Supplemental Material for "*ANISOtime*: Traveltime computation software for laterally homogeneous, transversely isotropic, spherical media"

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S1 Theory for laterally homogeneous, transversely isotropic, media

We present a self-contained derivation of the traveltimes for laterally homogeneous, transversely isotropic (TI) media with a vertical symmetry axis (sometimes referred to as VTI). As noted in the main body of the paper, results for a flat-layered, laterally homogeneous, TI, medium were originally derived by Vlaar (1968), and results for a spherically symmetric TI medium were derived by Vlaar (1969) and restated concisely by Woodhouse (1981). (References cited in this supplemental material are listed in the bibliography in the main body of the paper.)

We begin by stating the constitutive relation for a TI medium.

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} A & H & F & & & \\ H & A & F & & & \\ F & F & C & & & \\ & & & L & & \\ & & & & L & \\ & & & & & N \end{pmatrix} \begin{pmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{xz} \\ 2e_{yz} \\ 2e_{xy} \end{pmatrix}, \tag{S1}$$

where

$$H = A - 2N. \tag{S2}$$

Thus there are five independent elastic constants, A, C, F, L, and N, in a TI medium. For an isotropic medium the relations between the above five constants and λ and μ are as follows:

$$\lambda + 2\mu = A = C \tag{S3}$$

$$\mu = L = N \tag{S4}$$

$$\lambda = F = H. \tag{S5}$$

We first derive the theory for a flat-layered, transversely isotropic (TI), medium for SH- and pseudo-P-SV-waves, and then extend the results to the spherical case.

S1.1 SH waves in a flat-layered, TI, medium

We consider an SH plane wave in a flat medium (the extension to spherical media is straightforward, and is given below in subsection S1.3), for which the displacement in cartesian coordinates $\mathbf{u} = (u_x,\ u_y,\ u_z)^T$ is given by

$$u_x = 0 \tag{S6}$$

$$u_y = D\exp\left(i(\omega t - k_x x - k_z z)\right) \tag{S7}$$

$$u_z = 0, \tag{S8}$$

where D is the amplitude of the displacement, ω is the angular frequency, and $\mathbf{k} = (k_x,\ k_y,\ k_z)^T$ is the wavenumber vector.

The non-zero strains e_{ij} are

$$e_{xy} = \frac{1}{2}u_{y,x} = \frac{1}{2}(-ik_x u_y) \tag{S9}$$

$$e_{yz} = \frac{1}{2}u_{y,z} = \frac{1}{2}(-ik_z u_y), \tag{S10}$$

where the subscript $,x$ denotes the spatial differentiation with respect to the x coordinate.

The explicit values of the non-zero components of the stress tensor for the 2-D SH case are as follows:

$$\sigma_{xy} = -Nik_x u_y \tag{S11}$$

$$\sigma_{yz} = -Lik_z u_y. \tag{S12}$$

The strain energy is

$$\begin{aligned} PE &= \frac{1}{2}e_{ij}^{*}\sigma_{ij} \\ &= \frac{1}{2}(2e_{xy}^{*}\sigma_{xy} + 2e_{yz}^{*}\sigma_{yz}) \\ &= \frac{1}{2}\mathbf{W}^{*}\mathbf{H}\,\mathbf{W}, \end{aligned} \tag{S13}$$

where

$$\mathbf{H} = \left(\begin{array}{c} Nk_x^2 + Lk_z^2 \end{array} \right), \tag{S14}$$

$$\mathbf{W} = \left(\begin{array}{c} D \end{array} \right), \tag{S15}$$

and $\mathbf{W}^{*}$ is the conjugate transpose of $\mathbf{W}$.

The kinetic energy is

$$\begin{aligned} KE &= \frac{1}{2}\omega^2 u_i^{*}\rho u_i \\ &= \frac{1}{2}\omega^2(u_x^{*}\rho u_x + u_z^{*}\rho u_z) \\ &= \frac{1}{2}\omega^2\mathbf{W}^{*}\mathbf{T}\,\mathbf{W}, \end{aligned} \tag{S16}$$

where

$$\mathbf{T} = \left(\begin{array}{c} \rho \end{array} \right). \tag{S17}$$

Because T and H are Hermitian, the Lagrangian, $\mathcal{L}(\mathbf{W})$, is given by

$$\mathcal{L} = \frac{1}{2}\omega^2\mathbf{W}^{*}\mathbf{T}\mathbf{W} - \frac{1}{2}\mathbf{W}^{*}\mathbf{H}\mathbf{W}. \tag{S18}$$

The equations of motion are obtained by setting the gradient of the Lagrangian to zero, which gives

$$\left(\omega^2 \mathbf{T} - \mathbf{H}\right) \mathbf{W} = 0. \tag{S19}$$

The dispersion relation is then obtained by solving the equation of motion (S19), giving:

$$\omega = \sqrt{\frac{N k_x^2 + L k_z^2}{\rho}}. \tag{S20}$$

The group velocity $\mathbf{U} = (U_x,\ U_z)$ is given as follows:

$$U_x = \frac{\partial \omega}{\partial k_x} = \frac{k_x N/\rho}{\sqrt{(N k_x^2 + L k_z^2)/\rho}} \tag{S21}$$

$$U_z = \frac{\partial \omega}{\partial k_z} = \frac{k_z L/\rho}{\sqrt{(N k_x^2 + L k_z^2)/\rho}}. \tag{S22}$$

Traveltime and epicentral distance kernels

By analogy to eqs. (1) and (2) in the main body of the paper, the distance and time are given by

$$X(p) = \int q_X(p,z)dz \tag{S23}$$

$$T(p) = \int q_T(p,z)dz, \tag{S24}$$

where z is the depth, and X is the distance from the epicenter to the receiver along the Earth's surface.

Given the group velocity $\mathbf{U} = (U_x, U_z)$, the kernels for the traveltime (q_T) and epicentral distance (q_X) for the flat-layered case (eqs. S23 and S24) are as follows:

$$q_T(p) = U_z^{-1} \tag{S25}$$

$$q_X(p) = \frac{U_x}{U_z}. \tag{S26}$$

To compute $q_T(p)$, we first write U_z as a function of $\left(\frac{k_z}{k_x}\right)^2$:

$$U_z = \frac{L}{\rho}\frac{k_z}{k_x}\left(\frac{N}{\rho} + \frac{L}{\rho}\frac{k_z^2}{k_x^2}\right)^{-1/2}. \tag{S27}$$

The ray parameter, p, is given by

$$p = \frac{\omega}{k_x}. \tag{S28}$$

Note that we are dealing with a flat-layered model through the end of subsection S1.2, and that p thus has units of s/km, but that for the spherical case (everywhere else in this paper) p has units of s/radian (see subsection S1.3 for details).

We then multiply both sides of the eq. (S28) by k_z and divide both sides by k_x to obtain

$$\frac{k_z}{\omega} = p\frac{k_z}{k_x}. \tag{S29}$$

We now use the above two equations and the dispersion relation (20) to write

$$k_x^2 = p^2\omega^2 = p^2\frac{Nk_x^2 + Lk_z^2}{\rho}, \tag{S30}$$

which can be re-written to obtain:

$$\frac{k_z^2}{k_x^2} = \frac{1}{p^2}\left(\frac{\rho}{L} - \frac{Np^2}{L}\right) \equiv \frac{1}{p^2}q_\tau^2, \tag{S31}$$

where q_τ is defined as in Woodhouse (1981).

Using eqs. (S25), (S27) and (S31), we find:

$$q_T(p) = U_z^{-1} = \frac{\rho}{L}\left(\frac{\rho}{L} - \frac{N}{L}p^2\right)^{-\frac{1}{2}}. \tag{S32}$$

A similar derivation leads to

$$q_X(p) = p\frac{N}{\rho}\left(\frac{\rho}{L} - \frac{N}{L}p^2\right)^{\frac{1}{2}}. \tag{S33}$$

S1.2 Pseudo-P-SV waves in a flat-layered TI medium

We consider a pseudo-P-SV plane wave, for which the displacement in cartesian coordinates $\mathbf{u} = (u_x,\ u_y,\ u_z)^T$ is

$$u_x = D_x \exp(i(\omega t - k_x x - k_y y)) \tag{S34}$$

$$u_y = 0 \tag{S35}$$

$$u_z = D_z \exp(i(\omega t - k_x x - k_y y)), \tag{S36}$$

where $\mathbf{D} = (D_x,\ D_y,\ D_z)^T$ gives the amplitude of the displacement, and other quantities are the same as for the SH case.

The non-zero strains e_{ij} are

$$e_{xx} = u_{x,x} = -ik_x u_x \tag{S37}$$

$$e_{zz} = u_{z,z} = -ik_z u_z \tag{S38}$$

$$e_{xz} = \frac{1}{2}(u_{x,z} + u_{z,x}) = -\frac{i}{2}(k_x u_z + k_z u_x). \tag{S39}$$

The explicit values of the non-zero components of the stress tensor for the 2-D pseudo-P-SV case are as follows:

$$\sigma_{xx} = -i(Ak_x u_x + Fk_z u_z) \tag{S40}$$

$$\sigma_{yy} = -i(Hk_x u_x + Fk_z u_z) \tag{S41}$$

$$\sigma_{zz} = -i(Fu_x k_x + Cu_z k_z) \tag{S42}$$

$$\sigma_{xz} = -iL(k_x u_z + k_z u_x). \tag{S43}$$

The strain energy is

$$\begin{aligned} PE &= \frac{1}{2} e^*_{ij} \sigma ij \\ &= \frac{1}{2}(e^*_{xx}\sigma_{xx} + e^*_{zz}\sigma_{zz} + e^*_{xz}\sigma_{xz}) \\ &= \frac{1}{2}\mathbf{W}^*\mathbf{H}\mathbf{W}, \end{aligned} \tag{S44}$$

where $\mathbf{W} = (A_x,\, A_z)^T$, $\mathbf{W}^*$ is the conjugate transpose of $\mathbf{W}$, and

$$\mathbf{H} = \begin{pmatrix} Ak_x^2 + Lk_z^2 & k_x k_z (F+L) \\ k_x k_z (F+L) & Lk_x^2 + Ck_z^2 \end{pmatrix}. \tag{S45}$$

The kinetic energy is

$$\begin{aligned} KE &= \frac{1}{2}\omega^2 u_i^* \rho u_i \\ &= \frac{1}{2}\omega^2 (u_x^* \rho u_x + u_z^* \rho u_z) \\ &= \frac{1}{2}\omega^2 \mathbf{W}^*\mathbf{T}\mathbf{W}, \end{aligned} \tag{S46}$$

where

$$\mathbf{T} = \begin{pmatrix} \rho & \\ & \rho \end{pmatrix}. \tag{S47}$$

The two eigenvalues $\omega_\pm$ for pseudo-P and pseudo-SV waves are, respectively, as follows:

$$\omega_+^2 = \frac{1}{2\rho}\left(k_x^2(A+L) + k_z^2(C+L) + \sqrt{\left[k_x^2(A-L) - k_z^2(C-L)\right]^2 + 4k_x^2k_z^2(F+L)^2}\right) \tag{S48}$$

$$\omega_-^2 = \frac{1}{2\rho}\left(k_x^2(A+L) + k_z^2(C+L) - \sqrt{\left[k_x^2(A-L) - k_z^2(C-L)\right]^2 + 4k_x^2k_z^2(F+L)^2}\right). \tag{S49}$$

The components of the group velocity $\mathbf{U}^{\pm} = (U_x^{\pm},\ U_z^{\pm})^T$ are then given as follows:

$$U_x^{\pm} = \frac{\partial \omega_{\pm}}{\partial k_x} = \frac{1}{4\rho\omega}\left(2k_x(A+L) \pm \frac{2k_x(A-L)\left[k_x^2(A-L)-k_z^2(C-L)\right]+4k_xk_z^2(F+L)^2}{\sqrt{\left[k_x^2(A-L)-k_z^2(C-L)\right]^2+4k_x^2k_z^2(F+L)^2}}\right) \tag{S50}$$

$$U_z^{\pm} = \frac{\partial \omega_{\pm}}{\partial k_z} = \frac{1}{4\rho\omega}\left(2k_z(C+L) \pm \frac{2k_z(L-C)\left[k_x^2(A-L)-k_z^2(C-L)\right]+4k_x^2k_z(F+L)^2}{\sqrt{\left[k_x^2(A-L)-k_z^2(C-L)\right]^2+4k_x^2k_z^2(F+L)^2}}\right). \tag{S51}$$

Traveltime and epicentral distance kernels

As for the SH case, the kernels for the traveltime and epicentral distance integrals are computed from the group velocity. To compute the traveltime kernel, we express below the group velocity U_z as a function of the ray parameter p and the TI elastic constants. We re-write the z component of the group velocity (eq. 49) as function of k_z/k_x:

$$U_z^{\pm} = \frac{p}{2\rho}\frac{k_z}{k_x}\left((C+L) \pm \frac{(L-C)\left[(A-L)-\frac{k_z^2}{k_x^2}(C-L)\right]+2(F+L)^2}{\sqrt{\left[(A-L)-\frac{k_z^2}{k_x^2}(C-L)\right]^2+4\frac{k_z^2}{k_x^2}(F+L)^2}}\right). \tag{S52}$$

Using eqs. (S28) and (S29) and the dispersion relation (eq. S48), we obtain the relation between k_x and k_z:

$$k_x^2 = p^2\omega^2 = \frac{p^2}{2\rho}\left(k_x^2(A+L)+k_z^2(C+L) \pm \sqrt{\left[k_x^2(A-L)-k_z^2(C-L)\right]^2+4k_x^2k_z^2(F+L)^2}\right), \tag{S53}$$

which can be written as a second-order equation in $(k_z/k_x)^2$:

$$\left(\frac{k_z}{k_x}\right)^4 CL + \left(\frac{k_z}{k_x}\right)^2\left(AC-F^2-2FL-\frac{\rho}{p^2}(C+L)\right)+\frac{\rho^2}{p^4}-\frac{\rho}{p^2}(A+L)+AL = 0, \tag{S54}$$

whose solution is given by

$$\frac{k_z^2}{k_x^2} = \frac{1}{2CL}\left[F^2+2FL-AC+\frac{\rho}{p^2}(C+L) \pm \sqrt{(F^2+2FL-AC)^2+\frac{\rho^2}{p^4}((C+L)^2-4CL)-2(AC-F^2-2FL)\frac{\rho}{p^2}(C+L)-4CL\left(-\frac{\rho}{p^2}(A+L)+AL\right)}\right]. \tag{S55}$$

Using the quantities s_1, s_2, s_3, s_4, s_5, and R defined in Woodhouse (1981),

$$s_1 = \rho \frac{C+L}{2CL} \tag{S56}$$

$$s_2 = \rho \frac{C-L}{2CL} \tag{S57}$$

$$-2LCs_3 = F^2 + 2FL - AC \tag{S58}$$

$$s_4 = s_3^2 - \frac{A}{C} \tag{S59}$$

$$s_5 = \frac{1}{2}\rho\left(\frac{A+L}{LC}\right) - s_1 s_3 \tag{S60}$$

$$R = \sqrt{s_4 p^4 + 2s_5 p^2 + s_2^2}, \tag{S61}$$

eq. (S55) can be simplified to

$$\begin{aligned}
\frac{k_z^2}{k_x^2} &= -s_3 + \frac{s_1}{p^2} \pm \frac{1}{2CL}\sqrt{4L^2C^2s_3^2 - 4L^2CA + 4L^2C^2\frac{s_2^2}{p^4} - 4LCs_3 2LC\frac{s_1}{p^2} + 4CL\frac{\rho}{p^2}(A+L)} \\
&= -s_3 + \frac{s_1}{p^2} \pm \sqrt{s_4 + \frac{s_2^2}{p^4} + 2\frac{s_5}{p^2}} \\
&= -s_3 + \frac{s_1}{p^2} \pm \frac{R}{p^2} \\
&\equiv q_\tau(p),
\end{aligned} \tag{S62}$$

where the last equality comes from comparison of eq. (S62) with the integrand q_τ for the intersect traveltime in Woodhouse (1981).

The denominator in the equation for the vertical component of the group velocity U_z (eq. S52) can be expressed using the dispersion relation (eq. S48),

$$\begin{aligned}
\pm\sqrt{\left[k_x^2(A-L) - k_z^2(C-L)\right]^2 + 4k_x^2k_z^2(F+L)^2} &= 2\rho\omega_\pm^2 - k_x^2(A+L) - k_z^2(C+L) \\
&= k_x^2\left[2\rho\frac{\omega_\pm^2}{k_x^2} - A - L - \frac{k_z^2}{k_x^2}(C+L)\right].
\end{aligned} \tag{S63}$$

Thus,

$$\pm\sqrt{\left[(A-L) - \frac{k_z^2}{k_x^2}(C-L)\right]^2 + 4\frac{k_z^2}{k_x^2}(F+L)^2} = \frac{2\rho}{p^2} - A - L - \frac{k_z^2}{k_x^2}(C+L), \tag{S64}$$

where we used eq. (S28).

Using eqs. (S62) and (S64) , we rewrite U_z as follows:

$$U_z = \frac{q_\tau}{2\rho}\frac{(C+L)\frac{2\rho}{p^2}-(A+L)(C+L)-\frac{k_z^2}{k_x^2}(C+L)^2+(L-C)\left[(A-L)-\frac{k_z^2}{k_x^2}(C-L)\right]+2(F+L)^2}{\frac{2\rho}{p^2}-A-L-\frac{k_z^2}{k_x^2}(C+L)}$$
$$= q_\tau \frac{(C+L)\frac{2\rho}{p^2}-4LCs_3-4CL\frac{k_z^2}{k_x^2}}{\frac{4\rho^2}{p^2}-2\rho(A+L)-\frac{k_z^2}{k_x^2}2\rho(C+L)}. \tag{S65}$$

Using eqs. (S56) and (S62), the numerator of eq. (S65) simplifies to

$$(C+L)\frac{2\rho}{p^2}-4LCs_3-4CL\frac{k_z^2}{k_x^2} = \mp\frac{4LC}{p^2}R, \tag{S66}$$

and the denominator is simplified as

$$\frac{4\rho^2}{p^2}-2\rho(A+L)-\frac{k_z^2}{k_x^2}2\rho(C+L) = \frac{4}{p^2}\left(\rho^2-LCs_1^2\pm LCRs_1\right)-4LCs_5$$
$$= \frac{4LC}{p^2}\left(-s_2^2\pm Rs_1\right)-4LCs_5. \tag{S67}$$

Thus, the traveltime kernel is given by

$$q_T(p) = \frac{1}{U_z} = \frac{1}{q_\tau}\left[s_1 \pm \frac{1}{R}\left(s_2^2+s_5p^2\right)\right], \tag{S68}$$

where the +, and - signs correspond to the pseudo-P, and pseudo-SV waves, respectively.

A similar derivation using the horizontal component of the group velocity U_x (eq. 48) leads to

$$q_X(p) = \frac{U_x}{U_z} = \frac{p}{q_\tau}\left[s_3 \pm \frac{1}{R}\left(s_5+s_4p^2\right)\right], \tag{S69}$$

S1.3 Extension to a spherical TI medium

Integrands for a spherical TI medium are obtained from integrands in a flat TI medium using the following transformations

$$p \to \frac{p}{r} \tag{S70}$$
$$q_\Delta = \frac{q_X}{r}. \tag{S71}$$

The traveltime and epicentral distance kernels for the spherical case for the SH, pseudo-SV, and pseudo-P waves are thus given by, respectively,

$$q_{T,SH}(p,r) = \frac{\rho}{L}\left(\frac{\rho}{L} - \frac{N}{L}\frac{p^2}{r^2}\right)^{-\frac{1}{2}} \tag{S72}$$

$$q_{T,pseudo-SV}(p,r) = \left(-s_3 + s_1\frac{r^2}{p^2} - R\frac{r^2}{p^2}\right)^{-1}\left[s_1 - \frac{1}{R}\left(s_2^2 + s_5\frac{p^2}{r^2}\right)\right] \tag{S73}$$

$$q_{T,pseudo-P}(p,r) = \left(-s_3 + s_1\frac{r^2}{p^2} + R\frac{r^2}{p^2}\right)^{-1}\left[s_1 + \frac{1}{R}\left(s_2^2 + s_5\frac{p^2}{r^2}\right)\right], \tag{S74}$$

and

$$q_{\Delta,SH}(p,r) = p\frac{N}{\rho}\left(\frac{\rho}{L} - \frac{N}{L}\frac{p^2}{r^2}\right)^{\frac{1}{2}} \tag{S75}$$

$$q_{\Delta,pseudo-SV}(p,r) = \frac{p}{r^2}\left(-s_3 + s_1\frac{r^2}{p^2} - R\frac{r^2}{p^2}\right)^{-1}\left[s_3 - \frac{1}{R}\left(s_5 + s_4\frac{p^2}{r^2}\right)\right] \tag{S76}$$

$$q_{\Delta,pseudo-P}(p,r) = \frac{p}{r^2}\left(-s_3 + s_1\frac{r^2}{p^2} + R\frac{r^2}{p^2}\right)^{-1}\left[s_3 + \frac{1}{R}\left(s_5 + s_4\frac{p^2}{r^2}\right)\right], \tag{S77}$$

which are identical to the traveltime and epicentral distance kernels in Woodhouse (1981).

S2 Integration near the turning point

In order to perform the integration accurately, we have to take special care in handling the integration near the turning point of a raypath, where $q_\tau = 0$, and the integrands become singular but integrable. In *ANISOtime*, the integration for the range between the turning point, at radius r_{turn}, and radius $r_{\text{jeff}} > r_{\text{turn}}$, is computed using the relations in Jeffreys and Jeffreys (1956, p. 288–290), which are also mentioned in Woodhouse (1981). The radius r_{jeff} is defined as the radius r_i closest to r_{turn} that satisfies the following inequality:

$$q_T(r_i)/q_T(r_{i+1}) < \text{threshold}, \tag{S78}$$

where $r_{i+1} < r_i$ is the node of the integration mesh adjacent to r_i, and the threshold takes the default value of 0.9.

For SH waves, the computation for T and Δ near a turning point can be written as:

$$T = \int q_T dr = \int \frac{1}{q_\tau}\frac{\rho}{L}dr = \int \frac{1}{x^{1/2}}\frac{\rho}{L}\frac{dr}{dx}dx, \tag{S79}$$

and

$$\Delta = \int q_\Delta dr = \int \frac{p}{r^2 q_\tau}\frac{N}{L}dr = \int \frac{\rho - Lx}{pLx^{1/2}}\frac{dr}{dx}dx, \tag{S80}$$

where $x = q_\tau^2$ near the turning point.

For the pseudo-P-SV case,

$$T = \int q_T dr = \frac{1}{x^{1/2}}\left[s_1 \mp \frac{1}{R}\left(s_5\frac{s_1 - x \mp R}{s_3}, + s_2^2\right)\right]\frac{dr}{dx}dx, \tag{S81}$$

and

$$\Delta = \int q_\Delta dr = \frac{s_1 - x \mp R}{p s_3 x^{1/2}} \left[s_3 \pm \frac{1}{R} \left(s_4 \frac{s_1 - x \mp R}{s_3} + s_5 \right) \right] \frac{dr}{dx} dx, \tag{S82}$$

where $x = q_\tau^2$ near the turning point. Note that the equations for the pseudo-P-SV case are for pseudo-P and pseudo-SV when the upper and lower signs are taken, respectively.

We assume that all the integrals for T and Δ can be approximated by $\alpha/\sqrt{x}$ and use the relations in Jeffreys and Jeffreys (1956, p. 288–290).

S3 "PolynomialStructure" file

A "PolynomialStructure" completely specifies the Earth model parameters as piecewise functions using third order polynomials within each layer. A given layer is specified as in Table S1,

Table S1: A single layer in a "PolynomialStructure" file

r_1	r_2	ρ_0	ρ_1	ρ_2	ρ_3		
		$V_{PV,0}$	$V_{PV,1}$	$V_{PV,2}$	$V_{PV,3}$		
		$V_{PH,0}$	$V_{PH,1}$	$V_{PH,2}$	$V_{PH,3}$		
		$V_{SV,0}$	$V_{SV,1}$	$V_{SV,2}$	$V_{SV,3}$		
		$V_{SH,0}$	$V_{SH,1}$	$V_{SH,2}$	$V_{SH,3}$		
		η_0	η_1	η_2	η_3	Q_μ	Q_κ

where the values at normalized radius $x = r/r_{EARTH}$ of the density ρ (in g cm^{-3}), velocities V_{PV}, V_{PH}, V_{SV}, V_{SH} (in km s^{-1}), and η (dimensionless) (see Panning and Romanowicz, 2006, for definitions) are given as

$$\rho(x) = \sum_{i=0}^{3} \rho_i x^i, \tag{S83}$$

$$v(x) = \sum_{i=0}^{3} V_i x^i, \tag{S84}$$

$$\eta(x) = \sum_{i=0}^{3} \eta_i x^i. \tag{S85}$$

Q_μ, and Q_κ are the shear, and bulk attenuation coefficients in each layer, respectively, and are not used in computations by *ANISOtime*. They are included in the input parameter file so that the same input file can be used in both *ANISOtime* and the DSM waveform calculation software.

The first line of the "PolynomialStructure" file has to contain the total number of layers. For the first layer, r_1 should be the radius of an Earth-like planet. For the last layer, r_2 has to be 0. At present, the "PolynomialStructure" file must contain, in this order, a “mantle” (i.e., a solid outer layer that can include a crust, upper mantle, transition zone, etc.), an outer core with $V_{SV} = V_{SH} = 0$, and an inner core.

As an example, the polynomial structure file for the anisotropic PREM (Dziewonski and Anderson, 1981) is given in Table S2.

Table S2: Complete ‘PolynomialStructure’ file for the anisotropic PREM

12							
0.0	1221.5	13.0885	0.0000	-8.8381	0.0000		
		11.2622	0.0000	-6.3640	0.0000		
		11.2622	0.0000	-6.3640	0.0000		
		3.6678	0.0000	-4.4475	0.0000		
		3.6678	0.0000	-4.4475	0.0000		
		1.0000	0.0000	0.0000	0.0000	84.6	1327.7
1221.5	3480.0	12.5815	-1.2638	-3.6426	-5.5281		
		11.0487	-4.0362	4.8023	-13.5732		
		11.0487	-4.0362	4.8023	-13.5732		
		0.0000	0.0000	0.0000	0.0000		
		0.0000	0.0000	0.0000	0.0000		
		1.0000	0.0000	0.0000	0.0000	-1.0	57823.0
3480.0	3630.0	7.9565	-6.4761	5.5283	-3.0807		
		15.3891	-5.3181	5.5242	-2.5514		
		15.3891	-5.3181	5.5242	-2.5514		
		6.9254	1.4672	-2.0834	0.9783		
		6.9254	1.4672	-2.0834	0.9783		
		1.0000	0.0000	0.0000	0.0000	312.0	57823.0
3630.0	5600.0	7.9565	-6.4761	5.5283	-3.0807		
		24.9520	-40.4673	51.4832	-26.6419		
		24.9520	-40.4673	51.4832	-26.6419		
		11.1671	-13.7818	17.4575	-9.2777		
		11.1671	-13.7818	17.4575	-9.2777		
		1.0000	0.0000	0.0000	0.0000	312.0	57823.0
5600.0	5701.0	7.9565	-6.4761	5.5283	-3.0807		
		29.2766	-23.6027	5.5242	-2.5514		
		29.2766	-23.6027	5.5242	-2.5514		
		22.3459	-17.2473	-2.0834	0.9783		
		22.3459	-17.2473	-2.0834	0.9783		
		1.0000	0.0000	0.0000	0.0000	312.0	57823.0

Table S2: (continued)

5701.0	5771.0	5.3197	-1.4836	0.0000	0.0000		
		19.0957	-9.8672	0.0000	0.0000		
		19.0957	-9.8672	0.0000	0.0000		
		9.9839	-4.9324	0.0000	0.0000		
		9.9839	-4.9324	0.0000	0.0000		
		1.0000	0.0000	0.0000	0.0000	143.0	57823.0
5771.0	5971.0	11.2494	-8.0298	0.0000	0.0000		
		39.7027	-32.6166	0.0000	0.0000		
		39.7027	-32.6166	0.0000	0.0000		
		22.3512	-18.5856	0.0000	0.0000		
		22.3512	-18.5856	0.0000	0.0000		
		1.0000	0.0000	0.0000	0.0000	143.0	57823.0
5971.0	6151.0	7.1089	-3.8045	0.0000	0.0000		
		20.3926	-12.2569	0.0000	0.0000		
		20.3926	-12.2569	0.0000	0.0000		
		8.9496	-4.4597	0.0000	0.0000		
		8.9496	-4.4597	0.0000	0.0000		
		1.0000	0.0000	0.0000	0.0000	143.0	57823.0
6151.0	6291.0	2.6910	0.6924	0.0000	0.0000		
		0.8317	7.2180	0.0000	0.0000		
		3.5908	4.6172	0.0000	0.0000		
		5.8582	-1.4678	0.0000	0.0000		
		-1.0839	5.7176	0.0000	0.0000		
		3.3687	-2.4778	0.0000	0.0000	80.0	57823.0
6291.0	6346.6	2.6910	0.6924	0.0000	0.0000		
		0.8317	7.2180	0.0000	0.0000		
		3.5908	4.6172	0.0000	0.0000		
		5.8582	-1.4678	0.0000	0.0000		
		-1.0839	5.7176	0.0000	0.0000		
		3.3687	-2.4778	0.0000	0.0000	600.0	57823.0
6346.6	6356.0	2.9000	0.0000	0.0000	0.0000		
		6.8000	0.0000	0.0000	0.0000		
		6.8000	0.0000	0.0000	0.0000		
		3.9000	0.0000	0.0000	0.0000		
		3.9000	0.0000	0.0000	0.0000		
		1.0000	0.0000	0.0000	0.0000	600.0	57823.0
6356.0	6371.0	2.6000	0.0000	0.0000	0.0000		
		5.8000	0.0000	0.0000	0.0000		
		5.8000	0.0000	0.0000	0.0000		
		3.2000	0.0000	0.0000	0.0000		
		3.2000	0.0000	0.0000	0.0000		
		1.0000	0.0000	0.0000	0.0000	600.0	57823.0

S4 “Named Discontinuity” file

A “Named Discontinuity” structure file completely specifies the Earth model parameters as piecewise functions within each layer, where the functions are given by the Bullen law $v(r) = Ar^B$. The “Named Discontinuity” file format is similar to that used in *TauP* (Crotwell *et al.*, 1999). A given layer is specified as in Table S3,

Table S3: A single layer in a ‘Named Discontinuity’ file

depth_1	ρ_1	$V_{PV,1}$	$V_{PH,1}$	$V_{SV,1}$	$V_{SH,1}$	η_1	$Q_{\kappa,1}$	$Q_{\mu,1}$
depth_2	ρ_2	$V_{PV,2}$	$V_{PH,2}$	$V_{SV,2}$	$V_{SH,2}$	η_2	$Q_{\kappa,2}$	$Q_{\mu,2}$

where the density ρ is in g cm^{-3}, the velocities V_{PV}, V_{PH}, V_{SV}, V_{SH} are in km s^{-1}, and η, Q_κ, and Q_μ are dimensionless.

For the first layer, depth_1 has to be 0. For the last layer, depth_2 should be equal to the radius of an Earth-like planet. At present, the “Named Discontinuity” structure file must contain, in this order a “mantle,” an outer core with $V_{SV} = V_{SH} = 0$, and an inner core, as specified in Table S4. The “outer-core” and “inner-core” keywords in Table S4 are mandatory, but the “mantle” keyword is optional. If the “mantle” keyword is used, the layer above it will be assumed as the crust. All layers can contain additional named discontinuities, although more discontinuities result in higher memory requirements.

Table S4: Layers in a “Named Discontinuity” file.

0	structure parameters
...	...
$r_{crust}(optional)$	structure parameters$_+$
mantle (optional)	
$r_{crust}(optional)$	structure parameters$_-$
...	...
r_{CMB}	structure parameters$_+$
outer-core	
r_{CMB}	structure parameters$_-$
...	...
r_{ICB}	structure parameters$_+$
inner-core	
r_{ICB}	structure parameters$_-$
...	...
r_{planet}	structure parameters

Note that structure parameters$_+$, and structure parameters$_-$ specify the density, velocities, η, Q_κ, and Q_μ above, and below the discontinuity, respectively, that lines specifying “outer-core,” and “inner-core” must be included, and that the radii above and below these lines must be equal. Lines specifying “mantle” and the radius of the crust are optional. As an example, a “Named Discontinuity” structure file for the isotropic PREM (Dziewonski and Anderson, 1981) is given in Table S5.

Table S5: Complete “Named Discontinuity” file for the isotropic PREM

0.00	2.60000	5.80000	5.80000	3.20000	3.20000	1.0	57823.0	600.0
15.00	2.60000	5.80000	5.80000	3.20000	3.20000	1.0	57823.0	600.0
15.00	2.90000	6.80000	6.80000	3.90000	3.90000	1.0	57823.0	600.0
24.40	2.90000	6.80000	6.80000	3.90000	3.90000	1.0	57823.0	600.0
mantle								
24.40	3.38076	8.11061	8.11061	4.49094	4.49094	1.0	57823.0	600.0
40.00	3.37906	8.10119	8.10119	4.48486	4.48486	1.0	57823.0	600.0
60.00	3.37688	8.08907	8.08907	4.47715	4.47715	1.0	57823.0	600.0
80.00	3.37471	8.07688	8.07688	4.46953	4.46953	1.0	57823.0	80.0
115.00	3.37091	8.05540	8.05540	4.45643	4.45643	1.0	57823.0	80.0
150.00	3.36710	8.03370	8.03370	4.44361	4.44361	1.0	57823.0	80.0
185.00	3.36330	8.01180	8.01180	4.43108	4.43108	1.0	57823.0	80.0
220.00	3.35950	7.98970	7.98970	4.41885	4.41885	1.0	57823.0	80.0
220.00	3.43578	8.55896	8.55896	4.64391	4.64391	1.0	57823.0	143.0
265.00	3.46264	8.64552	8.64552	4.67540	4.67540	1.0	57823.0	143.0
310.00	3.48951	8.73209	8.73209	4.70690	4.70690	1.0	57823.0	143.0
355.00	3.51639	8.81867	8.81867	4.73840	4.73840	1.0	57823.0	143.0
400.00	3.54325	8.90522	8.90522	4.76989	4.76989	1.0	57823.0	143.0
400.00	3.72378	9.13397	9.13397	4.93259	4.93259	1.0	57823.0	143.0
450.00	3.78678	9.38990	9.38990	5.07842	5.07842	1.0	57823.0	143.0
500.00	3.84980	9.64588	9.64588	5.22428	5.22428	1.0	57823.0	143.0
550.00	3.91282	9.90185	9.90185	5.37014	5.37014	1.0	57823.0	143.0
600.00	3.97584	10.15782	10.15782	5.51602	5.51602	1.0	57823.0	143.0
635.00	3.98399	10.21203	10.21203	5.54311	5.54311	1.0	57823.0	143.0
670.00	3.99214	10.26622	10.26622	5.57020	5.57020	1.0	57823.0	143.0
670.00	4.38071	10.75131	10.75131	5.94508	5.94508	1.0	57823.0	312.0
721.00	4.41241	10.91005	10.91005	6.09418	6.09418	1.0	57823.0	312.0
771.00	4.44317	11.06557	11.06557	6.24046	6.24046	1.0	57823.0	312.0
871.00	4.50372	11.24490	11.24490	6.31091	6.31091	1.0	57823.0	312.0
971.00	4.56307	11.41560	11.41560	6.37813	6.37813	1.0	57823.0	312.0
1071.00	4.62129	11.57828	11.57828	6.44232	6.44232	1.0	57823.0	312.0
1171.00	4.67844	11.73357	11.73357	6.50370	6.50370	1.0	57823.0	312.0
1271.00	4.73460	11.88209	11.88209	6.56250	6.56250	1.0	57823.0	312.0
1371.00	4.78983	12.02445	12.02445	6.61891	6.61891	1.0	57823.0	312.0
1471.00	4.84422	12.16126	12.16126	6.67317	6.67317	1.0	57823.0	312.0
1571.00	4.89783	12.29316	12.29316	6.72548	6.72548	1.0	57823.0	312.0
1671.00	4.95073	12.42075	12.42075	6.77606	6.77606	1.0	57823.0	312.0
1771.00	5.00299	12.54466	12.54466	6.82512	6.82512	1.0	57823.0	312.0
1871.00	5.05469	12.66550	12.66550	6.87289	6.87289	1.0	57823.0	312.0
1971.00	5.10590	12.78389	12.78389	6.91957	6.91957	1.0	57823.0	312.0
2071.00	5.15669	12.90045	12.90045	6.96538	6.96538	1.0	57823.0	312.0
2171.00	5.20713	13.01579	13.01579	7.01053	7.01053	1.0	57823.0	312.0
2271.00	5.25729	13.13055	13.13055	7.05525	7.05525	1.0	57823.0	312.0
2371.00	5.30724	13.24532	13.24532	7.09974	7.09974	1.0	57823.0	312.0
2471.00	5.35706	13.36074	13.36074	7.14423	7.14423	1.0	57823.0	312.0

Table S5: (continued)

2571.00	5.40681	13.47742	13.47742	7.18892	7.18892	1.0	57823.0	312.0
2671.00	5.45657	13.59597	13.59597	7.23403	7.23403	1.0	57823.0	312.0
2741.00	5.49145	13.68041	13.68041	7.26597	7.26597	1.0	57823.0	312.0
2771.00	5.50642	13.68753	13.68753	7.26575	7.26575	1.0	57823.0	312.0
2871.00	5.55641	13.71168	13.71168	7.26486	7.26486	1.0	57823.0	312.0
2891.00	5.56645	13.71660	13.71660	7.26466	7.26466	1.0	57823.0	312.0
outer-core								
2891.00	9.90349	8.06482	8.06482	0.00000	0.00000	1.0	57823.0	0.0
2971.00	10.02940	8.19939	8.19939	0.00000	0.00000	1.0	57823.0	0.0
3071.00	10.18134	8.36019	8.36019	0.00000	0.00000	1.0	57823.0	0.0
3171.00	10.32726	8.51298	8.51298	0.00000	0.00000	1.0	57823.0	0.0
3271.00	10.46727	8.65805	8.65805	0.00000	0.00000	1.0	57823.0	0.0
3371.00	10.60152	8.79573	8.79573	0.00000	0.00000	1.0	57823.0	0.0
3471.00	10.73012	8.92632	8.92632	0.00000	0.00000	1.0	57823.0	0.0
3571.00	10.85321	9.05015	9.05015	0.00000	0.00000	1.0	57823.0	0.0
3671.00	10.97091	9.16752	9.16752	0.00000	0.00000	1.0	57823.0	0.0
3771.00	11.08335	9.27867	9.27867	0.00000	0.00000	1.0	57823.0	0.0
3871.00	11.19067	9.38418	9.38418	0.00000	0.00000	1.0	57823.0	0.0
3971.00	11.29298	9.48409	9.48409	0.00000	0.00000	1.0	57823.0	0.0
4071.00	11.39042	9.57881	9.57881	0.00000	0.00000	1.0	57823.0	0.0
4171.00	11.48311	9.66865	9.66865	0.00000	0.00000	1.0	57823.0	0.0
4271.00	11.57119	9.75393	9.75393	0.00000	0.00000	1.0	57823.0	0.0
4371.00	11.65478	9.83496	9.83496	0.00000	0.00000	1.0	57823.0	0.0
4471.00	11.73401	9.91206	9.91206	0.00000	0.00000	1.0	57823.0	0.0
4571.00	11.80900	9.98554	9.98554	0.00000	0.00000	1.0	57823.0	0.0
4671.00	11.87990	10.05572	10.05572	0.00000	0.00000	1.0	57823.0	0.0
4771.00	11.94682	10.12291	10.12291	0.00000	0.00000	1.0	57823.0	0.0
4871.00	12.00989	10.18743	10.18743	0.00000	0.00000	1.0	57823.0	0.0
4971.00	12.06924	10.24959	10.24959	0.00000	0.00000	1.0	57823.0	0.0
5071.00	12.12500	10.30971	10.30971	0.00000	0.00000	1.0	57823.0	0.0
5149.50	12.16634	10.35568	10.35568	0.00000	0.00000	1.0	57823.0	0.0
inner-core								
5149.50	12.76360	11.02827	11.02827	3.50432	3.50432	1.0	57823.0	85.0
5171.00	12.77493	11.03643	11.03643	3.51002	3.51002	1.0	57823.0	85.0
5271.00	12.82501	11.07249	11.07249	3.53522	3.53522	1.0	57823.0	85.0
5371.00	12.87073	11.10542	11.10542	3.55823	3.55823	1.0	57823.0	85.0
5471.00	12.91211	11.13521	11.13521	3.57905	3.57905	1.0	57823.0	85.0
5571.00	12.94912	11.16186	11.16186	3.59767	3.59767	1.0	57823.0	85.0
5671.00	12.98178	11.18538	11.18538	3.61411	3.61411	1.0	57823.0	85.0
5771.00	13.01009	11.20576	11.20576	3.62835	3.62835	1.0	57823.0	85.0
5871.00	13.03404	11.22301	11.22301	3.64041	3.64041	1.0	57823.0	85.0
5971.00	13.05364	11.23712	11.23712	3.65027	3.65027	1.0	57823.0	85.0
6071.00	13.06888	11.24809	11.24809	3.65794	3.65794	1.0	57823.0	85.0
6171.00	13.07977	11.25593	11.25593	3.66342	3.66342	1.0	57823.0	85.0
6271.00	13.08630	11.26064	11.26064	3.66670	3.66670	1.0	57823.0	85.0
6371.00	13.08848	11.26220	11.26220	3.66780	3.66780	1.0	57823.0	85.0

S5 Analytical solution for a spherically symmetric, TI, medium with constant velocity gradient

A simple analytical solution can be found for the medium defined below:

$$A(r) = A_0 r^2$$
$$C(r) = C_0 r^2$$
$$L(r) = L_0 r^2$$
$$N(r) = N_0 r^2$$
$$F(r) = F_0 r^2$$
$$\rho = \text{constant.}$$

For the SH case, using the notation of Woodhouse (1981), we have

$$q_\tau = \sqrt{\frac{\rho}{r^2 L_0} - \frac{N_0}{L_0}\frac{p^2}{r^2}} \equiv \frac{q_{\tau 0}}{r}, \tag{S86}$$

$$q_T = \frac{r}{q_{\tau 0}}\frac{\rho}{r^2 L_0} = \frac{1}{r}\frac{\rho}{q_{\tau 0} L_0} \equiv \frac{q_{T0}}{r}, \tag{S87}$$

and the traveltime between radius r_1 and r_2 for the SH case is given as follows:

$$T(p) = \int_{r_1}^{r_2} q_T(r,p)dr = q_{T0}\ln\frac{r_2}{r_1}. \tag{S88}$$

Similarly, for the pseudo-P-SV case, the quantities s_1, s_2, s_3, s_4, s_5, and R are given as follows:

$$s_1 = \frac{\rho}{2r^2}\left(\frac{1}{L_0} + \frac{1}{C_0}\right) \equiv \frac{s_{10}}{r^2}$$
$$s_2 = \frac{\rho}{2r^2}\left(\frac{1}{L_0} - \frac{1}{C_0}\right) \equiv \frac{s_{20}}{r^2}$$
$$s_3 = \frac{1}{2L_0C_0}\left(A_0C_0 - F_0^2 - 2L_0F_0\right) \equiv s_{30}$$
$$s_4 = s_{30}^2 - \frac{A_0}{C_0} \equiv s_{40}$$
$$s_5 = \frac{1}{2}\frac{\rho}{C_0 r^2}\left(1 + \frac{A_0}{L_0}\right) - \frac{s_{10}}{r^2}s_{30} \equiv \frac{s_{50}}{r^2}$$
$$R = \sqrt{s_{40}\frac{p^4}{r^4} + 2s_{50}\frac{p^2}{r^4} + \frac{s_{20}^2}{r^4}} \equiv \frac{R_0}{r^2}$$

Using the relations defined above, the integrands for the one-way intercept can be written as

$$q_\tau = \sqrt{\frac{s_{10}}{r^2} - s_{30}\frac{p^2}{r^2} \mp \frac{R_0}{r^2}} \equiv \frac{q_{\tau 0}}{r}, \tag{S89}$$

131 and for the traveltime eq. (S90)

$$\begin{aligned} q_T &= \frac{r}{q_{\tau 0}}\left[\frac{s_{10}}{r^2} \mp \frac{r^2}{R_0}\left(s_{50}\frac{p^2}{r^4} + \frac{s_{20}^2}{r^4}\right)\right] \\ &= \frac{1}{r}\frac{1}{q_{\tau 0}}\left[s_{10} \mp \frac{1}{R_0}\left(s_{50}p^2 + s_{20}^2\right)\right] \\ &\equiv \frac{q_{T0}}{r}. \end{aligned} \tag{S90}$$

132 The traveltime between radius r_1 and r_2 for the quasi-P-SV case is thus given as follows:

$$T(p) = \int_{r_1}^{r_2} q_T(r,p)dr = q_{T0}\ln\frac{r_2}{r_1}. \tag{S91}$$