博士論文

論文題目　Convergence of some non-convex energies under various topology

（様々な位相による非凸エネルギーの収束）

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Preface

In this paper, we consider some singular limit problems for several non-convex functionals. This paper consists of three chapters, which characterize the limit of a sequence of functionals in the sense of Γ-convergence. In considering Γ-convergence, it is important to determine the topology in the function space, which is the domain of definition of the functional. Choosing an appropriate topology according to the functional, we obtain several interesting singular limits. We are able to find the relation between minimizers of discrete and continuous models. Moreover, the detailed shape of the minimizer depending on the problem.

In Chapter 1, we introduce a random discrete energy of the energy for knots called the O'Hara energy, and discuss the convergence to continuous energy and the compactness of the discrete energy. Moreover, we show convergence stronger than that for conventional discretization. Specifically, we are successful to show locally uniform convergence and compactness of discrete energy in a space involving the optimal transport theory. By introducing a random discrete approximation of O'Hara energy using random variables, we show the convergence from a minimizer to a minimizer.

In Chapter 2, we consider the singular limit problem of a single-well Modica-Mortola energy and the Kobayashi–Warren–Carter energy in a one-dimensional domain. In this study, we introduce a finer topology of "graph convergence of functions" into the function space, and derive the singular limit of a single-well Modica-Mortola energy and the Kobayashi–Warren–Carter energy energies in the one-dimensional domain in the sense of Γ-convergence. The energy functional obtained as this singular limit is also shown to have the remarkable property of a minimizing function that is concave concerning the strength of jumps of a function. To characterize the limit under graph convergence, a new idea that is especially useful for one-dimensional problems is introduced. It is a change of parameter of the variable by the arc-length parameter of its graph, which is called unfolding by the arc-length parameter in this chapter.

In Chapter 3, as a continuation of Chapter 2, we consider the singular limit problems of a single-well Modica-Mortola energy and the Kobayashi–Warren–Carter energy. However, unlike Chapter 2, the domain is multidimensional. We introduce a new convergence concept called sliced graph convergence. Sliced graph convergence is, roughly speaking, graph convergence in almost every slice line for dense direction. This is because the method used to show Γ-convergence in multi-dimensional domains, called the "slicing method," is also used for finer topology.

Chapter 1 has been published in [1], while Chapter 2 is essentially the same as the publication [2].

[1] J.Okamoto, *Random discretization of O'Hara knot energy*, Advances in Mathematical Sciences and Applications, Vol.30, pp.507-520, (2021)

[2] Y.Giga, J.Okamoto, M.Uesaka, *A finer singular limit of a single-well Modica–Mortola functional and its applications to the Kobayashi–Warren–Carter energy*, Advances in Calculus of Variations, DOI : 10.1515/acv- 2020-0113, (2021)

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Contents

Chapter 1

Random discretization of O'Hara knot energy

In this chapter, we consider the random discrete approximation of O'Hara energy. O'Hara energy is the energy defined for a knot, and O'Hara energy was introduced for defining the standard shape for each knot class (equivalence class by ambient isotopy) by variational method. If the exponent is taken so that the energy is invariant Möbius transformation, O'Hara energy is called Möbius energy. Although discretization for various Möbius energies has been defined to analyze the shape of the minimizer so far, only Γ-convergence to the original energy has been shown for a conventional discretization. In this study, we are successful to show locally uniform convergence and compactness of discrete energy in a space involving the optimal transport theory, by introducing random discrete approximation of O'Hara energy using random variable and we can show convergence from the minimizer to the minimizer.

1.1 Introduction

Let $\mathcal{A}$ be the set of all closed regular curves that is parametrized by arc length in $\mathbb{R}^d$, with no self-intersections and with total length $\mathcal{L}$ i.e. $\mathcal{A} := \{\gamma \in C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \mid |\gamma'(x)| = 1 \ \text{ a.e. } x \in \mathbb{R}/\mathcal{L}\mathbb{Z}\}$. For $\alpha, p \in (0, \infty)$, the O'Hara (α, p)-energy $\mathcal{E}^{\alpha,p} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is defined as follows:

$$\mathcal{E}^{\alpha,p}(\gamma) := \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{M}^{\alpha,p}(\gamma) dx dy, \tag{1.1}$$

where

$$\mathcal{M}^{\alpha,p}(\gamma)(x,y) = \left(\frac{1}{|\gamma(x) - \gamma(y)|^\alpha} - \frac{1}{\mathcal{D}(\gamma(x), \gamma(y))^\alpha} \right)^p, \qquad (x,y) \in (\mathbb{R}/\mathcal{L}\mathbb{Z})^2 \tag{1.2}$$

and $\mathcal{D}$ is the length of a shortest arc of the curve γ connecting the two points $\gamma(x)$ and $\gamma(y)$, i.e.

$$\mathcal{D}(\gamma(x), \gamma(y)) = \min\{\mathcal{L} - |x-y|, |x-y|\}. \tag{1.3}$$

This energy was introduced and investigated by O'Hara in [6]-[9] for defining the standard shape for each knot class by variational method. In the case of $\alpha = 2, p = 1$, due to energy invariance under Möbius transformation, this energy is called "Möbius energy".

It is possible to show the existence of minimizer in the "prime knot". R. Kusner and J. Sullivan conjectured the minimizer in composite knot class may not exist [13]. This conjecture was established by numerical calculation with discretization of Möbius energy.

In this paper, we introduce the weighted O'Hara energy $\mathcal{E}_\rho^{\alpha,p} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ with weight $\rho : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}_{\geq 0}$ defined

$$\mathcal{E}_\rho^{\alpha,p}(\gamma) := \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{M}^{\alpha,p}(\gamma)\rho(x)\rho(y)dxdy. \tag{1.4}$$

Our main goal is to construct random discretization of this O'Hara energy and to show Γ-convergence in a space involving the optimal transport theory. We even show **locally uniform convergence** and **compactness**.

1.1.1 Known results

Various discretizations of O'Hara energy are defined not only for numerical calculation but also for shape analysis of minimizer.

First, D. Kim and R. Kusner constructed a Möbius energy for polygonal knots in [3].

This energy, defined on the class of arc length parametrizations of polygons of length $\mathcal{L}$ with n segments, is given by

$$\mathcal{E}_n(P) := \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{1}{|P(a_j) - P(a_i)|^2} - \frac{1}{d(a_j, a_i)^2} \right) d(a_{i+1}, a_i) d(a_{j+1}, a_j),$$

where a_i's are consecutive points on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, or interval $[0, \mathcal{L}]$ if we consider the polygon parametrized over an interval. This energy scales invariant. A slight variant would be to take $2^{-1}(d(a_{k-1}, a_k) + d(a_k, a_{k+1}))$ instead of $d(a_{k+1}, a_k)$.

S. Scholtes proved that this discretization Γ-converges to the Möbius energy in [17]. He furthermore showed that this energy is minimized by regular n-gons.

Second, J. Simon [5] defined the so-called *minimal distance energy* for a polygon P by

$$\mathcal{E}_s^m(P) = \tilde{\mathcal{E}}_s^m(P) - \tilde{\mathcal{E}}_s^m(R_m) + 4$$

with

$$\tilde{\mathcal{E}}_s^m(P) = \sum_{|i-j|>1} \frac{|X_i||X_j|}{\operatorname{dist}(X_i, X_j)^2},$$

where R_n is the regular n-gon. Note, that this energy scales invariant. Third, Möbius invariant discrete energy is introduced in [15] and show Γ-convergence is established in $W^{1,q}$-metric sense. The definition of that energy is as follows:

$$\mathcal{E}_{\cos}^m(P) = \sum_{d_m(i,j)>1} \frac{|\Delta_i P||\Delta_j P|}{|\Delta_i^j P||\Delta_{i+1}^{j+1} P|} \left(1 - \frac{1}{2}(\cos(\alpha_{ij}) + \cos(\tilde{\alpha}_{ij}))\right),$$

where $P(\theta_i)$ is a vertex of a closed polygon, $i = 1, 2, ..., m$ and $\Delta_i^j P := P(\theta_j) - P(\theta_i)$, $\Delta_i P = \Delta_i^{i+1} P$, α_{ij} be the angle of the crossing of the circles through at the points $P(\theta_i), P(\theta_{i+1}), P(\theta_j)$ and $P(\theta_j), P(\theta_{j+1}), P(\theta_i)$ and $\tilde{\alpha}_{ij}$ be the angle of the crossing of the circles through at the points $P(\theta_i), P(\theta_{i+1}), P(\theta_{j+1})$ and $P(\theta_j), P(\theta_{i+1}), P(\theta_{j+1})$.

1.1.2 Main results

We introduce a new discretization of O'Hara (α, p)-energy using random variables on $\mathbb{R}/\mathcal{L}\mathbb{Z}$.

Definition 1.1.1 (Random O'Hara Energy)**.** Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ with probability density function ρ.

Random O'hara energy $R_{n,\rho}\mathcal{E}^{\alpha,p} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is defined as follow:

$$R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma) := \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{1}{|\gamma(X_i) - \gamma(X_j)|^\alpha} - \frac{1}{\mathcal{D}(\gamma(X_i), \gamma(X_j))^\alpha} \right)^p .$$

Remark 1. Since $\{X_i\}_{i\in\mathbb{N}}$ has the probability density function ρ, we always have $\mathbb{P}(X_i = X_j) = 0$ for any $i \neq j$. Therefore $R_{n,\rho}\mathcal{E}^{\alpha,p}$ is well-defined almost surely.

Then we introduce the space for comparing continuous model and discrete model as follows.

Definition 1.1.2 (The TL^q metric space [10])**.** TL^q metric is defined on particular spaces of the family

$$TL^q(\mathbb{R}/\mathcal{L}\mathbb{N}) := \{(\mu, f) \mid \mu \in \mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{N}), f \in L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}; \mu)\},$$

where $1 \leq q < \infty$ and $\mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{N})$ denotes the set of Borel probability measure on $\mathbb{R}/\mathcal{L}\mathbb{N}$. For (μ, f) and (ν, g) in TL^q we define the distance

$$d_{TL^q}\big((\mu, f), (\nu, g)\big) := \inf_{\pi\in\Gamma(\mu,\nu)} \left(\int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} (|x - y|^q + |f(x) - g(y)|^q) \, d\pi(x, y) \right)^{1/q},$$

where $\Gamma(\mu, \nu)$ is the set of all coupling (or transportation plans) between μ and ν, that is, the set of all Borel probability measures on $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$ for which the marginal on the first variable is μ and the marginal on the second variable is ν. It is shown in [11] that d_{TL^q} is actually a metric. The distance d_{TL^q} is called a transportation distance between functions defined on graph. The TL^q topology provides a general and versatile way to compare functions in a discrete setting with functions in a continuum setting. It is a generalization of the weak convergence of measures and L^q convergence of functions.

Definition 1.1.3. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables and let us denote by ν_n the *empirical measure* of $\{X_i\}_{i\in\mathbb{N}}$:

$$\nu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i},$$

where δ_X is the Dirac measure of X.

Definition 1.1.4. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ and ν_n be an empirical measure of $\{X_i\}_{i\in\mathbb{N}}$ and ν be a distribution measure of $\{X_i\}_{i\in\mathbb{N}}$. We use a slight abuse of notation: $\gamma_n \xrightarrow{TL^q} \gamma$ instead of $(\nu_n, \gamma_n) \xrightarrow{TL^q} (\nu, \gamma)$. (Note that $\mathcal{A} \subset L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$.) Thus $\mathcal{A}$ is not a metric space with the metric d_{TL^q} but only convergence is defined.

The main results of the paper are

Theorem 1.1.1 (Γ-convergence and locally uniform convergence)**.** *Let* $\{X_i\}_{i\in\mathbb{N}}$ *be a sequence of i.i.d. random variables with probability density function* ρ *on* $\mathbb{R}/\mathcal{L}\mathbb{Z}$*. Then* $R_{n,\rho}\mathcal{E}^{\alpha,p}$ *Γ-converge to* $\mathcal{E}^{\alpha,p}_{\rho}$ *as* $n\to\infty$ *in the* TL^1 *sense.*

Moreover, we set $\mathcal{F} := \{\gamma\in\mathcal{A} \mid \mathcal{E}^{\alpha,p}_{\rho}(\gamma) < \infty\}$*, then* $R_{n,\rho}\mathcal{E}^{\alpha,p}\lfloor_{\mathcal{F}}$ *locally uniformly converges to* $\mathcal{E}^{\alpha,p}_{\rho}\lfloor_{\mathcal{F}}$ *as* $n\to\infty$ *in the* TL^1 *sense a.s.* $\omega\in\Omega$*.*

Theorem 1.1.2 (Compactness)**.** *Let* ρ *be bounded from below by a positive constant and let* $2\le \alpha p < 2p+1$ *and* $1\le q<\infty$*,*

Assume $\{\gamma_n\}_{n\in\mathbb{N}}\subset TL^q(\mathbb{R}/\mathcal{L}\mathbb{Z})$ *satisfying*

$$\sup_{n\in\mathbb{N}} R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma_n) < \infty.$$

and we assume that there exists $x\in\mathbb{R}/\mathcal{L}\mathbb{Z}$ *and* $C\in\mathbb{R}^d$ *such that* $\gamma_n(x) = C$ *for all* $n\in\mathbb{N}$*.*

Then $\{\gamma_n\}_{n\in\mathbb{N}}$ *is relatively compact in the* $TL^q(\mathbb{R}/\mathcal{L}\mathbb{Z})$ *sense a.s.* $\omega\in\Omega$*.*

The metric used in Γ-convergence is the TL^1 sense. Locally uniform convergence is also in the TL^1 sense, which will be defined in Section 1.2.

Corollary 1 (Minimizer to minimizer)**.** Under the assumption of Theorem 1.1.1 and Theorem 1.1.2, let $\{\gamma_n\}_{n\in\mathbb{N}}\subset\mathcal{A}$ be minimizers of $R_{n,\rho}\mathcal{E}^{\alpha,p}$ then there exists $\tilde{\gamma}\in\mathcal{A}$ and subsequence $\{\gamma_{n_k}\}_{k\in\mathbb{N}}$ with $\gamma_{n_k}\xrightarrow{TL^q}\tilde{\gamma}$ such that $\tilde{\gamma}$ is a minimizer of $\mathcal{E}^{\alpha,p}$.

1.2 Preliminaries

1.2.1 Γ-convergence and locally uniformly convergence in the TL^q sense

We recall notation of general Γ-converge and locally uniform convergence in the TL^q sense.

Definition 1.2.1 (Γ-convergence in the TL^q sense)**.** Let $F_n : \mathcal{A}\to\mathbb{R}\cup\{+\infty\}$ be a sequence of functionals. The sequence $\{F_n\}_{n\in\mathbb{N}}$ Γ-converges to the functional $F:\mathcal{A}\to\mathbb{R}\cup\{+\infty\}$ as $n\to\infty$ in the TL^q sense if the following inequality hold:

i) For every $x\in X$ and every sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n\xrightarrow{TL^q}x$

$$\liminf_{n\to\infty} F_n(x_n)\ge F(x),$$

ii) For every $x\in X$ there exists a sequence $\{x_n\}_{n\in N}$ and $x_n\xrightarrow{TL^q}x$ satisfying

$$\limsup_{n\to\infty} F_n(x_n)\le F(x).$$

Definition 1.2.2 (Locally uniform convergence in the TL^q sense)**.** A set $K\subset\mathcal{A}$ *is sequentially compact in the* TL^q *sense* if it satisfies the following conditions. For all sequence $\{\gamma_n\}_{n\in\mathbb{N}}\subset K$, there is a subsequence $\{\gamma_{n_k}\}_{k\in\mathbb{N}}$ and a $\gamma\in K$ such that $\gamma_{n_k}\xrightarrow{TL^q}\gamma$ as $k\to\infty$. Let X be a space containing $L^q(\nu_n)$ and $L^q(\nu)$, and let $F_n : X\to\mathbb{R}$ and $F:X\to\mathbb{R}$. The sequence $\{F_n\}_{n\in\mathbb{N}}$ *locally uniformly converges to* F *in the* TL^q *sense* if it satisfies the following conditions. For any sequentially compact set $K\subset X$ in the TL^q sense,

$$\lim_{n\to\infty}\sup_{\gamma\in K}|F_n(\gamma)-F(\gamma)| = 0.$$

We first discuss an equivalent condition for locally uniform convergence in the TL^q sense.

Proposition 1.2.1. *Let $F : X \to \mathbb{R}$ is continuous in the TL^q sense, (i.e. $\gamma_n \xrightarrow{TL^q} \gamma$ then $\lim_{n\to\infty} F(\gamma_n) = F(\gamma)$.) and a sequence $\{F_n\}_{n\in\mathbb{N}}$ locally uniformly converge to F in the TL^q sense.*

if and only if

For any sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset X$ with $\gamma_n \xrightarrow{TL^q} \gamma$,

$$\lim_{n\to\infty} F_n(\gamma_n) = F(\gamma).$$

Proof. This can be proved in a similar way to prove Ascoli-Arzelà theorem [18, Theorem 7.25.]. For $\gamma \in \mathcal{A}$ and $r > 0$, we set $\mathcal{B}(\gamma, r) := \{\tilde{\gamma} \in X \mid$ There exists an $n \in \mathbb{N}$ such that $d_{TL^q}\big((\nu_n, \gamma), (\nu, \tilde{\gamma})\big) < r\}$.

First, we show that suppose $K \subset X$ be a sequentially compact set in the TL^q sense, then for any $\varepsilon > 0$, there is a sequence $\{\gamma_i\}_{i=1}^{N_\varepsilon} \subset K$, such that $\bigcup_{i=1}^{N_\varepsilon} \mathcal{B}(\gamma_i, \varepsilon) \supset K$. If not, there is a $r > 0$ such that for all $\{\gamma_i\}_{i=1}^{m} \subset K$, $K \setminus \bigcup_{i=1}^{m} \mathcal{B}(\gamma_i, r) \neq \emptyset$. We choose $\gamma_1 \in K$ and inductively choose $\gamma_n \in K \setminus \bigcup_{i=1}^{n-1} \mathcal{B}(\gamma_i, r)$, then for all $m, n \in \mathbb{N}$,

$$\begin{aligned} r &\le d_{TL^q}\big((\nu_n, \gamma_n), (\nu, \gamma_m)\big) \le d_{TL^q}\big((\nu_n, \gamma_n), (\nu_m, \gamma_m)\big) + d_{TL^q}\big((\nu_m, \gamma_m), (\nu, \gamma_m)\big) \\ &= d_{TL^q}\big((\nu_n, \gamma_n), (\nu_m, \gamma_m)\big). \end{aligned}$$

This is a contradiction to the fact that K is sequentially compact set in the TL^q sense.

Let $\varepsilon_m > 0$ with $\varepsilon_m \searrow 0$. and we set

$$K_0 := \left\{ \{\gamma_i^m\}_{m=1}^{N_m} \in K \;\middle|\; \bigcup_{i=1}^{N_m} \mathcal{B}(\gamma_i^m, \varepsilon_m) \supset K \right\}. \tag{1.5}$$

Second, we show that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\gamma_1, \gamma_2 \in K$ and all $n \in \mathbb{N}$, if $d_{TL^q}\big((\nu_n, \gamma_1), (\nu, \gamma_2)\big) < \delta$ then

$$|F_n(\gamma_1) - F_n(\gamma_2)| < \varepsilon. \tag{1.6}$$

If not, there exists an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $\gamma_1^n, \gamma_2^n \in K$ and $m_n \in \mathbb{N}$ such that $d_{TL^q}\big((\nu_n, \gamma_1^n), (\nu, \gamma_2^n)\big) < 1/n$, and $|F_{m_n}(\gamma_1^n) - F_{m_n}(\gamma_2^n)| \ge \varepsilon$. By the K is sequentially compact, there exists a subsequence $\{\gamma_1^{n_k}\}_{k\in\mathbb{N}}, \{\gamma_2^{n_k}\}_{k\in\mathbb{N}}$, and $\gamma \in K$ such that $\gamma_1^{n_k} \xrightarrow{TL^q} \gamma$, $\gamma_2^{n_k} \xrightarrow{TL^q} \gamma$. Then

$$\varepsilon < |F_{m_{n_k}}(\gamma_1^{n_k}) - F_{m_{n_k}}(\gamma_2^{n_k})| \le |F_{m_{n_k}}(\gamma_1^{n_k}) - F(\gamma)| + |F(\gamma) - F_{m_{n_k}}(\gamma_2^{n_k})| \xrightarrow{k\to\infty} 0.$$

This is a contradiction.

Third, we show that there exists a subsequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ such that for all j , $\{F_{n_k}(\gamma_j)\}_{k\in\mathbb{N}}$ is convergence sequence by a diagonal argument. By $\lim_{n\to\infty} F_n(\gamma_1) = F(\gamma_1)$, $\{F_n(\gamma_1)\}_{n\in N}$

is bounded sequence on $\mathbb{R}$. Therefore there exists a subsequence $\{F_{n(1,k)}\}_{k\in\mathbb{N}}$ such that $\{F_{n(1,k)}(\gamma_1)\}_{k\in\mathbb{N}}$ is a convergence sequence in $\mathbb{R}$. In the same way, there exists a subsequence $\{F_{n(2,k)}\}_{k\in\mathbb{N}}$ such that $\{F_{n(2,k)}(\gamma_2)\}_{k\in\mathbb{N}}$ is a convergence sequence on $\mathbb{R}$. Further, in the same way, we construct subsequence $\{F_{n(p,k)}\}_{k\in\mathbb{N}}, p=3,4,...,$ and we set $F_{n_k}=F_{n(k,k)}$.

Finally, let any $\eta>0$, for sufficiently large m such that for all $n\in\mathbb{N}$, if $d_{TL^q}\big((\nu_n,\gamma_1),(\nu,\gamma_2)\big)<\varepsilon_m$ then

$$|F_n(\gamma)-F_n(\gamma_i^m)|<\eta/3. \tag{1.7}$$

Since $\{F_{n_k}(\gamma_i^m)\}_{k\in\mathbb{N}}$ is a convergence sequence on $\mathbb{R}$, there exists a number N such that if $k,l>N$ then $|F_{n_k}(\gamma_i^m)-F_{n_l}(\gamma_i^m)|<\eta/2$ for $i=1,2,...,N_m$. Now, let $\gamma\in K$, by (1.5) there exist an i and n such that

$$d_{TL^q}\big((\nu_n,\gamma_i^m),(\nu,\gamma)\big)<\varepsilon_m.$$

By (1.6) and (1.7) if $k,l>N$ then

$$|F_{n_k}(\gamma)-F_{n_l}(\gamma)|\le|F_{n_k}(\gamma)-F_{n_k}(\gamma_i^m)|+|F_{n_k}(\gamma_i^m)-F_{n_l}(\gamma_i^m)|+|F_{n_l}(\gamma)-F_{n_l}(\gamma_i^m)| \tag{1.8}$$
$$\le\eta. \tag{1.9}$$

Therefore $\{F_{n_k}(\gamma)\}_{k\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of $\mathbb{R}$, there exists a $F(\gamma)$ such that $\lim_{k\to\infty}F_{n_k}(\gamma)=F(\gamma)$. In (1.8) and (1.9), suppose that l goes to infinity, then we get

$$|F_{n_k}(\gamma)-F(\gamma)|<\eta. \tag{1.10}$$

This indicates that $\{F_{n_k}\}_{k\in\mathbb{N}}$ uniformly converges to F on K.

Assume that F is continuous and that F_n locally uniformly converges to F in the TL^q sense and $\gamma_n\xrightarrow{TL^q}\gamma$.

Clearly, $\{\gamma_n\mid n\in\mathbb{N}\}\cup\{\gamma\}\subset X$ is sequentially compact in the TL^q sense.

Therefore

$$\begin{aligned}|F_n(\gamma_n)-F(\gamma)|&\le|F_n(\gamma_n)-F(\gamma_n)|+|F(\gamma_n)-F(\gamma)|\\&\le\sup_{y\in\{\gamma_n\}_{n\in\mathbb{N}}\cup\{\gamma\}}\big(|F_n(y)-F(y)|\big)+|F(\gamma_n)-F(\gamma)|\xrightarrow{n\to\infty}0.\end{aligned}$$

□

1.2.2 The property of TL^q space and empirical measure

In this subsection, we consider the space based on optimal transport theory to compare discrete and continuous models.

Given a Borel map $T:\mathbb{R}/\mathcal{L}\mathbb{Z}\to\mathbb{R}/\mathcal{L}\mathbb{Z}$ and $\mu\in\mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ the *push-forward* of μ by T, denoted by $T_\#\mu\in\mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ is given by:

$$T_\#\mu(A):=\mu(T^{-1}(A)),A\in\mathcal{B}(\mathbb{R}/\mathcal{L}\mathbb{Z}).$$

Definition 1.2.3 ([10]). We say that a sequence of transportation plans $\{\pi_n\}_{n\in\mathbb{N}} \subset \Gamma(\mu, \mu_n)$ is *stagnating* if it satisfies

$$\lim_{n\to\infty} \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} |x-y|^q d\pi_n(x,y) = 0.$$

$\mu, \mu_n \in \mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{Z}), T_n : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}/\mathcal{L}\mathbb{Z}$: transportation maps with $T_{n\#}\mu = \mu_n$.

We say that a sequence of transportation maps $\{T_n\}_{n\in\mathbb{N}}$ is *stagnating* if it satisfies

$$\lim_{n\to\infty} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} |x - T_n(x)|^q d\mu(x) = 0.$$

Accept the following proposition introduced by [10]

Proposition 1.2.2 ([10]). *Let $(\mu, \gamma) \in TL^q$ and let $\{(\mu_n, \gamma_n)\}_{n\in\mathbb{N}} \subset TL^q$. The following statements are equivalent;*

i) $(\mu_n, \gamma_n) \to (\mu, \gamma)$ *in the* TL^q.

ii) $\mu_n \rightharpoonup \mu$ *and for every stagnating sequence of transportation plans* $\{\pi_n\}_{n\in\mathbb{N}} \subset \Gamma(\mu, \mu_n)$

$$\int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} |\gamma(x) - \gamma_n(x)|^q d\pi_n(x,y) \to 0. \tag{1.11}$$

iii) $\mu_n \rightharpoonup \mu$ *and there exists a stagnating sequence of transportation plans* $\{\pi_n\}_{n\in\mathbb{N}} \subset \Gamma(\mu, \mu_n)$ *such that*

$$\int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} |\gamma(x) - \gamma_n(x)|^q d\pi_n(x,y) \to 0. \tag{1.12}$$

Moreover, if the measure μ is absolutely continuous with respect to the Lebesgue measure, the following are equivalent to the previous statement:

iv) $\mu_n \rightharpoonup \mu$ *and there exists a stagnating sequence of transportation maps* $\{T_n\}_{n\in\mathbb{N}}$ *(with* $T_{n\#}\mu = \mu_n$*) such that*

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} |\gamma(x) - \gamma_n(T_n(x))|^q d\mu(x) \to 0. \tag{1.13}$$

v) $\mu_n \rightharpoonup \mu$ *and for any stagnating sequence of transportation maps* $\{T_n\}_{n\in\mathbb{N}}$ *with* $T_{n\#}\mu = \mu_n$*),*

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} |\gamma(x) - \gamma_n(T_n(x))|^q d\mu(x) \to 0. \tag{1.14}$$

Remark 2. Thanks to Proposition 1.2.2 when μ is absolutely continuous with respect to the Lebesgue measure $\gamma_n \xrightarrow{TL^p} \gamma$ as $n \to \infty$ if and only if for every (or one) stagnating sequence $\{T_n\}_{n\in\mathbb{N}}$ of transportation maps (with $T_{n\#}\mu = \mu_n$) $\gamma_n \circ T_n \xrightarrow{L^p(\mu)} \gamma$ as $n \to \infty$. Also, $\{u_n\}_{n\in\mathbb{N}}$ is relatively compact in TL^p if and only if for every (or one) stagnating sequence $\{T_n\}_{n\in N}$ of transportation maps (with $T_{\#}\mu = \mu_n$) $\{u_n \circ T_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^p(\mu)$.

We recall the following proposition.

Proposition 1.2.3 (Glivenko-Cantelli's Theorem). *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, and let ν is the distribution measure of $\{X_i\}_{i\in\mathbb{N}}$, and ν_n is the empirical measure of $\{X_i\}_{i\in\mathbb{N}}$. $F_n(x) := \nu_n((-\infty, x])$ and F is the distribution function of $\{X_i\}_{i\in\mathbb{N}}$, then,*

$$\lim_{n\to\infty} \|F_n - F\|_\infty = 0 \qquad a.s.\ \omega \in \Omega.$$

Theorem 1.2.1 ([12]). *Let $D \subset \mathbb{R}^d$ be a bounded, connected, open set with Lipschitz boundary. Let ν be a probability measure on D with density $\rho : D \to (0, \infty)$ which is bounded from below and from above by positive constants. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent random points distributed on D according to measure ν and let ν_n be the associated empirical measures. Then there is a constant $C > 0$ such that for a.s. $\omega \in \Omega$ there exists a sequence of transportation maps $\{T_n\}_{n\in\mathbb{N}}$ from ν to ν_n ($T_{n\#}\nu = \nu_n$) and such that*

if $d = 2$ then

$$\limsup_{n\to\infty} \frac{n^{1/2}\|Id - T_n\|_\infty}{(\log n)^{3/4}} \leq C$$

and if $d \geq 3$ then

$$\limsup_{n\to\infty} \frac{n^{1/d}\|Id - T_n\|_\infty}{(\log n)^{1/d}} \leq C.$$

1.3 Γ-convergence and locally uniformly convergence

Proof of Theorem 1.1.1

Proof. Let ν_n be an empirical measure of $\mathcal{L}^1\lfloor\rho$ and $T_n : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}/\mathcal{L}\mathbb{Z}$ be transportation maps with $T_{n\#}\mathcal{L}^1\lfloor\rho = \nu_n$.

· liminf inequality

Assume that $\gamma_n \to \gamma$ in TL^1 as $n \to \infty$. Since $T_{n\#}\mathcal{L}^1\lfloor\rho = \nu_n$, we change variables to get

$$\begin{aligned} R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma_n) &= \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{M}^{\alpha,p}(\gamma_n) d\nu_n(x) d\nu_n(y) &(1.15)\\ &= \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{M}^{\alpha,p}(\gamma_n \circ T_n)\rho(x)\rho(y) dx dy. &(1.16)\end{aligned}$$

By the way, we notice that

$$|\gamma_n(T_n(x)) - \gamma_n(T_n(y))| \leq |\gamma_n(T_n(x)) - \gamma(x)| + |\gamma(x) - \gamma(y)| + |\gamma(y) - \gamma_n(T_n(y))|.$$

and

$$\begin{aligned}\mathcal{D}(\gamma(x), \gamma(y)) &\leq \mathcal{D}(\gamma(x), \gamma_n(T_n(x))) + \mathcal{D}(\gamma_n(T_n(x)), \gamma_n(T_n(y))) + \mathcal{D}(\gamma_n(T_n(y)), \gamma(y))\\ &\leq |x - T_n(x)| + \mathcal{D}(\gamma_n(T_n(x)), \gamma_n(T_n(y))) + |T_n(y) - y|\\ &\leq 2\|T_n - \mathrm{Id}\|_\infty + \mathcal{D}(\gamma_n(T_n(x)), \gamma_n(T_n(y))).\end{aligned}$$

By Proposition 1.2.2 we deduce $\gamma_n \circ T_n \to \gamma$ in $L^1(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Thus, by taking an appropriate subsequence $\{\gamma_{n_k} \circ T_{n_k}\}_{k\in\mathbb{N}}$ of $\{\gamma_n \circ T_n\}_{n\in\mathbb{N}}$, we deduce $\gamma_{n_k}(T_{n_k}(x)) \to \gamma(x)$ a.e. $x \in \mathbb{R}/\mathcal{L}\mathbb{Z}$, and therefore

$$\liminf_{n\to\infty} \mathcal{M}^{\alpha,p}(\gamma_n \circ T_n) \geq \mathcal{M}^{\alpha,p}(\gamma) \quad \text{a.e. } (x,y) \in (\mathbb{R}/\mathcal{L}\mathbb{Z})^2.$$

So that $\mathcal{M}^{\alpha,p}(\gamma_n \circ T_n) > 0$, using Fatou's lemma we get

$$\liminf_{n\to\infty} R_{n,\rho}\mathcal{E}_n^{\alpha,p}(\gamma_n) \geq \mathcal{E}_\rho^{\alpha,p}(\gamma).$$

· **limsup inequality**

Assume that $\gamma_n \to \gamma$ in TL^1 as $n \to \infty$. If $\mathcal{E}_\rho^{\alpha,p}(\gamma) = \infty$, we are done, and so we henceforth assume $\mathcal{E}_\rho^{\alpha,p}(\gamma) < \infty$. We observe that

$$|\gamma(x) - \gamma(y)| \leq |\gamma(x) - \gamma_n(T_n(x))| + |\gamma_n(T_n(x)) - \gamma_n(T_n(y))| + |\gamma_n(T_n(y)) - \gamma(y)|,$$

and

$$\mathcal{D}(\gamma_n(T_n(x)), \gamma_n(T_n(y))) \leq \mathcal{D}(\gamma_n(T_n(x)), \gamma(x)) + \mathcal{D}(\gamma(x), \gamma(y)) + \mathcal{D}(\gamma(y), \gamma_n(T_n(y))).$$

In the same way as liminf inequality, we get
$\limsup_{n\to\infty} \mathcal{M}^{\alpha,p}(\gamma_n \circ T_n) \leq \mathcal{M}^{\alpha,p}(\gamma)$ a.e. $(x,y) \in (\mathbb{R}/\mathbb{Z})^2$. Since $\mathcal{E}_\rho^{\alpha,p}(\gamma) < \infty$, using Fatou's lemma to $\mathcal{M}^{\alpha,p}(\gamma) - \mathcal{M}^{\alpha,p}(\gamma_n \circ T_n)$, we get

$$\limsup_{n\to\infty} R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma_n) \leq \mathcal{E}_\rho^{\alpha,p}(\gamma).$$

By Proposition 1.2.1, the proof is now complete.

□

1.4 Compactness

In this section, we would like to prove Theorem 1.1.2. We first recall several function spaces.

1.4.1 Function spaces

Definition 1.4.1 (Sobolev-Slobodeckij spaces). For $s \in (0,1)$ and $q \in [1,\infty)$ we set

$$[\gamma]_{W^{s,p}} := \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\mathcal{L}/2}^{\mathcal{L}/2} \frac{|\gamma(u+w) - \gamma(u)|^p}{|w|^{1+ps}} dw du \right)^{1/p}$$

$$W^{s,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) := \left\{ \gamma \in L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \mid [\gamma]_{W^{s,p}} < \infty \right\}$$

and equip this space with the norm

$$\|\gamma\|_{W^{s,q}} := \|\gamma\|_{L^q} + [\gamma]_{W^{s,p}}.$$

Furthermore, we let

$$W^{1+s,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) := \left\{ \gamma \in W^{1,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \mid \gamma' \in W^{s,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \right\}$$

and

$$\|\gamma\|_{W^{1+s,q}} := \left(\|\gamma\|_{W^{s,q}}^q + \|\gamma\|_{W^{1,q}}^q \right)^{1/q}.$$

Definition 1.4.2 (Besov spaces). For $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $\mathcal{S}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$ is the Schwartz space.

We set

$$\psi(\xi) = \begin{cases} 1 & (|\xi| \le 4) \\ 0 & (|\xi| \ge 8) \end{cases}, \quad \varphi(\xi) = \begin{cases} 1 & (2 \le |\xi| \le 4) \\ 0 & (|\xi| \le 1 \text{ or } |\xi| \ge 8) \end{cases}.$$

We set $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and $\tau(D)f := \mathcal{F}^{-1}[\tau \cdot \mathcal{F}f]$ for $\tau \in \mathcal{S}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$. We recall Besov norms:

$$\|f\|_{B^s_{p,q}} := \begin{cases} \|\psi(D)f\|_{L^p} + \left(\sum_{j=1}^{\infty} 2^{jqs} \|\varphi_j(D)f\|^q_{L^p} \right)^{1/q} & (1 \le q < \infty) \\ \|\psi(D)f\|_{L^p} + \sup_{j \in \mathbb{N}} 2^{js} \|\varphi_j(D)f\|_{L^p} & (q = \infty) \end{cases}$$

$$B^s_{p,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) := \{ f \in \mathcal{S}'(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \mid \|f\|_{B^s_{p,q}} < \infty \}.$$

Note that $W^{s,p}$ agrees with $B^s_{p,p}$.

We recall the following embedding results.

Proposition 1.4.1 (Embedding Besov spaces). *Let $0 < p_0, p_1 \le \infty$ and $0 < q_0, q_1 \le \infty$ and $-\infty < s_1 < s_0 < \infty$. Assume that $s_0 - \frac{1}{p_0} > s_1 - \frac{1}{p_1}$, then*

$$B^{s_0}_{p_0,q_0}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d).$$

Here $\hookrightarrow$ denotes a continuous embedding.

Theorem 1.4.1 (Kondrachov embedding theorem). *Let $1 \le p, q < \infty$ and $1 < k, s$.*

Assume that $k - \frac{1}{p} > s - \frac{1}{q}$, then the Sobolev embedding

$$W^{k,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \hookrightarrow W^{s,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$$

is completely continuous.

Proposition 1.4.2 (Embedding L^1 space for Besov space). *Let $s \in \mathbb{R}$ and $0 < q \le \infty$, then*

$$L^1 \hookrightarrow B^0_{1,q} \text{ if and only if } q = \infty.$$

The following theorem is based on [14] and explains conditions for which O'Hara energy becomes finite.

Theorem 1.4.2 ([14])**.** *Let $\gamma \in \mathcal{A}$ and $\alpha, p \in (0, \infty)$ with $\alpha p \ge 2$ and $s := \frac{\alpha p - 1}{2p} < 1$ and $p \ge 1$, then $\mathcal{E}^{\alpha,p}(\gamma) < \infty$ if and only if $\gamma \in W^{1+s,2p}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Moreover, there is a $C = C(\alpha, p)$ such that*

$$\|\gamma'\|^{2p}_{W^{s,2p}} \le C(\mathcal{E}^{\alpha,p}(\gamma) + \|\gamma'\|^{2p}_{L^{2p}}).$$

Lemma 1.4.3. *Let $\{\gamma_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and we assume that there exists a $x \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ and $C \in \mathbb{R}^d$ such that $\gamma_n(x) = C$ for all $n \in \mathbb{N}$.*

Then

$$\sup_{n\in\mathbb{N}} \|\gamma_n\|_{L^1(\nu_n)} = \sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{i=1}^{n} |\gamma_n(X_i)| < \infty.$$

Proof. First, we show that there exists a $M > 0$ such that

$$\max_{i=1,2,\dots,n} |\gamma_n(X_i) - C| \leq M.$$

If not, there exists a $n \in \mathbb{N}$ and $i \in \{1, 2, ..., n\}$ such that

$$|\gamma_n(X_i) - C| > \mathcal{L}.$$

Then a length of γ_n is more than $\mathcal{L}$. This is a contradiction.
Thus

$$\begin{aligned}\|\gamma_n\|_{L^1(\nu_n)} &= \frac{1}{n}\sum_{i=1}^{n} |\gamma_n(X_i)| \\ &\leq \frac{1}{n}\sum_{i=1}^{n} |\gamma_n(X_i) - C| + \frac{1}{n}\sum_{i=1}^{n} |C| \leq M + |C|.\end{aligned}$$

□

1.4.2 Proof of Theorem 1.2.

Proof. Let $s := \frac{\alpha p - 1}{2p}$. By Theorem 1.4.2 we see

$$\|\gamma\|^{2p}_{W^{1+s,2p}} \leq C\left(\mathcal{E}^{\alpha,p}(\gamma) + \|\gamma\|^{2p}_{W^{1,2p}}\right).$$

By the Gagliardo-Nirenberg interpolation inequality, we choose $\Theta > 0$ such that $\Theta < \frac{(2p-1)q+2p}{(2p+\alpha p-2)q+2p} \leq 1$, and t with $\frac{1}{2p} - 1 > t$, to get

$$\begin{aligned}\|\gamma\|_{W^{1,2p}} &\leq \|\gamma\|_{W^{1+s/2,2p}} && (1.17)\\ &= \|\gamma\|_{B^{1+s/2}_{2p,2p}} \leq C\|\gamma\|^{\Theta}_{B^{1+s}_{2p,2p}} \|\gamma\|^{1-\Theta}_{B^{t}_{2p,2p}}. && (1.18)\end{aligned}$$

Since $L^1 \hookrightarrow B^0_{1,\infty} \hookrightarrow B^t_{2p,2p}$, this yields

$$\|\gamma\|_{W^{1,2p}} \leq C\|\gamma\|^{\Theta}_{B^{1+s}_{2p,2p}} \|\gamma\|^{1-\Theta}_{L^1}.$$

Using Young's inequality, for all $\varepsilon > 0$, we get

$$\begin{aligned}\|\gamma\|^{2p}_{W^{1,2p}} &\leq C\|\gamma\|^{2p\theta}_{W^{1+s,2p}} \|\gamma\|^{2p(1-\Theta)}_{L^1} && (1.19)\\ &\leq C\varepsilon^{1/\Theta}\Theta\|\gamma\|^{2p}_{W^{1+s,2p}} + C(1-\Theta)\frac{\|\gamma\|^{2p}_{L^1}}{\varepsilon^{1/(1-\Theta)}}. && (1.20)\end{aligned}$$

Therefore, for sufficient small $\varepsilon > 0$, we conclude that

$$\|\gamma\|^{2p}_{W^{1+s,2p}} \leq C'\left(\mathcal{E}^{\alpha,p}(\gamma) + \|\gamma\|^{2p}_{L^1}\right). \tag{1.21}$$

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of $TL^q(\mathbb{R}/\mathbb{Z})$ with

$$\sup_{n\in\mathbb{N}} R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma_n) < \infty,$$

and let $T_n : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}/\mathcal{L}\mathbb{Z}$ be transportation maps with $T_{n\#}\mu = \mu_n$. Then

$$\sup_{n\in\mathbb{N}} \mathcal{E}^{\alpha,p}_{\rho}(\gamma_n \circ T_n) < \infty.$$

Since ρ is bounded from below by a positive constant we deduce by Lemma 1.4.3 that

$$\sup_{n\in\mathbb{N}} \left(\mathcal{E}^{\alpha,p}(\gamma_n \circ T_n) + \|\gamma_n \circ T_n\|_{L^1}\right) < \infty.$$

Therefore by (1.21) and Lemma 1.4.2, we see

$$\sup_{n\in\mathbb{N}} \|\gamma_n \circ T_n\|_{W^{1+s,2p}} < \infty.$$

Since $1 + s - \frac{1}{2p} = \frac{2p+\alpha p-2}{2p} \geq 0 > -\frac{1}{q}$, Theorem 1.4.1 yields a compact embedding

$$\iota : W^{1+s,2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \hookrightarrow\hookrightarrow L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d).$$

Therefore there exists an $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ and $\gamma \in L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}, \rho)$ such that $\gamma_{n_k} \circ T_{n_k} \to \gamma$ in $L^q(\mathbb{R}/\mathbb{Z}, \rho)$. By Proposition 1.2.2 we see $\gamma_{n_k} \to \gamma$ in $TL^q(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

□

Chapter 2

A finer singular limit of a single-well Modica-Mortla functional on one dimensional domain

An explicit representation of the Gamma limit of a single-well Modica–Mortola functional is given for one-dimensional space under the graph convergence which is finer than conventional L^1-convergence or convergence in measure. As an application, an explicit representation of a singular limit of the Kobayashi–Warren–Carter energy, which is popular in materials science, is given. Some compactness under the graph convergence is also established. Such formulas as well as compactness are useful to characterize the limit of minimizers of the Kobayashi–Warren–Carter energy. To characterize the Gamma limit under the graph convergence, a new idea which is especially useful for one-dimensional problem is introduced. It is a change of parameter of the variable by arc-length parameter of its graph, which is called unfolding by the arc-length parameter in this chapter.

2.1 Introduction

In this chapter, we are interested in a singular limit called the Gamma limit of a single-well Modica–Mortola functional under the graph convergence, the convergence with respect to the Hausdorff distance of graphs, which is finer than conventional L^1-convergence or convergence in measure. A single-well Modica–Mortola functional is introduced by Ambrosio and Tortorelli [3, 4] to approximate the Mumford–Shah functional [29]. A typical explicit form of their functional now called the Ambrosio–Tortorelli functional is

$$\mathcal{E}^\varepsilon(u,v) := \sigma \int_\Omega v^2 \left|\nabla u\right|^2 \,\mathrm{d}x + \lambda \int_\Omega (u-g)^2 \,\mathrm{d}x + E^\varepsilon(v),$$

with small parameter $\varepsilon > 0$, where E^ε is a single-well Modica–Mortola functional of the form

$$E^\varepsilon(v) := \frac{1}{2\varepsilon} \int_\Omega (v-1)^2 \,\mathrm{d}x + \frac{\varepsilon}{2} \int_\Omega \left|\nabla v\right|^2 \,\mathrm{d}x.$$

Here g is a given function defined in a bounded domain Ω in $\mathbb{R}^n$ and $\sigma \geq 0$, $\lambda \geq 0$ are given parameters. The potential energy part $(v-1)^2$ is a single-well potential. If it is replaced by a double-well potential like $(v^2-1)^2$, the corresponding energy E^ε well approximates (a constant multiple of) the surface area of the interface and this observation went back to Modica and Mortola [27, 28]. Even for the single-well potential if v is close to zero around some interface then it is expected that E^ε still approximates the surface area of the interface. This observation enables us to prove that for $\sigma > 0$, the Gamma limit of $\mathcal{E}^\varepsilon(u, v)$ in the convergence in measure is a Mumford–Shah functional; see [3, 4, 14].

If $E^\varepsilon(v_\varepsilon)$ is bounded for small $\varepsilon > 0$, then it is rather clear that $v_\varepsilon \to 1$ in L^1 as $\varepsilon \to 0$, so that $v_\varepsilon \to 1$ almost everywhere by taking a suitable subsequence. Therefore, it seems natural to consider the Gamma convergence in L^1-sense. However, if one considers

$$E_b^\varepsilon(v) = E^\varepsilon(v) + bv(0)^2 \tag{2.1}$$

for $b > 0$, where $\Omega = (-1, 1)$, then we see L^1-convergence is too weak because in the limit stage, the effect of the term involving b is invisible but this should be counted.

To illustrate the point, we calculate the unique minimizer w_ε of $E_b^\varepsilon(v)$, that is,

$$E_b^\varepsilon(w_\varepsilon) = \min\left\{E_b^\varepsilon(v) \mid v \in H^1(-1, 1)\right\}.$$

This is strict convex problem so that the minimizer exists and is unique. Moreover, its Euler–Lagrange equation is linear. A simple manipulation shows that the minimizer of E_b^ε with the Neumann boundary conditions $w_\varepsilon'(\pm 1) = 0$ is given by

$$w_\varepsilon(x) = 1 + \frac{b\left(-e^{-\frac{2}{\varepsilon}} - 1\right)}{1 - e^{-\frac{4}{\varepsilon}} + b\left(1 + e^{-\frac{2}{\varepsilon}}\right)^2} e^{-\frac{|x|}{\varepsilon}} + \frac{b\left(-e^{-\frac{2}{\varepsilon}} - e^{-\frac{4}{\varepsilon}}\right)}{1 - e^{-\frac{4}{\varepsilon}} + b\left(1 + e^{-\frac{2}{\varepsilon}}\right)^2} e^{\frac{|x|}{\varepsilon}}.$$

It converges to 1 locally uniformly outside zero but

$$\lim_{\varepsilon \to 0} w_\varepsilon(0) = \frac{1}{1+b} > 0,$$

and

$$\lim_{\varepsilon \to 0} E_b^\varepsilon(w_\varepsilon) = \frac{b}{1+b} < b.$$

Since $E_b^\varepsilon(1) = b$ for any $\varepsilon > 0$, the information that $w_\varepsilon(x) \to 1$ almost everywhere is insufficient to identify the behavior of minimizers w_ε.

We show the graph of w_ε for several $\varepsilon > 0$ in Figure 2.1. We see that the graph of w_ε is dropping sharply at $x = 0$ and its sharpness increases as $\varepsilon \to 0$. Hence, it is natural to consider the graph convergence of w_ε and its limit is a set-valued function Ξ so that $\Xi(x) = \{1\}$ for $x \neq 0$ and $\Xi(0) = [1/(1 + b), 1]$.

Our first goal is to give an explicit representation formula for the Gamma limit of E_b^ε under the graph convergence as well as compactness. We discuss such problems only in one-dimensional domain since the problem is already complicated. The graph convergence enables us to characterize the limit of above w_ε as a minimizer of the Gamma limit of E_b^ε.

Our second goal is to give an explicit representation formula for the Gamma limit of the Kobayashi–Warren–Carter energy. A typical form of the energy is

$$E_{\mathrm{KWC}}^\varepsilon(u, v) = \sigma \int_\Omega v^2 |\nabla u| + E^\varepsilon(v),$$

where $\int v^2 |\nabla u|$ denotes a weighted total variation of a Radon measure ∇u; see Section 5 for a precise definition. Here and hereafter we suppress dx unless ∇u is absolutely continuous with respect to the Lebesgue measure. This energy is first proposed by [21, 20] to model motion of multi-phase problems in materials sciences. This energy looks similar

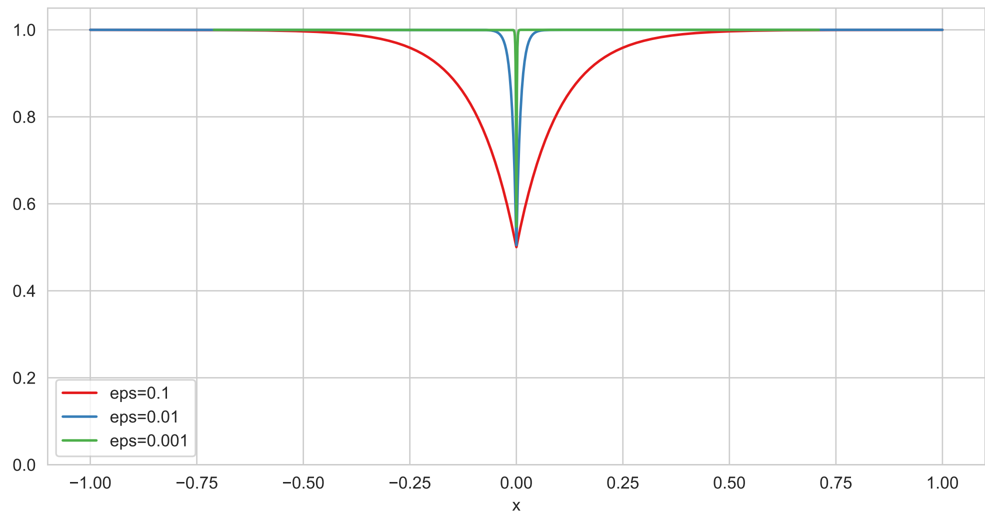

Figure 2.1: The graphs of w_ε as the minimizers of E_b^ε defined by (2.1) when $b = 1$ and $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$.

to the Ambrosio–Tortorelli functional $\mathcal{E}^\varepsilon$. It is obtained by inhomogenizing Dirichlet energy $\int |\nabla u|^2 \, dx$ by putting weights $\int v^2 |\nabla u|^2 \, dx$ with a single-well Modica–Mortola functional. By this observation, we call $\mathcal{E}^\varepsilon$ an Ambrosio–Tortorelli inhomogenization of the Dirichlet energy when $\lambda = 0$. From this point of view, the Kobayashi–Warren–Carter energy is interpreted as an Ambrosio–Tortorelli inhomogenization of the total variation. It turns out that natural topology for studying the limit of functionals as $\varepsilon \to 0$ is quite different.

For the Ambrosio–Tortorelli functional, it is enough to consider the $L^1 \times L^1$ convergence since $v_\varepsilon(x) \to 1$ except at most finitely many points where $\liminf_* v_\varepsilon(x) = 0$ if one assumes that $\mathcal{E}^\varepsilon(u_\varepsilon, v_\varepsilon)$ is bounded and $u_\varepsilon \to u$, $v_\varepsilon \to v$ in L^1. (see [3, 4, 14].) Here $\liminf_*$ denotes the relaxed liminf and we shall give its definition in Section 2.2. For the Kobayashi–Warren–Carter energy, however, the situation is quite different. Indeed, if one considers

$$u(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & -1 < x < 0, \end{cases}$$

then $E^\varepsilon_{\mathrm{KWC}}(u, v) = E^\varepsilon_\sigma(v)$ with $\Omega = (-1, 1)$. Thus the natural convergence for v must be in the graph convergence as we discussed before. Note that in our problem $v_\varepsilon \to 1$ except at most countably many points, where $\liminf_* v_\varepsilon$ may be nonzero. One merit of the graph convergence is that it is very strong so when we consider the Gamma limit problem, we don't need to restrict ourselves in the space of special BV functions as for the Ambrosio–Tortorelli functional.

Our first main result is a characterization of the Gamma limit of E_b^ε in the graph convergence (Theorem 2.2.1). To show the Gamma convergence, we need to prove the two types of inequalities often called liminf and limsup inequalities. To show liminf inequality, a key point is to study a general behavior near the set Σ of all exceptional points of the limit set-valued function Ξ; here, we say a point x is exceptional if $\Xi(x)$ is not a singleton. To describe behavior near Σ, a conventional method is to find a suitable accumulating sequence as in [14, proof of Proposition 3.3]. However, unfortunately, it seems that this argument does not apply to our setting, since Σ can be a countably infinite set. Thus we are forced to introduce a new method to show liminf inequality. When we study a absolutely continuous function u^ε on a bounded interval I, that is, $u^\varepsilon \in W^{1,1}(I)$, we associate its unfolding U^ε by replacing the variable by the arc-length parameter of the graph. Namely, we set

$$U^\varepsilon(s) = u^\varepsilon\left(x^\varepsilon(s)\right), \quad s \in J^\varepsilon = s^\varepsilon(I),$$

where $x^\varepsilon = x^\varepsilon(s)$ in the inverse function of the arc-length parameter

$$s^\varepsilon(x) = \int_0^x \left(1 + (u_x^\varepsilon(z))^2\right)^{1/2} dz.$$

If the total variation of u^ε is bounded, then the length of J^ε is bounded as $\varepsilon \to 0$. The unfolding U^ε has several merits compared with the original one. First, $\{U^\varepsilon\}$ and $\{x^\varepsilon\}$ are uniformly Lipschitz with constant 1. Second, the total variation of U^ε and u^ε are the same as expected. It is easy to study the convergence as $\varepsilon \to 0$ of unfolding U^ε compared with the original u^ε. Among other results, we are able to characterize the relaxed limits $\liminf_* u^\varepsilon$, $\limsup^* u^\varepsilon$ by the limit of U^ε and x^ε. We use this unfolding for $(v_\varepsilon - 1)^2/2$ in the case of E_b^ε to show liminf inequalities, where $\{v_\varepsilon\}$ is a given sequence with a bound for $E_b^\varepsilon(v_\varepsilon)$. The proof for limsup inequalities is not difficult although one has to be careful that there are countably many points where the limit of v^ε is not equal to one.

We also established a compactness under the graph convergence with a bound for E_b^ε (Theorem 2.2.2). This can be easily proved by use of unfoldings.

Based on results on E_b^ε, we are able to prove the Gamma convergence of the Kobayashi–Warren–Carter energy $E_{\mathrm{KWC}}^\varepsilon$ under the graph convergence (Theorem 2.2.3). If u is a piecewise constant function with countably many jump points $\{a_\ell\}_{\ell=1}^\infty \subset \Omega$ with positive jump $\{b_\ell\}_{\ell=1}^\infty$, we see that

$$E_{\mathrm{KWC}}^\varepsilon(u, v) = E^\varepsilon(v) + \sigma \sum_{\ell=1}^\infty b_\ell v^2(a_\ell).$$

The Gamma limit for such fixed u is easily reduced to the results of E_b^ε. However, to establish liminf inequality for $E_{\mathrm{KWC}}^\varepsilon$ for both u_ε and v_ε, we have to establish some lower estimate for a sequence $\int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon| \,\mathrm{d}x$ as $\varepsilon \to 0$, which is an additional difficulty. However, we still do not need to use SBV space here.

The Gamma convergence problem of the Modica–Mortola functional, which is the sum of Dirichlet type energy and potential energy was first studied by [27]. Since then, there is a large number of works discussing the Gamma convergence. However, the topology is either L^1 or convergence in measure. In our Gamma limit, the topology is the graph convergence, which is finer than previous studies. In [28], the L^1 Gamma limit of a double-well Modica–Mortola functional is characterized as a number of transition points in one-dimensional setting. Later in [26, 36], it was extended to multi-dimensional setting and the limit is a constant multiple of the surface area of the transition interface. This type of the Gamma convergence results as well as compactness is important to establish the convergence of local minimizer ([23]) as well as the global minimizer. However, the convergence of critical points are not in the framework of a general theory and a special treatment is necessary [17]. The double-well Modica–Mortola functional is by now well studied even in the level of gradient flow called the Allen–Cahn equation. The limit

$\varepsilon \to 0$ is often called the sharp interface limit and the resulting flow is known as the mean curvature flow. For early stage of development of the theory, see [7, 8, 9, 10].

A single-well Modica–Mortola functional is first used in [3] to approximate the Mumford–Shah functional. The Gamma limit of the Ambrosio–Tortorelli functional is by now well studied ([3, 4, 14]). However, convergence of critical points is studied only in one dimension ([12]). The Ambrosio–Tortorelli type approximation is now used in various problems. In [13], the Ambrosio–Tortorelli type approximation is introduced to describe brittle fractures. Its evolution is also described in [15]. For the Steiner problem, such approximation as also proposed ([24]) and its Gamma limit is established ([5]). However, all these problems is closer to the Ambrosio–Tortorelli inhomogenization of the Dirichlet energy, not of the total variation.

For the Kobayashi–Warren–Carter energy, its gradient flow for fixed ε is somewhat studied. Note that the well-posedness itself is non-trivial because even if one assumes $v \equiv 1$, the gradient flow of $E^{\varepsilon}_{\mathrm{KWC}}$ is the total variation flow and the definition of a solution itself is non trivial; see [19], for example. Apparently, there is no well-posedness result for the original system proposed by [20, 21, 22]. According to [21], its explicit form is

$$\tau_1 v_t = s\Delta v + (1-v) - 2sv\,|\nabla u|\,, \tag{2.2}$$

$$\tau_0 v^2 u_t = s\,\mathrm{div}\left(v^2 \frac{\nabla u}{|\nabla u|}\right), \tag{2.3}$$

where τ_0, τ_1, s are positive parameters. This system is regarded as the gradient flow of $E^{\varepsilon}_{\mathrm{KWC}}$ with $F(v) = (v-1)^2$, $\varepsilon = 1$, $\sigma = s$ with respect to a kind of weighted L^2 norm whose weight depends on the solution. If one replaces (2.3) by

$$\tau_0(v^2+\delta)u_t = s\ \mathrm{div}\left((v^2+\delta')\frac{\nabla u}{|\nabla u|} + \nu\nabla u\right)$$

with $\delta > 0$, $\delta' \geq 0$, and $\nu \geq 0$ satisfying $\delta' + \nu > 0$, then the studies of existence and large-time behavior of solutions are developed in [18, 30, 31, 33, 34, 35], under homogeneous settings of boundary conditions. However, the uniqueness question is almost open, and there is a few (only one) result [18, Theorem 2.2] for the one-dimensional solution, under $\nu > 0$. Meanwhile, the line of previous results can be extended to the studies of non-homogeneous cases of boundary conditions. For instance, if we impose the non-homogeneous Dirichlet boundary condition for (2.3), then we can further observe various structural patterns of steady-state solutions, under one-dimensional setting, two-dimensional radially-symmetric setting, and so on (cf. [32]).

This paper is organized as follows. In Section 2.2, we recall notion of the graph convergence and states our main Gamma convergence results as well as compactness. In Section 2.3, we introduce notion of unfoldings. Section 2.4 is devoted to the proof of the Gamma convergence of E^{ε}_{b} as well as the compactness in the graph convergence. Section 2.5 is devoted to the proof of the Gamma convergence of the Kobayashi–Warren–Carter energy.

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2.2 Singular limit under graph convergence

We first recall basic notion of set-valued functions; see [1] for example. Let (M, d_M) be a compact metric space. We consider a set-valued function Γ defined in M such that $\Gamma(x)$ is a compact set in $\mathbb{R}$ for each $x \in M$. If its graph Γ defined by

$$\operatorname{graph}\Gamma := \left\{(x,y) \in M \times \mathbb{R} \;\middle|\; y \in \Gamma(x),\ x \in M\right\}$$

is closed, we say that Γ is *upper semicontinuous*. Let $\mathcal{B}$ denote the totality of a bounded, upper semicontinuous set-valued functions. In other words,

$$\mathcal{B} := \{\Gamma \mid \operatorname{graph}\Gamma \text{ is compact in } M \times \mathbb{R}\}.$$

For $\Gamma_1, \Gamma_2 \in \mathcal{B}$, we set

$$d_g(\Gamma_1, \Gamma_2) := d_H(\operatorname{graph}\Gamma_1, \operatorname{graph}\Gamma_2),$$

where d_H denotes the Hausdorff distance of two sets in $M \times \mathbb{R}$. The Hausdorff distance d_H is defined as usual:

$$d_H(A,B) := \max\left\{\sup_{z\in A} \operatorname{dist}(z,B),\ \sup_{w\in B} \operatorname{dist}(w,A)\right\}$$

for $A, B \subset M \times \mathbb{R}$, where

$$\operatorname{dist}(z,B) := \inf_{w\in B} \operatorname{dist}(z,w), \quad \operatorname{dist}(z,w) := \left(d_M(z_1,w_1)^2 + |z_2 - w_2|^2\right)^{1/2}$$

for $z = (z_1, z_2)$ and $w = (w_1, w_2)$. It is easy to see that $(\mathcal{B}, d_g)$ is a complete metric space. The convergence with respect to d_g is called the *graph convergence*.

We next recall the notion of semi-convergent limit for sets. For a family of closed subsets $\{Z_\varepsilon\}_{0<\varepsilon<1}$ in $M \times \mathbb{R}$, we set

$$\limsup_{\varepsilon\to 0} Z_\varepsilon := \bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{0<\delta<\varepsilon} Z_\delta\right),$$

$$\liminf_{\varepsilon\to 0} Z_\varepsilon := \operatorname{cl}\left(\bigcup_{\varepsilon>0} \bigcap_{0<\delta<\varepsilon} Z_\delta\right),$$

where cl denotes the closure in $M \times \mathbb{R}$. These semi-limits can be defined for sequences like $\{Z_j\}_{j=1}^\infty$ with trivial modification.

Lemma 2.2.1. A sequence $\{\Gamma_j\}_{j=1}^\infty \subset \mathcal{B}$ converges to Γ in the sense of the graph convergence if and only if

$$\limsup_{j\to\infty} \operatorname{graph}\Gamma_j = \liminf_{j\to\infty} \operatorname{graph}\Gamma_j = \operatorname{graph}\Gamma.$$

Proof. Note that the Hausdorff convergence to A for a sequence $\{A_j\}_{j=1}^\infty$ of compact sets is equivalent to saying that

(i) for any $z \in A$, there is a sequence $z_j \in A_j$ such that $z_j \to z$ $(j \to \infty)$ and

(ii) if $w_j \in A_j$ converges to w, then $w \in A$.

Since (i) and (ii) are equivalent to

$$\liminf_{j\to\infty} A_j \supset A, \ \limsup_{j\to\infty} A_j \subset A,$$

respectively, the Hausdorff convergence is equivalent to saying that

$$A = \liminf_{j\to\infty} A_j = \limsup_{j\to\infty} A_j.$$

Thus the proof is complete. □

We next recall relaxed convergent limits of functions. Let $\{g_j\}$ be a sequence of real-valued function on M. For $x \in M$, We set

$$\begin{aligned} \limsup_{j\to\infty}{}^* g_j(x) &:= \lim_{j\to\infty} \sup\bigl\{g_k(y) \bigm| |y-x| < 1/j, \ k \geq j\bigr\}, \\ \liminf_{j\to\infty}{}_* g_j(x) &:= \lim_{j\to\infty} \inf\bigl\{g_k(y) \bigm| |y-x| < 1/j, \ k \geq j\bigr\}; \end{aligned}$$

see [16, Chapter 2] for more detail. By definition, the $\limsup^* w_j$ is upper semicontinuous and $\liminf_* w_j$ is lower semicontinuous.

Let $C(M)$ be the Banach space of all continuous real-valued functions on M equipped with the norm $\|f\|_\infty = \sup_{x\in M} |f(x)|$, $f \in C(M)$. For $g \in C(M)$, we associate a set-valued function Γ_g such that $\Gamma_g(x) = \{g(x)\}$ for $x \in M$. Clearly, $\Gamma_g \in \mathcal{B}$.

Lemma 2.2.2. Let $\{g_j\}_{j=1}^\infty \subset C(M)$ be a bounded sequence. Then the semi-limit $\Gamma_+ = \limsup_{j\to\infty} \Gamma_{g_j}$ still belongs to $\mathcal{B}$. Let K be the set-valued function of the form

$$K(x) := \left\{ y \in \mathbb{R} \,\middle|\, \liminf_{j\to\infty}{}_* g_j(x) \leq y \leq \limsup_{j\to\infty}{}^* g_j(x) \right\}.$$

Then $\Gamma_+(x) \subset K(x)$ for all $x \in M$.

Proof. The first statement is trivial. To prove $\Gamma_+ \subset K$, it suffices to prove that the limit $y = \lim_{j\to\infty} y_j$, $y_j \in \Gamma_{g_j}(x_j)$ belongs to $K(x)$ if $x_j \to x$. Since $y_j = g_j(x_j)$, by definition of relaxed limits $\limsup^*$ and $\liminf_*$ it is easy to see that

$$\liminf_{j\to\infty}{}_* g_j(x) \leq y \leq \limsup_{j\to\infty}{}^* g_j(x).$$

Thus $\Gamma_+(x) \subset K(x)$. □

We next discuss an equivalent condition of the graph convergence.

Lemma 2.2.3. 1. Let $\{g_j\}_{j=1}^\infty \subset C(M)$ be a bounded sequence. Then the semi-limit $\Gamma_- = \liminf_{j\to\infty} \Gamma_{g_j}$ belongs to $\mathcal{B}$.

2. Assume that M is locally arcwise connected. For each $x \in M$ following three conditions are equivalent.

   (a) $\Gamma_-(x)$ contains both $\liminf_* g_j(x)$ and $\limsup^* g_j(x)$.

(b) $K(x) \subset \Gamma_-(x)$.

(c) $\Gamma_-(x) = \Gamma_+(x) = K(x)$.

In particular, Γ_{g_j} converges to K in the graph sense if and only if one of (a), (b), (c) holds for all $x \in M$.

Proof. (1) follows from the definition and we focus on the proof of (2). Assume (a) so that $\Gamma_-(x)$ contains $\partial(K(x))$. Then there exist $x_j \in M$, $y_j = g_j(x_j)$ such that $x_j \to x$, $y_j \to \hat{y}$ for $\hat{y} = \liminf_* g_j(x)$ and that there exists $\overline{x}_j \in M$, $\overline{y}_j \in g_j(\overline{x}_j)$ such that $\overline{x}_j \to x$, $\overline{y}_j \to \overline{y}$ for $\overline{y} = \limsup^* g_j(x)$.

By assumption, for any $\delta > 0$ there exists an arc γ_j connecting x_j to $\overline{x}_j$, lying in a δ-neighborhood B_δ of x provided that j is sufficiently large. Since g_j is continuous on $\gamma_j \subset B_\delta$, the intermediate value theorem implies that $\left[y_j, \overline{y}_j\right] \subset g_j(B_\delta)$. Thus $K(x) \subset \Gamma_-(x)$. Thus (b) follows.

Assume (b). By Lemma 2.2.2, we know $\Gamma_+(x) \subset K(x)$. By definition of Γ_- we see $\Gamma_- \subset \Gamma_+$. Thus, we conclude (c). It is easy to see that (c) implies (a). The proof is now complete. □

We next consider an important subclass of $\mathcal{B}$. Let $\mathcal{A}$ be the family of $\Gamma \in \mathcal{B}$ such that $\Gamma(x)$ is a closed interval for all $x \in M$. Let $\mathcal{A}_0$ be the subfamily of $\mathcal{A}$ such that $\Gamma(x)$ is the singleton $\{1\}$ with at most countably many exceptions of $x \in M$. Such Γ is uniquely determined by $\{x_i\}_{i=1}^\infty$ where $\Gamma(x_i) = \left[\xi_i^-, \xi_i^+\right]$ with $\xi_i^- < \xi_i^+$ contains 1 and $\Gamma(x) = \{1\}$ if $x \notin \{x_i\}_{i=1}^\infty$. We call such a point x_i *an exeptional point* of $\Xi \in \mathcal{A}_0$, so that Σ is the set of all exceptional points of Ξ.

We next study the compactness in the graph convergence.

Lemma 2.2.4. Let $\{g_j\}_{j=1}^\infty \subset C(M)$ be a bounded sequence. Assume that

$$\begin{aligned} \eta^-(x) &< \eta^+(x) && \text{for } x \in S, \\ \eta^-(x) &= \eta^+(x) = 1 && \text{for } x \in M \setminus S, \end{aligned}$$

where S is a countable set and

$$\eta^-(x) = \liminf_{j\to\infty}{}_* \, g_j(x), \ \ \eta^+(x) = \limsup_{j\to\infty}{}^* \, g_j(x).$$

If $1 \in [\eta^-(x), \eta^+(x)]$, then there is a subsequence $\{g_{j_k}\}$ such that $\Gamma_{g_{j_k}}$ converges to some $\Gamma_0 \in \mathcal{A}_0$ in the graph sense.

Proof. We write $S = \{x_i\}_{i=1}^\infty$. By definition, there is a subsequence $\{g_{1,j}^-\}$ of $\{g_j\}$ such that

$$\eta^-(x_1) = \lim_{j\to\infty} g_{1,j}^-(y_{1,j})$$

with some $\{y_{1,j}\}$ converging to x_1. We set

$$\eta_1^+(x_1) := \limsup_{j\to\infty}{}^* \, g_{1,j}^-(x_1) \le \eta^+(x_1).$$

Since $\eta^- = \eta^+ = 1$ outside S, we see $\eta^+(x_1) \ge 1$. We take a further subsequence $\{g_{1,j}\}$ of $\{g_{1,j}^-\}$ so that

$$\eta_1^+(x_1) = \lim_{j\to\infty} g_{1,j}(z_{1,j})$$

with some $\{z_{1,j}\}$ converging to x_1. We repeat this procedure for $x_2, x_3, \dots$ and find a subsequence $\{g_{\ell,j}\}_{j=1}^{\infty}$ so that

$$\lim_{j\to\infty} g_{\ell,j}(y_{\ell,j}) = \liminf_{j\to\infty}{}_* \, g_{\ell,j}(x_\ell) \le 1,$$
$$\lim_{j\to\infty} g_{\ell,j}(z_{\ell,j}) = \limsup_{j\to\infty}{}^* \, g_{\ell,j}(x_\ell) \ge 1$$

with some $\{y_{\ell,j}\}$, $\{z_{\ell,j}\}$ converging to x_ℓ for $\ell = 1, 2, \dots, k$. By a diagonal argument, we see that $\{g_{k,k}\}_{k=1}^{\infty}$ has the property that

$$\xi^-(x) := \liminf_{k\to\infty}{}_* \, g_{k,k}(x), \ \xi^+(x) := \limsup_{k\to\infty}{}^* \, g_{k,k}(x)$$

belong to $\Gamma_-(x) = \liminf_{k\to\infty} \Gamma_{g_{k,k}}$ for $x \in M$. We now apply Lemma 2.2.3(2) to conclude that $\Gamma_{g_{k,k}}$ converges to Γ with

$$\Gamma(x) = [\xi^-(x), \xi^+(x)], \ x \in M.$$

By construction, $\Gamma(x) = \{1\}$ for $x \in M \setminus S$ and $\xi^-(x) \le 1 \le \xi^+(x)$ for $x \in S$. Thus, $\Gamma \in \mathcal{A}_0$ so the proof is now complete. □

We now define several functionals when $M = \overline{I}$ or $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, where I is a bounded open interval in $\mathbb{R}$ and $\overline{I} = \operatorname{cl} I$. For a real-valued function v on M and $\varepsilon > 0$, a single-well Modica–Mortola functional is defined by

$$E^{\varepsilon}_{\mathrm{sMM}}(v) := \frac{\varepsilon}{2} \int_M \left| \frac{\mathrm{d}v}{\mathrm{d}x} \right|^2 \mathrm{d}x + \frac{1}{2\varepsilon} \int_M F(v)\, \mathrm{d}x.$$

Here the potential energy F is a single-well potential. We shall assume that

(F1) $F \in C(\mathbb{R})$ is nonnegative and $F(v) = 0$ if and only if $v = 1$;

(F2) $\liminf_{|v|\to\infty} F(v) > 0$;

(F2') (growth condition) there are positive constants c_0, c_1 such that

$$F(v) \ge c_0 |v|^2 - c_1 \quad \text{for all} \quad v \in \mathbb{R}.$$

Remark 2.2.1. Obviously, (F2') implies (F2).

We are interested in a Gamma limit of $E^{\varepsilon}_{\mathrm{sMM}}$ not in usual L^1-convergence but the graph convergence which is of course finer than L^1 topology. As usual, we set

$$G(v) = \left| \int_1^v \sqrt{F(\tau)} d\tau \right|.$$

A typical example of $F(v)$ is $F(v) = (v-1)^2$. In this case,

$$G(v) = (v-1)^2/2.$$

To write the limit energy for $\Xi \in \mathcal{A}_0$, let $\Sigma = \{x_i\}_{i=1}^{\infty}$ denote the totality of points where $\Xi(x_i)$ is a nontrivial closed interval $\left[\xi_i^-, \xi_i^+\right]$ such that $\xi_i^- \le 1 \le \xi_i^+$. This set can be a finite set. By definition, $\Xi(x) = \{1\}$ if $x \notin \{x_i\}_{i=1}^{\infty}$. In the case that $M = \mathbb{T}$, we define

$$E^0_{\mathrm{sMM}}(\Xi, \mathbb{T}) := \begin{cases} 2 \displaystyle\sum_{i=1}^{\infty} \left\{ G(\xi_i^-) + G(\xi_i^+) \right\} & \text{for } \Xi \in \mathcal{A}_0, \\ \infty & \text{otherwise.} \end{cases}$$

In the case that $M = \overline{I}$, one has to modify the value when x_i is the end point of $\overline{I}$. The energy is defined by

$$E^0_{\mathrm{sMM}}(\Xi, \overline{I}) := \begin{cases} \displaystyle\sum_{i=1}^{\infty} \left\{2(G(\xi_i^-) + G(\xi_i^+)) - \kappa_i \max(G(\xi_i^-), G(\xi_i^+))\right\} & \text{for } \Xi \in \mathcal{A}_0, \\ \infty & \text{otherwise,} \end{cases}$$

where $\kappa_i = 0$ if $x_i \in I$ and $\kappa_i = 1$ if $x_i \in \partial I$. For brevity we simply write $v_j \xrightarrow{g} \Xi$ if $v_j \in C(M)$ is a sequence such that $v_j \to \Xi$ $(j \to \infty)$ in the sense of the graph convergence. We also use $v_\varepsilon \xrightarrow{g} \Xi$ as $\varepsilon \to 0$ if ε is a continuous parameter.

We shall state that the Gamma limit of $E^\varepsilon_{\mathrm{sMM}}$ is E^0_{sMM} as $\varepsilon \to 0$ under the graph convergence. For later applications, it is convenient to consider a slightly general functional of form $E^{\varepsilon,b}_{\mathrm{sMM}}(v) := E^\varepsilon_{\mathrm{sMM}}(v) + b\alpha\,(v(a))$, where $\alpha \in C(\mathbb{R})$ with $\alpha \geq 0$ and $a \in \mathring{M} = \operatorname{int} M$ and $b \geq 0$. The corresponding limit functional is

$$E^{0,b}_{\mathrm{sMM}}(\Xi, M) := E^0_{\mathrm{sMM}}(\Xi, M) + b \min_{\xi \in \Xi(a)} \alpha\,(\xi)\,.$$

Theorem 2.2.1 (Gamma limit under graph convergence)**.** Assume the following conditions:

- $M = \overline{I}$ or $\mathbb{T}$;
- F satisfies (F1) and (F2);
- $a \in \mathring{M} = \operatorname{int} M$ and $b \geq 0$.

Then the following inequalities hold:

(i) (liminf inequality) Let $\{v_\varepsilon\}_{0<\varepsilon<1}$ be in $H^1(M) \subset C(M)$. If $v_\varepsilon \xrightarrow{g} \Xi \in \mathcal{B}$, then

$$E^{0,b}_{\mathrm{sMM}}(\Xi, M) \leq \liminf_{\varepsilon \to 0} E^{\varepsilon,b}_{\mathrm{sMM}}(v_\varepsilon).$$

In particular, if the right-hand side is finite, then $\Xi \in \mathcal{A}_0$.

(ii) (limsup inequality) For any $\Xi \in \mathcal{A}_0$, there is $\{w_\varepsilon\}_{0<\varepsilon<1} \subset H^1(M) \subset C(M)$ such that $w_\varepsilon \xrightarrow{g} \Xi$ and

$$E^{0,b}_{\mathrm{sMM}}(\Xi, M) = \lim_{\varepsilon \to 0} E^{\varepsilon,b}_{\mathrm{sMM}}(w_\varepsilon).$$

We also have a compactness result.

Theorem 2.2.2 (Compactness)**.** Assume that $M = \overline{I}$ or $\mathbb{T}$. Assume that F satisfies (F1) and (F2'). Let $\{v_{\varepsilon_j}\}_{j=1}^\infty$ be in $H^1(M) \subset C(M)$. Assume that

$$\sup_j E^{\varepsilon_j}_{\mathrm{sMM}}(v_{\varepsilon_j}) < \infty$$

for $\varepsilon_j \to 0$ as $j \to \infty$. Then there exists a subsequence $\left\{v_{\varepsilon'_k}\right\}$ such that $v_{\varepsilon'_k} \xrightarrow{g} \Xi$ with some $\Xi \in \mathcal{A}_0$.

By combining the Gamma convergence result and the compactness, a general theory yields the convergence of a minimizer of $E^{\varepsilon,b}_{\mathrm{sMM}}$; see [6, Theorem 1.21] for example. Note that in the case of $b = 0$, the minimum of $E^\varepsilon_{\mathrm{sMM}}(v)$ is zero and is attained only at constant function $v = 1$ so the convergence of minimizers is trivial.

Corollary 2.2.1. Assume the same hypotheses of Theorem 2.2.1 and (F2'). Let v_ε be a minimizer of $E^{\varepsilon,b}_{\mathrm{sMM}}$ on $H^1(M)$. Then there is a subsequence $\{v_{\varepsilon_k}\}_{k=1}^\infty$ such that $v_{\varepsilon_k} \xrightarrow{g} \Xi_0$ with some $\Xi_0 \in \mathcal{A}_0$. Moreover, Ξ_0 is a minimizer of $E^{0,b}_{\mathrm{sMM}}$. Furthermore, $\Xi_0(x) = \{1\}$ if $x \neq a$ and $\Xi_0(a) = [p_0, 1]$, where p_0 is a minimizer of $2G(p) + bp^2$ with $p \in [0,1]$.

Remark 2.2.2. If $F'(v)(v-1) \geq 0$, then G is convex so that $2G(p)+bp^2$ is strictly convex for $b > 0$. In this case, the minimizer is unique. If $F(v) = (v-1)^2$ so that $G(v) = (v-1)^2/2$, then $2G(p) + bp^2 = (p-1)^2 + bp^2$ and its minimizer is $1/(b+1)$ and its minimal value is $E^{0,b}_{\mathrm{sMM}}(\Xi_0, M) = b/(b+1)$.

Our theory has an application to the Kobayashi–Warren–Carter energy [20, 21, 22] which can be interpreted as an Ambrosio–Tortorelli inhomogenization of the total variation energy. Its typical form is

$$E^\varepsilon_{\mathrm{KWC}}(u,v) := \sigma \int_{\mathring{M}} \alpha(v) \left| \frac{\mathrm{d}u}{\mathrm{d}x} \right| + E^\varepsilon_{\mathrm{sMM}}(v)$$

for a given $\alpha \in C(\mathbb{R})$ with $\alpha \geq 0$ and a constant $\sigma \geq 0$. The first integral denotes the total variation of u with weight $\alpha(v)$. See Section 2.5 for more rigorous definition. Note that if $u_x = 0$ outside a and u jumps at a with jump 1, then

$$E^\varepsilon_{\mathrm{KWC}}(u,v) = E^{\varepsilon,\sigma}_{\mathrm{sMM}}(v),$$

so our $E^{\varepsilon,\sigma}_{\mathrm{sMM}}(v)$ is considered a special value of $E^\varepsilon_{\mathrm{KWC}}(u,v)$ by fixing such u. For $\Xi \in \mathcal{A}_0$, let $\Sigma = \{x_i\}_{i=1}^\infty$ be the set of all exceptional points of Ξ. (Note that the set Σ can be finite.) Let $\xi_i^- = \min \Xi(x_i)$ for $x_i \in \Sigma$. For $u \in BV(\mathring{M})$, let J_u denote the set of jump discontinuities of u, i.e.,

$$J_u := \left\{ x \in \mathring{M} \;\middle|\; d(x) = \left| u(x+0) - u(x-0) \right| > 0 \right\},$$

where $u(x+0)$ (resp. $u(x-0)$) denotes the trace from right (resp. left). For $(u, \Xi) \in L^1(\mathring{M}) \times \mathcal{B}$, we set

$$E^0_{\mathrm{KWC}}(u,\Xi,M) := \begin{cases} \sigma \displaystyle\int_{\mathring{M}\setminus(J_u\cap\Sigma)} \alpha(1) \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| + \sigma \sum_{i=1}^\infty d_i \alpha_i + E^0_{\mathrm{sMM}}(\Xi, M) \\ \qquad\qquad\qquad\qquad \text{for } u \in BV(\mathring{M}) \text{ and } \Xi \in \mathcal{A}_0, \\ \infty \quad \text{otherwise,} \end{cases}$$

where $d_i = d(x_i)$ and

$$\alpha_i := \min \left\{ \alpha(\xi) \mid \xi_i^- \leq \xi \leq \xi_i^+ \right\}.$$

Here $\displaystyle\int_\Omega \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|$ denotes the total variation in $\Omega \subset \mathring{M}$. Since the measure $|u_x|$ is a continuous measure outside J_u so that $|u_x|(\Sigma \setminus J_u) = 0$, one may replace $\Sigma \cap J_u$ by Σ in the domain of integration in the definition of E^0_{KWC}.

Theorem 2.2.3 (Gamma limit)**.** Assume that the same hypotheses of Theorem 2.2.1 concerning M and F.

(i) (liminf inequality) Let $\{v_\varepsilon\}_{0<\varepsilon<1}$ be in $H^1(M) \subset C(M)$. Assume that $v_\varepsilon \xrightarrow{g} \Xi \in \mathcal{B}$ as $\varepsilon \to 0$. Let $\{u_\varepsilon\} \subset L^1(M)$ satisfy $u_\varepsilon \to u$ in $L^1(\mathring{M})$ as $\varepsilon \to 0$. Then

$$E^0_{\mathrm{KWC}}(u,\Xi,M) \leq \liminf_{\varepsilon\to 0} E^\varepsilon_{\mathrm{KWC}}(u_\varepsilon, v_\varepsilon).$$

(ii) (limsup inequality) For any $\Xi \in \mathcal{A}_0$ and $u \in BV(\mathring{M})$, there exists $\{w_\varepsilon\}_{0<\varepsilon<1} \subset H^1(\mathring{M}) \subset C(M)$ and $\{u_\varepsilon\}_{0<\varepsilon<1} \subset L^1(M)$ such that $w_\varepsilon \xrightarrow{g} \Xi$ and $u_\varepsilon \to u$ in L^1 satisfying

$$E^0_{\mathrm{KWC}}(u, \Xi, M) = \lim_{\varepsilon\to\infty} E^\varepsilon_{\mathrm{KWC}}(u_\varepsilon, w_\varepsilon).$$

Remark 2.2.3.

(i) From the proof of Lemma 2.2.1, it suffices to assume $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathring{M}\backslash\Sigma_0)$ where

$$\Sigma_0 = \{x \in M \mid \min \Xi(x) = 0\}$$

in the statement of Theorem 2.2.3 (i). Since Ξ must be in $\mathcal{A}_0$ and $E^0_{\mathrm{sMM}}(\Xi, M) < \infty$, this set Σ_0 must be a finite set.

(ii) We may add a fidelty term $\lambda\|u-g\|^2_{L^2(\mathring{M})}$ to energies $E^\varepsilon_{\mathrm{KWC}}$, E^0_{KWC} for $\lambda > 0$ with given $g \in L^2(\mathring{M})$ like the Ambrosio-Tortorelli functional $\mathcal{E}^\varepsilon$ and the Munford-Shah functional. More precisely, the statement of Theorem 2.2.3 is still valid for

$$E^{\varepsilon,\lambda}_{\mathrm{KWC}}(u, v) := E^\varepsilon_{\mathrm{KWC}}(u, v) + \lambda \int_{\mathring{M}} |u-g|^2 \, \mathrm{d}x,$$

$$E^{0,\lambda}_{\mathrm{KWC}}(u, \Xi, M) := E^0_{\mathrm{KWC}}(u, \Xi, M) + \lambda \int_{\mathring{M}} |u-g|^2 \, \mathrm{d}x.$$

The next compactness result easily follows from the compactness (Theorem 2.2.2) in $\mathcal{B}$ and L^1-compactness of $BV(\Omega)$, where Ω is an open set such that $\overline{\Omega} \subset \mathring{M}\backslash\Sigma_0$.

Theorem 2.2.4 (Compactness)**.** Assume the same hypothesis of Theorem 2.2.2 concerning M and F. Let $\lambda > 0$ be fixed. Let $\left\{v_{\varepsilon_j}\right\}_{j=1}^\infty$ be in $H^1(M) \subset C(M)$ and $\left\{u_{\varepsilon_j}\right\} \subset L^2(\mathring{M})$. Assume that

$$\sup_j E^{\varepsilon_j,\lambda}_{\mathrm{KWC}}\left(u_{\varepsilon_j}, v_{\varepsilon_j}\right) < \infty$$

for $\varepsilon_j \to 0$. Then there exists a subsequence $\left\{\left(u_{\varepsilon'_k}, v_{\varepsilon'_k}\right)\right\}$ such that $u_{\varepsilon'_k} \to u$ in $L^1_{\mathrm{loc}}(\mathring{M}\backslash\Sigma_0)$ with some $u \in L^1_{\mathrm{loc}}(\mathring{M}\backslash\Sigma_0)$ and that $v_{\varepsilon'_k} \xrightarrow{g} \Xi$ with some $\Xi \in \mathcal{A}_0$. Here

$$\Sigma_0 = \{x \in M \mid \min \Xi(x) = 0\}.$$

By combining the Gamma convergence result and the compactness, a general theory yields the convergence of a minimizer of $E^{\varepsilon,\lambda}_{\mathrm{KWC}}$; see [6, Theorem 1.21] for example.

Corollary 2.2.2. Assume the same hypothesis of Theorem 2.2.1. Let $(u_\varepsilon, v_\varepsilon)$ be a minimizer of $E^{\varepsilon,\lambda}_{\mathrm{KWC}}$. Then, there is a subsequence $\{(u_{\varepsilon_k}, v_{\varepsilon_k})\}_{k=1}^\infty$ such that $v_{\varepsilon_k} \xrightarrow{g} \Xi$, $u_{\varepsilon_k} \to u$ in $L^1_{\mathrm{loc}}(\mathring{M} \setminus \Sigma_0)$ and that the limit (u, Ξ_0) be a minimizer of $E^{\varepsilon,\lambda}_{\mathrm{KWC}}$. Here $\Sigma_0 = \{x \in M \mid \min \Xi_0(x) = 0\}$.

2.3 Unfolding by arc-length parameters

For a bounded open interval I let u be a real-valued C^1 function on $\overline{I}$, that is, $u \in C^1(\overline{I})$. To simplify notation, we set $I = (0, r)$. Then the arc-length parameter s of the graph curve $y = u(x)$ is defined as

$$s(x) = s_u(x) := \int_0^x \left(1 + u_x^2(z)\right)^{1/2} \mathrm{d}z.$$

One is able to extend this definition for general $u \in BV(I)$. By definition, $s(\cdot)$ is strictly monotone increasing. It is easy to see that $s(\cdot)$ is continuous if and only if the derivative u_x has no point mass, that is, u has no jump, which is equivalent to $u \in C(\overline{I})$. The inverse function $x = x(s)$ of $s = s(x)$ is always Lipschitz with Lipschitz constant 1, that is, $\mathrm{Lip}(x) \leq 1$. Indeed, since $\dfrac{\mathrm{d}x}{\mathrm{d}s} = (1 + u_x^2)^{-1/2}$, the inequality $\left|\dfrac{\mathrm{d}x}{\mathrm{d}s}\right| \leq 1$ always holds. For $u \in C(\overline{I}) \cap BV(I)$, we define an unfolding U by arc-length parameter of the form

$$U(s) = u\left(x(s)\right).$$

The function U is defined on $\overline{J}$ with $J = (0, L)$, where L is the length of the graph u on $\overline{I}$. Note that $L \geq r$, the length of I.

We begin with several basic properties of the unfoldings.

Lemma 2.3.1. Assume that $u \in W^{1,1}(\overline{I})$.

(i) U is Lipschitz continuous on J. More precisely, $\mathrm{Lip}(U) \leq 1$.

(ii) The total variation of U on J equals that of u in I, that is,

$$\mathrm{TV}(u) = \mathrm{TV}(U).$$

Proof. (i) Since

$$U_s = \frac{u_x}{(1 + u_x^2)^{1/2}},$$

$\mathrm{Lip}(U) \leq 1$ is rather clear.

(ii) By definition,

$$\mathrm{TV}(U) = \int_0^L |U_s| \,\mathrm{d}s = \int_0^r |u_x| \,\mathrm{d}x = \mathrm{TV}(u).$$

□

Since $q = p/(1 + p^2)^{1/2}$ is equivalent to $p = q/(1 - q^2)^{1/2}$, we see that

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{1}{(1 + u_x^2)^{1/2}} = \left(1 - U_s^2\right)^{1/2}.$$

We next discuss compactness for unfoldings and the lower semicontinuity of $\mathrm{TV}(\cdot)$.

Lemma 2.3.2. Assume that $\{u^\varepsilon\}_{0<\varepsilon<1} \subset W^{1,1}(I)$ with a bound for $\mathrm{TV}(u^\varepsilon)$ and $\|u^\varepsilon\|_\infty$. Then there is a subsequence such that U^ε tends to some function V with $\mathrm{Lip}(V) \leq 1$ uniformly in a domain of definition of V. Moreover, $\mathrm{TV}(V) \leq \liminf_{\varepsilon \to 0} \mathrm{TV}(u^\varepsilon)$.

Proof. Since $\mathrm{TV}(u^\varepsilon)$ is bounded, so is the length L_ε of the graph of u^ε. The existence of convergent subsequence follows from the Ascoli-Arzela theorem. A basic lower semicontinuity of $\mathrm{TV}(\cdot)$ yields

$$\mathrm{TV}(V) \leq \liminf_{\varepsilon \to 0} \mathrm{TV}(U^\varepsilon).$$

The right-hand side equals $\mathrm{TV}(u^\varepsilon)$ as proved in Lemma 2.3.1 (ii) so the proof is now complete. □

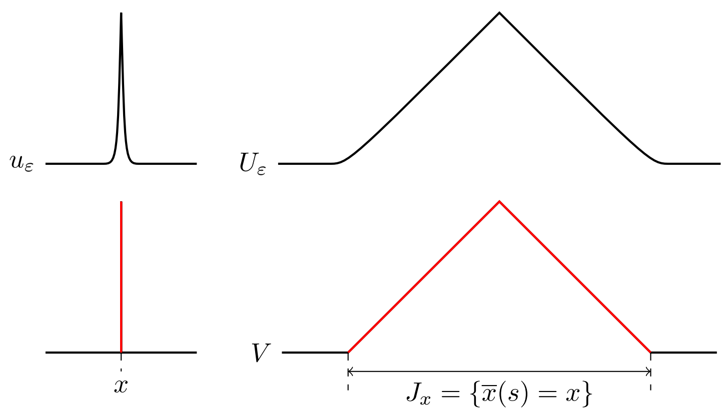

Figure 2.2: The visual example of Theorem 2.3.1. The sequence of function u^ε is unfolded to U^ε (the upper right figure). When U^ε converges uniformly to V, the corresponding limit of u^ε can be no longer captured as single-valued function but is possibly multi-valued. The red part of the graph of V (the lower right image), however, corresponds to the multi-valued part (the red part in the lower left image) and its maximum and minimum coincide with the upper and lower relaxed limit of u^ε, respectively.

We raise a question whether or not a Lipschitz function V on J with $\mathrm{Lip}(V) \leq 1$ can be written as $u\,(x(s))$. This is in general not true if there is a non trivial interval such that $U_s = 1$ (or $U_s = -1$). Indeed, if $U_s = \pm 1$, then $x(s)$ is not invertible.

In spite of this lack of the correspondence, however, the following lemma states that the limit of the unfolding contains the information on the pointwise behaviour of u^{ε_k}. (See also Figure 2.2.)

Theorem 2.3.1. Assume that $\{u^{\varepsilon_k}\}_{k=1}^\infty \subset W^{1,1}(I)$ is a sequence uniformly bounded in TV and its unfolding U^{ε_k} converges uniformly to V in the domain J of definition of V.

If x^{ε_k}, the inverse of the arc-length parameter of u^{ε_k}, converges uniformly to a limit $\overline{x}$ in J, then

$$\left(\limsup_{k\to\infty}{}^{*}\ u^{\varepsilon_k}\right)(x) = \max\left\{V(s) \mid \overline{x}(s) = x\right\},$$
$$\left(\liminf_{k\to\infty}{}_{*}\ u^{\varepsilon_k}\right)(x) = \min\left\{V(s) \mid \overline{x}(s) = x\right\}, \quad x \in \overline{I}.$$

Remark 2.3.1. The length of J is bounded by a bound of $\mathrm{TV}(u^\varepsilon)$ plus the length of I.

Proof. Since the proof is symmetric, we only give a proof for $\limsup^*$. Let $J_x = \{s \in J \mid \overline{x}(s) = x\}$. We take $s_* \in J_x$ such that

$$V(s_*) = \max_{J_x} V.$$

Since V is the limit of U^ε, we have

$$V(s_*) = \lim_{k\to\infty} u^{\varepsilon_k}\left(x^{\varepsilon_k}(s_*)\right) \leq \left(\limsup_{k\to\infty}{}^{*} u^{\varepsilon_k}\right)(x).$$

To prove the converse inequality, we set

$$J_x^\sigma = \left\{s \in J \;\middle|\; |\overline{x}(s) - x| \leq \sigma\right\}.$$

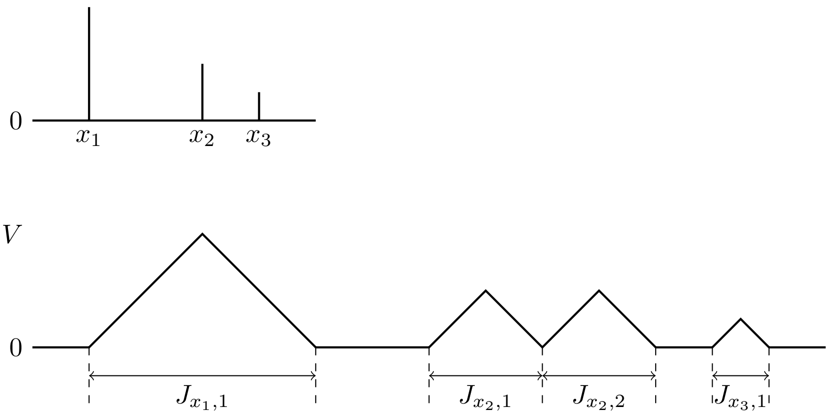

Figure 2.3: The visual explanation of Theorem 2.3.2. If the graph of $\{u^{\varepsilon_k}\}$ converges to the graph as the top, its unfolding converges to V, whose graph is like the bottom. Then $J_{x_i} = \{s \in J \mid \overline{x}(s) = x_i\}$ can be decomposed as the union of $\{J_{x_i,j}\}_{j=1,2,\ldots}$ by labelling the disjoint intervals where V does not vanish.

Since x^{ε_k} converges to $\overline{x}$ uniformly in J, for sufficiently large k, say $k > k_0(\sigma)$,

$$x^{\varepsilon_k}(J_x^{2\sigma}) \supset \left\{ y \in I \;\middle|\; |y - x| \le \sigma \right\};$$

here $k_0(\sigma)$ can be taken so that $k_0(\sigma) \to \infty$ as $\sigma \to \infty$ and $k_0(\sigma) > 1/\sigma$. We thus observe that

$$\sup_{|y-x|\le\sigma} u^{\varepsilon_k}(y) \le \sup \left\{ u^{\varepsilon_k}(y) \;\middle|\; y \in x^{\varepsilon_k}(J_x^{2\sigma}) \right\} = \sup \left\{ U^{\varepsilon_k}(s) \;\middle|\; s \in J_x^{2\sigma} \right\}$$

for $k > k_0(\sigma)$. Sending $\sigma \to 0$, we observe that

$$\lim_{\sigma \downarrow 0} \sup_{\substack{|y-x|\le\sigma \\ k > k_0(\sigma)}} u^{\varepsilon_k}(y) \le \max_{s \in J_x} V(s).$$

The left-hand side agrees with $\limsup^*_{k\to\infty} u^{\varepsilon_k}$ since

$$\begin{aligned} &\left\{ (y,k) \;\middle|\; |y-x| < 1/k_0(\sigma),\ k > k_0(\sigma) \right\} \\ \subset &\left\{ (y,k) \;\middle|\; |y-x| < \sigma,\ k > k_0(\sigma) \right\} \\ \subset &\left\{ (y,k) \;\middle|\; |y-x| < \sigma,\ k > 1/\sigma \right\}. \end{aligned}$$

We thus conclude that

$$\left(\limsup_{k\to\infty}{}^* u^{\varepsilon_k} \right)(x) \le \max \left\{ V(s) \;\middle|\; s \in J_x \right\}.$$

The proof is now complete. □

We next prove the inequality connecting the total variation and the relaxed limit in terms of the unfolding. (see Figure 2.3.)

Theorem 2.3.2. Assume the same hypothesis of Theorem 2.3.1. Then the set Σ of points x where

$$\limsup_{k\to\infty}{}^* u^{\varepsilon_k}(x) > \liminf_{k\to\infty}{}_* u^{\varepsilon_k}(x)$$

has at most countable cardinality. Assume furthermore that outside Σ the limit must be zero and $\liminf_{k\to\infty} {}_* u^{\varepsilon_k}(x) = 0$ for all $x \in I$. Then

$$\liminf_{k\to\infty} \mathrm{TV}(u^{\varepsilon_k}) \geq \sum_{x\in\Sigma} 2\chi(x) \limsup_{k\to\infty}{}^* u^{\varepsilon_k}(x),$$

where $\chi(x) = 1$ for $x \in I$ and $\chi(x) = 1/2$ for $x \in \partial I$.

Proof. If $\#\Sigma$ is uncountable, then there is an infinite number of intervals J_{x_i} such that $\max V - \min V > c_0$ with some $c_0 > 0$. This is impossible by Theorem 2.3.1, since $\mathrm{TV}(V) < \infty$. Thus, Σ is at most a countable set.

We write $\Sigma = \{x_i\}_{i=1}^{\infty}$ and $J_i = J_{x_i}$. We set $\rho_i = \max_{J_i} V$. The cases devided into two cases whether or not $\overline{J_i}$ contains a boundary point of J. The total variation is estimated so

$$\mathrm{TV}(V) \geq \sum_{i=1}^{\infty} 2\chi_i \rho_i,$$

where $\chi_i = \chi(x_i)$. Thus Theorem 2.3.1 and Lemma 2.3.2 yield the desired result. $\square$

We decompose J_i by

$$J_i := \mathrm{cl}\left(\bigcup_{j=1}^{\infty} J_{x_i,j}\right),$$

where $V > 0$ in an open interval $J_{x_i,j}$ and $V = 0$ on $\partial J_{x_i,j}$. The union can be finite.

We introduce $\overline{\chi}$ on subsets of J_{x_i} which reflects behavior finer than that of χ on the boundary. We set for $x = x_i \in \Sigma$,

$$\overline{\chi}(J_{x_i,j}) = \begin{cases} 1 & \text{if } J_{x_i,j} \cap \partial I = \emptyset, \\ 1/2 & \text{otherwise.} \end{cases}$$

By definition

$$\mathrm{TV}(V) \geq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 2\overline{\chi}(J_{x_i,j}) \rho_{x_i,j}$$

with

$$\rho_{x_i,j} = \max_{J_{x_i,j}} V.$$

Similarly to obtain Theorem 2.3.2, we are able to prove a stronger result.

Theorem 2.3.3. Assume the same hypothesis of Theorem 2.3.1. Then

$$\liminf_{k\to\infty} \mathrm{TV}(u^{\varepsilon_k}) \geq \sum_{x\in\Sigma} \sum_{j=1}^{\infty} 2\overline{\chi}(J_{x,j}) \rho_{x,j},$$

where $\rho_{x,j}$ and $\overline{\chi}$ are determined from V as above.

2.4 Proof of convergence of functional and compactness

We shall prove the characterization of the Gamma limit of the single-well Modica–Mortola functional by the results of the previous section on unfoldings.

Proof of Theorem 2.2.1.
(i) (liminf inequality) We discuss the case $M = \overline{I}$. We may assume $I = (0, r)$. Assume that $v_\varepsilon \xrightarrow{g} \Xi \in \mathcal{B}$ with $v_\varepsilon \in H^1(M)$. By the Modica–Mortola inequality which follows from $\alpha^2 + \beta^2 \geq 2\alpha\beta$ for numbers we have

$$E^\varepsilon_{\mathrm{sMM}}(v_\varepsilon) \geq \int_M \left| \frac{\mathrm{d}v_\varepsilon}{\mathrm{d}x} \right| \sqrt{F(v_\varepsilon)}\,\mathrm{d}x = \int_M |G(v_\varepsilon)_x|\,\mathrm{d}x.$$

The right-hand is equal to $\mathrm{TV}(u^\varepsilon)$ if one sets $u^\varepsilon = G(v_\varepsilon) \geq 0$. We may assume that $E^\varepsilon_{\mathrm{sMM}}(v_\varepsilon)$ is bounded for $\varepsilon \in (0,1)$ so that $\mathrm{TV}(u^\varepsilon)$ is bounded for $\varepsilon \in (0,1)$ and that $\int_M F(v_\varepsilon)\,\mathrm{d}x \to 0$ as $\varepsilon \to 0$. By (F2), the latter convergence implies that $v_\varepsilon \to 1$ in measure. By taking a subsequence, we see that $v_{\varepsilon'} \to 1$ a.e. so that $u^{\varepsilon'} \to 0$ a.e. This implies that

$$\liminf_{\varepsilon \to 0}{}_* u^\varepsilon(x) = 0 \quad \text{for all} \quad x \in M.$$

By taking a subsequence, we may assume that the inverse function x^{ε_k} of the arc-length parameter of u^{ε_k} converges to some $\overline{x}$. Applying Theorem 2.3.3, we see that

$$\liminf_{k\to\infty} E^{\varepsilon_k}_{\mathrm{sMM}}(v_{\varepsilon_k}) \geq \sum_{x\in\Sigma} \sum_{j=1}^{\infty} 2\overline{\chi}(J_{x,j})\rho_{x,j},$$

where Σ is the set where $\limsup^* u^{\varepsilon_k}(x) > 0$ and $\rho_{x,j}$ is determined by limit V of u^ε. Note that Σ is at most countable. If $v_\varepsilon \xrightarrow{g} \Xi$, then $u^\varepsilon \xrightarrow{g} \Theta$ and $\Theta(x) = \{0\}$ if $x \notin \Sigma$ and

$$\Theta(x_i) = \left[0, \max\left(G(\xi_i^+), G(\xi_i^-)\right)\right] \quad \text{for} \quad x_i \in \Sigma$$

by Lemma 2.2.3. By Theorem 2.3.1, at least one of $\rho_{x_i,j}$ should be equal to $\max\left(G(\xi_i^+), G(\xi_i^-)\right)$. However, if $\xi_i^- < 1 < \xi_i^+$, then $v_\varepsilon - 1$ is sign-changing near x_i. In this case, one of $\rho_{x_i,j}$'s must be equal to $\min\left(G(\xi_i^+), G(\xi_i^-)\right)$. Thus we observe that

$$\sum_{j=1}^{\infty} \overline{\chi}(J_{x_i,j})\rho_{x_i,j} \geq G(\xi_i^+) + G(\xi_i^-), \quad x_i \in \Sigma \cap I.$$

If $x \in \Sigma \cap \partial I$, one has to be more careful. For $x_i \in \Sigma \cap \partial I$, we see that

$$\sum_{j=1}^{\infty} \overline{\chi}(J_{x_i,j})\rho_{x,j} \geq \min(G(\xi_i^+), G(\xi_i^-)) + \frac{1}{2}\max(G(\xi_i^+), G(\xi_i^-)). \tag{2.4}$$

Indeed, without loss of generality, we assume that $G(\xi_i^-) < G(\xi_i^+)$. When $G(\xi_i^-) = 0$, (2.4) is rather easy to prove since the right hand side is equal to $\frac{1}{2}G(\xi_i^+)$, and then we may assume that $G(\xi_i^-) > 0$. Then there are at least two indices denoted by $j = 1, 2$, such that

$$\overline{\chi}(J_{x_i,1}) = \frac{1}{2},\ \overline{\chi}(J_{x_i,2}) = 1,\ \text{and}\ \{\rho_{x_i,1}, \rho_{x_i,2}\} = \{G(\xi_i^-), G(\xi_i^+)\}.$$

The left hand side is dominated from below by

$$\overline{\chi}(J_{x_i,1})\rho_{x_i,1} + \overline{\chi}(J_{x_i,2})\rho_{x_i,2} = \frac{1}{2}\rho_{x_i,1} + \rho_{x_i,2}.$$

The right hand side is minimized in the case that $\rho_{x_i,1} = G(\xi_i^+)$ and $\rho_{x_i,2} = G(\xi_i^-)$. We thus obtain the inequality (2.4).

We now conclude that

$$\begin{aligned}\liminf_{k\to\infty} E_{\mathrm{sMM}}^{\varepsilon_k}(v_{\varepsilon_k}) \geq \sum_{i=1}^{\infty} & 2(1-\kappa_i)(G(\xi_i^+) + G(\xi_i^-)) \\ & + \kappa_i \left[2\min(G(\xi_i^+), G(\xi_i^-)) + \max(G(\xi_i^+), G(\xi_i^-))\right] \\ = & E_{\mathrm{sMM}}^0(\Xi, \overline{I}),\end{aligned}$$

which is the desired liminf inequality for $b = 0$. Since $v_\varepsilon \xrightarrow{g} \Xi$, we see that

$$\liminf b\alpha\left(v_\varepsilon(a)\right) \geq b \min_{\xi\in\Xi(a)} \alpha\left(\xi\right).$$

Thus the desired liminf inequality follows for $b > 0$. The case $M = \mathbb{T}$ is easier since there is no boundary point.

(ii) (limsup inequality) This follows from explicit construction of function w_ε as for the standard double-well Modica–Mortola functional.

For $\xi < 1$ and $x > 0$, let $v(x, \xi)$ be a function determined by

$$\int_\xi^v \left(\frac{1}{\sqrt{F(\rho)}}\right) d\rho = x.$$

This equation is uniquely solvable by (F1) for all $x \in [0, x_*)$ with

$$x_* := \int_\xi^1 \left(\frac{1}{\sqrt{F(\rho)}}\right) d\rho.$$

Note that v solves the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{F(v)}, \quad x \in (0, x_*), \\ v(0,\xi) = \xi, \end{cases} \tag{2.5}$$

although this problem may admit many solutions.

although this problem may admit many solutions. For $\xi > 1$, we parallely define v by

$$\int_v^\xi \left(\frac{1}{\sqrt{F(z)}}\right) dz = x$$

for $x \in (0, x_*)$ with

$$x_* := \int_1^\xi \left(\frac{1}{\sqrt{F(z)}}\right) dz.$$

In this case, v also solves (2.5).

We also note that v is monotone and that

$$\lim_{x\to x_*} v(x,\xi)=1$$

including the case $x_*=\infty$. We consider the even extention of v and still denote by v, that is, $v(x,\xi)=v(-x,\xi)$ for $x\in(-x_*,0]$. We next translate and rescale v. Let v_ε be of the form

$$v_\varepsilon(x,z,\xi):=v\left(\frac{x-z}{\varepsilon},\xi\right),\quad x\in\mathbb{R}.$$

By the equality case of the Modica–Mortola functional, we see that

$$E^\varepsilon_{\mathrm{sMM}}(v_\varepsilon)=\int_M\left|\frac{\mathrm{d}v_\varepsilon}{\mathrm{d}x}\right|\sqrt{F(v_\varepsilon)}\,\mathrm{d}x=\int_M|G(v_\varepsilon)_x|\ \mathrm{d}x.$$

The right-hand side is estimated from above by

$$2\left(G(\xi)-G(1)\right)=2G(\xi)$$

and if z is a boundary point of M, we may replace $2G(\xi)$ by $G(\xi)$.

In order to explain the the main idea of the proof, we first study the case when all $\xi_i^+=1$ although logically we need not distinguish this case from general case. If all $\xi_i^+=1$, then it is easy to construct the desired w_ε by setting

$$w_\varepsilon(x)=\min_{x_i\in\Sigma} v(x,x_i,\xi_i^-).$$

Indeed, we still have

$$E^\varepsilon_{\mathrm{sMM}}(w_\varepsilon)=\int_M|G(w_\varepsilon)_x|\ \mathrm{d}x$$

and evidently this total variation is dominated from above by

$$\sum_{i=1}^\infty 2\chi_i G(\xi_i^-).$$

(The first identity can be proved by approximating w_ε by minimum of finitely many w_ε's.) We thus observe that $E^\varepsilon_{\mathrm{sMM}}(w_\varepsilon)\le E^0_{\mathrm{sMM}}(\Xi,\mathbb{T})$ for all $\varepsilon>0$. The graph convergence $w_\varepsilon\xrightarrow{g}\Xi$ is rather clear since

$$w_\varepsilon(x,0,\xi)\xrightarrow{g}\Xi_0$$

on any bounded closed interval as $\varepsilon\to0$, where

$$\Xi_0(x)=\begin{cases}1, & x\neq0,\\ [\xi,1]\,, & x=0.\end{cases}$$

The proof for general $\xi_i^\pm$ is more involved. For $\delta>0$, we cut off v by setting as follows: For $\xi<1$,

$$v^\delta(x,\xi)=\begin{cases}v(x,\xi) & \text{if}\quad v(x)\le1-\delta\beta,\ \ \beta=|\xi-1|,\\ (|x|+c)\wedge1 & \text{if}\quad v(x)\ge1-\delta\beta,\end{cases}$$

and for $\xi>1$,

$$v^\delta(x,\xi)=\begin{cases}v(x,\xi) & \text{if } v(x)\ge1+\delta\beta,\ \ \beta=|\xi-1|,\\ (-|x|+c')\vee1 & \text{if } v(x)\le1+\delta\beta,\end{cases}$$

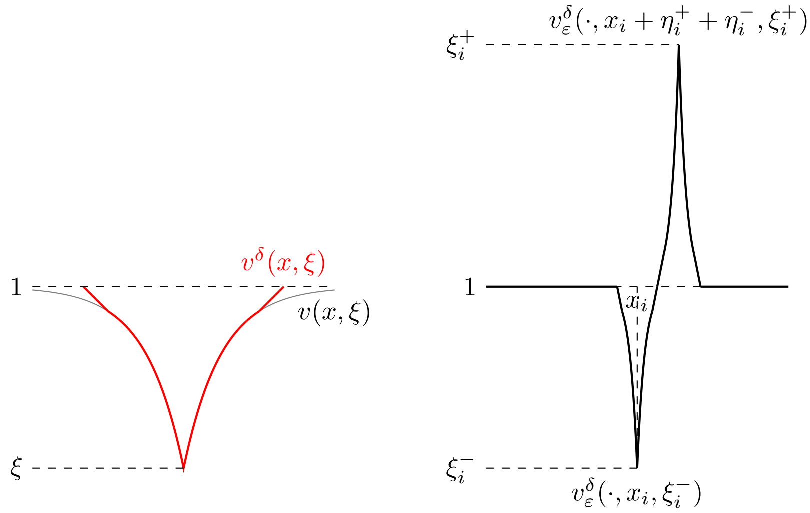

Figure 2.4: (left) The construction of $v^\delta(\cdot,\xi)$. In order to ensure the finiteness of the support, we take the cutoff by affine functions. (right) The construction of $v^\delta_{\varepsilon,i}$ for general $\xi_i^\pm$. It is constructed by combining $v_\varepsilon^\delta(\cdot, x_i, \xi_i^-)$ and $v_\varepsilon^\delta(\cdot, x_i, \xi_i^+)$ with shift in order that their supports touch at their endpoints.

where constants c, c' are taken so that v^δ is (Lipschitz) continuous. (See Figure 2.4.) We rescale and translate this v^δ and set

$$v_\varepsilon^\delta(x,z,\xi) := v^\delta\left(\frac{x-z}{\varepsilon}, \xi\right) \quad x \in \mathbb{R}.$$

We consider the case when $\xi < 1$. Since $\dfrac{\mathrm{d}v_\varepsilon^\delta}{\mathrm{d}x} = \dfrac{1}{\varepsilon}\sqrt{F(v_\varepsilon^\delta)}$ for $v_\varepsilon^\delta \le 1-\delta$, we see that for $z \in M$

$$\begin{aligned} E^\varepsilon_{\mathrm{sMM}}(v_\varepsilon^\delta) &= \frac{\varepsilon}{2}\int_M \left|\frac{\mathrm{d}v_\varepsilon^\delta}{\mathrm{d}x}\right|^2 \mathrm{d}x + \frac{1}{2\varepsilon}\int_M F(v_\varepsilon^\delta)\,\mathrm{d}x \\ &\le \int_{v_\varepsilon(x)<1-\delta} |G(v_\varepsilon)_x| \,\mathrm{d}x \\ &\quad + 2\left\{\frac{\varepsilon}{2}\left(\frac{1}{\varepsilon}\right)^2 \cdot \delta\beta\varepsilon + \frac{1}{2\varepsilon}\max\Big\{F(\rho)\,\Big|\, 1-\delta \le \rho \le 1\Big\}\delta\beta\varepsilon\right\} \\ &\le 2G(\xi) + 2\beta\delta \end{aligned} \tag{2.6}$$

for sufficiently small δ, say $\delta < \delta_F$, since $F(\rho) \to 0$ as $\rho \to 1$. This δ_F depends only on F. A similar argument for $\xi > 1$ yield the same estimate (2.6).

We first consider the case when $M = \mathbb{T}$. Let $\eta = \eta(\varepsilon,\delta,\xi)$ be a number such that $\operatorname{supp}\left(v_\varepsilon^\delta - 1\right) = [z-\eta, z+\eta]$. For $x_i \in \Sigma$, we set

$$v^\delta_{\varepsilon,i}(x) := \begin{cases} v_\varepsilon^\delta(x, x_i, \xi_i^-) & \text{if} \quad x \in (x_i - \eta_i^-, x_i + \eta_i^-), \\ v_\varepsilon^\delta(x, x_i + \eta_i^+ + \eta_i^-, \xi_i^+) & \text{otherwise,} \end{cases}$$

where $\eta_i^- = \eta(\varepsilon,\delta,\eta_i^-)$ and $\eta_i^+ = \eta(\varepsilon,\delta,\eta_i^+)$. (see Figure 2.4.) This function is (Lipschitz) continuous and is strictly monotone from $x_i - \eta_i^+ - \eta_i^-$ to x_i. For $v^\delta_{\varepsilon,i}(x)$ by (2.6), we see that

$$E^\varepsilon_{\mathrm{sMM}}\left(v^\delta_{\varepsilon,i}\right) \leq 2\left(G(\xi_i^+) + G(\xi_i^-)\right) + 4\beta_i\delta, \tag{2.7}$$

where $\beta_i = \max\left(|\xi_i^+ - 1|, |\xi_i^- - 1|\right)$.

Our goal is to construct w_ε such that $w_\varepsilon \xrightarrow{g} \Xi$ and for each $\mu > 0$, there is $\varepsilon_\mu > 0$ such that if $\varepsilon < \varepsilon_\mu$ then

$$E^\varepsilon_{\mathrm{sMM}}(w_\varepsilon) \leq E^0_{\mathrm{sMM}}(\Xi, \mathbb{T}) + \mu. \tag{2.8}$$

We order $x_i \in \Sigma$ so that β_i is decreasing. We note that $\{\beta_i\}$ must converge to zero because $\sum_{i=1}^\infty \left(G(\xi_i^+) + G(\xi_i^-)\right) < \infty$. For each $v^\delta_{\varepsilon,i}$, we set $\delta = \delta_i = \delta_i(\mu)$ such that $\sum_{i=1}^\infty 4\beta_i\delta_i < \mu$; this is, of course, possible for example by taking $\delta_i = 2^{-i-2}\mu$. Let $j(\mu,\varepsilon) > 0$ be the maximum number such that the support of $\left\{v^{\delta_i}_{\varepsilon,i} - 1\right\}_{i=1}^{j(\mu,\varepsilon)}$ is mutually disjoint. We set

$$w^\mu_\varepsilon(x) := 1 + \sum_{i=1}^{j(\mu,\varepsilon)} \left(v^{\delta_i(\mu)}_{\varepsilon,i}(x) - 1\right)$$

and observe by (2.7) that

$$\begin{aligned} E^\varepsilon_{\mathrm{sMM}}(w^\mu_\varepsilon) &\leq \sum_{i=1}^{j(\mu,\varepsilon)} 2\left(G(\xi_i^+) + G(\xi_i^-)\right) + \sum_{i=1}^{j(\mu,\varepsilon)} 4\beta_i\delta_i \\ &\leq E^0_{\mathrm{sMM}}(\Xi, \mathbb{T}) + \mu \quad \text{for all} \quad \varepsilon > 0. \end{aligned} \tag{2.9}$$

Since $j(\mu,\varepsilon) \to \infty$ as $\varepsilon \to 0$, we see that $w^\mu_\varepsilon \xrightarrow{g} \Xi$ as $\varepsilon \to 0$ for each $\mu > 0$. The desired w_ε is obtained through a diagonal argument. Indeed, for a given $\nu > 0$, we take $\varepsilon = \varepsilon(\nu,\mu)$ such that

$$d_H\left(\Gamma_{w^\mu_\varepsilon}, \Xi\right) < \nu$$

for $\varepsilon \in (0, \varepsilon(\nu,\mu))$. We may assume that $\varepsilon(\nu,\mu)$ is monotone in ν and μ, that is, $\varepsilon(\nu_2,\mu_2) \leq \varepsilon(\nu_1,\mu_1)$ if $\nu_1 \geq \nu_2$ and $\mu_1 \geq \mu_2$. We then set

$$w_\varepsilon := w^{\mu_\ell}_\varepsilon \quad \text{for} \quad \varepsilon \in \left[\varepsilon(\nu_{\ell+1}, \mu_{\ell+1}), \varepsilon(\nu_\ell, \mu_\ell)\right),$$

where $\nu_\ell,\ \mu_\ell \downarrow 0$ as $\ell \to \infty$. We now observe that $w_\varepsilon \xrightarrow{g} \Xi$ and by (2.8) the desired estimate (2.8) holds for $\varepsilon_\mu = \varepsilon(\nu_\ell, \mu_\ell)$ for $\mu_\ell < \mu$.

We thus proved the limsup inequality for $E^\varepsilon_{\mathrm{sMM}}$ for $M = \mathbb{T}$. If $b > 0$, we may assume that $\xi_i^- = \min \Xi(a) < 1$. It is easy to see that $w_\varepsilon(a) = \xi^i_-$ for all $\varepsilon > 0$ by construction. Thus the limsup inequality for $E^{\varepsilon,b}_{\mathrm{sMM}}$ for $b > 0$ is obtained.

It remains to handle the case for $M = \overline{I}$. Assume that $x_1 \in \Sigma$ is the right end point of $\overline{I}$. We first consider the case when $G(\xi_1^-) \leq G(\xi_1^+)$. Instead of (2.6), we have

$$E^\varepsilon_{\mathrm{sMM}}\left(v^\delta_{\varepsilon,1}\right) \leq 2G(\xi_1^+) + G(\xi_1^-) + 3\beta_1\delta.$$

If there is no other point of Σ on ∂I, arguing in the same way we obtain the desired limsup inequality by the same construction of w_ε. If $G(\xi_1^-) > G(\xi_1^+)$, then we modify the definition of $v^\delta_{\varepsilon,1}$ by

$$\overline{v}^\delta_{\varepsilon,1}(x) := \begin{cases} v^\delta_\varepsilon(x, x_1, \xi_i^+) & \text{of} \quad x \in (x_1 - \eta_i, x_1], \\ v^\delta_\varepsilon(x, x_1 + \eta_1^+ + \eta_1^-, \xi_i^-) & \text{otherwise.} \end{cases}$$

The remaining argument is similar. Symmetric argument yields the limsup inequality in the case that Σ has the left end point of $\overline{I}$. □

We next prove the compactness.

Proof of Theorem 2.2.2. As in the proof of Theorem 2.2.1 (i), we see that

$$\sup_j \int_M \left| G\left(v_{\varepsilon_j}\right)_x \right| \mathrm{d}x < \infty.$$

By (F2), we see that

$$G(v) \leq \sqrt{F(v)}|v-1| \leq \frac{F(v)}{2} + \frac{(v-1)^2}{2}, \quad v \in \mathbb{R}.$$

By (F2'), we see

$$F(v) \geq c_0(v-1)^2 - c_1'$$

so

$$G(v) \leq C'F(v)$$

for v such that $|v-1|$ is sufficiently large v with some content $C' > 0$. Since

$$\frac{1}{\varepsilon_j} \int_M F\left(v_{\varepsilon_j}\right) \mathrm{d}x$$

is bounded, so is $\int_M G\left(v_{\varepsilon_j}\right) \mathrm{d}x$. We set $u^{\varepsilon_j} = G\left(v_{\varepsilon_j}\right)$ and observe that $\mathrm{TV}\left(u^{\varepsilon_j}\right)$ is bounded and $\left\|u^{\varepsilon_j}\right\|_{L^1}$ is bounded. Since

$$\|f - f_{\mathrm{av}}\|_\infty \leq \|f_x\|_{L^1},$$

where f_{av} is the average of f over I, it follows that

$$\|f\|_\infty \leq \|f_x\|_{L^1} + \|f_x\|_{L^1}/|I|.$$

This interpolation inequality yields a bound for $\left\|u^{\varepsilon_j}\right\|_\infty$. Applying Lemma 2.3.2, there is a subsequence U^{ε_k} converges to V uniformly, where U^{ε_k} is the unfolding of u^{ε_k}. Since we may assume that x^{ε_k}, the inverse of arc-length of u^{ε_k}, converges to $\overline{x}$ uniformly in M by taking a subsequence, applying Theorem 2.3.2 yields that

$$\limsup_{k\to\infty}{}^{*} u^{\varepsilon_k}(x) > \liminf_{k\to\infty}{}_{*} u^{\varepsilon_k}(x), \quad x \in \Sigma$$

for at most countably many $x \in \Sigma$. Since $v_{\varepsilon_k} \to 1$ a.e. by taking a subsequence, we see that $\liminf_{*} u^{\varepsilon_k}(x) = 0$ for all $x \in M$. This implies that v_{ε_k} satisfies all assumptions on a sequence $\{g_j\}$ of the compactness lemma (Lemma 2.2.4) with $S = \Sigma$. Then by Lemma 2.2.4, we conclude that $v_{\varepsilon_k} \xrightarrow{g} \Xi$ with some $\Xi \in \mathcal{A}_0$. □

2.5 Singular limit of the Kobayashi–Warren–Carter energy

In this section, we shall study the Gamma limit of the Kobayashi–Warren–Carter energy.

We first derive an inequality for lower semicontinuity. Assume that M is either $\overline{I}$ or $\mathbb{T}$. Assume that

(C1) $v_\varepsilon \xrightarrow{g} \Xi$, $u_\varepsilon \to u$ in $L^1(\mathring{M})$ as $\varepsilon \to 0$, where $v_\varepsilon \in C(M)$, $u_\varepsilon \in L^1$.

For the limits, we assume that

(C2) $\Xi \in \mathcal{A}_0$, that is, there is a countable set $\Sigma = \{x_i\}_{i=1}^{\infty} \subset M$ such that $\Xi(x) = \{1\}$ for $x \notin \Sigma$ and $\Sigma(x_i) = \left[\xi_i^-, \xi_i^+\right] \ni 1$ with $\xi_i^- < \xi_i^+$ for $x_i \in \Sigma$. Moreover, $\sum_{i=1}^{\infty} G\left(\xi_i^-\right) < \infty$.

(C3) $u \in BV\big(\mathring{M} \setminus \Sigma_0\big)$, where $\Sigma_0 = \left\{x_i \in \Sigma \;\middle|\; \xi_i^- = 0\right\}$. (Since $\sum_{i=1}^{\infty} G\left(\xi_i^-\right) < \infty$, the set Σ_0 is a finite set.)

We define a weighted total variation

$$\int_{\mathring{M}} \alpha(v_\varepsilon) \left|\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right| := \sup\left\{ \int_{\mathring{M}} \varphi_x u_\varepsilon \,\mathrm{d}x \;\middle|\; |\varphi(x)| \leq \alpha(v_\varepsilon(x)),\ \varphi \in C_c^1(\mathring{M}) \right\},$$

where $C_c^1(\mathring{M})$ is the space of all C^1 functions in $\mathring{M}$ with compact support in $\mathring{M}$. For $u \in BV\big(\mathring{M} \setminus \Sigma_0\big)$, let J_u denote the set of jump discontinuities of u. In other words,

$$J_u = \left\{ x \in \mathring{M} \setminus \Sigma_0 \;\middle|\; d(x) = |u(x+0) - u(x-0)| > 0 \right\},$$

where $u(x \pm 0)$ is the trace from right $(+)$ and left $(-)$. It is at most a countable set.

Lemma 2.5.1. Assume conditions (C1) – (C3). Then

$$\int_{\mathring{M}\setminus(J_u\cap\Sigma)} \alpha(1)|u_x| + \sum_{x\in\Sigma'} d_i\alpha_i \leq \liminf_{\varepsilon\to 0} \int_{\mathring{M}} \alpha(v_\varepsilon) \left|\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right|,$$

where $d_i = d(x_i) \geq 0$, $\Sigma' = \Sigma \setminus \Sigma_0$.

Proof. It suffices to prove that for any $\delta \in (0,1)$,

$$\alpha(1-\delta) \int_{\mathring{M}\setminus(J_u\cap\Sigma_{1-\delta})} |u_x| + \sum_{x_i\in\Sigma'_{1-\delta}} d_i\alpha_i \leq \liminf_{\varepsilon\to 0} \int_{\mathring{M}} \alpha(v_\varepsilon) \left|\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right|,$$

where

$$\Sigma_{1-\delta} = \left\{ x_i \in \Sigma \;\middle|\; \xi_i^- < 1-\delta \right\}, \quad \Sigma'_{1-\delta} = \Sigma_{1-\delta} \setminus \Sigma_0.$$

By this notation $\Sigma_1 = \Sigma$. Note that the set $\Sigma_{1-\delta}$ is a finite set for $\delta > 0$ since $\sum_{i=1}^{\infty} G\left(\xi_i^-\right) < \infty$.

Since Σ_0 is a finite set and

$$\int_{\mathring{M}} \alpha(v_\varepsilon) \left|\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right| \geq \int_{\mathring{M}\setminus\Sigma_0} \alpha(v_\varepsilon) \left|\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right|,$$

it suffices to prove that for each interval $\{M_j\}_{j=1}^m$, which is a connected component of $\mathring{M} \setminus \Sigma_0$ the inequality

$$\alpha(1-\delta) \int_{M_j\setminus(J_u\cap\Sigma_{1-\delta})} |u_x| + \sum_{\substack{x_i\in\Sigma'_{1-\delta} \\ x_i\in M_j}} d_i\alpha_i \leq \liminf_{\varepsilon\to 0} \int_{M_j} v_\varepsilon^2 \left|\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right|.$$

Thus we may assume that $\Sigma_0 = \emptyset$.

We consider δ_1-open neighborhood of $\Sigma_{1-\delta}$, that is,

$$X_{\delta_1} = \left\{ x \in \mathring{M} \;\middle|\; \mathrm{dist}\,(x, \Sigma_{1-\delta}) < \delta_1 \right\}$$

and observe that

$$\int_{\mathring{M}} \alpha(v_\varepsilon) \left| \frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} \right| \geq \int_{\mathring{M} \setminus \overline{X}_{\delta_1}} \alpha(v_\varepsilon) \left| \frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} \right| + \int_{X_{\delta_1}} \alpha(v_\varepsilon) \left| \frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} \right|.$$

We may assume that X_{δ_1} consist of disjoint interval $B_{\delta_1}(x_i) = \left\{ x \in \mathring{M} \;\middle|\; |x - x_i| < \delta_1 \right\}$, $x_i \in \Sigma_{1-\delta}$ by taking δ_1 small. Since $v_\varepsilon \xrightarrow{g} \Xi$, for sufficiently small ε we observe that

$$\begin{aligned} &v_\varepsilon \geq 1 - \delta - \delta_1 && \text{in} \quad \mathring{M} \setminus \overline{X}_{\delta_1}, \\ &v_\varepsilon \geq \xi_i^- - \delta_1 && \text{in} \quad B_{\delta_1}(x_i),\ x_i \in \Sigma_{1-\delta}. \end{aligned}$$

We thus conclude that

$$\begin{aligned} &\liminf_{\varepsilon \to 0} \int_{\mathring{M}} \alpha(v_\varepsilon) \left| \frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} \right| \\ &\geq \liminf_{\varepsilon \to 0} \left\{ \alpha(1 - \delta - \delta_1) \int_{\mathring{M} \setminus \overline{X}_{\delta_1}} \left| \frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} \right| + \sum_{x_i \in \Sigma_{1-\delta}} \alpha\left(\xi_i^- - \delta_1\right) \int_{B_{\delta_1}(x_i)} \left| \frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} \right| \right\} \\ &\geq \alpha(1 - \delta - \delta_1) \int_{\mathring{M} \setminus \overline{X}_{\delta_1}} |u_x| + \sum_{x_i \in \Sigma_{1-\delta}} \alpha\left(\xi_i^- - \delta_1\right) \int_{B_{\delta_1}(x_i)} |u_x| \end{aligned}$$

by lower semicontinuity of $\mathrm{TV}(\cdot)$ with respect to L^1-convergence. The second term of the right-hand side is estimated from below by

$$\sum_{x_i \in \Sigma_{1-\delta}} \alpha\left(\xi_i^- - \delta_1\right) d_i.$$

Note that $\delta_2 < \delta_1$ implies $\mathring{M} \setminus \overline{X}_{\delta_1} \subset \mathring{M} \setminus \overline{X}_{\delta_2}$. Sending $\delta_1 \to 0$ and by definition α_0 we have

$$\alpha(1 - \delta) \int_{\mathring{M} \setminus \Sigma_{1-\delta}} |u_x| + \sum_{x_i \in \Sigma_{1-\delta}} d_i \alpha_i \leq \liminf_{\varepsilon \to 0} \int_{\mathring{M}} \alpha(v_\varepsilon) \left| \frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} \right|.$$

Replacement of $\mathring{M} \setminus \Sigma_{1-\delta}$ by $\mathring{M} \setminus (J_u \cap \Sigma_{1-\delta})$ is rather trivial because outside J_u the set $\Sigma_{1-\delta}$ has measure zero with respect to the measure $|u_x|$. □

We are now in position to give a proof for the Gamma limit of the Kobayashi–Warren–Carter energy.

Proof of Theorem 2.2.3.

(i) (liminf inequality) We may assume that

$$\liminf_{\varepsilon \to 0} E^\varepsilon_{\mathrm{KWC}}(u_\varepsilon, v_\varepsilon) < \infty.$$

By Theorem 2.2.1(i), we see that the limit Ξ satisfies (C2). Let Ω be an open set such that $\overline{\Omega}$ is compact and contained in $\mathring{M} \setminus \Sigma_0$. Assume that

$$\int_{\mathring{M}} \alpha(v_\varepsilon) \left| \frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} \right|$$

is bounded. Since $c := \min_{x\in\Sigma'} \Xi(x) > 0$ and $v_\varepsilon \xrightarrow{g} \Xi$, we set that $v_\varepsilon \geq c/2 > 0$ on $\overline{\Omega}$ for sufficiently small $\varepsilon > 0$. Thus $\displaystyle\int_\Omega \left|\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right|$ is bounded. This implies that the limit $u \in BV(\Omega)$. We now conclude that u satisfies (C3).

Applying Theorem 2.2.1 for $E^\varepsilon_{\mathrm{sMM}}$ and Lemma 2.5.1 for $\displaystyle\int \alpha(v_\varepsilon)\left|\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right|$, we see that

$$\sigma \int_{\mathring{M}\setminus(J_u\cap\Sigma)} \alpha(1)|u_x| + \sigma \sum_{x\in\Sigma'} d_i\alpha_i + E^0_{\mathrm{sMM}}(\Xi, M) \leq \liminf_{\varepsilon\to 0} E^\varepsilon_{\mathrm{KWC}}(u_\varepsilon, v_\varepsilon).$$

The second term in the left-hand side equals $\sigma\sum_{x\in\Sigma} d_i\alpha_i$ since $\xi^-_i = 0$ on Σ_0. Thus the left-hand side equals $E^0_{\mathrm{KWC}}(u, \Xi, M)$. The proof of liminf inequality is now complete.

(ii) (limsup inequality) We take $u_\varepsilon = u$. We notice that Theorem 2.2.1 extends to the case when $E^{0,b}_{\mathrm{sMM}}(\Xi, M)$, $E^{\varepsilon,b}_{\mathrm{sMM}}(v)$ are replaced by

$$E^{0,\{b_\ell\}}_{\mathrm{sMM}}(\Xi, M) := E^0_{\mathrm{sMM}}(\Xi, M) + \sum_{\ell=1}^\infty b_\ell \min_{\xi\in\Xi(a_\ell)} \alpha(\xi),$$

$$E^{\varepsilon,\{b_\ell\}}_{\mathrm{sMM}}(v_\varepsilon) := E^\varepsilon_{\mathrm{sMM}}(v_\varepsilon) + \sum_{\ell=1}^\infty b_\ell\alpha(v_\varepsilon(a_\ell)),$$

where we assume that $\sum_{\ell=1}^\infty b_\ell < \infty$ with $b_\ell \geq 0$ and $a_\ell \in \mathring{M}$ for $\ell = 1, 2, \ldots, m$. Let $\{a_\ell\}$ denote the jump discontinuity of u, that is, $J_u = \{a_\ell\}$. Let b_ℓ denote σ times the jump $d_\ell = |u(a_\ell + 0) - u(a_\ell - 0)|$, that is, $b_\ell = \sigma d_\ell$. Note that $\sum b_\ell < \infty$. By Theorem 2.2.1(ii) for $E^{\varepsilon,\{b_\ell\}}_{\mathrm{sMM}}$, we see that there exist $w_\varepsilon \xrightarrow{g} \Xi$ such that

$$E^{0,\{b_\ell\}}_{\mathrm{sMM}}(\Xi, M) = \lim_{\varepsilon\to 0} E^{\varepsilon,\{b_\ell\}}_{\mathrm{sMM}}(w_\varepsilon). \tag{2.10}$$

We notice that

$$\begin{aligned} E^\varepsilon_{\mathrm{KWC}}(u, w_\varepsilon) &= \sigma \int_{\mathring{M}} \alpha(w_\varepsilon)|u_x| + E^\varepsilon_{\mathrm{sMM}}(w_\varepsilon) \\ &= \sigma \int_{\mathring{M}\setminus\Sigma} \alpha(w_\varepsilon)|u_x| + \sum_{\ell=1}^\infty b_\ell\alpha(w_\varepsilon(a_\ell)) + E^\varepsilon_{\mathrm{sMM}}(w_\varepsilon) \\ &= \sigma \int_{\mathring{M}\setminus\Sigma} \alpha(w_\varepsilon)|u_x| + E^{\varepsilon,\{b_\ell\}}_{\mathrm{sMM}}(w_\varepsilon). \end{aligned}$$

By construction, w_ε is bounded and $w_\varepsilon \to 1$ almost everywhere with respect to the measure $|u_x|$ outside Σ, i.e. $|u_x|\lfloor_{\mathring{M}\setminus\Sigma}$. Since $\alpha(w_\varepsilon)-\alpha(1)$ tends to zero for all x outside Σ and it is bounded, the first term in the right-hand side converges to $\sigma\int_{M\setminus\Sigma}\alpha(1)\,|u_x|$ by a bounded convergence theorem. The convergence (2.10) yields the desired result. □

Chapter 3

A finer singular limit of a single-well Modica-Mortla functional on multi dimensional

In this chapter, we consider the Γ-convergence of a single-well Modica-Mortola energy in a multidimensional domain. We Introduce a new convergence concept called slice graph convergence. Slice graph convergence is, roughly speaking, graph convergence in almost every slice for dense direction. This is because the method used to show Γ-convergence in multidimensional domains, called the "slice method," is also used for finer topology.

3.1 Introduction

In this section, we introduce the notion of sliced graph convergence. We first recall a few basic notion of a set-valued function, especially on the measurability. Consequently, we review the notion of the slicing argument and introduce the concept of the sliced graph convergence.

3.1.1 A set-valued function and its measurability

We first recall a few basic notion of a set-valued function; see [1] for example. Let M be a Borel set in $\mathbb{R}^d$.

Let Γ be a set-valued function on M with values in $2^{\mathbb{R}^m} \backslash \{\emptyset\}$ such that $\Gamma(z)$ is closed in $\mathbb{R}^m$ for all $z \in M$. We simply say that such Γ is a closed set-valued function. We say that Γ is *Borel measurable* if $\Gamma^{-1}(U)$ is a Borel set whenever U is an open set in $\mathbb{R}^m$. Here the inverse $\Gamma^{-1}(U)$ is defined as

$$\Gamma^{-1}(U) := \left\{ z \in M \,\middle|\, \Gamma(z) \cap U \neq \emptyset \right\}.$$

Similarly, we say that Γ is *Lebesgue measurable* if $\Gamma^{-1}(U)$ is Lebesgue measurable whenever U is an open set. Assume that M is closed. We say that Γ is *upper semicontinuous* if $\operatorname{graph}\Gamma$ is closed in $M \times \mathbb{R}^m$, where

$$\operatorname{graph}\Gamma := \left\{ z = (x, y) \in M \times \mathbb{R}^m \,\middle|\, y \in \Gamma(z),\ x \in M \right\}.$$

If Γ is upper semicontinuous, Γ is Borel measurable [1]. Assume that M is compact. Then, $\operatorname{graph}\Gamma$ is compact if it is closed. We set

$$\mathcal{C} = \{\Gamma \mid \operatorname{graph}\Gamma \text{ is compact in } M \times \mathbb{R}^m \text{ and } \Gamma(x) \neq \emptyset \text{ for } z \in M\}.$$

For $\Gamma_1, \Gamma_2 \in \mathcal{C}$, we set

$$d_g(\Gamma_1, \Gamma_2) := d_H(\operatorname{graph}\Gamma_1, \operatorname{graph}\Gamma_2),$$

where d_H denotes the Hausdorff distance of two sets in $M \times \mathbb{R}^m$. Here, d_H is defined in Chapter 2.

We recall a basic properties of a Borel measurable set-valued function [1, Theorem 8.1.4].

Theorem 3.1.1. Let Γ be a closed set-valued function on a Borel set M in $\mathbb{R}^d$ with values in $2^{\mathbb{R}^m} \backslash \{\emptyset\}$. The following three statements are equivalent:

(i) Γ is Borel (resp. Lebesgue) measurable.

(ii) $\operatorname{graph}\Gamma$ is a Borel set ($\mathcal{M} \otimes \mathcal{B}$ measurable set) in $M \times \mathbb{R}^m$.

(iii) There is a sequence of Borel (Lebesgue) measurable functions $\{f_j\}_{j=1}^{\infty}$ such that

$$\Gamma(z) = \overline{\{f_j(z) \mid j = 1, 2, \ldots\}}.$$

Here $\mathcal{M}$ denotes the σ-algebra of Lebesgue measurable sets in M and $\mathcal{B}$ denotes the σ-algebra of Borel sets in $\mathbb{R}^m$.

3.1.2 Sliced graph convergence

We next recall the notation often used in the slicing argument [14].

Let S be a set in $\mathbb{R}^N$. Let $\mathbb{S}^{N-1}$ denote the unit sphere in $\mathbb{R}^N$ centered at the origin, i.e.,

$$\mathbb{S}^{N-1} = \left\{ \nu \in \mathbb{R}^N \;\middle|\; |\nu| = 1 \right\}.$$

For a given ν, let Π_ν denote the hyperplane whose normal equals ν. In other words,

$$\Pi_\nu := \left\{ x \in \mathbb{R}^N \;\middle|\; \langle x, \nu \rangle = 0 \right\}$$

where $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^N$. For $x \in \Pi_\nu$, let $S_{x,\nu}$ denote the intersection of Ω and the whole line with direction ν which contains x. In other words,

$$S_{x,\nu} := \left\{ x + t\nu \;\middle|\; t \in S^1_{x,\nu} \right\},$$

where

$$S^1_{x,\nu} = \left\{ t \in \mathbb{R} \;\middle|\; x + t\nu \in S \right\} \subset \mathbb{R}.$$

We also set

$$S_\nu := \left\{ x \in \Pi_\nu \;\middle|\; S_{x,\nu} \neq \emptyset \right\}.$$

For a given function f on Ω, we define the 1-d restriction function $f_{x,\nu}$ on $\Omega^1_{x,\nu}$ by

$$f_{x,\nu}(t) := f(x + t\nu) \quad (x \in \Omega, \nu \in \mathbb{S}^{N-1}).$$

Let Ω be a bounded domain in $\mathbb{R}^N$, and $\mathcal{T}$ denote the set of all Lebesgue measurable (closed) set-valued function $\Gamma : \Omega \to 2^{\mathbb{R}}$. For $\nu \in \mathbb{S}^{N-1}$, we consider $\Omega^1_{x,\nu} \subset \mathbb{R}$ and the (sliced) set-valued function $\Gamma_{x,\nu}$ on $\Omega^1_{x,\nu}$ defined by $\Gamma_{x,\nu}(t) = \Gamma(x + t\nu)$. Let $\overline{\Gamma_{x,\nu}}$ denote its closure defined on the closure of $\overline{\Omega^1_{x,\nu}}$, i.e., it is uniquely determined so that the graph of $\overline{\Gamma_{x,\nu}}$ equals the closure of graph $\Gamma_{x,\nu}$ in $\mathbb{R}\times\mathbb{R}$. As for a usual measurable function, we identify $\Gamma^{(1)},\Gamma^{(2)}\in\mathcal{T}$ if $\Gamma^{(1)}(z)=\Gamma^{(2)}(z)$ for $\mathcal{L}^N$-a.e. $z\in\Omega$, where $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure. By Fubini's theorem, $\Gamma^{(1)}_{x,\nu}(t)=\Gamma^{(2)}_{x,\nu}(t)$ for $\mathcal{L}^1$-a.e. t for $\mathcal{L}^{N-1}$-a.e. $x\in\Omega_\nu$. With this identification, we consider its equivalence class and we call each $\Gamma^{(1)}$, $\Gamma^{(2)}$ as a representative of this equivalence class. For $\nu\in\mathbb{S}^{N-1}$, we define the subset $\mathcal{B}_\nu\subset\mathcal{T}$ as follows: $\Gamma\in\mathcal{B}_\nu$ if

- There is a representative of $\Gamma_{x,\nu}$ such that $\overline{\Gamma_{x,\nu}}=\Gamma_{x,\nu}$ on $\Omega^1_{x,\nu}$;
- graph $\overline{\Gamma_{x,\nu}}$ is compact in $\overline{\Omega^1_{x,\nu}}\times\mathbb{R}$ for a.e. $x\in\Omega_\nu$.

We note that if $\Gamma^{(1)},\Gamma^{(2)}\in\mathcal{B}_\nu$, then $\overline{\Gamma^{(1)}_{x,\nu}},\overline{\Gamma^{(2)}_{x,\nu}}\in\mathcal{C}$ with $M=\overline{\Omega^1_{x,\nu}}$ by a suitable choice of representative of $\Gamma^{(1)}_{x,\nu},\Gamma^{(2)}_{x,\nu}$, which follows from definition.

In this situation, we have the following fact:

Lemma 3.1.1. The function

$$f(x)=d_g\left(\overline{\Gamma^{(1)}_{x,\nu}},\overline{\Gamma^{(2)}_{x,\nu}}\right)=d_H\left(\operatorname{graph}\Gamma^{(1)}_{x,\nu},\operatorname{graph}\Gamma^{(2)}_{x,\nu}\right)$$

is Lebesgue measurable in Ω_ν.

Proof. By Theorem 3.1.1 (iii), there is a representative of Γ which is Borel measurable. This is because that each Lebesgue measurable function f has a Borel measurable function $\overline{f}$ with $f(z)=\overline{f}(z)$ for $\mathcal{L}^N$-a.e. $z\in\Omega$. By Theorem 3.1.1 (ii), graph Γ is a Borel set for the Borel representative of Γ. Since the graph of the set-valued function $T: x\longmapsto\operatorname{graph}\overline{\Gamma_{x,\nu}}$ on Ω_ν equals graph Γ for $\Gamma\in\mathcal{B}_\nu$ by taking a suitable representative of Γ, we see that T should be Borel measurable if Γ is Borel measurable by Theorem 3.1.1 (ii). (Note that $T(x)$ is a compact set in $\mathbb{R}\times\mathbb{R}$.) Since d_H is continuous, the map $f(x)$ should be measurable. □

We now introduce a metric on $\mathcal{B}_\nu$ of the form

$$d_\nu\left(\Gamma^{(1)},\Gamma^{(2)}\right):=\int_{\Omega_\nu}\frac{d_g\left(\overline{\Gamma^{(1)}_{x,\nu}},\overline{\Gamma^{(2)}_{x,\nu}}\right)}{1+d_g\left(\overline{\Gamma^{(1)}_{x,\nu}},\overline{\Gamma^{(2)}_{x,\nu}}\right)}\,d\mathcal{H}^{N-1}(x)$$

for $\Gamma^1,\Gamma^2\in\mathcal{B}_\nu$, where $\mathcal{H}^{N-1}$ denotes the Hausdorff measure. From Lemma 3.1.2, we see that this is a well-defined quantity for all $\Gamma^{(1)},\Gamma^{(2)}\in\mathcal{B}_\nu$. We identify $\Gamma^{(1)},\Gamma^{(2)}\in\mathcal{B}_\nu$ if $\Gamma^{(1)}_{x,\nu}=\Gamma^{(2)}_{x,\nu}$ for a.e. x. With this identification, $(\mathcal{B}_\nu,d_\nu)$ is indeed a metric space. By a standard argument, we see that $(\mathcal{B}_\nu,d_\nu)$ is a complete metric space; we do not give a proof since we do not use this fact.

Let D be a countable dense set in $\mathbb{S}^{N-1}$. We set

$$\mathcal{B}_D:=\bigcap_{\nu\in D}B_\nu.$$

It is a metric space with metric

$$d_D\left(\Gamma^{(1)},\Gamma^{(2)}\right):=\sum_{j=1}^\infty\frac{1}{2^j}\frac{d_{\nu_j}\left(\Gamma^{(1)},\Gamma^{(2)}\right)}{1+d_{\nu_j}\left(\Gamma^{(1)},\Gamma^{(2)}\right)},$$

where $D = \{\nu_j\}_{j=1}^{\infty}$. (This is also a complete metric space.)

We shall fix D. The convergence with respect to d_D is called the *sliced graph convergence.* If $\{\Gamma_k\} \subset \mathcal{B}_D$ converges to $\Gamma \in \mathcal{B}_D$ with respect to d_D, we simply write $\Gamma_k \xrightarrow{sg} \Gamma$ (as $k \to \infty$). Roughly speaking, we say $\Gamma_k \xrightarrow{sg} \Gamma$ if the graph of the slice Γ_k converges to that of Γ for a.e. $x \in \Omega_\nu$ for any $\nu \in D$. For a function v on Ω, we associate a set-valued function Γ_v by $\Gamma_v(x) = \{v(x)\}$. If $\Gamma_k = \Gamma_{v_k}$ for some v_k, we shortly write, $v_k \xrightarrow{sg} \Gamma$ instead of $\Gamma_{v_k} \xrightarrow{sg} \Gamma$. We note that if $v \in H^1(\Omega)$, the L^2-Sobolev space of order 1, then $\Gamma_v \in \mathcal{B}_D$ for any D.

We conclude this subsection by giving an example that the graph convergence does not imply the sliced graph convergence. Let $C(r)$ denote the circle of radius $r > 0$ centered at the origin in $\mathbb{R}^2$. It is clear that $d_H(C(r), C(r-\varepsilon)) \to 0$ as $\varepsilon > 0$ tends to zero. However, for $\nu = (1,0)$, $C(r-\varepsilon)_{x,\nu}$ with $x = (0, \pm r)$ is empty and does not converge to a single point $C(r)_{x,\nu} = \{(0, \pm r)\}$. In this case, $C(r-\varepsilon)_{x,\nu}$ converges to $C(r)_{x,\nu}$ in the Hausdorff sense except the case $x = (0, \pm r)$. To make the exceptional set has a positive $\mathcal{L}^1$ measure in Π_ν, we recall a thick Cantor set defined by

$$\begin{aligned} G &:= [0,1] \backslash U \\ U &:= \bigcup \left\{ \left(\frac{a}{2^n} - \frac{1}{2^{2n+1}}, \frac{a}{2^n} + \frac{1}{2^{2n+1}} \right) \,\middle|\, n, a = 1, 2, \dots \right\}. \end{aligned}$$

This G is a compact set with positive $\mathcal{L}^1$ measure. We set

$$K := \bigcup_{r \in G} C(r), \quad K_\varepsilon := \bigcup_{r \in G} C(r - \varepsilon).$$

It is clear that K_ε converges to K as $\varepsilon \to 0$ in the Hausdorff distance sense. However, for any $\nu \in \mathbb{S}^1$, $(K_\varepsilon)_{x,\nu}$ does not converge to $(K)_{x,\nu}$ for $x \in \Pi_\nu$ with $|x| \in G$. Based on this set, it is easy to construct an example, that the graph convergence does not imply the sliced graph convergence. Let Ω be an open unit disk centered at the origin. We set

$$\Gamma_\varepsilon(x) := \left\{ \begin{array}{ll} [0,1], & z \in K_\varepsilon \\ \{1\}, & z \in \Omega \backslash K_\varepsilon \end{array} \right., \; \Gamma(x) := \left\{ \begin{array}{ll} [0,1], & z \in K \\ \{1\}, & z \in \Omega \backslash K \end{array} \right. .$$

The graph convergence is equivalent to the Hausdorff convergence of K_ε to K. The sliced graph convergence is equivalent to saying $(K_\varepsilon)_{x,\nu} \to (K)_{x,\nu}$ for $\nu \in D$ and a.e. x, where D is some dense set in S^1. However, because of construction of K_ε and K, we observe for any $\nu \in \mathbb{S}^1$, $K_{x,\nu}$ does not converge to K for x with $|x| \in G$, which has a positive $\mathcal{L}^1$ Lebesgue measure on Π_ν. Thus, Γ_ε does not converges to Γ in the sense of the sliced graph convergence while Γ_ε converges to Γ in the sense of graph convergence.

3.2 Liminf inequality

We recall a single-well Modica–Mortola function $\mathcal{E}^\varepsilon_{\mathrm{sMM}}$ on $H^1(\Omega)$, when Ω is a bounded multidimensional domain in $\mathbb{R}^N$. For $v \in H^1(\Omega)$, we set an integral

$$\mathcal{E}^\varepsilon_{\mathrm{sMM}}(v) := \int_\Omega \left\{ \frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{2\varepsilon} F(v) \right\} \, dx.$$

Here, the potential energy F is a single-well potential. We shall assume that

(F1) $F \in C^1(\mathbb{R})$ is non-negative and $F(v) = 0$ if and only if $v = 1$,

(F2) $\liminf_{|v|\to\infty} F(v) > 0$.

We occasionally impose stronger growth assumption than (F2):

(F2') (monotonicity condition) $F'(v)(v-1) \geq 0$ for all $v \in \mathbb{R}$.

We are interested in the Gamma limit of $\mathcal{E}^{\varepsilon}_{\mathrm{sMM}}$ as $\varepsilon \to 0$ under the sliced graph convergence. We define the subset $\mathcal{A}_0 := \mathcal{A}_0(\Omega) \subset \mathcal{B}_D$ as follows: $\Xi \in \mathcal{A}_0(\Omega)$ if there is a countably $N-1$ rectifiable set $\Sigma \subset \Omega$ such that

$$\Xi(z) = \begin{cases} 1, & z \in \Omega\backslash\Sigma \\ [\xi^-, \xi^+], & z \in \Sigma \end{cases} \tag{3.1}$$

with $\mathcal{H}^{N-1}$-measurable function $\xi_\pm$ on Σ and $\xi^-(z) \leq 1 \leq \xi^+(z)$ for $\mathcal{H}^{N-1}$-a.e. $z \in \Sigma$. For the definition of countably $N-1$ rectifiability, see the beginning of Section 3.2.1. Here $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure.

We briefly remark that the compactness of the elements in $\mathcal{A}_0$. By definition, if Ξ is of the form (3.1), then $\Xi \in \mathcal{J}$. However, there may be a chance that graph $\overline{\Gamma_{x,\nu}}$ is not compact. This happens even for the one-dimensional case ($N = 1$). Indeed, if a set-valued function on $(0,1)$ is of the form,

$$\Xi(z) = \begin{cases} [1, m] & \text{for } z = 1/m \\ \{1\} & \text{otherwise,} \end{cases}$$

then $\overline{\Xi}$ is not compact in $[0,1] \times \mathbb{R}$. It also possible to construct an example that $\overline{\Xi} \neq \Xi$ in $(0,1)$. This is a reason why we impose $\Xi \in \mathcal{B}_D$ in the definition of $\mathcal{A}_0$.

For $\Xi \in \mathcal{A}_0$, we define a functional

$$\mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega) := 2\int_\Sigma \left\{G(\xi^-) + G(\xi^+)\right\} \mathrm{d}\mathcal{H}^{N-1}, \quad \text{where} \quad G(\sigma) := \left|\int_1^\sigma \sqrt{F(\tau)}\,\mathrm{d}\tau\right|.$$

We shall state the liminf inequality for the convergence of $\mathcal{E}^{\varepsilon}_{\mathrm{sMM}}$.

Theorem 3.2.1. Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that F satisfies (F1) and (F2). Assume that $\alpha \in C(\mathbb{R})$ is non-negative. Let D be a countable dense set of $\mathbb{S}^{N-1}$. Let $\{v_\varepsilon\}_{0<\varepsilon<1}$ be in $H^1(\Omega)$ so that $\Gamma_{v_\varepsilon} \in \mathcal{B}_D$. If $v_\varepsilon \xrightarrow{sg} \Xi$ and $\Xi \in \mathcal{A}_0$, then

$$\liminf_{\varepsilon\to 0} \mathcal{E}^{\varepsilon}_{\mathrm{sMM}}(v_\varepsilon) \geq \mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega).$$

3.2.1 Basic properties of a countably $N-1$ rectifiable set

To prove Theorem 3.2.1, we begin with basic properties of a countably $N-1$ rectifiable set. A set S in $\mathbb{R}^N$ is said to be countably $N-1$ rectifiable if

$$S \subset S_0 \cup \left(\bigcup_{j=1}^{\infty} F_j\left(\mathbb{R}^{N-1}\right)\right)$$

where $\mathcal{H}^{N-1}(S_0) = 0$ and $F_j : \mathbb{R}^{N-1} \to \mathbb{R}^N$ are Lipschitz mapping for $j = 1, 2, \ldots$.

Definition 3.2.1. Let $\delta > 0$. A set K in $\mathbb{R}^N$ is δ-flat if there is $V \subset \mathbb{R}^{N-1}$ and a C^1 function $\psi\colon \mathbb{R}^{N-1} \to \mathbb{R}$ and a rotation $A \in SO(N)$ such that

$$K = \left\{ (x, \psi(x))\, A \;\middle|\; x \in V \right\}$$

and $\|\nabla\psi\|_\infty \le \delta$.

Lemma 3.2.1. Let Σ be countably $N-1$ rectifiable set. For any $\delta > 0$, there is a disjoint countable family $\{K_i\}_{i=1}^\infty$ of compact δ-flat sets and $\mathcal{H}^{N-1}$ measure zero N_0 such that

$$\Sigma = N_0 \cup \left(\bigcup_{i=1}^\infty K_i \right).$$

Proof. By [25, Lemma 11.1], there is a countable family of C^1 manifolds $\{M_i\}_{i=1}^\infty$ and N with $\mathcal{H}^{N-1}(N) = 0$ such that

$$\Sigma \subset N \cup \left(\bigcup_{i=1}^\infty M_i \right).$$

Since M_i is C^1 manifold, it can be written as a countable family of δ-flat sets. Thus, we may assume that M_i is δ-flat. We define $\{N_i, \Sigma_i\}_{k=1}^\infty$ inductively by

$$\begin{aligned} &N_i := \Sigma \cap M_1, \quad \Sigma_1 := \Sigma \backslash N_1 \\ &N_{i+1} := \Sigma_i \cap M_{i+1}, \quad \Sigma_{i+1} := \Sigma_i \backslash N_i \ (i = 1, \ldots, \ ). \end{aligned}$$

Here, N_i is $\mathcal{H}^{N-1}$-measurable and $\mathcal{H}^{N-1}(N_i) < \infty$. Since $\mathcal{H}^{N-1}$ is Borel regular, for any δ, there exists a compact set $C \subset N_i$ such that $\mathcal{H}^{N-1}(N_i \backslash C) < \delta$. Thus, there is a disjoint countable family $\{M_{ij}\}_{j=1}^\infty$ of compact sets and $\mathcal{H}^{N-1}$-zero set N_{i0} such that

$$N_i = N_{i0} \cup \left(\bigcup_{j=1}^\infty M_{ij} \right) \quad (i = 1, 2, \ldots).$$

Indeed, we define a sequence of compact sets $\{M_{ij}\}$ inductively by

$$\begin{aligned} &M_{ij+1} \subset N_i \backslash \bigcup_{k=1}^{j} M_{ij}, \ j = 1, 2, \ldots \\ &M_{i1} \subset N_i \end{aligned}$$

such that $\mathcal{H}^{N-1} \left(N_i \backslash \bigcup_{k=1}^{i} M_{ij} \right) < 1/2^i$. Then, setting $N_{i0} = N_i \backslash \bigcup_{j=1}^\infty M_{ij}$ yields the desired decomposition of N_i. Setting

$$N_0 = (N \cap \Sigma) \cup \left(\bigcup_{i=1}^\infty N_{i0} \right)$$

and renumbering $\{M_{ij}\}$ as $\{K_i\}$ to get the desired decomposition. □

3.2.2 Proof of liminf inequality

Proof of Theorem 3.2.1. Let $\{v_\varepsilon\}_{\varepsilon>0} \subset H^1(\Omega)$ be a sequence satisfying $v_\varepsilon \overset{sg}{\longrightarrow} \Xi$. By definition,

$$\overline{v_{\varepsilon,x,\nu}} \overset{g}{\to} \overline{\Xi_{x,\nu}}$$

for $\nu \in D$, $\mathcal{H}^{N-1}$-a.e.$x \in \Pi_\nu$. By Lemma 3.2.1, for any $\delta > 0$, there is a countable disjoint compact δ-flat family $\{K_i\}_{i\in\mathbb{N}}$ and zero set N_0 s.t.

$$\Sigma = \bigcup_{i\in\mathbb{N}} K_i \cup N_0.$$

For each $n \in \mathbb{N}$, we set

$$\Sigma_n := \bigcup_{i=1}^{n} K_i \cup N_0,$$

and take a disjoint open family $\{U_i^n\}_{i\in\mathbb{N}}$ such that $K_i \subset U_i^n$ for each $i = 1, \dots, m$. By definition, K_i is of the form

$$K_i = \{(x', \psi(x'))A \mid x \in V_i\}$$

for some $A_i \in SO(N)$, a compact set $V_i \subset \mathbb{R}^{N-1}$ and $\psi_i \in C^1(\mathbb{R}^{N-1})$ with $\|\nabla\psi_i\|_\infty < \delta$.

Since D is dense in S^{N-1}, we are able to take $\nu^i \subset D$, which is close to the normal of the hyperplane

$$P_i = \left\{(x,0)A_i \;\middle|\; x \in \mathbb{R}^{N-1}\right\}$$

for $i = 1, \dots, m$. By rotating slightly, we may assume that ν_i is a normal of P_i and $\|\nabla\psi_i\|_\infty \le 2\delta$.

By slicing and Fatou's lemma, we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_{U_i^n} \left\{\frac{\varepsilon}{2}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon}F(v_\varepsilon)\right\} dx \\
&= \liminf_{\varepsilon\to 0} \int_{(U_i^n)_{\nu^i}} \int_{(U_i^n)^1_{x,\nu^i}} \left\{\frac{\varepsilon}{2}|\nabla v_\varepsilon|^2_{x,\nu^i} + \frac{1}{2\varepsilon}F(v_{\varepsilon,x,\nu^i})\right\} dt d\mathcal{H}^{N-1}(x) \\
&\ge \liminf_{\varepsilon\to 0} \int_{(U_i^n)_{\nu^i}} \int_{(U_i^n)^1_{x,\nu^i}} \left\{\frac{\varepsilon}{2}|\partial_t(v_{\varepsilon,x,\nu^i})|^2 + \frac{1}{2\varepsilon}F(v_{\varepsilon,x,\nu^i})\right\} dt d\mathcal{H}^{N-1}(x) \\
&\ge \int_{(U_i^n)_{\nu^i}} \liminf_{\varepsilon\to 0} \int_{(U_i^n)^1_{x,\nu^i}} \left\{\frac{\varepsilon}{2}|\partial_t(v_{\varepsilon,x,\nu^i})|^2 + \frac{1}{2\varepsilon}F(v_{\varepsilon,x,\nu^i})\right\} dt d\mathcal{H}^{N-1}(x)
\end{aligned}$$

Applying one-dimensional results in Chapter 2, we get

$$\begin{aligned}
&\int_{(U_i^n)_{\nu^i}} \liminf_{\varepsilon\to 0} \int_{(U_i^n)^1_{x,\nu^i}} \left\{\frac{\varepsilon}{2}|\partial_t(v_{\varepsilon,x,\nu^i})|^2 + \frac{1}{2\varepsilon}F(v_{\varepsilon,x,\nu^i})\right\} dt d\mathcal{H}^{N-1}(x) \\
&\ge \int_{(U_i^n)_{\nu^i}} \sum_{t\in\Sigma^1_{x,\nu^i}\cap(U_i^n)^1_{x,\nu^i}} 2\left\{G(\xi^+_{x,\nu^i}(t)) + G(\xi^-_{x,\nu^i}(t))\right\} d\mathcal{H}^{N-1}(x) \\
&\ge \int_{(U_i^n)_{\nu^i}} \sum_{t\in(K_i)^1_{x,\nu^i}\cap(U_i^n)^1_{x,\nu^i}} 2\left\{G(\xi^+_{x,\nu^i}(t)) + G(\xi^-_{x,\nu^i}(t))\right\} d\mathcal{H}^{N-1}(x).
\end{aligned}$$

Here, we note that $\mathcal{H}^0((K_i)^1_{x,\nu^i} \cap (U_i^n)^1_{x,\nu^i}) = 1$, we set

$$(K_i)^1_{x,\nu^i} \cap (U_i^n)^1_{x,\nu^i} =: \{t_x\},$$

and we denote $\tilde{G}(x) := 2\{G(\xi^+(x)) + G(\xi^-(x))\}$ $(x \in \Sigma)$, then

$$\int_{(U_i^n)_{\nu^i}} \sum_{t \in (K_i)^1_{x,\nu^i} \cap (U_i^n)^1_{x,\nu^i}} 2\left\{G(\xi^+_{x,\nu^i}(t)) + G(\xi^-_{x,\nu^i}(t))\right\} d\mathcal{H}^{N-1}(x) = \int_{(U_i^n)_{\nu^i}} \tilde{G}(x+t_x\nu^i) d\mathcal{H}^{N-1}(x).$$

By the area formula, we see

$$\begin{aligned}
\int_{K_i} \tilde{G}(x) d\mathcal{H}^{N-1}(x) &= \int_{V_i} \tilde{G}((y, \psi(y))A) \cdot \sqrt{1 + |\nabla\psi(y)|^2} d\mathcal{L}^{N-1}(y) \\
&\leq \sqrt{1 + (2\delta)^2} \int_{V_i} \tilde{G}((y, \psi(y))A) d\mathcal{L}^{N-1}(y) \\
&= \sqrt{1 + (2\delta)^2} \int_{(U_i^n)_{\nu^i}} \tilde{G}(x + t_x\nu^i) d\mathcal{H}^{N-1}(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\liminf_{\varepsilon \to 0} \int_\Omega \left\{\frac{\varepsilon}{2}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon}F(v_\varepsilon)\right\} dx \\
&\geq \liminf_{\varepsilon \to 0} \int_{\bigcup_{i=1}^n U_i^n} \left\{\frac{\varepsilon}{2}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon}F(v_\varepsilon)\right\} dx \\
&\geq \sum_{i=1}^n \liminf_{\varepsilon \to 0} \int_{U_i^n} \left\{\frac{\varepsilon}{2}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon}F(v_\varepsilon)\right\} dx \\
&\geq \frac{1}{\sqrt{1 + (2\delta)^2}} \sum_{i=1}^n \int_{K_i} \tilde{G}(x)\ d\mathcal{H}^{N-1}(x) \\
&= \frac{1}{\sqrt{1 + (2\delta)^2}} \int_{\Sigma_n} \tilde{G}(x)\ d\mathcal{H}^{N-1}(x).
\end{aligned}$$

Sending $n \to \infty$ and $\delta \to 0$, we conclude

$$\liminf_{\varepsilon \to 0} \int_\Omega \left\{\frac{\varepsilon}{2}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon}F(v_\varepsilon)\right\} dx \geq \int_\Sigma \tilde{G}\ d\mathcal{H}^{N-1}.$$

□

3.3 Limsup inequality

In this section, we construct what is called $\{w_\varepsilon\}_{\varepsilon>0}$ to establish limsup inequality.

Theorem 3.3.1. Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that F satisfies (F1) and (F2'). Assume that $\alpha \in C(\mathbb{R})$ is non-negative. For any $\Xi \in \mathcal{A}_0$ with $\mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega) < \infty$, there exists a sequence $\{w_\varepsilon\} \subset H^1(\Omega)$ such that

$$\begin{aligned}
&\mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega) \geq \limsup_{\varepsilon \to 0} \mathcal{E}^\varepsilon_{\mathrm{sMM}}(w_\varepsilon), \\
&\lim_{\varepsilon \to 0} d_\nu(\Gamma_{w_\varepsilon}, \Xi) = 0 \quad \text{for all} \quad \nu \in \mathbb{S}^{N-1}.
\end{aligned}$$

In particular, $w_\varepsilon \overset{sg}{\longrightarrow} \Xi$ in $\mathcal{B}_D$ for any $D \subset \mathbb{S}^{n-1}$ with $\overline{D} = \mathbb{S}^{N-1}$. By Theorem 3.2.1,

$$\mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega) = \lim_{\varepsilon \to 0} \mathcal{E}^\varepsilon_{\mathrm{sMM}}(w_\varepsilon).$$

3.3.1 Approximation

We begin with various approximation.

Lemma 3.3.1. Assume that $\Xi \in \mathcal{A}_0$ so that its singular set $\Sigma = \left\{ y \in \Omega \;\middle|\; \Xi(y) \neq \{1\} \right\}$ is countably $N-1$ rectifiable. Let δ be positive. Assume (F1). Then, there exists a sequence $\{\Xi_m\}_{m=1}^\infty \subset \mathcal{A}_0$ such that the following properties hold.

(i) $\mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega) \geq \limsup_{m\to\infty} \mathcal{E}^0_{\mathrm{sMM}}(\Xi_m, \Omega)$,

(ii) $\lim_{m\to\infty} d_\nu(\Xi_m, \Xi) = 0$ for all $\nu \in \mathbb{S}^{N-1}$,

(iii) $\Xi_m(y) \subset \Xi(y)$ for all $y \in \Omega$,

(iv) the singular set $\Sigma_m = \left\{ y \in \Omega \;\middle|\; \Xi_m(y) \neq \{1\} \right\}$ consists of a disjoint finite union of compact δ-flat sets $\{K_j\}_{j=1}^k$ and ξ_m^+, ξ_m^- are constant functions on each K_j $(j = 1, \ldots, k)$, where $\Xi_m(y) = [\xi_m^-(y), \xi_m^+(y)] \ni 1$ on Σ_m. Here k may depend on m.

We recall an elementary fact.

Proposition 3.3.1. *Let $h \in C(\mathbb{R})$ be non-negative such that $h(1) = 0$ and strictly monotone increasing for $\sigma \geq 1$. Let $\{a_j\}_{j=1}^\infty$ be a sequence such that $a_j \geq 1$ $(j = 1, 2, \ldots)$ and*

$$\sum_{j=1}^\infty h(a_j) < \infty.$$

Then

$$\lim_{m\to\infty} \sup_{j\geq m} (a_j - 1) = 0.$$

Proof. By monotonicity of h for $\sigma \geq 1$, we observe that

$$h\left(\sup_{j\geq m} a_j \right) = \sup_{j\geq m} h(a_j) \leq \sum_{j\geq m} h(a_j) \to 0$$

as $m \to \infty$. This yields the desired result since $h(\sigma)$ is strictly monotone for $\sigma \geq 1$. $\square$

We next recall a special case of co-area formula [25, 12.7] for a countably rectifiable set.

Lemma 3.3.2. Let Σ be a countably $N-1$ rectifiable set on Ω. Let g be a $\mathcal{H}^{N-1}$-measurable function on Σ. For $\nu \in \mathbb{S}^{N-1}$, let π_ν denote the restriction on Σ of the orthogonal projection from $\mathbb{R}^N$ to Π_ν. Then

$$\int_\Sigma g J^* \pi_\nu \, d\mathcal{H}^{N-1} = \int_{\Omega_\nu} \left(\int_{\Sigma^1_{x,\nu}} g_{x,\nu}(t) d\mathcal{H}^0(t) \right) d\mathcal{L}^{N-1}(x).$$

Here $J^* f$ denote the Jacobian of a mapping f from Σ to Π_ν.

Proof of Lemma 3.3.1. Step 1. We shall construct Ξ_m satisfying (i) – (iv) except the property that ξ_m^+, ξ_m^- are constants on each K_j.

By Lemma 3.2.1, we found a disjoint family of compact δ-flat sets $\{K_j\}_{j=1}^\infty$ such that $\Sigma = \bigcup_{j=1}^\infty K_j$ up to $\mathcal{H}^{N-1}$-measure zero set for Σ associated with Ξ. By the co-area formula (Lemma 3.3.2) and $J^*\pi_\nu \leq 1$, we observe that

$$\int_{K_j} \tilde{G}(y)\, d\mathcal{H}^{N-1}(y) \geq \int_{K_j} \tilde{G} J^*\pi_\nu \, d\mathcal{H}^{N-1} = \int_{\Omega_\nu} \left(\int_{(K_j)^1_{x,\nu}} \tilde{G}_{x,\nu}(t) d\mathcal{H}^0(t) \right) d\mathcal{L}^{N-1}(x), \tag{3.2}$$

where $\tilde{G}(y) = 2\big(G\left(\xi^+(y)\right) + G\left(\xi^-(y)\right)\big)$. Since $\mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega) < \infty$, we see that

$$\sum_{j=1}^\infty \int_{K_j} \tilde{G}\, d\mathcal{H}^{N-1}(y) < \infty. \tag{3.3}$$

We then take

$$\Xi_m(y) = \begin{cases} [\xi^-(y), \xi^+(y)] & , \ y \in \Sigma_m = \bigcup_{j=1}^m K_j \\ \{1\} & , \ \text{otherwise.} \end{cases}$$

By definition, (i), (iii) are trivially fulfilled. The properly (iv) is fulfilled except the property that ξ^+, ξ^- are constant on each K_j.

It remains to prove (ii). By (3.2) and (3.3), we observe that

$$\sum_{j=1}^\infty \int_{\Omega_\nu} \left(\int_{(K_j)^1_{x,\nu}} \tilde{G}_{x,\nu}(t) d\mathcal{H}^0(t) \right) d\mathcal{L}^{N-1}(x) < \infty$$

for Ξ. Since all integrands are non-negative, the monotone convergence theorem implies that

$$\sum_{j=1}^\infty \int_{\Omega_\nu} \left(\int_{(K_j)^1_{x,\nu}} \tilde{G}_{x,\nu}\, d\mathcal{H}^0 \right) d\mathcal{L}^{N-1}(x) = \int_{\Omega_\nu} \left(\sum_{j=1}^\infty \int_{(K_j)^1_{x,\nu}} \tilde{G}_{x,\nu}\, d\mathcal{H}^0 \right) d\mathcal{L}^{N-1}(x).$$

Thus

$$\sum_{j=1}^\infty \int_{(K_j)^1_{x,\nu}} \tilde{G}_{x,\nu}\, d\mathcal{H}^0 < \infty$$

for $\mathcal{L}^{N-1}$-a.e. $x \in \Omega_\nu$. By Proposition 3.3.1, this yields

$$\lim_{m\to 0} \sup_{j \geq m} \sup_{t \in (K_j)^1_{x,\nu}} \left(\xi^+_{x,\nu}(t) - 1 \right) = 0$$

and similarly,

$$\lim_{m\to 0} \sup_{j \geq m} \sup_{t \in (K_j)^1_{x,\nu}} \left(1 - \xi^-_{x,\nu}(t) \right) = 0.$$

Since

$$d_H\left((\Xi_m)_{x,\nu}\, \Xi_{x,\nu} \right) = \sup_{j \geq m+1} \sup_{t \in (K_j)^1_{x,\nu}} \max\left\{ \left|\xi^+_{x,\nu}(t) - 1\right|, \left|\xi^-_{x,\nu}(t) - 1\right| \right\},$$

we conclude that

$$d_H\left((\Xi_m)_{x,\nu}, \Xi_{x,\nu} \right) \to 0$$

as $m \to \infty$ for a.e. $x \in \Omega_\nu$. Since the integrand of

$$d_\nu\left(\Xi_m, \Xi\right) = \int_{\Omega_\nu} \frac{d_H\left(\left(\Xi_m\right)_{x,\nu}, \Xi_{x,\nu}\right)}{1 + d_H\left(\left(\Xi_m\right)_{x,\nu}, \Xi_{x,\nu}\right)}\, d\mathcal{L}^{N-1}(x)$$

is bounded by 1, the Lebesgue dominated convergence theorem implies (ii).
Step 2. We next approximate Ξ_m constructed by Step 1 and construct a sequence $\{\Xi_{m_k}\}_{k=1}^\infty$ satisfying (i) – (iv) by replacing Ξ by Ξ_m. If such a sequence exists, a diagonal argument yields the desired sequence.

We may assume that

$$\Xi(y) = \begin{cases} \left[\xi^-(y), \xi^+(y)\right], & y \in \Sigma_m = \bigcup_{j=1}^m K_j \\ \{1\} & , \text{ otherwise.} \end{cases}$$

We approximate ξ^+ from below. For a given integer n, we set

$$\xi_n^+(y) = \inf\left\{\xi^+(z) \;\middle|\; z \in I_n^k\right\}, \quad I_n^k = \left\{y \in \Sigma_m \;\middle|\; \frac{k-1}{n} \le \xi^+(y) - 1 < \frac{k}{n}\right\}$$

for $k = 1, 2, \ldots$. Since I_n^k is $\mathcal{H}^{N-1}$-measurable set, as in the proof of Lemma 3.2.1, I_n^k is decomposed as a countably disjoint family of compact sets up to $\mathcal{H}^{N-1}$-measure zero set. We approximate ξ^- from above in a similar way. We set

$$\Xi_n(y) = \begin{cases} \left[\xi_n^-(y), \xi_n^+(y)\right] & , \ y \in \Sigma_m \\ \{1\} & , \text{ otherwise.} \end{cases}$$

It is easy to see that Ξ_n satisfies (iii), (iv) by replacing m by n. Since $\mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega) \ge \mathcal{E}^0_{\mathrm{sMM}}(\Xi_n, \Omega)$, the property (i) follows.

Since

$$d_H\left(\left(\Xi_n\right)_{x,\nu}, \Xi_{x,\nu}\right) = \sup_{t \in (\Sigma_m)^1_{x,\nu}} \max\left\{\left|\xi^+_{x,\nu} - \xi^+_{n,x,\nu}\right|, \left|\xi^-_{x,\nu} - \xi^-_{n,x,\nu}\right|\right\} \le 1/n,$$

we now conclude (ii) as discussed at the end of Step 1. □

3.3.2 Recovery sequences

In this subsection, we shall prove Theorem 3.3.1. A key step is construct a recovery sequence $\{w_\varepsilon\}$ when Ξ has a simple structure. The basic idea is similar to that of [3] and [14]. Besides generalization to general F satisfying (F1) and (F2') from $F(z) = (z-1)^2$, our situation is more involved because $\Xi(y) = [0, 1]$ on $y \in \Sigma$ in their case while in our case $\Xi(y) = [\xi^-(y), \xi^+(y)]$ for a general $\xi^- \le 1 \le \xi^+$. Moreover, we have to show the convergence in d_ν as well as to handle α-term.

Lemma 3.3.3. Assume the same hypotheses concerning Ω, F, and α. For $\Xi \in \mathcal{A}_0$, assume that its singular set $\Sigma = \{x \in \Omega \mid \Xi(x) \ne \{1\}\}$ consists of a disjoint finite union of compact δ-flat sets $\{K_j\}_{j=1}^k$ and ξ^-, ξ^+ are constant functions in each K_j $(j = 1, \ldots, k)$, where $\Xi(x) = [\xi^-, \xi^+]$ on Σ. Then there exists a sequence $\{w_\varepsilon\} \subset H^1(\Omega)$ such that

$$\mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega) \ge \limsup_{\varepsilon \to 0} \mathcal{E}^\varepsilon_{\mathrm{sMM}}(w_\varepsilon),$$
$$\lim_{\varepsilon \to 0} d_\nu(\Gamma_{w_\varepsilon}, \Xi) = 0 \quad \text{for all} \quad \nu \in \mathbb{S}^{N-1}.$$

This follows from explicit construction of function $\{w_\varepsilon\}$ as for the standard double-well Modica-Mortola functional.

Proof. We take a disjoint family of open sets $\{U_j\}_{j=1}^k$ with the property $K_j \subset U_j$. It suffices to construct a desired sequence $\{w_\varepsilon\}$ so that the support of $w_\varepsilon - 1$ is contained in $\bigcup_{j=1}^k U_j^k$. Thus we shall construct such w_ε in each U_j. We may assume $k = 1$ and simply write K_1, U_1 by K, U and ξ_-, ξ_+ by a, b ($a \leq 1 \leq b$) so that

$$\Xi(y) = \begin{cases} [a,b] & , \ y \in K, \\ \{1\} & , \ y \in U \backslash K. \end{cases}$$

For $c < 1$ and $s > 0$, let $\psi(s,c)$ be a function determined by

$$\int_c^\psi \frac{1}{\sqrt{F(z)}} dz = s.$$

By (F1), this equation is uniquely solvable for all $s \in [0, s_*)$ with

$$s_* := \int_c^1 \frac{1}{\sqrt{F(z)}} dz.$$

This $\psi(s,c)$ solves the initial value problem

$$\begin{cases} \dfrac{d\psi}{ds} = \sqrt{F(\psi)}, \quad s \in (0, s_*) \\ \psi(0,c) = c, \end{cases} \tag{3.4}$$

although this ODE may admit many solutions. For $c > 1$, we parallely define ψ by

$$\int_\psi^c \frac{1}{\sqrt{F(z)}} dz = s$$

for $s \in (0, s_*)$ with

$$s_* := \int_1^c \frac{1}{\sqrt{F(z)}} dz.$$

In this case, ψ also solves (3.4). We consider the even extension of ψ still denoted by ψ for $s < 0$ so that $\psi(s,c) = \psi(-s,c)$. For the case $c = 1$, we set $\psi(s,c) \equiv 1$. For a, b with $a \leq 1 \leq b$, we consider a rescaled function $\psi_\varepsilon(s,\cdot) = \psi(s/\varepsilon, \cdot)$ and then define

$$\Psi_\varepsilon(s,a,b) = \begin{cases} \psi_\varepsilon(s,a) & , \quad 0 \leq s \leq \sqrt{\varepsilon} \\ \alpha_1 s + \beta_1 & , \quad \sqrt{\varepsilon} \leq s \leq 2\sqrt{\varepsilon} \\ \psi_\varepsilon(s - 3\sqrt{\varepsilon}, b) & , \quad 2\sqrt{\varepsilon} \leq s \leq 4\sqrt{\varepsilon} \\ \alpha_2 s + \beta_2 & , \quad 4\sqrt{\varepsilon} \leq s \leq 5\sqrt{\varepsilon} \\ 1 & , \quad 5\sqrt{\varepsilon} \leq s \end{cases}$$

with $\alpha_i, \beta_i \in \mathbb{R}$ ($i = 1, 2$) so that Ψ_ε is Lipschitz continuous. We extend Ψ_ε for $s < 0$ so that the extended function (still denoted by Ψ_ε) is even in s. Let η be a minimizer of α in $[a,b]$. We first consider the case when $\eta < 1$ so that $a \leq \eta < 1$. In this case, by definition of Ψ_ε there is a unique $s_0 > 0$ such that $\Psi_\varepsilon(s_0,a,b) = \eta$. We then set

$$\varphi_\varepsilon(s,a,b) = \Psi_\varepsilon(s + s_0, a, b).$$

For the case $\eta \geq 1$, we take the smallest positive $s_0 > 0$ such that $\Psi_\varepsilon(s_0, a, b) = \eta$. This $s_0 = s_0(\varepsilon)$ is of order $\varepsilon^{3/2}$ as $\varepsilon \to 0$.

We then take

$$w_\varepsilon(z) = \varphi_\varepsilon\left(d(z), a, b\right),$$

where $d(z)$ is the distance of z from K. This is a desired sequence such that the support of $w_\varepsilon - 1$ is contained in U for sufficiently small $\varepsilon > 0$. Since w_ε is Lipschitz continuous, it is clear that $w_\varepsilon \in H^1(\Omega)$. Since

$$\nabla w_\varepsilon = \left(\partial_s \Psi_\varepsilon\right)\left(d(z) + s_0, a, b\right) \nabla d(z),$$

we have

$$\begin{aligned} \nabla w_\varepsilon(z) &= \left(\partial_s \psi_\varepsilon\right)\left(d(z) + s_0, a\right) \nabla d(z) \quad \text{for} \quad z, \quad d(z) + s_0 < \sqrt{\varepsilon} \\ &= \frac{1}{\varepsilon}\left(\partial_s \psi\right)\left(\left(d(z) + \delta_0\right)/\varepsilon, a\right) \nabla d(z). \end{aligned}$$

Thus, for z with $d(z) + s_0 < \sqrt{\varepsilon}$, we see that

$$\left|\nabla w_\varepsilon(z)\right|^2 = \frac{1}{\varepsilon^2}\left|\left(\partial_s \psi\right)\left(\left(d(z) + s_0\right)/\varepsilon, a\right)\right|^2.$$

Let U_ε denote the set

$$U_\varepsilon = \left\{z \in \Omega \;\middle|\; d(z) + s_0 < \sqrt{\varepsilon}\right\}.$$

Since s_0 is of order $\varepsilon^{3/2}$, $\overline{U}_\varepsilon$ converges to K in the sense of Hausdorff distance. We proceed

$$\begin{aligned} \mathcal{E}^\varepsilon_{\mathrm{sMM}}(w_\varepsilon, U_\varepsilon) &= \int_{U_\varepsilon} \left\{\frac{\varepsilon}{2}|\nabla w_\varepsilon|^2 + \frac{1}{2\varepsilon} F(w_\varepsilon)\right\} d\mathcal{L}^N \\ &= \frac{1}{2\varepsilon} \int_{U_\varepsilon} \left|\left(\partial_s \psi\right)\left(\left(d(z) + s_0\right)/\varepsilon, a\right)\right|^2 + F\big(\psi\left(\left(d(z) + s_0\right)/\varepsilon, a\right)\big) d\mathcal{L}^N(z) \\ &= \frac{1}{\varepsilon} \int_{U_\varepsilon} F\big(\psi\left(\left(d(z) + s_0\right)/\varepsilon, a\right)\big) d\mathcal{L}^N(z) \end{aligned}$$

by (3.4). To simplify the notation, we set

$$f_\varepsilon(t) = \frac{1}{\varepsilon} F\big(\psi\left((t + s_0)/\varepsilon, a\right)\big) \quad \text{for} \quad t > 0$$

and observe that

$$\mathcal{E}^\varepsilon_{\mathrm{sMM}}(w_\varepsilon, U_\varepsilon) = \int_{U_\varepsilon} f_\varepsilon\left(d(z)\right) d\mathcal{L}^N(z) = \int_0^{\beta_\varepsilon} f_\varepsilon(t) H(t) dt, \quad \beta(\varepsilon) := \sqrt{\varepsilon} - s_0(\varepsilon)$$

with $H(t) := \mathcal{H}^{N-1}\left(\left\{z \in U_\varepsilon \;\middle|\; d(z) = t\right\}\right)$ by the co-area formula.

We set $A(t) := \mathcal{L}^N\left(\left\{z \in U_\varepsilon \;\middle|\; d(z) < t\right\}\right)$ and observe that $A(t) = \int_0^t H(s) ds$ by the co-area formula. Integrating by parts, we observe that

$$\int_0^{\beta(\varepsilon)} f_\varepsilon(t) H(t) dt = f_\varepsilon\left(\beta(\varepsilon)\right) A\left(\beta(\varepsilon)\right) - \int_0^{\beta(\varepsilon)} f'_\varepsilon(t) A(t) dt.$$

By the relation of Minkowski contents and area [11, Theorem 3.2.39], we know that

$$\lim_{t \downarrow 0} A(t)/2t = \mathcal{H}^{N-1}(K).$$

In other words,

$$A(t) = 2\left(\mathcal{H}^{N-1}(K) + \rho(t)\right) t$$

with ρ such that $\rho(t) \to 0$ as $t \to 0$. Thus,

$$-\int_0^{\beta(\varepsilon)} f_\varepsilon'(t)A(t)dt \le -\int_0^{\beta(\varepsilon)} f_\varepsilon'(t)2tdt\left(\mathcal{H}^{N-1}(K) + \max_{0\le t\le\beta(\varepsilon)} \rho(t)_+\right)$$

since $f_\varepsilon'(t) \le 0$. Here we invoke (F2') so that $F'(\sigma) \le 0$ for $\sigma < 1$. We thus observe that

$$\mathcal{E}^\varepsilon_{\mathrm{sMM}}(w_\varepsilon, U_\varepsilon) \le f_\varepsilon\left(\beta(\varepsilon)\right) A\left(\beta(\varepsilon)\right) - \int_0^{\beta(\varepsilon)} f_\varepsilon'(t)2tdt\left(\mathcal{H}^{N-1}(K) + \max_{0\le t\le\beta(\varepsilon)} \rho(t)_+\right).$$

Integrating by parts yields

$$-\int_0^{\beta(\varepsilon)} f_\varepsilon'(t)2tdt = 2\int_0^{\beta(\varepsilon)} f_\varepsilon(t)dt - 2f_\varepsilon\left(\beta(\varepsilon)\right)\beta(\varepsilon).$$

Since $\psi(s) = \psi(s,a)$ solves (3.4), we see

$$\begin{aligned} f_\varepsilon(t - s_0) &= \frac{1}{\varepsilon}F\left(\psi(t/\varepsilon)\right) \\ &= \frac{1}{\varepsilon}(\partial_s\psi)(t/\varepsilon)\sqrt{F\left(\psi(t/\varepsilon)\right)} \\ &= -\frac{d}{dt}\big(G\left(\psi(t/\varepsilon)\right)\big). \end{aligned}$$

Thus

$$\int_0^{\beta(\varepsilon)} f_\varepsilon(t)dt = G\left(\psi(s_0/\varepsilon)\right) - G\left(\psi(1/\sqrt{\varepsilon})\right).$$

Since $s_0/\varepsilon \to 0$, $\psi(1/\sqrt{\varepsilon}, a) \to 1$ as $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon\to 0}\int_0^{\beta(\varepsilon)} f_\varepsilon(t)dt = G(a) - 0.$$

Combing these manipulations, we obtain that

$$\begin{aligned} \limsup_{\varepsilon\to 0}\mathcal{E}^\varepsilon_{\mathrm{sMM}}(w_\varepsilon, U_\varepsilon) \le \limsup_{\varepsilon\to 0} f_\varepsilon\left(\beta(s)\right)&\left\{A\left(\beta(\varepsilon)\right) - 2\left(\mathcal{H}^{N-1}(K) - \max_{0\le t\le\beta(\varepsilon)}|\rho(t)|\right)\beta(\varepsilon)\right\} \\ &+ 2\mathcal{H}^{N-1}(K)G(a) \end{aligned}$$

We thus conclude that

$$\limsup_{\varepsilon\to 0}\mathcal{E}^\varepsilon_{\mathrm{sMM}}(w_\varepsilon, U_\varepsilon) \le 2\mathcal{H}^{N-1}(K)G(a)$$

provided that

$$\lim_{\varepsilon\to 0} f_\varepsilon\left(\beta(\varepsilon)\right)\beta(\varepsilon) < \infty$$

since $\left(A(t) - 2\mathcal{H}^{N-1}(K)t\right) / t = \rho(t) \to 0$ as $t \to 0$. This follows from the next lemma by setting $\varepsilon^{1/2} = \delta$. Indeed, we obtain a stronger result

$$\limsup_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}} f_\varepsilon\left(\beta(\varepsilon)\right)\beta(\varepsilon) < \infty.$$

Lemma 3.3.4. Assume that F satisfies (F1), (F2'). Then, for $c \in \mathbb{R}$,

$$\frac{1}{\delta^2} F\left(\psi(1/\delta, c)\right) \leq (1-c)^2 \quad \text{for} \quad \delta > 0.$$

Proof of Lemma 3.3.4. We may assume $c < 1$ since the argument for $c > 1$ is symmetric and the case $c = 1$ is trivial. We simply write $\psi(s, a)$ by $\psi(s)$. By definition and monotonicity (F2') of F, we see that

$$\frac{1}{\delta} = \int_c^{\psi(1/\delta)} \frac{1}{\sqrt{F(z)}} dz \leq \frac{\psi(1/\delta) - c}{\sqrt{F\left(\psi(1/\delta)\right)}}.$$

Taking square of both sides, we end up with

$$\frac{1}{\delta^2} F\left(\psi(1/\delta)\right) \leq \left(\psi(1/\delta) - c\right)^2 \leq (1-c)^2.$$

□

The part corresponding to $\psi(s, b)$ is similar. The part where Ψ_ε is linear will be vanish as $\varepsilon \to 0$. So, we conclude

$$\lim_{\varepsilon \to 0} \mathcal{E}^{\varepsilon}_{\mathrm{sMM}}(w_\varepsilon, \Omega) \leq \mathcal{E}^{0}_{\mathrm{sMM}}(\Xi, \Omega).$$

The term related to α is independent of ε because of choice of s_0 so that $w_\varepsilon(x) = \eta$ on $x \in K$.

We prepare the following Lemma to prove $w_\varepsilon \xrightarrow{sg} \Xi$.

Lemma 3.3.5. Let $M \subset \mathbb{R}^N$ be a $\mathcal{H}^{N-1}$-measurable set satisfy $\mathcal{H}^{N-1}(M) < \infty$, then $\mathcal{H}^0(M_{x,\nu}) < \infty$ $\mathcal{H}^{N-1}$-a.e. $x \in \Pi_\nu$, for all $\nu \in \mathbb{S}^{N-1}$.

Proof. Let $\pi : \mathbb{R}^N \to \Pi_\nu$ be a orthogonal projection and $p : \Pi_\nu \to \mathbb{R}^{N-1}$ be a isometry map. We set $f := p \circ \pi$. By the co-area formula (Lemma 3.3.2), we see

$$\int_M J^* f \, d\mathcal{H}^{N-1} = \int_{\mathbb{R}^{N-1}} \mathcal{H}^0\left(f^{-1}(y) \cap M\right) \, d\mathcal{L}^{N-1}(y).$$

Since $J^* p = 1$, we get

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \mathcal{H}^0\left(f^{-1}(y) \cap M\right) \, d\mathcal{L}^{N-1}(y) &= \int_{\mathbb{R}^{N-1}} \mathcal{H}^0\left(\pi^{-1} \circ p^{-1}(y) \cap M\right) \, d\mathcal{L}^{N-1}(y) \\ &= \int_{\Pi_\nu} \mathcal{H}^0\left(\pi^{-1}(x) \cap M\right) \, d\mathcal{H}^{N-1}(x).\end{aligned}$$

Notice that $J^* f \leq 1$, we see

$$\int_M J^* f \, d\mathcal{H}^{N-1} \leq \mathcal{H}^{N-1}(M) < \infty.$$

Thus

$$\mathcal{H}^0\left(\pi^{-1}(x) \cap M\right) < \infty,$$

$\mathcal{H}^{N-1}$-a.e. $x \in \Pi_\nu$. By $\pi^{-1}(x) \cap M = M_{x,\nu}$, we now conclude it.

□

We introduce the following Proposition for $z \in K_{x,\nu}$.

Proposition 3.3.2. *Let K be a compact subset of a C^1 manifold M, and T_zK denote the tangent space of M at $z \in K$. Assume that $\Omega_{x,\nu} \not\subseteq T_zK$, then there are $\delta_0 > 0$, $\varepsilon_0 > 0$, and $t \in \Omega^1_{x,\nu}$ such that*

$$d(x + t'\nu, K) \geq 5\sqrt{\varepsilon}$$

for all $t' > t$ with $t' \in B_\delta(z)^1_{x,\nu}$, for all $\delta \in (0, \delta_0)$, for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. Let $p : \mathbb{R}^N \to T_zK$ be a orthogonal projection. For $z' \in \Omega_{x,\nu}$, we see

$$d(z', T_zK) \leq d\big(p(z'), K\big) + d(z', K).$$

and since $\Omega_{x,\nu} \not\subseteq T_zK$, for all $\varepsilon > 0$ there is a $t \in \Omega_{x,\nu}$ such that

$$d(x + t'\nu, T_zK) \geq 6\sqrt{\varepsilon} \tag{3.5}$$

for all t' with $t' > t$. By the implicit function theorem, there is C^1 function $\tilde{\psi} : T_zK \to \mathbb{R}$, $\delta > 0$ and $B \in SO(N)$ such that

$$\{(x', \tilde{\psi}(x'))B \mid x' \in T_zK\} \supset K \cap B_\delta(z).$$

We set

$$\tilde{K} := \{(x', \tilde{\psi}(x')) \mid x' \in T_zK\}.$$

Since $\nabla\tilde{\psi}(z) = 0$, for all $\varepsilon > 0$, there is a $\delta_0 > 0$ such that if $|z - z'| < \delta_0$ then $|\nabla\tilde{\psi}(z')| < 1/\sqrt{\varepsilon}$. We take $\varepsilon > 0$ such that $0 < \varepsilon < \delta_0$ and $z' \in B_\varepsilon(z)$ then

$$\begin{aligned}
d(z', \tilde{K}) &\leq |\tilde{\psi}(z')| \\
&= |\tilde{\psi}(z) - \tilde{\psi}(z')| \\
&\leq \nabla\tilde{\psi}(z') \cdot (z - z') \\
&\leq |\nabla\tilde{\psi}(z')||z - z'| < \sqrt{\varepsilon}.
\end{aligned}$$

Let θ_ν be an angle between $\Omega_{x,\nu}$ and T_zK, then

$$|z - p(z')| = |z - z'| \cos\theta_\nu.$$

Thus if $|z - z'| < \delta_0$ then $|z - p(z')| < \delta_0$. We set $z' = x + t'\nu$ in (3.5) then

$$\begin{aligned}
d(z', K) &\geq d(z', T_zK) - d(p(z'), K) \\
&> 6\sqrt{\varepsilon} - \sqrt{\varepsilon} \\
&= 5\sqrt{\varepsilon}.
\end{aligned}$$

□

Furthermore, we show in the following Lemma that almost all slices of K are transversal.

Lemma 3.3.6. Let $K \subset \mathbb{R}^N$ be a compact subset of a $(N-1)$-dimensional C^1 manifold. For all $\nu \in \mathbb{S}^{N-1}$ K and $\Omega_{x,\nu}$ intersect transversally $\mathcal{H}^{N-1}$-a.e. $x \in K_\nu$. i.e. For all $z \in K_{x,\nu}$, $\Omega_{x,\nu} \not\subseteq T_zK$.

Proof. $\pi : \mathbb{R} \to \Pi_\nu$ be a orthogonal projection and $p : \Pi_\nu \to \mathbb{R}^{N-1}$ be a isometry map. We set $f := p \circ \pi$ and $A = \{z \in K \mid J^* f(z) = 0\}$. By the co-are formula, we see

$$0 = \int_A J^* f \; d\mathcal{H}^{N-1} = \int_{\mathbb{R}^{N-1}} \mathcal{H}^0\big(f^{-1}(y) \cap A\big) \; d\mathcal{L}^{N-1}(y).$$

Thus we get

$$f^{-1}(y) \cap \{J^* f = 0\} = \emptyset$$

for $\mathcal{L}^{N-1}$-a.e. $y \in \mathbb{R}^{N-1}$. Therefore

$$\pi^{-1}(y) \cap \{J^* \pi = 0\} = \emptyset$$

for $\mathcal{H}^{N-1}$-a.e. $y \in \Pi_\nu$. For $z \in K$, we set $x = \pi(z)$. If $\Omega_{x,\nu} \subset T_z K$ then $J^*\pi(z) = 0$. We thus conclude that $\Omega_{x,\nu} \not\subseteq T_z K$ for $\mathcal{H}^{N-1}$-a.e. $x \in T_z K$.

□

We now prove that $w_\varepsilon \xrightarrow{sg} \Xi$. By Lemma 3.3.5, $K_{x,\nu}$ does not have accumulation point. therefore there is a $\delta > 0$ such that for all $z \in K_{x,\nu}$,

$$\mathcal{H}^0\left(B_\delta(z) \cap K_{x,\nu}\right) = 1,$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\nu$. For all $\delta > 0$, by Proposition 3.3.2 and Lemma 3.3.6, there exist $\varepsilon_0 > 0$ such that

$$\big(\text{graph } \Gamma_{w_\varepsilon x,\nu}\big)_\delta \supset \text{graph } \Xi_{x,\nu}$$

$$\big(\text{graph } \Xi_{x,\nu}\big)_\delta \supset \text{graph } \Gamma_{w_\varepsilon x,\nu}$$

on $B_\delta(z)$, for all $\varepsilon \in (0, \varepsilon_0)$. Thus

$$d_H\big(\text{graph } \Xi_{x,\nu}, \text{graph } \Gamma_{w_\varepsilon x,\nu}\big) \leq \delta,$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\nu$. Therefore we get

$$\lim_{\varepsilon \to 0} d_g\big(\overline{\Gamma_{w_\varepsilon x,\nu}}, \overline{\Xi_{x,\nu}}\big) = 0,$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\nu$. By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon \to 0} d_\nu\big(\Gamma_{w_\varepsilon}, \Xi\big) = \lim_{\varepsilon \to 0} \int_{\Omega_\nu} \frac{d_g\big(\overline{\Gamma_{w_\varepsilon x,\nu}}, \overline{\Xi_{x,\nu}}\big)}{1 + d_g\big(\overline{\Gamma_{w_\varepsilon x,\nu}}, \overline{\Xi_{x,\nu}}\big)} \; d\mathcal{H}^{N-1} = 0.$$

Thus for all dense set D in $\mathbb{S}^{N-1}$, we conclude

$$\lim_{\varepsilon \to 0} d_D\big(\Gamma_{w_\varepsilon}, \Xi\big) = 0.$$

□

Proof of Theorem 3.3.1. This follows from Lemma 3.3.1 and Lemma 3.3.3 by a diagonal argument. □

3.4 Gamma-limit of the Kobayashi–Warren–Carter energy

We first recall the Kobayashi–Warren–Carter energy. For a given $\alpha \in C(\mathbb{R})$ with $\alpha \geq 0$, we consider the Kobayashi-Warren-Carter energy of the form

$$\mathcal{E}^{\varepsilon}_{\mathrm{KWC}}(u,v) = \int_{\Omega} \alpha(v)\ d\|Du\| + \mathcal{E}^{\varepsilon}_{\mathrm{SMM}}(v)$$

for $u \in BV(\Omega)$ and $v \in H^1(\Omega)$. The first term is the weighted total variation of u with weight $w = \alpha(v)$. This is defined by

$$\int_{\Omega} w\ d\|Du\| := \sup\left\{ -\int_{\Omega} u \operatorname{div}\varphi \ \mathrm{d}\mathcal{L}^N \ \middle|\ |\varphi(z)| \leq w(z) \text{ a.e. } x,\ \varphi \in C^1_c(\Omega) \right\}$$

for any non-negative Lebesgue measurable function w on Ω.

We next define the functional, which turns to be a singular limit of the Kobayashi-Warren-Carter energy For $\Xi \in \mathcal{A}_0(\Omega)$ let Σ be its singular set in the sense that

$$\Sigma = \left\{ z \in \Omega \ \middle|\ \Xi(z) \neq \{1\} \right\}.$$

For $u \in BV(\Omega)$, u^+ and u^- are defined as follows

$$u^+(x) := \inf\left\{ t \in \mathbb{R} \ \middle|\ \lim_{r \to 0} \frac{\mathcal{L}^N\left(B_r(x) \cap \{u > t\}\right)}{r^N} = 0 \right\},$$

and

$$u^-(x) := \sup\left\{ t \in \mathbb{R} \ \middle|\ \lim_{r \to 0} \frac{\mathcal{L}^N\left(B_r(x) \cap \{u < t\}\right)}{r^N} = 0 \right\}.$$

We then define the limit Kobayashi-Warren-Carter energy:

$$\mathcal{E}^{0}_{\mathrm{KWC}}(u,\Xi,\Omega) = \int_{\Omega\setminus\Sigma} \alpha(1)\ d\|Du\| + \int_{\Sigma} \alpha_0(z)|u^+ - u^-|\ \mathrm{d}\mathcal{H}^{N-1}(z) + \mathcal{E}^{0}_{\mathrm{SMM}}(\Xi,\Omega).$$

with

$$\alpha_0(z) := \min\left\{ \alpha(\xi) \ \middle|\ \xi^-(z) \leq \xi \leq \xi^+(z) \right\}.$$

We are now in position to state our main results in a vigorous way.

Theorem 3.4.1. Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that F satisfies (F1) and (F2) and that $\alpha \in C(\mathbb{R})$ is non-negative.

(i) (liminf inequality) Assume that $\{u_\varepsilon\}_{0<\varepsilon<1} \subset BV(\Omega)$ converges to $u \in BV(\Omega)$ in L^1, i.e., $\|u_\varepsilon - u\|_{L^1} \to 0$. Assume that $\{u_\varepsilon\}_{0<\varepsilon<1} \subset H^1(\Omega)$. If $v_\varepsilon \xrightarrow{sg} \Xi$ and $\Xi \in \mathcal{A}_0$, then

$$\mathcal{E}^{0}_{\mathrm{KMC}}(u,\Xi\ \Omega) \leq \liminf_{\varepsilon \to 0} \mathcal{E}^{\varepsilon}_{\mathrm{KMC}}(u_\varepsilon, v_\varepsilon).$$

(ii) (limsup inequality) For any $\Xi \in \mathcal{A}_0$ and $u \in BM(\Omega)$ there exist a family of Lipschitz functions $\{w_\varepsilon\}_{0<\varepsilon<1}$ such that

$$\mathcal{E}^{0}_{\mathrm{KMC}}(u,\Xi,\Omega) = \lim_{\varepsilon \to 0} \mathcal{E}^{\varepsilon}_{\mathrm{KMC}}(u, w_\varepsilon).$$

Corollary 3.4.1. Assume the same hypotheses of Theorem 3.4.1. Assume that $f \in L^2(\Omega)$ and $\lambda \geq 0$. Then the results of Theorem 3.4.1 still holds of E^0_{KMC} as we replaced by

$$\mathcal{E}^0_{\mathrm{KMC}}(u, \Xi, \Omega) + \frac{\lambda}{2} \int_\Omega |u-f|^2 \, \mathrm{d}\mathcal{L}^N, \quad \mathcal{E}^\varepsilon_{\mathrm{KMC}}(u, v) + \frac{\lambda}{2} \int_\Omega |u-f|^2 \, \mathrm{d}\mathcal{L}^N,$$

respectively provided that $u \in L^2(\Omega)$.

3

3.4.1 liminf inequality of KWC

We recall a few properties of the measure $\langle Du, \nu \rangle$ for $u \in BV(\Omega)$, where Du denotes the distributional gradient of u and $\nu \in \mathbb{S}^{N-1}$. The next disintegration lemma is found in [[2], Theorem 3.107] .

Lemma 3.4.1. *For $u \in BV(\Omega)$ and $\nu \in \mathbb{S}^{N-1}$,*

$$|\langle Du, \nu \rangle| = \mathcal{H}^{N-1} \lfloor_{\Omega_\nu} \otimes \|Du_{x,\nu}\|.$$

In other words,

$$\int_\Omega \varphi \; d|\langle Du, \nu \rangle| = \int_{\Omega_\nu} \int_{\Omega^1_{x,\nu}} \varphi_{x,\nu} \; d\|Du_{x,\nu}\| \; d\mathcal{H}^{N-1}(x)$$

for any bounded Borel function $\varphi : \Omega \to \mathbb{R}$.

We also need a representation of total variation of a vector-valued measure and its component. We consider a division of $\mathbb{R}^N$ into a family of rectangles of the form

$$R^\tau_J = \prod_{i=1}^N [a_{j_i}, a_{j_i+1}), \quad J = (j_1, \ldots, j_N) \in \mathbb{Z}^N$$

with $a_{j_i+1} < a_{j_i} + \tau$ $(i = 1, \ldots, N)$ for a given $\tau > 0$. We say that the divition $\{R^\tau_J\}$ is a τ-rectangular division.

Lemma 3.4.2. *Let μ be an $\mathbb{R}^d$-valued finite Radon measure in a domain Ω in $\mathbb{R}^N$. Let $\{\tau_k\}$ be a decreasing sequence converging to zero as $k \to \infty$. Let $\{R^{\tau_k}_J\}$ be a fixed τ_k-rectangular division of $\mathbb{R}^N$. Let D be a dense subset of $\mathbb{S}^{N-1}$. Then*

$$|\mu|(A) = \sup \left\{ |\langle \mu, \nu_k \rangle| (A) \;\middle|\; \nu_k : \Omega \to D, \; \nu_k \textit{ is constant on } R^{\tau_k}_J \cap \Omega, \; J \in \mathbb{Z}^N, \; k = 1, 2, \ldots \right\},$$

where A is a Borel set.

We postpone its proof at the end of this section.

Proof of Theorem 3.4.1 (i). We recall the decomposition of Σ into a countable disjoin union of δ-flat compact sets K_i up to $\mathcal{H}^{N-1}$-measurable zero set and take the corresponding $\nu_i \in D$ as in Theorem 3.2. We use the notation in 3.2. We may assume that $\cap_{m=1}^\infty U_i^m = K_i$. By Lemma 3.4.1, we proceed

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_{U_i^m}\alpha(v_\varepsilon)\ d\|Du_\varepsilon\| \\
&\geq \liminf_{\varepsilon\to 0}\int_{U_i^m}\alpha(v_\varepsilon)\ d|\langle Du_\varepsilon,\nu_i\rangle| \\
&= \liminf_{\varepsilon\to 0}\int_{(U_i^m)_{\nu_i}}\int_{(U_i^m)^1_{x,\nu_i}}\alpha(v_{\varepsilon,x,\nu_i})\ d\|Du_{\varepsilon,x,\nu_0}\|\ d\mathcal{H}^{N-1}(x).
\end{aligned}
$$

Applying one-dimensional results of Theorem 2.2.3 in Chapter 2, we see that

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_{(U_i^m)^1_{x,\nu_i}}\alpha(v_{\varepsilon,x,\nu_0})\ d\|Du_{\varepsilon,x,\nu_i}\| \\
&\geq \int_{(U_i^m)^1_{x,\nu_i}\setminus\Sigma^1_{x,\nu_i}}\alpha(1)\ d\|Du\| + \sum_{t\in(\Sigma\cap U_i^m)^1_{x,\nu_i}}\left(\min_{\xi^-_{x,\nu_i}\leq\xi\leq\xi^+_{x,\nu_i}}\alpha(\xi)\right)|u^+_{x,\nu_i}-u^-_{x,\nu_i}|(t) \\
&\geq \int_{((U_i^m)_{\nu_i}}\sum_{t\in(\Sigma\cap U_i^m)^1_{x,\nu_i}}\alpha_{0,x,\nu_0}(t)|u^+_{x,\nu_0}-u^-_{x,\nu_0}|(t)\ d\mathcal{H}^{N-1}(x).
\end{aligned}
$$

We set $\Sigma^1_{x,\nu_0}=\{t_x\}$, then

$$
\begin{aligned}
&\int_{(U_i^m)_{\nu_i}}\sum_{t\in(\Sigma\cap U_i^m)^1_{x,\nu_i}}\alpha_{0,x,\nu_i}(t)|u^+_{x,\nu_0}-u^-_{x,\nu_0}|(t)\ d\mathcal{H}^{N-1}(x) \\
&= \int_{(\Sigma_\eta)_{\nu_0}}\alpha_0(x+t_x\nu_0)|u^+-u^-|(x+t_x\nu_0)\ d\mathcal{H}^{N-1}(x).
\end{aligned}
$$

By area formula, we see

$$
\int_{K_i^m}\alpha_0|u^+-u^-|\ d\mathcal{H}^{N-1}\leq\sqrt{1+(2\delta)^2}\int_{(K_i^m)_{\nu_i}}\alpha_0(x+t^i_x\nu_i)|u^+-u^-|(x+t^i_x\nu_i)\ d\mathcal{H}^{N-1}(x).
$$

Combining these observation, by Fatou's lemma we conclude that

$$
\liminf_{\varepsilon\to 0}\int_{U_i^m}\alpha(v_\varepsilon)\ d\|Du_\varepsilon\|\geq\frac{1}{\sqrt{1+(2\delta)^2}}\int_{K_i^m}\alpha_0|u^+-u^-|\ d\mathcal{H}^{N-1}.
$$

Adding from $i=1$ to m, we conclude that

$$
\liminf_{\varepsilon\to 0}\int_{\bigcup_{i=1}^m U_i^m}\alpha(v_\varepsilon)\ d\|Du_\varepsilon\|\geq\frac{1}{\sqrt{1+(2\delta)^2}}\int_{\Sigma^m}\alpha_0|u^+-u^-|\ d\mathcal{H}^{N-1}.
$$

For $W^m=\Omega\setminus\bigcup_{i=1}^m U_i^m$, we take $\nu\in D$ and argue in the same way to get

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_{W^m}\alpha(v_\varepsilon)\ d\|Du_\varepsilon\| \\
&\geq \int_{(W^m)_\nu}\int_{(W^m\setminus\Sigma)^1_{x,\nu}}\alpha(1)\ d\|Du_{x,\nu}\|\ d\mathcal{H}^{N-1}(x) \\
&= \int_{W^m\setminus\Sigma}\alpha(1)\ d\left|\langle Du,\nu\rangle\right|.
\end{aligned}
$$

The last equality follows from Lemma 3.2. Since $W^m \cap \Sigma_m = \emptyset$, combining the estimate of the integral on V^m, we now observe that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_\Omega \alpha(v_\varepsilon)\ d\|Du_\varepsilon\| &\geq \liminf_{\varepsilon\to 0} \int_{W^m\setminus(\Sigma\setminus\Sigma_m)} \alpha(v_\varepsilon) d\left|\langle Du, \nu\rangle\right| + \liminf_{\varepsilon\to 0}\int_{V^m} \alpha(v_\varepsilon)\ d\|Du_\varepsilon\| \\ &\geq \alpha(1) \int_{W^m\setminus(\Sigma\setminus\Sigma_m)} d\left|\langle Du, \nu\rangle\right| + \frac{1}{\sqrt{1+(2\delta)^2}} \int_{\Sigma_m} \alpha_0 \left|u^+ - u^-\right| d\mathcal{H}^{N-1}.\end{aligned}$$

Sending $m \to \infty$ yields

$$\liminf_{\varepsilon\to 0} \int_\Omega \alpha(v_\varepsilon)\ d\|Du_\varepsilon\| \geq \alpha(1) \int_{\Omega\setminus\Sigma} d\left|\langle Du, \nu\rangle\right| + \frac{1}{\sqrt{1+(2\delta)^2}} \int_{\Sigma} \alpha_0 \left|u^+ - u^-\right| d\mathcal{H}^{N-1}$$

by Fatou's lemma. Since $\delta > 0$ can be taken arbitrary, we now conclude that

$$\liminf_{\varepsilon\to 0} \int_\Omega \alpha(v_\varepsilon)\ d\|Du_\varepsilon\| \geq \alpha(1) \int_{\Omega\setminus\Sigma} d\left|\langle Du, \nu\rangle\right| + \int_{\Omega\cap\Sigma} \alpha_0 \left|u^+ - u^-\right| d\mathcal{H}^{N-1}.$$

For any $\nu \in D$, we may replace Ω by an open set in Ω, for example, $\Omega_0 \cap \Omega$ where Ω_0 is an open rectangle. Applying the co-area formula (or just the Fubini's theorem) to the projection $(x_1, \ldots, x_N) \longmapsto x_i$, for $\mathcal{L}^1$-a.e. q the $\mathcal{H}^{N-1}(\Sigma \cap \{x_i = q\}) = 0$, since otherwise $\mathcal{L}^N(\Sigma) > 0$. Thus, for any $\tau > 0$, there is a τ-rectangular division $\{R_J^\tau\}_J$ with $\mathcal{H}^{N-1}(\partial R_J^\tau \cap \Sigma) = 0$. Since $\mathcal{H}^{N-1}(\partial R_J^\tau \cap \Sigma) = 0$, by dividing Ω into $\{\Omega \cap R_J^\tau\}_J$, we conclude that

$$\liminf_{\varepsilon\to 0} \int_\Omega \alpha(v_\varepsilon)\ d\|Du_\varepsilon\| \geq \alpha(1) \int_{\Omega\setminus\Sigma} d\left|\langle Du, \nu(x)\rangle\right| + \int_{\Omega\cap\Sigma} \alpha_0 \left|u^+ - u^-\right| d\mathcal{H}^{N-1}$$

where $\nu : \Omega \to D$ is a constant on each rectangle. Applying Lemma 3.4.2, we now conclude that

$$\liminf_{\varepsilon\to 0} \int_\Omega \alpha(v_\varepsilon)\ d\|Du_\varepsilon\| \geq \alpha(1) \int_{\Omega\setminus\Sigma} d\|Du\| + \int_{\Omega} \alpha_0 \left|u^+ - u^-\right| d\mathcal{H}^{N-1}.$$

Since we already obtained

$$\liminf_{\varepsilon\to 0} \mathcal{E}^\varepsilon_{\mathrm{sMM}}(v_\varepsilon) \geq \mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega)$$

by Theorem 3.2.1 and since

$$\mathcal{E}^\varepsilon_{\mathrm{KWC}}(v) = \mathcal{E}^\varepsilon_{\mathrm{sMM}}(v) + \int_\Omega \alpha(v) d\|Du\|,$$

the desired liminf inequality follows.

□

Proof of Lemma 3.4.2. We may assume that A is open since μ is a Radon measure. By duality representation,

$$|\mu|(A) = \sup\left\{ \sum_{i=1}^d \int_A \varphi_i d\mu_i \ \middle|\ \varphi = (\varphi_1, \ldots, \varphi_d) \in C_c(A),\ \|\varphi\|_{L^\infty} \leq 1 \right\},$$

where $C_c(A)$ denotes the space of ($\mathbb{R}^d$-valued) continuous functions compactly supported in A and $\|\varphi\|_\infty := \sup_{x\in\Omega} |\varphi(x)|$ with the Euclidean norm $|a| = \langle a, a\rangle^{1/2}$ for $a \in \mathbb{R}^d$. Since $\mu(A) < \infty$, by this representation, we see that for any $\delta > 0$, there exists $\varphi \in C_c(A)$ with $\|\varphi\|_\infty \leq 1$ satisfying

$$|\mu|(A) \leq \sum_{i=1}^{d} \int_A \varphi_i d\mu_i + \delta.$$

Since φ is uniformly continuous in A and D is dense, for sufficiently large k there is τ_k-rectangular division $\{R_J^{\tau_k}\}$ and $\nu_k^\delta : \Omega \to D$ which is constant on $R_J^{\tau_k} \cap \Omega$ and that

$$\left|\varphi - \nu_k^\delta c_k\right| < \delta \quad \text{in} \quad R_J^{\tau_k} \cap \Omega$$

with some constant $0 \leq c_k \leq 1$. This implies that

$$\begin{aligned}\sum_{i=1}^{d} \int_A \varphi_i d\mu_i &\leq \sum_J \int_{R_J^{\tau_k}\cap A} c_k d\langle \mu, \nu_k^\delta\rangle + \delta|\mu|(A) \\ &\leq \left|\langle \mu, \nu_k^\delta\rangle\right|(A) + \delta|\mu|(A).\end{aligned}$$

Thus

$$|\mu|(A) \leq \left|\langle \mu, \nu_k^\delta\rangle\right|(A) + \delta + \delta|\mu|(A).$$

Since $\mu(A) < \infty$, and $\delta > 0$ is arbitrary, this implies

$$|\mu|(A) \leq \sup\left\{ |\langle \mu, \nu_k\rangle|(A) \;\middle|\; \nu_k : \Omega \to D,\ \nu_k \text{ is constant on } R_J^{\tau_k} \cap \Omega,\ J \in \mathbb{Z}^N,\ k = 1, 2, \ldots \right\}.$$

The reverse inequality is trivial, so the proof is now complete. □

3.4.2 limsup inequality of KWC

Proof of Theorem 3.4.1 (ii). We take w_ε in Theorem 3.3.1 we see

$$\mathcal{E}^0_{\mathrm{sMM}}(\Xi, \Omega) = \lim_{\varepsilon\to 0} \mathcal{E}^\varepsilon_{\mathrm{SMM}}(w_\varepsilon).$$

Since

$$\int_\Omega \alpha(w_\varepsilon)\ d\|Du\| = \int_{\Omega\setminus S_u} \alpha(w_\varepsilon)\ d\|Du\| + \int_{S_u} |u^+ - u^-|\alpha(w_\varepsilon)\ d\mathcal{H}^{N-1},$$

it suffices to prove that

$$\lim_{\varepsilon\to 0} \int_{\Omega\setminus\Sigma} \alpha(w_\varepsilon)\ d\|Du\| = \int_{\Omega\setminus\Sigma} \alpha(1)\ d\|Du\|.$$

We decompose

$$\limsup_{\varepsilon\to 0} \int_\Omega \alpha(w_\varepsilon)\ d\|Du\| = \limsup_{\varepsilon\to 0} \left(\int_{\Omega\setminus\Sigma_{5\sqrt{\varepsilon}}} \alpha(1)\ d\|Du\| + \int_{\Sigma_{5\sqrt{\varepsilon}}} \alpha(w_\varepsilon)\ d\|Du\| \right).$$

We have

$$\begin{aligned}\limsup_{\varepsilon\to 0} \int_{\Omega\setminus\Sigma_{5\sqrt{\varepsilon}}} \alpha(1)\ d\|Du\| &= \limsup_{\varepsilon\to 0} \alpha(1)\|Du\|(\Omega \setminus \Sigma_{5\sqrt{\varepsilon}}) \\ &= \alpha(1)\|Du\| \left(\Omega \setminus \bigcap_{\varepsilon>0} \Sigma_{5\sqrt{\varepsilon}}\right) \\ &= \alpha(1)\|Du\|(\Omega \setminus \Sigma).\end{aligned}$$

Since w_ε is bounded, $\alpha(w_\varepsilon)$ is also bounded. We assume $\alpha(w_\varepsilon) \le M$, then we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{\Sigma_{5\sqrt{\varepsilon}}} \alpha(w_\varepsilon)\ d\|Du\| \\
&= \limsup_{\varepsilon\to 0} \left(\int_{\Sigma_{5\sqrt{\varepsilon}}\setminus\Sigma} \alpha(w_\varepsilon)\ d\|Du\| + \int_{\Sigma} \alpha(w_\varepsilon)\ d\|Du\| \right) \\
&\le \limsup_{\varepsilon\to 0} \left(M\|Du\|(\Sigma_{5\sqrt{\varepsilon}} \setminus \Sigma) + \int_{\Sigma} \alpha(w_\varepsilon)\ d\|Du\| \right).
\end{aligned}$$

By the Lebesgue dominated convergence theorem we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \left(M\|Du\|(\Sigma_{5\sqrt{\varepsilon}} \setminus \Sigma) + \int_{\Sigma} \alpha(w_\varepsilon)\ d\|Du\| \right) \\
&= \limsup_{\varepsilon\to 0} \left(M\|Du\|(\Sigma_{5\sqrt{\varepsilon}} \setminus \Sigma) + \int_{\Sigma} \alpha_0\ d\|Du\| \right) \\
&= \limsup_{\varepsilon\to 0} \left(M\|Du\|(\Sigma_{5\sqrt{\varepsilon}} \setminus \Sigma) + \int_{\Sigma} \alpha_0 |u^+ - u^-|\ d\mathcal{H}^{N-1} \right) \\
&= \int_{\Sigma} \alpha_0 |u^+ - u^-|\ d\mathcal{H}^{N-1}.
\end{aligned}$$

The proof is now complete.

□

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