博士論文

論文題目 Studies on algebraic varieties admitting a polarized endomorphism and the minimal model program in mixed characteristic

(偏極自己準同型を持つ代数多様体と混標数の極小モデルプログラムの研究)

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Chapter 1

Introduction

In this thesis, we study the structure of varieties admitting a polarized endomorphism and the minimal model program in mixed characteristic. We will freely use the notation and terminology in [63].

A *polarized endomorphism* is a non-invertible endomorphism of a projective variety preserving some polarization. In Chapter 2, which is based on [98], [99],[101], we study the structure of varieties over $\mathbb{C}$ admitting a non-invertible polarized endomorphism. Such a study has its origin in the following conjecture about a characterization of projective spaces.

Conjecture 1.0.1 (cf. [29, Question 4.3], [78, Conjecture 6.5]). *Let X be a smooth Fano variety over $\mathbb{C}$ of Picard number one. Then X is a projective space if and only if X has a non-invertible endomorphism.*

Since every endomorphism of a Fano variety of Picard rank one is polarized, we can regard Conjecture 1.0.1 as a statement on the structure of varieties admitting a non-invertible polarized endomorphism. It was studied in [5], [51], [52], [85] and solved in dimension three. Moreover, Conjecture 1.0.1 is generalized to the case of toric varieties, which was solved in dimension two [82] and three [80].

Conjecture 1.0.2 (cf. [29, Question 4.4], [78, Question 6.6]). *Let X be a smooth and rationally connected variety over $\mathbb{C}$. Then X is toric if and only if X has a non-invertible polarized endomorphism.*

The *Minimal model program* (MMP, for short), which is a higher-dimensional analog of the classification method of surfaces, is a important tool to find a "simplest" variety in each birational equivalence class. In characteristic zero, this program holds for threefolds and varieties of general type (cf. [11]). In order to study the structure of (not necessarily rationally connected) varieties admitting an int-amplified endomorphism, Meng and Zhang established minimal model program that is equivariant with respect to non-invertible polarized endomorphism (equivariant MMP, for short). Meng [76] generalized this program for varieties admitting an *int-amplified* endomorphism, which is a surjective endomorphism $f\colon X \longrightarrow X$ of a projective variety X such that $f^*L \otimes L^{-1}$ is ample for some ample line bundle L

on X. We note that every non-invertible polarized endomorphism is int-amplified. Meng and Zhang obtained the following result using the equivariant MMP.

Theorem 1.0.3 ([76], cf. [77]). *Let X be a smooth projective variety over $\mathbb{C}$ admitting an int-amplified endomorphism. There exists an étale cover $\mu\colon \widetilde{X} \longrightarrow X$ such that the albanese morphism $\mathrm{alb}_{\widetilde{X}}$ is a fiber space whose every fiber is rationally connected.*

Following the above results, we propose the next conjecture, which is a generalization of Conjectures 1.0.1 and 1.0.2 and Theorem 1.0.3.

Conjecture 1.0.4. *Let X be a smooth projective variety over $\mathbb{C}$ admitting an int-amplified endomorphism. After replacing X with its étale cover, every fiber of the albanese morphism of X is toric.*

In order to state the main theorem in this chapter, we recall the notion of Fano type. Given a projective morphism $Z \longrightarrow B$ of normal varieties over $\mathbb{C}$, we say that Z is of *Fano type* over B if there exists an effective $\mathbb{Q}$-Weil divisor D on Z such that (Z, D) is klt and $-(K_Z + D)$ is ample over B. When B is a point, we simply say that Z is of Fano type. We note that if Z is of Fano type over B, then a general fiber is of Fano type. For example, toric varieties are of Fano type and projective bundles over B are of Fano type over B. Zhang [102] and Hacon-Mckernan [41] proved that varieties of Fano type are rationally connected. On the other hand, smooth and rationally connected varieties are not necessarily of Fano type in general. Therefore, the following theorem strengthens Theorem 1.0.3 and gives a partial answer to Conjecture 1.0.4.

Theorem A (Theorem 2.3.31, Remark 2.3.32). *Let X be a smooth projective variety over $\mathbb{C}$ admitting an int-amplified endomorphism. Then there exists an étale finite cover $\mu\colon \widetilde{X} \longrightarrow X$ such that the albanese morphism $\mathrm{alb}_{\widetilde{X}}\colon \widetilde{X} \longrightarrow A$ is a fiber space and $\widetilde{X}$ is of Fano type over A. Furthermore, every fiber of $\mathrm{alb}_{\widetilde{X}}$ is smooth and of Fano type. In particular, if X is rationally connected, then X is of Fano type.*

Theorem A has two interesting applications. The first application is the following characterization of toric varieties via an int-amplified endomorphism, which is an analog of results of Thomsen [95] and Achinger [1].

Theorem B (Theorem 2.4.6, Theorem 2.4.13). *Let X be a smooth projective variety over $\mathbb{C}$ admitting an int-amplified endomorphism f. Then X is toric if and only if f_*L is a direct sum of line bundles on X for every line bundle L on X.*

Remark 1.0.5. If f is a multiplication map, then the "only if" part of Theorem B essentially follows from the argument in [95]. Toric varieties, however, have other endomorphisms in general. For example, a projective space has many endomorphisms which do not preserve any toric structure.

As the second application of Theorem A, we study a connection between admitting an int-amplified endomorphism and Frobenius splitting. First, we recall the notion of global F-splitting. It is a global property of a projective variety over a perfect field of positive characteristic defined by the splitting of the absolute Frobenius morphism. Via reduction to positive characteristic, global F-splitting makes sense even in characteristic zero as well: X is said to be of *dense globally F-split type* if its modulo p reduction is globally F-split for infinitely many primes p. The second application is now stated as follows.

Theorem C (Theorem 2.5.7). *Let X be a normal surface over $\mathbb{C}$ admitting an int-amplified endomorphism. Then X is of dense globally F-split type.*

It is proved in [38] that if a normal surface X is of dense globally F-split type, then it is of *Calabi-Yau type*, that is, there exists an effective $\mathbb{Q}$-Weil divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-linearly trivial and (X, Δ) is log canonical. Thus, Theorem C gives an affirmative answer to the following conjecture in the surface case, which is a generalization of [16, Conjecture 1.2] to the int-amplified case.

Conjecture 1.0.6 (cf. [16, Conjecture 1.2]). *Let X be a normal projective variety of characteristic zero admitting an int-amplified endomorphism. Then X is of Calabi-Yau type.*

We remark that Conjecture 1.0.6 is wide open in the higher dimension except for the case where X is a Mori dream space or X is smooth and rationally connected. The first follows from essentially the same argument as the proof of [16]. The latter case follows from Theorem A.

In Chapter 3, we study the Minimal model theory in mixed characteristic. The minimal model theory in positive characteristic has been studied intensively in recent years, and the MMP is now known to hold for threefolds over a perfect field of characteristic $p > 3$ (see [45], [19], [10], [36], [12], [42]). The MMP is also studied for schemes not necessarily defined over a field. Such a generalization of the MMP plays an important role to construct a nice model $\mathcal{W}$ over an integer ring whose generic fiber is isomorphic to a given variety W (see [70] [66], [21], [20]). The MMP holds for excellent surfaces ([93]) and strictly semi-stable schemes over an excellent Dedekind scheme of relative dimension two whose each residue characteristic is neither 2 nor 3 ([56], [57]). In this chapter, we study the MMP for threefolds over an excellent Dedekind scheme of arbitrary residue characteristic.

The first main result in Chapter 3 is a generalization of the result of Kawamata. He used the classification of singularities, which depends on the residue characteristic, to prove the existence of flips. We use a completely different approach to prove the existence of flips without any assumption on the residue characteristic.

Theorem D. (Theorem 3.4.11) *Let V be an excellent Dedekind scheme. Let X be a scheme which is strictly semi-stable over V of relative dimension two. Let $X \longrightarrow Z$ be a projective morphism to a quasi-projective scheme Z over V. Then we can run a $K_{X/V}$-MMP over Z which terminates with a minimal model or a Mori fiber* *space. Furthermore, this program preserves good conditions (see Assumption 3.4.1), for example, the output Y of this MMP is Cohen-Macaulay and every irreducible component of each closed fiber of $Y \longrightarrow V$ is geometrically normal.*

Kawamata's result is used in several studies of reductions of varieties over an integer ring. Therefore, we also generalize such studies to the case where the residue characteristic is 2 or 3, as follows.

Good reduction criterion for K3 surfaces ([70], [66], [21])
Let K be a henselian discrete valuation field with perfect residue field of characteristic p and X a K3 surface over K. Suppose that X admits potentially semi-stable reduction. If the G_K-representation $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for some $\ell \neq p$, then X admits good reduction after an unramified extension of K.

Abelian surfaces have potentially combinatorial reduction ([20])
Let K be a henselian discrete valuation field with perfect residue field of characteristic p and X an abelian surface over K. Then X admits potentially combinatorial reduction in the sense of [20, Definition 10.1]. Such a model give a compactification of a Néron model, and the dual graph of the special fiber can be classified (see Theorem 3.4.14 and Proposition 3.4.15).

The second main result in Chapter 3 is a generalization of a result of Hacon-Witaszek [43, Theorem 1.1] to the mixed characteristic case.

Theorem E. (Theorem 3.3.6) *Let V be an excellent Dedekind scheme. Let (X, Δ) be a dlt pair, where X is a $\mathbb{Q}$-factorial integral scheme which is flat and quasi-projective over V of relative dimension two. Assume that there exists a projective birational morphism $\pi\colon X \longrightarrow Z$ to a normal $\mathbb{Q}$-factorial variety Z with* $\mathrm{Exc}(\pi) \subset \lfloor \Delta \rfloor$. *Then we can run a $(K_{X/V} + \Delta)$-MMP over Z which terminates with a minimal model.*

The existence of dlt modifications and the inversion of adjunction for klt pairs follow from Theorem E as a corollary (see Corollaries 3.3.9, 3.3.10).

One of the key ingredients of the proofs of Theorems D and E is to prove the existence of pl-flips with ample divisor in the boundary. Indeed, all flips appearing in the proof of Theorem E are of this type, and the existence of necessary flips for Theorem D is reduced to the existence of flips of this type by an argument in [43]. In positive characteristic, Hacon-Witaszek [44, Theorem 1.3] proved the existence of pl-flips with ample divisor in the boundary using global F-regularity and the vanishing theorem up to Frobenius twist. We employ the same strategy in mixed characteristic, replacing global F-regularity with *global T-regularity* and the Frobenius morphism with alterations. The global T-regularity of a log pair (X, Δ) over an excellent Dedekind scheme V is defined by the surjectivity of the map

$$H^0(\omega_{Y/V}(\lceil -\pi^*(K_{X/V} + \Delta) \rceil)) \longrightarrow H^0(\mathcal{O}_X)$$

induced by the Grothendieck trace map for every alteration $\pi\colon Y \longrightarrow X$. The vanishing theorem up to alterations is obtained as a corollary of [8, Theorem 6.28] (see Corollary 3.2.5). As a consequence, we obtain the following theorem, which is an analog of [44, Theorem 1.3].

Theorem F. (Theorem 3.2.29, cf. [44, Theorem 1.3]) *Let V be the spectrum of a complete discrete valuation ring. Let $(X, S+A+B)$ is a dlt pair such that S is an anti-ample $\mathbb{Q}$-Cartier Weil divisor on X, A is an ample $\mathbb{Q}$-Cartier Weil divisor on X, B is a Weil divisor on X. Let $f\colon X \longrightarrow Z$ be a $(K_{X/V}+S+A+B)$-flipping contraction with $\rho(X/Z)=1$ to an affine V-scheme Z. Assume that $(S^N, (1-\varepsilon)A_S+B_S)$ is globally T-regular for all $0<\varepsilon<1$ after localizing at all points of $f(\mathrm{Exc}(f))$. Further assume that the ring*

$$R(K_{S^N/V}+A_S+B_S) := \bigoplus_{m\in\mathbb{Z}_{\geqslant 0}} H^0(\mathcal{O}_{S^N}(m(K_{S^N/V}+A_S+B_S)))$$

is a finitely generated $\mathcal{O}_Z$-algebra, where $B_S := \mathrm{Diff}_{S^N}(B)$ and $A_S := A|_{S^N}$. Then the flip of f exists.

Theorem F can be applied for three-dimensional pl-flips with ample divisor in the boundary. Indeed, the finite generation of $R(K_{S^N/V}+A_S+B_S)$ is a consequence of the MMP for excellent surfaces, and the global T-regularity of $(S^N, (1-\varepsilon)A_S+B_S)$ follows from the inversion of adjunction for global T-regularity (see Proposition 3.2.21) and an argument similar to the proof of [43, Lemma 3.3].

Remark 1.0.7. After finishing this work, Jakub Witaszek taught us that he also write the paper about the MMP in mixed characteristic with Bhargav Bhatt, Linquan Ma, Zsolt Patakfalvi, Karl Schwede, Kevin Tucker, Joe Waldron (see [9]). In their article, they independently show that the MMP holds for threefolds whose each residue characteristic is greater than 5. They define and study the notion of global +-regularity which is very closely related to our global T-regularity (see [9, Lemma 4.7]). We can show that Theorems E and F hold when V is a spectrum of a regular excellent finite-dimensional domain, using [9, Proposition 3.6] instead of Theorem 3.2.3 and Cohen-Macaulayfications instead of regular alterations.

Chapter 2 is based on [99],[100], [101] and Chapter 3 is based on the joint work with Teppei Takamatsu ([89]).

Chapter 2

Structure of varieties admitting a polarized endomorphism

2.1 Notations and Terminologies

Throughout in this chapter, we use the following notations and terminologies.

- A variety over a field k is a geometrically integral separated scheme of finite type over k. A k-scheme X is essentially of finite type over k if X is a localization of some scheme of finite type over k. A $\mathbb{Q}$-Cartier divisor (resp. $\mathbb{R}$-Cartier divisor) on a variety X is an element of $(\mathrm{CDiv}X)\otimes_{\mathbb{Z}}\mathbb{Q}$ (resp. $(\mathrm{CDiv}X)\otimes_{\mathbb{Z}}\mathbb{R}$), where $\mathrm{CDiv}X$ is the group of Cartier divisors on X. When X is normal, these groups are embedded in $\mathrm{Div}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ (resp. $\mathrm{Div}(X)\otimes_{\mathbb{Z}}\mathbb{R}$) where $\mathrm{Div}(X)$ is the group of Weil divisors on X. An element of $\mathrm{Div}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ (resp. $\mathrm{Div}(X)\otimes_{\mathbb{Z}}\mathbb{R}$) is called $\mathbb{Q}$-Weil divisor (resp. $\mathbb{R}$-Weil divisor). Linear equivalence and $\mathbb{Q}$-linear equivalence are denoted by $\sim$ and $\sim_{\mathbb{Q}}$, respectively. For $\mathbb{Q}$-Weil divisors D and E, $D \sim E$ means $D - E$ is a principal divisor.

- Let X be a projective variety over an algebraically closed field.
  - $N^1(X)$ is the group of Cartier divisors modulo numerical equivalence (a Cartier divisor D is numerically equivalent to zero, which is denoted by $D \equiv 0$, if $(D \cdot C) = 0$ for all irreducible curves C on X).
  - $N_1(X)$ is the group of 1-cycles modulo numerical equivalence (a 1-cycle α is numerically zero if $(D \cdot \alpha) = 0$ for all Cartier divisors D). By definition, $N^1(X)$ and $N_1(X)$ are dual to each other.

- A morphism $f \colon X \longrightarrow X$ from a projective variety X to itself is called self-morphism of X or endomorphism on X. If it is surjective, then it is a finite morphism.

- A morphism $f \colon X \longrightarrow Y$ between varieties is called an algebraic fiber space if f is proper and $f_*\mathcal{O}_X = \mathcal{O}_Y$.

- A morphism $f\colon X \longrightarrow Y$ between varieties is called quasi-étale if f is étale at every codimension one point of X.
- Let $f\colon X \longrightarrow Y$ be a finite separable surjective morphism between normal varieties. The ramification divisor of f is denoted by R_f.
- The Picard number of a projective variety X is denoted by $\rho(X)$.
- The function field of a variety X is denoted by $K(X)$.
- Let $f\colon X \longrightarrow X$ be an endomorphism of a variety X. A subset $S \subset X$ is called totally invariant under f if $f^{-1}(S) = S$ as sets.
- Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\pi"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, dashed, "\pi"'] & Y, \end{tikzcd}$$

  where f, g are surjective morphisms and π is a dominant rational map. We write this diagram as

$$f \circlearrowright X \overset{\pi}{\dashrightarrow} Y \circlearrowleft g.$$

  We say a commutative diagram is equivariant if each object is equipped with an endomorphism and the morphisms are equivariant with respect to these endomorphisms.

- Let X be a normal projective variety. A K_X-negative extremal ray contraction $\pi\colon X \longrightarrow Y$ is called of fiber type or a Mori fiber space if $\dim Y < \dim X$.

2.2 Preliminaries

2.2.1 Varieties of Fano type and Calabi–Yau type

In this chapter, we use the following terminologies.

Definition 2.2.1 (cf. [63, Definition 2.34], [87, Remark 4.2]). Let X be a normal variety over a field k and Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi\colon Y \longrightarrow X$ be a birational morphism from a normal variety Y. Then we can write

$$K_Y = \pi^*(K_X + \Delta) + \sum_E (a_E(X, \Delta) - 1)E,$$

where E runs through all prime divisors on Y. We say that the pair (X, Δ) is *log canonical* or *lc*, for short (resp., *Kawamata log terminal* or *klt*, for short) if $a_E(X, \Delta) \geqslant 0$ (resp., $a_E(X, \Delta) > 0$) for every prime divisor E over X. If $\Delta = 0$, we simply say that X is log canonical (resp., klt).

Definition 2.2.2 (cf. [86, Lemma-Definition 2.6]). Let $\pi \colon X \longrightarrow B$ be a projective morphism of normal varieties over a field k and Δ be an effective $\mathbb{Q}$-Weil divisor on X.

1. We say that (X, Δ) is *log Fano over B* if $-(K_X + \Delta)$ is π-ample $\mathbb{Q}$-Cartier and (X, Δ) is klt. We say that (X, Δ) is of *Fano type over B* if there exists an effective $\mathbb{Q}$-Weil divisor Γ on X such that $(X, \Delta + \Gamma)$ is log Fano over B. If B is a point, we simply say that (X, Δ) is of Fano type.

2. We say that (X, Δ) is *log Calabi–Yau over B* if $K_X + \Delta \sim_{\mathbb{Q},B} 0$ and (X, Δ) is log canonical. We say that (X, Δ) is of *Calabi–Yau type over B* if there exists an effective $\mathbb{Q}$-Weil divisor Γ on X such that $(X, \Delta + \Gamma)$ is log Calabi–Yau over B. If B is a point, we say that (X, Δ) is of Calabi-Yau type.

We introduce the following basic properties.

Proposition 2.2.3. *Let $\pi \colon X \longrightarrow B$ be a surjective projective morphism of normal projective varieties over a field k of characteristic zero and Δ an effective $\mathbb{Q}$-Weil divisor on X. Then the following conditions are equivalent to each other.*

1. *(X, Δ) is of Fano type over B.*

2. *There exists an effective π-big $\mathbb{Q}$-Weil divisor Ω, that is, Ω is a sum of an π-ample $\mathbb{Q}$-Cartier divisor and an effective $\mathbb{Q}$-Weil divisor such that $(X, \Delta + \Omega)$ is klt and $K_X + \Delta + \Omega \sim_{\mathbb{Q},B} 0$.*

3. *There exists an effective π-big $\mathbb{Q}$-Weil divisor Ω such that $(X, \Delta + \Omega)$ is klt and $K_X + \Delta + \Omega \equiv_B 0$.*

Proof. First we assume that (X, Δ) is of Fano type over B. Then there exists an effective $\mathbb{Q}$-Wei divisor Γ such that $(X, \Delta + \Gamma)$ is klt and $-(K_X + \Delta + \Gamma)$ is π-ample. Then we can take an effective $\mathbb{Q}$-Weil divisor Ω' which is $\mathbb{Q}$-linear equivalent to $-(K_X + \Delta + \Gamma)$ over B such that $(X, \Delta + \Gamma + \Omega')$ is klt. Therefore, $\Omega = \Gamma + \Omega' \sim_{\mathbb{Q},B} -(K_X + \Delta)$ is big over B, $K_X + \Delta + \Omega \sim_{\mathbb{Q},B} 0$ and $(X, \Delta + \Omega)$ is klt.

It is clear that the second condition implies the third condition.

Next we assume that we can take Ω satisfying the third condition. Then there exist a π-ample $\mathbb{Q}$-Cartier divisor A and an effective $\mathbb{Q}$-Weil divisor D such that $\Omega = A + D$. Since $(X, \Delta + \Omega)$ is klt, for enough small $\varepsilon > 0$ such that $(X, \Delta + (1-\varepsilon)\Omega + \varepsilon D)$ is klt and

$$K_X + \Delta + (1-\varepsilon)\Omega + \varepsilon D \equiv_B -\varepsilon A$$

is anti-ample over B. It means that (X, Δ) is of Fano type over B. □

Proposition 2.2.4. *We consider the following commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\mu"] \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & B, & \end{tikzcd}$$

where X, Y, B are normal projective varieties over a field k of characteristic zero, π_X, π_Y are surjective projective morphisms and μ is a birational contraction, that is, μ^{-1} has no exceptional divisors. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that (X, Δ) is of Fano type over B and $\Delta' = \mu_*\Delta$. Then (Y, Δ') is also of Fano type over B.

Proof. By Proposition 2.2.3, there exists an effective π_X-big $\mathbb{Q}$-Weil divisor Ω on X such that $(X, \Delta + \Omega)$ is klt and $K_X + \Delta + \Omega \sim_{\mathbb{Q},B} 0$. Let $\Omega' = \mu_*\Omega$, then Ω' is π_Y-big $\mathbb{Q}$-Weil divisor. Indeed, since Ω is π_X-big, we have $\Omega = A + D$, where A is an π_X-ample $\mathbb{Q}$-Cartier divisor and D is an effective $\mathbb{Q}$-Weil divisor. Taking an π_Y-ample Cartier divisor H on Y, there exists an effective divisor $A' \sim_{\mathbb{Q},B} A$ such that $mA' \geqslant \mu_*^{-1}H$ for some positive integer m. Hence, we have $\Omega' \sim_{\mathbb{Q},B} \mu_*(A' + D) \geqslant \frac{1}{m}H$, and in particular, Ω' is π_Y-big. Furthermore, $K_Y + \Delta' + \Omega'$ is $\mathbb{Q}$-Cartier and $K_Y + \Delta' + \Omega' \sim_{\mathbb{Q},B} 0$. By the negativity lemma, $(Y, \Delta' + \Omega')$ is klt. □

The reader is referred to [86, Lemma-Definition 2.6] for more details.

2.2.2 Globally F-regular and F-split varieties

In this subsection, we review the definition and basic properties of *globally F-regularity* and *globally F-splitting.*

A field k of prime characteristic p is called *F-finite* if $[k : k^p] < \infty$.

Definition 2.2.5 ([87, Definition 3.1]). Let X be a normal projective variety defined over an F-finite field of characteristic $p > 0$.

1. We say that X is *globally F-split* if the Frobenius map

$$\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$$

splits as an $\mathcal{O}_X$-module homomorphism.

2. We say that X is *globally F-regular* if for every effective Weil divisor D on X, there exists $e \in \mathbb{Z}_{>0}$ such that the composition map

$$\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$$

of the e-times iterated Frobenius map $\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X$ with a natural inclusion $F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$ splits as an $\mathcal{O}_X$-module homomorphism.

Remark 2.2.6. Mehta-Ramanathan [74] introduced the notion of globally F-splitting and they call it by Frobenius splitting. In this thesis, we call it by globally F-splitting to distinguish from the local notion of F-pure (cf. Definition 2.5.3).

Now we briefly explain how to reduce things from characteristic zero to characteristic $p > 0$. The reader is referred to [50, Chapter 2] and [75, Section 3.2] for further details.

Let X be a normal variety over a field k of characteristic zero and $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-Weil divisor on X. Choosing a suitable finitely generated $\mathbb{Z}$-subalgebra A of k, we can construct a scheme X_A of finite type over A and closed subschemes $D_{i,A} \subsetneq X_A$ such that there exists isomorphisms

$$\begin{tikzcd} X \arrow[r, "\cong"] & X_A \times_{\mathrm{Spec}\ A} \mathrm{Spec}\ k \\ D_i \arrow[u, hook] \arrow[r, "\cong"] & D_{i,A} \times_{\mathrm{Spec}\ A} \mathrm{Spec}\ k. \arrow[u, hook] \end{tikzcd}$$

Note that we can enlarge A by localizing at a single nonzero element and replacing X_A and $D_{i,A}$ with the corresponding open subschemes. Thus, applying the generic freeness [50, (2.1.4)], we may assume that X_A and $D_{i,A}$ are flat over Spec A. Enlarging A if necessary, we may also assume that X_A is normal and $D_{i,A}$ is a prime divisor on X_A. Letting $D_A := \sum_i d_i D_{i,A}$, we refer to (X_A, D_A) as a *model* of (X, D) over A. Given a closed point $\mu \in \mathrm{Spec}\ A$, we denote by X_μ (resp., $D_{i,\mu}$) the fiber of X_A (resp., $D_{i,A}$) over μ. Then X_μ is a scheme of finite type over the residue field $\kappa(\mu)$ of μ, which is a finite field. Enlarging A if necessary, we may assume that X_μ is a normal variety over $\kappa(\mu)$, $D_{i,\mu}$ is a prime divisor on X_μ and consequently $D_\mu := \sum_i d_i D_{i,\mu}$ is a $\mathbb{Q}$-divisor on X_μ for all closed points $\mu \in \mathrm{Spec}\ A$.

Given a morphism $f : X \longrightarrow Y$ of varieties over k and a model (X_A, Y_A) of (X, Y) over A, after possibly enlarging A, we may assume that f is induced by a morphism $f_A : X_A \longrightarrow Y_A$ of schemes of finite type over A. Given a closed point $\mu \in \mathrm{Spec}\ A$, we obtain a corresponding morphism $f_\mu : X_\mu \longrightarrow Y_\mu$ of schemes of finite type over $\kappa(\mu)$. If f is projective (resp. finite), after possibly enlarging A, we may assume that f_μ is projective (resp. finite) for all closed points $\mu \in \mathrm{Spec}\ A$.

We denote by $X_{\overline{\mu}}$ the base change of X_μ to the algebraic closure $\overline{\kappa(\mu)}$ of $\kappa(\mu)$. Similarly for $D_{\overline{\mu}}$ and $f_{\overline{\mu}} : X_{\overline{\mu}} \longrightarrow Y_{\overline{\mu}}$. Note that (X_μ, D_μ) is globally F-regular (resp. globally F-split) if and only if so is $(X_{\overline{\mu}}, D_{\overline{\mu}})$.

Definition 2.2.7. Let the notation be as above. Suppose that X is a normal projective variety over a field of characteristic zero.

1. X is said to be of *dense globally F-split type* if for a model of X over a finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense subset of closed points $W \subseteq \mathrm{Spec}\ A$ such that X_μ is globally F-split for all $\mu \in W$.

2. X is said to be of *globally F-regular type* if for a model of X over a finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense open subset of closed points $W \subseteq \mathrm{Spec}\ A$ such that X_μ is globally F-regular for all $\mu \in W$.

This definition is independent of the choice of a model.

The following two theorems are very important in this chapter.

Theorem 2.2.8 ([87, Theorem 5.1]). *Let X be a normal projective variety defined over a field of characteristic zero. If X is of Fano type, then X is of globally F-regular type.*

Theorem 2.2.9 ([38, Theorem 1.1]). *Let S be a normal projective surface over an algebraically closed field of characteristic zero. If S is of dense globally F-split type (resp., globally F-regular type), then it is of Calabi–Yau type (resp., Fano type)*

2.2.3 Canonical modules and duality

Now we briefly explain canonical modules and duality. The reader is referred to [65], [74], [30], [75], [47], [88] for further details. Let $f\colon Y \longrightarrow X$ be a finite surjective morphism of normal integral schemes essentially of finite type over an F-finite field of characteristic $p > 0$. In this case, X and Y have canonical modules ω_X and ω_Y, respectively (cf. [48]), satisfying

$$\mathscr{H}om_X(f_*\mathcal{O}_Y, \omega_X) \simeq f_*\omega_Y$$

as $f_*\mathcal{O}_Y$-modules. We note that the natural $f_*\mathcal{O}_Y$-module structure on $\mathscr{H}om_X(f_*\mathcal{O}_Y, \omega_X)$ is induced by the multiplication of $f_*\mathcal{O}_Y$. Since X and Y are normal schemes, there exist Weil divisors K_X and K_Y such that $\mathcal{O}_X(K_X) \simeq \omega_X$ and $\mathcal{O}_Y(K_Y) \simeq \omega_Y$. We call K_X and K_Y canonical divisors on X and Y, respectively, and they are uniquely determined up to linear equivalence. By the above duality, we obtain

$$\mathcal{O}_Y(K_Y - f^*K_X) \simeq \mathscr{H}om_X(f_*\mathcal{O}_Y, \mathcal{O}_X).$$

In particular, if $\psi\colon f_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ is an $\mathcal{O}_X$-module homomorphism, then there exists a non-zero rational section $\alpha \in K(Y)$ such that

$$D_\psi := K_Y - f^*K_X + \mathrm{div}_Y(\alpha) \geqslant 0,$$

and regarding this as a global section of $\mathcal{O}_Y(K_Y - f^*K_X)$, this is corresponding to ψ by the above isomorphism. We say that ψ is corresponding to D_ψ. Furthermore, the above isomorphism induces

$$K(Y) \simeq \mathscr{H}om_X(f_*K(Y), K(X)).$$

Note that this depends on the choice of the canonical divisors K_X and K_Y. In particular, if $\psi\colon f_*K(Y) \longrightarrow K(X)$ is an $\mathcal{O}_X$-module homomorphism, then we can consider the corresponding divisor

$$D_\psi := K_Y - f^*K_X + \mathrm{div}(\alpha)$$

for some $\alpha \in K(Y)$.

Now, we obtain the following basic result.

Proposition 2.2.10. *Let $f\colon Y \longrightarrow X$ be a finite surjective morphism of normal integral schemes essentially of finite type over an F-finite field of characteristic $p > 0$. We fix canonical divisors K_X and K_Y on X and Y, respectively. Then $\mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ splits as an $\mathcal{O}_X$-module homomorphism if and only if there exists an $\mathcal{O}_X$-module homomorphism*

$$\psi\colon f_*K(Y) \longrightarrow K(X)$$

such that $\psi(1) = 1$ and ψ is corresponding to an effective divisor.

Example 2.2.11. If f is a separable morphism, then the ramification divisor R_f can be defined and R_f is linearly equivalent to $K_Y - f^*K_X$. Furthermore, we obtain the following isomorphism

$$\mathcal{O}_Y(R_f) \simeq \mathscr{H}om_X(f_*\mathcal{O}_Y, \mathcal{O}_X),$$

and the trace map Tr is corresponding to R_f. By using this isomorphism, for any $\psi\colon f_*K(Y) \longrightarrow K(X)$, there exists a non-zero rational section $\alpha \in K(Y)$ such that $\psi = \mathrm{Tr}(\alpha \cdot \_)$, where $\mathrm{Tr}(\alpha \cdot \_)$ sends $x \in K(Y)$ to $\mathrm{Tr}(\alpha x)$. In particular, ψ is contained in $\mathscr{H}om_X(f_*\mathcal{O}_Y, \mathcal{O}_X)$ if and only if

$$R_f + \mathrm{div}_Y(\alpha) \geqslant 0.$$

The following is a basic property of the global F-splitting.

Proposition 2.2.12. *Let $\rho\colon Y \longrightarrow X$ be a quasi-étale finite surjective morphism of normal varieties over a field of characteristic zero. Then X is of globally F-regular type or dense globally F-split type if and only if so is Y.*

Proof. We take models $\rho_A\colon Y_A \longrightarrow X_A$ over a finitely generated $\mathbb{Z}$-algebra A as in § 2.2 such that ρ_μ is quasi-étale finite surjective and $\deg(\rho_\mu)$ is coprime to $\mathrm{char}(\kappa(\mu))$ for every $\mu \in \mathrm{Spec}\, A$. We fix $\mu \in \mathrm{Spec}\, A$, and it is enough to show that X_μ is globally F-regular or globally F-split if and only if so is Y_μ.

First, we prove the "if" part. Since $\deg(\rho_\mu)$ is coprime to $\mathrm{char}(\kappa(\mu))$, $\mathcal{O}_{X_\mu} \longrightarrow (\rho_\mu)_*\mathcal{O}_{Y_\mu}$ splits by the trace map. Hence, if Y is globally F-split or globally F-regular, then so is X.

Next, we prove the "only if" part. Let D be a effective Weil divisor on Y and we assume that

$$\mathcal{O}_{X_\mu} \longrightarrow F_*\mathcal{O}_{X_\mu} \longrightarrow F^e_*\mathcal{O}_{X_\mu}((\rho_\mu)_*D)$$

splits for some positive integer e. Note that if $D = 0$, it means that X is globally F-split. We take a rational section $\alpha \in K(X)$ such that

$$(1 - p^e)K_{X_\mu} + \mathrm{div}_X(\alpha) - (\rho_\mu)_*D \geqslant 0$$

is corresponding to a splitting of the above homomorphism. Since $D' = (\rho_\mu)^*(\rho_\mu)_*D \geqslant D$, it is enough to show that

$$\mathcal{O}_{Y_\mu} \longrightarrow F_*\mathcal{O}_{Y_\mu} \longrightarrow F^e_*\mathcal{O}_{Y_\mu}(D')$$

splits. Since ρ_μ is quasi-étale, $\rho_\mu^*K_{X_\mu} = K_{Y_\mu}$. In particular, we obtain

$$(1 - p^e)K_{Y_\mu} + \mathrm{div}_Y(\alpha) - D' \geqslant 0,$$

and this is corresponding to a splitting of

$$\mathcal{O}_{Y_\mu} \longrightarrow F_*\mathcal{O}_{Y_\mu} \longrightarrow F^e_*\mathcal{O}_{Y_\mu}(D').$$

□

2.2.4 Int-amplified endomorphism

In this subsection, every variety is defined over an algebraically closed field of characteristic zero. Meng and Zhang established minimal model program equivariant with respect to endomorphisms in [79], for varieties admitting an int-amplified endomorphism. We summarize their results that we need later.

Definition 2.2.13. A surjective endomorphism $f\colon X \longrightarrow X$ of normal projective variety X is called *int-amplified* if there exists an ample Cartier divisor H on X such that $f^*H - H$ is ample.

We collect basic properties of int-amplified endomorphisms in the following lemma.

Proposition 2.2.14.

1. *Let X be a normal projective variety, $f\colon X \longrightarrow X$ a surjective morphism, and $n > 0$ a positive integer. Then, f is int-amplified if and only if so is f^n.*

2. *Let $\pi\colon X \longrightarrow Y$ be a surjective morphism between normal projective varieties. Let $f\colon X \longrightarrow X$, $g\colon Y \longrightarrow Y$ be surjective endomorphisms such that $\pi \circ f = g \circ \pi$. If f is int-amplified, then so is g.*

3. *Let $\pi\colon X \dashrightarrow Y$ be a dominant rational map between normal projective varieties of same dimension. Let $f\colon X \longrightarrow X$, $g\colon Y \longrightarrow Y$ be surjective endomorphisms such that $\pi \circ f = g \circ \pi$. Then f is int-amplified if and only if so is g.*

4. *Let $f\colon X \longrightarrow X$ be an int-amplified endomorphism of a normal projective variety and D a $\mathbb{Q}$-Cartier divisor on X. If $f^*D - D$ is numerically equivalent to an effective $\mathbb{Q}$-Weil divisor, then D is also numerically equivalent to an effective $\mathbb{Q}$-Weil divisor. In particular, if X is $\mathbb{Q}$-Gorenstein, then $-K_X$ is numerically equivalent to an effective $\mathbb{Q}$-Weil divisor.*

Proof. See [76, Theorem 3.3, Lemmas 3,4, 3.5, Theorem 1.5]. □

Theorem 2.2.15 (Meng-Zhang). *Let X be a $\mathbb{Q}$-factorial normal projective variety admitting an int-amplified endomorphism. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that (X, Δ) is klt.*

1. *There are only finitely many $(K_X + \Delta)$-negative extremal rays of $\overline{NE}(X)$. Moreover, let $f\colon X \longrightarrow X$ be a surjective endomorphism of X. Then every $(K_X+\Delta)$-negative extremal ray is fixed by the linear map $(f^n)_*$ for some $n > 0$.*

2. *Let $f\colon X \longrightarrow X$ be a surjective endomorphism of X. Let R be a $(K_X + \Delta)$-negative extremal ray and $\pi\colon X \longrightarrow Y$ its contraction. Suppose $f_*(R) = R$. Then,*

    (a) *f induces an endomorphism $g\colon Y \longrightarrow Y$ such that $g \circ \pi = \pi \circ f$;*

(b) *if π is a flipping contraction and X^+ is the flip, the induced rational self-map $h\colon X^+ \dashrightarrow X^+$ is a morphism.*

3. *In particular, for any finite sequence of $(K_X+\Delta)$-MMP and for any surjective endomorphism $f\colon X \longrightarrow X$, there exists a positive integer $n>0$ such that the sequence of MMPs is equivariant under f^n.*

Proof. (1) is a special case of [79, Theorem 4.6]. (2a) is true since the contraction is determined by the ray R. (2b) follows from [77, Lemma 6.6]. □

Theorem 2.2.16 (Equivariant MMP (Meng-Zhang))**.** *Let X be a $\mathbb{Q}$-factorial klt projective variety admitting an int-amplified endomorphism. Then for any surjective endomorphism $f\colon X \longrightarrow X$, there exists a positive integer $n>0$ and a sequence of rational maps*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_r$$

such that

1. *$X_i \dashrightarrow X_{i+1}$ is either a divisorial contraction, flip, or Mori fiber space of a K_{X_i}-negative extremal ray,*

2. *there exist surjective endomorphisms $g_i\colon X_i \longrightarrow X_i$ for $i=0,\ldots,r$ such that $g_0 = f^n$ and the following diagram commutes*

$$\begin{tikzcd} X_i \arrow[r, dashed] \arrow[d, "g_i"'] & X_{i+1} \arrow[d, "g_{i+1}"] \\ X_i \arrow[r, dashed] & X_{i+1}, \end{tikzcd}$$

3. *X_r is a Q-abelian variety (that is, there exists a quasi-étale finite surjective morphism $A \longrightarrow X_r$ from an abelian variety A, note that X_r might be a point). In this case, there exists a quasi-étale finite surjective morphism $A \longrightarrow X_r$ from an abelian variety A and an surjective endomorphism $h\colon A \longrightarrow A$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d] & A \arrow[d] \\ X_r \arrow[r, "g_r"'] & X_r \end{tikzcd}$$

commutes.

Proof. This is a part of [79, Theorem 1.2]. □

Remark 2.2.17. Surjective endomorphisms of a Q-abelian variety always lift to a certain quasi-étale cover by an abelian variety. See [18, Lemma 8.1 and Corollary 8.2], for example. The proof works over any algebraically closed field.

2.2.5 Index one covers for pairs with standard coefficients

In this subsection, we study index one covers for pairs with standard coefficients. In this subsection, every varieties defined over an algebraically closed field k of characteristic zero.

Definition 2.2.18. Let X be a normal variety and Δ an effective $\mathbb{Q}$-Weil divisor on X. We say that Δ has *standard coefficients* if for any prime divisor E on X, there exists a positive integer m such that $\mathrm{ord}_E(\Delta) = \frac{m-1}{m}$.

Lemma 2.2.19. *Let X be a normal projective variety and Δ an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to 0. Then there exists a finite surjective morphism $\mu\colon \widetilde{X} \longrightarrow X$ from normal projective variety such that the following conditions hold:*

- *$\mu^*(K_X + \Delta)$ is a principal divisor, that is, $\mu^*(K_X + \Delta) \sim 0$;*
- *if $\mu'\colon X' \longrightarrow X$ is a finite surjective morphism from a normal projective variety such that $\mu'^*(K_X + \Delta)$ is a principal divisor, then μ' factors through μ.*

Furthermore if Δ has standard coefficients, then $R_\mu = \mu^*(\Delta)$. In particular $K_{\widetilde{X}}$ is a principal divisor.

Proof. Let $m_0 = \min\{m \mid m(K_X + \Delta) \sim 0\}$ and take a non-zero rational section $\alpha \in K(X) = K$ such that $\mathrm{div}(\alpha) = m_0(K_X + \Delta)$. Let $L = K[T]/(T^{m_0} - \alpha)$. Note that L is a field. Let $\mu\colon \widetilde{X} \longrightarrow X$ be the normalization of X in L. Then we have

$$\mathrm{div}(T) = \frac{1}{m_0}\mathrm{div}(\alpha) = \mu^*(K_X + \Delta),$$

so $\mu^*(K_X+\Delta)$ is a principal divisor. Moreover, let $\mu'\colon X' \longrightarrow X$ be a finite surjective morphism from a normal projective variety such that $\mu'^*(K_X + \Delta)$ is a principal divisor. Then there exists a non-zero rational function $\beta \in K(X') = L'$ such that $\mathrm{div}(\beta) = \mu'^*(K_X + \Delta)$. In particular, we have $m_0\mathrm{div}(\beta) = \mathrm{div}(\alpha)$. Since the base field is algebraically closed, we may assume that $\beta^{m_0} = \alpha$. It means that there exists an injective K-algebra homomorphism from L to L', so μ' factors through μ.

Next we assume that Δ has standard coefficients. Let E be a prime divisor on X, m a positive integer, a an integer such that $\mathrm{ord}_E(K_X + \Delta) = \frac{m-1}{m} + a$. Let $(R, (\varpi))$ be the DVR associated to E and S the normalization of R in L. Then it is enough to show that the order of ϖ at every maximal ideal of S is equal to m. Since the order of α along E is equal to $m_0(\frac{m-1}{m} + a)$, there exists an unit u in R such that

$$\alpha = u\varpi^{m_0(\frac{m-1}{m}+a)}.$$

Since every coefficient of $\mathrm{div}(\alpha)$ is integer, there exists a positive integer b such that $mb = m_0$. Let $\rho = m - 1 + ma$. Then

$$\alpha = u\varpi^{b\rho}.$$

Now, we have

$$(\frac{T^m}{\varpi^\rho})^b = \frac{\alpha}{\varpi^{b\rho}} = u,$$

so S contains $\frac{T^m}{\varpi^\rho}$. In particular, $R \hookrightarrow S$ factors through $R' = R[Y]/(Y^b - u)$. Since $K' = K[Y]/(Y^b - u)$ is a field and $R' \longrightarrow K'$ is injective, R' is an integral domain and satisfies $R \subset R' \subset S$. Since R' is étale over R, ϖ is an uniformizer of $R'_i = R'_{\mathfrak{p}_i}$ for every maximal ideal $\mathfrak{p}_i$ of R'. We set $S_i = S \otimes_{R_i} R'_i \subset L$. Now, we have

$$(\frac{\varpi^{1+a}}{T})^m = \frac{\varpi^\rho}{T^m} \cdot \varpi = Y^{-1}\varpi.$$

Since Y is an unit in R', $R'_i \hookrightarrow S_i$ factors through $R''_i = R'_i[Z]/(Z^m - Y^{-1}\varpi)$. Since $K'' = K'[Z]/(Z^m - Y^{-1}\varpi)$ is a field and $R''_i \longrightarrow K''$ is injective, R''_i is an integral domain and satisfies $R'_i \subset R''_i \subset S_i$. Since we have

$$(\varpi^{1+a}Z^{-1})^{m_0} = (\varpi^{(1+a)m}Z^{-m})^b = (\varpi^\rho Y)^b = \alpha,$$

the quotient field of R''_i is L. Furthermore, since (Z) is the unique maximal ideal of R''_i, we obtain $R''_i = S_i$ and $\mathrm{ord}_{S_i}(\varpi) = m$. Therefore we obtain the last assertion. Furthermore,

$$K_{\tilde{X}} = \mu^* K_X - R_\mu = \mu^*(K_X + \Delta)$$

is a principal divisor. □

Lemma 2.2.20. *Let $f\colon X \longrightarrow X$ be a surjective endomorphism of a normal variety X and Σ a finite set of the prime divisors on X. Assume that for any $E \in \Sigma$ and an irreducible component F of $f^{-1}(E)$, we have $F \in \Sigma$. Then for any $E \in \Sigma$, $f^{-m}(E) = E$ as sets for some positive integer m.*

Proof. Let $\Sigma = \{E_1, \ldots, E_N\}$. Since $f^{-1}(E_i)$ and $f^{-1}(E_j)$ have no common irreducible component, we have $f^{-1}(E_i)$ is irreducible for all i. It means that f^{-1} induces the one-to-one corresponding of Σ. In particular, for some positive integer m, f^{-m} is identity. □

Proposition 2.2.21 (cf. Proposition 2.3.26). *Let X be a normal $\mathbb{Q}$-factorial projective variety admitting an int-amplified endomorphism f and Δ an effective $\mathbb{Q}$-Weil divisor on X such that (X, Δ) is a klt pair such that*

$$R_f + (\Delta - f^*\Delta) \geqslant 0$$

and $K_X + \Delta \sim_{\mathbb{Q}} 0$. Then there exits a finite morphism $\mu\colon A \longrightarrow X$ of (X, Δ) from an abelian variety A such that $\mu^*(K_X + \Delta)$ is a principal divisor. Furthermore, if $\varphi\colon X \longrightarrow X$ is a surjective endomorphism of X such that

$$R_\varphi + (\Delta - \varphi^*\Delta) \geqslant 0,$$

then φ lifts to A.

Proof. First, we prove that Δ has standard coefficients. Let Σ be the set of all prime divisors E on X such that $\mathrm{ord}_E(\Delta)$ is not standard. Since 0 is standard, Σ is a finite set. We take $E \in \Sigma$ and an irreducible component F of $f^{-1}(E)$.

Suppose that $F \notin \Sigma$, then $\mathrm{ord}_F(\Delta) = \frac{m-1}{m}$ for some positive integer m. We note that

$$0 \leqslant R_\Delta := R_f + \Delta - f^*\Delta \sim K_X + \Delta - f^*(K_X + \Delta) \sim_{\mathbb{Q}} 0,$$

hence, we have $R_\Delta = 0$. In particular, we have

$$f^*(K_X + \Delta) \sim K_X + \Delta$$

and

$$0 = \mathrm{ord}_F(R_f + \Delta - f^*\Delta) = r - 1 + \frac{m-1}{m} - r \cdot \mathrm{ord}_E(\Delta),$$

where $r = \mathrm{ord}_F(f^*(E))$. It means that

$$\mathrm{ord}_E(\Delta) = \frac{rm-1}{rm},$$

it is contradiction to $E \in \Sigma$.

Hence F is contained in Σ and Σ satisfies the assumption of Lemma 2.2.20. Next suppose that there exists an element $E \in \Sigma$. Then we have $f^{-l}(E) = E$ as sets for some positive integer l. Therefore, we have

$$0 = \mathrm{ord}_E(R_{f^l} + \Delta - (f^l)^*\Delta) = r' - 1 + \mathrm{ord}_E(\Delta) - r' \,\mathrm{ord}_E(\Delta),$$

where $(f^l)^*E = r'E$. Since f^l is int-amplified, r' is larger than one by [76, Theorem 3.3], so we have $\mathrm{ord}_E(\Delta) = 1$. However, it is contradiction to the fact that (X, Δ) is klt. In conclusion, we obtain $\Sigma = \varnothing$ and Δ has standard coefficients.

Thus, we obtain a finite surjective morphism $\mu' \colon \widetilde{X} \longrightarrow X$ such that $\mu'^*(K_X + \Delta) \sim 0$ and $K_{\widetilde{X}} \sim 0$ by Lemma 2.2.19. Furthermore, since

$$\mu'^* f^*(K_X + \Delta) \sim \mu'^*(K_X + \Delta) \sim 0,$$

f lifts to $\widetilde{X}$ as an int-amplified endomorphism by Lemma 2.2.19 and Proposition 2.2.14. Since (X, Δ) is klt, $\widetilde{X}$ is also klt. By Theorem 2.2.16, $\widetilde{X}$ has a cover by an abelian variety A.

Next, let $\varphi \colon X \longrightarrow X$ be a surjective endomorphism such that

$$R_\varphi + (\Delta - \varphi^*\Delta) \geqslant 0,$$

then we have $\varphi^*(K_X + \Delta) \sim K_X + \Delta$. By a similar argument as above, φ lifts to $\widetilde{X}$. By Remark 2.2.17, it lifts to A. □

Remark 2.2.22. In subsection 1.4.3, we reformulate Proposition 2.2.21 by using the notion of pairs with respect to an endomorphism and quasi-étale covers with respect to pairs.

2.3 Proof of Theorem A

In this section, every variety is defined over an algebraically closed field of characteristic zero.

2.3.1 Pairs with respect to an endomorphism

In this section, we study equivariant Mori fiber spaces. First we introduce the notion of the pair with respect to surjective endomorphisms. In Proposition 3.8, we construct such a pair on base varieties of equivariant Mori fiber spaces.

Definition 2.3.1. Let $f\colon X \longrightarrow X$ be a surjective endomorphism of a normal variety X. Then (X, Δ) is called an *f-pair* if

1. Δ is an effective $\mathbb{Q}$-Weil divisor, and
2. $R_\Delta := R_f + \Delta - f^*\Delta \geqslant 0$.

Remark 2.3.2. Let $f\colon X \longrightarrow X$ be a surjective endomorphism of a normal variety X.

- $(X, 0)$ is an f-pair.
- If (X, Δ) is an f-pair, then

$$R_\Delta = R_f + \Delta - f^*\Delta \sim K_X + \Delta - f^*(K_X + \Delta).$$

  Furthermore, for every positive integer m, we have

$$R_{\Delta,m+1} := R_{f^{m+1}} + \Delta - (f^{m+1})^*\Delta = R_\Delta + f^* R_{\Delta,m}.$$

  In particular, (X, Δ) is an f^m-pair for all m.

Example 2.3.3. Let E be an elliptic curve and $[m]$ a multiplication by m for all integers m. Since $[m]$ is $[-1]$-equivariant, we obtain the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "\mu"] \arrow[d, "{[m]}"'] & \mathbb{P}^1 \arrow[d, "h"] \\ E \arrow[r, "\mu"] & \mathbb{P}^1, \end{tikzcd}$$

where μ is the quotient map by $[-1]$ and h is the endomorphism induced by $[m]$. Let $Q_1, \ldots, Q_4$ be the 2-torsion points on E and $P_1 = \mu(Q_1), \ldots, P_4 = \mu(Q_4)$. Let

$$\Delta = \frac{1}{2}(P_1 + P_2 + P_3 + P_4),$$

then $(\mathbb{P}^1, \Delta)$ is an h-pair (see Example 2.3.10 and Example 2.3.25).

Proposition 2.3.4. *Let $f\colon X \longrightarrow X$ be an int-amplified endomorphism of a normal projective variety X and (X, Δ) an f-pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then $-(K_X + \Delta)$ is numerically equivalent to an effective $\mathbb{Q}$-Weil divisor.*

Proof. It follows from Proposition 2.2.14 and

$$0 \leqslant R_{\Delta} \sim K_X + \Delta - f^*(K_X + \Delta)$$

□

Remark 2.3.5. Let $f\colon X \longrightarrow X$ be an int-amplified endomorphism of a normal projective variety X. Then an f-pair (X, Δ) is valuative log canonical defined in [98], and the proof is similar to the proof of [98, Theorem 1.4]. In particular, if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then (X, Δ) is log canonical.

In order to introduce very important f-pairs, we define the following fiber spaces of pairs.

Definition 2.3.6. A morphism $\pi\colon (X, \Delta) \longrightarrow (Y, \Gamma)$ of pairs is called a *Mori fiber space of canonical bundle formula type* if

1. X is a normal $\mathbb{Q}$-factorial projective variety and Δ is an effective $\mathbb{Q}$-Weil divisor on X such that (X, Δ) is klt,
2. $\pi\colon X \longrightarrow Y$ is a $(K_X + \Delta)$-Mori fiber space, and
3. for any prime divisor E on Y, Γ satisfies
$$\mathrm{ord}_E(\Gamma) = \frac{m_E - 1 + \mathrm{ord}_F(\Delta)}{m_E},$$
where F is a prime divisor on X satisfying $\pi^*E = m_E F$ for some positive integer m_E.

Remark 2.3.7.

- If Y is a point, the third condition is always satisfied.
- Since π is a $(K_X + \Delta)$-Mori fiber space, we can take m_E and F as in Definition 2.3.6 (see [99, Lemma 4.10]).

Proposition 2.3.8. *Let $\pi\colon (X, \Delta) \longrightarrow (Y, \Gamma)$ be a Mori fiber space of canonical bundle formula type. Then (Y, Γ) is klt.*

Proof. Since $-(K_X + \Delta)$ is π-ample, we can take an ample Cartier divisor H on Y such that $-(K_X + \Delta) + \pi^*H$ is ample $\mathbb{Q}$-Cartier. There exists $0 \leqslant D \sim_{\mathbb{Q}} -(K_X + \Delta) + \pi^*H$ such that $(X, \Delta + D)$ is klt. By the Ambro's canonical bundle formula [2, Theorem 4.1], (Y, B) is klt, where B is an effective $\mathbb{Q}$-Weil divisor satisfying

$$\mathrm{ord}_E(B) = 1 - \mathrm{lct}_{\eta_E}(X, \Delta + D; \pi^*E)$$

for any prime divisor E with generic point η_E. Let $\pi^*E = m_E F$ for some positive integer m_E and prime divisor F, then we have

$$\mathrm{lct}_{\eta_E}(X, \Delta + D; \pi^*E) \leqslant \frac{1 - \mathrm{ord}_F(\Delta + D)}{m_E} \leqslant \frac{1 - \mathrm{ord}_F(\Delta)}{m_E}.$$

In particular, we have

$$\mathrm{ord}_E(B) \geqslant \frac{m_E - 1 + \mathrm{ord}_F(\Delta)}{m_E} = \mathrm{ord}_E(\Gamma).$$

It means that $B \geqslant \Gamma$, so (Y, Γ) is also klt. □

The following proposition gives very important pairs.

Proposition 2.3.9. *We consider the following commutative diagram*

$$\begin{array}{ccc} (X, \Delta) & \xrightarrow{\pi} & (Y, \Gamma) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (X, \Delta) & \xrightarrow[\pi]{} & (Y, \Gamma), \end{array}$$

where π is a Mori fiber space of canonical bundle formula type, f, g are surjective endomorphisms and (X, Δ) is an f-pair. Then

- *(Y, Γ) is a g-pair, and*
- *$R_\Delta - \pi^* R_\Gamma$ is effective and has no vertical components of π, that is, for every prime divisor F on X with $\pi(F) \neq Y$, we have $\mathrm{ord}_F(R_\Delta - \pi^* R_\Gamma) = 0$.*

Proof. We take a prime divisor E on Y, then we have $\pi^*E = m_E F$ for some positive integer m_E and prime divisor F on X. If the second assertion holds, then

$$m_E \,\mathrm{ord}_E(R_\Gamma) = \mathrm{ord}_F(\pi^* R_\Gamma) = \mathrm{ord}_F(R_\Delta) \geqslant 0,$$

so the first assertion holds. Therefore, it is enough to show the second assertion. Let

$$g^*E = a_1 E_1 + \cdots + a_r E_r,$$

and

$$f^*F = b_1 F_1 + \cdots + b_r F_r,$$

where all a_i, b_i are positive integers and E_i, F_i are prime divisors with $\pi^*E_i = m_{E_i} F_i$ for some positive integer m_{E_i}. Since $\pi^* g^* E = f^* \pi^* E$, we have $m_E b_i = a_i m_{E_i}$ for all i. Therefore, we have

$$\begin{aligned} \mathrm{ord}_{E_i}(R_\Gamma) &= \mathrm{ord}_{E_i}(R_g + \Gamma - g^*\Gamma) \\ &= a_i - 1 + \frac{m_{E_i} - 1 + \mathrm{ord}_{F_i}(\Delta)}{m_{E_i}} - a_i \frac{m_E - 1 + \mathrm{ord}_F(\Delta)}{m_E} \\ &= \frac{-1 + \mathrm{ord}_{F_i}(\Delta) + b_i - b_i \,\mathrm{ord}_F(\Delta)}{m_{E_i}} \\ &= \frac{\mathrm{ord}_{F_i}(R_\Delta)}{m_{E_i}}, \end{aligned}$$

it means that $\mathrm{ord}_{F_i}(R_\Delta - \pi^* R_\Gamma) = 0$. □

Example 2.3.10. In [72, Section 7], we give the following commutative diagram

$$\begin{tikzcd} g \circlearrowright Y \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{\mu}"'] & E \circlearrowleft [m] \arrow[d, "\mu"] \\ f \circlearrowright X \arrow[r, "\pi"'] & \mathbb{P}^1 \circlearrowleft h, \end{tikzcd}$$

where Y is a ruled surface over E, $\widetilde{\mu}$ is quasi-étale, π is a Mori fiber space, E, μ, and h are in Example 2.3.3. Then $\pi\colon (X, 0) \longrightarrow (\mathbb{P}^1, \Delta)$ is a Mori fiber space of canonical bundle formula type, where Δ is in Example 2.3.3. In particular, $(\mathbb{P}^1, \Delta)$ is an h-pair by Proposition 2.3.9.

2.3.2 Construction of the tower of Mori fiber spaces

In this section, we study Fano type assuming the existence of int-amplified endomorphisms. Corollary 2.3.13 means that if the variety has an int-amplified endomorphism and on the tower of Mori fiber spaces of canonical bundle formula type, then it is of Fano type over the bottom variety. Using this corollary, we replace a sequence of steps of MMP with a tower of Mori fiber spaces of canonical bundle formula type (see Theorem 2.3.21).

Lemma 2.3.11. *Let X be a normal $\mathbb{Q}$-factorial projective variety and Δ an effective $\mathbb{Q}$-Weil divisor. We consider the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "\pi"'] & Y, \end{tikzcd}$$

where π is a $(K_X + \Delta)$-negative extremal ray contraction, f, g are int-amplified endomorphisms and (X, Δ) is an f-pair. Then R_Δ is π-ample.

Proof. Since $N^1(X/Y)$ is 1-dimensional, there exists a positive number a such that $f^*|_{N^1(X/Y)} = a \cdot \mathrm{id}_{N^1(X/Y)}$. Hence we have

$$R_\Delta \sim K_X + \Delta - f^*(K_X + \Delta) \equiv_Y (1-a)(K_X + \Delta).$$

Since $-(K_X + \Delta)$ is π-ample, it is enough to show $a > 1$. By the definition of int-amplified endomorphisms, there exists an ample Cartier divisor H on X such that $f^*H - H$ is also ample. On the other hand, $f^*H - H \equiv_Y (a-1)H$, so we have $a > 1$. $\square$

Lemma 2.3.12. *We consider the following commutative diagram*

$$\begin{tikzcd} (X, \Delta) \arrow[r, "\pi"] \arrow[d, "f"'] & (Y, \Gamma) \arrow[r, "b"] \arrow[d, "g"] & B \arrow[d, "h"] \\ (X, \Delta) \arrow[r, "\pi"'] & (Y, \Gamma) \arrow[r, "b"'] & B, \end{tikzcd}$$

where π is a Mori fiber space of canonical bundle formula type, f, g, h are int-amplified endomorphisms, b is a surjective morphism to a normal projective variety B and (X, Δ) is an f-pair. Assume that Y is of Fano type over B and R_Γ contains an effective b-ample $\mathbb{Q}$-Cartier divisor H on Y. Then

- *R_Δ contains an effective $(b \circ \pi)$-ample $\mathbb{Q}$-Cartier divisor, and*
- *(X, Δ) is of Fano type over B.*

Proof. By Lemma 2.3.11, there exists a positive integer m such that

$$H_X := \frac{1}{m}(R_\Delta - \pi^* H) + \pi^* H = \frac{1}{m} R_\Delta + (1 - \frac{1}{m})\pi^* H$$

is $(b \circ \pi)$-ample. By Proposition 2.3.9, we have

$$0 \leqslant \pi^* H \leqslant \pi^* R_\Gamma \leqslant R_\Delta.$$

In particular, we have $0 \leqslant H_X \leqslant R_\Delta$, and we obtain the first assertion.

Next, we prove the second assertion. First, we prove the following claim.

Claim. There exists a positive integer m such that $-(K_X + \Delta) + (f^m)^* \pi^* H$ is $(b \circ \pi)$-ample.

Proof of Claim. Since $-(K_X + \Delta)$ is π-ample, there exists a positive integer k such that $-(K_X + \Delta) + k\pi^* H$ is $(b \circ \pi)$-ample. Since we have

$$\begin{aligned} -(K_X + \Delta) + (f^m)^* \pi^* H &= -(K_X + \Delta) + k\pi^* H + ((f^m)^* \pi^* H - k\pi^* H) \\ &= -(K_X + \Delta) + k\pi^* H + \pi^*((g^m)^* H - kH), \end{aligned}$$

it is enough to show that there exists a positive integer m such that $(g^m)^* H - kH$ is b-ample. Since Y is of Fano type over B, the number of the extremal rays of $\overline{NE}(Y/B)$ is finite by the cone theorem (cf. [63, Theorem 3.7]). Let $R_1, \ldots, R_N$ be all extremal rays of $\overline{NE}(Y/B)$. Replacing g by some iterate of g, we may assume that $g_* R_i = R_i$ for all i. Let $v_i \in R_i \setminus \{0\}$, then $g_*(v_i) = a_i v_i$ for some positive number a_i for all i. By the definition of int-amplified endomorphisms, there exists an ample Cartier divisor A on Y such that $g^* A - A$ is ample. Then we have

$$0 < (g^* A - A) \cdot v_i = (a_i - 1)(A \cdot v_i),$$

so we obtain $a_i > 1$ for all i. In particular, for enough large m, we have

$$((g^m)^* H - kH) \cdot v_i = (a_i^m - k)(H \cdot v_i) > 0,$$

and it means that $(g^m)^* H - kH$ is b-ample.

By the above claim, we may assume that $-(K_X + \Delta) + f^* \pi^* H$ is $(b \circ \pi)$-ample, replacing f by some iterate of f. In particular, there exists a Weil divisor $D \sim_{\mathbb{Q},B} -(K_X + \Delta) + f^* \pi^* H$ such that $(X, \Delta + D)$ is klt. Next, we define a $\mathbb{Q}$-Weil divisor D_1 as

$$D_1 = d^{-1} f_*(R_f + \Delta + D) - \Delta,$$

where $d = \deg(f)$. Then we have

$$\begin{aligned} D_1 = d^{-1}f_*(R_f + \Delta + D) - \Delta &= d^{-1}f_*(R_\Delta + f^*\Delta + D) - \Delta \\ &= d^{-1}f_*(R_\Delta + D) \geqslant 0, \end{aligned}$$

and

$$D_1 + \Delta = d^{-1}f_*(R_f + \Delta + D).$$

Since this construction is the same as the construction in [35, Lemma 1.1], $(X, \Delta + D_1)$ is klt and we have

$$f^*(K_X + \Delta + D_1) \equiv K_X + \Delta + D \equiv_B f^*\pi^*H$$

by an argument similar to the proof of [35]. In particular, we have

$$K_X + \Delta + D_1 \equiv_B \pi^*H.$$

Next, we construct D_2 by the same way, that is,

$$D_2 = d^{-1}f_*(R_f + \Delta + D_1) - \Delta = d^{-1}f_*(R_\Delta + D_1).$$

Then D_2 is effective, $(X, \Delta + D_2)$ is klt and

$$K_X + \Delta + D_2 \equiv_B d^{-1}f_*\pi^*H$$

by the same way as above. By the construction, we have

$$D_2 \geqslant d^{-1}f_*R_\Delta \geqslant d^{-1}f_*\pi^*H.$$

Let $\Omega = D_2 - d^{-1}f_*\pi^*H$, then $(X, \Delta + \Omega)$ is klt and we have

$$K_X + \Delta + \Omega \equiv_B d^{-1}f_*\pi^*H - d^{-1}f_*\pi^*H = 0.$$

Therefore it is enough to show that $-(K_X + \Delta)$ is $(b \circ \pi)$-big by Proposition 2.2.3. Since R_Δ contains $(b\circ\pi)$-ample $\mathbb{Q}$-Cartier divisor, R_Δ is $(b\circ\pi)$-big. Since $-(K_X+\Delta)$ is pseudo-effective by Proposition 2.3.4,

$$-f^*(K_X + \Delta) \sim R_\Delta - (K_X + \Delta)$$

is also $(b \circ \pi)$-big. In particular, $-(K_X + \Delta)$ is $(b \circ \pi)$-big, and we obtain Lemma 2.3.12. ☐

Corollary 2.3.13. *We consider the following sequence*

$$(X_0, \Delta_0) \xrightarrow{\pi_0} (X_1, \Delta_1) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{r-1}} (X_r, \Delta_r) \xrightarrow{\rho} B,$$

where

- $\pi_i \colon (X_i, \Delta_i) \longrightarrow (X_{i+1}, \Delta_{i+1})$ *is a Mori fiber space of canonical bundle formula type for* $0 \leqslant i \leqslant r - 1$,

- $\rho\colon X_r \longrightarrow B$ *is a* $(K_{X_r} + \Delta_r)$*-negative extremal ray contraction, and*

- X_0 *has an int-amplified endomorphism* f_0 *such that* (X_0, Δ_0) *is an* f_0*-pair.*

Then (X_0, Δ_0) *is of Fano type over* B.

Proof. By Theorem 2.2.15, we may assume that the above sequence is f_0-equivariant, replacing f_0 by some iterate. By Proposition 2.3.9, (X_i, Δ_i) is an f_i-pair, where f_i is the endomorphism of X_i induced by f_0. By Lemma 2.3.11, R_{Δ_r} is ρ-ample. Since $-(K_{X_r} + \Delta_r)$ is ρ-ample and (X_r, Δ_r) is klt, (X_r, Δ_r) is of Fano type over B. By using Lemma 2.3.12 inductively, we obtain R_{Δ_i} contains an effective $\mathbb{Q}$-Cartier divisor which is ample over B and (X_i, Δ_i) is of Fano type over B for all i. In conclusion, (X_0, Δ_0) is of Fano type over B. □

Lemma 2.3.14. [27, Lemma 3.3, 3.4] *Let* X *be a normal* $\mathbb{Q}$*-factorial projective variety and* Δ *an effective* $\mathbb{Q}$*-Weil divisor. Let* $\pi\colon X \longrightarrow Y$ *be a Mori fiber space and* $\rho\colon Y \longrightarrow W$ *an extremal ray contraction which is birational. Assume that* (X, Δ) *is of Fano type over* W*. Then we have the following commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\mu_X"] \arrow[d, "\pi"'] & & X' \arrow[d, "\pi'"] \\ Y \arrow[rr, dashed, "\mu_Y"] \arrow[dr, "\rho"'] & & Y' \arrow[dl] \\ & W, & \end{tikzcd}$$

such that μ_Y *is a flip of* ρ *or a divisorial contraction when* $\mu_Y = \rho$, μ_X *is a birational map obtained by a MMP over* W *for some divisor on* X*, and* π' *is* $(K'_X + \Delta')$*-Mori fiber space, where* $\Delta' = (\mu_X)_*\Delta$.

Remark 2.3.15. In [27], they only deal with K_Y-flips and K_Y-divisorial contractions. However, Lemma 2.3.14 follows from the same argument as [27, Lemma 3.3, 3.4] except for π'-ampleness of $-(K_{X'} + \Delta')$. In our case, since $-(K_X + \Delta)$ is big over W, $-(K_{X'} + \Delta')$ is also big over W. Since the relative Picard rank of X' over Y' is equal to one, $-(K_{X'} + \Delta')$ is ample over Y'.

Definition 2.3.16. A birational map $\mu\colon (Y, \Gamma) \dashrightarrow (Y'\Gamma')$ of pairs is called a *birational contraction over towers of Mori fiber spaces* if we have a following commutative diagram

$$\begin{tikzcd}
(Y,\Gamma) \arrow[rr, dashed, "\mu"] \arrow[d, "\pi_0"'] & & (Y',\Gamma') \arrow[d, "\pi'_0"] \\
(Z_1,\Omega_1) \arrow[d, "\pi_1"'] & & (Z'_1,\Omega'_1) \arrow[d, "\pi'_1"] \\
\vdots \arrow[d, "\pi_{r-1}"'] & & \vdots \arrow[d, "\pi'_{r-1}"] \\
(Z_r,\Omega_r) \arrow[rr, dashed, "\mu_r"] \arrow[dr, "\rho"'] & & (Z'_r,\Omega'_r) \arrow[dl] \\
& B, &
\end{tikzcd}$$

where,

- π_i, π'_i are Mori fiber spaces of canonical bundle formula type for all $0 \leqslant i \leqslant r-1$,
- ρ is a $(K_{Z_r} + \Omega_r)$-negative extremal ray contraction which is birational,
- μ_r is a flip of ρ or a divisorial contraction when $\mu_Y = \rho$,
- μ is a birational map which is a composition of steps of MMP over B, and
- $\Gamma' = \mu_* \Gamma$.

Remark 2.3.17. Divisorial contractions and flips obtained by a $(K_Y + \Gamma)$-negative extremal ray contraction are birational contractions over towers of Mori fiber spaces.

Lemma 2.3.18. *Let $\pi\colon (X,\Delta) \longrightarrow (Y,\Gamma)$ be a Mori fiber space of canonical bundle formula type and $\mu_Y\colon (Y,\Gamma) \dashrightarrow (Y',\Gamma')$ is a birational contraction over towers of Mori fiber spaces. Assume that X has an int-amplified endomorphism f such that (X,Δ) is an f-pair. Then we obtain a following commutative diagram*

$$\begin{tikzcd}
(X,\Delta) \arrow[r, dashed, "\mu_X"] \arrow[d, "\pi"'] & (X',\Delta') \arrow[d, "\pi'"] \\
(Y,\Gamma) \arrow[r, dashed, "\mu_Y"'] & (Y',\Gamma'),
\end{tikzcd}$$

where μ_X is a birational contraction over towers of Mori fiber spaces and π' is a Mori fiber space of canonical bundle formula type. Furthermore, f^m induces an int-amplified endomorphism f' of X' such that (X',Δ') is an f'-pair.

Proof. By the definition of birational contractions over towers of Mori fiber spaces, we obtain the following diagram

$$\begin{tikzcd}
(Y,\Gamma) \arrow[rr, dashed, "\mu_Y"] \arrow[d, "\pi_0"'] & & (Y',\Gamma') \arrow[d, "\pi'_0"] \\
(Z_1,\Omega_1) \arrow[d, "\pi_1"'] & & (Z'_1,\Omega'_1) \arrow[d, "\pi'_1"] \\
\vdots \arrow[d, "\pi_{r-1}"'] & & \vdots \arrow[d, "\pi'_{r-1}"] \\
(Z_r,\Omega_r) \arrow[rr, dashed, "\mu_r"] \arrow[dr, "\rho"'] & & (Z'_r,\Omega'_r) \arrow[dl] \\
& B &
\end{tikzcd}$$

as in Definition 2.3.16. Since we obtain the following sequence

$$(X,\Delta) \xrightarrow{\pi} (Y,\Gamma) \xrightarrow{\pi_0} \cdots \xrightarrow{\pi_{r-1}} (Z_r,\Omega_r) \xrightarrow{\rho} B,$$

and this sequence satisfies the assumption of Corollary 2.3.13, (X,Δ) is of Fano type over B. Since μ_Y is a composition of flips and divisorial contractions, we can take the first step of μ_Y as follows,

$$\begin{tikzcd}
Y \arrow[rr, dashed, "\mu_{Y_1}"] \arrow[dr, "\rho_1"'] & & Y_1 \arrow[dl] \\
& W, &
\end{tikzcd}$$

where ρ_1 is an extremal ray contraction over B and μ_{Y_1} is a flip of ρ_1 or divisorial contraction when $\rho_1 = \mu_{Y_1}$. Since ρ_1 is a morphism over B and (X,Δ) is of Fano type over B, (X,Δ) is also of Fano type over W. By Lemma 2.3.14, we obtain the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, dashed, "\mu_{X_1}"] \arrow[d, "\pi"'] & & X_1 \arrow[d, "\pi_{X_1}"] \\
Y \arrow[rr, dashed, "\mu_{Y_1}"] \arrow[dr, "\rho"'] & & Y_1 \arrow[dl] \\
& W &
\end{tikzcd}$$

as in Lemma 2.3.14. Let $\Delta_1 = (\mu_{X_1})_*\Delta$, then (X_1,Δ_1) is of Fano type over B by Proposition 2.2.4. Repeating the construction, we obtain the following commutative diagram

$$\begin{tikzcd}
(X,\Delta) \arrow[r, dashed, "\mu_X"] \arrow[d, "\pi"'] & (X',\Delta') \arrow[d, "\pi'"] \\
(Y,\Gamma) \arrow[r, dashed, "\mu_Y"'] & (Y',\Gamma'),
\end{tikzcd}$$

such that μ_X is a birational map which is a composition of steps of MMP over B and (X', Δ') is of Fano type over Y' by the construction, where $\Delta' = (\mu_X)_*\Delta$. In particular f^m induces an int-amplified endomorphism f' of X' for some m.

Finally, we prove that (X', Δ') is an f'-pair and $\pi' \colon (X', \Delta') \longrightarrow (Y, \Gamma)$ is a Mori fiber space of canonical bundle formula type. First, since we have

$$R_{\Delta'} = (\mu_X)_* R_{\Delta,m} = (\mu_X)_*(R_{f^m} + \Delta - (f^m)^*\Delta) \geqslant 0,$$

(X', Δ') is an f'-pair. Next, we take a prime divisor E' on Y' and $\pi'^*E' = m_{E'}F'$ for some $m_{E'}$ and prime divisor F'. Since μ_X and μ_Y are birational contractions, we can consider the strict transform E and F of E' and F', respectively. Since μ_X and μ_Y are isomorphism on the generic points of F and E, respectively, we have $\pi^*E = m_{E'}F$, $\operatorname{ord}_F(\Delta) = \operatorname{ord}_{F'}(\Delta')$ and $\operatorname{ord}_E(\Gamma) = \operatorname{ord}_{E'}(\Gamma')$. Hence π' is a Mori fiber space of canonical bundle formula type, since so is π. $\square$

Definition 2.3.19. A sequence of birational maps and morphisms of pairs

$$\begin{tikzcd}
(X,\Delta) \arrow[r, dashed, "\sigma_0"] & (X',\Delta') \arrow[d, "\pi_0"] \\
& (X_1,\Delta_1) \arrow[r, dashed, "\sigma_1"] & (X_1',\Delta_1') \arrow[d, "\pi_1"] \\
& & (X_2,\Delta_2) \arrow[r, dashed, "\sigma_1"] & {} \\
& & & \ddots \\
& & & {} \arrow[r, dashed, "\sigma_r"] & (X_r',\Delta_r') \arrow[d, "\pi_r"] \\
& & & & (W,\Delta_W),
\end{tikzcd}$$

is called *sequence of steps of MMP of canonical bundle formula type* if

- X is a normal $\mathbb{Q}$-factorial projective variety, Δ is an effective $\mathbb{Q}$-Weil divisor such that (X, Δ) is klt,
- $X_i \dashrightarrow X_i' \longrightarrow X_{i+1}$ is obtained by $(K_{X_i} + \Delta_i)$-MMP for all $0 \leqslant i \leqslant r$, and
- π_i is a Mori fiber space of canonical bundle formula type for all $0 \leqslant i \leqslant r$, where $\Delta_i' = (\sigma_i)_*\Delta_i$.

Furthermore, the above sequence is called *maximal* if $K_W + \Delta_W$ is pseudo-effective.

Remark 2.3.20. If $K_W + \Delta_W$ is not pseudo-effective, then we can run MMP for (W, Δ_W) and we obtain a Mori fiber space. In particular, there exists a maximal sequence of steps of MMP of canonical bundle formula type.

If X has an int-amplified endomorphism, then for every surjective endomorphism f, the above sequence is f^m-equivariant for some positive integer m by Theorem 2.2.15. We denote the induced endomorphisms on X_i, X_i' and W by f_i, f_i' and g, respectively. If (X, Δ) is an f-pair, then $(X_i, \Delta_i), (X_i', \Delta_i')$ and (W, Δ_W) are an f_i-pair, an f_i'-pair, a g-pair, respectively by Proposition 2.3.9 and the proof of Lemma 2.3.18. In particular, $-(K_W + \Delta_W)$ is pseudo-effective by Lemma 2.2.14. If we further assume that the sequence is maximal, then $K_W + \Delta_W \sim_{\mathbb{Q}} 0$.

Theorem 2.3.21. *Let $(X, \Delta) \dashrightarrow \cdots \dashrightarrow (W, \Delta_W)$ be a sequence of steps of MMP of canonical bundle formula type and f be an int-amplified endomorphism of X such that (X, Δ) is an f-pair. Then we obtain a sequence of birational contractions over towers of Mori fiber spaces*

$$\mu\colon (X, \Delta) \dashrightarrow (X', \Delta')$$

and a sequence of Mori fiber spaces of canonical bundle formula type

$$(X', \Delta') \longrightarrow (X_1', \Delta_1') \longrightarrow \cdots \longrightarrow (X_r', \Delta_r') \longrightarrow (W, \Delta_W).$$

Furthermore, (X', Δ') is of Fano type over W.

Proof. We prove Theorem 2.3.21 by the induction on the length of the sequences of steps of MMP of canonical bundle formula type. We take the first step $\pi\colon (X, \Delta) \dashrightarrow (Y, \Gamma)$ in the above sequence, in particular, π is a flip, a divisorial contraction or a Mori fiber space obtained by a $(K_X + \Delta)$-negative extremal ray contraction. By the induction hypothesis, we obtain a sequence of birational contractions over towers of Mori fiber spaces

$$\mu_Y\colon (Y, \Gamma) \dashrightarrow (Y', \Gamma')$$

and a sequence of Mori fiber spaces of canonical bundle formula type

$$(Y', \Gamma') \longrightarrow (Y_1', \Gamma_1') \longrightarrow \cdots \longrightarrow (Y_{r'-1}, \Gamma_{r-1}') \longrightarrow (W, \Delta_W).$$

If π is a birational map, $\mu_Y \circ \pi$ is also a composition over towers of canonical bundle formula type. Hence we may assume that $\pi\colon (X, \Delta) \longrightarrow (Y, \Gamma)$ is a Mori fiber space of canonical bundle formula type. By repeatedly applying Lemma 2.3.18, we obtain the following commutative diagram

$$\begin{tikzcd}
(X, \Delta) \arrow[r, dashed, "\mu"] \arrow[d, "\pi"'] & (X', \Delta') \arrow[d, "\pi'"] \\
(Y, \Gamma) \arrow[r, dashed, "\mu_Y"'] & (Y', \Gamma'),
\end{tikzcd}$$

where μ is a sequence of birational contractions over towers of canonical bundle formula type and π' is a Mori fiber space of canonical bundle formula type. Hence, we obtain the following composition of Mori fiber spaces of canonical bundle formula type

$$(X', \Delta') \longrightarrow (Y', \Gamma') \longrightarrow (Y_1', \Gamma_1') \longrightarrow \cdots \longrightarrow (Y_{r'-1}, \Gamma_{r-1}') \longrightarrow (W, \Delta_W).$$

By Lemma 2.2.15, μ is f^m-equivariant for some m, so f^m induces an int-amplified endomorphism f' of X' such that (X', Δ') is an f'-pair. By Corollary 2.3.13, we obtain (X', Δ') is of Fano type over W. $\square$

2.3.3 Maximal sequence of steps of MMP

In this section, we construct $\widetilde{X}$ and A in Theorem A by studying maximal sequences of steps of MMP of canonical bundle formula type. First, we construct A as a covering of the output of a sequence of steps of MMP. Lifting this covering, we construct $\widetilde{X}$.

Definition 2.3.22. A morphism $\mu\colon (\widetilde{X},\widetilde{\Delta}) \longrightarrow (X,\Delta)$ of pairs is called a quasi-étale cover of (X,Δ) if

- $\mu\colon \widetilde{X} \longrightarrow X$ is a finite surjective morphism of normal projective varieties,
- Δ is an effective $\mathbb{Q}$-Weil divisor on X, and
- $\widetilde{\Delta} = \mu^*\Delta - R_\mu$ is effective.

Remark 2.3.23. Let $\mu\colon (\widetilde{X},\widetilde{\Delta}) \longrightarrow (X,\Delta)$ be a quasi-étale cover of (X,Δ) and $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Then

- $K_{\widetilde{X}} + \widetilde{\Delta} = \mu^*(K_X+\Delta)$, and in particular, $K_{\widetilde{X}}+\widetilde{\Delta}$ is $\mathbb{Q}$-Cartier,
- (X,Δ) is klt if and only if $(\widetilde{X},\widetilde{\Delta})$ is klt by [63, Proposition 5.20], and
- if $\Delta = 0$, then μ is étale in codimension one.

Proposition 2.3.24. *We consider the following commutative diagram*

$$\begin{tikzcd} (\widetilde{X},\widetilde{\Delta}) \arrow[r, "\mu"] \arrow[d, "\widetilde{f}"'] & (X,\Delta) \arrow[d, "f"] \\ (\widetilde{X},\widetilde{\Delta}) \arrow[r, "\mu"'] & (X,\Delta), \end{tikzcd}$$

where μ is a quasi-étale cover of (X,Δ), f and $\widetilde{f}$ are surjective endomorphisms. Then we have

$$R_{\widetilde{\Delta}} = R_{\widetilde{f}} + \widetilde{\Delta} - \widetilde{f}^*\widetilde{\Delta} = \mu^*(R_f + \Delta - f^*\Delta) = \mu^* R_\Delta,$$

and in particular, (X,Δ) is an f-pair if and only if $(\widetilde{X},\widetilde{\Delta})$ is an $\widetilde{f}$-pair.

Proof. First, since $f\circ\mu = \mu\circ\widetilde{f}$, we have

$$R_{f\circ\mu} = R_\mu + \mu^*R_f = R_{\widetilde{f}} + \widetilde{f}^*R_\mu.$$

Hence we have

$$\begin{aligned} R_{\widetilde{\Delta}} &= R_{\widetilde{f}} + \widetilde{\Delta} - \widetilde{f}^*\widetilde{\Delta} \\ &= R_{\widetilde{f}} + \mu^*\Delta - R_\mu - \widetilde{f}^*(\mu^*\Delta - R_\mu) \\ &= \mu^*(R_f + \Delta - f^*\Delta) = \mu^*R_\Delta. \end{aligned}$$

□

Example 2.3.25. In Example 2.3.3, $\mu\colon (E,0) \longrightarrow (\mathbb{P}^1,\Delta)$ is a quasi-étale cover of $(\mathbb{P}^1,\Delta)$. In particular. $(\mathbb{P}^1,\Delta)$ is an h-pair by Proposition 2.3.24. Furthermore, in this case, $K_{\mathbb{P}^1}+\Delta$ is $\mathbb{Q}$-linearly trivial, so this is also an example of the following proposition.

The following Proposition is the reformulation of Proposition 2.2.21.

Proposition 2.3.26 (cf. Proposition 2.2.21, Example 2.3.25)**.** *Let X be a normal $\mathbb{Q}$-factorial projective variety admitting an int-amplified endomorphism f and Δ an effective $\mathbb{Q}$-Weil divisor on X such that (X,Δ) is a klt f-pair. Assume that $K_X+\Delta \sim_{\mathbb{Q}} 0$. Then there exits a quasi-étale cover $\mu\colon (A,0) \longrightarrow (X,\Delta)$ of (X,Δ) by an abelian variety A. Furthermore, if $\varphi\colon X \longrightarrow X$ is a surjective endomorphism of X such that (X,Δ) is a φ-pair, then φ lifts to A.*

Theorem 2.3.27 (cf. [73, Lemma 4.12])**.** *Let $(X,\Delta) \dashrightarrow \cdots \dashrightarrow (W,\Delta_W)$ be a maximal sequence of steps of MMP of canonical bundle formula type and f be an int-amplified endomorphism of X such that (X,Δ) is an f-pair. Then $\pi_X\colon X \dashrightarrow W$ is a morphism and we obtain the following commutative diagram*

$$\begin{tikzcd}
(\widetilde{X},\widetilde{\Delta}) \arrow[r, "\widetilde{\pi}"] \arrow[d, "\mu_X"'] & (A,0) \arrow[d, "\mu_W"] \\
(X,\Delta) \arrow[r, "\pi_X"'] & (W,\Delta_W),
\end{tikzcd}$$

where

- *A is an abelian variety,*
- *μ_X,μ_W are quasi-étale covers of $(X,\Delta),(W,\Delta_W)$, respectively,*
- *$\widetilde{X}$ is the normalization of the main component of $X\times_W A$, and*
- *R_Δ has no vertical components of π_X.*

Furthermore, if φ is a surjective endomorphism of X such that (X,Δ) is a φ-pair, then there exists the following commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\varphi^m"'] & \widetilde{X} \arrow[l, "\mu_X"'] \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{\varphi}"] & A \arrow[d, "\widetilde{\psi}"] \\
X & \widetilde{X} \arrow[l, "\mu_X"] \arrow[r, "\widetilde{\pi}"'] & A,
\end{tikzcd}$$

for some positive integer m, where $\widetilde{\varphi}$ and $\widetilde{\psi}$ are surjective endomorphism.

Proof. First, we prove the last assertion. Let φ be a surjective endomorphism of X such that (X,Δ) is a φ-pair. By Remark 2.3.20, φ^m induces the surjective endomorphism ψ of W such that (W,Δ_W) is a ψ-pair. By Proposition 2.3.26, ψ lifts to A denoted by $\widetilde{\psi}$. Since $\widetilde{X}$ is the normalization of the main component of $X \times_W A$, $\widetilde{X}$ has an endomorphism $\widetilde{\varphi}$ induced by φ^m, ψ and $\widetilde{\psi}$. This satisfies the conditions in Theorem 2.3.27.

Next, we prove Theorem 2.3.27 by the induction on the length of the maximal sequences of steps of MMP of canonical bundle formula type. We note that $(K_W + \Delta_W) \sim_{\mathbb{Q}} 0$ and (W, Δ_W) is a g-pair for some int-amplified endomorphism g on W by Remark 2.3.20. Hence by Proposition 2.3.26, we can construct a quasi-étale cover of (W, Δ_W) by an abelian variety A. This is the proof of the case where the length is equal to 0.

Let $\rho\colon (X, \Delta) \dashrightarrow (Y, \Gamma)$ be the first step of the above sequence. We may assume that ρ is f-equivariant replacing f by some iterate. By the induction hypothesis, we obtain the following commutative diagram

$$\begin{tikzcd}
(\widetilde{Y}, \widetilde{\Gamma}) \arrow[r, "\widetilde{\pi}_Y"] \arrow[d, "\mu_Y"'] & (A, 0) \arrow[d, "\mu_W"] \\
(Y, \Gamma) \arrow[r, "\pi_Y"'] & (W, \Delta_W),
\end{tikzcd}$$

where

- A is an abelian variety,
- μ_Y, μ_W are quasi-étale covers of $(Y, \Gamma), (W, \Delta_W)$, respectively,
- $\widetilde{Y}$ is the normalization of the main component of $Y \times_W A$, and
- R_Γ has no vertical components of π_Y.

First, we consider the case where ρ is a flip. Let $\widetilde{X}$ be the normalization of X in $K(\widetilde{Y})$. Since $\widetilde{X} \dashrightarrow \widetilde{Y}$ is isomorphism in codimension one,

$$\mu_X\colon (\widetilde{X}, \widetilde{\Delta}) \longrightarrow (X, \Delta)$$

is a quasi-étale cover of (X, Δ). In particular, $(\widetilde{X}, \widetilde{\Delta})$ is klt, so $\widetilde{X}$ has at worst rational singularities by [63, Theorem 5.22]. By [55, Lemma 8.1],

$$\widetilde{\pi}_X\colon \widetilde{X} \dashrightarrow A$$

is a morphism. By [73, Lemma 4.13], π_X is also a morphism. The other statements follow from the smallness of ρ and the proof of [73, Lemma 4.12].

Next, we consider the case where ρ is a divisorial contraction or a Mori fiber space. We obtain the following commutative diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{\rho}"] \arrow[d, "\mu_X"'] & \widetilde{Y} \arrow[r, "\widetilde{\pi}_Y"] \arrow[d, "\mu_Y"'] & A \arrow[d, "\mu_W"] \\
X \arrow[r, "\rho"'] & Y \arrow[r, "\pi_Y"'] & W,
\end{tikzcd}$$

where $\widetilde{X}$ is the normalization of the main component of $X \times_W A$.

Claim. R_Δ has no vertical components.

Proof. If ρ is a divisorial contraction, then there exists the unique exceptional prime divisor E on X. It is enough to show that $\pi_X(E) = W$. It follows from the fact that $\pi_X(E)$ is totally invariant (cf. [73, Lemma 4.12]).

If ρ is a Mori fiber space, then

$$\rho\colon (X, \Delta) \longrightarrow (Y, \Gamma)$$

is a Mori fiber space of canonical bundle formula type since ρ is in steps of MMP of canonical bundle formula type. By Proposition 2.3.9, $R_\Delta - \rho^* R_\Gamma$ has no vertical component of ρ. Since R_Γ has no vertical components of π_Y, R_Δ has no vertical component of π_X. □

Next, we prove that $\widetilde{\Delta} = (\mu_X)^*\Delta - R_{\mu_X}$ is effective, that is, μ_X is a quasi-étale cover of (X, Δ). Let Σ be the set of all prime divisors E on $\widetilde{X}$ with $\mathrm{ord}_E(\widetilde{\Delta}) < 0$. We may assume that f lifts to $\widetilde{X}$ denoted by $\widetilde{f}$ replacing f by some iterate. Suppose that $\Sigma \neq \varnothing$, and take $E \in \Sigma$ and an irreducible component F of $f^{-1}(E)$. Then $\mathrm{ord}_E(R_{\mu_X}) > 0$. By the construction of $\widetilde{X}$ (coming from base change), R_{μ_X} has only vertical components of $\widetilde{\pi}_X$. Hence E is a vertical component, so F is also a vertical component. Let

$$R_{\widetilde{\Delta}} = R_{\widetilde{f}} + \widetilde{\Delta} - \widetilde{f}^*\widetilde{\Delta}.$$

By Proposition 2.3.24, we obtain $(\mu_X)^* R_\Delta = R_{\widetilde{\Delta}}$, in particular, it is effective and has no vertical component of $\widetilde{\pi}_X$. Hence, $\mathrm{ord}_F(R_{\widetilde{\Delta}}) = 0$ and

$$\mathrm{ord}_F(\widetilde{\Delta}) \leqslant \mathrm{ord}_F(R_{\widetilde{f}} + \widetilde{\Delta}) = \mathrm{ord}_F(\widetilde{f}^*\widetilde{\Delta}) = r \cdot \mathrm{ord}_E(\widetilde{\Delta}) < 0,$$

where $r = \mathrm{ord}_F(\widetilde{f}^*(E))$. By Lemma 2.2.20, we may assume that $\widetilde{f}^*(E) = r'E$ for some $r' > 1$ replacing f by some iterate. Then we have

$$\mathrm{ord}_E(\widetilde{\Delta}) \leqslant r' \, \mathrm{ord}_E(\widetilde{\Delta}),$$

but it is contradiction to the fact that $\mathrm{ord}_E(\widetilde{\Delta}) < 0$. It means that $\Sigma = \varnothing$ and $\widetilde{\Delta}$ is effective. □

2.3.4 Structure of Fano fibrations for pairs with respect to an int-amplified endomorphism

In this section, we finish the proof of Theorem A.

Proposition 2.3.28 (cf. [35, Lemma 1.1]). *Let $\mu\colon (Y, \Gamma) \longrightarrow (X, \Delta)$ be a quasi-étale cover of (X, Δ) and $b\colon X \longrightarrow B$ a surjective morphism of projective varieties. Then (X, Δ) is of Fano type over B if and only if (Y, Γ) is of Fano type over B.*

Proof. First, we assume that (X, Δ) is of Fano type over B. Since $-(K_X + \Delta)$ is big over B, then

$$-(K_Y + \Gamma) = -\mu^*(K_X + \Delta)$$

is big over B. By Proposition 2.2.3, there exists an effective $\mathbb{Q}$-Weil divisor D on X such that $(X, \Delta + D)$ is klt and $K_X + \Delta + D \equiv_B 0$. Then $K_Y + \Gamma + \mu^* D \equiv_B 0$ and $(Y, \Gamma + \mu^* D)$ is klt. By Proposition 2.2.3, (Y, Γ) is of Fano type over B.

Next, we assume that (Y, Γ) is of Fano type over B. Then by the same argument as above, $-(K_X + \Delta)$ is big over B and we can take an effective $\mathbb{Q}$-Weil divisor D_Y on Y such that $K_Y + \Gamma + D_Y \equiv_B 0$ and $(Y, \Gamma + D_Y)$ is klt. Let

$$D_X = \deg(\mu)^{-1} \mu_*(R_\mu + \Gamma + D_Y) - \Delta.$$

Since $\Gamma = \mu^*\Delta - R_\mu$, we have

$$D_X = \deg(\mu)^{-1}\mu_* D_Y.$$

By the proof of [35, Lemma 1.1], $K_X + \Delta + D_X \equiv_B 0$ and $(X, \Delta + D_X)$ is klt, and in particular, (X, Δ) is of Fano type over B. $\square$

Proposition 2.3.29. *We consider the following commutative diagram*

$$\begin{tikzcd} f \circlearrowright X \arrow[rr, dashed, "\pi"] \arrow[dr, "b_X"'] & & Y \circlearrowleft g \arrow[dl, "b_Y"] \\ & h \circlearrowright B, & \end{tikzcd}$$

where

- X, Y, B *are normal projective varieties,*
- f, g, h *are int-amplified endomorphisms, and*
- π *is a birational map.*

Let Δ be an effective $\mathbb{Q}$-Weil divisors on X such that (X, Δ) is an f-pair and each coefficient of Δ is smaller that one. Assume that (Y, Γ) is of Fano type over B and (Y, Γ) is a g-pair, where $\Gamma = \pi_*\Delta$. Then (X, Δ) is also of Fano type over B.

Proof. We take W the normalization of the graph of π as follows,

$$\begin{tikzcd} & W \arrow[dl, "\mu_X"'] \arrow[dr, "\mu_Y"] & \\ X \arrow[rr, dashed, "\pi"'] & & Y. \end{tikzcd}$$

Then f and g induce the endomorphism f' of W such that $(W, \Omega = (\mu_X)_*^{-1}\Delta)$ is an f'-pair after iterating f'. Indeed, since $(\mu_X)_* R_\Omega = R_\Delta$ is effective, it is enough to show that $\mathrm{ord}_E(R_\Omega) \geqslant 0$ for every exceptional prime divisor E. Since we have

$$R_\Omega = R_{f'} + \Omega - (f')^*\Omega$$

and $\mathrm{ord}_E((f')^*\Omega) = 0$, we obtain $\mathrm{ord}_E(R_\Omega) \geqslant 0$. By Proposition 2.2.4, it is enough to show that (W, Ω) is of Fano type over B. Hence, we may assume that π is a morphism.

By Proposition 2.2.3, there exits an effective b_Y-big $\mathbb{Q}$-Weil divisor D_Y on Y such that $K_Y + \Gamma + D_Y \equiv_B 0$ and $(Y, \Gamma + D_Y)$ is klt. We define a $\mathbb{Q}$-Weil divisor D_X on X as

$$K_X + \Delta + D_X = \pi^*(K_Y + \Gamma + D_Y).$$

Then $K_X + \Delta + D_X \equiv_B 0$. Next, let

$$D_{Y.i} = \deg(g^i)^{-1}(g^i)_*(R_{g^i} + \Gamma + D_Y) - \Gamma$$

and

$$D_{X.i} = \deg(f^i)^{-1}(f^i)_*(R_{f^i} + \Delta + D_X) - \Delta.$$

Then by the same argument as in the proof of Lemma 2.3.12 and [35, Lemma 1.1], $D_{Y,i}$ is effective, $(Y, \Gamma + D_{Y,i})$ is klt, $K_Y + \Gamma + D_{Y,i} \equiv_B 0$ and

$$K_X + \Delta + D_{X,i} = \pi^*(K_Y + \Gamma + D_{Y,i}).$$

Claim. There exists a positive integer m such that $D_{X,m}$ is effective.

Proof of Claim. Since $\pi_* D_{X,i} = D_{Y,i}$, it is enough to show that for any exceptional prime divisor E on X such that $\mathrm{ord}_E(D_{X,m})$ is effective for some m. We fix an exceptional prime divisor E on X. We may assume that E is totally invariant under f replacing f by some iterate. Let $f^*E = rE$ for some $r > 1$ since f is int-amplified. Let $f_*E = eE$, then $re = \deg(f)$. Let

$$a = a_E(Y, \Gamma + D_Y) > 0,$$

where $a_E(Y, \Gamma + D_Y)$ is the log discrepancy of $(Y, \Gamma + D_Y)$ with respect to E. Then $\mathrm{ord}_E(\Delta + D_Y) = 1 - a$ by the definition of the log discrepancy. Then we have

$$\begin{aligned}
\mathrm{ord}_E(D_{X,i}) &= \mathrm{ord}_E(\deg(f^i)^{-1}(f^i)_*(R_{f^i} + \Delta + D_X) - \Delta) \\
&= \deg(f^i)^{-1} e^i (r^i - 1 + \mathrm{ord}_E(\Delta + D_X)) - \mathrm{ord}_E(\Delta) \\
&= \frac{r^i - a}{r^i} - \mathrm{ord}_E(\Delta) \\
&= 1 - \mathrm{ord}_E(\Delta) - \frac{a}{r^i} \geqslant 0
\end{aligned}$$

for enough large i.

By the claim, we obtain an effective $\mathbb{Q}$-Weil divisor $D_{X,m}$ such that $K_X + \Delta + D_{X,m} \equiv_B 0$ and $(X, \Delta + D_{X,m})$ is klt. It is enough to show that $D_{X,m}$ is big over B. Since we have

$$R_{\Delta,l} \sim K_X + \Delta - (f^l)^*(K_X + \Delta) \equiv_B (f^l)^* D_{X,m} - D_{X,m},$$

it is enough to show that $R_{\Delta,l} + D_{X,m}$ is big over B for some l. Since we have

$$R_\Gamma \sim K_Y + \Gamma - g^*(K_Y + \Gamma) \equiv_B g^* D_{Y,m} - D_{Y,m}$$

and $D_{Y,m} \equiv_B -(K_Y + \Gamma)$ is big over B, $R_\Gamma + D_{Y,m}$ is big over B. Hence it is enough to show that the following claim.

Claim. $R_{\Delta,l} + D_{X,m} \geqslant \pi^*(R_\Gamma + D_{Y,m})$ for some positive integer l.
Proof of Claim. Since we have

$$\pi_*(R_{\Delta,l} + D_{X,m}) \geqslant \pi_*(R_\Delta + D_{X,m}) = R_\Gamma + D_{Y,m},$$

it is enough to consider the coefficients with respect to the exceptional prime divisors. We take an exceptional prime divisor E on X and we may assume that E is totally invariant under f. Let $f^*E = rE$. Then we have

$$\begin{aligned} \mathrm{ord}_E(R_{\Delta,l} + D_{X,m}) &\geqslant \mathrm{ord}_E(R_{\Delta,l}) = \mathrm{ord}_E(R_{f^l} + \Delta - (f^l)^*\Delta) \\ &= r^l - 1 + \mathrm{ord}_E(\Delta) - r^l \,\mathrm{ord}_E(\Delta) \\ &= (r^l - 1)(1 - \mathrm{ord}_E(\Delta)) \geqslant \mathrm{ord}_E(\pi^*(R_\Gamma + D_{Y,m})) \end{aligned}$$

for large enough l, since $\mathrm{ord}_E(\pi^*(R_\Gamma + D_{Y,m}))$ does not depend on l.

□

Theorem 2.3.30. *Let X be a normal $\mathbb{Q}$-factorial projective variety admitting an int-amplified endomorphism f and Δ an effective $\mathbb{Q}$-Weil divisor on X such that (X, Δ) is a klt f-pair. Then there exist following morphisms and pairs*

$$(X, \Delta) \xleftarrow{\ \mu\ } (\widetilde{X}, \widetilde{\Delta}) \xrightarrow{\ \widetilde{\pi}\ } A,$$

such that

- *μ is a quasi-étale cover of (X, Δ),*
- *$\widetilde{\pi}$ is a fiber space,*
- *A is an abelian variety,*
- *$\widetilde{X}$ is a normal variety, and*
- *$(\widetilde{X}, \widetilde{\Delta})$ is of Fano type over A.*

In particular, $\widetilde{\pi}$ is the albanese morphism. Moreover, if φ is a surjective endomorphism of X such that (X, Δ) is a φ-pair, then we obtain the following diagram

$$\begin{tikzcd} X \arrow[d, "\varphi^m"'] & \widetilde{X} \arrow[l, "\mu"'] \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{\varphi}"'] & A \arrow[d, "\varphi_A"'] \\ X & \widetilde{X} \arrow[l, "\mu"] \arrow[r, "\widetilde{\pi}"'] & A \end{tikzcd}$$

for some m, where $\widetilde{\varphi}$ and φ_A are endomorphisms.

Proof. Let $(X,\Delta) \dashrightarrow \cdots \dashrightarrow (W,\Delta_W)$ be a maximal sequence of steps of MMP of canonical bundle formula type. By Theorem 2.3.27, we obtain the following commutative diagram

$$\begin{tikzcd} (\widetilde{X},\widetilde{\Delta}) \arrow[r, "\widetilde{\pi}"] \arrow[d, "\mu"'] & (A,0) \arrow[d, "\mu_W"] \\ (X,\Delta) \arrow[r, "\pi_X"'] & (W,\Delta_W) \end{tikzcd}$$

as in Theorem 2.3.27.

We prove that $(\widetilde{X},\widetilde{\Delta})$ is of Fano type over A. By Proposition 2.3.28, it is enough to show that (X,Δ) is of Fano type over W. By Theorem 2.3.21, we obtain a sequence of birational contractions over towers of Mori fiber spaces $(X,\Delta) \dashrightarrow (X',\Delta')$ such that (X',Δ') is of Fano type over W. By the construction, replacing f by some iterate, we obtain the following commutative diagram

$$\begin{tikzcd} f \circlearrowright X \arrow[rr, dashed] \arrow[dr] & & X' \circlearrowleft f' \arrow[dl] \\ & h \circlearrowright W. & \end{tikzcd}$$

By Proposition 2.3.29, (X,Δ) is of Fano type over W. Moreover, a general fiber of $\widetilde{\pi}$ is of Fano type, in particular, it is rationally connected by [41]. Since $\widetilde{\pi}$ is a fiber space, $\widetilde{\pi}$ is the albanese morphism. $\square$

Theorem 2.3.31 (Theorem A). *Let X be a normal $\mathbb{Q}$-factorial klt projective variety admitting an int-amplified endomorphism. Then there exists a quasi-étale finite cover $\mu\colon \widetilde{X} \longrightarrow X$ such that the albanese morphism $\mathrm{alb}_{\widetilde{X}}\colon \widetilde{X} \longrightarrow A$ is a fiber space and $\widetilde{X}$ is of Fano type over A.*

Moreover, if φ is a surjective endomorphism of X such that (X,Δ) is a φ-pair, then we obtain the following commutative diagram

$$\begin{tikzcd} X \arrow[d, "\varphi^m"'] & \widetilde{X} \arrow[l, "\mu"'] \arrow[r, "\pi"] \arrow[d, "\widetilde{\varphi}"'] & A \arrow[d, "\varphi_A"'] \\ X & \widetilde{X} \arrow[l, "\mu"] \arrow[r, "\pi"'] & A \end{tikzcd}$$

for some m, where $\widetilde{\varphi}$ and φ_A are endomorphisms.

Proof. Applying Theorem 2.3.30 for $(X,0)$, we obtain Theorem 2.3.31. We note that $(X,0)$ is a pair with respect to all surjective endomorphisms of X and quasi-étale cover of $(X,0)$ is étale in codimension one. $\square$

Remark 2.3.32. Let $\pi\colon X \longrightarrow A$ be a equivariant albanese morphism with respect to int-amplified endomorphisms such that X is of Fano type over A and smooth, then every fiber of π is of Fano type. Indeed, the locus of A where the fiber is not smooth is totally invariant under the int-amplified endomorphism g of A. Since every int-amplified endomorphism of an abelian variety has no non-trivial totally invariant closed subset, every fiber of π is smooth. Furthermore, since the locus of A where the fiber is not of Fano type is totally invariant under g, every fiber of π is of Fano type.

The following corollary gives a characterization of Fano type admitting an int-amplified endomorphism.

Corollary 2.3.33. *Let X be a normal $\mathbb{Q}$-factorial klt projective variety admitting an int-amplified endomorphism. The following conditions are equivalent to each other.*

1. *X is of Fano type.*
2. *$\pi_1^{\text{ét}}(X_{\text{sm}})$ is finite, where $\pi_1^{\text{ét}}(X_{\text{sm}})$ is the étale fundamental group of the smooth locus of X.*

Furthermore, if we further assume that X is smooth, then the following condition is also equivalent.
(3) X is rationally connected.

Proof. The condition (1) implies the condition (2) by [39, Theorem 1.13]. We assume that the condition (2). By Theorem A, we obtain the following diagram

$$X \xleftarrow{\mu} \widetilde{X} \xrightarrow{\pi} A$$

as in Theorem A. Since μ is quasi-étale, $\mu^{-1}(X_{\text{sm}}) \longrightarrow X_{\text{sm}}$ is étale by the Zariski-Nagata purity theorem [81, Theorem 41.1]. In particular, $\pi_1^{\text{ét}}(\widetilde{X}_{\text{sm}})$ is also finite. Hence A is a point, in particular, $\widetilde{X}$ is of Fano type. By Proposition 2.3.28, X is also of Fano type.

If we further assume that X is smooth rationally connected, then $\pi_1^{\text{ét}}(X)$ is trivial, so X is of Fano type. On the other hand, varieties of Fano type are rationally connected by [41]. $\square$

Corollary 2.3.33 is an affirmative answer for the following conjecture in the case where $\pi_1^{\text{ét}}(X_{\text{sm}})$ is finite and X is $\mathbb{Q}$-factorial klt.

Conjecture 2.3.34 ([16, Conjecture 1.2]). *Let X be a normal variety admitting a non-invertible polarized endomorphism. Then X is of Calabi-Yau type.*

2.4 Proof of Theorem B

In this section, every variety is defined over an algebraically closed field of characteristic zero.

2.4.1 Endomorphism on Cox rings

In this section, we recall the definition of Cox rings, and study properties of endomorphisms of them induced by endomorphisms of varieties. Lemma 2.4.5 gives a characterization of toric varieties via endomorphisms.

Definition 2.4.1. Let X be a normal projective variety such that $\mathrm{Pic}(X)$ is free. We take line bundles $L_1, \ldots L_r$ which are a basis of $\mathrm{Pic}(X)$. We define the *Cox ring* of X to be

$$\mathrm{Cox}(X) := \bigoplus_{v \in \mathbb{Z}^r} H^0(X, L^r)$$

with the multiplication induced by the product of rational functions, where

$$L^v := L_1^{\otimes n_1} \otimes \cdots \otimes L_r^{\otimes n_r}$$

for $v = (n_1, \ldots, n_r)$. The Cox ring of X is naturally graded by the Picard group of X and every homogeneous part is a k-algebra, so we regard $\mathrm{Cox}(X)$ as a graded k-algebra. Furthermore, $\mathrm{Cox}(X)$ is independent on the choice of basis.

Proposition 2.4.2. *Let X be a normal projective variety such that* $\mathrm{Pic}(X)$ *is free with basis $L_1, \ldots L_r$. Then*

$$\mathbf{m} := \bigoplus_{v \in \mathbb{Z}^r \setminus \{0\}} H^0(X, L^v)$$

is a graded maximal ideal of $\mathrm{Cox}(X)$*, it will be called the* canonical maximal ideal *of* $\mathrm{Cox}(X)$.

Proof. First we prove that $\mathbf{m}$ is a graded ideal. Taking homogeneous elements $\alpha \in \mathrm{Cox}(X)$ and $\beta \in \mathbf{m}$, then α and β are global sections of some line bundles L_α and L_β, respectively. If $\alpha\beta \in H^0(X, L_\alpha \otimes L_\beta)$ is not contained in $\mathfrak{m}$, then we have $L_\alpha \otimes L_\beta \simeq \mathcal{O}_X$. If α and β are non-zero global sections, then L_α and L_β are trivial, and it contradicts to $\beta \in \mathfrak{m}$. Hence $\mathfrak{m}$ is a graded ideal of $\mathrm{Cox}(X)$. Furthermore, we have

$$\mathrm{Cox}(X)/\mathfrak{m} \simeq H^0(X, \mathcal{O}_X) \simeq k,$$

so $\mathfrak{m}$ is a maximal ideal. □

Lemma 2.4.3. *Let X be a normal projective variety such that* $\mathrm{Pic}(X)$ *is free. Assume that* $\mathrm{Cox}(X)$ *is finitely generated k-algebra. Then X is smooth toric if and only if* $\mathrm{Cox}(X)_\mathfrak{m}$ *is regular local ring, where* $\mathrm{Cox}(X)_\mathfrak{m}$ *is the localization of the Cox ring by the canonical maximal ideal.*

Proof. By [58, Theorem 1.5], X is smooth toric if and only if $\mathrm{Cox}(X)$ is polynominal ring over k. By [69, Theorem 2.1], the regularity of $\mathrm{Cox}(X)_\mathfrak{m}$ implies the regularity of $\mathrm{Cox}(X)$. By [1, Lemma 5], $\mathrm{Cox}(X)$ is polynominal ring over k. □

Proposition 2.4.4. *Let X be a normal projective variety such that* $\mathrm{Pic}(X)$ *is free. Let $f\colon X \longrightarrow X$ be a finite morphism. Then f induces the injective k-algebra homomorphism $\varphi\colon \mathrm{Cox}(X) \longrightarrow \mathrm{Cox}(X)$ satisfying the following properties:*
(1) φ is finite if $\mathrm{Cox}(X)$ *is finitely generated k-algebra, and*
(2) $\varphi^{-1}(\mathfrak{m}) = \mathfrak{m}$ for the canonical maximal ideal $\mathfrak{m}$.

Proof. f induces an injective k-module homomorphism $H^0(X, L) \longrightarrow H^0(X, f^*L)$ for any line bundle L. Hence it induces the injective k-algebra homomorphism $\varphi\colon \mathrm{Cox}(X) \longrightarrow \mathrm{Cox}(X)$ such that the restriction of φ to $H^0(X, L)$ coincides with the above homomorphism.

By [84, Theorem 3.1], the submodule

$$\bigoplus_{L \in \mathrm{Pic}(X)} H^0(X, f^*L)$$

is a finite $\mathrm{Cox}(X)$-module. Since $f^*\colon \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X)$ has the finite cokernel, we obtain the first assertion.

Next we prove the second assertion. Let $s \in H^0(X, L)$ be a non-zero global section of a line bundle L. Then $\varphi(s)$ is contained in $\mathfrak{m}$ if and only if f^*L is not trivial. As L has a non-zero global section, f^*L is trivial if and only if L is trivial. Hence we have $\varphi^{-1}(\mathfrak{m}) = \mathfrak{m}$. ☐

Lemma 2.4.5. *Let X be a normal projective variety such that* $\mathrm{Pic}(X)$ *is free. Assume that* $\mathrm{Cox}(X)$ *is a finitely generated k-algebra. Let $f\colon X \longrightarrow X$ be a finite morphism. Assume that if $f^*L \simeq L$ for a line bundle L, then L is numerically trivial. Then X is smooth toric if and only if the induced homomorphism $\varphi\colon \mathrm{Cox}(X) \longrightarrow \mathrm{Cox}(X)$ is flat.*

Proof. If X is smooth toric, then $\mathrm{Cox}(X)$ is a polynominal ring. Since $\varphi\colon \mathrm{Cox}(X) \longrightarrow \mathrm{Cox}(X)$ is a finite morphism of regular rings, φ is flat.

Next, we assume that φ is flat. By Proposition 2.4.4, φ is finite and induces local endomorphism of $\mathrm{Cox}(X)_{\mathfrak{m}}$. By [4, Theorem 13.3], it is enough to show that φ is contracting, that is, $\varphi^e(\mathfrak{m}) \subset \mathfrak{m}^2$ for some e.

We denote $\mathrm{Cox}(X)$ by R and the degree λ part by R_λ for all $\lambda \in \mathrm{Pic}(X)$. We take a homogeneous generator $\{a_i \in R_{\lambda_i}\}_{i=1,\dots,m}$ of $\mathfrak{m}$. We take a homogeneous element $a \in R_\lambda \subset \mathfrak{m}$, then we have $a = h(a_1, \dots, a_m)$ for some polynominal h with $h(0, \dots, 0) = 0$. Since we have $\varphi^l(a) = h(\varphi^l(a_1), \dots, \varphi^l(a_m))$, it is enough to show that $\varphi^l(a_i)$ is contained in $\mathfrak{m}^2$ for all i for some l. We assume that $\varphi(a_i)$ is not contained in R_{λ_j} for $j = 1, \dots m$. Since we have $\varphi(a_i) = h_i(a_1, \dots, a_m)$ for some polynominal h_i and $\varphi(a_i)$ is homogeneous, the vanishing degree of h_i is grater than two, and in particular, $\varphi(a_i) \in \mathfrak{m}^2$. Hence it is enough to show that for some i, there exists a positive integer l such that $\varphi^l(a_i)$ is not contained in R_{λ_j} for all $j = 1, \dots m$. Otherwise, for some i, $(f^l)^*L_{\lambda_i}$ is isomorphic to L_{λ_i} for some l, where L_{λ_i} is a line bundle corresponding to λ_i. By the assumption of f, L_{λ_i} is numerically trivial. Since we have $0 \neq a_i \in H^0(X, L_{\lambda_i})$, L_{λ_i} is trivial, so it contradicts to $\lambda_i \neq 0$. ☐

2.4.2 Proof of the "only if" part

Let X be a smooth toric projective variety. Thomsen [95] proved that for every line bundle L on X, F_*L is a direct sum of line bundles, where F is the absolute Frobenius. In this section, we prove an analogue of this result for all finite endomorphisms which is not necessarily a toric morphism.

Theorem 2.4.6. (cf. [95, Theorem 1]) *Let X be a smooth toric projective variety and $f\colon X \longrightarrow X$ a finite morphism. Then for every line bundle M on X, f_*M is a direct sum of line bundles.*

Proof. Since X is smooth toric, $R := \mathrm{Cox}(X)$ is a polynominal ring with finite variables. f induces the endomorphism φ of R as in Proposition 2.4.4. Since φ is a finite homomorphism of regular rings, φ is flat. Thus φ_*R is flat and finite graded R-module. Let M be a line bundle on X. We define a graded R-module E_M by

$$E_M := \bigoplus_{L \in \mathrm{Pic}(X)} H^0(X, f_*M \otimes L).$$

Then we have a splitting injection

$$E_M \simeq \varphi_* \bigoplus_{L \in \mathrm{Pic}(X)} H^0(X, M \otimes f^*L) \longrightarrow \varphi_*R.$$

Since φ_*R is flat, E_M is a flat graded R-module, so E_M is free R-module. Hence we have $E_M \simeq R(\lambda_1) \oplus \cdots \oplus R(\lambda_d)$, where $\lambda_i \in \mathrm{Pic}(X)$ and $R(\lambda_i)$ is the shift of R. By [25, Proposition 6.A.3], f_*M is isomorphic to the sheaf associated to E_M, so we have

$$f_*M \simeq L_{\lambda_1} \oplus \cdots \oplus L_{\lambda_d}.$$

□

2.4.3 Proof of the "if" part

In this section, we prove an analogue of Achinger's result for int-amplified endomorphisms, which are generalization of polarized endomorphisms. It is the main difference from Achinger's proof that Lemma 4.5 and Lemma 4.6.

Definition 2.4.7. Let X be a normal projective variety and $f\colon X \longrightarrow X$ a finite endomorphism. f is called by *int-amplified* if $f^*H - H$ is ample for some ample Cartier divisor H on X.

Lemma 2.4.8. *Let X be a normal projective variety admitting an int-amplified endomorphism f. Let L be a line bundle with $f^*L \simeq L$. Then L is numerically trivial.*

Proof. It follows from [76, Theorem 3.3]. □

Lemma 2.4.9. (cf. [1, Lemma2]) *Let X be a normal projective variety admitting an endomorphism f. We assume that there exists a subgroup $\Lambda \subset \mathrm{Pic}(X)$ containing an ample line bundle such that for every line bundle $L \in \Lambda$, $f_*L \simeq L_1 \oplus \cdots \oplus L_d$ for some line bundles $L_1, \ldots L_d$ contained in Λ. If a line bundle M satisfies $f^*M \simeq \mathcal{O}_X$, then M is trivial.*

Proof. We take a line bundle $L \in \Lambda$. Let $m(L', L)$ denote the multiplicity of a line bundle L' as a direct summand of f_*L. Let M be a line bundle with $f^*M \simeq \mathcal{O}_X$. We prove $m(L' \otimes M, L) \geqslant m(L', L)$. We set $m := m(L', L)$, than $L'^{\oplus m}$ is a direct summand of f_*L. Hence, $(L' \otimes M)^{\oplus m}$ is a direct summand of $f_*L \otimes M$. Since we have

$$f_*L \otimes M \simeq f_*(L \otimes f^*M) \simeq f_*L,$$

$(L' \otimes M)^{\oplus m}$ is a direct summand of f_*L, and in particular, we have $m(L' \otimes M, L) \geqslant m(L', L)$. First we prove that M is a torsion element of $\mathrm{Pic}(X)$. By the assumption, there exists a line bundle L' with $m(L', L) \geqslant 1$. Hence, $L' \otimes M^k$ is a direct summand of f_*L for all k, so we have $M^{k_1} \simeq M^{k_2}$ for some positive integers $k_1 \neq k_2$. It means that M is a torsion element of $\mathrm{Pic}(X)$. Since M is a torsion element, we obtain $m(L' \otimes M, L) = m(L', L)$ for all line bundles L'. Let k be the minimum positive integer of all positive integers with $M^k \simeq \mathcal{O}_X$. Since we have $\chi(L' \otimes M) = \chi(L')$, we have

$$\chi(L) = k(\chi(L_1) + \cdots + \chi(L_m)),$$

, in particular, $\chi(L)$ is divided by k. Since $\chi(L_i)$ is also divided by k, $\chi(L)$ is divided by k^2. Repeating such a process, $\chi(L)$ is divided by k^e for all $L \in \Lambda$ and $e \in \mathbb{Z}_{\geqslant 0}$. Since Λ contains a very ample line bundle L with $\chi(L) \neq 0$, we have $k = 1$, and in particular, M is trivial. □

Lemma 2.4.10 means that any étale cover has the trivial Albanese variety in the setting in the proof of the "if" part of B. Combining this lemma and [101, Theorem 1.3, 1.5], the variety is of Fano type, in particular, it has the finitely generated Cox ring. It is a key ingredient of the proof of the "if" part.

Lemma 2.4.10. *Let X be a normal projective variety admitting an int-amplified endomorphism f. We assume that there exists a subgroup $\Lambda \subset \mathrm{Pic}(X)$ containing an ample line bundle such that for every line bundle $L \in \Lambda$, $f_*L \simeq L_1 \oplus \cdots \oplus L_d$ for some line bundles $L_1, \ldots L_d$ contained in Λ. Then the Albanese variety is a point.*

Proof. By [76, Theorem 1.8], we have the following commutative diagram;

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & A \arrow[d, "g"] \\ X \arrow[r, "a"'] & A, \end{tikzcd}$$

where A is the albanese variety of X, a is the albanese morphism of X and g is an int-amplified endomorphism. By Lemma 2.4.9, for $f^* \colon \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X)$, the preimage of 0 is a point, so f^* is an isomorphism. Since g is a composition of a translation and the dual of f^*, g is also an isomorphism. In conclusion, A is a point. $\square$

Lemma 2.4.11. *Let*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

be a commutative diagram of finite morphisms of smooth projective varieties such that π is étale, f and g are int-amplified endomorphisms. We assume that there exists a subgroup $\Lambda \subset \operatorname{Pic}(X)$ containing an ample line bundle such that for every line bundle $L \in \Lambda$, $f_*L \simeq L_1 \oplus \cdots \oplus L_d$ for some line bundles $L_1, \ldots L_d$ contained in Λ. Then the above diagram is a fiber product.

Proof. By the étaleness, it is enough to show that $Y' := X \times_X Y$ is connected, and in particular, it is enough to show that $h^0(\pi^* f_* \mathcal{O}_X) = 1$. We set $f_*\mathcal{O}_X \simeq M_1 \oplus \cdots \oplus M_d$ for line bundles $M_1, \ldots M_d$. We note that

$$h^0(f^*M_i^{-1}) = h^0(f_*\mathcal{O}_X \otimes M_i^{-1}) \geqslant h^0(\mathcal{O}_X) \geqslant 1.$$

We may assume $h^0(M_1) = 1$, then $h^0(f^*M_1^{-1}) \geqslant 1$ implies that M_1 is trivial. Hence we have $\pi^* f_* \mathcal{O}_X \simeq \mathcal{O}_Y \oplus \pi^* M_2 \oplus \cdots \oplus \pi^* M_d$ and it is enough to show $h^0(\pi^* M_i) = 0$ for all $i = 2, \ldots, d$.

Suppose we have $h^0(\pi^* M_2) \geqslant 1$. Since $h^0(\pi^* f^* M_2)$ and $h^0(\pi^* f^* M_2^{-1})$ is grater than zero, $\pi^* f^* M_2$ is trivial, and in particular, $f^* M_2$ is numerically trivial. Since $h^0(f^* M_2^{-1})$ is grater that zero, $f^* M_2$ is trivial. By Lemma 2.4.9, M_2 is also trivial, but it contradicts to $h^0(M_2) = 0$. $\square$

Lemma 2.4.12. *Let X be a smooth projective variety admitting an int-amplified endomorphism f. If for every line bundle L on X, f_*L is a direct sum of line bundles, then X is of Fano type, in particular,* $\operatorname{Pic}(X)$ *is free and* $\operatorname{Cox}(X)$ *is finitely generated.*

Proof. By [101, Theorem1.3, 1.5], we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & X, \end{tikzcd}$$

where π is an étale finite morphism, g is an int-amplified endomorphism, and Y is a smooth variety of Fano type over the Albanese variety of Y. By Lemma 2.4.11, the above diagram is cartesian. Let $\Lambda := \pi^* \operatorname{Pic}(X)$ be a subgroup of $\operatorname{Pic}(Y)$, then it contains an ample line bundle. Furthermore, for every $\pi^* L \in \Lambda$, we have

$$g_* \pi^* L \simeq \pi^* f_* L.$$

By the assumption, f_*L is a direct sum of line bundles. In particular, $g_*\pi^*L \simeq L_1 \oplus \cdots \oplus L_d$ for some $L_i \in \Lambda$. By Lemma 2.4.10, the Albanese variety of Y is a point, in particular, Y is of Fano type. Thus X is smooth and rationally connected, so π is trivial and X is also of Fano type. By [11, Corollary 1.1.9], $\mathrm{Cox}(X)$ is finitely generated. □

Theorem 2.4.13. *Let X be a smooth projective variety admitting an int-amplified endomorphism f. If for every line bundle L on X, f_*L is a direct sum of line bundles, then X is toric.*

Proof. By Lemma 2.4.12, $\mathrm{Pic}(X)$ is free and $R := \mathrm{Cox}(X)$ is finitely generated. By Lemma 2.4.5 and Lemma 2.4.8, it is enough to show that $\varphi\colon R \longrightarrow R$ induced by f is flat.

For $\mu \in \mathrm{Pic}(X)/f^*\mathrm{Pic}(X)$, we take a representation $\mu' \in \mathrm{Pic}(X)$ of μ. Then we have

$$\varphi_*R \simeq \bigoplus_{\mu \in \mathrm{Pic}(X)/f^*\mathrm{Pic}(X)} \varphi_*M_\mu$$

as an R-module, where M_μ is defined by

$$M_\mu := \bigoplus_{\lambda \in \mathrm{Pic}(X)} H^0(X, L_{\mu'} \otimes f^*L_\lambda)$$

By the projection formula, we have

$$\varphi_*M_\mu = \bigoplus_{\lambda \in \mathrm{Pic}(X)} H^0(X, f_*(L_{\mu'} \otimes f^*L_\lambda)) = \bigoplus_{\lambda \in \mathrm{Pic}(X)} H^0(X, f_*L_{\mu'} \otimes L_\lambda).$$

By the assumption, we have $f_*L_{\mu'} \simeq L_{\lambda_1} \oplus \cdots \oplus L_{\lambda_d}$. Hence we have

$$\varphi_*M_\mu = \bigoplus_{i=1}^{d} \bigoplus_{\lambda \in \mathrm{Pic}(X)} H^0(X, L_{\lambda_i+\lambda}) \simeq \bigoplus_{i=1}^{d} R.$$

It implies that φ is a flat finite homomorphism. □

2.5 Proof of Theorem C

2.5.1 Global F-splitting of varieties appearing in an equivariant MMP

In this section, we study the global F-splitting of varieties appearing in a minimal model program.

First, we consider the birational case. In this case, we obtain the following result.

Theorem 2.5.1. *Consider the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[g]{} & Y, \end{array}$$

where X, Y are normal projective varieties defined over an algebraically closed field k of characteristic 0, π is a birational morphism or a small birational map, f and g are int-amplified endomorphisms. Then X is of dense globally F-split type or of globally F-regular if and only if so is Y

Proof. First note that the "only if" part holds without the existence of int-amplified endomorphisms by [37, Lemma 2.14]. Furthermore, when π is small, Theorem 2.5.1 follows from [37, Lemma 2.14]. Thus we assume π is a birational morphism. We may assume that X, Y, f, g and π are defined over algebraically closed field of characteristic $p > 0$ and Y is globally F-split and $\deg(g)$ is coprime to p. It is enough to show that X is globally F-split to prove the first assertion. We fix canonical divisors K_X and K_Y on X and Y, respectively with $\pi_* K_X = K_Y$. Let $\psi\colon F_*\mathcal{O}_Y \longrightarrow \mathcal{O}_Y$ be an $\mathcal{O}_Y$-module homomorphism giving a splitting of the Frobenius morphism of Y. Then there exists a non-zero rational function $\alpha \in K(Y)$ such that

$$(1-p)K_Y + \mathrm{div}_Y(\alpha) \geqslant 0$$

is corresponding to ψ (see § 2.3). Note that ψ induces the homomorphism $F_*K(Y) \longrightarrow K(Y)$ with $\psi(1) = 1$. Since $K(Y) = K(X)$, ψ is also induces the homomorphism $F_*K(X) \longrightarrow K(X)$ and it is also denoted by ψ by the abuse of notations. Then ψ corresponds to the divisor

$$(1-p)K_X + \mathrm{div}_X(f^*\alpha).$$

Since π is a birational morphism, there exists an effective exceptional divisor E on X such that

$$(1-p)K_X + \mathrm{div}_X(f^*\alpha) \geqslant -E.$$

By the commutative diagram in the statement of Theorem 2.5.1, every exceptional prime divisor of π is totally invariant under f. Since f is an int-amplified endomorphism, we may assume that $E \leqslant R_f$ by replacing f with some iterate of f by [76, Theorem 3.3 (2)]. The homomorphism

$$\mathrm{Tr}_f \circ f_*\psi\colon f_*F_*K(X) = F_*f_*K(X) \longrightarrow K(X)$$

is corresponding to

$$(1-p)K_X + \mathrm{div}_X(f^*\alpha) + pR_f \geqslant -E + pR_f \geqslant 0.$$

It implies that $\mathrm{Tr}_f \circ f_*\psi$ defines the homomorphism $F_*f_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$. Since $\mathrm{Tr} \circ f_*\psi(1) = \deg(f)$ is an unit of $\mathcal{O}_X$, X is globally F-split.

Next, we assume that Y is globally F-regular. Let D be an effective Weil divisor on X. There exist a non-zero rational function $\alpha \in K(Y)$ and a positive integer e such that

$$(1-p^e)K_Y + \mathrm{div}_Y(\alpha) - \pi_*D \geqslant 0$$

and corresponding homomorphism ψ_1 gives a splitting of

$$\mathcal{O}_Y \longrightarrow F^e_*\mathcal{O}_Y \longrightarrow F^e_*\mathcal{O}_Y(\pi_*D).$$

ψ_1 induces the homomorphism $F^e_* K(X) \longrightarrow K(X)$ and it is also denoted by ψ_1 by the abuse of notations. Since R_f contains all exceptional prime divisors, there exists a positive integer e' such that

$$(1-p^e)K_X + \mathrm{div}_X(f^*\alpha) - D + p^{e+e'}R_f \geqslant 0$$

Since X is globally F-split, we can find a homomorphism ψ_2 giving a splitting $\mathcal{O}_X \longrightarrow F^{e'}_* \mathcal{O}_X$ and it corresponds to a divisor

$$(1-p^{e'})K_X + \mathrm{div}_X(\beta) \geqslant 0$$

for some non-zero rational function $\beta \in K(X)$. Hence $\mathrm{Tr}_f \circ f_*\psi_2 \circ f_* F^{e'}_* \psi_1$ is corresponding to

$$(1-p^e)K_X + \mathrm{div}_X(\alpha) - D + p^e((1-p^{e'})K_X + \mathrm{div}_X(\beta)) + p^{e+e'}R_f \geqslant 0.$$

Thus this homomorphism defines a homomorphism $F^{e+e'}_* f_* \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$ and it gives a splitting. Therefore X is globally F-regular. $\square$

Corollary 2.5.2. *Let X be a normal klt $\mathbb{Q}$-factorial projective variety defined over an algebraically closed field k of characteristic 0 and X admits an int-amplified endomorphism. Assume that some MMP of X ends up with a point. Then X is of globally F-regular type.*

Proof. We consider a MMP of X

$$X = X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_r$$

such that $X_r \longrightarrow \mathrm{Spec}(k)$ is a Mori fiber space. In particular, X_r is of Fano type. By Theorem 2.2.8, X_r is of globally F-regular type. Since X has an int-amplified endomorphism f, the above MMP is f^n-equivariant MMP for some $n \in \mathbb{Z}_{>0}$ by Theorem 2.2.15. By Theorem 2.5.1, X is also of globally F-regular type. $\square$

Let S be a normal surface defined over a field of characteristic zero admitting an int-amplified endomorphism. If some MMP for S ends up with a point, to prove the global F-splitting of S, we may assume that $-K_S$ is ample by Theorem 4.1. If S is klt, then S is globally F-regular type by Corollary 4.2. If S is log canonical, S is of Calabi-Yau type, but global F-splitting is not clear.

In order to prove global F-splitting of S, we discuss the local case.

Definition 2.5.3.

1. Let X be an integral scheme essentially of finite type over an F-finite field of positive characteristic. We say that X is *F-pure* if $F\colon \mathcal{O}_{X,x} \longrightarrow F_*\mathcal{O}_{X,x}$ splits as $\mathcal{O}_{X,x}$-module homomorphism for every point $x \in X$.

2. Let X be a normal integral scheme essentially of finite type over a field of characteristic zero. We say that X is of *dense F-pure type* if taking a model X_A over a finitely generated $\mathbb{Z}$-algebra A as in § 2.2, there exists a dense subset S of the closed points of $\mathrm{Spec}\, A$ such that X_s is F-pure for every $s \in S$.

Lemma 2.5.4. *Let $(R, \mathfrak{m})$ be a Noetherian local normal ring essentially of finite type over an F-finite field of characteristic $p > 0$ and $\varphi\colon R \longrightarrow R$ a injective finite local homomorphism such that p is coprime to* $\deg(\varphi)$. *Assume that* $\operatorname{Spec} R\backslash\{\mathfrak{m}\}$ *is F-pure and there exists a non-zero effective Cartier divisor D on* $\operatorname{Spec} R$ *such that $D \leqslant R_{\varphi}$. Then R is F-pure.*

Proof. We consider the evaluation map $\mathrm{Hom}_R(F_*R, R) \longrightarrow R$. Note that this map is surjective if and only if R is F-pure. Since $\operatorname{Spec}(R)\backslash\{\mathfrak{m}\}$ is F-pure, this evaluation map is surjective at any point of the punctured spectrum. Thus there exists a positive integer r such that the image of the evaluation map contains $\mathfrak{m}^r$. By the assumption, there exists an element $\alpha' \in \mathfrak{m}$ such that $\operatorname{div}(\alpha') \leqslant R_{\varphi}$. We set $\alpha = \varphi^{r-1}(\alpha') \cdot \cdots \cdot \varphi(\alpha') \cdot \alpha'$. Then we have

$$\operatorname{div}(\alpha) = (\varphi^{r-1})^*\operatorname{div}(\alpha') + \cdots + \operatorname{div}(\alpha') \leqslant R_{\varphi^r}$$

and $\alpha \in \mathfrak{m}^r$. We replace φ by φ^r. Since α is contained in the image of the evaluation map, there exists a homomorphism $\psi\colon F_*R \longrightarrow R$ such that $\psi(1) = \alpha$. Next we consider the homomorphism

$$\mathrm{Tr}_{\varphi}(\alpha^{-1} \cdot \_)\colon \varphi_*K(R) \longrightarrow K(R)$$

mapping $x \in \varphi_*K(R)$ to $\mathrm{Tr}_{\varphi}(\alpha^{-1}x)$. Note that the image of α by this map is an unit of R. Furthermore, this map is corresponding to

$$R_{\varphi} - \operatorname{div}(\alpha) \geqslant 0.$$

Thus $\mathrm{Tr}_{\varphi}(\alpha^{-1} \cdot \_)$ defines the homomorphism $\varphi_*R \longrightarrow R$ by Example 2.2.11. Hence

$$\frac{1}{\deg(\varphi)}\mathrm{Tr}_{\varphi}(\alpha^{-1} \cdot \_) \circ \varphi_*\psi\colon \varphi_*F_*R \longrightarrow \varphi_*R \longrightarrow R$$

gives a splitting of $F \circ \varphi$. In particular, R is F-pure. $\square$

Next, we prove the following global assertion by reducing to Lemma 2.5.4.

Proposition 2.5.5. *Let X be a normal projective variety defined over an algebraically closed field k of characteristic* 0 *with the Picard rank one and X admits a non-invertible endomorphism. Assume that $-K_X$ is ample $\mathbb{Q}$-Cartier divisor and X has at worst rational singularities. Furthermore assume that X is of dense F-pure type. Then X is of dense globally F-split type.*

Proof. Let f be a non-invertible endomorphism of X. Note that f is a polarized endomorphism because the Picard rank of X is equal to one. There exist an ample divisor H' on X and a positive integer q such that $f^*H' \sim qH'$. We note that q is larger than one since $q^{\dim X} = \deg(f)$ by the projection formula. Since X is of dense F-pure type and $\mathbb{Q}$-Gorenstein, X is log canonical by [47, Theorem]. By Kodaira type vanishing theorem [34, Corollary 2.42], $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Let $\pi\colon Y \longrightarrow X$ be a log resolution of X. Since X has at worst rational singularities, we have

$$\mathrm{H}^1(Y, \mathcal{O}_Y) = \mathrm{H}^1(X, \mathcal{O}_X) = 0.$$

It implies that $\mathrm{Pic}(Y)$ is finitely generated. Let U be the maximal open subset of X such that $\pi|_{\pi^{-1}(U)}$ is isomorphism. Thus, we have

$$\mathrm{CH}^1(X) \simeq \mathrm{CH}^1(U) \twoheadleftarrow \mathrm{CH}^1(Y) \simeq \mathrm{Pic}(Y),$$

in particular, $\mathrm{CH}^1(X)$ is finitely generated. Since R_f is $\mathbb{Q}$-Cartier, $\{R_{f^n} \mid n \in \mathbb{Z}_{>0}\}$ is a finite set in $\mathrm{Div}(X)/\mathrm{CDiv}(X)$.

It implies that there exist positive integers $m > n > 0$ such that $R_{f^m} - R_{f^n}$ is 0 in $\mathrm{Div}(X)/\mathrm{CDiv}(X)$. Thus, it is Cartier divisor. Furthermore,

$$R_{f^m} - R_{f^n} = (f^{m-1})^* R_f + \cdots (f^n)^* R_f$$

is effective. Replacing f be some iterate, we may assume that there exists an effective Cartier divisor A on X such that $R_f \geqslant A$. We define $A_n = (f^{n-1})^* A + \cdots f^* A + A$. Since $\mathrm{CH}^1(X)$ is finitely generated and $f^* A_n - qA_n$ is $\mathbb{Q}$-linearly trivial, we have $\{f^* A_n - qA_n \mid n \in \mathbb{Z}_{>0}\}$ is a finite set. Thus, there exist positive integers $m > n$ such that

$$f^* A_m - qA_m - (f^* A_n - qA_n) = f^* A' - qA'$$

is a principal divisor, where $A' = (f^{m-1})^* A + \cdots + (f^n)^* A$. Since $A \leqslant R_f$, we have

$$A' = (f^{m-1})^* A + \cdots + (f^n)^* A \leqslant (f^{m-1})^* R_f + \cdots + f^* R_f + R_f = R_{f^m}$$

and $f^* A' \sim qA'$. Hence replacing f by some iterate, we may assume that there exists an effective ample Cartier divisor H on X such that $R_f \geqslant H$ and $f^* H \sim qH$. Therefore f induces the graded endomorphism φ of the section ring $R = \bigoplus_{m \geqslant 0} \mathrm{H}^0(X, \mathcal{O}_X(mH))$. Let D be a corresponding effective Cartier divisor of H on R. Then $D \leqslant R_\varphi$, since $H \leqslant R_f$. Indeed, D is the pullback of H and R_φ is the pullback of R_f outside of the vertex 0 of $\mathrm{Spec}(R)$ via the natural projection $\mathrm{Spec}(R)\backslash\{0\} \longrightarrow X$. Next we take a model (X_A, H_A, f_A) of (X, H, f) over a suitable finitely generated $\mathbb{Z}$-subalgebra A of k as in § 2.2. Localizing A at a single element, we may assume that $(R_f)_\mu = R_{f_\mu}$ for every $\mu \in \mathrm{Spec}\, A$. In particular, we may assume that H_μ is an effective Cartier divisor such that $H_\mu \leqslant R_{f_\mu}$ and $f^* H_\mu \sim qH_\mu$. It means that f_μ induces an endomorphism φ_μ of the section ring

$$R_\mu = \bigoplus_{m \geqslant 0} \mathrm{H}^0(X_\mu, mH_\mu)$$

and there exists a non-zero effective Cartier divisor D on R_μ such that $D \leqslant R_\varphi$. Since X is of dense F-pure type, there exists a dense subset of closed points $W \subset \mathrm{Spec}\, A$ such that X_μ is F-pure and $\deg(\varphi_\mu)$ is coprime to p for all $\mu \in W$. Then R_μ satisfies the assumption of Lemma 2.5.4, thus R_μ is F-pure for all $\mu \in W$. By [87, Proposition 5.3], X_μ is globally F-split for all $\mu \in W$. $\square$

2.5.2 Surface case

In this section, we prove Theorem C (Theorem 2.5.7). This section is an only section we assume that X is a surface. In the smooth case, Theorem C follows from the classification by Nakayama and Fujimoto in [82], [32].

First, we consider ruled surfaces over an elliptic curve admitting an int-amplified endomorphism.

Lemma 2.5.6. *Let X be a minimal ruled surface over an elliptic curve defined over an algebraically closed field of characteristic zero and X admits an int-amplified endomorphism. Then X is of dense globally F-split type.*

Proof. By [3, Theorem 1.2] and Proposition 2.2.12, we may assume that the vector bundle defining X is decomposable. Then X is of dense globally F-split type since every elliptic curve is of dense globally F-split type by [28, Theorem 1.2, Theorem 7.1]. □

Theorem 2.5.7 (Theorem C)**.** *Let X be a normal projective surface defined over an algebraically closed field of characteristic zero and X admits an int-amplified endomorphism. Then X is of dense globally F-split type. In particular, X is of Calabi–Yau type.*

Proof. Let f be an int-amplified endomorphism of X. By [98, Theorem 1.4] or [17, Theorem 1.4], X is $\mathbb{Q}$-Gorenstein and log canonical. By [76, Theorem 1.5], $-K_X$ is pseudo-effective. If K_X is pseudo-effective, X is Q-abelian surface by [18, Lemma 9.3]. On the other hand, abelian surfaces are of dense globally F-split type by [83]. By Proposition 2.2.12, Q-abelian surfaces are of dense globally F-split type.

Thus we may assume that K_X is not pseudo-effective. Replacing f by some iterate, we may run an f-equivariant MMP:

$$X = X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_r \longrightarrow Y,$$

where $X_i \longrightarrow X_{i+1}$ is a birational contraction for all $1 \leqslant i \leqslant r-1$ and $X_r \longrightarrow Y$ is a Mori fiber space. By Theorem 2.5.1, it is enough to show that X_r is of dense globally F-split type. In particular, we may assume that $X = X_r$.

We obtain the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "g"'] & Y, \end{tikzcd}$$

where π is a Mori fiber space and Y is an elliptic curve, a projective line or a point, f and g are int-amplified endomorphisms. If Y is a point, then $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor and the Picard rank of X is equal to one. By the proof of [16, Theorem 5.1], X is a projective cone over an elliptic curve or X has at worst rational singularities. In the first case, X is of dense globally F-split type, and in the second case, Proposition 2.5.5 implies that X is of dense globally F-split type because log canonical surfaces are of dense F-pure type by [75] and [46].

Next, we assume that Y is not a point. Then by [72, Theorem 1.2], X is klt $\mathbb{Q}$-factorial. By the proof of Theorem 2.3.31 and the quasi-étale decent of being of dense globally F-split type (Proposition 2.2.12), we may assume that X is of Fano over the albanese variety A. If A is a point, then X is of Fano type, thus X is globally F-regular type by Theorem 2.2.8. Otherwise, A is an elliptic curve. In this case, X is a ruled surface over A by Theorem [72]. By Lemma 2.5.6, X is of dense globally F-split type. $\square$

Chapter 3

Minimal model theory in mixed characteristic

3.1 Preliminaries

3.1.1 Notations

In this subsection, we summarize notations used in this chapter.

- We will freely use the notation and terminology in [63] and [61].
- A morphism of schemes is *alteration* if it is projective, surjective, and generically finite.
- A Noetherian scheme X is *locally irreducible* if every connected component of X is irreducible.
- A scheme V is a *Dedekind scheme* if V is a Noetherian excellent 1-dimensional regular scheme.
- For a Dedekind scheme V, we say X is a *variety over* V or a V*-variety* if X is an integral scheme that is separated and of finite type over V. We say X is a *curve* over V or a V*-curve* (respectively a *surface* over V or a V*-surface*) if X is a V-variety of (absolute) dimension one (respectively two).
- Let V be a Dedekind scheme. Let $\alpha\colon X \longrightarrow V$ be a quasi-projective V-variety. The *dualizing complex* $\omega^{\bullet}_{X/V}$ is defined by $\alpha^{!}\mathcal{O}_V$, where $\alpha^{!}$ is defined as in [48, Ch. III, Theorem 8.7]. The *canonical sheaf* $\omega_{X/V}$ is defined by $(-d)$-th cohomology $h^{-d}(\omega^{\bullet}_{X/V})$ of the dualizing complex, where d is the integer such that $(-d)$-th cohomology is the lowest non-zero cohomology of $\omega^{\bullet}_{X/V}$, thus there exists a natural map $\omega_{X/V}[d] \longrightarrow \omega^{\bullet}_{X/V}$. We note that if α is flat, then d coincides with the relative dimension of X over V. If X is normal, there is a Weil divisor $K_{X/V}$, called a *canonical divisor*, such that $\omega_{X/V} \simeq \mathcal{O}_X(K_{X/V})$. Note that $K_{X/V}$ is uniquely determined up to linear equivalence. We note that

$\omega_{X/V}$ satisfies the condition (S_2) by [49, Lemma 1.3]. If the image of α is a closed point, then we denote the induced morphisms by

$$X \xrightarrow{\beta} \operatorname{Spec} k \xrightarrow{\theta} V.$$

Since $\theta^! \mathcal{O}_V[1] \simeq k$, we have $\omega^\bullet_{X/V}[1] \simeq \omega^\bullet_{X/k}$. In particular, we have $\omega_{X/V} \simeq \omega_{X/k}$. In this case, we denote $\omega^\bullet_{X/V}[1], \omega_{X/V}, K_{X/V}$ by $\omega^\bullet_X, \omega_X, K_X$, respectively, for simplicity.

- Let V be a Dedekind scheme. We say that (X, Δ) is a *log pair* over V if X is a quasi-projective normal V-variety and Δ is an effective $\mathbb{Q}$-Weil divisor on X such that $K_{X/V} + \Delta$ is $\mathbb{Q}$-Cartier. We will use the singularities of the MMP defined in [61, Definition 2.8]. Let S be a reduced divisor on X such that $\Delta' := \Delta - S$ and S have no common component. Then we denote the *different* of the pair (X, Δ) by $\mathrm{Diff}_{S^N}(\Delta')$ defined by [61, Definition 4.2], where S^N is the normalization of S. We will freely use the adjunction results in [61, Section 4].

- Let V be a Dedekind scheme. Let $\pi \colon Y \longrightarrow X$ be a proper morphism of V-schemes. The *trace map* $R\pi_*\omega^\bullet_{Y/V} \longrightarrow \omega^\bullet_{X/V}$ is defined as the following. Applying $R\mathscr{H}om(\_\_ , \omega^\bullet_{X/V})$ to the natural map $\mathcal{O}_X \longrightarrow R\pi_*\mathcal{O}_Y$, we have

$$R\mathscr{H}om(R\pi_*\mathcal{O}_Y, \omega^\bullet_{X/V}) \longrightarrow R\mathscr{H}om(\mathcal{O}_X, \omega^\bullet_{Y/V}) \simeq \omega^\bullet_{X/V}.$$

The left hand side is isomorphic to

$$R\pi_* R\mathscr{H}om(\mathcal{O}_Y, \omega^\bullet_{Y/V}) \simeq R\pi_*\omega^\bullet_{Y/V}$$

by the Grothendieck duality, thus we obtain the trace map $R\pi_*\omega^\bullet_{Y/V} \longrightarrow \omega^\bullet_{X/V}$. If X is of relative dimension d and π is an alteration, taking the $(-d)$-th cohomology and the composition with the natural map, we obtain the map $\pi_*\omega_{Y/V} \longrightarrow \omega_{X/V}$ is also called the trace map by abuse of notations. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on X and we further assume that X and Y are normal. Then we can extend the map $\pi_*\omega_{Y/V}(\lceil \pi^* D \rceil)|_U \longrightarrow \omega_{X/V}(\lceil D \rceil)|_U$ on the regular locus U of X to the map $\pi_*\omega_{Y/V}(\lceil \pi^* D \rceil) \longrightarrow \omega_{X/V}(\lceil D \rceil)$ on X because $\omega_{X/V}(\lceil D \rceil)$ satisfies the condition (S_2).

- Let R be a discrete valuation ring and $V = \operatorname{Spec} R$. R is of *characteristic* $(0, p)$ if the fractional field K is of characteristic zero and the residue field k is of characteristic $p > 0$. Let X be a V-variety. The *closed fiber* of X is $X \times_V \operatorname{Spec} k$ denoted by X_s and the *generic fiber* of X is $X \times_V \operatorname{Spec} K$ denoted by X_η.

3.1.2 Negativity lemma and finite generation of the Picard rank

In this subsection, we remark that the negativity lemma holds for the general setting. Originally, the negativity lemma follows from the Bertini's theorem and the negativity lemma for surfaces. However, in general setting, the Bertini type theorem is much harder. Thus, we use the alternative proof by [15].

Proposition 3.1.1. (Negativity lemma) *Let $\pi\colon Y \longrightarrow X$ be a projective morphism of Noetherian normal schemes. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-Weil π-nef divisor on Y. If $\pi_* D \leqslant 0$, then $D \leqslant 0$.*

Proof. The proof of [15, Proposition 2.12] also works in our setting. Thus, we obtain an analogous statement of [15, Proposition 2.12]. The negativity lemma follows from this statement. □

Proposition 3.1.2. *Let $X \to Y$ be a proper morphism of Noetherian schemes. Then, the relative Picard number* $\operatorname{rank}_{\mathbb{R}} N^1(X/Y)$ *is finite, where $N^1(X/Y)$ is defined by*

$$(\operatorname{Pic}(X)/\text{numerical equivalence over } Y) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Proof. The proof of [60, IV, §4] also works in our setting. □

3.1.3 Base change

In the proof of the existence of flips, we reduce to the case where V is the spectrum of a complete discrete valuation ring with an infinite residue field. To do this, we observe properties preserved under the base change via strictly henselization and completion. We note that $\mathbb{Q}$-factoriality is not preserved under the above base change.

Lemma 3.1.3. *Let V be an excellent Dedekind scheme. Let (X, Δ) be a dlt pair. Then every component D of $\lfloor \Delta \rfloor$ is normal up to universal homeomorphism if D is $\mathbb{Q}$-Cartier.*

Proof. We may assume that $D = \lfloor \Delta \rfloor$. The assertion follows from the same argument as in the proof of [44, Lemma 2.1]. □

Lemma 3.1.4. *Let V be the spectrum of an excellent discrete valuation ring. Let $\iota\colon V' \longrightarrow V$ be the completion of the strict henselization of V. Let (X, Δ) be a dlt pair over V. Let S be a component of $\lfloor \Delta \rfloor$ such that S is $\mathbb{Q}$-Cartier. Let (X', Δ') and S' be the base change of (X, Δ) and S via ι, respectively. Then (X', Δ') is dlt and S' is locally irreducible. In particular, every irreducible component of S' is $\mathbb{Q}$-Cartier.*

Proof. We note that ι is the composition of formally étale morphism and completion. Via both base change, being dlt and normality are preserved, we note that a strict henselization of an excellent local ring is also excellent [40, Crollary 5.6]. By Lemma 3.1.3, the normalization $S^N \longrightarrow S$ is a universal homeomorphism. By the base change via ι, we have $(S^N)' \longrightarrow S'$, then $(S^N)'$ is also normal and this map is a universal homeomorphism. Since normal schemes are locally irreducible and locally irreducible is preserved by homeomorphisms. Thus, S' is also locally irreducible. Since S is $\mathbb{Q}$-Cartier, S' is also $\mathbb{Q}$-Cartier. Being $\mathbb{Q}$-Cartier is a local property and S' is locally irreducible, every irreducible component is also $\mathbb{Q}$-Cartier. □

3.1.4 Alterations

Proposition 3.1.5. *Let V be an excellent Dedekind scheme. Let X be a V-variety and $x \in X$ be a point of X. Let $f_x \colon U' \longrightarrow U := \operatorname{Spec} \mathcal{O}_{X,x}$ be an alteration from an integral scheme. Then there exists an alteration $f \colon X' \longrightarrow X$ of V-varieties such that we have the following Cartesian diagram:*

$$\begin{tikzcd} U' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ X' \arrow[r] & X. \end{tikzcd}$$

Proof. Since f_x is projective, U' is embedded in some projective space $\mathbb{P}^N_U$. It extends to some open subset of X, thus we may assume that U is an open subset of X. We denote the closure of U' in $\mathbb{P}^N_X$ by X'. Then the natural morphism $X' \longrightarrow \mathbb{P}^N_X \longrightarrow X$ is projective. Since U' is integral and $\dim U' = \dim X$, the morphism $X' \longrightarrow X$ is also an alteration. By the construction, this satisfies the desire conditions. $\square$

Proposition 3.1.6. *Let V be an excellent Dedekind scheme. Let X be a V-variety. Let $f_i \colon Y_i \longrightarrow X$ be alterations of V-varieties for $i = 1, \ldots, r$. Then there exists an alteration $f \colon Y \longrightarrow X$ of V-varieties which factors through f_i for all i.*

Proof. By the induction on i, we may assume that $i = 2$. We denote the fiber product of Y_1 and Y_2 over X by $Y' := Y_1 \times_X Y_2$. Take an irreducible component Y of Y' which dominates X, then $f \colon Y \longrightarrow X$ is an alteration of V-varieties and f factors through f_i. $\square$

Proposition 3.1.7. *Let V be an excellent Dedekind scheme. Let X be a V-variety and $S_1, \ldots, S_r$ be closed sub-V-varieties of X. Let $f_i \colon T_i \longrightarrow S_i$ be an alteration of V-varieties. Then there exists an alteration $g \colon Y \longrightarrow X$ of V-varieties and a closed sub-V-varieties $S_{Y,1}, \ldots, S_{Y,r}$ of Y with $g(S_{Y,i}) = S_i$ for all i such that the induced morphism $g_{S,i} \colon S_{Y,i} \longrightarrow S_i$ factors through f_i.*

Proof. By Proposition 3.1.6, we may assume that $i = 1$. We set $f := f_1$, $S := S_1$, and $T := T_1$. We denote the generic point of S by x, then $\mathcal{O}_{X,x}$ is a local domain with residue field $K(S)$. Since $K(S) \subset K(T)$ is a finite extension of fields, it is a finite sequence of simple extensions. Thus we have a finite extension of domains $\mathcal{O}_{X,x} \longrightarrow A$ such that the residue field of some maximal ideal of A coincides with $K(T)$. It extends to a finite surjective morphism $\pi \colon X' \longrightarrow X$ such that it factors through $\mathcal{O}_{X,x} \longrightarrow A$ after localizing at x by taking the normalization of X in $K(A)$. In particular, there exists a closed sub-V-variety S' in X' such that $\pi(S') = S$ and there exists a rational dominant map $S' \dashrightarrow T$ over S. Replacing f into an elimination of indeterminacy $S' \dashrightarrow T$ and (X, S) into (X', S'), we may assume that f is birational. Since f is projective, it is blowing up with respect to some ideal sheaf $\mathscr{I}_S$ of $\mathcal{O}_S$. There exists an ideal sheaf $\mathscr{I}$ of $\mathcal{O}_X$ such that $\mathscr{I}$ contains a defining ideal of S and $\mathscr{I} \cdot \mathcal{O}_S = \mathscr{I}_S$. Then the strict transform S_Y of the blowing up $g \colon Y \longrightarrow X$ with respect to $\mathscr{I}$ of X coincides with T. $\square$

Proposition 3.1.8. *Let V be an excellent Dedekind scheme. Let X be a V-variety and $S \subset X$ be a closed V-variety. Let D be a Cartier divisor on S. Then for every positive integer n, there exists an alteration $f\colon Y \longrightarrow X$ from a V-variety and closed sub-V-variety S_Y of Y with $f(S_Y) = S$ such that $f_S^* D = nD'$ for some Cartier divisor D' on S_Y, where $f_S\colon S_Y \longrightarrow S$ is the induced morphism.*

Proof. Combining Proposition 3.1.5 and Proposition 3.1.6, we may assume that $D = \mathrm{div}(\varphi)$ for some non-zero section $0 \neq \varphi \in K(S)$ and X is affine by shrinking X. Let $g\colon T \longrightarrow S$ be the normalization of S in $K(S)[\varphi^{1/n}]$, then $g^*D = n\mathrm{div}(\varphi^{1/n})$. The assertion follows from Proposition 3.1.7. □

Theorem 3.1.9. ([26, Thorem 6.5]) *Let V be the spectrum of a complete discrete valuation ring. Let X be a flat V-variety, and let $Z \subset X$ be a proper closed subset. Then there exists a finite surjective morphism $V' \longrightarrow V$ and an alteration $\varphi\colon X' \longrightarrow X$ from a V'-variety such that the pair $(X', \varphi^{-1}(Z)_{\mathrm{red}})$ is a strictly semi-stable pair by means of [26, 6.3], and in particular, $(X', \varphi^{-1}(Z)_{\mathrm{red}})$ is a simple normal crossing pair.*

Remark 3.1.10. If X is not flat over V, then X is defined over a field. Thus, in this case, the last assertion of Theorem 3.1.9 follows from [26, Theorem 4.1].

3.1.5 Adjunction and Bertini type theorem

For the proof of the existence of flips, we discuss the adjunction of singularities related to local irreducibility. The reader is referred to [61, Section 4] for more details.

Proposition 3.1.11. *Let V be an excellent Dedekind scheme. Let $(X, S+A+B)$ is a dlt pair over V such that S and A are locally irreducible Weil divisors and $\lfloor B \rfloor = 0$. Then $(S^N, \mathrm{Diff}_{S^N}(A+B))$ is plt.*

Proof. Let $\pi\colon S^N \longrightarrow X$ be the composition of the normalization of S and the closed immersion $S \longrightarrow X$. Let E be an exceptional prime divisor over S^N centered at $x \in S^N$. If the log discrepancy $a_E(S^N, \mathrm{Diff}_{S^N}(A+B))$ is equal to 0, then $\pi(x)$ is the generic point of a stratum of $\lfloor S + A \rfloor$. Since E is exceptional, $\pi(x)$ is contained in at least three component of $S + A$. Since S and A are locally irreducible, it contradicts. □

Lemma 3.1.12. *Let V be the spectrum of a discrete valuation ring with an infinite residue field. Let (X, Δ) be a plt surface over V. Let H be an ample $\mathbb{Q}$-Cartier divisor. Then there exists an effective divisor $D \sim_{\mathbb{Q}} H$ such that $(X, \Delta + D)$ is also plt.*

Proof. First, we note that if X is not flat over V, then X is a surface over the residue field k of V. Then the assertion is well-known. Thus, we may assume that X is flat over V. We take a positive integer $m \geqslant 2$ such that mH is very ample. If there exists an effective divisor $D' \sim mH$ such that $(X, \Delta + D')$ is dlt, then $\frac{1}{m}D'$ is what we want. Let B_1 be the sum of the components of Δ contained in the closed fiber X_s and we write $B_2 := \Delta - B_1$. Then B_2 and X_s have no common components. Let Σ_s be the union of $B_2 \cap X_{s,\mathrm{red}}$ and the non-regular locus of $X_{s,\mathrm{red}}$. We note that Σ_s is a finite set as X_s is one-dimensional. Thus, for a general member D of $|m(H|_{X_{s,\mathrm{red}}})|$, D has no intersection with Σ_s, the pair $(X_{s,\mathrm{red}}, D)$ is a simple normal crossing pair and D has no common components with B_1 by [31, Corollary 3.4.14]. By the same argument, for a general member D of $|mH|_{X_\eta}$, D has no intersection with the non-regular locus of X_η and $B_2|_{X_\eta}$. By the proof of [53, Theorem 0], we find an effective divisor D' such that $D' \sim mH$ and $D'|_{X_\eta}$ and $D'|_{X_{s,\mathrm{red}}}$ satisfy the conditions as above. Thus $(X, \Delta + D')$ is dlt. Indeed, around the support of D, the pair $(X, \Delta + D)$ is a simple normal crossing pair. □

3.1.6 Rational singularities

In this subsection, we study properties of rational singularities. We will use Proposition 3.1.14 in Section 3.4 to prove the Cohen-Macaulayness of flips. The reader is referred to [64] for more details.

Definition 3.1.13. ([64]) Let X be a Noetherian excellent scheme. X has *rational singularities* if X is normal and Cohen-Macaulay, and for every birational projective morphism $f\colon Y \longrightarrow X$ from a Cohen-Macaulay scheme Y, the natural morphism $\mathcal{O}_X \longrightarrow Rf_*\mathcal{O}_Y$ is an isomorphism.

Proposition 3.1.14. *Let $\varphi\colon X \longrightarrow Z$ be a projective birational morphism from a normal klt scheme X to an excellent normal Cohen-Macaulay scheme Z admitting a normalized dualizing complex $\omega_Z^\bullet$.*

(i) *If X has rational singularities and $\mathcal{O}_Z \simeq R\pi_*\mathcal{O}_X$, then Z has rational singularities.*

(ii) *If $\dim \mathrm{Exc}(\varphi) \leqslant 1$, X has rational singularities except for finite closed points and Z has rational singularities, then X has rational singularities.*

Proof. Take a projective birational morphism $g\colon Y \longrightarrow Z$ from a Cohen-Macaulay scheme Y. We prove that the map $\mathcal{O}_Z \longrightarrow Rg_*\mathcal{O}_Y$ is an isomorphism. By [64, Lemma 7.4] and [64, Theorem 8.6], we may assume that g factors through φ. We denote the induced morphism by $f\colon Y \longrightarrow X$. We note that $f_*\mathcal{O}_Y = \mathcal{O}_X$ and $\pi_*\mathcal{O}_X = \mathcal{O}_Z$. First we assume that X has rational singularities and $\mathcal{O}_Z \simeq R\pi_*\mathcal{O}_X$, then we have $\mathcal{O}_X \simeq Rf_*\mathcal{O}_Y$. By the spectral sequence, we have $\mathcal{O}_Z \simeq R\pi_*\mathcal{O}_X \simeq Rg_*\mathcal{O}_Y$, so Z has rational singularities.

Next, we assume the conditions in (*ii*) and consider the spectral sequence

$$E_2^{i,j} = R^i\pi_*R^jf_*\mathcal{O}_Y \Longrightarrow E^{i+j} = R^{i+j}g_*\mathcal{O}_Y.$$

Since Z has rational singularities, we have $E^i = 0$ for $i > 0$. We have $E_2^{i,j} = 0$ for $i > 1$, since $\dim \mathrm{Exc}(\varphi) \leqslant 1$, and in particular, we obtain $E_2^{1,0} \simeq E_2^{0,1} \simeq E^2 = 0$. Thus, we have $E_2^{0,j} \simeq E^j = 0$ for all $j > 0$, so we have

$$\pi_*R^if_*\mathcal{O}_Y = 0$$

for all $i > 0$. Since X has rational singularities except for finite closed points, the support of $R^i f_* \mathcal{O}_Y$ is isolated, so we have $R^i f_* \mathcal{O}_Y = 0$ for every $i > 0$. Thus, it is enough to show that X is Cohen-Macaulay. To do this, we prove the natural map $Rf_* \omega_Y \longrightarrow \omega_X$ is an isomorphism. Since X has rational singularities except for finite closed points, $R^i f_* \omega_Y$ has the isolated support for $i > 0$ by [64, Theorem 1.4]. Since X is klt, we have $f_* \omega_Y \simeq \omega_X$. Thus, by replacing the structure sheaves into canonical sheaves in the above argument, we have the natural isomorphism $Rf_* \omega_Y \simeq \omega_X$. By the Grothendieck duality

$$Rf_* R\mathscr{H}om(\mathcal{O}_Y, \omega_Y^\bullet) \simeq R\mathscr{H}om(Rf_* \mathcal{O}_Y, \omega_X^\bullet),$$

we have

$$Rf_* \omega_Y^\bullet \simeq \omega_X^\bullet.$$

Since Y is Cohen-Macaulay and f is birational, $\omega_Y^\bullet$ is locally isomorphic to the shift of ω_Y over X, so $\omega_X^\bullet$ is locally isomorphic to the shift of ω_X. Thus X is Cohen-Macaulay. □

3.2 Existence of pl-flip with ample divisor in the boundary

In this section, we prove the existence of pl-flips with ample divisor in the boundary (cf. Theorem 3.2.29 and Corollary 3.2.33). In the first subsection, we study the vanishing theorem up to alterations. Next, we introduce the notion of global T-regularity and study properties of it, for example, the adjunction and the inversion of adjunction. Combining such arguments, we obtain the existence of flips in the special setting (cf. Theorem 3.2.29).

In this section, we basically work over a scheme V, satisfying the following properties.

Assumption 3.2.1. V is the spectrum of a complete discrete valuation ring of characteristic $(0, p)$.

Remark 3.2.2. Let V be a scheme satisfying Assumption 3.2.1. Let X be a V-variety. Then it is possible that X is a variety over a field, and in such a case, $X = X_s$ and $X_\eta = \emptyset$, or $X = X_\eta$ and $X_s = \emptyset$.

3.2.1 Kodaira type vanishing up to alterations

Bhatt [8] proved the killing of a local cohomology up to finite covers in mixed characteristic. Using this theorem, we obtain the Kodaira type vanishing up to alterations for semiample and big divisors (Corollary 3.2.5). This theorem plays an essential role to prove the existence of flips.

Theorem 3.2.3. ([8, Theorem 6.28], cf. [6, Proposition 5.5.3]) *Let V be a scheme satisfying Assumption 3.2.1. Let $f\colon X \longrightarrow Z$ be a projective surjective morphism of*

V-varieties to an affine scheme Z. Let L be a semiample and f-big line bundle on X. Let $x \in Z$ be a closed point with residue characteristic $p > 0$. Then there exists a finite surjective morphism $\pi \colon Y \longrightarrow X$ from a V-variety such that

$$R\Gamma_x R\Gamma(X_p, L^{-1}) \longrightarrow R\Gamma_x R\Gamma(Y_p, \pi^* L^{-1}),$$

is zero on h^i for all $i < \dim(X_p)$, where X_p and Y_p are closed subscheme of X and Y defined by $p = 0$.

Proof. If X is flat over V, it is [8, Theorem 6.28]. Otherwise, it follows from a similar argument to the argument in the proof of [6, Proposition 5.5.3]. □

Proposition 3.2.4. *Let V be a scheme satisfying Assumption 3.2.1. Let $f \colon X \longrightarrow Z$ be a projective surjective morphism from a flat V-variety X to an affine flat V-variety Z. Let L be a semiample and f-big line bundle on X. Then, for every positive integer m, there exists a finite surjective morphism $\pi \colon Y \longrightarrow X$ from a V-variety such that the image of the following map*

$$\mathrm{Tr}^i_\pi \colon R^{i-d}\Gamma(\omega^\bullet_{Y/V}(\pi^* L)) \longrightarrow R^{i-d}\Gamma(\omega^\bullet_{X/V}(L)),$$

is contained in $\varpi^m R^{i-d}\Gamma(\omega^\bullet_{X/V}(L))$ for all $i > 0$, where d is the relative dimension of X over Z and ϖ is a uniformizer of V.

Proof. We take a closed point $x \in Z_s$ of the closed fiber Z_s, it is enough to show that the assertion holds at x for some finite cover π from a V-variety, because any finitely many finite covers from V-varieties are factored by some finite cover from a V-variety. We set $A := \mathcal{O}_{Z_p,x}$. Let E be the injective hull of the residue field of A. By Theorem 3.2.3, there exists a finite surjective morphism $\pi \colon Y \longrightarrow X$ from a V-variety such that

$$h^i R\Gamma_x R\Gamma(X_p, L^{-1}) \longrightarrow h^i R\Gamma_x R\Gamma(Y_p, \pi^* L^{-1})$$

is zero for all $i < d$. It is enough to show that π satisfied the desired condition around x, because any finitely many finite covers from V-varieties are factored by some finite cover from a V-variety. We take base changes via $\operatorname{Spec} \mathcal{O}_{Z,x} \longrightarrow Z$ and we use the same notations by abuse of notations. By the local duality and the Grothendieck duality, we have

$$R\operatorname{Hom}_A(R\Gamma_x R\Gamma(X_p, L^{-1}), E)) \simeq R\Gamma \omega^\bullet_{X_p/V}(\widehat{L)}[1],$$

where the right hand side is the completion of the 1-shift of $R\Gamma\omega^\bullet_{X_p/V}(L)$. By the same equivalent holds for Y_p, we have the following diagram;

$$\begin{tikzcd}
R\operatorname{Hom}_A(R\Gamma_x R\Gamma(Y_p, \pi^* L^{-1}), E) \arrow[r] \arrow[d, "\simeq"'] & R\operatorname{Hom}_A(R\Gamma_x R\Gamma(X_p, L^{-1}), E) \arrow[d, "\simeq"] \\
R\Gamma \omega^\bullet_{Y_p/V}(\widehat{\pi^* L)}[1] \arrow[r] & R\Gamma \omega^\bullet_{X_p/V}(\widehat{L)}[1].
\end{tikzcd}$$

Taking the $(i-d)$-th cohomology, we obtain that the trace map

$$\mathrm{Tr}^i_{\pi_p}\colon R^{i+1-d}\Gamma(\omega^\bullet_{Y_p/V}(\pi^*L)) \longrightarrow R^{i+1-d}\Gamma(\omega^\bullet_{X_p/V}(L)),$$

is zero for all $i>0$. Next, we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_p} \longrightarrow 0,$$

and applying $R\Gamma R\mathscr{H}om(-, \omega^\bullet_{X/V}(L))$ and taking $(i-d)$-th cohomology, we have the exact sequence

$$R^{i-d}\Gamma(\omega^\bullet_{X/V}(L)) \longrightarrow R^{i-d}\Gamma(\omega^\bullet_{X/V}(L)) \longrightarrow R^{i+1-d}\Gamma(\omega^\bullet_{X_p/V}(L)).$$

Thus, we have the commutative diagram of exact sequences

$$\begin{tikzcd}
R^{i-d}\Gamma(\omega^\bullet_{X/V}(L)) \arrow[r, "\cdot p"] & R^{i-d}\Gamma(\omega^\bullet_{X/V}(L)) \arrow[r] & R^{i+1-d}\Gamma(\omega^\bullet_{X_p/V}(L)) \\
R^{i-d}\Gamma(\omega^\bullet_{Y/V}(\pi^*L)) \arrow[r, "\cdot p"] \arrow[u] & R^{i-d}\Gamma(\omega^\bullet_{Y/V}(\pi^*L)) \arrow[r] \arrow[u] & R^{i+1-d}\Gamma(\omega^\bullet_{Y_p/V}(\pi^*L)). \arrow[u, "0-\mathrm{map}"']
\end{tikzcd}$$

Therefore, $\mathrm{Im}(\mathrm{Tr}^i_\pi)$ is contained in $pR^{i-d}\Gamma(\omega^\bullet_{X/V}(L))$. By the same argument for Y, there exists a finite surjective morphism $g\colon W \longrightarrow Y$ such that $\mathrm{Im}(\mathrm{Tr}^i_g)$ is contained in $pR^{i-d}\Gamma(\omega^\bullet_{Y/V}(\pi^*L))$, thus we have

$$\mathrm{Im}(\mathrm{Tr}^i_{\pi\circ g}) \subset p^2 R^{i-d}\Gamma(\omega^\bullet_{X/V}(L)).$$

Repeating such a process, we obtain a finite surjective morphism of V-varieties such that the image of the trace map is contained in $p^m R^{i-d}\Gamma(\omega^\bullet_{X/V}(L))$. Since p is a multiple of ϖ, we have the assertion. $\square$

Corollary 3.2.5. (cf. [13, Theorem 5.5]) *Let V be a scheme satisfying Assumption 3.2.1. Let $f\colon X \longrightarrow Z$ be a projective surjective morphism from a normal flat V-variety X to an affine flat V-variety Z. Let D be a semiample π-big Cartier divisor on X. Then there exists an alteration $\pi\colon Y \longrightarrow X$ such that the trace map*

$$\mathrm{Tr}^i_\pi\colon R^{i-d}\Gamma(\omega^\bullet_{Y/V}(\pi^*D)) \longrightarrow R^{i-d}\Gamma(\omega^\bullet_{X/V}(D))$$

is zero for all $i>0$, where $d := \dim(X_s)$. Furthermore, if X is Cohen-Macaulay, there exists an alteration $\pi\colon Y \longrightarrow X$ such that the trace map

$$\mathrm{Tr}^i_\pi\colon H^i(\omega_{Y/V}(\pi^*D)) \longrightarrow H^i(\omega_{X/V}(D))$$

is zero for all $i>0$.

Proof. We may assume that X is regular by taking an alteration by Theorem 3.1.9. Then the generic fiber X_η is a variety over a field of characteristic zero and D_η is semiample and big over Z, thus we have

$$R^{i-d}\Gamma(\omega^\bullet_{X/V}(D))_\eta = 0$$

by the Kawamata-Viehweg vanishing, and in particular, $R^{i-d}\Gamma(\omega^{\bullet}_{X/V}(D))$ is a ϖ-torsion finite $\mathcal{O}_Z$-module. Thus, for some positive integer m, we have

$$\varpi^m R^{i-d}\Gamma(\omega^{\bullet}_{X/V}(D)) = 0.$$

By Proposition 3.2.4, there exists a finite surjective morphism $\pi\colon Y \longrightarrow X$ from a V-variety such that the image of the following map

$$\mathrm{Tr}^i_\pi\colon R^{i-d}\Gamma(\omega^{\bullet}_{Y/V}(\pi^*D)) \longrightarrow R^{i-d}\Gamma(\omega^{\bullet}_{X/V}(D)),$$

is contained in $\varpi^m R^{i-d}\Gamma(\omega^{\bullet}_{X/V}(D)) = 0$ for all $i > 0$. Thus, we obtain the first assertion.

Next, we assume that X is Cohen-Macaulay. By the first assertion, there exists an alteration $\pi\colon Y \longrightarrow X$ such that the trace map

$$\mathrm{Tr}^i_\pi\colon R^{i-d}\Gamma(\omega^{\bullet}_{Y/V}(\pi^*D)) \longrightarrow R^{i-d}\Gamma(\omega^{\bullet}_{X/V}(D))$$

is zero for all $i > 0$. Since X is Cohen-Macaulay, we have $R^{i-d}\Gamma(\omega^{\bullet}_{X/V}(D)) \simeq H^i(\omega_{X/V}(D))$. Thus, the trace map

$$H^i(\omega_{Y/V}(\pi^*D)) \longrightarrow R\Gamma^{i-d}(\omega^{\bullet}_{Y/V}(\pi^*D)) \longrightarrow H^i(\omega_{X/V}(D))$$

is zero for all $i > 0$. $\square$

Remark 3.2.6. In positive characteristic, an analogous statement of Corollary 3.2.5 holds and we can take Y in Corollary 3.2.5 as a finite cover by [13, Theorem 5.5].

3.2.2 Global T-regularity

In positive characteristic, global F-regularity is important in the proof of the existence of flips (see [45], [43], [44]). In mixed characteristic, we use global T-regularity instead of it.

Definition 3.2.7. Let $\pi\colon Y \longrightarrow X$ be an alteration of normal schemes.

- For a prime divisor E on X, a prime divisor E_Y is called a *strict transform of* E if $\pi(E_Y) = E$.
- For an $\mathbb{R}$-Weil divisor $D = \sum_i a_i E_i$, where E_i is a prime divisor, an $\mathbb{R}$-Weil divisor D_Y is called a *strict transform of* D if D_Y is denoted by $D_Y = \sum_i a_i E_{Y,i}$ such that each $E_{Y,i}$ is a strict transform of E_i.

Remark 3.2.8.

- Since π is an alteration, $\pi|_{E_Y}$ is also an alteration.
- If D is $\mathbb{Q}$-Weil or $\mathbb{Z}$-Weil, then so is D
- If D is a locally irreducible reduced divisor, then so is D_Y.

- If π is birational, then D_Y is the strict transform of D in the usual sense

Definition 3.2.9. (Globally T-regular, Purely globally T-regular) Let V be an excellent Dedekind scheme. Let (X, Δ) be a log pair, or a localization of a log pair over V.

- Let L be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on X. Then the submodule $T^0(X, \Delta; L)$ of $H^0(\mathcal{O}_X(\lceil L \rceil))$ is defined by

$$\bigcap_{\pi : Y \longrightarrow X} \operatorname{Im}(H^0(\omega_{Y/V}(\lceil \pi^*(L - K_{X/V} - \Delta) \rceil)) \longrightarrow H^0(\mathcal{O}_X(\lceil L \rceil))),$$

  where the map is a composition of a trace map and the natural injection and π runs over all alterations from a normal V-variety.

- The pair (X, Δ) is *globally T-regular* (globally Trace-regular) if for every alteration $\pi : Y \longrightarrow X$ from a normal V-variety, the trace map

$$H^0(\omega_{Y/V}(\lceil -\pi^*(K_{X/V} + \Delta) \rceil)) \longrightarrow H^0(\mathcal{O}_X) \qquad (3.1)$$

  is surjective, that is, $T^0(X, \Delta) := T^0(X, \Delta; 0) = H^0(\mathcal{O}_X)$.

- We set $\lfloor \Delta \rfloor = S$. Then (X, Δ) is *purely globally T-regular* if for every alteration $\pi : Y \longrightarrow X$ from a normal V-variety and every strict transform S_Y of S via π, the trace map

$$H^0(\omega_{Y/V}(S_Y + \lceil -\pi^*(K_{X/V} + \Delta) \rceil)) \longrightarrow H^0(\mathcal{O}_X) \qquad (3.2)$$

  is surjective. We note that the map in (3.2) is well-defined, indeed, by $S_Y \leqslant \pi^* S$ on the regular locus of X, the left hand side is contained in $H^0(\omega_Y(\lceil -\pi^*(K_{X/V} + \Delta') \rceil)$, where $\Delta' := \Delta - S$.

- If X is affine, we say that (X, Δ) is *T-regular* (resp. *purely T-regular*) if it is globally T-regular (resp. purely globally T-regular).

- Let $f : X \longrightarrow Z$ be a quasi-projective morphism of V-varieties. (X, Δ) is globally T-regular (resp. purely globally T-regular) over $z \in Z$ if (X, Δ) is globally T-regular (resp. purely globally T-regular) after localizing at z.

Proposition 3.2.10. ([13, Theorem 6.7]) *Let X be a quasi-projective variety over an F-finite field. Let (X, Δ) is a log pair and L a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on X. Then*

$$T^0(X, \Delta; L) = \bigcap_{\pi} \operatorname{Im}(H^0(\omega_Y(\lceil \pi^*(L - K_X - \Delta) \rceil)) \longrightarrow H^0(\mathcal{O}_X(\lceil L \rceil))),$$

where π runs over all alterations from a normal V-variety.

Remark 3.2.11. In [13], T^0 is defined as in Proposition 3.2.10, and they proved the equivalence of the definitions in positive characteristic. However, in mixed characteristic or in characteristic zero, it does not hold in general. Indeed, by Proposition 3.2.13, globally T-regular implies being klt.

Proposition 3.2.12. *Let V be a scheme satisfying Assumption 3.2.1. Let $f\colon X \longrightarrow Z$ be a projective surjective morphism of V-varieties. Let (X, Δ) is log pair over V. If (X, Δ) is globally T-regular, then so is (X', Δ'), where (X', Δ') is one of the following.*

- *A restriction of (X, Δ) over on some open subset X' of X, or*
- *A localization of (X, Δ) at some point of Z.*

Furthermore, the same assertion holds for purely globally T-regular log pairs.

Proof. We can extend every alteration of X' to it of X by Proposition 3.1.5, and the surjectivity is equivalent to the property that image of the trace map contains $1 \in H^0(\mathcal{O}_X)$. It does not change after localization or restriction. $\square$

Proposition 3.2.13. *Let V be an excellent Dedekind scheme. Let (X, Δ) be a log pair over V. If (X, Δ) is globally T-regular, then it is klt, and in particular, $\lfloor \Delta \rfloor = 0$. If (X, Δ) is purely globally T-regular, then it is plt, and in particular, $\lfloor \Delta \rfloor$ is reduced.*

Proof. By Proposition 3.2.12, we may assume that X is affine. We take a birational projective morphism $\pi\colon Y \longrightarrow X$ from a normal V-variety. If (X, Δ) is globally T-regular, then we have $\pi_*\omega_{Y/V}(\lceil -\pi^*(K_{X/V} + \Delta) \rceil) = \mathcal{O}_X$, as trace map is injective in the birational case. Thus, (X, Δ) is klt. If (X, Δ) is purely globally T-regular, we denote $S := \lfloor \Delta \rfloor$, then we have $\omega_{Y/V}(S_Y + \lceil -\pi^*(K_{X/V} + \Delta) \rceil) = \mathcal{O}_X$, where S_Y is the strict transform of S. Thus, (X, Δ) is plt. $\square$

Proposition 3.2.14. *Let V be an excellent Dedekind scheme. Let $(X, S + \Delta)$ be a log pair with $\lfloor S + \Delta \rfloor = S$. Assume that $(X, S + \Delta)$ is purely globally T-regular and S is $\mathbb{Q}$-Cartier. Then $(X, (1 - \varepsilon)S + \Delta)$ is globally T-regular for all $0 < \varepsilon < 1$.*

Proof. We fix a $0 < \varepsilon < 1$ and a positive integer m with $m\varepsilon > 1$. We take an alteration $\pi\colon Y \longrightarrow X$ from a normal V-variety and a strict transform S_Y of S such that $\pi^*S \geqslant mS_Y$ and $\pi^*(K_X + \Delta)$ is Cartier, the existence follows from Proposition 3.1.8. It is enough to show that the trace map

$$H^0(\omega_{Y/V}(\lceil -\pi^*(K_{X/V} + (1 - \varepsilon)S + D) \rceil)) \longrightarrow H^0(\mathcal{O}_X)$$

is surjective. Since $\pi^*(1-\varepsilon)S \leqslant \pi^*S - \varepsilon m S_Y \leqslant \pi^*S - S_Y$, the left hand side is larger than the left hand side of the map (3.2) . Thus, if $(X, S + \Delta)$ is purely globally T-regular, then $(X, (1 - \varepsilon)S_Y + \Delta)$ is globally T-regular. $\square$

Proposition 3.2.15. *Let V be a scheme satisfying Assumption 3.2.1. Let $f\colon X \longrightarrow Z$ be a projective surjective morphism from a normal V-variety X to an affine V-variety Z. Let (X, Δ) is a globally T-regular log pair over V. Then the following conditions hold.*

1. *For every alteration $\pi\colon Y \longrightarrow X$ from a normal V-variety, the trace map*

$$\pi_*\omega_{Y/V}(\lceil -\pi^*(K_{X/V} + \Delta)\rceil) \longrightarrow \mathcal{O}_X$$

is a splitting surjection.

2. *For every finite surjective morphism $\pi\colon Y \longrightarrow X$ from a normal V-variety, the canonical map*

$$\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y(\lfloor \pi^*\Delta \rfloor)$$

splits.

Proof. By the equation (3.1), there exists a global section

$$\alpha \in H^0(\omega_{Y/V}(\lceil -\pi^*(K_{X/V} + \Delta)\rceil))$$

mapped to $1 \in H^0(\mathcal{O}_X)$. The section α defines a morphism

$$\mathcal{O}_X \longrightarrow \pi_*\omega_{Y/V}(\lceil -\pi^*(K_{X/V} + \Delta)\rceil)$$

mapping 1 to α, thus it gives a splitting and we obtain (1). Applying $\mathscr{H}om(\mathcal{O}_X, \_\_)$ for the trace map in (1), we obtain the canonical map in (2) when π is finite surjective, thus we have (2). $\square$

Lemma 3.2.16. *Let V be a scheme satisfying Assumption 3.2.1. Let (X, Δ) be a globally T-regular log pair over V. Let D be a Weil $\mathbb{Q}$-Cartier divisor on X. Let $f\colon Y \longrightarrow X$ be a finite surjective morphism from a normal V-variety such that f^*D is Cartier. Then $\mathcal{O}_X(D) \longrightarrow f_*\mathcal{O}_Y(f^*D)$ splits.*

Proof. By Proposition 3.2.15 (2), $\mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ splits. Thus, $\mathcal{O}_X(D) \longrightarrow f_*\mathcal{O}_Y(f^*D)$ splits on the regular locus of X. Since $\mathcal{O}_X(D)$ is reflexive, the splitting extends to the splitting on X. $\square$

Proposition 3.2.17. *Let V be a scheme satisfying Assumption 3.2.1. Let $f\colon X \longrightarrow Z$ be a projective surjective morphism from a normal V-variety X to an affine V-variety Z. Let (X, Δ) is a globally T-regular log pair over V. Let D be a Weil $\mathbb{Q}$-Cartier divisor. Then the following hold.*

1. *If D is semiample, then $H^i(\mathcal{O}_X(D)) = 0$ around the closed fiber Z_s for all $i > 0$.*

2. *$\mathcal{O}_X(D)$ is maximal Cohen-Macaulay.*

3. *If D is semiample and f-big, then $H^i(\omega_{X/V}(D)) = 0$ for all $i > 0$.*

Proof. By [8, Theorem 6.28], there exists a finite surjective morphism $\pi\colon Y \longrightarrow X$ such that π^*D is Cartier and

$$H^i(\mathcal{O}_X(D)|_{X_p}) \longrightarrow H^i(\mathcal{O}_{Y_p}(\pi^*D))$$

is zero for all $i > 0$. Indeed, the assertion is reduced to the case where D is Cartier. By Lemma 3.2.16, we have $H^i(\mathcal{O}_X(D)|_{X_p}) = 0$. Thus, we have the map

$$H^i(\mathcal{O}_X(D)) \xrightarrow{\cdot p} H^i(\mathcal{O}_X(D))$$

is surjective for all $i > 0$. Thus, By the Nakayama's Lemma, $H^i(\mathcal{O}_X(D)) = 0$ around Z_s.

Next, we consider (2) and (3). Since the assertion (2) is a local question, we may assume that D is semiample and f-big. By [8, Theorem 6.28], for a closed point $z \in Z$, there exists a finite surjective morphism $\pi\colon Y \longrightarrow X$ such that π^*D is Cartier and the map

$$R^i\Gamma_z R\Gamma(\mathcal{O}_X(-D)|_{X_p}) \longrightarrow R^i\Gamma_z R\Gamma(\mathcal{O}_{Y_p}(-\pi^*D))$$

is zero for all $i < \dim X_p$. By Lemma 3.2.16, we have $R^i\Gamma_z R\Gamma(\mathcal{O}_X(-D)|_{X_p}) = 0$. If X is affine and f is the identity map, then $\mathcal{O}_X(D)$ is maximal Cohen-Macaulay around X_s. Since (X, Δ) is klt, $\mathcal{O}_X(D)$ is maximal Cohen-Macaulay on the generic fiber, thus we obtain the assertion (2). By the argument of the proof of Proposition 3.2.4, we have

$$H^i(\mathcal{H}om(\mathcal{O}_X(-D)|_{X_p}, \omega_{X_p/V})) = 0$$

for all $i > 0$. Thus, we have that

$$H^i(\omega_{X/V}(D)) \xrightarrow{\cdot p} H^i(\omega_{X/V}(D))$$

is surjective for all $i > 0$. Thus, $H^i(\omega_{X/V}(D)) = 0$ around Z_s. By Kawamata-Viehweg vanishing for the generic fiber X_η, the vanishing holds on Z_η. Thus we obtain the assertion (3). □

Proposition 3.2.18. *Let V be an excellent Dedekind scheme. Let $g\colon Y \longrightarrow X$ be a projective birational morphism of normal V-varieties. Let (X, Δ) and (Y, Γ) be log pairs such that $g^*(K_{X/V} + \Delta) = K_{Y/V} + \Gamma$. Then (X, Δ) is globally T-regular if and only if (Y, Γ) is globally T-regular. The same assertion holds for purely globally T-regular case if $\lfloor \Gamma \rfloor$ is the strict transform of $\lfloor \Delta \rfloor$.*

Proof. First, we consider the globally T-regular case. We note that (X, Δ) is klt if and only if (Y, Γ) is klt. By Proposition 3.2.13, we may assume that (X, Δ) and (Y, Γ) are klt, and in particular, $\lfloor \Gamma \rfloor = 0$. Thus, the trace map coincides with the following isomorphism

$$H^0(\omega_{Y/V}(\lceil -g^*(K_{X/V} + \Delta) \rceil) = H^0(\mathcal{O}_Y) \simeq H^0(\mathcal{O}_X).$$

Take an alteration $\pi\colon W \longrightarrow Y$. Then the composition of trace maps

$$H^0(\omega_{W/V}(\lceil -\pi^*(K_{Y/V} + \Gamma) \rceil)) \longrightarrow H^0(\mathcal{O}_Y) \simeq H^0(\mathcal{O}_X)$$

is the trace map with respect to $g \circ \pi$. Then the surjectivities of two trace maps are equivalent to each other.

Next, we consider the purely globally T-regular case. We denote $\lfloor\Delta\rfloor$ and $\lfloor\Gamma\rfloor$ by S and T, respectively. Then the corresponding trace map coincides with natural isomorphic $H^0(\mathcal{O}_Y) \longrightarrow H^0(\mathcal{O}_X)$. Indeed, $\lceil K_{Y/V}+T-g^*(K_{X/V}+\Delta)\rceil = \lceil T-\Gamma\rceil = 0$. Thus, by the same argument as above, we obtain the equivalence. $\square$

Lemma 3.2.19. *Let V be a scheme satisfying Assumption 3.2.1. Let $(X, S+B)$ be a log pair over V such that S is a reduced divisor and S and B have no common components. Let $\pi\colon X' \longrightarrow X$ be an alteration of normal V-varieties and S' be a strict transform of S on X'. Then there exists an alteration $f\colon Y \longrightarrow X$ from a normal V-variety and a strict transform S_Y of S such that the following hold.*

- *f factors through π and S_Y is a strict transform of S',*
- *S_Y is locally irreducible,*
- *(Y, S_Y) is a simple normal crossing pair,*
- *$f^*(K_{X/V}+S+B)$ and $f_S^*(K_{S^N/V}+B_S)$ are Cartier, and*
- *the following diagram commutes.*

$$\begin{array}{ccc} f_*\omega_{Y/V}(S_Y - f^*(K_{X/V}+S+B)) & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \\ j_*f_{S,*}\omega_{S_Y/V}(-f_S^*(K_{S^N/V}+B_S)) & \longrightarrow & j_*\mathcal{O}_{S^N}, \end{array}$$

where B_S is the different of $(X, S+B)$, $f_S\colon S_Y \longrightarrow S^N$ is the induced morphism and $j\colon S^N \longrightarrow X$ is the composition of the normalization and the inclusion.

Proof. By the definition of different, we have $(K_{X/V}+S+B)|_{S^N} \sim_{\mathbb{Q}} K_{S^N/V}+B_S$. By Theorem 3.1.9, there exists an alteration $f\colon Y \longrightarrow X$ and a strict transform S_Y of S such that (Y, S_Y) is a simple normal crossing pair and f factors through π. By taking blowing up along the stratum of S_Y, we may assume that S_Y is locally irreducible, thus S_Y is regular. In particular, $S_Y \longrightarrow S$ factors through the normalization $S^N \longrightarrow S$ denoted by $f_S\colon S_Y \longrightarrow S^N$. By Proposition 3.1.8, we may assume that $f^*(K_{X/V}+S+B)$ and $f^*(K_{S^N/V}+B_S)$ is Cartier and

$$f^*(K_{X/V}+S+B)|_{S_Y} \sim f_S^*((K_{X/V}+S+B)|_{S^N}) \sim f_S^*(K_{S^N/V}+B_S).$$

Thus, we define the morphism $\omega_{Y/V}(S_Y - f^*(K_{X/V}+S+B)) \longrightarrow \omega_{S_Y/V}(-f_S^*(K_{S^N/V}+B_S))$ induced by the adjunction formula $\omega_{Y/V}(S_Y)|_{S_Y} \simeq \omega_{S_Y/V}$. Then it is enough to show that the diagram

$$\begin{array}{ccc} f_*\omega_{Y/V}(S_Y - f^*(K_{X/V}+S+B)) & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \\ j_*f_{S,*}\omega_{S_Y/V}(-f_S^*(K_{S^N/V}+B_S)) & \longrightarrow & j_*\mathcal{O}_{S^N} \end{array}$$

commutes. Since $\mathscr{H}om(f_*\omega_{Y/V}(S_Y - f^*(K_{X/V} + S + B)), j_*\mathcal{O}_{S^N})$ is torsion-free as $\mathcal{O}_S$-module, it is enough to show that the above diagram commutes at every generic point of S. Thus, we may assume that (X, S) is a simple normal crossing pair and $B = 0$. By the construction of the residue map $\omega_{X/V}(S) \longrightarrow \omega_{S/V}$ and the trace map, we have the commutative diagram

$$\begin{array}{ccc} f_*\omega_{Y/V}(S_Y) & \longrightarrow & \omega_{X/V}(S) \\ \downarrow & & \downarrow \\ f_{S,*}\omega_{S_Y/V} & \longrightarrow & \omega_{S/V}. \end{array}$$

Applying $\otimes\mathcal{O}_X(-(K_{X/V} + S))$, we have the assertion. ☐

Proposition 3.2.20. (Adjunction for global T-regularity) *Let V be a scheme satisfying Assumption 3.2.1. Let $f\colon X \longrightarrow Z$ be a projective surjective morphism from a normal V-variety X to an affine V-variety Z and $(X, S + B)$ a purely globally T-regular pair with $\lfloor S + B \rfloor = S$. Then (S^N, B_S) is globally T-regular, where $B_S = \mathrm{Diff}_{S^N}(B)$ is the different of $(X, S + B)$.*

Proof. We take an alteration $g\colon T \longrightarrow S^N$. Combining Proposition 3.1.7 and Lemma 3.2.19, there exists an alteration $f\colon Y \longrightarrow X$ and a strict transform S_Y of S as in Lemma 3.2.19 and $f_S\colon S_Y \longrightarrow S^N$ factors through g. Then we have the commutative diagram

$$\begin{array}{ccc} H^0(\omega_{Y/V}(S_Y - f^*(K_{X/V} + S + B))) & \twoheadrightarrow & H^0(\mathcal{O}_X) \\ \downarrow & & \downarrow \\ H^0(\omega_{S_Y/V}(-f_S^*(K_{S^N/V} + B_S))) & \longrightarrow & H^0(\mathcal{O}_{S^N}). \end{array}$$

Thus, the image of the bottom map contains $1 \in H^0(\mathcal{O}_{S^N})$, and in particular, it is surjective. ☐

Proposition 3.2.21. (Inversion of Adjunction for global T-regularity) *Let V be a scheme satisfying Assumption 3.2.1. Let $f\colon X \longrightarrow Z$ be a projective surjective morphism from a normal V-variety X to an affine V-variety Z and $(X, S+\Delta)$ a log pair such that $\lfloor S+\Delta \rfloor = S$ is reduced and S has no common component with Δ. Let $X \longrightarrow Z' \longrightarrow Z$ be the Stein factorization and we denote the induced morphisms by $f'\colon X \longrightarrow Z'$ and $\varphi\colon Z' \longrightarrow Z$. Let $z \in Z$ be a point with $\varphi^{-1}(z) \subset f'(S)$. We assume that $-(K_{X/V} + S + \Delta)$ is semiample and f-big. If (S^N, D) is globally T-regular over z, then $(X, S + \Delta)$ is purely globally T-regular over z and S is normal, where S^N is the normalization of S and $D = \mathrm{Diff}_{S^N}(\Delta)$ is the different of the pair $(X, S + \Delta)$.*

Proof. We denote $H := -(K_{X/V} + S + \Delta)$ and $H_S := -(K_{S^N/V} + D)$. We take base changes via $\operatorname{Spec}\mathcal{O}_{Z,x} \longrightarrow Z$ and we use the same notations by abuse of notations. By the assumption $\varphi^{-1}(z) \subset f'(S)$, the ideal $H^0(\mathcal{O}_X(-S))$ of $H^0(\mathcal{O}_X) = H^0(\mathcal{O}_{Z'})$ is contained in all maximal ideals of $H^0(\mathcal{O}_X)$. We take an alteration $g\colon Y \longrightarrow X$ and a strict transform S_Y of S as in Lemma 3.2.19. In order to show that $(X, S+\Delta)$ is purely globally T-regular, it is enough to show that the image $I_g \subset H^0(\mathcal{O}_X)$ of the map

$$H^0(\omega_{Y/V}(S_Y + g^*H)) \longrightarrow H^0(\mathcal{O}_X)$$

is $H^0(\mathcal{O}_X)$. Since $H^0(\mathcal{O}_X)$ is Noetherian and $H^0(\mathcal{O}_X(-S))$ is contained in all maximal ideals, it is enough to show $\psi(I_g) = H^0(\mathcal{O}_{S^N})$, where $\psi\colon H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_{S^N})$. We take an element $\alpha \in H^0(\mathcal{O}_{S^N})$. Since g^*H is semiample and big over Z, there exists an alteration $h\colon W \longrightarrow Y$ and a strict transform S_W of S as in Lemma 3.2.19 and the trace map

$$H^1(\omega_{W/V}(h^*g^*H)) \longrightarrow H^1(\omega_{Y/V}(g^*H))$$

is zero by Corollary 3.2.5 and Remark 3.2.6. Since (S^N, D) is globally T-regular, there exists a section

$$\alpha_W \in H^0(\omega_{S_W/V}(h_S^*g_S^*H_S))$$

mapped to $\alpha \in H^0(\mathcal{O}_{S^N})$ by the trace map. Let $\alpha_Y \in H^0(\omega_{S_Y/V}(g_S^*H_S))$ be the image of α_W via the trace map. By the exact sequence

$$0 \longrightarrow \omega_{Y/V}(g^*H) \longrightarrow \omega_{Y/V}(S_Y + g^*H) \longrightarrow \omega_{S_Y/V}(g_S^*H_S) \longrightarrow 0,$$

we have the exact sequence

$$H^0(\omega_{Y/V}(S_Y + g^*H)) \longrightarrow H^0(\omega_{S_Y/V}(g_S^*H_S)) \longrightarrow H^1(\omega_{Y/V}(g^*H)).$$

Thus we have the following commutative diagram

$$\begin{tikzcd}
H^0(\omega_{S_W/V} - h_S^*g_S^*H_S)) \arrow[r] \arrow[d] & H^1(\omega_{W/V}(h^*g^*H)) \arrow[d, "0\text{-map}"] \\
H^0(\omega_{S_Y/V}(g_S^*H_S)) \arrow[r] & H^1(\omega_{Y/V}(g^*H))
\end{tikzcd}$$

by Lemma 3.2.19. Thus the image of α_Y via the connection map is zero, so α_Y extends to a section $\gamma \in H^0(\omega_{Y/V}(S_Y + g^*H))$ and its image is α in $H^0(\mathcal{O}_{S^N})$. Thus we have $\alpha \in \psi(I_g)$, and the equation $\psi(I_g) = H^0(\mathcal{O}_{S^N})$ holds.

Next, we prove the normality of S. We take an open affine covering $\{U_i\}$ of X. By Proposition 3.2.12, each $(U_i|_{S^N}, D|_{U_i|_{S^N}})$ is globally T-regular. This is a local problem, we may assume that X is affine by Proposition 3.2.12. By the above argument, we have $\psi(\mathcal{O}_X) = \mathcal{O}_{S^N}$, in particular, S is normal. □

Corollary 3.2.22. *Let V be a scheme satisfying Assumption 3.2.1. Let X be a normal affine V-variety and $(X, S+\Delta)$ be a log pair such that $\lfloor S+\Delta \rfloor = S$ is reduced. Then $(X, S+\Delta)$ is purely T-regular if and only if $(S^N, \mathrm{Diff}_{S^N}(\Delta))$ is T-regular. Furthermore, in both cases, S is normal, and in particular, S is locally irreducible.*

Proof. It follows from Proposition 3.2.20 and Proposition 3.2.21 for the case $f =$ id. □

Corollary 3.2.23. *Let V be a scheme satisfying Assumption 3.2.1. Let (X, Δ) be a simple normal crossing pair with $\lfloor \Delta \rfloor = 0$, where X is an affine V-variety. Then (X, Δ) is T-regular.*

Proof. We prove Corollary 3.2.23 by the induction on $d := \dim X$. We take an alteration $\pi \colon Y \longrightarrow X$ from a normal V-variety Y. It is enough to show that the trace map is surjective at each closed point $x \in X$. First, we consider the case where x is not contained in $\mathrm{Supp}(\Delta)$. By [7, Theorem 1.2],

$$\mathcal{O}_X \longrightarrow R\pi_*\mathcal{O}_Y$$

splits. By the Grothendieck duality, the map $\pi_*\omega_{Y/V} \longrightarrow \omega_{X/V}$ is surjective. Since X is Gorenstein, the trace map

$$\pi_*\omega_{Y/V}(-\pi^*K_{X/V}) \longrightarrow \mathcal{O}_X$$

is also surjective. Next, we assume that x is contained in a component S of $\mathrm{Supp}(\Delta)$. We take a positive rational number a with $\mathrm{ord}_S(\Delta + aS) = 1$. By Proposition 3.2.14, it is enough to show that $(X, \Delta + aS)$ is purely T-regular at x. By Corollary 3.2.22, it is enough to show that $(S, (\Delta - (1-a)S)|_S)$ is T-regular at x. Since this pair is a simple normal crossing pair, this pair is T-regular by the induction hypothesis on d. In conclusion, (X, Δ) is T-regular. $\square$

Remark 3.2.24. By the proof of [67, Theorem 6.21], if a log pair (X, Δ) is BCM-regular at x after completion, then (X, Δ) is T-regular at x. Thus, Corollary 3.2.23 also follows from [68, Theorem 4.1].

Proposition 3.2.25. *Let V be a scheme satisfying Assumption 3.2.1. Let $f \colon X \longrightarrow Z$ be a projective surjective morphism from a normal V-variety X to an affine V-variety Z and $(X, S + \Delta)$ a log pair such that $\lfloor S + \Delta \rfloor = S$ is a reduced divisor and S has no common component with Δ. Let L be a $\mathbb{Q}$-Cartier Weil divisor such that $L - (K_{X/V} + S + \Delta)$ is semiample and f-big and L is Cartier at all codimension two points of X contained in S. Let $g \colon Y \longrightarrow X$ be an alteration from a normal V-variety Y and S_Y a strict transform of S. Then we have*

$$T^0(S^N, D; L|_S) \subset I_g|_{S^N},$$

where $D := \mathrm{Diff}_{S^N}(\Delta)$ is the different, I_g is the image of the trace map

$$I_g := \mathrm{Im}(H^0(\omega_{Y/V}(S_Y + \lceil g^*(L - (K_{X/V} + S + \Delta)) \rceil)) \longrightarrow H^0(\mathcal{O}_X(L)))$$

and the right hand side is the image of I_g via the natural map

$$H^0(\mathcal{O}_X(L)) \longrightarrow H^0(\mathcal{O}_{S^N}(L|_{S^N})).$$

Proof. We denote $H := L - (K_{X/V} + S + \Delta)$ and $H_S := L_{S^N} - (K_{S^N/V} + D)$. We may assume that (Y, S_Y) is as in Lemma 3.2.19. By the proof of Proposition 3.2.21, we can contract an alteration $h \colon W \longrightarrow Y$ and a strict transform S_W of S as in

Lemma 3.2.19 such that the trace map on the first cohomology is zero. We obtain the commutative diagram

$$\begin{tikzcd}
H^0(\omega_{W/V}(S_W + h^*g^*H)) \arrow[r] \arrow[d] & H^0(\omega_{S_W/V}(h_S^*g_S^*H_S)) \arrow[r] \arrow[d] & H^1(\omega_{W/V}(h^*g^*H))) \arrow[d, "0-\mathrm{map}"] \\
H^0(\omega_{Y/V}(S_Y + g^*H)) \arrow[r] \arrow[d] & H^0(\omega_{S_Y/V}(g_S^*H_S)) \arrow[r] \arrow[d] & H^1(\omega_{Y/V}(g^*H)) \\
H^0(\mathcal{O}_X(L)) \arrow[r] & H^0(\mathcal{O}_{S^N}(L|_{S^N})).
\end{tikzcd}$$

Thus the proof is same as the proof of Proposition 3.2.21. $\square$

Proposition 3.2.26. *Let k be an F-finite field of positive characteristic. Let $f\colon X \longrightarrow Z$ be a projective surjective morphism from a normal k-variety X to an affine k-variety Z Let (X, B) be a log pair. Assume that (X, B) is globally F-regular. Let $\pi\colon Y \longrightarrow X$ be an alteration from a normal variety Y. If the trace map*

$$H^0(\omega_Y(\lceil -\pi^*(K_X + B)\rceil) \longrightarrow H^0(\mathcal{O}_X)$$

is non-zero, then it is surjective. In particular, if we further assume that the generic fiber of $(X, B) \longrightarrow Z$ is globally T-regular, then (X, B) is globally T-regular.

Proof. Let H be an ample Cartier divisor on X such that $\mathcal{O}_X(K_X + B + H)$ is globally generated. We take an effective divisor B' which is linearly equivalent to $K_X + B + H$. Then for divisible enough e, $(X, B + \frac{1}{p^e-1}B')$ is globally F-regular, the Cartier index of

$$(p^e - 1)(K_X + B + \frac{1}{p^e-1}B') \sim p^e(K_X + B) + H$$

is prime to p, and $\lceil -\pi^*(K_X + B + \frac{1}{p^e-1}B')\rceil = \lceil -\pi^*(K_X + B)\rceil$. Thus, we may assume that the Cartier index of $K_X + B$ is prime to p. We take an element $\alpha \in H^0(\omega_Y(\lceil -\pi^*(K_X + B)\rceil))$ such that $\beta := \mathrm{Tr}(\alpha) \in H^0(\mathcal{O}_X)$ is non-zero. Since (X, B) is globally F-regular, there exists a positive integer e such that $(p^e - 1)(K_X + B)$ is Cartier and the natural map

$$\mathcal{O}_X \longrightarrow F_*^e\mathcal{O}_X(\lfloor (p^e - 1)B\rfloor + \mathrm{div}(\beta))$$

splits. We take a section $\gamma \in H^0(\mathcal{O}_X((1 - p^e)(K_X + B)))$ corresponding to the splitting. Then the trace map

$$H^0(\mathcal{O}_X((1 - p^e)(K_X + B)) \longrightarrow H^0(\mathcal{O}_X)$$

maps $\gamma\beta$ to 1. Then the composition of morphisms

$$H^0(\omega_Y(\lceil -p^e\pi^*(K_X + B)\rceil) \longrightarrow H^0(\mathcal{O}_X((1 - p^e)(K_X + B))) \longrightarrow H^0(\mathcal{O}_X)$$

maps $\alpha\gamma$ to 1. Thus, the map

$$H^0(\omega_Y(\lceil -\pi^*(K_X + B)\rceil) \longrightarrow H^0(\mathcal{O}_X)$$

is surjective. $\square$

3.2.3 Restriction theorem

The goal of this subsection is to prove the existence of three-dimensional pl-flips with ample divisor in the boundary (Corollary 3.2.33). First, we prove the existence by assuming the global T-regularity of the boundary even in the higher-dimensional case (Theorem 3.2.29). It follows from the restriction theorem (Proposition 3.2.28) and Shokurov's reduction to pl-flips. Next, we show this condition in the three-dimensional case.

Lemma 3.2.27. (cf. [44, Lemma 3.2]) *Let V be a scheme satisfying Assumption 3.2.1. Let $f\colon X \longrightarrow Z$ be a projective birational morphism from a normal V-variety to an affine V-variety. Let (X, B) be a log pair which is globally T-regular over a point z of Z. Let L be a Weil $\mathbb{Q}$-Cartier divisor on X and Γ an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on X. If $g \in H^0(X, \mathcal{O}_X(L))$ corresponds to a divisor $G \in |L|$ such that $G \geqslant \Gamma$, then g is contained in $T^0(X, B + \Gamma; L)$ after localizing at z.*

Proof. We take base changes via $\operatorname{Spec} \mathcal{O}_{Z,x} \longrightarrow Z$ and we use the same notations by abuse of notations. It is enough to consider alterations $h\colon Y \longrightarrow X$ from a normal V-variety such that $h^*(K_{X/V} + B)$, h^*L, h^*G and $h^*\Gamma$ are Cartier. We consider the commutative diagram

$$\begin{array}{ccc}
H^0(\omega_{Y/V}(h^*(L - K_{X/V} - B - \Gamma))) & \longrightarrow & H^0(\mathcal{O}_X(L)) \\
\uparrow & & \uparrow \\
H^0(\omega_{Y/V}(h^*(L - K_{X/V} - B - G))) & \longrightarrow & H^0(\mathcal{O}_X(L - G)) \\
\simeq \uparrow \cdot g & & \simeq \uparrow \cdot g \\
H^0(\omega_Y(-h^*(K_{X/V} + B))) & \twoheadrightarrow & H^0(\mathcal{O}_X),
\end{array}$$

where the horizontal maps are trace maps and the surjectivity of the bottom map follows from the global T-regularity of (X, B). Thus, we have a section of $H^0(\omega_{Y/V}(h^*(L - K_{X/V} - B - \Gamma)))$ mapped to g. $\square$

Proposition 3.2.28. (cf. [44, Proposition 3.1]) *Let V be a scheme satisfying Assumption 3.2.1. Let $f\colon X \longrightarrow Z$ be a projective morphism from a normal V-variety X to an affine V-variety Z. Let $(X, S + A + B)$ be a dlt pair such that S and A are $\mathbb{Q}$-Cartier Weil divisors. Assume that A is ample and $\lfloor B \rfloor = 0$. Let $z \in Z$ be a point. If S is normal and $(S, (1 - \varepsilon)A_S + B_S)$ is globally T-regular over z for all $0 < \varepsilon < 1$, then for every $k \geqslant 1$ such that $k(K_{X/V} + S + A + B)$ is Cartier, we have*

$$|k(K_{X/V} + S + A + B)|_S = |k(K_{S/V} + A_S + B_S)|$$

after localization at z, where $B_S := \operatorname{Diff}_S(B)$ and $A_S := A|_S$.

Proof. We take base changes via $\operatorname{Spec} \mathcal{O}_{Z,x} \longrightarrow Z$ and we use the same notations by abuse of notations. We note that since $(X, S + A + B)$ is dlt and S and A are $\mathbb{Q}$-Cartier, $(X, S + A + B)$ is a simple normal crossing pair around $S \cap A$, and in particular, A is Cartier at all codimension two points on X contained in S. We take $F \in |k(K_{X/V}+S+A+B)|$. By the descending induction, we prove that there exists divisors $G_m \in |k(K_{X/V}+S+A+B)+mA|$ such that $G_m|_S = F+mA_S$. First, since A is ample, such G_m exists for large enough m by Serre vanishing. We assume that such G_{m+1} exists. We set

$$\begin{aligned} L &:= k(K_{X/V}+S+A+B)+mA \\ &= K_{X/V}+S+B+(k-1)(K_{X/V}+S+A+B)+(m+1)A \\ &\sim_{\mathbb{Q}} K_{X/V}+S+B+\frac{k-1}{k}G_{m+1}+\frac{m+1}{k}A. \end{aligned}$$

We write $H := L-(K_{X/V}+S+B+\frac{k-1}{k}G_{m+1})$, then it is ample. Since $(S, \frac{k-1}{k}A_S+B_S)$ is globally T-regular and

$$F+mA_S \geqslant \frac{k-1}{k}(F+mA_S) = \frac{k-1}{k}G_{m+1}|_S - \frac{k-1}{k}A_S,$$

a section $g \in H^0(L|_S)$ corresponding to $F+mA_S$ is contained in

$$T^0(S, B_S+\frac{k-1}{k}G_{m+1}|_S; L|_S)$$

by Lemma 3.2.27. By Proposition 3.2.25, g is contained in the image of $H^0(\mathcal{O}_X(L))$. □

The following theorem is the existence of pl-flips with ample divisor in the boundary in the special setting. The proof is an analog of the proof of [44, Theorem 1.3].

Theorem 3.2.29. (cf. [44, Theorem 1.3]) *Let V be a scheme satisfying Assumption 3.2.1. Let $(X, S+A+B)$ be a dlt pair such that S is an anti-ample $\mathbb{Q}$-Cartier Weil divisor and A is an ample $\mathbb{Q}$-Cartier Weil divisor. Let $f\colon X \longrightarrow Z$ be a $(K_{X/V}+S+A+B)$-flipping contraction with $\rho(X/Z)=1$ to an affine V-variety. Furthermore, we assume that $(S^N, (1-\varepsilon)A_S+B_S)$ is globally T-regular for all $0<\varepsilon<1$ over all points of $f(\mathrm{Exc}(f))$ and $R(K_{S^N/V}+A_S+B_S)$ is finitely generated, where $B_S := \mathrm{Diff}_{S^N}(B)$ and $A_S := A|_S$. Then the flip of f exists.*

Proof. Take a point $z \in Z$ contained in $f(\mathrm{Exc}(f))$. Since A is ample, we have $z \in f(A)$. We take base changes via $\operatorname{Spec} \mathcal{O}_{Z,x} \longrightarrow Z$ and we use the same notations by abuse of notations. We may assume that $\lfloor B \rfloor = 0$. By Proposition 3.2.21, the scheme S is normal. By Proposition 3.2.28, the restriction algebra

$$R_S(k(K_{X/V}+S+A+B)) := \mathrm{Im}(R(k(K_{X/V}+S+A+B)) \longrightarrow R(k(K_{S/V}+A_S+B_S)))$$

coincides with $R(k(K_{S/V}+A_S+B_S))$ for some positive integer k, and in particular, $R_S(K_{X/V}+S+A+B)$ is finitely generated. By Shokurov's reduction to pl-flips (see [22, Lemma 2.3.6]), the flip of f exists. □

Remark 3.2.30. If $(S^N, (1-\varepsilon)A_S + B_S)$ is globally F-regular over Z, then it is globally T-regular over Z by Proposition 3.2.26. Thus, applying Theorem 3.2.29 for the case $X = X_s$, we obtain [44, Theorem 1.3].

In order to use Theorem 3.2.29 for threefolds, we will show the pure global T-regularity of $(S^N, A_S + B_S)$.

Lemma 3.2.31. (cf. [43, Lemma 3.3]) *Let V be a scheme satisfying Assumption 3.2.1. We assume that the residue field of V is infinite. Let $f\colon S \longrightarrow T$ be a projective birational morphism from a normal V-surface S to an affine V-surface T. Let $(S, C+B)$ is plt pair with $\lfloor C+B \rfloor = C$. Assume that $-(K_{S/V}+C+B)$ and C are ample. Further assume that f has connected fibers. Then $(S, C+B)$ is purely globally T-regular over all points of $f(\mathrm{Exc}(f))$.*

Proof. Since f has connected fibers, the Stein factorization of f induces a homeomorphism $\varphi\colon T' \longrightarrow T$. Thus we have $\varphi^{-1}(f(\mathrm{Exc}(f)) = f'(\mathrm{Exc}(f'))$, so we may assume that T is normal and $f_*\mathcal{O}_X = \mathcal{O}_T$ by replacing T into T', where $f'\colon X \longrightarrow T'$ is the induced morphism. We take a point $t \in f(\mathrm{Exc}(f))$. First, we prove that C is irreducible after shrinking T around t. By shrinking T around t, the image of every irreducible component of C contains t. Since S is surface and $(S, C+B)$ is plt, then C is locally irreducible. By [93, Theorem 5.2], the intersection $C \cap f^{-1}(t)$ is connected, and in particular, the scheme C is irreducible. Furthermore, as C is ample, C is not an exceptional divisor of f. Since $-(K_{S/V}+C+B)$ is ample and S is a V-surface, there exists an effective $\mathbb{Q}$-Weil divisor D on S such that $(S, C+B+D)$ is plt and $K_{S/V}+C+B+D$ is $\mathbb{Q}$-linearly trivial by Lemma 3.1.12. We take base changes via $\operatorname{Spec}\mathcal{O}_{T,t} \longrightarrow T$ and we use the same notations by abuse of notations. We set $f_*C = C'$, $f_*B = B'$, and $f_*D = D'$. Since $K_{S/V}+C+B+D$ is $\mathbb{Q}$-linearly trivial, we have $f^*(K_{T/V}+C'+B'+D') = K_{S/V}+C+B+D$, thus $(T, C'+B'+D')$ is plt as C is not an exceptional divisor. By the adjunction, $(C', \mathrm{Diff}_{C'}(B'+D'))$ is a normal klt one-dimensional pair, thus this is a simple normal crossing pair and $\lfloor \mathrm{Diff}_{C'}(B'+D') \rfloor = 0$. By Corollary 3.2.23, $(C', \mathrm{Diff}_{C'}(B'+D'))$ is T-regular. By Proposition 3.2.21, the pair $(T', C'+B'+D')$ is purely T-regular. By Proposition 3.2.18, the pair $(S, C+B+D)$ is purely globally T-regular. We note that C is the strict transform of C'. $\square$

Lemma 3.2.32. *Let V be a scheme satisfying Assumption 3.2.1. Assume that the residue field of V is infinite. Let $f\colon X \longrightarrow Z$ be a small projective birational morphism from a normal V-variety X of dimension three to an affine V-variety Z. Let $(X, S+A+B)$ is a dlt pair such that $-(K_{X/V}+S+A+B)$ is ample, S and A are locally irreducible $\mathbb{Q}$-Cartier Weil divisors and $\lfloor B \rfloor = 0$. Assume that $-S$ and A are ample. Then $(S^N, \mathrm{Diff}_{S^N}(A+B))$ is purely globally T-regular over all points of $f(\mathrm{Exc}(f))$. In particular, S is normal over a neighborhood of $f(\mathrm{Exc}(f))$.*

Proof. Since $K_{X/V}+S+B$ is $\mathbb{Q}$-Cartier, $\mathrm{Diff}_{S^N}(A+B) = D + A|_{S^N}$, where $D := \mathrm{Diff}_{S^N}(B)$. We note that A is Cartier on the codimension two points of X contained in S. Since f is small, $f|_{S^N}\colon S^N \longrightarrow T$ is birational, where $T := f(S)$. Since $-S$

is ample, all exceptional curves of f are contained in S, thus $f|_S\colon S \longrightarrow T$ has connected fibers. Since $S^N \longrightarrow S$ is a universal homeomorphism by Lemma 3.1.3, $f|_{S^N}$ also has connected fibers. By Proposition 3.1.11, the pair $(S^N, D + A|_{S^N})$ is plt. By Lemma 3.2.31, $(S^N, D + A|_{S^N})$ is purely globally T-regular over all points of $f(\mathrm{Exc}(f))$. In particular, S is normal over a neighborhood of $f(\mathrm{Exc}(f))$ by Proposition 3.2.21. □

Corollary 3.2.33. (cf. [43, Proposition 3.4]) *Let V be an excellent Dedekind scheme. Let $(X, S + A + B)$ is a three-dimensional dlt pair over V. Let $f\colon X \longrightarrow Z$ be a $(K_{X/V} + S + A + B)$-flipping contraction with $\rho(X/Z) = 1$. Assume that S and A are locally irreducible Weil divisors such that $-S$ and A are ample $\mathbb{Q}$-Cartier. Then the flip of f exists.*

Proof. We may assume that V is the spectrum of an excellent discrete valuation ring. We may assume $\lfloor B \rfloor = 0$. By Shokurov's reduction to pl-flip (see [22, Lemma 2.3.6]), it is enough to show that $R_S(k(K_X + S + A + B)/Z)$ is finitely generated. This statement can be reduced to the case where V is complete and the residue field is infinite taking a strict henselization and completion. By Lemma 3.1.4, the assumption is preserved except for the condition that the relative Picard rank is one. By Lemma 3.2.32, S is normal and $(S, A_S + B_S)$ is purely globally T-regular over all points of $f(\mathrm{Exc}(f))$, where $B_S = \mathrm{Diff}_S(B)$ and $A_S = A|_S$. By Proposition 3.2.28, it is enough to show that $R(K_{S/V} + A_S + B_S)$ is finitely generated. We take an effective divisor A' on S with $A_S \sim_{\mathbb{Q}} A'$ such that A_S and A' have no common component. Since $(S, A_S + B_S)$ is plt, $(S, (1-\varepsilon)A_S + B_S + \varepsilon A')$ is klt for small enough positive rational number ε. By [93, Theorem 1.1, Theorem 4.2, Corollary 4.11], the canonical ring $R(K_{S/V} + (1-\varepsilon)A_S + B_S + \varepsilon A'/Z)$ is finitely generated. Since

$$K_{S/V} + (1-\varepsilon)A_S + B_S + \varepsilon A' \sim_{\mathbb{Q}} K_{S/V} + A_S + B_S,$$

$R(K_{S/V} + A_S + B_S/Z)$ is also finitely generated. □

Remark 3.2.34. The existence of necessary flips for [62, Theorem 6] follows from Corollary 3.2.33 over an excellent Dedekind scheme.

3.3 Proof of Theorem E and its applications

The goal of this section is to prove Theorem E and its applications. Proposition 3.3.12 is one of the applications and it will be used to prove Theorem D. To prove theorems, we establish the cone theorem for pseudo-effective pairs (Proposition 3.3.2), by following the method given by [59] [92]. We also prove the cone theorem for more general settings (Proposition 3.3.4) by using the method given by [56]. If every relative curve is contained in the special fiber, then the cone theorem is easily reduced to the case of surfaces, but in the relative setting, relative curves contained in the generic fiber may exist. Therefore, we should treat such cases carefully.

Proposition 3.3.1. *Let V be an excellent Dedekind scheme. Let $(X, S + B)$ be a three-dimensional dlt pair over V such that S is a $\mathbb{Q}$-Cartier Weil divisor. Let $\rho\colon X \longrightarrow U$ be a projective morphism over V. Let Σ be a $(K_{X/V} + S + B)$-negative extremal ray contracted by ρ. Let L be a ρ-nef Cartier divisor on X with $L^{\perp} = \mathbb{R}[\Sigma]$. Assume that S is a prime divisor and $S \cdot \Sigma < 0$. Then L is semiample over U.*

Proof. It follows from a similar argument to the argument in the proof of [HW, Proposition 4.4] by replacing [HW, Lemma 2.1] with Lemma 3.1.3 and using [96, Theorem 1.2]. ☐

Proposition 3.3.2. *Let V be an excellent Dedekind scheme. Let $\pi\colon X \to U$ be a projective V-morphism from a normal $\mathbb{Q}$-factorial quasi-projective V-threefold X to a quasi-projective V-variety U. Let B be an effective $\mathbb{R}$-Weil divisor on X satisfying the following.*

- *every coefficients c of B satisfy $0 \leqslant c \leqslant 1$, and*
- *$K_{X/V} + B$ is pseudo-effective.*

Let A be a π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then there exist finitely many π-relative curves $C_1, \ldots, C_r$ on X such that

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_{X/V}+B+A\geqslant 0} + \sum_{i=1}^{r} \mathbb{R}_{\geqslant 0}[C_i].$$

Proof. The assertion is proved by the same method as in [92, Theorem 7.6]. Here, we use [93, Theorem 2.14] after the reduction to the case of surfaces. ☐

In the proof of Proposition 3.3.12, we run an MMP with scaling. In order to do this, we prepare the following corollary.

Corollary 3.3.3. *Let V be an excellent Dedekind scheme. Let (X, Δ) be a dlt $\mathbb{Q}$-factorial pair over V satisfying that $\lfloor \Delta \rfloor = X_s$ as sets. Let $\pi\colon X \to U$ be a projective birational morphism over V from X to a quasi-projective V-variety U. Let H be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor such that $K_{X/V} + \Delta + H$ is π-nef. We put*

$$\lambda_H := \inf\{\lambda \in \mathbb{R}_{\geqslant 0} \mid K_{X/V} + \Delta + \lambda H \text{ is } \pi\text{-nef}\}.$$

Then there exists a $(K_{X/V} + \Delta)$-negative extremal ray $R \subset \overline{NE}(X/U)$ satisfying that

$$(K_{X/V} + \Delta + \lambda_H H) \cdot R = 0.$$

Proof. Take a rational number $a \in \mathbb{Q}_{>0}$ with $\Delta - aX_s \geqslant 0$. Since $K_{X/V} + \Delta$ is π-big, we have

$$K_{X/V} + \Delta \sim_{\mathbb{Q},\pi} A + E$$

for a π-ample $\mathbb{Q}$-Cartier divisor A and an effective $\mathbb{Q}$-Cartier divisor E. Take a rational number $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon << 1$ satisfying that $(X, \Delta - aX_s + \varepsilon E)$ is klt. Note that

$$K_{X/V} + \Delta - aX_s + \varepsilon E + \varepsilon A \sim_{\mathbb{R},\pi} (1 + \varepsilon)(K_{X/V} + \Delta).$$

Therefore, by using Proposition 3.3.2 for $B = \Delta - aX_s + \varepsilon E$, it finishes the proof. ☐

On the other hand, if the base scheme is local, then we can prove the cone theorem in a more general situation by reducing the problem to the special fiber. In relative setting, since relative curve is not necessarily contained in the special fiber (e.g. $X := \mathbb{A}^1_{\mathbb{Z}_p} \times \mathbb{P}^1_{\mathbb{Z}_p} \to \mathbb{A}^1_{\mathbb{Z}_p}$), we have to give an additional argument.

Proposition 3.3.4. *Let V be an excellent Dedekind scheme. Let $\pi\colon X \to U$ be a projective V-morphism from a normal $\mathbb{Q}$-factorial quasi-projective flat V-variety X of relative dimension two to a quasi-projective V-variety U. Let B be an effective $\mathbb{R}$-divisor on X such that every coefficient c of B satisfies $0 \leqslant c \leqslant 1$. Let A be a π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We suppose the one of the following.*

1. *The scheme V is the spectrum of a discrete valuation ring.*
2. *The scheme X is smooth over every generic point η of V, and B has no horizontal components.*

Then there exist finitely many π-relative curves $C_1, \ldots, C_r$ on X such that

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_{X/V}+B+A\geqslant 0} + \sum_{i=1}^{r} \mathbb{R}_{\geqslant 0}[C_i].$$

Proof. We may assume that V is connected. Moreover, replacing π by its Stein factorization, we may assume that $\pi_*(\mathcal{O}_X) = \mathcal{O}_U$. Therefore we may assume U is normal and flat over V. First, we prove the assertion in the case (1). If $U \to V$ is not surjective, then $X \to U$ is normal surface over a field, so the assertion follows from the cone theorem for surfaces (cf. [93, Theorem 2.14]). Therefore we may assume $U \to V$ is surjective. We denote the closed point of V by s. Let $S_1, \ldots, S_n$ be irreducible components of X_s. Let $\nu_i\colon S_i^N \to S_i$ be the normalization. By [61, Proposition 4.5], there exists an effective $\mathbb{R}$-divisor D_i such that

$$K_{S_i^N} + D_i \sim_{\mathbb{R}} (K_{X/V} + B)|_{S_i^N}.$$

Here, we put D_i as $\mathrm{Diff}_{S_i^N}(B - S_i + \alpha X_s)$ for suitable $\alpha \in \mathbb{R} \geqslant 0$. By [93, Theorem 2.14], there exists finitely many $\pi \circ \nu_i$-relative curves $\Gamma_{i,j}$ such that

$$\overline{NE}(S_i^N/U) = \overline{NE}(S_i^N/U)_{K_{S_i^N}+D_i+A|_{S_i^N}\geqslant 0} + \sum_j \mathbb{R}_{\geqslant 0}[\Gamma_{i,j}].$$

Now we will divide the case by the dimension of U. First, consider the case where U is of relative dimension 0 over V. In this case, a closed curve in X maps to a closed point in U, which maps to the closed point in V. Therefore, any π-relative curves are contained in X_s. Therefore we have

$$\begin{aligned}\overline{NE}(X/U) &= \sum_{i=1}^{n} \overline{NE}(S_i^N/U) \\ &= \overline{NE}(X/U)_{K_{X/V}+B+A\geqslant 0} + \sum_{i,j} \mathbb{R}_{\geqslant 0}[\Gamma_{i,j}].\end{aligned}$$

Next, we consider the case where U is of relative dimension 1 over V. Let $\pi_\eta : X_\eta \to U_\eta$ be the restriction of η to the generic fiber. Then the generic fiber of π_η is geometrically irreducible (cf. [92, Lemma 2.2]). Take an open subset $U_0 \subset U_\eta$ where π_η have geometrically irreducible fibers over U_0. Take a closed point $P_1 \in U_0$ which is also closed in U if exists. Let $P_2, \dots P_l \in U_\eta \backslash U_0$ be all the closed points which are also closed in U. Let $C_{s,t}$ be the irreducible components of $\pi_\eta^{-1}(P_s)$. Then any π-relative curve which is contained in X_η is generated by $[C_{s,t}] \subset \overline{NE}(X/U)$. Therefore, we have

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_{X/V}+B+A \geqslant 0} + \sum_{i,j} \mathbb{R}_{\geqslant 0}[\Gamma_{i,j}] + \sum_{s,t} \mathbb{R}_{\geqslant 0}[C_{s,t}].$$

Finally, we consider the case where U is of relative dimension 2 over V. In this case, π is birational morphism. Let $C_1, \dots, C_l$ be all the exceptional divisors of π which are contained in the generic fiber X_η. Then C_i are π-relative curves on X. Then we have

$$\begin{aligned} \overline{NE}(X/U) &= \overline{NE}(X_s/U) + \sum_s \mathbb{R}_{\geqslant 0}[C_s] \\ &= \overline{NE}(X/U)_{K_{X/V}+B+A \geqslant 0} + \sum_{i,j} \mathbb{R}_{\geqslant 0}[\Gamma_{i,j}] + \sum_s \mathbb{R}_{\geqslant 0}[C_s]. \end{aligned}$$

It finishes the proof of (1). The assertion in the case (2) follows from the argument in (1) and the lifting method in the proof of [56, Theorem 1.3]. □

Proposition 3.3.5. (cf. [33, Theorem 4.2.1]) *Let V be an excellent Dedekind scheme. Let (X, B) be a $\mathbb{Q}$-factorial three-dimensional dlt pair over V. Consider a sequence of log flips starting from $(X, B) = (X_0, B_0)$:*

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots,$$

where $\varphi_i : X_i \longrightarrow Z_i$ is a flipping contraction associated to an extremal ray and $\varphi^+ : X_i^+ = X_{i+1} \longrightarrow Z_i$ is the log flip. Then, after finitely many flips, the flipping locus is disjoint from $\lfloor B \rfloor$.

Proof. It follows from a similar argument to the argument in the proof of [33, Theorem 4.2.1]. □

Theorem 3.3.6. (Theorem E, cf. [43, Theorem 1.1]) *Let V be an excellent Dedekind scheme. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial dlt pair over V. Assume that there exists a projective birational morphism $\pi : X \longrightarrow Z$ to a normal $\mathbb{Q}$-factorial variety Z with* $\mathrm{Exc}(\pi) \subset \lfloor \Delta \rfloor$. *Then we can run a $(K_{X/V} + \Delta)$-MMP over Z which terminates with a minimal model.*

Proof. It follows from the same argument as in the proof of [43, Theorem 1.1] using The cone theorem (Proposition 3.3.2), the contraction theorem (Proposition 3.3.1), the existence of flips (Corollary 3.2.33), and the termination of flips (Proposition 3.3.5). □

Lemma 3.3.7. *Let $f\colon Y \longrightarrow X$ be a projective birational morphism of three dimensional separated excellent integral schemes. Let W be a closed subscheme of Y. If the singular locus of X is contained in some affine open subset of X, then there exists a projective birational morphism $\nu\colon Y' \longrightarrow Y$ such that Y' is regular and* $\mathrm{Exc}(\nu) \cup \nu^{-1}(W)$ *has simple normal crossing support.*

Proof. By [24, Theorem 1.1], the scheme X admits a projective resolution X_0. By [24, Theorem 4.4] and [23], an analog of [44, Conjecture 5.4] holds for regular three dimensional separated excellent integral schemes and its closed subschemes. By an analogous argument of the proof of [44, Proposition 5.5] for $X_0 \dashrightarrow Y$, we obtain the assertion. We note that we can take an elimination of $X_0 \dashrightarrow Y$ which is projective over X_0 and Y. □

Remark 3.3.8. Let U be a regular open subset of Y such that $f|_U$ is an isomorphism and $\mathrm{Exc}(\nu) \cup \nu^{-1}(W)$ has simple normal crossing support on U. We can take a morphism ν in Lemma 3.3.7 as a morphism whose isomorphic locus contains U. Indeed, there exists an elimination of $X_0 \dashrightarrow Y$ whose isomorphic locus contains U.

Corollary 3.3.9. ([43, Corollary 1.4]) *Let V be an excellent Dedekind separated scheme. Let (X, Δ) be a $\mathbb{Q}$-factorial three-dimensional log pair over V such that any coefficient of Δ is at most one. Assume that X is projective birational over an affine V-variety. Then there exits a projective birational morphism $\pi\colon Y \longrightarrow X$ such that the pair $(Y, \Delta_Y := \pi_*^{-1}\Delta + \mathrm{Exc}(\pi))$ satisfies the following conditions.*

1. *(Y, Δ_Y) is a $\mathbb{Q}$-factorial dlt pair over V, and*
2. *$K_{Y/V} + \Delta_Y$ is nef over X.*

Proof. By Lemma 3.3.7, we have a log resolution $f\colon W \longrightarrow X$ of (X, Δ). We write $\Delta_W := \pi_*^{-1}\Delta + \mathrm{Exc}(\pi)$, then (W, Δ_W) is dlt. By Theorem E, we can run a $(K_{W/V} + \Delta_W)$-MMP over X, and we get a minimal model $\pi\colon Y \longrightarrow X$. Then $(Y, \Delta_Y := \pi_*^{-1} + \mathrm{Exc}(\pi))$ is dlt and $K_{Y/V} + \Delta_Y$ is nef over X. □

Corollary 3.3.10. (cf. [43, Corollary 1.5]) *Let V be an excellent Dedekind separated scheme. Let $(X, S + B)$ be a $\mathbb{Q}$-factorial three-dimensional log pair over V such that S is a locally irreducible Weil divisor. Assume that X is projective birational over an affine V-variety. Then $(X, S + B)$ is plt on a neighborhood of S if and only if (S^N, B_S) is klt, where S^N is the normalization of S and $B_S :=$* $\mathrm{Diff}_{S^N}(B)$ *is the different.*

Proof. It follows from the same argument as in the proof of [43, Corollary 1.5] replacing [43, Corollary 1.4] into Corollary 3.3.9. □

Lemma 3.3.11. *Let V be an excellent Dedekind scheme. Let (X, Δ) be a three-dimensional dlt pair over V with $\lfloor \Delta \rfloor = S + S'$, where S and S' are locally irreducible $\mathbb{Q}$-Cartier divisors. Let $f\colon X \longrightarrow Z$ be a projective morphism over V such that $f(S)$ is two-dimensional. Let C be an irreducible component of $S \cap S'$. If $f(C)$ is a point, then $(S' \cdot C)$ is negative.*

Proof. We set $D := \mathrm{Diff}_{S^N}(\Delta - S')$, then (S^N, D) is plt by Proposition 3.1.11. As S^N is two-dimensional, $\lfloor D \rfloor$ is normal, and in particular, $\lfloor D \rfloor$ is locally irreducible. Since (X, Δ) is a simple normal crossing pair on the generic point of $S \cap S'$, we have $\lfloor D \rfloor = S'|_{S^N}$. By Lemma 3.1.3, $S^N \longrightarrow S$ is a universal homeomorphism, thus $S \cap S'$ is also locally irreducible. In particular, $C|_S$ is $\mathbb{Q}$-Cartier and $(C|_{S^N} \cdot D) = (C|_{S^N}^2)$. Thus, we have

$$(C \cdot S') = (C|_S \cdot S'|_S) = (C|_{S^N} \cdot D) = (C|_{S^N}^2) < 0,$$

because $C|_{S^N}$ is a contracted curve of S^N via generically finite morphism $f|_{S^N} \colon S^N \longrightarrow f(S)$. □

Proposition 3.3.12. (cf. [43, Proposition 4.1]) *Let V be the spectrum of an excellent discrete valuation ring. Let X be a flat V-variety of relative dimension two. Let (X, Δ) be a dlt $\mathbb{Q}$-factorial log pair over V. Let $f \colon X \longrightarrow Z$ be a $(K_{X/V} + \Delta)$-flipping contraction with $\rho(X/Z) = 1$. Suppose that X_s is contained in $\lfloor \Delta \rfloor$ as sets and every irreducible component of X_s is numerically trivial over Z. Then the flip of f exists.*

Proof. We may assume that $\lfloor \Delta \rfloor = X_{s,\mathrm{red}}$. We take a point $z \in f(\mathrm{Exc}(f))$. By shrinking Z, we may assume that Z is affine. First, we prove that $f^{-1}(z)$ intersects with only one irreducible component of X_s. Otherwise, there exist two irreducible components S and S' intersecting $f^{-1}(z)$. By the connectedness of $f^{-1}(z)$, there exists a flipping curve C intersecting with S and S'. By the assumption, S and S' are numerically trivial over Z. Since $(S \cdot C) = 0$ and $(S' \cdot C) = 0$, C is contained in S and S'. It contradicts Lemma 3.3.11.

Thus, we may assume that X_s is irreducible by shrinking Z around z, and in particular, (X, Δ) is plt. We take a reduced $\mathbb{Q}$-Cartier divisor H on X as in [44, Lemma 5.3], then $H \equiv_Z 0$ and it satisfies the conditions in the proof of [44, Theorem 1.2]. We note that as X_η is a surface and the quotient field of V is infinite, a general hyperplane preserves dlt singularities on the generic fiber. We take a dlt modification $Y \longrightarrow X$ of $(X, \Delta + H)$ by Corollary 3.3.9. We note that $f \colon X \longrightarrow Z$ is an isomorphism over the generic point of V, we may assume that $Y \longrightarrow X$ is an isomorphism over the generic point by Remark 3.3.8. We run a $(K_{Y/V} + \Delta_Y + H_Y)$-MMP by the same argument as in [44, Theorem 1.2]. Replacing $(Y, \Delta_Y + H_Y)$ into a minimal model, we may assume that $K_{Y/V} + \Delta + H_Y$ is nef over Z. By Corollary 3.3.3 and the same argument as in the proof of [44, Theorem 1.2], we can run a $(K_{Y/V} + \Delta)$-MMP over Z with scaling of H_Y. Replacing (Y, Δ_Y) into a minimal model, we may assume that $K_{Y/V} + \Delta_Y$ is nef over Z. We denote the map $Y \longrightarrow Z$ by h. Since (X, Δ) is plt, h is small by the negativity lemma (Proposition 3.1.1). Since the relative Picard rank of X over Z is one, Y is the flip of f. □

3.4 Proof of Theorem D and its applications

Our goal of this section is to prove Theorem D, which is a generalization of the result of Kawamata ([56], [57]). In this section, we deal with schemes satisfying the following conditions (Assumption 3.4.1), which are preserved under MMP-steps (cf. Proposition 3.4.8, 3.4.10). Kawamata proved this fact by the construction of flips, but it does not follow from our construction. Therefore, to prove the preservation, we precisely observe extremal ray contractions (cf. Proposition 3.4.7).

Assumption 3.4.1. Let V be an excellent Dedekind scheme. X is a V-variety satisfying the following conditions.

1. X is flat over V of relative dimension two.

2. Every generic fiber X_η is smooth.

3. The fibers X_s for the closed points $s \in V$ are geometrically reduced and satisfy the condition (S_2).

4. Each irreducible component S of every fiber X_s is geometrically irreducible, geometrically normal and a $\mathbb{Q}$-Cartier divisor on X.

5. (X, X_s) is dlt for all closed points $s \in V$.

6. For each closed point $s \in V$ and dominant morphism $\iota \colon V' = \mathrm{Spec} A \longrightarrow V$ with reduced fiber such that A is a complete discrete valuation ring with algebraically closed residue field k and s is contained in the image of ι, the base change $X' := X \times_V V'$ satisfies the condition (5).

Remark 3.4.2.
- The existence of an extension as in Assumption 3.4.1 (6) follows from [71, Theorem 29.1].
- Assumption 3.4.1 is preserved by taking a base change as in Assumption 3.4.1 (6).

Remark 3.4.3. In [56], it is additionally assumed that $\mathcal{O}_X(mK_{X/V})$ is maximal Cohen-Macaulay in order to prove the existence of flips. Kawamata proved that the condition is preserved under MMP-steps if each residue characteristic is larger than 3. However, in this chapter, we do not need this assumption. We note that if each residue characteristic is larger than 5, then such a condition is induced by Assumption 3.4.1. Indeed, we take a closed point $x \in X$ contained in an irreducible component S of some closed fiber X_s. Then by Assumption 3.4.1 (4) and (5), (X, S) is plt. Thus, by the adjunction, S is a klt surface. Since the characteristic of S is larger than 5, S is strongly F-regular. By Proposition 3.3.10, X is T-regular at x. In conclusion, X is T-regular. By Proposition 3.2.17, $\mathcal{O}_X(mK_{X/V})$ is maximal Cohen-Macaulay.

Definition 3.4.4. (strictly semi-stable)

1. Let V be the spectrum of a discrete valuation ring R. Let ϖ be a uniformizer of R. A flat V-variety X of relative dimension n is called *strictly semi-stable* if the following hold.

- The generic fiber X_η is smooth, where $\eta \in V$ is the generic point.
- For any closed point x in the special fiber X_s, there exists an open neighborhood U of x such that U is étale over the scheme

$$\operatorname{Spec} R[X_0, \dots, X_n]/(X_0 \cdots X_m - \varpi)$$

for some $m \leqslant n$.

As in [26, 2.16] if R has a perfect residue field, the above definition is equivalent to that (X, X_s) is a simple normal crossing pair.

2. Let V be a Dedekind scheme. An integral flat quasi-projective V-variety X of relative dimension n is called strictly semi-stable if $X_{\mathcal{O}_{V,s}} \to \operatorname{Spec} \mathcal{O}_{V,s}$ is strictly semi-stable for any closed point $s \in V$.

We note that a strictly semi-stable scheme over an excellent Dedekind scheme of relative dimension 2 satisfies Assumption 3.4.1.

Lemma 3.4.5. *Let V be an excellent Dedekind scheme and X be a V-variety satisfying Assumption 3.4.1. Let $s \in V$ be a closed point. Let $S_1, \dots, S_r$ be the irreducible components of X_s. We set $X_i := S_1 \cup \dots \cup S_i$ with reduced structure and the scheme-theoretic intersection $C_i := X_{i-1} \cap S_i$ for $1 \leqslant i \leqslant r$. Then X_i is reduced and satisfies the condition (S_2) and C_i is reduced and pure one-dimensional for every i.*

Proof. Since X_s is reduced and $X_s = X_r$ as sets, we have $X_s = X_r$. In particular, X_r satisfies the condition (S_2). Since X_{i-1} and S_i are $\mathbb{Q}$-Cartier divisors on X, the scheme-theoretic intersection $C_i = X_{i-1} \cap S_i$ is pure one-dimensional. In particular, each generic point of C_i is codimension two point in X. Since (X, X_s) is a simple normal crossing pair at each generic point of C_i, the scheme C_i satisfies the condition (R_0). Thus, in order to prove C_i is reduced, it is enough to show that C_i satisfies the condition (S_1).

We take a closed point P of C_i. Then P is contained in at least two components S_i and S_j for some $j < i$. If P is contained in three components, then (X, X_s) is a simple normal crossing pair at P, and in particular, C_i is reduced at P. Thus, we may assume that P is contained in only two components, so $X_s = S_i \cap S_j$ around P and we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_s} \longrightarrow \mathcal{O}_{S_i} \oplus \mathcal{O}_{S_j} \longrightarrow \mathcal{O}_{C_i} \longrightarrow 0$$

around P. Since X_s, S_i and S_j satisfy the condition (S_2), C_i satisfies the condition (S_1), so C_i is reduced for all i. The exact sequence

$$0 \longrightarrow \mathcal{O}_{X_i} \longrightarrow \mathcal{O}_{X_{i-1}} \oplus \mathcal{O}_{S_i} \longrightarrow \mathcal{O}_{C_i} \longrightarrow 0$$

implies that if X_{i-1} satisfies the condition (S_2), then so is X_i by the reducedness of C_i. □

Proposition 3.4.6. *Let $\pi\colon S \longrightarrow Z$ be a projective morphism from a surface to a variety over an algebraically closed field k. Let (S, D) be a dlt pair and L be a π-nef Cartier divisor such that $L-(K_S+D)$ is π-ample and C be a reduced Weil divisor with $C \leqslant D$. Then the following hold.*

(i) $\pi_*\mathcal{O}_S(mL) \longrightarrow \pi_*\mathcal{O}_C(mL)$ *is surjective for all i and divisible enough m.*

(ii) $R^i\pi_*\mathcal{O}_S(mL) = R^i\pi_*\mathcal{O}_C(mL) = 0$ *for every $i > 0$ and divisible enough m.*

(iii) *L is semiample over Z.*

Proof. We note that the semiampleness follows from the abundance, since $L-(K_S+D)$ is ample over π and k is infinite. By [94, Theorem 1.1], L is semiample over Z, thus we may assume that L is a pullback of an ample Cartier divisor on Z', where $\pi'\colon S \longrightarrow Z'$ is the morphism defined by L over Z. In particular, we may assume that L is trivial by replacing π into π'. First, we consider the case where the dimension of $\pi(S)$ is at least one. By a perturbation of coefficients of D and [56, Lemma 2.1], we have

$$R^i\pi_*\mathcal{O}_S = R^i\pi_*\mathcal{O}_S(-C) = 0$$

for all $i > 0$, thus we obtain the assertion. Next, we assume that $\pi(S)$ is a point. By [56, Lemma 2.2], we have $H^i(\mathcal{O}_S) = 0$ for all $i > 0$. By [93, Theorem 5.2], the scheme C is connected, so we have $H^0(\mathcal{O}_C) \simeq k$. Since we have $H^0(\mathcal{O}_S(-C)) = 0$ and $H^0(\mathcal{O}_S) \simeq k$, the map $H^0(\mathcal{O}_S) \longrightarrow H^0(\mathcal{O}_C)$ is surjective. Thus we have $H^1(\mathcal{O}_S(-C)) = 0$. Since $-(K_S+C)$ is big, we have $H^2(\mathcal{O}_S(-C)) = 0$, so we have $H^1(\mathcal{O}_C) = 0$. ☐

The following theorem is discussed in [56, Theorem 2.3]. However, we need more detailed observation of contractions, thus we use a bit different method form the method of [56, Theorem 2.3].

Proposition 3.4.7. (cf. [56, Theorem 2.3]) *Let V be an excellent Dedekind scheme and X is a V-variety satisfying Assumption 3.4.1. Let $\rho\colon X \longrightarrow U$ be a projective morphism to a V-variety. Let $s \in V$ be a closed point. Let $S_1, \dots S_r$ are the irreducible components of X_s. Let L be a ρ-nef Cartier divisor with $L - K_{X/V}$ is ρ-ample. Then L is semiample. Furthermore, the map $f\colon X \longrightarrow Z$ defined by L satisfies the following conditions.*

1. *$R^jf_*\mathcal{O}_X = 0$ for all $j > 0$.*

2. *$\pi_*\mathcal{O}_{S_i} = \mathcal{O}_{f(S_i)}$, where the images are equipped with the reduced structure.*

3. *Z is Cohen-Macaulay.*

Proof. Taking a base change via $\iota\colon V' \longrightarrow V$ as in Assumption 3.4.1 (6), we may assume that V is the spectrum of a complete discrete valuation ring with algebraically closed residue field. We set $X_i := S_1 \cup \dots \cup S_i$ and $C_i := X_{i-1} \cap S_i$ for $1 \leqslant i \leqslant r$. We may assume that the conditions $(i), (ii)$ in Proposition 3.4.6 for S_i are satisfied for $m = 1$ and $R^j\pi_*\mathcal{O}_{X_\eta}(L) = 0$ by replacing L into some power of L. We note that X_i is a pushout of X_{i-1} and S_i with C_i, and C_i is reduced and X_i satisfies (S_2) for all i by Lemma 3.4.5. Thus, we have the surjection $\pi_*\mathcal{O}_{X_i}(L) \longrightarrow \pi_*\mathcal{O}_{X_{i-1}}(L)$ and the isomorphism $R^j\pi_*\mathcal{O}_{X_i}(L) \simeq R^j\pi_*\mathcal{O}_{X_{i-1}}(L)$ by Proposition 3.4.6. By the induction on i and changing the order of $S_1, \dots, S_r$, we have $R^j\pi_*\mathcal{O}_{X_s}(L) = 0$ for all $j > 0$ and the surjection $\pi_*\mathcal{O}_{X_i}(L) \longrightarrow \pi_*\mathcal{O}_{S_i}(L)$ for all i. By the sujectivity of $\pi_*\mathcal{O}_{X_i}(L) \longrightarrow \pi_*\mathcal{O}_{S_i}(L)$ and $\pi_*\mathcal{O}_{X_i}(L) \longrightarrow \pi_*\mathcal{O}_{X_{i-1}}(L)$, if $L|_{S_i}$ and $L|_{X_{i-1}}$ is globally generated over U, then so is $L|_{X_i}$. By the induction on i, we may assume that $L|_{X_s}$ is globally generated.

By the exact sequence

$$0 \longrightarrow \mathcal{O}_X(L) \longrightarrow \mathcal{O}_X(L) \longrightarrow \mathcal{O}_{X_s}(L) \longrightarrow 0,$$

where the first map is the multiplication by a uniformizer ϖ, we have the surjection

$$R^j\pi_*\mathcal{O}_X(L) \xrightarrow{\cdot\varpi} R^j\pi_*\mathcal{O}_X(L)$$

for all $j > 0$. Thus, this vanishes around the closed fiber and the generic fiber, we have $R^j\pi_*\mathcal{O}_X(L) = 0$ for all $j > 0$. Since $L|_{X_s}$ and $L|_{X_\eta}$ is semiample and we have the surjection $\pi_*\mathcal{O}_X(L) \longrightarrow \pi_*\mathcal{O}_{X_s}(L)$, L is semiample over U.

Next, we prove the assertions (1) and (2), so we may assume that L is trivial. By the above argument, we have $R^jf_*\mathcal{O}_X = 0$ for all $j > 0$ and $f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_{S_i}$ is surjective for all i. By the commutative diagram

$$\begin{tikzcd} f_*\mathcal{O}_X \arrow[r, "\simeq"] \arrow[d, two heads] & \mathcal{O}_Z \arrow[d, two heads] \\ f_*\mathcal{O}_{S_i} \arrow[r, hook] & \mathcal{O}_{f(S_i)}, \end{tikzcd}$$

we have $f_*\mathcal{O}_{S_i} = \mathcal{O}_{f(S_i)}$. By the same argument as above, we have $f_*\mathcal{O}_{X_i} = \mathcal{O}_{f(X_i)}$ and $f_*\mathcal{O}_{C_i} = \mathcal{O}_{f(C_i)}$ for all i. By the vanishing $R^1f_*\mathcal{O}_{X_i} = 0$ for all i, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{f(X_i)} \longrightarrow \mathcal{O}_{f(S_i)} \oplus \mathcal{O}_{f(X_{i-1})} \longrightarrow \mathcal{O}_{f(C_i)} \longrightarrow 0.$$

By the induction and the condition (S_1) on $f(C_i)$, the scheme $f(X_i)$ satisfies the condition (S_2). In particular, Z_s satisfies the condition (S_2), thus Z is Cohen-Macaulay. $\square$

Proposition 3.4.8. *Let V be an excellent Dedekind scheme and X is a $\mathbb{Q}$-factorial V-variety satisfying Assumption 3.4.1. Let $f\colon X \longrightarrow Z$ be a K_X-negative extremal ray contraction which is a divisorial contraction. Then Z also satisfies Assumption 3.4.1.*

Proof. By Proposition 3.4.7, Z is Cohen-Macaulay, thus Z satisfies Assumption 3.4.1 (3). The other conditions follow from the standard argument. $\square$

Proposition 3.4.9. *Let V be the spectrum of an excellent discrete valuation ring and X is a $\mathbb{Q}$-factorial V-variety satisfying Assumption 3.4.1. Let*

$$X \xrightarrow{\varphi} Z \xleftarrow{\varphi^+} Y$$

be a K_X-flip with $\rho(X/Z) = 1$. Assume that there exists irreducible components S and A of the closed fiber X_s such that $-S$ and A are ample. Then the strict transforms S' and A' of S and A on Y, respectively, are geometrically normal.

Proof. Taking a base change via $V' \longrightarrow V$ in Assumption 3.4.1, we may assume that V is the spectrum of a complete discrete valuation ring with algebraically closed residue field. Since A and S are geometrically irreducible, the irreducibility is preserved. We take a point $z \in \varphi(\mathrm{Exc}(\varphi))$, by shrinking Z around z, we may assume that $\varphi(\mathrm{Exc}(\varphi)) = \{z\}$ and Z is affine. We denote the image of S and A via φ by T and B, respectively. By Proposition 3.4.7, $\varphi_*\mathcal{O}_S = \mathcal{O}_T$ and $\varphi_*\mathcal{O}_A = \mathcal{O}_B$, so T and B are normal. By the proof of Lemma 3.2.31, we may assume that the intersection $C := A \cap S$ is irreducible and not contracted by shrinking Z around z. By the perturbation of coefficients, there exists a dlt pair (X, Δ) such that $\lfloor \Delta \rfloor = A + S$ and $-(K_{X/V} + \Delta)$ is ample. The strict transform of Δ on Y is denoted by Δ_Y. Since X and Y have isolated singularities, the different coincides with $\Delta|_S$ and $\Delta|_{S'^N}$. In particular, $(\varphi|_S)_*\Delta|_S = (\varphi^+_{S'^N})_+\Delta|_{S'^N}$, it is denoted by Δ_T. Then by the same argument in the proof of Lemma 3.2.31, (T, Δ_T) is globally T-regular. Since $K_{S'^N} + \Delta|_{S'^N}$ is ample, $(S'^N, \Delta|_{S'^N})$ is globally T-regular, in particular, S' is normal.

Next, we consider the normality of A'. By the above argument, $S|_A$ is not exceptional, irreducible and anti-ample over B. Thus, $\varphi|_A$ is finite, and in particular, $\varphi|_A$ is an isomorphism because of $\varphi_*\mathcal{O}_A = \mathcal{O}_B$. By the same argument as above, $(A'^N, \Delta'|_{A'^N})$ is globally T-regular, thus A' is normal. □

Proposition 3.4.10. *Let V be an excellent Dedekind scheme and X is a $\mathbb{Q}$-factorial V-variety satisfying Assumption 3.4.1. Let*

$$X \xrightarrow{\varphi} Z \xleftarrow{\varphi^+} Y$$

be a K_X-flip with $\rho(X/Z) = 1$. Then Y is a scheme satisfying Assumption 3.4.1.

Proof. It is obvious that Y satisfies Assumption 3.4.1 except for the condition (S_2) for closed fibers in (3) and the geometric normality of irreducible components of closed fibers in (4). First, we prove that the (S_2) condition of closed fibers, it is enough to show that Y is Cohen-Macaulay. We take a point $z \in \varphi(\mathrm{Exc}(\varphi))$, by shrinking Z around z, we may assume that $\varphi(\mathrm{Exc}(\varphi)) = \{z\}$ and Z is affine. In particular, we may assume that V is a discrete valuation ring by localizing at the image s of z. By Proposition 3.4.7, we have $R^1\varphi_*\mathcal{O}_X = 0$ and Z is Cohen-Macaulay. Thus Z has rational singularities by Proposition 3.1.14. Since X is terminal, so is Y, and in particular, Y has isolated singularities. Since φ^+ is small, Y has also rational singularities by Proposition 3.1.14. Thus, Y is Cohen-Macaulay.

Next, we prove the geometric normality of irreducible components of X_s. First, we consider the case where $f^{-1}(z)$ is contained in only one irreducible component of X_s. Then, we may assume that X_s is irreducible. Thus, X_s is geometrically irreducible, and so is Y_s. After taking base change $V' \longrightarrow V$ as in Assumption 3.4.1 (6), the pair (Y, Y_s) is dlt and Y_s is irreducible. Thus, Y_s is normal in codimension one. Since Y is Cohen-Macaulay, so is Y_s, thus Y_s is normal. In conclusion, Y_s is geometrically normal.

Next, we consider the other case. The extremal ray contracted by φ is denoted by Σ. We take an irreducible component R of Y_s such that $\varphi^+(R)$ contains z. We write $T := \varphi^+_* R$ and $S := \varphi^{-1}_* T$. Suppose $(S \cdot \Sigma) = 0$, then $f^{-1}(z)$ is contained in S. By Lemma 3.3.11, every flipping curve is not contained in any other components. By the assumption, $f^{-1}(z)$ intersects another component S' of X_s. If $(S' \cdot \Sigma) \leqslant 0$, then S' contains flipping curves, so we have a contradiction. If $(S' \cdot \Sigma)$ is positive, then replacing S', we have $(S' \cdot \Sigma)$ is negative because of $X_s \sim 0$. In conclusion, we obtain $(S \cdot \Sigma) \neq 0$. First, we consider the case where $(S \cdot C)$ is positive. Then there exists an irreducible component S' of X_s such that $(S' \cdot C)$ is negative. In particular, the divisors S and $-S'$ are ample. By Proposition 3.4.9, the scheme R is geometrically normal. On the other hand, if $(S \cdot C)$ is negative, then there exists an irreducible component which is ample. Thus, by Proposition 3.4.9, the scheme R is geometrically normal. □

Theorem 3.4.11. (Theorem D, cf. [56]) *Let V be an excellent Dedekind scheme. Let X be a V-variety satisfying Assumption 3.4.1. Then we can run a $K_{X/V}$-MMP over Z preserving Assumption 3.4.1 which terminates with a minimal model or a Mori fiber space.*

Proof. We note that $(K_{X/V} + X_s)$-MMP is also $K_{X/V}$-MMP because X_s is linearly trivial. By the cone theorem (Proposition 3.3.4) and the contraction theorem (Proposition 3.4.7), we can contract any $K_{X/V}$-negative extremal ray. Let $f \colon X \longrightarrow Z$ be a K_X-negative extremal ray contraction. If f is divisorial contraction, Z also satisfies Assumption 3.4.1 by Proposition 3.4.8. If f is flipping contraction, the extremal ray contracted by f is denoted by Σ. In order to prove the existence of the flip, we may assume that V has the unique closed point s. If $(S \cdot \Sigma) = 0$ for every irreducible component S of X_s, then the flip of f exists by Proposition 3.3.12. Otherwise, as X_s is linearly trivial, there exists irreducible components S and A of X_s such that $(S \cdot \Sigma) < 0$ and $(A \cdot \Sigma) > 0$. By Corollary 3.2.33, the flip of f exists.

Then the flip $X \dashrightarrow Y$ of f exists and Y satisfies Assumption 3.4.1 by Proposition 3.4.10. Since X has terminal singularities, a sequence of flip terminates by the argument in [63, Theorem 6.17]. Thus, $K_{X/V}$-MMP terminates with a minimal model or a Mori fiber space. □

In the following, we review applications of Theorem D which are discussed in [20].

Definition 3.4.12 (cf. [20, Definition 5.1]). Let $\mathcal{O}_K$ be a Henselian discrete valuation ring with perfect residue field k. Let K be the fraction field of $\mathcal{O}_K$. Let X be a K3 surface over K or an abelian surface over K. Here, we note that an abelian surface does not necessarily admit a section. Then a *minimal strictly semi-stable model* of X is a proper algebraic space $\mathcal{X}$ over $\mathcal{O}_K$ satisfying the following.

1. The generic fiber $\mathcal{X}_K$ is isomorphic to X.

2. The special fiber $\mathcal{X}_s$ is a scheme whose irreducible components are smooth over k.

3. There exists an étale surjection $U \to \mathcal{X}$ such that U is a strictly semi-stable scheme in the sense of Definition 3.4.4.

4. The relative dualizing sheaf $\omega_{\mathcal{X}/\mathcal{O}_K}$ is trivial. Here, see [20, Section 5] for the definition of the relative dualizing sheaf.

Theorem 3.4.13. *Let $\mathcal{O}_K$, K, k and X be as in Definition 3.4.12. Suppose one of the following.*

1. *The scheme X is an abelian surface over K.*

2. *The scheme X is a K3 surface over K satisfying that X admits a projective strictly semi-stable scheme model over $\mathcal{O}_K$.*

Then there exists a finite separable extension K'/K such that there exists a minimal strictly semi-stable model over $\mathcal{O}_{K'}$ of $X_{K'}$.

Proof. By Theorem D, this theorem follows from the proof of [20, Theorem 10.3] and the proof of [66, Proposition 2.1]. □

The dual graph of the special fiber of a minimal strictly semi-stable model is classified in [20] in the case where $\operatorname{char} k \neq 2$. We will verify that their result holds even in $\operatorname{char} k = 2$.

Theorem 3.4.14. *Let $\mathcal{O}_K$, K, k, and X be as in Definition 3.4.12. Let $\mathcal{X}$ be a minimal strictly semi-stable model over $\mathcal{O}_K$. Then the special fiber $\mathcal{X}_k$ is combinatorial in the sense of [20, Definition 5.4, Definition 5.6].*

Proof. If X is a K3 surface, it follows from the same argument as in the proof of [20, Proposition 5.3, Theorem 6.1]. Therefore, we will treat the case where X is an abelian surface. We note that the case where $\operatorname{char} k \neq 2$ follows from the proof of [20, Proposition 5.3, Theorem 8.1]. In Chiarellotto and Lazda's argument, the assumption $\operatorname{char} k \neq 2$ is used only in the case where (2) (b) in [20, p.2253] holds for some irreducible component of the special fiber $\mathcal{X}_k$. We will review their arguments in this case. They show that the dual graph Γ of the special fiber $\mathcal{X}_k$ is a triangulation of a compact real surface M without border. The spectral sequence of coherent cohomologies shows that

$$\dim_k H^i_{\mathrm{sing}}(\Gamma, k) = \begin{cases} 1 & \text{if } i = 0, 2, \\ 2 & \text{otherwise.} \end{cases}$$

It implies that M is a torus if $\operatorname{char} k \neq 2$ since the left hand side is equal to $\mathbb{C}$-Betti number by the classification of real surfaces. On the other hand, in the weight spectral sequence as in the proof of [20, Theorem 8.3], we have an isomorphism $E_2^{-1,2} \simeq E_2^{1,0}$. Here, we take a prime number $\ell \in R^\times$. Moreover, we have $E_1^{0,1} = 0$ by [20, Lemma 4.2]. Since this spectral sequence degenerates at E_2, we have $\dim_{\mathbb{Q}_\ell} E_2^{1,0} = 2$. Since $E_2^{1,0} = H^1_{\mathrm{sing}}(\Gamma, \mathbb{Q}_\ell)$ as in the proof of [20, Theorem 8.3], we have $\dim_{\mathbb{Q}_\ell} H^1_{\mathrm{sing}}(\Gamma, \mathbb{Q}_\ell) = 2$. Therefore, the surface M is a torus even in $\operatorname{char} k = 2$. □

In the case where X is an abelian surface, it is well-known that there exists a Néron model of X (cf. [14]), which is a smooth model satisfying a useful extension property (see [14, Section 1.2, Definition 1] for the precise definition). The following proposition, which is proved in [54, Theorem 1.4] in the case where $\mathcal{X}$ is a scheme, shows that a minimal strictly semi-stable model gives a compactification of a Néron model.

Proposition 3.4.15. *Let $\mathcal{O}_K$, K, and k be as in Definition 3.4.12. Let X be an abelian surface over K. Let $\mathcal{X}$ be a minimal strictly semi-stable model of X over $\mathcal{O}_K$, and $\mathcal{X}^{\mathrm{sm}}$ the smooth locus of $\mathcal{X}$. Then $\mathcal{X}^{\mathrm{sm}}$ is a scheme and a Néron model of X over $\mathcal{O}_K$.*

Proof. Let $\mathcal{Y}$ be a Néron model of X over $\mathcal{O}_K$. By the descent argument, one can show that the Néron mapping property holds for algebraic spaces. Therefore, we have a morphism $f : \mathcal{X}^{\mathrm{sm}} \to \mathcal{Y}$ which extends an identity on X. It is enough to show that f is an isomorphism. We note that we have $\mathcal{X}(\mathcal{O}_K^{\mathrm{sh}}) \neq \varnothing$ by Hensel's Lemma, where $\mathcal{O}_K^{\mathrm{sh}}$ is the strict henselization of $\mathcal{O}_K$. By [14, Section 7.2, Theorem 1], we may assume that X admits a section. First, we will show that f is an étale morphism. It suffices to show that $f \circ u$ is étale for any étale morphism $u : U \to \mathcal{X}^{\mathrm{sm}}$. By [14, Section 2.2, Corollary 10], we want to show that $\Phi : (f \circ u)^* \Omega^2_{\mathcal{Y}/\operatorname{Spec} \mathcal{O}_K} \to \Omega^2_{U/\operatorname{Spec} \mathcal{O}_K}$ is an isomorphism. Since $\mathcal{X}^{\mathrm{sm}}$ is a scheme in codimension 1, by using [14, Section 4.3, Lemma 1], we have Φ is an isomorphism in codimension 1, so Φ is an isomorphism. Now we have f is étale. By Zariski's main theorem, the morphism f is an open immersion. Let Z be a complement $\mathcal{Y} \backslash \mathcal{X}^{\mathrm{sm}}$. Suppose that $Z \neq \varnothing$. Take a valued point $z \in Z(\overline{k})$. By Hensel's lemma, the valued point z lifts to $\widetilde{z} \in \mathcal{Y}(\mathcal{O}_K^{\mathrm{sh}}) = X(K^{\mathrm{sh}}) = \mathcal{X}(K^{\mathrm{sh}}) = \mathcal{X}(\mathcal{O}_K^{\mathrm{sh}})$, but it contradicts to the choice of z (cf. the proof of [54, Theorem 1.4]). Therefore, f is an isomorphism. □

3.5 More general relative MMP

The goal of this section is to prove the relative MMP in more general setting (Theorem 3.5.2) and the finite generation of relative canonical ring (Theorem 3.5.3). The first one is an analog of [43, Theorem 1.6] in mixed characteristic.

Proposition 3.5.1. *Let V be the spectrum of an excellent discrete valuation ring. Let X be a flat V-variety of relative dimension two. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt log pair with $X_s \subset \lfloor \Delta \rfloor$ as sets, where X is flat V-variety of relative dimension two. Let $\rho\colon X \longrightarrow U$ be a projective morphism to a quasi-projective V-variety U. Let L be a ρ-nef Cartier divisor on X with $L^{\perp} = \mathbb{R}[\Sigma]$, where Σ is a $(K_X + \Delta)$-negative extremal ray over ρ. Then L is semiample.*

Proof. Replacing L with mL for large enough m, we may assume that $L-(K_{X/V}+\Delta)$ is ample over U. Let $S_1, \dots, S_r$ be the irreducible components of X_s. We denote the different $\mathrm{Diff}_{S_i^N}(\Delta - S_i)$ by D_i, then (S_i^N, D_i) is dlt and $L - (K_{S_i^N} + D_i)$ is ample over U. We note that the normalization $S_i^N \longrightarrow S_i$ is a universal homeomorphism by Lemma 3.1.3. By [94, Theorem 1.1] and the proof of [59, Lemma 1.4], $L|_{S_i}$ is semiample for all i. We denote the map induced by $L|_{S_i}$ by $\varphi_i\colon S_i \longrightarrow T_i$. Since $-(K_{S_i^N} + D_i)$ is ample over T_i, the morphism $\varphi_i|_{S_1 \cup \cdots \cup S_{i-1}}$ has connected fibers by [93, Theorem 5.2]. By [59, Corollary 2.9] and the induction on i, $L|_{X_s}$ is semiample. Furthermore, $L|_{X_\eta}$ is semiample by the base point free theorem. Thus, by [97, Theorem 1.2], L is also semiample. □

Theorem 3.5.2. (cf. [43, Theorem 1.6]) *Let V be the spectrum of an excellent discrete valuation ring. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair over V, where X is a flat V-variety of relative dimension two. Let $X \longrightarrow Z$ be a projective morphism over V to a quasi-projective V-variety Z. Assume that $\lfloor \Delta \rfloor$ contains the closed fiber X_s as sets. Then we can run a $(K_{X/V} + \Delta)$-MMP over Z which terminates with a minimal model or a Mori fiber space.*

Proof. By the cone theorem (Proposition 3.3.4) and the contraction theorem (Proposition 3.5.1), we can contract any $(K_X + \Delta)$-negative extremal ray. If the contraction is a flipping contraction, then the flip exists by the argument in the proof of Theorem 3.4.11. The termination of flips follows from the special termination (Proposition 3.3.5). □

Theorem 3.5.3. *Let V be an excellent Dedekind scheme whose each residue field is perfect. Let X be a flat V-variety of relative dimension two. Let (X, Δ) be a dlt pair over V such that for each closed point $s \in V$, the pair $(X, \Delta + X_s)$ is dlt or X_s is contained in $\lfloor \Delta \rfloor$ as sets. Let $\rho\colon X \longrightarrow U$ be a projective morphism to a projective V-variety. Then*

$$R(K_{X/V} + \Delta/U) := \bigoplus_m \rho_* \mathcal{O}_X(m(K_{X/V} + \Delta))$$

is a finitely generated $\mathcal{O}_U$-algebra.

Proof. We may assume that V is the spectrum of an excellent discrete valuation ring and $\lfloor \Delta \rfloor$ contains the closed fiber X_s as sets. If $K_{X/V} + \Delta$ is not pseudo-effective, the theorem is trivial, thus, we may assume that $K_{X/V} + \Delta$ is pseudo-effective. By Theorem 3.5.2, we can run a $(K_{X/V} + \Delta)$-MMP over U, thus, we may assume that $L := K_{X/V} + \Delta$ is nef over U. By [97, Theorem 1.2], it is enough to show that $L|_{X_s}$ is semiample over U. We take an ample divisor H on U such that $\rho^* H + L$ is nef over V. By the argument of the proof of [43, Theorem 1.6], the pair (W, Δ_W) is a sdlt surface and π is a universal homeomorphism, where $\pi \colon W \longrightarrow X_s$ is the S_2-fication and $L_W := K_W + \Delta_W = \pi^*((K_X + \Delta)|_{X_s})$, thus it is enough to show that L_W is semiample over U. We set $H_W := \pi^* \rho^* H$. Then $L_W + H_W$ is nef and it is enough to show that $L_W + H_W$ is semiample. By the proof of [91, Theorem 1], there exists an effective $\mathbb{Q}$-Weil divisor D which is $\mathbb{Q}$-linearly equivalent to H_W such that $(W, \Delta_W + D)$ is also sdlt. By [90, Theorem 0.1], it is semiample. $\square$

Remark 3.5.4. In our proof, we need the projectivity of V to use [90, Theorem 0.1]. If we assume the existence of projective log resolutions, then we generalize the existence of dlt modifications to the general setting. Then, we can take a $\mathbb{Q}$-factorial dlt compactification for a pair (X, Δ) with $X_s \subset \lfloor \Delta \rfloor$ as sets and generalize Theorem 3.5.3 to the quasi-projective case.

Corollary 3.5.5. *Let V be an excellent Dedekind scheme. Let X be a V-variety satisfying Assumption 3.4.1. Let $\rho \colon X \longrightarrow U$ be a projective morphism to a projective V-variety U. Then*

$$R(K_{X/V}/U) := \bigoplus_m \rho_* \mathcal{O}_X(mK_{X/V})$$

is a finitely generated $\mathcal{O}_U$-algebra.

Proof. By Assumption 3.4.1 (6), we may assume that V is the spectrum of a discrete valuation ring with perfect residue field. Thus, Theorem 3.5.5 follows from Proposition 3.5.3. $\square$

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