博士論文

Differential models for the Anderson dual to bordism theories and invertible QFT's
(ボルディズム理論の Anderson 双対の微分モデルと可逆な場の理論)

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Abstract

This thesis is devoted to a study of differential extensions of the Anderson duals $(I\Omega^G)^*$ to the stable tangential G-bordism theories. The cohomology theory $(I\Omega^G)^*$ is conjectured by Freed and Hopkins [FH21] to classify deformation classes of possibly non-topological invertible quantum field theories (QFT's). This work is motivated by this conjecture, and each of the results has the corresponding physical interpretation.

This thesis consists of the following two parts.

- In Part 1, we construct new models for the Anderson duals $(I\Omega^G)^*$ to the stable tangential G-bordism theories and their differential extensions. In a physical interpretation, an element in the differential extension plays a role of the partition function of an invertible QFT. Using these models, we construct differential refinements of module structures by bordism cohomology theories and pushforwards in $(I\Omega^G)^*$. Physically, these maps corresponds to the compactifications of QFT's.
- In Part 2, we construct transformations between differential cohomology theories which is induced by the Anderson duals to multiplicative genera. This gives us a unified understanding of an important class of elements in the Anderson duals with physical origins.

Part 1 is based on the paper [YY21] with Kazuya Yonekura. Part 2 is based on the paper [Yam21].

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Contents

Part 1.

1. Introduction to Part 1

In Part 1, we construct new models for the Anderson duals $(I\Omega^G)^*$ to the stable tangential G-bordism theories and their differential extensions. Freed and Hopkins [FH21] conjectured that the generalized cohomology theory $(I\Omega^G)^*$ classifies deformation classes of possibly non-topological invertible quantum field theories (QFT's) on stable tangential G-manifolds. Our model is motivated by this conjecture, since it is made by abstractizing certain properties of invertible QFT's.

Associated to a generalized cohomology theory E^*, its *Anderson dual* ([HS05, Appendix B], [FMS07, Appendix B]) is a generalized cohomology theory which we denote by IE^*. The crucial property of this theory is that it fits into the following exact sequence for any spectra X.

(1.1)
$$\cdots \to \mathrm{Hom}(E_{n-1}(X), \mathbb{R}) \to \mathrm{Hom}(E_{n-1}(X), \mathbb{R}/\mathbb{Z}) \to IE^n(X) \\ \to \mathrm{Hom}(E_n(X), \mathbb{R}) \to \mathrm{Hom}(E_n(X), \mathbb{R}/\mathbb{Z}) \to \cdots \text{ (exact)}.$$

In this paper we are interested in the Anderson dual to *stable tangential G-bordism theories* Ω^G. Here $G = \{G_d, s_d, \rho_d\}_{d\in\mathbb{Z}_{\geq 0}}$ is a sequence of compact Lie groups equipped with homomorphisms $s_d \colon G_d \to G_{d+1}$ and $\rho_d \colon G_d \to \mathrm{O}(d, \mathbb{R})$ for each d which are compatible with the inclusion $\mathrm{O}(d, \mathbb{R}) \hookrightarrow \mathrm{O}(d+1, \mathbb{R})$. The homology theory corresponding to Ω^G is given by the stable tangential G-bordism groups $\Omega^G_*(X)$, and the exact sequence (1.1) becomes

(1.2)
$$\cdots \to \mathrm{Hom}(\Omega^G_{n-1}(X), \mathbb{R}) \to \mathrm{Hom}(\Omega^G_{n-1}(X), \mathbb{R}/\mathbb{Z}) \to (I\Omega^G)^n(X) \\ \to \mathrm{Hom}(\Omega^G_n(X), \mathbb{R}) \to \mathrm{Hom}(\Omega^G_n(X), \mathbb{R}/\mathbb{Z}) \to \cdots \text{ (exact)}.$$

The starting point of this work is the following conjecture by Freed and Hopkins.

Conjecture 1.3 ([FH21, Conjecture 8.37])**.** *There is a* $1:1$ *correspondence*[1]

$$(1.4)\qquad \left\{\begin{array}{l}\textit{deformation classes of reflection posi-}\\ \textit{tive invertible } n\textit{-dimensional fully extended}\\ \textit{field theories with symmetry type } G\end{array}\right\} \simeq (I\Omega^G)^{n+1}(\mathrm{pt}).$$

There are many difficulties in Conjecture 1.3, and here we point out two of them. First, we do not have the axioms for non-topological fully extended QFT's. Thus the left hand side of (1.4) is not a mathematically well-defined object.[2] Second, although the cohomology theory $(I\Omega^G)^*$ is mathematically defined, its definition is abstract. So the right hand side of (1.4) is difficult to treat directly, in particular from the physical point of view. Actually, those difficulties are overcome if we are interested in *topological* QFT's, and Freed and Hopkins proves the version of Conjecture 1.3 for *topological* QFT's, where the right hand side of (1.4) is replaced by its torsion part [FH21, Theorem 1.1].

This work is intended to overcome the second difficulty mentioned above, and to give a new approach to Conjecture 1.3. We construct a physically motivated model for the theory $(I\Omega^G)^*$, which is made by abstractizing certain properties of invertible QFT's. This result can be regarded as supporting Conjecture 1.3. On the other hand, our results also turn out to be mathematically interesting, in view of its relations to *differential cohomology theories.*

In the rest of the introduction, we first explain the main results in Subsection 1.1, and then explain its physical and mathematical significances in Subsections 1.2 and 1.3, respectively.

[1] Here *symmetry types* of QFT's in [FH21] are certain classes of G's in this paper which satisfy an additional set of conditions.

[2] On the other hand, there is the axiom system by Kontsevich and Segal [KS21] for non-extended QFT's which are physically reasonable. It would be interesting to prove the modified version of Conjecture 1.3 by using it.

1.1. **A sketch of the main results.** The first main result of Part 1 is the construction of models for the generalized cohomology theory $(I\Omega^G)^*$ and its differential extension. The model of $(I\Omega^G)^*$ is denoted by $(I\Omega^G_{\mathrm{dR}})^*$, and the differential extension is denoted by $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$. Both are defined on the category MfdPair of pairs of manifolds. The precise definition is given in Definition 4.22. The physical meaning of this construction is explained in Subsection 1.2 below. For simplicity, in this subsection we concentrate on the absolute case, and we only consider the case where G is oriented, i.e., the image of $\rho_d \colon G_d \to \mathrm{O}(d, \mathbb{R})$ lies in $\mathrm{SO}(d, \mathbb{R})$ for each d.

Let X be a manifold and let n be a nonnegative integer. The differential group $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$ consists of pairs (ω, h), where

- $\omega \in \Omega^n_{\mathrm{clo}}(X; \varprojlim_d (\mathrm{Sym}^{\bullet/2}\mathfrak{g}_d^*)^{G_d})$, i.e., ω is a closed differential form on X with values in invariant polynomials on $\mathfrak{g} := \varprojlim_d \mathfrak{g}_d$, of total degree n, where $\mathfrak{g}_d$ is the Lie algebra of G_d.
- h is a map which assigns an $\mathbb{R}/\mathbb{Z}$-value to a triple (M, g, f), where M is a closed $(n-1)$-dimensional manifold with a stable tangential G-structure with connection, which we call a *differential stable tangential G-structure* and symbolically denoted by g, and a smooth map $f \colon M \to X$.
- ω and h should satisfy the following compatibility condition. Suppose we have two triples (M_-, g_-, f_-) and (M_+, g_+, f_+) as above, and a bordism (W, g_W, f_W) between them, by a compact n-dimensional manifold with differential stable tangential G-structure and a map to X. The data of g_W allows us to define a top form on W,

$$\mathrm{cw}_{g_W}(f_W^*\omega) \in \Omega^n(W), \tag{1.5}$$

  by applying the Chern-Weil construction with respect to g_W to the coefficient of $f_W^*\omega$. We require that,

$$h([M_+, g_+, f_+]) - h([M_-, g_-, f_-]) = \int_W \mathrm{cw}_{g_W}(f_W^*\omega) \pmod{\mathbb{Z}}, \tag{1.6}$$

To define $(I\Omega^G_{\mathrm{dR}})^n(X)$, we introduce the equivalence relation $\sim$ on $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$. We set $(\omega, h) \sim (\omega', h')$ if there exists $\alpha \in \Omega^{n-1}(X; \varprojlim_d (\mathrm{Sym}^{\bullet/2}\mathfrak{g}_d^*)^{G_d})$ such that

$$\begin{aligned} \omega' &= \omega + d\alpha, \\ h'([M, g, f]) &= h([M, g, f]) + \int_M \mathrm{cw}_g(f^*\alpha). \end{aligned} \tag{1.7}$$

We define

$$(I\Omega^G_{\mathrm{dR}})^n(X) := (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)/\sim. \tag{1.8}$$

For the functor $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$, we construct the structure homomorphisms R, I and a (Definition 4.26) as well as the S^1-integration map $\int$ (Definition 4.39). The first main result of this paper concerning the differential model is the following.

Theorem 1.9 (Theorem 4.56)**.** *$(I\Omega^G_{\mathrm{dR}})^*$ gives a model for the generalized cohomology theory $(I\Omega^G)^*$, restricted to the category of manifolds. Moreover, $(\widehat{I\Omega^G_{\mathrm{dR}}}, R, I, a, \int)$ is a differential extension with S^1-integration of $\big((I\Omega^G)^*, \mathrm{ch}'\big)$.*

Here the homomorphism ch' is defined in (4.5) and coincides with the Chern-Dold homomorphism for examples of G we are usually interested in.

For example, in the case $G = \mathrm{SO}$, if we have a hermitian line bundle with unitary connection (L, ∇) over X, the pair of first chern form $c_1(\nabla) = \frac{\sqrt{-1}}{2\pi} F_\nabla \in \Omega^2_{\mathrm{clo}}(X)$ and the holonomy functional with respect to ∇ gives an element $(c_1(\nabla), \mathrm{Hol}_\nabla) \in (\widehat{I\Omega^{\mathrm{SO}}_{\mathrm{dR}}})^2(X)$. See Example 4.59 for details. The compatibility condition (1.6) follows from the relation of curvature and holonomy. For more examples, see Subsection 4.3.

We remark that in this paper we mainly focus on *tangential* G-bordism theories, as opposed to *normal* G-bordism theories which we denote by $\Omega^{G^\perp}$. But a straightforward modification of the tangential case gives a model for the Anderson dual to normal G-bordism theories $(I\Omega^{G^\perp})^*$, as explained in Subsection 4.5.

With these physically-motivated models at hand, we can sometimes understand known operations in physics regarding QFT's as natural transformations between differential cohomology theories, thus giving a mathematical understanding. As we now explain, the results in Section 5 and Section 6 are such examples. In Part 2 we will see another type of natural transformations.

Assume we have a homomorphism $\mu\colon G_1 \times G_2 \to G_3$ of tangential structure groups. Typical examples arise from multiplicative tangential structure groups G such as SO and Spin, where we set $G = G_1 = G_2 = G_3$. On the topological level, the homomorphism μ induces the following.

- The natural transformation

$$(I\Omega^{G_3})^n(-) \otimes (\Omega^{G_2})^{-r}(-) \to (I\Omega^{G_1})^{n-r}(-), \tag{1.10}$$

  where $(\Omega^{G_2})^{-r}(X)$ is the stable tangential G_2-bordism *co*homology theory group.
- The pushforward maps for tangentially stably G_2-oriented proper maps $(p\colon N \to X, g_p^{\mathrm{top}})$,

$$(p, g_p^{\mathrm{top}})_*\colon (I\Omega^{G_3})^n(N) \to (I\Omega^{G_1})^{n-r}(X). \tag{1.11}$$

  We remark that our terminology "pushforward", explained in Subsection 6.1, is a certain generalization of the most usual notion of pushforwards, which is associated to multiplicative genera.

In Section 5 and Section 6, we construct differential refinements of each of the above maps, respectively. Here, for the differential refinement $(\widehat{\Omega^{G_2}})$, we use a tangential variant of the cycle-based model constructed by Bunke and Schick Schröder and Wiethaup [BSSW09], which turns out to be very much suited to our model of Anderson duals. The model $(\widehat{\Omega^{G_2}})$ is defined in terms of *differential relative stable tangential G_2-cycles*, and the differential refinements of (1.10) and (1.11) are given in terms of fiber products between differential relative stable tangential G_2-cycles and differential stable tangential G_1-cycles. As we explain in Subsubsection 1.2.3, in the physical interpretation, these homomorphisms correspond to *compactifications* of invertible QFT's.

1.2. **Physical significance.** Our results are motivated by the problem of classification of invertible field theories, and this problem is also related to the classification of action functionals of background fields up to local counterterms. Let us explain some background in physics. In the following discussion, whenever we say "manifolds", they are always supposed to be equipped with some differential structure such as Riemannian metric, bundles and their connections, and so on. What differential structure we consider should be specified in advance. Another remark is that whenever we say "invertible field theory" in this subsection, we only consider non-extended versions of QFT's unless otherwise stated, as opposed to fully extended versions of QFT's as in [FH21]. In other words, we only consider Hilbert spaces, amplitudes, and partition functions as explained below.

1.2.1. *Some backgrounds on QFT and TQFT.* Very roughly speaking, a D-dimensional QFT (which is not extended) is a functor from some geometric bordism category to the (super)vector space category as follows. A QFT assigns a Hilbert space of physical states $\mathcal{H}(N)$ to each $(D-1)$-dimensional closed manifold N. In particular, we assume that for the empty manifold $N = \varnothing$, we have a canonical isomorphism $\mathcal{H}(\varnothing) \simeq \mathbb{C}$. It assigns a linear map $Z(M) : \mathcal{H}(N_1) \to \mathcal{H}(N_2)$ to each D-dimensional compact manifold M with boundaries $\partial M = \overline{N}_1 \sqcup N_2$ where $\overline{N}_1$ is a manifold which has the opposite structure to that of N_1 (such as orientation reversal), and we have assumed that M has appropriate collar structure near the boundaries, $[0, \epsilon) \times N_1$ and $(-\epsilon, 0] \times N_2$ for some $\epsilon > 0$. We do not try to make these axioms precise, but we remark that they are motivated by (Euclidean) path integrals in physics.

An invertible field theory is a QFT in which the Hilbert space of states $\mathcal{H}(N)$ on any closed manifold N is one-dimensional, $\dim \mathcal{H}(N) = 1$. Invertible field theories play crucial roles in the study of anomalies. (See e.g. [Fre14, Mon19] for overviews.) In fact, the classification of deformation classes of invertible QFT's in D-dimensions is believed to be the same as the classification of anomalies in $(D-1)$-dimensions.[3] (We will explain what we mean by "deformation classes" in a little more detail later.) In the context of condensed matter physics, deformation classes of invertible field theories are also called symmetry protected topological (SPT) phases or invertible phases of matter. Anomalous $(D-1)$-dimensional theories appear on the boundaries of these invertible phases and have various applications in physics. Therefore, it is an important problem to classify invertible phases.

In the case of topological QFT (TQFT), the classification of invertible phases has been conjectured to be given by certain cobordism groups [Kap14, KTTW14], and later proved at least for some physically motivated classes of structure types. It is proved under the axioms of fully extended TQFT in [FH21], and Atiyah-Segal axioms in [Yon18]. Let $\mathcal{S}$ be the structure type under consideration. For instance, we can consider manifolds equipped with

[3]We neglect anomalies which do not fit into the general framework, such as Weyl anomalies. Also, there may be subtleties in reflection non-positive theories [CL20].

Spin structures, and in that case we denote $\mathcal{S} = \mathrm{Spin}$. Then we may define a bordism group $\Omega^{\mathcal{S}}_D(\mathrm{pt})$ of D-dimensional manifolds equipped with structure of the type $\mathcal{S}$ roughly as follows. We introduce a monoid structure on the set of (isomorphism classes of) manifolds by disjoint union, $M_1 \sqcup M_2$. The empty manifold $\varnothing$ is the unit of this monoid since $M \sqcup \varnothing \simeq M$. Then we divide this monoid by an equivalence relation. If a closed D-manifold M is a boundary of some $(D+1)$-manifold W, $M = \partial W$, then it is defined to be equivalent to the empty set, $M \sim \varnothing$. By using the fact that $W = [0,1] \times M$ has the boundary $\partial W = M \sqcup \overline{M}$, one can see that we get a group whose elements are represented in terms of manifolds M as $[M]$. In particular, the inverse of $[M]$ is $[\overline{M}]$. This group is denoted as $\Omega^{\mathcal{S}}_D(\mathrm{pt})$.

According to [Kap14, KTTW14, FH21, Yon18], deformation classes of invertible TQFT's are classified by the group $\mathrm{Hom}((\Omega^{\mathcal{S}}_D(\mathrm{pt}))_{\mathrm{tor}}, \mathbb{R}/\mathbb{Z})$, where the subscript tor means to take the torsion part of the group. The reason that we take the torsion part is that we are considering deformation classes. To explain this point, let us first consider the group $\mathrm{Hom}(\Omega^{\mathcal{S}}_D(\mathrm{pt}), \mathbb{R}/\mathbb{Z})$. Then the relation between this group and the above axioms of QFT is the following. If we are given an element $h \in \mathrm{Hom}(\Omega^{\mathcal{S}}_D(\mathrm{pt}), \mathbb{R}/\mathbb{Z})$, it means that we can assign to each closed D-dimensional manifold M a number

$$Z(M) = \exp\left(2\pi\sqrt{-1}h([M])\right), \tag{1.12}$$

where $[M] \in \Omega^{\mathcal{S}}_D(\mathrm{pt})$ is the bordism class represented by M. Notice that for a closed manifold $\partial M = \varnothing$, a QFT should assign a linear map $Z(M) : \mathcal{H}(\varnothing) \to \mathcal{H}(\varnothing)$. Since $\mathcal{H}(\varnothing) \simeq \mathbb{C}$, the quantity $Z(M)$ can be regarded just as a number $Z(M) \in \mathbb{C}$. The function which assigns a number $Z(M) \in \mathbb{C}$ to each closed manifold M is called a partition function in physics. From an element $h \in \mathrm{Hom}(\Omega^{\mathcal{S}}_D(\mathrm{pt}), \mathbb{R}/\mathbb{Z})$, we can construct a partition function by (1.12).

A partition function itself does not give full data for the axioms of QFT. However, the theorems proved in [FH21, Yon18] imply that we can construct a TQFT from a given $h \in \mathrm{Hom}(\Omega^{\mathcal{S}}_D(\mathrm{pt}), \mathbb{R}/\mathbb{Z})$. (See Theorem 4.3 of [Yon18] for explicit construction in the case of Atiyah-Segal axioms.) Conversely, the partition function of any invertible TQFT can be deformed continuously to a partition function given by some $h \in \mathrm{Hom}(\Omega^{\mathcal{S}}_D(\mathrm{pt}), \mathbb{R}/\mathbb{Z})$. Among the elements of $\mathrm{Hom}(\Omega^{\mathcal{S}}_D(\mathrm{pt}), \mathbb{R}/\mathbb{Z})$, the ones which come from $\mathrm{Hom}(\Omega^{\mathcal{S}}_D(\mathrm{pt}), \mathbb{R})$ can be deformed continuously. In this way, we arrive at the classification of deformation classes of invertible TQFT's by $\mathrm{Hom}((\Omega^{\mathcal{S}}_D(\mathrm{pt}))_{\mathrm{tor}}, \mathbb{R}/\mathbb{Z})$.

1.2.2. *The exact sequence and classification of invertible phases.* How about the cases which are not necessarily topological? Freed and Hopkins have conjectured a classification of fully extended QFT's [FH21] in terms of the Anderson dual of bordism groups as stated in Conjecture 1.3. The author expect that the classification of QFT's which are not extended is given by the same group. Before going to discuss it, let us first explain some additional background.

Suppose we are given a manifold X. Then we can consider a new structure given as follows. In addition to the differential structure already present in a manifold M, we consider an additional datum $f : M \to X$ which is a map from M to X. In the context of invertible phases and anomalies in physics, we can consider various types of X. If the manifold X is taken to be the target space of a sigma model, it is relevant to sigma model anomalies [MN84, MN85, Tho17]. On the other hand, if X is taken to be the space of coupling constants, it is relevant to more subtle anomalies discussed in e.g. [TY17, STY18, ST19, LMSN18, CFLS19a, CFLS19b, HKT20]. Let us denote the new structure type as $(\mathcal{S}, X)$, where $\mathcal{S}$ is the original one already considered on M. Then we denote $\Omega^{\mathcal{S}}_D(X) := \Omega^{(\mathcal{S},X)}_D(\mathrm{pt})$. For appropriate structure types $\mathcal{S}$, it is known that $\Omega^{\mathcal{S}}_*$ gives a generalized homology theory, and $\Omega^{\mathcal{S}}_*(X)$ are the generalized homology groups of X.

Given a generalized homology theory E_*, we have the Anderson dual cohomology theory IE^* satisfying the exact sequence (1.1). This exact sequence is analogous to the one in the ordinary cohomology theory associated to the short exact sequence of coefficient groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$.

The conjecture mentioned above, with a generalization including sigma models and more general types of $\mathcal{S}$, is as follows; deformation classes of invertible field theories (extended or not) with the structure type $(\mathcal{S}, X)$ is given by $(I\Omega^{\mathcal{S}})^{D+1}(X)$, where $(I\Omega^{\mathcal{S}})^*$ is the Anderson dual of $\Omega^{\mathcal{S}}_*$. It fits into the exact sequence

$$\begin{aligned}\cdots \to \mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R}) \to \mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R}/\mathbb{Z}) \to (I\Omega^{\mathcal{S}})^{D+1}(X) \\ \to \mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X), \mathbb{R}) \to \mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X), \mathbb{R}/\mathbb{Z}) \to \cdots .\end{aligned}$$

Let us explain physical reasons to believe that this conjecture is reasonable, following [LOT20]. (See also [DL20] for some applications.)

First, let us consider elements of $(I\Omega^{\mathcal{S}})^{D+1}(X)$ which are in the kernel of the map $(I\Omega^{\mathcal{S}})^{D+1}(X) \to \mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X), \mathbb{R})$. We denote the homomorphism $\mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R}) \to \mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R}/\mathbb{Z})$ as p. By the exact sequence, the kernel is isomorphic to

$$\mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R}/\mathbb{Z})/\mathrm{Im}(p) \simeq \mathrm{Hom}((\Omega^{\mathcal{S}}_D(\mathrm{pt}))_{\mathrm{tor}}, \mathbb{R}/\mathbb{Z}). \tag{1.13}$$

This is what we have discussed before in the case of TQFT. Any element of $\mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R}/\mathbb{Z})$ gives a TQFT. The division by the image of the map $p : \mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R}) \to \mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R}/\mathbb{Z})$ is due to the fact that we are considering deformation classes. The group $\mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R})$ is a vector space over $\mathbb{R}$, and any two elements of this group can be continuously deformed into one another. Therefore, we should divide $\mathrm{Hom}(\Omega^{\mathcal{S}}_D(X), \mathbb{R}/\mathbb{Z})$ by $\mathrm{Im}(p)$ when we consider deformation classes of TQFT's.

Next, let us consider the physical meaning of the map $(I\Omega^{\mathcal{S}})^{D+1}(X) \to \mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X), \mathbb{R})$. In physics, we may expect the following property of invertible QFT. Suppose that a closed D-manifold M is the boundary of a $(D+1)$-manifold W with a collar structure $(-\epsilon, 0] \times M \subset W$ near the boundary, including geometric data such that $(-\epsilon, 0]$ has the trivial differential structure. We expect to have a closed differential $(D+1)$-form I_{D+1} on W (which is sometimes called an anomaly polynomial in the context of anomalies). It is constructed from geometric data on W. For example, W may have connections of some bundles from which we can construct characteristic forms. Also, W is equipped with a map $f_W : W \to X$ and hence we can pullback differential forms from X to W by using f_W. The closed form I_{D+1} is constructed by using such differential forms, and it is given by

$\mathrm{cw}_{g_W}(f_W^*\omega)$ which appeared in (1.5) in the case of $\mathcal{S}=G$. Then, physicists may expect that the partition function of an invertible QFT evaluated on $M=\partial W$ is given (after some continuous deformation of the theory[4]) by

$$Z(\partial W) = \exp\left(2\pi\sqrt{-1}\int_W I_{D+1}\right). \tag{1.14}$$

When $I_{D+1}=0$, this equation is precisely the bordism invariance of the partition function as implied by (1.12), since $[\partial W]=[\varnothing]$ by the definition of bordism groups.

Now, by using I_{D+1}, we can define an element of $\mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X),\mathbb{R})$ by

$$\Omega^{\mathcal{S}}_{D+1}(X)\ni[W]\mapsto\int_W I_{D+1}\in\mathbb{R} \tag{1.15}$$

for any closed $(D+1)$-manifold W. Its well-definedness (i.e. it only depends on the equivalence class $[W]$ rather than a representative W) is immediate from the Stokes theorem and $dI_{D+1}=0$. Moreover, for the partition function (1.14) to be well-defined, the integral $\int_W I_{D+1}$ on any closed W must be an integer since $\partial W=\varnothing$ implies $Z(\partial W)=1$. Therefore, (1.15) is actually an element of $\mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X),\mathbb{Z})$, and, equivalently, it is in the kernel of the map $\mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X),\mathbb{R})\to\mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X),\mathbb{R}/\mathbb{Z})$. For appropriate (but not all)[5] $\mathcal{S}$, the fact that any element of $\mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X),\mathbb{Z})$ can be realized in this way by some I_{D+1} will follow from the Chern-Weil theory and the Hurewicz theorem as we will see later in the paper. This gives the exactness at $\mathrm{Hom}(\Omega^{\mathcal{S}}_{D+1}(X),\mathbb{R})$. Also, if $I_{D+1}=dJ_D$ for some J_D which is constructed by geometric data, we get $Z(\partial W)=\exp\left(2\pi\sqrt{-1}\int_{\partial W}J_D\right)$. This kind of contribution can be continuously deformed to zero by considering a one parameter family of QFT's parametrized by $t\in[0,1]$ as

$$Z_t(M) = \exp\left(2\pi\sqrt{-1}\int_M tJ_D\right), \tag{1.16}$$

so it does not contribute to the deformation classes of QFT's. This corresponds to taking the equivalence classes as in (1.8). The $\mathbb{R}/\mathbb{Z}$-valued functions h which appeared in the definition of elements of $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$ correspond to partition functions as $Z=\exp(2\pi\sqrt{-1}h)$, and $\mathrm{cw}_g(f^*\alpha)$ in (1.7) corresponds to J_D.

The author is not aware of completely general proof of the expectation that the partition function can be expressed as (1.14). However, there are various evidence supporting this claim. First, (1.14) is exactly what is used in the construction of Wess-Zumino-Witten terms [WZ71] with the target

[4]In generic theories, there can be nonuniversal terms such as the cosmological constant of the background Riemannian metric and the Euler number term. We need to eliminate them by continuous deformation of the theory for the following claim to be valid. This deformation can be done by a procedure similar to (1.16) below.

[5]The following statement fails when the Chern-Weil construction does not give an isomorphism. It happens in some noncompact groups, such as $\mathrm{SL}(2,\mathbb{R})$. Thus we have assumed that the groups G_d in this paper are compact. However, there are also groups which are noncompact but the Chern-Weil isomorphism holds. An example is $\mathrm{SL}(2,\mathbb{Z})$ which has a trivial real cohomology $H^*(B\mathrm{SL}(2,\mathbb{Z}),\mathbb{R})$. This group can also have anomalies which are physically relevant [STY18, HTY19, HTY20].

space X by extending a manifold M to W [Wit83]. In physics literature, Chern-Simons invariants are also described by (1.14). (See Example 4.61 for more precise discussions.) Second, invertible field theories constructed from massive fermions in the large mass limit satisfy (1.14). (See e.g. [WY19] for a systematic discussion). Third, other nontrivial examples of invertible QFT's also satisfy (1.14), such as the one relevant for the anomalies of chiral p-form fields [HTY20]. Finally, it is possible to give a physically reasonable derivation of a weaker version of the claim as follows. The functional derivative of the log of the partition function $\log Z(M)$ in terms of a background field ϕ (i.e. geometric data such as Riemann metric, connections, etc.) is given by a one-point function of some local operator O,

$$\frac{\partial \log Z(M)}{\partial \phi(x)} = \langle O(x) \rangle. \qquad (x \in M) \tag{1.17}$$

In theories whose low energy limits are invertible QFT's, there is no light degrees of freedom and all Feynman propagators are short range. Thus we expect that the one-point function $\langle O(x) \rangle$ is given by local geometric data at the point $x \in M$. Therefore, if two manifolds equipped differential structure, M and M', are homotopy equivalent, the ratio of their partition functions is given by an integral of some local quantity. [6]

So far, we have argued that the exact sequence satisfied by $(I\Omega^{\mathcal{S}})^{D+1}(X)$ is physically reasonable. However, the above physical arguments do not tell us anything about the group $(I\Omega^{\mathcal{S}})^{D+1}(X)$ itself beyond the exact sequence. The Anderson dual is defined in a very abstract way, and it is hard to find a direct physical interpretation of the Anderson dual. One of our main results stated in Theorem 1.9 is a natural isomorphism of the cohomology theory $I\Omega^G$ to the theory $I\Omega^G_{\mathrm{dR}}$. Here the structure type $\mathcal{S}$ is taken to be a specific kind specified by G. The cohomology theory $I\Omega^G_{\mathrm{dR}}$ is constructed in a way which closely follows the above physical discussions. Therefore, our results give a very strong support of the conjecture that the deformation classes of invertible QFT's are given by $(I\Omega^{\mathcal{S}})^{D+1}(X)$.

Although $(I\Omega^G_{\mathrm{dR}})^n(X)$ is the group which is believed to classify the deformation classes of invertible field theories, the group $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$ before taking the deformation classes is also physically very relevant. When we make background fields to dynamical fields, invertible field theories give topologically interesting terms in the action of the dynamical fields. Examples of this kind include topological θ-terms in gauge theories and Wess-Zumino-Witten terms in sigma models. The elements of $(\widehat{I\Omega^G_{\mathrm{dR}}})^{D+1}(X)$ which are realized by $\alpha \in \Omega^{n-1}(X; \varprojlim_d (\mathrm{Sym}^{\bullet/2}\mathfrak{g}_d^*)^{G_d})$ in Theorem 1.9 are, in physics language, the terms in the Lagrangian which are manifestly gauge invariant and local in D-dimensions. These terms are gauge invariant even if we evaluate them on a D-manifold M with boundaries, so they do not contribute to anomalies via anomaly inflow. But they are interesting terms in the path integral.

Let us comment on reflection positivity which is an important physical condition. In the context of partition functions, reflection positivity is the following requirement. Consider a D-manifold M with a boundary. We

[6] This argument itself also applies to the case in which the theory is not invertible but is topologically ordered. Thus this argument does not give a complete proof of the claim.

glue M and its opposite $\overline{M}$ along their boundaries to get a closed manifold $M \cup \overline{M}$ which is called a double. Reflection positivity is a requirement that $Z(M \cup \overline{M})$ is a nonnegative real number. In the case of TQFT, reflection positivity is an important ingredient in the classification of [FH21, Yon18]. Indeed, there are counterexamples to the classification if we do not impose reflection positivity. See [FH21, HTY20] for these examples. The partition function on a sphere S^D becomes negative, $Z(S^D) = -1$, although S^D can be constructed as the double of a hemisphere. These examples have the property that I_{D+1} discussed above contains $\frac{1}{2}E$, where E is the Euler density which gives the Euler number of the manifold when it is integrated. The Euler density is excluded in this paper by imposing a stability condition which we will define later in this paper. But the stabilization is not important for the types of G considered in [FH21]. We will make more comments on this point in Subsection 4.1.2.

1.2.3. *Compactification.* We can get a QFT in $(D-d)$-dimensions by compactification of another QFT in D-dimensions on a d-dimensional manifold. Let us consider a d-manifold L on which we will compactify the theory. We put a geometric structure on L. Actually, it is better to consider a family of geometric structures on L. For example, we can consider various Riemannian metrics on L and connections of a G-bundle on L, and the parameter space may be called the moduli space of geometric structures. Usually, it is enough for many purposes to consider a finite dimensional approximation to the full moduli space of geometric structures. For instance, if L is a two dimensional torus T^2, we may only consider its complex structure modulus and the total area as a finite dimensional approximation to the full moduli space of Riemannian metrics on T^2. If there is an internal symmetry group, we may also consider holonomies of connections around cycles on T^2. From the point of view of the lower dimensional theory after compactification, the space of these parameters are the target space of a new sigma model in lower dimensions. If the target space of the D-dimensional theory is X, the new target space of the $(D-d)$-dimensional theory may be $X \times \mathcal{Y}$ where $\mathcal{Y}$ is such a moduli space.

Motivated by the above considerations, let us consider a fiber bundle $N \to Y$ in which the fiber is a d-manifold L and Y is a smooth base manifold. Here, Y is what we intuitively want to consider as a moduli space of geometric structures $\mathcal{Y}$, but we remark that Y is just an arbitrary smooth manifold. We assume that each fiber is equipped with a geometric structure, and Y may be intuitively regarded as parametrizing a family of geometric structures. (We will discuss more general situations in Sec. 5 than fiber bundles, where precise definitions are also given.) If we are given an element of $(\widehat{I\Omega^G_{\mathrm{dR}}})^{D+1}(X)$, then the compactification on L with a family of geometric structures Y may be expected to give an element of $(\widehat{I\Omega^G_{\mathrm{dR}}})^{(D-d)+1}(X \times Y)$.

This is an interpretation of (1.10). By making the target spaces more explicit, it is the transformation we construct in Section 5, [7]

$$(1.18) \qquad (\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^{D+1}(X) \otimes (\widehat{\Omega^{G_2}})^{-d}(Y) \dot{\to} (\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{(D-d)+1}(X \times Y),$$

Here, the group G_2 is the structure type of fiber manifolds L, and $(\widehat{\Omega^{G_2}})^{-d}(Y)$ is roughly the abelian group of fiber bundles $N \to Y$ equipped with geometric structures of fiber manifolds. The abelian group structure is given by fiber-wise disjoint union with the fixed base Y. The group G_1 is the structure type of $(D-d)$-manifolds M, and we assume that products of manifolds (or more generally fiber bundles $L \to M' \to M$ with base M and fiber L) with the G_1 and G_2 structures can be equipped with the G_3 structure. For example, we can obtain a Pin^+ D-manifold from a product of a Pin^+ d-manifold and a Spin $(D-d)$-manifold. This means that if a theory defined on Pin^+-manifolds is compactified on a Pin^+-manifold, we get a theory defined on Spin-manifolds. See [TY21, Section 2.2.7] for another example.

We can also consider a slightly different type of compactification, which corresponds to the construction in Section 6. Let M' be the D-manifold which is the fiber bundle with the base $(D-d)$-manifold M and the fiber d-manifold L. The compactification obtained by the above procedure is restricted to the case that the map $M' \to X$ factors as $M' \to M \to X$, where $M' \to M$ is the bundle projection. For some applications, it is also useful to have a variant of the above construction. Consider a fiber bundle $p : X' \to X$ where the fiber is a d-manifold L with G_2-structure. By a similar construction as above, we get a map

$$(1.19) \qquad (\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^{D+1}(X') \to (\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{(D-d)+1}(X).$$

Here, the map $M \to X$ is uplifted to $M' \to X'$. A simple example is given by compactification of brane actions in string theory. We interpret X' as a target spacetime manifold of string theory, M' as the worldvolume of a brane, and we have a map $f' : M' \to X'$ which describes how the brane is placed in X'. Now, we compactify the target spacetime from X' to X, and "wrap" the brane to the fiber L of $X' \to X$. This situation describes the above map (1.19). The left and right hand sides describe the (imaginary part of) the brane actions before and after the compactification, respectively.

Both of (1.18) and (1.19) are useful for describing compactification of different types. There can be more general situations, but we do not discuss them in this paper.

1.3. **Mathematical significance.** In this subsection we explain our results from a mathematical point of view, especially its relation with the *differential cohomology theories*. Given a generalized cohomology theory E^*, its *differential extension* $\widehat{E}^*$, defined on manifolds, refines E^* with additional

[7] This map is in the form of the external product, which is recovered by the internal product we use in Section 5 by pulling back to $X \times Y$ and multiply there. Conversely, we can recover the internal product from the external one by pulling back by the diagonal map.

differential-geometric data. $\widehat{E}^*$ itself is also called a *generalized differential cohomology theory*. For example, $H^2(X;\mathbb{Z})$ classifies line bundles on X, whereas $\widehat{H}^2(X;\mathbb{Z})$ classifies hermitian line bundles with connections on X. See 2.2 for necessary backgrounds.

One interesting point of the construction of $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ lies in its similarity with the *differential character group* of Cheeger-Simons [CS85], which is a model for differential ordinary cohomology theory. For the ordinary cohomology $H\mathbb{Z}^*$, there are several known models for its differential extension, and the relevant one for us is the *differential character group* $\widehat{H}^*_{\mathrm{CS}}(-;\mathbb{Z})$ of Cheeger-Simons [CS85] (Example 2.14). For a manifold X, $\widehat{H}^n_{\mathrm{CS}}(X;\mathbb{Z})$ consists of pairs (ω, k), where

- A closed form $\omega \in \Omega^n_{\mathrm{clo}}(X)$,
- A group homomorphism $k\colon Z_{\infty,n-1}(X;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$,
- ω and k satisfy the following compatibility condition. For any $c \in C_{\infty,n}(X;\mathbb{Z})$ we have
$$k(\partial c) \equiv \langle c, \omega\rangle_X \pmod{\mathbb{Z}}.$$

Here $Z_{\infty,*}$ and $C_{\infty,*}$ are the groups of smooth singular cycles and chains, respectively. We immediately see that the definition of the group $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$ explained in Subsection 1.1 is analogous to that of $\widehat{H}^n_{\mathrm{CS}}(X;\mathbb{Z})$. The essential difference is the domains of h and k, which, in the physical interpretation explained in Subsection 1.2, play the roles of the partition functions of invertible QFT's. The author feels that it is interesting that we found an analogy of differential characters out of classifications of invertible QFT's.

This is actually not just the analogy, but related to the *Anderson self-duality* of the ordinary cohomology. By the universal coefficient theorem we have $H\mathbb{Z}^* \simeq IH\mathbb{Z}^*$, with the *duality element* $\gamma_H \in IH\mathbb{Z}^0(\mathrm{pt})$, equivalently, the natural transformation $\gamma_H\colon H\mathbb{Z}^* \to (I\Omega^{\mathrm{fr}})^*(= I\mathbb{Z}^*)$. The above analogy allows us to construct the differential refinement $\widehat{\gamma}^n_{\mathrm{dR}}\colon \widehat{H}^*_{\mathrm{CS}}(-;\mathbb{Z}) \to (\widehat{I\mathbb{Z}_{\mathrm{dR}}})^*(-) := (\widehat{I\Omega^{\mathrm{fr}}_{\mathrm{dR}}})^*(-)$ of the above transformation in Subsection 4.4.

1.4. **The structure of Part 1.** Part 1 is organized as follows. We provide necessary preliminaries in Section 2. In Section 3 introduce the differential cycles and related constructions which we use throughout this paper. The first main result in this paper is in Section 4. We define the differential models $I\Omega^G_{\mathrm{dR}}$ and $\widehat{I\Omega_{\mathrm{dR}}}$ in Subsection 4.1, and prove the first main result, Theorem 1.9, in Subsection 4.2. Using these models, we give a refinement of the transformation (1.10) in Section 5 and of the pushforward map (1.11) in Section 6.

1.5. **Notations and Conventions.**

- By a topological space, we always mean a compactly generated topological space.
- By *manifolds*, we mean a smooth manifold with corners (Subsection 2.3). A *pair of manifolds* (X, Y) is a manifold X and its submanifold Y, which is a closed subset of X. We set
$$C^\infty((X,Y),(X',Y')) := \{f\colon X \to X' \mid f \text{ is smooth and } f(Y) \subset Y'\}.$$

The category of pairs of manifolds are denoted by MfdPair.

- The space of $\mathbb{R}$-valued differential forms on a manifold X is denoted by $\Omega^*(X)$. For a pair of manifolds (X,Y), we set
$$\Omega^n(X,Y) := \{\omega \in \Omega^n(X) \mid \omega|_Y = 0\}.$$
- We also deal with differential forms with values in a graded real vector space $V^\bullet$. In the notation $\Omega^n(-;V^\bullet)$, n means the *total* degree. In the case if $V^\bullet$ is infinite-dimensional, we topologize it as the colimit of all its finite-dimensional subspaces with the caonical topology, and set $\Omega^n(X;V^\bullet) := C^\infty(X;(\wedge T^*X \otimes_{\mathbb{R}} V^\bullet)^n)$. This means that, any element in $\Omega^n(X;V^\bullet)$ can locally be written as a finite sum $\sum_i \xi_i \otimes \phi_i$ with $\xi_i \in \Omega^{m_i}(X)$ and $\phi_i \in V^{n-m_i}$ for some m_i for each i. The space of closed forms are denoted by $\Omega^n_{\mathrm{clo}}(-;V^\bullet)$.
- For a manifold X and a real vector space V, we denote by $\underline{V}$ the trivial bundle $\underline{V} := X \times V$ over X.
- For a topological space X, we denote by $p_X \colon X \to \mathrm{pt}$ the map to pt. We set $X^+ := (X \sqcup \{*\}, \{*\})$.
- For two topological spaces X and Y, we denote by $\mathrm{pr}_X \colon X \times Y \to X$ the projection to X.
- We set $I = [0,1]$.
- For a real vector bundle V over a topological space, we denote its orientation line bundle (rank-1 real vector bundle) by $\mathrm{Ori}(V)$. For a manifold M, we set $\mathrm{Ori}(M) := \mathrm{Ori}(TM)$.
- For a *spectrum* $\{E_n\}_{n\in\mathbb{Z}}$, we require the adjoints $E_n \to \Omega E_{n+1}$ of the structure homomorphisms are homeomorphisms. For a sequence of pointed spaces $\{E'_n\}_{n\in\mathbb{Z}_{\geq a}}$ with maps $\Sigma E'_n \to E'_{n+1}$, we define its *spectrification* $LE' := \{(LE')_n\}_{n\in\mathbb{Z}}$ to be the spectrum given by
$$(LE')_n := \varinjlim_k \Omega^k E'_{n+k}.$$

2. Preliminaries

2.1. **The Anderson duals.** In this subsection we collect basics on the Anderson duals for generalized cohomology theories. For more details, see for example [HS05, Appendix B] and [FMS07, Appendix B]. In this subsection we entirely work with spectra. The corresponding statement for CW-pairs (X,Y) is obtained by considering the suspension spectrum $\Sigma^\infty(X/Y)$. Remark that $\pi_*^{\mathrm{st}}(X,Y) = \pi_*(\Sigma^\infty(X/Y))$.

First note that the functor $X \mapsto \mathrm{Hom}(\pi_*(X), \mathbb{R}/\mathbb{Z})$ on the stable homotopy category of spectra $\mathrm{Ho}(\mathsf{Sp})^{\mathrm{op}}$ satisfies the Eilenberg-Steenrood axioms, so it is represented by an Ω-spectrum denoted by $I(\mathbb{R}/\mathbb{Z})$. We also have the functor $X \mapsto \mathrm{Hom}(\pi_*(X), \mathbb{R})$, and the corresponding Ω-spectrum $I\mathbb{R}$. By the Hurewicz isomorphism we have $\mathrm{Hom}(\pi_*(X), \mathbb{R}) = H^*(X;\mathbb{R})$ and $I\mathbb{R}$ is isomorphic to the Eilenberg-MacLane spectrum $H\mathbb{R}$. Therefore we just take $I\mathbb{R} = H\mathbb{R}$. The morphism in $\mathrm{Ho}(\mathsf{Sp})$ representing the transformation $\mathrm{Hom}(\pi_*(-), \mathbb{R}) \to \mathrm{Hom}(\pi_*(-), \mathbb{R}/\mathbb{Z})$ is denoted by
$$\pi \colon H\mathbb{R} \to I(\mathbb{R}/\mathbb{Z}). \tag{2.1}$$

Definition 2.2 (The Anderson dual to the sphere spectrum)**.** The *Anderson dual to the sphere spectrum*, $I\mathbb{Z}$, is the homotopy fiber of the map (2.1).

Applying the homotopy fiber exact sequence, for each spectrum X we get the following exact sequence.

(2.3)
$$\cdots \to H^{n-1}(X;\mathbb{R}) \xrightarrow{\pi} \mathrm{Hom}(\pi_{n-1}(X), \mathbb{R}/\mathbb{Z}) \to I\mathbb{Z}^n(X) \\ \to H^n(X;\mathbb{R}) \xrightarrow{\pi} \mathrm{Hom}(\pi_n(X), \mathbb{R}/\mathbb{Z}) \to \cdots \text{ (exact).}$$

Definition 2.4 (The Anderson dual to a spectrum)**.** Let E be a spectrum. The *Anderson dual to* E, denoted by IE, is a spectrum defined as the function spectrum from E to $I\mathbb{Z}$,

$$IE := F(E, I\mathbb{Z}).$$

This implies that we have the following exact sequence.

(2.5)
$$\cdots \to \mathrm{Hom}(E_{n-1}(X), \mathbb{R}) \xrightarrow{\pi} \mathrm{Hom}(E_{n-1}(X), \mathbb{R}/\mathbb{Z}) \to IE^n(X) \\ \to \mathrm{Hom}(E_n(X), \mathbb{R}) \xrightarrow{\pi} \mathrm{Hom}(E_n(X), \mathbb{R}/\mathbb{Z}) \to \cdots \text{ (exact).}$$

Hopkins and Singer gave a model for $I\mathbb{Z}^*(X)$ in terms of functors between Picard groupoids in [HS05, Corollary B.17]. A symmetric monoidal category is called a *Picard groupoid* if all the objects are invertible under the monoidal product and all morphisms are invertible under the composition. For example, a homomorphism $\partial\colon A \to B$ between abelian groups associates a Picard groupoid $(A \xrightarrow{\partial} B)$. Namely, the objects are elements of B, and the morphism from b to b' is given by an element $a \in A$ such that $b' - b = \partial(a)$. The monoidal structure is given by the addition. Another class of examples comes from spectra $X = \{X_n\}_{n\in\mathbb{Z}}$. The fundamental groupoid $\pi_{\leq 1}(X_n)$ of its each n-th space X_n can be equipped with a structure of a Picard groupoid, uniquely up to equivalence ([HS05, Example B.7]).

Fact 2.6 ([HS05, Corollary B.17])**.** *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a spectrum and n be an integer. We have an isomorphism*

$$I\mathbb{Z}^n(X) \simeq \pi_0 \mathrm{Fun}_{\mathrm{Pic}}\left(\pi_{\leq 1}(X_{1-n}), (\mathbb{R} \xrightarrow{\bmod \mathbb{Z}} \mathbb{R}/\mathbb{Z})\right), \tag{2.7}$$

where the right hand side means the group of natural isomorphism class of functors of Picard groupoids[8]*. The isomorphism* (2.7) *fits into the commutative diagram,*

(2.8)
$$\begin{tikzcd}
\mathrm{Hom}(\pi_{n-1}(X), \mathbb{R}/\mathbb{Z}) \arrow[r] \arrow[d, equal] & I\mathbb{Z}^n(X) \arrow[r] \arrow[d, "\simeq"] & H^n(X;\mathbb{R}) \arrow[d, equal] \\
\mathrm{Hom}(\pi_{n-1}(X), \mathbb{R}/\mathbb{Z}) \arrow[r] & \pi_0 \mathrm{Fun}_{\mathrm{Pic}}\left(\pi_{\leq 1}(X_{1-n}), (\mathbb{R} \to \mathbb{R}/\mathbb{Z})\right) \arrow[r] & H^n(X;\mathbb{R})
\end{tikzcd}$$

Here the top row is (2.3) *and the bottom row is defined in the obvious way.*

[8] In [HS05, Corollary B.17] they use $(\mathbb{Q} \to \mathbb{Q}/\mathbb{Z})$ instead of $(\mathbb{R} \to \mathbb{R}/\mathbb{Z})$. We can use the latter because the inclusion is an equivalence. Also note that there is an obvious typo in the statement of [HS05, Corollary B.17], where X_{1-n} is written as X_{n-1}

In the above model, the Picard groupoid $(\mathbb{R} \to \mathbb{R}/\mathbb{Z})$ arises because we have

$$\pi_{\le 1}(I\mathbb{Z}_1) \simeq (\mathbb{Z} \xrightarrow{0} 0) \simeq (\mathbb{R} \to \mathbb{R}/\mathbb{Z}).$$

The isomorphism (2.7) assigns, to an element $I\mathbb{Z}^n(X) = [X, \Sigma^n I\mathbb{Z}]$, the induced functor between the fundamental Picard groupoids. The commutativity of (2.8) is implicit in the discussion in [HS05, Appendix B.3], but this commutativity easily follows by the fact that the homotopy fiber sequence $\Sigma^{-1} I\mathbb{R}/\mathbb{Z} \to I\mathbb{Z} \to H\mathbb{R}$ induces the sequence $(0 \to \mathbb{R}/\mathbb{Z}) \to (\mathbb{R} \to \mathbb{R}/\mathbb{Z}) \simeq (\mathbb{Z} \to 0) \to (\mathbb{R} \to 0)$ on the fundamental Picard groupoids of their first spaces. The above model for $I\mathbb{Z}$ is crucial to the proofs of our main results of this paper.

In this paper, we are interested in the case where E is the *stable tangential G-bordism theory*. Let $G = \{G_d, s_d, \rho_d\}_{d \in \mathbb{Z}_{\ge 0}}$ be a sequence of compact Lie groups equipped with homomorphisms $s_d \colon G_d \to G_{d+1}$ and $\rho_d \colon G_d \to \mathrm{O}(d, \mathbb{R})$ for each d such that the following diagram commutes.

$$\begin{tikzcd} G_d \arrow[r, "\rho_d"] \arrow[d, "s_d"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{O}(d,\mathbb{R}) \arrow[d] \\ G_{d+1} \arrow[r, "\rho_{d+1}"] & \mathrm{O}(d+1,\mathbb{R}) \end{tikzcd} .$$

Here we use the inclusion $\mathrm{O}(d, \mathbb{R}) \hookrightarrow \mathrm{O}(d+1, \mathbb{R})$ defined by

$$A \mapsto \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right]$$

throughout this paper. We call such G *tangential structure groups*. Given G, the stable tangential G-bordism homology theory assigns the stable tangential bordism group $(\Omega^G)_*(-)$. It is represented by the *Madsen-Tillmann spectrum MTG*, which is a variant of the Thom spectrum MG. For details see for example [Fre19, Section 6.6]. In this paper we take MTG and MG to be a spectrum (as opposed to a prespectrum) as in [HS05, (4.60)]. In this case, the exact sequence (2.5) becomes

(2.9)

$$\begin{aligned} \cdots \to \mathrm{Hom}(\Omega^G_{n-1}(X), \mathbb{R}) &\xrightarrow{\pi} \mathrm{Hom}(\Omega^G_{n-1}(X), \mathbb{R}/\mathbb{Z}) \to (I\Omega^G)^n(X) \\ &\xrightarrow{\mathrm{ch}'} \mathrm{Hom}(\Omega^G_n(X), \mathbb{R}) \xrightarrow{\pi} \mathrm{Hom}(\Omega^G_n(X), \mathbb{R}/\mathbb{Z}) \to \cdots \quad \text{(exact)}. \end{aligned}$$

Example 2.10. Here are some examples of tangential structure groups G and the corresponding stable tangential G-structures.

(1) The case where $G_d = \{1\}$ for all d. Since the stable tangential G-structure is a stable tangential framing in this case, we denote this G by fr.
(2) $\mathrm{SO} := \{\mathrm{SO}(d, \mathbb{R})\}_d$ with ρ_d and s_d given by the inclusions. A stable tangential SO-structure is an orientation with a Riemannian metric.
(3) $\mathrm{Spin} := \{\mathrm{Spin}(d)\}_d$ with $\rho_d \colon \mathrm{Spin}(d) \to \mathrm{O}(d, \mathbb{R})$ given by the double covering of $\mathrm{SO}(d, \mathbb{R})$ composed with the inclusion, and s_d given by the inclusion. A stable tangential Spin-structure is a Spin-structure in the usual sense.

(4) Let H be a compact Lie group, and set $\mathrm{SO} \times H := \{\mathrm{SO}(d, \mathbb{R}) \times H\}_d$ with ρ_d given by the composition of the projection $\mathrm{SO}(d, \mathbb{R}) \times H \to \mathrm{SO}(d, \mathbb{R})$ and the inclusion, and s_d given by the inclusion. A stable tangential $\mathrm{SO} \times H$-structure is an orientation with a Riemannian metric, together with a choice of principal H-bundle. This group H is called the *internal symmetry group* in physics.

2.2. **Generalized differential cohomology theories.** In this subsection we give a brief review of generalized differential cohomology theories, based on the axiomatic framework given in [BS12] (see also [BS10]). A *differential extension* (also called a *smooth extension*) of a generalized cohomology theory E^* is a refinement $\widehat{E}^*$ of the restriction of E^* to the category of smooth manifolds, which containes differential-geometric data.

Let E^* be a generalized cohomology theory. Let $N^\bullet$ be a graded vector space over $\mathbb{R}$ equipped with a transformation of cohomology theories

$$\mathrm{ch}\colon E^* \to H^*(-; N^\bullet). \tag{2.11}$$

The universal choice is $N^\bullet = E^\bullet(\mathrm{pt}) \otimes \mathbb{R} =: V_E^\bullet$ with ch the Chern-Dold homomorphism ([Rud98, Chapter II, 7.13]) for E.

For a manifold X, set $\Omega^*(X; N^\bullet) := C^\infty(X; \wedge T^*M \otimes_{\mathbb{R}} N^\bullet)$ with the $\mathbb{Z}$-grading by the total degree. Let $d\colon \Omega^*(X; N^\bullet) \to \Omega^{*+1}(X; N^\bullet)$ be the de Rham differential. We have the natural transformation

$$\mathrm{Rham}\colon \Omega^*_{\mathrm{clo}}(X; N^\bullet) \to H^*(X; N^\bullet).$$

Definition 2.12 (Differential extensions of a cohomology theory, [BS12, Definition 2.1])**.** A *differential extension* of the pair (E^*, ch) is a quadruple $(\widehat{E}, R, I, a)$, where

- $\widehat{E}$ is a contravariant functor $\widehat{E}\colon \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$.
- R, I and a are natural transformations

$$\begin{aligned} R&\colon \widehat{E}^* \to \Omega^*_{\mathrm{clo}}(-; N^\bullet) \\ I&\colon \widehat{E}^* \to E^* \\ a&\colon \Omega^{*-1}(-; N^\bullet)/\mathrm{im}(d) \to \widehat{E}^*. \end{aligned}$$

We require the following axioms.

- $R \circ a = d$.
- $\mathrm{ch} \circ I = \mathrm{Rham} \circ R$.
- For all manifolds X, the sequence

$$E^{*-1}(X) \xrightarrow{\mathrm{ch}} \Omega^{*-1}(M; N^\bullet)/\mathrm{im}(d) \xrightarrow{a} \widehat{E}(X) \xrightarrow{I} E^*(X) \to 0 \tag{2.13}$$

  is exact.

In the case $N^\bullet = V_E^\bullet$ and ch is the Chern-Dold homomorphism, we simply call it a *differential extension of* E^*.

Such a quadruple $(\widehat{E}, R, I, a)$ itself is also called a *generalized differential cohomology theory*. We usually abbreviate the notation and just write a generalized cohomology theory as $\widehat{E}^*$.

Example 2.14 (Differential characters). Here we explain a model for a differential extension of $H\mathbb{Z}$ given by Cheeger and Simons, in terms of *differential characters* [CS85]. Actually, our definition of the group $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ (Definition 4.22) is analogous to it. We relate them in Subsection 4.4. For later use, we explain the relative version. For another formulation see [BT06].

For a pair of manifolds (X, Y) and a nonnegative integer n, the group of *differential characters* $\widehat{H}^n_{\mathrm{CS}}(X, Y; \mathbb{Z})$ is the abelian group consisting of pairs (ω, k), where

- A closed differential form $\omega \in \Omega^n_{\mathrm{clo}}(X, Y)$,
- A group homomorphism[9] $k \colon Z_{\infty,n-1}(X, Y; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$,
- ω and k satisfy the following compatibility condition. For any $c \in C_{\infty,n}(X; \mathbb{Z})$ we have

$$k(\partial c) \equiv \langle \omega, c\rangle_X \pmod{\mathbb{Z}}. \tag{2.15}$$

We have homomorphisms

$$\begin{aligned} R_{\mathrm{CS}} \colon \widehat{H}^n_{\mathrm{CS}}(X, Y; \mathbb{Z}) &\to \Omega^n_{\mathrm{clo}}(X, Y), \quad (\omega, k) \mapsto \omega \\ a_{\mathrm{CS}} \colon \Omega^{n-1}(X, Y)/\mathrm{Im}(d) &\to \widehat{H}^n_{\mathrm{CS}}(X, Y; \mathbb{Z}), \quad \alpha \mapsto (d\alpha, \alpha). \end{aligned}$$

and the quotient map gives

$$I_{\mathrm{CS}} \colon \widehat{H}^n_{\mathrm{CS}}(X, Y; \mathbb{Z}) \to \widehat{H}^n_{\mathrm{CS}}(X, Y; \mathbb{Z})/\mathrm{Im}(a_{\mathrm{CS}}) \simeq H^n(X, Y; \mathbb{Z}).$$

The quadruple $(\widehat{H}^*_{\mathrm{CS}}, R_{\mathrm{CS}}, I_{\mathrm{CS}}, a_{\mathrm{CS}})$ is a differential extension of $H\mathbb{Z}$.

We can also consider the differential refinement of S^1-integration maps in E. We have the S^1-integration map for differential forms,

$$\int \colon \Omega^{n+1}(S^1 \times X; N^\bullet) \to \Omega^n(X; N^\bullet) \tag{2.16}$$

for any manifold X, which realizes the S^1-integration map in de Rham cohomology. The sign is defined so that

$$\int \mathrm{pr}^*_{S^1}\tau_{S^1} \wedge \mathrm{pr}^*_X \omega_X = \omega_X \tag{2.17}$$

for $\omega_X \in \Omega^*(X; V)$ and $\tau_{S^1} \in \Omega^1_{\mathrm{clo}}(S^1)$ which represents the fundamental class of S^1 for the standard orientation.

Definition 2.18 (Differential extensions with S^1-integrations, [BS12, Definition 2.12])**.** A differential extension *with integration* of E^* is a quintuple $(\widehat{E}, R, I, a, \int)$, where $(\widehat{E}, R, I, a)$ is a differential extension of E^* and $\int$ is a natural transformation

$$\int \colon \widehat{E}^{*+1}(S^1 \times -) \to \widehat{E}^*$$

such that

- $\int \circ (t \times \mathrm{id})^* = -\int$, where $t \colon S^1 \to S^1$ is given by $t(x) = -x$ for $x \in [-1, 1]/\{-1, 1\} = S^1$.
- $\int \circ (p_{S^1} \times \mathrm{id})^* = 0$.

[9] Precisely speaking in [CS85] they use normalized smooth singular cubic chains, but we can also work in terms of the usual smooth singular chains as in [BT06].

- The diagram

(2.19)

$$\begin{tikzcd} \Omega^*(S^1\times X;N^\bullet) \arrow[r, "a"] \arrow[d, "\int"] & \widehat{E}^{*+1}(S^1\times X) \arrow[r, "I"] \arrow[d, "\int"] \arrow[rr, bend left, "R"] & E^{*+1}(S^1\times X) \arrow[d, "\int"] & \Omega^{*+1}_{\mathrm{clo}}(S^1\times X;N^\bullet) \arrow[d, "\int"] \\ \Omega^{*-1}(X;N^\bullet) \arrow[r, "a"] & \widehat{E}^*(X) \arrow[r, "I"] \arrow[rr, bend right, "R"'] & E^*(X) & \Omega^*_{\mathrm{clo}}(X;N^\bullet) \end{tikzcd}$$

commutes for all manifolds X.

2.3. **Manifolds with corners.** In this paper, we need to deal with manifolds with corners. There are some variants in the definition among the literatures. In this paper the appropriate notion is $\langle k\rangle$-*manifolds* defined in [Jän68], which we recall here.

A *manifold with corners* of dimension n is a paracompact topological space M with an equivalence class of local coordinate system $\{V_i,\varphi_i\}_{i\in I}$, where V_i is an open subset of M for each i with $M=\cup_i V_i$,

$$\varphi_i\colon V_i\to\mathbb{R}^n_{\le 0}:=(-\infty,0]^n$$

is a homeomorphism onto an open subset of $\mathbb{R}^n_{\le 0}$ for each i, and

$$\varphi_i\varphi_j^{-1}\colon \varphi_j(V_i\cap V_j)\to\varphi_i(V_i\cap V_j)$$

is a diffeomorphism for all pairs (i,j). In this paper, by *manifolds* we always mean manifolds with corners. For a point $x\in M$, we define the *depth* of x, denoted by $\mathrm{depth}(x)$, to be the number of zeros in a local chart. For each nonnegative integer k, we set

$$S^k(M):=\{x\in M\mid \mathrm{depth}(x)=k\}. \tag{2.20}$$

Each $S^k(M)$ has the structure of $(n-k)$-dimensional manifold without boundary. The *tangent bundle* $TM\to M$ can be defined as a vector bundle of rank n over M.

A map $f\colon M\to N$ between manifolds with corners is called *smooth* if, taking a local coordinate system $\{V_i,\varphi_i\}_{i\in I}$ and $\{U_j,\psi_j\}_{j\in J}$ for M and N respectively, the map $\psi_j\circ f\circ\varphi_i^{-1}\colon \varphi_i^{-1}f^{-1}(U_j)\to\mathbb{R}^{\dim N}_{\le 0}$ is a restriction of a smooth map between open subsets of $\mathbb{R}^{\dim M}$ to $\mathbb{R}^{\dim N}$ for each $(i,j)\in I\times J$. A smooth map f induces a vector bundle map $df\colon TM\to TN$. f is called an *embedding* if f is injective and $df\colon T_xM\to T_{f(x)}N$ is injective for all $x\in M$. In such a case, we also regard M as a subspace of N and call M a *submanifold* of N. A smooth map $f\colon M\to N$ is called a *submersion* if $df\colon T_xM\to T_{f(x)}N$ and $df\colon T_xS^{\mathrm{depth}(x)}(X)\to T_{f(x)}S^{\mathrm{depth}(f(x))}(X)$ are surjective for all $x\in N$.

A manifold with corners is called a *manifold with faces* if each $x\in M$ belongs to the closure of $\mathrm{depth}(x)$-different components of $S^1(M)$[10]. For a manifold with faces M of dimension n, the closure of a connected component of $S^1(M)$ is called a *connected face* of M, which has the induced structure of

[10] For example, this excludes the case of the "teardrop", the 2-dimensional disk with a corner.

an $(n-1)$-dimensional manifold with faces. Any union of pairwise disjoint connected faces is called a *face* of M.

Definition 2.21 ($\langle k\rangle$-manifolds, [Jän68, Definition 1])**.** Let k be a nonnegative integer. A $\langle k\rangle$*-manifold* is a manifold with faces M together with an k-tuple $(\partial_0 M, \partial_1 M, \cdots, \partial_{k-1}M)$ of faces of M, satisfying

(1) $\partial_0 M \cup \cdots \cup \partial_{k-1}M = \partial M$.
(2) $\partial_i M \cap \partial_j M$ is a face of $\partial_i M$ and of $\partial_j M$ for $i \neq j$.

In particular, a $\langle 0\rangle$-manifold is equivalent to a manifold without boundary, and a $\langle 1\rangle$-manifold is equivalent to a manifold with boundary. See Figure 1 for $k=2$. Note that each $\partial_i M$ can be empty. For $I = \{0 \le i_1 < \cdots < i_m \le k-1\}$, the intersection $\partial_I M := \partial_{i_1} M \cap \cdots \cap \partial_{i_m} M$ is called an $\langle k-m\rangle$-face of M. It has the induced structure of an $\langle k-m\rangle$-manifold so that the order of the labels of the faces are preserved.

In this paper we are particularly interested in $\langle 0\rangle$, $\langle 1\rangle$ and $\langle 2\rangle$-manifolds.

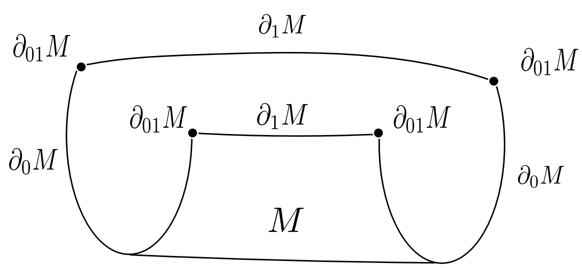

Figure 1. A $\langle 2\rangle$-manifold

3. Differential stable tangential G-structures

In this section we introduce the differential stable tnagential G-cycles and related constructions which we use throughout this paper. Let $G = \{G_d, s_d, \rho_d\}_{d \in \mathbb{Z}_{\ge 0}}$ be tangential structure groups.

Definition 3.1 (Differential stable G-structures on vector bundles)**.** Let V be a real vector bundle of rank n over a manifold M.

(1) A *representative of differential stable G-structure* on V is a quadruple $\widetilde{g} = (d, P, \nabla, \psi)$, where $d \ge n$ is an integer, (P, ∇) is a principal G_d-bundle with connection over M and $\psi \colon P \times_{\rho_d} \mathbb{R}^d \simeq \underline{\mathbb{R}}^{d-n} \oplus V$ is an isomorphism of vector bundles over M.
(2) We define the *stabilization* of such $\widetilde{g}$ by $\widetilde{g}(1) := (d+1, P(1) := P \times_{s_d} G_{d+1}, \nabla(1), \psi(1))$, where $\nabla(1)$ and $\psi(1)$ are naturally induced on $P(1)$ from ∇ and ψ, respectively.
(3) A *differential stable G-structure* g on V is a class of representatives $\widetilde{g}$ under the relation $\widetilde{g} \sim_{\mathrm{stab}} \widetilde{g}(1)$.
(4) Suppose we have two representatives of the forms $\widetilde{g} = (d, P, \nabla, \psi)$ and $\widetilde{g} = (d, P, \nabla, \psi')$, such that ψ and ψ' are homotopic. In this case, the resulting differential stable G-structures g and g' are called *homotopic*.

If we forget the information of the connection ∇, we get the corresponding notion of *(topological) differential stable G-structures*. For a differential stable G-structure g, we denote the underlying topological structure by g^{top}. Similar remarks apply to the various definitions below.

Definition 3.2 (Differential stable tangential G-structures)**.** Let M be a manifold. A *differential stable tangential G-structure* is a differential stable G-structure on the tangent bundle TM.

Definition 3.3 (Opposite differential stable tangential G-structure)**.** Let M be an n-dimensional manifold. Given a differential stable tangential G-structure g represented by $\widetilde{g} = (d, P, \nabla, \psi)$ on M with $d > n$, we define its *opposite differential stable tangential G-structure* g_{op} to be the one represented by $\widetilde{g}_{\mathrm{op}} := \left(d, P, \nabla, (\mathrm{id}_{\underline{\mathbb{R}}^{d-n-1}} \oplus -\mathrm{id}_{\underline{\mathbb{R}}} \oplus \mathrm{id}_{TM}) \circ \psi\right)$.

Definition 3.4 (Differential stable tangential G-cycles)**.** Let (X, Y) be a pair of manifolds. A *differential stable tangential G-cycle* of dimension n over (X, Y) is a triple (M, g, f), where M is an n-dimensional compact $\langle 1\rangle$-manifold, g is a differential stable tangential G-structure on M and $f \in C^\infty((M, \partial M), (X, Y))$. We define isomorphisms between two differential stable tangential G-cycles in an obvious way.

In the case $(X, Y) = \mathrm{pt}$, we use the notation $(M, g) := (M, g, p_M)$.

Definition 3.5 ($\mathcal{C}_n^{G_\nabla}(X, Y)$)**.** Let (X, Y) be a pair of manifolds. We introduce the equivalence relation $\sim$ on differential stable tangential G-cycles of dimension n over (X, Y), generated by

- Isomorphisms.
- $(M, g, f) \sqcup (M, g_{\mathrm{op}}, f) \sim \varnothing$.
- $(M, g, f) \sim (M, g', f)$ if g and g' are homotopic (Definition 3.1 (4)).

The set of equivalence classes under $\sim$ is denoted by $\mathcal{C}_n^{G_\nabla}(X, Y)$. The class of (M, g, f) is denoted by $[M, g, f]$. We introduce an abelian group structure on $\mathcal{C}_n^{G_\nabla}(X, Y)$ by disjoint union.

Next we proceed to define the bordism Picard groupoid of differential stable tangential G-cycles. Let $J \subset \mathbb{R}$ be an interval. For an M and a differential stable tangential G-structure g on it represented by $\widetilde{g} = (d, P, \nabla, \psi)$ with $d > \dim M$, we define g_J to be the differential stable tangential G-structure on $J \times M$ represented by

$$\widetilde{g}_J := (d, \mathrm{pr}_M^* P, \mathrm{pr}_M^* \nabla, \mathrm{pr}_M^* \psi). \tag{3.6}$$

Definition 3.7 (Bordism between differential stable tangential G-cycles)**.** Let (M_-, g_-, f_-) and (M_+, g_+, f_+) be two differential stable tangential G-cycles of dimension n over (X, Y). A *bordism* from (M_-, g_-, f_-) to (M_+, g_+, f_+) consists of the following set of data.

- An $(n+1)$-dimensional $\langle 2\rangle$-manifold W and a differential stable tangential G-structure g_W on W.
- A disjoint decomposition $\partial_0 W = \partial_{0,-} W \sqcup \partial_{0,+} W$ and open neighborhoods $U_\pm$ of $\partial_{0,\pm} W$, respectively, such that $U_+ \cap U_- = \varnothing$.
- Isomorphisms $\varphi_- \colon (U_-, g_W|_{U_-}) \simeq \left([0, \epsilon) \times M_-, (g_-)_{[0,\epsilon)}\right)$ and $\varphi_+ \colon (U_+, g_W|_{U_+}) \simeq \left((-\epsilon, 0] \times M_+, (g_+)_{(-\epsilon,0]}\right)$ of $\langle 2\rangle$-manifolds with differential stable tangential G-structures for some $\epsilon > 0$.

- A map $f_W \in C^\infty((W, \partial_1 W), (X, Y))$ such that $f_W|_{U_\pm} = f_\pm \circ \mathrm{pr}_{M_\pm} \circ \varphi_\pm$, respectively.

Given such a set of data, for any $0 < \epsilon' < \epsilon$ we get another set of data by restriction. We regard them as defining the same bordism. We denote a bordism data typically as (W, g_W, f_W), with the understanding that the collar structures $U_\pm$, $\varphi_\pm$ are also included.

Given (M_-, g_-, f_-) and (M_+, g_+, f_+), we introduce the bordism relation between two bordisms (W_-, g_{W_-}, f_{W_-}) and (W_+, g_{W_+}, f_{W_+}) between them in a routine way. Namely, they are called *bordant* if there exists the following set of data.

- an $(n+1)$-dimensional $\langle 3 \rangle$-manifold N equipped with a differential stable tangential G-structure g_N and $f_N \in C^\infty((N, \partial_2 W), (X, Y))$.
- a decomposition $\partial_0 N = \partial_{0,+} N \sqcup \partial_{0,-} N$ and $\partial_1 N = \partial_{1,+} N \sqcup \partial_{1,-} N$ with isomorphisms $\partial_{0,\pm} N \simeq W_\pm$ and $\partial_{1,\pm} N \simeq [0,1] \times M_\pm$ of $\langle 2 \rangle$-manifolds.
- Collar structures near $\partial_{0,\pm} N$ and $\partial_{1,\pm} N$ which restricts to the collar structures on $W_\pm$ in Definition 3.7.
- Corresponding isomorphisms of differential stable tangential G-structures on the collars, extending those on $W_\pm$ in Definition 3.7.

The bordism class of a bordism $(W, g_W, f_W)\colon (M_-, g_-, f_-) \to (M_+, g_+, f_+)$ is denoted by $[W, g_W, f_W]$

Definition 3.8 ($h\mathrm{Bord}_n^{G\nabla}(X, Y)$)**.** Let n be a positive integer and (X, Y) be a pair of manifolds. We define the symmetric monoidal category $h\mathrm{Bord}_n^{G\nabla}(X, Y)$ by the following.

- The objects are differential stable tangential G-cycles (M, g, f) of dimension n over (X, Y).
- The morphisms from (M_-, g_-, f_-) to (M_+, g_+, f_+) are the bordism classes $[W, g_W, f_W]$ of bordisms between them.
- The identity morphism on (M, g, f) is the cylinder $(I \times M, g_I, f \circ \mathrm{pr}_M)$ with the obvious collar structure.
- The monoidal product is the disjoint union, with the unit $\varnothing$.

Remark 3.9. The notation is due to the fact that the above category $h\mathrm{Bord}_n^{G\nabla}(X, Y)$ is the homotopy 1-category of an $(\infty, 0)$-category version of it. Although we do not introduce the higher categories, we use this notation.

The symmetric monoidal category $h\mathrm{Bord}_n^{G\nabla}(X, Y)$ is equivalent to the fundamental groupoid of the $(-n)$-th space of the Madsen-Tillmann spectrum via the Pontryagin-Thom construction, as follows.

Lemma 3.10. *Let (X, Y) be a pair of manifolds. There is a equivalence of symmetric monoidal categories*

$$h\mathrm{Bord}_n^{G\nabla}(X, Y) \simeq \pi_{\leq 1}(L((X/Y) \wedge MTG)_{-n}), \tag{3.11}$$

which is natural in (X, Y).

Proof. The theorem of Pontryagin-Thom implies there is an equivalence of symmetric monoidal categories which is natural in (X, Y),

$$\pi_{\leq 1}(L((X/Y) \wedge MTG)_{-n}) \simeq h\mathrm{Bord}_n^{G}(X, Y). \tag{3.12}$$

where $h\mathrm{Bord}_n^{G}(X,Y)$ is obtained from $h\mathrm{Bord}_n^{G\nabla}(X,Y)$ by forgeting the connections. In turn, the forgetful functor $h\mathrm{Bord}_n^{G\nabla}(X,Y) \to h\mathrm{Bord}_n^{G}(X,Y)$ is obviously an equivalence, so we get the result. □

By [HS05, Example B.7], the right hand side of (3.11) is a Picard groupoid. Thus we get the following.

Corollary 3.13. *The category $h\mathrm{Bord}_n^{G\nabla}(X,Y)$ is a Picard groupoid.*

Explicitly, the inverse under the monoidal product of an object (M,g,f) is (M,g_{op},f). The inverse under the composition of a morphism $[W,g_W,f_W]\colon (M_-,g_-,f_-) \to (M_+,g_+,f_+)$ is essentially given by "reversing g_W". If G is such that "reversing" $g_W \mapsto \overline{g}_W$ makes sense so that $(W,\overline{g}_W,f_W)$ is a bordism from (M_+,g_+,f_+) to (M_-,g_-,f_-), this represents the required inverse. Such G includes SO and Spin. However, for general G, for example $G=1$, such a reversal does not make sense, so the explicit description of the inverse is complicated (given by a suitable deformation of g_W on the collar). In any case, Corollary 3.13 implies that we can take the inverse $[W,g_W,f_W]^{-1}$ on the level of bordism classes.

4. Physically motivated models for the Anderson duals of G-bordisms

Let $G=\{G_d,s_d,\rho_d\}_{d\in\mathbb{Z}_{\geq 0}}$ be tangential structure groups. In this section, we give models $(I\Omega^G_{\mathrm{dR}})^*$ and $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ of the Anderson dual to G-bordism theory and its differential extension, respectively.

Let us define a G_d-module $\mathbb{R}_{G_d}$ with the underlying vector space $\mathbb{R}$, and the G_d-module structure given by the multiplication via $G_d \xrightarrow{\rho_d} \mathrm{O}(d,\mathbb{R}) \xrightarrow{\det} \{\pm 1\}$.

Lemma 4.1. *We have*

$$H^*(MTG;\mathbb{R}) = \varprojlim_d H^*(G_d;\mathbb{R}_{G_d}).$$

Here $H^*(G_d;\mathbb{R}_{G_d})$ is the group cohomology of G_d with coefficient in $\mathbb{R}_{G_d}$.

Proof. Recall that the Madsen-Tillmann spectrum MTG is defined as the direct limit of the Thom spaces of the normal bundles of the universal bundles over (approximations of) BG_d [Fre19, Section 6.6]. We see that their orientation bundle is the pullback of the bundle $EG_d\times_{G_d}\mathbb{R}_{G_d}$ over BG_d. The result follows by the Thom isomorphism and the isomorphism $H^n(BG_d;EG_d\times_{G_d}\mathbb{R}_{G_d})\simeq H^n(G_d;\mathbb{R}_{G_d})$. □

Lemma 4.2. *Fix a nonnegative integer d. For each integer n, we have*

$$\begin{aligned} H^{2n}(G_d;\mathbb{R}_{G_d}) &\simeq (\mathrm{Sym}^n\mathfrak{g}_d^*\otimes_{\mathbb{R}}\mathbb{R}_{G_d})^{G_d}, \\ H^{2n+1}(G_d;\mathbb{R}_{G_d}) &= 0. \end{aligned}$$

Here $\mathfrak{g}_d$ is the Lie algebra of G_d and $\mathfrak{g}_d^*$ is its dual. The notation $(-)^{G_d}$ means the G_d-invariant part, where G_d acts on $\mathfrak{g}_d$ by the adjoint.

Proof. Let K_d be the kernel of the homomorphism $\det\circ\rho_d\colon G_d\to\{\pm1\}$. In the case where $G_d=K_d$, the G_d-module $\mathbb{R}_{G_d}$ is trivial $\mathbb{R}$, and the desired results follow from the Chern-Weil isomorphism for G_d. In the case where $G_d \neq K_d$, apply the Hochschild-Serre spectral sequence for the group extension

$$1 \to K_d \to G_d \to \{\pm 1\} \to 1$$

and the G_d-module $\mathbb{R}_{G_d}$. We get the isomorphism $H^*(G_d; \mathbb{R}_{G_d}) \simeq (H^*(K_d;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}_{G_d})^{\{\pm 1\}} = (H^*(K_d;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}_{G_d})^{G_d}$. Using the Chern-Weil isomorphism for K_d, we get the result. □

We denote

(4.3)
$$N_G^\bullet := H^\bullet(MTG;\mathbb{R}) = \varprojlim_d H^\bullet(BG_d;\mathbb{R}_{G_d}) = \varprojlim_d (\mathrm{Sym}^{\bullet/2}\mathfrak{g}_d^* \otimes_{\mathbb{R}} \mathbb{R}_{G_d})^{G_d}.$$

In the case where G is oriented, i.e., the image of ρ_d lies in $\mathrm{SO}(d,\mathbb{R})$ for each d, the G_d-module $\mathbb{R}_{G_d}$ is trivial and $N_G^\bullet$ is the projective limit of invariant polynomials on $\mathfrak{g}_d$. In general cases, $N_G^\bullet$ can be regarded as the projective limit of polynomials on $\mathfrak{g}_d$ which change the sign by the action of G_d.

For any CW-complex X, by the Kunneth formula and the Hurewicz theorem we have

$$H^*(X; N_G^\bullet) \simeq \mathrm{Hom}(\Omega_*^G(X), \mathbb{R}), \tag{4.4}$$

so the fourth arrow in (2.9) gives the transformation

$$\mathrm{ch}' \colon (I\Omega^G)^* \to H^*(-; N_G^\bullet). \tag{4.5}$$

In general, $N_G^\bullet$ is not isomorphic to $V_{I\Omega^G}^\bullet = (I\Omega^G)^\bullet(\mathrm{pt}) \otimes \mathbb{R}$. The relation is the following. Let E be any spectrum. Applying the exact sequence (2.5) to $X = \mathrm{pt}$, we get the following short exact sequence,

$$0 \to \mathrm{Ext}(E_{n-1}(\mathrm{pt}), \mathbb{Z}) \to IE^n(\mathrm{pt}) \to \mathrm{Hom}(E_n(\mathrm{pt}), \mathbb{Z}) \to 0. \tag{4.6}$$

Since $\mathbb{R}$ is a flat $\mathbb{Z}$-module, we get the short exact sequence

$$0 \to \mathrm{Ext}(E_{n-1}(\mathrm{pt}), \mathbb{Z}) \otimes \mathbb{R} \to V_{IE}^n \to \mathrm{Hom}(E_n(\mathrm{pt}), \mathbb{Z}) \otimes \mathbb{R} \to 0. \tag{4.7}$$

The group $\mathrm{Ext}(E_{n-1}(\mathrm{pt}), \mathbb{Z}) \otimes \mathbb{R}$ does not vanish in general, but it vanishes for example if $E_{n-1}(\mathrm{pt})$ is finitely generated. Furthermore, we have a canonical map

$$\mathrm{Hom}(E_n(\mathrm{pt}), \mathbb{Z}) \otimes \mathbb{R} \to \mathrm{Hom}(E_n(\mathrm{pt}), \mathbb{R}), \tag{4.8}$$

and it is an isomorphism if $E_n(\mathrm{pt})$ is finitely generated. Thus we see that

Proposition 4.9. *For any tangential structure groups G, we have a canonical homomorphism*

$$q \colon V_{I\Omega^G}^\bullet \to N_G^\bullet. \tag{4.10}$$

It is an isomorphism if $\Omega_n^G(\mathrm{pt})$ is finitely generated for all n.

The finiteness condition in Proposition 4.9 is satisfied in examples we are usually interested in.

4.1. **The models.**

4.1.1. *The differential models.* In this subsubsection, we define the models. The definition uses a variant of Chern-Weil construction, as we now explain.

Definition 4.11. (1) Let W be a manifold and (P, ∇) be a principal G_d-bundle with connection. Let n be an even nonnegative integer and $\phi \in N_G^n$. Apply the forgetful map[11]

$$N_G^\bullet \to N_{G_d}^\bullet = (\mathrm{Sym}^{\bullet/2}\mathfrak{g}_d^* \otimes_{\mathbb{R}} \mathbb{R}_{G_d})^{G_d}. \tag{4.12}$$

to ϕ and denote by ϕ_d the resulting element. We set

$$\mathrm{cw}_\nabla(\phi) = \mathrm{cw}_\nabla(\phi_d) := \phi_d(F_\nabla) \in \Omega^n_{\mathrm{clo}}(W; P \times_{G_d} \mathbb{R}_{G_d}), \tag{4.13}$$

where $F_\nabla \in \Omega^2_{\mathrm{clo}}(W; P \times_{\mathrm{ad}} \mathfrak{g}_d)$ is the curvature form for (P, ∇), and we used the identification $\mathbb{R}_{G_d} = \mathbb{R}^*_{G_d}$ of G_d-modules. This construction gives a homomorphism $\mathrm{cw}_\nabla \colon N_G^n \to \Omega^n_{\mathrm{clo}}(W; P \times_{G_d} \mathbb{R}_{G_d})$. Extending it $\Omega^*(W)$-linearly, we get a homomorphism of $\mathbb{Z}$-graded real vector spaces,

$$\mathrm{cw}_\nabla \colon \Omega^*(W; N_G^\bullet) \to \Omega^*(W; P \times_{G_d} \mathbb{R}_{G_d}), \tag{4.14}$$

which restricts to a homomorphism $\mathrm{cw}_\nabla \colon \Omega^*_{\mathrm{clo}}(W; N_G^\bullet) \to \Omega^*_{\mathrm{clo}}(W; P \times_{G_d} \mathbb{R}_{G_d})$.

(2) If $V \to W$ is a real vector bundle with a differential stable G-structure g represented by $\widetilde{g} = (d, P, \nabla, \psi)$, the isomorphism ψ induces the isomorphism $\psi \colon P \times_{G_d} \mathbb{R}_{G_d} \simeq \mathrm{Ori}(V)$. Composing the homomorphism (4.14) with this ψ on the coefficient, we define a homomorphism of $\mathbb{Z}$-graded real vector spaces,

$$\mathrm{cw}_g := \psi \circ \mathrm{cw}_\nabla \colon \Omega^*(W; N_G^\bullet) \to \Omega^*(W; \mathrm{Ori}(V)), \tag{4.15}$$

which restricts to the homomorphism $\mathrm{cw}_g \colon \Omega^*_{\mathrm{clo}}(W; N_G^\bullet) \to \Omega^*_{\mathrm{clo}}(W; \mathrm{Ori}(V))$.

In particular, if we have a differential stable tangential G-structure g on W, the homomorphism (4.15) becomes

$$\mathrm{cw}_g \colon \Omega^*(W; N_G^\bullet) \to \Omega^*(W; \mathrm{Ori}(W)).$$

Remark 4.16. In Definition 4.11 (2), the homomorphism cw_g actually depends on g only through its *homotopy class* (Definition 3.1 (4)). This is because we only used ψ to construct the isomorphism $\psi \colon P \times_{G_d} \mathbb{R}_{G_d} \simeq \mathrm{Ori}(V)$, and it does not change when we replace ψ to a homotopic one.

Remark 4.17. Definition 4.11 admits the following generalization, which we use in Section 5 and Section 6, as well as in the next paper [Yam21]. Let $\mathcal{V}^*$ be any $\mathbb{Z}$-graded vector space over $\mathbb{R}$. We can generalize $N_G^\bullet = H^\bullet(MTG; \mathbb{R})$ in the above definition to $H^\bullet(MTG; \mathcal{V}^*)$. Then the homomorphism (4.15) becomes

$$\mathrm{cw}_g \colon \Omega^n(W; H^\bullet(MTG; \mathcal{V}^*)) \to \Omega^n(W; \mathrm{Ori}(V) \otimes_{\mathbb{R}} \mathcal{V}^*). \tag{4.18}$$

The construction is basically by just applying the above procedure $\mathcal{V}^*$-linearly. But we need some care because we need to allow $\mathcal{V}^*$ to be infinite-dimensional, since $H^\bullet(MTG_d; \mathcal{V}^*) \neq (N^{*'}_{G_d} \otimes_{\mathbb{R}} \mathcal{V}^*)^\bullet$ in general. To fix this

[11] Here we denoted by $G_d = \{(G_d)_{d'}, (s_d)_{d'}, (\rho_d)_{d'}\}_{d' \in \mathbb{Z}_{\geq 0}}$ the tangential structure groups defined by trancating G at degree d, i.e., $(G_d)_{d'} := G'_d$ for $d' \leq d$ and $(G_d)_{d'} := G_d$ for $d' \geq d$, with $(s_d)_{d'} = \mathrm{id}$ for $d' \geq d$ and the other structure maps are obvious ones.

point, in the construction corresponding Definition 4.11 (1), take a closed manifold $\mathcal{B}$ with a $(\dim W + 1)$-connected map $\mathcal{B} \to BG_d$. Then consider the pullback

(4.19)
$$H^\bullet(MTG_d; \mathcal{V}^*) \to H^\bullet(\mathcal{B}; (\mathcal{B} \times_{G_d} \mathbb{R}_{G_d}) \otimes_{\mathbb{R}} \mathcal{V}^*) = \oplus_{k=0}^{\dim \mathcal{B}} H^k(\mathcal{B}; \mathcal{B} \times_{G_d} \mathbb{R}_{G_d}) \otimes_{\mathbb{R}} \mathcal{V}^{*-k}.$$

We have $N^k_{G_d} \simeq H^k(\mathcal{B}; \mathcal{B} \times_{G_d} \mathbb{R}_{G_d})$ for $0 \le k \le \dim W$. Thus we can apply the image under the composition (4.19) to the curvature F_∇ to get the Chern-Weil form. The result does not depend on the choice of $\mathcal{B}$.

Suppose we are given a pair of manifolds (X, Y) and a form $\omega \in \Omega^*(X, Y; N^\bullet_G)$. If W is a manifold equipped with a differential stable tangential G-structure g, given a map $f \in C^\infty((W, \varnothing), (X, Y))$, the homomorphism (4.15) gives

$$\mathrm{cw}_g(f^*\omega) \in \Omega^*(W, f^{-1}(Y); \mathrm{Ori}(W)).$$

If $\omega \in \Omega^n_{\mathrm{clo}}(X, Y; N^\bullet_G)$, the resulting form is closed, so the above construction induces a homomorphism

(4.20)
$$\mathrm{cw}(\omega) \colon \mathrm{Hom}_{h\mathrm{Bord}^{G_\nabla}_{n-1}(X,Y)}((M_-, g_-, f_-), (M_+, g_+, f_+)) \to \mathbb{R}$$
$$[W, g_W, f_W] \mapsto \int_W \mathrm{cw}_{g_W}((f_W)^*\omega),$$

for each pair of objects $(M_\pm, g_\pm, f_\pm)$ in $h\mathrm{Bord}^{G_\nabla}_{n-1}(X, Y)$. In particular, applying this to $(M_\pm, g_\pm, f_\pm) = \varnothing$, we get

$$\mathrm{cw}(\omega) \colon \Omega^G_n(X, Y) \to \mathbb{R}, \quad [W, g_W, f_W] \mapsto \int_W \mathrm{cw}_{g_W}((f_W)^*\omega). \tag{4.21}$$

The map $\mathrm{Rham}(\omega) \mapsto \mathrm{cw}(\omega)$ gives the isomorphism $H^n(X, Y; N^\bullet_G) \simeq \mathrm{Hom}(\Omega^G_n(X, Y), \mathbb{R})$.

Definition 4.22 ($(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ **and** $(I\Omega^G_{\mathrm{dR}})^*$)**.** Let (X, Y) be a pair of manifolds and n be a nonnegative integer.

(1) Define $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X, Y)$ to be an abelian group consisting of pairs (ω, h), such that
   (a) ω is a closed n-form $\omega \in \Omega^n_{\mathrm{clo}}(X, Y; N^\bullet_G)$.
   (b) h is a group homomorphism $h \colon \mathcal{C}^{G_\nabla}_{n-1}(X, Y) \to \mathbb{R}/\mathbb{Z}$.
   (c) ω and h satisfy the following compatibility condition. Assume that we are given two objects (M_-, g_-, f_-) and (M_+, g_+, f_+) in $h\mathrm{Bord}^{G_\nabla}_{n-1}(X, Y)$ and a morphism $[W, g_W, f_W]$ from the former to the latter. Then we have

(4.23)
$$h([M_+, g_+, f_+]) - h([M_-, g_-, f_-]) = \mathrm{cw}(\omega)([W, g_W, f_W]) \pmod{\mathbb{Z}},$$

   where the right hand side is defined in (4.20).

Abelian group structure on $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X, Y)$ is defined in the obvious way.

(2) We define a homomorphsim of abelian groups,

$$a\colon \Omega^{n-1}(X,Y;N_G^\bullet)/\mathrm{Im}(d) \to (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X,Y) \qquad \alpha \mapsto (d\alpha, \mathrm{cw}(\alpha)). \tag{4.24}$$

Here the homomorphism $\mathrm{cw}(\alpha)\colon \mathcal{C}^{G_\nabla}_{n-1}(X,Y) \to \mathbb{R}/\mathbb{Z}$ is defined by (see Remark 4.16)

$$\mathrm{cw}(\alpha)([M,g,f]) := \int_M \mathrm{cw}_g(f^*\alpha) \pmod{\mathbb{Z}}. \tag{4.25}$$

We set

$$(I\Omega^G_{\mathrm{dR}})^n(X,Y) := (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X,Y)/\mathrm{Im}(a).$$

For negative integers n, we set $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X,Y) := 0$ and $(I\Omega^G_{\mathrm{dR}})^n(X,Y) := 0$.

For a smooth map $\phi \in C^\infty((X,Y),(X',Y'))$ between two pairs of manifolds, by the pullback we get the homomorphisms $\phi^*\colon (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X',Y') \to (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X,Y)$ and $\phi^*\colon (I\Omega^G_{\mathrm{dR}})^n(X',Y') \to (I\Omega^G_{\mathrm{dR}})^n(X,Y)$. Thus we get contravariant functors,

$$(\widehat{I\Omega^G_{\mathrm{dR}}})^*,\ (I\Omega^G_{\mathrm{dR}})^*\colon \mathrm{MfdPair}^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}.$$

In the case where $G_d = 1$ for all d, i.e., the case of the stably framed bordism theory Ω^{fr}, the corresponding Madsen-Tillmann spectrum is the sphere spectrum and $(I\Omega^{\mathrm{fr}})^* = I\mathbb{Z}^*$. So in this case, we also denote $(\widehat{I\mathbb{Z}}_{\mathrm{dR}})^* := (\widehat{I\Omega^{\mathrm{fr}}_{\mathrm{dR}}})^*$ and $(I\mathbb{Z}_{\mathrm{dR}})^* := (I\Omega^{\mathrm{fr}}_{\mathrm{dR}})^*$.

We sometimes say that "evaluate (ω,h) on $[M,g,f]$" to just mean getting the value $h([M,g,f])$.

The functor $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ is equipped with the following structure maps. In Theorem 4.56, we will see that the functors R, I and a makes $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ into a differential extension of $\big((I\Omega^G)^*, \mathrm{ch}'\big)$ where ch' is defined in (4.5).

Definition 4.26 (Structure maps for $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ and $(I\Omega^G_{\mathrm{dR}})^*$)**.** We define the following maps natural in (X,Y). The well-definedness is easy by Definition 4.22.

- We denote the quotient map by
$$I\colon (\widehat{I\Omega^G_{\mathrm{dR}}})^*(X,Y) \to (I\Omega^G_{\mathrm{dR}})^*(X,Y).$$
- We define
$$R\colon (\widehat{I\Omega^G_{\mathrm{dR}}})^*(X,Y) \to \Omega^*_{\mathrm{clo}}(X,Y;N_G^\bullet), \quad (\omega,h)\mapsto \omega.$$
- We define

$$\mathrm{ch}'\colon (I\Omega^G_{\mathrm{dR}})^*(X,Y) \to H^n(X,Y;N_G^\bullet)\left(\simeq \mathrm{Hom}(\Omega^G_*(X,Y),\mathbb{R})\right), \quad I((\omega,h)) \mapsto \mathrm{Rham}(\omega)\,(\mapsto \mathrm{cw}(\omega)).$$

- We define
$$p\colon \mathrm{Hom}(\Omega^G_{*-1}(X,Y),\mathbb{R}/\mathbb{Z}) \to (I\Omega^G_{\mathrm{dR}})^*(X,Y), \quad h \mapsto I((0,h)),$$
where we regard $h \in \mathrm{Hom}(\Omega^G_{n-1}(X,Y),\mathbb{R}/\mathbb{Z})$ as a group homomorphism $h\colon \mathcal{C}^{G_\nabla}_{n-1}(X,Y) \to \mathbb{R}/\mathbb{Z}$.

In the physical interpretation of the group $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$ explained in Subsection 1.2, an invertible $(n-1)$-dimensional QFT on manifolds equipped with differential stable tangential G-structures and maps to X gives an element $(\omega, h) \in (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$. In this picture, the value $\exp(2\pi\sqrt{-1}h([M,g,f]))$ corresponds to the value of partition function applied to $[M,g,f] \in \mathcal{C}^{G_\nabla}_{n-1}(X)$.

Now we make some easy observations on these models. An important consequence of the compatibility condition of ω and h in Definition 4.22 is the following integrality condition.

Lemma 4.27 (The integrality condition)**.** *We have*

$$\mathrm{Im}(\mathrm{ch}') \subset \mathrm{Hom}(\Omega^G_*(X,Y), \mathbb{Z}), \tag{4.28}$$

i.e., any element $(\omega, h) \in (\widehat{I\Omega^G_{\mathrm{dR}}})^*(X,Y)$ induces a $\mathbb{Z}$-valued homomorphism $\mathrm{cw}(\omega)\colon \Omega^G_*(X,Y) \to \mathbb{Z}$.

Remark 4.29. The inclusion (4.28) is actually an equality by Proposition 4.30.

Proof. This follows from the condition (c) in Definition 4.22 (1) applied to $M = \varnothing$. ☐

Now we show that $I\Omega^G_{\mathrm{dR}}$ fits into the same exact sequence as in (2.9), which is an important property of the Anderson dual.

Proposition 4.30.
For any pair of manifolds (X,Y) and integer n, the following sequence is exact.

$$\begin{aligned}\mathrm{Hom}(\Omega^G_{n-1}(X,Y), \mathbb{R}) \to \mathrm{Hom}(\Omega^G_{n-1}(X,Y), \mathbb{R}/\mathbb{Z}) \xrightarrow{p} (I\Omega^G_{\mathrm{dR}})^n(X,Y) \\ \xrightarrow{\mathrm{ch}'} \mathrm{Hom}(\Omega^G_n(X,Y), \mathbb{R}) \to \mathrm{Hom}(\Omega^G_n(X,Y), \mathbb{R}/\mathbb{Z}).\end{aligned} \tag{4.31}$$

To prove Proposition 4.30, we need the following lemma.

Lemma 4.32. *Let (X,Y) be a pair of manifolds and n be a nonnegative integer. Let $\omega \in \Omega^n_{\mathrm{clo}}(X,Y;N^\bullet_G)$ be a closed form such that the associated homomorphism $\mathrm{cw}(\omega)$ in (4.21) is $\mathbb{Z}$-valued, $\mathrm{cw}(\omega)\colon \Omega^G_n(X,Y) \to \mathbb{Z}$.*

Let (M_-, g_-, f_-) and (M_+, g_+, f_+) be two objects in $h\mathrm{Bord}^{G_\nabla}_{n-1}(X,Y)$. Given any two morphisms $[W, g_W, f_W], [W', g_{W'}, f_{W'}]\colon (M_-, g_-, f_-) \to (M_+, g_+, f_+)$, we have

$$\mathrm{cw}(\omega)([W, g_W, f_W]) = \mathrm{cw}(\omega)([W', g_{W'}, f_{W'}]) \pmod{\mathbb{Z}}. \tag{4.33}$$

Proof of Lemma 4.32. By Corollary 3.13 (also see the remarks following it), we can take the inverse $[W', g_{W'}, f_{W'}]^{-1}$ of the morphism. We have

$$\begin{aligned}&\mathrm{cw}(\omega)([W', g_{W'}, f_{W'}]) + \mathrm{cw}(\omega)([W', g_{W'}, f_{W'}]^{-1}) \\ &= \mathrm{cw}(\omega)\left([W', g_{W'}, f_{W'}] \circ [W', g_{W'}, f_{W'}]^{-1}\right) \\ &= \mathrm{cw}(\omega)\left(\mathrm{id}_{(M_+, g_+, f_+)}\right) \\ &= 0,\end{aligned}$$

by the obvious additivity of $\mathrm{cw}(\omega)$ under composition of morphisms. We also have

$$\begin{aligned}&\mathrm{cw}(\omega)([W, g_W, f_W]) + \mathrm{cw}(\omega)([W', g_{W'}, f_{W'}]^{-1})\\&= \mathrm{cw}(\omega)\left([W, g_W, f_W] \circ [W', g_{W'}, f_{W'}]^{-1}\right) \in \mathbb{Z},\end{aligned}$$

by the integrality assumption of ω. Combining these, we get Lemma 4.32. $\square$

Proof of Proposition 4.30. The composition at $\mathrm{Hom}(\Omega_n^G(X,Y),\mathbb{R})$ is zero by Lemma 4.27. The other compositions are obviously zero.

First we show the exactness at $\mathrm{Hom}(\Omega_{n-1}^G(X,Y),\mathbb{R}/\mathbb{Z})$. Suppose that $h \in \mathrm{Hom}(\Omega_{n-1}^G(X,Y),\mathbb{R}/\mathbb{Z})$ satisfies $I((0,h)) = 0$. Then there exists $\alpha \in \Omega_{\mathrm{clo}}^{n-1}(X,Y;N_G^\bullet)/\mathrm{Im}(d)$ with $h = \mathrm{cw}(\alpha)$. Since the homomorphism $\mathrm{cw}(\alpha)$ lifts to an $\mathbb{R}$-valued homomorphism defined by the same formula as (4.25), we see that h is in the image from $\mathrm{Hom}(\Omega_{n-1}^G(X,Y),\mathbb{R})$.

Next we show the exactness at $(I\Omega_{\mathrm{dR}}^G)^n(X,Y)$. Suppose that $I((\omega,h)) \in (I\Omega_{\mathrm{dR}}^G)^n(X,Y)$ satisfies $\mathrm{Rham}(\omega) = 0$. There exists $\alpha \in \Omega^{n-1}(X,Y;N_G^\bullet)$ such that $\omega = d\alpha$. Thus we have $I((\omega,h)) = I((0, h - \mathrm{cw}(\alpha)))$. This implies $p\,(h - \mathrm{cw}(\alpha)) = I((\omega,h))$.

Finally we show the exactness at $\mathrm{Hom}(\Omega_n^G(X,Y),\mathbb{R})$. It is equivalent to the claim that $\mathrm{ch}' \colon (I\Omega_{\mathrm{dR}}^G)^n(X,Y) \to \mathrm{Hom}(\Omega_n^G(X,Y),\mathbb{Z})$ is surjective. Take any element in $\mathrm{Hom}(\Omega_n^G(X,Y),\mathbb{Z}) \subset \mathrm{Hom}(\Omega_n^G(X,Y),\mathbb{R}) \simeq H^n(X,Y;N_G^\bullet)$ and take a representative $\omega \in \Omega_{\mathrm{clo}}^n(X,Y;N_G^\bullet)$. We would like to find a group homomorphism $h \colon \mathcal{C}_{n-1}^{G_\nabla}(X,Y) \to \mathbb{R}/\mathbb{Z}$ which satisfies the compatibility condition in Definition 4.22 (1) (c) with ω.

The compatibility condition with ω already determines the value of h on the kernel of the forgetful map $\mathcal{C}_{n-1}^{G_\nabla}(X,Y) \to \Omega_{n-1}^G(X,Y)$. Namely, given a differential smooth stable tangential G-cycle (M,g,f) over (X,Y) of dimension $(n-1)$ which is null-bordant, take any morphism $[W, g_W, f_W] \colon \varnothing \to (M,g,f)$ in $h\mathrm{Bord}_{n-1}^{G_\nabla}(X,Y)$ and set

$$h([M,g,f]) := \mathrm{cw}(\omega)([W,g_W,f_W]) \pmod{\mathbb{Z}}. \tag{4.34}$$

The right hand side does not depend on the choice of $[W,g_W,f_W]$ by Lemma 4.32. The fact that the formula (4.34) induces a group homomorphism h on the kernel of the forgetful map $\mathcal{C}_{n-1}^{G_\nabla}(X,Y) \to \Omega_{n-1}^G(X,Y)$ can be checked easily. Since $\mathbb{R}/\mathbb{Z}$ is an injective group, there exists a group homomorphism $h \colon \mathcal{C}_{n-1}^{G_\nabla}(X,Y) \to \mathbb{R}/\mathbb{Z}$ extending it, so we get the result. $\square$

In Subsection 4.2 we show that $I\Omega_{\mathrm{dR}}^G$ is a model for the Andeson dual to the G-bordism theory. The following result, combined with this result, implies that the quadruple $(\widehat{(I\Omega_{\mathrm{dR}}^G)}^*, R, I, a)$ is the differential extension of the pair $\left((I\Omega^G)^*, \mathrm{ch}'\right)$ with ch' in (4.5).

Proposition 4.35. *(1) We have $R \circ a = d$.*

(2) For any pair of manifolds (X, Y), the following diagram commutes.

$$\begin{tikzcd} (\widehat{I\Omega^G_{\mathrm{dR}}})^*(X,Y) \arrow[r, "R"] \arrow[d, "I"] & \Omega^*_{\mathrm{clo}}(X,Y;N^\bullet_G) \arrow[d, "\mathrm{Rham}"] \\ (I\Omega^G_{\mathrm{dR}})^*(X,Y) \arrow[r, "\mathrm{ch}'"] & H^*(X,Y;N^\bullet_G) \end{tikzcd} .$$

(3) For any pair of manifolds (X, Y), the following sequence is exact.

(4.36)

$$(I\Omega^G_{\mathrm{dR}})^{*-1}(X,Y) \xrightarrow{\mathrm{ch}'} \Omega^{*-1}(X,Y;N^\bullet_G)/\mathrm{Im}(d) \xrightarrow{a} (\widehat{I\Omega^G_{\mathrm{dR}}})^*(X,Y) \xrightarrow{I} (I\Omega^G_{\mathrm{dR}})^*(X,Y) \to 0.$$

Proof. (1) and (2) are obvious. For (3), the exactness at $\Omega^{*-1}(X,Y;N^\bullet_G)/\mathrm{Im}(d)$ easily follows from the exactness of (4.31) at $\mathrm{Hom}(\Omega^G_n(X,Y),\mathbb{R})$. The remaining parts are exact by definition. □

Our differential model $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ also has an S^1-integration. As shown in Theorem 4.56, it makes our model a *differential extension with S^1-integration* in the sense of [BS12, Definition 2.12] (Definition 2.18). Actually, as we will see in Remark 6.17, the S^1-integration map is a special case of the differential pushforwards which we introduce in Section 6.

In order to define the S^1-integration map, we need some preparation. Let M be an n-dimensional $\langle k\rangle$-manifold and g be a differential stable tangential G-structure on M represented by $\widetilde{g} = (d, P, \nabla, \psi)$ with $d \geq n+2$. For the 2-dimensional disk $D^2 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \leq 1\}$, let g_{D^2} be the differential stable tangential G-structure on $D^2 \times M$ represented by $\widetilde{g}_{D^2} := (d, \mathrm{pr}_M^*P, \mathrm{pr}_M^*\nabla, \mathrm{pr}_M^*\psi)$, where we identify $\underline{\mathbb{R}}^{d-n-2} \oplus T(D^2 \times M) \simeq \mathrm{pr}_M^*(\underline{\mathbb{R}}^{d-n-2} \oplus \underline{\mathbb{R}}^2 \oplus TM) = \mathrm{pr}_M^*(\underline{\mathbb{R}}^{d-n} \oplus TM)$. We can take the obvious collar structure near the boundary $\partial D^2 \times M$ which is induced by the polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$. Then we get an isomorphism

$$\psi_{S^1} \colon \mathrm{pr}_M^*P \times_{\rho_d} \underline{\mathbb{R}}^d \simeq \underline{\mathbb{R}}^{d-n-1} \oplus T(S^1 \times M) \tag{4.37}$$

such that $(D^2 \times M, g_{D^2}, f \circ \mathrm{pr}_M)$ is a bordism from $\varnothing$ to $(S^1 \times M, g_{S^1}, f \circ \mathrm{pr}_M)$ for any $f \colon M \to X$, where g_{S^1} is represented by $\widetilde{g}_{S^1} := (d, \mathrm{pr}_M^*P, \mathrm{pr}_M^*\nabla, \psi_{S^1})$.

Definition 4.38 (The bounding differential stable tangential structure)**.** Let M be an n-dimensional manifold and g be a differential stable tangential G-structure on M represented by $\widetilde{g} = (d, P, \nabla, \psi)$ with $d \geq n+2$. The *bounding differential stable tangential G-structure* g_{S^1} on $S^1 \times M$ is represented by

$$\widetilde{g}_{S^1} := (d, \mathrm{pr}_M^*P, \mathrm{pr}_M^*\nabla, \psi_{S^1}),$$

where ψ_{S^1} is defined in (4.37).

Definition 4.39 (The S^1-integration map for $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$)**.** Let n be a non-negative integer. We define the following map natural in (X, Y),

$$\int \colon (\widehat{I\Omega^G_{\mathrm{dR}}})^{n+1}(S^1 \times (X,Y)) \to (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X,Y),$$

by mapping (ω, h) to $(\int \omega, \int h)$, where (see Remark 4.42)

- $\int \omega$ is the image of ω under the S^1-integration of differential forms (2.16).
- We define the homomorphism $\int h \colon \mathcal{C}^{G\nabla}_{n-1}(X,Y) \to \mathbb{R}/\mathbb{Z}$ by

$$\left(\int h\right)([M,g,f]) := -h([S^1 \times M, g_{S^1}, \mathrm{id}_{S^1} \times f]). \tag{4.40}$$

  Here g_{S^1} is given by Definition 4.38.

The natural transformation $\int$ induces a natural transformation on the topological level, also denoted by

$$\int \colon (I\Omega^G_{\mathrm{dR}})^{n+1}(S^1 \times -) \to (I\Omega^G_{\mathrm{dR}})^n(-). \tag{4.41}$$

We call them the S^1*-integration map* for $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ and $(I\Omega^G_{\mathrm{dR}})^*$, respectively.

Remark 4.42. The minus sign in (4.40) is due to the fact that the assignment $g \mapsto g_{S^1}$ does not preserve the bordism relation. Rather, we need an additional automorphism on $\underline{\mathbb{R}}^{d-n-1} \oplus T(S^1 \times M)$ for objects which reverses the orientation. Also we have

$$\mathrm{cw}_{g_W}\left(\int \omega\right) = \int \mathrm{cw}_{(g_W)_{S^1}}(\omega). \tag{4.43}$$

Using these, the compatibility of the pair $(\int \omega, \int h)$ can be checked easily.

4.1.2. *The models in terms of equivalent Picard subgroupoids of* $h\mathrm{Bord}^{G\nabla}_{-}(-)$. In the definition of the model $I\Omega^G_{\mathrm{dR}}$ so far, we have used the Picard groupoid $h\mathrm{Bord}^{G\nabla}_{-}(-)$. However, in some cases the partition function h is naturally defined only for objects of a Picard subgroupoid of it, and it has enough information to define an element in $I\Omega^G_{\mathrm{dR}}$.

Let $\mathcal{D} \subset h\mathrm{Bord}^{G\nabla}_{n-1}(X,Y)$ be a Picard subcategory such that the inclusion is an equivalence. We define $\mathcal{C}_{\mathcal{D}} \subset \mathcal{C}^{G\nabla}_{n-1}(X,Y)$ to be the subgroup generated by the isomorphism classes of objects in $\mathcal{D}$. Then, consider the following group.

Definition 4.44 ($(\widehat{I\Omega^G_{\mathrm{dR},\mathcal{D}}})^n(X,Y)$ and $(I\Omega^G_{\mathrm{dR},\mathcal{D}})^n(X,Y)$)**.** In the above settings, we define $(\widehat{I\Omega^G_{\mathrm{dR},\mathcal{D}}})^n(X,Y)$ and $(I\Omega^G_{\mathrm{dR},\mathcal{D}})^n(X,Y)$ to be the abelian groups defined by replacing $h\mathrm{Bord}^{G\nabla}_{n-1}(X,Y)$ with $\mathcal{D}$ and $\mathcal{C}^{G\nabla}_{n-1}(X,Y)$ with $\mathcal{C}_{\mathcal{D}}$ in Definition 4.22.

Thus, an element in $(\widehat{I\Omega^G_{\mathrm{dR},\mathcal{D}}})^n(X,Y)$ is a pair $(\omega, h_{\mathcal{D}})$ where $\omega \in \Omega^n_{\mathrm{clo}}(X,Y;N^\bullet_G)$ as before, but the domain of $h_{\mathcal{D}}$ is now smaller, $h_{\mathcal{D}} \colon \mathcal{C}_{\mathcal{D}} \to \mathbb{R}/\mathbb{Z}$. We show that the resulting groups are isomorphic.

Proposition 4.45. *The obvious forgetful maps by the restriction of* h*,*

$$\mathrm{fgt} \colon (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X,Y) \to (\widehat{I\Omega^G_{\mathrm{dR},\mathcal{D}}})^n(X,Y) \tag{4.46}$$

$$\mathrm{fgt} \colon (I\Omega^G_{\mathrm{dR}})^n(X,Y) \to (I\Omega^G_{\mathrm{dR},\mathcal{D}})^n(X,Y) \tag{4.47}$$

are isomorphisms.

Proof. We can construct the inverse of (4.46) easily as follows. Given an element $(\omega, h_{\mathcal{D}}) \in (\widehat{I\Omega^G_{\mathrm{dR},\mathcal{D}}})^n(X,Y)$, we need to extend $h_{\mathcal{D}}$ to $h\colon \mathcal{C}^{G_\nabla}_{n-1}(X,Y) \to \mathbb{R}/\mathbb{Z}$. Since $\mathcal{D} \hookrightarrow h\mathrm{Bord}^{G_\nabla}_{n-1}(X,Y)$ is an equivalence, any object in $h\mathrm{Bord}^{G_\nabla}_{n-1}(X,Y)$ is bordant to an object in $\mathcal{D}$. By the compatibility condition in Definition 4.22 (1) (c), we are forced to define the value of h using such a bordism. The well-definedness follows from the fact that $\mathcal{D} \hookrightarrow h\mathrm{Bord}^{G_\nabla}_{n-1}(X,Y)$ is full, and the compatibility of $(\omega, h_{\mathcal{D}})$. It is obvious that this assignment gives the inverse of (4.46). The result for (4.47) also follows from this. $\square$

A typical class of examples of such situations is the following. In the context of unitary QFT's in physics, we are usually interested in G with the following properties. First we require that the image of ρ_d contains at least $\mathrm{SO}(d,\mathbb{R})$,

$$\mathrm{SO}(d,\mathbb{R}) \subset \rho_d(G_d). \tag{4.48}$$

Next we require that the following commutative diagram is a pullback diagram,

$$\begin{tikzcd} G_d \arrow[r, "\rho_d"] \arrow[d, "s_d"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{O}(d,\mathbb{R}) \arrow[d] \\ G_{d+1} \arrow[r, "\rho_{d+1}"] & \mathrm{O}(d+1,\mathbb{R}) \end{tikzcd} . \tag{4.49}$$

In Example 2.10, (2), (3) and (4) satisfy these assumtions, but (1) does not. For G satisfying these properties, we define

Definition 4.50. Let M be an n-dimensional manifold. A *physical tangential G-structure* on M is a triple $g_{\mathrm{ph}} = (P, \nabla, \psi)$, where

- The quadruple (n, P, ∇, ψ) is a representative of differential stable G-structure (Definition 3.1) for TM. In particular, there is no stabilization of TM.
- We have a Riemannian metric on TM induced from the standard metric on $P \times_{\rho_d} \mathbb{R}^{\dim M}$ by the isomorphism $\psi : P \times_{\rho_d} \mathbb{R}^{\dim M} \simeq TM$. The connection induced on $P \times_{\rho_d} \mathbb{R}^{\dim M} \simeq TM$ from ∇ coincides with the Levi-Civita connection of the Riemannian metric.

A physical tangential G-structure can be regarded as a differential stable tangential G-structure in the obvious way. Then, we define $h\mathrm{Bord}^{G_{\mathrm{ph}}}_{n-1}(X,Y)$ to be the full subcategory of $h\mathrm{Bord}^{G_\nabla}_{n-1}(X,Y)$ spanned by the objects with physical tangential G-structures. It is a standard fact that the inclusion is an equivalence, by the requirements (4.48) and (4.49). We remark that any morphism in $h\mathrm{Bord}^{G_{\mathrm{ph}}}_{n-1}(X,Y)$ can be represented by a bordism with physical tangential G-structure by the same reason.

We often encounter such situations. For example in [FH21], they require the conditions (4.48) and (4.49) in the definition of "symmetry types". Also see Examples 4.67 and 4.69 below. Since they are so typical, we use the notations $(\widehat{I\Omega^G_{\mathrm{ph}}})^*$ and $(I\Omega^G_{\mathrm{ph}})^*$ for the groups in Definition 4.44 in the case $\mathcal{D} = h\mathrm{Bord}^{G_{\mathrm{ph}}}_{-}(-)$.

We also encounter another type of $\mathcal{D}$ in Section 5. There, we use Picard subgroupoids spanned by objects (M, g, f) such that f satisfies certain transversality conditions.

4.2. **The proof of the isomorphism $(I\Omega^G_{\mathrm{dR}})^* \simeq (I\Omega^G)^*$.** In this subsection we prove the main result of this section, Theorem 4.56.

First we relate our models with functors from the bordism Picard categories. Recall that, as explained in Subsection 2.1, a homomorphism $\partial\colon A \to B$ between abelian groups associates a Picard groupoid $(A \xrightarrow{\partial} B)$. Given an element $(\omega, h) \in (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X, Y)$, we get the associated functor of Picard groupoids,

$$F_{(\omega,h)}\colon h\mathrm{Bord}^{G_\nabla}_{n-1}(X, Y) \to (\mathbb{R} \to \mathbb{R}/\mathbb{Z}) \tag{4.51}$$

by $F_{(\omega,h)}(M, g, f) := h([M, g, f])$ on objects and $F_{(\omega,h)}([W, g_W, f_W]) := \mathrm{cw}(\omega)([W, g_W, f_W])$ on morphisms. Moreover, given two elements (ω, h) and (ω', h'), and an element $\alpha \in \Omega^{n-1}(X, Y; N^\bullet_G)/\mathrm{Im}(d)$ so that $(\omega', h') - (\omega, h) = a(\alpha)$, we get the associated natural transformation,

$$F_\alpha\colon F_{(\omega,h)} \Rightarrow F_{(\omega',h')}, \tag{4.52}$$

by $F_\alpha(M, g, f) := \mathrm{cw}(\alpha)(M, g, f)$. Summarizing, we get the following.

Lemma 4.53. *The assignment (4.51) and (4.52) gives a symmetric monoidal functor*

(4.54)

$$F_{(X,Y)}\colon \left(\Omega^{n-1}(X, Y; N^\bullet_G)/\mathrm{Im}(d) \xrightarrow{a} (\widehat{I\Omega^G_{\mathrm{dR}}})^n(X, Y)\right) \to \mathrm{Fun}_{\mathrm{Pic}}\left(h\mathrm{Bord}^{G_\nabla}_{n-1}(X, Y), (\mathbb{R} \to \mathbb{R}/\mathbb{Z})\right)$$

which is natural in (X, Y). Here $\mathrm{Fun}_{\mathrm{Pic}}$ is regarded as a symmetric monoidal category. In particular, passing to the isomorphism classes of objects, we get the following natural transformation of functors $\mathrm{MfdPair}^{\mathrm{op}} \to \mathrm{Ab}$,

$$F\colon (I\Omega^G_{\mathrm{dR}})^n(-) \to \pi_0\mathrm{Fun}_{\mathrm{Pic}}\left(h\mathrm{Bord}^{G_\nabla}_{n-1}(-), (\mathbb{R} \to \mathbb{R}/\mathbb{Z})\right). \tag{4.55}$$

Now we show the main result of this section.

Theorem 4.56. *There is a natural isomorphism of the functors $\mathrm{MfdPair}^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$,*

$$F\colon I\Omega^G_{\mathrm{dR}} \simeq I\Omega^G,$$

which fits into the following commutative diagram.

(4.57)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(\Omega^G_{n-1}(-), \mathbb{Z}) \arrow[r, "p"] \arrow[d, equal] & (I\Omega^G_{\mathrm{dR}})^n \arrow[r, "\mathrm{ch}'"] \arrow[d, "F", "\simeq"'] & \mathrm{Hom}(\Omega^G_n(-), \mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ext}(\Omega^G_{n-1}(-), \mathbb{Z}) \arrow[r] & (I\Omega^G)^n \arrow[r] & \mathrm{Hom}(\Omega^G_n(-), \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Moreover, the quintuple $(\widehat{I\Omega^G_{\mathrm{dR}}}, R, I, a, \int)$ in Definitions 4.26 and 4.39 is a differential extension of $\left((I\Omega^G)^*, \mathrm{ch}'\right)$ with integration, where ch' is defined in (4.5). In particular, if $\Omega^G_n(\mathrm{pt})$ is finitely generated for all n, it gives a differential extension with integration, with respect to the Chern-Dold homomorphism $\mathrm{ch}\colon (I\Omega^G)^* \to H^*(-; V^\bullet_{I\Omega^G})$.

Proof. With Lemma 4.53 in hand, the proof is essentially the same as a part of the proof of [HS05, Proposition 5.24]. By Fact 2.6, there is an isomorphism

$$(I\Omega^G)^n(X,Y) \simeq \pi_0 \mathrm{Fun}_{\mathrm{Pic}}(\pi_{\le 1}(L((X/Y)\wedge MTG)_{1-n}), (\mathbb{R}\to\mathbb{R}/\mathbb{Z})) \tag{4.58}$$

natural in (X,Y). Combining the isomorphism (4.58) and Lemma 3.10 with the transformation (4.55), we get the natural transformation $F\colon I\Omega^G_{\mathrm{dR}} \simeq I\Omega^G$. Moreover, by construction it makes the diagram (4.57) commutative. Evaluating on each (X,Y), the bottom row of (4.57) is exact by (2.9), and the top row is also exact by Proposition 4.30. By the five lemma, we see that F gives the desired natural isomorphism.

For the remaining statement, the fact that $(\widehat{I\Omega^G_{\mathrm{dR}}}, R, I, a)$ is a differential extension follows from Proposition 4.35. For the S^1-integration $\int$, as we will see in Remark 6.17, $\int$ is a special case of the differential pushforwards in Section 6. The statement on the S^1-integration follows by Theorem 6.12. But we also remark that it is easy to give a direct proof in this case, using the fact that the bounding stable fr-structure g_{S^1} on $S^1 = S^1\times \mathrm{pt}$ in Definition 4.38 defines the element in $\Omega^{\mathrm{fr}}_1(S^1)$ which maps to the suspension of the unit in $\Omega^{\mathrm{fr}}_1(S^1,\mathrm{pt})$ and maps to the trivial element in $\Omega^{\mathrm{fr}}_1(\mathrm{pt})$. The last statement follows from Proposition 4.9. This completes the proof. ☐

4.3. **Examples of elements in $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$.** In this subsection, we give examples of elements in $(\widehat{I\Omega^G_{\mathrm{dR}}})^*(-)$ along with the corresponding invertible QFT's. In this subsection we only list examples. In the subsequent paper [Yam21] we will give topological characterization of some of the examples.

Example 4.59 (The holonomy theory (1)). In this example we consider $G = \mathrm{SO}$. Fix a manifold X and a hermitian line bundle with unitary connection (L,∇) over X. Then we get an element

$$(c_1(\nabla), \mathrm{Hol}_\nabla) \in (\widehat{I\Omega^{\mathrm{SO}}_{\mathrm{dR}}})^2(X).$$

Here,

- $c_1(\nabla) = \frac{\sqrt{-1}}{2\pi}F_\nabla \in \Omega^2_{\mathrm{clo}}(X)$ is the first Chern form of ∇. Identifying $\mathbb{R}$ with the degree-zero component of $N^\bullet_{\mathrm{SO}}$, we regard $\Omega^2_{\mathrm{clo}}(X) \subset \Omega^2_{\mathrm{clo}}(X; N^\bullet_{\mathrm{SO}})$.
- The homomorphism $\mathrm{Hol}_\nabla\colon \mathcal{C}^{\mathrm{SO}_\nabla}_1(X) \to \mathbb{R}/\mathbb{Z}$ is given by the holonomy along the closed curve in X. More precisely, an element $[M,g,f]$ in $\mathcal{C}^{\mathrm{SO}_\nabla}_1(X)$ consists of a closed oriented one-dimensional manifold M with a map $f\in C^\infty(M,X)$, together with additional information on metric and connection. Regarding it just as an oriented closed curve in X, we define $\mathrm{Hol}_\nabla([M,g,f])$ to be the holonomy of (L,∇) along the curve, by identifying $\mathbb{R}/\mathbb{Z}\simeq \mathrm{U}(1)$.

Example 4.60 (The holonomy theory (2)). In this example we consider $G = \mathrm{SO}\times\mathrm{U}(1)$. Here $\mathrm{U}(1)$ is the *internal symmetry group* explained in Example 2.10 (4). We have an element

$$(1\otimes c_1, \mathrm{Hol}) \in (\widehat{I\Omega^{\mathrm{SO}\times\mathrm{U}(1)}_{\mathrm{dR}}})^2(\mathrm{pt}).$$

Here,

- We have $N^\bullet_{\mathrm{SO}\times\mathrm{U}(1)} = \left(\varprojlim_d (\mathrm{Sym}(\mathfrak{so}(d,\mathbb{R}))^*)^{\mathrm{SO}(d;\mathbb{R})} \otimes_{\mathbb{R}} (\mathrm{Sym}(\mathfrak{u}(1))^*)^{\mathrm{U}(1)}\right)^\bullet$. The first Chern polynomial $c_1 \in \left((\mathrm{Sym}(\mathfrak{u}(1))^*)^{\mathrm{U}(1)}\right)^2$ gives the element $1 \otimes c_1 \in \Omega^2_{\mathrm{clo}}(\mathrm{pt}; N^\bullet_{\mathrm{SO}\times\mathrm{U}(1)}) = N^2_{\mathrm{SO}\times\mathrm{U}(1)}$.
- The homomorphism $\mathrm{Hol}\colon \mathcal{C}_1^{\mathrm{SO}\times\mathrm{U}(1)\nabla}(\mathrm{pt}) \to \mathbb{R}/\mathbb{Z}$ is given by the holonomy of the internal $\mathrm{U}(1)$-connection. More precisely, an element $[M,g]$ in $\mathcal{C}_1^{\mathrm{SO}\times\mathrm{U}(1)\nabla}(\mathrm{pt})$ consists of a closed oriented one-dimensional manifold M with a principal $\mathrm{U}(1)$-bundle with connection, together with other data. We define $\mathrm{Hol}([M,g])$ to be the holonomy of the $\mathrm{U}(1)$-connection, by identifying $\mathbb{R}/\mathbb{Z} \simeq \mathrm{U}(1)$.

Example 4.61 (The classical Chern-Simons theory). Fix a compact Lie group H and an element $\lambda \in H^n(BH;\mathbb{Z})$. The corresponding *classical Chern-Simons theory* ([Fre95], [Fre02]) is an invertible QFT on $(n-1)$-dimensional manifolds equipped with orientations and principal H-bundle with connection. This generalizes Example 4.60, which corresponds to $c_1 \in H^2(B\mathrm{U}(1);\mathbb{Z})$. Its partition functions are given by the *Chern-Simons invariants* of H-connections. Here we recall its definition.

Let $\lambda_{\mathbb{R}} \in H^*(BH;\mathbb{R})$ be the $\mathbb{R}$-reduction of the element λ. Consider the category $\mathcal{C}_H$ of triples (P,M,∇), where $P \to M$ is a smooth principal H-bundle over a manifold and ∇ is a H-connection on P. We fix the following data.

(1) An object $(\mathcal{E},\mathcal{B},\nabla_{\mathcal{E}})$ which is $(n+1)$-*classifying*, i.e., any object (P,M,∇) in $\mathcal{C}_H$ with $\dim M \le n$ admits a morphism to $(\mathcal{E},\mathcal{B},\nabla_{\mathcal{E}})$, and any such morphisms ϕ_1 and ϕ_2 are smoothly homotopic. By the theorem of Narasimhan-Ramanan [NR61] such an object exists.

(2) A differential lift $\widehat{\lambda} \in \widehat{H}^n(\mathcal{B};\mathbb{Z})$ of the element $\lambda \in H^n(\mathcal{B};\mathbb{Z}) \simeq H^n(BH;\mathbb{Z})$ such that $R(\widehat{\lambda}) = \mathrm{cw}_{\nabla_{\mathcal{E}}}(\lambda_{\mathbb{R}}) \in \Omega^n_{\mathrm{clo}}(\mathcal{B})$. Here $\widehat{H}^n(-;\mathbb{Z})$ is the differential ordinary cohomology group, for example given by the Cheeger-Simons model explained in Example 2.14.

The Chern-Simons invariants are defined using the pushforward in differential ordinary cohomology $\widehat{H\mathbb{Z}}$. In terms of the Cheeger-Simons model $\widehat{H\mathbb{Z}}_{\mathrm{CS}}$ in Example 2.14, the pushforward map $(p_M,o)_*\colon \widehat{H}^{\dim M+1}(M;\mathbb{Z}) \to \widehat{H}^1(\mathrm{pt};\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$ for a closed oriented manifold (M,o) is given by the evaluation on the fundamental cycle.

Definition 4.62 (The Chern-Simons invariants). Let $\lambda \in H^n(BH;\mathbb{Z})$ and fix the data (1) and (2) above. Let M be an $(n-1)$-dimensional closed manifold equipped with an orientation o_M and a principal H-bundle with connection (P,∇). Choose a morphism $\phi\colon (M,P,\nabla) \to (\mathcal{E},\mathcal{B},\nabla_{\mathcal{E}})$ in $\mathcal{C}_H$. We define the *Chern-Simons invariant* of (M,o,P,∇) by

$$h_{\mathrm{CS}_{\widehat{\lambda}}}(M,o,P,\nabla) := (p_M,o)_*\phi^*\widehat{\lambda} \in \widehat{H}^1(\mathrm{pt};\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}. \tag{4.63}$$

The value (4.63) does not depend on the choice of ϕ.

The classical Chern-Simons theory corresponds to the element

$$(1 \otimes \lambda_{\mathbb{R}}, h_{\mathrm{CS}_{\widehat{\lambda}}}) \in (\widehat{I\Omega^{\mathrm{SO}\times H}_{\mathrm{dR}}})^n(\mathrm{pt}). \tag{4.64}$$

Here $1\otimes\lambda_{\mathbb{R}}$ is as in Example 4.60, and $h_{\mathrm{CS}_{\widehat{\lambda}}}$ is regarded as a homomorphism from $\mathcal{C}^{\mathrm{SO}\times H\nabla}_{n-1}(\mathrm{pt})$.

Now we analyze the dependence on the choice of a lift $\widehat{\lambda}$ of λ in (2). By the axioms of differential cohomology (Definition 2.12), we see that two choices $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ differs by an element in $H^{n-1}(\mathcal{B};\mathbb{R})\simeq H^{n-1}(BH;\mathbb{R})$, i.e., there exists an element $\alpha\in H^{n-1}(\mathcal{B};\mathbb{R})$ with

$$a_{\mathrm{CS}}(\alpha)=\widehat{\lambda}_1-\widehat{\lambda}_2. \tag{4.65}$$

In particular, if n is even, the lift $\widehat{\lambda}$ is unique because $H^{\mathrm{odd}}(BH;\mathbb{R})=0$. In general it is possible that the difference (4.65) is nonzero, and in such a case the two elements (4.64) constructed from them are different. But they define the same element in $(I\Omega^{\mathrm{SO}\times H}_{\mathrm{dR}})^n(\mathrm{pt})$,

$$I(1\otimes\lambda_{\mathbb{R}},h_{\mathrm{CS}_{\widehat{\lambda}_1}})=I(1\otimes\lambda_{\mathbb{R}},h_{\mathrm{CS}_{\widehat{\lambda}_2}})\in(I\Omega^{\mathrm{SO}\times H}_{\mathrm{dR}})^n(\mathrm{pt}).$$

This is because

$$(1\otimes\lambda_{\mathbb{R}},h_{\mathrm{CS}_{\widehat{\lambda}_1}})-(1\otimes\lambda_{\mathbb{R}},h_{\mathrm{CS}_{\widehat{\lambda}_2}})=a(1\otimes\alpha).$$

Here the domain of a in (4.24) in this case is $\Omega^{n-1}(\mathrm{pt};N^{\bullet}_{\mathrm{SO}\times H})/\mathrm{Im}d=(H^*(B\mathrm{SO};\mathbb{R})\otimes_{\mathbb{R}}H^*(BH;\mathbb{R}))^{n-1}$. Thus we see that, the deformation class

$$I(1\otimes\lambda_{\mathbb{R}},h_{\mathrm{CS}_{\widehat{\lambda}}})\in(I\Omega^{\mathrm{SO}\times H}_{\mathrm{dR}})^n(\mathrm{pt}). \tag{4.66}$$

is independent of the choice of the lift $\widehat{\lambda}$.

Example 4.67 (The theory of massive free complex fermions). In this example we consider $G=\mathrm{Spin}^c$. Let k be a positive integer. Recall that we have constructed a model $(\widehat{I\Omega^G_{\mathrm{ph}}})^*$ in Subsubsection 4.1.2 which is isomorphic to $(\widehat{I\Omega^G_{\mathrm{dR}}})^*$ by Proposition 4.45. We are going to construct an element in $\left(\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{ph}}}\right)^{2k}$.

Given a closed $(2k-1)$-dimensional manifold M with a physical tangential Spin^c-structure g (Definition 4.50), we set

$$\overline{\eta}(M,g):=\overline{\eta}(D_M)=\frac{\eta(D_M)+\dim\ker D_M}{2}\in\mathbb{R}.$$

where D_M is the Spin^c-Dirac operator on M with respect to g and $\eta(D_M)\in\mathbb{R}$ is its eta invariant. Note that we have used the assumption that the connection in g is compatible with the Levi-Civita connection.

Recall that the Atiyah-Patodi-Singer index theorem ([APS76], [APS75a], [APS75b]) says that, if (W,g_W) is a compact $2k$-dimensional manifold with boundary with a collar structure equipped with a geometric Spin^c-structure which is compatible with Levi-Civita connection, we have

$$\mathrm{Ind}_{\mathrm{APS}}(D_W)=\int_W\mathrm{Todd}(g_W)-\overline{\eta}(\partial W,\partial g_W). \tag{4.68}$$

Here the left hand side of (4.68) is the Atiyah-Patodi-Singer index of the Dirac operator on W, which is an integer. Thus, regarding $\overline{\eta}$ as a homomorphism $\overline{\eta}\colon \mathcal{C}_{\mathcal{D}}(\mathrm{pt}) \to \mathbb{R}/\mathbb{Z}$ with $\mathcal{D} = h\mathrm{Bord}_{2k-1}^{\mathrm{Spin}^c_{\mathrm{ph}}}(\mathrm{pt})$, we get the element

$$((\mathrm{Todd})|_{2k}, \overline{\eta}) \in \left(\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{ph}}}\right)^{2k}(\mathrm{pt}) \simeq \left(I\Omega^{\mathrm{Spin}^c}_{\mathrm{ph}}\right)^{2k}(\mathrm{pt}) \simeq \left(I\Omega^{\mathrm{Spin}^c}_{\mathrm{dR}}\right)^{2k}(\mathrm{pt}).$$

This example can be generalized to include target spaces. Fix a manifold X and a hermitian vector bundle with unitary connection (E, h^E, ∇^E) over X. Then, using the reduced eta invariants $\overline{\eta}_{\nabla^E}$ for Dirac operators twisted by the pullback of (E, h^E, ∇^E), we get the element

$$\left((\mathrm{Ch}(\nabla^E) \otimes \mathrm{Todd})|_{2k}, \overline{\eta}_{\nabla^E}\right) \in \left(\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{ph}}}\right)^{2k}(X) \simeq \left(\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{dR}}}\right)^{2k}(X).$$

Its deformation class in $\left(I\Omega^{\mathrm{Spin}^c}_{\mathrm{ph}}\right)^{2k}(X)$ only depends on the class $[E] \in K^0(X)$.

Example 4.69 (The theory of massive free real fermions). Here we consider the real version of Example 4.67. Now $G = \mathrm{Spin}$. We consider the theory on $(8m+3)$-dimension with nonnegative integer m, the dimension where the difference from Example 4.67 appears. On Spin manifolds, the Atiyah-Patodi-Singer index theorem (4.68) becomes

$$\mathrm{Ind}_{\mathrm{APS}}(D_W) = \int_W \widehat{A}(g_W) - \overline{\eta}(\partial W, \partial g_W). \tag{4.70}$$

Moreover, if $\dim W \equiv 4 \pmod 8$, the APS index is an *even* integer. This allows us to define the element

$$\left(\frac{1}{2}\widehat{A}|_{8m+4}, \frac{1}{2}\overline{\eta}\right) \in \left(\widehat{I\Omega^{\mathrm{Spin}}_{\mathrm{ph}}}\right)^{8m+4}(\mathrm{pt}) \simeq \left(I\Omega^{\mathrm{Spin}}_{\mathrm{ph}}\right)^{8m+4}(\mathrm{pt}) \simeq \left(I\Omega^{\mathrm{Spin}}_{\mathrm{dR}}\right)^{8m+4}(\mathrm{pt}).$$

4.4. **The refinement of the Anderson self-duality in $H\mathbb{Z}$.** In this subsection, we relate our model $I\mathbb{Z}_{\mathrm{dR}}$ with the ordinary cohomology theory. The ordinary cohomology theory $H\mathbb{Z}$ is Anderson self-dual, with the self-duality element $\gamma_H \in [H\mathbb{Z}, I\mathbb{Z}]$, whose multiplication gives the isomorphism $H\mathbb{Z} \simeq IH\mathbb{Z}$. Using the obvious analogy of the Cheeger-Simons differential character model $\widehat{H\mathbb{Z}}_{\mathrm{CS}}$ (Example 2.14) and our differential model $\widehat{I\mathbb{Z}_{\mathrm{dR}}}$, we can refine $\gamma_H \colon H\mathbb{Z} \to I\mathbb{Z}$ to a transformation $\widehat{\gamma}_{\mathrm{dR}} \colon \widehat{H}^*_{\mathrm{CS}}(-;\mathbb{Z}) \to (\widehat{I\mathbb{Z}_{\mathrm{dR}}})^*(-)$.

In order to define $\widehat{\gamma}_{\mathrm{dR}}$, we remark the following. Let $(\omega, k) \in \widehat{H}^n_{\mathrm{CS}}(X, Y; \mathbb{Z})$. Then we get a group homomorphism also denoted by the same symbol k (here $\mathrm{fr}_\nabla = \mathrm{fr}$ in the obvious sense),

$$k\colon \mathcal{C}^{\mathrm{fr}}_{n-1}(X, Y) \to \mathbb{R}/\mathbb{Z}$$

by, given a (differential) smooth stable tangential fr-cycle (M, g, f) of dimension $(n-1)$ over (X, Y), choosing any representative $t_M \in Z_{\infty, n-1}(X, Y; \mathbb{Z})$ of the fundamental class and applying k to t_M. This value does not depend on the choice of t_M because of the compatibility condition for (ω, k).

Definition 4.71 ($\widehat{\gamma}_{\mathrm{dR}}$ and γ_{dR})**.** For a pair of manifolds (X, Y) and $n \in \mathbb{Z}$, we define a homomorphism

$$\widehat{\gamma}^n_{\mathrm{dR}} \colon \widehat{H}^n_{\mathrm{CS}}(X, Y; \mathbb{Z}) \to (\widehat{I\mathbb{Z}_{\mathrm{dR}}})^n(X, Y)$$

by sending an element (ω, k) to (ω, k). The compatibility condition (Definition 4.22 (1) (c)) for the pair (ω, k) follows from the compatibility condition (2.15) for the pair (ω, k).

We easily see that we have $a_{\mathrm{dR}} = \widehat{\gamma}^n_{\mathrm{dR}} \circ a_{\mathrm{CS}}$, so it induces the homomorphism on the quotient,

$$\gamma^n_{\mathrm{dR}} \colon H^n(X, Y; \mathbb{Z}) \to (I\mathbb{Z}_{\mathrm{dR}})^n(X, Y).$$

We also easily see that these homomorphisms are functorial, so gives natural transformations $\widehat{\gamma}_{\mathrm{dR}} \colon \widehat{H}^*_{\mathrm{CS}}(-; \mathbb{Z}) \to I\mathbb{Z}^*_{\mathrm{dR}}$ and $\gamma_{\mathrm{dR}} \colon H\mathbb{Z}^* \to I\mathbb{Z}^*_{\mathrm{dR}}$ between functors $\mathrm{MfdPair}^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$.

Proposition 4.72. *Under the isomorphism $I\mathbb{Z}_{\mathrm{dR}} \simeq I\mathbb{Z}$ in Theorem 4.56, the natural transformation γ_{dR} coincides with the self-duality map $\gamma_H \colon H\mathbb{Z} \to I\mathbb{Z}$.*

To show Proposition 4.72, we need to describe the Anderson self-duality homomorphism $\gamma_H \colon H\mathbb{Z} \to I\mathbb{Z}$ in the model of $I\mathbb{Z}$ by Hopkins-Singer (Fact 2.6). We work in the category of CW-pairs. We use the topological variant of the framed bordism Picard groupoid $h\mathrm{Bord}^{\mathrm{fr}}_{n-1}(X, Y)_{\mathrm{top}}$, which is defined for any CW-pair (X, Y), by requiring the map to (X, Y) continuous rather than smooth in Definition 3.8. By the theorem of Pontryagin-Thom, we have an equivalence

$$h\mathrm{Bord}^{\mathrm{fr}}_{n-1}(X, Y)_{\mathrm{top}} \simeq \pi_{\leq 1}((L(X/Y))_{1-n}). \tag{4.73}$$

Lemma 4.74. *Let (X, Y) be a CW-pair. Choose a functor of Picard groupoids,*

$$T_{X,Y} \colon h\mathrm{Bord}^{\mathrm{fr}}_{n-1}(X, Y)_{\mathrm{top}} \to \left(C_n(X, Y; \mathbb{Z})/B_n(X, Y; \mathbb{Z}) \xrightarrow{\partial} Z_{n-1}(X, Y; \mathbb{Z})\right), \tag{4.75}$$

where C_*, Z_* and B_* denote the singular chains, cycles and boundaries, by choosing fundamental cycles on objects and morphisms of $h\mathrm{Bord}^{\mathrm{fr}}_{n-1}(X, Y)_{\mathrm{top}}$.

Given a singular cohomology class $[c] \in H^n(X, Y; \mathbb{Z})$, Take a representative by a singular n-cocycle $c \in Z^n(X, Y; \mathbb{Z})$. Consider the functor of Picard groupoids,

$$\mathrm{ev}_c \colon \left(C_n(X, Y; \mathbb{Z})/B_n(X, Y; \mathbb{Z}) \xrightarrow{\partial} Z_{n-1}(X, Y; \mathbb{Z})\right) \to (\mathbb{Z} \to 0), \tag{4.76}$$

defined by the evaluation of c on morphisms. Then the natural isomorphism class of the composition of the functors $\mathrm{ev}_c \circ T_{X,Y}$ is independent of the choice of $T_{X,Y}$ and the cocycle c representing $[c]$, and defines a homomorphism

$$\begin{aligned} H^n(X, Y; \mathbb{Z}) &\to \pi_0 \mathrm{Fun}_{\mathrm{Pic}}\left(h\mathrm{Bord}^{\mathrm{fr}}_{n-1}(X, Y)_{\mathrm{top}}, (\mathbb{Z} \to 0)\right) \simeq I\mathbb{Z}^n(X, Y) \\ [c] &\mapsto [\mathrm{ev}_c \circ T_{X,Y}], \end{aligned} \tag{4.77}$$

where the last isomorphism use the Fact 2.6 and the equivalences $(\mathbb{Z} \to 0) \simeq (\mathbb{R} \to \mathbb{R}/\mathbb{Z})$ *and* (4.73)*. Moreover, the homomorphism* (4.77) *coincides with the transformation given by the Anderson self-duality element,*

$$\gamma_H \colon H^n(X, Y; \mathbb{Z}) \to I\mathbb{Z}^n(X, Y),$$

Proof. The first claim is easy, because both the natural isomorphism classes of the functors $T_{X,Y}$ and ev_c are independent of the choices. We can easily check that the homomorphism (4.77) is functorial and compatible with the relative coboundary maps, thus defining a transformation $H\mathbb{Z} \to I\mathbb{Z}$ of cohomology theories on CW-pairs.

Since we have $[H\mathbb{Z}, I\mathbb{Z}] = I\mathbb{Z}^0(H\mathbb{Z}) = \mathrm{Hom}(\pi_0(H\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}$ and we have $[H\mathbb{Z}, I\mathbb{Z}] = \varprojlim_n [H\mathbb{Z}_n, I\mathbb{Z}_n]$ by the vanishing of phantoms $\varprojlim_n [H\mathbb{Z}_n, I\mathbb{Z}_{n-1}] = 0$ ([Rud98, Chapter III, Theorem 4.21]), the transformation of cohomology theories $H\mathbb{Z} \to I\mathbb{Z}$ on CW-pairs are classified by its value on pt. We can easily check that the transformation given by (4.77) coincides with γ_H on pt, thus we conclude that it coincides with γ_H as a transformation of cohomology theories. This finishes the proof. □

Now we prove Proposition 4.72.

Proof of Proposition 4.72. Here we use the model $\widehat{H\mathbb{Z}}_{\mathrm{HS}}$ of the ordinary differential cohomology theory in terms of *differential cocycles* [HS05]. An element in $\widehat{H}^n_{\mathrm{HS}}(X, Y; \mathbb{Z})$ is represented by a triple $(c, h_{\mathbb{R}}, \omega) \in Z^n(X, Y; \mathbb{Z}) \times C^{n-1}(X, Y; \mathbb{R}) \times \Omega^n_{\mathrm{clo}}(X, Y)$ such that $\delta h_{\mathbb{R}} = c - \omega$ (as smooth singular $\mathbb{R}$-cochains). Such a triple is called a differential cocycle. The forgetful functor I is given by

$$I \colon \widehat{H}^n_{\mathrm{HS}}(X, Y; \mathbb{Z}) \to H^n(X, Y; \mathbb{Z}),\ ([c, h_{\mathbb{R}}, \omega]) \mapsto [c].$$

The models $\widehat{H\mathbb{Z}}_{\mathrm{HS}}$ and $\widehat{H\mathbb{Z}}_{\mathrm{CS}}$ are isomorphic with the isomorphism given by

$$\widehat{H}^n_{\mathrm{HS}}(X, Y; \mathbb{Z}) \simeq \widehat{H}^n_{\mathrm{CS}}(X, Y; \mathbb{Z}),\ [c, h_{\mathbb{R}}, \omega] \mapsto (\omega, h), \tag{4.78}$$

where we set $h := h_{\mathbb{R}} \pmod{\mathbb{Z}} \colon Z_{n-1}(X, Y; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$, and its restriction to $Z_{\infty, n-1}(X, Y; \mathbb{Z})$ is denoted by the same symbol.

Assume that we are given an element $(\omega, h) \in \widehat{H}^n_{\mathrm{CS}}(X, Y; \mathbb{Z})$. Take a differential cocycle $(c, h_{\mathbb{R}}, \omega)$ which maps to (ω, h) under the map (4.78). Then consider the diagram of functors,

(4.79)

$$\begin{tikzcd} \Big(C_{\infty,n}(X, Y; \mathbb{Z})/B_{\infty,n}(X, Y; \mathbb{Z}) \xrightarrow{\partial} Z_{\infty,n-1}(X, Y; \mathbb{Z})\Big) \arrow[r, "\mathrm{ev}_c"] \arrow[dr, "\mathrm{ev}_{(\omega,h)}"'] & (\mathbb{Z} \to 0) \arrow[d, "\simeq"] \\ & (\mathbb{R} \to \mathbb{R}/\mathbb{Z}) \end{tikzcd}$$

Here the top arrow is the restriction of (4.76) to smooth singular chains, and the rightdown arrow is given by the evaluation of h on objects and ω on morphisms. We can easily check that the two compositions of functors in (4.79) are naturally isomorphic, with natural transformation given by

$h_{\mathbb{R}}$. Moreover, by definition of $\widehat{\gamma}_{\mathrm{dR}}$ we see that the functor (4.51) $F_{\widehat{\gamma}_{\mathrm{dR}}(\omega,h)}$ associated to the element $\widehat{\gamma}_{\mathrm{dR}}(\omega,h) \in \widehat{I\mathbb{Z}}^n_{\mathrm{dR}}(X,Y)$ satisfies

$$F_{\widehat{\gamma}_{\mathrm{dR}}(\omega,h)} = \mathrm{ev}_{(\omega,h)} \circ T_{\infty,X,Y},$$

where $T_{\infty,X,Y}$ denotes an obvious smooth singular version of the fuctor (4.75). As explained in Subsection 4.2, the isomorphism $I\mathbb{Z}_{\mathrm{dR}} \simeq I\mathbb{Z}$ sends the class $\gamma_{\mathrm{dR}}([\omega,h])$ to the class of the associated functor $F_{\widehat{\gamma}_{\mathrm{dR}}(\omega,h)}$. By the natural isomorphism between two compositions in (4.79) and Lemma 4.74, together with the equivalence $(C_{\infty,n}(X,Y;\mathbb{Z})/B_{\infty,n}(X,Y;\mathbb{Z}) \xrightarrow{\partial} Z_{\infty,n-1}(X,Y;\mathbb{Z})) \simeq (C_n(X,Y;\mathbb{Z})/B_n(X,Y;\mathbb{Z}) \xrightarrow{\partial} Z_{n-1}(X,Y;\mathbb{Z}))$, we see that the class of the functor $F_{\widehat{\gamma}_{\mathrm{dR}}(\omega,h)}$ coincides with the class $\gamma_H([c])$. This completes the proof. $\square$

4.5. **The normal case.** So far we have focused on the *tangential* G-bordism theories and its Anderson duals. However, by a straightforward modification, we can construct the corresponding models for the Anderson duals $(I\Omega^{G^\perp})^*$ to the *normal* G-bordism theories $\Omega^{G^\perp}$ corresponding to the Thom spectrum MG. In this subsection we outline the construction.

Definition 4.80 (Differential stable normal G-structures on vector bundles)**.** Let V be a real vector bundle of rank n over a manifold M.

(1) A *representative of differential stable normal G-structure* on V is a quadruple $\widetilde{g}^\perp = (d, P, \nabla, \psi)$, where $d \geq n$ is an integer, (P,∇) is a principal G_{d-n}-bundle with connection over M and $\psi \colon (P \times_{\rho_{d-n}} \mathbb{R}^{d-n}) \oplus V \simeq \underline{\mathbb{R}}^d$ is an isomorphism of vector bundles over M.

(2), (3) We define the *stabilization* of such $\widetilde{g}^\perp$ in the same way as Definition 3.1, and a *differential stable normal G-structure* $g^\perp$ on V is defined to be a class of representatives under the stabilization relation.

(4) We define the *homotopy* relation between two such $g^\perp$'s also in the same way.

Definition 4.81 (Differential stable normal G-structures)**.** Let M be a manifold. A *differential stable normal G-structure* is a differential stable normal G-structure on the tangent bundle TM.

Then, the various objects introduced in Section 3 can be modified to the normal case easily. We get the notion of *differential stable normal G-cycles* $(M, g^\perp, f)$, the abelian groups $\mathcal{C}_n^{G^\perp_\nabla}(X,Y)$, the bordism relations and the Picard groupoids $h\mathrm{Bord}_n^{G^\perp_\nabla}(X,Y)$.

First note that we have

$$N^\bullet_{G^\perp} := H^*(MG;\mathbb{R}) = \varprojlim_d H^*(G_d;\mathbb{R}_{G_d}) = \varprojlim_d (\mathrm{Sym}^{\bullet/2}\mathfrak{g}_d^* \otimes_{\mathbb{R}} \mathbb{R}_{G_d})^{G_d}.$$

The proof is the same as that of Lemma 4.1, where now the Madsen-Tillmann spectrum MTG is replaced by the Thom spectrum MG. Note that we have $N^\bullet_G = N^\bullet_{G^\perp}$. This is because the orientation bundles of a vector bundle and its normal bundle are canonically identified. We use the transformation analogous to (4.5),

$$\mathrm{ch}' \colon (I\Omega^{G^\perp})^* \to H^*(-;N^\bullet_{G^\perp}) \simeq \mathrm{Hom}(\Omega_*^{G^\perp}(-),\mathbb{R}). \tag{4.82}$$

The variant of the Chern-Weil construction in Definition 4.11 also applies to the normal settings. Given a differential stable normal G-structure $g^\perp$ on a vector bundle $V \to W$, by the same procedure to the tangential case in Definition 4.11 we get a homomorphism

$$\mathrm{cw}_{g^\perp} \colon \Omega^*\left(W; N^\bullet_{G^\perp}\right) \to \Omega^*(W; \mathrm{Ori}(V)). \tag{4.83}$$

Applied to $V = TW$, (4.83) induces the homomorphisms corresponding to (4.20) and (4.21).

Definition 4.84 ($(\widehat{I\Omega^{G^\perp}_{\mathrm{dR}}})^*$ and $(I\Omega^{G^\perp}_{\mathrm{dR}})^*$). Let (X, Y) be a pair of manifolds and n be a nonnegative integer.

(1) Define $(\widehat{I\Omega^{G^\perp}_{\mathrm{dR}}})^n(X, Y)$ to be an abelian group consisting of pairs (ω, h), such that
  (a) ω is a closed n-form $\omega \in \Omega^n_{\mathrm{clo}}\left(X, Y; N^\bullet_{G^\perp}\right)$.
  (b) h is a group homomorphism $h \colon \mathcal{C}^{G^\perp_\nabla}_{n-1}(X, Y) \to \mathbb{R}/\mathbb{Z}$.
  (c) ω and h satisfy the compatibility condition analogous to Definition 4.22 (1) (c) with respect to morphisms in $h\mathrm{Bord}^{G^\perp_\nabla}_{n-1}(X, Y)$.

(2) We define a homomorphsim of abelian groups,

$$\begin{aligned} a \colon \Omega^{n-1}\left(X, Y; N^\bullet_{G^\perp}\right)/\mathrm{Im}(d) &\to (\widehat{I\Omega^{G^\perp}_{\mathrm{dR}}})^n(X, Y) \\ \alpha &\mapsto (d\alpha, \mathrm{cw}(\alpha)). \end{aligned}$$

We set

$$(I\Omega^{G^\perp}_{\mathrm{dR}})^n(X, Y) := (\widehat{I\Omega^{G^\perp}_{\mathrm{dR}}})^n(X, Y)/\mathrm{Im}(a).$$

For negative integer n we set $(\widehat{I\Omega^{G^\perp}_{\mathrm{dR}}})^n(X, Y) := 0$ and $(I\Omega^{G^\perp}_{\mathrm{dR}})^n(X, Y) := 0$.

The structure homomorphisms I, R, a and p, and the S^1-integration map $\int$, are also defined in the same way as Definitions 4.26 and 4.39. We can check that the following sequence is exact,

$$\begin{aligned} &\mathrm{Hom}(\Omega^{G^\perp}_{n-1}(X, Y), \mathbb{R}) \to \mathrm{Hom}(\Omega^{G^\perp}_{n-1}(X, Y), \mathbb{R}/\mathbb{Z}) \xrightarrow{p} (I\Omega^{G^\perp}_{\mathrm{dR}})^n(X, Y) \\ &\xrightarrow{\mathrm{ch}'} \mathrm{Hom}(\Omega^{G^\perp}_n(X, Y), \mathbb{R}) \to \mathrm{Hom}(\Omega^{G^\perp}_n(X, Y), \mathbb{R}/\mathbb{Z}) \qquad (\text{exact}). \end{aligned}$$

The normal version of Theorem 4.56, which says that the above $I\Omega^{G^\perp}_{\mathrm{dR}}$ is indeed a model for $I\Omega^{G^\perp}$, and that the quintuple $(\widehat{I\Omega^{G^\perp}_{\mathrm{dR}}}, R, I, a, \int)$ is its differential extension of $\left((I\Omega^{G^\perp})^*, \mathrm{ch}'\right)$ with S^1-integration, can be shown by the exactly same proof, replacing MTG with MG.

5. The multiplications by the bordism cohomology theories

Assume we are given three tangential structure groups $G_i := \{(G_i)_d, (s_i)_d, (\rho_i)_d\}_{d \in \mathbb{Z}_{\geq 0}}$ for $i = 1, 2, 3$, and a *homomorphism*

$$\mu \colon G_1 \times G_2 \to G_3 \tag{5.1}$$

of tangential structure groups, inducing a morphism between the Madsen-Tillmann spectra,

$$MTG_1 \wedge MTG_2 \to MTG_3. \tag{5.2}$$

Here a homomorphism (5.1) is defined in a fairly obvious way, whose precise definition is given in Remark 5.8 below. It consists of group homomorphisms $(G_1)_d \times (G_2)_{d'} \to (G_3)_{d+d'}$, which are compatible with the structure homomorphisms $(s_i)_d, (\rho_i)_d$'s and the multiplicative structure on O. There are many interesting examples as follows.

Example 5.3.
(1) An important class of examples arises from *multiplicative* G, where MTG is a ring spectrum. In this case we set $G = G_1 = G_2 = G_3$. For example O, SO, Spin and fr are equipped with the natural multiplicative structure.
(2) The case $G_1 = G_3 = \mathrm{Pin}^+$ and $G_2 = \mathrm{Spin}$.
(3) For any G, we have a homomorphism $G \times \mathrm{fr} \to G$. Here recall that $\mathrm{fr} = \{1\}_{d\in\mathbb{Z}}$. The group homomorphism $G_d \times \mathrm{fr}_{d'} = G_d \times 1 \to G_{d+d'}$ is the composition $s_{d+d'-1} \circ \cdots \circ s_d$ of the stabilization homomorphisms in G. Actually, as we will see in Remark 6.17, the differential pushforwards we introduce in Section 6 in this case recovers the S^1-integration map $\int$ of $\widehat{I\Omega^G_{\mathrm{dR}}}$ (Definition 4.39).

In general, if we have a morphism of spectra

$$t\colon E_1 \wedge E_2 \to E_3,$$

we get the following morphism on the Anderson duals.

$$IE_3 \wedge E_2 \xrightarrow{It\wedge \mathrm{id}_{E_2}} I(E_1 \wedge E_2) \wedge E_2 \xrightarrow{\mathrm{ev}_{E_2}} IE_1. \tag{5.4}$$

Here It is the Anderson dual to t, and the second arrow is the evaluation on E_2 (recall $I(E_1 \wedge E_2) = [E_1 \wedge E_2, I\mathbb{Z}]$). Applying (5.4) to (5.2), we get the morphism

$$IMTG_3 \wedge MTG_2 \to IMTG_1, \tag{5.5}$$

inducing the following homomorphism for each pair of integers (n, r),

$$(I\Omega^{G_3})^n(X, Y) \otimes (\Omega^{G_2})^{-r}(X) \to (I\Omega^{G_1})^{n-r}(X, Y), \tag{5.6}$$

which is natural in (X, Y). The purpose of this section is to get its differential refinement,

$$(\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^n(X, Y) \otimes (\widehat{\Omega^{G_2}})^{-r}(X) \dot{\to} (\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X, Y). \tag{5.7}$$

Here, for $(\widehat{\Omega^{G_2}})^{-r}(X)$ we use a cycle-model constructed by Bunke and Schick Schröder and Wiethaup [BSSW09], which we explain in Subsection 5.1.

Before proceeding, we explain the rough idea of the construction. A particularly nice class of elements in $(\widehat{\Omega^{G_2}})^{-r}(X)$ are represented by cycles of the form $(p\colon N \to X, g_p)$, where p is a fiber bundle whose fibers are closed r-dimensional manifold and g_p is a differential stable G_2-structure on the relative tangent bundle. The multiplication (5.7) by such an element is given as follows. For an object $[M, g_M, f] \in \mathcal{C}^{(G_1)_\nabla}_{n-r-1}(X, Y)$, we can define an object $\mu\left([M, g_M, f] \times_X [p\colon N \to X, g_p]\right) \in \mathcal{C}^{(G_3)_\nabla}_{n-1}(X, Y)$ by the fiber product over X (the pullback of the bundle) in a fairly obvious way. Then, for an element $(\omega, h) \in (\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^n(X, Y)$, we assign an element in $(\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X, Y)$ whose evaluation on $[M, g_M, f] \in \mathcal{C}^{(G_1)_\nabla}_{n-r-1}(X, Y)$ is given by the evaluation of h

on this fiber product. As explained in Subsubsection 1.2.3, in a physical interpretation, this process corresponds to *compactification* of QFT's.

Remark 5.8. Here we give the definition of *homomorphism* $\mu\colon G_1 \times G_2 \to G_3$ in (5.1). μ consists of group homomorphisms $\mu_{d,d'}\colon (G_1)_d \times (G_2)_{d'} \to (G_3)_{d+d'}$ for each (d,d') with the following conditions.

(1) For any (d,d'), the following diagram commutes.

$$\begin{tikzcd} (G_1)_d \times (G_2)_{d'} \arrow[r, "\mu_{d,d'}"] \arrow[d, "(\rho_1)_d \times (\rho_2)_{d'}"] & (G_3)_{d+d'} \arrow[d, "(\rho_3)_{d+d'}"] \\ \mathrm{O}(d,\mathbb{R}) \times \mathrm{O}(d',\mathbb{R}) \arrow[r] & \mathrm{O}(d+d',\mathbb{R}) \end{tikzcd}, \tag{5.9}$$

where the bottom arrow is the diagonal map in O.

(2) For any (d,d'), the left diagram below commutes, and the right diagram commute up to confugation by an element of $(G_3)_{d+d'+1}$ in the unit component.

(5.10)

$$\begin{tikzcd} (G_1)_d \times (G_2)_{d'} \arrow[r, "\mu_{d,d'}"] \arrow[d, "(s_1)_d \times \mathrm{id}"] & (G_3)_{d+d'} \arrow[d, "(s_3)_{d+d'}"] \\ (G_1)_{d+1} \times (G_2)_{d'} \arrow[r, "\mu_{d+1,d'}"] & (G_3)_{d+d'+1} \end{tikzcd} \qquad \begin{tikzcd} (G_1)_d \times (G_2)_{d'} \arrow[r, "\mu_{d,d'}"] \arrow[d, "\mathrm{id} \times (s_2)_{d'}"] & (G_3)_{d+d'} \arrow[d, "(s_3)_{d+d'}"] \\ (G_1)_{d} \times (G_2)_{d'+1} \arrow[r, "\mu_{d,d'+1}"] & (G_3)_{d+d'+1} \end{tikzcd}$$

Here, for Condition (2), we may as well assume only the homotopy-commutativity also for the left diagram in order to produce the morphism of Madsen-Tillmann spectra in (5.2). However, the strict-commutatibity is satisfied in most of the examples of interest. (In contrast to this, for the right diagram we can only expect the homotopy-commutatibity by the obvious reason.) We choose the above formulation to simplify the construction below.

5.1. A preliminary–A cycle-model for $(\widehat{\Omega^G})^*$ following [BSSW09].

Bunke and Schick Schröder and Wiethaup [BSSW09] gave a model for a differential extension of $(\widehat{\Omega^{G^\perp}})^*$ (*normal* G-bordism cohomology theory, represented by MG). They provide the detail for the case of complex bordisms, but as they note, their construction directly generalizes to any G. Moreover, their construction can be modified to give a model for $(\widehat{\Omega^G})^*$ (*tangential* G-bordisms). In this subsection we briefly explain it. For further details see [BSSW09]. Remark that sometimes we use different definitions from those for corresponding objects in [BSSW09] in order to make it compatible with the conventions in the main body of this paper. The differences are not essential.

Definition 5.11 (Stable relative tangent bundles)**.** Let $p\colon N \to X$ be a smooth map between manifolds with relative dimension $r := \dim N - \dim X$. Let us choose (k,ϕ), where $\phi\colon \underline{\mathbb{R}}^k \to p^*TX$ is a map of vector bundles over N such that $\phi \oplus dp\colon \underline{\mathbb{R}}^k \oplus T_xN \to T_{p(x)}X$ is surjective for all $x \in N$. Given such (k,ϕ), we define the associated *stable relative tangent bundle for* p *associated* *to* (k, ϕ) to be the following real vector bundle of rank $(k+r)$ over N.

$$T(\phi, p) := \ker(\phi \oplus dp \colon \underline{\mathbb{R}}^k \oplus TN \to p^*TX).$$

Using stable relative tangent bundles, we define the relative version of differential stable tangential G-structures as follows.

Definition 5.12 (Differential stable relative tangential G-structures)**.** Let $p\colon N \to X$ be a smooth map between manifolds.

(1) A *representative of differential stable relative tangential G-structure on p* consists of $\widetilde{g}_p = (k, \phi, P, \nabla, \psi)$, where (k, ϕ) is a choice as in Definition 5.11, P is a principal G_{k+r}-bundle over N and $\psi\colon P \times_{G_{k+r}} \mathbb{R}^{k+r} \simeq T(\phi, p)$ is an isomorphism of vector bundles over N.
(2) For such a $\widetilde{g}_p$, we can define its stabilization $\widetilde{g}_p(1)$ in the obvious way.
(3) A *differential stable relative tangential G-structure g_p on p* is a class of such representatives under the relation $\widetilde{g}_p \sim \widetilde{g}_p(1)$.

In particular, if $p\colon N \to X$ is a submersion, we can take $\phi = 0\colon \underline{\mathbb{R}}^k \to p^*TX$ and we have $T(0, p) = \underline{\mathbb{R}}^k \oplus T(p)$, the stabilization of the relative tangent bundle $T(p) := \ker dp$. Thus a differential stable G-structure (Definition 3.1) on $T(p)$ can be regarded as a special case of differential stable relative tangential G-structure on p. But note that the latter notion is more general.

Recall that our manifolds are allowed to have corners. To define the differential stable relative tangential G-cycles, we need to use the following class of maps between manifolds with corners.

Definition 5.13 (Neat maps, [HS05, Appendix C])**.** A smooth map $p\colon N \to X$ between manifolds is called *neat* if it preserves the depth of points, and the map

$$dp\colon T_xN/T_xS^k(N) \to T_{p(x)}(X)/T_{p(x)}S^k(X) \tag{5.14}$$

is an isomorphism for all points $x \in N$, where $k := \mathrm{depth}(x) = \mathrm{depth}(p(x))$.

For example, the map $p_X\colon X \to \mathrm{pt}$ is neat only if X has no boundary. The map $[0, \infty) \to [0, \infty)$, $x \mapsto x^2$, is not neat. We collect necessary facts on neat maps in Subsubsection 5.2.1 below. As we explain there, neatness guarantees a nice theory on fiber products.

Definition 5.15 (Differential stable relative tangential G-cycles)**.** Let X be a manifold and r be an integer. A *differential stable relative tangential G-cycle of dimension r* over X is a pair $\widehat{c} = (p\colon N \to X, g_p)$, where p is a proper neat map with relative dimension r and g_p is a differential stable relative tangential G-structure on p. A *representative of differential stable relative tangential G-cycle of dimension r* over X is a pair $\widetilde{c} := (p\colon N \to X, \widetilde{g}_p)$, where $\widetilde{g}_p$ is now a representative.

Definition 5.16 (Differential stable relative tangential G-bordism data)**.** Let X be a manifold and r be an integer. A *Differential stable relative tangential G-bordism data of dimension r* is a differential relative stable tangential G-cycle $\widehat{b} = (q\colon W \to \mathbb{R} \times X, g_q)$ of dimension r over $\mathbb{R} \times X$ such that q is transverse (Definition 5.28) to $\{0\} \times X$ and $q^{-1}((-\infty, 0] \times X)$ is compact. It defines a differential relative stable tangential G-cycle $\partial\widehat{b} := \left(q|_{q^{-1}(\{0\}\times X)}, g_q|_{q^{-1}(\{0\}\times X)}\right)$ over X by Proposition 5.29.

On the topological level, we have the Chern-Dold homomorphism

$$\mathrm{ch}\colon (\Omega^G)^{-r}(X) \to H^{-r}(X; V^\bullet_{\Omega^G}). \tag{5.17}$$

A differential relative stable tangential G-cycle $\widehat{c}$ over X represents a class $[\widehat{c}] \in (\Omega^G)^{-r}(X)$, so we get a class $\mathrm{ch}([\widehat{c}]) \in H^{-r}(X; V^\bullet_{\Omega^G})$.

On the differential level, the homomorphism (5.17) is refined as follows. Applying (5.17) to the identity element $\mathrm{id}_{MTG} \in (\Omega^G)^0(MTG)$, we have an element $\mathrm{ch}(\mathrm{id}_{MTG}) \in H^0(MTG; V^\bullet_{\Omega^G})$. Given a differential relative stable tangential G-cycle $\widehat{c} = (p\colon N \to X, g_p)$ over X, by the Chern-Weil construction in Remark 4.17 with coefficient $\mathcal{V}^* = V^*_{\Omega^G}$, we get[12]

$$\mathrm{cw}_{g_p}(\mathrm{ch}(\mathrm{id}_{MTG})) \in \Omega^0(N; \mathrm{Ori}(T(p)) \otimes_{\mathbb{R}} V^\bullet_{\Omega^G}). \tag{5.18}$$

Here we abuse the notation to write $\mathrm{Ori}(T(p)) := \mathrm{Ori}(T(\phi, p))$ for any choice of a representative of g_p, since the orientation bundle for stable relative tangent bundles do not depend on the choice of ϕ.

We would like to integrate it along the fiber of $p\colon N \to X$. Note that unless p is a submersion, the resulting form on X is singular, so we need to deal with differentiable currents $\Omega^*_{-\infty}$. Recall that in general the fiber integration of differentiable current along the map $p\colon N \to X$ is the following.

$$p_!\colon \Omega^*_{-\infty}(N; \mathrm{Ori}(T(p))) \to \Omega^{*-r}_{-\infty}(X). \tag{5.19}$$

Here we choose the sign of the integration (5.19) so that it is compatible with the *left* $\Omega^\bullet(X)$-module structure, i.e.,

$$\eta \wedge p_!(\omega) = p_!(p^*\eta \wedge \omega) \tag{5.20}$$

for any $\eta \in \Omega^\bullet(X)$ and $\Omega^*_{-\infty}(N; \mathrm{Ori}(T(p)))$. Remark that it is different from the sign convention on the S^1-integration in (2.16) and (2.17). We set

$$T(\widehat{c}) := p_!(\mathrm{cw}_{g_p}(\mathrm{ch}(\mathrm{id}_{MTG}))) \in \Omega^{-r}_{-\infty}(X; V^\bullet_{\Omega^G}). \tag{5.21}$$

This current represents the class $\mathrm{ch}([\widehat{c}])$ under the isomorphism between the de Rham and the currential cohomologies.

Definition 5.22 ($(\widehat{\Omega^G})^*$-cycles)**.** Let X be a manifold and r be an integer. An $(\widehat{\Omega^G})^{-r}$*-cycle* over X is a pair $(\widehat{c}, \alpha)$, where $\widehat{c}$ is an r-dimensional differential relative stable tangential G-cycle over X and $\alpha \in \Omega^{-r-1}_{-\infty}(X; V^\bullet_{\Omega^G})$ such that

$$R(\widehat{c}, \alpha) := T(\widehat{c}) - d\alpha \in \Omega^{-r}(X; V^\bullet_{\Omega^G}). \tag{5.23}$$

The role of α in Definition 5.22 is to replace $T(\widehat{c})$ with a smooth differential form, without changing the cohomology class.

The set of isomorphism classes of $(\widehat{\Omega^G})^{-r}$-cycles over X is denoted by $(Z\widehat{\Omega^G})^{-r}(X)$. It has a structure of an abelian semigroup by the disjoint union on cycles and the addition on currents.

[12] The element (5.18) can be understood as follows. It induces a degree-preserving $\mathbb{R}$-linear homomorphism from $N^\bullet_G$ to $\Omega^\bullet(N; \mathrm{Ori}(T(p)))$. This homomorphism coincides with cw_{g_p} in (4.15).

For an r-dimensional bordism data $\widehat{b} = (q\colon W \to \mathbb{R} \times X, g_q)$, we define

(5.24)
$$T(\widehat{b}) := \int_{(-\infty,0]} q_! \left(\mathrm{cw}_{g_q}(\mathrm{ch}(\mathrm{id}_{MTG}))|_{q^{-1}((-\infty,0]\times X)}\right) \in \Omega^{-r-1}_{-\infty}(X; V^{\bullet}_{\Omega^G}).$$

Then we can show that $(\partial\widehat{b}, T(\widehat{b})) \in (Z\widehat{\Omega^G})^{-r}(X)$.

Definition 5.25 ($(\widehat{\Omega^G})^*(X)$)**.** Let X be a manifold and r be an integer. On $(Z\widehat{\Omega^G})^{-r}(X)$ we introduce the equivalence relation $\sim$ generated by $(\partial\widehat{b}, T(\widehat{b})) \sim 0$ for a bordism data $\widehat{b}$. We define

$$(\widehat{\Omega^G})^{-r}(X) := (Z\widehat{\Omega^G})^{-r}(X)/\sim .$$

We denote the class of $(\widehat{c}, \alpha)$ in $(\widehat{\Omega^G})^{-r}(X)$ by $[\widehat{c}, \alpha]$.

We can define the structure maps R, a and I for $(\widehat{\Omega^G})^*(X)$ in an analogous way to [BSSW09]. The fact that the quadruple $(\widehat{\Omega^G}, R, a, I)$ is a differential extension of Ω^G can be easily checked as in the normal case.

5.2. **The differential multiplication by** $(\widehat{\Omega^{G_2}})^*$**.** Now assume we are given a homomorphism $\mu\colon G_1 \times G_2 \to G_3$. The definition of the transformation (5.7) uses the fiber products between differential relative stable tangential G_2-cycles and differential stable tangential G_1-cycles. To form the fiber products, we want to restrict our attention to differential stable tangential G_1-cycles (M, g, f) with f satisfying certain transversality conditions. For this, the result of Subsubsection 4.1.2 is useful. In Subsubsection 5.2.1 below, we construct a certain equivalent subcategory of $h\mathrm{Bord}^{(G_1)\triangledown}_{n-r}(X,Y)$ consisting of objects with a suitable transversality. This point is technical, and the reader who is willing to admit the existence of a nice subcategory to form fiber products can go directly to Subsubsection 5.2.2.

5.2.1. *A technical point : The construction of* $h\mathrm{Bord}^{(G_1)\triangledown}_{n-r}(X,Y)_{\pitchfork\widehat{c}}$. There are substantial technicalities concerning fiber products between manifolds with corners. For example see [Joy12][13]. But recall that we required the neatness (Definition 5.13) of the map in Definition 5.15. As we explain now, the theory on fiber products is very simple for neat maps.

First we explain a useful local picture of neat maps. The following lemma directly follows by Definition 5.13.

Lemma 5.26. *Let $p\colon N \to X$ be a neat map between manifolds. Let $x \in N$ be any point. Then there exist open neighborhoods $x \in V \subset N$ and $p(x) \in U \subset X$, manifolds without boundaries $\widehat{V}$ and $\widehat{U}$ with embeddings $V \hookrightarrow \widehat{V}$ and $U \hookrightarrow \widehat{U}$, a smooth map $\widehat{p}\colon \widehat{V} \to \widehat{U}$ extending $p|_V$ such that*

- *$\widehat{p}^{-1}(U) = V$, and*
- *$\widehat{p}$ is transverse to corners of U.*

[13]Be careful that "smooth" in this paper corresponds to "weakly smooth" in [Joy12]. The neatness in Definition 5.13 implies the "smoothness" in that paper. Using this, it is also possible to obtain the results in this subsubsection by applying the results in [Joy12, Section 6]. But since neat maps can be treated in an elementary way, we take a direct approach here.

Conversely, neatness is characterized by this local property.

In the following we call a set of data appearing in Lemma 5.26 a *local extension* of p. If we have a smooth map $f\colon M \to U$ from another manifold, the property $\widehat{p}^{-1}(U) = V$ implies that

$$M \times_U V \simeq M \times_{\widehat{U}} \widehat{V} \tag{5.27}$$

as a topological space. Using (5.27), we can reduce the local theory on fiber products between a smooth map $f\colon M \to X$ and a neat map $p\colon N \to X$ to the case where N and X are boundaryless.

Now we introduce the transversality condition.

Definition 5.28 (Transversality between smooth maps)**.** Let $f\colon M \to X$ and $p\colon N \to X$ be smooth maps. We say that f is *transverse* to p or f and p are *transverse*, if for any points $x \in M$ and $y \in N$ with $f(x) = p(y) = z$, we have

(1) The images of $df\colon T_xM \to T_zX$ and $dp\colon T_yN \to T_zX$ span T_zX, and
(2) The images of $df\colon T_xS^k(M) \to T_zS^j(X)$ and $dp\colon T_yS^l(N) \to T_zS^j(X)$ span $T_zS^j(X)$. Here k, j and l are the depths of the corresponding points.

If p is neat, the condition (2) in Definition 5.28 is equivalent to the condition that the images of $df\colon T_xS^k(M) \to T_zX$ and $dp\colon T_yN \to T_zX$ span T_zX. This implies that, in a local extension in Lemma 5.26, the transversality of f and p is equivalent to the transversality of f and $\widehat{p}$.

Proposition 5.29. *Let $f\colon M \to X$ be a smooth map and $p\colon N \to X$ be a neat map. Assume f is transverse to p, and we assume one of the following condition for f.*

(1) f is an embedding.
(2) M is equipped with a structure of $\langle k\rangle$-manifold with $(\partial_0 M, \partial_1 M, \cdots, \partial_{k-1}M)$, and there exists a collar structure near each $\partial_j M$ on which f is constant in the collar direction.

Then in the fiber product

$$\begin{tikzcd} M \times_X N \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{p}"] & N \arrow[d, "p"] \\ M \arrow[r, "f"] & X, \end{tikzcd} \tag{5.30}$$

the space $M \times_X N$ is equipped with a canonical structure of manifold with corners so that

$$S^k(M \times_X N) = S^k(M) \times_X N,$$

and the map $\widetilde{p}$ is neat. Moreover in the case (2) above, $M \times_X N$ is also equipped with a canonical structure of $\langle k\rangle$-manifold with $\partial_j(M \times_X N) = \partial_j M \times_X N$.

Proof. We can reduce to the case where N and X are boundaryless as explained above, and for such cases this result is well-known and easy to prove. We also remark that a part of this proposition follows from [HS05, Appendix C.22]. $\square$

Now we define the subcategory $h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}}$ of $h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)$.

Definition 5.31 ($h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}}$)**.** Let (X,Y) be a pair of manifolds and $\widehat{c} = (p\colon N \to X, g_p)$ be a differential stable relative tangential G_2-cycle over X. We define a Picard subcategory $h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}}$ of $h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)$ spanned by objects (M,g,f) with the following conditions.

1. (1) There exists a collar structure near ∂M of M on which the restriction of the map $f\colon (M,\partial M) \to (X,Y)$ is constant in the collar direction.
2. (2) The map $f\colon M \to X$ is transverse to $p\colon N \to X$.

In particular if p is a submersion, we have $h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}} = h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)$. Proposition 5.29 implies the following.

Corollary 5.32. *For any object $(M,g,f) \in h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}}$ with $\widehat{c} = (p\colon N \to X, g_p)$, the fiber product $M \times_X N$ is equipped with a structure of a $\langle 1\rangle$-manifold with $\partial(M\times_X N) = \partial M \times_X N$, with map $\tilde{f}\colon (M\times_X N, \partial(M\times_X N)) \to (N, p^{-1}(Y))$.*

Given a pair of manifolds (X,Y), We say that a differential stable relative tangential G_2-cycle $\widehat{c} = (p\colon N \to X, g_p)$ is *transverse to* Y if the underlying map p is transverse to Y. Remark that $(\widehat{\Omega^{G_2}})^{-r}(X)$ is generated by $(\widehat{c},\alpha)$ with $\widehat{c}$ satisfying this transversality.

Lemma 5.33. *If $\widehat{c}$ is transverse to Y, the inclusion $h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}} \subset h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)$ is an equivalence.*

Proof. It is enough to show that any elements in the relative bordism group $\Omega_m^{G_1}(X,Y)$ can be represented by an object in $h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}}$. Take any element $e \in \Omega_m^{G_1}(X,Y)$. First we consider its image of the boundary map, $\partial e \in \Omega_{m-1}^{G_1}(Y)$. Applying Proposition 5.29 case (1) to the embedding $\iota\colon Y \hookrightarrow X$ and $p\colon N \to X$, we get a canonical structure of a manifold with corners on $p^{-1}(Y)$ so that the map $p\colon p^{-1}(Y) \to Y$ is neat. We claim that we can represent ∂e by a smooth map $f_\partial\colon M_\partial \to Y$ transverse to $p\colon p^{-1}(Y) \to Y$. Indeed, using the homotopy equivalence $\mathring{Y} \sim Y$, it is easy to reduce to the genericity of transversality in the case without boundary.

By the transversality of $p\colon N \to X$ with $\iota\colon Y \hookrightarrow X$, we see that the composition $\iota \circ f_\partial\colon M_\partial \to X$ is transverse to $p\colon N \to X$. This also implies that the map $f_\partial \circ \mathrm{pr}_{M_\partial}\colon (-1,0] \times M_\partial \to X$ is transverse to $p\colon N \to X$, and we use it as a collar of the desired object. Now that we have defined the maps on the collar, the desired object in $h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}}$ which represents f_∂ can be obtained by reducing to the case where N and X are boundaryless as before, and using the usual genericity of transversality in the relative form. $\square$

By Lemma 5.33, fixing $\widehat{c}$ which is transverse to Y, we can apply the machinery of Subsubsection 4.1.2 to the equivalent Picard subcategory $h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}}$. We use the notation $\mathcal{C}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}}$ for the corresponding subgroup of $\mathcal{C}_m^{(G_1)\nabla}(X,Y)$, which is denoted by $\mathcal{C}_{\mathcal{D}}$ with $\mathcal{D} = h\mathrm{Bord}_m^{(G_1)\nabla}(X,Y)_{\pitchfork\widehat{c}}$ in Subsubsection 4.1.2.

5.2.2. *The differential multiplication by* $(\widehat{\Omega^{G_2}})^*$. Now we proceed to construct the transformation (5.7). Let (X, Y) be a pair of manifolds and let $\widehat{c} = (p\colon N \to X, g_p)$ be a differential stable relative tangential G_2-cycle of dimension r which is transverse to Y. First we define a functor between bordism Picard categories, essentially given by the fiber product over X composed with the homomorphism μ, but technically speaking we need to take a little care regarding stabilizations. In order to define a functor, we need the following additional choices (which eventually yields the same homomorphisms between $\mathcal{C}^{G_\nabla}$'s).

- A representative $\widetilde{c} = (p\colon N \to X, \widetilde{g}_p)$ of c with $\widetilde{g}_p = (k, \phi, P_p, \nabla_p, \psi_p)$, and we require that k is even.
- A subbundle $H_p \subset \underline{\mathbb{R}}^k \oplus TN$ over N which induces a splitting $\underline{\mathbb{R}}^k \oplus TN = H_p \oplus T(\phi, p)$.
- A Riemannian metric g_X^{met} on X.

Then we define a functor

$$(5.34)\quad \times_X(\widetilde{c}, H_p, g_X^{\mathrm{met}})\colon h\mathrm{Bord}_{n-r-1}^{(G_1)_\nabla}(X, Y)_{\pitchfork \widehat{c}} \to h\mathrm{Bord}_{n-1}^{(G_3)_\nabla}(N, p^{-1}(Y)),$$

as follows. For an object (M, g_M, f) in $h\mathrm{Bord}_{n-r-1}^{(G_1)_\nabla}(X, Y)_{\pitchfork \widehat{c}}$, we consider the fiber product $M \times_X N$ and use the notation of maps as in (5.30). Then by Corollary 5.32, $M \times_X N$ is a smooth $\langle 1 \rangle$-manifold with $\partial(M \times_X N) = \partial M \times_X N$ of dimension $(n-1)$. Moreover, the horizontal subbundle H_p and the Riemannian metric g_X^{met} gives the identification[14]

$$(5.36)\qquad \underline{\mathbb{R}}^k \oplus T(M \times_X N) \simeq \widetilde{p}^* TM \oplus \widetilde{f}^* T(\phi, p).$$

Take a representative $\widetilde{g}_M = (d, P_M, \nabla_M, \psi_M)$ for g_M so that $d \geq n - r = \dim M + 1$. We use the following isomorphism Ψ defined by the composition,

$$\begin{aligned}\Psi\colon \underline{\mathbb{R}}^{d-(n-r-1)+k} \oplus T(M \times_X N) &= \underline{\mathbb{R}}^{d-(n-r)} \oplus \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}} \oplus T(M \times_X N) \\ (5.37)\qquad &\xrightarrow{\text{flip}} \underline{\mathbb{R}}^{d-(n-r)} \oplus \underline{\mathbb{R}} \oplus \underline{\mathbb{R}}^k \oplus T(M \times_X N) \\ &\xrightarrow{(5.36)} (\underline{\mathbb{R}}^{d-(n-r)+1} \oplus \widetilde{p}^* TM) \oplus \widetilde{f}^* \ker(\phi, f).\end{aligned}$$

[14]The identification (5.36) is explicitely given as follows. Notice that TM and TN are equipped with Riemannian metrics by the data g_M, g_p, g_X^{met} and H_p. This gives a splitting of vector bundles over $M \times_X N$ of the form

$$(5.35)\qquad \widetilde{p}^* TM \oplus \widetilde{f}^* TN = T(M \times_X N) \oplus H'.$$

The restriction of $df - dp\colon \widetilde{p}^* TM \oplus \widetilde{f}^* TN \to (p \circ \tilde{f})^* TX$ to H' is an isomorphism,

$$(df - dp)|_{H'}\colon H' \simeq (p \circ \tilde{f})^* TX.$$

Denote by $h^\perp\colon \underline{\mathbb{R}}^k \oplus TN \to T(\phi, p)$ the projection induced by H_p. Then the map (5.36) is given by mapping an element

$$(v, m, n) \in \underline{\mathbb{R}}^k \oplus T(M \times_X N) \subset \underline{\mathbb{R}}^k \oplus TM \oplus TN$$

to the element

$$\left(m + \mathrm{pr}_{\widetilde{p}^* TM}((df - dp)|_{H'}^{-1}(\phi(v))),\ h^\perp\left(v, n + \mathrm{pr}_{\widetilde{f}^* TN}((df - dp)|_{H'}^{-1}(\phi(v)))\right)\right).$$

We can check that this is indeed an isomorphism. Here we note that when p is a submersion, the resulting identification (5.36) does not depend on the choice of g_X^{met} because the splitting (5.35) becomes the obvious one induced by H_p.

Here the second map flips $\underline{\mathbb{R}}^k$ and $\underline{\mathbb{R}}$ (the "flip" is necessary to make a functor). Then the representative $\widetilde{g}_M \times_X \widetilde{g}_p$ of differential stable tangential G_3-structure on $M \times_X N$ is defined to be

(5.38)
$$\widetilde{g}_M \times_X \widetilde{g}_p := \left(d+k, \mu_{d,k}\left(\widetilde{p}^*(P_M, \nabla_M) \times \widetilde{f}^*(P_p, \nabla_p)\right), \Psi^{-1} \circ \left(\widetilde{p}^*(\psi_M) \times \widetilde{f}^*(\psi_p)\right)\right).$$

Here for the last item we use the obvious isomorphism of the associated bundles given by the commutativity of (5.9). Then by the commutativity of the left square of (5.10) we have $(\widetilde{g}_M \times_X \widetilde{g}_p)(1) = (\widetilde{g}_M(1)) \times_X \widetilde{g}_p$. Thus, denoting the resulting differential stable tangential G_3-structure by $g_M \times_X \widetilde{g}_p$, we can define the following object of $h\mathrm{Bord}_{n-1}^{(G_3)\nabla}(N, p^{-1}(Y))$.

$$(M, g_M, f) \times_X (\widetilde{c}, H_p, g_X^{\mathrm{met}}) := (M \times_X N, g_M \times_X \widetilde{g}_p, \widetilde{f}).$$

For morphisms in $h\mathrm{Bord}_{n-r-1}^{(G_1)\nabla}(X, Y)_{\pitchfork \widehat{c}}$, note that we can always take representatives whose underlying maps to X are transverse to p and satisfy the condition (2) in Proposition 5.29. Then the corresponding morphisms in $h\mathrm{Bord}_{n-1}^{(G_1)\nabla}(N, p^{-1}(Y))$ is defined in a similar way by the fiber product over X using Proposition 5.29, but in this case we do *not* insert the "flip" in (5.37). Then we can easily check that we get the functor (5.34) as desired.

As a result, we get a group homomorphism between $\mathcal{C}^{G\nabla}$'s. Notice that the homotopy class (Definition 3.1 (4)) of the representative (5.38) does not depend on the choice of H_p or g_X^{met}. Moreover, recall that we have assumed that k is even. If we use the two-fold stabilization $\widetilde{g}(2) := (\widetilde{g}(1))(1)$, by the homotopy commutativity of (5.10) we see that $(\widetilde{g}_M \times_X \widetilde{g}_p)(2)$ and $\widetilde{g}_M \times_X (\widetilde{g}_p(2))$ are homotopic. Thus, for a differential stable relative tangential G_2-cycle $\widehat{c}$, we get a group homomorphism

$$\times_X \widehat{c} \colon \mathcal{C}_{n-r-1}^{(G_1)\nabla}(X, Y)_{\pitchfork \widehat{c}} \to \mathcal{C}_{n-1}^{(G_3)\nabla}(N, p^{-1}(Y)). \tag{5.39}$$

using any choice of $(\widetilde{c}, H_p, g_X^{\mathrm{met}})$ lifting $\widehat{c}$ as above. Recall that we have $N_G^\bullet = \mathrm{Hom}(\Omega_\bullet^G(\mathrm{pt}), \mathbb{R})$ and $V_{\Omega^G}^\bullet = \Omega_{-\bullet}^G(\mathrm{pt}) \otimes \mathbb{R}$. Thus the homomorphism $\mu \colon G_1 \times G_2 \to G_3$ induces the homomorphism

$$\mu \colon N_{G_3}^q \otimes V_{\Omega^{G_2}}^p \to N_{G_1}^{q-p} \tag{5.40}$$

for each p and q.

Definition 5.41. Let Z be a manifold. We define the linear maps

$$\begin{aligned} \wedge_\mu &\colon \Omega^n(Z; N_{G_3}^\bullet) \otimes \Omega^{-r}(Z; V_{\Omega^{G_2}}^\bullet) \to \Omega^{n-r}(Z; N_{G_1}^\bullet), \\ \wedge_\mu &\colon \Omega^n(Z; N_{G_3}^\bullet) \otimes \Omega_{-\infty}^{-r}(Z; V_{\Omega^{G_2}}^\bullet) \to \Omega_{-\infty}^{n-r}(Z; N_{G_1}^\bullet), \end{aligned} \tag{5.42}$$

by the wedge product on forms and currents, and the homomorphism μ in (5.40) on the coefficients.

By the definition of the functor (5.34), for any morphism $[W, g_W, f_W]$ in $h\mathrm{Bord}_{n-r-1}^{(G_1)\nabla}(X, Y)_{\pitchfork \widehat{c}}$ and $\omega \in \Omega_{\mathrm{clo}}^n(N, p^{-1}(Y); N_{G_3}^\bullet)$, we have

(5.43)
$$\mathrm{cw}(\omega)\left([W, g_W, f_W] \times_X (\widetilde{c}, H_p, g_X^{\mathrm{met}})\right) = \mathrm{cw}(p_!(\omega \wedge_\mu \mathrm{cw}_{g_p}(\mathrm{ch}(\mathrm{id}_{MTG_2}))))([W, g_W, f_W]),$$

where $(\widetilde{c}, H_p, g_X^{\mathrm{met}})$ is any choice lifting $\widehat{c}$, $\mathrm{cw}_{g_p}(\mathrm{ch}(\mathrm{id}_{MTG_2})) \in \Omega^0(N; \mathrm{Ori}(T(p)) \otimes_{\mathbb{R}} V_{\Omega^{G_2}}^\bullet)$ is defined in (5.18) and $p_!$ is the fiber integration of currents (5.19).

In the right hand side of (5.43), we use the obvious currential version of (4.20). We use this generalization throughout the rest of the paper.

In particular, for $\omega \in \Omega^n_{\mathrm{clo}}(X, Y; N^\bullet_{G_3})$, we have

$$\mathrm{cw}(\omega) \circ p_* \left([W, g_W, f_W] \times_X (\widetilde{c}, H_p, g_X^{\mathrm{met}})\right) = \mathrm{cw}(\omega \wedge_\mu T(\widehat{c}))([W, g_W, f_W]). \tag{5.44}$$

Here we denoted the functor p_* on the bordism Picard categories over $(N, p^{-1}(Y))$ to (X, Y) given by the composition of p. We will use the same notation for the corresponding homomorphism between $\mathcal{C}^{(G_1)\triangledown}$'s.

Now we proceed to define the multiplication (5.7). First, we define the multiplication of each element $(\widehat{c}, \alpha) \in (\widehat{Z\Omega^{G_2}})^{-r}(X)$ such that the underlying map p is transverse to $Y \subset X$. Given such an element, we set $(\widehat{I\Omega^{G_1}_{\pitchfork \widehat{c}}})^{n-r}(X, Y)$ to be the group in Definition 4.44 for $\mathcal{D} = h\mathrm{Bord}^{(G_1)\triangledown}_{n-r-1}(X, Y)_{\pitchfork \widehat{c}}$. It is isomorphic to $(\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X, Y)$ by Proposition 4.45 and Lemma 5.33.

Definition 5.45. Let (X, Y) be a pair of manifolds and r be an integer. Given $(\widehat{c}, \alpha) \in (\widehat{Z\Omega^{G_2}})^{-r}(X)$ such that $\widehat{c}$ is transverse to Y, we define a linear map

$$\times_X (\widehat{c}, \alpha) \colon (\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^{n}(X, Y) \to (\widehat{I\Omega^{G_1}_{\pitchfork \widehat{c}}})^{n-r}(X, Y), \tag{5.46}$$

by sending (ω, h) to $(\omega \wedge_\mu R([\widehat{c}, \alpha]), (\widehat{c}, \alpha)_* h)$, where

$$(\widehat{c}, \alpha)_* h := h \circ p_* \circ (\times_X \widehat{c}) - \mathrm{cw}(\omega \wedge_\mu \alpha) \colon \mathcal{C}^{(G_1)\triangledown}_{n-r-1}(X, Y)_{\pitchfork \widehat{c}} \to \mathbb{R}/\mathbb{Z}. \tag{5.47}$$

The compatibility condition for $(\omega \wedge_\mu R([\widehat{c}, \alpha]), (\widehat{c}, \alpha)_* h)$ is checked as follows. Take a morphism $[W, g_W, f_W] \colon (M_-, g_-, f_-) \to (M_+, g_+, f_+)$ in $h\mathrm{Bord}^{(G_1)\triangledown}_{n-r-1}(X, Y)_{\pitchfork \widehat{c}}$. We have, choosing any $(\widetilde{c}, H_p, g_X^{\mathrm{met}})$,

$$\begin{aligned} h \circ p_* \left(([M_+, g_+, h_+] - [M_-, g_-.f_-]) \times_X \widehat{c}\right) &= \mathrm{cw}(\omega) \circ p_* \left([W, g_W, f_W] \times_X (\widetilde{c}, H_p, g_X^{\mathrm{met}})\right) \\ &= \mathrm{cw}(\omega \wedge_\mu T(\widehat{c}))([W, g_W, f_W]) \end{aligned}$$

by the compatibility of (ω, h) and (5.44). Moreover we have

$$\mathrm{cw}(\omega \wedge_\mu \alpha) \left([M_+, g_+, h_+] - [M_-, g_-.f_-]\right) = \mathrm{cw}(\omega \wedge_\mu (T(\widehat{c}) - R([\widehat{c}, \alpha])))([W, g_W, f_W])$$

by (5.23). Thus we get the desired compatibility.

Lemma 5.48. *The composition of the map (5.46) with the isomorphism $(\widehat{I\Omega^{G_1}_{\pitchfork \widehat{c}}})^{n-r}(X, Y) \simeq (\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X, Y)$ in Proposition 4.45 only depends on the class $[\widehat{c}, \alpha] \in (\widehat{\Omega^{G_2}})^{-r}(X)$ of $(\widehat{c}, \alpha)$.*

Proof. By Definition 5.25, it is enough to check that for any r-dimensional bordism data $\widehat{b} = (q \colon W \to \mathbb{R} \times X, g_q)$ and any element $[M, g, f] \in \mathcal{C}^{(G_1)\triangledown}_{n-r-1}(X, Y)_{\pitchfork \partial \widehat{b}}$, we have

$$h \circ (q|_\partial)_* \left([M, g, f] \times_X \partial \widehat{b}\right) = \mathrm{cw}\left(\omega \wedge_\mu T(\widehat{b})\right)([M, g, f]) \pmod{\mathbb{Z}}. \tag{5.49}$$

To check it, take an object $(M, g, f) \in h\mathrm{Bord}^{(G_1)\triangledown}_{n-r-1}(X, Y)_{\pitchfork \partial \widehat{b}}$ representing $[M, g, f]$ and a data $(\widetilde{b}, H_q, g^{\mathrm{met}}_{\mathbb{R} \times X})$ lifting $\widehat{b}$, where we require $g^{\mathrm{met}}_{\mathbb{R} \times X}$ to be a cylindrical metric induced by some g_X^{met} on X. Then we can construct a bordism over (X,Y) (in the sense of Definition 3.7) from $\varnothing$ to $(M,g,f)\times_X (\partial\widetilde{b}, H_q|_\partial, g_X^{\mathrm{met}})$, essentially by "$(M,g,f)\times_X \left(\mathrm{pr}_X^*((\widetilde{b}, H_q, g^{\mathrm{met}}_{\mathbb{R}\times X})|_{(-\infty,0]\times X})\right)$". Here the fiber product over X is extended to this case in the obvious way, and we need a suitable deformation to have a collar structure. Since any choice are bordant, we abuse the notation and denote the resulting morphism in $h\mathrm{Bord}_{n-1}^{(G_3)\triangledown}(X,Y)$ by $q_*\left[(M,g,f)\times_X \left(\mathrm{pr}_X^*((\widetilde{b}, H_q, g^{\mathrm{met}}_{\mathbb{R}\times X})|_{(-\infty,0]\times X})\right)\right]$. Now the compatibility condition of (ω,h) implies that

$$h\circ (q|_\partial)_*\left([M,g,f]\times_X \partial\widehat{b}\right) = \mathrm{cw}(\omega)\left(q_*\left[(M,g,f)\times_X \left(\mathrm{pr}_X^*((\widetilde{b}, H_q, g^{\mathrm{met}}_{\mathbb{R}\times X})|_{(-\infty,0]\times X})\right)\right]\right) \pmod{\mathbb{Z}}.$$

On the other hand, (5.21) and (5.24) implies that the right hand side is equal to the right hand side of (5.49). This completes the proof. □

Thus, we can define the following.

Definition 5.50. Let (X,Y) be a pair of manifolds. We define a linear map

$$(5.51)\qquad (\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^n(X,Y)\otimes(\widehat{\Omega^{G_2}})^{-r}(X) \dot{\to} (\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X,Y).$$

by sending $(\omega,h)\otimes[\widehat{c},\alpha]$ to $(\omega,h)\times_X(\widehat{c},\alpha)\in(\widehat{I\Omega^{G_1}_{\pitchfork\widehat{c}}})^{n-r}(X,Y)\simeq(\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X,Y)$. This does not depend on the representative $(\widehat{c},\alpha)$ by Lemma 5.48.

Finally we show the following.

Theorem 5.52. *The map (5.51) refines the transformation (5.6) defined by (5.4).*

Proof. We use the arguments in Subsection 4.2. Recall that for an element $(\omega,h)\in(\widehat{I\Omega^{G}_{\mathrm{dR}}})^N(X,Y)$ we associated a functor $F_{(\omega,h)}$ in (4.51). By the proof of Theorem 4.56, the isomorphism $I\Omega^G_{\mathrm{dR}}\simeq I\Omega^G$ is given by mapping $I((\omega,h))$ to the natural isomorphism class of $F_{(\omega,h)}$.

Take any element $(\widehat{c},\alpha)\in(Z\widehat{\Omega^{G_2}})^{-r}(X)$ such that $\widehat{c}$ is transverse to Y and $(\omega,h)\in(\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^n(X,Y)$, so that we get $(\omega,h)\cdot[\widehat{c},\alpha]\in(\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X,Y)$. Then we claim that the restriction to $h\mathrm{Bord}^{(G_1)\triangledown}_{n-r-1}(X,Y)_{\pitchfork\widehat{c}}$ of the associated functor (4.51),

$$(5.53)\qquad F_{(\omega,h)\cdot[\widehat{c},\alpha]}\colon h\mathrm{Bord}^{(G_1)\triangledown}_{n-r-1}(X,Y)_{\pitchfork\widehat{c}}\to(\mathbb{R}\to\mathbb{R}/\mathbb{Z}),$$

is naturally isormorphic to the composition,

(5.54)

$$h\mathrm{Bord}^{(G_1)\triangledown}_{n-r-1}(X,Y)_{\pitchfork\widehat{c}}\xrightarrow{p_*\circ(\times_X(\widetilde{c},H_p,g_X^{\mathrm{met}}))} h\mathrm{Bord}^{(G_3)\triangledown}_{n-1}(X,Y)\xrightarrow{F_{(\omega,h)}}(\mathbb{R}\to\mathbb{R}/\mathbb{Z}),$$

for any choice of $(\widetilde{c},H_p,g_X^{\mathrm{met}})$ lifting $\widehat{c}$. Indeed, we have a natural isomorphism given by the transformation $F_{\omega\wedge_\mu\alpha}$. Here $\omega\wedge_\mu\alpha\in\Omega^{n-r-1}_{-\infty}(X,Y;N^\bullet_{G_1})$ is now a differential current, and the definition of the natural transformation (4.52) is extended to currents in the obvious way.

We use the equivalence of Picard groupoids in Lemma 3.10. By recalling the Pontryagin-Thom construction, we see that the first arrow in (5.54) is naturally isomorphic to the induced functor on $\pi_{\le 1}((-)_{-n+r+1})$ to the following compostion of maps of spectra.

$$
\begin{aligned}
(X/Y)\wedge MTG_1 &\xrightarrow{\mathrm{diag}\wedge \mathrm{id}} (X/Y)\wedge X^+ \wedge MTG_1 \\
&\xrightarrow{\mathrm{id}\wedge [c]\wedge \mathrm{id}} (X/Y)\wedge \Sigma^{-r} MTG_2 \wedge MTG_1 \\
&\xrightarrow{\mathrm{id}\wedge \mu} (X/Y)\wedge \Sigma^{-r} MTG_3.
\end{aligned}
\tag{5.55}
$$

The Anderson dual to this composition is the definition of the multiplication by $[c] \in (\Omega^{G_2})^{-r}(X)$ on the topological level (5.6) defined by (5.4). This completes the proof. $\square$

6. Differential pushforwards

Let $\mu\colon G_1 \times G_2 \to G_3$ be a homomorphism of tangential structure groups as in Section 5 (Remark 5.8). In this section, we introduce the differential refinement of *pushforward maps* associated to μ. Here, what we call the pushforwards here is a generalization of pushforward maps (also called *integrations*) associated to multiplicative genera, which we explain in Subsection 6.1. For each pair of manifolds (X, Y) and differential relative stable tangential G-cycle (Definition 5.15) $\widehat{c} = (p\colon N \to X, g_p)$ of dimension r over X whose underlying map p is a submersion, we define a homomorphism which we call the *differential pushforward map* along $\widehat{c}$,

$$
\widehat{c}_*\colon (\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^n(N, p^{-1}(Y)) \to (\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X, Y),
\tag{6.1}
$$

which refines the *topological pushforward map* defined for a topological relative stable tangential G_2-cycle $c = (p\colon N \to X, g_p^{\mathrm{top}})$,

$$
c_*\colon (I\Omega^{G_3})^n(N, p^{-1}(Y)) \to (I\Omega^{G_1})^{n-r}(X, Y),
\tag{6.2}
$$

by the general procedure in Subsection 6.1 applied to the morphism $IMTG_3 \wedge MTG_2 \to IMTG_1$ in (5.5) associated to μ. Actually, the homomorphism (6.1) is given by a straightforward modification of the multiplication by $\widehat{\Omega^{G_2}}$ in Section 5. As we explain in Remark 6.17, we can recover the S^1-integration map $\int$ of $\widehat{I\Omega^G_{\mathrm{dR}}}$ (Definition 4.39) as a special case of the construction in this section. Also we explain in Remark 6.18 that if we use the obvious *currential* version of $\widehat{I\Omega^G_{\mathrm{dR}}}$, we can generalize the results in this section to the case where p is not necessarily a submersion.

6.1. **Generalities on topological pushforwards.**

Here we introduce the definition of *(topological) pushforward maps* which we use in this paper. This is a generalization of the most common notion of pushforward maps associated to multiplicative genera. We only explain the tangential version, but the normal version also works by just replacing MTG to MG.

The setting is the following. Assume we have tangential structure groups G and two spectra E and F, together with a homomorphism of spectra,

$$
\mu\colon E \wedge MTG \to F.
\tag{6.3}
$$

We claim that (6.3) defines, for a topological relative stable G-cycle $c = (p\colon N \to X, g_p^{\mathrm{top}})$ of dimension r, a homomorphism

$$c_*\colon E^n(N, p^{-1}(Y)) \to F^{n-r}(X, Y), \tag{6.4}$$

which we call the *(topological) pushforward map* along c.

For simplicity we first explain the absolute case, $Y = \varnothing$. Given $c = (p\colon N \to X, g_p^{\mathrm{top}})$ as above, choose an embedding $\iota\colon N \hookrightarrow \mathbb{R}^k \times X$ over X (i.e., $\mathrm{pr}_X \circ \iota = p$) for k large enough. Choose a tubular neighborhood U of N in $\mathbb{R}^k \times X$ with a vector bundle structure $\pi\colon U \to N$. Then g_p^{top} induces a homotopy class of stable normal G-structures $g_\pi^{\perp,\mathrm{top}}$ on the vector bundle $\pi\colon U \to N$. Thus we get the (universal) Thom element for this vector bundle in Ω^G,

$$\nu(g_\pi^{\perp,\mathrm{top}}) \in (\Omega^G)^{k-r}(\mathrm{Thom}(\pi\colon U \to N)).$$

The pushforward map (6.4) is defined as the following composition.

(6.5)

$$E^n(N) \xrightarrow{\cdot\nu(g_\pi^{\perp,\mathrm{top}})} F^{n+k-r}(\mathrm{Thom}(\pi\colon U \to N)) \xrightarrow{\iota_*} F^{n+k-r}(\mathrm{Thom}(\mathbb{R}^k \times X \to X)) \xrightarrow{\mathrm{desusp}} F^{n-r}(X),$$

where the first multiplication uses μ, and the middle arrow is associated to the open embedding $\iota\colon U \hookrightarrow \mathbb{R}^k \times X$.

In the general case $Y \neq \varnothing$, the definition is basically the same, but we need to be careful because $(\mathbb{R}^k \times Y) \cap \pi^{-1}(N \setminus p^{-1}(Y)) \neq \varnothing$ in general. This subtlety does not arise when p is a submersion, because we can take π to be a map over X. In general, we need to perturb the map p by a homotopy so that p is transverse to the inclusion $Y \hookrightarrow X$. Then we can take the choices above so that $(\mathbb{R}^k \times Y) \cap \pi^{-1}(N \setminus p^{-1}(Y)) = \varnothing$. Applying this procedure to the morphism $IMTG_3 \wedge MTG_2 \to IMTG_1$ in (5.5), we get the topological pushforward maps (6.2).

An important class of the examples of homomorphisms (6.3) comes from *multiplicative genera*, i.e., homomorphisms of ring spectra

$$\mathcal{G}\colon MTG \to E, \tag{6.6}$$

for multiplicative G and E. In this case, $\mathcal{G}$ induces a MTG-module structure on E,

$$\mu_{\mathcal{G}}\colon E \wedge MTG \to E. \tag{6.7}$$

Applying the construction in this subsection to $\mu_{\mathcal{G}}$, we recover the usual pushforward in E for tangentially G-oriented proper maps.

6.2. **The differential pushforwards.** Now let us fix a homomorphism $\mu\colon G_1 \times G_2 \to G_3$. We construct the differential pushforward maps (6.1). Actually, we have already prepared the necessary ingredients in Section 5. Recall that, given a differential relative stable tangential G_2-cycle $\widehat{c} = (p\colon N \to X, g_p)$ of dimension r such that p is transverse to $Y \subset X$, we defined a group homomorphism (5.39)

$$\times_X \widehat{c}\colon \mathcal{C}^{(G_1)\nabla}_{n-r-1}(X, Y)_{\pitchfork\widehat{c}} \to \mathcal{C}^{(G_3)\nabla}_{n-1}(N, p^{-1}(Y)).$$

If p is a submersion, we have $\mathcal{C}^{(G_1)\triangledown}_{n-r-1}(X,Y)_{\pitchfork\widehat{c}} = \mathcal{C}^{(G_1)\triangledown}_{n-r-1}(X,Y)$. Also recall that we have (5.18)

$$\mathrm{cw}_{g_p}(\mathrm{ch}(\mathrm{id}_{MTG_2})) \in \Omega^0(N; \mathrm{Ori}(T(p)) \otimes_{\mathbb{R}} V^{\bullet}_{\Omega G_2}),$$

Given $\omega \in \Omega^n(N, p^{-1}(N); N^{\bullet}_{G_3})$, using the wedge product in Definition 5.41 and the fiber integration (5.19), we have

$$p_!(\omega \wedge_\mu \mathrm{cw}_{g_p}(\mathrm{ch}(\mathrm{id}_{MTG_2}))) \in \Omega^{n-r}_{-\infty}(X,Y; N^{\bullet}_{G_1}). \tag{6.8}$$

If p is a submersion, the element (6.8) is in $\Omega^{n-r}(X,Y;N^{\bullet}_{G_1})$.

Definition 6.9 (The differential pushforward maps associated to $\mu\colon G_1 \times G_2 \to G_3$)**.** Let (X,Y) be a pair of manifolds and n and r be nonnegative integers such that $n \geq r$. For each differential relative stable tangential G_2-cycle $\widehat{c} = (p\colon N \to X, g_p)$ of dimension r over X whose underlying map p is a submersion, we define a homomorphism which we call the *differential pushforward map* along $\widehat{c}$,

$$\widehat{c}_*\colon (\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^n(N, p^{-1}(Y)) \to (\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X,Y), \tag{6.10}$$

by mapping (ω, h) to $\left(p_!(\omega \wedge_\mu \mathrm{cw}_{g_p}(\mathrm{ch}(\mathrm{id}_{MTG_2}))), \widehat{c}_*h\right)$, where

$$\widehat{c}_*h := h \circ (\times_X \widehat{c})\colon \mathcal{C}^{(G_1)\triangledown}_{n-r-1}(X,Y) \to \mathbb{R}/\mathbb{Z}. \tag{6.11}$$

The compatibility condition for the pair $\left(p_!(\omega \wedge_\mu \mathrm{cw}_{g_p}(\mathrm{ch}(\mathrm{id}_{MTG_2}))), \widehat{c}_*h\right)$ can be checked by the same way as the corresponding compatibility checked for Definition 5.45, by using (5.43) instead of (5.44).

Theorem 6.12. *The differential pushforward map* (6.10) *refines the topological pushforward map* (6.2) *defined by* (6.5).

Proof. Let us fix an element $(\omega, h) \in (\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^n(N, p^{-1}(Y))$. As we did in the proof of Theorem 5.52, the proof is given by checking the natural isomorphism class of the functor (4.51) associated to the element $\widehat{c}_*(\omega,h) \in (\widehat{I\Omega^{G_1}_{\mathrm{dR}}})^{n-r}(X,Y)$,

$$F_{\widehat{c}_*(\omega,h)}\colon h\mathrm{Bord}^{(G_1)\triangledown}_{n-r-1}(X,Y) \to (\mathbb{R} \to \mathbb{R}/\mathbb{Z}), \tag{6.13}$$

coincides, under the equivalence in Lemma 3.10, with the element in

$$\pi_0\mathrm{Fun}_{\mathrm{Pic}}\left(\pi_{\leq 1}(L((X/Y) \wedge MTG_1)_{1-n+r}), (\mathbb{R} \to \mathbb{R}/\mathbb{Z})\right),$$

specified by the topological pushforward (6.2).

Recall that the group homomorphism (5.39) between $\mathcal{C}^{G\triangledown}$'s comes from the functor between the bordism Picard groupoids (5.34),

$$\times_X(\widetilde{c}, H_p)\colon h\mathrm{Bord}^{(G_1)\triangledown}_{n-r-1}(X,Y) \to h\mathrm{Bord}^{(G_3)\triangledown}_{n-1}(N, p^{-1}(Y)), \tag{6.14}$$

by choosing additional data of $(\widetilde{c}, H_p)$ lifting $\widehat{c}$ (In the case here p is submersion, so we do not need the choice of a Riemannian metric g_X^{met} as remarked at the end of Footnote 14). We easily see that the functor (6.13) coincides with the following composition.

(6.15)
$$h\mathrm{Bord}^{(G_1)\triangledown}_{n-r-1}(X,Y) \xrightarrow{\times_X(\widetilde{c},H_p)} h\mathrm{Bord}^{(G_3)\triangledown}_{n-1}(N, p^{-1}(Y)) \xrightarrow{F_{(\omega,h)}} (\mathbb{R} \to \mathbb{R}/\mathbb{Z}).$$

On the other hand, the topological pushforward map (6.2) is defined as the composition (6.5). On the level of spectra, this homomorphism is given by the Anderson dual of the composition

(6.16)
$$\begin{aligned} MTG_1 \wedge \Sigma^k(X/Y) &\xrightarrow{\mathrm{id}\wedge \iota} MTG_1 \wedge \left(\mathrm{Thom}(\pi\colon U \to N)/\pi^{-1}(p^{-1}(Y))\right) \\ &\to MTG_1 \wedge \Sigma^{k-r} MTG_2 \wedge (N/p^{-1}(Y)) \\ &\xrightarrow{\mu \wedge \mathrm{id}} \Sigma^{k-r} MTG_3 \wedge (N/p^{-1}(Y)), \end{aligned}$$

where the second morphism is the classifying map of the stable normal G-structure $g_\pi^{\perp,\mathrm{top}}$ on $\pi\colon U \to N$ and the identity on MTG_1. Recalling the Pontryagin-Thom construction, we see that the functor between the fundamental Picard groupoids

$$\pi_{\le 1}(L((X/Y) \wedge MTG_1)_{1-n+r}) \to \pi_{\le 1}(L((N/p^{-1}(N)) \wedge MTG_3)_{1-n})$$

induced by (6.16) is naturally isomorphic to the fiber product functor (6.14) under the equivalences in Lemma 3.10. This completes the proof. $\square$

Remark 6.17. We can recover the S^1-integration map $\int$ of $\widehat{I\Omega^G_{\mathrm{dR}}}$ (Definition 4.39) as a special case of the construction in this section. As explained in Example 5.3 (3), for any G we have a canonical homomorphism $G \times \mathrm{fr} \to G$. Let X be a manifold. The bounding (differential) stable tangential fr-structure g_{S^1} on $S^1 = S^1 \times \mathrm{pt}$ in Definition 4.38 induces the differential stable relative tangential fr-structure on $\mathrm{pr}_X\colon X \times S^1 \to X$. We easily see that the differential pushforward along this differential stable relative fr-cycle coincides with the S^1 integration map, but up to sign. The difference of the sign is due to the fact that the suspension multiplies S^1 from the left. This difference also appears in the difference between the signs for the S^1-integration on forms in (2.16), (2.17) and the fiber integration in (5.19), (5.20).

Remark 6.18. We have only defined differential pushforwards for proper *submersions*. This requirement is necessary for the element (6.8) to be a differential form. Actually, we can get refinements of pushforwards along general differential stable relative cycles if we introduce *currential* refinement of $I\Omega^G$'s. In general, currential refinements of cohomology theories are axiomatized by simply replacing forms to currents in Definition 2.12. Such refinements are used for example in [FL10] in the case of K-theory. In our case, it is obvious that we obtain the currential refinement of $I\Omega^G$, which we denote by $\widehat{I\Omega^G_{-\infty}}$, by just allowing ω in (ω, h) to be closed currents.

Then, given any differential stable relative G_2-cycle $\widehat{c} = (p\colon N \to X, g_p)$ of dimension r over X such that p is transverse to $Y \subset X$, by the same construction we get the refinement of the pushforward map,

$$\widehat{c}_*\colon (\widehat{I\Omega^{G_3}_{\mathrm{dR}}})^n(N, p^{-1}(Y)) \to (\widehat{I\Omega^{G_1}_{-\infty}})^{n-r}(X, Y). \tag{6.19}$$

We remark that in the construction we use the obvious currential version of Subsection 4.1.2.

Part 2.

7. Introduction to Part 2

In Part 1, we constructed a model $I\Omega^G_{\mathrm{dR}}$ of $I\Omega^G$ and its differential extension $\widehat{I\Omega^G_{\mathrm{dR}}}$ by abstractizing certain properties of invertible QFT's. Part 2 is devoted to their relations with *multiplicative genera*. We show that *pushforwards* (also called *integrations*) in generalized differential cohomology theories allow us to construct differential refinements of certain cohomology transformations which arise from the Anderson dual to multiplicative genera and the module structures of the Anderson duals. This gives us a unified understanding of an important class of elements in the Anderson duals with physical origins.

First, we explain the motivations of Part 2. As we saw in Part 1, the differential group $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$ consists of pairs (ω, h), where $\omega \in \Omega^*_{\mathrm{clo}}(X; H^\bullet(MTG;\mathbb{R}))$ with total degree n, and h is a map which assigns $\mathbb{R}/\mathbb{Z}$-values to *differential stable tangential G-cycles* of dimension $(n-1)$ over X, which satisfy a compatibility condition with respect to bordisms. For example, given a hermitian line bundle with unitary connection over X, the pair of the first Chern form and the holonomy function gives an element for $G = \mathrm{SO}$ and $n = 2$ (Example 4.59). Similarly, given a hermitian vector bundle with unitary connection, we can construct even-degree elements for $G = \mathrm{Spin}^c$ using the reduced eta invariants of twisted Dirac operators (Example 4.67). Then, a natural mathematical question arises: *what are these elements mathematically?* It is natural to expect a topological characterization of these elements. Questions of this kind also appears in [FH21, Conjecture 9.70]. Part 2 is devoted to this question. Actually, these examples are special cases of the general construction in this paper which we now explain.

Now we explain the general settings. In this paper, the tangential structure groups $G = \{G_d, s_d, \rho_d\}_{d\in\mathbb{Z}_{\geq 0}}$ is assumed to be *multiplicative*, i.e., the corresponding Madsen-Tillmann spectrum MTG is equipped with a structure of a ring spectrum. Assume we are given a ring spectrum E with a homomorphism of ring spectra,

$$\mathcal{G}\colon MTG \to E.$$

such $\mathcal{G}$ is also called a *multiplicative genus*, and examples include the usual orientation $\tau\colon MT\mathrm{SO} \to H\mathbb{Z}$ and the Atiyah-Bott-Shapiro orientations $\mathrm{ABS}\colon MT\mathrm{Spin}^c \to K$ and $\mathrm{ABS}\colon MT\mathrm{Spin} \to KO$.

On the topological level, a ring homomorphism $\mathcal{G}\colon MTG \to E$ gives *pushforwards* in E for proper G-oriented smooth maps. *Pushforwards in differential cohomology*, or *differential pushforwards*, are certain differential refinements of topological pushforwards. Basically, they consist of corresponding maps in $\widehat{E}$ for each proper map with a "differential E-orientation". The formulations depend on the context. To clarify this point, in Part 2 we use the differential extension $\widehat{E}_{\mathrm{HS}}$ of E constructed by Hopkins-Singer [HS05], and use the formulation of differential pushforwards in that paper. Throughout Part 2, we assume that E is *rationally even*, i.e., $E^{2k+1}(\mathrm{pt}) \otimes \mathbb{R} = 0$ for any integer k. In this case, by [Upm15] there exists a canonical multiplicative structure on the Hopkins-Singer's differential extension $\widehat{E}^*_{\mathrm{HS}}(-;\iota_E)$ associated to a fundamental cycle $\iota_E \in Z^0(E;V_E^\bullet)$. The theory of differential pushforwards gets simple in this case. This point is explained in Subsection 8.1 and Appendix A. Of course our result applies to any model of differential extension $\widehat{E}$ of E which is isomorphic to the Hopkins-Singer's model. Practically, most known examples of differential extensions are isomorphic to Hopkins-Singer's model (see Footnote 15). The holonomy functions are examples of differential pushforwards in the case $\tau\colon MT\mathrm{SO} \to H\mathbb{Z}$, and the reduced eta invariants are those for $\mathrm{ABS}\colon MT\mathrm{Spin}^c \to K$ by the result of Freed and Lott [FL10] and Klonoff [Klo08].

Let n be an integer such that $E^{1-n}(\mathrm{pt})\otimes\mathbb{R} = 0$. As we show in Subsection 8.3, the above data defines the following natural transformation,

$$\Phi_{\mathcal{G}}\colon \widehat{E}^*_{\mathrm{HS}}(-;\iota_E) \otimes IE^n(\mathrm{pt}) \to (\widehat{I\Omega^G_{\mathrm{dR}}})^{*+n}(-), \tag{7.1}$$

on $\mathrm{Mfd}^{\mathrm{op}}$ (Definition 8.29).

The main result of Part 2 is the following topological characterization of the transformation (7.1).

Theorem 7.2. *In the above settings, let X be a manifold and k be an integer. For $\widehat{e} \in \widehat{E}^k_{\mathrm{HS}}(X;\iota_E)$ and $\beta \in IE^n(\mathrm{pt})$, the element $I(\Phi_{\mathcal{G}}(\widehat{e}\otimes\beta)) \in (I\Omega^G)^{k+n}(X) = [X^+ \wedge MTG, \Sigma^{k+n}I\mathbb{Z}]$ coincides with the following composition,*

$$X^+ \wedge MTG \xrightarrow{e\wedge\mathcal{G}} \Sigma^k E \wedge E \xrightarrow{\text{multi}} \Sigma^k E \xrightarrow{\beta} \Sigma^{k+n} I\mathbb{Z}. \tag{7.3}$$

Here we denoted $e := I(\widehat{e}) \in E^k(X)$.

As we will see in Subsection 8.4, the transformations $\Phi_\tau(-\otimes\gamma_{H\mathbb{Z}})$ and $\Phi_{\mathrm{ABS}}(-\otimes\gamma_K)$ in (7.1), where $\gamma_{H\mathbb{Z}} \in IH\mathbb{Z}^0(\mathrm{pt})$ and $\gamma_K \in IK^0(\mathrm{pt})$ are the Anderson self-duality elements of $H\mathbb{Z}$ and K respectively, recovers the above mentioned examples. Applying Theorem 7.2, we get the desired topological characterization of such examples.

Part 2 is organized as follows. We construct the natural transformation (7.1) and prove Theorem 7.2 in Subsection 8.3. We explain some examples in Subsection 8.4. As we explain in Subsection 8.1, there are certain subtleties regarding the formulations of differential pushforwards. In Appendix A, we collect the necessary results concerning differential pushforwards for submersions when E is rationally even.

8. Pushforwards in differential cohomologies and the Anderson duality

8.1. **Preliminary–Differential pushforwards in the Hopkins-Singer model.** In this subsection, we briefly explain the differential extensions of generalized cohomology theories constructed by Hopkins-Singer and the differential pushforwards (called *integration* in [HS05]) in that model. We explain it in more detail in Appendix A.

On the topological level, a ring homomorphism $\mathcal{G}\colon MTG \to E$ gives *pushforwards* in E for G-oriented proper smooth maps. For proper smooth maps $p\colon N \to X$ of relative dimension $r := \dim N - \dim X$ with (topological) stable relative tangential G-structures g_p^{top}, we get the corresponding push-forward map,

$$(p, g_p^{\mathrm{top}})_* \colon E^*(N) \to E^{*-r}(X). \tag{8.1}$$

In particular in the case $X = \mathrm{pt}$, for a closed manifold M of dimension n with a stable tangential G-structure g^{top}, we get

$$(p_M, g^{\mathrm{top}})_* \colon E^*(M) \to E^{*-n}(\mathrm{pt}).$$

There are notions of differential refinements of the pushforward maps in $\widehat{E}$. For example see [HS05, Section 4.10], [BSSW09, Section 2] and [Bun12, Section 4.8 – 4.10]. Basically, they consist of corresponding maps in $\widehat{E}$ for each proper map with a "differential E-orientation". The formulations depend on the context. In this paper, we adopt the one by Hopkins-Singer[15].

Hopkins and Singer gave a model of differential extensions, which we denote by $\widehat{E}^*_{\mathrm{HS}}(-;\iota_E)$, for any spectrum E, in terms of *differential function complexes*. In general we choose a $\mathbb{Z}$-graded vector space $V^\bullet$, and a singular cocycle $\iota_E \in Z^0(E; V^\bullet) = \varprojlim_n Z^n(E_n; V^\bullet)$. Then for each n and for each manifold X, we get a simplicial complex called *differential function complex*,

$$(E_n; \iota_n)^X = (E; \iota)_n^X,$$

consisting of *differential functions* $X \times \Delta^\bullet \to (E_n; \iota_n)$. This complex has a filtration $\mathrm{filt}_s(E;\iota)_n^X$, $s \in \mathbb{Z}_{\geq 0}$. The differential cohomology group is defined as (it is denoted by $E(n)^n(X;\iota)$ in [HS05]),

$$\widehat{E}^n_{\mathrm{HS}}(X;\iota) := \pi_0 \mathrm{filt}_0(E;\iota)_n^X.$$

In particular this means that an element in $\widehat{E}^n_{\mathrm{HS}}(X;\iota)$ is represented by a *differential function* $(c, h, \omega) \colon X \to (E_n; \iota_n)$, consisting of a continuous map $c \colon X \to E_n$, a closed form $\omega \in \Omega^n_{\mathrm{clo}}(X; V^\bullet)$ and a singular cochain $h \in C^{n-1}(X; V^\bullet)$ such that $\delta h = c^*\iota_n - \omega$ as smooth singular cocycles.

A particularly important case is when $V = V_E^\bullet$ and $\iota_E \in Z^0(E; V_E^\bullet)$ is the *fundamental cocycle*, i.e., a singular cocycle representing the Chern-Dold character of E. In this case the associated differential cohomology groups $\widehat{E}^n_{\mathrm{HS}}(X;\iota_E)$ satisfies the axioms of differential cohomology theory in [BS10]. The isomorphism class of the resulting group is independent of the choice of the fundamental cocycle ι_E, with an isomorphism given by a cochain cobounding the difference.

In [HS05, Section 4.10], a differential pushforward is defined simply as maps of differential function spaces[16],

$$\widehat{\mathcal{G}} \colon \big(MTG_{-r} \wedge (E_n)^+; V_{\mathcal{G}}(\iota_{MTG})_{-r} \cup (\iota_E)_n\big) \to (E;\iota_E)_{n-r}, \tag{8.2}$$

[15] In particular we use the differential extension $\widehat{E}_{\mathrm{HS}}$. Practically this is not restrictive. We are assuming E is rationally even and multiplicative, so $\widehat{E}_{\mathrm{HS}}$ is equipped with a canonical multiplicative structure by [Upm15]. Thus, when the coefficients of E are countably generated, we can apply the uniqueness result in [BS10, Theorem 1.7] to conclude that any other multiplicative differential extension (defined on the category of all smooth manifolds) is isomorphic to $\widehat{E}_{\mathrm{HS}}$.

[16] This point is important in the proof of Proposition 8.17, which is the main ingredient of the proof of the main result (Theorem 7.2). This is the reason why we want to use the Hopkins-Singer's formulation.

refining the map $MTG \wedge (E_n)^+ \xrightarrow{\mathcal{G}\wedge \mathrm{id}} E \wedge (E_n)^+ \xrightarrow{\text{multi}} \Sigma^n E$. Here we are taking $V = V_E^\bullet$, and the cocycle $V_{\mathcal{G}}(\iota_{MTG}) \in Z^0(MTG; V_E^\bullet)$ is obtained by applying $V_{\mathcal{G}} \colon V_{MTG} \to V_E$ on the coefficient of ι_{MTG}. Then[17], the map $\widehat{\mathcal{G}}$ associates to every proper neat map of $p \colon N \to X$ of relative dimension r with a *differential (tangential) BG-orientation* g_p^{HS} with a map

$$(p, g_p^{\mathrm{HS}})_* \colon \widehat{E}^*_{\mathrm{HS}}(N; \iota_E) \to \widehat{E}^{*-r}_{\mathrm{HS}}(X; \iota_E), \tag{8.3}$$

called the *differential pushforward map*.

As we explain in Appendix A.2 and A.3, the definition of (the tangential version of) differential BG-oriented maps in [HS05] differs from the differential stable relative G-structure in Definition 5.12. Fix a fundamental cocycle $\iota_{MTG} \in Z^0(MTG; V_{MTG}^\bullet)$. Given a proper smooth map $p \colon N \to X$, a topological tangential BG-orientation consists of a choice of embedding $N \hookrightarrow \mathbb{R}^N \times X$ for some N, a tubular neighborhood W of N in $\mathbb{R}^N \times X$ with a vector bundle structure $W \to N$, and a classifying map $\overline{W} := \mathrm{Thom}(W) \to MTG_{N-r}$. A differential tangential BG-orientation g_p^{HS} consists of its lift to a differential function

$$t(g_p^{\mathrm{HS}}) = (c, h, \omega) \colon \overline{W} \to (MTG_{N-r}, (\iota_{MTG})_{N-r}), \tag{8.4}$$

Then the map (8.3) is given by (8.2) and the Pontryagin-Thom construction. The resulting pushforward maps depend on the various choices.

However, using the assumption that E is rationally even, in the case where p is a submersion the situation is simple. First of all, the relative tangent bundle $T(p) = \ker(TN \to TX)$ makes sense, and we restrict our attention to the case where we are given a differential stable G-structure g_p on $T(p)$ (as opposed to the more general notion of *differential stable relative tangential G-structure on p* in Definition 5.12). Then, associated to such g_p there is a canonical set of choices of g_q^{HS} which gives the same map (8.3). We explain this point in details in Appendix A. We call such g_p^{HS} a *lift* of g_p (Definition A.44). The map (8.3) defined by any choice of a lift g_p^{HS} of g_p is the unique map denoted by

$$(p, g_p)_* := (p, g_p^{\mathrm{HS}})_* \colon \widehat{E}^*_{\mathrm{HS}}(N; \iota_E) \to \widehat{E}^{*-r}_{\mathrm{HS}}(X; \iota_E). \tag{8.5}$$

We simply call it the differential pushforward map (Definition A.40 and Proposition A.45).

In the case where $p \colon N \to X$ is a submersion and equipped with a differential stable G-structure g_p on $T(p)$, there is also the corresponding pushforward map on the level of differential forms. The Chern-Dold character of the multiplicative genus $\mathcal{G} \in E^0(MTG)$ is the element

$$\mathrm{ch}(\mathcal{G}) \in H^0(MTG; V_E^\bullet). \tag{8.6}$$

For example, for $\mathcal{G} = \tau \colon MT\mathrm{SO} \to H\mathbb{Z}$, the Chern-Dold chacacter is trivial, 1. For $\mathcal{G} = \mathrm{ABS} \colon MT\mathrm{Spin}^c \to K$ and $\mathcal{G} = \mathrm{ABS} \colon MT\mathrm{Spin} \to K$, the Chern-Dold characters are the Todd polynomial and the $\widehat{A}$ polynomial, respectively.

[17] As we explain in Appendix A.2, this process needs some additional choices of cochains. By the assumption that E is rationally even, the resulting map on the differential cohomology level does not depend on the choices.

Applying the Chern-Weil construction in (4.18), we get the Chern-Dold character form for the relative tangent bundle,

$$\mathrm{cw}_{g_p}(\mathrm{ch}(\mathcal{G})) \in \Omega^0_{\mathrm{clo}}(W; \mathrm{Ori}(T(p)) \otimes_{\mathbb{R}} V_E^\bullet). \tag{8.7}$$

Using this, the pushforward map on $\Omega^*(-; V_E^\bullet)$ is given by

$$\int_{N/X} - \wedge \mathrm{cw}_{g_p}(\mathrm{ch}(\mathcal{G}))\colon \Omega^n(N; V_E^\bullet) \to \Omega^{n-r}(X; V_E^\bullet). \tag{8.8}$$

Restricted to the closed forms, the induced homomorphism on the cohomology, $H^n(N; V_E^\bullet) \to H^{n-r}(X; V_E^\bullet)$, is compatible with the Chern-Dold character for E and the topological pushforward (8.1). The differential pushforward map in (8.5) is compatible with the map (8.8) (tangential version of (A.19)).

In particular, if $X = \mathrm{pt}$, for every n-dimensional differential stable tangential G-cycle (M, g) over pt, the differential pushforward map (8.5) is

$$(p_M, g)_*\colon \widehat{E}^*_{\mathrm{HS}}(M; \iota_E) \to \widehat{E}^{*-n}_{\mathrm{HS}}(\mathrm{pt}; \iota_E). \tag{8.9}$$

As we explain in the last part of Appendix A.1, an important property of the pushforward is the following *Bordism formula*, relating the pushforward of differential forms (8.8) on the bulk and the differential pushforward (8.9) on the boundary.

Fact 8.10 (Bordism formula, [Bun12, Problem 4.235])**.** *For any morphism $[W, g_W]\colon (M_-, g_-) \to (M_+, g_+)$ in $h\mathrm{Bord}^{G\nabla}_{n-1}(\mathrm{pt})$, the following diagram commutes.*

$$\begin{tikzcd}
\widehat{E}^*_{\mathrm{HS}}(W; \iota_E) \arrow[r, "R"] \arrow[d, "(-i^*_{M_-}) \oplus i^*_{M_+}"] & \Omega^*(W; V_E^\bullet) \arrow[r, "\int_W - \wedge \mathrm{cw}_g(\mathrm{ch}(\mathcal{G}))"] & \Omega^{*-n}(\mathrm{pt}; V_E^\bullet) \arrow[d, "a"] \\
\widehat{E}^*_{\mathrm{HS}}(M_-; \iota_E) \oplus \widehat{E}^*_{\mathrm{HS}}(M_+; \iota_E) \arrow[rr, "(p_{M_-}, g_-)_* \oplus (p_{M_+}, g_+)_*"] & & \widehat{E}^{*-n+1}(\mathrm{pt})
\end{tikzcd}.$$

For example, in the case $\mathcal{G} = \tau\colon MT\mathrm{SO} \to H\mathbb{Z}$, the nontrivial degree of pushforwards $(p_M, g)_*\colon \widehat{H}^{\dim M+1}(M; \mathbb{Z}) \to \widehat{H}^1(\mathrm{pt}; \mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$ are called the *higher holonomy function* which appears in the definition of Chern-Simons invariants. In terms of the Cheeger-Simons model of $\widehat{H\mathbb{Z}}$ in terms of differential characters [CS85], it is given by the evaluation on the fundamental cycle. In particular for the case $\dim M = 1$ it is the usual holonomy, and the Bordism formula is satisfied because of the relation between the curvature and the holonomy for $U(1)$-connections.

In the case $\mathcal{G} = \mathrm{ABS}\colon MT\mathrm{Spin}^c \to K$, Freed and Lott [FL10] constructed a model of $\widehat{K}$ in terms of hermitian vector bundles with hermitian connections, and the refinement of pushforwards when $\dim M$ is odd, $(p_M, g)_*\colon \widehat{K}^0(M) \to \widehat{K}^{-\dim M}(\mathrm{pt}) \simeq \mathbb{R}/\mathbb{Z}$, is given by the reduced eta invariants. The Bordism formula is a consequence of the Atiyah-Patodi-Singer index theorem.

8.2. **Differential Pushforwards in terms of functors.** As a preparation to the main Subsection 8.3, in this subsection we translate the data of differential pushforwards into functors from $h\mathrm{Bord}^{G\nabla}_-(-)$.

Definition 8.11. In the above settings, let X be a manifold, k be an integer and $\widehat{e} \in \widehat{E}^k_{\mathrm{HS}}(X;\iota_E)$. Let n be an integer with $k+n-1 \geq 0$. Then define the functor of Picard groupoids,

$$T_{\mathcal{G},\widehat{e}} \colon h\mathrm{Bord}^{G_\nabla}_{k+n-1}(X) \to \left(V_E^{-n} \xrightarrow{a} \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E)\right), \tag{8.12}$$

by the following.

- On objects, we set

$$T_{\mathcal{G},\widehat{e}}(M,g,f) := (p_M,g)_* f^*(\widehat{e}) \in \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E) \tag{8.13}$$

- On morphisms, we set

$$T_{\mathcal{G},\widehat{e}}([W,g_W,f_W]) := \mathrm{cw}(R(\widehat{e}) \wedge \mathrm{ch}(\mathcal{G}))([W,g_W,f_W]).$$

  Here $R(\widehat{e}) \in \Omega^k_{\mathrm{clo}}(X;V_E^\bullet)$ is the curvature of $\widehat{e}$ and we use (4.20).

The well-definedness of the functor follows by the Bordism formula in Fact 8.10.

As is easily shown by the Bordism formula, the formula (8.13) defines the homomorphism

$$T_{\mathcal{G},\widehat{e}} \colon \mathcal{C}^{G_\nabla}_{k+n-1}(X) \to \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E). \tag{8.14}$$

As expected, the transformation (8.12) is induced by the first arrow in (7.3). To show this, first remark that for any spectrum F and its any fundamental cycle ι_F, the forgetful functor gives the equivalence of Picard groupoids,

$$\pi_{\leq 1}((F;\iota_F)^{\mathrm{pt}}_n) \simeq \pi_{\leq 1}(F_n), \tag{8.15}$$

where the left hand side means the simplicial fundamental groupoid, whose objects are differential functions $t_{\mathrm{pt}} \colon \mathrm{pt} \to (F;\iota_F)_n$, and morphisms are bordism classes of differential functions $t_I \colon I \to (F;\iota_F)_n$. The right hand side is the fundamental groupoid for the space F_n, which is equipped with the structure of a Picard groupoid by [HS05, Example B.7].

We have a functor of Picard groupoids,

$$\mathrm{ev} \colon \pi_{\leq 1}\left((E;\iota_E)^{\mathrm{pt}}_{1-n}\right) \to \left(V_E^{-n} \xrightarrow{a} \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E)\right), \tag{8.16}$$

given by assigning the element $[t_{\mathrm{pt}}] \in \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E)$ for an object and the integration of the curvature $R([t_I]) \in \Omega^{1-n}_{\mathrm{clo}}(I;V_E^\bullet)$ for a morphism.

Proposition 8.17. *The functor* (8.12) *of Picard groupoids is naturally isomorphic to the following composition,*

(8.18)
$$h\mathrm{Bord}^{G_\nabla}_{k+n-1}(X) \simeq \pi_{\leq 1}(L(X^+ \wedge MTG)_{1-k-n}) \xrightarrow{e \wedge \mathcal{G}} \pi_{\leq 1}(E_{1-n}) \to \left(V_E^{-n} \xrightarrow{a} \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E)\right),$$

where the first arrow is the equivalence in Lemma 3.10, and the last arrow is the composition of (8.15) *and* (8.16).

Proof. Choose a differential function $t(\widehat{e}) \colon X \to (E_k;(\iota_E)_k)$ representing $\widehat{e}$. For each object (M,g,f) in $h\mathrm{Bord}^{G_\nabla}_{k+n-1}(X)$, choose a Hopkins-Singer's differential G-structure g^{HS} lifting g. By the discussion in Appendix A.2 and its tangential variant in Appendix A.3, we get a functor

(8.19)
$$h\mathrm{Bord}_{k+n-1}^{G_\nabla}(X) \to \pi_{\le 1}((E_k)^+ \wedge MTG_{1-k-n}; (\iota_E)_k \cup V_{\mathcal{G}}(\iota_{MTG})_{1-k-n})^{\mathrm{pt}}).$$

Indeed, for objects, given (M, g, f) with the chosen lift g^{HS}, denote the underlying embedding and tubular neighborhood by $M \subset U \subset \mathbb{R}^N$. We have differential functions $f^*t(\widehat{e}) \colon M \to (E; \iota_E)_k$ and $t(g^{\mathrm{HS}}) \colon \overline{U} \to (MTG; \iota_{MTG})_{N-(k+n-1)}$. Applying the ($MTG$-version of the) left vertical arrow of (A.28) to them and using the open embedding $U \hookrightarrow \mathbb{R}^N$ (the Pontryagin-Thom collapse), we get the differential function $\mathrm{pt} \to (E_k \wedge MTG_{1-k-n}; (\iota_E)_k \cup V_{\mathcal{G}}(\iota_{MTG})_{1-k-n})$.

For morphisms $[W, g_W, f_W] \colon (M_-, g_-, f_-) \to (M_+, g_+, f_+)$, choose any representative (W, g_W, f_W) and smooth map $p_W \colon W \to I$ (not necessarily a submersion) so that it coincides with a collar coordinates of each objects $(M_\pm, g_\pm, f_\pm)$ near the endpoints, respectively. The structure g_W induces g_{p_W}, in particular the topological structure $g_{p_W}^{\mathrm{top}}$, on p_W. Take any Hopkins-Singer's differential tangential BG-oorientation $g_{p_W}^{\mathrm{HS}}$ (Appendix A.3) for p_W which coincides with the chosen lifts at the boundary, and whose underlying map classifies $g_{p_W}^{\mathrm{top}}$. Then applying the same procedure as that we did for objects above, we get a differential function $I \to (E_k \wedge MTG_{1-k-n}; (\iota_E)_k \cup V_{\mathcal{G}}(\iota_{MTG})_{1-k-n})$ which restricts at the boundary to the ones assigned to objects above. Since any of the choices we have made is unique up to bordisms, the resulting morphism in the right hand side of (8.19) is uniquely determined. This gives the desired functor.

By Definition 8.11 and Proposition A.45, the functor $T_{\mathcal{G},\widehat{e}}$ coincides with the composition of (8.19) with

(8.20)
$$\begin{aligned}\pi_{\le 1}\left((E_k \wedge MTG_{1-k-n}; (\iota_E)_k \cup V_{\mathcal{G}}(\iota_{MTG})_{1-k-n})^{\mathrm{pt}}\right) &\xrightarrow{\widehat{\mathcal{G}}} \pi_{\le 1}((E; \iota_E)_{1-n}^{\mathrm{pt}}) \\ &\xrightarrow{\mathrm{ev}} \left(V_E^{-n} \xrightarrow{a} \widehat{E}_{\mathrm{HS}}^{1-n}(\mathrm{pt}; \iota_E)\right).\end{aligned}$$

The fact that it is naturally isomorphic to (8.18) is just the cosequence of the fact that $\widehat{e}$ and $\widehat{\mathcal{G}}$ are refinements of e and $\mathcal{G}$, respectively. This completes the proof. □

8.3. **The construction and the proof.** In this main subsection, we construct the transformation 7.1 and give a proof to Theorem 7.2. In this subsection, we fix an integer n so that $V_E^{1-n} = 0$. As a preparation, we show that there exists a canonical homomorphism[18]

$$s \colon IE^n(\mathrm{pt}) \to \mathrm{Hom}_{\mathrm{Ch}}\left(\left(V_E^{-n} \xrightarrow{a} \widehat{E}_{\mathrm{HS}}^{1-n}(\mathrm{pt}; \iota_E)\right), (\mathbb{R} \to \mathbb{R}/\mathbb{Z})\right), \tag{8.21}$$

[18]The existsnce of a canonical pairing $IE^{-n}(\mathrm{pt}) \otimes \widehat{E}_{\mathrm{HS}}^{1-n}(\mathrm{pt}; \iota_E) \to \mathbb{R}/\mathbb{Z}$ is used in [FMS07, Proposition 6], in particular in the last arrow of the second line of the proof of that proposition. They do not state any condition on E, but they use the assumption $V_E^{1-n} = 0$ implicitely.

where $\mathrm{Hom}_{\mathrm{Ch}}$ denotes the group of chain maps of complexes of abelian groups. Indeed, by [HS05, (4.57)], we have a canonical isomorphism[19]

$$\ker\left(R\colon \widehat{E}^*_{\mathrm{HS}}(-;\iota_E) \to \Omega^*_{\mathrm{clo}}(-;V^\bullet_E)\right) \simeq E^{*-1}(-;\mathbb{R}/\mathbb{Z}). \tag{8.22}$$

Here, for any abelian group $\mathbb{G}$, the cohomology theory $E^*(-;\mathbb{G})$ is represented by the spectrum $E\mathbb{G} := E \wedge S\mathbb{G}$, where $S\mathbb{G}$ is the Moore spectrum. As explained there, this is because the differential function complexes can be fits into the homotopy Cartesian square [HS05, (4.12)]. Applied to pt and $* = 1-n$, we get the identification

$$\widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E) = \ker\left(R\colon \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E) \to V^{1-n}_E\right) \simeq E^{-n}(\mathrm{pt};\mathbb{R}/\mathbb{Z}). \tag{8.23}$$

An element $\beta \in IE^n(\mathrm{pt}) = [E, \Sigma^n I\mathbb{Z}]$ induces the element $\beta_{\mathbb{G}} \in [E\mathbb{G}, \Sigma^n I\mathbb{Z} \wedge S\mathbb{G}]$ for any $\mathbb{G}$, and using $I\mathbb{Z} \wedge S\mathbb{R} \simeq H\mathbb{R}$ and $I\mathbb{Z} \wedge S\mathbb{R}/\mathbb{Z} \simeq I\mathbb{R}/\mathbb{Z}$, we get the induced homomorphisms on pt, which we also denote as

$$\beta_{\mathbb{R}}\colon V^{-n}_E = E^{-n}(\mathrm{pt};\mathbb{R}) \to \mathbb{R}, \tag{8.24}$$

$$\beta_{\mathbb{R}/\mathbb{Z}}\colon \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E) \xrightarrow[(8.23)]{\simeq} E^{-n}(\mathrm{pt};\mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}, \tag{8.25}$$

The homomorphism (8.24) coincides with the one obtained by the map $IE^n(X) \to \mathrm{Hom}(E_n(X), \mathbb{R})$ in (2.5). The homomorphism (8.21) is given by mapping β to the pair $(\beta_{\mathbb{R}}, \beta_{\mathbb{R}/\mathbb{Z}})$. The well-definedness follows by the construction.

On the other hand, by Fact 2.6, we have an isomorphism for any spectra E,

$$IE^n(\mathrm{pt}) \simeq \pi_0\mathrm{Fun}_{\mathrm{Pic}}\left(\pi_{\le 1}(E_{1-n}), (\mathbb{R} \to \mathbb{R}/\mathbb{Z})\right). \tag{8.26}$$

By (8.15), (8.16) and (8.26), we get a homomorphism

$$\mathrm{ev}_*\colon \pi_0\mathrm{Fun}_{\mathrm{Pic}}\left(\left(V^{-n}_E \xrightarrow{a} \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt};\iota_E)\right), (\mathbb{R} \to \mathbb{R}/\mathbb{Z})\right) \to IE^n(\mathrm{pt}). \tag{8.27}$$

It directly follows from the definition of the identification (8.22) that we have

$$\mathrm{id} = \mathrm{ev}_* \circ s\colon IE^n(\mathrm{pt}) \to IE^n(\mathrm{pt}). \tag{8.28}$$

Definition 8.29 ($\Phi_{\mathcal{G}}$)**.** In the settings explained in the introduction[20], for each manifold X we define a homomorphism of abelian groups

$$\Phi_{\mathcal{G}}\colon \widehat{E}^*_{\mathrm{HS}}(X;\iota_E) \otimes IE^n(\mathrm{pt}) \to (\widehat{I\Omega^G_{\mathrm{dR}}})^{*+n}(X), \tag{8.30}$$

by the following. For $\widehat{e} \in \widehat{E}^k_{\mathrm{HS}}(X;\iota_E)$ and $\beta \in IE^n(\mathrm{pt})$, set $\Phi_{\mathcal{G}}(\widehat{e} \otimes \beta) := (\beta_{\mathbb{R}}(R(\widehat{e}) \wedge \mathrm{ch}(\mathcal{G})), \beta_{\mathbb{R}/\mathbb{Z}} \circ T_{\mathcal{G},\widehat{e}}) \in (\widehat{I\Omega^G_{\mathrm{dR}}})^{k+n}(X)$, where

- The element $\beta_{\mathbb{R}}(R(\widehat{e}) \wedge \mathrm{ch}(\mathcal{G})) \in \Omega^{n+k}_{\mathrm{clo}}(X; V^\bullet_{I\Omega^G})$ is obtained by applying (8.24) on the coefficient of $R(e) \wedge \mathrm{ch}(\mathcal{G}) \in \Omega^k_{\mathrm{clo}}(X; H^*(MTG; V^\bullet_E))$.
- $\beta_{\mathbb{R}/\mathbb{Z}} \circ T_{\mathcal{G},\widehat{e}}$ is the composition of (8.14) and (8.25).

[19] This isomorphism does not follow from the axiom of differential cohomology theory in [BS10]. For more on this point, see [BS10, Section 5].

[20] Recall that we assumed $E^{1-n}(\mathrm{pt}) \otimes \mathbb{R} = 0$ there.

The fact that the pair $(\beta_{\mathbb{R}}(R(\widehat{e}) \wedge \mathrm{ch}(\mathcal{G})), \beta_{\mathbb{R}/\mathbb{Z}}(T_{\mathcal{G},\widehat{e}}))$ satisfies the compatibility condition follows from the well-definedness of (8.21) and the fact that $T_{\mathcal{G},e}$ in Definition 8.11 is a functor.

Now we prove Theorem 7.2.

Proof of Theorem 7.2. We use the argument in Subsection 4.2. Recall that, for an element $(\omega, h) \in (\widehat{I\Omega^G_{\mathrm{dR}}})^N(X)$ we associated a functor $F_{(\omega,h)} \colon h\mathrm{Bord}^{G_\nabla}_{N-1}(X) \to (\mathbb{R} \to \mathbb{R}/\mathbb{Z})$ in (4.51). The natural isomorphism

$$I\Omega^G \simeq I\Omega^G_{\mathrm{dR}}, \tag{8.31}$$

where for the former we use the model of $I\mathbb{Z}$ by Fact 2.6, was given as follows. Using Lemma 3.10, we have $(I\Omega^G)^N(X) = \pi_0\mathrm{Fun}_{\mathrm{Pic}}\left(\pi_{\le 1}(h\mathrm{Bord}^{G_\nabla}_{N-1}(X) \to (\mathbb{R} \to \mathbb{R}/\mathbb{Z})\right)$. The map (8.31) is given by mapping the isomorphism class of the functor $F_{(\omega,h)}$ to $I(\omega, h) \in (I\Omega^G_{\mathrm{dR}})^N(X)$.

Now fix $\widehat{e} \in \widehat{E}^k_{\mathrm{HS}}(X; \iota_E)$ and $\beta \in IE^n(\mathrm{pt})$. By Definitions 8.29 and 8.11, the functor associated to $\Phi_{\mathcal{G}}(\widehat{e} \otimes \beta)$ coincides with the following composition.

(8.32)

$$F_{\Phi_{\mathcal{G}}(\widehat{e}\otimes\beta)} \colon h\mathrm{Bord}^{G_\nabla}_{k+n-1}(X) \xrightarrow{T_{\mathcal{G},\widehat{e}}} \left(V^{-n}_E \xrightarrow{a} \widehat{E}^{1-n}_{\mathrm{HS}}(\mathrm{pt}; \iota_E)\right) \xrightarrow{s(\beta)=(\beta_{\mathbb{R}}, \beta_{\mathbb{R}/\mathbb{Z}})} (\mathbb{R} \to \mathbb{R}/\mathbb{Z}).$$

Combining this with Proposition 8.17 and (8.28), we see that, under the equivalence $h\mathrm{Bord}^{G_\nabla}_{k+n-1}(X) \simeq \pi_{\le 1}(L(X^+ \wedge MTG)_{1-k-n})$, (8.32) coincides with

$$\pi_{\le 1}(L(X^+ \wedge MTG)_{1-k-n}) \xrightarrow{e\wedge\mathcal{G}} \pi_{\le 1}(E_{1-n}) \xrightarrow{\beta} (\mathbb{R} \to \mathbb{R}/\mathbb{Z}),$$

up to a natural isomorphism. This completes the proof of Theorem 7.2. □

8.4. **Examples.**

8.4.1. *The holonomy theory (1) : Example 4.59.* Here we explain the "Holonomy theory (1)" in Example 4.59. This corresponds to the case $E = H\mathbb{Z}$, $\mathcal{G} = \tau \colon MT\mathrm{SO} \to H\mathbb{Z}$ is the usual orientation, and $n = 0$.

Recall that, given a manifold X and a hermitian line bundle with unitary connection (L, ∇) over X, we get the element

$$(c_1(\nabla), \mathrm{Hol}_\nabla) \in (\widehat{I\Omega^{\mathrm{SO}}_{\mathrm{dR}}})^2(X). \tag{8.33}$$

On the other hand, in the case $E = H\mathbb{Z}$ we have the canonical choice of an element in $IH\mathbb{Z}^0(\mathrm{pt})$, namely the Anderson self-duality element $\gamma_H \in IH\mathbb{Z}^0(\mathrm{pt})$. Thus we have the homomorphism

$$\Phi_\tau(- \otimes \gamma_H) \colon \widehat{H}^2(X; \mathbb{Z}) \to (\widehat{I\Omega^{\mathrm{SO}}_{\mathrm{dR}}})^2(X).$$

Using the model of $\widehat{H\mathbb{Z}}^2$ in terms of hermitian vector bundles with $U(1)$-connections (for example see [HS05, Example 2.7]), the pair (L, ∇) defines a class $[L, \nabla] \in \widehat{H}^2(X; \mathbb{Z})$. We have the following.

Proposition 8.34. *We have the following equality in* $(\widehat{I\Omega^{\mathrm{SO}}_{\mathrm{dR}}})^2(X)$*,*

$$(c_1(\nabla), \mathrm{Hol}_\nabla) = \Phi_\tau([L, \nabla] \otimes \gamma_H). \tag{8.35}$$

Moreover, the element $I(c_1(\nabla), \mathrm{Hol}_\nabla) \in (I\Omega^{\mathrm{SO}}_{\mathrm{dR}})^2(X)$ *coincides with the following composition,*

$$X^+ \wedge MT\mathrm{SO} \xrightarrow{c_1(L)\wedge\tau} \Sigma^2 H\mathbb{Z} \wedge H\mathbb{Z} \xrightarrow{\text{multi}} \Sigma^2 H\mathbb{Z} \xrightarrow{\gamma_H} \Sigma^2 I\mathbb{Z}.$$

Proof. The last statement follows from (8.35) and Theorem 7.2. The equality (8.35) follows from the fact that the self-duality element γ_H induces the canonical isomorphism $\widehat{H}^1(\mathrm{pt};\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$ and $H^0(\mathrm{pt};\mathbb{Z}) \simeq \mathbb{R}$, together with the following well-known facts about $\widehat{H\mathbb{Z}}$ (for example see [HS05, Section 2.4]). The element $[L, \nabla] \in \widehat{H}^2(X;\mathbb{Z})$ satisfies

$$\gamma_H \circ R([L,\nabla]) = c_1(\nabla) \in \Omega^2(X),$$

and, given a map $f\colon M \to X$ from an oriented 1-dimensional closed manifold (M, g), the pushforward $(p_M, g)_*\colon \widehat{H}^2(M;\mathbb{Z}) \to \widehat{H}^1(\mathrm{pt};\mathbb{Z}) \underset{\simeq}{\xrightarrow{\gamma_H}} \mathbb{R}/\mathbb{Z}$

$$\gamma_H \circ (p_M, g)_* f^*[L,\nabla] = \mathrm{Hol}_{f^*\nabla}.$$

$\square$

8.4.2. *The classical Chern-Simons theory : Example 4.61.* Here we explain the classical Chern-Simons theory which appeared in Example 4.61. This is essentially a generalization of Subsection 8.4.1, corresponding to the case $E = H\mathbb{Z}$, $\mathcal{G} = \tau\colon MT\mathrm{SO} \to H\mathbb{Z}$ is the usual orientation, and $n = 0$.

Recall that, given a compact Lie group H and an element $\lambda \in H^n(BH;\mathbb{Z})$, the corresponding classical Chern-Simons theory of level λ is defined by choosing an $(n+1)$-classifying object $(\mathcal{E}, \mathcal{B}, \nabla_{\mathcal{E}})$ in the category of manifolds with principal H-bundles with connections, and fixing an element $\widehat{\lambda} \in \widehat{H}^n(\mathcal{B};\mathbb{Z})$ lifting λ. Then we have the element

$$(1 \otimes \lambda_{\mathbb{R}}, h_{\mathrm{CS}_{\widehat{\lambda}}}) \in (\widehat{I\Omega^{\mathrm{SO}\times H}_{\mathrm{dR}}})^n(\mathrm{pt}), \tag{8.36}$$

whose equivalence class in $(I\Omega^{\mathrm{SO}\times H}_{\mathrm{dR}})^n(\mathrm{pt})$ does not depend on the lift $\widehat{\lambda}$.

Proposition 8.37. *The element* $I(1\otimes\lambda_{\mathbb{R}}, h_{\mathrm{CS}_{\widehat{\lambda}}}) \in (I\Omega^{\mathrm{SO}\times H}_{\mathrm{dR}})^n(\mathrm{pt})$ *coincides with the following composition,*

$$BH^+ \wedge MT\mathrm{SO} \xrightarrow{\lambda\wedge\tau} \Sigma^n H\mathbb{Z} \wedge H\mathbb{Z} \xrightarrow{\text{multi}} \Sigma^n H\mathbb{Z} \xrightarrow{\gamma} \Sigma^n I\mathbb{Z}.$$

Proof. The classifying map induces an equivalence $\pi_{\le 1}(L(\mathcal{B}^+ \wedge MT\mathrm{SO})_{n-1}) \simeq \pi_{\le 1}(L(BH^+ \wedge MT\mathrm{SO})_{n-1})$. Moreover, by the pullback of the universal connection $\nabla_{\mathcal{E}}$ it is refined to a functor of Picard groupoids,

$$h\mathrm{Bord}^{\mathrm{SO}}_{n-1}(\mathcal{B}) \xrightarrow{\simeq} h\mathrm{Bord}^{\mathrm{SO}\times H}_{n-1}(\mathrm{pt}) \tag{8.38}$$

which is naturally isomorphic to the above one under the equivalences $h\mathrm{Bord}^{\mathrm{SO}}_{n-1}(X) \simeq \pi_{\le 1}(L(X^+ \wedge MT\mathrm{SO})_{n-1})$.

We have the element

$$\Phi_\tau(\widehat{\lambda} \otimes \gamma_H) \in (\widehat{I\Omega^{\mathrm{SO}}_{\mathrm{dR}}})^n(\mathcal{B}). \tag{8.39}$$

Recall that an element $(\omega, h) \in (\widehat{I\Omega^{G}_{\mathrm{dR}}})^n(X)$ associates a functor $F_{(\omega,h)}\colon h\mathrm{Bord}^{G_\nabla}_{n-1}(X) \to (\mathbb{R} \to \mathbb{R}/\mathbb{Z})$ by (4.51). We claim that the functors associated to the elements

(8.39) and (8.36) are related by

$$F_{\Phi_\tau(\widehat{\lambda}\otimes\gamma_H)}\colon h\mathrm{Bord}_{n-1}^{\mathrm{SO}}(\mathcal{B}) \xrightarrow{(8.38)} h\mathrm{Bord}_{n-1}^{\mathrm{SO}\times H}(\mathrm{pt}) \xrightarrow{F_{(1\otimes\lambda_\mathbb{R}, h_{\mathrm{CS}_{\widehat{\lambda}}})}} (\mathbb{R}\to\mathbb{R}/\mathbb{Z}).$$

Indeed, this follows from the fact that the Chern-Simons invariants are given by the pushforward in differential ordinary cohomology (4.63). Applying Theorem 7.2 to the element (8.39), we get the result. □

8.4.3. *The theory of massive free complex fermions : Example 4.67.* Here we explain the example of the theory on massive free complex fermions which appeared in Example 4.67. This example corresponds to the case $E = K$, $\mathcal{G} = \mathrm{ABS}\colon MT\mathrm{Spin}^c \to K$ and $n = 0$.

Recall that, given a hermitian vector bundle with unitary connection (W, ∇^W) over a manifold X, we get an element

$$\left((\mathrm{Ch}(\nabla^W)\otimes \mathrm{Todd})|_{2k}, \overline{\eta}_{\nabla^W}\right) \in \left(\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{ph}}}\right)^{2k}(X) \simeq \left(\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{dR}}}\right)^{2k}(X).$$

On the other hand, in the case $E = K$ we have the canonical choice of an element in $IK^0(\mathrm{pt})$, namely the self-duality element $\gamma_K \in IK^0(\mathrm{pt})$. Thus we have the homomorphism

$$\Phi_{\mathrm{ABS}}(-\otimes\gamma_K)\colon \widehat{K}^{2k}(X) \to (\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{dR}}})^{2k}(X). \tag{8.40}$$

Using the model of $\widehat{K}$ in terms of hermitian vector bundles with unitary connections by Freed-Lott ([FL10]), we have the class $[W, h^W, \nabla^W, 0] \in \widehat{K}^0(X) \simeq \widehat{K}^{2k}(X)$.

Proposition 8.41. *We have the following equality in* $(\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{dR}}})^{2k}(X)$,

$$((\mathrm{Ch}(\nabla^W)\otimes\mathrm{Todd})|_{2k}, \overline{\eta}_{\nabla^W}) = \Phi_{\mathrm{ABS}}([W, h^E, \nabla^E, 0]\otimes\gamma_K). \tag{8.42}$$

Moreover, the element $I((\mathrm{Ch}(\nabla^W)\otimes\mathrm{Todd})|_{2k}, \overline{\eta}_{\nabla^W}) \in (I\Omega^{\mathrm{Spin}^c})^{2k}(X)$ *coincides with the following composition,*

$$X^+ \wedge MT\mathrm{Spin}^c \xrightarrow{[E]\wedge\mathrm{ABS}} K\wedge K \xrightarrow{\mathrm{multi}} K \underset{\simeq}{\xrightarrow{\mathrm{Bott}}} \Sigma^{2k}K \xrightarrow{\gamma_K} \Sigma^{2k} I\mathbb{Z}.$$

Proof. The last statement follows from (8.42) and Theorem 7.2. Denote the Bott element by $u \in K^{-2}(\mathrm{pt})$. The equality (8.42) follows from the fact that the self-duality element γ_K induces the canonical isomorphism $\widehat{K}^1(\mathrm{pt}) \simeq \mathbb{R}/\mathbb{Z}$ and $K^0(\mathrm{pt}) \simeq \mathbb{Z}$, together with the following facts about $\widehat{K}$ in [FL10]. The element $[W, h^W, \nabla^W, 0] \in \widehat{K}^0(X)$ satisfies

$$R([W, h^E, \nabla^E, 0]) = \mathrm{Ch}(\nabla^W) \in \Omega^0(X; V_K^\bullet) = \Omega^0(X; \mathbb{R}[u, u^{-1}]),$$

and, given a map $f\colon M \to X$ from an oriented $(2k-1)$-dimensional closed manifold with a physical Spin^c-structure (M, g), the pushforward $(p_M, g)_*\colon \widehat{K}^0(M) \to \widehat{K}^{-2k+1}(\mathrm{pt})$ is given by

$$(p_M, g)_* f^*[W, h^W, \nabla^W, 0] = \bar{\eta}_{\nabla^W}(M, g, f)\cdot u^k \in \widehat{K}^{-2k+1}(\mathrm{pt}) = (\mathbb{R}/\mathbb{Z})\cdot u^k.$$

□

Remark 8.43. In the examples in this subsection, we used the Anderson self-duality elements in $IE^n(\mathrm{pt})$ for $E = H\mathbb{Z}, K$. However, the results in this subsection do *not* use the self-duality, and indeed there are many other interesting examples given by non-self-duality elements in $IE^n(\mathrm{pt})$. For example, in the analysis of anomalies of the heterotic string theories in [TY21], we encounter such examples when $E = \mathrm{TMF}$ and $E = \mathrm{KO}((q))$ with the Witten genus $\mathcal{G} = \mathrm{Wit} \colon MT\mathrm{String} \to \mathrm{TMF}$ and $\mathcal{G} = \mathrm{Wit}_{\mathrm{Spin}} \colon MT\mathrm{Spin} \to \mathrm{KO}((q))$.

Appendix A. Differential pushforwards for proper submersions

As mentioned in Subsection 8.1, there are certain subtleties regarding the formulations of differential pushforwards. In this appendix, we explain that there is a nice theory on differential pushforwards for proper submersions under the assumption that E is rationally even. The author believe that the results in this Appendix well-known among experts. It is convenient to start with multiplicative differential extensions $\widehat{E}$ which are not necessarily the one given by the Hopkins-Singer. The minimal requirements for the differential extension $\widehat{E}$ are,

- For real vector bundles $V \to X$ over manifolds, the *properly supported* differential cohomology groups

$$\widehat{E}^*_{\mathrm{prop}/X}(V) \tag{A.1}$$

  are defined with a module structure over $\widehat{E}^*(X)$, so that they refine properly supported cohomologies and forms.
- If we have a vector bundle $W \to N$ and we have an *open* embedding $\iota \colon W \hookrightarrow V$ in the total space of another vector bundle $V \to X$, we have the corresponding map

$$\iota_* \colon \widehat{E}^*_{\mathrm{prop}/N}(W) \to \widehat{E}^*_{\mathrm{prop}/X}(V),$$

  refining the topological and form counterparts.
- We have the *desuspension map*,

$$\mathrm{desusp} \colon \widehat{E}^*_{\mathrm{prop}/X}(\mathbb{R}^k \times X) \to \widehat{E}^{*-k}(X),$$

  refining the topological and form counterparts.

Since we are assuming E is rationally even, the Hopkins-Singer's differential extension $\widehat{E}^*_{\mathrm{HS}}(-;\iota_E)$ admits a canonical multiplicative structure by [Upm15], and the above properties are also satisfied.

A.1. The normal case.

In this subsection we explain the normal case. The content of this subsection basically follows the unpublished survey by Bunke [Bun12, Section 4.8–4.10]. Let G and E be multiplicative with E rationally even, and assume we are given a homomorphism of ring spectra,

$$\mathcal{G} \colon MG \to E, \tag{A.2}$$

where MG is the Thom spectrum. Then for each real vector bundle V of rank r over a topological space X equipped with a stable G-structure g^{top}, we get the *Thom class* $\nu \in E^r(\overline{V})$, where we denote $\overline{V} := \mathrm{Thom}(V)$. Its multiplication gives the *Thom isomorphism* $E^*(X) \simeq E^{*+r}(\overline{V})$. Its Chern-Dold character is an element $\mathrm{ch}(\nu) \in H^r(\overline{V}; V_E^\bullet)$. We set

$$\mathrm{Td}(\nu) := \int_{V/X} \mathrm{ch}(\nu) \in H^0(X; \mathrm{Ori}(V) \otimes_{\mathbb{R}} V_E^\bullet).$$

Definition A.3 (Differential Thom classes, $\mathrm{Td}(\widehat{\nu})$, homotopy)**.** Let V be a smooth real vector bundle over a manifold M of rank r equipped with a stable G-structure g^{top}.

(1) A *differential Thom class* $\widehat{\nu} \in \widehat{E}^r_{\mathrm{prop}/M}(V)$ is an element such that $I(\widehat{\nu}) \in \widehat{E}^r_{\mathrm{prop}/M}(V)$ is the Thom class for (V, g^{top}).

(2) For such a $\widehat{\nu}$, we define

$$\mathrm{Td}(\widehat{\nu}) := \int_{V/M} R(\widehat{\nu}) \in \Omega^0_{\mathrm{clo}}(M; \mathrm{Ori}(V) \otimes_{\mathbb{R}} V_E^\bullet). \tag{A.4}$$

(3) A *homotopy* between two differential Thom classes $\widehat{\nu}_0$ and $\widehat{\nu}_1$ is a differential Thom class $\widehat{\nu}_I \in \widehat{E}^r_{\mathrm{prop}/(I\times M)}(I \times V)$ for $\mathrm{pr}_M^* V$ with $\widehat{\nu}_I|_{\{i\}\times V} = \widehat{\nu}_i$ for $i = 0, 1$ such that

$$\mathrm{Td}(\widehat{\nu}_I) = \mathrm{pr}_M^* \mathrm{Td}(\widehat{\nu}_0). \tag{A.5}$$

The homotopy class of $\widehat{\nu}$ is denoted by $[\widehat{\nu}]$.

In particular, if $\widehat{\nu}_0$ and $\widehat{\nu}_1$ are homotopic, we have $\mathrm{Td}(\widehat{\nu}_0) = \mathrm{Td}(\widehat{\nu}_1)$. Thus we use the notation $\mathrm{Td}([\widehat{\nu}]) \in \Omega^0_{\mathrm{clo}}(M; \mathrm{Ori}(V) \otimes_{\mathbb{R}} V_E^\bullet)$.

Lemma A.6. *Let M and (V, g^{top}) be as before, and ν be the Thom class for (V, g^{top}). Assume we are given an element $\omega \in \Omega^0_{\mathrm{clo}}(M; \mathrm{Ori}(V) \otimes_{\mathbb{R}} V_E^\bullet)$ such that* $\mathrm{Rham}(\omega) = \mathrm{Td}(\nu)$*.*

(1) There exists a differential Thom class $\widehat{\nu}$ with $Td(\widehat{\nu}) = \omega$.

(2) The set of homotopy classes $[\widehat{\nu}]$ of differential Thom classes with $\mathrm{Td}([\widehat{\nu}]) = \omega$ *is a torsor over*

$$\frac{H^{-1}(M; \mathrm{Ori}(V) \otimes_{\mathbb{R}} V_E^\bullet)}{\mathrm{Td}(\nu) \cup a(E^{-1}(M))}. \tag{A.7}$$

Proof. The proof is in [Bun12, Problem 4.186], and essentially the same proof appears in [GS21, Proposition 49] in the case of KO-theory. We need the orientation bundles here because we allow G to be un-oriented. □

If V is equipped with a stable differential G-structure g, applying the Chern-Weil construction (4.18) to $\mathrm{ch}(\mathcal{G}) \in H^0(MG; V_E^\bullet)$, we have

$$\mathrm{cw}_g(\mathrm{ch}(\mathcal{G})) \in \Omega^0_{\mathrm{clo}}(M; \mathrm{Ori}(V) \otimes_{\mathbb{R}} V_E^\bullet). \tag{A.8}$$

This satisfies $\mathrm{Rham}(\mathrm{cw}_g(\mathrm{ch}(\mathcal{G}))) = \mathrm{Td}(\nu)$.

For (V, g_V) of rank r represented by $\widetilde{g}_V = (d, P, \nabla, \psi \colon P \times_{\rho_d} \mathbb{R}^d \simeq \underline{\mathbb{R}}^{d-r} \oplus V)$ with $d \geq r+1$, we associate a differential stable G-structure $g_{\underline{\mathbb{R}}\oplus V}$ on $\underline{\mathbb{R}} \oplus V$ which is represented by $(d, P, \nabla, \psi \colon P \times_{\rho_d} \mathbb{R}^d \simeq \underline{\mathbb{R}}^{d-r-1} \oplus (\underline{\mathbb{R}} \oplus V))$. For a topological stable G-structure g_V^{top}, we define $g^{\mathrm{top}}_{\underline{\mathbb{R}}\oplus V}$ in the same way.

If we have a homotopy class of diffential Thom classes $[\widehat{\nu}_{\underline{\mathbb{R}}\oplus V}]$ for $(\underline{\mathbb{R}}\oplus V, g^{\mathrm{top}}_{\underline{\mathbb{R}}\oplus V})$, the integration

$$\int_{\mathbb{R}}[\widehat{\nu}_{\underline{\mathbb{R}}\oplus V}]$$

defines a well-defined homotopy class of differential Thom classes for (V, g_V^{top}). Moreover, by Lemma A.6, the above integration gives a bijection between the sets of homotopy classes of diffential Thom classes for $(\underline{\mathbb{R}}\oplus V, g^{\mathrm{top}}_{\underline{\mathbb{R}}\oplus V})$ and for (V, g_V^{top}).

Proposition A.9. [21] *There exists a unique way to assign a homotopy class $[\widehat{\nu}(g)]$ of differential Thom classes $\widehat{\nu}(g)\in \widehat{E}^{\mathrm{rank}V}_{\mathrm{prop}/M}(V)$ to every real vector bundle with differential stable G-structure $(V,g)\to M$ such that the following three conditions hold.*

(1) It is compatible with pullbacks.
(2) We have $\int_{\mathbb{R}}[\widehat{\nu}(g_{\underline{\mathbb{R}}\oplus V})] = [\widehat{\nu}(g_V)]$.
(3) We have $\mathrm{cw}_g(\mathrm{ch}(\mathcal{G})) = \mathrm{Td}([\widehat{\nu}(g)])$.

Moreover, the resulting homotopy class $[\widehat{\nu}(g)]$ only depends on the homotopy class (Definition 3.1 (4)) of differential stable G-structure g.

Proof. By the condition (2), it is enough to consider only (V,g) such that g is represented by a representative of the form $\widetilde{g} = (\mathrm{rank}(V), P, \nabla, \psi)$, i.e., without stabilization.

The proof basically follows that for [Bun12, Problem 4.197]. Suppose we have (V,g) of rank r over M with $\dim M = n$ with a representative $\widetilde{g} = (r, P, \nabla, \psi)$. Take a manifold $\mathcal{B}$ with an $(n{+}1)$-connected map $\mathcal{B}\to BG_r$. We can factor the classifying map for P as $M\xrightarrow{f}\mathcal{B}\to BG_r$ with f smooth. Take a G_r-connection $\nabla_{\mathcal{B}}$ on the pullback $\mathcal{P}\to\mathcal{B}$ of the universal bundle, and denote by the resulting differential G-structure on $\mathcal{V} := \mathcal{P}\times_{G_r}\mathbb{R}^r$ by $g_{\mathcal{V}}$. We have maps $f_P\colon P\to\mathcal{P}$ and $f_V\colon V\to\mathcal{V}$ covering f. We may assume that $g_{\mathcal{V}}$ pulls back to g by (f, f_P, f_V).

The difference of any two choices of the homotopy classes $[\widehat{\nu}(g_{\mathcal{V}})]$ of differential Thom classes on $(\mathcal{V}, g_{\mathcal{V}})$ is measured by an element in $\frac{H^{-1}(\mathcal{B};\mathrm{Ori}(\mathcal{V})\otimes_{\mathbb{R}}V_E^\bullet)}{\mathrm{Td}(\nu(g_{\mathcal{V}}))\cup a(E^{-1}(\mathcal{B}))}$ by Proposition A.6. The pullback map $f^*\colon H^{-1}(\mathcal{B};\mathrm{Ori}(\mathcal{V})\otimes_{\mathbb{R}}V_E^\bullet)\to H^{-1}(M;\mathrm{Ori}(V)\otimes_{\mathbb{R}} V_E^\bullet)$ is zero because $\mathcal{B}\to BG_r$ is $(n{+}1)$-connected and we have $H^{-1}(BG_r;(EG_r\times_{G_r}\mathbb{R}_{G_r})\otimes_{\mathbb{R}}V_E^\bullet) = 0$ since E is rationally even. Thus, taking any homotopy class $[\widehat{\nu}(g_{\mathcal{V}})]$ of differential Thom classes for $(\mathcal{V}, g_{\mathcal{V}})$, the pullback

$$f_V^*[\widehat{\nu}(g_{\mathcal{V}})] \tag{A.10}$$

defines a homotopy class of differential Thom classes for (V,g) which does not depend on the choice of $[\widehat{\nu}(g_{\mathcal{V}})]$. By the condition (1) and (2), we are forced to define the required homotopy class as

$$[\widehat{\nu}(g)] := f_V^*[\widehat{\nu}(g_{\mathcal{V}})], \tag{A.11}$$

[21] In the proof we use the assumption that E is rationally even. However, by a small modification of the proof, this assumption can be weakened to $H^{-1}(MG;V_E^\bullet)=0$. As a result, the results in this subsection hold under this weaker condition. The same remark applies to Proposition A.35.

by taking any $[\widehat{\nu}(g_{\mathcal{V}})]$ on $(\mathcal{V}, g_{\mathcal{V}})$.

We need to check that the element (A.11) does not depend on the other choices made above. But this easily follows from the cofinality of such choices. Namely, given two choices with the underlying manifolds $f_i\colon M \to \mathcal{B}_i$ for $i = 1, 2$, we may take another $\mathcal{B}$ with maps $g_i\colon \mathcal{B}_i \to \mathcal{B}$ so that $g_1 \circ f_1 = g_2 \circ f_2$, and other data on $\mathcal{B}$ which pulls back to those given on $\mathcal{B}_i$. From this, we conclude that the elements (A.11) defined using $\mathcal{B}_1$ and $\mathcal{B}_2$ coincide with the one defined using $\mathcal{B}$, so the element (A.11) is well-defined. By the arguments so far, they satisfy the required conditions and the uniqueness.

For the last statement, changing a differential stable G-structure g on V to a homotopic one amounts to changing the vector bundle map $f_V\colon V \to \mathcal{V}$ by a homotopy while fixing f and f_P in the above procedure. Pulling back the homotopy class $[\widehat{\nu}(g_{\mathcal{V}})]$ by such a homotopy, we get a homotopy of differential Thom classes between the differential Thom classes pulled back at the endpoints. This completes the proof. □

Now we turn to differential pushforwards for proper submersions. Let $p\colon N \to X$ be a proper submersion between manifolds of relative dimension r, and assume it is equipped with a differential stable normal G-structure $g_p^{\perp}$ (Definition 4.80) on the relative tangent bundle $T(p)$. Take a representative $\widetilde{g}_p^{\perp} = (k, P, \nabla, \psi)$ of $g_p^{\perp}$. It induces a differntial stable G-structure on $P \times_{G_{k-r}} \mathbb{R}^{k-r}$ which we denote g_P, represented by $\widetilde{g}_P = (k-r, P, \nabla, \mathrm{id})$. By Proposition A.9 we have a differential Thom class whose homotopy class $[\widehat{\nu}(g_P)]$ is canonically determined,

$$\widehat{\nu}(g_P) \in \widehat{E}^{k-r}_{\mathrm{prop}/N}(P \times_{G_{k-r}} \mathbb{R}^{k-r}) \tag{A.12}$$

If we stabilize k to $k+1$, the homotopy classes of (A.11) are related as Proposition A.9 (2).

Now, choose an embedding $\iota\colon N \hookrightarrow \mathbb{R}^k \times X$ over X (i.e., $\mathrm{pr}_X \circ \iota = p$) for k large enough, a tubular neighborhood W of N in $\mathbb{R}^k \times X$ with a vector bundle structure $W \to N$ so that it is a map over X (this is possible because p is a submersion). Replacing k larger if necessary, choose an isomorphism $\psi_W\colon W \simeq P \times_{G_{k-r}} \mathbb{R}^{k-r}$ of vector bundles over N so that the isomorphism $(P \times_{G_{k-r}} \mathbb{R}^{k-r}) \oplus T(p) \xrightarrow{\psi_W^{-1} \oplus \mathrm{id}} W \oplus T(p) \simeq \underline{\mathbb{R}}^k$ is homotopic to ψ. The isomorphism ψ_W induces a differential stable G-structure g_W on W, and the element (A.12) induces a differential Thom class on (W, g_W) denoted by

$$\widehat{\nu}(g_W) := \psi_W^* \widehat{\nu}(g_P) \in \widehat{E}^{k-r}_{\mathrm{prop}/N}(W). \tag{A.13}$$

We consider the composition,

$$\widehat{E}^n(N) \xrightarrow{\cdot \widehat{\nu}(g_W)} \widehat{E}^{n+k-r}_{\mathrm{prop}/N}(W) \xrightarrow{\iota_*} \widehat{E}^{n+k-r}_{\mathrm{prop}/X}(\mathbb{R}^k \times X) \xrightarrow{\mathrm{desusp}} \widehat{E}^{n-r}(X), \tag{A.14}$$

where the first map uses the module structure of the properly supported $\widehat{E}$, and the middle arrow is induced by the open embedding $W \hookrightarrow \mathbb{R}^k \times X$.

Proposition A.15. *The composition (A.14) only depends on the differential stable normal G-structure $g_p^{\perp}$ on $T(p)$.*

Proof. The above procedure includes the following ambigiuities : the choice of $\widehat{\nu}(g_P)$ representing $[\widehat{\nu}(g_P)]$ and the choice of the data of embedding with a tubular neighborhood and an isomorphism ψ_W. The independence on ψ_W directly follows from the last statement of Proposition A.9.

First we show the independence on the choice of $\widehat{\nu}(g_P)$, with the other data fixed. Since its homotopy class $[\widehat{\nu}(g_P)]$ is fixed by Proposition A.9, any two choices $\widehat{\nu}_i(g_P)$, $i = 0, 1$, are connected by a homotopy $\widehat{\nu}_I \in \widehat{E}^{k-r}_{\mathrm{prop}/(I\times N)}(I \times (P \times_{G_{k-r}} \mathbb{R}^{k-r}))$. Its pullback by ψ_W gives a homotopy $\widehat{\nu}_{I\times W} := \psi_W^* \widehat{\nu}_I$ between the corresponding differential Thom classes on (W, g_W). Denote the inclusion by $i_t \colon N \simeq \{t\} \times N \hookrightarrow I \times N$ for $t = 0, 1$. Consider the following commutative diagram,

(A.16)

$$\begin{tikzcd}
\Omega^n(I\times N; V_E^\bullet) \arrow[r, "\wedge R(\widehat{\nu}_{I\times W})"] & \Omega^{n+k-r}_{\mathrm{prop}/(I\times N)}(I\times W; V_E^\bullet) \arrow[r, "\int_{(I\times W)/(I\times X)}"] & \Omega^{n-r}(I\times X; V_E^\bullet) \arrow[r, "\int_{(I\times X)/X}"] & \Omega^{n-r-1}(X; V_E^\bullet) \arrow[d, "a"] \\
\widehat{E}^n(I\times N) \arrow[u, "R"] \arrow[r, "\cdot\widehat{\nu}_{I\times W}"] & \widehat{E}^{n+k-r}_{\mathrm{prop}/(I\times N)}(I\times W) \arrow[u, "R"] \arrow[r, "(\mathrm{desusp})\circ(\mathrm{id}_I\times\iota)_*"] & \widehat{E}^{n-r}(I\times X) \arrow[u, "R"] \arrow[r, "i_1^*-i_0^*"] & \widehat{E}^{n-r}(X)
\end{tikzcd}$$

The commutativity of the middle square is because the vector bundle structure $W \to N$ is a map over X. The commutativity of the right square is by the homotopy formula ([BS10, Lemma 1]).

Take any element $\widehat{e} \in \widehat{E}^n(N)$. Then the image of $\mathrm{pr}_N^* \widehat{e} \in \widehat{E}^n(I \times N)$ under the composition of the bottom arrows in (A.16) is equal to the difference of the elements in $\widehat{E}^{n-r}(X)$ obtained by applying to $\widehat{e}$ the composition (A.14) using $\widehat{\nu}_0(g_P)$ and $\widehat{\nu}_1(g_P)$. By the commutativity of (A.16), it is enough to check that the element $R(\mathrm{pr}_N^* \widehat{e}) \in \Omega^n_{\mathrm{clo}}(I \times N; V_E^\bullet)$ maps to zero under the composition of the top arrows in (A.16). Indeed, since $W \to N$ is a map over X, we can factor the upper middle horizontal integration in (A.16) on $I \times N$, and the result is equal to

$$\int_{(I\times X)/X} \int_{(I\times N)/(I\times X)} \mathrm{pr}_N^* R(\widehat{e}) \wedge \int_{(I\times W)/(I\times N)} R(\widehat{\nu}_{I\times W}), \tag{A.17}$$

and by (recall (A.5))

$$\int_{(I\times W)/(I\times N)} R(\widehat{\nu}_{I\times W}) = \mathrm{Td}(\widehat{\nu}_{I\times W}) = \mathrm{pr}_N^* \mathrm{Td}(\widehat{\nu}_0(g_P)),$$

so (A.17) is equal to

$$\int_{(I\times X)/X} \mathrm{pr}_X^* \int_{N/X} R(\widehat{e}) \wedge \mathrm{Td}(\widehat{\nu}_0(g_P)) = 0,$$

as desired. Thus we conclude that, fixing the data of an embedding with a tubular neighborhood, the composition (A.14) only depends on the homotopy class $[\widehat{\nu}(g_P)]$.

Now consider the stabilization of the embeddings, increasing k to $(k+1)$ and W to $\underline{\mathbb{R}} \oplus W$. By the condition (2) in Proposition A.9 and the result so far, we also conclude that the composition (A.14) is invariant under this stabilization.

The desired independence of (A.14) on the remaining choices is also proved in a parallel way, by choosing corresponding objects on the cylinder so that they restrict to stabilizations of the given ones on the endpoints. This completes the proof of Proposition A.15.

□

Thus we define the following.

Definition A.18. Let $p\colon N \to X$ be a proper submersion of relative dimension r, equipped with a differential stable normal G-structure $g_p^{\perp}$ on the relative tangent bundle $T(p)$. We define the *differential pushforward map*,

$$(p, g_p^{\perp})_*\colon \widehat{E}^n(N) \to \widehat{E}^{n-r}(X)$$

to be the composition (A.14). This does not depend on any choices by Proposition A.15.

By the construction, the following diagram commutes.

(A.19)

$$\begin{tikzcd}
\Omega^{n-1}(N; V_E^\bullet)/\mathrm{im}(d) \arrow[r, "a"] \arrow[d, "\int_{N/X} - \wedge \mathrm{cw}_{g_p^{\perp}}(\mathrm{ch}(\mathcal{G}))"] & \widehat{E}^n(N) \arrow[r, "I"] \arrow[d, "(p, g_p^{\perp})_*"] \arrow[rr, bend left, "R"] & E^n(N) \arrow[d, "(p, g_p^{\perp, \mathrm{top}})_*"] & \Omega^n_{\mathrm{clo}}(N; V_E^\bullet) \arrow[d, "\int_{N/X} - \wedge \mathrm{cw}_{g_p^{\perp}}(\mathrm{ch}(\mathcal{G}))"] \\
\Omega^{n-r-1}(X; V_E^\bullet)/\mathrm{im}(d) \arrow[r, "a"] & \widehat{E}^{n-r}(X) \arrow[r, "I"] \arrow[rr, bend right, "R"'] & E^{n-r}(X) & \Omega^{n-r}_{\mathrm{clo}}(X; V_E^\bullet)
\end{tikzcd}.$$

In this sense, Definition A.18 refines the pushforwards on E^* and $\Omega^*(-; V_E^\bullet)$.

An important property of differential pushforwards is the *Bordism formula* [Bun12, Problem 4.235], which says that if we have a bordism $(W, g_W^{\perp})\colon (M_-, g_-^{\perp}) \to (M_+, g_+^{\perp})$, the differential pushforwards at the boundary can be computed by the integration of the characteristic form on the bordism. Its normal variant is stated in the form we use in this paper as Fact 8.10. To prove it, we need to consider differential pushforwards for proper maps which is not submersions, namely boundary defining functions $W \to I$. The result easily follows by the homotopy formula ([BS10, Lemma 1]). For the details of the proof we refer [Bun12, Problem 4.235].

A.2. **Differential pushforwards in Hopkins-Singer's differential extensions.** Now we turn to the Hopkins-Singer's differential extensions. As we explain, the definition of differential pushforwards in [HS05] differs from the one in Subsection A.3. In this subsubsection, we clarify their relation in the settings of our interest (Proposition A.33).

Fix fundamental cocycles $\iota_E \in Z^0(E; V_E^\bullet)$ and $\iota_{MG} \in Z^0(MG; V_{MG}^\bullet)$ for E and MG, respectively. Since E is rationally even, the Hopkins-Singer's model $\widehat{E}^*_{\mathrm{HS}}(-; \iota_E)$ admits a canonical multiplicative structure by [Upm15]. We briefly explain it here. We only explain the even-degrees. The remaining cases are induced by requiring the compatibility with the S^1-integration. Let n and m be even integers, and denote by $\mu_{nm}\colon E_n \wedge E_m \to E_{n+m}$ a multiplication map. We need to choose a reduced cochain $c_{nm} \in \widetilde{C}^{n+m-1}(E_n \wedge$

$E_m; V_E^\bullet)$ so that

$$\delta c_{nm} = \iota_n \cup \iota_m - \mu_{nm}^* \iota_{n+m}. \tag{A.20}$$

Since E is rationally even, we have $\widetilde{H}^{n+m-1}(E_n \wedge E_m; V_E^\bullet) = 0$ by the proof of [BS10, Lemma 3.8]. Thus any two choices of such cochains c_{nm} differ by a coboundary. Using c_{nm} we get the map of differential function spaces ([HS05, Remark 4.17]),

$$(E_n \wedge E_m; (\iota_E)_n \cup (\iota_E)_m)^M \to (E; \iota_E)_{n+m}^M, \tag{A.21}$$

for any manifold M. Also choose a natural cochain homotopy $B\colon \Omega^n(-) \otimes \Omega^m(-) \to C^{n+m-1}(-)$ cobounding the difference between $\wedge$ on forms and $\cup$ on singular cochains as in [HS05, (3.8)], [Upm15, Section 6]. Any two such choices are naturally cochain homotopic. It induces the map

$$(E; \iota_E)_n^M \times (E; \iota_E)_m^M \to (E_n \wedge E_m; (\iota_E)_n \cup (\iota_E)_m)^M, \tag{A.22}$$

for any M. Combining (A.21) and (A.22), we get the multiplication map,

$$\cdot\colon \widehat{E}_{\mathrm{HS}}^n(M; \iota_E) \otimes \widehat{E}_{\mathrm{HS}}^m(M; \iota_E) \to \widehat{E}_{\mathrm{HS}}^{n+m}(M; \iota_E). \tag{A.23}$$

This does not depend on any of the choices above. For a real vector bundle $V \to M$, in the same way we get a map using the properly supported differential functions ([HS05, Section 4.3])

$$(E; \iota_E)_n^M \times (E; \iota_E)_m^{\overline{V}} \to (E; \iota_E)_{n+m}^{\overline{V}}, \tag{A.24}$$

which gives the module structure,

$$\cdot\colon \widehat{E}_{\mathrm{HS}}^n(M; \iota_E) \otimes \widehat{E}_{\mathrm{HS,prop}/M}^m(V; \iota_E) \to \widehat{E}_{\mathrm{HS,prop}/M}^{n+m}(V; \iota_E). \tag{A.25}$$

As we mentioned in Subsection 8.1, Hopkins-Singer's normal differential BG-orientations are defined in terms of differential functions to $(MG; \iota_{MG})$. A differential pushforward is defined by fixing a map of differential function spaces

$$\widehat{\mathcal{G}}\colon \left(MG_{-r} \wedge (E_n)^+; V_{\mathcal{G}}(\iota_{MG})_{-r} \cup \iota_E\right) \to (E; \iota_E)_{n-r}. \tag{A.26}$$

whose underlying map factors as $MG_{-r} \wedge (E_n)^+ \xrightarrow{\mathcal{G}\wedge \mathrm{id}} E_{-r} \wedge (E_n)^+ \xrightarrow{\mu_{-r,n}} E_{n-r}$. Here $V_{\mathcal{G}}\colon V_{MG}^\bullet \to V_E^\bullet$ is induced by $\mathcal{G}$, so that $V_{\mathcal{G}}(\iota_{MG}) \in Z^0(MG; V_E^\bullet)$ represents $\mathrm{ch}(\mathcal{G})$. We can take a $c_{\mathcal{G}} \in C^{-1}(MG; V_E^\bullet)$ so that $\delta c_{\mathcal{G}} = \mathcal{G}^* \iota_E - V_{\mathcal{G}}(\iota_{MG})$, and it is determined up to coboundary because $H^{-1}(MG; V_E^\bullet) = 0$. We may take (A.26) to be the composition

(A.27)

$$\widehat{\mathcal{G}}\colon \left(MG_{-r} \wedge (E_n)^+; V_{\mathcal{G}}(\iota_{MG})_{-r} \cup (\iota_E)_n\right) \to (E_{-r} \wedge E_n; (\iota_E)_{-r} \cup (\iota_E)_n) \xrightarrow{\text{(A.21)}} (E; \iota_E)_{n-r},$$

where the first map uses $c_{\mathcal{G}}$. Let $V \to M$ be a real vector bundle, and consider the following diagram.

(A.28)
$$\begin{tikzcd}
(MG;\iota_{MG})^{\overline{V}}_{-r} \times (E;\iota_E)^M_n \arrow[r] \arrow[d] & (E;\iota_E)^{\overline{V}}_{-r}\times(E;\iota_E)^M_n \arrow[d] \arrow[dr, "(A.24)"] & \\
(MG_{-r}\wedge (E_n)^+; V_{\mathcal{G}}(\iota_{MG})_{-r}\cup(\iota_E)_n)^{\overline{V}} \arrow[r] \arrow[rr, bend right, "\widehat{\mathcal{G}}"'] & (E_{-r}\wedge E_n; (\iota_E)_{-r}\cup(\iota_E)_n)^{\overline{V}} \arrow[r] & (E;\iota_E)^{\overline{V}}_{n-r}
\end{tikzcd}$$

Here the top horizontal arrow uses $c_{\mathcal{G}}$, the left vertical arrow uses the cochain homotopy B, and the remaining arrows are as before. The two triangles commute. The square does not commute on the level of differential function spaces, but we can easily check that the difference is a coboundary so the induced maps on the differential cohomology level,

$$(\widehat{MG})^{-r}_{\mathrm{HS},\mathrm{prop}/M}(V;\iota_{MG}) \otimes \widehat{E}^n_{\mathrm{HS}}(M;\iota_E) \to \widehat{E}^{n-r}_{\mathrm{HS},\mathrm{prop}/M}(V;\iota_E) \tag{A.29}$$

are the same. Using the top factorization of (A.28), we see that (A.29) factors as

(A.30)
$$(\widehat{MG})^{-r}_{\mathrm{HS},\mathrm{prop}/M}(V;\iota_{MG}) \otimes \widehat{E}^n_{\mathrm{HS}}(M;\iota_E) \to \widehat{E}^{-r}_{\mathrm{HS},\mathrm{prop}/M}(V;\iota_E) \otimes \widehat{E}^n_{\mathrm{HS}}(M;\iota_E) \dot{\to} \widehat{E}^{n-r}_{\mathrm{HS},\mathrm{prop}/M}(V;\iota_E)$$

To put them into the picture in Subsection A.1, apply the discussions there in the case $E = MG$ and $\mathcal{G} = \mathrm{id}\colon MG \to MG$. Assume that we have a proper submersion $p\colon N \to X$ equipped with a differential stable normal G-structure $g_p^\perp$ on the relative tangent bundle $T(p)$, represented by $\widetilde{g}_p^\perp = (k, P, \nabla, \psi)$. A *Hopkins-Singer's normal differential BG-orientation* ([HS05, Section 4.9.2]) $g_p^{\perp,\mathrm{HS}}$ consists of choices of an embedding $N \hookrightarrow \mathbb{R}^k \times X$ over X, a tubular neighborhood with a vector bundle structure $W \to N$, an isomorphism $\psi_W\colon W \simeq P \times_{G_{k-r}} \mathbb{R}^{k-r}$ as in Subsection A.1 (in general $W \to N$ is not required to be a map over X), and a lift of a classifying map for the induced G-structure (W, g_W^{top}) on $W \to N$ to a differential function $t(g_p^{\perp,\mathrm{HS}})\colon \overline{W} \to (MG_{k-r}; (\iota_{MG})_{k-r})$. Then, the differential function $t(g_p^{\perp,\mathrm{HS}})$ represents a differential Thom class for (W, g_W^{top}),

$$\left\langle t(g_p^{\perp,\mathrm{HS}}) \right\rangle \in (\widehat{MG})^{k-r}_{\mathrm{HS},\mathrm{prop}/N}(W;\iota_{MG}), \tag{A.31}$$

where we denoted by $\langle (c,h,\omega)\rangle$ the differential cohomology class represented by a differential function (c,h,ω). Now we define the following.

Definition A.32. Let $p\colon N \to X$ be a proper submersion between manifolds of relative dimension r, and assume it is equipped with a differential stable normal G-structure $g_p^\perp$ on $T(p)$. A Hopkins-Singer's normal differential BG-orientation $g_p^{\perp,\mathrm{HS}}$ is said to be a *lift* of $g_p^\perp$ if, in the notations above,

- The vector bundle structure $W \to N$ is a map over X, and

- The homotopy class $\left[\left\langle t(g_p^{\perp,\mathrm{HS}})\right\rangle\right]$ of the differential Thom class $\left\langle t(g_p^{\perp,\mathrm{HS}})\right\rangle$ is the one associated to g_W by Proposition A.9 (applied to $E = MG$).

In particular, this means that,

$$\mathrm{cw}_{g_W}(\mathrm{ch}(\mathrm{id}_{MG})) = \mathrm{Td}\left(\left\langle t(g_p^{\perp,\mathrm{HS}})\right\rangle\right) := \int_{W/N} R\left(\left\langle t(g_p^{\perp,\mathrm{HS}})\right\rangle\right).$$

where $\mathrm{ch}(\mathrm{id}_{MG}) \in H^0(MG; V_{MG}^{\bullet})$.

Now assume we are given an element $\widehat{e} \in \widehat{E}^n_{\mathrm{HS}}(N; \iota_E)$. Then, in [HS05, Section 4.10] the differential pushforward of $\widehat{e}$ is formulated as follows. Take a differential function $t(\widehat{e})\colon N \to (E; \iota)_n$ representing $\widehat{e}$. Apply the left bottom composition of (A.28) for the vector bundle $W \to N$ to the pair $(t(g_p^{\perp,\mathrm{HS}}), t(\widehat{e}))$ to get a differential function in $(E; \iota_E)^{\overline{W}}_{n+k-r}$. By the open embedding $W \hookrightarrow \mathbb{R}^k \times X$ we get a differential function in $(E, \iota_E)^{\overline{\mathbb{R}^k} \times X}_{n+k-r}$, and this represents the desired element $(p, g_p^{\perp,\mathrm{HS}})_* \widehat{e} \in \widehat{E}^{n-r}_{\mathrm{HS}}(X; \iota_E)$. By the discussion so far, the result is the same if we use the top right composition in (A.28), and it is given by the composition (A.30). Then we see that the above definition of $(p, g_p^{\perp,\mathrm{HS}})_* \widehat{e}$ exactly translates into the definition of differential pushforwards (A.14) in the Subsection A.1. Thus we conclude that

Proposition A.33. *In the settings of Definition A.32, the differential pushforward map $(p, g_p^{\perp})_*\colon \widehat{E}^*_{\mathrm{HS}}(N; \iota_E) \to \widehat{E}^{n-r}_{\mathrm{HS}}(X; \iota_E)$ in Definition A.18 applied to $\widehat{E}^n_{\mathrm{HS}}(-; \iota_E)$ coincides with the differential pushforward map $(p, g_p^{\perp,\mathrm{HS}})_*$ in [HS05] as long as we use $g_p^{\perp,\mathrm{HS}}$ lifting $g_p^{\perp}$.*

A.3. **The tangential case.** Now we explain the tangential variants of the last Subsections A.1 and A.2. The constructions and verifications are parallel to the normal case, so we go briefly.

In this case, we are given a homomorphism of ring spectra,

$$\mathcal{G}\colon MTG \to E, \tag{A.34}$$

where MTG is the Madsen-Tillmann spectrum. MTG is constructed as a direct limit of Thom spaces of stable normal bundles to the universal bundles over approximations of BG_d's, so classifies vector bundles with stable normal G-structures. Then for each real vector bundle V of rank r over a topological space X equipped with a topological stable normal G-structure $g^{\perp,\mathrm{top}}$, we get the *Thom class* $\nu \in E^r(\overline{V})$, whose multiplication gives the *Thom isomorphism* $E^*(X) \simeq E^{*+r}(\overline{V})$.

We formulate the notion of *differential Thom classes* as a differential refinements of the Thom classes, as well as differential forms $\mathrm{Td}(\widehat{\nu})$ and homotopies in the same way as Definition A.3. By the exactly the same proof, the classification result of differential Thom classes corresponding to Lemma A.6 also holds in the case here. The Chern-Dold character for (A.34) is an element $\mathrm{ch}(\mathcal{G}) \in H^0(MTG; V_E^{\bullet})$. If $V \to M$ is equipped with a stable normal G-structure $g^{\perp}$, the characteristic form (A.8) is replaced by the form

$\mathrm{cw}_{g^\perp}(\mathrm{ch}(\mathcal{G}))$, where we use the Chern-Weil construction in (4.83). Then, the same proof as that of Proposition A.9 shows the following.

Proposition A.35. *There exists a unique way to assign a homotopy class $[\widehat{\nu}(g^\perp)]$ of differential Thom classes $\widehat{\nu}(g^\perp) \in \widehat{E}^{\mathrm{rank} V}_{\mathrm{prop}/M}(V)$ to every real vector bundle with differential stable normal G-structure $(V, g^\perp) \to M$ such that the following three conditions hold.*

(1) It is compatible with pullbacks.
(2) We have $\int_{\mathbb{R}} [\widehat{\nu}(g^\perp_{\underline{\mathbb{R}} \oplus V})] = [\widehat{\nu}(g^\perp_V)]$.
(3) We have $\mathrm{cw}_{g^\perp}(\mathrm{ch}(\mathcal{G})) = \mathrm{Td}([\widehat{\nu}(g^\perp)]) := \int_{V/M} R(\widehat{\nu}(g^\perp))$.

Moreover, the resulting homotopy class $[\widehat{\nu}(g^\perp)]$ only depends on the homotopy class (Definition 4.80 (4)) of differential stable normal G-structure $g^\perp$.

Let $p\colon N \to X$ be a proper submersion between manifolds of relative dimension r, equipped with a differential stable G-structure g_p on the relative tangent bundle $T(p)$ represented by $\widetilde{g}_p = (d, P, \nabla, \psi)$. Choose an embedding $\iota\colon N \hookrightarrow \mathbb{R}^k \times X$ over X for k large enough, a tubular neighborhood W of N in $\mathbb{R}^k \times X$ with a vector bundle structure $W \to N$ so that it is a map over X (this is possible because p is a submersion). Then we get an isomorphism

$$\psi_W\colon (P \times_{G_d} \mathbb{R}^d) \oplus W \simeq \underline{\mathbb{R}}^{d-n} \oplus T(p) \oplus W \simeq \underline{\mathbb{R}}^{d-n+k} \tag{A.36}$$

of vector bundles over N. As a result, we get a differential stable normal G-structure $g^\perp_W$ on the vector bundle $W \to N$, represented by $\widetilde{g}^\perp_W = (d - n + k, P, \nabla, \psi_W)$. For $g^\perp_W$, Proposition A.35 assigns a differential Thom class whose homotopy class is canonically determined,

$$\widehat{\nu}(g^\perp_W) \in \widehat{E}^{k-r}_{\mathrm{prop}/N}(W). \tag{A.37}$$

We consider the composition,

$$\widehat{E}^n(N) \xrightarrow{\cdot \widehat{\nu}(\widetilde{g}^\perp_W)} \widehat{E}^{n+k-r}_{\mathrm{prop}/N}(W) \xrightarrow{\iota_*} \widehat{E}^{n+k-r}_{\mathrm{prop}/X}(\mathbb{R}^k \times X) \xrightarrow{\mathrm{desusp}} \widehat{E}^{n-r}(X). \tag{A.38}$$

The following proposition can be shown in the same way as Proposition A.15.

Proposition A.39. *The composition (A.38) only depends on the differential stable G-structure g_p on $T(p)$.*

Proposition A.39 allows us to define the following.

Definition A.40. Let $p\colon N \to X$ be a proper submersion between manifolds of relative dimension r, equipped with a differential stable G-structure g_p on the relative tangent bundle $T(p)$. We define the *differential pushforward map*,

$$(p, g_p)_*\colon \widehat{E}^n(N) \to \widehat{E}^{n-r}(X)$$

to be the composition (A.38).

Now we turn to the Hopkins-Singer's models as in Subsection A.2. Take fundamental cocycles ι_E and ι_{MTG} for E and MTG, respectively. As explained there, $\widehat{E}^*_{\mathrm{HS}}(-;\iota_E)$ admits a canonical multiplicative structure. In the normal case, a differential pushforward is defined by a map of differential function spaces

$$\widehat{\mathcal{G}}\colon \big(MTG_{-r} \wedge (E_n)^+; V_{\mathcal{G}}(\iota_{MTG})_{-r} \cup \iota_E\big) \to (E;\iota_E)_{n-r}. \tag{A.41}$$

whose underlying map factors as $MTG_{-r} \wedge (E_n)^+ \xrightarrow{\mathcal{G}\wedge \mathrm{id}} E_{-r} \wedge (E_n)^+ \xrightarrow{\mu_{-r,n}} E_{n-r}$. By the same argument to the normal case, the map (A.41) induces the map of differential cohomologies for any real vector bundle $V \to M$,

$$(\widehat{MTG})^{-r}_{\mathrm{HS,prop}/M}(V;\iota_{MTG}) \otimes \widehat{E}^n_{\mathrm{HS}}(M;\iota_E) \to \widehat{E}^{-r}_{\mathrm{HS,prop}/M}(V;\iota_E) \otimes \widehat{E}^n_{\mathrm{HS}}(M;\iota_E) \xrightarrow{\cdot} \widehat{E}^{n-r}_{\mathrm{HS,prop}/M}(V;\iota_E). \tag{A.42}$$

Let $p\colon N \to X$ be a proper submersion between manifolds of relative dimension r, equipped with a differential stable G-structure g_p on the relative tangent bundle $T(p)$ represented by $\widetilde{g}_p = (d, P, \nabla, \psi)$. A *Hopkins-Singer's tangential differential BG-orientation* g_p^{HS} consists of choices of an embedding $N \subset \mathbb{R}^k \times X$, a tubular neighborhood W, a vector bundle structure $W \to N$ as in the first part of this subsection (in general $W \to N$ is not required to be a map over X), and a lift of a classifying map for $(W, g_W^{\perp,\mathrm{top}})$ of the induced normal structure to a differential function $t(g_p^{\mathrm{HS}})\colon \overline{W} \to (MTG_{k-r};(\iota_{MTG})_{k-r})$. Then, the differential function $t(g_p^{\mathrm{HS}})$ represents a differential Thom class for $(W, g_W^{\perp,\mathrm{top}})$,

$$\big\langle t(g_p^{\mathrm{HS}})\big\rangle \in (\widehat{MTG})^{k-r}_{\mathrm{HS,prop}/N}(W;\iota_{MTG}). \tag{A.43}$$

Now we define the following.

Definition A.44. In the above settings, Hopkins-Singer's tangential differential BG-orientation g_p^{HS} is said to be a *lift* of g_p if, in the notations above,

- The vector bundle structure $W \to N$ is a map over X, and
- The homotopy class $\big[\big\langle t(g_p^{\mathrm{HS}})\big\rangle\big]$ of the differential Thom class $\big\langle t(g_p^{\mathrm{HS}})\big\rangle$ is the one associated to $g_W^{\perp}$ by Proposition A.35 (applied to $E = MTG$).

In particular, this means that,

$$\mathrm{cw}_{g_W^{\perp}}(\mathrm{ch}(\mathrm{id}_{MTG})) = \mathrm{Td}\left(\big[t(g_p^{\mathrm{HS}})\big]\right) := \int_{W/N} R\left(\big[t(g_p^{\mathrm{HS}})\big]\right).$$

where $\mathrm{ch}(\mathrm{id}_{MTG}) \in H^0(MTG; V^{\bullet}_{MTG})$.

Let us take $\widehat{e} \in \widehat{E}^n_{\mathrm{HS}}(N;\iota_E)$. By the same procedure as in the last paragraph of Subsection A.2, the tangential variant of [HS05, Section 4.10] using the map (A.41) and the open embedding $W \hookrightarrow \mathbb{R}^k \times X$ produces the element $(p, g_p^{\mathrm{HS}})_*\widehat{e} \in \widehat{E}^{n-r}_{\mathrm{HS}}(X;\iota_E)$. We get

Proposition A.45. *In the settings of Definition A.44, the differential pushforward map $(p, g_p)_*\colon \widehat{E}^n_{\mathrm{HS}}(N;\iota_E) \to \widehat{E}^{n-r}_{\mathrm{HS}}(X;\iota_E)$ in Definition A.40 applied to $\widehat{E}^*_{\mathrm{HS}}(-;\iota_E)$ coincides with the tangential variant of the differential pushforward map $(p, g_p^{\mathrm{HS}})_*$ in [HS05] as long as we use g_p^{HS} lifting g_p.*

Thus we conclude that the differential pushforward maps in Definition A.40 for $\widehat{E}^*_{\mathrm{HS}}(-;\iota_E)$ comes from maps between differential function spaces (A.41). As we mentioned in Footnote 16, this is the reason why we want to use the Hopkins-Singer's formulation.

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