–(1.4.10) yield (1.4.6). The proof is complete. □

Lemma 1.4.3. *Assume that $Q_{\tau_1}(\alpha) = 0$ for some $\alpha \in \overline{\mathcal{U}_{\tau_0,\tau_1}}$. Then:*

$$|a_n(\tau)| \le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu} \tag{1.4.11}$$

for $\tau_0 \le \tau \le \tau_1$ and $n = 0, 1, \ldots, \ell - 1$, where $\mu > 0$ is as in (1.4.7).

Proof. Due to $Q_{\tau_1}(\alpha) = 0$, we see that

$$\begin{aligned} |a_n(\tau)| &= \left| -\int_\tau^{\tau_1} e^{-\lambda_n(\tau-s)} \langle f(v(s)), \phi_n\rangle_N\, ds \right| \le C \int_\tau^{\infty} e^{-\lambda_n(\tau-s)} \varepsilon_0(s)^{2|\lambda_0|+\mu}\, ds \\ &\le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu} \int_\tau^{\infty} e^{-(\ell-n+\mu\omega_\ell)(s-\tau)}\, ds \le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu}, \end{aligned}$$

whence (1.4.11). The proof is complete. □

Lemma 1.4.4. *Assume that $Q_{\tau_1}(\alpha) = 0$ for some $\alpha \in \overline{\mathcal{U}_{\tau_0,\tau_1}}$ and $\widetilde{\theta} \in (2\theta, 1)$ as in* (1.3.3) *is sufficiently small such that $\widetilde{\theta} \le \mu/(2 + \sqrt{D})$. Then:*

$$|\alpha_n| \le C\varepsilon_0(\tau_0)^{2|\lambda_0|+\widetilde{\theta}(2+\sqrt{D})} \quad \text{for} \quad n = 0, 1, \ldots, \ell - 1\,; \tag{1.4.12a}$$

$$|a_\ell(\tau_0) - a_\ell^*(\tau_0)| \le C\varepsilon_0(\tau_0)^{2|\lambda_0|+\widetilde{\theta}(2+\sqrt{D})}\,; \tag{1.4.12b}$$

$$|a_n(\tau_0)| \le Cc_n\varepsilon_0(\tau_0)^{2|\lambda_0|+\widetilde{\theta}(2+\sqrt{D})} \quad \text{for} \quad n = \ell + 1, \ell + 2 \ldots. \tag{1.4.12c}$$

Proof. Set $v_0(r) := v(r, \tau_0) = \Phi_0(r) - U_\infty(r)$. Notice that (1.3.3) implies that

$$\begin{aligned} &\int_{\varepsilon_0(\tau_0)^{\widetilde{\theta}}}^{e^{(1/2-\widetilde{\omega})\tau_0}} v_0(r)\phi_n(r) r^{N-1} e^{-r^2/4}\, dr \\ &= \int_{\varepsilon_0(\tau_0)^{\widetilde{\theta}}}^{e^{(1/2-\widetilde{\omega})\tau_0}} \left(a_\ell^*(\tau_0)\phi_\ell(r) + \sum_{k=0}^{\ell-1} \alpha_k\phi_k(r) \right) \phi_n(r) r^{N-1} e^{-r^2/4}\, dr \\ &= \mathbf{a}_n - \left(\int_0^{\varepsilon_0(\tau_0)^{\widetilde{\theta}}} + \int_{e^{(1/2-\widetilde{\omega})\tau_0}}^{\infty} \right) \left(a_\ell^*(\tau_0)\phi_\ell(r) + \sum_{k=0}^{\ell-1} \alpha_k\phi_k(r) \right) \phi_n(r) r^{N-1} e^{-r^2/4}\, dr, \end{aligned}$$

where

$$\mathbf{a}_n := \begin{cases} \alpha_n, & n = 0, 1, \ldots, \ell - 1, \\ a_\ell^*(\tau_0), & n = \ell, \\ 0, & n = \ell + 1, \ell + 2, \ldots. \end{cases} \tag{1.4.13}$$

Then, due to $a_n(\tau_0) = \langle v_0(\tau_0), \phi_n \rangle_N$, we have

$$\begin{aligned} a_n(\tau_0) &= \left(\int_0^{\varepsilon_0(\tau_0)^{\widetilde{\theta}}} + \int_{\varepsilon_0(\tau_0)^{\widetilde{\theta}}}^{e^{(1/2-\widetilde{\omega})\tau_0}} + \int_{e^{(1/2-\widetilde{\omega})\tau_0}}^{\infty} \right) v_0(r) \phi_n(r) r^{N-1} e^{-r^2/4} \, dr \\ &= \mathbf{a}_n + \left(\int_0^{\varepsilon_0(\tau_0)^{\widetilde{\theta}}} + \int_{e^{(1/2-\widetilde{\omega})\tau_0}}^{\infty} \right) \left(v_0(r) - a_\ell^*(\tau_0)\phi_\ell(r) - \sum_{n=0}^{\ell-1} \alpha_n \phi_n(r) \right) \phi_n(r) r^{N-1} e^{-r^2/4} \, dr \\ &=: \mathbf{a}_n + H'_{1;\,n} + H'_{2;\,n}. \end{aligned} \tag{1.4.14}$$

The same argument of Lemma 1.3.2 shows that

$$\begin{aligned} &\left| v_0(r) - a_\ell^*(\tau_0)\phi_\ell(r) - \sum_{n=0}^{\ell-1} \alpha_n \phi_n(r) \right| \\ &\le \begin{cases} C\varepsilon_0(\tau_0)^{2|\lambda_0|} r^{-\gamma} + C r^{-2\beta}, & r \le \varepsilon_0(\tau_0)^{1-\theta'}, \\ C\varepsilon_0(\tau_0)^{2|\lambda_0|} r^{-\gamma}, & \varepsilon_0(\tau_0)^{1-\theta'} < r \le \varepsilon_0(\tau_0)^{\widetilde{\theta}}, \\ C r^{-2\beta} G(r, \tau_0) + C\varepsilon_0(\tau_0)^{2|\lambda_0|} r^{-\gamma+2\ell}, & e^{(1/2-\widetilde{\omega})\tau_0} < r, \end{cases} \end{aligned}$$

where $\theta' \in (0, 1 - \widetilde{\theta})$ as in Lemma 1.3.2.

Estimates for $H'_{1;\,n}$. Due to this and identity $-2\gamma + N = \sqrt{D} + 2$, we have

$$\begin{aligned} |H'_{1,\,n}| &\le C c_n \int_0^{\varepsilon_0(\tau_0)^{1-\theta'}} r^{-\gamma-2\beta+N-1} \, dr + C c_n \varepsilon_0(\tau_0)^{2|\lambda_0|} \int_0^{\varepsilon_0(\tau_0)^{\widetilde{\theta}}} r^{-2\gamma+N-1} \, dr \\ &\le C c_n \int_0^{\varepsilon_0(\tau_0)^{1-\theta'}} r^{2|\lambda_0|+\sqrt{D}+1} \, dr + C c_n \varepsilon_0(\tau_0)^{2|\lambda_0|} \int_0^{\varepsilon_0(\tau_0)^{\widetilde{\theta}}} r^{\sqrt{D}+1} \, dr \\ &\le C c_n \varepsilon_0(\tau_0)^{2|\lambda_0|+\sqrt{D}+2-\theta'(2|\lambda_0|+\sqrt{D}+2)} + C c_n \varepsilon_0(\tau_0)^{2|\lambda_0|+\widetilde{\theta}(\sqrt{D}+2)}\,; \end{aligned} \tag{1.4.15}$$

Estimate for $H'_{2;\,n}$. We see from $|\tilde{c}_n| \le c_n$ that

$$\begin{aligned} |H'_{2;\,n}| &\le C|\tilde{c}_n| \int_{e^{(1/2-\widetilde{\omega})\tau_0}}^{\infty} \left(\frac{r}{e^{(1/2-\widetilde{\omega})\tau_0}} \right)^{2\lambda_\ell/(1/2-\widetilde{\omega})} r^{-2\beta-\gamma+2\ell+2n+N-1} e^{-r^2/4} \, dr \\ &\quad + C|\tilde{c}_n| \varepsilon_0(\tau_0)^{2|\lambda_0|} \int_{e^{(1/2-\widetilde{\omega})\tau_0}}^{\infty} \left(\frac{r}{e^{(1/2-\widetilde{\omega})\tau_0}} \right)^{\lambda_\ell/(1/2-\widetilde{\omega})} r^{-2\gamma+2\ell+2n+N-1} e^{-r^2/4} \, dr \\ &\le C c_n \varepsilon_0(\tau_0)^{4|\lambda_0|} \int_1^{\infty} r^{2\lambda_\ell/(1/2-\widetilde{\omega})+2|\lambda_0|+2n+\sqrt{D}+1} e^{-r^2/4} \, dr \\ &\quad + C c_n \varepsilon_0(\tau_0)^{4|\lambda_0|} \int_1^{\infty} r^{\lambda_\ell/(1/2-\widetilde{\omega})+2\ell+2n+\sqrt{D}+1} e^{-r^2/4} \, dr \le C c_n \varepsilon_0(\tau_0)^{4|\lambda_0|}. \end{aligned} \tag{1.4.16}$$

The claim (1.4.12) then follows from (1.4.11) and (1.4.14)–(1.4.16) with sufficiently small θ' such that

$$0 < \theta' < \frac{\sqrt{D}+2}{2|\lambda_0| + \sqrt{D} + 2}(1-\widetilde{\theta}) \in (0, 1-\widetilde{\theta}).$$

The proof is complete. □

1.4.2 Estimates of remainder terms

Our next goal is to estimate the higher Fourier mode: $v(r,\tau) - \sum_{n=0}^{\ell} a_n(\tau)\phi_n(r)$. To this end, it is convenient to introduce a new dependent variable

$$W(r,\tau;\alpha) := r^\gamma v(r,\tau;\alpha) \quad \text{and} \quad \psi_n(r) := r^\gamma \phi_n(r),$$

where $v(r,\tau;\alpha) := \Phi(r,\tau;\alpha) - U_\infty(r)$. Then:

$$\begin{aligned} W_0(r;\alpha) &:= W(r,\tau_0;\alpha) \\ &= \begin{cases} \varepsilon_0(\tau_0)^{-2\beta} r^\gamma \left[U_1\left(\dfrac{r}{\varepsilon_0(\tau_0)}\right) - U_\infty\left(\dfrac{r}{\varepsilon_0(\tau_0)}\right)\right], & r \le \varepsilon_0(\tau_0)^{\widetilde{\theta}}, \\ a_\ell^*(\tau_0)\psi_\ell(r) + \displaystyle\sum_{n=0}^{\ell-1} \alpha_n \psi_n(r), & \varepsilon_0(\tau_0)^{\widetilde{\theta}} < r \le e^{(1/2-\widetilde{\omega})\tau_0}, \\ r^\gamma U_\infty(r) G(r;\tau_0), & e^{(1/2-\widetilde{\omega})\tau_0} < r, \end{cases} \end{aligned}$$

and function W satisfies $W_\tau = -\mathcal{L}W + g(W)$ where

$$-\mathcal{L}W := W'' + \left(\frac{N-2\gamma-1}{r} - \frac{r}{2}\right)W' - \lambda_0 W \quad \text{and} \quad g(W) := r^\gamma f(v).$$

We set

$$m := N - 2\gamma = 2 + \sqrt{D} > 2.$$

For a while, we consider the case where m is an integer (the general case is discussed at the end of this section). Let us write

$$W(r,\tau) = \sum_{n=0}^{\ell} a_n(\tau)\psi_n(r) + R(r,\tau) \quad \text{with} \quad R(r,\tau) := \sum_{n=\ell+1}^{\infty} a_n(\tau)\psi_n(r), \tag{1.4.17}$$

$$\langle W, \psi_n \rangle_m := \int_0^\infty W(r)\psi_n(r) r^{m-1} e^{-r^2/4}\, dr = \langle v, \phi_n \rangle_N = a_n(\tau).$$

Function R satisfies

$$\langle R(\,\cdot\,,\tau), \psi_n \rangle_m = 0 \quad \text{for} \quad n = 0, 1, \ldots, \ell,$$

$$R_\tau = -\mathcal{L}R + g(W) - \sum_{n=0}^{\ell} \langle g(W), \psi_n \rangle_m \psi_n.$$

We denote by $S(\tau)$ the semigroup for $-\mathcal{L}$, which is expressed as

$$[S(\tau)W](y) = \sum_{n=0}^{\infty} e^{-\lambda_n\tau}\langle W, \psi_n\rangle_m \psi_n(|y|) \tag{1.4.18a}$$

$$= \frac{C_m e^{\Lambda_1\tau}|y|^{-m/2+1}}{1-e^{-\tau}} \int_0^\infty I_{\Lambda_2}\left\{\frac{|y|e^{-\tau/2}r}{2(1-e^{-\tau})}\right\} \exp\left\{-\frac{|y|^2e^{-\tau}+r^2}{4(1-e^{-\tau})}\right\} r^{m/2}W(r)\,dr \tag{1.4.18b}$$

$$= \frac{e^{|\lambda_0|\tau}}{(4\pi(1-e^{-\tau}))^{m/2}} \int_{\mathbf{R}^m} \exp\left\{-\frac{|ye^{-\tau/2}-z|^2}{4(1-e^{-\tau})}\right\} W(z)\,dz \quad (m\in\mathbf{N}) \tag{1.4.18c}$$

with $\Lambda_1 := |\lambda_0| + (m-2)/4$ and $\Lambda_2 := \gamma + N/2 - 1 = m/2 - 1$, where I_Λ denotes the modified Bessel function with order Λ. The bounds for the modified Bessel function

$$|I_\Lambda(z)| \le \frac{Cz^\Lambda e^z}{(1+z)^{\Lambda+1/2}}, \quad z\in\mathbf{R}_+, \tag{1.4.19}$$

yields the following estimate :

$$\big|[S(\tau)W](y)\big| \le \frac{Ce^{|\lambda_0|\tau}}{(1-e^{-\tau})^{m/2}} \int_0^\infty \exp\left\{-\frac{\big||y|e^{-\tau/2}-r\big|^2}{4(1-e^{-\tau})}\right\} |W(r)|\mathcal{B}_y(r,\tau)r^{m-1}\,dr. \tag{1.4.20}$$

where

$$\mathcal{B}_y(r,\tau) := \left(1+\frac{|y|e^{-\tau/2}r}{1-e^{-\tau}}\right)^{-(m-1)/2}.$$

Remark 1.4.1. The series (1.4.18a) converges in the norm of $L^2_{\rho,\mathrm{rad}}(\mathbf{R}^m)$ for every $\tau \ge 0$ and, moreover, absolutely for every $y\in\mathbf{R}^m$ and each $\tau>0$. This is because eigenfunctions $\psi_n(r)$ are algebraically bounded with respect to n uniformly in every compact set of $[0,\infty)$ due to Remark 1.2.1, which is much slower than $e^{\lambda_n\tau}$. Furthermore, it converges for $|y|\le e^{\tau/2}$, since $\psi_n(r)\sim \tilde{c}_n r^{2n}$ as $r\to\infty$, which is canceled out there by the exponential factor of (1.4.18a), and (1.2.19) involves adequate rate of decay.

The series expression (1.4.18) implies

$$a^*_\ell(\tau)\psi_\ell(r) = -\frac{h}{c_\ell}e^{-\lambda_\ell\tau}\psi_\ell(r) = e^{-\lambda_\ell(\tau-\tau_0)}a^*_\ell(\tau_0)\psi_\ell(r) = \big[S(\tau-\tau_0)\{a^*_\ell(\tau_0)\psi_\ell\}\big](r),$$

whence :

$$\begin{aligned} &W(r,\tau) - a^*_\ell(\tau)\psi_\ell(r) \\ &= \big[S(\tau-\tau_0)\{W_0 - a^*_\ell(\tau_0)\psi_\ell\}\big] + \int_{\tau_0}^{\tau} \big[S(\tau-s)g(W(s))\big](r)\,ds. \end{aligned} \tag{1.4.21}$$

We note some useful estimates for $S(\tau)$. Let us set

$$B_1(y,\sigma) := \{r\in\mathbf{R}_+\,;\, r\le 2|y|e^{-(\tau-\sigma)/2}\}, \tag{1.4.22a}$$

$$B_2(y,\sigma) := \{r\in\mathbf{R}_+\,;\, 2|y|e^{-(\tau-\sigma)/2} < r\}, \tag{1.4.22b}$$

and

$$\mathcal{S} := \frac{1}{(1-e^{-(\tau-\sigma)})^{m/2}} \int_{B_1(y,\sigma)} \exp\left\{-\frac{\left||y|e^{-(\tau-\sigma)/2}-r\right|^2}{4(1-e^{-(\tau-\sigma)})}\right\} \mathcal{B}_y(r,\tau-\sigma) r^{m-1}\, dr, \tag{1.4.23}$$

$$\mathcal{S}_k := \frac{1}{(1-e^{-(\tau-\sigma)})^{m/2}} \int_{B_2(y,\sigma)} \exp\left\{-\frac{\left||y|e^{-(\tau-\sigma)/2}-r\right|^2}{4(1-e^{-(\tau-\sigma)})}\right\} \mathcal{B}_y(r,\tau-\sigma) r^{k+m-1}\, dr \tag{1.4.24}$$

with $k \geq 1$, for $y \in \mathbf{R}^m$ and $\tau_0 \leq \sigma \leq \tau \leq \tau_1$. Due to

$$\frac{r^{m-1}\mathcal{B}_y(r,\tau-\sigma)}{(1-e^{-(\tau-\sigma)})^{(m-1)/2}} = \left(\frac{r^2}{1-e^{-(\tau-\sigma)}+|y|e^{-(\tau-\sigma)/2}r}\right)^{(m-1)/2} \leq 1$$

for $r \in B_1(y,\sigma)$ and $\tau_0 \leq \sigma \leq \tau \leq \tau_1$, we have

$$\mathcal{S} \leq \frac{1}{(1-e^{-(\tau-\sigma)})^{1/2}} \int_{B_1(y,\sigma)} \exp\left\{-\frac{\left||y|e^{-(\tau-\sigma)/2}-r\right|^2}{4(1-e^{-(\tau-\sigma)})}\right\} dr \leq C. \tag{1.4.25}$$

On the other hand, because of

$$||y|e^{-(\tau-\sigma)/2}-r| = r-|y|e^{-(\tau-\sigma)/2} > \frac{r}{2} > |y|e^{-(\tau-\sigma)/2} \tag{1.4.26}$$

for $r \in B_2(y,\sigma)$ and $\tau_0 \leq \sigma \leq \tau \leq \tau_1$ and $\mathcal{B}_y(r,\tau-\sigma) \leq 1$, we obtain that

$$\begin{aligned}\mathcal{S}_k &\leq \frac{1}{(1-e^{-(\tau-\sigma)})^{m/2}} \int_{B_2(y,\sigma)} \exp\left\{-\frac{r^2}{16(1-e^{-(\tau-\sigma)})}\right\} r^{k+m-1}\, dr \\ &\leq C(1-e^{-(\tau-\sigma)})^{k/2} \leq C.\end{aligned} \tag{1.4.27}$$

Furthermore, we see that

$$\begin{aligned}&\frac{1}{(1-e^{-(\tau-\sigma)})^{m/2}} \exp\left\{-\frac{\left||y|e^{-(\tau-\sigma)/2}-r\right|^2}{4(1-e^{-(\tau-\sigma)})}\right\} \\ &\leq \frac{1}{(1-e^{-(\tau-\sigma)})^{m/2}} \exp\left\{-\frac{(|y|e^{-(\tau-\sigma)/2})^2}{16(1-e^{-(\tau-\sigma)})}\right\} \\ &\leq C\left(\sup_{L\geq 0} L^m e^{-L^2}\right)(|y|e^{-(\tau-\sigma)/2})^{-m} \leq C|y|^{-m}e^{m(\tau-\sigma)/2}\end{aligned} \tag{1.4.28}$$

for $r \in \mathbf{R}_+$ satisfying $||y|e^{-(\tau-\sigma)/2}-r| > |y|e^{-(\tau-\sigma)/2}/2$ and $\tau_0 \leq \sigma \leq \tau \leq \tau_1$.

A priori estimates in the short-time case

We first discuss the short-time case, that is, $\tau_0 \leq \tau \leq \tau_0+1$. Notice that

$$e^{-1} \leq e^{-(\tau-\sigma)} \leq 1 \quad \text{for} \quad \tau_0 \leq \sigma \leq \tau \leq \tau_0+1. \tag{1.4.29}$$

For simplicity, we shall abuse some notation such as $W(|y|,\tau) = W(y,\tau)$, $\psi_n(|y|) = \psi_n(y)$.

Lemma 1.4.5. *There holds*

$$\left|S(\tau-\tau_0)\{W_0 - a_\ell^*(\tau_0)\psi_\ell\}(y)\right| \le C\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}(1+|y|^{2\ell}) \tag{1.4.30}$$

for $\varepsilon_0(\tau)^\theta < |y| \le e^{-\omega\tau_0}e^{\tau/2}$ *and* $\tau_0 \le \tau \le \tau_0+1$ *with sufficiently large* τ_0.

Proof. For $i = 1, 2, 3$, we set

$$I_i := \frac{e^{|\lambda_0|(\tau-\tau_0)}}{(1-e^{-(\tau-\tau_0)})^{m/2}} \int_{D_i(\tau_0)} \exp\left\{-\frac{\left||y|e^{-(\tau-\tau_0)/2} - r\right|^2}{4(1-e^{-(\tau-\tau_0)})}\right\} \times \left|W_0(r) - a_\ell^*(\tau_0)\psi_\ell(r)\right| \mathcal{B}_y(r, \tau-\tau_0) r^{m-1}\, dr,$$

where

$$D_1(\sigma) := \left\{r \in \mathbf{R}_+ \,;\, r \le \varepsilon_0(\sigma)^{\widetilde{\theta}}\right\}, \tag{1.4.31a}$$

$$D_2(\sigma) := \left\{r \in \mathbf{R}_+ \,;\, \varepsilon_0(\sigma)^{\widetilde{\theta}} < r \le e^{(1/2-\widetilde{\omega})\sigma}\right\}, \tag{1.4.31b}$$

$$D_3(\sigma) := \left\{r \in \mathbf{R}_+ \,;\, e^{(1/2-\widetilde{\omega})\sigma} < r\right\} \tag{1.4.31c}$$

for $\tau_0 \le \sigma \le \tau \le \tau_1$. Hereafter we always assume $\varepsilon_0(\tau)^\theta < |y| \le e^{-\omega\tau_0}e^{\tau/2}$ and $\tau_0 \le \tau \le \tau_0+1$.

Estimate for I_1. Let us divide the region $D_1(\sigma)$ in (1.4.31a) as the disjoint union of

$$D_{1,1}(\sigma) := \left\{r \in \mathbf{R}_+ \,;\, r \le \varepsilon_0(\sigma)^{1-\theta'}\right\}, \tag{1.4.32a}$$

$$D_{1,2}(\sigma) := \left\{r \in \mathbf{R}_+ \,;\, \varepsilon_0(\sigma)^{1-\theta'} < r \le \varepsilon_0(\sigma)^{\widetilde{\theta}}\right\}, \tag{1.4.32b}$$

for $\tau_0 \le \sigma \le \tau \le \tau_1$, where $\theta' \in (0, 1-\widetilde{\theta})$ is as in (1.3.10). The corresponding integrals are denoted as $I_{1,1}$ and $I_{1,2}$, respectively. For $\varepsilon_0(\tau)^\theta < |y|$ and $r \in D_1(\tau_0)$, there holds

$$\left||y|e^{-(\tau-\tau_0)/2} - r\right| \ge |y|e^{-(\tau-\tau_0)/2} - r \ge \frac{1}{2}|y|e^{-(\tau-\tau_0)/2} \tag{1.4.33}$$

if $\tau_0 \ge (2\log 2 + 1 + \theta\omega_\ell)/2\theta\omega_\ell$, where $\omega_\ell = \lambda_\ell/2|\lambda_0|$. Because of $\mathcal{B}_y(r, \tau-\tau_0) \le 1$, (1.3.10), (1.4.28), (1.4.29), and (1.4.33), we have

$$\begin{aligned} I_{1,1} &\le C|y|^{-m}e^{m(\tau-\tau_0)/2}\int_{D_{1,1}(\tau_0)} \left|W_0(r) - a_\ell^*(\tau_0)\psi_\ell(r)\right| r^{m-1}\,dr \\ &\le C\varepsilon_0(\tau_0)^{2|\lambda_0|}|y|^{-m}\int_0^{\varepsilon_0(\tau_0)^{1-\theta'}} r^{m-1}\,dr + C|y|^{-m}\int_0^{\varepsilon_0(\tau_0)^{1-\theta'}} r^{2|\lambda_0|+m-1}\,dr \\ &\le C\varepsilon_0(\tau_0)^{2|\lambda_0|+(1-\theta')m}\varepsilon_0(\tau)^{-\theta m} + C\varepsilon_0(\tau_0)^{(1-\theta')(2|\lambda_0|+m)}\varepsilon_0(\tau)^{-\theta m} \\ &\le C\varepsilon_0(\tau)^{2|\lambda_0|+m-\theta'(2|\lambda_0|+m)-\theta m}. \end{aligned} \tag{1.4.34}$$

For $r \in D_{1,2}(\tau_0)$, (1.3.11) implies that

$$|W_0(r) - a_\ell^*(\tau_0)\psi_\ell(r)| \le C\varepsilon_0(\tau_0)^{2|\lambda_0|}\left[\left(\frac{r}{\varepsilon_0(\tau_0)}\right)^{-\mu} + r^2\right] \le C\varepsilon_0(\tau_0)^{2|\lambda_0|+\theta'\mu} + C\varepsilon_0(\tau_0)^{2|\lambda_0|+2\widetilde{\theta}}.$$

Then, due to (1.4.29), we obtain

$$I_{1,2} \le C\varepsilon_0(\tau_0)^{2|\lambda_0|}(\varepsilon_0(\tau_0)^{\theta'\mu} + \varepsilon_0(\tau_0)^{2\widetilde{\theta}})\big|[S(\tau-\tau_0)1](y)\big| \le C\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}. \tag{1.4.35}$$

for $\widetilde{\theta} \in (2\theta, 1)$ and $\theta' = 3\theta/\mu$ ($< 1 - \widetilde{\theta}$ if θ is sufficiently small).

Estimate for I_2. We set

$$D_{2,1}(\sigma) := B_1(y,\sigma) \cap D_2(\sigma) \quad \text{and} \quad D_{2,2}(\sigma) := B_2(y,\sigma) \cap D_2(\sigma) \tag{1.4.36}$$

for $\tau_0 \le \sigma \le \tau \le \tau_1$, and split I_2 as $I_2 = I_{2,1} + I_{2,2}$, accordingly. It then follows from (1.3.10), (1.4.25), (1.4.27), and (1.4.29) that

$$\begin{aligned} I_{2,1} \le &\frac{C\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}}{(1-e^{-(\tau-\tau_0)})^{m/2}} \int_{D_{2,1}(\tau_0)} \exp\left\{-\frac{\big||y|e^{-(\tau-\tau_0)/2} - r\big|^2}{4(1-e^{-(\tau-\tau_0)})}\right\} \\ &\qquad\qquad\qquad \times \mathcal{B}_y(r,\tau-\tau_0)(1+r^{2\ell})r^{m-1}\,dr \\ &\le C\mathcal{S}\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}\big(1 + (2|y|e^{-(\tau-\tau_0)/2})^{2\ell}\big) \le C\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}(1+|y|^{2\ell}) \end{aligned} \tag{1.4.37}$$

and

$$I_{2,2} \le C(\mathcal{S}_0 + \mathcal{S}_{2\ell})\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta} \le C\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}, \tag{1.4.38}$$

where $\mathcal{S}$ and $\mathcal{S}_k$ are as in (1.4.23) and (1.4.24), respectively;

Estimate for I_3. We see from (1.3.10) that

$$\big|W_0(r) - a_\ell^*(\tau_0)\psi_\ell(r)\big| \le C\varepsilon_0(\tau_0)^{2|\lambda_\ell|}r^{2\ell}$$

for $r \in D_3(\tau_0)$, and

$$2|y|e^{-(\tau-\tau_0)/2} \le 2e^{(1/2-\omega)\tau_0} < e^{(1/2-\widetilde{\omega})\tau_0} \le r \tag{1.4.39}$$

for $|y| \le e^{-\omega\tau_0}e^{\tau/2}$, $r \in D_3(\tau_0)$, and sufficiently large τ_0. It then follows from (1.4.27), (1.4.29), and (1.4.39) that

$$\begin{aligned} I_3 \le &\frac{C\varepsilon_0(\tau_0)^{2|\lambda_0|}}{(1-e^{-(\tau-\tau_0)})^{m/2}} \int_{D_3(\tau_0)} \exp\left\{-\frac{\big||y|e^{-(\tau-\tau_0)/2} - r\big|^2}{4(1-e^{-(\tau-\tau_0)})}\right\} \\ &\qquad\qquad\qquad \times \left(\frac{r}{e^{(1/2-\widetilde{\omega})\tau_0}}\right)^{\lambda_\ell/(1/2-\widetilde{\omega})} \mathcal{B}_y(r,\tau-\tau_0)r^{2\ell+m-1}\,dr \\ &\le C\mathcal{S}_{2\ell+\lambda_\ell/(1/2-\widetilde{\omega})}\varepsilon_0(\tau_0)^{4|\lambda_0|} \le C\varepsilon_0(\tau)^{4|\lambda_0|}, \end{aligned} \tag{1.4.40}$$

where S_k is as in (1.4.24).

Due to (1.4.34), (1.4.35), (1.4.37), (1.4.38), and (1.4.40), we obtain (1.4.30) and the proof is complete. □

Lemma 1.4.6. *There hold*

$$\left|\int_{\tau_0}^{\tau}\big[S(\tau-s)g(W(s))\big](y)\,ds\right| \le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu}(1+|y|^{2\ell}), \tag{1.4.41}$$

$$\left|\int_{\tau_0}^{\tau}\left[S(\tau-s)\left\{\sum_{n=0}^{\ell}\langle g(W(s)),\psi_n\rangle_m\psi_n\right\}\right](y)\,ds\right| \le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu}(1+|y|^{2\ell}), \tag{1.4.42}$$

for $\varepsilon_0(\tau)^\theta < |y|$ *and* $\tau_0 \le \tau \le \tau_0+1$ *with sufficiently large* τ_0*, where* $\mu > 0$ *is as in* (1.4.7).

Proof. For $i = 1, 2, 3$, we set

$$J_i := \int_{\tau_0}^{\tau}\frac{e^{|\lambda_0|(\tau-s)}}{(1-e^{-(\tau-s)})^{m/2}}\int_{E_i(s)}\exp\left\{-\frac{\big||y|e^{-(\tau-s)/2}-r\big|^2}{4(1-e^{-(\tau-s)})}\right\}|g(W(r,s))|\mathcal{B}_y(r,\tau)r^{m-1}\,drds,$$

where

$$E_1(\sigma) := \big\{r\in\mathbf{R}_+\,;\, r\le\varepsilon_0(\sigma)^{\widehat{\theta}}\big\}, \tag{1.4.43a}$$

$$E_2(\sigma) := \big\{r\in\mathbf{R}_+\,;\, \varepsilon_0(\sigma)^{\widehat{\theta}} < r \le e^{-\widehat{\omega}\sigma}\big\}, \tag{1.4.43b}$$

$$E_3(\sigma) := \big\{z\in\mathbf{R}_+\,;\, e^{\widehat{\omega}\sigma} < r\big\}. \tag{1.4.43c}$$

for $\tau_0 \le \sigma \le \tau \le \tau_1$, $\widehat{\theta}\in(\theta,1)$ and $\widehat{\omega}\in(0,1/2)$ are as in (1.4.2). We recall that $m = 2+\sqrt{D} > 2$, $\langle f(v),\phi_n\rangle_N = \langle g(v),\psi_n\rangle_m$, and $r^\gamma f(v) = g(W)$.

Estimate for J_1**.** We see from (1.4.2) with $\widehat{\theta} = 1-\theta$, (1.4.29), and (1.4.33) that

$$\begin{aligned} J_1 &\le C|y|^{-m}\int_{\tau_0}^{\tau}e^{m(\tau-s)/2}\int_{E_1(s)}|f(v(r,\tau))|r^{\gamma+m-1}\,drds \\ &\le C\varepsilon_0(\tau)^{-\theta m}\int_{\tau_0}^{\tau}\int_0^{\varepsilon_0(s)^{1-\theta}} r^{\gamma-2\beta+m-3}\,drds \\ &\le C\varepsilon_0(\tau)^{-\theta m}\int_{\tau_0}^{\tau}\varepsilon_0(s)^{(1-\theta)(2|\lambda_0|+\sqrt{D})}\,drds \le C\varepsilon_0(\tau)^{2|\lambda_0|+\sqrt{D}-2\theta(|\lambda_0|+\sqrt{D}+1)}\,; \end{aligned} \tag{1.4.44}$$

Estimate for J_2**.** We divide the region E_2 in (1.4.43b) with $\widehat{\theta} = 1-\theta$ into

$$E_{2,1}(\sigma) := \big\{r\in\mathbf{R}_+\,;\, \varepsilon_0(\sigma)^{1-\theta} < r \le \varepsilon_0(\sigma)^{2\theta}\big\}, \tag{1.4.45a}$$

$$E_{2,2}(y,\sigma) := B_1(y,\sigma)\cap\big\{r\in\mathbf{R}_+\,;\, \varepsilon_0(\sigma)^{2\theta} < r \le e^{\widehat{\omega}\sigma}\big\}, \tag{1.4.45b}$$

$$E_{2,3}(y,\sigma) := B_2(y,\sigma)\cap\big\{r\in\mathbf{R}_+\,;\, \varepsilon_0(\sigma)^{2\theta} < r \le e^{\widehat{\omega}\sigma}\big\}, \tag{1.4.45c}$$

for $y\in\mathbf{R}^m$ and $\tau_0 \le \sigma \le \tau \le \tau_1$, where B_1 and B_2 are as in (1.4.22), and split J_2 as $J_2 = J_{2,1} + J_{2,2} + J_{2,3}$ accordingly. Arguing as the estimate for I_2, we see that

$$E_{2,2}(y,s) \ne \emptyset \quad\text{for}\quad |y| > \varepsilon_0(\tau)^\theta,$$

$$E_{2,3}(y,s) = \emptyset \quad\text{for}\quad |y| > \frac{1}{2}e^{-(1/2-\widehat{\omega})s}e^{\tau/2}.$$

Owing to $|\lambda_0|>1$, (1.4.2), (1.4.28), (1.4.29), and (1.4.33), we obtain

$$\begin{aligned}
J_{2,1} &\le C|y|^{-m}\int_{\tau_0}^{\tau} e^{m(\tau-s)/2}\int_{E_{2,1}(s)} |f(v(r,s))|r^{\gamma+m-1}\,drds\\
&\le C\varepsilon_0(\tau)^{-\theta m}\int_{\tau_0}^{\tau}\varepsilon_0(s)^{4|\lambda_0|}\int_{\varepsilon_0(s)^{1-\theta}}^{\varepsilon_0(s)^{2\theta}} r^{-2|\lambda_0|+\sqrt{D}-1}\,drds\\
&\le C\varepsilon_0(\tau)^{-\theta m}\int_{\tau_0}^{\tau}\begin{cases}\varepsilon_0(s)^{4|\lambda_0|+2\theta(-2|\lambda_0|+\sqrt{D})}\,ds & \text{if}\quad 2|\lambda_0|<\sqrt{D},\\ \varepsilon_0(s)^{4|\lambda_0|-(1-\theta)}\,ds & \text{if}\quad 2|\lambda_0|=\sqrt{D},\\ \varepsilon_0(s)^{2|\lambda_0|+\sqrt{D}-\theta(-2|\lambda_0|+\sqrt{D})}\,ds & \text{if}\quad 2|\lambda_0|>\sqrt{D}\end{cases}\\
&\le C\varepsilon_0(\tau)^{2|\lambda_0|+2\mu-\theta(2+\sqrt{D})}\le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu}.
\end{aligned} \tag{1.4.46}$$

Recall $\ell>1$ and $\lambda_\ell=-|\lambda_0|+\ell>0$. By (1.4.2) with $\widehat{\omega}=1/2-1/4(\lambda_\ell+1)$ and (1.4.29), we have

$$\begin{aligned}
J_{2,2} &\le \int_{\tau_0}^{\tau}\frac{C\varepsilon_0(s)^{4|\lambda_0|}}{(1-e^{-(\tau-s)})^{m/2}}\int_{E_{2,2}(y,s)}\exp\left\{-\frac{\big||y|e^{-(\tau-s)/2}-r\big|^2}{4(1-e^{-(\tau-s)})}\right\}\\
&\qquad\qquad\times\mathcal{B}_y(r,\tau-s)r^{-2|\lambda_0|+m-3}(1+r^{4\ell})\,drds\\
&\le C\mathcal{S}\int_{\tau_0}^{\tau}\varepsilon_0(s)^{4|\lambda_0|-4\theta(|\lambda_0|+1)}\,ds + C\mathcal{S}|y|^{2\ell}\int_{\tau_0}^{\tau}\begin{cases}\varepsilon_0(s)^{4|\lambda_0|-4\theta|\lambda_\ell-1|}\,ds & \text{if}\quad \lambda_\ell\le 1,\\ \varepsilon_0(s)^{4|\lambda_0|}e^{2\widehat{\omega}|\lambda_\ell-1|s}\,ds & \text{if}\quad \lambda_\ell>1\end{cases}\\
&\le C\varepsilon_0(\tau)^{4|\lambda_0|-4\theta(|\lambda_0|+1)}+\begin{cases}\varepsilon_0(\tau)^{4|\lambda_0|-4\theta|\lambda_\ell-1|}|y|^{2\ell} & \text{if}\quad \lambda_\ell\le 1,\\ \varepsilon_0(\tau)^{2|\lambda_0|(1+1/\lambda_\ell)}|y|^{2\ell} & \text{if}\quad \lambda_\ell>1.\end{cases}
\end{aligned} \tag{1.4.47}$$

On the other hand, (1.4.2) with $\widehat{\omega}=1/2-1/4(\lambda_\ell+1)$, (1.4.26), and (1.4.29) imply that

$$\begin{aligned}
J_{2,3} &\le \int_{\tau_0}^{\tau}\frac{C\varepsilon_0(s)^{4|\lambda_0|}}{(1-e^{-(\tau-s)})^{m/2}}\int_{E_{2,3}(y,s)}\exp\left\{-\frac{\big||y|e^{-(\tau-s)/2}-r\big|^2}{4(1-e^{-(\tau-s)})}\right\}\\
&\qquad\qquad\times\mathcal{B}_y(r,\tau-s)r^{-2|\lambda_0|+m-3}(1+r^{4\ell})\,drds\\
&\le C(\mathcal{S}_0+\mathcal{S}_{4\ell})\int_{\tau_0}^{\tau}\varepsilon_0(s)^{4|\lambda_0|-4\theta(|\lambda_0|+1)}\le C\varepsilon_0(\tau)^{4|\lambda_0|-4\theta(|\lambda_0|+1)}\,;
\end{aligned} \tag{1.4.48}$$

Estimate for J_3. Let us divide the region E_3 in (1.4.43c) with

$$E_{3,1}(y,\sigma):=B_1(y,\sigma)\cap E_3(\sigma)\quad\text{and}\quad E_{3,2}(y,\sigma):=B_2(y,\sigma)\cap E_3(\sigma), \tag{1.4.49}$$

for $y\in\mathbf{R}^m$ and $\tau_0\le\sigma\le\tau\le\tau_1$, and split J_3 as $J_3=J_{3,1}+J_{3,2}$, accordingly. We note that

$$E_{3,1}(y,s)=\emptyset\quad\text{for}\quad \varepsilon_0(s)^\theta<|y|\le\frac{1}{2}e^{-(1/2-\widehat{\omega})s}e^{\tau/2}.$$

Due to (1.4.2) with $\widehat{\omega} = 1/2 - 1/4(\lambda_\ell + 1)$, (1.4.25), and (1.4.29), we obtain

$$\begin{aligned} J_{3,1} &\le \int_{\tau_0}^{\tau} \frac{C}{(1-e^{-(\tau-s)})^{m/2}} \int_{E_{3,1}(y,s)} \exp\left\{ -\frac{\left||y|e^{-(\tau-s)/2} - r\right|^2}{4(1-e^{-(\tau-s)})} \right\} \\ &\qquad\qquad \times \mathcal{B}_y(r,\tau) r^{2|\lambda_0|-2\ell+m-3} r^{2\ell}\, drds \\ &\le C\mathcal{S}|y|^{2\ell} \int_{\tau_0}^{\tau} e^{-2\widehat{\omega}(\lambda_\ell+1)s}\, ds \le C\varepsilon_0(\tau)^{2|\lambda_0|+|\lambda_0|/\lambda_\ell} |y|^{2\ell}. \end{aligned} \tag{1.4.50}$$

It follows from $|\lambda_0| > 1$, (1.4.2), (1.4.27), and (1.4.29) that

$$\begin{aligned} J_{3,2} &\le \int_{\tau_0}^{\tau} \frac{C}{(1-e^{-(\tau-s)})^{m/2}} \int_{E_{3,2}(y,s)} \exp\left\{ -\frac{\left||y|e^{-(\tau-s)/2} - r\right|^2}{4(1-e^{-(\tau-s)})} \right\} \\ &\qquad\qquad \times \left(\frac{r}{e^{\widehat{\omega}s}}\right)^{2\lambda_\ell/\widehat{\omega}} \mathcal{B}_y(r,\tau) r^{2|\lambda_0|+m-3}\, drds \\ &\le C\mathcal{S}_{2\lambda_\ell/\widehat{\omega}+2|\lambda_0|-2} \int_{\tau_0}^{\tau} e^{-2\lambda_\ell s}\, ds \le C\varepsilon_0(\tau)^{4|\lambda_\ell|}. \end{aligned} \tag{1.4.51}$$

Because of (1.3.1), (1.4.44), (1.4.46)–(1.4.48), (1.4.50), and (1.4.51), we obtain (1.4.41) and the proof is complete.

To show (1.4.42), we use Lemma 1.4.2, (1.4.25), (1.4.27), and (1.4.29), to get

$$\begin{aligned} &\left|[S(\tau-s)\{\langle g(W(s)), \psi_n\rangle_m \psi_n\}](y)\right| \\ &\le \int_{\tau_0}^{\tau} \frac{C\varepsilon_0(s)^{2|\lambda_0|+\mu}}{(1-e^{-(\tau-s)})^{m/2}} \int_0^\infty \exp\left\{ -\frac{\left||y|e^{-(\tau-s)/2} - r\right|^2}{4(1-e^{-(\tau-s)})} \right\} \mathcal{B}_y(r,\tau) r^{m-1}(1+r^{2n})\, drds \\ &\le C\big(\mathcal{S}(1+|y|^{2n}) + \mathcal{S}_0 + \mathcal{S}_{2n}\big) \int_{\tau_0}^{\tau} \varepsilon_0(s)^{2|\lambda_0|+\mu}\, ds \le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu}(1+|y|^{2\ell}), \end{aligned}$$

for $n = 0, 1, \ldots, \ell$. We obtain (1.4.42) and the proof is complete. □

Lemma 1.4.7. *There holds*

$$\left|W(r,\tau) - a_\ell^*(\tau)\psi_\ell(r)\right| \le C\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}(1+|y|^{2\ell}) \tag{1.4.52}$$

for $\varepsilon_0(\tau)^\theta < |y| \le e^{-\omega\tau_0}e^{\tau/2}$, $\tau_0 \le \tau \le \tau_0 + 1$ *with sufficiently large* τ_0.

Proof. To apply the estimates (1.4.30) and (1.4.41) for the representation of solution (1.4.21), we have (1.4.52) in short time case and the proof is complete. □

A priori estimate in the long-time case

Next, we show the estimate in the long-time $\tau_0 + 1 \le \tau \le \tau_1$. Notice that

$$1 - e^{-1} \le 1 - e^{-(\tau-\sigma)} \le 1 \quad \text{for } \tau_0 + 1 \le \sigma \le \tau \le \tau_1. \tag{1.4.53}$$

Let $R_0(|y|) := R(|y|, \tau_0)$ where R is as in (1.4.17).

Lemma 1.4.8. *There holds*

$$\big|[S(\tau-\tau_0)R_0](y)\big| \le C\varepsilon_0(\tau_0)^{\widetilde{\theta}(2+\sqrt{D})}\varepsilon_0(\tau)^{2|\lambda_0|}(1+|y|^{2\ell}), \tag{1.4.54}$$

for $\varepsilon_0(\tau)^\theta < |y| \le e^{(\tau-\tau_0-1)/2}$ *and* $\tau_0+1\le\tau\le\tau_1$ *with sufficiently large* τ_0*, where* $\widetilde{\theta}\in(2\theta,1)$ *is as in* (1.3.3).

Proof. Since R_0 is orthogonal to the eigenfunctions ψ_n for $n=0,1,\dots,\ell$ in $L^2_\rho(\mathbf{R}^m)$, the series expansion (1.4.18a) with (1.2.16) and (1.4.12c) (cf. Remark 1.4.1) implies

$$\begin{aligned}\big|[S(\tau-\tau_0)R_0](y)\big| &\le \sum_{n=\ell+1}^{\infty} e^{-\lambda_n(\tau-\tau_0)}|a_n(\tau_0)||\psi_n(|y|)|\\ &\le C\varepsilon_0(\tau)^{2|\lambda_0|+\widetilde{\theta}(2+\sqrt{D})}\sum_{n=\ell+1}^{\infty} c_n^2 e^{-(\lambda_n-\lambda_\ell-\widetilde{\theta}(2+\sqrt{D}))(\tau-\tau_0)}\\ &\le C\varepsilon_0(\tau)^{2|\lambda_0|+\widetilde{\theta}(2+\sqrt{D})}\sum_{n=\ell+1}^{\infty} c_n^2 e^{-(n-\ell-\widetilde{\theta}(2+\sqrt{D}))} \le C\varepsilon_0(\tau)^{2|\lambda_0|+\widetilde{\theta}(2+\sqrt{D})}\end{aligned} \tag{1.4.55}$$

for $\varepsilon_0(\tau)^\theta < |y| \le 1$ and $\tau_0+1\le\tau\le\tau_1$ and

$$\begin{aligned}\big|[S(\tau-\tau_0)R_0](y)\big| &\le \sum_{n=\ell+1}^{\infty} e^{-\lambda_n(\tau-\tau_0)}|a_n(\tau_0)||\psi_n(|y|)|\\ &\le C\varepsilon_0(\tau_0)^{\widetilde{\theta}(2+\sqrt{D})}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}\sum_{n=\ell+1}^{\infty} c_n|\tilde{c}_n|e^{-(n-\ell)(\tau-\tau_0)}|y|^{2(n-\ell)}\\ &\le C\varepsilon_0(\tau_0)^{\widetilde{\theta}(2+\sqrt{D})}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}\sum_{n=\ell+1}^{\infty} c_n^2 e^{-(n-\ell)} \le C\varepsilon_0(\tau_0)^{\widetilde{\theta}(2+\sqrt{D})}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}\end{aligned} \tag{1.4.56}$$

for $1<|y|\le e^{(\tau-\tau_0-1)/2}$ and $\tau_0+1\le\tau\le\tau_1$, where the exact formulas of c_n and $\tilde{c}_n$ as in (1.2.17a) and (1.2.17b) have been used as well. Because of

$$\left|\frac{c_{n+1}^2 e^{-(n+1)}}{c_n^2 e^{-n}}\right| = \frac{n!\Gamma(m/2+n+1)}{e(n+1)!\Gamma(m/2+n)} = \frac{m/2+n}{e(n+1)} \to \frac{1}{e} < 1 \quad \text{as} \quad n\to\infty, \tag{1.4.57}$$

we see from the ratio test that the series in the last line of (1.4.55) and (1.4.56) converges. We then obtain (1.4.54) and the proof is complete. □

Lemma 1.4.9. *There holds*

$$\big|[S(\tau-\tau_0)\{W_0-a_\ell^*(\tau_0)\psi_\ell\}](y)\big| \le C\varepsilon_0(\tau_0)^{3\theta}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}, \tag{1.4.58}$$

for $e^{(\tau-\tau_0-1)/2} < |y| \le e^{-\omega\tau_0}e^{\tau/2}$ *and* $\tau_0+1\le\tau\le\tau_1$ *with sufficiently large* τ_0*, where* $\widetilde{\theta}\in(2\theta,1)$ *is as in* (1.3.3).

Proof. Let $e^{(\tau-\tau_0-1)/2} < |y| \le e^{-\omega\tau_0}e^{\tau/2}$. We remark that

$$2|y|e^{-(\tau-\tau_0)/2} \in (1, 2e^{(1/2-\omega)\tau_0}) \quad \text{and} \quad |y| \ge 1$$

for $e^{(\tau-\tau_0-1)/2} < |y| \le e^{-\omega\tau_0}e^{\tau/2}$ and $\tau_0 + 1 \le \tau \le \tau_1$, and

$$1 \le e^{\ell}e^{-\ell(\tau-\tau_0)}|y|^{2\ell} \quad \text{for} \quad |y| \ge e^{(\tau-\tau_0-1)/2}. \tag{1.4.59}$$

We use the same notation D_i, $D_{i,j}$, I_i, and $I_{i,j}$ in Lemma 1.4.5.

Estimate for I_1. Similarly to the proof of (1.4.34), it follows from (1.3.10) with $\theta' = 3\theta/\mu$, (1.4.28), (1.4.33), and (1.4.59) that

$$\begin{aligned} I_{1,1} &\le Ce^{|\lambda_0|(\tau-\tau_0)}(|y|e^{-(\tau-\tau_0)/2})^{-m}\int_{D_{1,1}(\tau_0)}(\varepsilon_0(\tau_0)^{2|\lambda_0|} + r^{2|\lambda_0|})r^{m-1}\,dr \\ &\le Ce^{|\lambda_0|(\tau-\tau_0)}(\varepsilon_0(\tau_0)^{2|\lambda_0|+(1-3\theta/\mu)m} + \varepsilon_0(\tau_0)^{(1-3\theta/\mu)(2|\lambda_0|+m)})(|y|e^{-(\tau-\tau_0)/2})^{2\ell} \\ &\le Ce^{-\lambda_\ell(\tau-\tau_0)}\varepsilon_0(\tau_0)^{2|\lambda_0|}\varepsilon_0(\tau_0)^{(1-3\theta/\mu)m}|y|^{2\ell} \le C\varepsilon_0(\tau_0)^{(1-3\theta/\mu)m}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}. \end{aligned} \tag{1.4.60}$$

A similar argument shows

$$\begin{aligned} I_{1,2} &\le Ce^{|\lambda_0|(\tau-\tau_0)}(|y|e^{-(\tau-\tau_0)/2})^{2\ell}\big(\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta} + \varepsilon_0(\tau_0)^{2|\lambda_0|+2\widetilde{\theta}}\big)|[S(\tau-\tau_0)1](y)| \\ &\le Ce^{-\lambda_\ell(\tau-\tau_0)}\varepsilon_0(\tau_0)^{2|\lambda_0|}\varepsilon_0(\tau_0)^{3\theta}|y|^{2\ell} \le C\varepsilon_0(\tau_0)^{3\theta}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}. \end{aligned} \tag{1.4.61}$$

Estimate for I_2. Because of (1.3.10), (1.4.25), (1.4.27), and (1.4.59), we have

$$I_{2,1} \le C\mathcal{S}\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}e^{|\lambda_0|(\tau-\tau_0)}\big(1 + (|y|e^{-(\tau-\tau_0)/2})^{2\ell}\big) \le C\varepsilon_0(\tau_0)^{3\theta}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell} \tag{1.4.62}$$

and

$$I_{2,2} \le C(\mathcal{S}_0 + \mathcal{S}_{2\ell})\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}e^{|\lambda_0|(\tau-\tau_0)} \le C\varepsilon_0(\tau_0)^{3\theta}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}. \tag{1.4.63}$$

Estimate for I_3. Similar argument to (1.4.40) implies

$$I_3 \le C\mathcal{S}_{2\ell+\lambda_\ell(1/2-\widetilde{\omega})}\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}e^{|\lambda_0|(\tau-\tau_0)} \le C\varepsilon_0(\tau_0)^{3\theta}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}. \tag{1.4.64}$$

Because of (1.3.1) and (1.4.60)-(1.4.64), we obtain (1.4.58) and the proof is complete. □

Lemma 1.4.10. *Let $\tau_0 + 1 \le \tau \le \tau_1$ and $\tau_0 \le s \le \tau - 1$ with sufficiently large τ_0. Then:*

$$\big|[S(\tau-s)g(W(s))](y)\big| \le C\varepsilon_0(s)^{\mu}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell} \tag{1.4.65}$$

for $|y| \ge e^{(\tau-s-1)/2}$ and

$$\left|\left[S(\tau-s)\left\{g(W(s)) - \sum_{n=0}^{\ell}\langle g(W(s)), \psi_n\rangle_m\psi_n\right\}\right](y)\right| \le C\varepsilon_0(s)^{\mu}\varepsilon_0(\tau)^{2|\lambda_0|}(1 + |y|^{2\ell}), \tag{1.4.66}$$

for $|y| \ge \varepsilon_0(\tau)^{\theta}$, where $\mu > 0$ is as in (1.4.7).

Proof. Recall that $\langle f(v), \phi_n\rangle_N = \langle g(W), \psi_n\rangle_m$. Arguing as in the proof of Lemma 1.4.8, we obtain from (1.4.6) and (1.4.18a) that

$$\begin{aligned}
&\left|\left[S(\tau-s)\left\{g(W(s)) - \sum_{n=0}^{\ell}\langle g(W(s)), \psi_n\rangle_m \psi_n\right\}\right](y)\right| \\
&\le \sum_{n=\ell+1}^{\infty} e^{-\lambda_n(\tau-s)}|\langle g(W(s)), \psi_n\rangle_m||\psi_n(|y|)| \le C\varepsilon_0(s)^{\mu}\varepsilon_0(\tau)^{2|\lambda_0|}(1+|y|^{2\ell})
\end{aligned}$$

for $|y| < e^{(\tau-s-1)/2}$. The estimate (1.4.66) is archived for $\varepsilon_0(\tau)^{\theta} \le |y| < e^{(\tau-s-1)/2}$.

We next prove (1.4.65) and (1.4.66) for $|y| \ge e^{(\tau-s-1)/2}$. To this end, we assume $|y| \ge e^{(\tau-s-1)/2}$ hereafter without stating explicitly. We use the same notations J_i and E_i in the proof of Lemma 1.4.6. Remark that

$$2|y|e^{-(\tau-s)/2} > 1, \quad |y| \ge 1, \quad \text{and} \quad e^{\ell}e^{-\ell(\tau-s)}|y|^{2\ell} \ge 1,$$

for $|y| > e^{(\tau-s-1)/2}$, $\tau_0 + 1 \le \tau \le \tau_1$, and $\tau_0 \le s \le \tau - 1$, and $r^{\gamma} f(v) = g(W)$.

Estimate for J_1. By $|\lambda_0| - \ell = -\lambda_\ell$ and (1.4.2) with $\widehat{\theta} = 1 - \theta$, we see that

$$\begin{aligned}
J_1 &\le Ce^{|\lambda_0|(\tau-s)} \int_{E_1(s)} r^{\gamma-2\beta+m-3}\,dr \\
&\le Ce^{|\lambda_0|(\tau-s)}(e^{-(\tau-s-1)/2}|y|)^{2\ell} \int_0^{\varepsilon_0(s)^{1-\theta}} r^{2|\lambda_0|+\sqrt{D}-1}\,dr \\
&\le C\varepsilon_0(s)^{\sqrt{D}-\theta(2|\lambda_0|+\sqrt{D})}e^{-\lambda_\ell\tau}|y|^{2\ell}.
\end{aligned} \tag{1.4.67}$$

Estimate for J_2. Let us divide further the region E_2 with $\widehat{\theta} = 1 - \theta$ as

$$\begin{aligned}
\widetilde{E}_{2,1}(s) &:= \{r \in \mathbf{R}_+\,;\, \varepsilon_0(s)^{1-\theta} < r \le 1\}, \\
\widetilde{E}_{2,2}(y,s) &:= B_1(y,s) \cap \{r \in \mathbf{R}_+\,;\, 1 < r \le e^{\widehat{\omega}s}\}, \\
\widetilde{E}_{2,3}(y,s) &:= B_2(y,s) \cap \{r \in \mathbf{R}_+\,;\, 1 < r \le e^{\widehat{\omega}s}\},
\end{aligned}$$

and denote the corresponding integrals by $\widetilde{J}_{2,j}$ for $j = 1, 2, 3$, accordingly. We note that

$$\begin{aligned}
&\widetilde{E}_{2,2}(y,s) \ne \emptyset \quad \text{for} \quad |y| > e^{(\tau-s-1)/2}, \\
&\widetilde{E}_{2,3}(y,s) = \emptyset \quad \text{for} \quad |y| > \frac{1}{2}e^{-(1/2-\widehat{\omega})s}e^{\tau/2}.
\end{aligned}$$

The estimate (1.4.2) of $f(v(r,s))$ in $\widetilde{E}_{2,1}(s)$ and $|\lambda_0|>1$ implies that

$$\begin{aligned}
\widetilde{J}_{2,1} &\le C\varepsilon_0(s)^{4|\lambda_0|}e^{|\lambda_0|(\tau-s)}\int_{\widetilde{E}_{2,1}(s)} r^{-2|\lambda_0|+m-3}\,dr \\
&\le C\varepsilon_0(s)^{4|\lambda_0|}e^{|\lambda_0|(\tau-s)}(e^{-(\tau-s-1)/2}|y|)^{2\ell}\int_{\varepsilon_0(s)^{1-\theta}}^{1} r^{-2|\lambda_0|+\sqrt{D}-1}\,dr \\
&\le C\varepsilon_0(s)^{2|\lambda_0|}e^{-\lambda_\ell\tau}|y|^{2\ell}\times\begin{cases}1 & \text{if}\quad 2|\lambda_0|<\sqrt{D},\\ \varepsilon_0(s)^{-1+\theta} & \text{if}\quad 2|\lambda_0|=\sqrt{D},\\ \varepsilon_0(s)^{(1-\theta)(-2|\lambda_0|+\sqrt{D})} & \text{if}\quad 2|\lambda_0|>\sqrt{D}\end{cases} \\
&\le \begin{cases}C\varepsilon_0(s)^{|\lambda_0|}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell} & \text{if}\quad 2|\lambda_0|\le\sqrt{D},\\ \varepsilon_0(s)^{\sqrt{D}}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell} & \text{if}\quad 2|\lambda_0|>\sqrt{D}.\end{cases}
\end{aligned} \tag{1.4.68}$$

Due to $-|\lambda_0|+\ell=\lambda_\ell$, (1.4.2) with $\widehat{\omega}=1/2-1/4(\lambda_\ell+1)$, and (1.4.25), we have

$$\begin{aligned}
\widetilde{J}_{2,2} &\le \frac{C\varepsilon_0(s)^{4|\lambda_0|}e^{|\lambda_0|(\tau-s)}}{(1-e^{-(\tau-s)})^{m/2}}\int_{\widetilde{E}_{2,2}(y,s)}\exp\left\{-\frac{\big||y|e^{-(\tau-s)/2}-r\big|^2}{4(1-e^{-(\tau-s)})}\right\} \\
&\qquad\qquad\times\mathcal{B}_y(r,\tau-s)r^{4\ell-2|\lambda_0|+m-3}\,dr \\
&\le C\mathcal{S}\varepsilon_0(s)^{4|\lambda_0|}e^{|\lambda_0|(\tau-s)}(2|y|e^{-(\tau-s)/2})^{2\ell}\times\begin{cases}1 & \text{if}\quad \lambda_\ell\le 1,\\ e^{2\widehat{\omega}|\lambda_\ell-1|s} & \text{if}\quad \lambda_\ell>1\end{cases} \\
&\le \begin{cases}C\varepsilon_0(\tau)^{2|\lambda_0|}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell} & \text{if}\quad \lambda_\ell\le 1,\\ C\varepsilon_0(s)^{2|\lambda_0|/\lambda_\ell}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell} & \text{if}\quad \lambda_\ell>1.\end{cases}
\end{aligned} \tag{1.4.69}$$

To use a similar argument to (1.4.48), we have

$$\widetilde{J}_{2,3}\le C\mathcal{S}_{4\ell}\varepsilon_0(s)^{4|\lambda_0|}e^{|\lambda_0|(\tau-s)}(e^{-(\tau-s-1)/2}|y|)^{2\ell}\le C\varepsilon_0(s)^{2|\lambda_0|}\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}. \tag{1.4.70}$$

Estimate for J_3. We note that

$$E_{3,1}(y,s)=\emptyset\quad\text{for}\quad e^{(\tau-s-1)/2}<|y|\le\frac{1}{2}e^{-(1/2-\widehat{\omega})s}e^{\tau/2}=\frac{1}{2}e^{-s/4(\lambda_\ell+1)}e^{\tau/2}$$

with $\widehat{\omega}=1/2-1/4(\lambda_\ell+1)$. The fact of $|\lambda_0|>1$, (1.4.2), and (1.4.25) imply that

$$\begin{aligned}
J_{3,1} &\le C\mathcal{S}e^{|\lambda_0|(\tau-s)}(2|y|e^{-(\tau-s)/2})^{2|\lambda_0|-2}\le Ce^{\tau-s}|y|^{2|\lambda_0|-2} \\
&\le Ce^{\tau-s}|y|^{-2(\lambda_\ell+1)}|y|^{2\ell}\le Ce^{\tau-s}e^{s/2}e^{-(\lambda_\ell+1)\tau}|y|^{2\ell}\le Ce^{-s/2}e^{-\lambda_\ell\tau}|y|^{2\ell}
\end{aligned} \tag{1.4.71}$$

for $|y|>e^{-s/4(\lambda_\ell+1)}e^{\tau/2}/2$. Similarly to the argument in (1.4.40), we see that

$$J_{3,2}\le C\mathcal{S}_{2\lambda_\ell/\widehat{\omega}+2|\lambda_0|-2}e^{-2\lambda_\ell s}e^{|\lambda_0|(\tau-s)}(e^{-(\tau-s-1)/2}|y|)^{2\ell}\le Ce^{-\lambda_\ell s}e^{-\lambda_\ell\tau}|y|^{2\ell}. \tag{1.4.72}$$

Due to (1.3.1), (1.4.67)–(1.4.72), we have (1.4.65).

To obtain the inequality (1.4.66) for $|y| \geq e^{-(\tau-s-1)/2}$, it is sufficient to show

$$\left|\left[S(\tau-s)\left\{\sum_{n=0}^{\ell}\langle g(W(s)),\psi_n\rangle_m\psi_n\right\}\right](y)\right| \leq C\varepsilon_0(s)^\mu\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}$$

for $|y| \geq e^{(\tau-s-1)/2}$. Notice that $|y| \geq 1$ and $\lambda_\ell - \lambda_n = \ell - n$. Recalling Lemma 1.4.2 and the series expression (1.4.18a) of $S(\tau)$, we easily obtain

$$\begin{aligned}\big|[S(\tau-s)\{\langle g(W(s)),\psi_n\rangle_m\psi_n\}](y)\big| &\leq C\varepsilon_0(s)^\mu e^{-\lambda_\ell s}e^{-\lambda_n(\tau-s)}|y|^{2n}\\ &\leq C\varepsilon_0(s)^\mu e^{-\lambda_\ell\tau}|y|^{2\ell}(e^{-(\tau-s)/2}|y|)^{-2(\ell-n)} \leq C\varepsilon_0(s)^\mu e^{-\lambda_\ell\tau}|y|^{2\ell}\end{aligned}$$

for $n = 0, 1, \ldots, \ell$ and $|y| \geq e^{-(\tau-s-1)/2}$. It then follows the estimate (1.4.66) and the proof is complete. ☐

Lemma 1.4.11. *Let $\tau_0 + 1 < \tau \leq \tau_1$ with sufficiently large τ_0. Then:*

$$\left|\int_{\tau_0}^{\tau}[S(\tau-s)g(W(s))](y)\,ds\right| \leq C\varepsilon_0(\tau_0)^\mu\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell} \tag{1.4.73}$$

for $|y| > e^{(\tau-\tau_0-1)/2}$ and

$$\begin{aligned}&\left|\int_{\tau_0}^{\tau}\left[S(\tau-s)\left\{g(W(s))-\sum_{n=0}^{\ell}\langle g(W(s)),\psi_n\rangle_m\psi_n\right\}\right](y)\,ds\right|\\ &\leq C\varepsilon_0(\tau_0)^\mu\varepsilon_0(\tau)^{2|\lambda_0|}(1+|y|^{2\ell}),\end{aligned} \tag{1.4.74}$$

for $|y| > \varepsilon_0(\tau)^\theta$, where $\mu > 0$ is as in (1.4.7).

Proof. We first divide integral interval in time as $[\tau_0, \tau] = [\tau_0, \tau-1] \cup [\tau-1, \tau]$. Clearly, integration with the later interval (that is short time) may be estimated as in (1.4.41) and (1.4.42). It thus suffices to consider the former integral interval. Because of $e^{(\tau-\tau_0-1)/2} \geq e^{(\tau-s-1)/2}$ for $\tau_0 \leq s \leq \tau-1$, we obtain from (1.4.65) that

$$\left|\int_{\tau_0}^{\tau-1}[S(\tau-s)g(W(s))](y)\,ds\right| \leq C\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}\int_{\tau_0}^{\tau-1}\varepsilon_0(s)^\mu\,ds \leq C\varepsilon_0(\tau_0)^\mu\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}$$

for $|y| \geq e^{(\tau-\tau_0-1)/2}$. Then, the estimate (1.4.73) is archived. The same calculation with (1.4.66) implies the inequality (1.4.74). The proof is complete. ☐

Lemma 1.4.12. *There holds*

$$\big|W(r,\tau) - a_\ell^*(\tau)\psi_\ell(r)\big| \leq C\varepsilon_0(\tau_0)^{3\theta}\varepsilon_0(\tau)^{2|\lambda_0|}(1+|y|^{2\ell}) \tag{1.4.75}$$

for $\varepsilon_0(\tau)^\theta < |y| \leq e^{-\omega\tau_0}e^{\tau/2}$, $\tau_0 \leq \tau \leq \tau_0 + 1$ with sufficiently large τ_0.

Proof. We see (1.4.75) for $\varepsilon_0(\tau) < |y| \leq e^{(\tau-\tau_0-1)/2}$ to apply the estimates (1.4.11), (1.4.54), and (1.4.74) to (1.4.21), In additional, combining (1.4.58), (1.4.73), and (1.4.21) give us the estimate (1.4.75) for $e^{(\tau-\tau_0-1)/2} < |y| \leq e^{-\omega\tau_0}e^{\tau/2}$. The proof is complete. ☐

1.5 A priori estimates in the inner region

In this section, we will prove a priori estimates in the region where $|y| \le \varepsilon_0(\tau)^\theta$ using the idea of [44, 46]. This together with the lemmas in §1.4 complete the proof of Lemma 1.3.1.

Lemma 1.5.1. *Assume that $p > p_{\mathrm{JL}}$, $N > 10 + 8a$. Assume also that*

$$\big|\Phi_0(r) - \Phi_{\mathrm{med}}(r, \tau_0)\big| \le C\varepsilon_0(\tau_0)^{\gamma - 2\beta + 2\theta} r^{-\gamma} \tag{1.5.1}$$

for $r \le \varepsilon_0(\tau_0)^\theta$, where

$$\Phi_{\mathrm{med}}(r, \tau_0) = U_\infty(r) + a_\ell^*(\tau)\phi_\ell(r).$$

If there exists a constant $M > 0$ such that

$$\left|\Phi(r,\tau) - U_\infty(r) + \frac{h}{c_\ell}\varepsilon_0(\tau)^{\gamma-2\beta}\phi_\ell(r)\right| \le M\varepsilon_0(\tau)^{\gamma-2\beta+2\theta} r^{-\gamma} \tag{1.5.2}$$

for $r = \varepsilon_0(\tau)^\theta$ and $\tau_0 \le \tau \le \tau_1$, then there exists a positive smooth function $H(\eta)$ with

$$H(\eta) = \begin{cases} O(\eta^{-\gamma}) & \text{as} \quad \eta \to \infty, \\ O(1) & \text{as} \quad \eta \to 0, \end{cases}$$

such that

$$\big|\Phi(r,\tau) - \Phi_{\mathrm{inn}}(r,\tau)\big| \le \varepsilon_0(\tau)^{-2\beta+2\theta} H\left(\frac{r}{\varepsilon_0(\tau)}\right) \tag{1.5.3}$$

for $r \le \varepsilon_0(\tau)^\theta$ and $\tau_0 \le \tau \le \tau_1$ with τ_0 large enough.

Proof. We extend the idea of the proof of [44, Proposition 2.1] for the neutral eigenvalues to that of stable ones. We shall recall equation (1.2.5):

$$\varepsilon_0(\tau)^2 U_\tau = \Delta_\xi U + |\xi|^{2a} U^p - \mathcal{E}(\tau)\left(\frac{\xi \cdot \nabla_\xi U}{2} + \beta U\right),$$

where $\xi = y/\varepsilon_0(\tau)$, $U(\xi,\tau) = \varepsilon_0(\tau)^{2\beta}\Phi(y,\tau)$, and $\mathcal{E}(\tau) := \varepsilon_0(\tau)^2 - 2\varepsilon_0(\tau)\dot{\varepsilon}_0(\tau)$. Consider the new dependent variable:

$$V(\xi,\tau) := U(\xi,\tau) - U_1(\xi).$$

Equation (1.2.5) for $U(\xi,\tau)$ is then converted to the one for $V(\xi,\tau)$ as follows:

$$\begin{aligned} \mathcal{N}V := {} & -\varepsilon_0(\tau)^2\frac{\partial V}{\partial \tau} + \Delta_\xi V - \mathcal{E}(\tau)\left[T_1(\xi) + \left(\frac{\xi\cdot\nabla_\xi V}{2} + \beta V\right)\right] \\ & + |\xi|^{2a}\Big[pU_2(\xi)^{p-1}V + (U_1(\xi) + V)^p - U_1(\xi)^p \\ & \qquad - pU_1(\xi)^{p-1}V + p\big(U_1(\xi)^{p-1} - U_2(\xi)^{p-1}\big)V\Big] \end{aligned} \tag{1.5.4}$$

where

$$T_1(\xi) := \frac{sU_1'(\xi)}{2} + \beta U_1(\xi). \tag{1.5.5}$$

Here and henceforth, we shall abuse the notation such as $U_1(\xi) = U_1(|\xi|)$ for simplicity. Set $\eta := |\xi|$. The ordered structure of the family $\{U_\kappa\}_{\kappa>0}$ implies that

$$T_1(\eta) = \beta \frac{\partial}{\partial \kappa} U_\kappa(\eta)\bigg|_{\kappa=1} > 0 \quad \text{for any} \quad \eta > 0\,; \quad T_1(0) = \beta.$$

Let us write

$$H_\Lambda(\eta) := \frac{\partial}{\partial \kappa} U_\kappa(\eta)\bigg|_{\kappa=\Lambda} = U_1(\Lambda^{1/2\beta}\eta) + \frac{1}{2\beta}\Lambda^{1/2\beta}\eta U_1'(\Lambda^{1/2\beta}\eta) \quad \text{for} \quad \Lambda > 0,$$

where we use $U_\kappa(\eta) = \kappa U_1(\kappa^{1/2\beta}\eta)$ for any $\kappa > 0$. The function H_Λ solves

$$\begin{cases} 0 = H_\Lambda'' + \dfrac{N-1}{\eta} H_\Lambda' + p\eta^{2a} U_\Lambda(\eta)^{p-1} H_\Lambda, \quad \eta > 0, \\ H_\Lambda(0) = 1, \quad H_\Lambda'(0) = 0. \end{cases}$$

Taking advantage of the asymptotics of U_1 and U_1' as in Proposition 1.2.1, we obtain

$$H_1(\eta) = \beta T_1(\eta) = \frac{h(\gamma - 2\beta)}{2}\eta^{-\gamma} + o(\eta^{-\gamma}), \tag{1.5.6}$$

$$H_2(\eta) = C_0 \eta^{-\gamma} + o(\eta^{-\gamma}) \quad \text{with} \quad C_0 := \frac{h(\gamma-2\beta)}{2^{\gamma/2\beta+1}\beta}, \tag{1.5.7}$$

as $\eta \to \infty$. Let $H(\eta)$ be a solution of the inhomogeneous ODE:

$$H'' + \frac{N-1}{\eta} H' + p\eta^{2a} U_1(\eta)^{p-1} H = T_1(\eta),$$

satisfying $H(0) = H'(0) = 0$. A standard computation then shows that

$$\begin{aligned} H(\eta) &= H_1(\eta) \int_0^\eta \frac{1}{H_1(\rho)^2 \rho^{N-1}} \int_0^\rho H_1(r) T_1(r) r^{N-1}\, dr d\rho \\ &= C_1 \eta^{-\gamma+2} + o(\eta^{-\gamma+2}) \quad \text{with} \quad C_1 := \frac{h(\gamma-2\beta)}{4(2+\sqrt{D})} \end{aligned} \tag{1.5.8}$$

as $\eta \to \infty$. Let $k > 0$ be a constant to be chosen later. We will construct sub- and supersolutions of (1.5.4), using auxiliary functions

$$Z_\pm(\eta, \tau) := \mathcal{E}_\pm(\tau) H(\eta) \pm k\varepsilon_0(\tau)^{2\theta} H_2(\eta) \tag{1.5.9a}$$

$$\text{with} \quad \mathcal{E}_\pm(\tau) := \mathcal{E}(\tau) \mp \sqrt{k}\varepsilon_0(\tau)^{2+2\theta}. \tag{1.5.9b}$$

The functions $Z_\pm$ satisfy

$$\begin{aligned} \mathcal{N} Z_\pm = \eta^{2a}\Big[&(U_1(\eta) + Z_\pm)^p - U_1(\eta)^p - pU_1(\eta)^{p-1} Z_\pm \\ &+ p\big(U_1(\eta)^{p-1} - U_2(\eta)^{p-1}\big) Z_\pm\Big] + \big(\mathcal{E}_\pm(\tau) - \mathcal{E}(\tau)\big) T_1(\eta) \\ &- \mathcal{E}(\tau)\mathcal{E}_\pm(\tau)\left(\frac{\eta}{2}H'(\eta) + \beta H(\eta)\right) - \varepsilon_0(\tau)^2 \frac{d}{d\tau}\big(\mathcal{E}_\pm(\tau)\big) H(\eta) \\ &\mp k\varepsilon_0(\tau)^{2\theta}\mathcal{E}(\tau)\left(\frac{\eta}{2}H_2'(\eta) + \beta H_2(\eta)\right) \mp k\varepsilon_0(\tau)^2 \frac{d}{d\tau}\big(\varepsilon_0(\tau)^{2\theta}\big) H_2(\eta). \end{aligned} \tag{1.5.10}$$

Notice that the last two terms in (1.5.10) are roughly of oder $\varepsilon(\tau)^{2+2\theta}$ as $\tau \to \infty$, which is the same as of $\big(\mathcal{E}_\pm(\tau) - \mathcal{E}(\tau)\big)T_1(\xi)$. To cancel out the terms proportional to

$$T_2(\eta) := \frac{\eta}{2}H_2'(\eta) + \beta H_2(\eta) \quad \text{and} \quad kH_2(\eta)$$

in (1.5.10), respectively, we introduce the functions

$$\mathcal{J}_1(\eta) := H_2(\eta)\int_0^\eta \frac{1}{H_2(\rho)^2\rho^{N-1}}\int_0^r H_2(r)T_2(r)r^{N-1}\,drd\rho,$$
$$\mathcal{J}_2(\eta) := H_2(\eta)\int_0^\eta \frac{1}{H_2(\rho)^2\rho^{N-1}}\int_0^r H_2(r)^2r^{N-1}\,drd\rho,$$

Notice that they solve ODEs

$$\mathcal{J}_1'' + \frac{N-1}{\eta}\mathcal{J}_1' + p\eta^{2a}U_2(\eta)^{p-1}\mathcal{J}_1 = T_2(\eta),$$
$$\mathcal{J}_2'' + \frac{N-1}{\eta}\mathcal{J}_2' + p\eta^{2a}U_2(\eta)^{p-1}\mathcal{J}_2 = H_2(\eta),$$

with boundary conditions $\mathcal{J}_1(0) = \mathcal{J}_1'(0) = 0$, $\mathcal{J}_2(0) = \mathcal{J}_2'(0) = 0$. We now set

$$z_\pm(\eta,\tau) := \pm k\mathcal{E}(\tau)\varepsilon_0(\tau)^{2\theta}\mathcal{J}_1(\eta) \pm k\varepsilon_0(\tau)^2\frac{d}{d\tau}\big(\varepsilon_0(\tau)^{2\theta}\big)\mathcal{J}_2(\eta).$$

Using L'Hôpital's rule, we readily obtain

$$T_2(\eta) = \left(-\frac{\gamma}{2} + \beta\right)C_0\eta^{-\gamma} + o(\eta^{-\gamma}), \tag{1.5.11a}$$
$$\mathcal{J}_1(\eta) = \frac{(-\gamma/2+\beta)}{2(2+\sqrt{D})}C_0\eta^{-\gamma+2} + o(\eta^{-\gamma+2}), \tag{1.5.11b}$$
$$T_3(\eta) := \frac{\eta}{2}\mathcal{J}_1'(\eta) + \beta\mathcal{J}_1(\eta) = \frac{(-\gamma/2+\beta)(-\gamma/2+\beta+1)}{2(2+\sqrt{D})}C_0\eta^{-\gamma+2} + o(\eta^{-\gamma+2}), \tag{1.5.11c}$$

as $\eta \to \infty$. Notice that $(-\gamma/2+\beta)(-\gamma/2+\beta+1) = \lambda_0\lambda_1 > 0$ for $p > p_{\mathrm{JL}}$. The redefined function

$$\mathcal{Z}_\pm(\eta,\tau) := Z_\pm(\eta,\tau) + z_\pm(\eta,\tau)$$

satisfies

$$\begin{aligned}\mathcal{N}\mathcal{Z}_\pm = \eta^{2a}\Big[&\big(U_1(\eta)+\mathcal{Z}_\pm\big)^p - U_1(\eta)^p - pU_1(\eta)^{p-1}\mathcal{Z}_\pm \\ &+ p\big(U_1(\eta)^{p-1} - U_2(\eta)^{p-1}\big)\mathcal{Z}_\pm\Big] + \big(\mathcal{E}_\pm(\tau) - \mathcal{E}(\tau)\big)T_1(\eta) \\ &- \mathcal{E}(\tau)\mathcal{E}_\pm(\tau)\left(\frac{\eta}{2}H'(\eta) + \beta H(\eta)\right) - \varepsilon_0(\tau)^2\frac{d}{d\tau}(\mathcal{E}_\pm(\tau))H(\eta) + \mathcal{R}_\pm(\eta,\tau),\end{aligned} \tag{1.5.12}$$

where

$$\mathcal{R}_\pm(\eta,\tau) = \pm k\mathcal{E}(\tau)^2\varepsilon_0(\tau)^{2\theta}T_3(\eta) \mp k\varepsilon_0(\tau)^2\frac{d}{d\tau}(\mathcal{E}(\tau)\varepsilon_0(\tau)^{2\theta})\mathcal{J}_1(\eta)$$
$$\pm k\mathcal{E}(\tau)\varepsilon_0(\tau)^2\frac{d}{d\tau}(\varepsilon_0(\tau)^{2\theta})\left(\frac{\eta}{2}\mathcal{J}_2'(\eta)+\beta\mathcal{J}_2(\eta)\right)$$
$$\mp k\varepsilon_0(\tau)^2\frac{d}{d\tau}\left(\varepsilon_0(\tau)^2\frac{d}{d\tau}\left(\varepsilon_0(\tau)^{2\theta}\right)\right)\mathcal{J}_2(\eta).$$

As is readily seen, there is a constant $\widetilde{C}_k > 0$ such that

$$|\mathcal{R}(\eta,\tau)| \le \widetilde{C}_k\varepsilon_0(\tau)^{4+2\theta}(1+\eta)^{-\gamma+2}.$$

This is smaller than

$$\big(\mathcal{E}_\pm(\tau)-\mathcal{E}(\tau)\big)T_1(\eta) = \mp\sqrt{k}\varepsilon_0(\tau)^{2+2\theta}T_1(\eta)$$

in its modulus. Due to (1.5.8), we have

$$\begin{aligned}&\left|-\mathcal{E}(\tau)\mathcal{E}_\pm(\tau)\left(\frac{\eta}{2}H'(\eta)+\beta H(\eta)\right)-\varepsilon_0(\tau)^2\frac{d}{d\tau}(\mathcal{E}_\pm(\tau))H(\eta)\right|\\&\le C\varepsilon_0(\tau)^4(1+\eta)^{-\gamma+2}\le C\varepsilon(\tau)^{2+2\theta}(1+\eta)^{-\gamma}\end{aligned}\tag{1.5.13}$$

for $\eta \le \varepsilon_0(\tau)^{\theta-1}$. We now choose k large enough, so that the last quantity of (1.5.13) is dominated by $\big(\mathcal{E}_\pm(\tau)-\mathcal{E}(\tau)\big)T_1(\eta)$ as well.

Consider the case where the plus sign of $\mathcal{Z}_\pm$ is selected. Since $T_1(\eta)$ is positive, it follows from (1.5.6), (1.5.9b), (1.5.12), and (1.5.13) that

$$\begin{aligned}\mathcal{N}\mathcal{Z}_+ \le \eta^{2a}\Big[&\big(U_1(\eta)+\mathcal{Z}_+\big)^p - U_1(\eta)^p - pU_1(\eta)^{p-1}\mathcal{Z}_+\\&+p\big(U_1(\eta)^{p-1}-U_2(\eta)^{p-1}\big)\mathcal{Z}_+\Big] - \frac{1}{3}\varepsilon_0(\tau)^{2+2\theta}T_1(\eta)\end{aligned}$$

holds for $\eta \le \varepsilon_0(\tau)^{\theta-1}$. Moreover, it is easily seen that this last term dominates for $1 \ll \eta \le \varepsilon_0(\tau)^{\theta-1}$, whereas the negative term $p\big(U_1(\eta)^{p-1}-U_2(\eta)^{p-1}\big)\mathcal{Z}_+$ dominates for $\eta = O(1)$. Therefore the function $\mathcal{Z}_+$ is a supersolution. The case where the negative sign of $\mathcal{Z}_\pm$ is selected is similar. In this case the both $\big(U_1(\eta)+\mathcal{Z}_-\big)^p - U_1(\eta)^p - pU_1(\eta)^{p-1}\mathcal{Z}_-$ and $p\big(U_1(\eta)^{p-1}-U_2(\eta)^{p-1}\big)\mathcal{Z}_-$ are positive. Consequently, the function $\mathcal{Z}_-$ is a subsolution.

Next, we verify

$$\mathcal{Z}_-(\eta,\tau) \le U(\eta,\tau)-U_1(\eta) \le \mathcal{Z}_+(\eta,\tau) \quad \text{for} \quad \eta=\varepsilon(\tau)^{\theta-1},\ \tau_0\le\tau\le\tau_1. \tag{1.5.14}$$

To this end, we recall

$$\Phi_{\mathrm{med}}(r,\tau) = U_\infty(r)+a_\ell^*(\tau)\phi_\ell(r) \quad \text{and} \quad a_\ell^*(\tau) = -\frac{h}{c_\ell}\varepsilon(\tau)^{\gamma-2\beta}.$$

Due to (1.2.8a), we obtain, as $r = \varepsilon_0(\tau)^\theta \to 0$,

$$\begin{aligned}&\Phi_{\rm med}(r,\tau) - \Phi_{\rm inn}(r,\tau)\\ &= C_2\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma+2}\big(1+O(r^2)\big) + \varepsilon(\tau)^{\gamma-2\beta+\min\{2|\lambda_0|,\sqrt{D}\}}O\big(r^{-\gamma-\min\{2|\lambda_0|,\sqrt{D}\}}\big),\end{aligned} \tag{1.5.15}$$

where $C_2 := h\ell/2(2+\sqrt{D}) = C_1(1+2\omega_\ell)$ (cf. (1.5.8)). Combining (1.5.2) with (1.5.15), we get

$$\big|\Phi(r,\tau) - \Phi_{\rm inn}(r,\tau) - C_2\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma+2}\big| \le 2K\varepsilon_0(\tau)^{\gamma-2\beta+2\theta}r^{-\gamma} \tag{1.5.16}$$

for $r = \varepsilon_0(\tau)^\theta$ and $\tau_0 \le \tau \le \tau_1$. Rewriting this estimate by the inner variables, we obtain

$$\big|U(\eta,\tau) - U_1(\eta) - \mathcal{E}(\tau)H(\eta)\big| \le 3K\varepsilon_0(\tau)^{2\theta}\eta^{-\gamma}.$$

It then follows that

$$\big|U(\eta,\tau) - U_1(\eta)\big| \le \mathcal{E}(\tau)H(\eta) + 3K\varepsilon_0(\tau)^{2\theta}\eta^{-\gamma} \le \mathcal{Z}_\pm(\eta,\tau) + K\varepsilon(\tau)^{2\theta}H_0(\xi).$$

We thus obtain (1.5.14) with $M_1 = 5M_0C_0^{-\gamma}$ if τ_0 is large enough (cf. (1.5.7)). We finally verify if the bound corresponding to (1.5.14) at $\tau = \tau_0$ is true for $|\xi| \le \varepsilon(\tau_0)^{\theta-1}$, which amounts to asking if there holds

$$\left|\Phi_0(r) - \Phi_{\rm inn}(r,\tau_0) - \varepsilon(\tau_0)^{-2\beta}\mu(\tau_0)H_1\left(\frac{r}{\varepsilon(\tau_0)}\right)\right| \le M_1\varepsilon(\tau_0)^{-2\beta+2\theta}H_0\left(\frac{r}{\varepsilon(\tau_0)}\right) \tag{1.5.17}$$

for $r \le \varepsilon(\tau_0)^\theta$. This is clearly satisfied for $r \le \varepsilon(\tau_0)^{2\theta}$, since $\Phi_0(r) = \Phi_{\rm inn}(r,\tau_0)$ there and $\varepsilon(\tau_0)^{2(1-\theta)}H_1(\xi) \ll H_0(\xi)$ with $\xi = r/\varepsilon(\tau_0) \le 2\varepsilon(\tau_0)^{2\theta-1}$. As for the region $\{\varepsilon(\tau_0)^{2\theta} < r \le \varepsilon(\tau_0)^\theta\}$, an estimate similar to (1.5.15) with $\tau = \tau_0$ implies

$$\left|\Phi_{\rm out}(r,\tau_0) - \Phi_{\rm inn}(r,\tau_0) - \widetilde{C_1}\varepsilon(\tau_0)^{\gamma-2\beta}r^{-\gamma+2}\right| \le C\varepsilon(\tau_0)^{\gamma-2\beta+4\theta}r^{-\gamma}.$$

Combining this with the assumption (1.5.1), we readily obtain (1.5.17). Comparison principle completes the proof. □

1.6 Completion of the key a priori estimate

We now prove Lemma 1.3.1. Due to Lemmas 1.4.3 and 1.5.1 below, it suffices to show

$$|W(r,\tau) - a_\ell^*(\tau)\psi_\ell(r)| < \frac{1}{2}\varepsilon_0(\tau)^{\gamma-2\beta+2\theta}, \qquad \varepsilon(\tau)^\theta \le r \le 1, \tag{1.6.1a}$$

$$|W(r,\tau) - a_\ell^*(\tau)\psi_\ell(r)| < \frac{1}{2}\varepsilon_0(\tau_0)^{2\theta}\varepsilon_0(\tau)^{\gamma-2\beta}r^{2\ell}, \quad 1 \le r \le e^{-\omega\tau_0}e^{\tau/2}, \tag{1.6.1b}$$

$$|W(r,\tau)| < \frac{1}{2}e^{-\Theta\tau_0}r^{\gamma-2\beta}, \qquad r > e^{-\omega\tau_0}e^{\tau/2}, \tag{1.6.1c}$$

as long as $\tau_0 \le \tau \le \tau_1$ with τ_0 large enough. When $\tau_0 \le \tau_1 \le \tau_0 + 1$, the estimates (1.6.1a) and (1.6.1b) follow from Lemma 1.4.7. When $\tau_0 + 1 < \tau_1$, we obtain (1.6.1a) and (1.6.1b) from Lemma 1.4.12 as well as the estimate (1.4.11) of Fourier coefficients. We will prove the estimate (1.6.1c) in Lemma 1.6.1 below. Then, the proof of Lemma 1.3.1 is now complete.

1.6.1 A priori estimates in the outer region

Lemma 1.6.1. *There exists a constant* $\widetilde{\Theta} > 0$ *such that*

$$|W(y,\tau)| \le Ce^{-\widetilde{\Theta}\tau_0}|y|^{2|\lambda_0|} \tag{1.6.2}$$

for $|y| > e^{-\widetilde{\omega}\tau_0}e^{\tau/2}$ *and* $\tau_0 \le \tau \le \tau_1$ *with sufficiently large* τ_0.

Proof. Similarly to (1.3.10), there exists a positive constant C such that

$$|W_0(r)| \le \begin{cases} Cr^{2|\lambda_0|} & r \le \varepsilon_0(\tau_0)^{\widetilde{\theta}}, \\ Ce^{-2\widetilde{\omega}\lambda_\ell\tau_0}r^{2|\lambda_0|} & \varepsilon_0(\tau_0)^{\widetilde{\theta}} < r \le e^{(1/2-\widetilde{\omega})\tau_0}, \\ Cr^{2|\lambda_0|-\kappa}, & e^{(1/2-\widetilde{\omega})\tau_0} < r, \end{cases}$$

We define functions W_1 and W_2 as

$$W(y,\tau;\alpha) = [S(\tau-\tau_0)W_0](y) + \int_{\tau_0}^{\tau}[S(\tau-s)g(W(s))](y)\,ds =: W_1 + W_2. \tag{1.6.3}$$

The estimate (1.4.20) implies that

$$W_1 \le \frac{Ce^{|\lambda_0|(\tau-\tau_0)}}{(1-e^{-(\tau-\tau_0)})^{m/2}}\int_0^\infty \exp\left\{-\frac{\left||y|e^{-(\tau-\tau_0)/2}-r\right|^2}{4(1-e^{-(\tau-\tau_0)})}\right\}|W_0(r)|\mathcal{B}_y(r,\tau-\tau_0)r^{m-1}\,dr.$$

We split the integral region as $\mathbf{R}_+ = D_1(\tau_0)\cup D_2(\tau_0)\cup D_3(\tau_0)$ and $W_1 = W_{1,1}+W_{1,2}+W_{1,3}$, accordingly, where D_i is as in (1.4.31).

Estimate for $W_{1,1}$. We remark that

$$\left||y|e^{-(\tau-\tau_0)/2}-r\right| \ge \frac{1}{2}|y|e^{-(\tau-\tau_0)/2} + \frac{1}{2}e^{(1/2-\omega)\tau_0} - \varepsilon_0(\tau_0)^{\widetilde{\theta}} \ge \frac{1}{2}|y|e^{-(\tau-\tau_0)/2}$$

for $|y| > e^{-\omega\tau_0}e^{\tau/2}$ and $r \in D_1(\tau_0)$. Similar argument to (1.4.34), we have

$$\begin{aligned} W_{1,1} &\le Ce^{(|\lambda_0|+m/2)(\tau-\tau_0)}|y|^{-2|\lambda_0|-m}|y|^{2|\lambda_0|}\int_{D_1(\tau_0)} r^{2|\lambda_0|+m-1}\,dr \\ &\le Ce^{-(1/2-\omega)(2|\lambda_0|+m)\tau_0}\varepsilon_0(\tau_0)^{(1-\theta)(2|\lambda_0|+m)}|y|^{2|\lambda_0|}. \end{aligned}$$

Estimate for $W_{1,2}$. We split $D_2(\tau_0) = D_{2,1}(\tau_0)\cup D_{2,2}(\tau_0)$ and $W_{1,2} = W_{1,2,1}+W_{1,2,2}$, accordingly, where $D_{2,j}$ is as in (1.4.36). It then follows from (1.4.25) that

$$\begin{aligned} W_{1,2,1} &\le C\mathcal{S}e^{-2\widetilde{\omega}\lambda_\ell\tau_0}e^{2|\lambda_0|(1/2-\widetilde{\omega})\tau_0}e^{|\lambda_0|(\tau-\tau_0)}|y|^{-2|\lambda_0|}|y|^{2|\lambda_0|} \\ &\le Ce^{2|\lambda_0|(\omega-\widetilde{\omega})}e^{-2\widetilde{\omega}\lambda_\ell\tau_0}|y|^{2|\lambda_0|} \le Ce^{-\widetilde{\omega}\lambda_\ell\tau_0}|y|^{2|\lambda_0|} \end{aligned}$$

for sufficiently small $\widetilde{\omega} \in (0,\omega)$. To use (1.4.27), we obtain

$$W_{1,2,1} \le C\mathcal{S}_{2|\lambda_0|}e^{-2\widetilde{\omega}\lambda_\ell\tau_0}e^{|\lambda_0|(\tau-\tau_0)}|y|^{-2|\lambda_0|}|y|^{2|\lambda_0|} \le Ce^{-(1-2\omega)|\lambda_0|\tau_0}e^{-2\widetilde{\omega}\lambda_\ell\tau_0}|y|^{2|\lambda_0|}.$$

Estimate for $W_{1,3}$. We Set

$$D_{3,1}(y,\tau_0) := D_3(\tau_0) \cap B_1(y,\tau_0) \quad \text{and} \quad D_{3,2}(y,\tau_0) := D_3(\tau_0) \cap B_2(y,\tau_0),$$

and split $W_{1,3} = W_{1,3,1} + W_{1,3,2}$, accordingly, where B_i is as in (1.4.22). Due to (1.4.25), we see that

$$W_{1,3,1} \leq C\mathcal{S}e^{|\lambda_0|(\tau-\tau_0)}(2|y|e^{-(\tau-\tau_0)/2})^{2|\lambda_0|}e^{-\kappa(1/2-\widetilde{\omega})\tau_0} \leq Ce^{-\kappa(1/2-\widetilde{\omega})\tau_0}|y|^{2|\lambda_0|}.$$

In additional, (1.4.27) implies that

$$W_{1,3,2} \leq C\mathcal{S}_{2|\lambda_0|}e^{|\lambda_0|(\tau-\tau_0)}e^{-\kappa(1/2-\widetilde{\omega})\tau_0}|y|^{-2|\lambda_0|}|y|^{2|\lambda_0|} \leq Ce^{-(|\lambda_0|+\kappa)(1/2-\widetilde{\omega})\tau_0}|y|^{2|\lambda_0|}.$$

It follows from similar argument to Lemmas 1.4.6 and 1.4.11 that $|W_2| \leq C\varepsilon_0(\tau_0)^{\mu}|y|^{2|\lambda_0|}$ for $|y| > e^{-\widetilde{\omega}\tau_0}e^{\tau/2}$. The proof is complete. $\square$

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Chapter 2

Large time behavior of solutions of the heat equationwith inverse square potential

2.1 Introduction and main results

Let $L := -\Delta + V$ be a nonnegative Schrödinger operator on $L^2(\mathbf{R}^N)$, where $N \geq 2$ and V is a radially symmetric inverse square potential, that is

$$\begin{aligned} V(r) &= \lambda_1 r^{-2} + o\big(r^{-2+\theta}\big) \quad \text{as} \quad r \to 0, \\ V(r) &= \lambda_2 r^{-2} + o\big(r^{-2-\theta}\big) \quad \text{as} \quad r \to \infty, \end{aligned}$$

for some $\lambda_1, \lambda_2 \in [\lambda_*, \infty)$ with $\lambda_* := -(N-2)^2/4$ and $\theta > 0$. We are interested in the precise description of the large time behavior of $u = e^{-tL}\varphi$, which is a solution of

$$\begin{cases} u_t = \Delta u - V(|x|)u & \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \\ u(x,0) = \varphi(x) & \text{in} \quad \mathbf{R}^N \end{cases} \tag{2.1.1}$$

Nonnegative Schrödinger operators and their heat semigroups appear in various fields and have been studied intensively by many authors since the pioneering work due to Simon [40] (see e.g., [2], [4], [6], [8], [11], [12], [16]–[22], [26]–[30], [32], [33], [35]–[44] and references therein). See also the monographs of Davies [7], Grigor'yan [9] and Ouhabaz [34]. The inverse square potential is a typical one appearing in the study of the Schrödinger operators and it arises in the linearized analysis for nonlinear diffusion equations and in the asymptotic analysis for diffusion equations.

Throughout this Chapter we assume the following condition on the potential V :

$$
\begin{aligned}
&(\mathrm{i}) \quad V = V(r) \in C^1(\mathbf{R}_+)\,; \\
&(\mathrm{ii}) \quad \lim_{r\to 0} r^{-\theta}\big|r^2V(r)-\lambda_1\big| = 0 \quad \text{and} \quad \lim_{r\to\infty} r^{\theta}\big|r^2V(r)-\lambda_2\big| = 0, \\
&\qquad\ \text{for some } \lambda_1,\lambda_2\in[\lambda_*,\infty) \text{ with } \lambda_* := -\frac{(N-2)^2}{4} \text{ and } \theta>0\,; \\
&(\mathrm{iii}) \quad \sup_{r\geq 1}\big|r^3V'(r)\big| < \infty.
\end{aligned}
\tag{V}
$$

We say that $L := -\Delta + V(|x|)$ is nonnegative on $L^2(\mathbf{R}^N)$ if

$$
\int_{\mathbf{R}^N} \big[|\nabla\phi|^2 + V(|x|)\phi^2\big]\,dx \geq 0, \quad \phi\in C_0^\infty\big(\mathbf{R}^N\setminus\{0\}\big).
$$

When L is nonnegative, we say that

- L is subcritical if, for any $W\in C_0(\mathbf{R}^N)$, $L-\varepsilon W$ is nonnegative for all sufficiently small $\varepsilon>0$:
- L is critical if L is not subcritical.

On the other hand, L is said to be supercritical if L is not nonnegative.

Consider the ordinary differential equation

$$
U'' + \frac{N-1}{r}U' - V(r)U = 0 \quad \text{in} \quad \mathbf{R}_+ \tag{O}
$$

under condition (V). Equation (O) has two linearly independent solutions U (a *regular* solution) and $\widetilde{U}$ (a *singular* solution) such that

$$
U(r)\sim r^{A^+(\lambda_1)} \quad \text{and} \quad \widetilde{U}(r)\sim \begin{cases} r^{A^-(\lambda_1)} & \text{if} \quad \lambda_1>\lambda_*, \\ r^{-(N-2)/2}|\log r| & \text{if} \quad \lambda_1=\lambda_*, \end{cases} \tag{2.1.2}
$$

as $r\to 0$, where

$$
A^\pm(\lambda) := \frac{-(N-2)\pm\sqrt{(N-2)^2+4\lambda}}{2} \quad \text{for} \quad \lambda\geq\lambda_*. \tag{2.1.3}
$$

In particular, $U\in L^2_{\mathrm{loc}}(\mathbf{R}^N)$. Assume that L is nonnegative on $L^2(\mathbf{R}^N)$. Then it follows from [22, Theorem 1.1] that U is positive in $\mathbf{R}_+$ and

$$
U(r)\sim c_*v(r) \quad \text{as} \quad r\to\infty \tag{2.1.4}
$$

for some positive constant c_*, where

$$
v(r) := \begin{cases} r^{A^+(\lambda_2)} & \text{if} \quad L \text{ is subcritical and } \lambda_2>\lambda_*, \\ r^{-(N-2)2}\log r & \text{if} \quad L \text{ is subcritical and } \lambda_2=\lambda_*, \\ r^{A^-(\lambda_2)} & \text{if} \quad L \text{ is critical.} \end{cases} \tag{2.1.5}
$$

(See also [33] for the case $\lambda_1 = 0$.) We often call U a positive harmonic function for the operator L. When L is critical, following [38], we say that L is positive-critical if $U \in L^2(\mathbf{R}^N)$ and that L is null-critical if $U \notin L^2(\mathbf{R}^N)$. Generally, the behavior of the fundamental solution $p = p(x, y, t)$ corresponding to e^{-tL} can be classified by whether L is either subcritical, null-critical or positive-critical. Indeed, in the case of $\lambda_1 = 0$, by [38, Theorem 1.2], we have:

(L1) If L is subcritical, then

$$\lim_{t\to\infty} p(x,y,t) = 0 \quad \text{and} \quad \int_0^\infty p(x,y,t)\,dt < \infty,$$

for $x, y \in \mathbf{R}^N$ with $x \neq y$;

(L2) If L is null-critical, that is $A^-(\lambda_2) \geq -N/2$, then

$$\lim_{t\to\infty} p(x,y,t) = 0 \quad \text{and} \quad \int_0^\infty p(x,y,t)\,dt = \infty,$$

for $x, y \in \mathbf{R}^N$ with $x \neq y$;

(L3) If L is positive-critical, that is $A^-(\lambda_2) < -N/2$, then

$$\lim_{t\to\infty} p(x,y,t) = \frac{U(|x|)U(|y|)}{\|U\|^2_{L^2(\mathbf{R}^N)}},$$

for $x, y \in \mathbf{R}^N$.

See Corollary 2.1.5 for (L1) and (L2) in the case of $\lambda_1 \neq 0$.

On the other hand, under condition (V), Ishige, Kabeya, and Ouhabaz recently studied in [22] the Gaussian estimate of the fundamental solution $p = p(x, y, t)$ in the subcritical case and in the critical case with $A^-(\lambda_2) > -N/2$. They proved that

$$0 < p(x,y,t) \leq Ct^{-N/2} \frac{U\big(\min\{|x|,\sqrt{t}\}\big)U\big(\min\{|y|,\sqrt{t}\}\big)}{U(\sqrt{t})^2} \exp\left\{-\frac{|x-y|^2}{Ct}\right\} \tag{2.1.6}$$

holds for all $x, y \in \mathbf{R}^N$ and $t > 0$, where C is a positive constant (see [22, Theorem 1.3]). For related results, see e.g., [2], [6], [10], [26], [28], [29], [30], [43], [44] and references therein.

The precise description of the large time behavior of $e^{-tL}\varphi$ with $\varphi \in L^2(\mathbf{R}^N, e^{|x|^2/4}\,dx)$ has been studied in a series of papers [16]–[19] only in the subcritical case with some additional restrictions such as $V \in C^1([0,\infty))$, $\lambda_2 > \lambda_*$ and the sign of the potential. See also [20].

The purpose of this Chapter is to establish a method for obtaining the precise description of the large time behavior of $e^{-tL}\varphi$ with $\varphi \in L^2(\mathbf{R}^N, e^{|x|^2/4}\,dx)$ in the subcritical case and in the null-critical case with $A^-(\lambda_2) > -N/2$, under condition (V). In particular, we show that the solution u of (2.1.1) behaves as a suitable multiple of

$$\begin{cases} v_{\text{reg}}(x,t) & \text{if} \quad L \text{ is subcritical and } \lambda > \lambda_*, \\ \dfrac{v_{\text{reg}}(x,t)}{\log(1+t)} & \text{if} \quad L \text{ is subcritical and } \lambda = \lambda_*, \\ v_{\text{sing}}(x,t) & \text{if} \quad L \text{ is critical and } A^-(\lambda_2) > -\dfrac{N}{2}, \end{cases}$$

as $t \to \infty$ on all parabolic cones $\{x \in \mathbf{R}^N;\, \sqrt{t}/R \le |x| \le R\sqrt{t}\}$ with $R > 1$. (See Theorem 2.1.4.) Here

$$v_{\rm reg}(x,t) := t^{-N/2-A^+(\lambda_2)}|x|^{A^+(\lambda_2)}\exp\left\{-\frac{|x|^2}{4t}\right\},$$
$$v_{\rm sing}(x,t) := t^{-N/2-A^-(\lambda_2)}|x|^{A^-(\lambda_2)}\exp\left\{-\frac{|x|^2}{4t}\right\},$$

which are self-similar solutions of

$$v_t = \Delta v - \lambda_2|x|^{-2}v \quad \text{in} \quad \left[\mathbf{R}^N\setminus\{0\}\right]\times\mathbf{R}_+.$$

However, due to the fact that $v_{\rm sing}(t) \notin H^1(\mathbf{R}^N)$ for any $t > 0$, the arguments in [16]–[19] are not applicable to the critical case. In this Chapter we study the large time behavior of the function $|x|^{-A}e^{-tL}\varphi$, instead of $e^{-tL}\varphi$, with

$$A := A^+(\lambda_2) \text{ if } L \text{ is subcritical} \quad \text{and} \quad A := A^-(\lambda_2) \text{ if } L \text{ is critical,} \tag{2.1.7}$$

and overcome the difficulty arising from the fact that $v_{\rm sing}(t) \notin H^1(\mathbf{R}^N)$.

2.1.1 Radial solutions

In this subsection we focus on radially symmetric solutions of (2.1.1). Divide the operator L into the following three cases:

(S) : L is subcritical and $\lambda_2 > \lambda_*$; (S$_*$) : L is subcritical and $\lambda_2 = \lambda_*$;
(C) : L is critical and $A^-(\lambda_2) > -N/2$.

Set

$$\begin{aligned} &d := N + 2A, \qquad \rho_d(\xi) := \xi^{d-1}e^{\xi^2/4}, \qquad \psi_d(\xi) := c_d e^{-\frac{\xi^2}{4}},\\ &\left|\mathbf{S}^{d-1}\right| := \frac{2\pi^{d/2}}{\Gamma(d/2)}, \qquad c_d = \sqrt{\frac{\left|\mathbf{S}^{d-1}\right|}{2^d\pi^{d/2}}} = \sqrt{\frac{1}{2^{d-1}\Gamma(d/2)}} \end{aligned} \tag{2.1.8}$$

where Γ is the Gamma function. Then $\|\psi_d\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} = 1$. If d is an integer such that $d \ge 2$, then $\left|\mathbf{S}^{d-1}\right|$ coincides with the volume of $(d-1)$-dimensional unit sphere.

Let φ be radially symmetric and $\varphi \in L^2(\mathbf{R}^N, e^{|x|^2/4}\,dx)$. Then $e^{-tL}\varphi$ is radially symmetric with respect to x and set

$$u(|x|,t) = \left[e^{-tL}\varphi\right](x) \quad \text{and} \quad v(|x|,t) := |x|^{-A}u(|x|,t),$$

for $x \in \mathbf{R}^N$ and $t > 0$. Then v satisfies the Cauchy problem for a *d-dimensional* parabolic equation

$$\begin{cases} v_t = \dfrac{1}{r^{d-1}}\partial_r\left(r^{d-1}\partial_r v\right) - V_{\lambda_2}(r)v & \text{in} \quad \mathbf{R}_+\times\mathbf{R}_+,\\ v(r,0) = r^{-A}\varphi(r) & \text{in} \quad \mathbf{R}_+, \end{cases}$$

where $V_{\lambda_2}(r) := V(r) - \lambda_2 r^{-2}$.

In the first and the second theorems we obtain the precise description of the large time behavior of the radially symmetric solutions of (2.1.1) in either (S) or (C).

Theorem 2.1.1. *Let* $N \geq 2$ *and assume condition* (V). *Let* L *satisfy either* (S) *or* (C). *Let* $u = u(|x|, t)$ *be a radially symmetric solution of* (2.1.1) *such that* $\varphi \in L^2(\mathbf{R}^N, e^{|x|^2/4}\,dx)$. *Define* $w = w(\xi, s)$ *by*

$$\begin{aligned} & w(\xi, s) := (1+t)^{d/2} r^{-A} u(r, t) \\ & \text{with} \quad \xi = \frac{r}{\sqrt{1+t}} \geq 0 \quad \text{and} \quad s = \log(1+t) \geq 0. \end{aligned} \tag{2.1.9}$$

Then there exists a positive constant C *such that*

$$\sup_{s>0} \|w(s)\|_{L^2(\mathbf{R}_+,\, \rho_d\, d\xi)} \leq C \|w(0)\|_{L^2(\mathbf{R}_+,\, \rho_d\, d\xi)}.$$

Furthermore,

$$\lim_{s\to\infty} w(\xi, s) = m(\varphi)\psi_d(\xi) \quad \text{in} \quad L^2(\mathbf{R}_+,\, \rho_d\, d\xi) \cap C^2(K) \tag{2.1.10}$$

for any compact set K *in* $\mathbf{R}^N \setminus \{0\}$, *where*

$$m(\varphi) := \frac{c_d}{c_*} \int_0^\infty \varphi(r) U(r) r^{N-1}\, dr. \tag{2.1.11}$$

In particular, if $m(\varphi) = 0$, *then*

$$\|w(s)\|_{L^2(\mathbf{R}_+,\, \rho_d\, d\xi)} + \|w(s)\|_{C^2(K)} = O(e^{-s}) \quad \text{as} \quad s \to \infty. \tag{2.1.12}$$

Theorem 2.1.2. *Assume the same conditions as in Theorem* 2.1.1. *Set* $u_*(r, t) := u(r, t)/U(r)$.

(a) *For any* $j \in \{0, 1, \dots\}$, $\partial_t^j u_* \in C\big([0, \infty) \times \mathbf{R}_+\big)$.

(b) $\displaystyle\lim_{t\to\infty} t^{d/2} u_*(0, t) = \frac{c_d}{c_*} m(\varphi)$ and $\displaystyle\lim_{t\to\infty} t^{d/2+1} (\partial_t u_*)(0, t) = -\frac{d c_d}{2 c_*} m(\varphi)$.

(c) *Let* $T > 0$ *and* ε *be a sufficiently small positive constant. Define*

$$G_d(r, t) := u_*(r, t) - \big[u_*(0, t) + (\partial_t u_*)(0, t) F_d(r)\big] \tag{2.1.13}$$

for $r \geq 0$ *and* $t > 0$ *with*

$$U_d(s) := r^{-A} U(r) \quad \text{and} \quad F_d(r) := \int_0^r \frac{1}{U_d(s)^2 s^{d-1}} \int_0^s U_d(\tau)^2 \tau^{d-1}\, d\tau ds.$$

Then there exists a positive constant C *such that*

$$\big|(\partial_r^\ell G_d)(r, t)\big| \leq C t^{-d/2-2} r^{4-\ell} \|\varphi\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\, dx)} \tag{2.1.14}$$

for $\ell \in \{0, 1, 2\}$, $0 \leq r \leq \varepsilon\sqrt{1+t}$, *and* $t \geq T$.

In case (S_*) we have:

Theorem 2.1.3. *Let $N \geq 2$ and assume condition* (V). *Let L satisfy* (S_*). *Let $u = u(|x|, t)$ be a radially symmetric solution of* (2.1.1) *such that $\varphi \in L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)$.*

(I) *Let w be as in Theorem* 2.1.1 *and K a compact set in $\mathbf{R}^N \setminus \{0\}$. Then there exists a positive constant C_1 such that*

$$\sup_{s>0}\,(1+s)\|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_2\,d\xi)} \leq C_1\|w(0)\|_{L^2(\mathbf{R}_+,\,\rho_2\,d\xi)}.$$

Furthermore,

$$\lim_{s\to\infty} sw(\xi, s) = 2m(\varphi)\psi_2(\xi) \quad \text{in} \quad L^2(\mathbf{R}_+,\, \rho_2\,d\xi) \cap C^2(K),$$

where $m(\varphi)$ is as in (2.1.11).

(II) *Let u_*, U_2, F_2 and G_2 be as in Theorem* 2.1.2 *with $d = 2$. Then*

$$\partial_t^j u_* \in C\big([0,\infty)\times\mathbf{R}_+\big) \quad \text{for} \quad j \in \{0, 1, \dots\},$$

$$\lim_{t\to\infty} t(\log t)^2 u_*(0,t) = \frac{2\sqrt{2}}{c_*}m(\varphi) \quad \text{and} \quad \lim_{t\to\infty} t^2(\log t)^2(\partial_t u_*)(0,t) = -\frac{2\sqrt{2}}{c_*}m(\varphi).$$

Furthermore, for any $T > 0$ and any sufficiently small $\varepsilon > 0$, there exists a positive constant C_2 such that

$$\big|(\partial_r^\ell G_2)(r,t)\big| \leq C_2 t^{-3}\big(\log(1+t)\big)^{-2} r^{4-\ell}\|\varphi\|_{L^2(\mathbf{R}^N,\,e^{|x|^2/4}\,dx)} \tag{2.1.15}$$

for $\ell \in \{0,1,2\}$, $0 \leq r \leq \varepsilon\sqrt{1+t}$, and $t \geq T$.

The function w defined by (2.1.9) satisfies

$$w_s = -\mathcal{L}_d w - \widetilde{V}(\xi, s)w \quad \text{for} \quad \xi \in [0,\infty),\ s > 0, \tag{2.1.16}$$

where

$$\mathcal{L}_d w := -\frac{1}{\rho_d(\xi)}\partial_\xi\big(\rho_d(\xi)\partial_\xi w\big) - \frac{d}{2}w \quad \text{and} \quad \widetilde{V}(\xi, s) := e^s V_{\lambda_2}(e^{s/2}\xi).$$

For the proofs of Theorems 2.1.1–2.1.3, we regard the operator $\mathcal{L}_d$ as a *d-dimensional* elliptic operator with

$$\begin{cases} d > 2 & \text{in the case of (S)}, \\ d = 2 & \text{in the case of } \lambda_2 = \lambda_*, \\ 0 < d < 2 & \text{in the case of (C) with } \lambda_2 > \lambda_*, \end{cases}$$

and study the large time behavior of $w = w(\xi, s)$ by developing the arguments in a series of papers [13]–[19]. The function ψ_d defined by (2.1.8) is the first eigenfunction of the eigenvalue problem

$$\mathcal{L}_d\phi = \mu\phi \quad \text{in} \quad \mathbf{R}_+, \quad \phi \in H^1(\mathbf{R}_+,\, \rho_d(\xi)\,d\xi) \tag{E}$$

and the corresponding eigenvalue is 0 (see Lemma 2.2.7). We show that w behaves like a suitable multiple of ψ_d as $s \to \infty$. Furthermore, combining the radially symmetry of u with the behavior of w, we prove Theorems 2.1.1–2.1.3.

The eigenfunction ψ_d corresponds to $v_{\rm reg}$ in the subcritical case and $v_{\rm sing}$ in the null-critical case, respectively. In the null-critical case, $v_{\rm reg}$ is transformed by (2.1.9) into

$$e^{-(A^+(\lambda_2)-A^-(\lambda_2))s/2}\widetilde{\psi}_d \quad \text{with} \quad \widetilde{\psi}_d := \xi^{A^+(\lambda_2)-A^-(\lambda_2)}e^{-\xi^2/4}.$$

Here $\widetilde{\psi}_d$ is the first eigenfunction of the eigenvalue problem

$$\mathcal{L}_d\phi = \mu\phi \quad \text{in} \quad \mathbf{R}_+, \quad \phi \in H_0^1(\mathbf{R}_+,\, \rho_d(\xi)\, d\xi)$$

and the corresponding eigenvalue is $\big(A^+(\lambda_2) - A^-(\lambda_2)\big)/2 > 0$. In the null-critical case with $\lambda_2 > \lambda_*$, we see that $0 < d < 2$ and $H_0^1(\mathbf{R}_+,\, \rho_d(\xi)\, d\xi) \neq H^1(\mathbf{R}_+,\, \rho_d(\xi)\, d\xi)$. This justifies that the operator $\mathcal{L}_d$ has two positive eigenfunctions ψ_d and $\widetilde{\psi}_d$.

The case of $d = 0$ is on borderline where L is null-critical and it is not treated in this Chapter. Indeed, it seems difficult to apply the arguments of this Chapter to the case of $d = 0$ since $\rho_d(\xi) \sim \xi^{-1}$ as $\xi \to 0$ and $\rho_d \notin L^1(\mathbf{R}_+)$.

2.1.2 Nonradial solutions

We discuss the large time behavior of solutions of (2.1.1) without the radially symmetry of the solutions.

Let $\Delta_{\mathbf{S}^{N-1}}$ be the Laplace-Beltrami operator on $\mathbf{S}^{N-1}$. Let $\{\omega_k\}_{k=0}^{\infty}$ be the eigenvalues of

$$-\Delta_{\mathbf{S}^{N-1}}Q = \omega Q \quad \text{on} \quad \mathbf{S}^{N-1}, \quad Q \in L^2(\mathbf{S}^{N-1}).$$

Then $\omega_k = k(N + k - 2)$ for $k = 0, 1, \ldots.$ Let $\{Q_{k,i}\}_{i=1}^{\ell_k}$ and ℓ_k be the orthonormal system and the dimension of the eigenspace corresponding to ω_k, respectively. In particular, $\ell_0 = 1$, $\ell_1 = N$, and

$$Q_{0,1} = q_* := \left|\mathbf{S}^{N-1}\right|^{-1/2}, \quad Q_{1,i} = q_N\frac{x_i}{|x|} \quad \text{with} \quad q_N := \sqrt{N}q_*. \tag{2.1.17}$$

For any $\varphi \in L^2(\mathbf{R}^N,\, e^{|x|^2/4}\, dx)$, we can find radially symmetric functions $\{\phi^{k,i}\} \subset L^2(\mathbf{R}^N,\, e^{|x|^2/4}\, dx)$ such that

$$\varphi = \sum_{k=0}^{\infty}\sum_{i=1}^{\ell_k}\varphi^{k,i} \quad \text{in} \quad L^2(\mathbf{R}^N,\, e^{|x|^2/4}\, dx), \quad \varphi^{k,i}(x) := \phi^{k,i}(|x|)Q_{k,i}\left(\frac{x}{|x|}\right)$$

(see [14] and [16]). Define $L_k := -\Delta + V_k(|x|)$ and $V_k(r) := V(r) + \omega_k r^{-2}$. Then

$$\begin{aligned} \left[e^{-tL}\varphi^{k,i}\right](x) &= \left[e^{-tL_k}\phi^{k,i}\right](x)Q_{k,i}\left(\frac{x}{|x|}\right), \\ \left[e^{-tL}\varphi\right](x) &= \sum_{k=0}^{\infty}\sum_{i=1}^{\ell_k}\left[e^{-tL_k}\phi^{k,i}\right](x)Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in } L^2(\mathbf{R}^N) \text{ for any } t > 0. \end{aligned} \tag{2.1.18}$$

Therefore the behavior of $e^{-tL}\varphi$ is described by a series of the radially symmetric solutions $e^{-tL_k}\phi^{k,i}$. Furthermore, V_k satisfies condition (V) with λ_1 and λ_2 replaced by $\lambda_1+\omega_k$ and $\lambda_2+\omega_k$, respectively. In particular, L_k is subcritical if $k \ge 1$. Therefore, applying our results in §§2.1.1, we can obtain the precise description of the large time behavior of $e^{-tL}\varphi$.

As an application of the above argument, we obtain the following result.

Theorem 2.1.4. *Let $N \ge 2$ and $\varphi \in L^2(\mathbf{R}^N, e^{|x|^2/4}\,dx)$. Assume condition* (V). *Let*

$$M(\varphi) := \frac{1}{c_*\kappa}\int_{\mathbf{R}^N} \varphi(x)U(|x|)\,dx, \quad \kappa := \frac{|\mathbf{S}^{N-1}|}{c_d^2} = 2^{N+2A}\pi^{N/2}\Gamma\left(\frac{N+2A}{2}\right)\Big/\Gamma\left(\frac{N}{2}\right).$$

(a) *In cases* (S) *and* (C),

$$\lim_{t\to\infty} t^{(N+A)/2}[e^{-tL}\varphi](\sqrt{t}y) = M(\varphi)|y|^A e^{-|y|^2/4}$$

in $L^2(\mathbf{R}^N,\, e^{|y|^2/4}\,dy)$ and in $L^\infty(K)$ for any compact set $K \subset \mathbf{R}^N \setminus \{0\}$. Furthermore,

$$\lim_{t\to\infty} t^{N/2+A}\frac{\left[e^{-tL}\varphi\right](x)}{U(|x|)} = \frac{M(\varphi)}{c_*}$$

uniformly on $B(0,R)$ for any $R>0$.

(b) *In case* (S_*),

$$\lim_{t\to\infty} t^{(N+A)2}(\log t)[e^{-tL}\varphi](\sqrt{t}y) = 2M(\varphi)|y|^A e^{-|y|^2/4}$$

in $L^2(\mathbf{R}^N,\, e^{|y|^2/4}\,dy)$ and in $L^\infty(K)$ for any compact set $K \subset \mathbf{R}^N \setminus \{0\}$. Furthermore,

$$\lim_{t\to\infty} t^{N/2+A}(\log t)^2\frac{\left[e^{-tL}\varphi\right](x)}{U(|x|)} = \frac{4M(\varphi)}{c_*}$$

uniformly on $B(0,R)$ for any $R>0$.

As a corollary of Theorem 2.1.4, we have:

Corollary 2.1.5. *Let $N \ge 2$ and assume condition* (V). *Let $x,\, y \in \mathbf{R}^N$. Then*

$$\lim_{t\to\infty} t^{N/2+A}\frac{p(x,y,t)}{U(|x|)U(|y|)} = \frac{1}{c_*^2\kappa} \quad \text{in cases (S) and (C),}$$

$$\lim_{t\to\infty} t(\log t)^2\frac{p(x,y,t)}{U(|x|)U(|y|)} = \frac{4}{c_*^2\kappa} \quad \text{in case (S}_*\text{).}$$

Corollary 2.1.5 implies the same conclusion as in (L1) and (L2). For related results, see e.g., [4], [28], [32], [36] and [38].

The above argument also enables us to obtain the higher order asymptotic expansions of $e^{-tL}\varphi$. Furthermore, similarly to [13]–[19], it is useful for the study the large time behavior of the *hot spots* of $e^{-tL}\varphi$. (See Chapter 3.)

The rest of this Chapter is organized as follows. In §2.2 we formulate the definition of the solution of (2.1.1) and prove some preliminary lemmas. In §2.3 we obtain a priori estimates of radially symmetric solutions of (2.1.1) by using the comparison principle. In §2.4 we obtain the precise description of the large time behavior of radially symmetric solutions of (2.1.1) and complete the proofs of Theorems 2.1.1–2.1.3. In §2.5, by the argument in §1.2 we apply Theorems 2.1.1–2.1.3 to prove Theorem 2.1.4 and Corollary 2.1.5.

2.2 Preliminaries

We formulate the definition of the solution of (2.1.1) and obtain some properties related to the operator L.

2.2.1 Definition of the solution

Assume condition (V) and let $L := -\Delta + V$ be nonnegative. In this subsection we consider the Cauchy problem

$$\begin{cases} \partial_t u_* = -L_* u_* & \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \\ u_*(x,0) = \varphi_*(x) & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{P}$$

where

$$L_* u_* := -\frac{1}{\nu}\mathrm{div}\,(\nu \nabla u_*), \quad \nu := U^2 \in L^1_{\mathrm{loc}}(\mathbf{R}^N), \quad \varphi_* \in L^2(\mathbf{R}^N,\, \nu\, dx).$$

Definition 2.2.1. Let $\varphi_* \in L^2(\mathbf{R}^N,\, \nu\, dx)$. We say that u_* is a solution of (P) if

$$u_* \in C([0,\infty)) : L^2(\mathbf{R}^N,\, \nu\, dx) \cap L^2\big(\mathbf{R}_+ : H^1(\mathbf{R}^N,\, \nu\, dx)\big),$$

$$\int_0^\infty \int_{\mathbf{R}^N} \big(-u_* h_t + \nabla u_* \nabla h \big) \nu\, dx\, d\tau = 0 \quad \text{for any} \quad h \in C_0^\infty\big(\mathbf{R}^N \times \mathbf{R}_+\big),$$

$$\lim_{t \to +0} \|u_*(t) - \varphi_*\|_{L^2(\mathbf{R}^N,\, \nu\, dx)} = 0.$$

Problem (P) possesses a unique solution u_* such that

$$\|u_*(t)\|_{L^2(\mathbf{R}^N,\, \nu\, dx)} \le \|\varphi_*\|_{L^2(\mathbf{R}^N,\, \nu\, dx)}, \quad t > 0, \tag{2.2.1}$$

and we often denote by $e^{-tL_*}\varphi_*$ the unique solution u_*. Since $U \in C^2(\mathbf{R}^N \setminus \{0\})$ and $U > 0$ in $\mathbf{R}^N \setminus \{0\}$, applying the parabolic regularity theorems (see e.g., [25, Chapter IV]) to (P), we see that

$$\partial_t^j u_* \in C^{2,1}\big((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}_+\big), \quad j = 0, 1, \dots. \tag{2.2.2}$$

Lemma 2.2.2. *Assume condition* (V) *and that L is nonnegative. Let $\varphi_* \in L^2(\mathbf{R}^N,\, \nu\, dx)$ and $u_* := e^{-tL_*}\varphi_*$.*

(i) *For any* $j \in \{1, 2, \dots\}$, *there exists* $C > 0$ *such that*

$$\|(\partial_t^j u_*)(t)\|_{L^2(\mathbf{R}^N,\, \nu\, dx)} \le C t^{-j} \|\varphi_*\|_{L^2(\mathbf{R}^N,\, \nu\, dx)}, \quad t > 0.$$

(ii) *If* $\varphi_* \in L^2(\mathbf{R}^N,\, e^{|x|^2/4}\nu\, dx)$, *then*

$$\sup_{t>0} \|u_*(t)\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4(1+t)}\nu\, dx)} \le \|\varphi_*\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\nu\, dx)}.$$

Proof. Assertion (i) follows from the same argument as in the proof of [14, Lemma 2.1]. We prove assertion (ii). It follows that

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^N} (\partial_t u_*)(x,\tau) u_*(x,\tau) \exp\left\{\frac{|x|^2}{4(1+\tau)}\right\} \nu(x)\, dx d\tau \\
&= \frac{1}{2}\int_{\mathbf{R}^N} u_*(x,t)^2 \exp\left\{\frac{|x|^2}{4(1+t)}\right\} \nu(x)\, dx - \frac{1}{2}\int_{\mathbf{R}^N} u_*(x,0)^2 e^{|x|^2/4} \nu(x)\, dx \\
&\quad + \frac{1}{8}\int_0^t \int_{\mathbf{R}^N} \frac{|x|^2}{(1+\tau)^2} u_*(x,\tau)^2 \exp\left\{\frac{|x|^2}{4(1+\tau)}\right\} \nu(x)\, dx d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^N} \nabla u_*(x,\tau) \cdot \nabla \left(u_*(x,\tau) \exp\left\{\frac{|x|^2}{4(1+\tau)}\right\}\right) \nu(x)\, dx d\tau \\
&= \int_0^t \int_{\mathbf{R}^N} |\nabla u_*(x,\tau)|^2 \exp\left\{\frac{|x|^2}{4(1+\tau)}\right\} \nu(x)\, dx d\tau \\
&\quad + \int_0^t \int_{\mathbf{R}^N} u_*(x,\tau) \nabla u_*(x,\tau) \cdot \frac{x}{2(1+\tau)} \exp\left\{\frac{|x|^2}{4(1+\tau)}\right\} \nu(x)\, dx d\tau \\
&\ge -\frac{1}{8}\int_0^t \int_{\mathbf{R}^N} \frac{|x|^2}{(1+\tau)^2} u_*(x,\tau)^2 \exp\left\{\frac{|x|^2}{4(1+\tau)}\right\} \nu(x)\, dx d\tau.
\end{aligned}$$

Then, multiplying (P) by $u_*(x,\tau) \exp\left\{|x|^2/4(1+\tau)\right\}$ and integrating it in $\mathbf{R}^N \times \mathbf{R}_+$, we obtain

$$\int_{\mathbf{R}^N} u_*(x,t)^2 \exp\left\{\frac{|x|^2}{4(1+\tau)}\right\} \nu(x)\, dx \le \int_{\mathbf{R}^N} \varphi_*(x)^2 \exp\left\{\frac{|x|^2}{4(1+\tau)}\right\} \nu(x)\, dx$$

for $t > 0$. Thus assertion (ii) follows. (The proof of assertion (ii) is somewhat formal, however it is justified by use of approximate solutions.) □

Furthermore, we have:

Lemma 2.2.3. *Assume condition* (V) *and that* L *is nonnegative. Let* u_* *be a radially symmetric solution of* (P). *Then* $\partial_t^j u_*$ *is continuous in* $\mathbf{R}^N \times \mathbf{R}_+$, *where* $j \in \{0, 1, \dots\}$.

Proof. Let $j \in \{0, 1, \dots\}$ and set $v_j = \partial_t^j u_*$. By (2.2.2), it suffices to prove the continuity of v_j at $(0, t) \in \mathbf{R}^N \times \mathbf{R}_+$. Since v_j is radially symmetric, v_j satisfies

$$\begin{aligned} \partial_t v_j &= \frac{1}{r^{N-1}\nu(r)} \partial_r \big(r^{N-1} \nu(r) \partial_r v_j \big) \\ &= \frac{1}{r^{N+k-1} r^{-k} \nu(r)} \partial_r \big(r^{N+k-1} r^{-k} \nu(r) \partial_r v_j \big), \quad r > 0, \ t > 0, \end{aligned} \tag{2.2.3}$$

for any $k \in \mathbf{R}$. Since $A^+(\lambda_1) > -N/2$, we can find $k \in \{1, 2, \dots\}$ such that

$$-N - k < 2A^+(\lambda_1) - k < N + k. \tag{2.2.4}$$

Set $\tilde{v}_j(\mathbf{x}, t) := v_j(|\mathbf{x}|, t)$ and $\tilde{\nu}(\mathbf{x}) := |\mathbf{x}|^{-k} \nu(|\mathbf{x}|)$ for $\mathbf{x} \in \mathbf{R}^{N+k}$ and $t > 0$. By Definition 2.2.1, Lemma 2.2.2 (i), and (2.2.3), we see that $\tilde{v}_j$ satisfies

$$\begin{aligned} &\partial_t \tilde{v}_j = \frac{1}{\tilde{\nu}} \mathrm{div}_{N+k} \, (\tilde{\nu} \, \nabla_{N+k} \tilde{v}_j) \quad \text{in} \quad \mathbf{R}^{N+k} \times \mathbf{R}_+, \\ &\|\tilde{v}_j(t)\|_{L^2(\mathbf{R}^{N+k}, \, \tilde{\nu} \, d\mathbf{x})} = \sqrt{\frac{|\mathbf{S}^{N+k-1}|}{|\mathbf{S}^{N-1}|}} \|v_j(t)\|_{L^2(\mathbf{R}^N, \, \nu \, dx)} \le C t^{-j} \|\varphi_*\|_{L^2(\mathbf{R}^N, \, \nu \, dx)}, \end{aligned}$$

where div_{N+k} is the $(N+k)$-dimensional divergence operator. Furthermore, it follows from (2.1.2) that $\tilde{\nu}(\mathbf{x}) \sim |\mathbf{x}|^{2A^+(\lambda_1) - k}$ as $|\mathbf{x}| \to 0$. This together with (2.2.4) implies that $\tilde{\nu}$ is an A_2 weight in a neighborhood of $\mathbf{0} \in \mathbf{R}^{N+k}$. By Lemma 2.2.2 (i), applying the regularity theorems for parabolic equations with A_2 weight (see e.g., [5] and [13]), we see that $\tilde{v}_j$ is continuous at $(\mathbf{0}, t) \in \mathbf{R}^{N+k} \times \mathbf{R}_+$. This means that $\partial_t^j u_*$ is continuous at $(0, t) \in \mathbf{R}^N \times \mathbf{R}_+$. Thus Lemma 2.2.3 follows. □

We formulate the definition of the solution of (2.1.1). See also [29] and [30].

Definition 2.2.4. Let u be a measurable function in $\mathbf{R}^N \times \mathbf{R}_+$ and $\varphi \in L^2(\mathbf{R}^N)$. Define

$$u_*(x, t) := \frac{u(x, t)}{U(|x|)} \quad \text{and} \quad \varphi_*(x) := \frac{\varphi(x)}{U(|x|)}.$$

Then we say that u is a solution of (2.1.1) if u_* is a solution of (P).

In the case where $\lambda_1, \lambda_2 > \lambda_*$, we can deduce from (2.1.2) and (2.1.3) that $U \in H^1(\mathbf{R}^N)$ and that a solution u of (2.1.1) satisfies

$$u \in C([0, \infty) : L^2(\mathbf{R}^N)) \cap L^2(\mathbf{R}_+ : H^1(\mathbf{R}^N)).$$

We remark that $\varphi \in L^2(\mathbf{R}^N)$ if and only if $\varphi_* \in L^2(\mathbf{R}^N, \, \nu \, dx)$. Furthermore, by (2.1.6), we have the following lemma (see also [11, Theorem 1.2] and [12, Theorem 1.1]).

Lemma 2.2.5. *Let u be a solution of* (2.1.1) *under condition* (V). *Assume either L is subcritical or L is critical with $A^-(\lambda_2) > -N/2$. Then, for any $T > 0$, there exists $C > 0$ such that*

$$\frac{|u(x, t)|}{U\big(\min\{|x|, \sqrt{t}\}\big)} \le \frac{C \|\varphi\|_{L^2(\mathbf{R}^N)}}{t^{N/4} U(\sqrt{t})}, \quad x \in \mathbf{R}^N, \ t \ge T. \tag{2.2.5}$$

Proof. It follows from (2.1.6) that

$$\begin{aligned}
\frac{|u(x,t)|}{U\big(\min\{|x|,\sqrt{t}\}\big)} &\le \frac{1}{U\big(\min\{|x|,\sqrt{t}\}\big)}\left(\int_{\{|y|\le\sqrt{t}\}}+\int_{\{|y|>\sqrt{t}\}}\right)p(x,y,t)|\varphi(y)|\,dy\\
&\le Ct^{-N/2}U(\sqrt{t})^{-2}\int_{\{|y|\le\sqrt{t}\}}|\varphi(y)|U(|y|)\,dy\\
&\quad + Ct^{-N/2}U(\sqrt{t})^{-1}\int_{\{|y|>\sqrt{t}\}}\exp\left\{-\frac{|x-y|^2}{Ct}\right\}|\varphi(y)|\,dy\\
&\le Ct^{-N/2}U(\sqrt{t})^{-2}\|U\|_{L^2(\{|y|\le\sqrt{t}\})}\|\varphi\|_{L^2(\mathbf{R}^N)}+Ct^{-N/4}U(\sqrt{t})^{-1}\|\varphi\|_{L^2(\mathbf{R}^N)}
\end{aligned}$$

for $x\in\mathbf{R}^N$ and $t>0$. On the other hand, by (2.1.4) and (2.1.5), we have

$$\|U\|_{L^2(\{|y|\le\sqrt{t}\})}\le Ct^{N/4}U(\sqrt{t})$$

for $t\ge T$ (see also (2.3.7)). These imply (2.2.5) and Lemma 2.2.5 follows. □

2.2.2 Preliminary lemmas

We prove a lemma on the decay of U' as $r\to\infty$.

Lemma 2.2.6. *Let $N\ge 2$. Assume condition* (V) *and that $L=-\Delta+V(|x|)$ is nonnegative. Let U and v be as in* (2.1.2) *and* (2.1.5)*, respectively. In cases* (S) *and* (C) *there exists $\delta>0$ such that*

$$\partial_r\left(\frac{U(r)}{v(r)}\right)=O(r^{-1-\delta})\quad\text{as}\quad r\to\infty. \tag{2.2.6}$$

Proof. Let $V_{\lambda_2}(r):=V(r)-\lambda_2 r^{-2}$. Set

$$v^+(r):=\begin{cases} r^{-(N-2)/2}\log r & \text{if } L \text{ is subcritical and } \lambda=\lambda_*,\\ r^{A^+(\lambda_2)} & \text{otherwise,}\end{cases}\qquad v^-(r):=r^{A^-(\lambda_2)}.$$

It follows from (2.1.4) and (V) (ii) that

$$\begin{aligned}
\tau^{N-1}v^-(\tau)V_{\lambda_2}(\tau)U(\tau) &= O\big(\tau^{N-3-\theta+A^-(\lambda_2)}v(\tau)\big)\\
&=\begin{cases} O(\tau^{-1-\theta}) & \text{if } L \text{ is subcritical and } \lambda_2>\lambda_*,\\ O(\tau^{-1-\theta-\sqrt{Q}}) & \text{if } L \text{ is critical and } \lambda_2>\lambda_*,\\ O(\tau^{-1-\theta}) & \text{if } L \text{ is critical and } \lambda_2=\lambda_*,\end{cases}
\end{aligned}\tag{2.2.7}$$

as $\tau\to\infty$, where $Q=(N-2)^2+4\lambda_2$. Then the function

$$G(r):=v^-(r)\int_1^r\frac{1}{v^-(s)^2s^{N-1}}\int_s^\infty v^-(\tau)V_{\lambda_2}(\tau)U(\tau)\tau^{N-1}\,d\tau ds$$

can be defined for any $r > 0$ and satisfies

$$\begin{aligned} &G''(r) + \frac{N-1}{r}G'(r) - \lambda_2 r^{-2}G(r) = V_{\lambda_2}(r)U(r) \quad \text{in} \quad \mathbf{R}_+, \\ &G(r) = o(v^+(r)) \quad \text{as} \quad r \to \infty. \end{aligned} \tag{2.2.8}$$

Since

$$U''(r) + \frac{N-1}{r}U'(r) - \lambda_2 r^{-2}U(r) = V_{\lambda_2}(r)U(r) \quad \text{in} \quad \mathbf{R}_+,$$

the function $\tilde{v}(r) := U(r) - G(r)$ satisfies

$$\tilde{v}''(r) + \frac{N-1}{r}\tilde{v}'(r) - \lambda_2 r^{-2}\tilde{v}(r) = 0 \quad \text{in} \quad \mathbf{R}_+. \tag{2.2.9}$$

On the other hand, $v^\pm$ satisfy (2.2.9) and are linearly independent. Therefore, applying the standard theory for ordinary differential equations, we can find a, $b \in \mathbf{R}$ such that $\tilde{v}(r) = av^+(r) + bv^-(r)$ in $\mathbf{R}_+$, that is

$$U(r) = av^+(r) + bv^-(r) + G(r) \quad \text{in} \quad \mathbf{R}_+. \tag{2.2.10}$$

Assume that L is subcritical. By (2.1.4), (2.2.8), and (2.2.10), we have

$$\frac{U(r)}{v(r)} = c_* + r^{-\sqrt{Q}}\left[b + \int_1^r \frac{1}{v^-(s)^2 s^{N-1}} \int_s^\infty v^-(\tau)V_{\lambda_2}(\tau)U(\tau)\tau^{N-1}\,d\tau ds\right].$$

Since $Q = (N-2)^2 + 4\lambda_2 > 0$, by (2.2.7), we can find $\delta' > 0$ such that

$$\begin{aligned} \partial_r\left(\frac{U(r)}{v(r)}\right) &= \frac{r^{-\sqrt{Q}}}{v^-(r)^2 r^{N-1}} \int_r^\infty v^-(\tau)V_{\lambda_2}(\tau)U(\tau)\tau^{N-1}\,d\tau \\ &\quad - \sqrt{Q}r^{-\sqrt{Q}-1}\left[b + \int_1^r \frac{1}{v^-(s)^2 s^{N-1}} \int_s^\infty v^-(\tau)V_{\lambda_2}(\tau)U(\tau)\tau^{N-1}\,d\tau ds\right] \\ &= O(r^{-1-\delta'}) \quad \text{as} \quad r \to \infty. \end{aligned}$$

This implies (2.2.6) in the subcritical case.

Next we assume that L is critical. By (2.1.4) and (2.2.8), we see that $a = 0$ and

$$\frac{U(r)}{v(r)} = b + \int_1^r \frac{1}{v^-(s)^2 s^{N-1}} \int_s^\infty v^-(\tau)V_{\lambda_2}(\tau)U(\tau)\tau^{N-1}\,d\tau ds.$$

This together with (2.2.7) implies that

$$\partial_r\left(\frac{U(r)}{v(r)}\right) = \frac{1}{v^-(r)^2 r^{N-1}} \int_r^\infty v^-(\tau)V_{\lambda_2}(\tau)U(\tau)\tau^{N-1}\,d\tau ds = O(r^{-1-\theta})$$

as $r \to \infty$, and (2.2.6) holds with $\delta = \theta$. Thus Lemma 2.2.6 follows. □

At the end of this section we state the following lemma on eigenvalue problem (E).

Lemma 2.2.7. *Let* $\{\mu_i\}_{i=0}^\infty$ *be the eigenvalues of* (E) *such that* $\mu_0 \le \mu_1 \le \dots$. *Then, for any* $i \in \{0, 1, \dots\}$, $\mu_i = i$ *and* μ_i *is simple. Furthermore,* ψ_d *given in* (2.1.8) *is the first eigenfunction of* (E).

Proof. We leave the proof to the reader since it is proved by the same argument as in [31, Lemma 2.1]. □

2.3 A priori estimates of radial solutions

Let $T > 0$ and $\varepsilon > 0$. Define

$$D_\varepsilon(T) := \left\{ (x,t) \in \mathbf{R}^N \times (T,\infty) \,:\, |x| < \varepsilon\sqrt{t} \right\}.$$

In this section we prove the following proposition.

Proposition 2.3.1. *Assume condition* (V). *Let L satisfy either* (S), (S$_*$) *or* (C). *Let $u_* = u_*(|x|,t)$ be a radially symmetric solution of* (P) *such that $\|\varphi_*\|_{L^2(\mathbf{R}^N,\,\nu\,dx)} = 1$. Assume that*

$$\sup_{t>0} t^D \big(\log(2+t)\big)^{D'} \|u_*(t)\|_{L^2(\mathbf{R}^N,\,\nu\,dx)} < \infty \tag{2.3.1}$$

for some $D \geq 0$ and $D' \geq 0$. Let $j \in \{0,1,\dots\}$. Then the following holds for any $T > 0$ and any sufficiently small $\varepsilon > 0$.

(i) *There exists $C_1 > 0$ such that*

$$\big|(\partial_t^j u_*)(|x|,t)\big| \leq C_1 \Gamma_{D,\,D',\,j}(t)$$

for $(x,t) \in D_\varepsilon(T)$, where

$$\Gamma_{D,\,D',\,j}(t) := \begin{cases} t^{-D-d/4-j}\big(\log(2+t)\big)^{-D'} & \text{in the case of (S),} \\ t^{-D-d/4-j}\big(\log(2+t)\big)^{-D'-1} & \text{in the case of (S}_*\text{),} \\ t^{-D-d/4-j}\big(\log(2+t)\big)^{-D'} & \text{in the case of (C).} \end{cases} \tag{2.3.2}$$

(ii) *Let*

$$F_N^j(r,t) := \int_0^r \frac{1}{\nu(s)s^{N-1}} \int_0^s \nu(\tau)(\partial_t^{j+1}u_*)(\tau,t)\tau^{N-1}\,d\tau ds.$$

Then

$$(\partial_t^j u_*)(|x|,t) = (\partial_t^j u_*)(0,t) + F_N^j(|x|,t) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+.$$

Furthermore, there exists $C_2 > 0$ such that

$$\big|F_N^j(|x|,t)\big| \leq C_2 \Gamma_{D,\,D',\,j+1}(t)|x|^2, \qquad \big|(\partial_r F_N^j)(|x|,t)\big| \leq C\Gamma_{D,\,D',\,j+1}(t)|x|,$$

for $(x,t) \in D_\varepsilon(T)$.

For the proof, we construct supersolutions of problem (P) in $D_\varepsilon(T)$.

Lemma 2.3.2. *Assume condition* (V). *Let $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$. Set*

$$\zeta(t) := t^{-\gamma_1}\big(\log(2+t)\big)^{-\gamma_2}.$$

Then, for any $T > 0$ and any sufficiently small $\varepsilon > 0$, there exists a function $W_* = W_*(x,t)$ such that

$$\partial_t W_* + L_* W_* \geq 0 \qquad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \tag{2.3.3}$$

$$\zeta(t) \leq W_*(x,t) \leq 2\zeta(t) \quad \text{in} \quad D_\varepsilon(T). \tag{2.3.4}$$

Proof. Let $T > 0$ and $\varepsilon > 0$. Let κ be a positive constant such that

$$|\zeta'(t)| \le \kappa t^{-1}\zeta(t), \qquad t > 0. \tag{2.3.5}$$

Let

$$F(x) := \int_0^{|x|} \frac{1}{\nu(s)s^{N-1}} \int_0^s \nu(\tau)\tau^{N-1}\, d\tau ds,$$

which satisfies $-L_*F = 1$ in $\mathbf{R}^N$. Set

$$W_*(x,t) := 2\zeta(t)\big(1 - \kappa t^{-1}F(x)\big). \tag{2.3.6}$$

Since ζ is monotone decreasing, by (2.3.5), we have

$$\begin{aligned} \partial_t W_* + L_* W_* &\ge 2\zeta'(t)\left[1 - \kappa t^{-1}F(x)\right] + 2\kappa\zeta(t)t^{-2}F(x) + 2\kappa t^{-1}\zeta(t) \\ &\ge 2\zeta'(t) + 2\kappa t^{-1}\zeta(t) \ge 0 \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+. \end{aligned}$$

This implies (2.3.3). On the other hand, by (2.1.2), (2.1.4), and (2.1.5), we have

$$\int_0^s \tau^{N-1}\nu(\tau)\, d\tau \le C s^{2A^+(\lambda_1)+N} \quad \text{for} \quad 0 < s \le 1,$$

$$\int_0^s \tau^{N-1}\nu(\tau)\, d\tau \le \begin{cases} s^{2A^+(\lambda_2)+N} & \text{in the case of (S)}, \\ s^2\big(\log(2+s)\big)^2 & \text{in the case of } (\mathrm{S}_*), \\ s^{2A^-(\lambda_2)+N} & \text{in the cases of (C)}, \end{cases} \qquad \text{for} \quad s > 1.$$

These imply that

$$\int_0^s \tau^{N-1}\nu(\tau)\, d\tau \le C s^N \nu(s), \quad s \ge 0. \tag{2.3.7}$$

Then it follows that $0 \le F(x) \le C|x|^2$ for $x \in \mathbf{R}^N$. Taking a sufficiently small $\varepsilon > 0$ if necessary, we obtain

$$0 \le \kappa t^{-1}F(x) \le C\varepsilon^2\kappa \le \frac{1}{2}, \quad (x,t) \in D_\varepsilon(T).$$

This together with (2.3.6) implies (2.3.4). Thus Lemma 2.3.2 follows. $\square$

Applying the same argument as in [19, Lemma 3.2], we have:

Lemma 2.3.3. *Assume the same conditions as in Proposition* 2.3.1. *Furthermore, assume* (2.3.1) *for some $D \ge 0$ and $D' \ge 0$. Let $T > 0$ and let ε be a sufficiently small positive constant. Then, for any $j \in \{0, 1, \dots\}$, there exists $C > 0$ such that*

$$\big|(\partial_t^j u_*)(|x|,t)\big| \le C\Gamma_{D,D',j}(t) \quad \text{in} \quad D_\varepsilon(T). \tag{2.3.8}$$

Proof. Let $j \in \{0, 1, \dots\}$. Set $v_j := \partial_t^j u_*$ and $u_j := U(|x|)v_j(x,t)$. Since

$$v_j(\,\cdot\,,t) = \partial_t^j\big(e^{-(t/2)L_*}u_*(t/2)\big), \quad t > 0,$$

Lemma 2.2.2 together with (2.3.1) implies that

$$\sup_{t>0} t^{D+j}\big(\log(2+t)\big)^{D'}\|u_j(t)\|_{L^2(\mathbf{R}^N)} = \sup_{t>0} t^{D+j}\big(\log(2+t)\big)^{D'}\|v_j(t)\|_{L^2(\mathbf{R}^N,\,\nu\,dx)} < \infty.$$

Let $T > 0$ and let ε be a sufficiently small positive constant. Since u_j satisfies

$$\partial_t u_j = \Delta u_j - V(|x|)u_j \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+,$$

by Lemma 2.2.5, we have

$$|u_j(|x|,t)| \le Ct^{-N/4}\|u_j(t/2)\|_{L^2(\mathbf{R}^N)} \le Ct^{-D-N/4-j}\big(\log(2+t)\big)^{-D'}$$

for all $x \in \mathbf{R}^N$ and $t > T$ with $|x| \ge \varepsilon\sqrt{1+t}$. This together with (2.1.4), (2.1.5), (2.1.7), (2.1.8) and (2.3.2) implies that

$$|v_j(|x|,t)| \le \frac{Cu_j(|x|,t)}{U(\varepsilon\sqrt{1+t})} \le C\Gamma_{D,D',j}(t) \tag{2.3.9}$$

for all $(x,t) \in \mathbf{R}^N \times [T,\infty)$ with $|x| = \varepsilon\sqrt{1+t}$. On the other hand, it follows from Lemma 2.2.3 that

$$|v_j(|x|,T)| \le C \tag{2.3.10}$$

for $x \in \mathbf{R}^N$ with $|x| \le \varepsilon\sqrt{1+T}$ Let W_* be as in Lemma 2.3.2 with ζ replaced by $\Gamma_{D,D',j}$. Then, by Lemma 2.3.2, (2.3.9), and (2.3.10), we apply the comparison principle to obtain

$$|v_j(|x|,t)| \le CW_*(x,t) \le 2C\Gamma_{D,D',j}(t) \quad \text{in} \quad D_\varepsilon(T).$$

This implies (2.3.8), and the proof is complete. □

Now we are ready to complete the proof of Proposition 2.3.1.

Proof of Proposition 2.3.1. By Lemma 2.3.3, it suffices to prove assertion (ii). Let $T > 0$ and let ε be a sufficiently small positive constant. By (2.3.7) and (2.3.8), we obtain

$$\begin{aligned}
&|F_N^j(|x|,t)| \le C\Gamma_{D,D',j+1}(t)\int_0^{|x|}\frac{1}{\nu(s)s^{N-1}}\int_0^s \nu(\tau)\tau^{N-1}\,d\tau ds \le C\Gamma_{D,D',j+1}(t)|x|^2,\\
&\big|(\partial_r F_N^j)(|x|,t)\big| \le C\Gamma_{D,D',j+1}(t)|x|,
\end{aligned}$$

for $(x,t) \in D_\varepsilon(T)$. Set

$$\hat{v}_j(|x|,t) := (\partial_t^j u_*)(|x|,t) - F_N^j(|x|,t) \quad \text{and} \quad \hat{u}_j(|x|,t) := U(|x|)\hat{v}_j(|x|,t).$$

Since F_N^j satisfies

$$\frac{1}{\nu(r)r^{N-1}}\partial_r\big(\nu(r)r^{N-1}\partial_r F_N^j\big) = (\partial_t^{j+1}u_*)(r,t) \quad \text{for } r>0 \text{ and } t>0,$$

by Lemma 2.2.3 and (2.2.3), we have

$$\frac{1}{\nu(r)r^{N-1}}\partial_r\big(\nu(r)r^{N-1}\partial_r \hat{v}_j\big) = 0 \quad \text{for} \quad r>0,\ t>0, \tag{2.3.11}$$

$$\limsup_{r\to 0}|\hat{v}_j(r,t)|<\infty \qquad \text{for any} \quad t>0. \tag{2.3.12}$$

It follows from (2.3.11) that $\hat{u}_j$ satisfies (O) for any fixed $t>0$. On the other hand, since U and $\widetilde{U}$ are linearly independent solutions of (O), for any $t>0$, we can find constants $c_j(t)$ and $\hat{c}_j(t)$ such that

$$\hat{u}_j(r,t) = c_j(t)U(r) + \hat{c}_j(t)\widetilde{U}(r) \quad \text{for} \quad r>0.$$

This implies that

$$\hat{v}_j(r,t) = \frac{\hat{u}_j(r,t)}{U(r)} = c_j(t) + \frac{\hat{c}_j(t)\widetilde{U}(r)}{U(r)} \quad \text{for} \quad r>0.$$

By (2.1.2) and (2.3.12), we then have $\hat{c}_j(t)=0$ and see that $\hat{v}_j(r,t)\equiv c_j(t)$ for $r\ge 0$. Therefore we have

$$(\partial_t^j u_*)(|x|,t) = c_j(t) + F_N^j(|x|,t) \quad \text{in} \quad \mathbf{R}^N\times\mathbf{R}_+ \quad \text{and} \quad c_j(t) = (\partial_t^j u_*)(0,t).$$

Thus assertion (ii) follows, and the proof of Proposition 2.3.1 is complete. □

2.4 Large time behavior of radially symmetric solutions

In this section, under condition (V), we study the large time behavior of radially symmetric solution $u=u(|x|,t)$ of (2.1.1) and prove Theorems 2.1.1–2.1.3.

Let A, d and w be as in §§ 2.1.1. Set $U_d(r) := r^{-A}U(r)$ and $\nu_d := U_d^2$. By (2.1.2), (2.1.4), and Lemma 2.2.6, we have:

$$\frac{1}{r^{d-1}}\partial_r\left(r^{d-1}\partial_r U_d\right) - V_{\lambda_2}(r)U_d = 0 \quad \text{in} \quad \mathbf{R}_+\,;$$

$$U_d(r) \sim r^{A^+(\lambda_1)-A} = r^{A_d^+(\lambda)} \qquad \text{as} \quad r\to 0\,;$$

$$U_d(r)\sim c_*, \quad U_d'(r) = O(r^{-1-\delta}) \qquad \text{as} \quad r\to\infty \quad \text{in cases (S) and (C).} \tag{2.4.1}$$

Here c_* is as in (2.1.4), $\lambda := \lambda_1-\lambda_2$ and

$$A_d^+(\lambda) := \frac{-(d-2)+\sqrt{(d-2)^2+4\lambda}}{2}.$$

Furthermore, similarly to Lemma 2.2.6, we see that

$$U_d(r) \sim c_* \log r, \quad U_d'(r) = O(r^{-1}) \quad \text{as} \quad r \to \infty \quad \text{in case of } (\mathrm{S}_*). \tag{2.4.2}$$

Then the function F_N^j given in Proposition 2.3.1 satisfies

$$F_N^j(r,t) = F_d^j(r,t) := \int_0^r \frac{1}{\nu_d(s)s^{d-1}} \int_0^s \nu_d(\tau)(\partial_t^{j+1}u_*)(\tau,t)\tau^{d-1}\,d\tau ds, \tag{2.4.3}$$

where $j \in \{0,1,\dots\}$. Furthermore, it follows from (2.3.7) that

$$\int_0^s \tau^{d-1}\nu_d(\tau)\,d\tau \le Cs^d\nu_d(s), \quad s \ge 0. \tag{2.4.4}$$

Assume the same conditions as in Theorem 2.1.1. Let θ be the constant given in condition (V) and set

$$\theta_* = \frac{\theta}{4(2+\theta)} < \frac{\theta}{8}.$$

Since $\widetilde{V}(\xi,s) = e^s V_{\lambda_2}(e^{s/2}\xi)$, it follows from (V) (ii) that

$$|\widetilde{V}(\xi,s)| \le C\xi^{-2}|e^{s/2}\xi|^{-\theta} \le C\exp\left\{-\frac{\theta}{2}s + (2+\theta)\theta_* s\right\} = Ce^{-\theta s/4} \tag{2.4.5}$$

for $\xi \in (e^{-\theta_* s}, \infty)$ and $s > 0$. Let δ be as in Lemma 2.2.6. Then, taking a sufficiently small $\theta > 0$ if necessary, we have

$$0 < \theta < \min\{1, d, d^{-1}\}, \quad \sigma := \left(\frac{1}{2} - \theta_*\right)(1+\delta) - \frac{1}{2} > \theta_* > 0. \tag{2.4.6}$$

We prepare some lemmas on estimates of w.

Lemma 2.4.1. *Let* $\|\varphi_*\|_{L^2(\mathbf{R}^N,\,\nu e^{|x|^2/4}\,dx)} = 1$. *Assume the same conditions as in Theorem* 2.1.1. *Then*

(i) $\sup_{s>0} e^{-ds/4}\|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} < \infty$;

(ii) *Assume that*

$$\sup_{s>0} e^{\gamma s}\|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} < \infty \tag{2.4.7}$$

for some $\gamma \ge -d/4$. *Then*

$$w(e^{-\theta_* s}, s) = O(e^{-\gamma s}), \tag{2.4.8}$$

$$(\partial_\xi w)(e^{-\theta_* s}, s) = O(e^{-\gamma s - \theta_* s}), \tag{2.4.9}$$

$$\int_0^{e^{-\theta_* s}} |w(\xi,s)|^2 \rho_d\,d\xi = O\big(e^{-2\gamma s - d\theta_* s}\big), \tag{2.4.10}$$

for all sufficiently large $s > 0$.

Proof. Since

$$\begin{aligned} w(\xi,s) &= (1+t)^{d/2} r^{-A} u(r,t) \\ &= (1+t)^{d/2} r^{-A} U(r) u_*(r,t) = (1+t)^{d/2} U_d(r) u_*(r,t) \end{aligned} \tag{2.4.11}$$

with $\xi = r/\sqrt{1+t}$ and $s = \log(1+t)$, it follows from Lemma 2.2.2 (ii) that

$$\begin{aligned} \|w(s)\|^2_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} &= (1+t)^{d/2} \int_0^\infty |u_*(r,t)|^2 U(r)^2 r^{N-1} \exp\left\{\frac{r^2}{4(1+t)}\right\} dr \\ &= \left|\mathbf{S}^{N-1}\right|^{-1} (1+t)^{d/2} \|u_*(t)\|^2_{L^2(\mathbf{R}^N,\, e^{|x|^2/4(1+t)}\nu\,dx)} \\ &\le \left|\mathbf{S}^{N-1}\right|^{-1} (1+t)^{d/2} \|\varphi_*\|^2_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\nu\,dx)} \\ &= \left|\mathbf{S}^{N-1}\right|^{-1} (1+t)^{d/2} \|\varphi\|^2_{L^2(\mathbf{R}_+,\, e^{|x|^2/4}\,dx)} < \infty \end{aligned} \tag{2.4.12}$$

for $s > 0$ and $t > 0$ with $s = \log(1+t)$, where $\left|\mathbf{S}^{N-1}\right|$ is the volume of $(N-1)$-dimensional unit sphere, that is $\left|\mathbf{S}^{N-1}\right| = 2\pi^{N/2}/\Gamma(N/2)$. Thus assertion (i) follows.

We prove assertion (ii). It follows from (2.4.12) that

$$\begin{aligned} \|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} &= (1+t)^{d/4} \left|\mathbf{S}^{N-1}\right|^{-1/2} \|u_*(t)\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4(1+t)}\nu\,dx)} \\ &\ge (1+t)^{d/4} \left|\mathbf{S}^{N-1}\right|^{-1/2} \|u_*(t)\|_{L^2(\mathbf{R}^N,\,\nu\,dx)} \end{aligned} \tag{2.4.13}$$

for $s > 0$ and $t > 0$ with $s = \log(1+t)$. Assume (2.4.7) for some $\gamma \ge -d/4$. Then

$$\sup_{t>0} (1+t)^{\gamma + d/4} \|u_*(t)\|_{L^2(\mathbf{R}^N,\,\nu\,dx)} < \infty.$$

Applying Proposition 2.3.1 with $D = \gamma + d/4$ and $D' = 0$, we obtain

$$u_*(|x|,t) = u_*(0,t) + F^0_N(|x|,t) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+. \tag{2.4.14}$$

Furthermore, for any $T > 0$ and any sufficiently small $\varepsilon > 0$,

$$\begin{aligned} &|u_*(|x|,t)| \le Ct^{-\gamma - d/2}, \\ &|F^0_N(|x|,t)| \le Ct^{-\gamma - d/2 - 1}|x|^2 \le C\varepsilon^2 t^{-\gamma - d/2}, \\ &\left|(\partial_r F^0_N)(|x|,t)\right| \le Ct^{-\gamma - d/2 - 1}|x| \le C\varepsilon t^{-\gamma - d/2 - 1/2}, \end{aligned} \tag{2.4.15}$$

for $(x,t) \in D_\varepsilon(T)$. By (2.4.1), (2.4.11), and (2.4.15), we then have $w(e^{-\theta_* s}, s) = O(e^{-\gamma s})$ for all sufficiently large $s > 0$. Furthermore,

$$\begin{aligned} (\partial_\xi w)(e^{-\theta_* s}, s) &= (1+t)^{(d+1)/2} \big(U_d'(r) u_*(r,t) + U_d(r)(\partial_r u_*)(r,t)\big) \\ &= (1+t)^{(d+1)/2} \big(O(r^{-1-\delta}) u_*(r,t) + (c_* + o(1))(\partial_r F^0_N)(r,t)\big) \\ &= (1+t)^{-\gamma + 1/2} O(r^{-1-\delta}) + (1+t)^{-\gamma - 1/2} O(r) \\ &= O(e^{-\gamma s - \sigma s}) + O(e^{-\gamma s - \theta_* s}) = O(e^{-\gamma s - \theta_* s}) \end{aligned}$$

for all sufficiently large $s > 0$, where $r = e^{(1/2-\theta_*)s}$, $s = \log(1+t)$ and σ is as in (2.4.6). So we have (2.4.8) and (2.4.9).

On the other hand, by (2.4.1), (2.4.4), (2.4.11), (2.4.14), and (2.4.15), we have

$$\begin{aligned}\int_0^{e^{-\theta_* s}} |w(\xi,s)|^2\rho_d\,d\xi &= (1+t)^{d/2}\int_0^{(1+t)^{1/2-\theta_*}} |u_*(r,t)|^2 U_d(r)^2 r^{d-1}\exp\left\{\frac{r^2}{4(1+t)}\right\}dr\\ \le Ct^{-2\gamma-d/2}&\int_0^{(1+t)^{1/2-\theta_*}} \nu_d(r)r^{d-1}\,dr \le Ct^{-2\gamma-d\theta_*}U_d\big((1+t)^{1/2-\theta_*}\big)^2 = O(e^{-2\gamma s-d\theta_* s})\end{aligned}$$

for all sufficiently large $s > 0$ and $t > 0$ with $s = \log(1+t)$. This implies (2.4.10). Thus assertion (ii) follows, and the proof is complete. $\square$

Lemma 2.4.2. *Assume the same conditions as in Lemma* 2.4.1. *Then*

$$\sup_{s>0}\|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} < \infty. \tag{2.4.16}$$

Proof. Assume that (2.4.7) holds for some $\gamma \ge -d/4$. Let $I(s) := (e^{-\theta_* s},\infty)$. It follows from (2.1.16) that

$$\begin{aligned}\frac{d}{ds}\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi &= 2\int_{I(s)} w(\partial_s w)\rho_d\,d\xi + \theta_* e^{-\theta_* s}\big|w(e^{-\theta_* s},s)\big|^2\rho_d(e^{-\theta_* s})\\ &= 2\int_{I(s)} w\partial_\xi(\rho_d\partial_\xi w)\,d\xi + d\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi\\ &\quad - 2\int_{I(s)}\widetilde{V}w^2\rho_d\,d\xi + \theta_* e^{-\theta_* s}\big|w(e^{-\theta_* s},s)\big|^2\rho_d(e^{-\theta_* s})\\ &= -2w(e^{-\theta_* s},s)\rho_d(e^{-\theta_* s})(\partial_\xi w)(e^{-\theta_* s},s) - 2\int_{I(s)}|\partial_\xi w|^2\rho_d\,d\xi\\ &\quad + d\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi - 2\int_{I(s)}\widetilde{V}w^2\rho_d\,d\xi + \theta_* e^{-\theta_* s}|w(e^{-\theta_* s},s)|^2\rho_d(e^{-\theta_* s})\end{aligned}$$

for $s > 0$. This together with Lemma 2.4.1 and (2.4.5) implies that

$$\begin{aligned}\frac{d}{ds}\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi &\le -2\int_{I(s)}|\partial_\xi w|^2\rho_d\,d\xi + d\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi\\ &\qquad + Ce^{-\theta s/4}\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi + O(e^{-2\gamma s}e^{-d\theta_* s})\end{aligned} \tag{2.4.17}$$

for all sufficiently large $s > 0$.

Set

$$\widehat{w}(\xi,s) := \begin{cases} w(\xi,s) & \text{if} \quad \xi \ge e^{-\theta_* s},\\ w(e^{-\theta_* s},s) & \text{if} \quad 0 \le \xi < e^{-\theta_* s}.\end{cases} \tag{2.4.18}$$

It follows from Lemmas 2.2.7 and 2.4.1 that

$$\begin{aligned}
&-2\int_{I(s)}|(\partial_\xi w)(\xi,s)|^2\rho_d\,d\xi + d\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi\\
&= -2\int_0^\infty |(\partial_\xi \widehat{w})(\xi,s)|^2\rho_d\,d\xi + d\int_0^\infty |\widehat{w}(\xi,s)|^2\rho_d\,d\xi + O(e^{-d\theta_* s}w(e^{-\theta_* s},s)^2) \qquad (2.4.19)\\
&\le -2\mu_0\int_0^\infty |\widehat{w}(\xi,s)|^2\rho_d\,d\xi + O\big(e^{-d\theta_* s}w(e^{-\theta_* s},s)^2\big) = O\big(e^{-2\gamma s-d\theta_* s}\big)
\end{aligned}$$

for all sufficiently large $s>0$. This together with (2.4.17) implies that

$$\frac{d}{ds}\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi \le Ce^{-\theta s/4}\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi + O(e^{-2\gamma s}e^{-d\theta_* s}) \qquad (2.4.20)$$

for all sufficiently large $s>0$.

On the other hand, by Lemma 2.4.1 (i), we see that (2.4.7) holds with $\gamma=-d/4$. Without loss of generality, we can find $j\in\{0,1,\dots\}$ such that

$$j\theta_* < \frac{1}{2} < (j+1)\theta_*. \qquad (2.4.21)$$

Since $\theta_*<1/4$, applying (2.4.20) with $\gamma=-d/4$, we have

$$\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi = O(e^{ds/2-d\theta_* s})$$

for all sufficiently large $s>0$. This together with Lemma 2.4.1 implies that

$$\sup_{s>0} e^{-ds/4+d\theta_* s/2}\|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)}<\infty. \qquad (2.4.22)$$

By (2.4.21), repeating this arguments, we obtain

$$\sup_{s>0} e^{-ds/4+jd\theta_* s/2}\|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)}<\infty.$$

Applying (2.4.20) with $\gamma=-d/4-jd\theta_*/2$ again, by (2.4.21), we have

$$\sup_{s\ge1}\int_{I(s)}|w(\xi,s)|^2\rho_d\,d\xi<\infty.$$

Then, similarly to (2.4.22), we obtain (2.4.16). Thus Lemma 2.4.2 follows. □

Combining Lemma 2.4.2 with assertion (ii) of Lemma 2.4.1, we have:

Lemma 2.4.3. *Assume the same conditions as in Lemma* 2.4.1. *Let* $\widehat{w}$ *be as in* (2.4.18). *Then*

$$\sup_{s\ge1}|w(e^{-\theta_* s},s)|<\infty, \qquad \sup_{s\ge1} e^{\theta_* s}\big|(\partial_\xi w)(e^{-\theta_* s},s)\big|<\infty,$$

$$\sup_{s\ge1}\|\widehat{w}(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)}<\infty, \quad \sup_{s\ge1} e^{d\theta_* s}\int_0^{e^{-\theta_* s}}|w(\xi,s)|^2\rho_d\,d\xi<\infty.$$

Next we study the large time behavior of $\widehat{w}$ and prove the following proposition.

Proposition 2.4.4. *Let* $\|\varphi_*\|_{L^2(\mathbf{R}^N,\,\nu e^{|x|^2/4}\,dx)} = 1$. *Assume the same conditions as in Theorem* 2.1.1. *Let* $\widehat{w}$ *be as in* (2.4.18). *Set*

$$a(s) := \int_0^\infty \widehat{w}\psi_d\rho_d\,d\xi = c_d\int_0^\infty \widehat{w}(\xi,s)\xi^{d-1}\,d\xi.$$

Then $\|\widehat{w} - a(s)\psi_d\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} = O(e^{-\theta' s})$ *as* $s\to\infty$, *where* $\theta' := \min\{d\theta_*/2, \theta/8\}$.

For the proof of Proposition 2.4.4, we prepare the following lemma.

Lemma 2.4.5. *Assume the same conditions as in Proposition* 2.4.4. *Then*

$$\sup_{s\ge 1}|a(s)| < \infty, \tag{2.4.23}$$

$$\sup_{s\ge 1} e^{2\theta' s}|a'(s)| < \infty, \tag{2.4.24}$$

$$\sup_{s\ge 1}\left|\frac{d}{ds}w(e^{-\theta_* s},s)\right| < \infty. \tag{2.4.25}$$

Proof. It follows from Lemma 2.4.3 that

$$\sup_{s\ge 1}|a(s)| \le \sup_{s\ge 1}\|\widehat{w}\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)}\|\psi_d\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} < \infty.$$

So we have (2.4.23).

We prove (2.4.25). By Proposition 2.3.1 (ii), and (2.4.11), we have

$$w(e^{-\theta_* s},s) = e^{ds/2}U_d(r(s))u_*(r(s),t(s)) = e^{ds/2}U_d(r(s))\big[u_*(0,t(s)) + F_N^0(r(s),t(s))\big]$$

for $s>0$, where $r(s) = e^{s/2-\theta_* s}$ and $t(s) = e^s - 1$. Then

$$\begin{aligned}\frac{d}{ds}w(e^{-\theta_* s},s) = {} & \frac{d}{2}w(e^{-\theta_* s},s) + \frac{U_d'(r(s))}{U_d(r(s))}r'(s)w(e^{-\theta_* s},s)\\ & + e^{ds/2}U_d(r(s))(\partial_t u_*)(0,t(s))t'(s)\\ & \quad + e^{ds/2}U_d(r(s))\big[(\partial_r F_N^0)(r(s),t(s))r'(s) + (\partial_t F_N^0)(r(s),t(s))t'(s)\big]\end{aligned} \tag{2.4.26}$$

for $s>0$. It follows from (2.4.1) that

$$U_d(r(s)) \sim c_* \quad\text{and}\quad U_d'(r(s))r'(s) = O(r(s)^{-1-\delta}r'(s)) = O(e^{-\delta(1/2-\theta_*)s}), \tag{2.4.27}$$

for all sufficiently large $s>0$. On the other hand, by Lemma 2.4.2 and (2.4.13), we have

$$\sup_{t>0}(1+t)^{d/4}\|u_*(t)\|_{L^2(\mathbf{R}^N,\,\nu\,dx)} < \infty. \tag{2.4.28}$$

Then we apply Proposition 2.3.1 with $D = d/4$ and $D' = 0$ to obtain

$$\begin{aligned} &(\partial_t u_*)(0, t(s)) = O(e^{-ds/2-s}),\\ &(\partial_r F_N^0)(r(s), t(s)) = O\big(e^{-ds/2-s} r(s)\big),\\ &(\partial_t F_N^0)(r(s), t(s)) = F_N^1(r(s), t(s)) = O\big(e^{-ds/2-2s} r(s)^2\big), \end{aligned} \tag{2.4.29}$$

for all sufficiently large $s > 0$. By Lemma 2.4.3, (2.4.26), (2.4.27), and (2.4.29), we have (2.4.25). Furthermore, by Lemma 2.2.7, Lemma 2.4.3, (2.4.5), (2.4.16) and (2.4.25) we obtain

$$\begin{aligned} a'(s) &= \frac{c_d}{d} e^{-d\theta_* s}\big(w(e^{-\theta_* s}, s)\big)' - c_d\theta_* e^{-d\theta_* s} w(e^{-\theta_* s}, s)\\ &\quad + c_d\theta_* e^{-d\theta_* s} w(e^{-\theta_* s}, s) + \int_{I(s)} \partial_s w \psi_d \rho_d \, d\xi\\ &= O(e^{-d\theta_* s}) + \int_{I(s)} \partial_\xi(\rho_d \partial_\xi w)\psi_d \, d\xi + \frac{d}{2}\int_{I(s)} w\psi_d\rho_d \, d\xi - \int_{I(s)} \widetilde{V} w \psi_d \rho_d \, d\xi\\ &= O(e^{-d\theta_* s}) + \int_{I(s)} w \partial_\xi(\rho_d \partial_\xi \psi_d) \, d\xi + \frac{d}{2}\int_{I(s)} w\psi_d\rho_d \, d\xi + O(e^{-\theta s/4})\\ &= O(e^{-d\theta_* s}) + O(e^{-\theta s/4}) \end{aligned}$$

for all sufficiently large $s > 0$. This implies (2.4.24). Thus Lemma 2.4.5 follows. □

Proof of Proposition 2.4.4. Set $\widetilde{w}(\xi, s) := \widehat{w}(\xi, s) - a(s)\psi_d(\xi)$. It follows from Lemma 2.2.7 and (2.1.16) that

$$\partial_s \widetilde{w} = \partial_s \widehat{w} - a'(s)\psi_d = -\mathcal{L}_d \widehat{w} - \widetilde{V}\widehat{w} - a'(s)\psi_d = -\mathcal{L}_d \widetilde{w} - \widetilde{V}\widehat{w} - a'(s)\psi_d$$

for $\xi \in I(s)$ and $s > 0$. By Lemma 2.4.3, Lemma 2.4.5, and (2.4.5), we have

$$\begin{aligned} &\frac{d}{ds}\int_{I(s)} |\widetilde{w}(\xi, s)|^2 \rho_d \, d\xi = 2\int_{I(s)} \widetilde{w}(\partial_s \widetilde{w})\rho_d \, d\xi + \theta_* e^{-\theta_* s} |\widetilde{w}(e^{-\theta_* s}, s)|^2 \rho_d(e^{-\theta_* s})\\ &= 2\int_{I(s)} \left(\widetilde{w}\partial_\xi(\rho_d \partial_\xi \widetilde{w}) + \frac{d}{2}\widetilde{w}^2\rho_d - \widetilde{V}\widehat{w}\widetilde{w}\rho_d - a'(s)\psi_d \widetilde{w}\rho_d\right) d\xi + O(e^{-d\theta_* s})\\ &= -2\widetilde{w}(e^{-\theta_* s}, s)\rho_d(e^{-\theta_* s})(\partial_\xi \widetilde{w})(e^{-\theta_* s}, s) - 2\int_{I(s)} |(\partial_\xi \widetilde{w})(\xi, s)|^2 \rho_d \, d\xi\\ &\quad + d\int_{I(s)} |\widetilde{w}(\xi, s)|^2 \rho_d \, d\xi - 2\int_{I(s)} \widetilde{V}\widehat{w}\widetilde{w}\rho_d \, d\xi - 2a'(s)\int_{I(s)} \widetilde{w}\psi_d\rho_d \, d\xi + O(e^{-d\theta_* s})\\ &= -2\int_{I(s)} |(\partial_\xi \widetilde{w})(\xi, s)|^2 \rho_d \, d\xi + d\int_{I(s)} |\widetilde{w}(\xi, s)|^2 \rho_d \, d\xi + O(e^{-d\theta_* s}) + O(e^{-\theta s/4}) \end{aligned} \tag{2.4.30}$$

for all sufficiently large $s > 0$. Furthermore, similarly to (2.4.19), by Lemmas 2.2.7, 2.4.3, and 2.4.5, we obtain

$$
\begin{aligned}
&\int_{I(s)} |(\partial_\xi \widetilde{w})(\xi,s)|^2 \rho_d \, d\xi - \frac{d}{2}\int_{I(s)} |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi \\
&\ge \int_0^\infty |(\partial_\xi \widetilde{w})(\xi,s)|^2 \rho_d \, d\xi - \frac{d}{2}\int_0^\infty |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi - a(s)^2 \int_0^{e^{-\theta_* s}} |\partial_\xi \psi_d|^2 \rho_d \, d\xi \\
&= \int_0^\infty |(\partial_\xi \widetilde{w})(\xi,s)|^2 \rho_d \, d\xi - \frac{d}{2}\int_0^\infty |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi + O(e^{-(d+2)\theta_* s}) \\
&\ge \int_0^\infty |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi + O(e^{-(d+2)\theta_* s}) \ge \int_{I(s)} |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi + O(e^{-(d+2)\theta_* s})
\end{aligned}
\tag{2.4.31}
$$

for all sufficiently large $s > 0$. Therefore we deduce from (2.4.30) and (2.4.31) that

$$
\begin{aligned}
\frac{d}{ds}\int_{I(s)} |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi &\le -2\int_{I(s)} |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi + O(e^{-d\theta_* s}) + O(e^{-\theta s/4}) \\
&= -2\int_{I(s)} |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi + O(e^{-2\theta' s})
\end{aligned}
\tag{2.4.32}
$$

for all sufficiently large $s > 0$. Since $d\theta_* < d\theta < 1$ (see (2.4.6)), by (2.4.32), we have

$$
\int_{I(s)} |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi = O(e^{-2\theta' s}) \tag{2.4.33}
$$

for all sufficiently large $s > 0$. Combining (2.4.33) with Lemmas 2.4.3 and 2.4.5, we obtain

$$
\int_0^\infty |\widetilde{w}(\xi,s)|^2 \rho_d \, d\xi = O(e^{-2\theta' s})
$$

for all sufficiently large $s > 0$. Thus Proposition 2.4.4 follows. □

Proposition 2.4.6. *Let $\|\varphi_*\|_{L^2(\mathbf{R}^N,\, \nu e^{|x|^2/4}\, dx)} = 1$. Assume the same conditions as in Theorem 2.1.1. Then*

$$
|a(s) - m(\varphi)| = O(e^{-2\theta' s}), \tag{2.4.34}
$$

$$
\|\widehat{w}(s) - m(\varphi)\psi_d\|_{L^2(\mathbf{R}_+,\, \rho_d \, d\xi)} = O(e^{-\theta' s}), \tag{2.4.35}
$$

for all sufficiently large $s > 0$. Furthermore, if $m(\varphi) = 0$, then

$$
\|\widehat{w}(s)\|_{L^2(\mathbf{R}_+,\, \rho_d \, d\xi)} = O(e^{-s}), \quad \|w(s)\|_{L^2(\mathbf{R}_+,\, \rho_d \, d\xi)} = O(e^{-s}), \tag{2.4.36}
$$

for all sufficiently large $s > 0$.

Proof. By (2.4.24), we can find a constant a_∞ such that

$$
|a(s) - a_\infty| = O(e^{-2\theta' s}) \quad \text{as} \quad s \to \infty. \tag{2.4.37}
$$

On the other hand, by Lemma 2.4.3, we have

$$\left|\int_{I(s)^c} \widehat{w}\xi^{d-1}\,d\xi\right| \le \left(\int_{I(s)^c} \widehat{w}^2\rho_d\,d\xi\right)^{1/2}\left(\int_{I(s)^c} \xi^{d-1}e^{-\xi^2/4}\,d\xi\right)^{1/2} = O(e^{-d\theta_* s}) \tag{2.4.38}$$

for all sufficiently large $s>0$, where $I(s)^c := \mathbf{R}_+ \setminus I(s)$. By Lemma 2.4.3, (2.4.1), (2.4.11), and (2.4.38), we obtain

$$\begin{aligned} a(s) &= c_d\int_{I(s)} w\xi^{d-1}\,d\xi + O(e^{-d\theta_* s}) \\ &= c_d\int_{(1+t)^{\frac{1}{2}-\theta_*}}^{\infty} u_*(r,t)U_d(r)r^{d-1}\,dr + O(e^{-d\theta_* s}) \\ &= \frac{c_d}{c_*}\int_{(1+t)^{1/2-\theta_*}}^{\infty} u_*(r,t)\nu_d(r)r^{d-1}\,dr + o(1) \end{aligned} \tag{2.4.39}$$

for all sufficiently large $s>0$ and $t>0$ with $s=\log(1+t)$.

On the other hand, by (2.4.28), we apply Proposition 2.3.1 with $D=d/4$ and $D'=0$ to obtain

$$\sup_{0\le r\le (1+t)^{1/2-\theta_*}} |u_*(r,t)| = O(t^{-d/2}) \tag{2.4.40}$$

for all sufficiently large $t>0$. Combining (2.4.40) with (2.4.4), we see that

$$\begin{aligned} &\int_0^{(1+t)^{1/2-\theta_*}} u_*(r,t)\nu_d(r)r^{d-1}\,dr \\ &= O(t^{-d/2})\int_0^{(1+t)^{1/2-\theta_*}} \nu_d(r)r^{d-1}\,dr = O(t^{-d/2})O(t^{d/2-d\theta_*}) = O(t^{-d\theta_*}) \end{aligned} \tag{2.4.41}$$

for all sufficiently large $t>0$. Therefore, by (2.4.39) and (2.4.41), we obtain

$$a_\infty = \lim_{s\to\infty} a(s) = \lim_{t\to\infty}\frac{c_d}{c_*}\int_0^\infty u_*(r,t)\nu_d(r)r^{d-1}\,dr. \tag{2.4.42}$$

On the other hand, since u_* is a radial solution of problem (P), we have

$$\int_0^\infty u_*(r,t)\nu_d(r)r^{d-1}\,dr = \int_0^\infty u_*(r,t)\nu(r)r^{N-1}\,dr = \int_0^\infty \varphi_*(r)\nu(r)r^{N-1}\,dr. \tag{2.4.43}$$

We deduce from (2.4.42) and (2.4.43) that $a_\infty = m(\varphi)$. This together with (2.4.37) implies (2.4.34). Furthermore, by Proposition 2.4.4 and (2.4.34), we have (2.4.35).

It remains to prove (2.4.36). Assume that $m(\varphi)=0$. Then it follows from (2.4.35) and Lemma 2.4.3 that

$$\|\widehat{w}(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} = O(e^{-\theta' s}), \qquad \|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} = O(e^{-\theta' s}), \tag{2.4.44}$$

for all sufficiently large $s > 0$. Applying the same argument as in the proof of (2.4.32), we see that

$$\frac{d}{ds}\int_{I(s)} |\widetilde{w}(\xi,s)|^2\rho_d\,d\xi \le -\int_{I(s)} |\widetilde{w}(\xi,s)|^2\rho_d\,d\xi + O(e^{-4\theta' s})$$

for all sufficiently large $s > 0$. Furthermore, similarly to (2.4.44), we have

$$\|\widehat{w}(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} = O(e^{-2\theta' s}) \quad \text{and} \quad \|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} = O(e^{-2\theta' s}),$$

for all sufficiently large $s > 0$. Repeating this argument, we can find $\tilde{\theta} > 1$ such that

$$\frac{d}{ds}\int_{I(s)} |\widetilde{w}(\xi,s)|^2\rho_d\,d\xi \le -\int_{I(s)} |\widetilde{w}(\xi,s)|^2\rho_d\,d\xi + O(e^{-\tilde{\theta} s})$$

for all sufficiently large $s > 0$, instead of (2.4.32). This implies that

$$\|\widehat{w}(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} = O(e^{-s}) \quad \text{and} \quad \|w(s)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)} = O(e^{-s}),$$

for all sufficiently large $s > 0$. Thus (2.4.36) holds. Therefore the proof of Proposition 2.4.6 is complete. □

We are ready to complete the proof of Theorems 2.1.1 and 2.1.2.

Proof of Theorem 2.1.1. By the linearity of the operator L it suffices to consider only the case

$$1 = \|\varphi\|_{L^2(\mathbf{R}^N,\,e^{|x|^2/4}\,dx)} = \|\varphi_*\|_{L^2(\mathbf{R}^N,\,\nu e^{|x|^2/4}\,dx)} = \left|\mathbf{S}^{N-1}\right|^{1/2}\|w(0)\|_{L^2(\mathbf{R}_+,\,\rho_d\,d\xi)}. \tag{2.4.45}$$

Let $R > 1$. By Lemma 2.4.2, we apply the parabolic regularity theorems (see e.g., [25]) to (2.1.16). Then we can find $\alpha \in (0,1)$ such that

$$\begin{aligned}\|w\|_{C^{2,\,\alpha\,;1,\,\alpha/2}(I_R\times(S,\infty))} &:= \sum_{0\le \ell+2j\le 2}\ \sup_{\xi\in I_R,\,s\in(0,S)} |(\partial_s^j\partial_\xi^\ell w)(\xi,s)| \\ &\quad + \sum_{\ell+2j=2}\ \sup_{\substack{(\xi_1,\,s_1),\,(\xi_2,\,s_2)\in I_R\times(0,S),\\ (\xi_1,\,s_1)\neq(\xi_2,\,s_2)}} \frac{\left|(\partial_s^j\partial_\xi^\ell w)(\xi_1,s_1) - (\partial_s^j\partial_\xi^\ell w)(\xi_2,s_2)\right|}{|\xi_1-\xi_2|^\alpha + |s_1-s_2|^{\alpha/2}} < \infty\end{aligned} \tag{2.4.46}$$

for any $R > 1$ and $S > 0$, where $I_R := [R^{-1}, R]$. Therefore, for any sequence $\{s_i\} \subset \mathbf{R}_+$ with $\lim_{i\to\infty} s_i = \infty$, by Proposition 2.4.6 and (2.1.16), we apply the Ascoli-Arzelà theorem and the diagonal argument to find a subsequence $\{s_i'\} \subset \{s_i\}$ such that

$$\lim_{i\to\infty} \|w(s_i') - m(\varphi)\psi_d\|_{C^2(I_R)} = 0 \quad \text{and} \quad \lim_{i\to\infty} \|(\partial_s w)(s_i')\|_{C^2(I_R)} = 0,$$

for any $R > 1$. Since $m(\varphi)\psi_d$ is independent of the choice of $\{s_i'\}$, we see that

$$\lim_{s\to\infty} \|w(s) - m(\varphi)\psi_d\|_{C^2(I_R)} = 0 \quad \text{and} \quad \lim_{s\to\infty} \|(\partial_s w)(s)\|_{C^2(I_R)} = 0, \tag{2.4.47}$$

for any $R > 1$. Furthermore, if $a_\infty = m(\varphi) = 0$, then, similarly to (2.4.47), by (2.4.36), we have

$$\sup\left\{\left|(\partial_\xi^\ell w)(\xi, s)\right|;\, \xi \in I_R,\, s \geq S\right\} = O(e^{-s}) \quad \text{as} \quad s \to \infty$$

for any $R > 1$, where $\ell = 0, 1, 2$. These together with Proposition 2.4.6 imply (2.1.10) and (2.1.12). Thus Theorem 2.1.1 follows. $\square$

Proof of Theorem 2.1.2. Similarly to the proof of Theorem 2.1.1, we can assume (2.4.45) without loss of generality. Let $T > 0$ and let ε be any sufficiently small positive constant. By Lemma 2.4.2 and (2.4.3), applying Proposition 2.3.1 with $D = d/4$ and $D' = 0$, we obtain

$$(\partial_t^j u_*)(|x|, t) = (\partial_t^j u_*)(0, t) + F_d^j(|x|, t) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \tag{2.4.48}$$

where $j \in \{0, 1, \dots\}$. Furthermore,

$$|F_d^j(r, t)| \leq Ct^{-d/2-j-1} r^2 \quad \text{and} \quad |(\partial_r F_d^j)(r, t)| \leq Ct^{-d/2-j-1} r \tag{2.4.49}$$

for $0 \leq r \leq \varepsilon\sqrt{1+t}$ and $t \geq T$. Then it follows from (2.4.48) and (2.4.49) that

$$|(\partial_r u_*)(r, t)| \leq C_2 t^{-d/2-1} r \tag{2.4.50}$$

for $0 \leq r \leq \varepsilon\sqrt{1+t}$ and $t \geq T$. Furthermore, by (2.4.3) and (2.4.48), we have

$$\begin{aligned} F_d^0(r, t) &= \int_0^r \frac{1}{\nu(s)s^{d-1}} \int_0^s \nu_d(\tau)(\partial_t u_*)(\tau, t)\tau^{d-1}\, d\tau ds \\ &= \int_0^r \frac{1}{\nu(s)s^{d-1}} \int_0^s \nu_d(\tau)\big((\partial_t u_*)(0, t) + F_d^1(\tau, t)\big)\tau^{d-1}\, d\tau ds \\ &= (\partial_t u_*)(0, t)F_d(r) + G_d(r, t) \end{aligned} \tag{2.4.51}$$

for $r \geq 0$ and $t > 0$, where F_d is given in Theorem 2.1.2 and

$$G_d(r, t) = \int_0^r \frac{1}{\nu(s)s^{d-1}} \left(\int_0^s \tau^{d-1}\nu_d(\tau)F_d^1(\tau, t)\, d\tau\right) ds. \tag{2.4.52}$$

Then (2.1.13) holds. In addition, by (2.4.4), (2.4.49) and (2.4.52) we have

$$\begin{aligned} |G_d(r, t)| &\leq Ct^{-d/2-2} \int_0^r \frac{1}{\nu(s)s^{d-1}} \int_0^s \nu_d(\tau)\tau^{d+1}\, d\tau ds \\ &\leq Ct^{-d/2-2} \int_0^r \frac{1}{\nu(s)s^{d-1}} \cdot s^{d+2}\nu_d(s)\, ds \leq Ct^{-d/2-2} r^4 \end{aligned} \tag{2.4.53}$$

for $0 \leq r \leq \varepsilon\sqrt{1+t}$ and $t \geq T$. A similar argument with (2.4.1) implies that

$$\left|(\partial_r^\ell G_d)(r, t)\right| \leq Ct^{-d/2-2} r^{4-\ell}$$

for $0 \leq r \leq \varepsilon\sqrt{1+t}$ and $t \geq T$, where $\ell \in \{1, 2\}$. Thus (2.1.14) holds for $\ell \in \{0, 1, 2\}$.

It remains to prove assertion (b). By (2.4.11) and (2.4.48), we have

$$w(\xi,s)=(1+t)^{d/2}U_d(r)u_*(r,t)=(1+t)^{d/2}U_d(r)\left[u_*(0,t)+F_d^0(r,t)\right] \tag{2.4.54}$$

for $\xi\in\mathbf{R}_+$ and $s>0$ with $\xi=r/\sqrt{1+t}$ and $s=\log(1+t)$. Let $0<\xi<\varepsilon$. By (2.4.1), (2.4.49), and (2.4.54), we obtain

$$\left|w(\xi,s)-(1+t)^{d/2}(c_*+o(1))u_*(0,t)\right|\le C\xi^2 \tag{2.4.55}$$

for all sufficiently large $s>0$ and $t>0$ with $s=\log(1+t)$ and $0<\xi<\varepsilon$. On the other hand, it follows from (2.4.47) that

$$\lim_{s\to\infty}w(\xi,s)=c_dm(\varphi)e^{-\xi^2/4}. \tag{2.4.56}$$

Then we deduce from (2.4.55) and (2.4.56) that

$$\lim_{t\to\infty}t^{d/2}u_*(0,t)=\frac{c_d}{c_*}m(\varphi). \tag{2.4.57}$$

Furthermore, it follows from (2.4.11) that

$$\begin{aligned}(\partial_sw)(\xi,s)&=\frac{d}{2}w(\xi,s)+e^{(d+1)s/2}U_d'(e^{s/2}\xi)\frac{\xi}{2}u_*(e^{s/2}\xi,t)\\&\quad+e^{(d+1)s/2}U_d(e^{s/2}\xi)\frac{\xi}{2}(\partial_ru_*)(e^{s/2}\xi,t)+e^{(d+2)s/2}U_d(e^{s/2}\xi)(\partial_tu_*)(e^{s/2}\xi,e^s-1).\end{aligned}$$

This together with (2.4.1), (2.4.50) and (2.4.56) implies that

$$\begin{aligned}(\partial_sw)(\xi,s)&=\frac{d}{2}w(\xi,s)+e^{s/2}\frac{U_d'(e^{s/2}\xi)}{U_d(e^{s/2}\xi)}\frac{\xi}{2}w(\xi,s)\\&\quad+O(\xi^2)+e^{(d+2)s/2}(c_*+o(1))(\partial_tu_*)(e^{s/2}\xi,e^s-1)\\&=\frac{d}{2}m(\varphi)c_de^{-\xi^2/4}+o(1)+O\big((e^{s/2}\xi)^{-\delta}\big)\\&\quad+O(\xi^2)+e^{(d+2)s/2}\big(c_*+o(1)\big)(\partial_tu_*)(e^{s/2}\xi,e^s-1)\end{aligned} \tag{2.4.58}$$

for all sufficiently large $s>0$. On the other hand, by (2.4.48) and (2.4.49), we have

$$e^{(d+2)s/2}(\partial_tu_*)(e^{s/2}\xi,e^s-1)=e^{(d+2)s/2}(\partial_tu_*)(0,e^s-1)+O(\xi^2) \tag{2.4.59}$$

for all sufficiently large $s>0$. Therefore, by (2.4.47), (2.4.58), and (2.4.59), we obtain

$$\limsup_{s\to\infty}\left|(c_*+o(1))e^{(d+2)s/2}(\partial_tu_*)(0,e^s-1)+\frac{d}{2}c_dm(\varphi)\right|\le C\xi^2.$$

Since $0<\xi<\varepsilon$, we deduce that

$$\limsup_{s\to\infty}\left|e^{(d+2)s/2}(\partial_tu_*)(0,e^s-1)+\frac{dc_d}{2c_*}m(\varphi)\right|=0.$$

This together with (2.4.57) implies assertion (b). Thus Theorem 2.1.2 follows. □

Proof of Theorem 2.1.3. Similarly to the proof of Theorem 2.1.1, we can assume (2.4.45) without loss of generality. Assume the same conditions as in Theorem 2.1.3. Then $d = 2$ and V_{λ_2} satisfies condition (V) with λ_1 and λ_2 replaced by $\lambda_1 - \lambda_2\,(\geq 0)$ and 0, respectively. Applying a similar argument as in the proof of argument as in [18, Proposition 3.1], we have

$$\lim_{s\to\infty} sw(\xi,s) = \frac{1}{c_*}\left[\int_0^\infty w(r,0)U_d(r)r\,dr\right]e^{-\xi^2/4} = 2m(\varphi)\psi_d(\xi) \tag{2.4.60}$$

in $L^2(\mathbf{R}_+,\rho_2\,d\xi)\cap C^2(K)$, for any compact set K in $\mathbf{R}^2\setminus\{0\}$. Furthermore,

$$\lim_{t\to\infty} t(\log t)^2 u_*(0,t) = \frac{2\sqrt{2}}{c^*}m(\varphi) \quad \text{and} \quad \lim_{t\to\infty} t^2(\log t)^2(\partial_t u_*)(0,t) = -\frac{2\sqrt{2}}{c^*}m(\varphi)$$

On the other hand, similarly to (2.4.48), we have

$$(\partial_t^j u_*)(|x|,t) = (\partial_t^j u_*)(0,t) + F_2^j(|x|,t) \quad \text{in} \quad \mathbf{R}^N\times\mathbf{R}_+,$$

where $j\in\{0,1,\dots\}$. It follows from (2.4.60) that

$$\sup_{t>0}(1+t)^{d/4}\log(2+t)\,\|u_*(t)\|_{L^2(\mathbf{R}^N,\,\nu e^{|x|^2/4(1+t)}\,dx)} < \infty.$$

Let $T > 0$ and ε be a sufficiently small positive constant. By (2.4.3), we then apply Proposition 2.3.1 with $D = d/4$ and $D' = 1$ to obtain

$$\left|F_2^j(r,t)\right| \leq Ct^{-1-j-1}\big(\log(2+t)\big)^{-2}r^2$$

for $0\leq r\leq\varepsilon\sqrt{1+t}$ and $t\geq T$. Similarly to (2.4.51) and (2.4.52), we have

$$\begin{aligned} F_2^0(r,t) &= (\partial_t u_*)(0,t)F_2(r) + G_2(r,t),\\ G_2(r,t) &= \int_0^r \frac{1}{\nu_2(s)s}\int_0^s \nu_2(\tau)F_2^1(\tau,t)\tau\,d\tau ds, \end{aligned}$$

for $r\geq 0$ and $t>0$. Furthermore, similarly to (2.4.53), we obtain

$$|G_2(r,t)| \leq Ct^{-3}\big(\log(2+t)\big)^{-2}\int_0^r\frac{1}{\nu_2(s)s}\int_0^s \nu_2(\tau)\tau^3\,d\tau ds \leq Ct^{-3}\big(\log(2+t)\big)^{-2}r^4$$

for $0\leq r\leq\varepsilon\sqrt{1+t}$ and $t\geq T$. A similar argument with (2.4.2) implies that

$$\left|(\partial_r^\ell G_2)(r,t)\right| \leq Ct^{-3}\big(\log(2+t)\big)^{-2}r^{4-\ell}, \quad \ell=1,2,$$

for $0\leq r\leq\varepsilon\sqrt{1+t}$ and $t\geq T$. So we see that (2.1.15) holds for $\ell\in\{0,1,2\}$. Thus Theorem 2.1.3 follows. □

2.5 Proof of Theorem 2.1.4

We use the same notation as in §§ 2.1.2. Let $m \in \{1, 2, \dots\}$. Then

$$L_m := -\Delta + V(|x|) + \frac{\omega_m}{|x|^2}$$

is subcritical and problem (O) corresponding to L_m possesses a positive solution U_m satisfying

$$U_m(r) \sim \begin{cases} r^{A^+(\lambda_1+\omega_m)} & \text{as} \quad r \to 0, \\ c_m\, r^{A^+(\lambda_2+\omega_m)} & \text{as} \quad r \to \infty, \end{cases} \tag{2.5.1}$$

for some positive constant c_m. Set

$$u(x,t) := e^{-tL}\varphi \quad \text{and} \quad u_m(x,t) := u(x,t) - \sum_{k=0}^{m-1} \sum_{i=1}^{\ell_k} e^{-tL}\varphi^{k,i}.$$

Lemma 2.5.1. *Let $m \in \{1, 2, \dots\}$. Then there exists $C_1 > 0$ such that*

$$\|u_m(t)\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4(1+t)}\,dx)} \le C_1 t^{-d_m/4} \|u_m(0)\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)} \tag{2.5.2}$$

for $t > 0$, where $d_m := N + 2A^+(\lambda_2 + \omega_m)$. Furthermore, there exists $C_2 > 0$ such that

$$\left| \frac{u_m(x,t)}{U\big(\min\{|x|, \sqrt{t}\}\big)} \right| \le \frac{C_2 t^{-(N+d_m)/4}}{U(\sqrt{t})} \|u_m(0)\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)} \tag{2.5.3}$$

for $x \in \mathbf{R}^N$ and $t > 0$.

Proof. Let $m \in \{1, 2, \dots\}$. The comparison principle implies that

$$\left| \left[e^{-tL_k}\phi^{k,i}\right](x) \right| \le \left[e^{-tL_k}|\phi^{k,i}|\right](x) \le \left[e^{-tL_m}|\phi^{k,i}|\right](x) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+$$

for $k \in \{m, m+1, \dots\}$ and $i \in \{1, \dots, \ell_k\}$. On the other hand, by Theorem 2.1.1 and (2.5.1) (see also (2.4.28)), we have

$$\left\| e^{-tL_m}|\phi^{k,i}| \right\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4(1+t)}\,dx)} \le C(1+t)^{-d_m/4} \|\phi^{k,i}\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)}, \quad t > 0,$$

for $k \in \{m, m+1, \dots\}$ and $i \in \{1, \dots, \ell_k\}$. These together with (2.1.18) implies that

$$\begin{aligned} \left\| e^{-tL}\varphi^{k,i} \right\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4(1+t)}\,dx)} &= \left|\mathbf{S}^{N-1}\right|^{-1/2} \left\| e^{-tL_k}\phi^{k,i} \right\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4(1+t)}\,dx)} \\ &\le C(1+t)^{-d_m/4} \left\|\phi^{k,i}\right\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)} \le C(1+t)^{-d_m/4} \left\|\varphi^{k,i}\right\|_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)} \end{aligned}$$

for $t > 0$. Therefore we deduce from the orthogonality of $\{Q_{k,i}\}$ that

$$\begin{aligned} \|u_m(t)\|^2_{L^2(\mathbf{R}^N,\, e^{|x|^2/4(1+t)}\,dx)} &= \sum_{k=m}^{\infty} \sum_{i=1}^{\ell_k} \left\| e^{-tL}\varphi^{k,i} \right\|^2_{L^2(\mathbf{R}^N,\, e^{|x|^2/4(1+t)}\,dx)} \\ &\le C(1+t)^{-d_m/2} \sum_{k=m}^{\infty} \sum_{i=1}^{\ell_k} \|\varphi^{k,i}\|^2_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)} \le C(1+t)^{-d_m/2} \left\|u_m(0)\right\|^2_{L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)} \end{aligned}$$

for $t>0$. This implies (2.5.2). On the other hand, by Lemma 2.2.5, we have

$$\frac{|u_m(x,2t)|}{U\big(\min\{|x|,\sqrt{t}\}\big)}\le \frac{Ct^{-N/4}}{U(\sqrt{t})}\|u_m(t)\|_{L^2(\mathbf{R}^N)},\quad x\in\mathbf{R}^N,\ t>0.$$

This together with (2.5.2) implies (2.5.3). Thus Lemma 2.5.1 follows. □

Proof of Theorem 2.1.4. Let $\varphi\in L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)$ and $v:=e^{-tL_0}\varphi^{0,1}$. Let K be any compact set in $\mathbf{R}^N\setminus\{0\}$ and $R>0$. In cases (S) and (C), recalling that $d=2N+A$, by Theorems 2.1.1 and 2.1.2, we have

$$\lim_{t\to\infty} t^{(N+A)2}v\big(\sqrt{t}y,t\big)=c_d m(\varphi^{0,1})|y|^A e^{-|y|^2/4}\quad\text{in}\quad L^2(\mathbf{R}^N,\, e^{|y|^2/4}\,dy)\cap L^\infty(K), \tag{2.5.4a}$$

$$\lim_{t\to\infty} t^{(N+2A)/2}\frac{v(x,t)}{U(|x|)}=\frac{c_d}{c_*}m(\varphi^{0,1})\quad\text{in}\quad L^\infty(B(0,R)). \tag{2.5.4b}$$

In case (S_*), Theorem 2.1.3 implies that

$$\lim_{t\to\infty} t^{(N+A)/2}(\log t)v\big(\sqrt{t}y,t\big)=2c_d m(\varphi^{0,1})|y|^A e^{-|y|^2/4}\quad\text{in}\quad L^2(\mathbf{R}^N, e^{|y|^2/4}\,dy)\cap L^\infty(K), \tag{2.5.5a}$$

$$\lim_{t\to\infty} t^{(N+2A)/2}(\log t)^2\frac{v(x,t)}{U(|x|)}=\frac{2\sqrt{2}}{c_*}m(\varphi^{0,1})=\frac{4c_d}{c_*}m(\varphi^{0,1})\quad\text{in}\quad L^\infty(B(0,R)). \tag{2.5.5b}$$

Here

$$\begin{aligned}c_d m(\varphi^{0,1})&=\frac{c_d^2}{c_*}\int_0^\infty \varphi^{0,1}(r)U(r)r^{N-1}\,dr\\&=\frac{c_d^2}{c_*\big|\mathbf{S}^{N-1}\big|}\int_{\mathbf{R}^N}\varphi^{0,1}(|x|)U(|x|)\,dx=\frac{c_d^2}{c_*\big|\mathbf{S}^{N-1}\big|}\int_{\mathbf{R}^N}\varphi(x)U(|x|)\,dx=M(\varphi).\end{aligned} \tag{2.5.6}$$

Taking a sufficiently large integer m, by Lemma 2.5.1, we have

$$\lim_{t\to\infty} t^{(N+A)/2}u_m\big(\sqrt{t}y,t\big)=0\quad\text{in}\quad L^2(\mathbf{R}^N,\, e^{|y|^2/4}\,dy)\cap L^\infty(K), \tag{2.5.7a}$$

$$\lim_{t\to\infty} t^{(N+2A)/2}\frac{u_m(x,t)}{U(|x|)}=0\quad\text{in}\quad L^\infty(B(0,R)), \tag{2.5.7b}$$

for any compact set $K\subset\mathbf{R}^N\setminus\{0\}$ and $R>0$. On the other hand, L_k is subcritical and $A^+(\lambda_2+\omega_k)>A$ for $k\in\{1,2,\dots,m-1\}$. Then, taking a sufficiently small $\varepsilon>0$ if necessary, by Theorems 2.1.1 and 2.1.2, we obtain

$$\lim_{t\to\infty} t^{(N+A)/2}\left[e^{-tL_k}\phi^{k,i}\right]\big(\sqrt{t}y,t\big)=0\quad\text{in}\quad L^2(\mathbf{R}^N,\, e^{|y|^2/4}\,dy)\cap L^\infty(K), \tag{2.5.8a}$$

$$\lim_{t\to\infty} t^{(N+2A)/2}\frac{\left[e^{-tL_k}\phi^{k,i}\right](x)}{U_k(|x|)}=0\quad\text{in}\quad L^\infty(B(0,R)), \tag{2.5.8b}$$

for any compact set $K \subset \mathbf{R}^N \setminus \{0\}$ and $R > 0$. On the other hand, it follows from (2.1.2) that (2.5.1) that $U_k(r)/U(r)$ is bounded on $(0, R)$ for any $R > 0$. This together with (2.5.8) implies that

$$\lim_{t\to\infty} t^{(N+2A)/2} \frac{\left[e^{-tL_k}\phi^{k,\,i}\right](|x|)}{U(|x|)} = 0 \quad \text{in} \quad L^\infty(B(0,R)) \tag{2.5.9}$$

for any $R > 0$. Since

$$[e^{-tL}\varphi](x) = v(x,t) + \sum_{k=1}^{m-1}\sum_{i=1}^{\ell_k} \left[e^{-tL_k}\phi^{k,\,i}\right](|x|)Q_{k,\,i}\left(\frac{x}{|x|}\right) + u_m(x,t),$$

by (2.5.4)–(2.5.9), we obtain assertions (a) and (b). Thus the proof is complete. $\square$

Proof of Corollary 2.1.5. Let $p = p(x,y,t)$ be the fundamental solution corresponding to e^{-tL}. Let $y \in \mathbf{R}^N$ and $\tau > 0$. Set $\varphi(x) = p(x,y,\tau)$ for $x \in \mathbf{R}^N$. Taking a sufficiently small $\tau > 0$ if necessary, by (2.1.6), we see that $\varphi \in L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)$. On the other hand, since $p(x,y,t) = p(y,x,t)$, we have

$$\int_{\mathbf{R}^N} \varphi(x)U(|x|)\,dx = \int_{\mathbf{R}^N} p(x,y,\tau)U(|x|)\,dx = \int_{\mathbf{R}^N} p(y,x,\tau)U(|x|)\,dx = U(|y|)$$

for $y \in \mathbf{R}^N$ and $\tau > 0$. Then, applying Theorem 2.1.4 and letting $\tau \to 0$, we obtain the desired results. Thus Corollary 2.1.5 follows. $\square$

Chapter 3

Hot spots of solutions to the heat equation with inverse square potential

3.1 Introduction

In this chapter, we investigate the large time behavior of the hot spots

$$H(u(t)) := \left\{ x \in \mathbf{R}^N \,:\, u(x,t) = \sup_{y \in \mathbf{R}^N} u(y,t) \right\}$$

for the solution u of (2.1.1). The study of the large time behavior of the hot spots is delicate and it is obtained by the higher order asymptotic expansion of the solutions. Combing the arguments in [16]–[19], and [23] (Chapter 2), we study the large time behavior of $H(u(t))$ in the cases (S), (S_*), and (C) and reveal the relationship between the large time behavior of $H(u(t))$ and the corresponding harmonic functions (see §§3.2.1). We remark that L is not necessarily subcritical.

The behavior of the hot spots for parabolic equations in unbounded domains has been studied since the pioneering work by Chavel and Karp [4], who studied the behavior of the hot spots for the heat equations on some non-compact Riemannian manifolds. In particular, for the heat equation on $\mathbf{R}^N$ with nonnegative initial data $\varphi \in L^\infty_c(\mathbf{R}^N)$, they proved:

(H1) $H(e^{t\Delta}\varphi)$ is a subset of the closed convex hull of the support of the initial function φ;

(H2) There exists $T > 0$ such that $H(e^{t\Delta}\varphi)$ consists of only one point and moves along a smooth curve for any $t \geq T$;

(H3) $\displaystyle\lim_{t\to\infty} H(e^{t\Delta}\varphi) = \int_{\mathbf{R}^N} x\varphi(x)\,dx \Big/ \int_{\mathbf{R}^N} \varphi(x)\,dx.$

(See also Remark 3.3.3.) The behavior of the hot spots for the heat equation on the half space of $\mathbf{R}^N$ and on the exterior domain of a ball was studied in [14], [15] and [24]. Subsequently, in [16]–[19], Ishige–Kabeya developed the arguments in [14] and [15] and studied the large time behavior of the hot spots for the solution of (2.1.1) under condition (V) in the subcritical case with some additional assumptions.

Our arguments in this chapter are based on [23] (Chapter 2), where the precise description of the large time behavior of the solution of (2.1.1) was discussed under condition (V). Applying the arguments in Chapter 2, we modify the arguments in [16]–[19] and study the large time behavior of the hot spots. We study the following subjects when the hots spots tend to the space infinity as $t \to \infty$:

(a) The rate and the direction for the hot spots to tend to the space infinity as $t \to \infty$;

(b) The number of the hot spots for sufficiently large t.

On the other hand, when the hots spots accumulate to a point x_*, we characterize the limit point x_* by the positive harmonic function U. Furthermore, we give a sufficient condition for the hot spots to consist of only one point and to move along a smooth curve.

The rest of this chapter is organized as follows. In §3.2 we recall some preliminary results on the behavior of the solution of (2.1.1) and prove some lemmas. In §3.3 we study the large time behavior of the hot spots for problem (2.1.1).

3.2 Preliminaries

Throughout this chapter, we use the notations L_k, V_k, $\phi^{k,i}$, $Q_{k,i}$, q_*, and q_N as in subsection 2.1.2. Let U_k be a (unique) solution of

$$U_k'' + \frac{N-1}{r}U_k' - V_k(r)U_k = 0 \quad \text{in} \quad \mathbf{R}^N. \tag{3.2.1}$$

Then

$$U_k(r) \sim \begin{cases} r^{A^+(\lambda_1+\omega_k)} & \text{as} \quad r \to 0, \\ c_k\, r^{A^+(\lambda_2+\omega_k)} & \text{as} \quad r \to \infty, \end{cases} \qquad U_k'(r) = O\big(r^{A^+(\lambda_2+\omega_k)-1}\big) \quad \text{as} \quad r \to \infty, \tag{3.2.2}$$

for some positive constant c_k. By Theorems 2.1.1, 2.1.2, and 2.1.3, we then obtain the precise description of the large time behavior of $e^{-tL^k}\phi^{k,i}$, where $k = 0, 1, \dots$. In particular, for $k = 0, 1, \dots$, for any sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} & t^{N/2+A_k}\partial_r^\ell \frac{[e^{-tL^k}\phi^{k,i}](|x|)}{U_k(|x|)} \\ &= \big[M_{k,i} + o(1)\big]\delta_{0\ell} - \left[\frac{N+2A_k}{2}M_{k,i} + o(1)\right] t^{-1}(\partial_r^\ell F_k)(|x|) + t^{-2}O(|x|^{4-\ell}) \\ &= \big[M_{k,i} + o(1)\big]\delta_{0\ell} + O(t^{-1}|x|^{2-\ell}) \quad \text{as} \quad t \to \infty, \end{aligned} \tag{3.2.3}$$

uniformly for $x \in \mathbf{R}^N$ with $|x| \le \varepsilon\sqrt{1+t}$, where $\ell = 0, 1, 2$ and $\delta_{0,\ell}$ is the Kronecker symbol. Here

$$A_k := \begin{cases} A & \text{for} \quad k = 0, \\ A^+(\lambda_2 + \omega_k) & \text{for} \quad k = 1, 2, \dots, \end{cases} \qquad c_{A_k} := \left(2^{N+2A_k-1} \Gamma\left(\frac{N+2A_k}{2} \right) \right)^{-1/2},$$

$$M_{k,i} := \frac{c_{A_k}^2}{c_k^2} \int_0^\infty \phi^{k,i}(r) U_k(r) r^{N-1}\, dr = \frac{c_{A_k}^2}{c_k^2} \int_{\mathbf{R}^N} U_k(|y|) Q_{k,i}\left(\frac{y}{|y|}\right) \varphi(y)\, dy, \tag{3.2.4}$$

$$F_k(r) := \int_0^r \frac{1}{U_k(s)^2 s^{N-1}} \int_0^s U_k(\tau)^2 \tau^{N-1}\, d\tau ds.$$

Here we used

$$\int_{\mathbf{R}^N} U_k(|y|) Q_{k,i}\left(\frac{y}{|y|}\right) \varphi(y)\, dy = \int_{\mathbf{S}^{N-1}} Q_{k,i}(\theta)^2\, d\theta \int_0^\infty U_k(r) \phi^{k,i}(r) r^{N-1}\, dr$$
$$= \int_0^\infty U_k(r) \phi^{k,i}(r) r^{N-1}\, dr,$$

which follows from the orthonormality of $\{Q_{k,i}\}$ on $L^2(\mathbf{S}^{N-1})$.

3.2.1 Gaussian estimates and the hot spots

Let $p = p(x, y, t)$ be the fundamental solution generated by e^{-tL}. The upper Gaussian estimates of $p = p(x, y, t)$ (see (2.1.6)) implies the following Lemma which ensures the existence of hot spots.

Lemma 3.2.1. *Let u be a solution of* (2.1.1) *under condition* (V), *where $\varphi \in L^2(\mathbf{R}^N, e^{|x|^2/4}\, dx)$. Assume that*

$$\int_{\mathbf{R}^N} \varphi(y) U(|y|)\, dy > 0.$$

Then $H(u(t)) \neq \emptyset$ for $t > 0$. Furthermore, there exist $L > 0$ and $T > 0$ such that

$$H(u(t)) \subset B(0, L\sqrt{t}) \quad \text{for} \quad t \ge T. \tag{3.2.5}$$

Proof. Let $t > 0$. Since U is a harmonic function for L, we see that

$$U(|x|) = \int_{\mathbf{R}^N} p(x, y, t) U(|y|)\, dy = \int_{\mathbf{R}^N} p(y, x, t) U(|y|)\, dy, \quad x \in \mathbf{R}^N.$$

Then the Fubini theorem implies that

$$\int_{\mathbf{R}^N} u(x, t) U(|x|)\, dx = \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} p(x, y, t) U(|x|) \varphi(y)\, dy\, dx = \int_{\mathbf{R}^N} \varphi(y) U(|y|)\, dy > 0.$$

Therefore we can find $x_t \in \mathbf{R}^N$ such that $u(x_t, t) > 0$. On the other hand, by (2.1.6), we can find $R > 0$ such that

$$\sup_{x \in \mathbf{R}^N \setminus B(0,R)} u(x, t) < u(x_t, t).$$

This together with (2.2.1) implies that $H(u(t)) \neq \emptyset$.

We show (3.2.5) in the cases of (S) and (C). Since

$$|x-y|^2 \geq \frac{1}{2}|x|^2 - |y|^2 \quad \text{for} \quad x, y \in \mathbf{R}^N,$$

by (2.1.4) and (2.1.6), we have

$$\begin{aligned} |u(x,t)| &\leq Ct^{-N/2}\left(\int_{B(0,\sqrt{t})} + \int_{\mathbf{R}^N\setminus B(0,\sqrt{t})}\right)\frac{U\big(\min\{|y|,\sqrt{t}\}\big)}{U(\sqrt{t})}\exp\left\{-\frac{|x-y|^2}{Ct}\right\}|\varphi(y)|\,dy \\ &\leq Ct^{-(N+A)/2}e^{-|x|^2/2Ct}\int_{B(0,\sqrt{t})} e^{|y|^2/Ct}U(|y|)|\varphi(y)|\,dy \\ &\quad + Ct^{-N/2}e^{-|x|^2/2Ct}\int_{\mathbf{R}^N\setminus B(0,\sqrt{t})} e^{|y|^2/Ct}|\varphi(y)|\,dy \end{aligned} \tag{3.2.6}$$

for $x \in \mathbf{R}^N$ and $t \geq 1$ with $|x| \geq \sqrt{t}$. Recalling that $\varphi \in L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx)$, by the Cauchy-Schwarz inequality we see that

$$\begin{aligned} &\int_{\mathbf{R}^N\setminus B(0,\sqrt{t})} e^{|y|^2/Ct}|\varphi(y)|\,dy \\ &\leq \left(\int_{\mathbf{R}^N\setminus B(0,\sqrt{t})} e^{2|y|^2/Ct}e^{-|y|^2/4}\,dy\right)^{1/2}\left(\int_{\mathbf{R}^N} e^{|y|^2/4}|\varphi(y)|^2\,dy\right)^{1/2} \\ &\leq C\left(\int_{\mathbf{R}^N\setminus B(0,\sqrt{t})} e^{-|y|^2/8}\,dy\right)^{1/2} \leq Ce^{-t/C} \end{aligned} \tag{3.2.7}$$

for all sufficiently large t. Then, for any $\varepsilon > 0$, by (2.1.4), (3.2.6), and (3.2.7), we can find constants $L \geq 1$ such that

$$\sup_{x\in\mathbf{R}^N\setminus B(0,L\sqrt{t})} |u(x,t)| \leq \varepsilon t^{-(N+A)/2} \tag{3.2.8}$$

for all sufficiently large t. On the other hand, by Theorem 2.1.4 (a), we see that

$$\liminf_{t\to\infty} t^{(N+A)/2}\max_{y\in\partial B(0,1)} u(\sqrt{t}y,t) \geq CM(\varphi) > 0. \tag{3.2.9}$$

Taking a sufficiently small $\varepsilon > 0$ if necessary, we deduce from (3.2.8) and (3.2.9) that $H(u(t)) \subset B(0, L\sqrt{t})$ for all sufficiently large t. Thus (3.2.5) holds in the cases of (S) and (C). Similarly, we can prove (3.2.5) in the case of (S_*). Thus Lemma 3.2.1 follows. □

By Theorem 2.1.4 and Lemma 3.2.1, we have:

Theorem 3.2.2. *Let $N \geq 3$ and L be a nonnegative Schrödinger operator under condition* (V) *with $\lambda_1 < 0$. Let u be a solution of* (2.1.1) *such that*

$$\varphi \in L^2(\mathbf{R}^N,\, e^{|x|^2/4}\,dx) \quad \text{and} \quad \int_{\mathbf{R}^N}\varphi(y)U(|y|)\,dy > 0. \tag{3.2.10}$$

Then $H(u(t)) = \{0\}$ for all sufficiently large $t > 0$.

In the case of $\lambda_1 < 0$, Theorem 2.1.4 together with (3.2.1) implies that $A^+(\lambda_1) < 0$, $U(r) \sim r^{A^+(\lambda_1)}$ as $r \to 0$ and $u(0,t) = \infty$ for all sufficiently large t. Thus Theorem 3.2.2 follows.

3.3 Large time behavior of the hot spots

We study the large time behavior of the hot spots for problem (2.1.1) in the case of $\lambda_1 \geq 0$. Set

$$\Pi := \left\{ r \in [0,\infty)\,;\, U(r) = \sup_{\tau \in [0,\infty)} U(\tau) \right\} \quad \text{and} \quad \Xi(\varphi) := \int_{\mathbf{R}^N} \varphi(y) U_1(|y|) \frac{y}{|y|} dy.$$

For the reader's convenience, we give a correspondence table of our theorems. We recall $A = A^+(\lambda_2)$ in the cases of (S) and (S$_*$) and $A = A^-(\lambda_2)$ in the case of (C).

- (I) $\lambda_1 < 0$ (see Theorem 3.2.2) ;
- (II) $\lambda_1 \geq 0$
  - (1) $A > 0$ (see Theorem 3.3.1) ;
  - (2) $A = 0$
    - (a) $\Pi = \emptyset$ and $N = 2$ (see Theorems 3.3.2 and 3.3.3) ;
    - (b) $\Pi = \emptyset$ and $N \geq 3$ (see Theorem 3.3.3 and Corollaries 3.3.4 and 3.3.5) ;
    - (c) $\Pi \neq \emptyset$ (see Theorems 3.3.8 and 3.3.9) ;
  - (3) $A < 0$ (see Theorems 3.3.6 and 3.3.7).

We consider the case of $A > 0$. Theorem 3.3.1 is proved by the same arguments as in [16, §4] with the aid of the results in §3.2. See also Theorem 2.1.4 and [16, Theorem 1.2].

Theorem 3.3.1. *Let L be a nonnegative Schrödinger operator under condition* (V) *with $\lambda_1 \geq 0$ and $A > 0$. Assume* (3.2.10) *and let u be a solution of* (2.1.1). *Then the following holds*:

(a) $\displaystyle \lim_{t\to\infty} \sup_{x \in H(u(t))} \left| \frac{|x|}{\sqrt{t}} - \sqrt{2A} \right| = 0\,;$

(b) *Assume that $\Xi(\varphi) \neq 0$. Then there exist a constant $T > 0$ and a curve $x = x(t) \in C^1\big([T,\infty) : \mathbf{R}^N\big)$ such that $H(u(t)) = \{x(t)\}$ for $t \geq T$ and*

$$\lim_{t\to\infty} \frac{x(t)}{|x(t)|} = \frac{\Xi(\varphi)}{|\Xi(\varphi)|}.$$

Secondly, we consider the case where $A = 0$ and $\Pi = \emptyset$. Theorems 3.3.2 and 3.3.3 are obtained by the same arguments as in [18, §4] and [17, §4, §5], respectively, with the aid of the results in §3.2. See also [18, Theorem 1.2] and [17, Theorem 1.2]. We remark that, in the case of $N = 2$, $A = 0$ if and only if $\lambda_2 = 0$. Furthermore, if L is subcritical, then $U(r) \asymp \log r$ as $r \to \infty$ and $\Pi = \emptyset$.

Theorem 3.3.2. *Let $N=2$ and L be a subcritical Schrödinger operator under condition* (V) *with $\lambda_1 \ge 0$ and $A=0$. Assume* (3.2.10) *and let u be a solution of* (2.1.1). *Then*

$$\lim_{t\to\infty} \sup_{x\in H(u(t))} \left| \frac{\log t}{t}|x|^2 - 2 \right| = 0.$$

Furthermore, assertion (b) *of Theorem* 3.3.1 *holds.*

Theorem 3.3.3. *Let L be a nonnegative Schrödinger operator under condition* (V) *with $\lambda_1 \ge 0$, $A=0$ and $\Pi = \emptyset$. Assume that L is critical if $N=2$. Assume* (3.2.10) *and let u be a solution of* (2.1.1). *Then*

$$\lim_{t\to\infty} \sup_{x\in H(u(t))} \left| \frac{tU'(|x|)}{c_*|x|} - \frac{1}{2} \right| = 0.$$

Furthermore, assertion (b) *of Theorem* 3.3.1 *holds.*

Let $\lambda_1 \ge 0$ and

$$H_k(r) := r^k \int_0^r \frac{1}{s^{N+2k-1}} \int_0^s V(\tau)U_k(\tau)\tau^{N+k-1}\, d\tau ds \quad \text{for} \quad k=0,1,\dots.$$

Since $W_k := U_k - H_k$ satisfies

$$W_k'' + \frac{N-1}{r}W_k' - \frac{\omega_k}{r^2}W_k = 0 \quad \text{in} \quad \mathbf{R}_+ \quad \text{and} \quad W_k(r) \asymp r^{A^+(\lambda_1+\omega_k)} \quad \text{as} \quad r\to 0,$$

by the uniqueness of the solution of (3.2.1) with V replaced by $\omega_k r^{-2}$, we see that

$$U_k(r) = \begin{cases} r^k + H_k(r) & \text{if} \quad \lambda_1 = 0, \\ H_k(r) & \text{if} \quad \lambda_1 > 0. \end{cases} \tag{3.3.1}$$

In particular, in the case of $A=0$, we have

$$c_* \equiv \lim_{r\to\infty} U(r) = \begin{cases} H_0(\infty)+1 & \text{if} \quad \lambda_1 = 0, \\ H_0(\infty) & \text{if} \quad \lambda_1 > 0. \end{cases}$$

As a corollary of Theorem 3.3.3, we have the following result, which revises [17, Corollary 1.1] and [18, Remark 1.1].

Corollary 3.3.4. *Assume the same conditions as in Theorem* 3.3.3 *with $\lambda_1 = 0$. Furthermore, assume that $V(r) \sim \mu r^{-d}$ as $r\to\infty$ for some $\mu \ne 0$ and $d > 2$.*

(a) *Let $\mu > 0$. Then*

$$|x| = \begin{cases} \left(\dfrac{2\mu t}{(H_0(\infty)+1)(N-d)} \right)^{1/d} (1+o(1)) & \text{if} \quad 2 < d < N, \\ \left(\dfrac{2\mu\, t \log t}{(H_0(\infty)+1)N} \right)^{1/N} (1+o(1)) & \text{if} \quad d = N, \\ \left(\dfrac{2\Lambda t}{H_0(\infty)+1} \right)^{1/N} (1+o(1)) & \text{if} \quad d > N, \quad \Lambda > 0, \end{cases}$$

as $t \to \infty$ *uniformly for* $x \in H(u(t))$. *Here* $\Lambda := \displaystyle\int_0^\infty \tau^{N-1} V(\tau) U(\tau)\, d\tau$.

(b) *Let* $\mu < 0$ *and* $d > N$. *Then*

$$|x| = \begin{cases} \left(\dfrac{2\Lambda t}{H_0(\infty) + 1} \right)^{1/N} (1 + o(1)) & \text{if} \quad \Lambda > 0, \\ \left(\dfrac{2|\mu| t}{(H_0(\infty) + 1)(d - N)} \right)^{1/d} (1 + o(1)) & \text{if} \quad \Lambda = 0, \end{cases}$$

as $t \to \infty$ *uniformly for* $x \in H(u(t))$.

Furthermore, we have :

Corollary 3.3.5. *Assume the same conditions as in Theorem* 3.3.3 *with* $\lambda_1 > 0$. *Then the same assertions of Corollary* 3.3.4 *holds with* $H_0(\infty) + 1$ *replaced by* $H_0(\infty)$.

Remark 3.3.1. Assume the same conditions as in Theorem 3.3.3. Let $V(r) \sim \mu r^{-d}$ as $r \to \infty$ for some $\mu \neq 0$ and $d > 2$.
(i) Consider the case where $\mu > 0$ and $d > N$. Since $U(r) \sim c_* > 0$ as $r \to \infty$, Λ can be defined. If $\Lambda \leq 0$ and $\mu > 0$, then it follows from (3.3.1) that

$$U'(r) = r^{1-N} \int_0^r V(\tau) U(\tau) \tau^{N-1}\, d\tau = r^{1-N} \left(\Lambda - \int_r^\infty V(\tau) U(\tau) \tau^{N-1}\, d\tau \right) < 0$$

for all sufficiently large $r > 0$. This implies that $\Pi \neq \emptyset$.
(ii) Consider the case where $\mu < 0$. By (3.3.1), we see that $r^{N-1} U'(r) \to -\infty$ as $r \to \infty$ if $2 < d \leq N$. Similarly, if $d > N$ and $\Lambda < 0$, then $U'(r) < 0$ for all sufficiently large $r > 0$. In the both cases, it follows that $\Pi \neq \emptyset$.

Next we study the large time behavior of the hot spots in the case where $\lambda_1 \geq 0$ and $A < 0$. It follows from $A < 0$ that $U(r) \to 0$ as $r \to \infty$ and $\Pi \neq \emptyset$.

Theorem 3.3.6. *Let* L *be a nonnegative Schrödinger operator under condition* (V) *with* $\lambda_1 \geq 0$ *and* $A < 0$. *Assume* (3.2.10) *and let* u *be a solution of* (2.1.1). *Then*

$$\lim_{t \to \infty} \sup_{x \in H(u(t))} \big| |x| - \min \Pi \big| = 0. \tag{3.3.2}$$

Furthermore, if $\Xi(\varphi) \neq 0$, *then*

$$\lim_{t \to \infty} \sup_{x \in H(u(t))} \left| x - \min \Pi \frac{\Xi(\varphi)}{|\Xi(\varphi)|} \right| = 0. \tag{3.3.3}$$

Proof. For any $\varepsilon > 0$, by Theorem 2.1.4 with $A < 0$ and Lemma 3.2.1, we see that

$$H(u(t)) \subset B(0, \varepsilon\sqrt{t}) \tag{3.3.4}$$

for all sufficiently large t.

We consider the cases of (S) and (C). In the case of $\Xi(\varphi) \neq 0$ we can assume, without loss of generality, that $\Xi(\varphi) = \big(|\Xi(\varphi)|, 0, \dots, 0\big)$. By (3.2.4), we have

$$M_{0,1} > 0 \quad \text{and} \quad M_{1,i} = q_N \frac{c_{A_1}^2}{c_1^2} \Xi_i(\varphi) = q_N \frac{c_{A_1}^2}{c_1^2} |\Xi(\varphi)| \delta_{1,i} \quad \text{for} \quad i = 1, \dots, N, \tag{3.3.5}$$

where $\delta_{1,i}$ is the Kronecker symbol. Let $\varepsilon > 0$ be sufficiently small. By Lemma 2.5.1, (2.1.4), (2.1.17), and (3.2.3), we take a sufficiently large $m \in \{1, 2, \dots\}$ so that

$$\begin{aligned}
t^{N/2+A} \frac{u_0(x,t)}{q_* U(|x|)} &= \big(M_{0,1} + o(1)\big) - \left(\frac{N+2A}{2} M_{0,1} + o(1)\right) t^{-1} F(|x|) + t^{-2} O(|x|^4) \\
&= M_{0,1} + o(1) + t^{-1} O(|x|^2), \\
u_{1,i}(x,t) &= q_N \big(M_{1,i} + o(1)\big) t^{-N/2-A_1} U_1(|x|) \frac{x_i}{|x|} + t^{-N/2-A_1} U_1(|x|) O(t^{-1}|x|^2) \\
&= O\big(t^{-N/2-A_1} (1+|x|)^{A_1}\big), \\
u_2(x,t) &= \sum_{k=2}^{m-1} \sum_{i=1}^{\ell_k} u_{k,i}(x,t) + u_m(x,t) = O\big(t^{-N/2-A_2} (1+|x|)^{A_2}\big),
\end{aligned} \tag{3.3.6}$$

as $t \to \infty$ uniformly for $x \in \mathbf{R}^N$ with $|x| \le \varepsilon\sqrt{1+t}$, where $i = 1, \dots, N$. Let ν be a sufficiently small positive constant. Since $A < 0$, we can find $R > 0$ such that

$$\begin{aligned}
&|u_0(x,t)| \le C t^{-N/2-A} U(|x|) \le \nu t^{-N/2-A}, \\
&|u_{1,i}(x,t)| \le C t^{-(N+A_1)/2} = C t^{-N/2-A} t^{(A-A_1)/2} t^{A/2} \le \nu t^{-N/2-A}, \\
&|u_2(x,t)| \le C t^{-(N+A_2)/2} = C t^{-N/2-A} t^{(A-A_2)/2} t^{A/2} \le \nu t^{-N/2-A},
\end{aligned} \tag{3.3.7}$$

for $x \in \mathbf{R}^N$ and all sufficiently large $t > 0$ with $R \le |x| \le \varepsilon\sqrt{1+t}$, where $i = 1, \dots, N$. On the other hand, Theorem 2.1.4 implies that

$$\liminf_{t \to \infty} t^{N/2+A} \sup_{x \in \mathbf{R}^N} u(x,t) > 0. \tag{3.3.8}$$

By (3.3.4), (3.3.7), and (3.3.8), we can find $R > 0$ such that

$$H(u(t)) \subset B(0, R) \tag{3.3.9}$$

for all sufficiently large t.

It follows from (2.1.3) and $\lambda_2 < 0$ that

$$A + 1 \le A^+(\lambda_2) + 1 < A^+(\lambda_2 + \omega_1) = A_1. \tag{3.3.10}$$

By (3.3.6) and (3.3.10), we have

$$
\begin{aligned}
&t^{N/2+A}u(x,t)\\
&= q_*\big(M_{0,1}+o(1)\big)U(|x|) - q_*\left(\frac{N+2A}{2}M_{0,1}+o(1)\right)t^{-1}U(|x|)F(|x|)+o(t^{-1}) \qquad (3.3.11)\\
&= q_*\big(M_{0,1}+o(1)\big)U(|x|)+O(t^{-1})
\end{aligned}
$$

as $t\to\infty$ uniformly for $x\in B(0,R)$. Since F is strictly monotone increasing in $\mathbf{R}_+$, by (3.3.5), (3.3.9), and (3.3.11), we obtain (3.3.2) and (3.3.3). Therefore Theorem 3.3.6 follows in the cases of (S) and (C). Similarly, Theorem 3.3.6 also follows in the case of (S_*). Thus the proof is complete. □

We give sufficient conditions for the hot spots to consist of only one point and to move along a smooth curve. We denote by $\nabla^2 f$ the Hessian matrix of a function f. For any real symmetric $N\times N$ matrix M, by $M\ge 0$ and $M\le 0$ we mean that M is positive semi-definite and negative semi-definite, respectively.

Theorem 3.3.7. *Let L be a nonnegative Schrödinger operator under condition* (V) *with $\lambda_1\ge 0$ and $A<0$. Assume* (3.2.10) *and let u be a solution of* (2.1.1)*. Let $x_*\in\mathbf{R}^N$ be such that $|x_*|\in\Pi$ and*

$$
\lim_{t\to\infty}\sup_{x\in H(u(t))}|x-x_*|=0.
$$

Then there exist a constant $T>0$ and a curve $x=x(t)\in C^1([T,\infty):\mathbf{R}^N)$ such that $H(u(t))=\{x(t)\}$ for $t\ge T$ in the following cases:

(a) *$|x_*|=0$, $V\in C^\gamma([0,\infty))$ for some $\gamma\in(0,1)$ and $\nabla^2U(|x|)\le 0$ in a neighborhood of the origin;*

(b) *$|x_*|>0$, $U''\le 0$ in a neighborhood of $r=|x_*|$ and $\Xi(\varphi)\ne 0$.*

Proof. We consider the cases of (S) and (C). Let $r_*:=|x_*|$ and $\varepsilon>0$. The proof is divided into the following four cases:

$$
\begin{array}{llll}
(\mathrm{i})\ \ r_*=0 \ \text{ and } \ U''(0)<0; & & (\mathrm{ii})\ \ r_*=0 \ \text{ and } \ U''(0)=0;\\
(\mathrm{iii})\ \ r_*>0 \ \text{ and } \ U''(r_*)<0; & & (\mathrm{iv})\ \ r_*>0 \ \text{ and } \ U''(r_*)=0.
\end{array}
$$

In case (i)**.** Since $U'(0)=0$ and $U''(0)<0$, by (2.1.17) and (3.2.3), we can find $\eta_1>0$ such that

$$
-\frac{1}{q_*}t^{N/2+A}(\nabla^2u_0)(x,t)=-\big(M_{0,1}+o(1)\big)\nabla^2U(|x|)+O(t^{-1})\ge-\frac{M_{0,1}U''(0)}{2}I_N-\varepsilon I_N \qquad (3.3.12)
$$

for $x\in B(0,\eta_1)$ and all sufficiently large t, where I_N is the N-dimensional identity matrix. On the other hand, by condition (a), (3.2.3) and (2.5.3) we apply the parabolic regularity theorems to see that $u_{1,i}$, $R_2\in C^{2,\gamma;1,\gamma/2}(\mathbf{R}^N\times\mathbf{R}_+)$ and

$$
\big\|\nabla^2u_{1,i}\big\|_{L^\infty(B(0,\eta_1))}+\big\|\nabla^2R_2\big\|_{L^\infty(B(0,\eta_1))}=O(t^{-N/2-A_1}) \qquad (3.3.13)
$$

for all sufficiently large t, where $i = 1, 2, \ldots, N$. Since ε is arbitrary, by (3.3.12) and (3.3.13), we see that $-(\nabla^2 u)(x,t)$ is positive definite in $B(0,\eta_1)$ for all sufficiently large $t > 0$. Then Theorem 3.3.7 in case (i) follows from the implicit function theorem.

In case (ii)**.** By condition (a), (2.1.17), and (3.2.3), we have

$$
\begin{aligned}
&-\frac{1}{q_*} t^{N/2+A+1} (\nabla^2 u_0)(x,t) \\
&= -t\big(M_{0,1} + o(1)\big)\nabla^2 U(|x|) + \left(\frac{N+2A}{2} M_{0,1} + o(1)\right) \nabla^2 (UF_0)(|x|) + O(t^{-1}) \\
&\geq \left(\frac{N+2A}{2} M_{0,1} + o(1)\right) \nabla^2 (UF_0)(|x|) + O(t^{-1})
\end{aligned}
\tag{3.3.14}
$$

in a neighborhood of $x = 0$ and all sufficiently large t. On the other hand, it follows from (3.2.1) and (3.2.4) that

$$
\lim_{r\to 0} F_0''(r) = \lim_{r\to 0} \left(-\frac{N-1}{U(r)^2 r^N} - \frac{2U'(r)}{U(r)^3 r^{N-1}} \right) \int_0^r U(\tau)^2 \tau^{N-1}\, d\tau + 1 = \frac{1}{N}. \tag{3.3.15}
$$

This implies that $(UF_0)''(0) = 1/N$. Therefore, by (3.3.14) and (3.3.15), we can find $\eta_2 > 0$ such that

$$
-\frac{1}{q_*} t^{N/2+A+1} (\nabla^2 u_0)(x,t) \geq \frac{N+2A}{4N} M_{0,1} I_N - \varepsilon I_N \tag{3.3.16}
$$

for $x \in B(0,\eta_2)$ and all sufficiently large t. Similarly to case (i), since ε is arbitrary, by (3.3.10), (3.3.13), and (3.3.16), we see that $-(\nabla^2 u)(x,t)$ is positive definite in $B(0,\eta_2)$ for all sufficiently large $t > 0$. Similarly to case (i), Theorem 3.3.7 in case (ii) follows from the implicit function theorem.

In case (iii)**.** By Theorem 3.3.6, we can assume, without loss of generality, that $x_* = (r_*, 0, \ldots, 0)$. Then $M_{1,1} > 0$ and $M_{1,i} = 0$ for $i \in \{2, \ldots, N\}$. Let $\theta_\alpha := x_\alpha/|x|$ for $\alpha = 1, 2, \ldots, N$. Then $(r, \theta_2, \ldots, \theta_N)$ gives a local coordinate of $\mathbf{R}^N$ in a neighborhood of x_*. We study the large time behavior of $\widetilde{\nabla}^2 u$ in a neighborhood of x_*, where $\widetilde{\nabla} := (\partial_r, \partial_{\theta_2}, \ldots, \partial_{\theta_N})$. Since $U''(r_*) < 0$, similarly to (3.3.12), we can find $\eta_3 > 0$ such that

$$
-\frac{1}{q_*} t^{N/2+A} (\partial_r^2 u_0)(x,t) = -\big(M_{0,1} + o(1)\big)(\partial_r^2 U)(|x|) + O(t^{-1}) \geq -\frac{M_{0,1}}{2} U''(r_*) \tag{3.3.17}
$$

for $x \in B(x_*, \eta_3)$ and all sufficiently large t. Furthermore,

$$
(\partial_r \partial_{\theta_\alpha} u_0)(x,t) = (\partial_{\theta_\alpha} \partial_{\theta_\beta} u_0)(x,t) = 0 \tag{3.3.18}
$$

for $x \in B(x_*, \eta_3)$ and all sufficiently large t, where α, $\beta \in \{2, \ldots, N\}$.

On the other hand, by (2.1.17), (3.2.3), and (3.3.6), we have

$$
\frac{1}{q_N} (\widetilde{\nabla}^2 u_{1,i})(x,t) = \big(M_{1,i} + o(1)\big) t^{-N/2+A_1} \widetilde{\nabla}^2 (U_1(|x|)\theta_i) + O(t^{-N/2-A_1-1}) \tag{3.3.19}
$$

for $x \in B(x_*, \eta_3)$ and all sufficiently large t. Since

$$\theta_1 = \sqrt{1 - \sum_{\alpha=2}^{N} \theta_\alpha^2}, \quad \frac{\partial \theta_1}{\partial \theta_\alpha} = -\frac{\theta_\alpha}{\theta_1}, \quad \frac{\partial^2 \theta_1}{\partial \theta_\alpha \partial \theta_\beta} = -\frac{\delta_{\alpha,\beta}}{\theta_1} - \frac{\theta_\alpha \theta_\beta}{\theta_1^3}, \tag{3.3.20}$$

for α, $\beta \in \{2, \dots, N\}$, combining $M_{1,1} = q_N c_{A_1}^2 |\Xi(\varphi)|/c_1^2 > 0$ and $U'(r_*) = 0$, we can find $\eta_4 > 0$ and $C > 0$ such that

$$\begin{aligned} &- t^{N/2+A_1} (\partial_r^2 u_{1,1})(x,t) \geq -C, \\ &- t^{N/2+A_1} (\partial_{\theta_\alpha} \partial_{\theta_\beta} u_{1,1})(x,t) \geq \frac{q_N M_{1,1}}{2} U_1(r_*) \delta_{\alpha,\beta} - \varepsilon, \\ &- t^{N/2+A_1} (\partial_r \partial_{\theta_\alpha} u_{1,1})(x,t) \geq -C|\theta_\alpha| U_1'(r) + O(t^{-1}) \geq -\varepsilon, \end{aligned} \tag{3.3.21}$$

for $x \in B(x_*, \eta_4)$ and all sufficiently large t. Furthermore, for $i = 2, \dots, N$, it follows that $M_{1,i} = 0$ and we have

$$\big(\widetilde{\nabla}^2 u_{1,i}\big)(x,t) = o(t^{-N/2-A_1}) \tag{3.3.22}$$

for $x \in B(x_*, \eta_4)$ and all sufficiently large t. Similarly to (3.3.13), by (3.3.6), we apply the parabolic regularity theorems to obtain

$$\big(\widetilde{\nabla}^2 R_2\big)(x,t) = O(t^{-N/2-A_2}) \tag{3.3.23}$$

for $x \in B(x_*, \eta_4)$ and all sufficiently large t. On the other hand, $A_1 > A + 1$ holds by $A < 0$. Then, by (3.3.17), (3.3.18), (3.3.19), (3.3.21), (3.3.22), and (3.3.23), we see that $-\big(\widetilde{\nabla}^2 u\big)(x,t)$ is positive definite in a neighborhood of $x_* = (r_*, 0)$ for all sufficiently large $t > 0$. Therefore Theorem 3.3.7 in case (iii) follows from the implicit function theorem.

In case (iv)**.** Similarly to the case (iii), without loss of generality, we can assume that $\Xi(\varphi)/|\Xi(\varphi)| = (1, 0, \dots, 0)$. It follows from $U'(r_*) = U''(r_*) = 0$ and $r_* \in \Pi$ that

$$\begin{aligned} (UF_0)''(r_*) &= U(r_*) F_0''(r_*) = U(r_*) \left(-\frac{N-1}{U(r_*)^2 r_*^N} \int_0^{r_*} U(\tau)^2 \tau^{N-1} \, d\tau + 1 \right) \\ &\geq U(r_*) \left(-\frac{N-1}{U(r_*)^2 r_*^N} \int_0^{r_*} U(r_*)^2 \tau^{N-1} \, d\tau + 1 \right) = \frac{1}{N} U(r_*) > 0. \end{aligned}$$

Then, by condition (b) we have

$$\begin{aligned} &- \frac{1}{q_*} t^{N/2+A+1} (\partial_r^2 u_0)(x,t) \\ &= -\big(M_{0,1} + o(1)\big) t (\partial_r^2 U)(|x|) + \left(\frac{N+2A}{2} M_{0,1} + o(1) \right) \big(\partial_r^2 (UF)\big)(|x|) + O(t^{-1}) \\ &\geq \left(\frac{N+2A}{2} M_{0,1} + o(1) \right) \big(\partial_r^2 (UF)\big)(|x|) + O(t^{-1}) \geq \frac{N+2A}{4N} M_{0,1} U(r_*) > 0 \end{aligned} \tag{3.3.24}$$

in a neighborhood of $x_* = (r_*, 0, \dots, 0)$. Furthermore, by the same argument as in case 3.3.7 in case (iii) we obtain (3.3.18), (3.3.21), (3.3.22) and (3.3.23). Therefore, since $A_1 > A+1$, we see that $-\big(\widetilde{\nabla}^2 u\big)(x,t)$ is positive definite in a neighborhood of $x_* = (r_*, 0)$ for all sufficiently large $t > 0$. Therefore Theorem 3.3.7 in case (iv) follows from the implicit function theorem. Thus Theorem 3.3.7 follows in the cases of (S) and (C). Similarly, Theorem 3.3.7 also follows in case (S_*). Therefore the proof of Theorem 3.3.7 is complete. □

Finally we study the large time behavior of the hot spots in the cases where $\lambda_1 \ge 0$, $A = 0$, and $\Pi \ne \emptyset$.

Theorem 3.3.8. *Let L be a nonnegative Schrödinger operator under condition* (V) *with $\lambda_1 \ge 0$, $A = 0$, and $\Pi \ne \emptyset$. Assume* (3.2.10) *and let u be a solution of* (2.1.1)*. Then there exists $R > 0$ such that*

$$H(u(t)) \subset B(0,R) \tag{3.3.25}$$

for all sufficiently large t.

Let x_* be an accumulating point of $H(u(t))$ as $t \to \infty$. If $\Xi(\varphi) \ne 0$, then $x_*/|x_*| = \Xi(\varphi)/|\Xi(\varphi)|$. Furthermore, $r_* := |x_*|$ is a maximum point of

$$S(r) := -\frac{\kappa N}{c_*} M(\varphi) U(r) F_0(r) + \frac{1}{c_1^2} |\Xi(\varphi)|\, U_1(r) \quad \text{on} \quad \Pi.$$

Remark 3.3.2. Assume the same conditions as in Theorem 3.3.8. It follows from $A = 0$ that $\lambda_2 = 0$ and $A_1 = 1$. By (2.2.3) and (3.2.2), we see that

$$F_0(r) \sim \frac{1}{2N} r^2, \qquad U(r) \sim c_*, \qquad U_1(r) \sim c_1 r, \tag{3.3.26}$$

as $r \to \infty$. Then $S(|x|) \to -\infty$ as $|x| \to \infty$ and the maximum point of S on Π exists.

Proof. Let ε be a sufficiently small positive constant. Similarly to (3.3.4), by Theorem 2.1.4 with $A = 0$, we see that

$$H(u(t)) \subset B(0, \varepsilon\sqrt{t}) \tag{3.3.27}$$

for all sufficiently large t. On the other hand, similarly to the proof of Theorem 3.3.6, we have

$$\begin{aligned} t^{N/2} u(x,t) = {} & q_*\big(M_{0,1} + o(1)\big) U(|x|) - q_* \left(\frac{N}{2} M_{0,1} + o(1)\right) t^{-1} (UF_0)(|x|) \\ & + t^{-2} O(|x|^4 U(|x|)) + \sum_{i=1}^{N} \big(M_{1,i} + o(1)\big) t^{-1} U_1(|x|) Q_{1,i}\left(\frac{x}{|x|}\right) \\ & + t^{-2} U_1(|x|) O(|x|^2) + o(t^{-1}) \end{aligned} \tag{3.3.28}$$

as $t \to \infty$ uniformly for $x \in \mathbf{R}^N$ with $|x| \le \varepsilon\sqrt{t}$. Since $A_1 = A^+(\omega_1) = 1$, it follows from (2.1.17), (3.2.4) and (3.2.10) that

$$q_* M_{0,1} = \frac{q_*^2 \kappa}{2^{N-1} c_* \Gamma(N/2)} M(\varphi),$$

and

$$\sum_{i=1}^{N} M_{1,i}Q_{1,i}\left(\frac{x}{|x|}\right) = \frac{q_N^2}{2^{N+1}c_1^2\Gamma\big((N+2)/2\big)}\left(\frac{x}{|x|}\cdot\int_{\mathbf{R}^N}\varphi(y)U_1(|y|)\frac{y}{|y|}\,dy\right)$$
$$= \frac{q_N^2}{2^N c_1^2\Gamma(N/2)}\left(\frac{x}{|x|}\cdot\Xi(\varphi)\right).$$

Then we have

$$\begin{aligned}&-q_*\frac{N}{2}M_{0,1}U(|x|)F_0(|x|) + \sum_{i=1}^{N} M_{1,i}U_1(|x|)Q_{1,i}\left(\frac{x}{|x|}\right)\\ &= \frac{q_*^2}{2^N\Gamma(N/2)}\left(-\frac{\kappa N}{c_*}M(\varphi) + \frac{1}{c_1^2}\left(\frac{x}{|x|}\cdot\Xi(\varphi)\right)U_1(|x|)\right).\end{aligned} \tag{3.3.29}$$

Then Theorem 3.3.8 follows from (3.3.26), (3.3.27), (3.3.28) and (3.3.29). □

Modifying Theorem 3.3.7, we give sufficient conditions for the hot spots to consist of only one point and to move along a smooth curve in the case where $A = 0$ and $\Pi \neq \emptyset$. We remark that $A_1 = A + 1$ if $A = 0$.

Theorem 3.3.9. *Let L be a nonnegative Schrödinger operator under condition* (V) *with $\lambda_1 \geq 0$, $A = 0$ and $\Pi \neq \emptyset$. Assume* (3.2.10) *and let u be a solution of* (2.1.1). *Let $x_* \in \mathbf{R}^N$ be such that $|x_*| \in \Pi$ and*

$$\lim_{t\to\infty}\sup_{x\in H(u(t))}|x - x_*| = 0.$$

Then there exist a constant $T > 0$ and a curve $x = x(t) \in C^1\big([T,\infty) : \mathbf{R}^N\big)$ such that $H(u(t)) = \{x(t)\}$ for $t \geq T$ in the following cases:

(a) *$|x_*| = 0$, $V \in C^\gamma([0,\infty))$ for some $\gamma \in (0,1)$ and $\nabla^2 U(|x|) \leq 0$ in a neighborhood of the origin;*

(b) *$|x_*| > 0$, $U''(r_*) < 0$ and $\Xi(\varphi) \neq 0$;*

(c) *$|x_*| > 0$, $U''(r) \leq 0$ in a neighborhood of $r = r_*$, $S''(r_*) < 0$ and $\Xi(\varphi) \neq 0$.*

Here $S = S(r)$ is as Theorem 3.3.8.

Proof. The proofs in case (a) with $U''(0) < 0$ and case (b) are obtained by the same argument as the proof of Theorem 3.3.7 in cases (i) and (iii), respectively. So it suffices to consider case (a) with $U''(0) = 0$ and case (c).

Let us consider case (a) with $U''(0) = 0$. Let $\varepsilon > 0$. It follows from (3.3.1) that $U_1'(0) = 0$ and

$$U_1''(r) = -\frac{N-1}{r^{N+1}}\int_0^r V(\tau)U_1(\tau)\tau^N\,d\tau + V(r)U_1(r) \to 0 \quad \text{as} \quad r \to 0.$$

These imply that $U_1 \in C^2\big([0,\infty)\big)$ and $U_1''(0) = 0$. Then, similarly to (3.3.21), we have

$$-t^{N/2+1}(\partial_r^2 u_{1,1})(x,t) \geq -\varepsilon$$

in a neighborhood of $x = 0$ for all sufficiently large $t > 0$. Then, applying a similar argument as in the proof of Theorem 3.3.7 in case (ii), we obtain Theorem 3.3.9 in case (a) with $U''(0) = 0$.

Let us consider case (c). Similarly to the proof of Theorem 3.3.7 in case (iii), without loss of generality, we can assume that $\Xi(\varphi)/|\Xi(\varphi)| = (1,0,\dots,0)$ and $x_* = (r_*,0,\dots,0)$ and we introduce the coordinate $(r,\theta_2,\dots,\theta_N)$ in a neighborhood of $x = x_*$. Then, by (2.1.17) and (3.2.3), we have

$$\begin{aligned}
&-\frac{1}{q_*}t^{N/2}(\partial_r^2 u_0)(x,t) \\
&= -\big(M_{1,0}+o(1)\big)(\partial_r^2 U)(|x|) + \left(\frac{N}{2}M_{0,1}+o(1)\right)t^{-1}\big(\partial_r^2(UF_0)\big)(|x|) + O(t^{-2}) \qquad (3.3.30)\\
&\geq \left(\frac{N}{2}M_{0,1}+o(1)\right)t^{-1}\big(\partial_r^2(UF_0)\big)(|x|) + O(t^{-2})
\end{aligned}$$

and

$$-\frac{1}{q_N}t^{N/2+1}(\partial_r^2 u_{1,1})(x,t) = -\big(M_{1,1}+o(1)\big)(\partial_r^2 U_1)(|x|)\theta_i + O(t^{-1}), \qquad (3.3.31)$$

in a neighborhood of $x = x_*$ for all sufficiently large $t > 0$, where $i = 1,\dots,N$. By condition (c), (3.3.29), and (3.3.30), we obtain

$$\begin{aligned}
&-t^{N/2+1}[(\partial_r^2 u_0)(x,t) + (\partial_r^2 u_{1,1})(x,t)] \\
&\geq -\frac{q_*^2}{2^N\Gamma(N/2)}S''(r) + o(1) + O(t^{-1}) \geq -\frac{q_*^2}{2^{N+1}\Gamma(N/2)}S''(r_*) > 0
\end{aligned}$$

in a neighborhood of $x = x_*$ for all sufficiently large $t > 0$. Similarly, we have

$$-t^{N/2+1}(\partial_r\partial_{\theta_\alpha}u_{1,1})(x,t) = O(|\theta_\alpha|) + O(t^{-1}) \qquad (3.3.32)$$

in a neighborhood of $x = x_*$ for all sufficiently large $t > 0$, where $\alpha = 2,\dots,N$. Furthermore, similarly to the proof of Theorem 3.3.7 in case (iii), we have (3.3.18), (3.3.22) and (3.3.23). Then, combining (3.3.30), (3.3.31), and (3.3.32), we see that $-\big(\widetilde{\nabla}^2 u\big)(x,t)$ is positive definite in a neighborhood of $x_* = (r_*,0)$ for all sufficiently large $t > 0$. Thus Theorem 3.3.9 in case (c) follows from the implicit function theorem. Therefore the proof of Theorem 3.3.9 is complete. □

Remark 3.3.3. Consider the case of the heat equation under condition (3.2.10). Then $V \equiv 0$, $c_* = 1$, $c_1 = 1$, $U(r) = 1$, $U_1(r) = r$, $F_0(r) = r^2/2N$, and $\Pi = [0,\infty)$. Since

$$S'(r) = -rM(\varphi) + |\Xi(\varphi)| = -r\int_{\mathbf{R}^N}\varphi(y)\,dy + \left|\int_{\mathbf{R}^N} y\varphi(y)\,dy\right|,$$

it follows from Theorem 3.3.8 that the hot spots converges to $\int_{\mathbf{R}^N} y\varphi(y)\,dy / \int_{\mathbf{R}^N} \varphi(y)\,dy$. Furthermore, if $\Xi(\varphi) \neq 0$, then, by Theorem 3.3.9 (c), we see that the hot spots consist of only one point and move along a smooth curve. These coincide with statements (H2) and (H3) in §3.1.

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