博士論文

論文題目

On the geometry of projections of von Neumann algebras

（ von Neumann 環の射影束の幾何構造について ）

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Abstract

We study the geometry of projections of von Neumann algebras from two different viewpoints.

In the first part, we consider the metric structure of projections, and investigate surjective isometries between projection lattices of two von Neumann algebras. We show that such mappings are characterized by means of Jordan *-isomorphisms. In particular, we prove that two von Neumann algebras without type I_1 direct summands are Jordan *-isomorphic if and only if their projection lattices are isometric. Our theorem extends a result for type I factors by G.P. Gehér and P. Šemrl, which is a generalization of Wigner's theorem.

In the second part, we consider the lattice structure of projections. Generalizing von Neumann's result on type II_1 von Neumann algebras, we characterize lattice isomorphisms between projection lattices of arbitrary von Neumann algebras by means of ring isomorphisms between the algebras of locally measurable operators. Moreover, we give a complete description of ring isomorphisms of locally measurable operator algebras when the von Neumann algebras are without type II direct summands.

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ON THE GEOMETRY OF PROJECTIONS OF VON NEUMANN ALGEBRAS

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1. Introduction

Since the very first work [31] by Murray and von Neumann more than 80 years ago, the geometry of projections has played the central role in understanding the structure of von Neumann algebras (rings of operators). For a von Neumann algebra M, let $\mathcal{P}(M)$ denote the projection lattice of M, that is, $\mathcal{P}(M) := \{p \in M \mid p = p^* = p^2\}$. In this thesis, we would like to study the geometry of projection lattices. This thesis is based on two papers [27, 29].

In the first part of the thesis, we investigate the metric structure of projection lattices. In particular, we consider surjective isometries between projection lattices of two von Neumann algebras. The study of isometries between operator algebras has a long history. The first achievement in this field dates back to 1951 by Kadison [17]. He proved that if $\phi\colon A \to B$ is a complex-linear surjective isometry between two unital C*-algebras, then $\phi(1)$ is a unitary operator in B and the mapping $x \mapsto \phi(1)^{-1}\phi(x)$, $x \in A$ is a Jordan *-isomorphism. (A linear bijection $J\colon A \to B$ between two C*-algebras is called a *Jordan *-isomorphism* if it satisfies $J(x^*) = J(x)^*$ and $J(xy + yx) = J(x)J(y) + J(y)J(x)$ for any $x, y \in A$.) On the other hand, recall that the celebrated Mazur–Ulam theorem asserts that every surjective isometry between two Banach spaces is affine. Also, Mankiewicz's generalization [22] of this theorem states that every surjective isometry between open connected subsets of Banach spaces is affine. This gives rise to a question which asks whether an analogous result holds for isometries between substructures of operator algebras. In recent years, there have been several great developments in such a study. Hatori and Molnár proved that every surjective isometry between unitary groups of two von Neumann algebras extends uniquely to a real-linear surjective isometry [16]. Tanaka applied this theorem to consider Tingley's problem for finite von Neumann algebras [44]. Tingley's problem asks whether every surjective isometry between unit spheres of two Banach spaces admits a real-linear extension. Stimulated by Tanaka's research, Tingley's problem began to be considered in various settings of operator algebras. See [26], [37] and [30] for latest progresses in such a study.

Since projection lattices play very important roles in the theory of von Neumann algebras, it is natural to ask whether a result similar to Hatori and Molnár's theorem holds for isometries between projection lattices. Here we give an observation which seems to imply an affirmative answer to this question. Let M be a von Neumann algebra. The symbol $\mathcal{U}(M)$ means the unitary group of M, that is, $\mathcal{U}(M) := \{u \in$

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$M \mid u^*u = 1 = uu^*\}$. Consider two projections $p_1 := \mathrm{diag}(1,0)$, $p_2 := \mathrm{diag}(0,1) \in \mathcal{P}(\mathbb{M}_2(M))$. Then we have

$$\left\{ p \in \mathcal{P}(\mathbb{M}_2(M)) \,\middle|\, \|p - p_1\| = \frac{1}{\sqrt{2}} = \|p - p_2\| \right\} = \left\{ \frac{1}{2}\begin{pmatrix} 1 & u \\ u^* & 1 \end{pmatrix} \,\middle|\, u \in \mathcal{U}(M) \right\}.$$

This set is isometric to $\mathcal{U}(M)/2 = \{u/2 \mid u \in \mathcal{U}(M)\}$. By the Hatori–Molnár theorem, this set contains much information about M.

It is well known that the distance between two distinct connected components in the projection lattice of a von Neumann algebra is always 1. Thus, in order to consider surjective isometries between projection lattices of von Neumann algebras, it suffices to consider isometries between connected components. In this thesis, a connected component in $\mathcal{P}(M)$ which contains more than one element is called a *Grassmann space* in M. We know that every Jordan *-isomorphism between two von Neumann algebras restricts to isometries between Grassmann spaces. Another example of an isometry between Grassmann spaces on M can be obtained by the mapping $p \mapsto p^\perp$ $(:= 1 - p)$. In the first part of the thesis, we show that every surjective isometry between Grassmann spaces can be decomposed to such mappings (Theorem 3.1).

As for the case $M = B(H)$, the research of isometries between Grassmann spaces is motivated by Wigner's unitary-antiunitary theorem. Wigner's theorem plays an important role in the mathematical foundation of quantum mechanics. Let $\mathcal{P}_1(H)$ stand for the collection of rank 1 projections on a complex Hilbert space H. Note that $\mathcal{P}_1(H)$ is a Grassmann space in $B(H)$. Wigner's theorem shows that every surjective isometry from $\mathcal{P}_1(H)$ onto itself extends to a Jordan *-automorphism on $B(H)$. See Introduction of [3]. After several attempts (e.g. [3], [12]) to generalize this result, Gehér and Šemrl recently gave a complete description of surjective isometries between two Grassmann spaces in $B(H)$ [13]. They made use of the idea of geodesics between two projections, which is also essential in our proof of Theorem 3.1. See also [42], [10], [36] and [25], [39], in which mappings between projection lattices with an assumption which is different from ours are studied.

In Section 3, we give the proof of Theorem 3.1. Throughout the proof, we depend on the idea by Gehér and Šemrl for $B(H)$ in [13], but we need more discussions in order to consider general von Neumann algebras. Our strategy is as follows. We see that we may assume every projection in the Grassmann spaces is finite or properly infinite, and the mapping preserves orthogonality in both directions. By the Hatori–Molnár theorem combined with the idea about $\mathbb{M}_2(M)$ as above, we can construct a Jordan *-isomorphism between small subspaces. Using that, we extend the given mapping to a bijection between whole projection lattices which preserves orthogonality in both directions. Finally, we make use of Theorem 1.4 below due to Dye [6] to complete the proof.

In Section 4, by means of Theorem 3.1, we consider surjective isometries between projection lattices. We show that two von Neumann algebras without type I_1 direct summands are Jordan *-isomorphic if and only if their projection lattices are isometric (Theorem 4.1). We also consider concrete cases when two von Neumann algebras are factors.

In the second part of the thesis, we study the lattice structure of projection lattices. We consider the following question: What is the general form of lattice isomorphisms between projection lattices of von Neumann algebras?

There are several important results related to this question. Let us first think about finite dimensional factors. The case $M = N = \mathbb{M}_n(\mathbb{C})$ for $n = 1, 2$ is not interesting at all. Indeed, if $n = 1$, then $\mathcal{P}(\mathbb{M}_n(\mathbb{C}))$ is $\{0, 1\}$, and a lattice automorphism of it is the identity mapping. If $n = 2$, then a bijection Φ on $\mathcal{P}(\mathbb{M}_n(\mathbb{C}))$ is a lattice automorphism if and only if $\Phi(0) = 0$ and $\Phi(1) = 1$. If $M = N = \mathbb{M}_n(\mathbb{C})$ for $3 \leq n < \infty$, then the fundamental theorem of projective geometry gives an answer to our question. Recall that a function $f\colon X \to Y$ between complex vector spaces is said to be *semilinear* if it is additive and there exists a ring homomorphism $\sigma\colon \mathbb{C} \to \mathbb{C}$ satisfying $f(cx) = \sigma(c)f(x)$ for all $c \in \mathbb{C}$ and $x \in X$.

Theorem 1.1 (Fundamental theorem of projective geometry)**.** *Let $3 \leq n < \infty$. Suppose that $\Phi\colon \mathcal{P}(\mathbb{M}_n(\mathbb{C})) \to \mathcal{P}(\mathbb{M}_n(\mathbb{C}))$ is a lattice automorphism. Then there exists a semilinear bijection $f\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\Phi(p_\xi) = p_{f(\xi)}$ for every $\xi \in \mathbb{C}^n$, where p_ξ denotes the projection from $\mathbb{C}^n$ onto $\mathbb{C}\xi$ for a vector $\xi \in \mathbb{C}^n$.*

In the case of type I_∞ factors, we can make use of a result below by Fillmore and Longstaff in 1984. Recall that a projection $p \in \mathcal{P}(B(H))$ can be identified with its range pH, which is a closed subspace of H.

Theorem 1.2 ([8, Theorem 1])**.** *Let X and Y be infinite dimensional complex normed spaces. Let $\mathcal{C}(X)$ (resp. $\mathcal{C}(Y)$) denote the lattice of all closed subspaces of X (resp. Y), ordered by inclusion. Suppose that $\Phi\colon \mathcal{C}(X) \to \mathcal{C}(Y)$ is a lattice isomorphism. Then there exists a bicontinuous linear or conjugate-linear bijection $f\colon X \to Y$ such that $\Phi(C) = f(C)$ for any $C \in \mathcal{C}(X)$.*

See also the classical result [19, Theorem 1], in which Kakutani and Mackey studied orthocomplementation on the lattice $\mathcal{P}(B(H))$.

For type I factors, we may observe a correspondence between lattices and rings. Let H be a Hilbert space with $\dim H \geq 3$. For any lattice automorphism $\Phi\colon \mathcal{P}(B(H)) \to \mathcal{P}(B(H))$, take a mapping $f\colon H \to H$ as above. It is a semilinear bijection if $\dim H < \infty$; a linear or conjugate-linear bounded bijection if $\dim H = \infty$. Hence we may construct a ring automorphism $\Psi\colon B(H) \to B(H)$ such that $\Phi(l(x)) = l(\Psi(x))$ for every $x \in B(H)$ (namely, $\Psi(x) := f \circ x \circ f^{-1}$), where $l(x)$ denotes the left support projection of x. It is easy to see that the converse also holds. That is, any ring automorphism $\Psi\colon B(H) \to B(H)$ determines a lattice automorphism Φ of $\mathcal{P}(B(H))$ such that $\Phi(l(x)) = l(\Psi(x))$ for every $x \in B(H)$.

We next consider finite von Neumann algebras. In 1930's, motivated by the geometry of projection lattices of type II_1 factors, von Neumann produced the beautiful theory on the correspondence between complemented modular lattices and regular rings. One of his achievements [34, Part II, Theorem 4.2], applied to the case of arbitrary type II_1 von Neumann algebras, reads as follows.

Theorem 1.3 (von Neumann)**.** *Let M and N be von Neumann algebras of type II_1. Suppose that $\Phi\colon \mathcal{P}(M) \to \mathcal{P}(N)$ is a lattice isomorphism. Then there exists a unique ring isomorphism $\Psi\colon S(M) \to S(N)$ between the algebras of measurable operators such that $\Phi(l(x)) = l(\Psi(x))$ for any $x \in S(M)$.*

See Section 5 for the definition of undefined terms and see also Section 8 for further details about von Neumann's theory.

In the general setting of von Neumann algebras, with an additional assumption, Dye obtained the following result in 1955.

Theorem 1.4 ([6, Corollary of Theorem 1], see also [7, Theorem 1])**.** *Let M and N be von Neumann algebras without type I_2 direct summands. Suppose that $\Phi\colon \mathcal{P}(M) \to \mathcal{P}(N)$ is a lattice isomorphism with*

$$pq = 0 \iff \Phi(p)\Phi(q) = 0$$

for any $p, q \in \mathcal{P}(M)$. Then there exists a real *-isomorphism $\Psi\colon M \to N$ that extends Φ.

Each of the above results implies that lattice isomorphisms between projection lattices are closely related to ring isomorphisms. See also McAsey's survey [24] which discusses projection lattice isomorphisms in various settings. It is natural to imagine that we can give a similar result for arbitrary lattice isomorphisms in the general setting of von Neumann algebras. The following theorem realizes it.

Theorem A. *Let M and N be two von Neumann algebras. Suppose that M does not admit type I_1 nor I_2 direct summands, and that $\Phi\colon \mathcal{P}(M) \to \mathcal{P}(N)$ is a lattice isomorphism. Then there exists a unique ring isomorphism $\Psi\colon LS(M) \to LS(N)$ such that $\Phi(l(x)) = l(\Psi(x))$ for all $x \in LS(M)$.*

Here, $LS(M)$ and $LS(N)$ mean the algebras of locally measurable operators of M and N, respectively (see Section 5.2). We remark that the converse of Theorem A can be verified without difficulty. Namely, any ring isomorphism $\Psi\colon LS(M) \to LS(N)$ determines a unique lattice isomorphism $\Phi\colon \mathcal{P}(M) \to \mathcal{P}(N)$ such that $\Phi(l(x)) = l(\Psi(x))$ for all $x \in LS(M)$ (Proposition 6.1). Therefore, Theorem A naturally gives rise to the following

Question. *Let M, N be von Neumann algebras. What is the general form of ring isomorphisms from $LS(M)$ onto $LS(N)$?*

We may answer this question for finite type I von Neumann algebras using ring isomorphisms of their centers (Proposition 7.2). Moreover, we obtain

Theorem B. *Let M, N be von Neumann algebras of type I_∞ or III. If $\Psi\colon LS(M) \to LS(N)$ is a ring isomorphism, then there exist a real *-isomorphism $\psi\colon M \to N$ (which extends to a real *-isomorphism from $LS(M)$ onto $LS(N)$) and an invertible element $y \in LS(N)$ such that $\Psi(x) = y\psi(x)y^{-1}$, $x \in LS(M)$.*

In Section 5, we introduce some tools we use later. Section 6 is devoted to the proof of Theorem A. The proof is based on the combination of von Neumann's strategy in [34, Part II, Chapter IV] and a binary relation on the projection lattice which we call LS-orthogonality. After that we give a proof of Dye's theorem as an application of Theorem A. We consider Question in Section 7, and prove Theorem B. This thesis ends with comparison of our result with von Neumann's theory and several suggestions of further research directions (Section 8).

Notation Throughout the thesis, we use standard terminology and basic properties concerning the geometry of projection lattices. See for example [18, Chapter 6] or [43, Chapter V.1]. Let $M \subset B(H)$ be a von Neumann algebra. We use the symbol $\sim$ to mean the Murray–von Neumann equivalence relation on $\mathcal{P}(M)$. That is, for $p, q \in \mathcal{P}(M)$, $p \sim q$ means that there exists a partial isometry $v \in M$ such that $p = vv^*$ and $q = v^*v$. In addition, we write $p \prec q$ when there exists a partial isometry $v \in M$ such that $vv^* = p$ and $v^*v \leq q$. As usual, for $p, q \in \mathcal{P}(M)$, $p \perp q$ means that p and q are orthogonal. That is, $pq = qp = 0$, or equivalently, $pH \perp qH$

in the Hilbert space H. We use the symbol $p^\perp := 1-p$ for $p \in \mathcal{P}(M)$. The symbol $\mathcal{Z}(M) = \{x \in M \mid xy = yx \text{ for all } y \in M\}$ means the center of M, and $z(p)$ denotes the central support of p for a projection $p \in \mathcal{P}(M)$.

2. Basic tool: Halmos's two projection theorem

In order to play with projection lattices, it is useful to look at the relative position of a pair of projections. For it, we make use of Halmos's two projection theorem [15] (see also [4]) from the viewpoint of von Neumann algebra theory. Here we recapitulate the argument in [27, Lemma 2.2]. (A similar argument can be found in, for example, [43, pp. 306–308].)

Let $M \subset B(H)$ be a von Neumann algebra and $p, q \in \mathcal{P}(M)$. Put

$$e_1 = p - p\wedge q - p \wedge q^\perp, \quad e_2 = p^\perp - p^\perp \wedge q - p^\perp \wedge q^\perp,$$

and $x := e_1(q - p\wedge q - p^\perp \wedge q)e_2$. By an elementary calculation, we see that the left and right support projections of x are e_1 and e_2, respectively. By polar decomposition, we may take a partial isometry $v = v_{p,q} \in M$ such that $x = v|x| = |x^*|v$, $vv^* = e_1$ and $v^*v = e_2$.

We can identify each $y \in (e_1+e_2)M(e_1+e_2)$ with $\begin{pmatrix} e_1ye_1 & e_1yv^* \\ vye_1 & vyv^* \end{pmatrix} \in \mathbb{M}_2(e_1Me_1)$. Then $q - p\wedge q - p^\perp\wedge q \,(\leq e_1+e_2)$ is identified with

$$\begin{aligned}
&\begin{pmatrix} e_1(q - p\wedge q - p^\perp \wedge q)e_1 & e_1(q - p\wedge q - p^\perp \wedge q)v^* \\ v(q - p\wedge q - p^\perp \wedge q)e_1 & v(q - p\wedge q - p^\perp \wedge q)v^* \end{pmatrix} \\
&= \begin{pmatrix} e_1(q - p\wedge q - p^\perp \wedge q)e_1 & |x^*| \\ |x^*| & v(q - p\wedge q - p^\perp \wedge q)v^* \end{pmatrix} \in \mathbb{M}_2(e_1Me_1).
\end{aligned}$$

Put $a := (e_1(q-p\wedge q-p^\perp\wedge q)e_1)^{1/2}$ and $b := (v(q-p\wedge q-p^\perp\wedge q)v^*)^{1/2}$, which are positive injective operators in $M_{p,q} := e_1Me_1$. Since $\begin{pmatrix} a^2 & |x^*| \\ |x^*| & b^2 \end{pmatrix}$ is a projection, some calculations show that a, b and $|x^*|$ commute, $a^2+b^2 = e_1$ and $|x^*| = ab$. Thus $q - p\wedge q - p^\perp \wedge q$ corresponds to $\begin{pmatrix} a^2 & ab \\ ba & b^2 \end{pmatrix}$.

Therefore, we may decompose p and q in the following manner:

$$p = 1 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = 0 \oplus 1 \oplus 1 \oplus 0 \oplus \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix},$$

where H is decomposed as $H = (p\wedge q^\perp)H \oplus (p^\perp \wedge q)H \oplus (p\wedge q)H \oplus (p^\perp\wedge q^\perp)H \oplus (e_1+e_2)H$, and a and b are positive injective operators in $M_{p,q}$ $(= e_1Me_1)$ such that $a^2+b^2 = 1_{M_{p,q}}$.

Part 1. Isometries between projection lattices of von Neumann algebras

3. Isometries between Grassmann spaces

Let M be a von Neumann algebra. Let $\mathcal{P}$ be a Grassmann space in M. That is, $\mathcal{P}$ is a connected component in $\mathcal{P}(M)$ with more than one element. Let $p \in \mathcal{P}$. It is an elementary exercise to show that a projection $q \in \mathcal{P}(M)$ belongs to $\mathcal{P}$ if and only if p is unitarily equivalent to q in M. Thus the pair $(z(p), z(p^\perp))$ of central projections does not depend on the choice of $p \in \mathcal{P}$. In this thesis, a Grassmann space $\mathcal{P}$ in M is said to be *proper* if $z(p) = 1 = z(p^\perp)$ for every $p \in \mathcal{P}$. Fix a projection $p_0 \in \mathcal{P}$. The mapping $p \mapsto pz(p_0)z(p_0^\perp)$ determines a bijection from $\mathcal{P}$ onto a proper Grassmann space in the von Neumann algebra $Mz(p_0)z(p_0^\perp)$. Therefore, in order to consider surjective isometries between Grassmann spaces, we may assume that these Grassmann spaces are proper.

The main theorem of this section is the following one:

Theorem 3.1. *Let M, N be von Neumann algebras and $\mathcal{P} \subset M$, $\mathcal{Q} \subset N$ be proper Grassmann spaces. Suppose $T\colon \mathcal{P} \to \mathcal{Q}$ is a surjective isometry. Then there exist a Jordan *-isomorphism $J\colon M \to N$ and a central projection $r \in \mathcal{P}(N)$ which satisfy*

$$T(p) = J(p)r + J(p^\perp)r^\perp, \quad p \in \mathcal{P}.$$

We construct this section to some extent along the lines of the paper [13] by Gehér and Šemrl. For two projections $p, q \in \mathcal{P}(M)$, we write $p \bigtriangleup q$ if there exists a central projection $r \in M$ such that $pr \perp qr$ and $p^\perp r^\perp \perp q^\perp r^\perp$. Note that this relation is a generalization of the relation which is written as "$\sim$" in the paper [13]. (We save the symbol $\sim$ for the Murray–von Neumann equivalence.)

Proposition 3.2. *Let $M \subset B(H)$ be a von Neumann algebra, $\mathcal{P}$ be a Grassmann space in M and $p, q \in \mathcal{P}$ with $\|p - q\| = 1$. Then we have $p \bigtriangleup q$ if and only if the following holds.*

<u>Condition</u> *Set $m(p, q) := \{e \in \mathcal{P} \mid \|e - p\| = \|e - q\| = 1/\sqrt{2}\}$. Then $m(p, q)$ is not empty, and for every $p_0 \in m(p, q)$, there exists a unique path $\gamma\colon [0, \pi/2] \to \mathcal{P}$ which satisfies*

$$\gamma(0) = p, \quad \gamma(\pi/2) = q, \quad \gamma(\pi/4) = p_0$$

and

$$\|\gamma(\theta_1) - \gamma(\theta_2)\| = \sin|\theta_1 - \theta_2|$$

for all $\theta_1, \theta_2 \in [0, \pi/2]$.

Proof. The discussion in the paper [13] can be applied almost verbatim, so we give only a sketch of the proof.

Suppose $p \bigtriangleup q$. It suffices to consider the case $p \perp q$. Fix a partial isometry $v \in M$ which satisfies $vv^* = p$ and $v^*v = q$. We can identify $x \in (p+q)M(p+q)(\subset M)$ with $\begin{pmatrix} pxp & pxv^* \\ vxp & vxv^* \end{pmatrix} \in \mathbb{M}_2(pMp)$. Then, it follows

$$m(p, q) = \left\{ \frac{1}{2} \begin{pmatrix} 1 & u \\ u^* & 1 \end{pmatrix} \middle| \, u \in \mathcal{U}(pMp) \right\} \subset (p+q)M(p+q) \subset M.$$

Let $u \in \mathcal{U}(pMp)$ and put $e := \frac{1}{2} \begin{pmatrix} 1 & u \\ u^* & 1 \end{pmatrix} \in m(p, q)$. Then the same discussion as in [13, Lemma 2.5] shows that, the only path $\gamma\colon [0, \pi/2] \to \mathcal{P}$ as in Condition is given by

$$\gamma(\theta) = \begin{pmatrix} \cos^2\theta & u\cos\theta\sin\theta \\ u^*\cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \quad \theta \in [0, \pi/2].$$

Suppose p and q satisfy Condition. We decompose p and q by means of Halmos's two projection theorem:

$$H = (p\wedge q^\perp)H \oplus (p^\perp \wedge q)H \oplus (p\wedge q)H \oplus (p^\perp \wedge q^\perp)H \oplus e_1H \oplus e_2H,$$

$$p = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0 \end{pmatrix}, \quad q = \begin{pmatrix} 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&a^2&ab\\ 0&0&0&0&ba&b^2 \end{pmatrix}.$$

Since $m(p, q) \neq 0$, by the same discussion as in [13, Lemma 2.4], there exists a partial isometry $v \in M$ which satisfies $vv^* = p \wedge q^\perp$ and $v^*v = p^\perp \wedge q$. If $p \wedge q^\perp = 0$, then the condition $\|p - q\| = 1$ implies $\|b\| = 1$. As [13, Lemma 2.9], there exists a projection $p_0 \in m(p, q)$ which admits more than one path with the property as in Condition. We can also show by [13, Lemmas 2.8 and 2.9] that the following does not happen: $p \wedge q^\perp \neq 0$ and $e_1 \neq 0$. Thus we have $p \wedge q^\perp \neq 0$ and $0 = e_1\,(\sim e_2)$. Then p and q commutes. If there exist subprojections $0 \neq p_1 \le p\wedge q$ and $q_1 \le p^\perp \wedge q^\perp$ in M which satisfy $p_1 \sim q_1$, then we can easily construct more than one path for the projection $p_0 = (p\wedge q^\perp + v + v^* + p^\perp \wedge q)/2 + p\wedge q \in m(p, q)$, which contradicts Condition. Hence there exists a central projection $r \in M$ with $p \wedge q \le r^\perp$ and $p^\perp \wedge q^\perp \le r$. It follows $pr \perp qr$ and $p^\perp r^\perp \perp q^\perp r^\perp$. $\square$

We begin the proof of Theorem 3.1. Let $\mathcal{P} \subset M$ and $\mathcal{Q} \subset N$ be proper Grassmann spaces and suppose $T\colon \mathcal{P} \to \mathcal{Q}$ is a surjective isometry. The preceding proposition implies that, for $p, q \in \mathcal{P}$, $p \bigtriangleup q$ if and only if $T(p) \bigtriangleup T(q)$.

By the comparison theorem, there exists a central projection $r_0 \in \mathcal{P}(M)$ which satisfies $pr_0 \prec p^\perp r_0$ and $pr_0^\perp \succ p^\perp r_0^\perp$ for some (and thus every) $p \in \mathcal{P}$. We say that a mapping between Grassmann spaces (or between von Neumann algebras) is *typical* if it can be written as in the equation in the statement of Theorem 3.1. Since the composition of two typical mappings is also typical, in order to show Theorem 3.1, we may and do assume that $p \prec p^\perp$ for every $p \in \mathcal{P}$ and $q \prec q^\perp$ for every $q \in \mathcal{Q}$.

Our next task is to decompose T into two mappings. We need preliminaries.

Lemma 3.3. *Let $\mathcal{P} \subset M$ be a proper Grassmann space in a von Neumann algebra $M \subset B(H)$ with $p \prec p^\perp$ for every $p \in \mathcal{P}$.*

- (a) *If $e, f \in \mathcal{P}$ and $e \bigtriangleup f$, then $m(e, f)$ is isometric to $\mathcal{U}(eMe)/2$ $(= \{u/2 \mid u \in \mathcal{U}(eMe)\})$.*
- (b) *Suppose $p_1, p_2 \in \mathcal{P}$ satisfy $\|p_1 - p_2\| < 1$. Then there exist projections $e, f \in \mathcal{P}$ such that $e \bigtriangleup f$ and $p_1, p_2 \in m(e, f)$.*

Proof. (a) It suffices to consider two cases: $e \perp f$ or $e^\perp \perp f^\perp$. In the former case, there exists a partial isometry $v \in M$ such that $vv^* = e$ and $v^*v = f$. It follows $m(e, f) = \{(e + uv + v^*u^* + f)/2 \mid u \in \mathcal{U}(eMe)\}$, which is isometric to $\mathcal{U}(eMe)/2$.

In the latter case, we similarly obtain that $m(e,f)$ is isometric to $\mathcal{U}(e^\perp Me^\perp)/2$. In addition, we have $e \prec e^\perp \leq f \sim e$, thus $\mathcal{U}(e^\perp Me^\perp)$ is isometric to $\mathcal{U}(eMe)$.

(b) By Halmos's theorem applied to the pair $(p,q) = (p_1,p_2)$, we can consider p_1 and p_2 as

$$p_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & ab \\ 0 & 0 & ba & b^2 \end{pmatrix}$$

through the decomposition $H = (p \wedge q)H \oplus (p^\perp \wedge q^\perp)H \oplus e_1H \oplus e_2H$. By the comparison theorem, we may assume $p \wedge q \prec p^\perp \wedge q^\perp$ or $p^\perp \wedge q^\perp \prec p \wedge q$. In the former case, take a partial isometry $v \in M$ with $vv^* = p \wedge q$ and $v^*v \leq p^\perp \wedge q^\perp$. Put

$$e := \frac{1}{2}\begin{pmatrix} 1 & v & 0 & 0 \\ v^* & v^*v & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \end{pmatrix}, \quad f := \frac{1}{2}\begin{pmatrix} 1 & -v & 0 & 0 \\ -v^* & v^*v & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \end{pmatrix}.$$

Then it is not difficult to see $e \perp f$ and $p_1, p_2 \in m(e,f)$. The latter case can be proved similarly. ☐

In addition, we recall Hatori and Molnár's theorem. We remark that every Jordan *-isomorphism between von Neumann algebras decomposes to the direct sum of a *-isomorphism and a *-antiisomorphism [18, Exercise 10.5.26].

Theorem 3.4 (Hatori and Molnár, [16, Corollary 3])**.** *Let M and N be von Neumann algebras. Suppose that $\tau\colon \mathcal{U}(M) \to \mathcal{U}(N)$ is a surjective isometry. Then there exist a central projection $e \in \mathcal{P}(N)$ and a Jordan *-isomorphism $j\colon M \to N$ which satisfy $\tau(u) = \tau(1)(j(u)e + j(u)^*e^\perp)$, $u \in \mathcal{U}(M)$.*

We return to the proof of Theorem 3.1. There exists a unique central projection $r_1 \in \mathcal{P}(M)$ which satisfies pr_1 is a finite projection and $pr_1^\perp$ is a properly infinite projection in M for every $p \in \mathcal{P}$. We define $\mathcal{P}_{\mathrm{fin}} := \{pr_1 \mid p \in \mathcal{P}\}$ and $\mathcal{P}_{\mathrm{infin}} := \{pr_1^\perp \mid p \in \mathcal{P}\}$. Note that, if $r_1 \neq 0$ (resp. $r_1 \neq 1$), $\mathcal{P}_{\mathrm{fin}}$ (resp. $\mathcal{P}_{\mathrm{infin}}$) is a proper Grassmann space in Mr_1 (resp. $Mr_1^\perp$) and every projection in $\mathcal{P}_{\mathrm{fin}}$ (resp. $\mathcal{P}_{\mathrm{infin}}$) is a finite (resp. properly infinite) projection.

Lemma 3.5. *There exist surjective isometries $T_{\mathrm{fin}}\colon \mathcal{P}_{\mathrm{fin}} \to \mathcal{Q}_{\mathrm{fin}}$ and $T_{\mathrm{infin}}\colon \mathcal{P}_{\mathrm{infin}} \to \mathcal{Q}_{\mathrm{infin}}$ which are uniquely determined by the equation*

$$T(p) = T_{\mathrm{fin}}(pr_1) + T_{\mathrm{infin}}(pr_1^\perp), \quad p \in \mathcal{P}.$$

Proof. Take the central projection $r_2 \in \mathcal{P}(N)$ such that $\mathcal{Q}_{\mathrm{fin}} = \{qr_2 \mid q \in \mathcal{Q}\}$ and $\mathcal{Q}_{\mathrm{infin}} = \{qr_2^\perp \mid q \in \mathcal{Q}\}$. Let $p_1, p_2 \in \mathcal{P}$. What we have to show are the following:

(a) If $p_1r_1 = p_2r_1$, then $T(p_1)r_2 = T(p_2)r_2$.
(b) If $p_1r_1^\perp = p_2r_1^\perp$, then $T(p_1)r_2^\perp = T(p_2)r_2^\perp$.

We show (a) and (b) at the same time. Since every Grassmann space is path-connected, it suffices to show them in the case $\|p_1 - p_2\| < 1$. In this case, take projections e, f as in the proof of the preceding lemma. It follows $e \mathbin{\triangle} f$, $p_1, p_2 \in m(e,f)$ and thus $T(e) \mathbin{\triangle} T(f)$, $T(p_1), T(p_2) \in m(T(e), T(f))$. Then T restricts to a bijection from $m(e,f)$ onto $m(T(e),T(f))$. By (a) of the preceding lemma, it determines a surjective isometry T_1 from $\mathcal{U}(eMe)$ onto $\mathcal{U}(T(e)NT(e))$. Then we can apply the theorem due to Hatori and Molnár. By the fact that every Jordan *-isomorphism between two von Neumann algebras preserves finite (properly infinite) projections, it follows that T_1 is decomposed to the direct sum of two surjective isometries $T_2\colon \mathcal{U}(r_1eMe) \to \mathcal{U}(r_2T(e)NT(e))$ and $T_3\colon \mathcal{U}(r_1^\perp eMe) \to \mathcal{U}(r_2^\perp T(e)NT(e))$. Now it is easy to see that (a) and (b) hold. $\square$

We say $\mathcal{P}$ is *finite* if every $p \in \mathcal{P}$ is finite, and $\mathcal{P}$ is *properly infinite* if every $p \in \mathcal{P}$ is properly infinite. By the preceding lemma, what we have to do is to prove Theorem 3.1 in the case both $\mathcal{P}$ and $\mathcal{Q}$ are finite, or both $\mathcal{P}$ and $\mathcal{Q}$ are properly infinite.

First we consider the case $\mathcal{P}$ and $\mathcal{Q}$ are finite. Thus the setting is as follows: Let $\mathcal{P} \subset M$ and $\mathcal{Q} \subset N$ be finite proper Grassmann spaces. Assume $p \prec p^\perp$ for every $p \in \mathcal{P}$ and $q \prec q^\perp$ for every $q \in \mathcal{Q}$. Suppose that $T\colon \mathcal{P} \to \mathcal{Q}$ is a surjective isometry.

A key to the proof is the following lemma.

Lemma 3.6. *In the above setting, suppose $p_1, p_2 \in \mathcal{P}$ are mutually orthogonal elements. By our assumption, we have $T(p_1) \perp T(p_2)$. Then, T restricts to a bijection $T_0\colon \{p \in \mathcal{P} \mid p \leq p_1 + p_2\} \to \{q \in \mathcal{Q} \mid q \leq T(p_1) + T(p_2)\}$. Moreover, T_0 extends to a typical mapping from $(p_1+p_2)M(p_1+p_2)$ onto $(T(p_1)+T(p_2))N(T(p_1)+T(p_2))$.*

Proof. Since $p_1 \sim p_2$, using the way as before, we can identify: $p_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $p_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\begin{aligned} m(p_1, p_2) &= \left\{ \frac{1}{2} \begin{pmatrix} 1 & u \\ u^* & 1 \end{pmatrix} \middle| u \in \mathcal{U}(p_1Mp_1) \right\} \\ &\subset \mathbb{M}_2(p_1Mp_1) = (p_1+p_2)M(p_1+p_2) \subset M. \end{aligned}$$

Similarly, we identify $(T(p_1)+T(p_2))N(T(p_1)+T(p_2))$ with $\mathbb{M}_2(T(p_1)NT(p_1))$. We may assume

$$T\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Consider the restriction of T to $m(p_1, p_2)$ and define a surjective isometry $\tau\colon \mathcal{U}(p_1Mp_1) \to \mathcal{U}(T(p_1)NT(p_1))$ by

$$T\left(\frac{1}{2}\begin{pmatrix} 1 & u \\ u^* & 1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & \tau(u) \\ \tau(u)^* & 1 \end{pmatrix}, \quad u \in \mathcal{U}(p_1Mp_1).$$

By the theorem due to Hatori and Molnár, there exist central projections $r_1, r_2, r_3, r_4 \in \mathcal{P}(p_1Mp_1)$ and $r_1', r_2', r_3', r_4' \in \mathcal{P}(T(p_1)NT(p_1))$ which satisfy

$$r_1 + r_2 + r_3 + r_4 = p_1, \quad r_1' + r_2' + r_3' + r_4' = T(p_1)$$

and a *-isomorphism $\varphi_1\colon Mr_1 \to Nr_1'$, a *-antiisomorphism $\varphi_2\colon Mr_2 \to Nr_2'$, a conjugate-linear *-isomorphism $\varphi_3\colon Mr_3 \to Nr_3'$, a conjugate-linear *-antiisomorphism $\varphi_4\colon Mr_4 \to Nr_4'$ such that $\tau(u) = \varphi_1(ur_1) + \varphi_2(ur_2) + \varphi_3(ur_3) + \varphi_4(ur_4)$, $u \in \mathcal{U}(p_1Mp_1)$. We define a typical mapping $\widetilde{T}$ from $\mathbb{M}_2(p_1Mp_1)$ onto $\mathbb{M}_2(T(p_1)NT(p_1))$ by

$$\begin{aligned} \widetilde{T}\begin{pmatrix} x & y \\ z & w \end{pmatrix} &:= \begin{pmatrix} \varphi_1(xr_1) & \varphi_1(yr_1) \\ \varphi_1(zr_1) & \varphi_1(wr_1) \end{pmatrix} + \begin{pmatrix} r_2' - \varphi_2(wr_2) & \varphi_2(yr_2) \\ \varphi_2(zr_2) & r_2' - \varphi_2(xr_2) \end{pmatrix} \\ &\quad + \begin{pmatrix} \varphi_3(xr_3) & \varphi_3(yr_3) \\ \varphi_3(zr_3) & \varphi_3(wr_3) \end{pmatrix}^* + \begin{pmatrix} r_4' - \varphi_4(wr_4) & \varphi_4(yr_4) \\ \varphi_4(zr_4) & r_4' - \varphi_4(xr_4) \end{pmatrix}^*, \end{aligned}$$

$x, y, z, w \in p_1Mp_1$. We show that this is an extension of T_0.

Let p be an element in $\mathcal{P}$ with the property $p \leq p_1 + p_2$. By the finiteness of $\mathcal{P}$, there exist positive elements $a, b \in p_1Mp_1$ and a unitary $w \in \mathcal{U}(p_1Mp_1)$ with the property

$$a^2 + b^2 = p_1, \quad p = \begin{pmatrix} a^2 & abw \\ w^*ba & w^*b^2w \end{pmatrix}.$$

Then p is an element of

$$m := m\left(\frac{1}{2}\begin{pmatrix} 1 & iw \\ -iw^* & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -iw \\ iw^* & 1 \end{pmatrix}\right),$$

so it follows $T(p) \leq T(p_1) + T(p_2)$. We have to show that the mapping Φ from $\{p \in \mathcal{P} \mid p \leq p_1 + p_2\}$ onto itself which is defined by $\Phi(p) = \widetilde{T}^{-1} \circ T(p)$ is the identity mapping. We already know that the projections

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} 1 & u \\ u^* & 1 \end{pmatrix}, \ u \in \mathcal{U}(p_1Mp_1)$$

are all fixed under Φ.

It follows Φ restricts to a bijection from m (as above) onto itself. It suffices to show that Φ restricts to the identity mapping on m. The self-adjoint unitary $U := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & iw \\ -iw^* & -1 \end{pmatrix}$ gives rise to an isometry $\mathrm{Ad}(U)$ on $\mathbb{M}_2(p_1Mp_1)$. Then m is isometric to

$$\begin{aligned}\mathrm{Ad}(U)m &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & iw \\ -iw^* & -1 \end{pmatrix} m\left(\frac{1}{2}\begin{pmatrix} 1 & iw \\ -iw^* & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -iw \\ iw^* & 1 \end{pmatrix}\right) \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & iw \\ -iw^* & -1 \end{pmatrix} \\ &= m\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \left\{\frac{1}{2}\begin{pmatrix} 1 & v \\ v^* & 1 \end{pmatrix} \middle| v \in \mathcal{U}(p_1Mp_1)\right\}.\end{aligned}$$

Our task is to show that the mapping $\mathrm{Ad}(U) \circ \Phi \circ \mathrm{Ad}(U)$ is equal to the identity mapping on $\left\{\frac{1}{2}\begin{pmatrix} 1 & v \\ v^* & 1 \end{pmatrix} \middle| v \in \mathcal{U}(p_1Mp_1)\right\}$. We have

$$\mathrm{Ad}(U)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & iw \\ -iw^* & 1 \end{pmatrix}$$

and

$$\mathrm{Ad}(U)\left(\frac{1}{2}\begin{pmatrix} 1 & u \\ u^* & 1 \end{pmatrix}\right) = \frac{1}{4}\begin{pmatrix} 2 - iuw^* + iwu^* & -u - wu^*w \\ -u^* - w^*uw^* & 2 - iu^*w + iw^*u \end{pmatrix}$$

for every $u \in \mathcal{U}(p_1Mp_1)$. In particular, for every self-adjoint unitary $a \in \mathcal{U}(p_1Mp_1)$, we have

$$\mathrm{Ad}(U)\left(\frac{1}{2}\begin{pmatrix} 1 & -aw \\ -w^*a & 1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & aw \\ w^*a & 1 \end{pmatrix}.$$

Therefore, if $v = iw$ or $v = aw$ for some self-adjoint unitary a, then

$$\mathrm{Ad}(U) \circ \Phi \circ \mathrm{Ad}(U)\left(\frac{1}{2}\begin{pmatrix} 1 & v \\ v^* & 1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & v \\ v^* & 1 \end{pmatrix}.$$

By the Hatori–Molnár theorem, the same equation holds for every $v \in \mathcal{U}(p_1Mp_1)$. □

In fact, we may assume that the above typical mapping is always a Jordan *-isomorphism. We explain this.

First, take central projections $r_a, r_b, r_c \in \mathcal{P}(M)$ with $r_a + r_b + r_c = 1$ such that

- $r_a p$ is an abelian projection for every $p \in \mathcal{P}$,
- $r_b p \sim r_b p^{\perp}$ for every $p \in \mathcal{P}$, and
- $r_c pMp$ does not admit a type I_1 direct summand for every $p \in \mathcal{P}$, and $z(1-p_1-p_2)r_c = r_c$ for arbitrary $p_1, p_2 \in \mathcal{P}$ with $p_1 \perp p_2$.

Fix $p_1, p_2 \in \mathcal{P}$ with $p_1 \perp p_2$. Since $r_a p$ is an abelian projection, we can take $\widetilde{T}$ as in the above proof so that it is a Jordan *-homomorphism on $r_a(p_1+p_2)M(p_1+p_2)$. We show that $\widetilde{T}$ is a Jordan *-homomorphism on $r_c(p_1+p_2)M(p_1+p_2)$. By the condition of r_c, we can take a projection $e \in \mathcal{P}(M)$ such that $e \leq r_c p_2$, $r_c z(e) = r_c = r_c z(p_2 - e)$ and $e \prec (1-p_1-p_2)$. Consider the restriction of T to the subset $S = \{p \in \mathcal{P} \mid p \leq p_1 + e\}$. Note that T is equal to $\widetilde{T}$ on this subset. Put $S_1 := \{p \in \mathcal{P} \mid p \perp (p_1+e)\}$. It follows $S = \{p \in \mathcal{P} \mid p \perp S_1\}$. Since T preserves orthogonality, we have $\widetilde{T}(S) = T(S) = \{q \in \mathcal{Q} \mid q \perp T(S_1)\}$. If $\widetilde{T}$ is not a Jordan *-homomorphism on $r_c(p_1+p_2)M(p_1+p_2)$, then $\widetilde{T}(S)$ cannot be written as above. Hence $\widetilde{T}$ is a Jordan *-homomorphism on $r_c(p_1+p_2)M(p_1+p_2)$.

Note that $r_b(p_1+p_2) = r_b$. We can take a typical mapping $\psi\colon r_bM \to r_bM$ with the property that $\widetilde{T} \circ \psi\colon r_bM \to N$ is a Jordan *-homomorphism. Define the typical mapping $\Psi\colon M \to M$ by $\Psi(x) := \psi(r_bx) + (r_a + r_c)x$, $x \in M$. By the assumption concerning r_b, we have $\Psi(\mathcal{P}) = \mathcal{P}$. Considering the composition $T \circ \Psi$ instead of T, we may assume $\widetilde{T}$ is a Jordan *-isomorphism.

Let $p_3, p_4 \in \mathcal{P}$ satisfy $p_3 \perp p_4$. There exists $p_5 \in \mathcal{P}$ such that $p_1 \perp p_5$ and $p_3 \leq p_1 + p_5$. Note that $r_b(p_1+p_5) = r_b$. Considering the restriction of T to the set $\{p \in \mathcal{P} \mid p \leq (r_a+r_c)p_1 + r_b\}$ and using the same discussion as above, we see that the restriction of T to the subset $\{p \in \mathcal{P} \mid p \leq p_1 + p_5\}$ extends to a Jordan *-isomorphism from $(p_1+p_5)M(p_1+p_5)$ onto $(T(p_1)+T(p_5))N(T(p_1)+T(p_5))$. Since $p_3 \leq p_1+p_5$, considering the restriction of T to the set $\{p \in \mathcal{P} \mid p \leq (r_a+r_c)p_3+r_b\}$, we also see that the restriction of T to the subset $\{p \in \mathcal{P} \mid p \leq p_3 + p_4\}$ extends to a Jordan *-isomorphism from $(p_3+p_4)M(p_3+p_4)$ onto $(T(p_3)+T(p_4))N(T(p_3)+T(p_4))$.

Recall that a bijection F from $\mathcal{P}(M)$ onto $\mathcal{P}(N)$ is called an *orthoisomorphism* when it satisfies $pq = 0$ if and only if $F(p)F(q) = 0$, for $p, q \in \mathcal{P}(M)$.

We show that, under the above assumptions, the mapping T extends uniquely to an orthoisomorphism from $\mathcal{P}(M)$ onto $\mathcal{P}(N)$.

First, we extend T to a mapping T_1 from $\{e \in \mathcal{P}(M) \mid e \leq p \text{ for some } p \in \mathcal{P}\}$ to $\{f \in \mathcal{P}(N) \mid f \leq q \text{ for some } q \in \mathcal{Q}\}$ by

$$T_1(e) := \bigwedge \{T(p) \mid p \in \mathcal{P}, e \leq p\}.$$

We show that T_1 is a bijection which preserves orthogonality in both directions. Fix e. Take some $p_0 \in \mathcal{P}$ with $e \leq p_0$ and $f \in \mathcal{P}(M)$ with $e \sim f \leq p_0^{\perp}$. We prove $T_1(e) = T(p_0) - T(p_0)T((p_0-e)+f)$. Suppose $p_1 \in \mathcal{P}$ satisfies $e \leq p_1$. There exists a projection $p_2 \in \mathcal{P}$ with the property $p_2 \perp p_0$ and $f, p_1 \leq p_0 + p_2$. Then T restricts to a bijection $T_0\colon \{p \in \mathcal{P} \mid p \leq p_0 + p_2\} \to \{q \in \mathcal{Q} \mid q \leq T(p_0) + T(p_2)\}$ and T_0 extends to a Jordan *-isomorphism J_0 from $(p_0+p_2)M(p_0+p_2)$ onto $(T(p_0)+T(p_2))N(T(p_0)+T(p_2))$. Hence we obtain $T(p_1) = J_0(p_1) \geq J_0(e) = J_0(p_0) - J_0(p_0)J_0((p_0-e)+f) = T(p_0) - T(p_0)T((p_0-e)+f)$ for any $p_1 \in \mathcal{P}$ with $e \leq p_1$ and thus $T_1(e) \geq T(p_0) - T(p_0)T((p_0-e)+f)$. In addition, we have $T(p_0) - T(p_0)T((p_0-e)+f) = J_0(e) = J_0(p_0)J_0((p_2-f)+e) = T(p_0)T((p_2-f)+e) \geq T_1(e)$. It follows $T_1(e) = T(p_0) - T(p_0)T((p_0-e)+f)$.

Let $p_3, p_4 \in \mathcal{P}$ be mutually commuting projections. Put $e = p_3p_4$, $p_0 = p_3$, take some $f \in \mathcal{P}(M)$ so that $e \sim f \leq 1 - p_3 \vee p_4$ and put $p_2 = (p_4 - e) + f$. Then the above discussion shows that $T_1(p_3p_4) = T_1(e) = T(p_0)T((p_2 - f) + e) = T(p_3)T(p_4)$. Thus T_1 is determined uniquely by the condition $T_1(p_3p_4) = T(p_3)T(p_4)$ for an arbitrary pair of mutually commuting projections $p_3, p_4 \in \mathcal{P}$. It follows T_1 is a bijection with its inverse $T_1^{-1} \colon \{f \in \mathcal{P}(N) \mid f \leq q \text{ for some } q \in \mathcal{Q}\} \to \{e \in \mathcal{P}(M) \mid e \leq p \text{ for some } p \in \mathcal{P}\}$ which is defined by $T_1^{-1}(f) := \bigwedge\{q \mid q \in \mathcal{Q}, f \leq q\}$. Since T preserves orthogonality in both directions, so does T_1.

We define a mapping $T_2 \colon \mathcal{P}(M) \to \mathcal{P}(N)$ by

$$\begin{aligned} T_2(p) &:= \bigvee\{T_1(e) \mid e \leq p,\ e \leq p_0 \text{ for some } p_0 \in \mathcal{P}\} \\ &= \bigwedge\{T_1(e)^\perp \mid e \perp p,\ e \leq p_0 \text{ for some } p_0 \in \mathcal{P}\}. \end{aligned}$$

It follows T_2 is an orthoisomorphism which extends T.

Lastly, we rely on the following proposition by the author in his master's thesis, which slightly extends Theorem 1.4 by Dye.

Proposition 3.7 ([26, Proposition 5.2]). *Let M and N be two von Neumann algebras. Suppose $T \colon \mathcal{P}(M) \to \mathcal{P}(N)$ is an orthoisomorphism which preserves the distances between maximal abelian projections in the type I_2 direct summands. Then there exists a Jordan *-isomorphism from M onto N which extends T.*

Since our assumption shows that T_2 restricts to a surjective isometry between the classes of maximal abelian projections in the type I_2 direct summands, T_2 extends to a Jordan *-isomorphism from M onto N. This completes the proof of Theorem 3.1 when $\mathcal{P}$ and $\mathcal{Q}$ are finite.

Next we consider the case both $\mathcal{P}$ and $\mathcal{Q}$ are properly infinite. Thus the setting is as follows: Let $\mathcal{P} \subset M$ and $\mathcal{Q} \subset N$ be properly infinite proper Grassmann spaces. Assume $p \prec p^\perp$ for every $p \in \mathcal{P}$ and $q \prec q^\perp$ for every $q \in \mathcal{Q}$. Suppose that $T \colon \mathcal{P} \to \mathcal{Q}$ is a surjective isometry.

The first step is to show that we may assume T preserves orthogonality in both directions. As in [13], for two projections $p_1, p_2 \in \mathcal{P}$, we write $p_1 \sharp p_2$ when $p_1 \perp p_2$ and $p_1 \prec (1 - p_1 - p_2)$.

Since $\mathcal{P}$ is properly infinite, we can take mutually orthogonal projections $p_1, p_2, p_3 \in \mathcal{P}$. We have $p_1 \vartriangle p_2$, $p_2 \vartriangle p_3$, $p_3 \vartriangle p_1$, thus $T(p_1) \vartriangle T(p_2)$, $T(p_2) \vartriangle T(p_3)$, $T(p_3) \vartriangle T(p_1)$. It follows there exists a central projection $r \in \mathcal{P}(M)$ such that $T(p_1)r, T(p_2)r, T(p_3)r$ are mutually orthogonal and $T(p_1)^\perp r^\perp, T(p_2)^\perp r^\perp, T(p_3)^\perp r^\perp$ are mutually orthogonal. Composing T with the typical mapping $q \mapsto qr + q^\perp r^\perp$ on $\mathcal{Q}$, we may assume that $T(p_1), T(p_2), T(p_3)$ are mutually orthogonal.

Under this assumption, we show that, for any projections $p, p_0 \in \mathcal{P}$, we have $p \sharp p_0$ if and only if $T(p) \sharp T(p_0)$. Suppose $p \sharp p_0$. We have $p \sim p_1$, $p_0 \sim p_2$. Since $\mathcal{P}$ is properly infinite, we obtain $(1 - p - p_0) \sim ((1 - p - p_0) + p + p_0) = 1$ and similarly $(1 - p_1 - p_2) \sim 1$, thus $(1 - p - p_0) \sim (1 - p_1 - p_2)$. Therefore there exists a unitary $u \in \mathcal{U}(M)$ which satisfies $upu^* = p_1$ and $up_0u^* = p_2$. By the functional calculus on M, there exists a self-adjoint operator $a \in M_{sa}$ with $u = e^{ia}$. We show $T(e^{ita}pe^{-ita}) \sharp T(e^{ita}p_0e^{-ita})$ for every $t \in [0, 1]$. It suffices to show $T(p) \sharp T(p_0)$ when

$\|p-p_1\|<1/2$ and $\|p_0-p_2\|<1/2$. In that case, we have

$$\begin{aligned}&\|(1-T(p)-T(p_0))-(1-T(p_1)-T(p_2))\|\\ \le\ &\|T(p_1)-T(p)\|+\|T(p_2)-T(p_0)\|=\|p_1-p\|+\|p_2-p_0\|<1.\end{aligned}$$

Combine this inequality with $T(p)\triangle T(p_0)$ to obtain $T(p)\perp T(p_0)$. Moreover, we can apply the generalization of Halmos's theorem to the two projections $1-T(p)-T(p_0)$ and $1-T(p_1)-T(p_2)$ to obtain $(1-T(p)-T(p_0))\sim(1-T(p_1)-T(p_2))$. Thus we have $T(p)\sharp T(p_1)$.

We have shown that T preserves the relation $\sharp$ in both directions. It is easy to see that for $p_1,p_2\in\mathcal{P}$, we have $p_1\le p_2$ if and only if $\{p\in\mathcal{P}\mid p\sharp p_1\}\supset\{p\in\mathcal{P}\mid p\sharp p_2\}$. Thus we obtain $p_1\le p_2$ if and only if $T(p_1)\le T(p_2)$.

Let $p_1,p_2\in\mathcal{P}$ satisfy $p_1\vee p_2\in\mathcal{P}$. Since $p_1\vee p_2$ is the minimal projection in $\mathcal{P}$ which majorizes both p_1 and p_2, we have $T(p_1\vee p_2)=T(p_1)\vee T(p_2)$. Similarly, if $p_1,p_2\in\mathcal{P}$ satisfy $p_1\wedge p_2\in\mathcal{P}$, then $T(p_1\wedge p_2)=T(p_1)\wedge T(p_2)$.

Let $p_1,p_2\in\mathcal{P}$ satisfy $p_1\perp p_2$. Since $\mathcal{P}$ is properly infinite, there exist mutually orthogonal subprojections $p_{11},p_{12}\in\mathcal{P}$ of p_1 which satisfy $p_1=p_{11}+p_{12}$. Since $p_{11}\sharp p_2$ and $p_{12}\sharp p_2$, we have $T(p_{11})\sharp T(p_2)$ and $T(p_{12})\sharp T(p_2)$. Hence we obtain $T(p_2)\perp(T(p_{11})\vee T(p_{12}))=T(p_{11}\vee p_{12})=T(p_1)$. Therefore, T preserves orthogonality in both directions.

We show a version of Lemma 3.6.

Lemma 3.8. *Under the above assumptions, suppose $p_1,p_2\in\mathcal{P}$ are mutually orthogonal. Then, T restricts to a bijection $T_0\colon\{p\in\mathcal{P}\mid p\le p_1+p_2\}\to\{q\in\mathcal{Q}\mid q\le T(p_1)+T(p_2)\}$. Moreover, T_0 extends (uniquely) to a Jordan *-isomorphism from $(p_1+p_2)M(p_1+p_2)$ onto $(T(p_1)+T(p_2))N(T(p_1)+T(p_2))$.*

Proof. Using the same notations and discussions as in the proof of Lemma 3.6, we can construct a typical mapping $\widetilde{T}$ from $(p_1+p_2)M(p_1+p_2)$ onto $(T(p_1)+T(p_2))N(T(p_1)+T(p_2))$. Take projections $p,\tilde{p}_1,\tilde{p}_2\in\mathcal{P}$ such that $p\le p_1$, $p\sim(p_1-p)$ and $\tilde{p}_1\le\tilde{p}_2\le p_2$, $\tilde{p}_1\sim(\tilde{p}_2-\tilde{p}_1)\sim(p_2-\tilde{p}_2)$. By the same discussion as in Lemma 3.6, we see that $T(p+\tilde{p}_1)=\widetilde{T}(p+\tilde{p}_1)$ and $T(p+\tilde{p}_2)=\widetilde{T}(p+\tilde{p}_2)$. It follows $\widetilde{T}(p+\tilde{p}_1)\le\widetilde{T}(p+\tilde{p}_2)$, which shows that $\widetilde{T}$ is actually a Jordan *-isomorphism.

We show $T(p)=\widetilde{T}(p)$ for every $p\in\mathcal{P}$ with $p\le p_1+p_2$. Since $p\sim p_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, there exist $x,y\in p_1Mp_1$ which satisfy

$$x^*x+y^*y=p_1,\quad p=\begin{pmatrix}xx^*&xy^*\\yx^*&yy^*\end{pmatrix}.$$

Let $x=v|x|$, $y=w|y|$ be polar decompositions. By the spectral theorem, we may assume that the spectrum $\sigma(|x|)$ of $|x|$ is a finite set. Thus $|x|=\sum_{k=1}^n\lambda_ke_k$ for some $0=\lambda_1<\lambda_2<\cdots<\lambda_n=1$ and mutually orthogonal projections $e_k\in\mathcal{P}(p_1Mp_1)$ such that $\sum_{k=1}^ne_k=p_1$. (Projections e_1 and e_n may be 0.) We have $|y|=\sum_{k=1}^n\sqrt{1-\lambda_k^2}e_k$. Since p_1 is properly infinite, there exist subprojections $f_k\le e_k$ in $\mathcal{P}(p_1Mp_1)$, $k=1,\dots,n$, which satisfy the following property: $\sum_{k=1}^nf_k\sim\sum_{k=1}^n(e_k-f_k)\sim p_1$, and partial isometries $v\sum_{k=2}^nf_k$, $w\sum_{k=1}^{n-1}f_k$, $v\sum_{k=2}^n(e_k-f_k)$ and $w\sum_{k=1}^{n-1}(e_k-f_k)$ admit unitary extensions v_0,w_0,v_1 and $w_1\in\mathcal{U}(p_1Mp_1)$, respectively. We show that the projection

$$p_0 := \begin{pmatrix} v(\sum_{k=1}^n \lambda_k f_k)^2 v^* & v(\sum_{k=1}^n \lambda_k f_k)(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k) w^* \\ w(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k)(\sum_{k=1}^n \lambda_k f_k) v^* & w(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k)^2 w^* \end{pmatrix}$$
$$= \begin{pmatrix} v_0(\sum_{k=1}^n \lambda_k f_k)^2 v_0^* & v_0(\sum_{k=1}^n \lambda_k f_k)(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k) w_0^* \\ w_0(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k)(\sum_{k=1}^n \lambda_k f_k) v_0^* & w_0(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k)^2 w_0^* \end{pmatrix}$$

in $\mathcal{P}$ satisfies $T(p_0) = \widetilde{T}(p_0)$. Consider the projection

$$p_0 + \begin{pmatrix} v_0(\sum_{k=1}^n (e_k - f_k)) v_0^* & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} v_0((\sum_{k=1}^n \lambda_k f_k)^2 + \sum_{k=1}^n (e_k - f_k)) v_0^* & v_0(\sum_{k=1}^n \lambda_k f_k)(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k) w_0^* \\ w_0(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k)(\sum_{k=1}^n \lambda_k f_k) v_0^* & w_0(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k)^2 w_0^* \end{pmatrix}$$
$$= \begin{pmatrix} a^2 & ab v_0 w_0^* \\ w_0 v_0^* ba & w_0 v_0^* b^2 v_0 w_0^* \end{pmatrix},$$

where $a := v_0(\sum_{k=1}^n \lambda_k f_k + \sum_{k=1}^n (e_k - f_k)) v_0^*$ and $b := v_0(\sum_{k=1}^n \sqrt{1-\lambda_k^2} f_k) v_0^*$. It follows $a, b \geq 0$, $a^2 + b^2 = p_0$. By the same discussion as in Lemma 3.6, we obtain

$$T\left(p_0 + \begin{pmatrix} v_0(\sum_{k=1}^n (e_k - f_k)) v_0^* & 0 \\ 0 & 0 \end{pmatrix}\right) = \widetilde{T}\left(p_0 + \begin{pmatrix} v_0(\sum_{k=1}^n (e_k - f_k)) v_0^* & 0 \\ 0 & 0 \end{pmatrix}\right)$$

Similarly, we obtain

$$T\left(p_0 + \begin{pmatrix} 0 & 0 \\ 0 & w_0(\sum_{k=1}^n (e_k - f_k)) w_0^* \end{pmatrix}\right) = \widetilde{T}\left(p_0 + \begin{pmatrix} 0 & 0 \\ 0 & w_0(\sum_{k=1}^n (e_k - f_k)) w_0^* \end{pmatrix}\right).$$

Since

$$\left(p_0 + \begin{pmatrix} v_0(\sum_{k=1}^n (e_k - f_k)) v_0^* & 0 \\ 0 & 0 \end{pmatrix}\right) \wedge \left(p_0 + \begin{pmatrix} 0 & 0 \\ 0 & w_0(\sum_{k=1}^n (e_k - f_k)) w_0^* \end{pmatrix}\right) = p_0,$$

we have $T(p_0) = \widetilde{T}(p_0)$. Similarly, we have $T(p - p_0) = \widetilde{T}(p - p_0)$. Finally, we have $T(p) = T(p_0) \vee T(p - p_0) = \widetilde{T}(p_0) \vee \widetilde{T}(p - p_0) = \widetilde{T}(p)$. □

A discussion which is similar to (or simpler than) that in finite cases shows that it is possible to extend T to an orthoisomorphism from $\mathcal{P}(M)$ onto $\mathcal{P}(N)$. By Dye's theorem, T extends to a Jordan *-isomorphism from M onto N. □

4. Isometries between projection lattices

In this section, we write $M \cong N$ when two von Neumann algebras M and N are Jordan *-isomorphic.

Theorem 4.1. *Let M, N be von Neumann algebras without type I_1 direct summands. Then M and N are Jordan *-isomorphic if and only if $\mathcal{P}(M)$ and $\mathcal{P}(N)$ are isometric.*

Suppose $T\colon \mathcal{P}(M) \to \mathcal{P}(N)$ is a surjective isometry. Since M does not admit a type I_1 direct summand, there exists a projection $p \in \mathcal{P}(M)$ which satisfies $z(p) = z(p^\perp) = 1$. Take the (proper) Grassmann space $\mathcal{P}$ in M which contains p. Then $T(\mathcal{P})$ is a proper Grassmann space in $Nz(T(p))z(T(p)^\perp)$. By Theorem 3.1, it follows that M is Jordan *-isomorphic to $Nz(T(p))z(T(p)^\perp)$, which is a direct summand of N. Similarly, N is Jordan *-isomorphic to a direct summand of M. Therefore, it suffices to show the following lemma.

Lemma 4.2. *Let M, N be von Neumann algebras. Suppose that M is Jordan *-isomorphic to a direct summand of N, and N is Jordan *-isomorphic to a direct summand of M. Then M is Jordan *-isomorphic to N.*

Proof. There exist central projections $p \in \mathcal{P}(M)$ and $q \in \mathcal{P}(N)$ such that M, N are Jordan *-isomorphic to Nq, Mp, respectively. It follows

$$M = Mp \oplus Mp^\perp \cong N \oplus Mp^\perp = Nq \oplus Nq^\perp \oplus Mp^\perp \cong M \oplus Nq^\perp \oplus Mp^\perp.$$

Take a Jordan *-isomorphism $\Phi\colon M \oplus Nq^\perp \oplus Mp^\perp \to M$. We define $i\colon M \to M \oplus Nq^\perp \oplus Mp^\perp$ by $i(x) := x \oplus 0 \oplus 0$, $x \in M$. Put $p_0 := \Phi(0 \oplus q^\perp \oplus p^\perp)$ and $p_n := (\Phi \circ i)^n(p_0)$, $n \in \mathbb{N}$. Then $\{p_n\}_{n\geq 0}$ is an orthogonal family of central projections in M and $Mp_n \cong Nq^\perp \oplus Mp^\perp \cong Mp_0$, $n \geq 0$. Put $p_\infty := \vee_{n\geq 0}\, p_n$. We have

$$\begin{aligned} M &= Mp_\infty^\perp \oplus Mp_\infty \\ &\cong Mp_\infty^\perp \oplus Mp_0 \mathbin{\overline{\otimes}} \ell^\infty \\ &\cong Mp_\infty^\perp \oplus Mp_0 \mathbin{\overline{\otimes}} \ell^\infty \oplus Mp_0 \mathbin{\overline{\otimes}} \ell^\infty \\ &\cong M \oplus Mp_0 \mathbin{\overline{\otimes}} \ell^\infty. \end{aligned}$$

Similarly, we obtain $N \cong N \oplus Mp_0 \mathbin{\overline{\otimes}} \ell^\infty$. Lastly, we have

$$\begin{aligned} M \oplus Mp_0 \mathbin{\overline{\otimes}} \ell^\infty &\cong Nq \oplus (Nq^\perp \oplus Mp^\perp) \mathbin{\overline{\otimes}} \ell^\infty \\ &\cong Nq \oplus Nq^\perp \oplus (Nq^\perp \oplus Mp^\perp) \mathbin{\overline{\otimes}} \ell^\infty \\ &= N \oplus Mp_0 \mathbin{\overline{\otimes}} \ell^\infty. \end{aligned}$$

□

If in the above theorem we drop the condition concerning type I_1 summand, then we can find a counterexample. Indeed, any bijection between $\mathcal{P}(L^\infty([0,1]))$ and $\mathcal{P}(L^\infty([0,1]) \oplus \mathbb{C})$ is isometric, but $L^\infty([0,1])$ and $L^\infty([0,1]) \oplus \mathbb{C}$ are not isomorphic.

Theorem 3.1 also gives a complete description of surjective isometries between projection lattices of two von Neumann algebras. However, to give such a description in concrete situations is a complicated work. In the rest of this part, we consider factor cases.

Let M, N be countably decomposable factors and suppose $T\colon \mathcal{P}(M) \to \mathcal{P}(N)$ is a surjective isometry. Then Theorem 3.1 implies that M and N are Jordan *-isomorphic, and thus M and N are *-isomorphic or *-antiisomorphic. We assume $M = N$. Note that only two points 0 and 1 are isolated in $\mathcal{P}(M)$, and thus T restricts to a bijection on $\{0, 1\}$.

First we consider type I factors. Let H be a separable complex Hilbert space. For $n \in \mathbb{N} = \{1, 2, \dots\}$, the symbol $\mathcal{P}_n(H)$ denotes the collection of rank n projections in $B(H)$, and we put $\mathcal{P}^n(H) := \{p^\perp \mid p \in \mathcal{P}_n(H)\}$. The symbol $\mathcal{P}_\infty(H)$ denotes the set of projections in $B(H)$ whose range and kernel are both infinite dimensional.

Example 4.3. If $M = B(H)$ is a type I_N factor with $N \in \mathbb{N}$, then Grassmann spaces of M are $\mathcal{P}_n(H)$, $n = 1, 2, \dots, N-1$. In this case, there exists a mapping σ from $\{1, 2, \dots, N-1\}$ to $\{1, -1\}$ which satisfies the following conditions:

- For $n = 1, \dots, N-1$, $\sigma(n)\sigma(N-n) = 1$.

- If $\sigma(n) = 1$, the mapping T restricts to a bijection T_n from $\mathcal{P}_n(H)$ onto itself. Moreover, T_n extends uniquely to a *-automorphism or a *-antiautomorphism on $B(H)$.
- If $\sigma(n) = -1$, the mapping T restricts to a bijection T_n from $\mathcal{P}_n(H)$ onto $\mathcal{P}_{N-n}(H)$. Moreover, the mapping $p \mapsto 1 - T_n(p)$, $p \in \mathcal{P}_n(H)$ extends uniquely to a *-automorphism or a *-antiautomorphism on $B(H)$.

Example 4.4. If $M = B(H)$ is a type I_∞ factor, then Grassmann spaces of M are $\mathcal{P}_n(H)$, $\mathcal{P}^n(H)$, $n \in \mathbb{N}$ and $\mathcal{P}_\infty(H)$. In this case, T restricts to a bijection T_∞ from $\mathcal{P}_\infty(H)$ onto itself. Thus T_∞ extends uniquely to a *-automorphism or a *-antiautomorphism, or the mapping $p \mapsto 1 - T_\infty(p)$, $p \in \mathcal{P}_\infty(H)$ extends uniquely to a *-automorphism or a *-antiautomorphism on $B(H)$. In addition, there exists a unique mapping σ from $\mathbb{N}$ to $\{1, -1\}$ which satisfies the following conditions:

- If $\sigma(n) = 1$, the mapping T restricts to a bijection T_n from $\mathcal{P}_n(H)$ onto itself, and T also restricts to a bijection T^n from $\mathcal{P}^n(H)$ onto itself. Each mapping extends uniquely to a *-automorphism or a *-antiautomorphism on $B(H)$.
- If $\sigma(n) = -1$, the mapping T restricts to a bijection T_n from $\mathcal{P}_n(H)$ onto $\mathcal{P}^n(H)$, and T also restricts to a bijection T^n from $\mathcal{P}^n(H)$ onto $\mathcal{P}_n(H)$. Thus the mappings $1 - T_n$ and $1 - T^n$ extend to a *-automorphism or a *-antiautomorphism on $B(H)$.

Note that, for every *-automorphism (resp. *-antiautomorphism) Φ on $B(H)$, there exists a unitary (resp. antiunitary) u on H which satisfies $\Phi(x) = uxu^*$ (resp. $\Phi(x) = ux^*u^*$), $x \in B(H)$. Thus we see that our result actually generalizes the theorem due to Gehér and Šemrl [13, Theorem 1.2].

Example 4.5. If M is a type II_1 factor with a normal tracial state τ, then Grassmann spaces of M are $\mathcal{P}_\lambda(M) := \{p \in \mathcal{P}(M) \mid \tau(p) = \lambda\}$, $0 < \lambda < 1$. In this case, we can use the fact that every Jordan *-automorphism on a tracial factor preserves the trace. It follows there exists a unique mapping $\sigma\colon (0,1) \to \{1, -1\}$ which satisfies the following conditions:

- For $\lambda \in (0,1)$, $\sigma(\lambda)\sigma(1-\lambda) = 1$.
- If $\sigma(\lambda) = 1$, the mapping T restricts to a bijection T_λ from $\mathcal{P}_\lambda(M)$ onto itself. Moreover, T_λ extends uniquely to a *-automorphism or a *-antiautomorphism on M.
- If $\sigma(\lambda) = -1$, the mapping T restricts to a bijection T_λ from $\mathcal{P}_\lambda(M)$ onto $\mathcal{P}_{1-\lambda}(M)$. Moreover, the mapping $p \mapsto 1 - T_\lambda(p)$, $p \in \mathcal{P}_\lambda(M)$ extends uniquely to a *-automorphism or a *-antiautomorphism on M.

Example 4.6. If M is a type II_∞ factor with a normal semifinite faithful tracial weight τ, then Grassmann spaces of M are $\mathcal{P}_{(\lambda,1)} := \{p \in \mathcal{P}(M) \mid \tau(p) = \lambda\}$, $\mathcal{P}_{(\lambda,-1)} := \{p^\perp \mid p \in \mathcal{P}_{(\lambda,1)}\}$, $0 < \lambda < \infty$, and $\mathcal{P}_\infty = \{p \in \mathcal{P}(M) \mid \tau(p) = \infty = \tau(p^\perp)\}$.

This case is the most complicated. First, T restricts to a bijection T_∞ from $\mathcal{P}_\infty$ onto itself, and T_∞ or $1 - T_\infty$ extends to a *-automorphism or a *-antiautomorphism on M. In order to consider the other Grassmann spaces, we need to take the following multiplicative group into account:

$$\mathcal{F} := \{\lambda \in (0,\infty) \mid pMp \cong qMq \text{ for some } p \in \mathcal{P}_{(1,1)},\ q \in \mathcal{P}_{(\lambda,1)}\}$$

(Note that the symbol $\cong$ means that two algebras are Jordan *-isomorphic. cf. The fundamental group of the II_1 factor pMp is a subgroup of $\mathcal{F}$.)

There exists a bijection f from $(0,\infty)\times\{1,-1\}$ onto itself which satisfies the following condition: Let $(\lambda,s),(\mu,t)\in(0,\infty)\times\{1,-1\}$ satisfy $f(\lambda,s)=(\mu,t)$. Then

- $\lambda/\mu\in\mathcal{F}$.
- The mapping T restricts to a bijection $T_{(\lambda,s)}$ from $\mathcal{P}_{(\lambda,s)}$ onto $\mathcal{P}_{(\mu,t)}$.
- If $st=1$, then $T_{(\lambda,s)}$ extends uniquely to a *-automorphism or a *-antiautomorphism on M.
- If $st=-1$, the mapping $p\mapsto 1-T_{(\lambda,s)}(p)$, $p\in\mathcal{P}_{(\lambda,s)}$ extends uniquely to a *-automorphism or a *-antiautomorphism on M.

Example 4.7. If M is a type III factor, then the unique Grassmann space of M is $\mathcal{P}:=\mathcal{P}(M)\setminus\{0,1\}$. It follows that the restriction T_0 of T on $\mathcal{P}$ is described as one and only one of the following four options: it extends uniquely to a *-automorphism or a *-antiautomorphism, or the mapping $p\mapsto 1-T_0(p)$, $p\in\mathcal{P}$ extends uniquely to a *-automorphism or a *-antiautomorphism.

Part 2. Lattice isomorphisms between projection lattices of von Neumann algebras

5. Preliminaries

Let M be a von Neumann algebra. For $n \in \mathbb{N} = \{1, 2, \ldots\}$, we say that M has *order* n if there exists a collection $p_1, \ldots, p_n$ of mutually orthogonal projections in M such that $p_1 \sim p_2 \sim \cdots \sim p_n$ and $\sum_{k=1}^{n} p_k = 1$. It is well known that every von Neumann algebra without finite type I direct summands has order n for any $n \in \mathbb{N}$ [18, Lemma 6.5.6]. In particular, such an algebra has order 3. It follows that every von Neumann algebra M without type I_1 and I_2 direct summands can be decomposed into the (ℓ^∞-)direct sum of von Neumann algebras M_n, $3 \leq n < \infty$, such that M_n has order n for every n. If M has order $n \in \mathbb{N}$, then M can be identified with the algebra $\mathbb{M}_n(\hat{M})$ of $n \times n$ matrices with entries in some von Neumann algebra $\hat{M}$.

5.1. **Various isomorphisms of von Neumann algebras.** For *-algebras A and B, a (not necessarily linear) bijection $\psi \colon A \to B$ is called

- a *semigroup isomorphism* if it is multiplicative,
- a *ring isomorphism* if it is additive and multiplicative,
- a *real algebra isomorphism* if it is a real-linear ring isomorphism,
- an *algebra isomorphism* if it is a complex-linear ring isomorphism,
- a *real *-isomorphism* if it is a real algebra isomorphism and satisfies $\psi(x^*) = \psi(x)^*$ for any $x \in A$,
- a **-isomorphism* if it is a complex-linear real *-isomorphism, and
- a *conjugate-linear *-isomorphism* if it is a conjugate-linear real *-isomorphism.

Lemma 5.1. *Let M and N be von Neumann algebras. Suppose that $\psi \colon M \to N$ is a bijection.*

(1) If M is without type I_1 direct summands and ψ is a semigroup isomorphism, then ψ is a ring isomorphism.
(2) If M does not admit a finite dimensional ideal and ψ is a ring isomorphism, then ψ is a real algebra isomorphism.
(3) If ψ is a real algebra isomorphism, then there exist a real *-isomorphism $\psi_0 \colon M \to N$ and an invertible element $y \in N$ such that $\psi(x) = y\psi_0(x)y^{-1}$ for any $x \in M$.
(4) If ψ is a real *-isomorphism, then there exist central projections $p \in M$ $q \in N$, a *-isomorphism $\psi_1 \colon Mp \to Nq$, and a conjugate-linear *-isomorphism $\psi_2 \colon Mp^\perp \to Nq^\perp$ such that $\psi(x) = \psi_1(xp) + \psi_2(xp^\perp)$ for any $x \in M$.

Proof. Each item is easily obtained by known results.

(1) We may take a projection $p \in \mathcal{P}(M)$ such that both of the central supports of p and $1 - p$ are equal to 1. It is easy to see that the following hold: (a) If $x \in M$ satisfies $xM = \{0\}$, then $x = 0$; (b) If $x \in M$ satisfies $pMx = \{0\}$, then $x = 0$; (c) If $x \in M$ satisfies $pxpMp^\perp = \{0\}$, then $pxp = 0$. Hence we may apply Martindale's theorem [23, Theorem] to obtain the desired conclusion.

The item (2) is a consequence of Kaplansky's result [20, Theorem].

We prove (3) and (4) at the same time. Let $\psi \colon M \to N$ be a real algebra isomorphism. We know that $\psi(i)^2 = \psi(i^2) = \psi(-1) = -1$ and that $\psi(i)$ is central in N. It follows that $\psi(i) = qi - q^\perp i$ for some central projection q of N. Put

$p := \psi^{-1}(q)$, which is a central projection of M. If ψ is a real *-isomorphism, then ψ restricted to Mp is a *-isomorphism from Mp onto Nq, and ψ restricted to $Mp^\perp$ is a conjugate-linear *-isomorphism from $Mp^\perp$ onto $Nq^\perp$, hence the proof of (4) is complete. If ψ is merely a real algebra isomorphism, then ψ restricted to Mp is an algebra isomorphism from Mp onto Nq, and ψ restricted to $Mp^\perp$ determines an algebra isomorphism from $Mp^\perp$ onto $\overline{Nq^\perp}$, where $\overline{Nq^\perp}$ means the complex conjugation of the von Neumann algebra $Nq^\perp$. See e.g. [38, Section 2.3] for the definition of complex conjugation of von Neumann algebras. Lastly, we may use the result on the general form of algebra isomorphisms between von Neumann algebras ([35, Theorem I], see also [9] and [40, Section 4.1]) to obtain the desired conclusion. □

5.2. **The algebra of locally measurable operators.** Let $M \subset B(H)$ be a von Neumann algebra. In this part, the algebra $LS(M)$ of locally measurable operators with respect to M, which we briefly describe below, plays a crucial role.

A densely defined closed operator x on H is said to be *affiliated with* M (and we write $x\eta M$) if $yx \subset xy$ for any $y \in M'$, where $M' := \{y \in B(H) \mid ay = ya$ for any $a \in M\}$ denotes the commutant of M. An operator $x\eta M$ is said to be *measurable* with respect to M if the spectral projection $\chi_{(c,\infty)}(|x|) \in \mathcal{P}(M)$ is a finite projection in M for some real number $c > 0$. An operator $x\eta M$ is said to be *locally measurable* with respect to M if there exists an increasing sequence $\{p_n\}_{n\geq 1}$ of central projections in M such that $p_n \nearrow 1$ and xp_n is measurable with respect to M for any n. We write $S(M)$ (resp. $LS(M)$) to mean the collection of all measurable (resp. locally measurable) operators with respect to M. If $x, y \in S(M)$ (resp. $LS(M)$), then x^* and the closures of xy, $x + y$ are in $S(M)$ (resp. $LS(M)$). Using this fact, we can consider $S(M)$ and $LS(M)$ as *-algebras that contain M. In what follows, we abbreviate the symbol of the closure of an unbounded operator unless it is confusing. We remark that $LS(M) = M$ holds if and only if M is the direct sum of finite number of type I and III factors. We also remark that if M is finite then $LS(M) = S(M)$ is the collection of all affiliated operators. See [45] and [41] for more details of (locally) measurable operators.

The following lemma and its proof by the author are taken from [28, Subsection 2.2].

Lemma 5.2. *Let M be a von Neumann algebra and $a \in M_+$. Take the central projections $p_i \in \mathcal{P}(\mathcal{Z}(M))$, $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$, which are determined by the condition that $p_{\mathrm{I}} + p_{\mathrm{II}} + p_{\mathrm{III}} = 1$ and either $p_i = 0$ or Mp_i is of type i, $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$. Then the following three conditions are equivalent.*

(1) *The operator a is invertible in the algebra $LS(M)$.*
(2) *There exists no element $b \in M_+ \setminus \{0\}$ with the following property: If $x \in M_+$ satisfies $x \leq a$ and $x \leq b$, then $x = 0$.*
(3) *There exists a sequence $\{q_n\}_{n\geq 1} \subset \mathcal{P}(\mathcal{Z}(M))$ of central projections in M such that $\sum_{n\geq 1} q_n = 1$, $aq_n(p_{\mathrm{I}} + p_{\mathrm{III}})$ is invertible in $Mq_n(p_{\mathrm{I}} + p_{\mathrm{III}})$, and aq_np_{II} is an invertible element in $S(Mq_np_{\mathrm{II}})$, $n \geq 1$.*

For the proof we need an additional lemma.

Lemma 5.3. *Let M be a von Neumann algebra and $p \in M$ be a finite projection. Suppose that an increasing sequence $\{p_n\}_{n\geq 1}$ of projections in M satisfies $\bigvee_{n\geq 1} p_n \succ p$. Then there exists an increasing sequence $\{\tilde{p}_n\}_{n\geq 1}$ of projections in M such that $\tilde{p}_n \leq p_n$ and $\bigvee_{n\geq 1} \tilde{p}_n \sim p$.*

Proof. For $n \geq 1$, take the maximal central projection $e_n \in \mathcal{P}(\mathcal{Z}(M))$ that satisfies $p_n e_n \succ pe_n$. Then $\{e_n\}_{n\geq 1}$ is an increasing sequence and $p_n e_n^\perp \prec pe_n^\perp$. Put $e := \bigvee_{n\geq 1} e_n$. Take a sequence $\{q_n\}_{n\geq 1} \subset \mathcal{P}(M)$ such that $pe_1 \sim q_1 \leq p_1 e_1$ and $p(e_n - e_{n-1}) \sim q_n \leq p_n(e_n - e_{n-1})$ for $n \geq 2$. Put $\tilde{p}_n := \sum_{k=1}^n q_k + p_n e^\perp$ $(\leq p_n)$. Then $\{\tilde{p}_n\}_{n\geq 1}$ is an increasing sequence and satisfies $\bigvee_{n\geq 1} \tilde{p}_n e \sim pe$. The sequence $\{\tilde{p}_n e^\perp\}_{n\geq 1}$ is an increasing sequence and satisfies $\tilde{p}_n e^\perp \prec pe^\perp$, $n \geq 1$. Take a projection $\hat{p}_1 \in \mathcal{P}(M)$ such that $\tilde{p}_1 e^\perp \sim \hat{p}_1 \leq pe^\perp$. By finiteness of $pe^\perp$, we can take a sequence $\{\hat{p}_n\}_{n\geq 2}$ of projections in M such that $\{\hat{p}_n\}_{n\geq 1}$ is mutually orthogonal and $(\tilde{p}_n - \tilde{p}_{n-1})e^\perp \sim \hat{p}_n \leq pe^\perp$, $n \geq 2$. Then $\bigvee_{n\geq 1} \tilde{p}_n e^\perp \sim \sum_{n\geq 1} \hat{p}_n \leq pe^\perp$. Since $\bigvee_{n\geq 1} \tilde{p}_n e^\perp = \bigvee_{n\geq 1} p_n e^\perp \succ pe^\perp$, we have $\bigvee_{n\geq 1} \tilde{p}_n e^\perp \sim pe^\perp$, and thus $\bigvee_{n\geq 1} \tilde{p}_n \sim p$. $\square$

Proof of Lemma 5.2. (3) $\Rightarrow$ (1) We can take $b_n \in S(Mq_n)$ such that $b_n a q_n = q_n$ for each $n \geq 1$. Then the sum $\sum_{n\geq 1} b_n$ is the inverse of a in $LS(M)$.

(1) $\Rightarrow$ (2) Take the inverse $a^{-1} \in LS(M)$ of a and its positive square root $a^{-1/2} \in LS(M)$. The mapping $x \mapsto a^{-1/2} x a^{-1/2}$ is an order isomorphism from M_+ onto $\{x \in LS(M) \mid 0 \leq x \leq ca^{-1}$ for some positive real number $c\}$ and a is mapped to 1. We define a function $f\colon [0,\infty) \to \mathbb{R}$ by $f(t) := \min\{t, 1\}$, $t \in [0,\infty)$. For every $0 \neq b \in LS(M)$ with $0 \leq b \leq ca^{-1}$, $c > 0$ real, the element $(0 \neq)$ $x := f(b) \in M_+$ satisfies both $x \leq b$ and $x \leq 1$. Thus the condition (2) holds.

(2) $\Rightarrow$ (3) If we decompose M into a direct sum, it suffices to consider each direct summand. First we consider the cases of type I or III. It suffices to show that $\bigwedge_{n\geq 1} z(\chi_{[0,1/n)}(a)) = 0$. Assume $r := \bigwedge_{n\geq 1} z(\chi_{[0,1/n)}(a)) \neq 0$. Considering the pair (Mr, ar) instead of (M, a), we may assume $\bigwedge_{n\geq 1} z(\chi_{[0,1/n)}(a)) = 1$. Take a normal (tracial) state τ on $\mathcal{Z}(M)$. We may also assume $\mathrm{supp}(\tau) = 1 \in \mathcal{P}(\mathcal{Z}(M))$. By our assumptions, we may take a strictly decreasing sequence $\{c_n\}_{n\geq 1}$ of positive real numbers that satisfies $c_1 > \|a\|$, $c_n \to 0$ $(n \to 0)$ and $\tau(z(\chi_{[c_{n+1},c_n)}(a))) \geq 1 - 3^{-n}$, $n \geq 1$. Then $\tau(\bigwedge_{n\geq 1} z(\chi_{[c_{n+1},c_n)}(a))) \geq 1 - \sum_{n\geq 1} 3^{-n} > 0$ and thus $\bigwedge_{n\geq 1} z(\chi_{[c_{n+1},c_n)}(a)) \neq 0$. We may assume $\bigwedge_{n\geq 1} z(\chi_{[c_{n+1},c_n)}(a)) = 1$.

If M is of type I, we can take an abelian projection $p_n \leq \chi_{[c_{n+1},c_n)}(a)$ with $z(p_n) = 1$ for each $n \geq 1$. If M is of type III, by the assumption that $\mathcal{Z}(M)$ has a normal faithful state, we can take a countably decomposable projection $p_n \leq \chi_{[c_{n+1},c_n)}(a)$ with $z(p_n) = 1$ for each $n \geq 1$. In both cases, $\{p_n\}_{n\geq 1}$ is a family of mutually orthogonal equivalent nonzero projections. We consider the operator $\tilde{a} := \sum_{n\geq 1} c_n p_n + c_1(1 - \sum_{n\geq 1} p_n)$. We have $a \leq \sum_{n\geq 1} c_n \chi_{[c_{n+1},c_n)}(a) \leq \tilde{a}$, so $\tilde{a}$ also satisfies the condition (2). We can identify $\sum_{n\geq 1} c_n p_n \in (\sum_{n\geq 1} p_n) M (\sum_{n\geq 1} p_n)$ with $(\sum_{n\geq 1} c_n e_n) \otimes p_1 \in B(\ell^2) \mathbin{\overline{\otimes}} p_1 M p_1$, where e_n is the projection onto n-th coordinate of $\ell^2 = \ell^2(\mathbb{N})$, $\mathbb{N} = \{1, 2, \ldots\}$. Put $T := \sum_{n\geq 1} c_n e_n \in B(\ell^2)$. Since $c_n \to 0$ as $n \to \infty$, the positive operator $T^{1/2} \in B(\ell^2)$ is not invertible. Thus there exists a vector $\xi \in \ell^2 \setminus T^{1/2}\ell^2$. Take the projection $e \in B(\ell^2)$ whose range is $\mathbb{C}\xi$. It is easy to see that, if $x \in B(\ell^2)_+$ satisfies $x \leq T$ and $x \leq e$, then $x = 0$. Take the projection $p \in (\sum_{n\geq 1} p_n) M (\sum_{n\geq 1} p_n)$ that corresponds to $e \otimes p_1 \in B(\ell^2) \mathbin{\overline{\otimes}} p_1 M p_1$. By [18, Proposition 11.2.24], there exists a family $\{\Phi_i\colon B(\ell^2) \mathbin{\overline{\otimes}} p_1 M p_1 \to B(\ell^2) \otimes \mathbb{C}p_1\}_{i\in I}$ of normal conditional expectations with the property that if $x \in (B(\ell^2) \mathbin{\overline{\otimes}} p_1 M p_1)_+$ satisfies $\Phi_i(x) = 0$ for every $i \in I$, then $x = 0$. Suppose that $x \in M_+$ satisfies both $x \leq \tilde{a}$ and $x \leq p$. It follows that $x \in (\sum_{n\geq 1} p_n) M (\sum_{n\geq 1} p_n)$ and $\Phi_i(x) \leq T \otimes p_1$,

$\Phi_i(x) \leq e \otimes p_1$ in $B(\ell^2) \otimes \mathbb{C}p_1$ for every $i \in I$. Thus we have $\Phi_i(x) = 0$ for every $i \in I$ and hence $x = 0$. We obtain a contradiction.

Next we consider the case where M is of type II. For a projection $p \in \mathcal{P}(M)$, we take the central projection $z_{\mathrm{infin}}(p) \in \mathcal{P}(\mathcal{Z}(M))$ that is defined as the maximal projection in $\{z \in \mathcal{P}(\mathcal{Z}(M)) \mid pz \text{ is properly infinite, } z \leq z(p)\}$. It suffices to show that $\bigwedge_{n\geq 1} z_{\mathrm{infin}}(\chi_{[0,1/n)}(a)) = 0$. Hence we assume that $\bigwedge_{n\geq 1} z_{\mathrm{infin}}(\chi_{[0,1/n)}(a)) = 1$. Take a normal semifinite faithful tracial weight τ on M with $\tau(1) \geq 1$ and a (finite) projection $p \in M$ with $\tau(p) = 1$. It follows that $\chi_{[0,1/n)}(a) \succ p$ for every $n \geq 1$. By Lemma 5.3, there exist a strictly decreasing sequence $\{c_n\}_{n\geq 1}$ of positive real numbers and a sequence $\{p_n\}_{n\geq 1}$ of projections in M such that $c_1 > \|a\|$, $c_n \to 0$ $(n \to \infty)$, $p_n \leq \chi_{[c_{n+1},c_n)}(a)$, $p_n \prec p$ and $\tau(p_n) \geq 1 - 3^{-n}$, $n \geq 1$. Take a projection $\tilde{p}_n \in \mathcal{P}(M)$ such that $p_n \sim \tilde{p}_n \leq p$, $n \geq 1$. Put $\tilde{p} := \bigwedge_{n\geq 1} \tilde{p}_n$. Then $\tau(\tilde{p}) = \tau(\bigwedge_{n\geq 1} \tilde{p}_n) \geq 1 - \sum_{n\geq 1} 3^{-n} > 0$. Hence $\tilde{p} \neq 0$. Take a projection $\hat{p}_n \in \mathcal{P}(M)$ such that $\hat{p}_n \leq p_n$ and $\hat{p}_n \sim \tilde{p}$, for $n \geq 1$. Then $\{\hat{p}_n\}_{n\geq 1}$ is a family of mutually orthogonal equivalent nonzero projections. Put $\tilde{a} := \sum_{n\geq 1} c_n\hat{p}_n + c_1(1 - \sum_{n\geq 1} \hat{p}_n)$ $(\geq a)$. By a discussion similar to that in the preceding paragraph, we can obtain a contradiction. $\square$

For $x \in LS(M)$, let $l(x) \in \mathcal{P}(M)$ denote the left support of x. That is, $l(x) := \bigwedge\{p \in \mathcal{P}(M) \mid px = x\}$. Similarly, we write $r(x) := \bigwedge\{p \in \mathcal{P}(M) \mid x = xp\}$. Then $l(x) = \chi_{(0,\infty)}(|x^*|)$ and $r(x) = \chi_{(0,\infty)}(|x|)$ hold. We remark that, for $x, y \in LS(M)$, we have $xy = 0$ if and only if $r(x)l(y) = 0$. Indeed, if $r(x)l(y) = 0$, then $xy = xr(x)l(y)y = 0$. If $xy = 0$, then we have $|x||y^*| = 0$, which implies $\chi_{(\varepsilon,\infty)}(|x|)\chi_{(\varepsilon,\infty)}(|y^*|) = 0$ for every $\varepsilon > 0$. Take the limit $\varepsilon \to 0$ in the strong operator topology to obtain $r(x)l(y) = 0$.

5.3. **Center-valued norm.** Let M be a von Neumann algebra of type I or III and $x \in LS(M)$. Then there exists a unique minimal element $|\!|\!|x|\!|\!| \in LS(\mathcal{Z}(M))_+$ $(\subset LS(M))$ with $|x| \leq |\!|\!|x|\!|\!|$. The mapping $|\!|\!|\cdot|\!|\!| \colon LS(M) \to LS(\mathcal{Z}(M))_+$ is called the *center-valued norm*. Remark that if M is a factor, then $\mathcal{Z}(M)$ can be identified with $\mathbb{C}$ and we have $|\!|\!|x|\!|\!| = \|x\| \in \mathbb{R}$ for every $x \in M$. Be cautious of the fact that we cannot take such a mapping for a type II von Neumann algebra. That's why we will need to exclude type II cases in the proof of Theorem B.

As is expected, the center-valued norm possesses e.g. the following properties: For any $x, y \in LS(M)$ and $a \in LS(\mathcal{Z}(M))$, we have (i) $|\!|\!|x|\!|\!| = 0 \Longrightarrow x = 0$. (ii) $|\!|\!|x + y|\!|\!| \leq |\!|\!|x|\!|\!| + |\!|\!|y|\!|\!|$. (iii) $|\!|\!|a|\!|\!| = |a|$. (iv) $|\!|\!|ax|\!|\!| = |a||\!|\!|x|\!|\!|$. (v) $|\!|\!|xy|\!|\!| \leq |\!|\!|x|\!|\!||\!|\!|y|\!|\!|$. See for example [1, Section 2] and references therein for further information about the center-valued norm.

6. Lattice isomorphisms of projection lattices

Part of this section heavily depends on von Neumann's argument in [34, Part II, Chapter IV]. The aim of this section is to give a proof of

Theorem A. *Let M and N be two von Neumann algebras. Suppose that M does not admit type I_1 nor I_2 direct summands, and that $\Phi \colon \mathcal{P}(M) \to \mathcal{P}(N)$ is a lattice isomorphism. Then there exists a unique ring isomorphism $\Psi \colon LS(M) \to LS(N)$ such that $\Phi(l(x)) = l(\Psi(x))$ for all $x \in LS(M)$.*

Before its proof, we consider the converse of Theorem A.

Proposition 6.1. *Let M and N be von Neumann algebras. Suppose that $\Psi\colon LS(M)\to LS(N)$ is a ring isomorphism. Then there exists a unique lattice isomorphism $\Phi\colon \mathcal{P}(M)\to\mathcal{P}(N)$ such that $\Phi(l(x))=l(\Psi(x))$ for any $x\in LS(M)$.*

Proof. It is easy to see that $\Psi(0)=0$. Let $x,y\in LS(M)$ satisfy $l(x)\le l(y)$. Then we have $\{z\in LS(M)\mid zx\neq 0\}\subset\{z\in LS(M)\mid zy\neq 0\}$ and hence $\{z\in LS(N)\mid z\Psi(x)\neq 0\}\subset\{z\in LS(N)\mid z\Psi(y)\neq 0\}$, which in turn leads to $l(\Psi(x))\le l(\Psi(y))$. We obtain $l(x)\le l(y)\iff l(\Psi(x))\le l(\Psi(y))$ for any $x,y\in LS(M)$. Therefore, the mapping $\Phi\colon\mathcal{P}(M)\to\mathcal{P}(N)$ defined by $\Phi(p)=l(\Psi(p))$, $p\in\mathcal{P}(M)$, satisfies the desired condition. $\square$

Remark 6.2. The same proof is valid even if we replace a ring isomorphism with a semigroup isomorphism. However, Martindale's result [23] implies that a semigroup isomorphism $\Psi\colon LS(M)\to LS(N)$ is automatically a ring isomorphism if M is without type I_1 direct summands.

We begin the proof of Theorem A. Let us first check the uniqueness of Ψ.

Lemma 6.3. *Let M be a von Neumann algebra without type I_1 direct summands. For any $x\in M$, there exists a subset $F\subset M$ with $\#F\le 9$, $\sum_{y\in F}y=x$, and the following property: For any $y\in F$, there exists a pair $p,q\in\mathcal{P}(M)$ of mutually orthogonal projections such that $p\sim q$ and either $pyp=y$ or $pyq=y$.*

Proof. It suffices to consider the case where M has fixed order $2\le n<\infty$. Then we may identify M with $\mathbb{M}_n(\hat{M})$ for some von Neumann algebra $\hat{M}$. We may write $x\in M$ as $x=(x_{ij})_{1\le i,j\le n}\in\mathbb{M}_n(\hat{M})$. It is easy to see that we can take integers $n_0:=0\le n_1\le n_2\le n=:n_3$ such that $n_1,n_2-n_1,n_3-n_2\le n/2$. For $1\le k,l\le 3$, define $x^{kl}=(x^{kl}_{ij})_{1\le i,j\le n}\in\mathbb{M}_n(\hat{M})$ by $x^{kl}_{ij}=x_{ij}$ if $n_{k-1}+1\le i\le n_k$ and $n_{l-1}+1\le j\le n_l$, and $x^{kl}_{ij}=0$ otherwise. (Here, we are decomposing x into 3×3 blocks.) Then the nine operators x^{kl}, $1\le k,l\le 3$, (some of which may be 0) satisfy the desired condition. $\square$

Lemma 6.4. *Let M be a von Neumann algebra without type I_1 direct summands. Suppose that $\Psi\colon LS(M)\to LS(M)$ is a ring isomorphism with $l(\Psi(x))=l(x)$ for all $x\in LS(M)$. Then Ψ is the identity mapping on $LS(M)$.*

Proof. Let $p\in\mathcal{P}(M)$. We prove $\Psi(p)=p$. Since $pp^\perp=0$, we have $\Psi(p)\Psi(p^\perp)=0$, which implies $0=r(\Psi(p))l(\Psi(p^\perp))=r(\Psi(p))p^\perp$. We obtain $r(\Psi(p))\le p$. We also have the equation $\Psi(p)^2=\Psi(p^2)=\Psi(p)$. Hence we obtain $(p-\Psi(p))\Psi(p)=0$, which implies $0=(p-\Psi(p))l(\Psi(p))=(p-\Psi(p))p$ and $p-\Psi(p)=0$.

In what follows, let $p,q\in\mathcal{P}(M)$ be mutually orthogonal mutually Murray–von Neumann equivalent projections. We next prove that $\Psi(x)=x$ if $x\in M$ ($\subset LS(M)$) satisfies $pxq=x$. By additivity, we may assume $\|x\|\le 1/2$. Then there exists a projection $e\in\mathcal{P}(M)$ such that $e\le p+q$, $peq=x$. Indeed, let $x=v|x|=|x^*|v$ be the polar decomposition. Take an operator $a\in(pMp)_+$ such that $\|a\|\le\pi/4$ and $|x^*|=\sin a\cos a=(\sin 2a)/2$. Then

$$e:=\cos^2 a+v^*(\sin a\cos a)+(\sin a\cos a)v+v^*(\sin^2 a)v$$

satisfies this property. We obtain $\Psi(x)=\Psi(peq)=\Psi(p)\Psi(e)\Psi(q)=peq=x$.

Suppose that $x\in M$ satisfies $pxp=x$. Take a partial isometry $v\in M$ such that $vv^*=p$ and $v^*v=q$. Then we have $p(xv)q=xv$ and $qv^*p=v^*$. Hence $\Psi(x)=\Psi(xvv^*)=\Psi(xv)\Psi(v^*)=xvv^*=x$.

By the additivity of Ψ and the preceding lemma, we see that Ψ fixes every element in M. Let $x \in LS(M)$ and let $x = v|x|$ be its polar decomposition. It is clear that $\Psi(1) = 1$. Since $v, (|x|+1)^{-1} \in M$, we obtain

$$\begin{aligned}\Psi(x) = \Psi(v|x|) &= \Psi(v)\Psi(|x|)\\ &= v(\Psi(|x|+1) - 1) = v(\Psi((|x|+1)^{-1})^{-1} - 1)\\ &= v((|x|+1) - 1) = v|x| = x\end{aligned}$$

□

Hence we obtain the uniqueness of Ψ in Theorem A. Indeed, if two ring isomorphisms $\Psi, \Psi' \colon LS(M) \to LS(N)$ satisfies $l(\Psi(x)) = l(\Psi'(x))$ for all $x \in LS(M)$, then we have $l(\Psi^{-1} \circ \Psi'(x)) = l(x)$ for all $x \in LS(M)$, hence the preceding lemma implies $\Psi^{-1} \circ \Psi'(x) = x$ for all $x \in LS(M)$.

We introduce a binary relation on $\mathcal{P}(M)$, which is a key to the proof of Theorem A. Let $p, q \in \mathcal{P}(M)$ be two projections with $p \wedge q = 0$. By Section 2, we decompose p and q:

$$p = 1 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}. \tag{6.1}$$

We say that p is *LS-orthogonal* to q if the operator $b \in M_{p,q}$ is invertible in $LS(M_{p,q})$.

Lemma 6.5. *Let M be a von Neumann algebra and $p, q \in \mathcal{P}(M)$. Suppose that p is LS-orthogonal to q. Then there exists an invertible element $S = S_{p,q} \in LS(M)$ such that $S(p \vee q)^\perp = (p \vee q)^\perp S = (p \vee q)^\perp$, $Sp = p$ and $l(SqS^{-1}) = p \vee q - p$.*

Proof. Put $S := 1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} 1 & -ab^{-1} \\ 0 & b^{-1} \end{pmatrix}$ with respect to the decomposition as above. Then S is an element in $LS(M)$ with inverse $S^{-1} = 1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$. It is easy to see that

$$S(p \vee q)^\perp = (p \vee q)^\perp S = (p \vee q)^\perp = 0 \oplus 0 \oplus 1 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We also have

$$Sp = 1 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p$$

and

$$l(SqS^{-1}) = l\left(0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}\right) = 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = p \vee q - p.$$

□

Lemma 6.6. *Let M be a von Neumann algebra and $p, q \in \mathcal{P}(M)$ be two projections with $p \wedge q = 0$. Then the following are equivalent.*

1. *(1) The projection p is LS-orthogonal to q.*
2. *(2) There exists a lattice automorphism Φ of $\mathcal{P}(M)$ such that $\Phi(p) \perp \Phi(q)$.*
3. *(3) If a projection $p_0 \in \mathcal{P}(M)$ satisfies $p_0 \leq p$ and $p_0 \vee q = p \vee q$, then $p_0 = p$.*
4. *(4) The projection q is LS-orthogonal to p.*

Proof. (1) $\Rightarrow$ (2) Take $S \in LS(M)$ as in the preceding lemma and let Φ be the unique lattice isomorphism such that $\Phi(l(x)) = l(SxS^{-1})$, $x \in LS(X)$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) We use the decomposition (6.1). By Lemma 5.2, if (1) does not hold, then there exists an element $d \in M_{p,q+} \setminus \{0\}$ such that $\{x \in M_{p,q+} \mid x \leq b,\ x \leq d\} = \{0\}$. Take the nonzero spectral projection $p_1 := \chi_{(\|d\|/2, \|d\|]}(d) \in \mathcal{P}(M_{p,q})$. It follows that

$$\{x \in M_{p,q+} \mid x \leq b,\ x \leq p_1\} = \{0\}. \tag{6.2}$$

Indeed, if $x \in M_{p,q+}$ satisfies $x \leq b$ and $x \leq p_1$, take a positive real number c with $c \leq 1$ and $c \leq \|d\|/2$, then $cx \leq cb \leq b$ and $cx \leq cp_1 \leq d$, hence $cx = 0$ and we obtain $x = 0$. Put $p_0 := 1 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 1-p_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{P}(M)$. Then $p_0 \leq p$ and $p_0 \neq p$. We prove that $p_0 \vee q = p \vee q$, or equivalently, $\begin{pmatrix} 1-p_1 & 0 \\ 0 & 0 \end{pmatrix} \vee \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} = 1_{\mathbb{M}_2(M_{p,q})}$, which is in turn equivalent to

$$\begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} = 0_{\mathbb{M}_2(M_{p,q})}. \tag{6.3}$$

We have

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\leq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p_1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\leq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b^2 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Since the square root mapping preserves the order of positive operators, (6.2) implies that the square root of the operator

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is equal to 0. Hence

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

or equivalently, $\begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ holds. However, we know $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} = 0$, so we finally obtain (6.3).

Exchanging the roles of p and q, we also obtain (2) $\Leftrightarrow$ (4). $\square$

Let us recall the setting of Theorem A: Let M, N be von Neumann algebras. Suppose that M is without type I_1 and I_2 direct summands and $\Phi\colon \mathcal{P}(M) \to \mathcal{P}(N)$

is a lattice isomorphism. By the preceding lemma, we see that Φ preserves LS-orthogonality in both directions, that is, for any $p, q \in \mathcal{P}(M)$, p and q are LS-orthogonal if and only if $\Phi(p)$ and $\Phi(q)$ are LS-orthogonal.

In what follows, we show the existence of Ψ as in the statement of Theorem A in the case M has order 3. Thus M can be identified with $\mathbb{M}_3(\hat{M})$ for some von Neumann algebra $\hat{M}$. Put

$$e_1^M := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_2^M := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_3^M := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{P}(\mathbb{M}_3(\hat{M})).$$

Put $e_1 := \Phi(e_1^M)$, $e_2 := \Phi(e_2^M)$, $e_3 := \Phi(e_3^M)$. We know that $e_1 \vee e_2$ is LS-orthogonal to e_3, and e_1 is LS-orthogonal to e_2. In addition, we know $e_1 \vee e_2 \vee e_3 = 1$. Take $S_{e_1 \vee e_2, e_3}$ and S_{e_1, e_2} as in the statement of Lemma 6.5. Consider the lattice automorphism $\varphi\colon \mathcal{P}(N) \to \mathcal{P}(N)$ determined by the condition $\varphi(l(x)) = l(S_{e_1,e_2} S_{e_1 \vee e_2, e_3} x S_{e_1 \vee e_2, e_3}^{-1} S_{e_1,e_2}^{-1}) \,(= l(S_{e_1,e_2} S_{e_1 \vee e_2, e_3} x))$, $x \in LS(N)$. Then a moment's calculation shows that $\varphi(e_1), \varphi(e_2), \varphi(e_3)$ are mutually orthogonal and $\varphi(e_1) + \varphi(e_2) + \varphi(e_3) = 1_N$.

Lemma 6.7. *We have $\varphi(e_1) \sim \varphi(e_2) \sim \varphi(e_3)$ in N.*

Proof. Section 2 implies: For $p, q \in \mathcal{P}(N)$, if $p \vee q = 1$ and $p \wedge q = 0$, then $p^\perp \sim q$. Since $\varphi \circ \Phi$ is a lattice isomorphism, we obtain

$$\varphi(e_1) = \varphi \circ \Phi(e_1^M) \sim \left(\varphi \circ \Phi \left(\frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right) \right)^\perp \sim \varphi \circ \Phi(e_2^M) = \varphi(e_2).$$

Similarly, we obtain $\varphi(e_1) \sim \varphi(e_3)$. □

It suffices to consider $\varphi \circ \Phi$ instead of Φ. Hence we may identify N with $\mathbb{M}_3(\hat{N})$ for some von Neumann algebra $\hat{N}$, and we may assume $\Phi(e_1^M) = e_1^N$, $\Phi(e_2^M) = e_2^N$ and $\Phi(e_3^M) = e_3^N$, where

$$e_1^N := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_2^N := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_3^N := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{P}(\mathbb{M}_3(\hat{N})).$$

Let $x \in LS(\hat{M})$. Suppose that $\hat{M} \subset B(K)$. Viewing x as a closed operator, we see that the collection

$$\left\{ \begin{pmatrix} \xi \\ x\xi \\ 0 \end{pmatrix} \in K \oplus K \oplus K \,\middle|\, \xi \in \operatorname{dom} x \right\}$$

is a closed subspace in $K \oplus K \oplus K$. Take the projection $P_{12}[x] \in \mathcal{P}(B(K \oplus K \oplus K))$ onto this subspace. Then we have

$$P_{12}[x] = \begin{pmatrix} (1 + x^*x)^{-1} & (1 + x^*x)^{-1} x^* & 0 \\ x(1 + x^*x)^{-1} & x(1 + x^*x)^{-1} x^* & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.4}$$

and hence $P_{12}[x] \in \mathcal{P}(\mathbb{M}_3(\hat{M}))$. Similarly, let $P_{13}[x], P_{23}[x] \in \mathcal{P}(\mathbb{M}_3(\hat{M}))$ denote the projections onto

$$\left\{ \begin{pmatrix} \xi \\ 0 \\ x\xi \end{pmatrix} \in K \oplus K \oplus K \,\middle|\, \xi \in \operatorname{dom} x \right\}, \quad \left\{ \begin{pmatrix} 0 \\ \xi \\ x\xi \end{pmatrix} \in K \oplus K \oplus K \,\middle|\, \xi \in \operatorname{dom} x \right\},$$

respectively.

Lemma 6.8. *Let $Q \in \mathcal{P}(\mathbb{M}_3(\hat{M}))$. Then the following conditions are equivalent:*

(1) There exists an $x \in LS(\hat{M})$ such that $Q = P_{12}[x]$.
(2) $Q \vee e_2^M = e_1^M \vee e_2^M$, and Q is LS-orthogonal to e_2^M.

Proof. (1) $\Rightarrow$ (2) Let $Q = P_{12}[x]$. Since $(1 + x^*x)^{-1}$ is a positive injective operator, we have $Q \vee e_2^M = e_1^M \vee e_2^M$ by (6.4). Let $x = v|x|$ be the polar decomposition. By (6.4), we have

$$Q = P_{12}[x] = \begin{pmatrix} (1+|x|^2)^{-1} & (1+|x|^2)^{-1}|x|v^* & 0 \\ v|x|(1+|x|^2)^{-1} & v|x|(1+|x|^2)^{-1}|x|v^* & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$Q \wedge e_2^M \leq \begin{pmatrix} 0 & 0 & 0 \\ 0 & v|x|(1+|x|^2)^{-1}|x|v^* & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $1 - v|x|(1+|x|^2)^{-1}|x|v^*$ is a positive injective operator, we see that $Q \wedge e_2^M = 0$. As in (6.1), we may decompose e_2^M and Q in the following form:

$$e_2^M = 1 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; Q = 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}.$$

We also have

$$e_1^M = 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with respect to the same decomposition. Recall that $(1 + x^*x)^{-1}$ is invertible in $LS(\hat{M})$, or equivalently, $e_1^M Q e_1^M$ is invertible in $LS(e_1^M M e_1^M)$. This means that

$$0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & b^2 \end{pmatrix}$$

is invertible in $LS(e_1^M M e_1^M)$, which in particular implies the invertibility of b in $LS(M_{e_2^M, Q})$.

(2) $\Rightarrow$ (1) As in (6.1), we may decompose e_2^M and Q in the following form:

$$e_2^M = 1 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; Q = 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}.$$

Note that b is invertible as a locally measurable operator. It follows that

$$e_1^M = 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the partial isometry

$$w = 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}.$$

We have $ww^* = e_1^M$ and $w^*w = Q$. Moreover, $e_1^M w e_1^M$ is a positive invertible element in $LS(e_1^M M e_1^M)$. Thus a moment's reflection shows that there exist $w_1, w_2 \in \hat{M}$ such that $w_1 \geq 0$, w_1 is invertible in $LS(\hat{M})$ and $w = \begin{pmatrix} w_1 & w_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}_3(\hat{M})$. (Here w_1 corresponds to $e_1^M w e_1^M$.) Put $x = w_2^* w_1^{-1}$. Since $ww^* = e_1^M$, we obtain $w_1^2 + w_2 w_2^* = 1_{\hat{M}}$. Hence

$$1 + x^*x = 1 + w_1^{-1} w_2 w_2^* w_1^{-1} = 1 + w_1^{-1}(1 - w_1^2) w_1^{-1} = w_1^{-2}.$$

It follows by (6.4) that

$$P_{12}[x] = \begin{pmatrix} (1+x^*x)^{-1} & (1+x^*x)^{-1}x^* & 0 \\ x(1+x^*x)^{-1} & x(1+x^*x)^{-1}x^* & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} w_1^2 & w_1 w_2 & 0 \\ w_2^* w_1 & w_2^* w_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = w^*w = Q.$$

□

Corollary 6.9. *Let $k \in \{12, 13, 23\}$. There exists a bijection $\psi_k \colon LS(\hat{M}) \to LS(\hat{N})$ such that $\Phi(P_k[x]) = P_k[\psi_k(x)]$. Moreover, $x \in LS(\hat{M})$ is invertible in $LS(\hat{M})$ if and only if $\psi_k(x)$ is invertible in $LS(\hat{N})$*

Proof. Since Φ is a lattice isomorphism with $\Phi(e_1^M) = e_1^N$ and $\Phi(e_2^M) = e_2^N$, the first half of the case $k = 12$ follows from the preceding lemma. For $x \in LS(\hat{M})$, let $P_{21}[x] \in \mathcal{P}(\mathbb{M}_3(\hat{M}))$ denote the projection onto

$$\left\{ \begin{pmatrix} x\xi \\ \xi \\ 0 \end{pmatrix} \in K \oplus K \oplus K \;\middle|\; \xi \in \operatorname{dom} x \right\},$$

thus

$$P_{21}[x] = \begin{pmatrix} x(1+x^*x)^{-1}x^* & x(1+x^*x)^{-1} & 0 \\ (1+x^*x)^{-1}x^* & (1+x^*x)^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that, for $x, y \in LS(\hat{M})$, the equation $P_{12}[x] = P_{21}[y]$ holds if and only if x is invertible in $LS(\hat{M})$ and $y = x^{-1}$. Therefore, Lemma 6.8 implies that an operator $x \in LS(\hat{M})$ is invertible in $LS(\hat{M})$ if and only if $P_{12}[x]$ is LS-orthogonal to e_1^M and $P_{12}[x] \vee e_1^M = e_1^M \vee e_2^M$. Thus ψ_{12} preserves invertibility. The other cases can be shown similarly. □

In particular, the operators $\psi_{12}(1), \psi_{13}(1)$ are invertible in $LS(\hat{N})$. Consider the lattice automorphism ϕ of $\mathcal{P}(\mathbb{M}_3(\hat{N}))$ determined by $\phi(l(x)) = l(SxS^{-1})$, where $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \psi_{12}(1)^{-1} & 0 \\ 0 & 0 & \psi_{13}(1)^{-1} \end{pmatrix}$. We see that $\phi(e_i^N) = e_i^N$, $i = 1, 2, 3$, and

$$\phi \circ \Phi(P_{12}[1_{\hat{M}}]) = P_{12}[1_{\hat{N}}], \;\; \phi \circ \Phi(P_{13}[1_{\hat{M}}]) = P_{13}[1_{\hat{N}}].$$

Considering $\phi \circ \Phi$ instead of Φ, we may assume $\Phi(P_{12}[1_{\hat{M}}]) = P_{12}[1_{\hat{N}}]$ and $\Phi(P_{13}[1_{\hat{M}}]) = P_{13}[1_{\hat{N}}]$, or equivalently, $\psi_{12}(1) = \psi_{13}(1) = 1$.

Lemma 6.10. *For any $x, y \in LS(\hat{M})$, we have*

$$P_{13}[xy] = (P_{23}[-x] \vee P_{12}[y]) \wedge (e_1^M \vee e_3^M).$$

Proof. Let $\hat{M} \subset B(K)$. We know that the range of $P_{23}[-x] \vee P_{12}[y]$ is the closure of

$$V := \left\{ \begin{pmatrix} \eta \\ \xi + y\eta \\ -x\xi \end{pmatrix} \in K \oplus K \oplus K \,\middle|\, \xi \in \operatorname{dom} x,\ \eta \in \operatorname{dom} y \right\}.$$

In particular, we have $\begin{pmatrix} \eta \\ 0 \\ xy\eta \end{pmatrix} \in V$ for any $\eta \in \operatorname{dom} y$ with $y\eta \in \operatorname{dom} x$. Since the collection $\{\eta \in \operatorname{dom} y \mid y\eta \in \operatorname{dom} x\}$ is a core of the operator $xy \in LS(\hat{M})$, we have $P_{13}[xy] \leq (P_{23}[-x] \vee P_{12}[y]) \wedge (e_1^M \vee e_3^M)$.

We claim that the orthogonal complement $V^\perp$ of V is

$$\left\{ \begin{pmatrix} -y^*x^*\zeta \\ x^*\zeta \\ \zeta \end{pmatrix} \in K \oplus K \oplus K \,\middle|\, \zeta \in \operatorname{dom} x^*,\ x^*\zeta \in \operatorname{dom} y^* \right\}.$$

It is clear that any $\begin{pmatrix} -y^*x^*\zeta \\ x^*\zeta \\ \zeta \end{pmatrix}$ as above is an element in $V^\perp$. If $\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} \in V^\perp$, then

$$0 = \left\langle \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}, \begin{pmatrix} 0 \\ \xi \\ -x\xi \end{pmatrix} \right\rangle = \langle \zeta_2, \xi \rangle - \langle \zeta_3, x\xi \rangle$$

for any $\xi \in \operatorname{dom} x$, and hence we obtain $\zeta_3 \in \operatorname{dom} x^*, \zeta_2 = x^*\zeta_3$. By the equation

$$0 = \left\langle \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}, \begin{pmatrix} \eta \\ y\eta \\ 0 \end{pmatrix} \right\rangle$$

for $\eta \in \operatorname{dom} y$, we obtain the claim. Let $\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$ belong to the range of $(P_{23}[-x] \vee P_{12}[y]) \wedge (e_1^M \vee e_3^M)$, which is equal to the orthogonal complement of $V^\perp \cup \left\{ \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} \in K \oplus K \oplus K \,\middle|\, k \in K \right\}$. Then we have $h_2 = 0$ and

$$0 = \left\langle \begin{pmatrix} h_1 \\ 0 \\ h_3 \end{pmatrix}, \begin{pmatrix} -y^*x^*\zeta \\ x^*\zeta \\ \zeta \end{pmatrix} \right\rangle = -\langle h_1, y^*x^*\zeta \rangle + \langle h_3, \zeta \rangle$$

for any $\zeta \in \operatorname{dom} x^*$ with $x^*\zeta \in \operatorname{dom} y^*$. We know that $\{\zeta \in \operatorname{dom} x^* \mid x^*\zeta \in \operatorname{dom} y^*\}$ is a core of the operator $y^*x^* \in LS(\hat{M})$. Thus we obtain $h_1 \in \operatorname{dom}(y^*x^*)^* = \operatorname{dom}(xy)$ and $h_3 = (xy)h_1$ (here we view xy as a closed operator in $LS(\hat{M})$). □

Lemma 6.11. *We have $\psi_{12} = \psi_{13} = \psi_{23} =: \psi$. Moreover, $\psi\colon LS(\hat{M}) \to LS(\hat{N})$ is multiplicative.*

Proof. Let $x, y \in LS(\hat{M})$. By the preceding lemma, we have

$$P_{13}[xy] = (P_{23}[-x] \vee P_{12}[y]) \wedge (e_1^M \vee e_3^M)$$

and hence

$$\begin{aligned}P_{13}[\psi_{13}(xy)] = \Phi(P_{13}[xy]) &= \Phi\left((P_{23}[-x] \vee P_{12}[y]) \wedge (e_1^M \vee e_3^M)\right)\\ &= (\Phi(P_{23}[-x]) \vee \Phi(P_{12}[y])) \wedge (\Phi(e_1^M) \vee \Phi(e_3^M))\\ &= (P_{23}[\psi_{23}(-x)] \vee P_{12}[\psi_{12}(y)]) \wedge (e_1^N \vee e_3^N).\end{aligned}$$

It follows by the preceding lemma again (applied to N instead of M) that

$$(P_{23}[\psi_{23}(-x)] \vee P_{12}[\psi_{12}(y)]) \wedge (e_1^N \vee e_3^N) = P_{13}[-\psi_{23}(-x)\psi_{12}(y)].$$

Thus we obtain $P_{13}[-\psi_{23}(-x)\psi_{12}(y)] = P_{13}[\psi_{13}(xy)]$, which implies $-\psi_{23}(-x)\psi_{12}(y) = \psi_{13}(xy)$.

In particular, putting $x = y = 1$, we obtain $\psi_{23}(-1) = -1$. Putting $x = 1$, we obtain $-\psi_{23}(-1)\psi_{12}(y) = \psi_{13}(y)$, hence $\psi_{12}(y) = \psi_{13}(y)$. Moreover, putting $y = 1$, we obtain $-\psi_{23}(-x)\psi_{12}(1) = \psi_{13}(x)$, hence $-\psi_{23}(-x) = \psi_{13}(x)$. Thus $\psi_{12}(x)\psi_{12}(y) = -\psi_{23}(-x)\psi_{12}(y) = \psi_{13}(xy) = \psi_{12}(xy)$. Therefore, ψ_{12} is multiplicative. It follows that $\psi_{12}(-1)$ is central in $LS(\hat{N})$, $\psi_{12}(-1)^2 = 1$ and $\psi_{12}(-1)y \neq y$ for any $y \neq 0$, and hence we obtain $\psi_{12}(-1) = -1$. We reach the equation $\psi_{13} = \psi_{12} = \psi_{23}$. □

Lemma 6.12. *The mapping ψ is additive.*

Proof. Let $x, y \in LS(\hat{M})$. Consider the projections

$$f = (P_{12}[x] \vee e_3^M) \wedge (P_{13}[1] \vee e_2^M) \quad \text{and} \quad g = (P_{12}[y] \vee P_{13}[1]) \wedge (e_2^M \vee e_3^M).$$

By an argument similar to that in the proof of Lemma 6.10, we can check the following: The range of f is equal to

$$\left\{ \begin{pmatrix} \xi \\ x\xi \\ \xi \end{pmatrix} \in K \oplus K \oplus K \;\middle|\; \xi \in \operatorname{dom} x \right\}$$

and the range of g is equal to

$$\left\{ \begin{pmatrix} 0 \\ -y\eta \\ \eta \end{pmatrix} \in K \oplus K \oplus K \;\middle|\; \eta \in \operatorname{dom} y \right\},$$

hence $(f \vee g) \wedge (e_1^M \vee e_2^M) = P_{12}[x+y]$. Apply Φ to both sides to obtain the desired conclusion. □

We define a mapping $\Psi\colon LS(\mathbb{M}_3(\hat{M})) \to LS(\mathbb{M}_3(\hat{N}))$ by $\Psi((x_{ij})_{ij}) := (\psi(x_{ij}))_{ij}$, $x_{ij} \in LS(\hat{M})$, $i, j = 1, 2, 3$. The preceding lemmas imply that Ψ is a ring isomorphism from $LS(\mathbb{M}_3(\hat{M}))$ onto $LS(\mathbb{M}_3(\hat{N}))$.

Lemma 6.13. *We have $\Phi(l(x)) = l(\Psi(x))$ for any $x \in LS(\mathbb{M}_3(\hat{M}))$.*

Proof. We partly imitate Dye's argument in the proof of [6, Lemma 7]. By Lemma 6.4, it suffices to show that the lattice isomorphism $\Phi'\colon \mathcal{P}(\mathbb{M}_3(\hat{M})) \to \mathcal{P}(\mathbb{M}_3(\hat{N}))$ determined by $l(\Psi(x)) = \Phi'(l(x))$, $x \in LS(\mathbb{M}_3(\hat{M}))$, satisfies $\Phi = \Phi'$. For $x \in$

$LS(\hat{M})$, we have

$$\begin{aligned}\Phi(P_{12}[x]) = P_{12}[\psi(x)] &= l\begin{pmatrix}1 & 0 & 0\\ \psi(x) & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\ &= l\left(\Psi\begin{pmatrix}1 & 0 & 0\\ x & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right) = \Phi'\left(l\begin{pmatrix}1 & 0 & 0\\ x & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right) = \Phi'(P_{12}[x]).\end{aligned}$$

Similarly, we see that $\Phi(p) = \Phi'(p)$ for any $p \in \{P_k[x] \mid x \in LS(\hat{M}),\ k = 12,\ 23,\ 13\}$.

Let $x_2, x_3 \in LS(\hat{M})$. We consider the projection $P_{x_2,x_3} \in \mathcal{P}(\mathbb{M}_3(\hat{M}))$ onto the closed subspace

$$\left\{\begin{pmatrix}\xi\\ x_2\xi\\ x_3\xi\end{pmatrix} \in K \oplus K \oplus K \;\middle|\; \xi \in \operatorname{dom} x_2 \cap \operatorname{dom} x_3\right\}.$$

It is not difficult to see that this projection is equal to $(P_{12}[x_2] \vee e_3^M) \wedge (P_{13}[x_3] \vee e_2^M)$. It follows that $\Phi(P_{x_2,x_3}) = \Phi'(P_{x_2,x_3})$.

Consider an arbitrary nonzero projection $p = (p_{i,j})_{1\le i,j\le 3} \in \mathcal{P}(\mathbb{M}_3(\hat{M}))$. By Zorn's lemma, to show that $\Phi(p) = \Phi'(p)$, it suffices to find a nonzero subprojection $(p \geq)\ q \in \mathcal{P}(\mathbb{M}_3(\hat{M}))$ such that $\Phi(q) = \Phi'(q)$. Note that $p_{ii} = \sum_{1\le k\le 3} p_{ik}p_{ik}^*$, hence we see that $p_{ii} \neq 0$ for some $i \in \{1, 2, 3\}$.

If $p_{11} \neq 0$, put $e := \chi_{(\|p_{11}\|/2, \|p_{11}\|]}(p_{11}) \in \mathcal{P}(\hat{M}) \setminus \{0\}$ and $x_1 := p_{11}^{-1}e \in \hat{M}$. It follows that the projection $q \in \mathcal{P}(\mathbb{M}_3(\hat{M}))$ onto the subspace

$$\left\{\begin{pmatrix}p_{11}\xi\\ p_{21}\xi\\ p_{31}\xi\end{pmatrix} \in K \oplus K \oplus K \;\middle|\; \xi \in eK\right\} = \left\{\begin{pmatrix}\eta\\ p_{21}x_1\eta\\ p_{31}x_1\eta\end{pmatrix} \in K \oplus K \oplus K \;\middle|\; \eta \in eK\right\}$$

is a nonzero subprojection of p. Since $q = P_{p_{21}x_1, p_{31}x_1} \wedge ((P_{12}[e^\perp] \wedge e_1^M) \vee e_2^M \vee e_3^M)$, we obtain $\Phi(q) = \Phi'(q)$.

If $p_{11} = 0$ and $p_{22} \neq 0$, we have $p \leq e_2^M \vee e_3^M$. Then a similar discussion applies. If $p_{11} = p_{22} = 0$, then $p_{33} \in \mathcal{P}(\hat{M})$. Use the equation $(P_{13}[1] \vee P_{13}[p_{33}^\perp]) \wedge e_3^M = p$, which can be verified easily, to obtain the desired conclusion. $\square$

Therefore, the proof of Theorem A is complete in the case M has order 3. The same discussion with a slight modification is valid in any case M has order n with $3 \leq n < \infty$. We know that a projection lattice isomorphism preserves central projections because a projection p in a von Neumann algebra M is central if and only if $\{q \in \mathcal{P}(M) \mid p \vee q = 1, p \wedge q = 0\} = \{p^\perp\}$. Since every von Neumann algebra without type I_1 and I_2 direct summands decomposes into the direct sum of algebras of order $3 \leq n < \infty$, now it easy to complete the proof of Theorem A in the general case.

In what follows, let us give a proof of Theorem 1.4 by Dye (in the case the von Neumann algebras are without commutative direct summands) as an application of Theorem A. The proof below is partly based on Feldman's argument in [7, Proof of Theorem 3].

Let M and N be von Neumann algebra without type I_1 and I_2 direct summands and suppose that $\Phi\colon \mathcal{P}(M) \to \mathcal{P}(N)$ is a lattice isomorphism. Suppose further that we have $pq = 0$ if and only if $\Phi(p)\Phi(q) = 0$ for any pair $p, q \in \mathcal{P}(M)$. By Theorem A, there exists a unique ring isomorphism $\Psi\colon LS(M) \to LS(N)$ such that $\Phi(l(x)) = l(\Psi(x))$ for any $x \in LS(M)$.

Then we have $\Psi(p) = \Phi(p) \in \mathcal{P}(N)$ for every $p \in \mathcal{P}(M)$. Indeed, since $p^2 = p$ and $pp^\perp = 0$, we have $\Psi(p)^2 = \Psi(p)$ and $\Psi(p)\Psi(p^\perp) = 0$. Thus we have $r(\Psi(p))l(\Psi(p^\perp)) = 0$. Our assumption implies $l(\Psi(p^\perp)) = \Phi(p^\perp) = \Phi(p)^\perp = l(\Psi(p))^\perp$, thus we obtain $r(\Psi(p)) \leq l(\Psi(p))$. By the equation $(l(\Psi(p)) - \Psi(p))\Psi(p) = 0$, we obtain $0 = (l(\Psi(p)) - \Psi(p))l(\Psi(p)) = l(\Psi(p)) - \Psi(p)$. Hence $\Psi(p) = l(\Psi(p)) = \Phi(p) \in \mathcal{P}(N)$.

Consider the ring automorphism $x \mapsto \Psi^{-1}(\Psi(x^*)^*)$ of $LS(M)$. This fixes every projection, hence Lemma 6.4 implies that $x = \Psi^{-1}(\Psi(x^*)^*)$, or equivalently, $\Psi(x)^* = \Psi(x^*)$ for each $x \in LS(M)$. It follows that Ψ maps the self-adjoint part of $LS(M)$ onto that of $LS(N)$. Since Ψ preserves squares, Ψ restricted to self-adjoint parts preserves order in both directions. Since $\Psi(1) = 1$, Ψ restricts to a real *-isomorphism from M onto N and extends Φ, which is the desired conclusion.

7. Ring isomorphisms of locally measurable operator algebras

By the preceding section, lattice isomorphisms between projection lattices are in one-to-one correspondence with ring isomorphisms between the algebras of locally measurable operators. Hence the following question is well motivated.

Question. *Let M, N be von Neumann algebras. What is the general form of ring isomorphisms from $LS(M)$ onto $LS(N)$?*

Lemma 7.1. *Let M, N be general von Neumann algebras. Let*

$$M = \left(\bigoplus_{n\geq 1} M_{\mathrm{I}_n}\right) \oplus M_{\mathrm{I}_\infty} \oplus M_{\mathrm{II}_1} \oplus M_{\mathrm{II}_\infty} \oplus M_{\mathrm{III}},$$

$$N = \left(\bigoplus_{n\geq 1} N_{\mathrm{I}_n}\right) \oplus N_{\mathrm{I}_\infty} \oplus N_{\mathrm{II}_1} \oplus N_{\mathrm{II}_\infty} \oplus N_{\mathrm{III}}$$

be the type decompositions, where M_j, N_j are von Neumann algebras of type j. Suppose that $\Psi\colon LS(M) \to LS(N)$ is a ring isomorphism. Then there exist ring isomorphisms $\psi_j\colon LS(M_j) \to LS(N_j)$ such that $\Psi(x) = \psi_j(x)$ for any $x \in LS(M_j)$ $(\subset LS(M))$.

Proof. It is easy to see that Ψ maps the collection of central projections in M onto that in N. Hence it suffices to show: If M, N are of type $j, k \in \{\mathrm{I}_n \mid n \geq 1\} \cup \{\mathrm{I}_\infty, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$, respectively, then $j = k$. We consider the lattice isomorphism $\Phi\colon \mathcal{P}(M) \to \mathcal{P}(N)$ as in Proposition 6.1. It is easy to see that a projection $p \in \mathcal{P}(M)$ is abelian (namely, pMp is an abelian von Neumann algebra) if and only if $\Phi(p)$ is abelian. Moreover, a projection $p \in \mathcal{P}(M)$ is finite if and only if $\Phi(p) \in \mathcal{P}(N)$ is finite. Indeed, if p is not finite, then there exist mutually orthogonal nonzero subprojections p_1, p_2, p_3 of p such that $p_1 \sim p_2 \sim p_3 \sim p_1 + p_2$. The same argument as in the proof of Lemma 6.7 implies $\Phi(p_1) \sim \Phi(p_3) \sim \Phi(p_1 + p_2)$, which shows that $\Phi(p)$ is not finite. Similarly, if $\Phi(p)$ is not finite, then p is not finite. The rest of the proof is a standard argument of von Neumann algebra theory, and we omit the details. See e.g. [18, Chapter 6]. $\square$

Therefore, Question reduces to the case both M and N are of type j, $j \in \{\mathrm{I}_n \mid n \geq 1\} \cup \{\mathrm{I}_\infty, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$.

We first consider Question in the case M, N are von Neumann algebras of type I_n. Suppose that $LS(M)$ is ring isomorphic to $LS(N)$. Since the central projection lattices of M and N are lattice isomorphic, we see that the center of M is *-isomorphic to that of N. Hence there exists a commutative von Neumann algebra A such that $M \cong N \cong \mathbb{M}_n(A)$. Therefore, it suffices to think about ring automorphisms of $LS(\mathbb{M}_n(A))$, which can be identified with the collection of all $n \times n$ matrices with entries in $LS(A)$. Note that A can be identified with the algebra $L^\infty(\mu)$ of all complex-valued essentially bounded measurable functions (modulo almost-everywhere equivalence) for some measure μ. Then $LS(A)$ corresponds to $L^0(\mu)$, which denotes the collection of all complex-valued measurable functions. Remark that any ring automorphism ψ of $LS(A)$ determines a ring automorphism ψ' of $LS(\mathbb{M}_n(A))$ by the formula $\psi'((x_{ij})) = (\psi(x_{ij}))_{ij}$. The following proposition slightly generalizes (but can be shown by exactly the same argument as in) [1, Theorem 3.3] by Albeverio, Ayupov, Kudaybergenov and Djumamuratov.

Proposition 7.2. *Let $n \geq 1$ be an integer and A be a commutative von Neumann algebra. Suppose that Ψ is a ring automorphism of $LS(\mathbb{M}_n(A))$. Then there exist a ring automorphism $\psi\colon LS(A) \to LS(A)$ and an invertible element y in $LS(\mathbb{M}_n(A))$ such that $\Psi(x) = y\psi'(x)y^{-1}$, $x \in LS(\mathbb{M}_n(A))$.*

Proof. Note that Ψ restricts to a ring automorphism ψ of the center of $LS(\mathbb{M}_n(A))$, which is canonically isomorphic to $LS(A)$. Then $\Psi \circ {\psi'}^{-1}$ fixes every element in the center of $LS(\mathbb{M}_n(A))$. We may apply [1, Theorem 3.1] to obtain the desired conclusion. $\square$

There exist highly nontrivial examples of ring automorphisms of $LS(A) = L^0(\mu)$ for a commutative von Neumann algebra A. For example, consider the case $A = \mathbb{C} = LS(A)$. There are many ring automorphisms of $\mathbb{C}$ that are far from real-linear. Consider the case μ is an atomless measure. It is known [21, (1)$\Leftrightarrow$(6) of Theorem 3.4] (see also [21, Remark 6.3]) that there exists a (complex-linear) algebra automorphism ψ of $L^0(\mu)$ such that $\psi(p) = p$ for any $p \in \mathcal{P}(A)$ but $\psi \neq \mathrm{id}_{L^0(\mu)}$. It seems that these examples are beyond the scope of the theory of operator algebras.

In contrast, we may give a purely operator algebraic solution to Question for type I_∞ or III in the following manner. This improves [1, Theorem 3.8], in which algebra isomorphisms of the case of type I_∞ were considered.

Theorem B. *Let M, N be von Neumann algebras of type I_∞ or III. If $\Psi\colon LS(M) \to LS(N)$ is a ring isomorphism, then there exist a real *-isomorphism $\psi\colon M \to N$ (which extends to a real *-isomorphism from $LS(M)$ onto $LS(N)$) and an invertible element $y \in LS(N)$ such that $\Psi(x) = y\psi(x)y^{-1}$, $x \in LS(M)$.*

Proof. Beware of the fact that Ψ restricts to a lattice isomorphism between the central projection lattices of M and N. We first prove:

<u>Claim</u> There exists an operator $a \in LS(\mathcal{Z}(N))_+$ such that $|||\Psi(x)||| \leq a$ for any $x \in M$ $(\subset LS(M))$ with $\|x\| \leq 1$.

Assume that this claim does not hold. We will obtain a contradiction in Step 4.

<u>Step 1</u> We prove that there exists a central projection e in M such that for any $n \geq 1$ there exists some $x \in M$ with $\|x\| \leq 1$ and $|||\Psi(x)||| \geq n\Psi(e)$.

Assume for a while that the center $\mathcal{Z}(M)$ of M admits a faithful normal state $\tau\colon \mathcal{Z}(M) \to \mathbb{C}$. For each positive integer n, consider the collection

$$E_n := \{e \in \mathcal{P}(\mathcal{Z}(M)) \mid \text{ there exists } x \in M \text{ with } \|x\| \leq 1 \text{ and } |||\Psi(x)||| \geq n\Psi(e)\}.$$

Suppose that e, f belong to this collection. Take $x, y \in M$ such that $\|x\|, \|y\| \leq 1$ and $|||\Psi(x)||| \geq n\Psi(e)$, $|||\Psi(y)||| \geq n\Psi(f)$. Then the element $x' := xe + ye^{\perp}$ satisfies $\|x'\| \leq 1$ and

$$\begin{aligned} |||\Psi(x')||| &= \left|\left|\left|\Psi(xe + ye^{\perp})\right|\right|\right| \\ &= \left|\left|\left|\Psi(x)\Psi(e) + \Psi(y)\Psi(e)^{\perp}\right|\right|\right| \\ &= |||\Psi(x)|||\Psi(e) + |||\Psi(y)|||\Psi(e)^{\perp} \\ &\geq n\Psi(e) + n\Psi(f)\Psi(e)^{\perp} \\ &= n\Psi(e) \vee \Psi(f) = n\Psi(e \vee f). \end{aligned}$$

Hence we have $e \vee f \in E_n$, which implies that E_n is upward directed. Put $c_n := \sup\{\tau(e) \mid e \in E_n\}$. We may take an increasing sequence $\{e^{(k)}\} \subset E_n$ such that $\tau(e^{(k)}) \to c_n$ as $k \to \infty$. For each k take $x^{(k)} \in M$ such that $\|x^{(k)}\| \leq 1$ and $\left|\left|\left|\Psi(x^{(k)})\right|\right|\right| \geq n\Psi(e^{(k)})$. Some calculations show that the element

$$x'' := x^{(1)}e^{(1)} + \sum_{k\geq 2} x^{(k)}(e^{(k)} - e^{(k-1)}) \in M$$

satisfies $\|x''\| \leq 1$ and $|||\Psi(x'')||| \geq n\Psi(e^{(k)})$ for every k. This implies that for the projection $e_n := \bigvee E_n \in \mathcal{P}(\mathcal{Z}(M))$ there exists $x_n \in \mathcal{P}(\mathcal{Z}(M))$ such that $\|x_n\| \leq 1$ and $|||\Psi(x_n)||| \geq n\Psi(e_n)$.

Clearly, $\{e_n\}$ is a decreasing sequence. Assume that $e_n \to 0$ as $n \to \infty$, then the element $a = \Psi(1 + \sum_{n\geq 1} e_n) \in LS(\mathcal{Z}(N))_+$ satisfies the property of Claim, which contradicts our assumption. Hence we have $e_n \to e \in \mathcal{P}(\mathcal{Z}(M)) \setminus \{0\}$ as $n \to \infty$, and e satisfies the desired property. Since every von Neumann algebra can be decomposed into the direct sum of von Neumann algebras whose centers admit faithful normal states, the same holds for arbitrary M and N.

Considering the restriction of Ψ to a ring isomorphism from $LS(Me)$ onto $LS(N\Psi(e))$, we may assume that for any $n \geq 1$ there exists some $x \in M$ with $\|x\| \leq 1$ and $|||\Psi(x)||| \geq n$.

<u>Step 2</u> We prove that for any $a \in LS(\mathcal{Z}(N))_+$ there exists some $x \in M$ with $\|x\| \leq 1$ and $|||\Psi(x)||| \geq a$.

Let $a \in LS(\mathcal{Z}(N))_+$. We may take a sequence of mutually orthogonal central projections $\{f_n\}$ such that $a \leq \sum_{n\geq 1} nf_n$. For each n, take $x_n \in M$ such that $\|x_n\| \leq 1$ and $|||\Psi(x_n)||| \geq nf_n$. Some calculations show that the element $x := \sum_{n\geq 1} x_n\Psi^{-1}(f_n)$ satisfies $\|x\| \leq 1$ and $|||\Psi(x)||| \geq \sum_{n\geq 1} nf_n \geq a$.

<u>Step 3</u> We prove: For any $p \in \mathcal{P}(M)$ with $p \sim p^{\perp}$ and any $a \in LS(\mathcal{Z}(N))_+$, there exists an element $x \in M$ with $pxp = x$, $\|x\| \leq 1$ and $|||\Psi(x)||| \geq a$.

Take a partial isometry $v \in M$ such that $vv^* = p$ and $v^*v = p^{\perp}$. Since Ψ is a ring isomorphism, for any $x \in M$, we have

$$\begin{aligned} \Psi(x) &= \Psi(pxp + pxp^{\perp} + p^{\perp}xp + p^{\perp}xp^{\perp}) \\ &= \Psi(pxp) + \Psi(pxv^*)\Psi(v) + \Psi(v^*)\Psi(vxp) + \Psi(v^*)\Psi(vxv^*)\Psi(v). \end{aligned}$$

For a given $a \in LS(\mathcal{Z}(N))_+$, put

$$b := 4a + 4a|||\Psi(v)||| + 4a|||\Psi(v^*)||| + 4a|||\Psi(v)||||||\Psi(v^*)||| \in LS(\mathcal{Z}(N))_+.$$

The preceding step implies there exists $x \in M$ with $\|x\| \leq 1$ and

$$\begin{aligned} b &\leq |||\Psi(x)||| \\ &\leq |||\Psi(pxp)||| + |||\Psi(pxv^*)||||||\Psi(v)||| + |||\Psi(v^*)||||||\Psi(vxp)||| + |||\Psi(v^*)||||||\Psi(vxv^*)||||||\Psi(v)|||. \end{aligned}$$

Hence there exists a quadruple f_1, f_2, f_3, f_4 of central projections in N such that $f_1 + f_2 + f_3 + f_4 = 1$, $|||\Psi(pxp)|||f_1 \geq bf_1/4$, $|||\Psi(pxv^*)||||||\Psi(v)|||f_2 \geq bf_2/4$, $|||\Psi(v^*)||||||\Psi(vxp)|||f_3 \geq bf_3/4$ and $|||\Psi(v^*)||||||\Psi(vxv^*)||||||\Psi(v)|||f_4 \geq bf_4/4$. Put

$$x' := pxp\Psi^{-1}(f_1) + pxv^*\Psi^{-1}(f_2) + vxp\Psi^{-1}(f_3) + vxv^*\Psi^{-1}(f_4).$$

Then we have $px'p = x'$, $\|x'\| \leq 1$ and

$$\begin{aligned} |||\Psi(x')||| &= |||\Psi(pxp)f_1 + \Psi(pxv^*)f_2 + \Psi(vxp)f_3 + \Psi(vxv^*)f_4||| \\ &= |||\Psi(pxp)|||f_1 + |||\Psi(pxv^*)|||f_2 + |||\Psi(vxp)|||f_3 + |||\Psi(vxv^*)|||f_4 \\ &\geq \frac{1}{4}b(f_1 + |||\Psi(v)|||^{-1}f_2 + |||\Psi(v^*)|||^{-1}f_3 + |||\Psi(v)|||^{-1}|||\Psi(v^*)|||^{-1}f_4) \geq a. \end{aligned}$$

(Note that $|||\Psi(v)|||$, $|||\Psi(v^*)|||$ are invertible in $LS(\mathcal{Z}(N))$.)

Step 4 Since M is properly infinite, we may take a sequence $\{p_n\}_{n\geq 1}$ of mutually orthogonal projections in M such that $p_n \sim p_n^\perp$, $n \geq 1$. By Step 3, for each $n \geq 1$, we may take an element $x_n \in M$ with $p_n x_n p_n = x_n$, $\|x_n\| \leq 1$ and $|||\Psi(x_n)||| \geq n|||\Psi(p_n)|||$. Put $x := \sum_{n\geq 1} x_n \in M$ (which is well-defined since p_n, $n \geq 1$, are mutually orthogonal). For every $n \geq 1$, we have

$$|||\Psi(x)||||||\Psi(p_n)||| \geq |||\Psi(x)\Psi(p_n)||| = |||\Psi(xp_n)||| = |||\Psi(x_n)||| \geq n|||\Psi(p_n)|||.$$

Since $|||\Psi(p_n)|||$ is invertible in $LS(\mathcal{Z}(N))$, we obtain $|||\Psi(x)||| \geq n$ for all $n \in \mathbb{N}$, a contradiction. This completes the proof of Claim.

Step 5 It follows that there exists an element $a \in LS(\mathcal{Z}(N))_+$ such that $|||\Psi(x)||| \leq a$ for any $x \in M$ with $\|x\| \leq 1$. By the same discussion applied to Ψ^{-1}, we also obtain an element $a' \in LS(\mathcal{Z}(M))_+$ such that $\left|\left|\left|\Psi^{-1}(y)\right|\right|\right| \leq a'$ for any $y \in N$ with $\|y\| \leq 1$. We may take a sequence e_n of central projections in M such that $e_n \nearrow 1$ and Ψ restricts to a norm-bicontinuous ring isomorphism Ψ_n from Me_n onto $N\Psi(e_n)$, $n \geq 1$. By Lemma 5.1 we may verify the statement for each Ψ_n, which suffices to complete the proof. □

Corollary 7.3. *Let M, N be von Neumann algebras of type I_∞ or III. Suppose that $\Phi\colon \mathcal{P}(M) \to \mathcal{P}(N)$ is a lattice isomorphism. Then there exist a real *-isomorphism $\psi\colon M \to N$ and an invertible element $y \in LS(N)$ such that $\Phi(p) = l(y\psi(p))$, $p \in \mathcal{P}(M)$.*

8. Questions

The author skeptically conjectures that the same as Theorem B holds for type II von Neumann algebras:

Conjecture 8.1. *Let M and N be von Neumann algebras of type II. Suppose that $\Psi\colon LS(M) \to LS(N)$ is a ring isomorphism. Then there exist an invertible operator $y \in LS(N)$ and a real *-isomorphism $\psi\colon M \to N$ such that $\Psi(x) = y\psi(x)y^{-1}$ for any $x \in LS(M)$.*

Not much is known about the structure of the algebra $LS(M)$ for a type II (in particular, II_1) von Neumann algebra M. The author does not know whether or not such a Ψ is automatically real-linear even in the case M and N are (say, approximately finite dimensional) II_1 factors. Note that $LS(M)$ cannot have a Banach algebra structure because of the fact that an element of $LS(M)$ can have an empty or dense spectrum. Hence it seems to be difficult to make use of *automatic continuity* results on algebra isomorphisms as in [5]. However, the author suspects that at least the following weaker statement holds:

Conjecture 8.2. *Let M and N be von Neumann algebras of type II. If $\mathcal{P}(M)$ and $\mathcal{P}(N)$ are lattice isomorphic, or equivalently, if $LS(M)$ and $LS(N)$ are ring isomorphic, then M and N are real *-isomorphic (or equivalently, M and N are Jordan *-isomorphic).*

Remark. After the post of this part to the arXiv, Ayupov and Kudaybergenov [2] *gave an affirmative solution to Conjecture 8.1 (and 8.2) for type II_1 von Neumann algebras.*

In another direction, we compare Theorem A with von Neumann's theory of complemented modular lattices and regular rings. Von Neumann axiomatized projection lattices of type II_1 von Neumann algebras, and completed the amazing theory on the correspondence between the vast classes of complemented modular lattices and regular rings. Let us briefly recall this theory in [34, Part II].

Definition 8.3. A lattice L with greatest element 1 and least element 0 is *complemented* if for each $a \in L$ there exists $b \in L$ such that $a \vee b = 1$, $a \wedge b = 0$. A lattice L is *modular* if the equation $(a \vee b) \wedge c = a \vee (b \wedge c)$ holds for any $a, b, c \in L$ with $a \leq c$.

Let L be a complemented modular lattice. Two elements $a, b \in L$ are said to be *perspective* if there exists $c \in L$ such that $a \vee c = 1 = b \vee c$ and $a \wedge c = 0 = b \wedge c$. Let n be a positive integer. We say L has *order* n if there exist pairwise perspective elements $a_1, a_2, \dots, a_n \in L$ with $a_1 \vee a_2 \vee \cdots \vee a_n = 1$ and $\left(\bigvee_{i \in I_1} a_i\right) \wedge \left(\bigvee_{j \in I_2} a_j\right) = 0$ for any disjoint $I_1, I_2 \subset \{1, 2, \dots, n\}$.

Definition 8.4. A (*von Neumann*) *regular ring* is a ring R with unit such that for each $x \in R$ there exists $y \in R$ such that $xyx = x$.

Let R be a regular ring. A right ideal $\mathfrak{a}$ of R is *principal* if it is generated by one element of R.

Let M be a von Neumann algebra. Then $\mathcal{P}(M)$ is a complemented lattice. It is not difficult to show that the following three conditions are equivalent.

- The von Neumann algebra M is finite.
- The lattice $\mathcal{P}(M)$ is modular.
- The ring $LS(M)$ is regular.

Theorem 8.5 (von Neumann)**.** *If R is a regular ring, then the collection L of all principal right ideals of R, ordered by inclusion, forms a complemented modular lattice.*

We call L in the statement of the preceding theorem the *right ideal lattice of* R.

Theorem 8.6 (von Neumann)**.** *Let R_1, R_2 be regular rings with right ideal lattices L_1, L_2, respectively. Suppose that L_1 has order $n \geq 3$. If $\Phi\colon L_1 \to L_2$ is a lattice* *isomorphism, then there exists a unique ring isomorphism* $\Psi\colon R_1 \to R_2$ *such that* $\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$, $\mathfrak{a} \in L_1$.

Theorem 8.7 (von Neumann)**.** *Let* L *be a complemented modular lattice with order* $n \geq 4$. *Then there exists a regular ring* R *such that the right ideal lattice of* R *is lattice isomorphic to* L.

Let M be a finite von Neumann algebra. Let $\mathfrak{a} \subset LS(M)$ be a principal right ideal generated by $a \in LS(M)$. It is an easy exercise to show that $\mathfrak{a} = \{x \in LS(M) \mid l(x) \leq l(a)\}$. Hence we obtain an identification of the right ideal lattice of $LS(M)$ with the projection lattice $\mathcal{P}(M)$. In particular, Theorem 1.3 is a corollary of von Neumann's results above. See also the article [14] by Goodearl, which deals with the history of the study of regular rings in connection with functional analysis.

Von Neumann's theory, applied to the setting of von Neumann algebras, is valid only for finite von Neumann algebras. In this thesis, we proved that there exists a complete correspondence between lattice isomorphisms and ring isomorphisms in the general setting of von Neumann algebras. Hence the author believes that one might be able to generalize von Neumann's theory to a broader class of lattices that covers projection lattices of any von Neumann algebras (of fixed order $n \geq 3$ or 4). This is left as a research program in the future.

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