Doctoral Dissertation

博士論文

Bordism Analyses of Symmetries and Anomalies in Quantum Field Theories

(場の量子論の対称性とアノマリーのボルディズムによる解析)

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Abstract

In this thesis, we study certain topological terms called the Wess-Zumino-Witten (WZW) terms appearing in non-linear sigma models. The WZW terms must be present if the sigma models are realized as low-energy effective description of four-dimensional massless quantum chromodynamics (QCD), as they are responsible for appropriately reproducing the 't Hooft anomalies of global symmetries in the QCD.

This anomaly matching determines the overall coefficient of the WZW terms, but it further reveals that they are not well-defined on arbitrary spacetime manifolds, since they do not obey the required quantization conditions. For the simplest case of the IR non-linear sigma models of SU QCD, it was pointed out by [Fre06] that the underlying spacetime manifolds need to be equipped with spin structure allowing spinors to be defined on them, which is indeed natural as the original QCD one started with contain fermions. However, the method used to verify this was rather *ad hoc* and was not applicable to QCD with other gauge groups of interest such as SO.

We will explain that the WZW terms should be described in terms of (co)bordism instead of naïve ordinary (co)homology, and show that this description nicely makes sense of those subtleties concerning overall coefficients, not only for SU QCD but also for SO QCD. The solution to the former case has been known as mentioned above, but we newly provide a more sophisticated argument based on (co)bordism with an advantage that it also applies to the latter case which was previously intractable.

Also, SO QCD have another interesting twist concerning the "generalized" global symmetries. It was recently found by [HL20] that $\mathrm{SO}(2n_c)$ QCD with even number of colors can have mixed 't Hooft anomaly between ordinary symmetries and "higher-form" symmetries, while it remained unclear how this anomaly is matched in the IR non-linear sigma models. By examining solitonic strings in the sigma models, we find that the WZW terms are also responsible for reproducing this novel anomaly too.

Contents

Chapter 1

Introduction

Quantum chromo-dynamics (QCD) is a gauge theory with a number of fermions charged under the gauge group G. In $(3+1)\,d$, when the ratio between the number of fermions ("flavors") N_f and the number of colors N_c is sufficiently small, the theory is known to be asymptotically free, meaning that it is weakly coupled at high-energy (UV) and gradually becomes strongly coupled as one flows down to low-energy (IR). As a result, one experiences various unexpected phenomena in deep IR such as

- confinement of the gauge charge, allowing only neutral particles to appear in isolation
- generation of the gauge-boson(-ball) mass, in spite of the absence of a mass term

whose mechanisms are notoriously difficult to explain for ones with only perturbative handles. In Nature, the strong interaction seems to be well-described by SU QCD (i.e. $G = \mathrm{SU}(N_c)$) with $N_c = 3$ and $N_f = 3$ (or 2) for example, and it confirms the actual occurrence of these phenomena, together with the results of numerical simulations. Although a rather satisfactory explanation has been provided in models with sufficient amounts of supersymmetry [SW94], their complete understanding in the original non-supersymmetric setup is still lacking at the moment.

To proceed, one is naturally urged to exploit non-perturbative aspects of the theory. Although one usually does not have much control of them, there are (at least) two lucky exceptions, namely the **global symmetries** and their **('t Hooft) anomalies**. The latter represents the impossibility of gauging (which can be regarded as "mild violation" of) the former, and can be described in terms of topological quantities, which is invariant under the renormalization-group (RG) flow. This invariance is the key property, as it implies that by computing the anomaly in a weakly-coupled region of the RG flow, one can also obtain full knowledge of the anomaly in a strongly-coupled region, circumventing the direct computation which would have been troublesome. Applied not only to ordinary symmetries but also to "generalized" symmetries which will be explained later, this 't Hooft anomaly matching [tH80] has been a powerful tool to impose non-trivial constraints on strongly-coupled region of gauge theories.

In practice, when a theory is strongly-coupled, it is often the case that the theory is described as another weakly-coupled theory. For the case of QCD, if fermions are massless, it is believed that fermions form condensates at some point along the RG flow, leading to the spontaneously breaking of the flavor symmetry, and as a result, the theory is effectively described in terms of a non-linear sigma model parameterizing the Nambu-Goldstone bosons associated to the spontaneous symmetry breaking (SSB). Since the flavor symmetry is chiral, i.e. acts separately on left / right-handed fermions, the symmetry has non-zero 't Hooft anomaly and cannot be gauged. An important point is that this 't Hooft anomaly has to be reproduced in the low-energy sigma model, according to the 't Hooft's anomaly matching argument. For the case at hand, the Nambu-Goldstone bosons being the only massless degrees of freedom, this is provided by a certain topological term defined on the sigma model, which was originally identified by Wess and Zumino [WZ71]. Its topological significance was later brought to the fore by Witten [Wit83a], where it was also noted that this term remains non-trivial even after turning off the background gauge field for the flavor symmetry, while the terms of this general form were already described independently by Novikov [Nov82]. Following the convention, we call it the **(un)gauged Wess-Zumino-Witten (WZW) term**, and the aim of this thesis is to review various subtleties on this topological term for SU and SO QCD.[1]

The first issue to be addressed is the quantization of overall coefficients of the WZW terms. They were initially determined by considering sigma model configurations on simple spacetime manifolds such as a flat space $\mathbb{R}^4$ or a sphere S^4, and therefore it had not been clear whether the WZW terms are well-defined on an arbitrary spacetime M_4 with an arbitrary sigma model configuration in the first place. It turns out that the consistency of the ungauged WZW terms for SU QCD in fact requires the spacetime manifold to be equipped with spin structure, as first pointed out by Freed [Fre06]. We will revisit this problem in the context of the recent improved understanding of topological terms as **invertible QFT**s, which are QFTs depending on some set of background fields such that the Hilbert space is one-dimensional on any closed spatial slice [FM04]. Invertible QFTs depending on background *gauge fields* are essentially equivalent to what is called the symmetry-protected topological (SPT) phases, which were originally introduced in the condensed-matter community and have been extensively studied. In contrast, the WZW terms we are interested in here are examples of invertible QFTs depending on background *scalar fields*, i.e. maps from the spacetime manifold to the sigma model target space. Such invertible QFTs (and the corresponding anomalies) have only recently begun to be studied in the literature (see e.g. [FKS17, TY17, Tho17, STY18, CFLS19a, CFLS19b, HKT20]), but the general classification of invertible QFTs based on **bordism**[2] established in the last several years [KTTW14, FH16, GJF17, Yon18, Fre19] is equally applicable and will facilitate our analysis.

[1] Here we abused the notation; in addition to the ordinary one with $G = \mathrm{SO}(N_c)$, "SO QCD" includes the one with the discrete theta angle and also the one with $G = \mathrm{Spin}(N_c)$ i.e. Spin QCD.

[2] In particular, the reference [Fre06] used a generalized cohomology called $E^\bullet$, which is actually a truncation of another generalized cohomology $(D\Omega_{\text{spin}})^\bullet$, namely the Anderson dual of the spin bordism. As we will describe in the main part, the latter is now understood to be the correct theory classifying the spin invertible QFTs.

Summary of the results (1)

Showed that the WZW terms are appropriately described in terms of bordism, and as a result clarified the necessity of spin structures for the definition of WZW terms in the IR of

- $\mathrm{SU}(N_c)$ QCD (refinement of Freed's result [Fre06])
- $\mathrm{SO}(N_c)$ QCD (completely novel)

on *generic* spacetime manifolds, by computing relevant bordism groups.

Another issue to be discussed is about topological solitons in the low-energy sigma models. It is well known that the baryons in SU QCD are described as solitonic particles from the sigma model point of view [Sky61, Wit83b], and the $\mathrm{U}(1)_B$ baryonic symmetry in the UV QCD having a mixed anomaly between the flavor symmetry is also nicely reproduced in the IR sigma model [GW81, BNRS82, CL85]. Similarly, it is known that there are $\mathbb{Z}_2$-valued *electric* flux tubes in Spin QCD, which are again to be reproduced in the sigma model as solitonic strings [Wit83b]. We would like to describe its analog in the case of $\mathbb{Z}_2$-valued *magnetic* flux tubes of SO QCD. For this we need the concept of ***p*-form symmetries**: while point-like operators are charged under *ordinary* symmetries, higher-dimensional operators are charged under *higher-form* symmetries. Denoting the dimensionality of charged operators by p, they can be treated uniformly [GKSW14]. In this language, solitonic particles are charged under the $\mathrm{U}(1)_B$ 0-form symmetry of SU QCD, and solitonic strings are charged under the $\mathbb{Z}_2$ 1-form symmetries of (Spin and) SO QCD. Then, what we need to do is :

- to understand the mixed anomaly of the $\mathbb{Z}_2$ 1-form symmetry between other symmetries in the UV QCD, and
- to describe how it is represented in the IR non-linear sigma model.

The first task was very recently performed in [HL20], and the second task was done in our paper [LOT20], both of which will be reviewed in this thesis.

Summary of the results (2)

Successfully reproduced the newly-found mixed 't Hooft anomaly of $\mathrm{SO}(2n_c)$ QCD [HL20] (involving higher-form symmetries) in the IR non-linear sigma model by the WZW term.

Outline

The rest of the thesis is organized as follows. First three chapters in the main part are preliminary. In Chap. 2, we will describe the notion of bordism and display some of the elementary examples. In Chap. 3, we will introduce the notion of invertible QFT and see how they are related to bordism. In Chap. 4, we will review the notion of symmetries and anomalies from a modern point of view, especially with emphases on the relation between invertible QFT. We will then look into the concrete examples of four-dimensional massless SU and SO QCD in turn in Chap. 5 and 6, where the WZW terms in low-energy non-linear sigma models are studied in detail.

In Appendix A, we give a brief introduction to spectral sequences in general. In Appendix B, we compute the relevant bordism groups of the classifying spaces and the homogeneous spaces, which are used extensively in the main part. We mostly employ the Atiyah-Hirzebruch spectral sequence, but some of the computations are supplemented by uses of the Adams spectral sequence.

Special notes

This thesis is based on the following paper

[LOT20] Y. Lee, K. Ohmori, Y. Tachikawa, "Revisiting Wess-Zumino-Witten terms" arXiv:2009.00033 [hep-th], SciPost Phys. **10** (2021) 061.

Chapter 2

Bordism

In short, the bordism is an equivalence relation between d-dimensional closed manifolds M_d (i.e. compact without boundary) equipped with maps $f : M_d \to X$ to another topological space X. One can think of this as a relaxed "generalized" version of isomorphism between two manifolds, where (very roughly speaking) two manifolds are identified and regarded as equivalent if they can be smoothly deformed to each other.

It turns out that the bordism equivalence classes form a group, just as more familiar relatives do[1]:

$$\begin{aligned} &\text{homotopy} \quad \pi_d(X) \ : \ \text{equivalence classes of maps } S^d \to X, \\ &\text{homology} \quad H_d(X) \ : \ \text{equivalence classes of maps } \Delta^d \to X, \\ &\text{bordism} \quad\ \ \Omega_d(X) \ : \ \text{equivalence classes of maps } M_d \to X, \end{aligned}$$

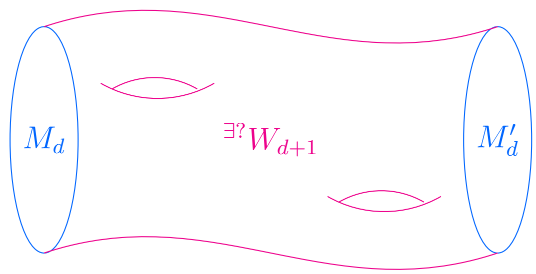
and is indeed a "generalized" (co)homology not only in this naïve sense but also in a more rigorous sense of what is called the Eilenberg-Steenrod axioms.[2] Let us first elaborate on the definition, and then briefly take a look at some elementary examples.

[1]Strictly speaking, the domain of the homology case should be a formal sum of standard d-simplices Δ^d.

[2]The curious reader is referred to e.g. [Fre13, DK01] for more details on unfamiliar jargons popping up hereafter, although the bulk of this thesis should be (hopefully) readable without fully understanding them, which is exactly the author's intent. The author believes that he is not expected to pad his thesis by duplicating definitions and explanations of every single notion from existing mathematics textbooks, and readers can safely ignore alien symbols or diagrams as they wish.

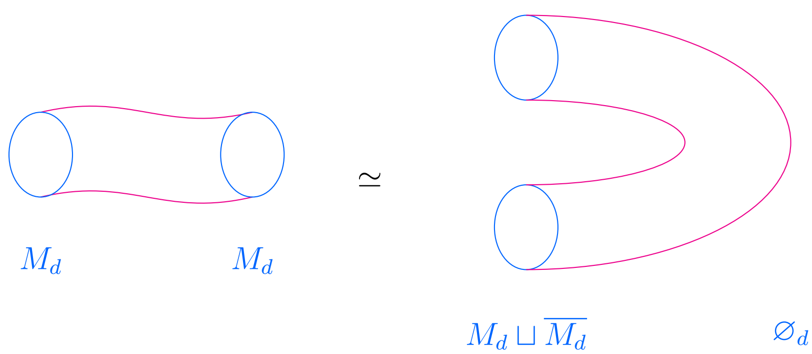
2.1 Definition

Closed d-manifolds M_d and M'_d are defined to be equivalent (*bordant*) if there exists a compact $(d+1)$-manifold W_{d+1} with boundary $\partial W_{d+1} = M_d \sqcup \overline{M'_d}$, where $\overline{M'_d}$ denotes the orientation-reversal of M'_d.



(2.1)

Recall that closed d-manifolds forms a (commutative) monoid under the disjoint union $\sqcup$ of manifolds; the identity element is an empty d-manifold $\varnothing_d$, as $M_d \sqcup \varnothing_d = M_d$. Further passing the disjoint union operation $\sqcup$ to the set of bordism classes $\Omega_d^{\text{oriented}}$, these classes have not only an identity element but also inverse elements under the operation, and thus form a (Abelian) group; the identity equivalence class is the one to which $\varnothing_d$ belongs, and the inverse of the equivalence class to which M_d belongs is an equivalence class to which $\overline{M_d}$ belongs, as can be seen below.



(2.2)

One can further take various (tangential) structures into account, such as spin structures which allows one to define spinors on the manifold, or more generally maps $f : M_d \to X$ from the manifold to a given topological space X. The resulting group is denoted $\Omega_d^{\text{structure}}(X)$. Note that, when X is connected, the bordism group splits as

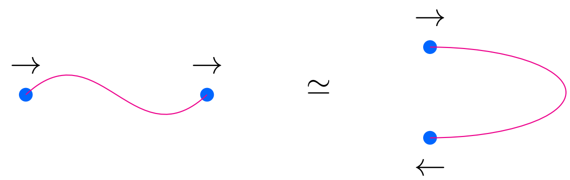
$$\Omega_d^{\text{structure}}(X) = \Omega_d^{\text{structure}}(pt) \oplus \widetilde{\Omega}_d^{\text{structure}}(X). \tag{2.3}$$

The first direct summand on the right hand side is the bordism group of a point, and the second direct summand is the *reduced* bordism group. This splitting comes straightforwardly from the fact that any class $[f : M_d \to X] \in \Omega_d^{\text{structure}}(X)$ determines $[M_d] \in \Omega_d^{\text{structure}}(pt)$ by forgetting the map f, and vice versa $[M_d] \in \Omega_d^{\text{structure}}(pt)$ determines $[f_0 : M_d \to X] \in \Omega_d^{\text{structure}}(X)$ where f_0 sends M_d to a single point on X.

2.2 Examples

$d = 0$, oriented

Closed oriented 0-manifolds are points, and as can be seen easily,

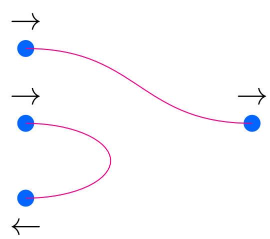


(2.4)

the inverse of (the equivalence class of) an oriented point is a point with the opposite orientation. Therefore, one can immediately identify the bordism group

$$\Omega_0^{\text{oriented}}(pt) = \mathbb{Z}, \tag{2.5}$$

where each element $n \in \mathbb{Z}$ corresponds to "multiple points" (which is indeed a closed 0-manifold) with n being the net number of points with the chosen orientation. An example of bordant pairs is



(2.6)

where two closed oriented 0-manifolds on both sides are connected by 1-manifold in the bulk, and belong to the equivalence class labeled by $(+1) \cdot 2 + (-1) \cdot 1 = (+1) \cdot 1 = 1 \in \mathbb{Z}$.

$d = 1$, oriented

Closed oriented 1-manifolds are circles S^1, but as they can bound two-dimensional disks D^2, they are bordant to an empty 1-manifold $\varnothing_1$. One can also understand this by starting from a pair of pants and then dropping two of the three circles.

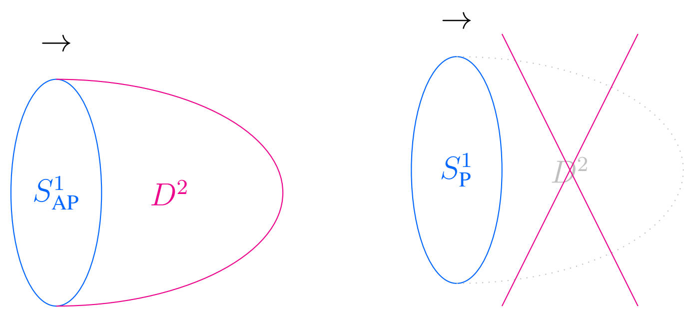
(2.7)

Either way, one can see that the bordism group is trivial

$$\Omega_1^{\text{oriented}}(pt) = 0. \tag{2.8}$$

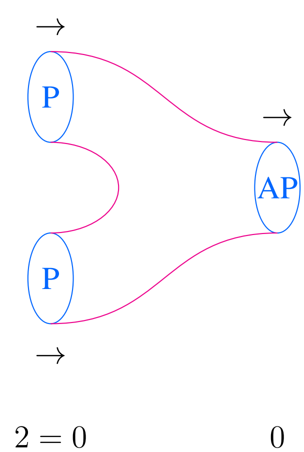
$d = 1$, spin

Let us next consider the case concerning *spin structures*, which are structures allowing spinors to be defined on the manifold. The situation is almost the same as the above example, but this time the difference in required (tangential) structure plays a crucial role. There are in fact two types of S^1; the one with periodic boundary condition and the other with anti-periodic boundary condition. Furthermore, it is known that two-dimensional disk D^2 equipped with spin structure is bounded by the anti-periodic one, but not by the periodic one:



(2.9)

This can be understood from a mod-2 index of a Dirac operator on S^1. While S^1_{AP} has a trivial index, S^1_{P} has a non-trivial index, and therefore cannot be bordant to an empty spin 1-manifold $\varnothing_1$, which obviously has a trivial index. This is an example of bordism invariant. Again, one can also understand this by considering a pair of pants



(2.10)

where a disjoint union of two S^1_{P}'s has a trivial mod-2 index as a whole, and it can be bordant to S^1_{AP}, which was further bordant to an empty 1-manifold $\varnothing$. Anyway, the bordism group becomes

$$\Omega_1^{\mathrm{spin}}(pt) = \mathbb{Z}_2 \tag{2.11}$$

with the generator (representative) manifold being S^1_{P}.

Generic results

One can further go on and will find [Wal60, ABP67, BG87]

d	$\Omega_d^{\text{oriented}}(pt)$	$\Omega_d^{\text{spin}}(pt)$	$\Omega_d^{\text{spin}^c}(pt)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	0	$\mathbb{Z}_2$	0
2	0	$\mathbb{Z}_2$	$\mathbb{Z}$
3	0	0	0
4	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$
5	$\mathbb{Z}_2$	0	0
6	0	0	$\mathbb{Z} \oplus \mathbb{Z}$
7	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Combining these data with the (co)homology information on a topological space X, one can compute the bordism groups $\Omega_d^{\text{structure}}(X)$ by certain tools called the Atiyah-Hirzebruch spectral sequence (AHSS) or the Adams spectral sequence. See Appendix B for the details.

Chapter 3

Invertible QFT

3.1 Basics

A standard lore to start with is that, for any physical systems (typically lattice models in mind), their long-range behavior can be described by scale-invariant QFT. In particular, if the original system has a mass gap above the lowest energy states in the spectrum of the Hamiltonian, excited states are practically invisible to low-energy observers, and it would be fair to say that the Hilbert space of the effective field theory consists only of ground states.

The invertible QFT is a special class of such theories whose ground state is non-degenerate. In other words, a QFT is defined to be invertible if its Hilbert space is one-dimensional

$$\dim \mathcal{H} = 1 \tag{3.1}$$

on any closed spatial slice.

These theories are called "invertible" for the following reason. Given two QFTs Q and Q', one can produce a new QFT $Q \star Q'$ by stacking them, namely by taking a product of partition functions and taking a tensor product of Hilbert spaces

$$\begin{aligned} Z_{Q\star Q'} &= Z_Q \cdot Z_{Q'}, \\ \mathcal{H}_{Q\star Q'} &= \mathcal{H}_Q \otimes \mathcal{H}_{Q'}. \end{aligned} \tag{3.2}$$

Since the trivial QFT Q_{trivial} has

$$\begin{aligned} Z_{Q_{\text{trivial}}} &= 1, \\ \mathcal{H}_{Q_{\text{trivial}}} &= \mathbb{C}, \end{aligned} \tag{3.3}$$

generic QFTs form a (commutative) monoid under the stacking operation $\star$, where the identity element is Q_{trivial}. However, QFTs with one-dimensional Hilbert space further form a (Abelian) group under $\star$, since every such element Q has an inverse element Q^{-1} such that $Q \star Q^{-1} = Q_{\text{trivial}}$

with

$$\begin{aligned} Z_{Q^{-1}} &= Z_Q^{-1}, \\ \mathcal{H}_{Q^{-1}} &= \mathcal{H}_Q^{\vee}, \end{aligned} \tag{3.4}$$

whose Hilbert space $\mathcal{H}_{Q^{-1}}$ is also one-dimensional. Therefore, QFTs with one-dimensional Hilbert space are indeed "invertible" in this sense.

Restricting QFTs of interest to those requiring the spacetime manifold to be equipped with certain (tangential) structures or fields $\phi : M_d \to X$ taking values in the target space X, one can consider various classes of special invertible QFTs. In particular, invertible QFTs whose target space are taken to be a classifying space BG of a group G is essentially the same thing as what is called the G-symmetry protected topological (G-SPT) phases, and has been under extensive investigation in the condensed matter physics community for the last decade.

One of the main questions was about their classification, and initially they were thought to be classified by *ordinary (co)homology* very much like other "topological" notions in QFT. However, it turned out that some of them do not fit in this naïve (co)homology classification, and [Kap14] proposed a classification based on *(co)bordism* to incorporate those beyond-(co)homology phases. This conjectural classification was proved to be correct by [FH16, Yon18], which further suggested a natural extension to the full classification of invertible QFTs (see also [YY21]).

On the other hand, topological terms appearing in QFTs by itself can be regarded as (actions of) almost-empty QFTs with sparse spectrum with isolated ground states, as already described. In the following, we will discuss various "topological" terms in sigma models, all of which can be uniformly thought of as generalizations of WZW terms. Conventionally, they have been described in terms of *ordinary (co)homology* as already mentioned, but careful examinations indicate that the correct description should be based on *(co)bordism*, which in fact reduces to ordinary (co)homology in some cases, including those in low spacetime dimensions with simple enough target spaces. Their classification reproduces that proposed for invertible QFTs, and therefore serves as a strong evidence of validity (and at the same time as a demonstration of useful application) of the latter.

3.2 Classification

Let us first discuss ungauged WZW terms from a modern perspective. In particular, we would like to explain why one needs (co)bordism, rather than (co)homology, to describe these terms.

3.2.1 $d = 1 : \mathrm{U}(1)$ connection

Let us start with the simplest example, as was also done in [Wit83a]. Consider a $(0+1)\,d$ theory with a scalar field ϕ taking values in a manifold X. This is just a convoluted way of referring to a quantum mechanical particle moving on X. Then, make the particle electrically charged by coupling to a $\mathrm{U}(1)$ gauge connection A on X. Here we denote the worldline of the particle by a scalar field $\phi : S^1 \to X$. The contribution $e^{-S[\phi,A]}$ of the connection to the exponentiated action is given by the holonomy along S^1 of the pull-back $\phi^*(A)$ of the gauge connection. Physicists often write this somewhat imprecisely as

$$\begin{aligned} e^{-S[\phi,A]} &= \exp\left(i\int_{S^1}\phi^*(A)\right) \\ &= \exp\left(i\int_{\phi(S^1)} A\right), \end{aligned} \tag{3.5}$$

as if A were always a globally well-defined 1-form.

It is useful to interpret the holonomy as follows. On one hand, suppose that the loop $\phi : S^1 \to X$ is contractible within X, or equivalently, that it can be extended to $\phi : D^2 \to X$, where D^2 is a two-dimensional disk. One can then write the holonomy as

$$e^{-S[\phi,A]} = \exp\left(i\int_{\phi(D^2)} F\right) \tag{3.6}$$

where F is the field strength, or equivalently the curvature of the $\mathrm{U}(1)$ connection A.[1] In a similar manner, one can compare the holonomy along two configurations $\phi_0 : S^1 \to X$ and $\phi_1 : S^1 \to X$ deformable to each other, in the sense that there is a map $\phi_x : S^1 \times [0,1] \to X$ such that the

[1]This does not depend on the choice of the extension ϕ, since the difference between another extension $\phi' : D^2 \to X$ is given by

$$\frac{\exp\left(i\int_{\phi(D^2)} F\right)}{\exp\left(i\int_{\phi'(D^2)} F\right)} = \exp\left(i\int_{\phi(D^2)\sqcup\overline{\phi'(D^2)}} F\right),$$

where $\overline{M}$ denotes the orientation reversal of M and the union $\sqcup$ is taken by identifying the boundaries. This makes the union an image of a two-sphere S^2 in X, and $\int_{\phi(S^2)} F \in 2\pi\mathbb{Z}$ shows that the right hand side is indeed 1.

boundary values are equal to $\phi_{0,1}$, respectively. Then one has

$$\frac{e^{-S[\phi_0,A]}}{e^{-S[\phi_1,A]}} = \exp\left(i\int_{S^1\times[0,1]}\phi^*(F)\right) = \exp\left(i\int_{\phi_x(S^1\times[0,1])}F\right).$$

The right hand side is independent of the choice of ϕ_x interpolating ϕ_0 and ϕ_1.[2]

On the other hand, suppose that the $\mathrm{U}(1)$ connection A is flat and the field strength F vanishes. Then the holonomy does not depend on deformations of the loop $\phi(S^1)\subset X$, which means that the holonomy in fact determines a character

$$\chi_A : H_1(X;\mathbb{Z}) \to \mathrm{U}(1),$$

and as a result one has

$$e^{-S[\phi,A]} = \chi_A\Big([\phi(S^1)]\Big). \tag{3.7}$$

In the end, a general $\mathrm{U}(1)$ connection can be regarded as a certain combination of two extremes (3.6) and (3.7).

3.2.2 $d=2$: B-field

As a next example, let us consider a $(1+1)\,d$ QFT describing a string moving within a manifold. We denote the worldsheet of the string by M_2, the target manifold by X, and the embedding by $\phi : M_2 \to X$. An important ingredient of string theory is the B-field, whose contribution to the exponentiated action is its holonomy $e^{-S[\phi(M_2),B]}$. This can again be written as

$$e^{-S[\phi(M_2),B]} = \exp\left(i\int_{\phi(M_2)}B\right), \tag{3.8}$$

when B is a globally well-defined 2-form. More generally, a B-field has a field strength H which is a closed 3-form, such that we have

$$\int_{[Y_3]} H \in 2\pi\mathbb{Z} \tag{3.9}$$

[2] Indeed, given another $\phi'_x : S^1\times[0,1]\to X$ interpolating $\phi_{0,1}$, we can consider

$$\frac{\exp\left(i\int_{\phi_x(S^1\times[0,1])}F\right)}{\exp\left(i\int_{\phi'_x(S^1\times[0,1])}F\right)} = \exp\left(i\int_{\phi_x(S^1\times[0,1])\sqcup\overline{\phi'_x(S^1\times[0,1])}}F\right)$$

where the integral is over a 2-cycle in X. Again, it is quantized and guarantees that this is indeed 1.

for any 3-cycle $[Y_3] \in H_3(X;\mathbb{Z})$. Now, for two embeddings $\phi : M_2 \to X$ and $\phi' : M_2' \to X$, suppose that there exists a three-dimensional compact manifold W_3 such that $\partial W_3 = M_2 \sqcup \overline{M_2'}$ so that there is a map $\phi : W_3 \to X$ whose restrictions on the boundaries give ϕ and ϕ' respectively. Then,

$$\frac{e^{-S[\phi,B]}}{e^{-S[\phi',B]}} = \exp\left(i\int_{\phi(W_3)} H\right) = 1,$$

and in particular, when $\phi : M_2 \to X$ is contractible and extensible to $\phi : W_3 \to X$ such that $\partial W_3 = M_2$, the property above suffices to determine the holonomy, and indeed one has

$$e^{-S[\phi(M_2),B]} = \exp\left(i\int_{\phi(W_3)} H\right). \tag{3.10}$$

For the $2d$ WZW model, X is a group manifold of a compact Lie group G. Assuming G is simple and simply-connected, it is known that $H^3(X;\mathbb{R}) = \mathbb{R}$ and there is a G-invariant 3-form Γ_3 generating it. Let us normalize it so that $\int_{Y_3}\Gamma_3 = 2\pi$, where $[Y_3]$ is a generator of $H_3(X;\mathbb{Z}) \simeq \mathbb{Z}$. Choosing an extension $\phi : W_3 \to X$ such that $\partial W_3 = M_2$ (which is always possible since $\Omega_2^{\text{spin}}(X) = 0$), the WZW term is given by

$$e^{-S[\phi,B]} = \exp\left(ik\int_{W_3} \phi^*(\Gamma_3)\right) \tag{3.11}$$

where $k \in \mathbb{Z}$ is called the *level*.

In the other extreme, when the field strength H vanishes, the B-field determines a character $\chi_B : H_2(X;\mathbb{Z}) \to \mathrm{U}(1)$, and therefore the holonomy is given by

$$e^{-S[\phi(M_2),B]} = \chi_B\Big([\phi(M_2)]\Big). \tag{3.12}$$

3.2.3 Generic WZW terms at the level of (co)homology

The two constructions above can be generalized to arbitrary spacetime dimensions as follows [Nov82, CS85]. Consider a d-dimensional theory with a scalar field ϕ taking values in a manifold X. We can now consider a d-form gauge field C on X, which has the following features. First, it has an associated closed $(d+1)$-form field strength G, such that when the scalar field $\phi : M_d \to X$ is extensible to $\phi : W_{d+1} \to X$ with $\partial W_{d+1} = M_d$, the coupling is given by

$$e^{-S[\phi(M_d),C]} = \exp\left(i\int_{\phi(W_{d+1})} G\right). \tag{3.13}$$

For this coupling to be well-defined independent of the extension, one has to require

$$\int_{[Y_{d+1}]} G \in 2\pi\mathbb{Z} \tag{3.14}$$

for all $[Y_{d+1}] \in H_{d+1}(X;\mathbb{Z})$. Second, when the field strength G vanishes so that the d-form gauge field is flat, the coupling is given by

$$e^{-S[\phi(M_d),C]} = \chi_C\Big([\phi(M_d)]\Big), \tag{3.15}$$

where $\chi_C : H_d(X;\mathbb{Z}) \to \mathrm{U}(1)$ is a character. The mathematically precise formulation of these ideas is known as *differential characters/cohomology*.

The topological class of a d-form gauge field C is given by a class

$$c = \left[\frac{G}{2\pi}\right] \in H^{d+1}(X;\mathbb{Z}); \tag{3.16}$$

when $d = 1$, this c is the *first Chern class* of the $\mathrm{U}(1)$ gauge connection. The important fact is that the information contained in c can be decomposed to pieces corresponding to (3.13) and (3.15). To see this, use the universal coefficient theorem of the ordinary (co)homology[3]

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}(H_d(X;\mathbb{Z}),\mathbb{Z}) \xrightarrow{\ b\ } H^{d+1}(X;\mathbb{Z}) \xrightarrow{\ a\ } \mathrm{Hom}_{\mathbb{Z}}(H_{d+1}(X;\mathbb{Z}),\mathbb{Z}) \longrightarrow 0. \tag{3.17}$$

Then,

$$a(c) : H_{d+1}(X;\mathbb{Z}) \to \mathbb{Z}$$

is the mapping $\int_{[Y_{d+1}]} G/2\pi$ for $[Y_{d+1}] \in H_{d+1}(X;\mathbb{Z})$, which exactly corresponds to the (3.13) part. When $a(c)$ vanishes, the gauge field can be continuously deformed to a flat one, whose information is captured by (3.15). The holonomy assigned to the free part of $H_d(X;\mathbb{Z})$ can be continuously deformed to a trivial one, and therefore the topological class of c when $a(c) = 0$ is specified by $\mathrm{Hom}(H_d(X;\mathbb{Z})_{\text{torsion}}, \mathrm{U}(1))$.

Note that the exact sequence (3.17) can be identified with a part of the long exact sequence associated with the change of coefficients

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\ \pi\ } \mathrm{U}(1) \longrightarrow 0. \tag{3.18}$$

Indeed, one can write

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}_{\mathbb{Z}}(H_d(X;\mathbb{Z}),\mathbb{Z}) \arrow[r, "b"] & H^{d+1}(X;\mathbb{Z}) \arrow[r, "a"] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}}(H_{d+1}(X;\mathbb{Z}),\mathbb{Z}) \arrow[r] \arrow[d, hook] & 0 \\
\cdots \arrow[r, "\pi_d"] & H^d(X;\mathrm{U}(1)) \arrow[r, "\beta"] \arrow[u, two heads] & H^{d+1}(X;\mathbb{Z}) \arrow[r, "\iota"] & H^{d+1}(X;\mathbb{R}) \arrow[r, "\pi_{d+1}"] & \cdots
\end{tikzcd}$$

where the homomorphism β is what is called the Bockstein associated with (3.18).

[3] For a finitely-generated Abelian group $A = \mathbb{Z}^n \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, it is

$$\begin{array}{rclcl}
\mathrm{Ext}_{\mathbb{Z}}(A,\mathbb{Z}) & = & \mathrm{Hom}(A_{\text{torsion}}, \mathrm{U}(1)) & \simeq & A_{\text{torsion}}, \\
\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Z}) & = & \mathrm{Hom}(A_{\text{free}}, \mathbb{Z}) & \simeq & A_{\text{free}},
\end{array}$$

where the rightmost isomorphisms are non-canonical.

3.2.4 Invertible phases and (the Anderson dual of) the bordism group

As one might have noticed, the discussion in the previous section is in fact not fully general, due to the imperfect condition (3.14); we will revise the relevant part and discuss its consequence in the following. Suppose we have a d-dimensional spin theory with a scalar field taking values in a manifold X. Here, a spin theory means that the spacetime manifold M_d is required to be equipped with a spin structure. Now, we would like to specify a $\mathrm{U}(1)$-valued phase in the exponentiated action. Physics imposes various conditions, and consistent such phases are called the *invertible phases*.

Free part

As before, assume that there is a closed $(d+1)$-form field strength G, such that, when the scalar field $\phi : M_d \to X$ is extensible to $\phi : W_{d+1} \to X$ with $\partial W_{d+1} = M_d$, the coupling is given by

$$e^{-S[\phi]} = \exp\left(i \int_{\phi(W_{d+1})} G \right). \tag{3.19}$$

Here we allow G to consist not only of differential forms on X but also of the Pontrjagin classes p_i of $\phi(W_{d+1})$. Since $\mathbb{Q}[p_1, p_2, \ldots] = H^*(B\mathrm{Spin}; \mathbb{Q})$, this means that we regard G as an element of $H^{d+1}(B\mathrm{Spin} \times X; \mathbb{Q})$. For this coupling to be well-defined independent of the extension, one must require

$$\int_{\phi(W^{\text{closed}}_{d+1})} G \in 2\pi\mathbb{Z} \tag{3.20}$$

for all maps from a closed spin manifold $\phi : W^{\text{closed}}_{d+1} \to X$. This determines a homomorphism

$$\Omega^{\text{spin}}_{d+1}(X) \to \mathbb{Z},$$

opposed to our discussion in (3.14) where we had the ordinary homology group $H_{d+1}(X; \mathbb{Z})$ instead of the bordism group $\Omega^{\text{spin}}_{d+1}(X)$. Such a homomorphism is specified by an element of

$$\mathrm{Hom}_{\mathbb{Z}}(\Omega^{\text{spin}}_{d+1}(X), \mathbb{Z}). \tag{3.21}$$

Torsion part

When G vanishes, the relation (3.19) implies that $e^{-S[\phi]}$ only depends on the bordism class $[\phi]$ of $\phi : M_d \to X$. Then, the coupling is given by

$$e^{-S[\phi]} = \chi\big([\phi]\big), \tag{3.22}$$

where χ is now a character $\chi : \Omega^{\text{spin}}_d(X) \to \mathrm{U}(1)$. Again this is different from what we had in (3.15) where we encountered $H_d(X; \mathbb{Z})$ instead of $\Omega^{\text{spin}}_d(X)$. Such characters up to continuous deformation are classified by

$$\mathrm{Hom}(\Omega^{\text{spin}}_d(X)_{\text{torsion}}, \mathrm{U}(1)) = \mathrm{Ext}_{\mathbb{Z}}(\Omega^{\text{spin}}_d(X), \mathbb{Z}). \tag{3.23}$$

Combining the two and the Anderson dual

A general invertible phase is a certain combination of these two extremes. This means that the group $\mathrm{Inv}^d_{\text{spin}}(X)$ of deformation classes of d-dimensional spin invertible phases sits in the middle of a short exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}(\Omega^{\text{spin}}_d(X), \mathbb{Z}) \longrightarrow \mathrm{Inv}^d_{\text{spin}}(X) \xrightarrow{a} \mathrm{Hom}_{\mathbb{Z}}(\Omega^{\text{spin}}_{d+1}(X), \mathbb{Z}) \longrightarrow 0. \tag{3.24}$$

Note the difference with respect to (3.17), where we had ordinary homology groups instead of bordism groups. As before, for a class $c \in \mathrm{Inv}^d_{\text{spin}}(X)$, the image $a(c)$ specifies the pairing (3.20). When $a(c)$ vanishes, the invertible phase is continuously deformable to a flat one, which is then given by a character (3.22).

The explanations so far should have clarified why one needs to use (co)bordism instead of (co)homology; the spin QFT only deals with spacetime manifolds equipped with spin structure, but not with arbitrary representatives of homology classes which might be non-spin, unoriented, or even not be expressed as an image from manifolds. It is true that the ordinary homology groups are the easiest algebraic-topological invariants of spaces, but they are not necessarily natural for the purpose of spin QFT.

Let us discuss more about the sequence (3.24). Mathematically, $\Omega^{\text{spin}}_\bullet(X)$ is an example of *generalized* homology theory. For any generalized homology theory $E_\bullet(X)$, there is a (generalized) cohomology theory $DE^\bullet(X)$ called the *Anderson dual* [And69], satisfying

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}(E_d(X), \mathbb{Z}) \longrightarrow DE^{d+1}(X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(E_{d+1}(X), \mathbb{Z}) \longrightarrow 0. \tag{3.25}$$

This can be viewed as a generalization of the universal coefficient theorem (3.17) for ordinary homology theory $H_\bullet(-;\mathbb{Z})$, where it was Anderson-self-dual $DH^\bullet(-;\mathbb{Z}) = H^\bullet(-;\mathbb{Z})$. Comparing it with (3.24), one can conclude that

$$\mathrm{Inv}^d_{\text{spin}}(X) = D\Omega^{d+1}_{\text{spin}}(X), \tag{3.26}$$

and this is the meaning of the statement that invertible phases are classified by the Anderson dual of the bordism group whose degree is shifted by one, originally formulated in [FH16].[4]

[4]While the Anderson dual of the bordism group describes the deformation classes of the invertible phases, the invertible phases themselves, not their deformation classes, should be described by the differential version thereof.

Chapter 4

Symmetry and Anomaly

4.1 Global symmetry

Symmetry is one of the most fundamental characteristics of a given theory. Conventionally, it was regarded mostly as the invariance of the action (and the path-integral measure), but the existence of "non-Lagrangian" theories or dualities which provide multiple Lagrangian descriptions to the same theory suggests that it is not the only legitimate way to treat symmetries. In this section, we will review the modern perspective where the symmetry actions are implemented by topological operators, following [GKSW14].

Basics

The existence of faithful symmetry implies that there are charged operators which transform non-trivially under the symmetry. For ordinary continuous symmetry, where the infinitesimal transformations leave the action invariant, one has the associated Noether current $j = j_\mu dx^\mu$ which is conserved

$$\partial_\mu j^\mu = 0 \quad \leftrightarrow \quad d(*j) = 0. \tag{4.1}$$

Correspondingly, the Noether charge is

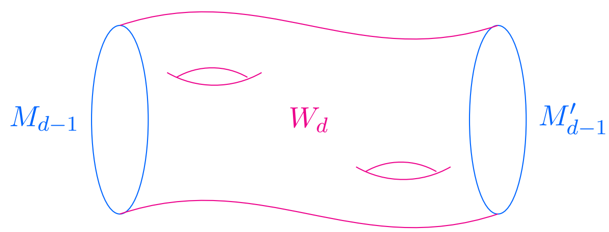
$$\begin{aligned} Q &= \int_{\text{space}} d^{d-1}x\, j_0 \\ &= \int_{\text{space}} *j \end{aligned} \tag{4.2}$$

and this serves as a generator of the symmetry algebra, where the equal-time commutation relation between operator $\mathcal{O}$ is given as

$$[Q, \mathcal{O}] = \delta_Q \mathcal{O}. \tag{4.3}$$

Generic symmetry actions can be implemented by an exponential of the generators; note that this exponentiated operator is topological and is invariant under small deformations of the space $M_{d-1} \to M'_{d-1}$ (more precisely is bordism-invariant), due to the Stokes' theorem (and the current conservation)

$$Q_{M'_{d-1}} - Q_{M_{d-1}} = \int_{W_d} d(*j) = 0. \tag{4.4}$$



Now let us generalize this and introduce the notion of p-form symmetry. It is a symmetry whose charged operators are p-dimensional (in spacetime). For a theory in d spacetime dimensions with symmetry group G, its action is defined to be implemented by a topological operator $U_g(M_{d-p-1})$ labeled by the group element $g \in G$, which is associated with a codimension-$(p+1)$ closed manifold M_{d-p-1}. (See figures below for example.) By definition, the symmetry operators obey the multiplication law[1]

$$U_g(M_{d-p-1})\, U_{g'}(M_{d-p-1}) = U_{gg'}(M_{d-p-1}). \tag{4.5}$$

The operator $\mathcal{O}$ acted by a symmetry transformation $g \in G$ is turned into $\mathcal{O}^g$, or in terms of a commutation relation

$$U_g \mathcal{O} = \mathcal{O}^g U_g$$

and thus

$$\delta_g \mathcal{O} = [U_g, \mathcal{O}]. \tag{4.6}$$

From this point of view, ordinary symmetries correspond to 0-form symmetry, but a nice thing here is that this generalized notion is also capable of describing discrete-group symmetries.

Note that, while two codimension-1 symmetry operators extended along the space direction cannot be commuted with each other in the time direction, two higher-codimension operators can, and this implies that the group G must be Abelian for $p \geq 1$.

[1] In more generic cases where the symmetry is not necessarily a group, operators obeys the "fusion" rule

$$U_a U_b = \sum_c U_c$$

which is not invertible opposed to the group-symmetry cases [BT17, CLS+18]. These are called the "non-invertible" or "categorical" symmetries.

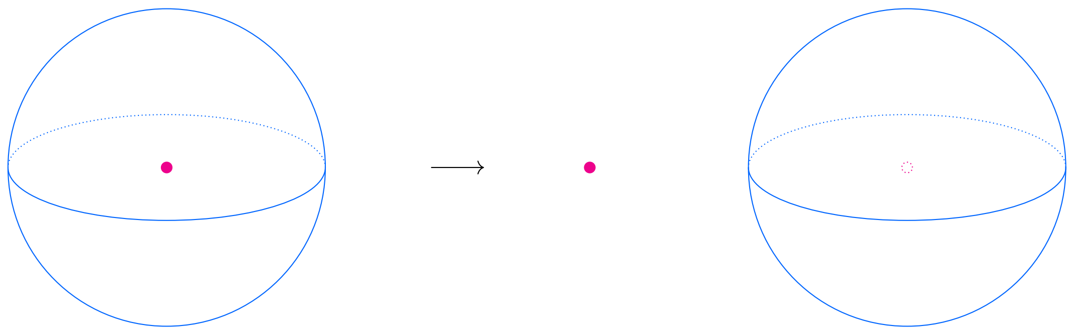


Figure 4.1: In $p = 0$ case, the local operator $\mathcal{O}$ is acted by the codimension-1 symmetry operator U_g. The symmetry operator U_g after the transformation can be completely shrunk and collapsed if there are no other charged operator left inside.

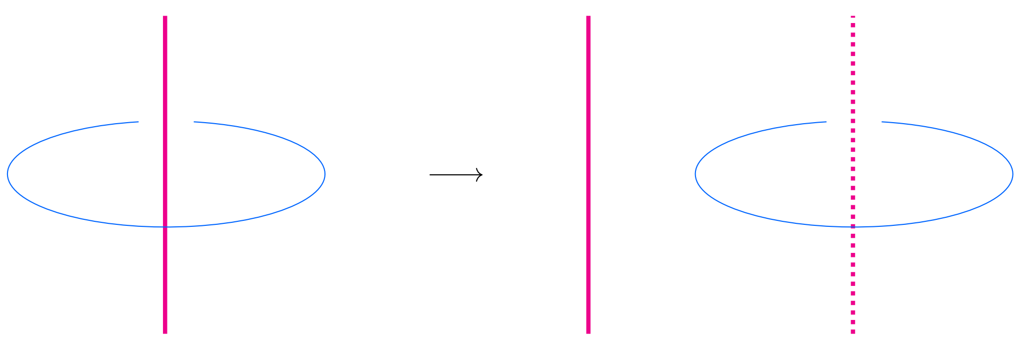


Figure 4.2: In $p = 1$ case, the non-local line operator $\mathcal{L}$ is acted by the codimension-2 symmetry operator U_g.

Gauging

The first step to gauging a global symmetry is to couple the theory to background (non-dynamical) gauge fields. For ordinary 0-form symmetries, a flat background gauge field in a spatial slice M_{d-1} can be described in terms of transition functions of a G-bundle; an intersection of patches $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ to which $g_{\alpha\beta} \in G$ is associated can be interpreted as a symmetry operator $U_{g_{\alpha\beta}}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Also, multiple intersections of patches can be resolved into triple intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$ with $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$, and these correspond to trivalent junctions of the symmetry operators. One can immediately generalize this description to p-form symmetries, where symmetry operators are defined on $(p+2)$-tuple intersections of patches, which are effectively codimension-$(p+1)$.

Summing over all possible configuration of background gauge fields, one obtains a gauged theory. Let us take a gauge theory with gauge group G as an example. As will be elaborated in Sec. 5.1.1, the Γ subgroup of the center $Z(G)$ becomes a global 1-form symmetry of the theory when none of the matter fields transform non-trivially under it. Gauging Γ, the center reduces to $Z(G)/\Gamma$, which implies that the gauge group itself is also reduced to G/Γ. In this case, non-trivial background gauge field A characterizes the equivalence classes of G/Γ-bundles which are not G-bundles, as then coupling a G gauge theory to background gauge fields A and summing over them corresponds to summing over G/Γ-bundle, which by definition gives a G/Γ gauge theory

$$Z_{G/\Gamma} = \sum_A Z_G[A]. \tag{4.7}$$

Note that one can also incorporate non-trivial weights when summing over background gauge fields, which will give rise to different gauged theories; this is namely the discrete torsion or the discrete theta angle.

4.2 Anomaly

Basics

Given a theory, even when it seems to be symmetric at first sight (at the level of classical action), the symmetries are sometimes actually violated by quantum effects. For gauge symmetries, this leads to inconsistency of the theory by definition, and therefore gauge anomalies must be canceled as a whole. On the other hand, for global symmetries, non-trivial anomalies have no problem by itself. However, there is an interesting case where the global symmetries themselves are anomaly-free but they cannot be gauged (or coupled to background gauge fields) as doing so leads to anomalies of gauged symmetries. Such form of the anomaly is called the 't Hooft anomaly. These gauge anomalies or 't Hooft anomalies can be described as a non-invariance of the partition function $Z[A]$ under the gauge transformation $g : M_d \to G$

$$Z[A^g] \neq Z[A].$$

Naively, the invariance under infinitesimal gauge transformations would be extended to the invariance under arbitrary gauge transformations. However, even if the theory is free of perturbative anomalies, it is in fact still in danger of suffering from non-perturbative anomalies as it might not be invariant under "large" gauge transformations, which cannot be smoothly deformed to the trivial gauge transformation.

't Hooft anomaly matching

The 't Hooft anomalies are believed to be invariant under the renormalization group (RG) flow from high-energy to low-energy, for the following reason. Given a theory with certain amount of 't Hooft anomaly, adding whatever degrees of freedom ("spectator") which cancel this 't Hooft anomaly (and at the same time do not strongly interact with the original theory) by hand makes the symmetry gauge-able. Gauging the symmetry, its anomaly should remain trivial as one flows down the RG flow, and at the end, ungauging the symmetry and removing the added degrees of freedom (which is possible by construction) must reproduce the same amount of 't Hooft anomaly as that one started with. This property allows us to "match" 't Hooft anomalies between the weakly-coupled and strongly-coupled regions of the theory at different energy scales, and provides us a probe into the latter which is otherwise difficult to analyze in general.

Anomaly inflow

An anomalous QFT with G global symmetry in $(d+1)$ spacetime dimensions is considered to be realized on the boundary of an invertible QFT with G symmetry in $(d+1)+1 = (d+2)$ spacetime dimensions; the deformation class $\alpha \in \mathrm{Inv}^{d+1}_{\text{spin}}(BG)$ of the invertible QFT is believed to be the anomaly of the original theory on the boundary.

From the short exact sequence (3.24), one can extract the quantity

$$a(\alpha) \in \mathrm{Hom}_{\mathbb{Z}}(\Omega^{\text{spin}}_{d+2}(BG), \mathbb{Z}) \tag{4.8}$$

which corresponds to the anomaly polynomial. It is usually given as an element of

$$\mathrm{Hom}_{\mathbb{Z}}(\Omega^{\text{spin}}_{d+2}(BG), \mathbb{Z}) \otimes \mathbb{Q} \;\simeq\; H^{d+2}(B\mathrm{Spin} \times BG; \mathbb{Q}), \tag{4.9}$$

which is given by a polynomial of spacetime Pontrjagin classes and the differential forms on BG. Furthermore, for a chiral fermion in the representation V of G, the anomaly polynomial is given by

$$a(\alpha(V)) = \left[\hat{A}\,\mathrm{ch}(V)\right]_{d+2} \tag{4.10}$$

where $\hat{A} \in H^*(B\mathrm{Spin})$ is the *A-roof polynomial* and $\mathrm{ch}\,(V) = \mathrm{tr}_V e^{iF/(2\pi)}$ is the *Chern character* where F is the curvature of the G-bundle. More generally, the anomaly of a fermion system is given by an η invariant, which carries not only the data of the anomaly polynomial but also those of the torsion part (see e.g. [WY19]).

Given a fermion system charged under G, we would like to gauge its normal subgroup G_{gauge}. This requires G_{gauge} to be anomaly free. The flavor symmetry group is then $G_{\text{flavor}} = G/G_{\text{gauge}}$, *if* there is no mixed anomaly between G_{gauge} and G_{flavor}, that is, *if* the anomaly $\alpha_G \in \mathrm{Inv}^{d+1}_{\text{spin}}(BG)$ of the fermion system is pulled back from an element $\alpha_{G_{\text{flavor}}} \in \mathrm{Inv}^{d+1}_{\text{spin}}(BG_{\text{flavor}})$ via the projection $p : G \to G_{\text{flavor}}$ as

$$\alpha_G = p^*(\alpha_{G_{\text{flavor}}}).$$

Then, the G_{flavor}-anomaly of the gauged theory is simply given by $\alpha_{G_{\text{flavor}}}$.

In the presence of the mixed anomaly, it is not even guaranteed that F is the flavor symmetry group [Tac17]. In this thesis we only consider the simpler cases where there is no mixed anomaly in the original ungauged fermion system.

Chapter 5

$4d$ $\mathrm{SU}(N_c)$ gauge theories

5.1 UV

5.1.1 Pure Yang-Mills theory

When matters are absent, the Euclidean action is simply given by a kinetic term of the gauge field

$$\int_{M_4} \mathrm{tr}\, F \wedge (*F) = \int_{M_4} d^4x \,\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}.$$

The important observables in this theory are the Wilson lines

$$W_\gamma = \mathrm{tr}_R \left[P \exp\left(i \oint_\gamma A \right) \right]$$

which are defined on a loop γ and represent the holonomy along γ. Under gauge transformations, the path-ordered exponential inside the trace transforms according to its representation R under the gauge group G, but from the invariance of a trace under cyclic permutations, one can see that these operators are actually gauge-invariant. Thus, the Wilson lines seem to be labeled by R, which can be identified with a point on the weight lattice Λ_{weight}. However, this is not strictly the case as the dynamical gauge bosons which are charged under adjoint representation screen them; the net charge of the Wilson lines take values in

$$\Lambda_{\text{weight}}/\Lambda_{\text{root}} \simeq Z(G),$$

and as a result they are charged under the center 1-form $\widehat{Z(G)}$ symmetry. For $G = \mathrm{SU}(N_c)$, it is namely $\mathbb{Z}_{N_c}$ symmetry, generated by the lines in the fundamental representation $\boldsymbol{N_c}$.

If one gives up the invariance under the parity transformation (and allows oneself to use an epsilon tensor), one can also incorporate a topological term

$$\theta \int_{M_4} \mathrm{tr}\, F \wedge F = \theta \int_{M_4} d^4x \,\mathrm{tr}\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}.$$

The integral detects non-triviality of the gauge bundle, namely the instanton number $H^4(B\mathrm{SU}(n);\mathbb{Z}) = \mathbb{Z}$ in this case.

5.1.2 QCD

Let us now add fermions in fundamental representation $\boldsymbol{N_c}$ of the gauge group $G = \mathrm{SU}(N_c)$

$$\int_{M_4} d^4x \left(\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}(\not{D} + m)\psi\right).$$

For the case of massless fermions in which we are mostly interested, one must include the same number N_f of left-handed Weyl fermions and right-handed Weyl fermions, in order to avoid the gauge anomaly. Then, the flavor symmetry is $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$. Wilson lines in fundamental representation are screened away by dynamical quarks, and therefore the center 1-form symmetry becomes trivial in this standard SU QCD.

Also, as is well-known, the gauge coupling g runs along the energy scale μ as

$$\begin{aligned}\frac{dg}{d\log\mu} &= -\frac{g^3}{16\pi^2}\left[\frac{11}{3}C(\text{adj.}) - \sum_f^{\text{fermions}}\frac{2}{3}C(R_f) - \sum_s^{\text{scalars}}\frac{1}{3}C(R_s)\right] + \mathcal{O}(g^4)\\ &= -\frac{g^3}{16\pi^2}\left[\frac{11}{3}\cdot N_c - \frac{2}{3}\cdot\frac{1}{2}\cdot 2N_f - 0\right] + \mathcal{O}(g^4)\\ &= -\frac{g^3}{48\pi^2}(11N_c - 2N_f) + \mathcal{O}(g^4),\end{aligned}$$

where $C(R)$'s are certain group-theoretic constants depending on representations R, whose values are suitably substituted. As a result, QCD is asymptotically free for small enough N_f. For such N_f, the gauge coupling g grows as one flows down to the low energy, and it is believed that QCD eventually undergoes a spontaneous breaking of the flavor symmetry $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ to its diagonal subgroup $H = \mathrm{SU}(N_f)_{\text{diag.}}$, due to the strong gauge interaction.

5.1.3 Anomaly

Gauge symmetry

For $N_c \geq 3$, there exist perturbative gauge anomalies indicated by $\Omega_6^{\text{spin}}(B\mathrm{SU}(N_c)) = \mathbb{Z}$, where the generator corresponds to the anomaly polynomial of a Weyl fermion in fundamental representation. As briefly mentioned above, the anomaly cancellation requires the numbers of left-handed and right-handed fermions to be identical.

This is not the case for $N_c = 2$, since the fundamental representation of $\mathrm{SU}(2)$ is pseudo-real and thus fermions do not contribute to the perturbative gauge anomaly, which is indeed consistent with $\Omega_6^{\text{spin}}(B\mathrm{SU}(2)) = 0$. Instead, one now has to be careful of Witten's global (non-perturbative) gauge anomaly [Wit82], which forbids the number of Weyl fermions (in any even-dimensional representations in general) to be odd. Conventionally this was attributed to the fact that the gauge group has non-trivial homotopy

$$\pi_4(\mathrm{SU}(2)) \simeq \pi_5(B\mathrm{SU}(2)) = \mathbb{Z}_2,$$

but the correct indicator to look at is actually the bordism

$$\Omega_5^{\text{spin}}(B\text{SU}(2)) = \mathbb{Z}_2,$$

which happens to be naturally isomorphic.[1]

Anyway, in the QCD case at hand, there are always even number of Weyl fermions since we have had the same number of "left-handed" ones and "right-handed" ones by construction, and therefore this anomaly is safely avoided.[2]

Global symmetry

The chiral fermions are in $\boldsymbol{N_c} \otimes \boldsymbol{N_f^{(L)}} \otimes \boldsymbol{1^{(R)}}$ and $\boldsymbol{\bar{N}_c} \otimes \boldsymbol{1^{(L)}} \otimes \boldsymbol{N_f^{(R)}}$ representations, and for $N_f \geq 3$, their anomaly polynomials are

$$N_c \cdot \text{ch}\, \boldsymbol{N_f^{(L)}} + N_f \cdot \text{ch}\, \boldsymbol{N_c}$$
$$-N_c \cdot \text{ch}\, \boldsymbol{N_f^{(R)}} + N_f \cdot \text{ch}\, \boldsymbol{\bar{N}_c}$$

respectively, which add up to

$$N_c \cdot (\text{ch}\, \boldsymbol{N_f^{(L)}} - \text{ch}\, \boldsymbol{N_f^{(R)}}) = \frac{N_c}{2} \cdot (c_3^{(L)} - c_3^{(R)}). \tag{5.1}$$

Here $c_i^{(L,R)} \in H^{2i}(B\text{SU}(N_f)_{L,R}; \mathbb{Z})$ denote the Chern classes. Note that due to the fractional coefficient, these are not elements of $H^6(B\text{SU}(N_f); \mathbb{Z})$, but the expression (5.1) still integrates to an integer on a spin manifold.[3] As $\Omega_5^{\text{spin}}(B(\text{SU}(N_c) \times \text{SU}(N_f))) = 0$, the anomaly polynomial completely determines the anomaly.

Now if we actually gauge $\text{SU}(N_c)$, as is clear from its form, the anomaly (5.1) is pulled back from the anomaly of the quotient group $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$. Therefore, there is no mixed anomaly between the gauge symmetry $\text{SU}(N_c)$, and the flavor symmetry of the theory is indeed the quotient $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$.

We can also consider the $\text{U}(1)_B$ baryon number symmetry, which assigns charge ± 1 to the left-handed and right-handed fermions respectively. The contribution to the anomaly polynomial can be similarly obtained and is given by

$$N_c \cdot \left[\frac{F_B}{2\pi}\right] (c_2^{(L)} - c_2^{(R)}). \tag{5.2}$$

As $(-1) \in \text{U}(1)_B$ acts on the fermions in the same way as the 2π rotation, the spin structure of the spacetime can in fact be upgraded to the spin^c structure.

[1] The absence of Witten-type anomaly for $N_c \geq 3$ can be inferred from $\Omega_5^{\text{spin}}(B\text{SU}(N_c)) = 0$.

[2] However, as the flavor symmetry becomes $\text{SU}(2N_f)$ rather than $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$, the SU(2) QCD is somewhat exceptional and therefore we will restrict ourselves to $N_c \geq 3$ hereafter.

[3] One can see this from its appearance in the index theorem applied to the fermion bundle. Alternatively, referring to Appendix B, one can use $c_3 = Sq^2 c_2 = w_2(TM) c_2$ mod 2 which guarantees that the integral of c_3's on a spin manifold M (which has $w_2(TM) = 0$) to be even.

5.2 IR

As already mentioned, QCD is believed to undergo a spontaneous breaking of the flavor symmetry $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ to $H = \mathrm{SU}(N_f)_{\text{diag.}}$ along the RG flow for small enough N_f. The theory is then effectively described in terms of a non-linear sigma model in the low-energy limit, whose target space is

$$G/H = \frac{\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R}{\mathrm{SU}(N_f)_{\text{diag.}}} = \mathrm{SU}(N_f)_{\text{target}},$$

in which the Nambu-Goldstone bosons σ associated to the spontaneous symmetry breaking (SSB) take value. For $N_f \geq 3$, assuming that they are the only massless degrees of freedom, the anomaly matching requires these Nambu-Goldstone fields to reproduce the anomaly (5.1) in the UV QCD, which is in fact achieved by a certain topological term called the Wess-Zumino-Witten (WZW) term. For $N_f = 2$, the anomaly (5.1) does not exist, and correspondingly the ordinary WZW term is also absent, but there is a discrete analog of the WZW term which also plays an important role as we will see in a moment.

5.2.1 WZW term: $N_f \geq 3$

Somewhat surprisingly, a proper description of WZW term on general manifolds requires the use of the spin structure, as already emphasized in [Fre06].

Given a spacetime four-manifold M_4 and a field configuration $\sigma : M_4 \to \mathrm{SU}(N_f)_{\text{target}}$, suppose one can pick an auxiliary five-dimensional manifold W_5 whose boundary is M_4 with a suitable extension $\sigma : W_5 \to \mathrm{SU}(N_f)_{\text{target}}$. Pick a closed 5-form on the group manifold $\mathrm{SU}(N_f)_{\text{target}}$, generating $H^5(\mathrm{SU}(N_f);\mathbb{R}) = \mathbb{R}$. There is a natural $\mathrm{SU}(N_f)$-invariant one, which is $\mathrm{tr}(\sigma^{-1}d\sigma)^5$. Then we define the WZW term as $e^{ik\int_{W_5}\mathrm{tr}(\sigma^{-1}d\sigma)^5}$ with a suitable coefficient k.

It was argued in [Wit83b] that the proper coefficient is given by

$$e^{-S_{\mathrm{WZW}}[\sigma]} := \exp\left(2\pi i \cdot N_c \int_{W_5} \Gamma_5(\sigma)\right) \tag{5.3}$$

where

$$\Gamma_{2n-1}(\sigma) := \left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!}\,\mathrm{tr}(\sigma^{-1}d\sigma)^{2n-1} \tag{5.4}$$

is normalized to integrate to 1 on the generator of $\pi_{2n-1}(\mathrm{SU}(N_f)) = \mathbb{Z}$ [BS78]. According to [Bot58], the map

$$\pi_{2n-1}(\mathrm{SU}(N_f)) \to H_{2n-1}(\mathrm{SU}(N_f);\mathbb{Z}) \tag{5.5}$$

sends 1 to $(n-1)!$ times a generator. This means that Γ_5 integrates to $1/2$ on the generator of $H_5(\mathrm{SU}(N_f);\mathbb{Z})$, which in turn implies that the coupling (5.3) violates the quantization condition

(3.14) based on (co)homology, when N_c is odd.[4] The way out is to require spin structures on the manifolds.

First, it is known that $H^*(\mathrm{SU}(N_f);\mathbb{Z}) = \bigwedge_{\mathbb{Z}}[x_3, x_5, \cdots]$, where $x_i \in H^i(\mathrm{SU}(N_f);\mathbb{Z})$. In particular, x_5 is the generator of $H^5(\mathrm{SU}(N_f);\mathbb{Z}) = \mathbb{Z}$. Denoting the corresponding elements in $H^i(\mathrm{SU}(N_f);\mathbb{R})$ by the same symbol x_i, one has $\Gamma_5 = x_5/2$, since Γ_5 integrates to $1/2$ on the generator Wu (see the footnote 4) of $H_5(\mathrm{SU}(N_f);\mathbb{Z})$. Also denoting the mod 2 reductions of x_i by the same symbol, it is known that $x_5 = Sq^2 x_3$ [BS53]. Then, for any closed manifold W_5 equipped with $\sigma : W_5 \to \mathrm{SU}(N_f)_{\text{target}}$, one has

$$
\begin{aligned}
\int_{W_5} \sigma^*(x_5) &= \int_{W_5} \sigma^*(Sq^2 x_3) && \pmod 2 \\
&= \int_{W_5} Sq^2 \sigma^*(x_3) && \\
&= \int_{W_5} \nu_2(TW_5)\, \sigma^*(x_3) && \pmod 2 \\
&= \int_{W_5} \left[w_2(TW_5) + w_1(TW_5)^2\right]\, \sigma^*(x_3)
\end{aligned}
\tag{5.6}
$$

where $\nu_i(T)$ is the Wu class of the tangent bundle TW_5. As we assume our W_5 to be oriented (i.e. $w_1(TW_5) = 0$) and spin (i.e. $w_2(TW_5) = 0$), this means $\int_{W_5} \sigma^*(x_5) \in 2\mathbb{Z}$ in our case, and therefore implying $\int_{W_5} \sigma^*(\Gamma_5) \in \mathbb{Z}$. This makes the $4d$ WZW coupling (5.3) for odd N_c well-defined on spin manifolds, at least when we can find a W_5 and $\sigma : W_5 \to \mathrm{SU}(N_f)_{\text{target}}$ extending a given $\sigma : M_4 \to \mathrm{SU}(N_f)_{\text{target}}$, with both M_4 and W_5 equipped with spin structure.

Now one needs to discuss whether such an extension really exists, and this can be answered using the theory of bordisms. The (reduced) bordism group relevant for our question is

$$\widetilde{\Omega}_4^{\text{spin}}(\mathrm{SU}(N_f)) = 0 \quad \text{for } N_f \geq 3.$$

This means that any $\sigma : M_4 \to \mathrm{SU}(N_f)$ is bordant to $\sigma_0 : M_4 \to \mathrm{SU}(N_f)$ where σ_0 sends the entire spacetime to a single point. In other words, we can find W_5 with $\partial W_5 = M_4 \sqcup \overline{M_4}$ such that there is a map $\sigma : W_5 \to \mathrm{SU}(N_f)$ which extends σ and σ_0 on both boundaries. Declaring that the WZW term is trivial for the topologically trivial configuration σ_0, the WZW term for a non-trivial σ is simply given by the integral (5.3) over W_5.

[4] This can be confirmed explicitly for example by taking $\sigma : W_5 \to \mathrm{SU}(N_f)_{\text{target}}$ to be the inclusion

$$\sigma : \mathrm{Wu} \hookrightarrow \mathrm{SU}(3) \subset \mathrm{SU}(N_f)$$

where $\mathrm{Wu} = \{\sigma \in \mathrm{SU}(3) \mid \sigma = \sigma^{\mathsf{T}}\} \simeq \mathrm{SU}(3)/\mathrm{SO}(3)$ is the so-called Wu manifold. For example, introduce a basis of $\mathfrak{su}(3)$ such that $\mathrm{tr}\lambda_a\lambda_b = 2\delta_{ab}$, so that $\lambda_{6,7,8}$ belongs to $\mathfrak{so}(3)$. Let $U = 1 + \lambda_a x^a \in \mathrm{SU}(3)$ and let $\sigma = UU^{\mathsf{T}}$. We find $\Gamma_5 = 4/(\sqrt{3}\pi^3)\, dx^1 \cdots dx^5$ at the origin, while it is known [BST02] that the volume of $\mathrm{SU}(3)/\mathrm{SO}(3)$ in this metric is $\sqrt{3}\pi^3/8$. Therefore Γ_5 integrates to $1/2$ in the end. Also, note that the Wu manifold is in fact a generator of $\Omega_5^{\text{oriented}} = \mathbb{Z}_2$, which can be detected by a product of second and third Stiefel-Whitney classes $w_2(TW_5)w_3(TW_5)$.

5.2.2 WZW term: $N_f = 2$

Let us now consider the special case of $N_f = 2$. There is no appropriate 5-form since $\dim \mathrm{SU}(2) = 3$, but instead the bordism group is

$$\widetilde{\Omega}_4^{\text{spin}}(\mathrm{SU}(2)) = \mathbb{Z}_2$$

opposed to the $N_f \geq 3$ case. Using this, one can introduce the coupling

$$e^{-S_{\mathrm{WZW}}[\sigma]} := (-1)^{N_c[\sigma:M_4 \to \mathrm{SU}(2)]} \tag{5.7}$$

where $[\sigma : M_4 \to \mathrm{SU}(2)]$ represents the equivalence class of the map σ and takes value in $\mathbb{Z}_2$.

This sign has a more explicit description: a map σ can be viewed as a collection of Skyrmions. The worldline of (cores of) Skyrmions can be defined as an inverse image of a point on $\mathrm{SU}(2)$. After deforming σ slightly if necessary, this inverse image is a collection of circles embedded in M_4. As M_4 itself is assumed to be a spin manifold, its spin structure can also be used to define spin structures on these circles. Then one can assign a weight ± 1 on each circle depending on the spin structure, and we multiply them.[5] That this construction detects $\widetilde{\Omega}_4^{\text{spin}}(\mathrm{SU}(2)) = \mathbb{Z}_2$ can be seen by studying the Atiyah-Hirzebruch spectral sequence (AHSS) computing it.

In the end, this means that a Skyrmion behaves as a fermion if there is a non-trivial discrete WZW term. This fact was first explained more elementarily in [Wit83b]. The preceding discussions show that it is essential to have spin structures on M_4 to define the $\mathrm{SU}(2)$ WZW term.

Relation between WZW terms for $N_f \geq 3$

So far, we have learned that the WZW terms (5.3) and (5.7) for SU QCD look rather different depending on whether $N_f \geq 3$ or $N_f = 2$. However, there is in fact a close relationship between them. By composing with the standard inclusion $\mathrm{SU}(2) \subset \mathrm{SU}(N_f)$, a map $\sigma : M_4 \to \mathrm{SU}(2)$ can also be thought of as a map $\tilde{\sigma} : M_4 \to \mathrm{SU}(N_f \geq 3)$. Then we have an equality

$$(-1)^{N_c[\sigma:M_4 \to \mathrm{SU}(2)]} = \exp\left(2\pi i \cdot N_c \int_{W_5} \Gamma_5(\tilde{\sigma})\right) \tag{5.8}$$

where W_5 is a spin manifold such that $\partial W_5 = M_4$. This equality was shown in [Wit83b] by explicitly constructing W_5 and σ, and then evaluating Γ_5 on it, but it can also be shown using algebraic topology, see Appendix B.3.

[5] Similar methods were used extensively in [GOP+18].

5.2.3 WZW terms as invertible phases

First when $N_f \geq 3$, one has a short exact sequence

$$0 \to \underbrace{\mathrm{Ext}_{\mathbb{Z}}(\Omega_4^{\text{spin}}(\mathrm{SU}(N_f)), \mathbb{Z})}_{=0} \to (D\Omega_{\text{spin}})^5(\mathrm{SU}(N_f)) \to \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\Omega_5^{\text{spin}}(\mathrm{SU}(N_f)), \mathbb{Z})}_{=\mathbb{Z}} \to 0$$

showing that $D\Omega^5_{\text{spin}}(\mathrm{SU}(N_f)) = \mathbb{Z}$ whose elements are labeled by the number N_c of colors. Note that $\Omega_5^{\text{spin}}(\mathrm{SU}(N_f)) = \mathbb{Z} \to H_5(\mathrm{SU}(N_f); \mathbb{Z}) = \mathbb{Z}$ is a multiplication by two, and therefore dually, $H^5(\mathrm{SU}(N_f); \mathbb{Z}) = \mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}}(\Omega_5^{\text{spin}}(\mathrm{SU}(N_f)), \mathbb{Z}) = \mathbb{Z}$ is also a multiplication by two. This corresponds to the fact that the generator x_5 of $H^5(\mathrm{SU}(N_f); \mathbb{Z})$ integrates to even integers on the image of spin manifolds in $\mathrm{SU}(N_f)$, which can be seen from $x_5 = Sq^2 x_3$ as in (5.6). This allows us to put the latter multiplication by two into a short exact sequence

$$0 \to \underbrace{H^5(\mathrm{SU}(N_f); \mathbb{Z})}_{=\mathbb{Z}} \to \underbrace{D\Omega^5_{\text{spin}}(\mathrm{SU}(N_f))}_{=\mathbb{Z}} \to \underbrace{H^3(\mathrm{SU}(N_f); \mathbb{Z}_2)}_{=\mathbb{Z}_2} \to 0. \tag{5.9}$$

As analyzed in detail in Appendix B.3, this sequence naturally arises from the computation of the Atiyah-Hirzebruch spectral sequence (AHSS) determining $D\Omega^5_{\text{spin}}(\mathrm{SU}(N_f))$.

On the other hand, when $N_f = 2$, the sequences are modified to

$$0 \to \underbrace{\mathrm{Ext}_{\mathbb{Z}}(\Omega_4^{\text{spin}}(\mathrm{SU}(2)), \mathbb{Z})}_{=\mathbb{Z}_2} \to D\Omega^5_{\text{spin}}(\mathrm{SU}(2)) \to \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\Omega_5^{\text{spin}}(\mathrm{SU}(2)), \mathbb{Z})}_{=0} \to 0$$

and similarly ends up with

$$0 \to \underbrace{H^5(\mathrm{SU}(2); \mathbb{Z})}_{=0} \to \underbrace{D\Omega^5_{\text{spin}}(\mathrm{SU}(2))}_{=\mathbb{Z}_2} \to \underbrace{H^3(\mathrm{SU}(2); \mathbb{Z}_2)}_{=\mathbb{Z}_2} \to 0. \tag{5.10}$$

Comparing (5.9) and (5.10), one finds that the pull-back along $\mathrm{SU}(2) \subset \mathrm{SU}(N_f \geq 3)$ is the mod 2 reduction

$$\underbrace{D\Omega^5_{\text{spin}}(\mathrm{SU}(N_f))}_{=\mathbb{Z}} \to \underbrace{D\Omega^5_{\text{spin}}(\mathrm{SU}(2))}_{=\mathbb{Z}_2},$$

which explains the relation (5.8) from a different, more abstract point of view.

Aside

One can also consider spin^c structure instead of spin structure. Physically, this corresponds to considering the non-chiral baryonic $\mathrm{U}(1)$ charge of the fermions in QCD. The relevant bordism group vanishes $\widetilde{\Omega}_4^{\text{spin}^c}(\mathrm{SU}(2)) = 0$, see Appendix B.3. Therefore one can find a spin^c manifold W_5 together with $\sigma : W_5 \to \mathrm{SU}(2)$ which extends the σ on $M_4 = \partial W_5$. One can then write the coupling

$$\exp\left(2\pi i \cdot N_c \int_{W_5} \frac{F}{2\pi} \wedge \Gamma_3\right), \tag{5.11}$$

where F is the curvature of the $\mathrm{U}(1)$ part of the spin^c connection, or equivalently the background gauge field for the baryonic $\mathrm{U}(1)$ symmetry, and Γ_3 was introduced in (5.4) and measures the Skyrmion number. Once one allows W_5 to be spin^c, it is rather natural to introduce the spin^c structure on the boundary M_4 itself. When $F = dA$, the expression above can be partially integrated to give

$$\exp\left(i \cdot N_c \int_{M_4} A \wedge \Gamma_3\right), \tag{5.12}$$

which simply means that this term induces baryonic charge N_c on a single Skyrmion. In particular, because of the spin-charge relation imposed by the spin^c structure, this term makes a Skyrmion a fermion when N_c is odd.

5.2.4 Gauged WZW terms

Let us now discuss a few extra complications which arise when we introduce gauge fields into the discussions. Our approach here is to try to define and discuss the gauged WZW terms for the sigma model target space X with an isometry group G in general, with the later use in Chap. 6 in mind.

At the level of differential forms, the anomaly of a d-dimensional system with G symmetry is given by its anomaly polynomial $\alpha_{d+2}(A, \omega)$, which is a gauge-invariant closed $(d+2)$-form constructed from the background G-gauge field A and the spin connection ω of the spacetime. Below we leave the possible ω-dependence implicit and focus on A-dependence.

Suppose that the symmetry G spontaneously breaks down to its subgroup H in the infrared, resulting in the Nambu-Goldstone scalar field $\sigma : M_d \to G/H$. The crucial insight of [Wit83a] is that this process induces the WZW term for σ. We assume that the Nambu-Goldstone field is the sole massless field in the IR. Then, while the torsion part of the anomaly can be carried by other topological parts of the theory, the sigma model part needs to reproduce the anomaly polynomial.[6] This can be achieved as follows.

The WZW term when $\sigma : M_d \to X$ extends to $\tilde{\sigma} : W_{d+1} \to X$ was given by the integral

$$\exp\left(2\pi i \int_{W_{d+1}} \Gamma_{d+1}(\sigma)\right). \tag{5.13}$$

Introducing the background gauge field A for the flavor symmetry, the coupling is generalized to

$$\exp\left(2\pi i \int_{W_{d+1}} \underline{\Gamma}_{d+1}(\sigma, A)\right), \tag{5.14}$$

[6] For SU QCD with $N_c = 2$ and N_f odd, the UV theory has the Witten's global $\mathrm{SU}(2)$ anomaly [Wit82]. Although this is a torsion part of the anomaly, this cannot be cancelled by gapped modes in the theory [GEHO$^+$17, Sec. 5] and has to be reproduced also by the sigma model part. More generally, any torsional anomaly not cancellable by gapped modes, recently discussed e.g. in [CO19a, CO19b], needs to be reproduced by the sigma model.

where $\underline{\Gamma}_{d+1}(\sigma, A)$ is a possibly-non-closed but gauge-invariant $(d+1)$-form such that it reduces to $\Gamma_{d+1}(\sigma)$ when the background gauge field is turned off

$$\Gamma_{d+1}(\sigma) := \underline{\Gamma}_{d+1}(\sigma, A=0). \tag{5.15}$$

As we assumed that the sigma model field is the sole massless degree of freedom in the IR, this coupling needs to reproduce the anomaly polynomial (5.1) of the flavor symmetry in the UV, meaning that it should have the same variation under the changes of W_{d+1} and the gauge field A on it as the expression

$$\exp\left(2\pi i \int_{W_{d+1}} \mathrm{CS}_{d+1}(A)\right), \tag{5.16}$$

where $\mathrm{CS}_{d+1}(A)$ is the Chern-Simons term satisfying $\alpha_{d+2}(A) = d\,\mathrm{CS}_{d+1}(A)$. This condition can be achieved by postulating

$$\begin{aligned} d\underline{\Gamma}_{d+1}(\sigma, A) &= d\,\mathrm{CS}_{d+1}(A) \\ &= \quad \alpha_{d+2}(A). \end{aligned} \tag{5.17}$$

Running the argument in reverse, this allows us to determine the ungauged WZW term starting from the anomaly by solving (5.17) and then setting $A = 0$.

More mathematically, a gauge-invariant closed $(d+2)$-form $\alpha_{d+2}(A)$ constructed from the background G-gauge field A determines an element $\alpha \in H^{d+2}(BG;\mathbb{R})$. Similarly, the closed $(d+1)$-form $\Gamma_{d+1}(\sigma)$ comes from an element $\Gamma \in H^{d+1}(G/H;\mathbb{R})$. The equation (5.17) means that α trivializes cohomologically when pulled back to the total space of the universal G/H bundle over BG. From the definition of the classifying spaces, this universal bundle is homotopy-equivalent to BH, and one has a fibration

$$G/H \longrightarrow BH \xrightarrow{p} BG. \tag{5.18}$$

More generally, associated to a fibration

$$F \longrightarrow E \xrightarrow{p} B \tag{5.19}$$

one can consider the following operation. Take an element $\alpha \in H^{d+2}(B)$. Assume its pull-back to E trivializes: $p^*(\alpha) = 0 \in H^{d+2}(E)$. At the cochain level, this means that there is an element $\underline{\Gamma} \in C^{d+1}(E)$ such that $\delta\underline{\Gamma} = p^*(\alpha)$. If we now restrict $\underline{\Gamma}$ to the fiber F, then $\delta\underline{\Gamma}|_F = 0 \in C^{d+2}(F)$, and therefore correspondingly we have an element $\Gamma \in H^{d+1}(F)$. This operation is known as the *transgression* in algebraic topology, and Γ is said to transgress to α.[7,8]

[7]That the relation between the WZW term and the anomaly is the transgression is of course long known. See e.g. [DW90, Sec. 4], [Wit92, Appendix] and [Fre06, Sec. 5]. In particular, the reference [DW90, Sec. 4] already contains a detailed explanation of the transgression associated to the special but important case $G \to EG \to BG$.

[8]Note that we here made our analysis at the level of differential forms. This does not allow us, for example, to obtain the WZW term for $N_f = 2$ from the Witten's global anomaly via transgression. This was done in [Fre06] using the differential version of the generalized cohomology theory $E^\bullet$, for SU WZW terms. An abstract version of the transgression map in the bordism case, constructing elements $\Gamma \in \mathrm{Inv}^d_{\mathrm{spin}}(F)$ from $\alpha \in \mathrm{Inv}^{d+1}_{\mathrm{spin}}(B)$ assuming that it trivializes in $\mathrm{Inv}^{d+1}_{\mathrm{spin}}(E)$, was also discussed in [KOT19, Sec. 2.5].

Normalization of the ungauged WZW terms

From the construction of ungauged WZW terms discussed above, one can also check its normalization as follows. First of all, the Leray-Serre spectral sequence (LSSS) associated to the fibration (5.19), whose E_2 page is given by

$$E_2^{p,q} = H^p\big(B; H^q(F;\mathbb{Z})\big), \tag{5.20}$$

converges to $H^{p+q}(E;\mathbb{Z})$ (see Appendix A for the brief introduction to spectral sequences). The elements in $H^r(F;\mathbb{Z})$ which transgress to $H^{r+1}(B;\mathbb{Z})$ are known to be those elements which survive to the $(r+1)$-st page $E_{r+1}^{0,r}$, and the transgression is known to coincide with the differential

$$d_{r+1} : E_{r+1}^{0,r} \to E_{r+1}^{r+1,0}. \tag{5.21}$$

For the case of interest, the fibration is (5.18) where $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ and $H = \mathrm{SU}(N_f)_{\text{diag.}}$ with $N_f \geq 3$, and correspondingly the convergence is as

$$E_2^{p,q} = H^p\big(B(\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R); H^q(\mathrm{SU}(N_f)_{\text{target}};\mathbb{Z})\big) \qquad\qquad H^{p+q}\big(B\mathrm{SU}(N_f)_{\text{diag.}};\mathbb{Z}\big)$$

$$\begin{array}{c|ccccccc}
6 & & & & & & & \\
5 & \boxed{\mathbb{Z}} & & & & * & & * \\
4 & & & & & & & \\
3 & \mathbb{Z} & & & & * & & * \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & & \mathbb{Z}^{\oplus 2} & & \boxed{\mathbb{Z}^{\oplus 2}} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\quad \Longrightarrow \quad
\begin{array}{c|c}
6 & \mathbb{Z} \\
5 & \\
4 & \mathbb{Z} \\
3 & \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array} \tag{5.22}$$

To realize correct convergence in degree $p+q = 5, 6$, the differential $d_6 : E_6^{0,5} \to E_6^{6,0}$ must be an injection so that $E_7^{0,5} = E_\infty^{0,5} = 0$ and $E_7^{6,0} = E_\infty^{6,0} = \mathbb{Z}$. Considering the symmetry of exchanging two $\mathrm{SU}(N_f)_{L,R}$ factors, this means that the generator $x_5 \in H^5(\mathrm{SU}(N_f)_{\text{target}};\mathbb{Z})$ transgresses to $c_3 - c_3' \in H^6(B\mathrm{SU}(N_f)_L \times B\mathrm{SU}(N_f)_R;\mathbb{Z})$. Since $\Gamma_5 = x_5/2$, the normalization (5.3) is verified.

Topological consistency of the gauged WZW term

Let us now consider possible global topological issues associated to the gauged WZW term (5.14). By construction, the combination

$$\exp\left(2\pi i \int_{W_{d+1}} \Big(\mathrm{CS}(A) - \underline{\Gamma}(\sigma, A)\Big)\right) \tag{5.23}$$

for closed manifolds W_{d+1} determines a bordism invariant

$$\gamma : \Omega_{d+1}^{\text{spin}}(BH) \to \mathrm{U}(1). \tag{5.24}$$

When (the torsion part of) γ is non-zero, it signifies that the gauged WZW term $\underline{\Gamma}_{d+1}(\sigma, A)$ itself is still anomalous, and at the same time it determines a $(d+1)$-dimensional spin invertible phase, and therefore gives an element of $\mathrm{Inv}^{d+1}_{\text{spin}}(BH)$. Also, the deformation class of the expression (5.23) is the same as that of the Chern-Simons term alone, since $\underline{\Gamma}(\sigma, A)$ is a globally well-defined differential form. This means that γ as an element of $\mathrm{Inv}^{d+1}_{\text{spin}}(BH)$ is simply a pull-back of the original anomaly $\alpha \in \mathrm{Inv}^{d+1}_{\text{spin}}(BG)$.

Mathematically, the situation can be summarized as follows. First we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}_{\mathbb{Z}}(\Omega^{\text{spin}}_{d+1}(BH), \mathbb{Z}) & \xrightarrow{b} & \mathrm{Inv}^{d+1}_{\text{spin}}(BH) & \xrightarrow{a} & \mathrm{Hom}_{\mathbb{Z}}(\Omega^{\text{spin}}_{d+2}(BH), \mathbb{Z}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Ext}_{\mathbb{Z}}(\Omega^{\text{spin}}_{d+1}(BG), \mathbb{Z}) & \xrightarrow{b} & \mathrm{Inv}^{d+1}_{\text{spin}}(BG) & \xrightarrow{a} & \mathrm{Hom}_{\mathbb{Z}}(\Omega^{\text{spin}}_{d+2}(BG), \mathbb{Z}) & \longrightarrow & 0. \end{array}$$

Starting from the anomaly $\alpha \in \mathrm{Inv}^{d+1}_{\text{spin}}(BG)$ in the lower middle part, one obtains $p^*(\alpha) \in \mathrm{Inv}^{d+1}_{\text{spin}}(BH)$ by pulling it back to the upper middle part. Since we assumed that the anomaly polynomial restricted to the unbroken subgroup H is zero i.e. $a(p^*(\alpha))$ vanishes, $p^*(\alpha)$ is in the image of b from the exactness of the sequence, so one can write $p^*(\alpha) = b(\gamma)$, where

$$\gamma \in \mathrm{Ext}_{\mathbb{Z}}(\Omega^{\text{spin}}_{d+1}(BH), \mathbb{Z}) = \mathrm{Hom}(\Omega^{\text{spin}}_{d+1}(BH)_{\text{torsion}}, \mathrm{U}(1)).$$

This γ being non-zero signifies that there is a residual global anomaly in the gauged WZW term.

In the concrete case of SU WZW terms,

$$\Omega^{\text{spin}}_5(B\mathrm{SU}(N_f)_{\text{diag.}}) = 0 \quad \text{for } N_f \geq 3$$

and one can conclude that this γ is actually zero. This is consistent with the physics point of view, since $\gamma \in \mathrm{Inv}^5_{\text{spin}}(BH)$ would be the anomaly of fermions with respect to a symmetry H, under which they can be given a non-zero mass; this immediately guarantees that not only the free part characterized by the anomaly polynomial vanishes, but also the subtler torsion part does.

5.2.5 Solitonic symmetries

As $\pi_3(\mathrm{SU}(N_f)) = \mathbb{Z}$ and $H_3(\mathrm{SU}(N_f)) = \mathbb{Z}$ are naturally isomorphic by the Hurewicz theorem, the low-energy sigma model has a single type of point-like solitons whose number as an integer is conserved. This quantum number in the IR has been identified as the baryon number in the UV [Sky61, Wit83b]. Let us recall why this should be the case.

As reviewed in Sec. 5.1.3, the $\mathrm{U}(1)_B$ baryon number and the $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ flavor symmetries have an anomaly (5.2). By inspecting the LSSS (5.22), one finds that the generator $\Gamma_3 = x_3$ of $H^3(\mathrm{SU}(N_f); \mathbb{Z}) = \mathbb{Z}$ transgresses to $c_2^{(L)} - c_2^{(R)}$. This means that the anomaly (5.2) transgresses from the WZW-type coupling

$$\exp\left(2\pi i \cdot N_c \int_{W_5} \frac{F_B}{2\pi} \wedge \Gamma_3\right) \tag{5.25}$$

which can be partially integrated to

$$\exp\left(i \cdot N_c \int_{M_4} A_B \wedge \Gamma_3\right). \tag{5.26}$$

As A_B is the background gauge field for the baryonic symmetry, Γ_3 should be the sigma-model expression of the baryonic number current [GW81, BNRS82, CL85].

Chapter 6

$4d$ $\mathrm{SO}(N_c)$ gauge theories

6.1 UV

6.1.1 Pure Yang-Mills theory

When $N_c = 2n_c$ is even, the center of the gauge group is

$$Z(\mathrm{SO}(2n_c)) = \mathbb{Z}_2,$$

and as before there are non-trivial Wilson lines in the vector representation of the gauge group, charged under the center $\mathbb{Z}_2$ 1-form symmetry. On the other hand, when N_c is odd, the center is trivial and correspondingly the theory does not have non-trivially charged Wilson lines.

't Hooft line

The fact that the group $\mathrm{SO}(N_c)$ is not simply-connected brings several twists into the story. The first twist is that there is an additional type of non-triviality for $\mathrm{SO}(N_c)$ gauge bundles, characterized by

$$H^2(B\mathrm{SO}(N_c); \mathbb{Z}_2) = \mathbb{Z}_2$$

which was absent in the $\mathrm{SU}(N_c)$ gauge theories. Correspondingly, there are magnetic monopoles in $\mathrm{SO}(N_c)$ gauge theories in addition to ordinary gauge instantons. This singularity of a gauge bundle c is detected by the second Stiefel-Whitney class $w_2(c) \in H^2(B\mathrm{SO}(N_c); \mathbb{Z}_2)$, evaluated on an enclosing two-sphere S^2. One can view this S^2 as a two-dimensional manifold on which a codimension-2 symmetry operator measuring the charge of 1-form symmetry is defined; the charged operator is what is called the 't Hooft line. The theory can be coupled to the background gauge field B of this magnetic $\mathbb{Z}_2$ 1-form symmetry [AST13, GKSW14] via

$$\exp\left(2\pi i \cdot \frac{1}{2}\int_{M_4} B w_2(c)\right). \tag{6.1}$$

Discrete theta angle

Another twist is that there is also an additional type of topological term due to

$$\mathrm{Hom}(\Omega_4^{\text{spin}}(B\mathrm{SO}(N_c))_{\text{torsion}}, \mathrm{U}(1)) = \mathbb{Z}_2 \quad \text{for } N_f \geq 5,$$

which is a discrete analog of the ordinary theta term. Its explicit form is given as

$$\frac{1}{2}\int_{M_4} \mathfrak{P}(w_2(c))$$

where $\mathfrak{P} : H^2(-;\mathbb{Z}_2) \to H^4(-;\mathbb{Z}_2)$ is a cohomology operation called the Pontrjagin square.[1] Opposed to the ordinary one, a theory with discrete theta term and a theory without it are not continuously deformable to each other, and thus they are two distinct theories; the former is called the $\mathrm{SO}(N_c)_-$ theory while the latter ordinary theory is called the $\mathrm{SO}(N_c)_+$ theory [AST13].

Global structure of the gauge group

Yet another twist is that there is a further variant of the theory, namely the $\mathrm{Spin}(N_c)$ gauge theory. A Spin group is a simply-connected double cover of SO group. For even $N_c = 2n_c$, its center is either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on the parity of n_c, and correspondingly there are various quotients thereof, associated with characteristic classes as follows:

$$\begin{tikzcd}
 & \mathrm{Spin}(2n_c) & \\
\mathrm{Ss}(2n_c) \arrow[ur, "v_2=w_2'"] & \mathrm{SO}(2n_c) \arrow[u, "w_2=w_2'"'] & \mathrm{Ss}'(2n_c) \arrow[ul, "v_2=w_2"'] \\
 & \mathrm{SO}(2n_c)/\mathbb{Z}_2 \arrow[ul, "w_2'"] \arrow[u, "v_2"'] \arrow[ur, "w_2"'] &
\end{tikzcd}$$

<u>n_c even</u>

$$\begin{tikzcd}
\mathrm{Spin}(2n_c) \\
\mathrm{SO}(2n_c) \arrow[u, dashed, "w_2"] \\
\mathrm{SO}(2n_c)/\mathbb{Z}_2 \arrow[u, "v_2"] \arrow[uu, bend right=90, "x_2"']
\end{tikzcd}$$

<u>n_c odd</u>

Characteristic classes obstruct the lifting of corresponding principal bundles. For example, $\mathrm{SO}(2n_c)/\mathbb{Z}_2$-bundles can be uplifted to $\mathrm{SO}(2n_c)$-bundles if the class $v_2 \in H^2(B(\mathrm{SO}(2n_c)/\mathbb{Z}_2);\mathbb{Z}_2)$ is trivial, and it can be further uplifted to $\mathrm{Spin}(2n_c)$-bundles if the class $w_2 \in H^2(B\mathrm{SO}(2n_c);\mathbb{Z}_2)$ is also trivial. Note that for n_c odd, w_2 might not be a cocycle from the beginning; starting from an $\mathrm{SO}(2n_c)/\mathbb{Z}_2$-bundle, there are $\mathbb{Z}_4$-valued cocycle x_2 and its $\mathbb{Z}_2$-reduction v_2, while the would-be cocycle w_2 is actually a cochain satisfying

$$\delta w_2 = \beta v_2, \tag{6.2}$$

[1] Note that the codomain is not necessarily $\mathbb{Z}_2$-valued but rather $\mathbb{Z}_4$-valued in general when the underlying spacetime manifold is not equipped with spin structure. Also there is a subtlety concerning non-closedness of $w_2(c)$ for n_c odd cases, see e.g. [LOT21] for further details.

where β is a Bockstein homomorphism associated with the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

As a result, w_2 becomes a cocycle only when v_2 is trivial, or equivalently, when x_2 takes value in $\{0,2\} \subset \mathbb{Z}_4$, and is given by $w_2 = \frac{1}{2}x_2$ taking values in $\{0,1\} = \mathbb{Z}_2$. For n_c even, there is no such issue, and w_2 is a genuine cocycle. Combining the two cases, one can write

$$\delta w_2(c) = n_c \cdot \beta v_2(c). \tag{6.3}$$

Now, from what we have discussed in Sec. 4.1, one can obtain an SO_+ gauge theory by gauging the $\Gamma = \mathbb{Z}_2$ subgroup 1-form symmetry of a Spin gauge theory. From the $G = \mathrm{Spin}$ gauge theory point of view, the corresponding background gauge field is $w_2(c)$ which characterizes $G/\Gamma = \mathrm{SO}$ bundles, coupled to the Spin gauge theory through

$$\frac{1}{2}\int Bw_2(c)$$

where B is another $\mathbb{Z}_2$-valued degree-2 closed cochain, and is exactly the background gauge field of the magnetic $\mathbb{Z}_2$ 1-form symmetry of the SO_+ gauge theory appeared in (6.1).

Conversely, starting from an SO_+ gauge theory, one can also obtain a Spin gauge theory by gauging the magnetic $\mathbb{Z}_2$ 1-form symmetry, as B serves as a Lagrange multiplier and summing over B forces $w_2(c)$ to be trivial, where $w_2(c) = 0$ meant that the bundle is actually lifted to a Spin bundle.

6.1.2 QCD

Let us now add massless fermions in vector representation of the gauge group $\mathrm{SO}(N_c)$. When $N_c = 2n_c$ is even, the center $\mathbb{Z}_2$ actions of the gauge $\mathrm{SO}(2n_c)$ group and the spacetime Spin group on them are identical. If we really identify them, the fermions are actually charged under

$$\frac{\mathrm{Spin(spacetime)} \times \mathrm{SO}(2n_c)}{\mathbb{Z}_2} \times \mathrm{SU}(N_f), \tag{6.4}$$

which has a subgroup $\mathrm{SO}(2n_c) = \frac{\{1,(-1)^F\}\times \mathrm{SO}(2n_c)}{\mathbb{Z}_2}$ we are going to gauge. Therefore, whether the $\mathrm{SO}(2n_c)/\mathbb{Z}_2$-gauge bundle lifts to an $\mathrm{SO}(2n_c)$-bundle is synchronized with whether the spacetime SO tangent bundle lifts to a Spin bundle. In terms of cohomology classes, this means

$$v_2(c) = w_2(TM_4). \tag{6.5}$$

6.1.3 Anomaly

Global symmetry (ordinary 0-form)

For $N_f \geq 3$, the chiral fermions are in $\boldsymbol{N_c} \otimes \boldsymbol{N_f}$ representation, and their anomaly polynomial is given by

$$N_c \cdot \mathrm{ch}\, \boldsymbol{N_f} = \frac{N_c}{2} \cdot c_3 \tag{6.6}$$

as before. Since $\Omega_5^{\text{spin}}(B(\mathrm{SO}(N_c) \times \mathrm{SU}(N_f))) = 0$, this is sufficient to completely specify the anomaly of the fermion system. This anomaly is pulled back from the anomaly of $\mathrm{SU}(N_f)$ symmetry. Therefore, there is no mixed anomaly between the gauge symmetry $\mathrm{SO}(N_c)$, and the flavor symmetry of the theory is $\mathrm{SU}(N_f)$, whose 't Hooft anomaly is given by (6.6).

Furthermore, as computed in Appendix B.2, all possible anomalies under the structure (6.4) are pull-backs from either $\frac{\mathrm{Spin(spacetime)} \times \mathrm{SO}(2n_c)}{\mathbb{Z}_2}$ or $\mathrm{SU}(N_f)$. However, since the fermions can be made massive under the former, the anomaly is again a pull-back from the latter, specified by (6.6). After the gauging, therefore, we have the spacetime structure

$$\underbrace{\mathrm{Spin(spacetime)}/\mathbb{Z}_2}_{=\mathrm{SO(spacetime)}} \times \mathrm{SU}(N_f). \tag{6.7}$$

Novel mixed anomaly

When the right hand side of (6.2) is non-zero, $w_2(c)$ is not a cocycle anymore, and the coupling (6.1) is not well-defined. Indeed, the integrand is not closed:

$$\begin{aligned} \delta(Bw_2(c)) &= n_c \cdot B\beta v_2(c) \\ &= n_c \cdot B\beta w_2(TM_4) \end{aligned}$$

where we used (6.5). This depends only on the background fields, and therefore one can safely add the bulk $5d$ action

$$\exp\left(2\pi i \cdot \frac{n_c}{2} \int_{W_5} B\beta w_2(TW_5)\right) \tag{6.8}$$

to make the combined bulk-boundary system non-anomalous. In other words, the gauge theory on the $4d$ boundary has a mixed 't Hooft anomaly between the $\mathbb{Z}_2$ 1-form symmetry and the spacetime rotation symmetry. The existence of this mixed anomaly was first pointed out in [HL20], where $\frac{\mathrm{SO}(N_c) \times \mathrm{SU}(N_f)}{\mathbb{Z}_2}$ was used instead.[2]

[2] It was also pointed out there that this mixed anomaly is transformed into a 2-group structure between the electric $\mathbb{Z}_2$ 1-form symmetry and the rest of the symmetry in the $\mathrm{Spin}(2n_c)$ gauge theory. Let us briefly recall how this 2-group arises. The $\mathrm{Spin}(2n_c)$ gauge theory has an electric $\mathbb{Z}_2$ 1-form symmetry, whose background field E sets the second Stiefel-Whitney class $w_2(c) \in H^2(X; \mathbb{Z}_2)$ of the $\mathrm{SO}(2n_c)$ gauge bundle to be $E = w_2(c)$. Then, the relation (6.3) together with (6.5) results in

$$\delta E = n_c \cdot \beta w_2(TM_4). \tag{6.9}$$

6.2 IR

6.2.1 WZW term: $N_f \geq 3$

The $\mathrm{Spin}(N_c)$ QCD contains fermions $\psi_{\alpha a i}$ with the spacetime spinor index $\alpha = 1, 2$, the color index $a = 1, \ldots, N_c$, and the flavor index $i = 1, \ldots, N_f$. It is expected that, when N_f is not too large with respect to N_c, the strongly-coupled dynamics generates the condensate

$$\Lambda^3 \sigma_{ij} := \langle \epsilon^{\alpha\beta} \delta^{ab} \psi_{\alpha a i} \psi_{\beta b j} \rangle$$

where σ_{ij} is a complex symmetric matrix and Λ is the dynamical scale. The σ field then takes values in the subset

$$\{\sigma \in \mathrm{SU}(N_f) \mid \sigma = \sigma^{\mathsf{T}}\}$$

which can be identified with the homogeneous space $\mathrm{SU}(N_f)/\mathrm{SO}(N_f)$.

Let us first discuss the $N_f \geq 3$ case. As in the SU QCD case in Sec. 5.2.1, given a configuration $\sigma : M_4 \to \mathrm{SU}(N_f)/\mathrm{SO}(N_f)$ which can be extended to a map $\sigma : W_5 \to \mathrm{SU}(N_f)/\mathrm{SO}(N_f)$ such that $\partial W_5 = M_4$, one can pull back the differential form $\Gamma_5(\sigma)$ on $\mathrm{SU}(N_f)$ to $\mathrm{SU}(N_f)/\mathrm{SO}(N_f)$ and define the WZW term as $e^{ik \int_{W_5} \Gamma_5(\sigma)}$ with a suitable coefficient k. As we will recall later in Sec. 6.2.4, the normalization coming from physics consideration is

$$\exp\left(2\pi i \cdot N_c \int_{W_5} \frac{\Gamma_5(\sigma)}{2}\right). \tag{6.10}$$

The integrand $\Gamma_5/2$ integrates to $1/4$ on the generator of $H_5(\mathrm{SU}(N_f)/\mathrm{SO}(N_f); \mathbb{Z}) \simeq \mathbb{Z}$, which can be taken to be the Wu manifold Wu. Therefore this coupling is not well-defined if we allow arbitrary oriented W_5.[3] Stated differently, denoting the generator of $\mathbb{Z} \subset H^5(\mathrm{SU}(N_f)/\mathrm{SO}(N_f); \mathbb{Z})$ by y_5, one needs to show that $\int_{W_5} y_5$ is a multiple of four once we impose some constraint on the allowed manifold W_5. As QCD requires spin structure, it is a natural condition for W_5 to be equipped with spin structure. However, the argument we used for SU QCD in Sec. 5.2.1 does not suffice, since it only shows that $\int_{W_5} y_5$ is a multiple of two. Fortunately, by a more detailed analysis, one can show that the integral on any spin manifold is in fact 0 mod 4 as required. We provide one method utilizing Adams spectral sequence in Appendix B.4.

This means that the electric $\mathbb{Z}_2$ 1-form symmetry extends the spacetime symmetry. This extension can also be understood as the combination of the following two facts, namely that the $\mathrm{Spin}(2n_c)$ theory is obtained by gauging B of the $\mathrm{SO}(2n_c)$ theory [KS14, GKSW14], and that a theory with an anomaly (6.8) turns into a theory with an extension (6.9) under the gauging of the 1-form symmetry [Tac17].

[3] As we have mentioned earlier, when $N_c = 2n_c$ is even, one can put the theory on a non-spin manifold, and in this case, the consistency of the WZW term is realized in a subtler way. We will come back to this question in Sec. 6.2.5.

Then, we now need to ask whether one can find such an extension $\sigma : W_5 \to \mathrm{SU}(N_f)/\mathrm{SO}(N_f)$ in the first place. This information is encoded in the reduced bordism group

$$\widetilde{\Omega}_4^{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)) = \begin{cases} \mathbb{Z}_2 & (N_f \geq 5), \\ \mathbb{Z} & (N_f = 4), \\ 0 & (N_f = 3). \end{cases}$$

$N_f \geq 5$

For $N_f \geq 5$, the reduced bordism group is non-trivial, and $\sigma : M_4 \to \mathrm{SU}(N_f)/\mathrm{SO}(N_f)$ in the generating bordism class cannot be extended to a bulk W_5 with $\partial W_5 = M_4$. However, as the group is $\mathbb{Z}_2$, two copies of them can be extended to a bulk W_5 with $\partial W_5 = M_4 \sqcup M_4$, which implies[4]

$$\left(e^{-S_{\text{WZW}}[\sigma : M_4 \to \mathrm{SU}(N_f)/\mathrm{SO}(N_f)]}\right)^2 = \exp\left(2\pi i \cdot N_c \int_{W_5} \frac{\Gamma_5(\sigma)}{2}\right) \tag{6.11}$$

for which there are two solutions differing by a sign. Let us fix a representative σ of the generating bordism class and pick a particular solution anyway. Then, one can define a WZW term for generic $\sigma' : M_4' \to \mathrm{SU}(N_f)/\mathrm{SO}(N_f)$ in the generating bordism class as

$$e^{-S^{(1)}_{\text{WZW}}[\sigma']} := \exp\left(2\pi i \cdot N_c \int_{W_5'} \frac{\Gamma_5(\sigma)}{2}\right) e^{-S^{(1)}_{\text{WZW}}[\sigma]}$$

as M_4 and M_4' are bordant such that $\partial W_5' = M_4 \sqcup \overline{M_4'}$ with σ suitably extended. The other solution for the fixed representative leads to

$$e^{-S^{(2)}_{\text{WZW}}[\sigma]} = e^{-S^{(1)}_{\text{WZW}}[\sigma]} \cdot \chi\Big([\sigma : M_4 \to \mathrm{SU}(N_f)/\mathrm{SO}(N_f)]\Big)$$

where $\chi \in \mathrm{Hom}(\widetilde{\Omega}_4^{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)), \mathrm{U}(1)) = \mathbb{Z}_2$ is a non-trivial character. Here, neither $S^{(1)}_{\text{WZW}}$ nor $S^{(2)}_{\text{WZW}}$ is privileged at this point; the solutions to (6.11) form an affine space (or a torsor) over $\mathrm{Hom}(\widetilde{\Omega}_4^{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)), \mathrm{U}(1))$. The path integral of SO QCD should provide one specific solution among them, and determining it in any meaningful manner would be an interesting question.

The non-trivial character χ can be explicitly determined and is given by

$$\chi\Big([\sigma : M_4 \to \mathrm{SU}(N_f)/\mathrm{SO}(N_f)]\Big) = \exp\left(2\pi i \cdot \frac{1}{2} \int_{M_4} \frac{1}{2}\mathfrak{P}(\sigma^* w_2)\right) \tag{6.12}$$

where w_2 is the generator of $H^2(\mathrm{SU}(N_f)/\mathrm{SO}(N_f); \mathbb{Z}_2)$, and $\mathfrak{P}$ is the Pontrjagin square. For the derivation, see Appendix B.2. As $\sigma^* w_2$ represents the worldsheet of a electric flux tube, the character above adds a factor -1 at each intersection of two flux tubes.

[4] In [Yon20, Sec. 4], a nice representative was constructed, where $\underline{M_4} = T^2 \times T^2$ and the configuration $\underline{\sigma}$ is invariant under the spacetime parity transformation. This guarantees that the right hand side of (6.11) is trivial, and $e^{-S_{\text{WZW}}[\sigma]} = \pm 1$.

$N_f = 4$

For $N_f = 4$, one can assign an arbitrary phase $e^{-S_{\rm WZW}[\sigma]} = e^{i\theta}$ instead of a sign for the generator of the reduced bordism group. As θ can be continuously deformed, it does not affect the deformation class of the WZW term, but the actual WZW term depends on the value of θ, and should be fixed by the QCD path integral, similarly as in the $N_f \geq 5$ case.[5]

$N_f = 3$

For $N_f = 3$, the reduced bordism group is trivial, and any σ can be extended to that in the bulk. Therefore, there are no subtleties as in $N_f \geq 4$, and the WZW term is simply given by (6.10).

6.2.2 WZW term: $N_f = 2$

This time, the sigma model target space is $\mathrm{SU}(2)/\mathrm{SO}(2) \simeq S^2$, for which we find that $\widetilde{\Omega}_4^{\rm spin}(S^2) = \mathbb{Z}_2$. Quite similarly as in the SU QCD case, this allows us to write down a WZW term

$$(-1)^{N_c[\sigma:M_4 \to S^2]} \tag{6.13}$$

where $[\sigma : M_4 \to S^2]$ represents the (reduced) bordism class of the map σ and takes value in $\mathbb{Z}_2$.

Again this sign has an explicit description. Let us perturb σ slightly to make it generic, and then take the inverse image of a point on S^2 under σ. This defines a union of surfaces Σ (which is codimension-2) within M_4. Given a spin structure on M_4, it induces a spin structure on Σ. We then take the Arf invariant of the spin surface Σ. Physically, this surface can be interpreted as a worldsheet of the color flux tube associated to $\pi_2(S^2) = \mathbb{Z}$, and the $2d$ effective theory on it is a non-trivial fermionic invertible phase corresponding to the non-trivial element of $\mathrm{Hom}(\Omega_2^{\rm spin}(pt), \mathrm{U}(1)) = \mathbb{Z}_2$, which is known as the Arf theory or the Kitaev chain [Kit00].

The boundary of the Arf theory famously carries an odd number of Majorana fermion zero modes. Therefore, this means that the boundary of the electric flux tube with odd (resp. even) N_c carries an odd (resp. even) number of Majorana zero modes. To see this in the UV description, recall that electric flux tubes of the $\mathrm{Spin}(N_c)$ gauge theory are charged under the spinor representation, and therefore ends on the Wilson line in the spinor representation. In our $\mathrm{Spin}(N_c)$ QCD, the dynamical fermions ψ_a were in the vector representation. The Wilson lines in the spinor representation of $\mathrm{Spin}(N_c)$ then have an action of the gamma matrices Γ_a ($a = 1, \ldots, N_c$), which can also be considered as Majorana fermions, and there are clearly N_c of them.

[5] A proposal was recently given in [Yon20, Sec. 4], which uses gauged WZW terms in an essential manner.

Relation between WZW terms for $N_f \geq 3$

The map $\widetilde{\Omega}_4^{\text{spin}}(S^2) \to \widetilde{\Omega}_4^{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f))$ for $N_f \geq 3$ is a zero map, as shown in Appendix B.4. This means that the SO WZW term for $N_f = 2$ can be expressed as an SO WZW term for $N_f \geq 3$ using (6.10) as

$$(-1)^{N_c[\sigma:M_4\to S^2]} = \exp\left(2\pi i \cdot N_c \int_{W_5} \frac{\Gamma_5(\tilde{\sigma})}{2}\right), \tag{6.14}$$

where W_5 is found by considering $N_f = 2$ configurations as $N_f \geq 3$ configurations. This equality can be shown using algebraic topology, see Appendix B.4. The two-fold ambiguity of the $N_f \geq 3$ WZW term given by (6.12) is immaterial here, since $\mathfrak{P}(w_2) = 0$ on S^2 because of $H^4(S^2) = 0$.

6.2.3 WZW terms as invertible phases

First when $N_f \geq 3$, one has a short exact sequence

$$\begin{aligned} 0 \to \underbrace{\mathrm{Ext}_{\mathbb{Z}}(\Omega_4^{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)), \mathbb{Z})}_{=\mathbb{Z}_2 \text{ or } 0} & \\ \to (D\Omega_{\text{spin}})^5(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)) & \\ \to \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\Omega_5^{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)), \mathbb{Z})}_{=\mathbb{Z}} \to 0, & \end{aligned}$$

and the ordinary WZW term (6.10) for the SO(N_c) QCD corresponds to N_c times the generator of the free part $\mathbb{Z}$. Recall that

$$H^5(\mathrm{SU}(N_f)/\mathrm{SO}(N_f); \mathbb{Z}) \supset \mathbb{Z} \ \to\ \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\Omega_5^{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)), \mathbb{Z})}_{=\mathbb{Z}}$$

is a multiplication by four. This cannot be understood by just noting that the generator y_5 of $H^5(\mathrm{SU}(N_f)/\mathrm{SO}(N_f); \mathbb{Z})$ is in the image of Sq^2 as before; it actually comes from the extension by $H^3(\mathrm{SU}(N_f)/\mathrm{SO}(N_f); \mathbb{Z}_2) = \mathbb{Z}_2$ and then another extension by $H^2(\mathrm{SU}(N_f)/\mathrm{SO}(N_f); \mathbb{Z}_2) = \mathbb{Z}_2$, as can be seen from the analysis of the Atiyah-Hirzebruch spectral sequence (AHSS) or Adams spectral sequence, see Appendix B.4. Meanwhile, the torsion part the WZW term for SO QCD has not been determined successfully.

On the other hand, when $N_f = 2$, we instead have $D\Omega^5_{\text{spin}}(\mathrm{SU}(2)/\mathrm{SO}(2)) = \mathbb{Z}_2$. The pull-back along $\mathrm{SU}(2)/\mathrm{SO}(2) \subset \mathrm{SU}(N_f \geq 3)/\mathrm{SO}(N_f)$ is a mod 2 reduction (of the free part)

$$\underbrace{D\Omega^5_{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f))}_{\supset\mathbb{Z}} \ \to\ \underbrace{D\Omega^5_{\text{spin}}(\mathrm{SU}(2)/\mathrm{SO}(2))}_{=\mathbb{Z}_2}.$$

6.2.4 Gauged WZW terms

Let us now turn on the gauge fields for the flavor symmetry and apply the generic argument given in Sec. 5.2.4.

Normalization of the ungauged WZW terms

For $G = \mathrm{SU}(N_f)$ and $H = \mathrm{SO}(N_f)$ with $N_f \geq 5$, the Leray-Serre spectral sequence (LSSS) associated with the fibration (5.18) is given as

$$
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{E_2^{p,q} = H^p\big(B\mathrm{SU}(N_f); H^q(\mathrm{SU}(N_f)/\mathrm{SO}(N_f);\mathbb{Z})\big)} \\
6 & & & & & & & \\
5 & \boxed{\mathbb{Z}\oplus\mathbb{Z}_2} & & & & * & & * \\
4 & & & & & & & \\
3 & \mathbb{Z}_2 & & & & * & & * \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & & \mathbb{Z} & & \boxed{\mathbb{Z}} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\quad \Longrightarrow \quad
\begin{array}{c|c}
\multicolumn{2}{c}{H^{p+q}\big(B\mathrm{SO}(N_f);\mathbb{Z}\big)} \\
6 & \mathbb{Z}_2 \\
5 & \mathbb{Z}_2 \\
4 & \mathbb{Z} \\
3 & \mathbb{Z}_2 \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array}
\tag{6.15}
$$

To realize correct convergence in degree $p + q = 5, 6$, the differential $d_6 : E_6^{0,5} \to E_6^{6,0}$ must be a multiplication by two on the summand $\mathbb{Z}$ and the zero map on the summand $\mathbb{Z}_2$. Therefore the generator $y_5 \in H^5(\mathrm{SU}(N_f)/\mathrm{SO}(N_f);\mathbb{Z})$ transgresses to $2c_3 \in H^6(B\mathrm{SU}(N_f);\mathbb{Z})$. Since $\Gamma_5 = y_5/2$, the normalization (6.10) is verified.

Topological consistency of the gauged WZW term

As for the symmetry $H = \mathrm{SO}(N_f)$ after the spontaneous breaking one has

$$\Omega_5^{\text{spin}}(B\mathrm{SO}(N_f)) = 0,$$

there is no residual global anomaly in the gauged WZW term. This is again consistent with the physics point of view, since $\gamma \in \mathrm{Inv}^5_{\text{spin}}(BH)$ would be the anomaly of fermions with respect to a symmetry H, under which they can be given a non-zero mass; this immediately guarantees that not only the free part characterized by the anomaly polynomial vanishes, but also the subtler torsion part does.

However, if we consider $\mathrm{SO}(2n_c)$ QCD on non-spin manifolds (see Sec. 6.1.2), things become complicated. In this case, the anomaly of the QCD takes values in $\mathrm{Inv}^5_{\text{oriented}}(B\mathrm{SU}(2n_c))$. Its pull-back is in $\mathrm{Inv}^5_{\text{oriented}}(B\mathrm{SO}(2n_c))$, which is not directly the anomaly of fermions which can be made massive, and therefore we cannot argue that it vanishes. To seek a way out, let us continue the discussion of the general case and study how we can actually determine γ in the non-zero case.

Here we make a simplifying assumption that the anomaly α_{d+2} comes from the cohomology class $\alpha \in H^{d+2}(BG;\mathbb{Z})$, rather than the bordism class. In this case, instead of the commutative diagram (5.2.4), one can use the following:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}_{\mathbb{Z}}(H_{d+1}(BH;\mathbb{Z}),\mathbb{Z}) & \xrightarrow{\beta} & H^{d+2}(BH;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(H_{d+2}(BH;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Ext}_{\mathbb{Z}}(H_{d+1}(BG;\mathbb{Z}),\mathbb{Z}) & \xrightarrow{\beta} & H^{d+2}(BG;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(H_{d+2}(BG;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0. \end{array}$$

As recalled around (3.2.3), β is simply the Bockstein homomorphism acting on

$$\mathrm{Ext}_{\mathbb{Z}}(H_{d+1}(BH;\mathbb{Z}),\mathbb{Z}) = H^{d+1}(BH,\mathrm{U}(1))_{\text{torsion}}.$$

Therefore, determining γ reduces to finding the element $\gamma \in H^{d+1}(BH,\mathrm{U}(1))_{\text{torsion}}$ whose Bockstein $\beta(\gamma)$ equals the original anomaly $\alpha \in H^{d+2}(BG;\mathbb{Z})$ pulled back to $H^{d+2}(BH;\mathbb{Z})$.

Now let us examine the concrete case of of $\mathrm{SO}(N_c)$ QCD. The gauged WZW term was $(N_c/4)\,y_5$, where y_5 is a generator of $\mathbb{Z} \subset H^5(\mathrm{SU}(N_f)/\mathrm{SO}(N_f);\mathbb{Z})$. Recall also that y_5 transgresses to $2c_3$. Then, when $N_c = 2n_c$ is even, the anomaly $n_c c_3$ is well-defined without the spin structure. Referring to the Appendix B.2, one can immediately read off the following:

- $c_3 \in H^6(B\mathrm{SU}(N_f);\mathbb{Z})$ pulls back to $(W_3)^2 \in H^6(B\mathrm{SO}(N_f);\mathbb{Z})$,
- which reduces to $(w_3)^2 \in H^6(B\mathrm{SO}(N_f);\mathbb{Z}_2)$,
- which is the image of the Bockstein $\beta = Sq^1$ of $w_2 w_3 \in H^5(B\mathrm{SO}(N_f);\mathbb{Z}_2)$.

This implies that the possible inconsistency of the gauged WZW term for the anomaly $\alpha = n_c c_3$ is given by

$$\gamma = n_c \cdot w_2(f) w_3(f), \tag{6.16}$$

where $w_i(f)$ is the i-th Stiefel-Whitney classes of the $\mathrm{SO}(N_f)$ bundle. This seems non-zero and therefore the gauged WZW term might not be well-defined at this point, on generic non-spin manifolds.[6] However, it turns out that this γ is actually trivial, as will be discussed in Sec. 6.2.5.

[6]This possible inconsistency disappears on spin manifolds, as we have $w_2 w_3 = w_2 Sq^1 w_2 = Sq^2 w_3$ and therefore

$$\int_{W_5} w_2(f) w_3(f) = \int_{W_5} Sq^2 w_3(f) = \int_{W_5} \nu_2(TW_5) w_3(f)$$

modulo 2, which is 0 mod 2 on a spin manifold as in (5.6). Therefore, the gauged WZW term is well-defined on a spin manifold, and this conclusion agrees with the general discussion we made at the beginning.

Torsion part

Recall from our discussion in Sec. 6.2.1 that, for $N_f \geq 5$, the part $(N_c/4)\, y_5 = (N_c/2)\, \Gamma_5$ only specifies the ungauged SO WZW term up to the addition of the torsion part specified by a character

$$\chi : \widetilde{\Omega}_4^{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)) \to \mathrm{U}(1).$$

Correspondingly, the gauged SO WZW term also suffers this indeterminacy.

To describe the gauged WZW term, one needs to describe its behavior for the sigma model fields taking values not only in $\mathrm{SU}(N_f)/\mathrm{SO}(N_f)$, but also in the total space $B\mathrm{SO}(N_f)$ fibered over $B\mathrm{SU}(N_f)$. As computed in Appendix B.4, one has the equality

$$\widetilde{\Omega}_4^{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)) \simeq \underbrace{\widetilde{\Omega}_4^{\text{spin}}(B\mathrm{SO}(N_f))_{\text{torsion}}}_{=\mathbb{Z}_2} \quad \text{for } N_f \geq 5.$$

This means that the unfixed torsion part of the gauged SO WZW term simply comes from the character

$$\chi \in \mathbb{Z}_2 \subset \mathrm{Hom}(\widetilde{\Omega}_4^{\text{spin}}(B\mathrm{SO}(N_f)), \mathrm{U}(1)).$$

Its non-trivial element is given by [AST13]

$$\exp\left(2\pi i \int_{M_4} \frac{1}{2}\mathfrak{P}(w_2(f))\right) \tag{6.17}$$

where $w_2(f)$ is the second Stiefel-Whitney class of the $\mathrm{SO}(N_f)$ bundle f and $\mathfrak{P}$ is the Pontrjagin square. Also, note that the pull-back of $w_2(f)$ to $\mathrm{SU}(N_f)/\mathrm{SO}(N_f)$ is simply the generator of $H^2(\mathrm{SU}(N_f)/\mathrm{SO}(N_f);\mathbb{Z}_2) = \mathbb{Z}_2$. The analysis so far only determines the gauged WZW term up to the addition of this torsion WZW term (6.17). It is at present difficult to specify exactly which.[7]

6.2.5 Solitonic symmetries

The lowest non-trivial homotopy group is $\pi_2(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)) = \mathbb{Z}_2$, which is naturally isomorphic to $H_2(\mathrm{SU}(N_f)/\mathrm{SO}(N_f);\mathbb{Z})$ via Hurewicz theorem. This gives rise to a string-like soliton in the low-energy sigma model, whose tension is controlled by the dynamical scale of the QCD. Since the group is $\mathbb{Z}_2$, two copies of such a flux tube can annihilate together. This indeed matches the property of the electric flux tube in the confining phase, generated by a charge in the spinor representation of $\mathrm{Spin}(N_c)$ [Wit83b], and is an IR counterpart of the electric $\mathbb{Z}_2$ 1-form symmetry of $\mathrm{Spin}(N_c)$ QCD in the UV; the operator charge is measured through the generator w_2 of $H^2(\mathrm{SU}(N_f)/\mathrm{SO}(N_f);\mathbb{Z}_2) = \mathbb{Z}_2$.

This further allows us to determine the coupling of the $\mathbb{Z}_2$ gauge theory with the low-energy sigma model. As already mentioned in Sec. 6.1.1, the $\mathrm{SO}(N_c)$ gauge theory can be obtained by

[7] A proposal was recently made in [Yon20, Sec. 4].

gauging the electric $\mathbb{Z}_2$ 1-form symmetry of the $\mathrm{Spin}(N_c)$ gauge theory. This means that the low-energy limit of $\mathrm{SO}(N_c)$ gauge theory has the coupling

$$\exp\left(2\pi i\cdot\frac{1}{2}\int_{M_4}(a\cup\delta b+b\cup\sigma^*(w_2)+b\cup B)\right), \tag{6.18}$$

where $a\in C^1(M_4;\mathbb{Z}_2)$ and $b\in C^2(M_4;\mathbb{Z}_2)$; $a\delta b$ is the kinetic term of the $\mathbb{Z}_2$ gauge theory, and $B\in Z^2(M_4;\mathbb{Z}_2)$ is the background for the magnetic $\mathbb{Z}_2$ 1-form symmetry of the $\mathrm{SO}(N_c)$ gauge theory. The equation of motion of (6.18) forces

$$B=\sigma^*(w_2) \tag{6.19}$$

at the level of cohomology classes. In other words, the space of the sigma model field has disconnected components labelled by the cohomology class $w_2\in H^2(M_4;\mathbb{Z}_2)$, and every sector other than the one specified by the background field B through (6.19) is projected out from the path integral by the $\mathbb{Z}_2$ gauge theory.

On non-spin manifolds

As we saw in Sec. 6.1.3, when $N_c=2n_c$ is even, one can relax the constraint that our spacetime manifold is spin.

Let us start from considering $\mathrm{Spin}(2n_c)$ gauge theory. The electric $\mathbb{Z}_2$ 1-form symmetry extends the spacetime rotation group non-trivially as we saw in (6.9), which we reproduce here:

$$\delta E=n_c\cdot\beta w_2(T). \tag{6.20}$$

In the following consider the more interesting case of odd n_c. For the ungauged WZW term (6.10)

$$\exp\left(2\pi i\cdot N_c\int_{W_5}\frac{\Gamma_5}{2}\right)=\exp\left(2\pi i\cdot n_c\int_{W_5}\frac{y_5}{2}\right) \tag{6.21}$$

to be well-defined, one needs to show that the integral of the generator y_5 of $H^5(\mathrm{SU}/\mathrm{SO};\mathbb{Z})$ is even on a closed oriented (non-spin) 5-manifold with E satisfying (6.20) is specified. Using the information gathered in Appendix B.4, this can be shown as follows.

The mod 2 reduction of y_5 is $w_2 w_3$, where $w_i \in H^i(\mathrm{SU}/\mathrm{SO};\mathbb{Z}_2)$. Then we have

$$\begin{aligned}
\int_{W_5} \sigma^*(y_5) &= \int_{W_5} \sigma^*(w_2 w_3) && (\mathrm{mod}\ 2) \\
&= \int_{W_5} \sigma^*(Sq^2 \beta w_2) \\
&= \int_{W_5} Sq^2 \sigma^*(\beta w_2) \\
&= \int_{W_5} \nu_2(TW_5)\, \sigma^*(\beta w_2) && (\mathrm{mod}\ 2) \\
&= \int_{W_5} w_2(TW_5)\, \sigma^*(\beta w_2) && \text{as } w_1(TW_5) = 0 \\
&= \int_{W_5} w_2(TW_5)\, \beta\sigma^*(w_2) \\
&= \int_{W_5} [\beta w_2(TW_5)]\, \sigma^*(w_2) && (\mathrm{mod}\ 2)
\end{aligned} \tag{6.22}$$

where in the last equality we used the formula[8] $\int a\beta b = \int b\beta a \bmod 2$. Now, our assumption (6.20) means that $\beta w_2(TW_5)$ is cohomologically trivial, making the integral on the left hand side even.

Now let us turn to $\mathrm{SO}(2n_c)$ gauge theory. The integral of y_5 modulo 2 is still given by (6.22), which is now

$$= \int_{W_5} B \beta w_2(TW_5). \tag{6.23}$$

Since this final expression only depends on the external background fields and not on the WZW sigma model fields σ which are path integrated, it gives the mixed anomaly of the system, and indeed reproduces the anomaly (6.8) of the $\mathrm{SO}(2n_c)$ QCD. The consistency of the gauged WZW term also follows in a similar manner; the possible inconsistency is given by (6.16), which is also $w_2 w_3$, this time of the total space $B\mathrm{SO}$ of the fibration $\mathrm{SU}/\mathrm{SO} \to B\mathrm{SO} \to B\mathrm{SU}$.

[8] This follows from

$$\int \beta(ab) = \int \nu_1(TW_5) ab = \int w_1(TW_5) ab = 0$$

on an oriented W_5.

Chapter 7

Conclusion

In this thesis, we studied the WZW terms in non-linear sigma models from the modern point of view based on bordism classification of topological terms and invertible QFT. Let us briefly review and summarize the content.

In Chap. 2, we encountered the notion of bordism. Bordism was a rather coarse equivalence relation between closed manifolds, and the equivalence classes formed a (Abelian) group under the operation of disjoint union of manifolds. We familiarized ourselves with these bordism groups through some elementary examples in low dimensions.

In Chap. 3, we discussed various WZW-like topological terms in sigma models and saw that they are appropriately described in terms of (co)bordism rather than ordinary (co)homology which had been conventionally used. We also introduced the notion of invertible QFT which was defined to be QFT with one-dimensional Hilbert space, and saw that their conjectured classification seems to be identical to that of WZW-like terms, which strongly supports the validity of the former.

In Chap. 4, we reviewed the recent understanding of global symmetries. It had various advantages over the conventional understanding based on the invariance of Lagrangians of QFT, including the unified incorporation of "generalized" higher-form symmetries under which higher-dimensional operators are charged. Although these higher-form symmetries can also be coupled to background gauge fields, this gauging procedure might lead to inconsistency of the gauged theory, represented in a form of 't Hooft anomaly as with ordinary symmetries. We explained that these anomalies serve as a powerful tool to extract information on dynamics in strongly-coupled region of theories, and that they are believed to be realized as boundaries of corresponding invertible QFT in one-higher dimensions, which indeed sounds plausible from our knowledge on anomaly inflow examples.

In Chap. 5, we finally started our investigation of the WZW terms appearing in the low-energy effective description of SU QCD in $(3+1)\,d$. We succeeded in confirming that the underlying spacetime manifolds are required to be equipped with spin structure, or in other words able to define spinors on them, for the WZW terms to be well-defined. This was already known for the SU QCD case to some extent, but our analysis based on bordism can be regarded as a refinement of the previous analysis.

In Chap. 6, we continued our investigation of the WZW terms appearing in the low-energy effective description of SO QCD in $(3+1)\,d$, and saw that our refined analysis is also applicable opposed to the previous *ad hoc* analysis. Again we found the necessity of spin structure on underlying spacetime manifolds. Furthermore, for $\mathrm{SO}(2n_c)$ QCD with even number of colors, theory is known to possess higher-form global symmetries in addition to ordinary ones, and there was a novel kind of (mixed) 't Hooft anomaly involving them. We newly found that the WZW terms are again responsible for its reproduction in the IR non-linear sigma models, which was deduced by examining the solitonic strings. In spite of these successes, there were also some subtleties concerning the torsion part, or equivalently, discrete WZW terms for the SO QCD case, which definitely deserve further investigation in the future.

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After the Miniluv in Oakeania (apparently) ceased telescreening and became inactive enough, the author was finally (both physically and mentally) able to come back to his office for the first time in an year and a half, and he is grateful to Big Brother and the party for reminding him of the overwhelming advantage of a large desk (which he sadly could not afford in his small pigpen), significantly affecting his quality of life and its sustainable development.

Thus writing this thesis alone in the student room, the author missed those good old days of enjoying chats and discussions, playing games, and going out for dinner with his colleagues. He would like to take this opportunity to thank them, including Yuichi Enoki, Hajime Fukuda, Masayuki Fukuda, Mocho Go, Keita Kanno, Nozomu Kobayashi, Masataka Watanabe, and Takemasa Yamaura from the hep-th/ph group, and also Kazuyuki Akitsu, Yosuke Kobayashi, Toshiki Kurita, and Sunao Sugiyama from the astro-ph group.

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Appendix A

Spectral sequence

Spectral sequences are the tools to compute graded algebraic objects A^* from their smaller pieces. In general, it is rather difficult to obtain the former while one has far easier access to the latter, and the *philosopher's sequences* allow alchemists today for variety of glorious transmutations. As noble objects A^*, we typically have generalized (co)homology $h^*(X)$ of some topological spaces X in mind. The arcana of *magnum opus* presented below are mostly based on [McC01], and interested readers are further referred to e.g. [DK01, Hat04, Tam13] and references therein.

Phase 1. Nigredo

First of all, suppose that the graded object A^* is *filtered* and the filtration is bounded, i.e.

$$\{0\} \subset \cdots \subset F^p A^* \subset \cdots \subset F^1 A^* \subset F^0 A^* = A^*.$$

Then, one can try to recover the original A^* from the *associated graded* $E_\infty^*(A^*)$ where

$$E_\infty^p(A^*) = F^p A^* \big/ F^{p+1} A^*,$$

by summing over as

$$A^* \sim \bigoplus_{p=0}^{\infty} E_\infty^p(A^*).$$

Since A^* itself is graded, one can carry out a further decomposition based on a filtration

$$\{0\} \subset \cdots \subset \left(F^p A^* \cap A^r\right) \subset \cdots \subset \left(F^1 A^* \cap A^r\right) \subset \left(F^0 A^* \cap A^r\right) = A^r.$$

As a result, one can recover the original A^* as

$$A^* \sim \bigoplus_{p+q=r} E_\infty^{p,q} = \bigoplus_{p+q=r} \frac{F^p A^* \cap A^{p+q}}{F^{p+1} A^* \cap A^{p+q}} \tag{A.1}$$

Although the equivalences are up to isomorphism and possibly miss non-trivial extensions, if any, (A.1) at least serves as a first-order approximation to A^* and is useful enough in some cases.

Phase 2. Albedo

Here comes the central definition. A *spectral sequence* $\{E_r^{*,*}, d_r\}$ is a sequence of bi-graded $E_r^{*,*}$ equipped with differentials d_r satisfying $d_r \circ d_r = 0$ (together consists of a *complex*), which map

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and each $E_{r+1}^{*,*}$ is given by the "(co)homology" of $E_r^{*,*}$, namely

$$E_{r+1}^{p,q} = \frac{\mathrm{Ker}\, d_r : \qquad E_r^{p,q} \to \bullet}{\mathrm{Im}\, d_r : \bullet \to E_r^{p,q}}$$

As alluded by the notation, our goal is to find a nice $\{E_r^{*,*}\}$ which converges to the desired $E_\infty^{*,*}$ reproducing A^* as in (A.1).

Phase 3. Citrinitas

Actually, the shadow above is casted by a light of more general setting as follows. Given a long sequence of objects D, E

$$\cdots \xrightarrow{k} D \xrightarrow{i} D \xrightarrow{j} E \xrightarrow{k} D \xrightarrow{i} \cdots$$

or more conventionally written as

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

the tuple $\{D, E, i, j, k\}$ composes an *exact couple* if i, j, k are all exact, i.e.

$$\begin{aligned} \mathrm{Ker}\, j &= \mathrm{Im}\, i, \\ \mathrm{Ker}\, k &= \mathrm{Im}\, j, \\ \mathrm{Ker}\, i &= \mathrm{Im}\, k. \end{aligned}$$

If so, one can immediately equip E with a differential $d : E \to E$ satisfying $d \circ d = 0$, namely $d = j \circ k$. This differential naturally leads us to consider the "(co)homology"

$$E' = \mathrm{Ker}\, d \big/ \mathrm{Im}\, d$$

and by defining

$$\begin{aligned} D' &:= i(D) \\ i' &:= i|_{D'} \\ j'(i(x)) &:= j(x) + d(E) \\ k'(y + d(E)) &:= k(y) \end{aligned}$$

the tuple $\{D', E', i', j', k'\}$ again composes an exact couple, which is called the *derived couple*. As one might have guessed by now, it turns out that $\{E_r, d_r\}$ collected from r-th derived couples gives rise to a spectral sequence when objects D, E are bi-graded.

Phase 4. Rubedo

The typical base objects given as input to start with is $E_2^{*,*}$. For important subclasses of interest, one has $E_2^{p,q} \neq 0$ only for a finite number of (p,q)'s, which makes d_r trivial for large enough r. In general, some of the $E_{r+1}^{p,q}$'s become trivial for exact d_r with $\operatorname{Ker} d_r = \operatorname{Im} d_r$ as one turn *pages*, and at some point, all of them inevitably stabilize as $E_r^{*,*} = E_{r+1}^{*,*} = \cdots = E_\infty^{*,*}$; in extreme cases, $E_2^{*,*}$ is so sparse that nothing can happen and the sequence is already stabilized at $r = 2$.

For example, the spectral sequences we encounter in this thesis are mostly Leray-Serre type, whose E_2 pages are given by

$$E_2^{p,q} = H^p(B; h^q(F))$$

and converges to

$$h^*(E)$$

for a fibration $F \to E \to B$ with suitable assumptions. For the ordinary Leray-Serre spectral sequences, h is taken to be the ordinary (co)homology H, while the Atiyah-Hirzebruch spectral sequences treat more generic h, including the (co)bordism Ω.

Also, note that one might be able to exploit (additional) structure with which A^* is equipped to fully recover A^* from $E_\infty^{*,*}$, although not always successful. Two major examples are when

- A^* is a (graded) *algebra*, equipped with a *product*
  $\boxed{\text{ex.}}$ cohomology $H^*(X)$: cup product
- A^* is a (graded) *module*, equipped with an *action* of another (graded) algebra
  $\boxed{\text{ex.}}$ cohomology $H^*(X; \mathbb{Z}_p)$: action of mod-p Steenrod algebra $\mathcal{A}_p$

Prima materia

We will also encounter another type of spectral sequences in Appendix B.4, which have far more complicated E_2 pages. Below, we collect miscellaneous notions and facts including ones on which they are based, although it is not necessary for the readers to understand all these Greeks as we have repeatedly warned.

Module

Let R be a commutative ring (with a unit element 1). An R*-module* is an Abelian group $(M, +)$ equipped with scalar multiplication $R \times M \to M$ such that, for all $x, y \in M$ and $a, b \in R$,

$$\begin{aligned} (a+b)x &= ax + bx, \\ (ab)x &= a(bx), \\ a(x+y) &= ax + ay, \\ 1x &= x. \end{aligned}$$

A *homomorphism* between two R-modules M, N is a function $f : M \to N$ which preserves the structure of two operations

$$\begin{aligned} f(x+y) &= f(x) + f(y), \\ f(ax) &= a(f(x)). \end{aligned}$$

Now, let us consider the following type of (very short) exact sequence of modules

$$A' \xrightarrow{\alpha} A \longrightarrow 0$$

so that α is surjective (i.e. $\operatorname{Im}\alpha = A$). Then, an R-module P is defined to be *projective* if there exists a homomorphism $f' : P \to A'$ such that $\alpha \circ f' = f$

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists? f'"'] \arrow[d, "f"] & \\ A' \arrow[r, "\alpha"'] & A \arrow[r] & 0 \end{tikzcd}$$

for any A, A', α, and f.[1] A prominent example of P is *free* R-modules, which are the modules with a basis (i.e. linearly-independent elements generating the whole module), and one can *resolve* any R-module by a free module over it, which is projective.

[1]Note that one can similarly define a notion of *injective* modules by reversing the arrows in the above diagrams.

Ext

Given two R-modules M and N, the homomorphisms between them form a group $\mathrm{Hom}_R(M,N)$. One can think of $\mathrm{Hom}_R(-,M)$ as a function taking a module as an input and returning a group as an output. Applying $\mathrm{Hom}_R(-,M)$ to a short exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

one partially loses its exactness in general, but instead has a natural long exact sequence as

$$\begin{array}{ccccccccc} \cdots & \longleftarrow & \mathrm{Ext}^n_R(A,M) & \longleftarrow & \mathrm{Ext}^n_R(B,M) & \longleftarrow & \mathrm{Ext}^n_R(C,M) & \longleftarrow & \\ & & \vdots & & \vdots & & \vdots & \vdots & \\ & & \mathrm{Ext}^1_R(A,M) & \longleftarrow & \mathrm{Ext}^1_R(B,M) & \longleftarrow & \mathrm{Ext}^1_R(C,M) & \longleftarrow & \\ & & \mathrm{Hom}_R(A,M) & \longleftarrow & \mathrm{Hom}_R(B,M) & \longleftarrow & \mathrm{Hom}_R(C,M) & \longleftarrow & 0 \end{array}$$

with a condition that $\mathrm{Ext}^{n\geq 1}_R(P,M)=0$ for projective modules P guaranteeing its uniqueness.

Tor

A *tensor product* between two R-modules M,N is an Abelian group $M\otimes N$ whose elements are (m,n) for $m\in M, n\in N$ with the original operations naturally inherited as

$$\begin{array}{rcccl} (m+m',n) & = & (m,n)+(m',n), & & \\ (m,n+n') & = & (m,n)+(m,n'), & & \\ (am,n) & = & a(m,n) & = & (m,an). \end{array}$$

Again, one can think of $-\otimes M$ as a function taking a module as an input and returning a group as an output, and applying it to the same short exact sequence as before, one partially loses its exactness in a slightly different way as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A\otimes M & \longrightarrow & B\otimes M & \longrightarrow & C\otimes M & & \\ & \longrightarrow & \mathrm{Tor}^R_1(A,M) & \longrightarrow & \mathrm{Tor}^R_1(B,M) & \longrightarrow & \mathrm{Tor}^R_1(C,M) & & \\ & \vdots & \vdots & & \vdots & & \vdots & & \\ & \longrightarrow & \mathrm{Tor}^R_n(A,M) & \longrightarrow & \mathrm{Tor}^R_n(B,M) & \longrightarrow & \mathrm{Tor}^R_n(C,M) & \longrightarrow & \cdots \end{array}$$

with a condition that $\mathrm{Tor}^{n\geq 1}_R(P,M)=0$ for projective modules P guaranteeing its uniqueness.

From the conditions on projective modules, it turns out that both $\mathrm{Ext}^{n\geq 2}_{\mathbb{Z}}(-,M)$ and $\mathrm{Tor}^{n\geq 2}_{\mathbb{Z}}(-,M)$ are trivial for $R=\mathbb{Z}$,[2] and one usually just drops the super / subscripts for $n=1$ in this case.

[2]More generally, this is true for any *principal ideal domains (PID)* which are commutative rings with no *zero divisors* (i.e. integral domain) and every ideal being generated by a single element (i.e. principal).

Appendix B

Bordism group computation

The anomalies of QFT are believed to be captured by a one-higher dimensional invertible QFT, which is nicely described in terms of bordism groups, as described in Sec. 3. In particular, the information on the anomalies d-dimensional spin QFT with target space X are encoded in the bordism group $\Omega^{\text{spin}}_{d+1}(X)$. In this section, we compute this bordism group for some X's using various types of spectral sequences. For more details which are omitted (and notions which are undefined) in the following, refer to e.g.

- [BT82, Hat02], mathematics textbooks on algebraic topology in general
- [GEM18], an introduction to the Atiyah-Hirzebruch spectral sequence aimed at physicists
- [BC18], an introduction to the Adams spectral sequence

and references therein.

B.1 $B\mathrm{SU}(n)$

First of all, the ordinary cohomology of the classifying space $B\mathrm{SU}(n)$ is known to be

$$\begin{aligned} H^*\big(B\mathrm{SU}(n);\mathbb{Z}\big) &= \mathbb{Z}[c_2, c_3, \ldots, c_n], \\ H^*\big(B\mathrm{SU}(n);\mathbb{Z}_2\big) &= \mathbb{Z}_2[c_2, c_3, \ldots, c_n], \end{aligned} \tag{B.1}$$

where c_j's are the j-th Chern classes (and their mod-2 reductions) which have degree $2j$ [MT91]. For $\mathbb{Z}_2$ cohomology, there are certain cohomology operations called the Steenrod powers Sq^i, whose actions are given by the Wu formula

$$Sq^{2i}(c_j) = \sum_{k=0}^{i} \binom{j-k-1}{i-k} c_{i+j-k} c_k \quad (0 \le i \le j). \tag{B.2}$$

Using these results, one can compute the bordism group by a machinery called the Atiyah-Hirzebruch spectral sequence (AHSS). It can be regarded as a generalization of the Leray-Serre spectral sequence (LSSS) appeared in the main part e.g. Sec. 5.2.4; this time the E^2 page is given by

$$E^2_{p,q} = H_p\big(B; \Omega^{\text{spin}}_q(F)\big),$$

and converges to $\Omega^{\text{spin}}_{p+q}(E)$ for the fibration $F \longrightarrow E \xrightarrow{p} B$. For our purpose, it is sufficient to use the trivial fibration

$$pt \longrightarrow X \xrightarrow{p} X.$$

Anyway, let us take a look at how it works. One can deduce the necessary homology groups from the cohomology information (B.1) by using the universal coefficient theorem, and the E^2 page can be easily filled as follows

$$E^2_{p,q} = H_p\big(BSU(n); \Omega^{\text{spin}}_q(pt)\big)$$

$$\begin{array}{c|cccccc:cc}
6 & & & & & & & & \\
5 & & & & & & & & \\
4 & \mathbb{Z} & & & & * & & * & \\
3 & & & & & & & & \\
2 & \mathbb{Z}_2 & & & & \boxed{\mathbb{Z}_2} & & * & \\
1 & \mathbb{Z}_2 & & & & \boxed{\mathbb{Z}_2} & & \boxed{\mathbb{Z}_2} & \\
0 & \mathbb{Z} & & & & \mathbb{Z} & & \boxed{\mathbb{Z}} & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array} \tag{B.3}$$

where the horizontal and vertical axes correspond to p and q respectively. Note that when $n = 2$, the columns to the right of the dotted lines are to be discarded. Also, the differentials going into the $p = 0$ column are all zero (see e.g. [DK01, below Theorem 9.10]), which follows from the splitting of $\Omega^{\text{spin}}_d(X) = \Omega^{\text{spin}}_d(pt) \oplus \widetilde{\Omega}^{\text{spin}}_d(X)$ explained in (2.3).

A key property specific to the spin bordism is that the differentials $d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$ for $q = 0, 1$ are known [Tei93] to be the duals of the Steenrod square Sq^2 (composed with mod-2 reduction for $q = 0$). From the knowledge on the cohomology (B.2), $\boxed{d_2 : E^2_{6,0} \to E^2_{4,1}}$ and $\boxed{d_2 : E^2_{6,1} \to E^2_{4,2}}$ are both non-trivial for $n \geq 3$ due to $Sq^2 c_2 = c_3$, and the E^3 page results in

$$\begin{array}{c|cccccccc}
6 & & & & & & & & \\
5 & & & & & & & & \\
4 & \mathbb{Z} & & & & * & & * & \\
3 & & & & & & & & \\
2 & \mathbb{Z}_2 & & & & & & * & \\
1 & \mathbb{Z}_2 & & & & & & & \\
0 & \mathbb{Z} & & & & \mathbb{Z} & & \mathbb{Z} & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array} \tag{B.4}$$

As there are no more non-trivial differentials at $p+q \leq 6$, the spectral sequence collapses at this point and $E^3_{p,q} = E^\infty_{p,q}$ for this region. Therefore, one can read off that

$$\Omega_5^{\text{spin}}(B\text{SU}(n \geq 3)) = 0, \tag{B.5}$$

and

$$\Omega_6^{\text{spin}}(B\text{SU}(n \geq 3)) = \mathbb{Z}, \tag{B.6}$$

and notably the map $\Omega_6^{\text{spin}}(B\text{SU}(n)) = \mathbb{Z} \to H_6(B\text{SU}(n);\mathbb{Z}) = \mathbb{Z}$ is a multiplication by two.

On the other hand, for $n = 2$, the entries $E_{6,0}$ and $E_{6,1}$ are empty, and correspondingly the spectral sequence already collapses at the E^2 page. Therefore, the bordism groups of interest are

$$\Omega_5^{\text{spin}}(B\text{SU}(2)) = \mathbb{Z}_2, \tag{B.7}$$

and

$$\Omega_6^{\text{spin}}(B\text{SU}(2)) = 0. \tag{B.8}$$

B.2 $B\text{SO}(n)$

The $\mathbb{Z}_2$ cohomology of the classifying space $B\text{SO}(n)$ is known to be

$$H^*\big(B\text{SO}(n);\mathbb{Z}_2\big) = \mathbb{Z}_2[w_2,\ldots,w_n],$$

where w_i's are the i-th Stiefel-Whitney classes which have degree i. As before, there are Steenrod square operations, whose actions are given by the Wu formula of a similar form

$$Sq^i(w_j) = \sum_{k=0}^{i} \binom{j-k-1}{i-k} w_{i+j-k} w_k \quad (0 \leq i \leq j). \tag{B.9}$$

The integral cohomology is more involved [Bro82, Fes83]; up to degree 6 one has

d	0	1	2	3	4	5	6	$\cdots$
$H^d(B\text{SO}(3);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
$H^d(B\text{SO}(4);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}^{\oplus 2}$	0	$\mathbb{Z}_2$	$\cdots$
$H^d(B\text{SO}(n \geq 5);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
generator	1	0	0	W_3	p_1 (e_4)	W_5	W_3^2 (e_6)	$\cdots$

(B.10)

where W_i is an integral lift of w_i, p_1 is the first Pontrjagin class which reduces to w_2^2, and e_{2k}'s are the Euler classes generating $\mathbb{Z}$ which reduces to w_{2k}. (Accordingly, the $n = 6$ case is somewhat exceptional, but here we omit them for simplicity.)

One can now fill the E^2 pages as follows:

$$
E^2_{p,q} = H_p\big(BSO(3);\Omega^{\text{spin}}_q\big)
\qquad
\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z} & \mathbb{Z}_2 & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

$$
E^2_{p,q} = H_p\big(BSO(4);\Omega^{\text{spin}}_q\big)
\qquad
\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^{\oplus 2} & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z}^{\oplus 2} & \mathbb{Z}_2 & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

$$
E^2_{p,q} = H_p\big(BSO(n \geq 5);\Omega^{\text{spin}}_q\big)
\qquad
\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^{\oplus 2} & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z} \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\tag{B.11}
$$

Since $Sq^2 w_2 = (w_2)^2$, $Sq^2 w_3 = w_5 + w_3 w_2$, $Sq^2 (w_2)^2 = (w_3)^2$, and $Sq^2 w_4 = w_6 + w_4 w_2$, the differentials marked above are all non-trivial. Resulting E^3 pages are given as follows:

$$
n = 3
\qquad
\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & & & * & * & * \\
1 & \mathbb{Z}_2 & & & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z} & & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

$$
n = 4
\qquad
\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & & & * & * & * \\
1 & \mathbb{Z}_2 & & & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z}^{\oplus 2} & & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

$$
n \geq 5
\qquad
\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & & & * & * & * \\
1 & \mathbb{Z}_2 & & & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z} \oplus \mathbb{Z}_2 & & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\tag{B.12}
$$

As a result, one can read off

$$\widetilde{\Omega}_4^{\text{spin}}(B\mathrm{SO}(n)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & (n \geq 5) \\ \mathbb{Z}^{\oplus 2} & (n = 4) \\ \mathbb{Z} & (n = 3) \end{cases} \tag{B.13}$$

and

$$\Omega_5^{\text{spin}}(B\mathrm{SO}(n)) = 0. \tag{B.14}$$

Note that the result (B.13) corresponds to the fact that $4d$ SO gauge theories have a discrete theta angle (for the $\mathbb{Z}_2$ summand), in addition to the standard theta angle (for the $\mathbb{Z}$ summand) [AST13]. Using characteristic classes, this comes from the fact [Tho60] that

$$p_1 \equiv 2w_4 + \mathfrak{P}(w_2) \pmod 4$$

where 2 is a map sending $\{0,1\} = \mathbb{Z}_2$ to $\{0,2\} \subset \mathbb{Z}_4$, and $\mathfrak{P} : H^2(-;\mathbb{Z}_2) \to H^4(-;\mathbb{Z}_4)$ is the Pontrjagin square operation. On a spin manifold, $\mathfrak{P}(w_2)$ is even, and one can "divide" by two

$$\frac{p_1}{2} \equiv w_4 + \frac{1}{2}\mathfrak{P}(w_2) \pmod 2.$$

The left hand side is the dual of the generator of the $\mathbb{Z}$ summand, and either of the terms on the right hand side can be taken to be the dual of the generator of the $\mathbb{Z}_2$ summand.

Also, note that there are relations between Chern classes and Stiefel-Whitney classes induced from a projection $\psi : B\mathrm{SO}(n) \longrightarrow B\mathrm{SU}(n)$,

$$\begin{array}{ccc} H^*(B\mathrm{SU}(n);\mathbb{Z}_2) & \xrightarrow{\psi^*} & H^*(B\mathrm{SO}(n);\mathbb{Z}_2) \\ \Cup & & \Cup \\ c_i & \longmapsto & (w_i)^2, \end{array} \tag{B.15}$$

and

$$\begin{array}{ccc} H^*(B\mathrm{SU}(n);\mathbb{Z}) & \xrightarrow{(-1)^i \cdot \psi^*} & H^*(B\mathrm{SO}(n);\mathbb{Z}) \\ \Cup & & \Cup \\ c_{2i} & \longmapsto & p_i. \end{array} \tag{B.16}$$

Absence of mixed anomalies

Here we examine the mixed anomaly between $\mathrm{SO}(N_c)$ and $\mathrm{SU}(N_f)$ symmetries for fermion systems charged under $\mathrm{SO}(N_c) \times \mathrm{SU}(N_f)$. This assures the naive intuition that the flavor symmetry of $4d$ SO QCD is SU.

The E_2 page of the relevant AHSS for the Anderson-dual of bordism can be filled by applying the Künneth formula[1] and is given as follows:

$$
E_2^{p,q} = H^p\big(B\mathrm{SO} \times B\mathrm{SU}; \mathrm{Inv}^q_{\mathrm{spin}}\big)
$$

$$
\begin{array}{r|ccccccc}
4 & & & & & & & \\
3 & \mathbb{Z} & & & * & * & * & * \\
2 & \mathbb{Z}_2 & & \mathbb{Z}_2 & * & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\
0 & & & & & & & \\
-1 & \mathbb{Z} & & & \mathbb{Z}_2 & \mathbb{Z}^{\oplus 2} & \mathbb{Z}_2 & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\tag{B.17}
$$

Fortunately, since the non-trivial mixing between $H^*(B\mathrm{SO};\mathbb{Z})$ and $H^*(B\mathrm{SU};\mathbb{Z})$ occurs above degree 7 (similarly above degree 6 for $\mathbb{Z}_2$-cohomology), elements of $\mathrm{Inv}^{p+q\leq 5}_{\mathrm{spin}}(B\mathrm{SO} \times B\mathrm{SU})$ should be exhausted by the pull-backs from those of $\mathrm{Inv}^{d\leq 5}_{\mathrm{spin}}(B\mathrm{SO})$ and $\mathrm{Inv}^{d\leq 5}_{\mathrm{spin}}(B\mathrm{SU})$. This means that there is no mixed anomaly between SO and SU at least for spacetime dimensions ≤ 4. Also, note that since there is no interference between cohomologies of $B\mathrm{SU}$ and $B\mathrm{SO}$, one can stack two AHSS (B.3) and (B.11) for the region $p+q \leq 5$ and deduce

$$
\Omega^{\mathrm{spin}}_d(B\mathrm{SO} \times B\mathrm{SU}) = \Omega^{\mathrm{spin}}_d(B\mathrm{SO}) \oplus \Omega^{\mathrm{spin}}_d(B\mathrm{SU}) \quad (d \leq 5).
$$

Furthermore, a similar argument applies to the $\frac{\mathrm{Spin(spacetime)} \times \mathrm{SO}(2n_c)}{\mathbb{Z}_2} \times \mathrm{SU}(2n_f)$ case. The relevant bordism in this case is *twisted* spin bordism

$$
\mathrm{Inv}^d_{\frac{\mathrm{Spin}\times\mathrm{SO}}{\mathbb{Z}_2}}(B\mathrm{SU}),
$$

where the E_2 page of the AHSS converging to it is given by $E_2^{p,q} = H^p(B(\mathrm{SO}/\mathbb{Z}_2) \times B\mathrm{SU}; \mathrm{Inv}^d_{\mathrm{spin}})$. From the LSSS associated to the fibration $B\mathrm{SO} \to B(\mathrm{SO}/\mathbb{Z}_2) \to K(\mathbb{Z}_2, 2)$ (which we omit the detail), one finds that the lowest degree of non-trivial elements in $H^*(B(\mathrm{SO}(2n_c)/\mathbb{Z}); \mathbb{Z})$ is 3 (and accordingly that in $H^*(B(\mathrm{SO}(2n_c)/\mathbb{Z}); \mathbb{Z}_2)$ is 2). Therefore, there is no non-trivial mixing between $B(\mathrm{SO}/\mathbb{Z}_2)$ and $B\mathrm{SU}$ at $p+q \leq 5$, as in the previous case.

[1] When the coefficient is a field F it is simply

$$
H^d(X \times Y; F) = \bigoplus_{p+q=d} H^p(X;F) \otimes H^q(Y;F).
$$

For the $\mathbb{Z}$-coefficient case it is given by

$$
0 \longrightarrow \bigoplus_{p+q=d} H^p(X;\mathbb{Z}) \otimes H^q(Y;\mathbb{Z}) \longrightarrow H^d(X \times Y;\mathbb{Z}) \longrightarrow \bigoplus_{p+q=d+1} \mathrm{Tor}_{\mathbb{Z}}(H^p(X;\mathbb{Z}), H^q(Y;\mathbb{Z}) \longrightarrow 0
$$

which is known to split non-canonically, see e.g. [Spa81, Theorem 5.5.11].

B.3 $\mathrm{SU}(n)$

The ordinary cohomology of the group manifold $\mathrm{SU}(n)$ is known to be

$$\begin{aligned} H^*\big(\mathrm{SU}(n);\mathbb{Z}\big) &= \textstyle\bigwedge_{\mathbb{Z}}[x_3, x_5, \ldots, x_{2n-1}], \\ H^*\big(\mathrm{SU}(n);\mathbb{Z}_2\big) &= \textstyle\bigwedge_{\mathbb{Z}_2}[x_3, x_5, \ldots, x_{2n-1}], \end{aligned} \tag{B.18}$$

where x_i's have degree i. Here, the exterior algebra $\bigwedge[a_1, a_2, \ldots]$ is defined to be the polynomial algebra modulo the relations $a_1^2 = a_2^2 = \cdots = 0$ and $a_i a_j = -a_j a_i$; note that the former relation does not follow from the latter over $\mathbb{Z}_2$. Again, there are Steenrod square operations for $\mathbb{Z}_2$ cohomology [BS53]

$$Sq^{2i} x_{2j-1} = \binom{j-1}{i} x_{2i+2j-1}. \tag{B.19}$$

Let us first compute the spin bordism of $\mathrm{SU}(n)$, whose E^2 page is the following:

$$E^2_{p,q} = H_p\big(\mathrm{SU}(n);\Omega^{\text{spin}}_q(pt)\big)$$
$$\begin{array}{c|ccccc:cc}
5 & & & & & & & \\
4 & \mathbb{Z} & & & * & & * & \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & & \boxed{\mathbb{Z}_2} & & * & \\
1 & \mathbb{Z}_2 & & & \boxed{\mathbb{Z}_2} & & \boxed{\mathbb{Z}_2} & \\
0 & \mathbb{Z} & & & \mathbb{Z} & & \boxed{\mathbb{Z}} & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{B.20}$$

As always, the columns to the right of the vertical dotted lines are to be discarded when $n = 2$. The differential $\boxed{d_2 : E^2_{5,0} \to E^2_{3,1}}$ is non-trivial for $n \geq 3$ since $Sq^2 x_3 = x_5$, and one obtains

$$\widetilde{\Omega}^{\text{spin}}_4(\mathrm{SU}(n \geq 3)) = 0, \tag{B.21}$$

while it is trivial for $n = 2$ as x_5 does not exist, leading to

$$\widetilde{\Omega}^{\text{spin}}_4(\mathrm{SU}(2)) = \mathbb{Z}_2. \tag{B.22}$$

Furthermore, the differential $\boxed{d_2 : E^2_{5,1} \to E^2_{3,2}}$ is also non-trivial for $n \geq 3$, and one has

$$\Omega^{\text{spin}}_5(\mathrm{SU}(n \geq 3)) = \mathbb{Z} \tag{B.23}$$

and

$$\Omega^{\text{spin}}_6(\mathrm{SU}(n \geq 3)) = 0. \tag{B.24}$$

and notably the map $\Omega^{\text{spin}}_5(\mathrm{SU}(n)) = \mathbb{Z} \to H_5(\mathrm{SU}(n);\mathbb{Z}) = \mathbb{Z}$ is a multiplication by two.

One can similarly compute the spin^c bordism of SU, starting from the following E^2 page:

$$
E^2_{p,q} = H_p\big(\mathrm{SU}(n); \Omega_q^{\mathrm{spin}^c}(pt)\big)
$$

$$
\begin{array}{c|ccccc:c}
5 & & & & & & \\
4 & \mathbb{Z}^{\oplus 2} & & & * & & * \\
3 & & & & & & \\
2 & \mathbb{Z} & & & \mathbb{Z} & & \mathbb{Z} \\
1 & & & & & & \\
0 & \mathbb{Z} & & & \mathbb{Z} & & \mathbb{Z} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
\tag{B.25}
$$

Since degree $p + q = 4$ entries (except $E^2_{0,4}$) are empty, one immediately obtains

$$
\widetilde{\Omega}_4^{\mathrm{spin}^c}(\mathrm{SU}(n)) = 0, \tag{B.26}
$$

independent of whether $n = 2$ or $n \geq 3$. This means that the WZW term for $N_f = 2$ with spin^c structure is not of the torsion type but of the free type, contrary to the genuine spin structure case.

WZW terms for SU QCD from cobordism

Let us see how discrete WZW terms for $N_f = 2$ are related to ordinary WZW terms for $N_f \geq 3$, from the cobordism point of view. Again considering the AHSS for the Anderson-dual, the E_2 page is given by

$$
E_2^{p,q} = H^p\big(\mathrm{SU}(N_f); \mathrm{Inv}^q_{\text{spin}}\big)
$$

$$
\begin{array}{c|ccccc:cc}
4 & & & & & & & \\
3 & \mathbb{Z} & & & * & & * & \\
2 & \mathbb{Z}_2 & & & * & & * & \\
1 & \mathbb{Z}_2 & & & \mathbb{Z}_2 & & * & \\
0 & & & & & & & \\
-1 & \mathbb{Z} & & & \mathbb{Z} & & \mathbb{Z} & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\tag{B.27}
$$

which indeed shows

$$
0 \to \underbrace{H^5(\mathrm{SU}(N_f);\mathbb{Z})}_{E_2^{5,-1}=\mathbb{Z}} \to \underbrace{\mathrm{Inv}^4_{\text{spin}}(\mathrm{SU}(N_f))}_{\mathbb{Z}} \to \underbrace{\mathrm{Inv}^4_{\text{spin}}(\mathrm{SU}(2))}_{E_2^{3,1}=\mathbb{Z}_2} \to 0.
$$

This reads: the $4d$ WZW term for general $\mathrm{SU}(N_f)$ on *oriented* manifolds is given by $H^5(\mathrm{SU}(N_f);\mathbb{Z})$, and this maps to twice the minimal WZW term for generic $\mathrm{SU}(N_f)$ on *spin* manifolds given by $\mathrm{Inv}^4_{\text{spin}}(\mathrm{SU}(N_f))$, where the difference comes from the discrete WZW term for $\mathrm{SU}(2)$ on *spin* manifolds.

B.4 $\mathrm{SU}(n)/\mathrm{SO}(n)$

The $\mathbb{Z}_2$ cohomology of the homogeneous space $\mathrm{SU}(n)/\mathrm{SO}(n)$ is known to be

$$H^*\big(\mathrm{SU}(n)/\mathrm{SO}(n);\mathbb{Z}_2\big) = \textstyle\bigwedge_{\mathbb{Z}_2}[w_2, w_3, \ldots, w_n]. \tag{B.28}$$

where w_i's have degree i and the same Steenrod square actions as $B\mathrm{SO}(n)$. The integral cohomology of SU/SO is similarly complicated [Car60]

(B.29)

d	0	1	2	3	4	5	6	7	$\cdots$
$H^d(\mathrm{SU}(3)/\mathrm{SO}(3);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	$\cdots$
$H^d(\mathrm{SU}(4)/\mathrm{SO}(4);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
$H^d(\mathrm{SU}(5)/\mathrm{SO}(5);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}\oplus\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\cdots$
$H^d(\mathrm{SU}(6)/\mathrm{SO}(6);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\cdots$
$H^d(\mathrm{SU}(n\geq 7)/\mathrm{SO}(n);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}\oplus\mathbb{Z}_2$	0	$\mathbb{Z}_2^{\oplus 2}$	$\cdots$
generator	1	0	0	W_3	(e_4)	y_5, W_5	(e_6)	a_7 W_7	$\cdots$

where W_{2k+1} is the integral lift of w_{2k+1}, and a_7 is the integral lift of $Sq^1(w_2w_4) = w_2w_5 + w_3w_4$ (therefore $a_7 = W_3e_4$ for $n = 4$). Furthermore, e_{2k} only exists when $n = 2k$ is even and reduces to w_{2k}, while y_5 generates $\mathbb{Z}$ and reduces to w_2w_3.

Also, note that the Stiefel-Whitney classes w_i of $B\mathrm{SO}(n)$ pull back to w_i of $\mathrm{SU}(n)/\mathrm{SO}(n)$ along the map $\iota : \mathrm{SU}(n)/\mathrm{SO}(n) \to B\mathrm{SO}(n)$; that is, we have

$$\begin{array}{ccc} H^*(B\mathrm{SO}(n);\mathbb{Z}_2) & \xrightarrow{\iota^*} & H^*(\mathrm{SU}(n)/\mathrm{SO}(n);\mathbb{Z}_2) \\ \Cup & & \Cup \\ w_i & \longmapsto & w_i. \end{array} \tag{B.30}$$

Now, one can compute the spin bordism of $\mathrm{SU}(n)/\mathrm{SO}(n)$ in a similar manner. As $\mathrm{SU}(2)/\mathrm{SO}(2) = S^2$ and therefore $\widetilde{\Omega}^{\text{spin}}_d(\mathrm{SU}(2)/\mathrm{SO}(2)) = \Omega^{\text{spin}}_{d-2}(pt)$ from the suspension isomorphism, let us focus on the $n \geq 3$ cases. The E^2 pages are now as follows:

$$E^2_{p,q} = H_p\big(\mathrm{SU}(3)/\mathrm{SO}(3);\Omega^{\mathrm{spin}}_q\big)$$
$$\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & & * & \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \boxed{\mathbb{Z}_2} & & * & \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \boxed{\mathbb{Z}_2} & & \boxed{\mathbb{Z}_2} & \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & & \boxed{\mathbb{Z}} & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$$E^2_{p,q} = H_p\big(\mathrm{SU}(4)/\mathrm{SO}(4);\Omega^{\mathrm{spin}}_q\big)$$
$$\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \mathbb{Z}_2 & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \boxed{\boxed{\mathbb{Z}_2}} & \boxed{\mathbb{Z}_2} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \boxed{\mathbb{Z}} & \boxed{\mathbb{Z}} & \boxed{\mathbb{Z}_2} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$$E^2_{p,q} = H_p\big(\mathrm{SU}(5)/\mathrm{SO}(5);\Omega^{\mathrm{spin}}_q\big)$$
$$\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \mathbb{Z}_2 & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \boxed{\boxed{\mathbb{Z}_2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}} & \boxed{\mathbb{Z}_2} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$$E^2_{p,q} = H_p\big(\mathrm{SU}(6)/\mathrm{SO}(6);\Omega^{\mathrm{spin}}_q\big)$$
$$\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \mathbb{Z}_2 & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \boxed{\boxed{\mathbb{Z}_2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}} & \boxed{\mathbb{Z} \oplus \mathbb{Z}_2} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$$E^2_{p,q} = H_p\big(\mathrm{SU}(n \geq 7)/\mathrm{SO}(n);\Omega^{\mathrm{spin}}_q\big)$$
$$\begin{array}{c|ccccccc}
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \mathbb{Z}_2 & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \boxed{\boxed{\mathbb{Z}_2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}} & \boxed{\mathbb{Z}_2^{\oplus 2}} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{B.31}$$

The differentials $\boxed{d_2 : E^2_{4,0} \to E^2_{2,1}}$, $\boxed{d_2 : E^2_{5,0} \to E^2_{3,1}}$ and $\boxed{d_2 : E^2_{6,0} \to E^2_{4,1}}$ are again given by the mod 2 reduction composed with the dual of Sq^2. The first one turns out to be zero since $Sq^2 w_2 = (w_2)^2 = 0$, while the remaining two are possibly non-trivial since $Sq^2 w_3 = w_5 + w_3 w_2$ and $Sq^2 w_4 = w_6 + w_4 w_2$. The other differentials $\boxed{d_2 : E^2_{4,1} \to E^2_{2,2}}$ and $\boxed{d_2 : E^2_{5,1} \to E^2_{3,2}}$ are again the dual of Sq^2, and the same argument tells that the former is always trivial while the latter can be non-trivial. Therefore, we arrive at the E^3 pages given as follows:

$$
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{n=3} \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & & * & \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & & & * & \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & & & & \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & & \boxed{\mathbb{Z}} & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\qquad
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{n=4} \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & & & & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z} & \boxed{\mathbb{Z}} & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

$$
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{n=5} \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z}_2 & \boxed{\mathbb{Z}} & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\qquad
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{n=6} \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z}_2 & \boxed{\mathbb{Z}} & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

$$
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{n\geq 7} \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z}_2 & \boxed{\mathbb{Z}} & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\tag{B.32}
$$

Unfortunately, one cannot tell how the differential d_3's act on in general, and one is stuck in a dead end here from the AHSS alone. However, it turns out that $\boxed{d_3 : E^3_{5,0} \to E^3_{2,2}}$ has to be non-trivial[2] in order to match the results with those computed by using the Adams spectral sequence, which we will see later. In the end, one is led to

$$\widetilde{\Omega}^{\text{spin}}_4(\text{SU}(n)/\text{SO}(n)) = \begin{cases} \mathbb{Z}_2 & (n \geq 5) \\ \mathbb{Z} & (n = 4) \\ 0 & (n = 3) \end{cases} \tag{B.33}$$

and

$$\Omega^{\text{spin}}_5(\text{SU}(n)/\text{SO}(n)) = \mathbb{Z}, \tag{B.34}$$

where $\Omega^{\text{spin}}_5(\text{SU}(n)/\text{SO}(n)) = \mathbb{Z} \to H_5(\text{SU}(n)/\text{SO}(n); \mathbb{Z}) = \mathbb{Z}$ is a multiplication by four.

[2]It should also be possible to check by using the general form of d_3 in terms of cochains determined in [BM18].

WZW terms for SO **QCD from cobordism**

As in Sec. B.3, let us see how the discrete WZW term for $N_f = 2$ is related to the ordinary WZW term for $N_f \geq 3$, from the cobordism point of view. The E_2 page of the relevant AHSS for the Anderson dual of spin bordism is as follows:

$$E_2^{p,q} = H^p\big(\mathrm{SU}(N_f)/\mathrm{SO}(N_f); \mathrm{Inv}^q_{\text{spin}}\big)$$
$$\begin{array}{r|ccc|cccc}
4 & & & & & & & \\
3 & \mathbb{Z} & & & * & & * & * \\
2 & \mathbb{Z}_2 & & \mathbb{Z}_2 & * & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\
0 & & & & & & & \\
-1 & \mathbb{Z} & & & \mathbb{Z}_2 & (\mathbb{Z}) & \mathbb{Z}\,(\oplus\mathbb{Z}_2) & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{B.35}$$

As we have discussed repeatedly, the free part of the $4d$ WZW term for general $\mathrm{SU}(N_f)/\mathrm{SO}(N_f)$ on *oriented* manifolds which is given by $H^5(\mathrm{SU}(N_f)/\mathrm{SO}(N_f); \mathbb{Z})$, maps to four times the (free part of the) minimal WZW term for general $\mathrm{SU}(N_f)/\mathrm{SO}(N_f)$ on *spin* manifolds. The E_2 page above says that this factor of four arises due to the extension of the direct summand $\mathbb{Z}$ in $E^2_{5,-1}$ once by $\mathbb{Z}_2 = E_2^{3,1}$ and then again by $\mathbb{Z}_2 = E_2^{2,2}$. This last $\mathbb{Z}_2$ is already present when $N_f = 2$, meaning that the generator of the direct summand $\mathbb{Z}$ of $\mathrm{Inv}^4_{\text{spin}}(\mathrm{SU}(N_f)/\mathrm{SO}(N_f))$ pulls back to the generator of $\mathbb{Z}_2$ of $\mathrm{Inv}^4_{\text{spin}}(\mathrm{SU}(2)/\mathrm{SO}(2))$.

Adams spectral sequence

There is another way to compute the bordism group of interest, namely the Adams spectral sequence. For the case of interest, the E_2 page is given as

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}\big(\widetilde{H}^*(M\mathrm{Spin} \wedge X; \mathbb{Z}_2), \mathbb{Z}_2\big) \implies \widetilde{\Omega}_{t-s}^{\mathrm{spin}}(X)_2^\wedge, \tag{B.36}$$

and converges to the 2-completion[3] of the desired (reduced) bordism group. Here, $\mathcal{A}$ is the mod-2 Steenrod algebra generated by certain cohomology operations, Ext_R is a certain functor in the category of (graded) R-modules which takes values in Abelian groups, and $M\mathrm{Spin}$ is the Thom spectrum of the universal bundle over $B\mathrm{Spin}$.

Using the Künneth formula, the (reduced) cohomology of a smash product is decomposed as

$$\widetilde{H}^*(X \wedge Y; \mathbb{Z}_2) \simeq \widetilde{H}^*(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \widetilde{H}^*(Y; \mathbb{Z}_2).$$

Note that it is known [ABP67, FH16, Guo18] that

$$\widetilde{H}^*(M\mathrm{Spin}; \mathbb{Z}_2) \simeq \mathcal{A} \otimes_{\mathcal{A}(1)} (\mathbb{Z}_2 \oplus M_{\geq 8}) \tag{B.37}$$

where $\mathcal{A}(1)$ is the subalgebra of $\mathcal{A}$ generated by Sq^1 and Sq^2 and $M_{\geq 8}$ is an $\mathcal{A}(1)$-module which is trivial in degrees less than 8. Then, the combination of the shearing isomorphism and the adjunction formula allows us to rewrite the E_2-terms as

$$\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}\big((\mathbb{Z}_2 \oplus M_{\geq 8}) \otimes_{\mathbb{Z}_2} \widetilde{H}^*(X; \mathbb{Z}_2), \mathbb{Z}_2\big). \tag{B.38}$$

Fortunately, things become significantly easier in low degree in which we are interested. Namely, for $t - s \leq 7$, the $M_{\geq 8}$ part has no effect, so the E_2-terms can actually be reduced to

$$\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}\big(\mathbb{Z}_2 \otimes_{\mathbb{Z}_2} \widetilde{H}^*(X; \mathbb{Z}_2), \mathbb{Z}_2\big) = \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}\big(\widetilde{H}^*(X; \mathbb{Z}_2), \mathbb{Z}_2\big). \tag{B.39}$$

To compute $\mathrm{Ext}_{\mathcal{A}(1)}^{*,*}(M, \mathbb{Z}_2)$ for an $\mathcal{A}(1)$-module M, first take the minimal projective resolution

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots$$

where each module P_s is free up to a grade-shift, namely $P_s = \bigoplus_{j=1}^{N_s} \mathcal{A}(1)[t_{s,j}]$ with N_s's minimized sequentially from $s = 0$. A general theorem on Hopf algebra asserts that the morphisms between P_s's become trivial after passed to the functor $\mathrm{Hom}(-, \mathbb{Z}_2)$, and thus

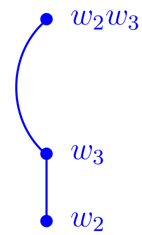
$$\mathrm{Ext}_{\mathcal{A}(1)}^{*,*}(M, \mathbb{Z}_2) = \bigoplus_s \bigoplus_j^{N_s} \mathbb{Z}_2[s, t_{s,j}], \tag{B.40}$$

where $[s, t]$ denotes a bi-degree shift. Once the E_2 page is determined, it is convenient to draw the *Adams chart* in which, for each summand in (B.40), a dot is drawn at $(t_{s,j} - s, s)$ in the $(t - s, s)$ coordinate.

[3] In practice, the 2-completion removes odd-torsion parts and replaces each $\mathbb{Z}$ by a module of 2-adic integers $\mathbb{Z}_2^\wedge$.

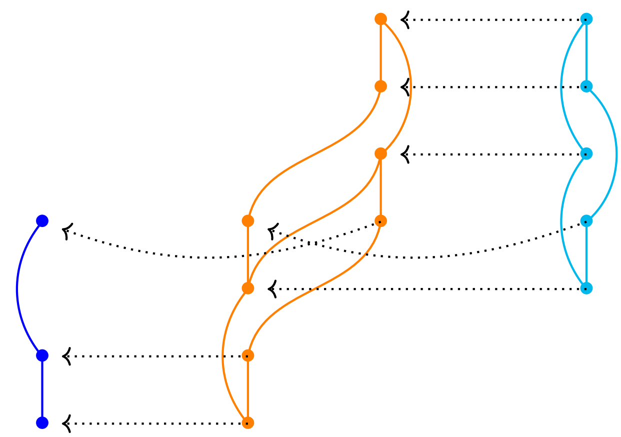
$\mathbf{SU(3)/SO(3)}$

Let us see how it works. The $\mathcal{A}(1)$-module structure of $\widetilde{H}^*(\mathrm{SU}(3)/\mathrm{SO}(3);\mathbb{Z}_2)$ is represented as



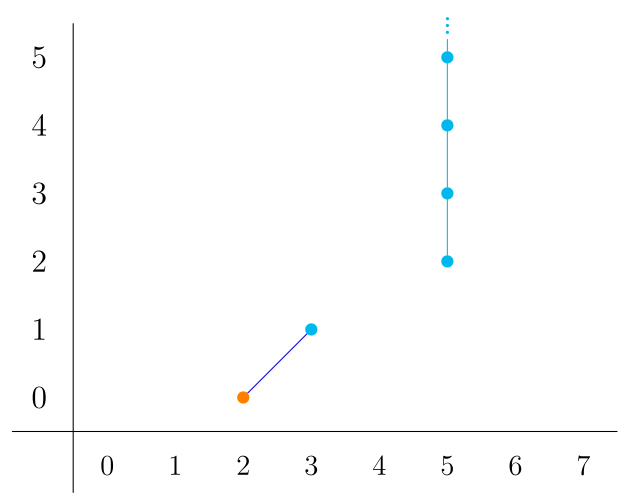
(B.41)

where the straight lines and curved lines represent the actions of Sq^1 and Sq^2 respectively. Noting that the bottom element w_2 has (t-)degree 2, one finds the following exact sequence



$$0 \longleftarrow \tilde{H}^*(\mathrm{SU}(3)/\mathrm{SO}(3)) \longleftarrow \mathcal{A}(1)[2] \longleftarrow J[4] \longleftarrow 0$$

where $P_0 = \mathcal{A}(1)[2]$, which corresponds to a dot at $(t-s,s) = (2,0)$ in the Adams chart, and also $P_1 = J[4]$, where J is a named $\mathcal{A}(1)$-module "Joker." Migrating the Adams chart of J from [BC18, Fig. 29] with a shift $(t-s,s) = (3,1)$, one obtains the following Adams chart



where the horizontal and vertical axes correspond to $t-s$ and s respectively. The vertical lines in the chart represent the action by $h_0 \in \mathrm{Ext}^{1,1}_{\mathcal{A}(1)}(\mathbb{Z}_2, \mathbb{Z}_2)$ with degree $(t-s, s) = (0, 1)$, and the "h_0 tower" remaining in the E_∞ page indicates the extension. Similarly, the sloped lines in the chart represent the action by $h_1 \in \mathrm{Ext}^{1,2}_{\mathcal{A}(1)}(\mathbb{Z}_2, \mathbb{Z}_2)$ with degree $(t-s, s) = (1, 1)$.

Fortunately, this E_2 page is too sparse for any differential, as the degree of a differential d_r is $(t-s, s) = (-1, *)$. All in all, the Adams spectral sequence converges as follows:

$$\begin{array}{c|cccccccc} d & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \hline \widetilde{\Omega}^{\text{spin}}_d(\mathrm{SU}(3)/\mathrm{SO}(3)) & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z} & 0 & \cdots \\ \hline \end{array} \tag{B.42}$$

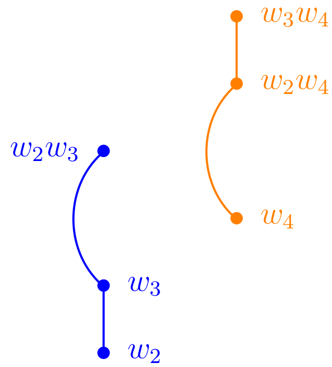
The fact that the h_0 tower representing the free part starts from $s = 2$ indicates that the generator of the free part is $2^2 = 4$ times the generator of the free part of the ordinary homology.[4] This is indeed consistent with the AHSS computation, since the $d = 4$ part must be trivial according to the Adams spectral sequence computation. In this way, one can sometimes obtain further information by combining two spectral sequences.[5]

[4] Replacing $\mathcal{A}$ in (B.36) not by $\mathcal{A}(1)$ but by $\mathcal{A}(0) = \mathbb{F}_2[Sq^1]/(Sq^1)^2$, one can consider a (Bockstein) spectral sequence converging to the (2-localized) ordinary homology $H(-;\mathbb{Z})$. For $X = \mathrm{SU}(3)/\mathrm{SO}(3)$, the h_0 tower there starts from $(t-s, s) = (5, 0)$, indicating the generator y_5 in the cohomology with $\mathbb{Z}$ coefficient is the $\mathbb{Z}$ uplift of $w_2 w_3$. From the naturality of the Adams spectral sequence, the map $\Omega \to H\mathbb{Z}$ is induced, in the way compatible with h_0, from the map between the corresponding map between the E_2 pages. Therefore, the image of the map $\Omega \to H\mathbb{Z}$ is generated by four times the dual of y_5. However, note that this argument does not determine the possible further multiplicity with prime factors other than 2.

[5] See e.g. [LT20] for another interesting example of using the Adams spectral sequence to compute bordism groups.

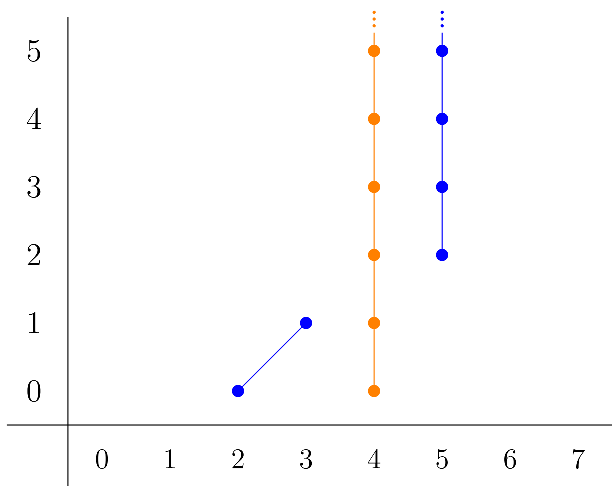
SU(4)/SO(4)

Introducing a new generator w_4, the $\mathcal{A}(1)$-module structure of $\widetilde{H}^*(\mathrm{SU}(4)/\mathrm{SO}(4);\mathbb{Z}_2)$ up to degree 7 is represented as



(B.43)

with an additional module compared to the previous case. This module is also a named one (Q, "question mark upside-down") and its Adams chart can again be found in [BC18, Fig. 29]. Therefore one has



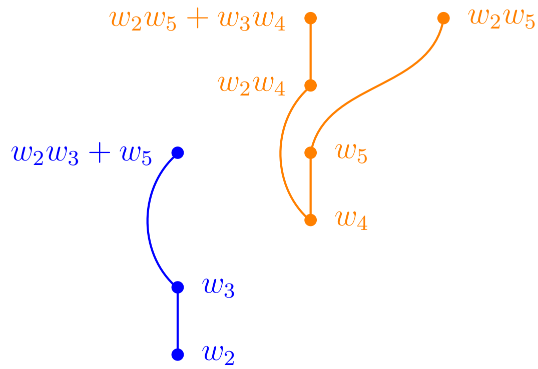
Although this E_2 page alone does not determine the differential from the h_0 tower at $t-s=5$ to that at $t-s=4$, one can infer from the AHSS computation that it should be trivial, since otherwise it will kill the $\mathbb{Z}$ at degree 5. In the end, one can conclude that the spectral sequence converges as follows:

d	0	1	2	3	4	5	6	$\cdots$
$\widetilde{\Omega}^{\mathrm{spin}}_d(\mathrm{SU}(4)/\mathrm{SO}(4))$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\cdots$

(B.44)

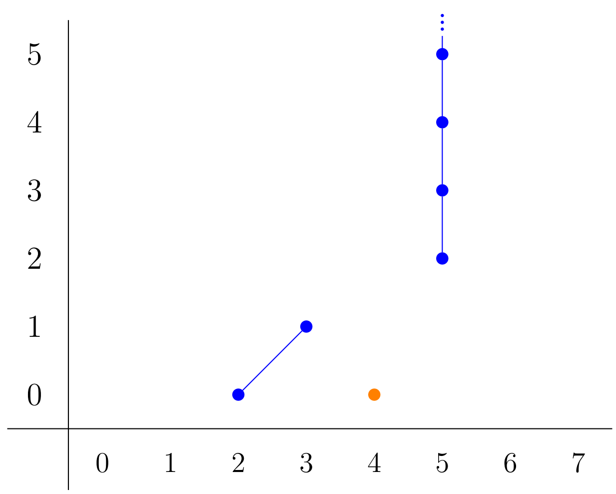
SU(5)/SO(5)

Introducing a new generator w_5, the $\mathcal{A}(1)$-module structure of $\widetilde{H}^*(\mathrm{SU}(5)/\mathrm{SO}(5);\mathbb{Z}_2)$ up to degree 7 is represented as



(B.45)

where the second module is (although not fully drawn) in fact turned into $\mathcal{A}(1)[4]$ from Q. Correspondingly, the Adams chart becomes



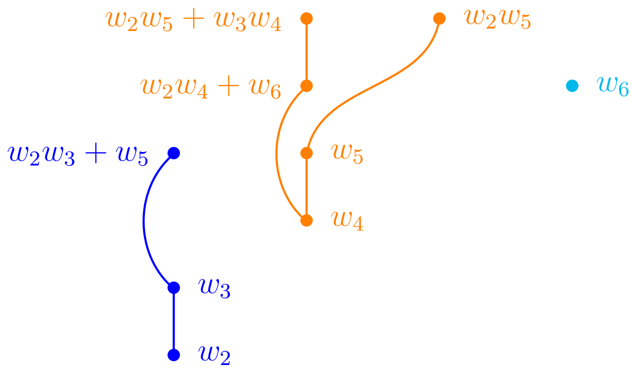
A possibly-nontrivial differential is the one with the source at $(t-s,s)=(4,0)$ and would hit the class in $(t-s,s)=(3,1)$, but again one knows that this is forbidden due to the AHSS result; otherwise it in particular kills $\mathbb{Z}_2$ at degree 3 which we know to survive on the AHSS side. Therefore, the spectral sequence converges as follows:

d	0	1	2	3	4	5	6	$\cdots$
$\widetilde{\Omega}^{\text{spin}}_d(\mathrm{SU}(5)/\mathrm{SO}(5))$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

(B.46)

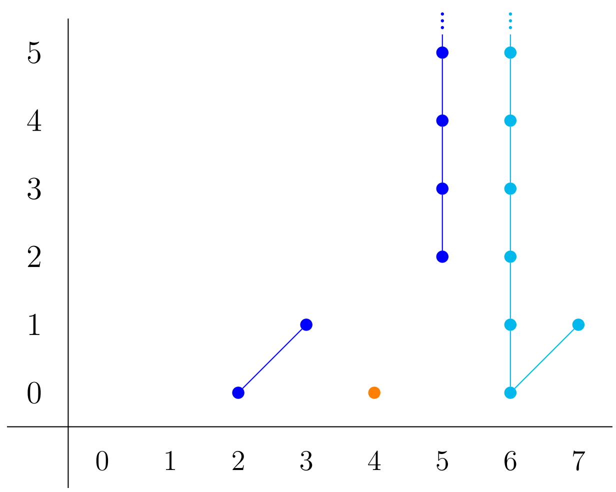
$\mathbf{SU(6)/SO(6)}$

Introducing a new generator w_6, the $\mathcal{A}(1)$-module structure of $\widetilde{H}^*(\mathrm{SU}(6)/\mathrm{SO}(6);\mathbb{Z}_2)$ up to degree 7 is represented as



(B.47)

and one now has an additional trivial $\mathcal{A}(1)$-module $\mathbb{Z}_2$. Its Adams chart can again be found in [BC18, Fig. 20], and as a result one obtains



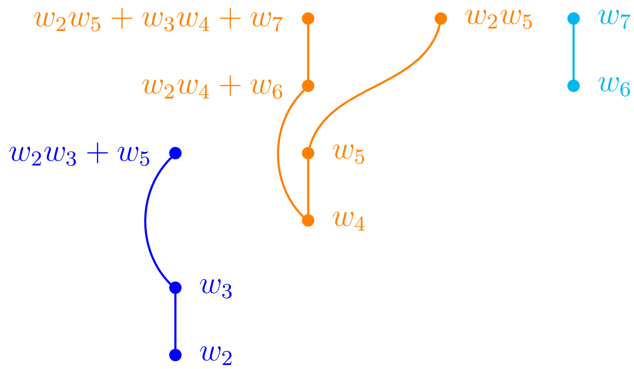
Similar reasonings as before guarantees all possibly-nontrivial differentials to be trivial, and the spectral sequence converges as

d	0	1	2	3	4	5	6	$\cdots$
$\widetilde{\Omega}^{\mathrm{spin}}_d(\mathrm{SU}(6)/\mathrm{SO}(6))$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$

(B.48)

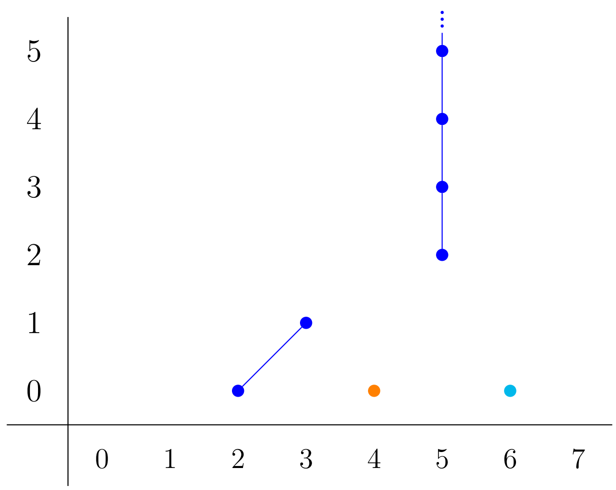
$\mathbf{SU(n \geq 7)/SO(n)}$

Finally, introducing a new generator w_7, the $\mathcal{A}(1)$-module structure of $\widetilde{H}^*(\mathrm{SU}(n \geq 7)/\mathrm{SO}(n); \mathbb{Z}_2)$ up to degree 7 is represented as



(B.49)

where the third module is in fact turned into $\mathcal{A}(1)[6]$ from a trivial module. Therefore, the resulting Adams chart is



The reasoning we have been using cannot be applied to the differential from $(t-s, s) = (6, 0)$ to $(5, 2)$, but it is not consistent with the h_0 action and hence should be trivial. At last, the spectral sequence converges as

d	0	1	2	3	4	5	6	$\cdots$
$\widetilde{\Omega}^{\mathrm{spin}}_d(\mathrm{SU}(n \geq 7)/\mathrm{SO}(n))$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\cdots$

(B.50)

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