

Cellular Chain Complexes of Universal Covers of Some 3-Manifolds

By Takefumi NOSAKA

Abstract. For a closed 3-manifold M in a certain class, we give a presentation of the cellular chain complex of the universal cover of M . The class includes all surface bundles, some surgeries of knots in S^3 , some cyclic branched cover of S^3 , and some Seifert manifolds. In application, we establish a formula for calculating the linking form of a cyclic branched cover of S^3 , and develop procedures of computing some Dijkgraaf-Witten invariants.

1. Introduction

In order to investigate a connected CW-complex X with a non-trivial fundamental group $\pi_1(X)$, it is important to give a concrete presentation of the cellular chain complex, $C_*(\tilde{X}; \mathbb{Z})$, and the cup-products of the universal cover \tilde{X} . In fact, the homology of X with local coefficients and the (twisted) Reidemeister torsion of X are defined from $C_*(\tilde{X}; \mathbb{Z})$. If X is a $K(\pi, 1)$ -space, the chain complex means a projective resolution of the group ring $\mathbb{Z}[\pi_1(X)]$. Thus, it is also of use for computing many invariants to concretely present $C_*(\tilde{X}; \mathbb{Z})$.

This paper focuses on a class of closed 3-manifolds satisfying the following condition:

ASSUMPTION (\dagger). *A closed oriented 3-manifold M satisfies that any closed 3-manifold M' with a group isomorphism $\pi_1(M) \cong \pi_1(M')$ admits a homotopy equivalence $M \simeq M'$.*

For example, M satisfies this assumption if M is an Eilenberg-MacLane space of type $(\pi_1(M), 1)$, which is equivalent to that M is irreducible and has an infinite fundamental group. In Section 3, we examine many 3-manifolds,

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including all surface bundles, some surgeries of knots in S^3 , spliced sums, cyclic branched covers of S^3 with Assumption (†), and some Seifert manifolds. For when M is one of these, we describe presentations of the complex $C_*(\widetilde{M}; \mathbb{Z})$ and of the cup-product $H^1(M; N) \otimes H^2(M; N') \rightarrow H^3(M; N \otimes N')$ for any local coefficient modules N, N' . The procedure for obtaining such descriptions essentially follows from the work of [Sie, Tro] in terms of “identity”, which we review in Section 2. This procedure can also be used to describe the fundamental homology 3-class, $[M]$ of M ; see Remark 2.4.

In application, we give a formula for the linking forms of cyclic branched covers of S^3 with Assumption (†) (see Propositions 4.1). Furthermore, we develop procedures of computing some Dijkgraaf-Witten invariants from the above descriptions; see §5. In addition, such descriptions of identities are used for computing knot concordance groups, Reidemeister torsions, and Casson invariants; see [MP, No1, Waki]. There might be other applications from the above presentations of the complexes $C_*(\widetilde{M}; \mathbb{Z})$.

Conventional notation. In this paper, every manifold is understood to be smooth, connected, and orientable. By M , we mean a closed 3-manifold with orientation $[M]$.

2. Taut Identities and Cup-Products

2.1. Review: identities and cup-products

Let us recall the procedure of obtaining cellular chain complexes of some universal covers, as described in the papers [Sie] and [Tro]. There is nothing new in this section.

We will start by reviewing identities. Take a finitely presented group $\langle x_1, \dots, x_m \mid r_1, \dots, r_m \rangle$ of deficiency zero. Setting up the free groups $F := \langle x_1, \dots, x_m \mid \rangle$ and $P := \langle \rho_1, \dots, \rho_m \mid \rangle$, let us consider the homomorphism,

$$\psi : P * F \longrightarrow F \quad \text{defined by} \quad \psi(\rho_j) = r_j, \quad \psi(x_i) = x_i.$$

An element $s \in P * F$ is an *identity* if $s \in \text{Ker}(\psi)$ and s can be written as $\prod_{k=1}^n \omega_k \rho_{j_k}^{\epsilon_k} \omega_k^{-1}$ for some $w_k \in F$, $\epsilon_k \in \{\pm 1\}$ and indices j_k 's.

Given a closed 3-manifold M with a genus- m Heegaard splitting, let us review the cellular complex of the universal cover, \widetilde{M} , of M . A CW-complex structure of M induced by the splitting consists of a single zero-cell, m one-handles, m two-handles, and a single three-handle. Therefore, $\pi_1(M)$ has a

group presentation $\langle x_1, \dots, x_m \mid r_1, \dots, r_m \rangle$, and the cellular complex of \widetilde{M} is described as

$$(1) \quad C_*(\widetilde{M}; \mathbb{Z}) : 0 \rightarrow \mathbb{Z}[\pi_1(M)] \xrightarrow{\partial_3} \mathbb{Z}[\pi_1(M)]^m \xrightarrow{\partial_2} \mathbb{Z}[\pi_1(M)]^m \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(M)] \rightarrow 0.$$

Here, $\mathbb{Z}[\pi_1(M)]$ is the group ring of $\pi_1(M)$. We will explain the boundary maps ∂_* in detail. Let $\{a_1, \dots, a_m\}$, $\{b_1, \dots, b_m\}$, and $\{c\}$ denote the canonical bases of $C_1(\widetilde{M}; \mathbb{Z})$, $C_2(\widetilde{M}; \mathbb{Z})$, and $C_3(\widetilde{M}; \mathbb{Z})$ as left $\mathbb{Z}[\pi_1(M)]$ -modules, respectively. Then, as is shown in [Lyn], $\partial_1(a_i) = 1 - x_i$, and $\partial_2(b_i) = \sum_{k=1}^m [\frac{\partial r_i}{\partial x_k}] a_k$, where $\frac{\partial r_i}{\partial x_k}$ is the Fox derivative. Moreover, the main result in [Sie] is that there exists an identity s such that $\partial_3(c) = \sum_k [\psi(\frac{\partial s}{\partial \rho_k})] b_k$.

Next, we will briefly give a formula for the cup-product in terms of the identity, which is a result of [Tro, §2.4]. Let N and N' be left $\mathbb{Z}[\pi_1(M)]$ -modules. We can define the cochain complex on $C^*(M; N) := \text{Hom}_{\mathbb{Z}[\pi_1(M)]}(C_*(\widetilde{M}; \mathbb{Z}), N)$ with local coefficients. Recalling the definition of the identity $s = \prod_{k=1}^n \omega_k \rho_{j_k}^{\epsilon_k} \omega_k^{-1}$, define

$$(2) \quad D^\sharp(c) = \sum_{k=1}^n \epsilon_k \left(\sum_{\ell=1}^m [\frac{\partial \omega_k}{\partial x_\ell}] a_\ell \otimes \omega_k b_{j_k} \right) \in C_1(\widetilde{M}; \mathbb{Z}) \otimes C_2(\widetilde{M}; \mathbb{Z}).$$

Then, for cochains $p \in C^1(M; N)$ and $q \in C^2(M; N')$, we define a 3-cochain $p \smile q$ by

$$p \smile q(uc) := (p \otimes q)(uD^\sharp(c)) \in N \otimes_{\mathbb{Z}} N'.$$

Here, $u \in \mathbb{Z}[\pi_1(M)]$. Then, the map

$$\smile : C^1(M; N) \otimes C^2(M; N') \rightarrow C^3(M; N \otimes_{\mathbb{Z}} N'); \quad (p, q) \mapsto p \smile q,$$

induces the bilinear map on cohomology, which is known to be equal to the usual cup-product. Here, notice that, since the third $\partial_3 \otimes_{\mathbb{Z}[\pi_1(M)]} \text{id}_{\mathbb{Z}}$ is zero, the 3-class $s \otimes 1 \in C_*(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z}$ is a generator of $H_3(C_*(\widetilde{M}) \otimes \mathbb{Z}) \cong H_3(M; \mathbb{Z}) \cong \mathbb{Z}$, which represents the fundamental 3-class $[M]$; thus, given a $\pi_1(M)$ -invariant bilinear map $\psi : N \otimes N' \rightarrow A$ for some abelian group A , we have the following equality on the pairing of $[M]$:

$$(3) \quad \psi \circ \smile (p, q) = \psi \langle p \smile q, [M] \rangle \in A,$$

for any cochains $p \in C^1(M; N)$ and $q \in C^2(M; N')$.

In summary, for a description of the complex $C_*(\widetilde{M})$ and the cup-product, it is important to describe an identity from M .

2.2. Taut identities

In order to find such identities giving the complex (1), we review tautness from [Sie]; see also [Waki, Appendix] for a brief explanation. Fix a finite presentation $\langle x_1, \dots, x_m \mid r_1, \dots, r_m \rangle$. Let $s = \prod_{k=1}^{2m} w_k \rho_{j_k}^{\epsilon_k} w_k^{-1} \in P * F$ be an identity, where ρ_{j_k} and w_k can be written in

$$\rho_{j_k} = a_{k,1}^{\epsilon_{k,1}} \cdots a_{k,\ell_k}^{\epsilon_{k,\ell_k}}, \quad w_k = b_{k,1}^{\eta_{k,1}} \cdots b_{k,n_k}^{\eta_{k,n_k}}, \quad (\epsilon_{i,j}, \eta_{i,j} \in \{\pm 1\}).$$

Here, $a_{k,\ell}$ and $b_{k,\ell}$ lie in $\{x_1, \dots, x_m\}$. For each $w_k \rho_{j_k}^{\epsilon_k} w_k^{-1}$, take the ℓ_k -gon D_{j_k} whose i -th edge is labeled by $a_{k,i}^{\epsilon_{k,i}}$, and the segment $I_k = [0, n_k]$ such that $[i-1, i]$ is labeled by $b_{k,i}^{\eta_{k,i}}$.

DEFINITION 2.1 ([Sie]).

(1) A self-bijection

$$\mathcal{I} : \cup_{k=1}^{2m} \{(k, 1), \dots, (k, \ell_k)\} \rightarrow \cup_{k=1}^{2m} \{(k, 1), \dots, (k, \ell_k)\}$$

is called a *syllable* if $a_{\mathcal{I}(i,j)} = a_{i,j} \in F$ and $\epsilon_{i,j} = -\epsilon_{\mathcal{I}(i,j)} \in \{\pm 1\}$.

(2) For a syllable \mathcal{I} , consider the following equivalence on the disjoint union $\sqcup_{i=1}^{2m} D_{r_i}$: the interval with labeling $a_{i,j}$ is identified with those with labeling $a_{\mathcal{I}(i,j)}$.

(3) An identity s is said to be *taut* if there is a syllable \mathcal{I} such that the quotient space $\sqcup_{i=1}^{2m} D_{r_i} / \sim$ of $\sqcup_{i=1}^{2m} D_{r_i}$ subject to the above equivalence \sim is homeomorphic to S^2 , and if there are injective continuous maps

$$\kappa_k : I_k = [0, n_k] \rightarrow \sqcup_{i=1}^{2m} \partial D_{r_i} / \sim, \quad \lambda_k : [0, \ell_k) \rightarrow \partial D_{r_k} / \sim$$

satisfying the following condition (*).

(*) For each k , the image $\kappa_k([i-1, i])$ coincides with an edge labeled by $b_{k,i}$ compatible with the orientations, and $\lambda_k([j-1, j])$ coincides with the j -th edge of D_{r_k} compatible with the orientations. Furthermore, $\kappa_k(n_k) = \lambda_k(0) = \lambda_k(\ell_k)$.

This paper is mainly based on the following theorem of Sieradski:

THEOREM 2.2 ([Sie]). *Given a group presentation $\langle x_1, \dots, x_m \mid r_1, \dots, r_m \rangle$ with a taut identity s , there exists a closed 3-manifold M with*

a genus- m Heegaard splitting such that the complex $C_*(\widetilde{M}; \mathbb{Z})$ is isomorphic to the complex (1).

In a concrete situation where an identity s is explicitly described, it is not so hard to find such a \mathcal{I} and show the tautness of s (in fact, this check is to construct a 2-sphere from the disjoint union $\sqcup_{i=1}^n D_{r_i}$ as a naive pasting). In all the statements in §3, we will claim that some identities satisfy the taut condition; however, we will also omit the check by elementary complexity, as in other papers on taut identities [BH, Sie, Tro].

Example 2.3. As an easy example of the pasting, we focus on the 3-dimensional torus $M = (S^1)^3$ with presentation $\pi_1(M) = \langle x, y, z \mid r, s, u \rangle$, where $r = [x, y], s = [y, z], u = [z, x]$. As in [Sie], consider the following identity.

$$W_{(S^1)^3} = r(y^{-1}u^{-1}y)s(z^{-1}rz)u(x^{-1}s^{-1}x).$$

Then, Figure 1 gives a self-bijection and λ_m, κ_m satisfy the tautness. Moreover, if we attach a 3-ball in the right hand side in the figure along the boundary of the 3-cube, the resulting space is equal to $(S^1)^3$.

REMARK 2.4. Suppose that we find a taut identity s from $\langle x_1, \dots, x_m \mid r_1, \dots, r_m \rangle$, and the resulting 3-manifold M satisfies Assumption (†). Then, by Assumption (†), the resulting 3-manifold up to homotopy does not depend on the choice of s . In particular, we emphasize that, if M satisfies Assumption (†) and we find a taut identity from $\pi_1(M) = \langle x_1, \dots, x_m \mid$

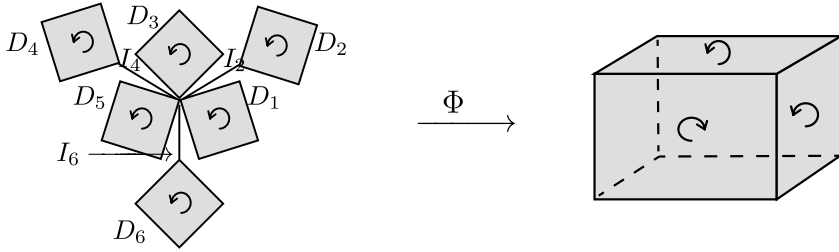


Fig. 1. The tautness of $(S^1)^3$. The right side means the 2-sphere obtained as the quotient $\sqcup_{i=1}^6 D_i / \sim$. Here, the restriction map on I_i of Φ means λ_i , and the restriction map on ∂D_i of Φ means κ_i .

$r_1, \dots, r_m\rangle$, then the third ∂_3 and the cup-product are uniquely determined, up to homotopy, by the identity. In fact, if we have another identity ω' and consider the associated $C_*(\widetilde{M})'$, Assumption (\dagger) ensures a chain map $C_*(\widetilde{M}) \rightarrow C_*(\widetilde{M})'$, which induces a homotopy equivalence.

3. Descriptions of Taut Identities of Various 3-Manifolds

In this section, we give several examples of identities from some classes of 3-manifolds. We will describe the cellular complexes of some universal covers.

3.1. Fibered 3-manifolds with surface fibers over the circle

First, we will focus on surface bundles over S^1 . Let Σ_g be an oriented closed surface of genus g and $f : \Sigma_g \rightarrow \Sigma_g$ an orientation-preserving diffeomorphism. The mapping torus, T_f , is the quotient space of $\Sigma_g \times [0, 1]$ subject to the relation $(y, 0) \sim (f(y), 1)$ for any $y \in \Sigma_g$. The homeomorphism type of T_f depends on the mapping class of f . Conversely, if a closed 3-manifold M is a fibered space over S^1 , then M is homeomorphic to T_f for some f . Since T_f is a Σ_g -bundle over S^1 , it is a $K(\pi, 1)$ -space and therefore satisfies Assumption (\dagger) .

We will construct an identity. Choose a generating set $\{x_1, \dots, x_{2g}\}$ of $\pi_1(\Sigma_g)$, which gives the isomorphism $\pi_1(\Sigma_g) \cong \langle x_1, \dots, x_{2g} \mid [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \rangle$. Following a van Kampen argument, we can verify the presentation of $\pi_1(T_f)$ as

$$(4) \quad \langle x_1, \dots, x_{2g}, \gamma \mid r_i := \gamma f_*(x_i) \gamma^{-1} x_i^{-1}, \quad (i \leq 2g), \\ r_{2g+1} := [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \rangle.$$

Here, γ represents a generator of $\pi_1(S^1)$. For $i \leq 2g$, define $w_i = \prod_{j=1}^i [x_{2j-1}, x_{2j}] \in F$, and

$$W_i := w_{i-1} \rho_{2i-1} w_{i-1}^{-1} \cdot (w_{i-1} x_{2i-1}) \rho_{2i} (w_{i-1} x_{2i-1})^{-1} \\ \cdot (w_i x_{2i}) \rho_{2i-1}^{-1} (w_i x_{2i})^{-1} \cdot w_i \rho_{2i}^{-1} w_i^{-1}.$$

Since f can be isotoped so as to preserve a point $z \in \Sigma_g$, we regard the induced map f_* as a homomorphism $\pi_1(\Sigma_g \setminus \{z\}) \rightarrow \pi_1(\Sigma_g \setminus \{z\})$. Since

f_* is a group isomorphism, there exists a unique element $q_f \in \langle x_1, \dots, x_{2g} \rangle$ satisfying

$$f_*([x_1, x_2] \cdots [x_{2g-1}, x_{2g}]) = q_f([x_1, x_2] \cdots [x_{2g-1}, x_{2g}])q_f^{-1} \in \langle x_1, \dots, x_{2g} \rangle.$$

THEOREM 3.1. *Let W be $(\Pi_{i=1}^g W_i)\rho_{2g+1}(\gamma q_f \rho_{2g+1}^{-1} q_f^{-1} \gamma^{-1}) \in F * P$. Then, W is an identity.*

PROOF. Direct calculation gives $\psi(W_i) = w_{i-1}\gamma[f_*(x_{2i-1}), f_*(x_{2i})]\gamma^{-1}w_i^{-1}$, which implies

$$\begin{aligned} \psi(\Pi_{i=1}^g W_i) &= \gamma(\Pi_{i=1}^g [f_*(x_{2i-1}), f_*(x_{2i})])\gamma^{-1}w_g^{-1} \\ &= \gamma\Pi_{i=1}^g [f_*(x_{2i-1}), f_*(x_{2i})]\gamma^{-1}(\Pi_{i=1}^g [x_{2i-1}, x_{2i}])^{-1}. \end{aligned}$$

Hence, $\psi(W) = 1$ by definition; that is, W turns out to be an identity. \square

Furthermore, we can verify that W is taut by the definition of W . Hence, from the discussion in §2, we can readily prove the following corollary.

COROLLARY 3.2. *Under the above terminology, the cellular chain complex of \widetilde{T}_f is given by*

$$\begin{aligned} C_*(\widetilde{T}_f; \mathbb{Z}) : 0 \rightarrow \mathbb{Z}[\pi_1(T_f)] &\xrightarrow{\partial_3} \mathbb{Z}[\pi_1(T_f)]^{2g+1} \\ &\xrightarrow{\partial_2} \mathbb{Z}[\pi_1(T_f)]^{2g+1} \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(T_f)] \rightarrow 0. \end{aligned}$$

Here, $\partial_1(a_i) = 1 - x_i$, $\partial_1(\gamma) = 1 - \gamma$, and ∂_2 and ∂_3 have the matrix presentations,

$$\begin{pmatrix} \left\{ \gamma \frac{\partial f_*(x_i)}{\partial x_j} - \delta_{ij} \right\}_{1 \leq i, j \leq 2g} & \left\{ 1 - x_i \right\}_{1 \leq i \leq 2g}^{\text{transpose}} \\ \left\{ \frac{\partial r_{2g+1}}{\partial x_j} \right\}_{1 \leq j \leq 2g} & 0 \end{pmatrix},$$

$$\left(\left\{ w_{j-1} - w_j x_{2j}, w_{j-1} x_{2j-1} - w_j \right\}_{1 \leq j \leq g}, \quad 1 - \gamma q_f \right).$$

Furthermore, the diagonal map $D^\sharp(c)$ is represented by

$$\begin{aligned} & \left(\sum_{i=1}^g \sum_{k=1}^{2g} \frac{\partial w_{i-1}}{\partial x_k} a_k \otimes w_{i-1} b_{2i-1} - \frac{\partial(w_i x_{2i-1})}{\partial x_k} a_k \otimes w_i x_{2i} b_{2i-1} \right. \\ & \quad \left. - \frac{\partial(w_{i-1} x_{2i-1})}{\partial x_k} a_k \otimes w_{-1i} x_{2i-1} b_{2i} + \frac{\partial w_i}{\partial x_k} a_k \otimes w_i b_{2i} \right) \\ & - \left(\sum_{k=1}^{2g} \frac{\partial(\gamma q_f)}{\partial x_k} a_k \otimes \gamma q_f b_{2g+1} \right) + a_{2g+1} \otimes (1 - \gamma q_f) b_{2g+1}. \end{aligned}$$

REMARK 3.3. Corollary 3.2 for every g is a generalization of the result of [Mar]; the paper gives the cellular complexes of \widetilde{T}_f only in the case $g = 1$. We can verify that Corollary 3.2 with $g = 1$ coincides with the results in [Mar].

Finally, we mention the virtually fibered conjecture, which was eventually proven by Wise; see, e.g., [Wise]. This conjecture states that every closed, irreducible, atoroidal 3-manifold M with an infinite fundamental group has a finite cover, which is homeomorphic to T_f for some f . Let $d \in \mathbb{N}$ be the degree of the covering. Then, if we can find such a cover $p : T_f \rightarrow M$, the pushforward of the above identity W gives an algebraic presentation of $d[M]$.

3.2. Spliced sums and $(p/1)$ - and $(1/q)$ -surgeries of S^3 along knots

We will focus on spliced sums and some surgeries of S^3 along knots and construct taut identities. This section supposes that the reader has basic knowledge of knot theory, as in [Lic, Chapters 1–11].

Let us review spliced sums. Take two knots $K, K' \subset S^3$ and an orientation-reversing homeomorphism $h : \partial(S^3 \setminus \nu K) \rightarrow \partial(S^3 \setminus \nu K')$, where νK means an open tubular neighborhood of K . Then, we can define a closed 3-manifold, $\Sigma_h(K, K')$, as the attaching space $(S^3 \setminus \nu K) \cup_h (S^3 \setminus \nu K')$ with $\partial(S^3 \setminus \nu K)$ glued to $\partial(S^3 \setminus \nu K')$ by h . This space is commonly referred to as *the spliced sum of (K, K') via h* . Spliced sums sometimes appear in discussions on additivity of topological invariants; see, e.g., [BC]. Further, choose the preferred meridian-longitude pair (\mathbf{m}, \mathbf{l}) (resp. $(\mathbf{m}', \mathbf{l}')$) as a generating set of $\pi_1 \partial(S^3 \setminus \nu K)$ (resp. of $\partial(S^3 \setminus \nu K')$). If $h_* : \pi_1 \partial(S^3 \setminus \nu K) \rightarrow$

$\pi_1 \partial(S^3 \setminus \nu K')$ is represented by $\begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ q & -1 \end{pmatrix}$) for some $p, q \in \mathbb{Z}$, we denote $\Sigma_h(K, K')$ by $\Sigma_{p/1}(K, K')$ (resp. $\Sigma_{1/q}(K, K')$). In particular, if K' is the unknot, then $\Sigma_{p/1}(K, K')$ and $\Sigma_{1/q}(K, K')$ are the closed 3-manifolds obtained by $(p/1)$ - and $(1/q)$ -Dehn surgery on K in S^3 , respectively.

Since the identities of $\Sigma_{p/1}(K, K')$ and $\Sigma_{1/q}(K, K')$ will be constructed in an analogous way to [Tro, Page 481], let us review the terminology in [Tro]. Choose a Seifert surface Σ of genus g and a bouquet of circles $W \subset \Sigma$ such that W is a deformation retract of Σ and $\pi_1(S^3 \setminus \Sigma)$ is a free group. 21. Page. For example, any Seifert surface obtained by a Seifert algorithm admits such a bouquet. Choose a bicollar $\Sigma \times [-1, 1]$ of Σ such that $\Sigma \times \{0\} = \Sigma$. Let $\iota_{\pm} : \Sigma \rightarrow S^3 \setminus \Sigma$ be embeddings whose images are $\Sigma \times \{\pm 1\}$. Take generating sets $\{v_1, \dots, v_{2g}\}$ of $\pi_1 \Sigma$ and $\{x_1, \dots, x_{2g}\}$ of $\pi_1(S^3 \setminus \Sigma)$, and set $u_i^{\sharp} := (\iota_+)_*(v_i)$ and $u_i^{\flat} := (\iota_-)_*(v_i)$, where we may suppose that $[v_1, v_2] \cdots [v_{2g-1}, v_{2g}]$ represents a loop of $\pi_1 \partial \Sigma$; a van Kampen argument yields a presentation

$$(5) \quad \langle x_1, \dots, x_{2g}, \mathbf{m} \mid r_i := \mathbf{m} u_i^{\sharp} \mathbf{m}^{-1} (u_i^{\flat})^{-1} \quad (1 \leq i \leq 2g) \rangle$$

of $\pi_1(S^3 \setminus K)$. Here, \mathbf{m} is a representative of a meridian in $\pi_1(S^3 \setminus K)$, and the x_i 's lie in the commutator subgroup of $\pi_1(S^3 \setminus K)$. Since the boundary loops of $\pi_1 \Sigma$ and $\pi_1(S^3 \setminus \Sigma)$ are equal by definition, we should notice

$$(6) \quad [u_1^{\flat}, u_2^{\flat}] \cdots [u_{2g-1}^{\flat}, u_{2g}^{\flat}] = [u_1^{\sharp}, u_2^{\sharp}] \cdots [u_{2g-1}^{\sharp}, u_{2g}^{\sharp}] \in \pi_1(S^3 \setminus \Sigma),$$

which we denote by \mathfrak{l} . In other words, \mathfrak{l} means a preferred longitude of K .

In a parallel way, concerning the other K' , we have a generating set $\{x'_1, \dots, x'_{2g'}\}$ of $\pi_1(S^3 \setminus \Sigma')$ and can define appropriate words \mathbf{u}_i^{\sharp} and \mathbf{u}_i^{\flat} such that

$$\pi_1(S^3 \setminus K') \cong \langle x'_1, \dots, x'_{2g'}, \mathbf{m}' \mid r'_i := \mathbf{m}' \mathbf{u}_i^{\sharp} (\mathbf{m}')^{-1} (\mathbf{u}_i^{\flat})^{-1} \quad (1 \leq i \leq 2g') \rangle.$$

We also redefine \mathfrak{l}' by $[\mathbf{u}_1^{\flat}, \mathbf{u}_2^{\flat}] \cdots [\mathbf{u}_{2g-1}^{\flat}, \mathbf{u}_{2g}^{\flat}]$.

Before we state Theorem 3.4, we should notice from the van Kampen theorem that the fundamental groups $\pi_1(\Sigma_{p/1}(K, K'))$ and $\pi_1(\Sigma_{1/q}(K, K'))$ are presented by

$$(7) \quad \langle x_1, \dots, x_{2g}, \mathbf{m}, x'_1, \dots, x'_{2g'} \mid r_1, r_2, \dots, r_{2g}, r'_1, \dots, r'_{2g'}, \\ r_{2g+1} := \mathfrak{l} \mathbf{m}^p (\mathfrak{l}')^{-1} \rangle,$$

$$\langle x_1, \dots, x_{2g}, \mathbf{m} \ x'_1, \dots, x'_{2g'}, \mathbf{m}' \mid r_1, r_2, \dots, r_{2g}, r'_1, \dots, r'_{2g'}, \\ r_{\dagger} := \mathbf{m} \mathfrak{l}^q (\mathfrak{l}')^{-1}, r_{\star} := \mathbf{m}' \mathfrak{l}'^{-1} \rangle.$$

Here, in (7), we identify \mathbf{m} with \mathbf{m}' . Define w_i to be $\prod_{j=1}^i [u_{2j-1}^b, u_{2j}^b]$, and

$$W_i := w_{i-1} \rho_{2i-1} w_{i-1}^{-1} \cdot (w_{i-1} u_{2i-1}^b) \rho_{2i} (w_{i-1} u_{2i-1}^b)^{-1} \cdot \\ (w_i u_{2i}^b) \rho_{2i-1}^{-1} (w_i u_{2i}^b)^{-1} \cdot w_i \rho_{2i}^{-1} w_i^{-1}.$$

Likewise, we also define words w'_i and W'_i . We consider the two words,

$$W_{p/1}^{K,K'} := (\Pi_{i=1}^g W_i) \cdot \rho_{2g+1} \cdot (\Pi_{i=1}^{g'} W'_i)^{-1} (\mathbf{m} \rho_{2g+1}^{-1} \mathbf{m}^{-1}), \\ W_{1/q}^{K,K'} := (\Pi_{i=1}^g W_i) \cdot \rho_{\star}^{-1} \cdot (\mathbf{m}' \rho_{\dagger} (\mathbf{m}')^{-1}) \cdot (\Pi_{i=1}^{g'} W'_i) \cdot (\mathfrak{l}' \rho_{\star} (\mathfrak{l}')^{-1}) \cdot \rho_{\dagger}^{-1}.$$

THEOREM 3.4. *Then, $W_{p/1}^{K,K'}$ and $W_{1/q}^{K,K'}$ are taut identities with respect to the presentations (7) of $\pi_1(\Sigma_{p/1}(K, K'))$ and $\pi_1(\Sigma_{1/q}(K, K'))$, respectively.*

PROOF. An immediate computation gives $\psi(W_i) = w_{i-1} \mathbf{m} [u_{2i-1}^{\sharp}, u_{2i}^{\sharp}] \mathbf{m}^{-1} w_i^{-1}$, so that $\psi(\Pi_{i=1}^g W_i) = \mathbf{m} \Pi_{i=1}^g [u_{2i-1}^{\sharp}, u_{2i}^{\sharp}] \mathbf{m}^{-1} w_g^{-1}$. Then, $W_{p/1}^{K,K'}$ and $W_{1/q}^{K,K'}$ turn out to be identities by (7). Furthermore, by the definition of $W_{\bullet}^{K,K'}$, we verify that $W_{\bullet}^{K,K'}$ are taut. \square

As a corollary, if K' is the unknot, we have the complex $C_*(\widetilde{M}; \mathbb{Z})$, where M is the 3-manifold, $M_{p/1}(K)$, obtained by $p/1$ -surgery of S^3 along K :

COROLLARY 3.5. *If $M := M_{p/1}(K)$ satisfies Assumption (\dagger) , then the boundary maps ∂_2 and ∂_3 in the associated complex $C_*(\widetilde{M}; \mathbb{Z})$ in (1) are given by the following matrix presentations:*

$$\begin{pmatrix} \left\{ \mathbf{m} \frac{\partial u_i^{\sharp}}{\partial x_j} - \frac{\partial u_i^b}{\partial x_j} \right\}_{1 \leq i, j \leq 2g} & \left\{ 1 - \frac{\partial \mathfrak{l}}{\partial x_j} \right\}_{1 \leq j \leq 2g}^{\text{transpose}} \\ \left\{ \frac{\partial \mathfrak{l}}{\partial x_j} \mathbf{m}^p \right\}_{1 \leq j \leq 2g} & \mathfrak{l} \frac{\partial \mathbf{m}^p}{\partial \mathbf{m}} \end{pmatrix}, \\ \partial_3(s) = (1 - \mathbf{m}) b_{2g+1} + \sum_{i=1}^g (w_{i-1} - w_i u_{2i}^b) b_{2i-1} + (w_{i-1} u_{2i-1}^b - w_i) b_{2i}.$$

REMARK 3.6. We give a comparison to Theorem 3.9 in [MP]. The authors give an expression of the chain complex $C_*(\widetilde{M}; \mathbb{Z})$, where $M = M_{0/1}(K)$. However, the numbers of basis of C_3, C_2, C_1 are $2, c+1, c$, respectively, where c is the crossing number of K , while those in Corollary 3.5 are fewer.

Let us recall the cabling conjecture, which predicts that if K is not a cabling knot, then $M_0(K)$ is irreducible; this conjecture has been proven for some classes of knots. Since $\pi_1(M_0(K))$ is of infinite order, it is fair to say that most $M_0(K)$ satisfy Assumption (\dagger) . Incidentally, it is a problem for the future to clarify a taut identity for the (p/q) -surgery for any $p/q \in \mathbb{Q}$.

3.3. Branched covering spaces of S^3 branched over a knot

Take a knot K in S^3 , and $d \in \mathbb{N}$. In this subsection, we will give a taut identity of $\pi_1(B_K^d)$, where we mean by B_K^d the d -fold cyclic covering space of S^3 branched over K . We should remark the fact that, if K is a prime knot and $\pi_1(B_K^d)$ is of infinite order, then B_K^d is aspherical and therefore admits Assumption (\dagger) . Let $p : E_K^d \rightarrow S^3 \setminus K$ be the d -fold cyclic covering. For $k \in \mathbb{Z}/d$, let $x_i^{(k)}$ be a copy of x_i and $u_{i,k}^\sharp$ be the word obtained by replacing x_i with $x_i^{(k)}$ in the word u_i^\sharp . We similarly define the word $u_{i,k}^b$. Then, by using the Reidemeister-Schreier method (see, e.g., [Kab, Proposition 3.1]), it follows from presentation (5) that $\pi_1(E_K^d)$ is presented by

$$(8) \quad \langle x_1^{(k)}, \dots, x_{2g}^{(k)}, \overline{\mathbf{m}} \quad (k \in \mathbb{Z}/d) \mid \overline{\mathbf{m}} u_{i,k}^\sharp \overline{\mathbf{m}}^{-1} (u_{i,k+1}^b)^{-1} \\ (1 \leq i \leq 2g, k \in \mathbb{Z}/d) \rangle.$$

Since B_K^d is obtained from E_K^d by attaching a solid torus which annihilates the meridian $\overline{\mathbf{m}}$, $\pi_1(B_K^d)$ is presented by the quotient of $\pi_1(E_K^d)$ subject to $\overline{\mathbf{m}} = 1$; that is,

$$(9) \quad \pi_1(B_K^d) \cong \langle x_1^{(k)}, \dots, x_{2g}^{(k)} \quad (k \in \mathbb{Z}/d) \mid r_{i,k} := u_{i,k}^\sharp (u_{i,k+1}^b)^{-1} \\ (1 \leq i \leq 2g, k \in \mathbb{Z}/d) \rangle.$$

Let F be the free group $\langle x_1^{(k)}, \dots, x_{2g}^{(k)} \quad (k \in \mathbb{Z}/d) \rangle$. From (6), we should notice that $[u_{1,k}^b, u_{2,k}^b] \cdots [u_{2g-1,k}^b, u_{2g,k}^b] = [u_{1,k}^\sharp, u_{2,k}^\sharp] \cdots [u_{2g-1,k}^\sharp, u_{2g,k}^\sharp] \in F$ for any $k \in \mathbb{Z}/d$.

Similarly to §3.2., we will give an identity with respect to the presentation (9). For $1 \leq i \leq g, 1 \leq k \leq d$, define $w_{i,k} = \prod_{j=1}^i [u_{2j-1,k+1}^b, u_{2j,k+1}^b]$, and

$$W_{i,k} = w_{i-1,k} \rho_{2i-1,k} w_{i-1,k}^{-1} \cdot (w_{i-1,k} u_{2i-1,k+1}^b) \rho_{2i,k} (w_{i-1,k} u_{2i-1,k+1}^b)^{-1} \cdot \\ (w_{i,k} u_{2i,k+1}^b) \rho_{2i-1,k}^{-1} (w_{i,k} u_{2i,k+1}^b)^{-1} \cdot w_{i,k} \rho_{2i,k}^{-1} w_{i,k}^{-1}.$$

PROPOSITION 3.7. *Define W to be $\prod_{k=1}^d W_{1,k} W_{2,k} \cdots W_{g,k}$, by the above equality in F . Then, W is a taut identity. In particular, if B_K^d satisfies Assumption (\dagger) , the associated complex in (1) is isomorphic to the cellular chain complex of the universal cover of B_K^d .*

PROOF. Direct calculation gives $\psi(W_{i,k}) = w_{i-1,k} [u_{2i-1,k}^\sharp, u_{2i,k}^\sharp] w_{i,k}^{-1}$, which deduces

$$\psi(\prod_{i=1}^g W_{i,k}) = (\prod_{i=1}^g [u_{2i-1,k}^\sharp, u_{2i,k}^\sharp]) w_{g,k}^{-1} \\ = \prod_{i=1}^g [u_{2i-1,k}^\sharp, u_{2i,k}^\sharp] (\prod_{i=1}^g [u_{2i-1,k+1}^b, u_{2i,k+1}^b])^{-1}.$$

Thus, W turns out to be an identity. Furthermore, since we can verify that W is taut by the definition of W , Remark 2.2 readily leads to the latter part. \square

Example 3.8. Let K be the figure-eight knot. It can be verified that the presentation (5) can be written as

$$\langle x_1, x_2, \mathbf{m} \mid \mathbf{m} x_1 x_2 \mathbf{m}^{-1} = x_1, \mathbf{m} x_2 x_1 x_2 \mathbf{m}^{-1} = x_2 \rangle.$$

Thus, by (9), we have

$$\pi_1(B_K^d) \cong \langle x_1^{(i)}, x_2^{(i)} \mid x_1^{(i)} x_2^{(i)} = x_1^{(i+1)}, \\ x_2^{(i)} x_1^{(i)} x_2^{(i)} = x_2^{(i+1)} \mid (1 \leq i \leq d) \rangle.$$

Annihilating $x_2^{(i)}$ by using the relation $x_1^{(i)} x_2^{(i)} = x_1^{(i+1)}$, we have

$$\pi_1(B_K^d) \cong \langle x_1^{(1)}, \dots, x_1^{(d)} \mid (x_1^{(i)})^{-1} (x_1^{(i+1)})^2 (x_1^{(i+2)})^{-1} x_1^{(i+1)} \mid (1 \leq i \leq d) \rangle.$$

This isomorphism coincides exactly with the result in [KKV, Page 963].

Likewise, we can verify that some groups, called “cyclically presented groups” in [KKV] and references therein, are isomorphic to $\pi_1(B_K^d)$ for some K and d .

REMARK 3.9. As the referee points out, it is reasonable to hope that Proposition 3.7 is true without Assumption (\dagger). In fact, as seen in [Sie], given a Heegaard diagram, we can construct a “squashing map” and a taut identity compatible with the complex (1). Thus, it is a conjecture that we can find an appropriate Heegaard diagram of B_K^d such that the associated taut identity is equal to the above W .

3.4. 0-Surgery-like spaces from branched covering spaces of S^3

Using the notation in the preceding subsection, we can examine the 3-manifold obtained by the 0-surgery on the knot $p^{-1}(K) \subset B_K^d$. The 0-surgery appears in the topic of the concordance group including the Casson-Gordon invariant [CG]. More precisely, regarding the boundary of E_K^d as a knot in B_K^d , we consider the 3-manifold obtained by 0-surgery on the knot in B_K^d . Notice from (8) that the fundamental group canonically has a group presentation

$$(10) \quad \langle x_1^{(k)}, \dots, x_{2g}^{(k)} \quad (k \in \mathbb{Z}/d), \overline{\mathbf{m}} \mid r_i^{(k)} \quad (i \leq 2g, k \in \mathbb{Z}/d), \\ \Pi_{i=1}^g [u_{2i-1,1}^b, u_{2i,1}^b] \rangle.$$

Let $\mathfrak{l}^{(k)} := \Pi_{i=1}^g [u_{2i-1,k}^b, u_{2i,k}^b]$, and consider an analogous presentation

$$(11) \quad \langle x_1^{(k)}, \dots, x_{2g}^{(k)} \quad (k \in \mathbb{Z}/d), \overline{\mathbf{m}} \mid r_i^{(k)} \quad (i \leq 2g, k \in \mathbb{Z}/d), \\ r_\ell := \mathfrak{l}^{(1)} \mathfrak{l}^{(2)} \dots \mathfrak{l}^{(d)} \rangle.$$

Similarly to §3.3, we can construct an identity. For $i \leq 2g, k \leq d$, define $z_k = \mathfrak{l}^{(1)} \mathfrak{l}^{(2)} \dots \mathfrak{l}^{(k)}$ and

$$W_{i,k} = z_k w_{i-1,k} \rho_{2i-1,k} w_{i-1,k}^{-1} z_k^{-1} \cdot \\ (z_k w_{i-1,k} u_{2i-1,k+1}^b) \rho_{2i,k} (z_k w_{i-1,k} u_{2i-1,k+1}^b)^{-1} \cdot \\ (z_k w_{i,k} u_{2i,k+1}^b) \rho_{2i-1,k}^{-1} (z_k w_{i,k} u_{2i,k+1}^b)^{-1} \cdot z_k w_{i,k} \rho_{2i,k}^{-1} w_{i,k}^{-1} z_k^{-1}.$$

In the usual way, we can easily show the following:

PROPOSITION 3.10. *Define W to be $(\Pi_{k=1}^d \Pi_{i=1}^g W_{i,k}) \cdot \rho_\ell \cdot (\mathfrak{m} \rho_\ell^{-1} \mathfrak{m}^{-1})$. Then, W is a taut identity. In particular, Remark 2.2 ensures that if the fundamental group of a closed 3-manifold satisfying Assumption (\dagger) is isomorphic to (11), then the cellular chain complex of the universal cover is isomorphic to the complex (1).*

3.5. Some Seifert fibered spaces over S^2

In the last subsection, we will discuss some of the Seifert fibered spaces and Brieskorn manifolds. The theorem of Scott [Sc] shows that the homeomorphism types of such spaces with infinite π_1 can be detected by the fundamental groups; thus, the spaces satisfy Assumption (\dagger) .

Let us state Proposition 3.11. Take integers a_1, \dots, a_{n+1} with $a_i \geq 2$, and $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$. Let M be a Seifert fibered space of the form

$$\Sigma(0; (0, 1), (a_1, \epsilon_1), (a_2, \epsilon_2), \dots, (a_n, \epsilon_n), (a_{n+1}, 1)).$$

Then, as is classically known, the fundamental group has the presentation

$$\langle x_1, \dots, x_{n+1}, h \mid hx_i h^{-1} x_i^{-1}, x_i^{a_i} h^{\epsilon_i} \ (i \leq n) \ , x_{n+1}^{a_{n+1}} h, x_1 \cdots x_{n+1} \ \rangle.$$

Furthermore, let us consider a group G with the presentation

$$(12) \quad \langle x_1, \dots, x_n \mid r_i := (x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1})^{-a_{n+1}} x_i^{\epsilon_i a_i} \ (i \leq n) \ \rangle.$$

We can easily check that the correspondence $x_i \mapsto x_i$, $x_{n+1} \mapsto (x_1 \cdots x_n)^{-1}$, $h \mapsto x_1^{\epsilon_1 a_1}$ gives rise to a group isomorphism $\pi_1(M) \cong G$. Therefore, we shall define a taut identity on the presentation (12):

PROPOSITION 3.11. *Suppose that $\pi_1(M)$ is of infinite order. Define W to be*

$$\rho_1(x_1^{-1} \rho_1^{-1} x_1) \rho_2(x_2^{-1} \rho_2^{-1} x_2) \cdots \rho_n(x_n^{-1} \rho_n^{-1} x_n).$$

Then, W is a taut identity of the presentation (12).

The proof is similar to the ones above, so we will omit the details.

REMARK 3.12. The taut identity when $n = 2$ is presented in [Sie, p. 127]. The paper does not mention the homeomorphism type of the associated 3-manifold; however, Proposition 3.11 implies that the homeomorphism type can be detected by a Seifert structure.

Finally, let us turn to the topic of Brieskorn 3-manifolds. Choose integers $a, b, p, q, m \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$ satisfying $ap + bq = 1$ and $p, q, m > 1$. We will focus on the Brieskorn 3-manifold of the form,

$$\begin{aligned} M &:= \Sigma(p, q, mpq + \varepsilon) \\ &:= \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^{mpq+\varepsilon} = 0, \quad |x|^2 + |y|^2 + |z|^2 = 1\}, \end{aligned}$$

which is an Eilenberg-MacLane space if $1/p + 1/q + 1/(mpq + \varepsilon) < 1$. The manifold is known to be homeomorphic to a 3-manifold obtained from (ε/m) -surgery on the (p, q) -torus knot $T_{p,q}$. Recall the presentation of $\pi_1(S^3 \setminus T_{p,q})$ as $\pi_1(S^3 \setminus T_{p,q}) \cong \langle x, y \mid x^q = y^p \rangle$, and that the meridian \mathfrak{m} and the preferred longitude \mathfrak{l} are identified with $x^a y^b$ and $(x^a y^b)^{-pq} x^q$, respectively. Therefore, $\pi_1(M)$ admits a genus-two Heegaard decomposition and has the group presentation,

$$(13) \quad \pi_1(M) \cong \langle x, y \mid r_1 := x^{qm} (x^a y^b)^{-mpq-\varepsilon}, \quad r_2 := (x^a y^b)^{mpq+\varepsilon} y^{-p} x^{-qm-q} \rangle.$$

Likewise, we can show the following result:

PROPOSITION 3.13. *Suppose $1/p + 1/q + 1/(mpq + \varepsilon) < 1$ as above. Then the following word is a taut identity of the presentation (13).*

$$\rho_1 \rho_2^{-1} \rho_1^{-1} (x^{qm} y^{-p} x^{-qm-q} \rho_2 x^{qm+q} y^p x^{-qm}).$$

4. First Application to the Linking Forms of Branched Covers

4.1. Review of the linking form and a theorem

Here, we will review the linking form of M for a closed 3-manifold M with $H_*(M; \mathbb{Q}) \cong H_*(S^3; \mathbb{Q})$. Considering the short exact sequence

$$(14) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

we can easily check that the Bockstein maps

$$\beta : H_2(M; \mathbb{Q}/\mathbb{Z}) \cong H_1(M; \mathbb{Z}), \quad \beta : H^1(M; \mathbb{Q}/\mathbb{Z}) \cong H^2(M; \mathbb{Z}),$$

are isomorphisms from the long exact homology sequences. Let $\text{PD}_M^{\mathbb{Z}}$ be the Poincaré duality on the integral (co)-homology. We denote by Ω the composite map defined by setting

$$H_1(M; \mathbb{Z}) \xrightarrow{\text{PD}_M^{\mathbb{Z}}} H^2(M; \mathbb{Z}) \xrightarrow{\beta^{-1}} H^1(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{ev}} \text{Hom}(H_1(M; \mathbb{Z}); \mathbb{Q}/\mathbb{Z}),$$

where the last map is the Kronecker evaluation map. Then, *the linking form of M* is

$$\lambda_M : H_1(M; \mathbb{Z}) \times H_1(M; \mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined by $\lambda_M(a, b) = \Omega(a)(b)$. This bilinear map is known to be symmetric and non-singular. This definition goes back to Seifert [Sei], and the form has sometimes appeared in the study of algebraic surgery theory (see, e.g., [Wall]) and the concordance groups of knots [CG]. Recently, the linking form of M can be computed in terms of Heegaard splittings [CFH].

Of particular interest to us is an application to the Casson-Gordon invariant [CG] and a procedure for computing λ_M in another way. In what follows, let B_K^d be the d -fold cyclic covering space of S^3 branched over a knot K . In the context of the invariant, the linking form of B_K^d plays an important role: more precisely, it is important to calculate metabolizers of the form; see, e.g., [CG].

Now let us give a matrix presentation of the homology $H_1(B_K^d; \mathbb{Z})$ and state the main theorem. Choose a Seifert surface Σ of K whose genus is g , as in §3.2.. Then, we have the Seifert form $\alpha : H_1(\Sigma; \mathbb{Z}) \otimes H_1(\Sigma; \mathbb{Z}) \rightarrow \mathbb{Z}$; see [Lic, Chapter 6] for the definition. Let J be the inverse matrix $(V - {}^tV)^{-1}$, where $\det(V - {}^tV) = 1$ is known (see [Lic, Theorem 6.10]). The matrix presentation is often written as $V \in \text{Mat}(2g \times 2g; \mathbb{Z})$ and is called *the Seifert matrix*. Consider the following matrices of size $(2gd \times 2gd)$:

$$A := \begin{pmatrix} -V & 0 & \cdots & 0 & {}^tV \\ {}^tV & -V & \cdots & 0 & 0 \\ 0 & {}^tV & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & {}^tV & -V \end{pmatrix},$$

$$B := \begin{pmatrix} 0 & 0 & \cdots & 0 & J^t V \\ J^t V & 0 & \cdots & 0 & 0 \\ 0 & J^t V & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & J^t V & 0 \end{pmatrix},$$

which appear in [Tro, Page 494]. As is known (see [Sei, Tro] or [Lic, Theorem 9.7]), the first homology $H_1(B_K^d; \mathbb{Z})$ is isomorphic to the cokernel of A , i.e., $H_1(B_K^d; \mathbb{Z}) \cong \mathbb{Z}^{2gd} / {}^t A \mathbb{Z}^{2gd}$. In particular, $\det(A) \neq 0$ if and only if $H_1(B_K^d; \mathbb{Q}) \cong 0$. The linking formula of B_K^d can be algebraically formulated in the above notation as follows:

THEOREM 4.1. *Suppose that B_K^d satisfies Assumption (\dagger) and $H_1(B_K^d; \mathbb{Q}) \cong 0$. Then, the matrix multiplication $B : \mathbb{Z}^{2gd} \rightarrow \mathbb{Z}^{2gd}$ induces an isomorphism $\mathcal{B} : \mathbb{Z}^{2gd} / {}^t A \mathbb{Z}^{2gd} \rightarrow \mathbb{Z}^{2gd} / {}^t A \mathbb{Z}^{2gd}$ and the linking form $\lambda_{B_K^d}$ of B_K^d is equal to the form,*

$$(15) \quad \begin{aligned} \mathbb{Z}^{2gd} / {}^t A \mathbb{Z}^{2gd} \times \mathbb{Z}^{2gd} / {}^t A \mathbb{Z}^{2gd} &\longrightarrow \mathbb{Q} / \mathbb{Z}; \\ (v, w) &\longmapsto {}^t v \operatorname{adj}(A) {}^t \mathcal{B}^{-1} w / \Delta. \end{aligned}$$

Here, $\operatorname{adj}(A)$ is the adjugate matrix of A , and Δ is the order $|H_1(B_K^d; \mathbb{Z})| \in \mathbb{N}$.

This statement is implicitly connoted in [Sei, Satz I] and [Tro, p. 496]¹; however, there is no complete proof for this statement in the literature.

Here, let us make a few remarks. Whereas the matrix $\operatorname{adj}(A) {}^t \mathcal{B}^{-1}$ is not always symmetric, the quotient on $\mathbb{Z}^{2gd} / {}^t A \mathbb{Z}^{2gd}$ is symmetric. Next, the second condition of $H_1(B_K^d; \mathbb{Q}) \cong 0$ is not so strong: indeed, according to [Lic, Corollary 9.8], if any d -th root of unity is not a zero point of the Alexander polynomial of K (e.g., the case d is a prime power), then $H_1(B_K^d; \mathbb{Q}) \cong 0$. Furthermore, as the proof and Remark 3.9 imply, one may hope that the theorem is true even if we drop the condition of Assumption (\dagger) .

¹To be precise, the original statements implicitly claim that the linking form $\lambda_{B_K^d}$ is equal to the matrix presentation $B \operatorname{adj}(A)$ up to isomorphisms. However, for applications to the Casson-Gordon invariants, we should describe the linking form from a basis of $H_1(B_K^d; \mathbb{Z})$.

PROOF OF THEOREM 4.1. It is known [CFH, Lemma 2.5] that the linking form can be formulated in the terminology of cohomology as

$$(16) \quad \lambda_M(a, b) = \langle (\beta^{-1} \circ \text{PD}_M^{\mathbb{Z}})(a) \smile \text{PD}_M^{\mathbb{Z}}(b), [M] \rangle.$$

Here, \smile is the cup-product $H^1(M; \mathbb{Q}/\mathbb{Z}) \otimes H^2(M; \mathbb{Z}) \rightarrow H^3(M; \mathbb{Q}/\mathbb{Z})$.

Let M be B_K^d , and let R be one of \mathbb{Z}, \mathbb{Q} or \mathbb{Q}/\mathbb{Z} as trivial coefficients. Let $\varepsilon : \mathbb{Z}[\pi_1(M)] \rightarrow \mathbb{Z}$ be the augmentation map. Then, as is known (see [Tro, Proposition 4.1]), by choosing a Seifert surface, the integral matrices $\{\varepsilon(\frac{\partial u_i^{\sharp}}{\partial x_j})\}_{1 \leq i, j \leq 2g}$ and $\{\varepsilon(\frac{\partial u_i^{\flat}}{\partial x_j})\}_{1 \leq i, j \leq 2g}$ are equal to V and tV , respectively. Let us identify the complex $C^*(M; R)$ in the coefficients R with $C^*(\widetilde{M}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} R$ via ε . Then, by presentation (9), the complex $C^*(M; R)$ reduces to

$$(17) \quad 0 \rightarrow C^0(M; \mathbb{Z}) \xrightarrow{0} C^1(M; R) \xrightarrow{A} C^2(M; R) \xrightarrow{0} C^3(M; R) \rightarrow 0.$$

If $R = \mathbb{Q}$, the matrix A is an isomorphism because of $H^*(M; \mathbb{Q}) \cong H^*(S^3; \mathbb{Q})$. Therefore, from the definition of the Bockstein inverse map $\beta^{-1} : C^2(M; \mathbb{Z}) \rightarrow C^1(M; \mathbb{Q}/\mathbb{Z})$ is identified with $\mathbb{Z}^{2gd} \rightarrow (\mathbb{Q}/\mathbb{Z})^{2gd}; v \mapsto \text{adj}(A)v/\Delta$.

Meanwhile, from the formula for the identity W in Proposition 3.7 and the formula (3), the cup-product $\smile : C^1(M; \mathbb{Q}/\mathbb{Z}) \times C^2(M; \mathbb{Z}) \rightarrow C^3(M; \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ is considered to be $(\mathbb{Q}/\mathbb{Z})^{2gd} \times \mathbb{Z}^{2gd} \rightarrow \mathbb{Q}/\mathbb{Z}; (v, w) \mapsto {}^t v B w$. The Poincaré duality ensures the non-degeneracy of the cup product on cohomology. In particular, the desired induced map \mathcal{B} is an isomorphism, and is identified with the duality $H_1(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z})$, where $H^2(M; \mathbb{Z})$ is canonically regarded as $\text{Coker}(A) = \mathbb{Z}^{2gd}/A\mathbb{Z}^{2gd}$ by (17). Hence, upon the identification $H^2(M; \mathbb{Z}) \cong H_1(M; \mathbb{Z}) \cong \mathbb{Z}^{2gd}/{}^t A\mathbb{Z}^{2gd}$, the formula (16) immediately implies that the linking form is equal to the required (15). \square

4.2. Example computations

It is easier to quantitatively compute kernels rather than cokernels. Let us examine Corollary 4.2 below. Let $\text{Ker}(A)_{\mathbb{Z}/\Delta}$ be $\{v \in (\mathbb{Z}/\Delta\mathbb{Z})^{2gd} \mid Av = 0 \in (\mathbb{Z}/\Delta\mathbb{Z})^{2gd}\}$. Consider the linear map

$$\mathbb{Z}^{2gd}/A\mathbb{Z}^{2gd} \longrightarrow \text{Ker}(A)_{\mathbb{Z}/\Delta}; \quad v \longmapsto \text{adj}(A)v.$$

This map is an isomorphism if $|\Delta| \neq 0$: in fact, with a choice of the section $\mathfrak{s} : \mathbb{Z}^{2gd}/A\mathbb{Z}^{2gd} \rightarrow \mathbb{Z}^{2gd}$, the inverse map is defined by $w \mapsto (A\mathfrak{s}(w))/\Delta$. In summary, from Theorem 4.1, we immediately have the following:

COROLLARY 4.2. *Let Δ , A , B and $\text{adj}(A)$ be as in Theorem 4.1. Under the supposition in Theorem 4.1, the linking form $\lambda_{B_K^d}$ of B_K^d is isomorphic to the bilinear form*

$$\text{Ker}(A)_{\mathbb{Z}/\Delta} \times \text{Ker}(A)_{\mathbb{Z}/\Delta} \longrightarrow \mathbb{Q}/\mathbb{Z}; \quad (v, w) \longmapsto {}^t \mathbf{s}(v) {}^t ABw / \Delta^2.$$

Example 4.3. Let $p, q, r \in \mathbb{Z}$ be odd numbers. Let K be the Pretzel knot $P(p, q, r)$. When $d = 2$, the branched cover B_K^2 is known to be a Seifert fibered space of type $\Sigma(p, q, r)$ over S^2 . Furthermore, we can choose a Seifert matrix of the form $V = \frac{1}{2} \begin{pmatrix} p+q & q+1 \\ q-1 & q+r \end{pmatrix}$, and $\Delta = pq + qr + rp$; see [Lic, Example 6.9].

First, consider the case where p, q, r are relatively prime. Then, $\text{Ker}(A)$ is generated by $(-r-q, q, -r-q, q)$, and we can easily verify that the linking form equal to $2(q+r)/\Delta$.

However, if p, q, r are not relatively prime, $\text{Ker}(A)$ and the linking form are complicated. For example, if $(p, q, r) = (p, -p, p)$, then $\text{Ker}(A) \cong (\mathbb{Z}/p)^2$ is generated by $(0, p, 0, p)$ and $(p, 0, p, 0)$; the linking matrix is equal to $\frac{2}{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Meanwhile, if $(p, q, r) = (p, p, p)$ and p is not divisible by 3, then $\text{Ker}(A) \cong \mathbb{Z}/p \oplus \mathbb{Z}/3p$ possesses a basis, $v = (0, 3p, 0, 3p)$, $w = (3p + p^2, p^2, 3p + p^2, p^2)$. Hence, $\begin{pmatrix} \text{lk}(v, v) & \text{lk}(v, w) \\ \text{lk}(w, v) & \text{lk}(w, w) \end{pmatrix}$ can be computed as $\frac{2}{p} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

In a similar way, we can compute many linking forms of d -fold branched covering spaces for small d with the help of a computer program.

5. Second Application to Dijkgraaf-Witten Invariants

As another application, we develop procedures of computing some Dijkgraaf-Witten invariants in terms of identities.

We start by reviewing the Dijkgraaf-Witten invariant [DW]. Let G be a finite group, A a commutative ring, and ψ a group 3-cocycle of G . Denoting by BG an Eilenberg-MacLane space of type $(G, 1)$, we have a classifying map $\iota : M \hookrightarrow B\pi_1(M)$ uniquely up to homotopy. Then, as is known, ψ can

be regarded as a 3-cocycle of $H^3(BG; A)$, and any group homomorphism $f : M \rightarrow G$ canonically gives rise to the composite

$$\iota^* \circ f^* : H^*(BG; A) \rightarrow H^*(B\pi_1(M); A) = H^*(\pi_1(M); A) \rightarrow H^*(M; A).$$

Then, the *Dijkgraaf-Witten invariant* of M is defined as a formal sum in the group ring $\mathbb{Z}[A]$ by setting

$$\text{DW}_\psi(M) := \sum_{f \in \text{Hom}(\pi_1(M), G)} \langle \iota^* \circ f^*(\psi), [M] \rangle \in \mathbb{Z}[A].$$

Although the definition seems rather simple or direct, it is not easy to compute $\text{DW}_\psi(M)$ except in the case where G is abelian, because it is not trivial to explicitly express $[M]$ and f^* (however, see [DW, Wakui] for the abelian case and [No2] for a partially non-abelian case). To the knowledge of the author, there are few examples of Dijkgraaf-Witten invariants when G is non-abelian.

This section develops a method for computing the invariants, and gives non-abelian examples. First, for simplicity, we now restrict on the case $\psi = \gamma \smile \delta$ for some $\gamma \in H^1(G; A)$ and $\delta \in H^2(G; A)$. Take a group homomorphism $f : \pi_1(M) \rightarrow G$ and a group presentation $G = \langle y_1, \dots, y_n \mid s_1, \dots, s_\ell \rangle$. Then, as in (1), we have a commutative diagram:

$$\begin{array}{ccccccc} C_*(M; A) & \xrightarrow{\partial_3} & A \otimes \mathbb{Z}[\pi_1(M)]^m & \xrightarrow{\partial_2} & A \otimes \mathbb{Z}[\pi_1(M)]^m & \xrightarrow{0} & A \otimes \mathbb{Z}[\pi_1(M)] \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ C_*(G; A) & \xrightarrow{\partial'_3} & A \otimes \mathbb{Z}[G]^\ell & \xrightarrow{\partial'_2} & A \otimes \mathbb{Z}[G]^n & \xrightarrow{\partial'_1=0} & A \otimes \mathbb{Z}[G]. \end{array}$$

Here, the tensors are over $\mathbb{Z}[G]$, and $\partial'_2(b'_i) = \sum_{k=1}^n [\frac{\partial s_i}{\partial y_k}] a'_k$.

Example 5.1. Suppose $p, q \in \mathbb{N}$ such that $(p, q) = 1$. Let $A = G = \mathbb{Z}/p$, and M be the lens space $L(p, q)$. Then, as is known, $H^*(G; A) \cong \mathbb{Z}/p$, and we can choose appropriate generators $\alpha_i \in H^i(G; A) \cong \mathbb{Z}/p$ such that $\alpha_3 = \alpha_1 \smile \alpha_2$. We fix a presentation $G = \pi_1(M) = \langle x \mid |s := x^p \rangle$. Then, the taut identity of $L(p, q)$ is known to be $W_{p,q} = sx^{-q}s^{-1}x^q$; see [Sie]. Then, for $i \leq 3$, we can regard α_i as a map $\mathbb{Z}/p = C_*(M; \mathbb{Z}/p) \rightarrow \mathbb{Z}/p$ that sends a generator to 1. Then it follows from (2) that the cup product $\smile : H^1(L(p, q); \mathbb{Z}/p) \times H^2(L(p, q); \mathbb{Z}/p) \rightarrow \mathbb{Z}/p$ is computed as $(a, b) \mapsto qab$.

Moreover, for $a \in \mathbb{Z}/p$, if we define $f_a : \pi_1(M) \rightarrow G$ by setting $x \mapsto a$, then $\text{Hom}(\pi_1(M), G)$ is equal to $\{f_a | a \in \mathbb{Z}/p\}$, and we can compute

$$\langle f_a^*(\alpha_3), \iota_*[M] \rangle = \langle f_a^*(\alpha_1 \smile \alpha_2), \iota_*[M] \rangle = \langle a\alpha_1 \smile a\alpha_2, \iota_*[M] \rangle = qa^2.$$

In conclusion,

$$\text{DW}_{\alpha_3}(L(p, q)) = \sum_{a \in \mathbb{Z}/p} 1\{qa^2\} \in \mathbb{Z}[\mathbb{Z}/p].$$

In a similar way, if M is another manifold such that the cohomology ring is known, we can compute $\text{DW}_{\alpha_3}(M)$ for $G = \mathbb{Z}/p$. Comparing with [DW, Wakui] as original computations, the above computation seems easier.

Example 5.2. Let m, n be natural numbers such that m is relatively prime to $6n$. Let G be the non-abelian group of order m^3 which has a group presentation

$$(18) \quad \langle x, y, z \mid x^m, y^m, z^m, s := xzx^{-1}z^{-1}, t := yzy^{-1}z^{-1}, u := zyx y^{-1}x^{-1} \rangle.$$

The (co)-homology of G is known (see, e.g., [Lea]). As a result, $H_1(G; \mathbb{Z}) \cong (\mathbb{Z}/m)^2$. Dually, the first cohomology $H^1(G; \mathbb{Z}/m) \cong (\mathbb{Z}/m)^2$ is generated by the maps α and β defined by $\alpha(x) = \beta(y) = 1$ and $\alpha(y) = \beta(x) = 0$. Furthermore, the Massey product $\langle \alpha, \beta, \alpha \rangle$ and the product $\psi := \beta \smile \langle \alpha, \beta, \alpha \rangle$ are known to be non-trivial. The equality $\psi = -\alpha \smile \langle \beta, \alpha, \beta \rangle$ is also known. Since the cup product $C^1 \otimes C^1 \rightarrow C^2$ is well described in [Tro, §2.4], the Massey product $\langle \alpha, \beta, \alpha \rangle$ can be, by definition, regarded as the map $C_2(G; \mathbb{Z}/m) \rightarrow \mathbb{Z}/m$ by setting

$$(19) \quad x^m \mapsto 0, \quad y^m \mapsto 0, \quad z^m \mapsto 0, \quad s \mapsto 0, \quad t \mapsto 0, \quad u \mapsto 2.$$

On the other hand, for simplicity, we specialize to the Seifert manifolds of type $M_{m,n} := \Sigma(0, (1, 0), (m, 1), (m, -1), (n, -1))$ over S^2 , whose fundamental groups are presented by

$$\pi_1(M_m) = \langle x_1, x_2 \mid r_1 := x_1^m(x_1^{-1}x_2^{-1})^n, r_2 := x_2^m(x_2^{-1}x_1^{-1})^n \rangle.$$

By Proposition 3.11, the identity is $W := r_2x_2r_2^{-1}x_2^{-1}r_1x_1r_1^{-1}x_1^{-1}$. We further analyze the set $\text{Hom}(\pi_1(M_{m,n}), G)$. For $a, b, c \in \mathbb{Z}/m$, consider the homomorphism $f_{a,b,c} : \pi_1(M_{m,n}) \rightarrow G$ defined by

$$f_{a,b,c}(x_1) := x^a y^b z^c, \quad f_{a,b,c}(x_2) := x^{-a} y^{-b} z^{-c+ab}.$$

It is not so hard to check the bijectivity of $(\mathbb{Z}/m)^3 \leftrightarrow \text{Hom}(\pi_1(M_{m,n}), G)$ which sends (a, b, c) to $f_{a,b,c}$. Then, the conclusion is as follows:

PROPOSITION 5.3. *Let ψ be $\beta \smile \langle \alpha, \beta, \alpha \rangle \in H^3(G, \mathbb{Z}/m)$. Let $m \in \mathbb{Z}$ be relatively prime to $6n$. Then, upon the identification $(\mathbb{Z}/m)^3 \leftrightarrow \text{Hom}(\pi_1(M_{m,n}), G)$, the Dijkgraaf-Witten invariant is equal to*

$$\text{DW}_\psi(M_{m,n}) = \sum_{(a,b,c) \in (\mathbb{Z}/m)^3} 1\{n(2abc - a(a-1)b(b-1))\} \in \mathbb{Z}[\mathbb{Z}/m].$$

PROOF. Recall from (1) that the basis of $C_2(\widetilde{M_{m,n}}) \cong \mathbb{Z}[\pi_1(M_m)]^2$ is denoted by b_1, b_2 , where b_i corresponds to the relator r_i . We now analyse $(f_{a,b,c})_*(b_1) \in C_2(G; \mathbb{Z}/m)$. We can easily check that $f_{a,b,c}(r_i)$ is transformed to $x^{am}y^{bm}z^{cm(m+1)/2}$ by the above relators s, t, u . Let us define $N_{b_i} \in \mathbb{Z}$ to be the numbers of applying u when we transform $(f_{a,b,c})(r_i)$ by $x^{ma}y^{bm}z^{cm(m+1)/2}$. Then, by (19), the pairing $\langle \langle \alpha, \beta, \alpha \rangle, (f_{a,b,c})_*(b_1) \rangle$ is equal to $2N_{b_1}$. From the definition of N_{b_1} , a little complicated computation can lead to

$$N_{b_1} = \frac{m(m+1)ac}{2} + \left(\sum_{i=1}^{m-1} \frac{ia(ia-1)}{2} \right) + nac - \frac{na(a-1)(b-1)}{2} \in \mathbb{Z}.$$

Since m is relatively prime to $6n$, we can easily check the first and second terms to be zero modulo m . Hence, using the above description of W and the formula (2), we have

$$\begin{aligned} \langle \psi, (f_{a,b,c})_*[M_{m,n}] \rangle &= 0 + b \cdot 2N_{b_1} - 0 \cdot N_{b_2} + 0 \\ &= b(2nac - na(a-1)(b-1)) \in \mathbb{Z}/m, \end{aligned}$$

which immediately leads to the conclusion. \square

The above computation is relatively simple, since so are the presentations of $\pi_1(M)$ and G ; however, a similar computation seems to be harder if $\pi_1(M)$ is complicated.

In contrast, we conclude this paper by suggesting another procedure of computing $\text{DW}_\psi(M)$, which is implicitly discussed in [No1, §4]. Hereafter $\psi \in H^3(G; A)$ may be an arbitrary 3-cocycle.

Let $C_*^{\text{nh}}(G; \mathbb{Z})$ be the normalized homogenous complex of G , which is defined as the quotient \mathbb{Z} -free module of $\mathbb{Z}[G^{n+1}]$ subject to the relation $(g_0, \dots, g_n) \sim 0$ if $g_i = g_{i+1}$ for some i ; see [Bro, 19 page]. Assume that we know an explicit expression of $\psi : G^4 \rightarrow A$ as an element of $C_{\text{nh}}^3(G, A)$. When $* \leq 3$, we now define a chain map $c_* : C_n(\widetilde{M}; \mathbb{Z}) \rightarrow C_n^{\text{nh}}(\pi_1(M); \mathbb{Z})$ as follows. Let c_0 be the identity map. Let $A \in \mathbb{Z}[\pi_1(M)]$ be any element. Define $c_1(Ax_i) := (A, Ax_i)$. If r_i is expanded as $x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_n}^{\epsilon_n}$ for some $\epsilon_k \in \{\pm 1\}$, we define

$$c_2(Ar_i) = \sum_{m: 1 \leq m \leq n} \epsilon_m(A, Ax_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_{m-1}}^{\epsilon_{m-1}} x_{i_m}^{(\epsilon_m-1)/2}, \\ Ax_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_{m-1}}^{\epsilon_{m-1}} x_{i_m}^{(\epsilon_m+1)/2}) \in C_2^{\text{nh}}(\pi_1(M); \mathbb{Z}).$$

Then, we can easily verify $\partial_1^\Delta \circ c_1 = c_0 \circ \partial_1$ and $\partial_2^\Delta \circ c_2 = c_1 \circ \partial_2$. Let $\mathcal{O}_M \in C_3(\widetilde{M}; \mathbb{Z})$ be the basis. Notice that $\partial_2^\Delta \circ c_2 \circ \partial_3(\mathcal{O}_M) = c_1 \circ \partial_2 \circ \partial_3(\mathcal{O}_M) = 0$, that is, $c_2 \circ \partial_3(\mathcal{O}_M)$ is a 2-cycle. If we expand $c_2 \circ \partial_3(\mathcal{O}_M)$ as $\sum n_i(g_0^i, g_1^i, g_2^i)$ for some $n_i \in \mathbb{Z}, g_j^i \in G$, then $\mathcal{O}'_M := -\sum n_i(1, g_0^i, g_1^i, g_2^i)$ satisfies $\partial_3^\Delta(\mathcal{O}'_M) = c_2 \circ \partial_3(\mathcal{O}_M)$. Therefore, the correspondence $\mathcal{O}_M \mapsto \mathcal{O}'_M$ gives rise to a chain map $c_3 : C_*(\widetilde{M}) \rightarrow C_*^{\text{Nor}}(\pi_1(M); \mathbb{Z})$, as desired. In conclusion, the above discussion can be summarized as follows:

PROPOSITION 5.4. *For any homomorphism $f : \pi_1(M) \rightarrow G$, the push-forward $f_* \circ \iota_*[M]$ is equal to $1 \otimes_{\pi_1(M)} f_* \circ c_3(\mathcal{O}_M)$ in $H_3^{\text{nh}}(G; \mathbb{Z})$.*

To conclude, if we know an explicit presentation of $\pi_1(M)$ and a representative of the 3-cocycle $\psi : G^4 \rightarrow A$, in principle, we can compute $\text{DW}_\psi(M)$ in terms of the chain map c_* (with the help of computer program).

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Department of Mathematics
Tokyo Institute of Technology
2-12-1 Ookayama, Meguro-ku
Tokyo 152-8551, Japan
E-mail: nosaka@math.titech.ac.jp