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Deep Learing and Hydrodynamic Limits

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Abstract

Using the properties of the Hydrodynamic limits, with the powerful recognition ability of deep learning for images, we have carried out some results on the changes from micro to macro.

深層学習と流体力学極限について

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概要

画像の深層学習の強力な認識能力を備えた流体力学極限の特性を使用して、微視的から巨視的への変化について考察した。

1 Introduction

The topic of hydrodynamic limits can be traced back to the work of J. Clerk Maxwell and L. Boltzmann, the founders of gas dynamics theory. At a time when the existence of atoms was controversial, kinetic theory could explain how the size of gas molecules could be estimated from macroscopic data, such as the viscosity of a gas. Later, D. Hilbert formulated the hydrodynamic limit problem as a mathematical problem.

So what is the hydrodynamic limits? To use a simple example, water flow can be observed from two types, one is the flow of macroscopic water, and the other is the complex and chaotic microscopic world formed by H2O particles, obviously no matter what way Observation, the water flow does not change, we can establish the collision equation describing the microscopic H2O particles, and we can also establish the equation describing the macroscopic water flow.

We first introduce the Boltzmann equation, which describes the motion and collision of microscopic particles. Its form is：

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad F|_{t=0} = F_0, \tag{1}$$

where $F(t, x, v)$ is a probability density with a given datum F_0 at $t = 0$, and x and v stand respectively for the spatial and velocity variables, and we consider here the important physical dimension three and suppose both vary in the whole space $\mathbb{R}^3$.

The bilinear operator Q on the right-hand side of (1) stands for the collision part acting only on the velocity, so the spatially inhomogeneous Boltzmann equation degenerates in x, which is one of the main difficulties in the regularity theory. In addition to the degeneracy, another major difficulty arises from the nonlocal property of the collision operator Q, which is defined for suitable functions F and G by

$$Q(G, F)(t, x, v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)(G'_* F' - G_* F) d\sigma dv_*.$$

Next we introduce the Navier-Stokes equation, named after the French engineer and physicist Claude-Louis Navier, and the Irish physicist and mathematician George Stokes, are a set of equations. Partial differential equations that describe the motion of fluids such as liquids and air. Its form is：

$$\partial_{\widetilde{t}} u + u \cdot \nabla_{\widetilde{x}} u + \nabla_{\widetilde{x}} p = \nu \triangle_{\widetilde{x}} u, \tag{2}$$

where $u(\widetilde{t}, \widetilde{x})$ is the velocity of the fluid at spatial $\widetilde{x}$ at time $\widetilde{t}$; p is the pressure; is the external force per unit volume of the fluid ν constant is the dynamic viscosity.

Next we will briefly describe the connection of these two equations and our method.

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2 Mathematical Explanation

Now we introduce the connection between the Navier–Stokes equations and the Boltzmann equation, firstly, we can naturally find that there is a difference in spatial scale between macro and micro:

$$x = \widetilde{x} \times \frac{1}{\varepsilon}.$$

Here $\varepsilon > 0$ is a constant, representing a multiple on the spatial scale.

We found that if we only scaled the size of space, the velocity on the microscopic equations would become insignificant after scaling down, so we also need to make a similar change in the scale of time, so that we can see the flow of water on a macro scale:

$$t = \widetilde{t} \times \frac{1}{\varepsilon^2}.$$

Then we must think of a question next, if we make a scale change to the microscopic Boltzmann equation, can it become a macroscopic N-S equation? The answer is yes, under certain conditions we can do it, Next we start with the scaled microscopic equations and assume：

$$\begin{aligned} \rho_\varepsilon &= \int_{\mathbb{R}^3_v} F^\varepsilon dv, \\ u_\varepsilon &= \int_{\mathbb{R}^3_v} vF^\varepsilon dv, \\ \theta_\varepsilon &= \int_{\mathbb{R}^3_v} (\frac{|v|^2 - 3}{3})F^\varepsilon dv. \end{aligned}$$

Where $F^\varepsilon(t, x, v)$ is the solution of scaled Boltzmann equation:

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \tfrac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon), \quad F^\varepsilon|_{t=0} = F_0. \tag{3}$$

After some calculations, we have the following lemma：

Lemma 1 *For the Cauchy problem* (3)*, There exist a* $\delta_0 > 0$*, let* $\{F^\varepsilon\}$ *be a family of solutions to* (3)*, if the initial data* $||F_0||_{H^2(\mathbb{R}^6_{x,v})} \le \delta_0$*, then there exists an* $F \in L^\infty([0, \infty), H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v)))$ *satisfies*

$$F^\varepsilon \to F = \rho + u \cdot v + \theta(\frac{|v|^2}{2} - \frac{3}{2}) \ (\varepsilon \to 0), \tag{4}$$

where $\rho, \theta \in L^\infty([0, \infty), H^N(\mathbb{R}^3_x))$ *and the convergence is weak-*$\star$ *for* t*, strongly in* $H^{N-\eta}\left(\mathbb{R}^3_x\right)$ *for any* $\eta > 0$*, and weakly in* $L^2(\mathbb{R}^3_v)$*,* u *is the solution of*

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \triangle_x u, \tag{5}$$

with initial data:

$$u|_{t=0} = \mathbb{P} u_0(x), \tag{6}$$

where $\mathbb{P}$ *is the Leray projection.*

3 Idea and Methods

Since the main idea of hydrodynamic limits is the scaling of time and space, then as long as we perform the same transformation, we can theoretically change from the image of microscopic equations to the image of macroscopic equations.

First, let's consider a small area in the microscopic fields.

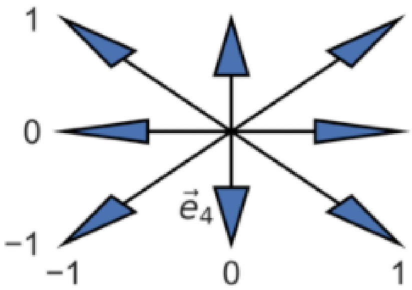

図 1: LBM velocity directions

Step 1: We use the Lattice Boltzmann Method(LBM):
We consider the 16×16 Lattice, and each small Lattice is as shown above, each lattice stores the particle distribution function f, which is used to describe the distribution of fluid particles. It is reasonable to think that adjacent regions are similar, so we can think of boundaries as reflective. $f(x,t)$ is a function of spatial position x and time t. It should be noted that the particle distribution function has one in each direction, so an LBM lattice has 9 particle distribution function, written as $f_i(x,t)$, And the direction we record as $c_i, \quad i = 0, 1, \cdots, 8$. In general, we combine the two steps of flow and collision to achieve, that is:

$$f_i(x + c_i\triangle t, t + \triangle t) = f_i(x,t) - \frac{\triangle t}{\tau_f}(f_i(x,t) - f_i^{eq}(x,t)). \tag{7}$$

Where $f_i^{eq}(x,t)$ is the distribution function in equilibrium state of flow field as follows:

$$f_i^{eq}(x,t) = \omega_i \rho(x,t)(1 + \frac{u(x,t) \cdot c_i}{c_s^2} + \frac{(u(x,t) \cdot c_i)}{2c_s^4} - \frac{u(x,t) \cdot u(x,t)}{2c_s^2}).$$

$\omega_i,\ c_s$ are weights and physical constants and ρ, u are the linear combination of f_i for $i = 0, 1, \cdots, 8$. With this formula, we can simulate the equation of motion on the microscopic fields.
Step 2: We are going to reproduce the hydrodynamic limits from microscopic to macroscopic. Although the LBM can indeed simulate the flow of water on the macroscopic level to some extent, but we know that this simple model is actually difficult to reproduce the macroscopic water flow ' s complexity, so we use some steps:
We repeat step 1 for 4 times and write down the data obtained from the step 1 as $f_i^j(x,t)$, $j = 1, \cdots, 4$. Then put these 2×2 lattice together into one big lattice:

$$f_i^{new}(x,t) = \frac{1}{4}\sum_j f_i^j(x,t).$$

Doubling the time, this formula will also multiply, but no matter what, this is always a linear term, and if it still satisfies a similar equation as (7), then it means that τ_f will change, and τ_f is closely related to the properties of the fluid.
Step 3: We want to know if we do this step 2 so that the 25 by 25 particle model produces the properties of a fluid. Finally we use the knowledge of deep learning to construct a convolution neural network for recognizing images and finding patterns in images:

- Repeat step2 multiple times to get multiple graphs of different scales;
- We download a turbulent database and make a 25×25 map;
- Use the trained neural network to test our images.

4 Main Result

Theorem 2 *The result given by our neural network is after going through step2 4 times,* 2% *pictures can be identified as having certain turbulent properties. After more iterations, the proportion soon increases to* 11%*, but after continuing to iterate many times, the percent of the pictures which can be identified as having certain turbulent properties downs to 4*

This theorem shows that our thinking is correct. Expanding the time scale and space scale will show the characteristics of the fluid, but the calculation method still needs to be improved, for the noise cannot be suppressed.

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