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Rich by accident: the second welfare theorem with a redundant asset under imperfect foresight*

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Abstract

We consider a multiperiod (T-period) model with no uncertainty where short term bonds co-exist with a long term bond. Markets are complete with just the short term bonds so that under the usual hypothesis of perfect foresight, the long term bond is redundant by no arbitrage in that it has no allocational implications. We dispense with perfect foresight, derive appropriate no arbitrage conditions and show that the presence of the long term bond has significant allocational implications. Specifically, in the model with just the short term bond, we show that a T dimensional subset of efficient allocations can arise as Walrasian equilibria whereas the dimension of efficient allocations is one less than the number of households (assumed to be much larger than T). In the model with the both types of bonds, essentially all efficient allocations might arise as Walrasian equilibria; minute errors in forecasting prices might generate all income transfers that are consistent with efficiency. We argue that the beneficiaries of such unanticipated income transfers are determined not by the superiority of forecasts but rather by accident. (JEL classification numbers: D51, D53, D61)

Keywords:General equilibrium; Efficient temporary equilibrium; Endogenous price forecasts; Redundant Assets

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1 Introduction

What allocational role might a redundant financial asset play in an intertemporal Walrasian setting? Traditional wisdom would suggest none, since by definition, a redundant financial asset can be replicated by trading other assets dynamically at market prices so that any trader is indifferent between holding it and ignoring it, and so its presence in no way alters the possibilities of income transfers across periods/states. But notice that this conclusion might not be valid if the market prices are not correctly anticipated. That is, this conclusion relies entirely on the feature that the axiom of perfect foresight is built into the particular equilibrium concept, Radner equilibrium, used in the analysis. We dispense with perfect foresight and show that essentially *all* intertemporally efficient allocations can arise as Walrasian equilibria when a redundant asset is traded.

To fix ideas, consider a discount bond which matures in a few years. If there is no uncertainty about the fundamentals of the economy at all, then its yield must be given by the compound one year interest rates, since the long term bond can be replicated by an iterative one year saving. Then the bond is a redundant asset which does not add any new saving opportunity and the law of one price, or the no-arbitrage principle, will immediately determine its market value. However, notice that this argument implicitly assumes that the one year rates are known, which in particular means that the forecasts are perfectly aligned and thus homogenous across agents. Even though there is no uncertainty, the one year rates that will prevail in the future years are not realized yet, leaving some room for heterogenous forecasts. The long term bond might not be redundant even in an idealized world of frictionless trading, unless perfect foresight is exogenously imposed.

Imposing that forecasts be perfectly aligned (as is implied by perfect foresight), however, does not sit well with the Walrasian paradigm since decentralized households cannot be expected to coordinate on prices that are not commonly observed.[1] One might still think that if the degree of heterogeneity of forecasts is small enough, the market outcome will be close to the one predicted in a perfect foresight model, and the presence of the long term bond like above will not add much qualitatively. We will argue that such a

[1]See Radner [1982] and Grandmont [1988] for instance. Various kinds of evidence against alignment of forecasts can be supplied even in more restrictive contexts of asset pricing (see e.g., Bossaerts [2002]).

conjecture is incorrect, and that there are profound welfare effects which are completely missed out in rational expectations models.

We formulate our question in a one good economy which lasts for T periods, and which has two nominal assets. The first of these is a discount bond traded in periods $t = 0, 1, ..., T-1$, which matures in one period, i.e., the bond traded in period t pays out \$1 in period $t+1$. The second is a discount bond with a longer maturity, which will be called the L-bond: the L-bond can be traded in every period, but pays \$1 in period T, and nothing in other periods. In every period, the good and the bonds are traded competitively. We shall compare two models which only differ in their asset structures: In Model I, the discount bond is traded in every period while in Model II, in addition to the bond, the L-bond is traded in every period.

As is known, *if perfect foresight (rational expectation) is assumed*, the markets are already complete with bonds with one period maturity, and the payoffs of the L-bond can be replicated by a plan of dynamic transactions of the bonds. It means that the market for the L-bond has no additional implication on the allocation of the good except for indeterminacy of asset trade, which arises since the L-bond and the dynamic plan are perfect substitutes at any time. In this sense, models I and II are equivalent in terms of the allocations they generate.

We allow for heterogeneous forecasts, however, which leads us to inspect temporary equilibria. To avoid the myriad of temporary equilibrium allocations that emerge when allowing for heterogenous forecasts, and to bring out the allocative implications in a sharper manner, one needs to put some more discipline into the analysis.[2] One way of doing so would be to put restrictions directly on the sort of forecasts agents are allowed to hold.[3] We advocate instead an allocation based approach that induces restrictions

[2]Allocative implications did not receive attention in the earlier literature on temporary equilibrium (Grandmont [1988]), which was focussed on existence (achieved by restricting the dependence of forecasts on current period prices). Subsequently, the literature examined learning behavior wherein forecasts were updated in a structured way (using OLS or Bayesian updating for instance (Chatterji [1995])) with a view to investigating the stability of a perfect foresight equilibrium. Our interest is in scenarios where heterogeneity of forecasts persists and we investigate its allocational implications.

[3]Recent literature in macroeconomics and finance explores implications of various kinds of forecasts. For instance, Woodford and Xie [2022] and Woodford, [2013]) consider bounded rationality in the design of fiscal and monetary policy and inflation targeting. Adam et al [2016] considers the CAPM model to

on forecasts implicitly: in so doing we are motivated by the concerns that underlie the fundamental welfare theorems, namely characterizing the parts of the Pareto set that are attainable via Walrasian markets when allowing heterogeneity in forecasts. These allocations are then a natural benchmark for any sort of welfare analysis in these markets to say the least. Thus, we postulate that markets allocate resources in an intertemporally efficient manner and characterize the set of efficient temporary equilibrium allocations (ETE) that can arise in Model I or Model II.

Taking an allocation based approach to its extreme, we choose to impose no restrictions on forecasts a priori. However our methodology can in principle allow one to incorporate additional restrictions in forecasts. Specifically, we separate the analysis into two parts. The first part identifies efficient allocations that are "price supportable" in that they satisfy the budget and feasibility conditions necessary for a Walrasian equilibrium. The second part investigates which of these allocations are "justifiable", that is, arise as solutions to optimization for some specification of forecasts. If there are a priori reasons to restrict the sort of forecasts that are admissible in the model, these restrictions will only impinge on the justifiability part of our analysis and can be incorporated therein.

We show that in Model I, the dimension of ETE allocations is at most T, one less than the number of trading periods, while, as is well known, the set of efficient allocations is $H - 1$ dimensional. While the case $H < T$ appears to be of interest in modelling scenarios where trading opportunities arise frequently, it is at odds with the spirit of perfect competition since it means in effect that there are more markets than traders. Our emphasis will therefore be on the case $H > T$, where ETE induces discipline on allocations, in that while accommodating heterogeneous forecasts does expand the set of efficient market outcomes, significant parts of the Pareto set can never be realized.

Moving to Model II, recall that the L-bond is redundant under perfect foresight. Even when perfect foresight is not assumed, if a trader finds an arbitrage opportunity with its price forecasts, the markets cannot be in a temporary equilibrium: Hence at

show that small deviations from the rational expectations generate realistic amounts of stock price volatility. This literature allows some sorts of deviations of forecasts from the rational expectations hypothesis that cause inefficiencies and studies the scope of policy interventions in making welfare improvements.

every trading opportunity, every household must have price forecasts which do not admit any arbitrage opportunity in the markets to take advantage of. It means that the two types of the bonds are deemed perfect substitutes at any time by any household, i.e., a household will never find it advantageous to trade the L-bond in equilibrium. A plausible conjecture might then be that the set of ETE allocations is the same as in Model I. But surprisingly, the set of ETE allocations is of dimension $H-1$, which includes all intertemporally efficient allocations in the vicinity of the Arrow Debreu allocation. When $H > T$ in particular, the set of ETE expands rather dramatically with the additional bond since without it the set of ETE is at most of dimension T.

Roughly speaking, the result holds since in perfectly competitive markets, any income transfer that is consistent with efficiency as well as individual optimization can occur owing to *ex post* forecasting errors that lead to *ex post* arbitrage opportunities. Note that unless forecasts are perfectly correct, such ex post errors are inevitable, although they might be small for sophisticated traders. Interestingly enough, for the result in Model II, the errors might be arbitrarily small: the ex post prices that sustain an ETE different from the perfect foresight allocation, may resemble arbitrarily closely the sequence of perfect foresight prices to an observer. This brings into question the interpretation of the determinacy of the perfect foresight allocation, at least in Model II, since as far as we are aware of, (publicly known) asset pricing models about derivative securities are built upon an arguably stringent assumption that the assumed price processes of underlying assets are correct.

Notice that an ETE by definition yields an efficient outcome and in this sense the markets function ideally, fulfilling their mandates. Every household' mathematical model explains the bond prices perfectly at any time under its forecast prices. In our simple world, each household is no worse than a financial firm which uses very sophisticated model to find out the correct valuations of redundant assets in every trading opportunity. In spite of these aspects, our analysis concludes that any kind of income transfer can arise implicitly in competitive markets, i.e., there will be winners and losers. We argue that the beneficiaries of such unanticipated income transfers are determined not by the superiority of forecasts but rather by accident.

The point will be seen clearly if the households are identical, and they might trade only when their forecasts differ from each other. One can view this special case as an adequate setup to explore a version of purely speculative trade. We show that the no-trade outcome is a unique ETE in Model I, but any efficient allocation with some lower bound can arise as an ETE in Model II; that is, it is the presence of redundant assets which bring about income transfers of all sorts.

The remainder of the paper is organized as follows. Section 2 introduces the model, the ex post budget constraints and the no arbitrage condition. Sections 3 studies wealth transfers using the key intermediary concept of price supportability. Section 4 studies conditions for justifiability and characterizes the set of efficient allocations that are obtained in the model. Section 5 briefly explores the possibility of speculative trade in this set up while Section 6 concludes.

2 The Model and Temporary Equilibrium

2.1 Set up

We consider a very simple market economy whose properties are well-known, but we nonetheless summarize its key properties for completeness. Let there be $T+1$ periods starting with period 0, where $T > 0$. A single non-storable good is available in every period. There is no uncertainty in the economy.

There are H households, labeled by $h = 1, 2, ..., H$. Household h is endowed with e_h^t units of the good in period t, $t = 0, 1, ..., T$, which is known to household h. To avoid triviality and zero income, we assume $H > 1$ and $e_h >> 0$ for every h. We shall write $x_h^t \geq 0$ for the consumption of household h in period t, and $x_h = \left(\cdots, x_h^t, \cdots\right)$ for the sequence of consumption. An allocation of the goods, $x = (\cdots, x_h, \cdots) \in \left(\mathbb{R}_+^{T+1}\right)^H$, is feasible if $\sum_{h=1}^{H} \left(x_h^t - e_h^t\right) = 0$ for $t = 0, 1, ..., T$. Household h's preferences are represented by an additively time separable utility function $u_h(x_h) = u_h^0\left(x_h^0\right) + u_h^1\left(x_h^1\right) + \cdots + u_h^T\left(x_h^T\right)$. The additive structure allows us to provide a clean analysis of the efficient allocations which arise as temporary equilibria. It also eliminates the conceptual issues about continuation utility which might appear as a potential hazard for the intertemporal general equilibrium analysis. However, it will become clear that the additive structure is not needed mathematically for the general point about induced income transfers.

We will consider two kinds of assets in the economy. The first of these is a discount bond traded in periods $t = 0, 1, ..., T-1$, which matures in one period. That is, the bond traded in period t pays out \$1 in period $t+1$. The second is a discount bond with a longer maturity, called the L-bond, which can be traded in every period. The L-bond pays \$1 in period T, nothing in other periods. Note that the payout is fixed in units of account, not in units of good. The net supply of any of these bonds is zero. In every period, the good and the bonds are traded competitively. By assumption no default occurs.

We shall compare two models which only differ in their asset structures: In Model I, the discount bond is traded in every period while in Model II, in addition to the bond, the L-bond is traded in every period. In both models, we write p^t and q^t for the respective prevailing prices of the good and the bond in units of account in period t. Write b_h^t for the amount of the bond household h holds at the end of period t. Write q_L^t for the prevailing price of the L-bond in period t and l_h^t for the amount of the L-bond held by household h at the end of period t. Thus $l_h^t - l_h^{t-1}$ is the amount traded in period t, which costs $q_L^t \left(l_h^t - l_h^{t-1} \right)$. Unlimited short sales are allowed, so b_h^t and l_h^t are possibly any negative number, but recall that default is not allowed. We write $b^t = (..., b_h^t, ...)$ and $l^t = (..., l_h^t, ...)$ for allocations of the bond and the L-bond in period t, respectively, and write $\mathbf{b} = \left(\cdots, b^t, \cdots \right)$ and $\mathbf{l} = \left(\cdots, l^t, \cdots \right)$ for a sequence of such allocations. Since the bonds are in zero net supply, we say $\mathbf{b}$ (resp. $\mathbf{l}$) is feasible if $\sum_{h=1}^{H} b_h^t = 0$ (resp. $\sum_{h=1}^{H} l_h^t = 0$) holds in every period t.

As is known, *if perfect foresight (rational expectation) is assumed*, the markets are already complete with bonds with one period maturity, and the L-bond is a redundant asset since its payoffs can be replicated by a plan of dynamic transactions of the bonds: buy $q^t \cdots q^{T-1}$ units of the bond in period $t-1$, which costs $q^{t-1} q^t \cdots q^{T-1}$, and use the payout of the bond to buy $q^{t+1} \cdots q^{T-1}$ units of the bond in the next period t. This trading plan has exactly the same yield as a unit of the long term bond in period T, and hence by the no arbitrage principle or the law of one price, the market price of the L-bond in period t, q_L^t, must be the same as the cost of the plan. If they are different, an arbitrarily large amount of profits will be extracted with no cost, i.e., there is free lunch in the bond markets. It can be readily verified that the converse holds, too: the bond markets admit *no free lunch if and only if* $q_L^t = q^t q^{t+1} \cdots q^{T-1}$ holds for $t = 0, 1, ..., T-1$.[4] Thus we shall use the following convention:

Definition 1 *For a sequence of bond prices starting in period t, $q^t, q^{t+1}, ..., q^{T-1}$ and $q_L^t, q_L^{t+1}, , , , q_L^{T-1}$, the no arbitrage condition is satisfied in period t if $q_L^s = q^{s+1} \cdots q^{T-1}$ holds for $s = t, t+1, ..., T-1$.*

In other words, the no arbitrage condition holds in period t if and only if the L-bond and the bond are perfect substitutes under those prices in period t as well as the following periods. Notice in particular that $q_L^{T-1} = q^{T-1}$ holds since the two bonds are indistinguishable perfect substitutes in period $T-1$ by construction. Consequently, under perfect foresight, the market for the L-bond has no additional implication on the allocation of the good except for indeterminacy of asset trade, which arises since the L-bond and the dynamic plan are perfect substitutes in Model II: the two models are equivalent in terms of the allocations they generate.

To ask to what extent the conclusion above depends on perfect foresight, we shall first define the temporary equilibrium which accommodates heterogeneous forecasts in each model. In both models, household h will trade with some forecast prices in mind, which are not necessarily correct ex post, given market prices prevailing in period t, i.e., p^t q^t and q_L^t. It means in particular that households might anticipate different rate of real returns of the bond. For t, $t = 0, 1, ..., T$, write $\hat{p}_{h|t} = \left(\hat{p}_{h|t}^{t+1}, ..., \hat{p}_{h|t}^{T}\right)$, $\hat{q}_{h|t} = \left(\hat{q}_{h|t}^{t+1}, ..., \hat{q}_{h|t}^{T-1}\right)$ and $\hat{q}_{Lh|t} = \left(\hat{q}_{Lh|t}^{t+1}, ..., \hat{q}_{Lh|t}^{T-1}\right)$ for the forecast prices of the good and respectively the prices of the short and the L-bond, where the subscript $h|t$ indicates that it is the forecast of household h made in period t. As mentioned in the introduction, in what follows we do not restrict forecasts a priori; thus in particular we do not impose any specific learning procedure at this point, though requiring markets to be in a temporary equilibrium will dictate that forecasts cannot admit arbitrage opportunities.

[4]It is clear that the bonds have linearly independent payoffs and so any (positive) prices are consistent. Then no free lunch is equivalent to the L-bond being properly priced, which is exactly the condition states. More generally, as is well known (e.g., Lemma 19.E.1 of Mas-Collel et al [1995]), in the rational expectations model, no free lunch is equivalent to the existence of state prices, and asset prices are derived using the state prices. In our context, the (normalized) state prices are discounted values of forecast future bond prices.

In Model I, in every period t, $t = 0, 1, ..., T$, household h optimizes given prices p^t and q^t as well as the outstanding bond holdings b_h^{t-1} (where $b_h^{-1} = 0$ by convention) under the following constraints,

$$\begin{aligned} p^t x^t + q^t b^t &\leq p^t e_h^t + b_h^{t-1}, \\ \hat{p}_{h|t}^{t+1} \hat{x}^{t+1} + \hat{q}_{h|t}^{t+1} \hat{b}^{t+1} &\leq \hat{p}_{h|t}^{t+1} e_h^{t+1} + b^t, \\ &\vdots \\ \hat{p}_{h|t}^{T-1} \hat{x}^{T-1} + \hat{q}_{h|t}^{T-1} \hat{b}^{T-1} &\leq \hat{p}_{h|t}^{T-1} e_h^{T-1} + \hat{b}^{T-2}, \\ \hat{p}_{h|t}^{T} \hat{x}^{T} &\leq \hat{p}_{h|t}^{T} e_h^{T} + \hat{b}^{T-1}, \end{aligned} \tag{1}$$

with variables x^t, b^t, $\hat{x}^{t+1}, ..., \hat{x}^T$, and $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$. Note that the choice variables with hats are also forecasts (where these variables are written without the subscript $h|t$ for convenience) that are yet to be realized at the time household h trades x^t and b^t, and they do not necessarily coincide with the actual trades carried out in future. Moreover, forecasts need not constitute feasible allocations.

The inequalities in (1) exhibits some homogeneity, and one might wonder if the price of the good each period can be normalized to 1 without loss of generality. Indeed if all future prices were perfectly anticipated, it would be natural to set p_t and $\hat{p}_{h|t}^{t+s}$, $s = 1, ..., T - t$ equal to one. In our setting, the price forecasts are heterogeneous and presumably updated each period. Setting the price of the good to one each period, though technically feasible, is somewhat problematic conceptually as it does not sit well with the heterogeneity of forecasts and their updating across households. We therefore choose to proceed without normalization.

In Model II, in every period t, $t = 0, 1, ..., T$, household h optimizes given prices p^t, q^t and q_L^t as well as the outstanding bond holdings b_h^{t-1} and l_h^{t-1} (where $l_h^{-1} = b_h^{-1} = 0$

by convention) under the following constraints,

$$\begin{aligned} p^t x^t + q^t b^t + q_L^t \left(l^t - l_h^{t-1}\right) &\le p^t e_h^t + b_h^{t-1}, \qquad (2)\\ \hat{p}_{h|t}^{t+1} \hat{x}^{t+1} + \hat{q}_{h|t}^{t+1} \hat{b}^{t+1} + \hat{q}_{Lh|t}^{t+1} \left(\hat{l}^{t+1} - l^t\right) &\le \hat{p}_{h|t}^{t+1} e_h^{t+1} + b^t,\\ &\vdots\\ \hat{p}_{h|t}^{T-1} \hat{x}^{T-1} + \hat{q}_{h|t}^{T-1} \hat{b}^{T-1} + \hat{q}_{Lh|t}^{T-1} \left(\hat{l}^{T-1} - \hat{l}^{T-2}\right) &\le \hat{p}_{h|t}^{T-1} e_h^{T-1} + \hat{b}^{T-2},\\ \hat{p}_{h|t}^{T} \hat{x}^{T} &\le \hat{p}_{h|t}^{T} e_h^{T} + \hat{b}^{T-1} + l^{T-1}, \end{aligned}$$

with variables x^t, b^t, l^t, $\hat{x}^{t+1}, ..., \hat{x}^T$, $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$ and $\hat{l}^{t+1}, ..., \hat{l}^{T-1}$. Like in Model I, the choice variables with hats are also forecasts and yet to be realized at the time household h trades x^t, b^t and l^t. Here too, forecasts do not necessarily coincide with the actual trades carried out in future, and they need not constitute feasible allocations. For the same reason as in Model I, we proceed without normalizing forecast prices.

We say that choice variables in period t are *justifiable* if there are forecasts such that the chosen value of the variables is part of an optimal trade, given period t market prices. That is, the current choice is justifiable if they constitute household h's demand for the good and the bonds in period t for some forecasts. In Model I, $\left(x^t, b^t\right)$ is justifiable at prices $\left(p^t, q^t\right)$ (and b_h^{t-1}) in period t for household h if there exist forecasts $\hat{p}_{h|t}$ and $\hat{q}_{h|t}$ about future prices such that household h's utility is maximized at x^t, b^t, $\hat{x}^{t+1}, ..., \hat{x}^T$, and $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$ for some $\hat{x}^{t+1}, ..., \hat{x}^T$, and $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$. In Model II, $\left(x^t, b^t, l^t\right)$ is justifiable at prices $\left(p^t, q^t, q_L^t\right)$ (and b_h^{t-1}, l_h^{t-1}) in period t for household h if there exist forecasts $\hat{p}_{h|t}$, $\hat{q}_{h|t}$, and $\hat{q}_{Lh|t}$ about future prices such that household h's utility is maximized at x^t, b^t, l^t, $\hat{x}^{t+1}, ..., \hat{x}^T$, $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$and $\hat{l}^{t+1}, ..., \hat{l}^{T-1}$ for some $\hat{x}^{t+1}, ..., \hat{x}^T$, $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$ and $\hat{l}^{t+1}, ..., \hat{l}^{T-1}$.

Sequential markets are in temporary equilibrium if the demand meets the supply in every market. In our models, a temporary equilibrium (TE) is defined as follows:

Definition 2 *Model I: Prices p^t and q^t, $t = 0, 1, ..., T$, and an allocation $(x, \mathbf{b})$ constitute a temporary equilibrium if (1) x, and $\mathbf{b}$ are feasible, (2) for every household h, in every period $t = 0, 1, ..., T$, $\left(x_h^t, b_h^t\right)$ is* justifiable *at prices $\left(p^t, q^t\right)$ (and b_h^{t-1}).*
Model II: Prices p^t, q^t and q_L^t, $t = 0, 1, ..., T$, and an allocation $(x, \mathbf{b}, \mathbf{l})$ constitute a temporary equilibrium if (1) x, $\mathbf{b}$, and $\mathbf{l}$ are feasible, (2) for every household h, in every

period $t = 0, 1, ..., T$, $\left(x_h^t, b_h^t, l_h^t\right)$ *is* justifiable *at prices* $\left(p^t, q^t, q_L^t\right)$ *(and* b_h^{t-1}, l_h^{t-1}*). The prices* p^t, q^t *in Model I and* p^t, q^t *and* q_L^t, $t = 0, 1, ..., T$ *in Model II, will be referred to as ex post temporary equilibrium (ex post TE) prices.*

Remark 3 *The justifiability requirement (2) above has two implications. First, household h's forecasts must not allow any arbitrage opportunity to itself at any time, or else the utility maximization problem has no solution since there is no limit on the volume of trade. Thus in particular, each household must have forecasts for which the L-bond is a redundant asset at every trading opportunity, or else it will provide itself with a free lunch at some point. Secondly, even though justifiability is an individualistic exercise, since the ex post prices are common across the households, the households will end up being subject to some common constraints induced by the budget constraint with the ex post prices. The constraints take different forms in the two models, which will be scrutinized in the following subsection.*

2.2 Ex post budget constraint and no arbitrage condition

For a sequence of prices p^t, q^t and a feasible allocation $(x, \mathbf{b})$ to constitute a TE in Model I, a necessary condition is that the constraint (1) must hold in any period t with these prices and allocations. With the monotonicity of the utility function, we therefore have the following system of equations that is required to sustain TE allocations for every household h. We shall refer to this system as the ex post budget constraint for Model I.

$$\begin{aligned}
p^0 x_h^0 + q^0 b_h^0 &= p^0 e_h^0 \\
p^1 x_h^1 + q^1 b_h^1 &= p^1 e_h^0 + b_h^0 \\
&\vdots \\
p^t x_h^t + q^t b_h^t &= p^t e_h^t + b_h^{t-1} \\
&\vdots \\
p^{T-1} x_h^{T-1} + q^{T-1} b_h^{T-1} &= p^{T-1} e_h^{T-1} + b_h^{T-2} \\
p^T x_h^T &= p^T e_h^T + b_h^{T-1}
\end{aligned} \tag{3}$$

Although households do not necessarily anticipate the prices correctly, the well known technique for the rational expectation models can be adopted to show that these dynamic constraints can be reduced to a single equation. Specifically, multiplying the period t equation by $q^0 q^1 \cdots q^{t-1}$ and summing up, we obtain a single ex post budget constraint:

$$\sum_{t=0}^{T} \tilde{p}^t \left(x_h^t - e_h^t\right) = 0 \tag{4}$$

where $\tilde{p}^t$ is the discounted period t price, i.e., $\tilde{p}^t = q^0 q^1 \cdots q^{t-1} p^t$. It can be readily seen that a feasible allocation x of the goods satisfies (4) for every h if and only if there exist a feasible allocation $\mathbf{b}$ of bond such that (3) holds for every h.

We emphasize that constraint (4) holds in any TE *ex post*. Therefore, if perfect foresight is assumed in addition, constraint (4) holds ex ante with forecast prices, i.e., household h plans to choose a utility maximizing x_h given constraint (4), and trades the bonds to finance, i.e., to satisfy (3). If the markets of the good clear with a stream of discount prices, we have an Arrow Debreu equilibrium (AD equilibrium). This is of course the reason why a perfect foresight equilibrium (PFE) is equivalent to an AD equilibrium in this model.

But even without perfect foresight, constraint (4) means that household h's income is the market value of the endowments ex post. Thus in any TE, the wealth of a household is determined by the discounted market prices only, and in this sense, there is no income transfer among the households except for implicit ones induced by market prices.

In Model II, for prices p^t, q^t and q_L^t and a feasible allocation x, b, and l to arise as TE, since the total expenditure must be equal to the total income in every period, the following budget equations, which we shall refer to as the ex post budget constraint for Model II, must hold for x_h, b_h, l_h for every household h:

$$\begin{aligned}
p^0 x_h^0 + q^0 b_h^0 + q_L^0 l_h^0 &= p^0 e_h^0 \\
p^1 x_h^1 + q^1 b_h^1 + q_L^1 \left(l_h^1 - l_h^0\right) &= p^1 e_h^0 + b_h^0 \\
&\vdots \\
p^t x_h^t + q^t b_h^t + q_L^t \left(l_h^t - l_h^{t-1}\right) &= p^t e_h^t + b_h^{t-1} \\
&\vdots \\
p^{T-1} x_h^{T-1} + q^{T-1} b_h^{T-1} + q_L^{T-1} \left(l_h^{T-1} - l_h^{T-2}\right) &= p^{T-1} e_h^{T-1} + b_h^{T-2} \\
p^T x_h^T &= p^T e_h^T + b_h^{T-1} + l_h^{T-1}
\end{aligned} \tag{5}$$

We ask if these equations can be reduced to a single constraint as in Model II. Recall that in a TE, the forecasts must not allow any free lunch, which means that in every period, the forecast prices must respect the no arbitrage condition (Definition 1). We shall report these observations as a Lemma below:[5] recall that $\hat{q}^k_{h|t}$ is household h's forecast period k bond price in period t, where $k > t$. By convention, we set $\hat{q}^T_{h|t} = 1$. We have:

Lemma 4 *Consider a TE. (1) if q^t, q^t_L, $t = 0, 1, 2..., T-1$, arise in the TE, then $q^{T-1}_L = q^{T-1}$ must hold. For period $t = 0, 1, ..., T-2$, and household $h = 1, ..., H$, forecasts satisfy the no arbitrage condition in period t if and only if (a) $q^t_L = q^t \hat{q}^{t+1}_{h|t} \cdots \hat{q}^{T-1}_{h|t}$ for $t = 0, 1, ..., T-1$ and (b) $\hat{q}^{t+j}_{Lh|t} = \hat{q}^{t+j}_{h|t} \hat{q}^{t+j+1}_{h|t} \cdots \hat{q}^{T-1}_{h|t}$, $t = 0, ..., T-2$, $j = 1, ..., T-1-t$. (2) for any household, in every period, the bond and the L-bond are perfect substitutes.*

Proof. (1) $q^{T-1}_L = q^{T-1}$ must hold since the two bonds are perfect substitutes in period $T-1$ in any TE. Conditions (a) and (b) are exactly the no arbitrage condition (Definition 1) applied to the sequence of forecasts.

(2) Since a household must have forecasts which satisfies the no arbitrage condition (Definition 1) in period t, it must find that the L-bond and the synthetic trading strategy of the bond are an identical saving instrument. Hence in particular, forecasts must be formed in such a way that the bond and the L-bond in period t markets are perfect substitutes. ■

Note that condition (a) in Lemma 4 can be written as

$$\frac{q^t_L}{q^t} = \hat{q}^{t+1}_{h|t} \cdots \hat{q}^{T-1}_{h|t} \text{ for } t = 0, 1, ...T-1,$$

which means in any TE, every household must forecast future bond prices in a way which is consistent to the relative L-bond price commonly observed in the markets. That is, the no arbitrage condition requires that households' forecasts are aligned to the ex post relative price. Setting $t = T-1$, we have $\frac{q^{T-1}_L}{q^{T-1}} = 1$, which is a condition we have already pointed out. One might wonder if there are more constraints about ex post prices owing to the no arbitrage condition of the forecasts.

[5] In Model I, there is no effective constraint like the no arbitrage condition. The reason is, as long as the forecast prices are positive, the maximization problem is well defined and has a solution.

Suppose that for the price sequence p^t, q^t and q_L^t under consideration, the no arbitrage condition $q_L^t = q^t q^{t+1} \cdots q^{T-1}$ holds for $t = 0, 1, ..., T-1$. Then multiplying the period t equation in (5) by $q^0 q^1 \cdots q^{t-1}$ and summing them up, we get the same equation (4) as for Model I, and hence Model II does not add anything to Model I as far as the allocation of the good is concerned. But note that although the L-bond must be a perfect substitute of the synthetic strategy at the forecast prices, it need not be so at the ex post TE prices, i.e., the no arbitrage condition $q_L^t = q^t q^{t+1} \cdots q^{T-1}$ of ex post TE prices is not warranted.

Indeed, we argue that $q_L^{T-1} = q^{T-1}$ is the *only* binding implication of the no arbitrage condition on ex post prices under heterogeneous forecasts. Indeed, for $t < T-1$, condition (a) says $\frac{q_L^t}{q^t}$ might be any number, since the forecasts about future prices are free variables. Clearly, for any given stream of forecast bond prices, there is a unique L-bond price given by the no arbitrage condition, satisfying condition (b) in Lemma 4. In this argument, it is important that the forecasts are updated completely freely.

In conclusion, ex post prices of a TE tend not to satisfy the no arbitrage condition. It turns out that the failure of the no arbitrage condition for ex post prices has significant implications on income transfers, which in our view are completely overlooked in rational expectations models, and which we shall explore in the next section.

Remark 5 *Since $q_L^{T-1} = q^{T-1}$ is a common requirement in both models, we shall automatically assume $q_L^{T-1} = q^{T-1}$ whenever ex post TE prices are considered.*

Remark 6 *A TE in Model I can be naturally seen as a TE in Model II where households never trade the L-bond: in a TE in Model I, by definition, in every period, every household has forecasts which justify their choices in period t. By Lemma 4, the bond and the L-bond are perfect substitutes in the view of household h throughout the remaining periods and hence in particular it is optimal to choose no trade of the L-bond in period t, and to trade the bond for desired saving and borrowing. In this sense, justifiability in Model I implies justifiability in Model II. This observation in particular shows that any TE in Model I can be identified with a TE in Model II where the L-bond is never traded.*

3 Price supportability and Wealth Transfer by the Redundant Asset

3.1 Set up

Since TE can accommodate a large class of feasible allocations, the aforementioned redistribution role associated with the L-bond, a redundant asset, will be hard to identify if all TE allocations are considered. An obvious modelling choice is to focus on the efficient allocations; if two distinct efficient allocations are compared, one can be deemed as a result of an (efficient) wealth transfer operated on the other. Thus we are primarily interested in an efficient temporary equilibrium (ETE) in the following analysis.

It will then be useful to work with an economy where the set of efficient allocations has a simple structure so that the purely distributional effects can be observed in a transparent manner. Therefore, in addition to the additively time separable utility function $u_h\left(x_h^0\right)+u_h\left(x_h^1\right)+\cdots+u_h\left(x_h^T\right)$, with $u_h'>0$, $u_h''<0$ and $u_h'(0)=+\infty$, we assume that the total endowment is one in every period, i.e., $\sum_{h=1}^{H} e_h^t=1$ for $t=0,1,...,T$. Consequently, a feasible allocation of goods is efficient intertemporally if and only if assigns a time invariant consumption to every household.[6] An efficient allocation can therefore be parameterized by a tuple of positive numbers $\xi_1,\xi_2,\cdots,\xi_H$ with $\sum_{h=1}^{H}\xi_h=1$, where ξ_h is the time invariant consumption level of household h. We shall identify an efficient allocation with a tuple $(\cdots,\xi_h,\cdots)$ of H positive numbers summing up to one, which might be viewed as a wealth distribution, which will help us to identify the redistribution role.

As a first benchmark, consider an AD equilibrium $(p,x)\in\mathbb{R}^{T+1}\times\left(\mathbb{R}^{T+1}\right)^H$ of this economy, where $p=\left(\cdots,p^t,\cdots\right)$ are positive prices of the goods and $x=(\cdots,x_h,\cdots)$ is the associated allocation of the goods; that is, each household is maximizing utility at x_h given prices p and income $p\cdot e_h$, and x is feasible. As is discussed earlier, if perfect foresight is assumed in Model I or Model II, the sequential budget constraint is in effect given by a single constraint (4), and hence an AD equilibrium allocation constitutes an

[6] Indeed, it can be readily confirmed that a time invariant feasible allocation is efficient. Conversely, for any consumption path $\left(x^0,x^1,...,x^T\right)$, every household prefers the path which provides its average consumption $\frac{1}{T+1}\sum_{t=0}^{T}x^t$ every period, so an efficient allocation must be time invariant.

equilibrium allocation in a TE with perfect foresight, a PFE. As far as the allocational property is concerned, we will use AD and PFE interchangeably. Recall that in a PFE in Model II, the L-bond is redundant, and the set of PFE allocations remains the same if the L-bond markets are all closed (i.e., only the bonds are available), which is effectively the set up of Model I.

The allocation x in an AD equilibrium is of course efficient by the first fundamental theorem of welfare economics. From utility maximization and the additive time separability of the utility function, prices p must be proportional to the gradient vector $u_h'(x_h) = \left(u_h'\left(x_h^0\right), u_h'\left(x_h^1\right), \cdots, u_h'\left(x_h^T\right)\right)$ for every household h. The observation about time invariance of the efficient allocations above implies that $\left(u_h'\left(x_h^0\right), u_h'\left(x_h^1\right), \cdots, u_h'\left(x_h^T\right)\right) = (\cdots, u_h'(\xi_h), \cdots)$, which is also time invariant. Thus the AD equilibrium price system must also be time invariant.

By the homogeneity of equilibrium prices, the AD equilibrium price of the good can be normalized to be $\frac{1}{T+1}$ in each period. With the normalized AD equilibrium prices, the value of the total endowments (one in every period) is one, and the market value of household h's endowments is $\frac{1}{T+1}\sum_{t=0}^{T} e_h^t$, and the market value of the consumption consuming ξ_h in every period is ξ_h. So from the budget constraint, we conclude that the AD equilibrium allocation is unique and household h consumes $\xi_h = \frac{\sum_{t=0}^{T} e_h^t}{T+1}$ in every period.

Example 7 *$H = 4$ and $T = 2$. The endowments are given as in the following table:*

$h \backslash t$	$t=0$	$t=1$	$t=2$
$h=1$	$\frac{1}{4}+2\eta$	$\frac{1}{4}-\eta$	$\frac{1}{4}-\eta$
$h=2$	$\frac{1}{4}-\eta$	$\frac{1}{4}+2\eta$	$\frac{1}{4}-\eta$
$h=3$	$\frac{1}{4}-\eta$	$\frac{1}{4}-\eta$	$\frac{1}{4}+2\eta$
$h=4$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

where $0 \leq \eta < \frac{1}{4}$. That is, household h, $h = 1, 2, 3$, has a high endowment in period $t = h - 1$, and a low endowment in the other periods, whereas household $h = 4$ has a constant endowment $\frac{1}{4}$ in every period. Notice that $e_h^0 + e_h^1 + e_h^2 = \frac{3}{4}$ for all households. Thus in a unique AD equilibrium, every household consumes $\frac{1}{4}$ in every period.

3.2 Price Supportable Allocations

To address the issue of wealth transfer with heterogeneous forecasts in ETE, it will turn out to be convenient to first examine allocations which satisfy the ex post budget constraints with some stream of ex post market prices. Such an allocation will be called *price supportable* (PS allocation). Since the ex post budget must hold in any TE, it will certainly identify the largest set of allocations which can arise in equilibrium. Price supportability of an efficient allocation is especially important since it is necessary for a ETE. If the justifiability requirement is met for all households with some forecasts, then we have a ETE.[7]

We shall first study the structure of price supportable allocations for models I and II in this section, and the issue of justifiability will be taken up in the next section. It will become evident that keeping the justifiability requirement separate from price supportability in this way makes the analysis more tractable and transparent. The concept of price supportability has little to do with efficiency, but we will see that the set of efficient and price supportable allocations has a simple structure in Model I. In contrast, the implication of price supportability will be very different in Model II, indicating that the L-bond might induce an additional channel of wealth transfers which does not exist in Model I, let alone under rational expectations.

3.2.1 Price supportable efficient allocations of Model I

Recall that in Model I, the ex post sequential budget constraints can be reduced to a single budget: an allocation is price supportable if and only if there are (discounted) ex post prices for which (4) is satisfied. Hence an allocation x is feasible and price supportable if and only if there are positive prices $p^0, p^1, ...p^T$ which are the discounted

[7] As we have seen, the ex post budget constaint is implied by the justifiability requirement of TE, and so there is a bit of logical repetition in separating only the budget constraints. Furthermore, since there is no requirement that the forecast values of future variables that are used in justifying an allocation actually clear markets, there could be many different profiles of forecasts that may justify the same TE allocation.

prices of the goods, satisfying the following equations:

$$\begin{array}{rcl} \sum_{t=0}^{T} p^t \left(x_h^t - e_h^t \right) & = & 0 \quad \text{for } h = 1, 2, .., H \\ \left(\sum_{h=1}^{H} x_h^t \right) - 1 & = & 0 \quad \text{for } t = 0, 1, .., T \end{array} \tag{6}$$

The first set of H equations implies that the ex post budget balances for all households: mathematically, the net trade vector is perpendicular to the price vector for each household, and a familiar geometric intuition can be given as in the standard theory of compete markets. The second set of T equations implies feasibility.

When restricted to efficient allocations, i.e., time invariant allocations, the system of equations (6) can be significantly simplified, and hence the set of efficient and price supportable (EPS) allocations under the maintained assumptions has a clean structure. Notice that the last feasibility equation is redundant in (6) since the first H equations imply $\sum_{t=0}^{T} p^t \left(x_h - e_h^t \right) = \left(\sum_{t=0}^{T} p^t \right) x_h - \sum_{t=0}^{T} p^t e_h^t = 0$ for every h, and hence

$$\begin{aligned} \sum_{h=1}^{H} x_h &= \sum_{h=1}^{H} \frac{\sum_{t=0}^{T} p^t e_h^t}{\left(\sum_{t=0}^{T} p^t \right)} \\ &= \frac{1}{\sum_{t=0}^{T} p^t} \sum_{t=0}^{T} p^t \left(\sum_{h=1}^{H} e_h^t \right) \\ &= \frac{1}{\sum_{t=0}^{T} p^t} \sum_{t=0}^{T} p^t \\ &= 1. \end{aligned}$$

Therefore, the system of equations (6) is equivalent to

$$\sum_{t=0}^{T} p^t e_h^t \quad = \quad x_h \quad \text{for } h = 1, 2, .., H \ . \tag{7}$$

Or equivalently, let E be a $(T+1) \times H$ matrix given by the rule

$$E = [e_1, e_2, ..., e_H]$$

where e_h is the column vector of endowments for household h, and then (7) can be written as

$$pE = x, \tag{8}$$

where $p = \left(p^0, p^1, \cdots, p^T \right)$ and $x = (x_1, x_2, \cdots, x_H)$ are row vectors. Notice that if (7) is satisfied, prices are automatically normalized, since $\sum_{h=1}^{H} x_h = 1$, $1 = \sum_{h=1}^{H} \left(\sum_{t=0}^{T} p^t e_h^t \right) = \sum_{t=0}^{T} p^t \left(\sum_{h=1}^{H} e_h^t \right) = \sum_{t=0}^{T} p^t$.

To sum up, a row vector of an efficient time invariant consumption allocation, $x = (\cdots, x_h, \cdots) \in \mathbb{R}^H$, is a EPS allocation if and only if the simultaneous equations (7), or equivalently (8), have a positive solution $p = \left(p^0, p^1, ..., p^T\right) \in \mathbb{R}^T$.

It means that the set of EPS allocations is the image of the set of normalized discounted positive prices for the linear function $p \mapsto pE$. The dimension of this set is the rank of matrix E minus one. More specifically, notice that pE can be expressed as a convex combination of row vectors of initially endowed goods among households in period t:

$$pE = \sum_{t=0}^{T} p^t e^t$$

where $e^t = \left(\cdots, e_h^t, \cdots\right)$ is the row vector of initially endowed goods among households in period t. Recall that a unique AD equilibrium normalized prices are $p^t = \frac{1}{T+1}$ for $t = 0, 1, ..., T$, which certainly satisfies the equation above. Thus we obtain the following result:

Proposition 8 *In Model I, the set of EPS allocation is* $\left\{\sum_{t=0}^{T} p^t e^t : \sum_{t=0}^{T} p^t = 1, p^t > 0, \text{ for } t = 0, 1, ..., T\right\}$. *That is, the set of EPS allocations is the relative interior of the convex hull of* $T+1$ *vectors* $e^0, e^1, ..., e^T$ *in* $\mathbb{R}^H$.

Since the total resource is one for any t, the dimension of the convex set in question is at most $\min(T, H-1)$. If any choice of $\min(T+1, H)$ vectors among $e^0, e^1, e^2, \cdots, e^T$ are affine independent, then the dimension is exactly $\min(T, H-1)$. Since such affine independence is a generic property, we conclude that generically in endowments (with total resource equal to one in every period), the dimension of EPS allocation is $\min(T, H-1)$. In comparison with the set of PFE, which is a singleton set, we see that heterogeneity of forecasts alone might create a great deal of wealth transfers among the households, potentially. Also notice that if $T < H - 1$, which is the case we focus on, there are efficient allocations which are not price supportable. Thus efficiency and price supportability do have some explanatory power on market outcomes even in the absence of perfect foresight.

Example 9 *In the economy of Example 7, in any EPS allocation, household* 4 *consumes* $\frac{1}{4}$. *For the other households,* $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ *is contained in the relative interior of*

$\left(\frac{1}{4}+2\eta, \frac{1}{4}-\eta, \frac{1}{4}-\eta\right)$, $\left(\frac{1}{4}-\eta, \frac{1}{4}+2\eta, \frac{1}{4}-\eta\right)$, *and* $\left(\frac{1}{4}-\eta, \frac{1}{4}-\eta, \frac{1}{4}+2\eta\right)$, *which is a two dimensional set. More explicitly, with the corresponding normalized discounted prices* p^0, p^1, p^2 *with* $p^0+p^1+p^2=1$, *consumption* $\bar{x}_h$ *can be written as* $\bar{x}_h = \sum_{t=0}^{2} p^t e_h^t = \frac{1}{4} + \left(2p^h - \left(1-p^h\right)\right)\eta = \frac{1}{4} + \left(3p^h - 1\right)\eta$, *for* $h = 1,2,3$. *Of course if those prices are equal,* $\bar{x}_h = \frac{1}{4}$, *i.e., it is the PFE allocation.*

3.2.2 Price supportable allocations in Model II

Fix any ex post prices $p^0, p^1 ..., p^T$, $q^0, q^1, ..., q^{T-1}$, and $q_L^0, q_L^1, ..., q_L^{T-1}$ such that $q^{T-1} = q_L^{T-1}$ and any (possibly not constant) stream of consumption x_h for a household h. The discounted prices are written as $\tilde{p}^t := q^0 q^1 \cdots q^{t-1} p^t$, $\tilde{q}^t := q^0 q^1 \cdots q^{t-1} q^t$, and $\tilde{q}_L^t := q^0 q^1 \cdots q^{t-1} q_L^t$. Recall that the no arbitrage condition of these bond prices is equivalent to the discounted price $\tilde{q}_L^t$ being invariant of time t and is equal to $q^0 q^1 \cdots q^{T-1} = \tilde{q}_L^{T-1}$.

Our goal here is to show that if the no arbitrage condition above is *not* satisfied, for any household h, there is a dynamic transaction of bonds such that the consumption stream x_h is budget feasible. Notice that the prices as well as the consumption are completely arbitrary. This in particular shows that any allocation is price supportable in Model II, let alone efficient ones, in stark contrast to Model I.

Assume that there is a period when the no arbitrage condition fails, and so there is a period T^*, $0 \leq T^* < T-1$ where $\tilde{q}_L^{T-1} - \tilde{q}_L^{T^*} \neq 0$. Consider the following dynamic trading plan, which we shall refer to as a *canonical trading plan*, where the L-bond is traded only in period T^*: $l_h^t = 0$ for $t < T^*$, and then keep household h from any additional transaction of the L-bond after period T^*, so that $\left(l_h^t - l_h^{t-1}\right) = 0$ for $t = T^*+1, ..., T^*-1$. Hence by construction, $q_L^t\left(l_h^t - l_h^{t-1}\right) = 0$ holds for $t = 1, ...T-1$, except $t = T^*$ when $l_h^{T^*}$ units of the L-bond is bought. The amount $l_h^{T^*}$ will be specified later. For $t = 0, 1, ..., T^*-1$, set b_h^t iteratively to meet period t budget, $p^t x_h^t + q^t b_h^t = p^t e_t^t + b_h^{t-1}$, where $b_h^{-1} = 0$. Let $b_h^{T^*}$ solve the period T^* budget, $p^{T^*} x_h^{T^*} + q^{T^*} b_h^{T^*} + q_L^{T^*} l_h^{T^*} = p^{T^*} e_h^{T^*} + b_h^{T^*-1}$, where $b_h^{T^*-1}$ and $l_h^{T^*}$ have already been determined as above. For $t = T^*+1, ..., T-1$, set b_h^t iteratively to meet period t budget, $p^t x_h^t + q^t b_h^t = p^t e_t^0 + b_h^{t-1}$.

Then by construction the budget in each period $t = 0, 1, ..., T-1$ is satisfied, i.e., the plan finances the given stream of consumption at the given prices up to period $T-1$, regardless of the choice of $l_h^{T^*}$:

$$\begin{aligned} p^0 x_h^0 + q^0 b_h^0 &= p^0 e_h^0 \qquad (9)\\ p^1 x_h^1 + q^1 b_h^1 &= p^1 e_h^0 + b_h^0 \\ &\vdots \\ p^{T^*} x_h^{T^*} + q^{T^*} b_h^{T^*} + q_L^{T^*} l_h^{T^*} &= p^{T^*} e_h^{T^*} + b_h^{T^*-1} \\ p^{T^*+1} x_h^{T^*+1} + q^{T^*+1} b_h^{T^*+1} &= p^t e_h^t + b_h^{T^*} \\ &\vdots \\ p^{T-1} x_h^{T-1} + q^{T-1} b_h^{T-1} &= p^{T-1} e_h^{T-1} + b_h^{T-2} \end{aligned}$$

Consequently, the constructed trading plan $b_h^0, b_h^1, ..., b_h^{T-1}$ and $l_h^0, l_h^1, ..., l_h^{T-1}$ is budget feasible if the period T budget equation is satisfied in addition, which is

$$p^T x_h^T = p^T e_h^T + b_h^{T-1} + l_h^{T^*},$$

since $l_h^t = l_h^{T^*}$ for $t > T^*$.

Assuming that the period T budget equation holds, multiply the period t budget with $q^0 q^1 \cdots q^{t-1}$ for $t = 1, 2, ..., T$, summing them from period 0 to T, we have

$$\sum_{t=0}^{T} \tilde{p}^t \left(x_h^t - e_h^t \right) = (\tilde{q}_L^{T-1} - \tilde{q}_L^{T^*}) l_h^{T^*}, \qquad (10)$$

and hence the period T budget equation is satisfied as well if and only if (10) holds. Since $\tilde{q}_L^{T-1} - \tilde{q}_L^{T^*} \neq 0$ by assumption, (10) holds for $l_h^{T^*}$ given by the rule:

$$l_h^{T^*} = \frac{\sum_{t=0}^{T} \tilde{p}^t \left(x_h^t - e_h^t \right)}{\tilde{q}_L^{T-1} - \tilde{q}_L^{T^*}} \qquad (11)$$

To sum up, we have established the following result:

Proposition 10 *In Model II, for any ex post prices, any stream of consumption is budget feasible for any household, with a canonical trading plan. In particular, in any efficient allocation, a household's consumption is budget feasible for any ex post prices.*

Compare this result with Proposition 8: the set of EPS allocations is expressed as the convex hull of the endowment vectors, and so in particular, an efficient allocation which assigns to household h an amount $\bar{x}_h$ smaller than the minimum of $e_h^0, e_h^1, ..., e_h^T$ is not a EPS in such an environment. On the other hand, Proposition 10 shows that any feasible allocation can arise in a TE, let alone efficient allocations.

The construction of the trading plan used in the proof of Proposition 10 might become clearer if one recalls the AD budget constraint: notice that equation (11) says that consumption x_h is AD budget feasible if household h is provided with an extra income of $(\tilde{q}_L^{T-1} - \tilde{q}_L^{T^*}) l_h^{T^*}$, which might be negative of course. If allocation $(\cdots, x_h, \cdots)$ is feasible, $\sum_{h=1}^{H} \sum_{t=0}^{T} \tilde{p}^t \left(x_h^t - e_h^t \right) = 0$, and so $\left(\tilde{q}_L^{T^*} - \tilde{q}_L^{T-1} \right) l_h^{T^*}$, $h = 1, 2, ..., H$, constitute income transfers among households. Recall that the L-bond and the bond are perfect substitutes under its forecasts, and household h chooses this particular amount by accident in period T^*. That is, the ones who are "rich by accident", are subsidized by the ones who are "poor by accident", and there is no particular link to the quality of their forecasts. From this viewpoint, the essence of Proposition 10 is that the failure of no arbitrage condition in any one period ex post, no matter how minor it might be, is consistent with any amount of income transfer. Therefore, the L-bond, which is redundant under perfect foresight, can have a significant distributional role.

One might wonder why there is no constraint other than failure of the no arbitrage condition on ex post TE bond prices. Indeed, if the L-bond is not traded as in Model I, to sustain a particular efficient allocation, these prices must be configured in a certain way depending on the allocation. But in Model II, required income transfers effectively occur through trade of the L-bond in only one period when the no arbitrage condition fails ex post. Intuitively, if the no arbitrage condition fails ex post in a period, the law of one price is broken ex post in that period: if the same object has two prices simultaneously, any kind of transfers can be established in competitive markets.

We emphasize that the existence of such income transfer does not depend on the magnitude of failure, i.e., the prescribed income transfer might arise if $q^t q^{t+1} \cdots q^{T-1}$ differs from q_L^t by any slightest amount ex post. It is not owing to households' improper forecasts either: in Model II, in every moment, each household behaves rationally with a perfectly sensible forecast which does not give itself any free lunch. The canonical trading strategy proposed here is not the unique one that would work for establishing the Proposition; we chose it as it will be convenient to work with it for the subsequent justifiability argument.

4 Justifiability and Structure of ETE

This section studies justifiability in Models I and II. We maintain the assumption of history-free updating of forecasts, and hence there is no ad hoc learning procedure.[8]

4.1 Justifiability in Model I

Given a price supportable allocation, we discuss if and when it is justifiable for all households. This is more than we need to do for Model I where our primary interest is on EPS allocations where consumption is time invariant. However, the idea of justifiability itself is individualistic and hence it does not depend on efficiency directly. More importantly, a slightly modified idea will be used in Model II where we will argue any price supportable allocation is justifiable.

We will not formally demonstrate the justifiability of EPS allocations in Model I here as it is not the central finding of this paper. We provide below an informal discussion of how justifiability is obtained in this framework.

To begin with, we shall ask if $\left(x^t, b^t\right)$ is justifiable in period t; that is, $\left(x^t, b^t\right)$ maximizes (continuation) utility with some forecast prices and trading plan subject to the (continuation) dynamic budget (1) in period t, where b_h^{t-1}, the bond purchased in period $t-1$, is taken as a exogenously fixed parameter[9]. Recall that maximization implies forecast prices must not allow any arbitrage opportunity, and hence the dynamic budget constraint (1) can be reduced to the AD budget

$$\tilde{p}^t \left(x_h^t - e_h^t\right) + \sum_{s=t+1}^{T} \hat{p}^s \left(\hat{x}_h^s - e_h^s\right) = \tilde{q}^{t-1} b_h^{t-1} \tag{12}$$

where $\hat{p}^s$ and $\hat{x}^s$ are forecasts of the discounted price of the good and consumption respectively, for period s. In principle we need to describe the forecast bond prices, but they can be readily identified from the discount prices with the no arbitrage condition,

[8] We refer to Chatterji-Kajii [2023] for general techniques and issues therein. In particular, stochastic forecasts would make justifiability easier to satisfy, and hence the set of ETE might be larger than what we shall report. But we restrict attention to point forecasts since it suffices to make our point, besides expositional simplicity.

[9] Since the saving decision in period $t-1$ might be done with very incorrect forecasts, it is possible that the household is practically bankrupt. However, as long as endowments are positive, there always exist (very optimisitic) forecasts with which household can repay the debt in future.

and hence we omit them. In short, the utility maximization can be solved by first finding a consumption bundle which maximizes utility $u_h\left(x^t\right)+\sum_{s=t+1}^{T} u_h\left(\hat{x}^s\right)$ under the AD budget (12), and then determining bond transactions which finances the consumption $\left(x^t, \hat{x}^{t+1}, ..., \hat{x}^T\right)$. Then $\left(x_h^t, b_h^t\right)$ of household h is justified in period t if and only if x_h^t is the quantity demanded of the good in period t under the AD budget (12) for some forecasts about the discounted prices, $\tilde{p}^{t+1}, ..., \tilde{p}^T$, and the associated b_h^t is found to satisfy the dynamic budget.

Suppose that the demand function under the AD budget is responsive to forecasts in the sense that as a function of price forecasts, the demand changes in any direction. This property will be generically true under some mild and plausible conditions on utility functions and endowments, at least if x_h^t is the AD (thus PFE) consumption in period t. We therefore contend that any price supportable feasible allocation close enough to the AD equilibrium allocation is justifiable in any period. Therefore, a fortiori, any EPS allocation near the AD equilibrium allocation is an ETE allocation.

For a specific case of additive log function, i.e., $u_h(z)=\ln(z)$, we can give a more constructive argument. In this case, in period t, $t<T$, household h spends its AD income evenly in the remaining periods including period t, and the quantity demand in period t is found by dividing the intended expenditure by the prevailing market price of the good. From (12) we see that the AD income is $\tilde{p}^t e_h^t+\tilde{q}^{t-1} b_h^{t-1}$ plus the forecast (discounted) income $\hat{m}:=\sum_{s=t+1}^{T} \hat{p}^s e_h^s$, which depends on price forecasts, but not time t market variables. Thus the problem of justifiability is reduced to find a forecast income $\hat{m}$ such that $\tilde{p}^t x_h^t=\frac{\tilde{p}^t e^t+\tilde{q}^{t-1} b^{t-1}+\hat{m}}{(T-t+1)}$.

It is clear that $\hat{m}$ can be made arbitrarily small by forecasting low prices of the good, and also arbitrarily large by forecasting high prices of the good as long as endowments are strictly positive. Then by continuity, the existence of a suitable $\hat{m}$ is warranted if and only if

$$(T-t+1)\,\tilde{p}^t x_h^t>\tilde{p}^t e_h^t+\tilde{q}^{t-1} b^{t-1} \tag{13}$$

The condition above does not take advantage of price supportability: suppose that the consumption plan meets the budget with ex post prices, i.e., (3) holds. Summing up equations from period t to period T with the appropriate weights, we have $\tilde{p}^t\left(x_h^t-e_h^t\right)+\sum_{s=t+1}^{T} \hat{p}^s\left(x_h^s-e_h^s\right)=\tilde{q}^{t-1} b_h^{t-1}$. Note that from the budget equations till $t-1$, we also have $\sum_{s=0}^{t-1} \tilde{p}^s \left(x_h^s - e_h^s\right) + \tilde{q}^{t-1} b_h^{t-1} = 0$; in words, since transactions before t are already completed and they meet the budget equation (3), b_h^{t-1} is equal to the total net saving accumulated before t. This relation can also be readily confirmed since by summing up all equations we have $\tilde{p}^t \left(x_h^t - e_h^t\right) + \sum_{s=t+1}^{T} \hat{p}^s \left(\hat{x}_h^s - e_h^s\right) + \sum_{s=0}^{t-1} \tilde{p}^s \left(x_h^s - e_h^s\right) = 0$. Using this relation, we see that (13) is equivalent to

$$\sum_{s=0}^{t} \tilde{p}^s \left(x_h^s - e_h^s\right) + (T - t)\, \tilde{p}^t x_h^t > 0 \tag{14}$$

Inequality (14) is trivially satisfied if $\sum_{s=0}^{t} \tilde{p}^s \left(x_h^s - e_h^s\right) \geq 0$, i.e., the discounted value of household h's consumption is non-negative, which means household h is a borrower at the end of period t. In general the inequality will hold if household h's lending is not excessively large.

Summing up the discussion above, we have,

Lemma 11 *Assume $u_h(z) = \ln(z)$ and $e_h^t > 0$ for every t. Then consumption x_h^t with outstanding bond holding b_h^{t-1} is justifiable in period t if and only if (13) holds. A PS consumption stream x_h^0, x_h^1, ..., x_h^T is justifiable if and only if inequality (14) is satisfied for every $t = 0, 1, ...T$.*

Note that this result is not restricted to being in a neighborhood of the AD equilibrium consumption.

4.2 Justifiability in Model II

Fix a consumption bundle which is budget feasible under given ex post prices, i.e., (5) holds, and fix a portfolio of the bond and the L-bond associated with the consumption. Thus (5) is satisfied.

We shall first ask if $\left(x^t, b^t, l^t\right)$ is justifiable in period t; Household h maximizes (continuation) utility with some forecast prices and trading plan subject to the (continuation) dynamic budget (2) in period t, given b_h^{t-1} and l_h^{t-1}. Recall from remark 3 that in a TE household h's forecast prices must satisfy the no arbitrage condition, which means that the dynamic continuation budget equation can be reduced to a single AD budget constraint under the forecasts. Therefore, just like in Model I, with such forecasts in mind, household h should be able to reduce the dynamic budget constraint (2) to the AD budget

$$\tilde{p}^t \left(x^t - e_h^t\right) + \sum_{s=t+1}^{T} \hat{p}^s \left(\hat{x}^s - e_h^s\right) = \tilde{q}^{t-1} b_h^{t-1} + \tilde{q}_L^t l_h^{t-1} \tag{15}$$

where $\hat{p}^s$ are forecasts of the discounted price of the good for period s, which arise with forecasts of the bond prices consistent with the no arbitrage condition from period $t+1$ and after.

The utility maximization can be solved by first finding a consumption bundle which maximizes utility $u_h\left(x^t\right) + \sum_{s=t+1}^{T} u_h\left(\hat{x}^s\right)$ under the AD budget (15), and then determining bond transactions which finances the consumption $\left(x^t, \hat{x}^{t+1}, ..., \hat{x}^T\right)$. Then $\left(x_h^t, b_h^t, l_h^t\right)$ of household h is justified in period t if and only if x_h^t is the quantity demanded of the good in period t under the AD budget (15) for some forecasts about the discounted prices, $\tilde{p}^{t+1}, ..., \tilde{p}^T$, and the associated $\left(b_h^t, l_h^t\right)$ is found to satisfy the dynamic budget (2) for period t.

In principle we need to describe the forecast prices of the bond and the L-bond, but since the forecasts must satisfy the no arbitrage condition, the two types of bonds must be perfect substitutes from the viewpoint of household h when solving the utility maximization problem. It means that when consumption x^t is demanded, any combination $\left(b_h^t, l_h^t\right)$ which satisfies the period t budget can be demanded. Thus if we want to induce the household to choose a particular position on the L-bond in period t, the period t budget is a necessary and sufficient condition, and it is not necessary to fix the details of the forecast bond prices. To sum up, all we need to show is x^t is the quantity demanded under the AD budget (15) for some forecast $\hat{p}^s$, $s = t+1, ..., T$. Hence the issue of justifiability is essentially the same as in Model I.

Suppose that the demand function under the AD budget is responsive to forecasts in the sense that as a function of price forecasts, the demand changes in any direction. This property will be generically true under some mild and plausible conditions on utility functions and endowments, at least if x_h^t is the AD (thus PFE) consumption in period t and l_h^s is zero for $t = 0, 1, ..., t-1$, and given that the ex post prices are PFE prices. We therefore contend that any price supportable allocation close enough to the AD equilibrium is a TE in Model II if the underlying trade of the L-bond is kept small enough.

In general, when the position of the L-bond is large, it is hard to tell if the demand function has a desired responsiveness property. We therefore choose to illustrate it for a specific case of additive log function, i.e., $u_h(z) = \ln(z)$. Just as in Model I, from (15) we see that the AD income is $\tilde{p}^t e_h^t + \tilde{q}^{t-1} b_h^{t-1} + \tilde{q}_L^t l_h^{t-1}$ plus the forecast (discounted) income $\hat{m} := \sum_{s=t+1}^{T} \hat{p}^s e_h^s$, which depends on price forecasts, but not time t market variables. Thus the problem of justifiability is reduced to find a forecast income $\hat{m}$ such that $\tilde{p}^t x_h^t = \frac{\tilde{p}^t e^t + \tilde{q}^{t-1} b_h^{t-1} + \tilde{q}_L^t l_h^{t-1} + \hat{m}}{(T-t+1)}$, and a necessary and sufficient condition for the existence is

$$(T-t+1)\,\tilde{p}^t x_h^t > \tilde{p}^t e_h^t + \tilde{q}^{t-1} b^{t-1} + \tilde{q}_L^t l_h^{t-1}, \tag{16}$$

which corresponds to (13) in Model I.

In summary, we have

Lemma 12 *When $u_h(z) = \ln(z)$ and $e_h^t > 0$ for every t, a budget feasible consumption stream x_h^0, x_h^1, ..., x_h^T is justifiable if and only if inequality (16) is satisfied for every $t = 0, 1, ..., T-1$.*

However, since the ex post prices might allow arbitrage between the two types of the bonds, the dynamic budget constraint is not reduced to a single equation. Thus there is no general counterpart to condition (14) in Model II, which makes the justifiability issue more complex in Model II. Nonetheless we attempt to make some connection which is specific to the canonical trading strategy: consider the canonical trading strategy we constructed to support an arbitrarily given consumption in section 3.2.2. Recall that with the canonical strategy, the dynamic budget equations (5) is simplified to (9). For t with $t \leq T^*$, the problem of justifiability is identical to Model I, hence (16) can be reduced to (14) for $t \leq T^*$. For $t > T^*$, multiplying period s equation in (9) with $\tilde{q}^{s-1} := q^0 q^1 \cdots q^{s-1}$, and summing them up from period 0 to $t-1$, we get $\sum_{s=0}^{t-1} \tilde{p}^s (x_h^s - e_h^s) + \tilde{q}^{t-1} b_h^{t-1} + \tilde{q}_L^{T^*} l_h^{T^*} = 0$, and hence $\tilde{q}^{t-1} b_h^{t-1} = -\left(\sum_{s=0}^{t-1} \tilde{p}^s (x_h^s - e_h^s) + \tilde{q}_L^{T^*} l_h^{T^*}\right)$, where $l_h^{T^*}$ was found in (11). Then (16) reads

$$\sum_{s=0}^{t} \tilde{p}^s (x_h^s - e_h^s) + (T-t)\,\tilde{p}^t x_h^t + l_h^{T^*} \left(\tilde{q}_L^{T^*} - \tilde{q}_L^t\right) > 0 \tag{17}$$

The condition above still contains an endogenous variable $l_h^{T^*}$ and so it does not necessarily give a definitive answer for justifiability. But for a constant consumption stream it can be decided since $l_h^{T^*}$ can be explicitly computed. We shall provide here an example of $T = 2$ where household h consumes a constant amount $\bar{x}_h$ in every period, which will be required in ETE.

Fix ex post prices, and assume that the period when the no arbitrage condition fails is $T^* = 0$, so that the condition $\tilde{q}_L^{T^*} - \tilde{q}_L^{T-1} \neq 0$ becomes $q^0 q^1 - q_L^0 \neq 0$. We shall consider a canonical trading strategy, where the L-bond is traded only in $T^* = 0$. To ensure price supportability of the constant consumption $\bar{x}_h$ at the ex post prices, we set b_h^0, b_h^1 in accordance with the iterative procedure[10] specified earlier, and specify l_h^0 $(= l_h^1)$ according to (11); that is,

$$\begin{aligned} l_h^0 &= \frac{\left(p^0 + \tilde{p}^1 + \tilde{p}^2\right) \bar{x}_h - \left(p^0 e_h^0 + \tilde{p}^1 e_h^1 + \tilde{p}^2 e_h^2\right)}{\left(q^0 q^1 - q_L^0\right)}, \\ b_h^0 &= \frac{1}{q^0} \left(p^0 e_h^0 - p^0 \bar{x}_h - q_L^0 l_h^0\right), \\ b_h^1 &= \frac{1}{q^1} \left(p^1 e_h^1 + b_h^0 - p^1 \bar{x}_h\right). \end{aligned}$$

Now we shall examine the justifiability for each period.

Period 2: This holds automatically from the period 2 budget feasibility.

Period 1: Applying condition (16) with $t = 1$ and $T = 2$, we find that the necessary and sufficient condition for justifiability is $2\tilde{p}^1 \bar{x}_h > \tilde{p}^1 e_h^1 + \tilde{q}^0 b_h^0 + \tilde{q}_L^1 l_h^0$. Substituting b_h^0 found above (recall $\tilde{q}^0 = q^0$), it is equivalent to $\left(p^0 + 2\tilde{p}^1\right) \bar{x}_h > p^0 e_h^0 + \tilde{p}^1 e_h^1 + \left(\tilde{q}_L^1 - q_L^0\right) l_h^0$. Substituting l^0 found above (recall $\tilde{q}_L^1 = q^0 q_L^1$ by definition, and $q^1 = q_L^1$ since the two bonds are identical in period 1), it is equivalent to $\left(p^0 + 2\tilde{p}^1\right) \bar{x}_h > p^0 e_h^0 + \tilde{p}^1 e_h^1 + \left(\left(p^0 + \tilde{p}^1 + \tilde{p}^2\right) \bar{x}_h - \left(p^0 e_h^0 + \tilde{p}^1 e_h^1 + \tilde{p}^2 e_h^2\right)\right)$, which is reduced to:

$$\tilde{p}^1 \bar{x}_h > \tilde{p}^2 \left(\bar{x}_h - e_h^2\right). \tag{18}$$

Period 0: Applying condition (17) with $T = 2$ and $t = T^* = 0$, we find that the necessary and sufficient condition for justifiability is $\tilde{p}^0 \left(\bar{x}_h - e_h^0\right) + 2\tilde{p}^0 \bar{x}_h + l_h^0 \left(\tilde{q}_L^0 - \tilde{q}_L^0\right) > 0$, which is readily simplified as

$$3\bar{x}_h > e_h^0 \tag{19}$$

In summary, constant consumption $\bar{x}_h$ is justifiable in every period if and only if conditions (18) and (19) are satisfied.

[10]Recall we had set b_h^t iteratively to meet period t budget, $p^t x_h^t + q^t b_h^t = p^t e_t^t + b_h^{t-1}$, where $b_h^{-1} = 0$.

Notice that condition (18) holds if $\tilde{p}^1 = \tilde{p}^2$, which is true at PFE where the discounted prices are constant. Therefore, if the ex post prices are close enough to the PFE prices, then any $\bar{x}_h > 0$ can be justified in period 1. Recall that at PFE, household h consumes $\bar{x}_h = \frac{1}{3}\left(e_h^0 + e_h^1 + e_h^2\right)$, and so (19) is satisfied.

Remark 13 *Assuming the common log utility function, in conditions (16) and (17), forecasts are implicitly determined. Although we take forecasts as purely individualistic, it is curious if they might be aligned across the household to justify an efficient allocation: e.g., every household correctly forecasts period 2 price in period 0, so that the wealth transfers induced in the ETE are driven only by forecasting errors of period 1 prices. Indeed, it is possible to construct such forecasts in the log example, which is demonstrated by direct computation in Appendix.*

4.3 Structure of ETE

As is discussed in the previous section, we contend that a EPS allocation is, at least in the vicinity of a PFE, tends to be justifiable for all household in Model I, and then it is a ETE allocation. As is pointed out earlier (Remark 6), a TE in Model I can be seen as a special case of a TE in Model II where households choose not to trade the L-bond in any period. Also under its forecasts the L-bond is a perfect substitute of the bond, and hence any position on the L-bond is consistent with utility maximization. The proximity to a Model I ETE allocation assures that it can be generated with a low volume of trade in the L-bond. Therefore, a EPS allocation in Model II which is close enough to a ETE allocation in Model I is also justifiable, and hence it is a ETE allocation.

Recall that the EPS allocations in Model I constitute a T dimensional set when H is large (Proposition 8) whereas the set of EPS allocations in Model II is $H-1$ dimensional (Proposition 10). The argument above shows that at least around a PFE, EPS allocations are also ETE allocations. Then, the set of ETE in Model I is T dimensional, whereas the set of ETE in Model II is $H-1$ dimensional set, revealing the distributional effect of a redundant asset.

We believe that a formal analysis can be carried out with differential topology technique to verify the assertions above, but we also expect that it will be technically involved.

We therefore choose to focus on the log utility case we explored in the previous subsections where we can clearly see the structure of ETE explicitly with more elementary technique. So assume $u_h(z) = \ln(z)$ from now on and fix an efficient allocation and write $\bar{x}_h$ for the time invariant consumption of household h. Recall that the allocation is an ETE allocation if it is an EPS allocation and justifiable for every h.

In Model I, an EPS allocation can be expressed as a convex combination of (normalized) discounted prices, $p^0, p^1, ..., p^T$ (Proposition 8). When condition (14) is applied to an EPS consumption where household h's consumption is a constant stream of $\bar{x}_h = \sum_{t=0}^{T} p^t e_h^t$ it reads:

$$\left(\sum_{s=0}^{t} p^s + (T-t)\, p^t\right) \sum_{t=0}^{T} p^t e_h^t - \sum_{s=0}^{t} p^s e_h^s > 0 \tag{20}$$

where the first term is the value of the whole consumption stream assuming that the discounted prices stays the same as $\tilde{p}^t$ after period t. Therefore, we have

Proposition 14 *In Model I, an efficient allocation $(\cdots, \bar{x}_h, \cdots)$ is an ETE allocation if and only if there are $\sum_{t=0}^{T} p^t = 1$, $p^t > 0$, for $t = 0, 1, ..., T$, such that for every h, $\bar{x}_h = \sum_{t=0}^{T} p^t e_h^t$ and (20) are satisfied for every $t = 0, 1, ...T$.*

Example 15 *In the economy of Example 7, as was seen in Example 9, household h consumes $\bar{x}_h = \sum_{t=0}^{2} p^t e_h^t = \frac{1}{4} + \left(3p^h - 1\right)\eta$, $h = 1, 2, 3$. Household 4 consume $\bar{x}_4 = \frac{1}{4}$. For household 1, conditions (20) for $t = 0$ and 1 are*

$$3p^0\left(\frac{1}{4} + \left(3p^0 - 1\right)\eta\right) - p^0\left(\frac{1}{4} + 2\eta\right) > 0$$
$$\left(p^0 + 2p^1\right)\left(\frac{1}{4} + \left(3p^0 - 1\right)\eta\right) - \left(p^0\left(\frac{1}{4} + 2\eta\right) + p^1\left(\frac{1}{4} - \eta\right)\right) > 0$$

and the set of EPS for which household 1's consumption is justifiable can be found by solving the simultaneous (quadratic) inequalities above. Notice that when η is small enough, both inequalities are satisfied for any prices. A similar computations show that the other households' consumption streams are justifiable if η is small enough. Thus when η is small enough, the set of ETE is identical to the set of ETE in this economy.

In Model II, any feasible allocation arises as a PS allocation by Proposition 10, hence an efficient allocation $(\cdots, \bar{x}_h, \cdots)$ is a EPS allocation. The justifiability condition for the log case follows from Lemma 12. Therefore we have:

Proposition 16 *In Model II, an efficient allocation* $(\cdots, \bar{x}_h, \cdots)$ *is an ETE allocation with given ex post prices if and only if inequality (16) is satisfied for every* $t = 0, ..., T-1$.

Remark 17 *Recall that in the two period case, if the given ex post prices are close enough to the PFE prices, then any efficient allocation where (19) holds for every household* h *is justifiable, and hence it is an ETE allocation. Thus the set of ETE allocations is a* $H-1$ *dimensional set containing the PFE allocation.*

Example 18 *In the economy of Example 7 with the common log utility function, using (19), it can be shown that the set of ETE contains all positive* $\bar{x}_1, \bar{x}_2, \bar{x}_3$ *and* $\bar{x}_4$ *with* $\sum_h \bar{x}_h = 1$ *such that* $3\bar{x}_1 > \frac{1}{4} + 2\eta$, $3\bar{x}_2 > \frac{1}{4} - \eta$, $3\bar{x}_3 > \frac{1}{4} - \eta$, *and* $3\bar{x}_4 > \frac{1}{4}$. *In particular, household 4's consumption may be different from* $\frac{1}{4}$ *by a large amount whereas it must consume* $\bar{x}_3 = \frac{1}{4}$ *in any of ETE in Model I (Example 15). Hence the predictability retained in Model I is lost in Model II. In the Appendix we compute the forecasts that justify the ETE identified here.*

Since an ETE in Model I can be seen as an ETE of Model II (Remark 6), there are at least two types of ETE in Model II: the first involves no trading of the L-bond, and the ex post prices other than the L-bond prices must be configured so that the allocation is budget feasible for all households, and the ex post prices of the L-bond is induced from the bond prices by the no arbitrage condition. The second rests on the existence of an ex post arbitrage opportunity between the two kinds of bonds in some period, and the configuration of the ex post prices matters only to the extent that the discounted (continuation) income must be kept positive for the justifiability argument. This observation has the following implication about the predictability: an analyst who uses Model I would find some relation between ex post prices and the consumption allocation. On the other hand, an analyst who uses Model II cannot relate ex post prices to the underlying allocation at all: the analyst who observed prices which are very close to PFE prices cannot infer that the allocation is close to that in PFE.

Finally, we comment on the accuracy of forecasts and welfare. Since the construction of a EPS and its justifiability issue can be established separately, it can be readily inferred that those who are benefitted from the implicit transfers do not necessarily have forecasts which are accurate ex post; the former is determined by the construction of an EPS, whereas the latter is related to the issue of justifiability. This observation is indeed valid, even in the common log utility model where the quality of a forecast might appear to be the only source for an advantageous trade.

Example 19 *Consider Model I for the economy of Example 7 with the common log utility function. Household 4 is induced to consume* $\frac{1}{4}$ *in every period in any ETE, which can be sustained by time invariant price forecasts, just as in the unique PFE (see Remark 13). In this sense, household 4' forecasts are always on the right track. But it does not imply household 4's forecasts are necessarily correct ex post. Indeed, as is seen in the Appendix, there are ETE where discounted ex post prices* $\tilde{p}^1$ *and* $\tilde{p}^2$ *are different from* p^0*, and then household 4's forecasts cannot be correct ex post.*

5 On Speculative Trade

The so called no-trade theorem asks if a purely speculative trade based on private information is possible in a rational expectation equilibrium. Since it is hard to distinguish speculative motive from other genuine motives based on perceived gains from trade, work in this literature typically start with an ex ante efficient allocation and ask if there is an equilibrium where trade takes place. If there is one, it can be regarded as a result of pure speculation. A general conclusion in this literature is that there tends not to be any purely speculative trade, which is referred to as the no (speculative) trade theorem.[11]

We can carry out the following exercise with a similar motivation in spirit in our framework. Suppose that there are many, identical households. The initial allocation is efficient by construction, and a unique perfect foresight equilibrium occurs with no trade. The question is whether or not there is a non-trivial ETE where households trade in this

[11] The literature was initiated by Milgrom and Stokey [1982], and a clean, efficiency based formulation was given in Morris [1995]. See Kajii and Ui [2009] for its relation with the fundamental theorems of welfare economics.

economy. If there is, one might interpret that the trade is driven by heterogeneous (and incorrect) forecasts, i.e., lack of rational expectation.

Of course, there are many inefficient TE, i.e., households might choose trades that distort intertemporal efficiency. One might think that there might also be trades based on heterogeneity of forecasts that preserve efficiency: households whose price forecasts disagree seem to find (incorrectly) that they have mutually beneficial trading opportunities. Even if the initial endowments are efficient, a household which thinks the price will be very low is willing to sell the good today to another household which thinks the price will be very high. This process might induce effective income transfers among households from the ones with good forecasts to the ones with bad forecasts, without distorting efficiency. But in general this is not straightforward even in Model I since good forecasts are not necessarily recipient of transfers, as is seen in Example 19.

It turns out that there is no ETE other than PFE for Model I. To see this recall the characterization result Proposition 8: The set of EPS allocation is $\left\{\sum_{t=0}^{T} p^t \mathbf{e}^t : \sum_{t=0}^{T} p^t = 1, p^t > 0, \text{ for } t = 0, 1, ..., T\right\}$, where $\mathbf{e}^t = \left(\cdots, e^t, \cdots\right)$. Since $\mathbf{e}^t$ is equal to a constant vector $\bar{\mathbf{e}}$ for $t = 0, 1, ..., T$, and since the total resource is time invariant, it can be readily seen that $\sum_{t=0}^{T} p^t \mathbf{e}^t = \left(\sum_{t=0}^{T} p^t\right) \bar{\mathbf{e}} = \bar{\mathbf{e}}$ for any element of this set; that is, it is a singleton set consisting of the initial, no trade allocation, and it is exactly the set of ETE allocations consisting of the PFE allocation. In conclusion, lack of rational expectation does not necessarily invoke trade that leads to another efficient allocation.

The conclusion, however, is sensitive to the structure of the market: to be more specific, let $T = 2$ and consider Model II. There are H identical households, and each gets $\frac{1}{H}$ in every period. Assume that the common utility function is $u_h(x) = \ln(x)$. The analysis of Model II in the previous section has shown that any efficient allocation where each household's consumption is greater than $\frac{1}{3H}$ can arise as an ETE, as reported in Proposition 16 and Remark 17. That is, lack of rational expectation might invoke income transfers among the households with the L-bond which is a redundant asset under perfect foresight.

The analysis of this section so far suggests first that heterogeneity of forecasts alone might not constitute motives for market trade without distortion of efficiency. Secondly, it suggests that heterogeneity of forecasts combined with ex post arbitrage opportunity might generate market trades which preserve efficiency. However, households are engaged in trade not because they see arbitrage opportunities for easy profits. Recall that by the construction of TE each household holds forecasts which do not permit itself any arbitrage opportunity. Each household thinks, at any time, there is no such thing as free lunch in the markets, but nevertheless their trade creates income transfers from the winners to the losers who are determined "by accident". We conclude this section with an example to illustrate this point.

Example 20 *In the economy of Example 7, set* $\eta = 0$, *and assume the log utility function for all households. In Model I, the set of ETE coincides with PFE, but in Model II any efficient allocation which gives more than* $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ *to every household can arise as an ETE. When* $q_L^0 < 1$, *i.e., where the long term bond is inexpensive according to the ex post prices, those who enjoy consumption higher than* $\frac{1}{4}$ *every period, i.e., the winners, buy the long term bond in period 0 as in (11), whereas those who sell the long term bond consume less than* $\frac{1}{4}$ *to be the losers. Details about forecasts that justify these allocations can be found in the Appendix.*

6 Final Remarks

We first comment on the generality of our findings. For Model II, the key observation about the distributional role of redundant assets is very general, as it does not rely on anything but budget equations. It therefore holds in a model with more goods, a longer period, or with general utility functions. The justifiability problem can be more complex, because it delicately relies on the property of the demand functions. When the demand functions can be explicitly calculated, we will be able to derive sufficient conditions for justifiability analogous to the ones we found for the log utility case, but the conditions will not be clean and attractive. For general time-separable utility functions, we expect that generically around a PFE, any PS allocation can be justified for all household, and we shall leave it for future research.

The message from the speculative trade seems very general. The works in this literature typically asks possibility of speculative trade under rational expectations, and thus in particular, the way individual forecasts might be related to private information is common knowledge. One can view the forecasts in our model being related to some unmodeled private information, and from this perspective our finding can be understood as possibility of speculation without common knowledge of forecasts: trade occurs not only by lack of common knowledge about forecasts but also the structure of (redundant) assets.

The analysis can be extended to an infinite-horizon setup to show that the main message is general in this respect as well. In Model I, with a suitable requirement, the dynamic budget constraint can still be reduced to a one-shot AD budget constraint, and hence the idea of price supportability remains the same. Although we no longer obtain a clear-cut result about the dimension of PS allocations, it can be verified that not all efficient allocations around the AD equilibrium are price supportable. In Model II, by requiring that the forecast budget is reduced to a one-shot AD budget, which holds automatically from no arbitrage condition in the finite horizon setup, one can demonstrate that any kind of income transfers can be generated if the ex-post prices do not satisfy the no-arbitrage condition, utilizing suitably modified canonical trading strategies.

A natural and important extension is to accommodate uncertainty in the model. It is especially important in the context of our interest in studying the role of redundant assets. In particular, derivative securities, which constitute a rich class of financial assets, can only be studied in models that explicitly incorporate uncertainty.[12] It would be interesting to investigate whether our observation regarding the indeterminacy of wealth transfers under efficiency can be made by studying a model with uncertainty which accommodates a wider variety of financial assets. We expect that under our formulation, the presence of assets which are equivalent under rational expectations would provide some channels of income transfers, and thus would expand the set of attainable (ex ante)

[12] The classical Black-Scholes option pricing formula finds the theoretical price of an option contract as a derivative asset assuming the relevant price processes are rationally expected. It has also been argued (first by Ross [1976] and subsequently elaborated by Polemarchakis and Ku [1990], Krasa and Werner [1991], Kajii [1997], among others) that under rational expectations, the presence of options might complete the markets, and consequently a rational expectations equilibrium with options is efficient and determinate.

efficient allocations beyond the set of rational expectations allocations.

We make some observations on the quality of forecasts in our framework.[13] For instance, in both Model I and Model II, an ex post correct forecast might not be a good forecasts from the view point of individual utility level. In particular, in Model II with log utilities, households forecasts about period 2 can be set to be ex post correct. In fact, one can write an example with longer periods, where households' forecasts are ex post correct from period 2 and after (so they learn to be perfect forecasters, but welfare shifts takes place before they become perfect). Thus the forecasts are correct for most periods for all households, and in this sense there is little difference in the quality of forecasts, but any kind of income transfer might occur in a ETE, and winners and losers will emerge by accident. While this is an artifact of the log form of utilities, it is nonetheless interesting that efficiency preserving transfers can arise due to short term forecasting problems which resolve themselves over the longer term. It suggests that the phenomenon we have identified need not be incompatible with some sort of learning behavior that leads to improvements in forecasts over time. This (in)dependence of the quality of forecasts to the quality of life appears to be very general, but more research is needed to articulate this phenomenon.

We are agnostic about forecasts and provide a methodology that will allow learning and forecasting rules to be incorporated in the justifiability part without impinging on PS allocations. One can readily accommodate stochastic forecasts, which enlarge the set of justifiable allocations while the structure of PS allocations is unaltered, and hence our main points about the wealth transfers remain unchanged. One can choose to require specific learning and forecasting rules as part of TE, which changes the set of justifiable allocations, but it does not alter the set of PS allocations.

Finally, we point out that the model we use can be seen as a multiperiod extension of an earlier model (Chatterji and Kajii [2023]) where we have studied the structure of intertemporally efficient allocations which can arise as a sequence of temporary equilibria (ETE) in a two period set up with multiple non-storable goods with one nominal asset

[13] We have observed in earlier work (Chatterji *et al* [2018]) that households whose forecasts turn out more accurate need not be the beneficiaries in the ETE induced by heterogeneous forecasts. The same remains true here.

(bond), and shown that there is a one dimensional set of ETE allocations around each perfect foresight equilibrium (PFE) allocation, generically in endowments. In this one good set up, Proposition 14 generalizes the indeterminacy finding of that paper to a multi-period set up and shows the dimension of the set of ETE is generically one less than the number of periods. As we wrote above, we expect this finding to hold more generally with multiple goods and general utilities. But for such a generalization, as is pointed out our the earlier model, there are additional complications about justifiability even in a two period model when utilities are non time-separable.

References

[1] Adam K., Marcet A., and J.P. Nicolini, 2016, Stock market volatility and learning, *Journal of Finance*, 71, 33-82.

[2] Bossaerts P., *The paradox of asset pricing*, Princeton University Press 2002.

[3] Chatterji S. (1995), "Temporary equilibrium dynamics with bayesian learning ", *Journal of Economic Theory*, 67, 2, 590-598.

[4] Chatterji S., Kajii, A. and H. Zeng (2018), "Intertemporal efficiency does not imply a common price forecast", KIER discussion papers No.999, August 2018

[5] Chatterji S. and Kajii, A. (2023), "Decentralizability of efficient allocations with heterogeneous forecasts" *Journal of Economic Theory* vol 207, article number 105592. doi.org/10.1016/j.jet.2022.105592

[6] Grandmont, J-M., *Temporary equilibrium selected readings*, Academic Press, 1988.

[7] Kajii, A. (1997), "On the Role of Options in Sunspot Equilibria", *Econometrica* 65, 977-986.

[8] Kajii, A. and Ui, T. (2009) "Interim Efficient Allocations under Uncertainty", *Journal of Economic Theory* 144 no. 1, 337-353. doi.org/10.1016/j.jet.2008.05.006

[9] Krasa, S. and J. Werner (1991), "Equilibria with Options: Existence and Indeterminacy", *Journal of Economic Theory*, 54, 305-320.

[10] Milgrom, P., Stokey, N., (1982) "Information, trade and common knowledge". *J. Econ. Theory* 59, 17–27.

[11] Morris, S., (1995), Trade with heterogeneous prior beliefs and asymmetric information. *Econometrica* 62, 1327–1348.

[12] Polemarchakis, H.M. and Ku, B.-I., (1990): "Options and Equilibrium," *Journal of Mathematical Economics*, 19, 107-112.

[13] Radner, R., (1972), Existence of Equilibrium of plans, prices, and price expectations in a sequence of markets, *Econometrica*, 40, 289-303.

[14] Radner, R., (1982), Equilibrium under uncertainty, in *Handbook of Mathematical Economics*, edited by K.J. Arrow and M.D. Intrilligator, North-Holland publishing company, vol. 2, chap. 20, 923–1006.

[15] Ross, S. (1976), "Options and Efficiency" *Quarterly Journal of Economics*, 90, 75-89.

[16] Woodford, M., 2013, Macroeconomic analysis without the rational expectations hypothesis, *Annual Review of Economics*, 5, 303-346.

[17] Woodford M., and Y. Xie, 2022, Fiscal and monetary stabilization policy at the zero lower bound: consequences of limited foresight, *Journal of Monetary Economics*, 125, 18-35.

Appendix

1. We first provide the details of the forecasts used in the example in subsection 4.2, mentioned in Remark 13 where we assumed a common log utility function:

Fix a household $h \in \{1, 2, 3, 4\}$. We construct a canonical trading strategy to justify the constant consumption $\bar{x}_h$ where household h trades the L-bond only in period 0. Note that from the ex post budget constraint we have:

$$
\begin{aligned}
p^0 \bar{x}_h + q^0 b_h^0 + q_L^0 l_h^0 &= p^0 e_h^0 \\
p^1 \bar{x}_h + q^1 b_h^1 &= p^1 e_h^1 + b_h^0 \\
p^2 \bar{x}_h &= p^2 e_h^2 + b_h^1 + l_h^0
\end{aligned}
$$

First, we explicitly find the forecast $\hat{p}_h^2$, which satisfies

$$
p^1 \bar{x}_h = \frac{p^1 e_h^1 + q^1 \hat{p}_h^2 e_h^2 + b_h^0 + q^1 l_h^0}{2},
$$

so that $\bar{x}_h$ is justified in period 1 given p^1 and q^1. Multiply both sides by $2q^0$ and substituting $q^1 l_h^0 = q^1 p^2 \bar{x}_h - q^1 \left(p^2 e_h^2 + b_h^1\right)$ (period 2 budget) and $q^1 b_h^1 = \left(p^1 e_h^1 + b_h^0\right) - p^1 \bar{x}_h$ (period 1 budget), we have

$$
\begin{aligned}
\hat{p}_h^2 &= \frac{2q^0 p^1 \bar{x}_h - \left(q^0 p^1 e_h^1 + q^0 \left(b_h^0 + q^1 l_h^0\right)\right)}{q^0 q^1 e_h^2} \\
&= \frac{2q^0 p^1 \bar{x}_h - \left(q^0 p^1 e_h^1 + \left(q^0 p^1 \left(\bar{x}_h - e_h^1\right) + q^0 q^1 p^2 \left(\bar{x}_h - e_h^2\right)\right)\right)}{q^0 q^1 e_h^2} \\
&= \frac{q^0 p^1 \bar{x}_h - q^0 q^1 p^2 \left(\bar{x}_h - e_h^2\right)}{q^0 q^1 e_h^2} \qquad (21)
\end{aligned}
$$

To construct an example of an ETE where the forecast about period 2 price are all correct ex post, we will normalize $q^0 = p^1 = p^2 = 1$ and $\hat{p}_h^2 = 1$ to simplify calculation from now on throughout the appendix. Then we find $q^1 e_h^2 = \bar{x}_h - q^1 \left(\bar{x}_h - e_h^2\right)$ from (21), that is, (21) holds if $q^1 = 1$. So set $q^1 = q_L^1 = 1$, and then the consumption in period 1 is justifiable. Of course, unless we are at PFE, the forecast $\hat{p}_h^1$ about period 1 price held in period 0 might not be correct ex post.

2. Next, for illustration, we explicitly compute forecasts about period 1 price which justify the ETE found in Example 18 where we assumed a common log utility function:

h	$\hat{p}_h^1$
1	$\frac{1}{1-4\eta}\left(-q_L^0 + 12\bar{x}_1 - 8\eta + 4q_L^0\eta - 1\right)$
2	$\frac{1}{1+8\eta}\left(-q_L^0 + 12\bar{x}_2 + 4\eta + 4q_L^0\eta - 1\right)$
3	$\frac{1}{1-4\eta}\left(-q_L^0 + 12\bar{x}_3 + 4\eta - 8q_L^0\eta - 1\right)$
4	$2 - q_L^0$

(22)

With the price normalization, a perfectly accurate period 1 forecast is $\hat{p}_h^1 = 1$. Household 4, who must consume $\frac{1}{4}$ in every period, has a correct forecast 1 only when $q_L^0 = 1$, i.e., there is no ex post arbitrage opportunity. Observe that forecast $\hat{p}_h^1$ is increasing in $\bar{x}_h$ for $h = 1, 2, 3$, and thus a household who is benefited from the implicit income transfer tends to forecast a higher price, not an accurate forecast 1.

3. Finally, we provide the computation of forecasts for Example 20 where we assumed a common log utility function for each household h. With the price normalization, we can compute period 1 forecasts explicitly as in (22) by setting $\hat{p}_h^1 = -q_L^0 + 12\bar{x}_h - 1$ for household h. Since $\hat{p}_h^1$ is increasing in $\bar{x}_h$, for these two households,

1. if $q_L^0 < 1$ (thus $2 - q_L^0 > 1$), a household with $\bar{x}_h > \frac{1}{4}$ must have $\hat{p}_h^1 > 2 - q_L^0 > 1$, and a household with $\bar{x}_h < \frac{1}{4}$ must have $\hat{p}_h^1 < 2 - q_L^0$. Thus a household who is worse off than in the PFE may be ex post correct, but a household who is better off than in the PFE never has a correct forecast.

2. if $q_L^0 > 1$ (thus $2 - q_L^0 < 1$), a household with $\bar{x}_h > \frac{1}{4}$ must have $\hat{p}_h^1 > 2 - q_L^0$, and a household with $\bar{x}_h < \frac{1}{4}$ must have $\hat{p}_h^1 < 2 - q_L^0 < 1$. Thus a household who is better off than in the PFE may also be ex post correct, but a household who is worse off than in the PFE never has a correct forecast.