博士論文

論文題目　　Studies on semiclassical analysis and resonance theory
（半古典解析と共鳴理論の研究）

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Studies on semiclassical analysis and resonance theory

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Abstract

In this thesis, we describe several new results in semiclassical analysis and resonance theory. We first discuss semiclassical analysis and the Agmon-Finsler metric for discrete Schrödinger operators. We next discuss resonances and viscosity limit for the Wigner-von Neumann-type Hamiltonians. We finally discuss the complex absorbing potential method for Stark resonances.

1 Introduction

Semiclassical analysis and resonance theory are rich areas in spectral theory and mathematical quantum mechanics. Semiclassical analysis studies the quantum-classical correspondence in the semiclassical limit in quantum mechanics. A parameter-dependent formalism of microlocal analysis is a powerful method in semiclassical analysis (see [61]). While the classical microlocal analysis emphasizes the study of singularities, the semiclassical microlocal analysis is useful in the asymptotic analysis. It is richer than the classical theory since lower order parts of (pseudo)differential operators also play the principal role in the semiclassical limit. Resonances correspond to quasi-steady states and are closely related to scattering theory. They are hidden complex eigenvalues of self-adjoint Schrödinger operators and are closely related to the spectral theory of non-self-adjoint operators. Mathematical theory of resonances is full of interesting ideas and methods (see [14]).

We first discuss semiclassical analysis and the Agmon-Finsler metric for discrete Schrödinger operators. This part is based on [32]. We discuss discrete Schrödinger operators in the semiclassical setting where semiclassical continuous Schrödinger operators are discretized with the mesh size proportional to the semiclassical parameter. In this setting, we first prove the Weyl law for the number of eigenvalues. We also prove the semiclassical Agmon estimate for the eigenfunctions with the Agmon metric defined as a Finsler metric rather than a Riemannian metric. We call it Agmon-Finsler metric. We then construct approximate eigenfunctions by the WKB method near a local potential minimum in terms of the Agmon-Finsler metric. We also prove the Agmon estimate and the exponential decay of eigenfunctions for non-semiclassical discrete Schrödinger operators. This exponential decay is optimal for a concrete example.

We next discuss resonances and the complex absorbing potential method for the Wigner-von Neumann-type Hamiltonian. This part is based on [31], which is a joint work with Shu Nakamura. The complex absorbing potential method is also called as "resonances as viscosity limits". In this method we consider the Hamiltonian with a complex absorbing potential added and take the limit where the absorbing coefficient tends to zero. Then the resonances for the original Hamiltonian are characterized as limit points of discrete eigenvalues of Hamiltonian with complex absorbing potential. Wigner-von Neumann-type Hamiltonians are Schrödinger operators with oscillatory and slowly decaying potentials. We define the resonances and justify the complex absorbing potential method for the Wigner-von Neumann-type Hamiltonian by introducing the periodic complex distortion in the Fourier space.

We finally discuss the complex absorbing method for Stark resonances. This part is based on [33]. Stark Hamiltonians describe the particles in the external electric field. We characterize the resonances for Stark Hamiltonians by the complex absorbing potential method. The proof is based on the complex distortion outside a cone introduced in the master thesis by the author. Perturbations may be potentials with local singularities such as the Coulomb potential.

This thesis is organized as follows. In Section 2, we discuss semiclassical analysis and the Agmon-Finsler metric for discrete Schrödinger operators. In Section 3, we discuss resonances and the complex absorbing potential method for the Wigner-von Neumann-type Hamiltonian. In Section 4, we discuss the complex absorbing potential method for Stark resonances.

2 Semiclassical analysis and the Agmon-Finsler metric for discrete Schrödinger operators

2.1 Introduction to Section 2

In this section, we mainly consider the following semiclassical setting for discrete Schrödinger operators. In the appendix to this section, we also discuss the non-semiclassical standard setting. We first set a continuous semiclassical Schrödinger operator

$$H^{\mathrm{cont}} = H^{\mathrm{cont}}(h) = -h^2\Delta + V(x) \quad \text{on} \quad L^2(\mathbb{R}^d),$$

where $V \in C^\infty(\mathbb{R}^d;\mathbb{R})$ is a potential. The dimension $d \in \mathbb{Z}_{>0}$ is fixed throughout this section. We discretize this operator with mesh size $\tau > 0$ and obtain a discrete Schrödinger operator $H^\tau(h)$ on $\ell^2(\tau\mathbb{Z}^d)$ defined by

$$H^\tau(h)u(x) = -\left(\frac{h}{\tau}\right)^2 \sum_{|x-y|=\tau} (u(y)-u(x)) + V(x)u(x).$$

Here $x, y \in \tau\mathbb{Z}^d \subset \mathbb{R}^d$ and $u \in \ell^2(\tau\mathbb{Z}^d)$.

In the limit $h \to 0$ for fixed $\tau > 0$, the $H^\tau(h)$ converges to $V(x)$ on $\ell^2(\tau\mathbb{Z}^d)$ since the difference Laplacian is a bounded operator. We note that $h^{-2}H^\tau(h)$ when $h \to 0$ for fixed $\tau > 0$ is studied in [3]. The limit $\tau \to 0$ for fixed $h > 0$ is the continuum limit problem and we expect that "$\lim_{\tau\to 0} H^\tau(h) = H^{\mathrm{cont}}(h)$" (see for instance, [28] [40]). This formally corresponds to $\tau = h^\infty$. In this section, we put $\tau = h$. It will be interesting to study the case of $\tau = h^\mu$ with $1 < \mu < \infty$.

Then our semiclassical discrete Schrödinger operator $H(h)$ on $\ell^2(h\mathbb{Z}^d)$ is defined by

$$H(h)u(x) = -\sum_{|x-y|=h} (u(y) - u(x)) + V(x)u(x).$$

Here $x, y \in h\mathbb{Z}^d \subset \mathbb{R}^d$ and $u \in \ell^2(h\mathbb{Z}^d)$. We set $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$. In this section, the semiclassical discrete Fourier transform $\mathcal{F}_h : \ell^2(h\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$ is defined by

$$\mathcal{F}_h u(\xi) = (2\pi)^{-d/2} \sum_{x\in h\mathbb{Z}^d} u(x)e^{i\langle x,\xi\rangle/h}.$$

Then we have

$$\widetilde{H}(h) := \mathcal{F}_h H(h)\mathcal{F}_h^{-1} = \sum_{j=1}^{d} (2 - 2\cos\xi_j) + V(hD_\xi).$$

Here $V(hD_\xi)$ is the semiclassical pseudodifferential operator on $\mathbb{T}^d$ with the symbol $V(x)$ (see subsection 2.2 for the definition). We interpret $x \in \mathbb{R}^d$ as the dual variable of $\xi \in \mathbb{T}^d$ on $T^*\mathbb{T}^d$. Then $\widetilde{H}(h)$ is the semiclassical quantization of $p(\xi, x) = \sum_{j=1}^d (2 - 2\cos\xi_j) + V(x) \in C^\infty(T^*\mathbb{T}^d)$ on the torus. A quantum-classical correspondence is obtained and it is expected that various quantities related to $H(h)$ are described in terms of $p(\xi, x)$ when $h \to 0$, that is, "$\lim_{h\to 0} H(h) = p(\xi, x)$".

We observe the Weyl law in this semiclassical setting. We denote the number of eigenvalues of $H(h)$ in $[a, b]$ by $N_{[a,b]}(h)$. We denote $\langle x\rangle = (1 + x^2)^{1/2}$.

Theorem 1. *Assume $V \in C^\infty(\mathbb{R}^d; \mathbb{R})$, $\underline{\lim}_{|x|\to\infty} V(x) \geq 0$. Moreover assume that there exits $0 < \theta \leq 1$ such that*

$$|\partial^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-\theta|\alpha|}$$

for any $\alpha \in \mathbb{Z}^d_{\geq 0}$. Then for any fixed $a < b < 0$,

$$N_{[a,b]}(h) = (2\pi h)^{-d}\mathrm{Vol}(\{(\xi, x) \in T^*\mathbb{T}^d |\, a \leq p(\xi, x) \leq b\}) + o(h^{-d})$$

when $h \to 0$.

We then discuss the Agmon estimate in our semiclassical setting. We set $\mathcal{G}_E = \{x \in \mathbb{R}^d |\, V(x) \leq E\}$. We also set the δ-neighborhood of $\mathcal{G}_E$ as $\mathcal{G}_{E,\delta} = \{x \in \mathbb{R}^d |\, \mathrm{dist}(x, \mathcal{G}_E) < \delta\}$. Here $\mathrm{dist}(\cdot, \cdot)$ is the Euclidean distance. We write $\mathcal{G}^c_{E,\delta} = \mathbb{R}^d \setminus \mathcal{G}_{E,\delta}$. We denote the space of smooth functions which are bounded with their all derivatives by $C_b^\infty(\mathbb{R}^d)$.

Assumption 1. The potential $V \in C_b^\infty(\mathbb{R}^d;\mathbb{R})$ and $E \in \mathbb{R}$ satisfy the following:

$$\inf_{x\in\mathcal{G}_{E,\delta}^c} V(x) > E \quad \text{for any } \delta > 0.$$

We note that the compactness of $\mathcal{G}_E$ is not assumed. We set

$$K_x = \big\{\xi \in \mathbb{R}^d \mid \sum_{j=1}^d \sinh^2 \frac{\xi_j}{2} \le \frac{(V(x)-E)_+}{4}\big\},$$

where $(\cdot)_+ = \max\{\cdot, 0\}$. We introduce the Agmon-Finsler metric for discrete Schrödinger operators

$$L(x,v) = \sup_{\xi\in K_x} \langle \xi, v\rangle.$$

This is the length of $v \in T_x\mathbb{R}^d = \mathbb{R}^d$ in this metric.

The Finsler metric $L(x,v)$ induces the (pseudo-)distance $d_E(x,y)$ between $x, y \in \mathbb{R}^d$ (see subsubsection 2.3.2 for details). We set

$$d_E(x) = \inf_{y\in\mathcal{G}_E} d_E(x,y).$$

Our semiclassical Agmon estimate for discrete Schrödinger operators is the following.

Theorem 2. *Under Assumption 1 and the above notation, for any $C_0 > 0$, $\delta_0 > 0$ and $\varepsilon > 0$, there exist $C > 0$, $h_0 > 0$, $0 < \delta < \delta_0$, $\chi, \tilde{\chi} \in C_b^\infty(\mathbb{R}^d; [0,1])$ with*

$$\operatorname{supp}(1-\chi) \subset \mathcal{G}_{E,\delta}, \quad \operatorname{supp}\tilde{\chi} \subset \mathcal{G}_{E,\delta} \setminus \mathcal{G}_{E,\delta/2}$$

and $\rho \in C^\infty(\mathbb{R}^d;\mathbb{R}_{\ge 0})$ with $|(1-\varepsilon)d_E(x) - \rho(x)| \le \varepsilon$ such that for $0 < h < h_0$,

$$\|\chi e^{\rho(x)/h} u\|_{\ell^2} \le C\|\tilde{\chi} u\|_{\ell^2} + C\|\chi e^{\rho(x)/h}(H(h)-z)u\|_{\ell^2}$$

for any $u \in \ell^2(h\mathbb{Z}^d)$ and any $z \in [E - C_0, E + C_0 h] + i[-C_0, C_0]$.

We prove the Agmon estimate and the optimal exponential decay of eigenfunctions which are valid for $h = 1$ in the appendix to this section.

We finally construct local approximate eigenfunctions of $H(h)$ near a non-degenerate potential minimum.

Assumption 2. The potential $V \in C^\infty(\mathbb{R}^d;\mathbb{R})$ satisfies

$$V(0) = 0, \;\; \partial V(0) = 0 \;\text{ and }\; \partial^2 V(0) > 0.$$

Moreover, a positive number E_0 satisfies the following. There exists a unique $\alpha \in \mathbb{Z}_{\ge 0}^d$ such that $E_0 = \sum_{j=1}^d \lambda_j(\alpha_j + 1/2)$, where $\lambda_1, \dots \lambda_d$ are positive square roots of eigenvalues of $\frac{1}{2}\partial^2 V(0)$.

We denote the Agmon-Finsler distance to 0 at energy 0 for this potential by $d(x) = d_0(x,0)$.

Theorem 3. *Under Assumption 2, there exist unique* $E_j \in \mathbb{R}$, $(j \geq 1)$, *such that the following holds. There exist* $a_j(x) \in C^\infty(\mathbb{R}^d)$, $(j \geq 0)$, *such that if* $a \sim \sum_{j=0}^\infty h^j a_j$ *and* $E(h) \sim \sum_{j=0}^\infty h^j E_j$ *then*

$$(H(h) - hE(h))(a(x)e^{-d(x)/h}) = r(x)e^{-d(x)/h}, \quad r(x) = \mathcal{O}(h^\infty)$$

near $x = 0$. *The formal power series* $\sum_{j=0}^\infty h^j a_j$ *is essentially uniquely defined in the sense that any other solution is given by* $(\sum_{j=0}^\infty h^j c_j)(\sum_{j=0}^\infty h^j a_j)$ *near* $x = 0$ *for some* $c_j \in \mathbb{C}$.

The microlocal analysis on the torus for discrete Schrödinger operators is also discussed in the study of the long-range scattering theory (see [39] [54]).

For the history of the semiclassical Weyl law for continuous Schrödinger operators, see [11]. The proof of Theorem 1 is analogous to the usual continuous case employing the pseudodifferential operators on the torus.

The Agmon estimate was introduced by Agmon (see [1]). We follow the strategy in [36]. Since we work in the Fourier space (torus), we study the operator conjugated with the exponential of a Fourier multiplier and the calculations are more complicated than those in [36]. See [36] for the history of the semiclassical Agmon estimate for continuous case.

Theorem 3 for the continuous Schrödinger operator case was proved by Helffer-Sjöstrand [20]. Helffer-Sjöstrand [21] considered the Harper operator

$$H_{\theta,h}u(n) = \frac{1}{2}(u(n+1) + u(n-1)) + \cos(hn + \theta)u(n)$$

on $\ell^2(\mathbb{Z})$. For fixed θ, this is a special case of our setting. They studied $\bigcup_\theta \sigma(H_{\theta,h})$ by considering

$$P(h) = \cos(hD_x) + \cos x$$

on $L^2(\mathbb{R})$. Among many things, they proved the Agmon estimate based on the Agmon-type Riemann metric

$$ds_E = 2\text{arsinh}\frac{\sqrt{(V(x) - E)_+}}{2} ds,$$

where ds is the length of the standard metric on $\mathbb{R}$. Our Agmon-Finsler metric reduces to this metric when $d = 1$. They also discussed the one-dimensional case of Theorem 3 employing this metric. The general strategy of our proof of Theorem 3 is similar to Dimassi-Sjöstrand [11, section 3]. Modifications are needed for treating a Finsler metric.

We noticed that Rabinovich [44] already studied the same semiclassical setting of discrete Schrödinger operators in higher dimensions and proved the exponential decay of eigenfunctions. Nevertheless, the Agmon-Finsler metric is not introduced in this paper and the exponential decay in [44] is weaker than ours. We note that our Theorem 3 suggests that the Agmon-Finsler distance to $\mathcal{G}_E = \{x \in \mathbb{R}^d | \, V(x) \leq E\}$ is the natural function for estimating the exponential decay of eigenfunctions.

This section is organized as follows. In subsection 2.2, we recall basic facts about pseudodifferential operators on the torus and prove Theorem 1. In subsection 2.3, we discuss the Agmon-Finsler metric and prove Theorem 2. In subsection 2.4, we construct WKB solutions near a nondegenerate potential minimum and prove Theorem 3. In the appendix to this section, we prove the Agmon estimate for discrete Schrödinger operators in the non-semiclassical setting. We also discuss the optimality of this estimate.

2.2 Preliminaries for Section 2

We recall basic facts on microlocal analysis on the torus in this subsection. Functions on $T^*\mathbb{T}^d$ or $\mathbb{T}^d$ are identified with those on $T^*\mathbb{R}^d$ or $\mathbb{R}^d$ which are $2\pi\mathbb{Z}$-periodic. We recall the notation

$$S^m_{\theta,0}(T^*\mathbb{T}^d) = \{a(\cdot;h) \in C^\infty(T^*\mathbb{T}^d) \,|\, |\partial_\xi^\alpha \partial_x^\beta a(\xi,x;h)| \leq C_{\alpha,\beta}\langle x\rangle^{m-\theta|\beta|}\}.$$

Here $\alpha,\beta \in \mathbb{Z}^d_{\geq 0}$. We write $S^m_{\theta,0} = S^m_{\theta,0}(T^*\mathbb{T}^d)$, $S^m = S^m_{1,0}$, $S = S^0_{0,0}$ and $S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m$. For $a \in S^m_{\theta,0}$ we define a pseudodifferential operator $a(\xi, hD_\xi)$ on $\mathbb{T}^d$ by the expression

$$a(\xi, hD_\xi)u(\xi) = (2\pi h)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(\xi,x)e^{i\langle \xi-\eta, x\rangle/h}u(\eta)d\eta dx$$

in the sense of oscillatory integral, where $u \in C^\infty(\mathbb{T}^d)$. The class of pseudodifferential operators corresponding to $S^m_{\theta,0}$ is denoted by $\mathrm{Op}S^m_{\theta,0}$.

We have $V(hD_\xi) = \mathcal{F}_h V(x)\mathcal{F}_h^{-1}$ for $V \in C_b^\infty(\mathbb{R}^d)$ since

$$\begin{aligned} V(hD_\xi)u(\xi) &= (2\pi h)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V(x)e^{i\langle\xi-\eta,x\rangle/h}u(\eta)d\eta dx \\ &= (2\pi h)^{-d}\int_{\mathbb{R}^d} V(x)\left(\sum_{x\in h\mathbb{Z}^d}(2\pi)^{d/2}h^d(\mathcal{F}_h^{-1}u)(x)\delta_x\right)e^{i\langle\xi,x\rangle/h}dx \\ &= (2\pi)^{-d/2}\sum_{x\in h\mathbb{Z}^d} V(x)(\mathcal{F}_h^{-1}u)(x)e^{\langle\xi,x\rangle/h} \end{aligned}$$

for $u \in C^\infty(\mathbb{T}^d)$. Thus $\widetilde{H}(h) = p(\xi, hD_\xi)$.

While the definition of $a(\xi, hD_\xi)$ is based on the special structure of the torus, the pseudolocality of pseudodifferential operators implies that we can employ the general theory of pseudodifferential operators on manifolds (see [61, Chapter 5, 14]). In particular, the functional calculus and the trace formula for pseudodifferential operators are valid. We give a proof of Theorem 1 using these method.

Proof of Theorem 1. For small $\varepsilon > 0$, we take $\chi_{1,\varepsilon}, \chi_{2,\varepsilon} \in C_c^\infty(\mathbb{R};[0,1])$ such that $\chi_{1,\varepsilon} = 1$ on $[a-\varepsilon, b+\varepsilon]$, $\operatorname{supp}\chi_{1,\varepsilon} \subset [a-2\varepsilon, b+2\varepsilon]$, $\chi_{2,\varepsilon} = 1$ on $[a+2\varepsilon, b-2\varepsilon]$

and supp $\chi_{2,\varepsilon} \subset [a+\varepsilon, b-\varepsilon]$. Since $N_{[a,b]}(\widetilde{H}(h)) = \operatorname{tr}(\chi_{[a,b]}(\widetilde{H}(h)))$ and $\chi_{2,\varepsilon} \leq \chi_{[a,b]} \leq \chi_{1,\varepsilon}$, we have

$$\operatorname{tr}(\chi_{2,\varepsilon}(\widetilde{H}(h))) \leq N_{[a,b]}(\widetilde{H}(h)) \leq \operatorname{tr}(\chi_{1,\varepsilon}(\widetilde{H}(h))).$$

By the functional calculus and the trace formula for pseudodifferential operators, we have

$$\operatorname{tr}(\chi_{j,\varepsilon}(\widetilde{H}(h))) = (2\pi h)^{-d} \int_{T^*\mathbb{T}^d} \chi_{j,\varepsilon}(p(\xi,x)) d\xi dx + \mathcal{O}_\varepsilon(h^{-d+1})$$

for $j = 1, 2$. By Fubini's theorem and the definition of $p(\xi, x)$, We have $\mathrm{Vol}_{2d}(\{(\xi,x) |\, p(\xi,x) = a, b\}) = 0$. Then we have

$$\lim_{\varepsilon \to 0} \int_{T^*\mathbb{T}^d} \chi_{j,\varepsilon}(p(\xi,x)) d\xi dx = \mathrm{Vol}(\{(\xi,x) \in T^*\mathbb{T}^d |\, a \leq p(\xi,x) \leq b\})$$

for $j = 1, 2$.

Take any $\delta > 0$. Then we have for small $\varepsilon > 0$

$$-\delta - \mathcal{O}_\varepsilon(h) \leq (2\pi h)^d N_{[a,b]}(\widetilde{H}(h)) - \mathrm{Vol}(\{(\xi,x) |\, a \leq p(\xi,x) \leq b\}) \leq \delta + \mathcal{O}_\varepsilon(h)$$

by the above arguments. We take $h \to 0$ and then $\delta \to 0$, which completes the proof. □

2.3 The Agmon estimate

We prove Theorem 2 in this subsection.

2.3.1 Calculation of exponentially conjugated operator

For $\rho \in C_b^\infty(\mathbb{R}^d; \mathbb{R})$, we compute $\widetilde{H}_\rho(h) = e^{\rho(hD_\xi)/h} \widetilde{H}(h) e^{-\rho(hD_\xi)/h}$. Note that $e^{\rho(hD_\xi)/h} V(hD_\xi) e^{-\rho(hD_\xi)/h} = V(hD_\xi)$. We thus consider $e^{\rho(hD_\xi)/h} p_0(\xi) e^{-\rho(hD_\xi)/h}$, where $p_0(\xi) = \sum_{j=1}^d (2 - 2\cos\xi_j)$.

Lemma 2.1. *For any* $\rho \in C_b^\infty(\mathbb{R}^d; \mathbb{R})$*,*

$$e^{\rho(hD_\xi)/h} p_0(\xi) e^{-\rho(hD_\xi)/h} = a_\rho(\xi, hD_\xi; h) \in \mathrm{Op}S,$$

where $a_\rho \sim \sum_{k=0}^\infty h^k a_{\rho,k}(\xi,x)$, $a_{\rho,k} \in S$ *and*

$$a_{\rho,0}(\xi,x) = p_0(\xi - i\partial\rho(x), x).$$

If moreover

$$|\partial_x^\alpha \rho(x)| \leq C_\alpha \langle x \rangle^{1-|\alpha|} \text{ for any } \alpha \in \mathbb{Z}_{\geq 0}^d, \tag{1}$$

then $a_\rho \in S^0$ *and* $a_{\rho,k} \in S^{-k}$.

Proof. We set $g(x) = e^{-|x|^2}$. Then we have

$$\begin{aligned}
&e^{-\rho(hD_{\hat{\eta}})/h} u(\hat{\eta}) \\
&= \lim_{\varepsilon\to 0} (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{i\langle \hat{\eta}-\eta, x\rangle/h} e^{-\rho(x)/h} u(\eta) g(\varepsilon x) g(\varepsilon \eta) dx d\eta \\
&= (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{i\langle \hat{\eta}-\eta, x\rangle/h}\ {}^t L_1^N \left(e^{-\rho(x)/h} u(\eta)\right) dx d\eta
\end{aligned}$$

by integration by parts, where $N \geq 2d+1$ and

$$L_1 = \frac{1 - x \cdot hD_\eta + (\hat{\eta} - \eta) \cdot hD_x}{1 + |x|^2 + |\eta - \hat{\eta}|^2}.$$

Thus

$$\begin{aligned}
&e^{\rho(hD_\xi)/h} p_0(\xi) e^{-\rho(hD_\xi)/h} u(\xi) \\
&= (2\pi h)^{-2d} \int_{\mathbb{R}^{2d}} e^{i\langle \xi - \hat{\eta}, y\rangle/h}\ {}^t L_2^N e^{\rho(y)/h} p_0(\hat{\eta}) \int_{\mathbb{R}^{2d}} e^{i\langle \hat{\eta}-\eta, x\rangle/h} \\
&\qquad\qquad {}^t L_1^{2N} \left(e^{-\rho(x)/h} u(\eta)\right) dx d\eta dy d\hat{\eta} \\
&= \lim_{\varepsilon\to 0} (2\pi h)^{-2d} \int_{\mathbb{R}^{4d}} e^{i\langle \xi-\hat{\eta}, y\rangle/h} e^{(\rho(y)-\rho(x))/h} e^{i\langle \hat{\eta}-\eta, x\rangle/h} \\
&\qquad\qquad u(\eta) g(\varepsilon x) g(\varepsilon \eta) g(\varepsilon y) g(\varepsilon \hat{\eta}) dx d\eta dy d\hat{\eta},
\end{aligned}$$

where

$$L_2 = \frac{1 - y \cdot hD_{\hat{\eta}} + (\xi - \hat{\eta}) \cdot hD_y}{1 + |y|^2 + |\hat{\eta} - \xi|^2}.$$

We set $\rho(y) - \rho(x) = (y - x) \cdot \Phi(x, y)$, where $\Phi(x, y) = \int_0^1 \partial\rho(y + t(x - y)) dt$. We deform the integral by Cauchy's theorem and obtain

$$\begin{aligned}
&e^{\rho(hD_\xi)/h} p_0(\xi) e^{-\rho(hD_\xi)/h} u(\xi) \\
&= \lim_{\varepsilon\to 0} (2\pi h)^{-2d} \int_{\mathbb{R}^{4d}} e^{i\langle \xi-\hat{\eta}, y\rangle/h} e^{i\langle \hat{\eta}-\eta, x\rangle/h} p_0(\hat{\eta} - i\Phi(x, y)) u(\eta) \\
&\qquad\qquad g(\varepsilon x) g(\varepsilon \eta) g(\varepsilon y) g(\varepsilon\hat{\eta} - \varepsilon i \Phi(x, y)) d\eta dx d\hat{\eta} dy.
\end{aligned}$$

Using ${}^t L_2^N$ and ${}^t L_1^{2N}$ as above, we see that

$$\begin{aligned}
&e^{\rho(hD_\xi)/h} p_0(\xi) e^{-\rho(hD_\xi)/h} u(\xi) \\
&= \lim_{\varepsilon\to 0} \lim_{\varepsilon'\to 0} (2\pi h)^{-2d} \int_{\mathbb{R}^{4d}} e^{i\langle \xi-\hat{\eta}, y\rangle/h} e^{i\langle \hat{\eta}-\eta, x\rangle/h} p_0(\hat{\eta} - i\Phi(x, y)) u(\eta) \\
&\qquad\qquad \psi(\varepsilon x) \psi(\varepsilon \eta) \psi(\varepsilon' y) \psi(\varepsilon' \hat{\eta}) d\eta dx d\hat{\eta} dy \\
&= \lim_{\varepsilon\to 0} \lim_{\varepsilon'\to 0} (2\pi h)^{-2d} \int_{\mathbb{R}^{4d}} e^{i\langle \xi-\eta, x\rangle/h} e^{-i\langle y, \hat{\eta}\rangle/h} p_0(\hat{\eta} + \xi - i\Phi(x, y + x)) u(\eta) \\
&\qquad\qquad \psi(\varepsilon x) \psi(\varepsilon \eta) \psi(\varepsilon' y + \varepsilon' x) \psi(\varepsilon' \hat{\eta} + \varepsilon' \xi) d\hat{\eta} dy d\eta dx.
\end{aligned}$$

Here $\psi \in C_c^\infty(\mathbb{R}^d)$ is a cutoff near 0 with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^d \,|\, |x| < 1/4\}$. We also changed the variables from y and $\hat{\eta}$ to $y + x$ and $\hat{\eta} + \xi$.

We next insert

$$1 = (1 - \psi(y)\psi(\hat{\eta})) + \psi(y)\psi(\hat{\eta})$$

into the integrand. Then we estimate the $\lim_{\varepsilon' \to 0} (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} \cdots d\hat{\eta} dy$ part. We set

$$L_3 = \frac{-\hat{\eta} h D_y - y h D_{\hat{\eta}}}{|\hat{\eta}|^2 + |y|^2}.$$

We see that the $1 - \psi(y)\psi(\hat{\eta})$ term contributes as $h^\infty S$ by using ${}^t L_3^N$ with $N \gg 1$. To estimate the $\psi(y)\psi(\hat{\eta})$ term, we apply the stationary phase method ([19, Theorem 7.7.6]) with respect to $(\hat{\eta}, y)$. The stationary point $(\partial_{\hat{\eta},y}\phi = 0)$ is $(\hat{\eta}, y) = (0, 0)$. We have $\operatorname{sgn}\partial^2_{\hat{\eta},y}\phi = 0$ and $|\det \partial^2_{\hat{\eta},y}\phi| = 1$ at $(\hat{\eta}, y) = (0, 0)$. We then obtain an asymptotic expansion with respect to h in S with the leading term $p_0(\xi - i\partial\rho(x))$. Here we used $\Phi(x, x) = \partial\rho(x)$.

Finally we assume (1) and prove the asymptotic expansion in S^0. For this, we change the variables from y to $\langle x \rangle y$ and set $\tilde{h} = h\langle x \rangle^{-1}$. Then we have

$$\begin{aligned}
&e^{\rho(hD_\xi)/h} p_0(\xi) e^{-\rho(hD_\xi)/h} u(\xi) \\
&= \lim_{\varepsilon \to 0} \lim_{\varepsilon' \to 0} (2\pi h)^{-d} (2\pi\tilde{h})^{-d} \int_{\mathbb{R}^{4d}} e^{i\langle \xi - \eta, x \rangle/h} e^{-i\langle y, \hat{\eta} \rangle/\tilde{h}} p_0(\hat{\eta} + \xi - i\Phi(x, \langle x \rangle y + x)) \\
&\qquad u(\eta)\psi(\varepsilon x)\psi(\varepsilon\eta)\psi(\varepsilon'\langle x \rangle y + \varepsilon' x)\psi(\varepsilon'\hat{\eta} + \varepsilon'\xi) d\hat{\eta} dy d\eta dx.
\end{aligned}$$

We insert

$$1 = (1 - \psi(y)) + \psi(y)(1 - \psi(\hat{\eta})) + \psi(y)\psi(\hat{\eta})$$

into the integrand and estimate the $\lim_{\varepsilon' \to 0} (2\pi\tilde{h})^{-d} \int_{\mathbb{R}^{2d}} \cdots d\hat{\eta} dy$ part. We set

$$\tilde{L}_3 = \frac{-\hat{\eta}\tilde{h} D_y - y\tilde{h} D_{\hat{\eta}}}{|\hat{\eta}|^2 + |y|^2} \quad \text{and} \quad \tilde{L}_4 = \frac{-y\tilde{h} D_{\hat{\eta}}}{|y|^2}.$$

We see that the $1 - \psi(y)$ term contributes as $h^\infty S^{-\infty}$ by using ${}^t\tilde{L}_3^{d+1}$ and ${}^t\tilde{L}_4^N$ with $N \gg 1$. We also see that the $\psi(y)(1 - \psi(\hat{\eta}))$ term contributes as $h^\infty S^{-\infty}$ by using ${}^t\tilde{L}_3^N$ with $N \gg 1$. To see this, we note that for $|y| \leq 1/4$

$$|\partial_y^\alpha \Phi(x, \langle x \rangle y + x)| \leq C_\alpha \quad \text{for any} \quad \alpha \in \mathbb{Z}^d_{\geq 0}$$

since $|\langle x \rangle y + x| \geq |x|/2$ for $|x| \geq 1$ and $|y| \leq 1/4$. We then apply the stationary phase method to the $\psi(y)\psi(\hat{\eta})$ term and obtain asymptotic expansion with respect to $h\langle x \rangle^{-1}$ in S^0 using the above estimate on $\partial_y^\alpha \Phi(x, \langle x \rangle y + x)$. This completes the proof. □

Remark 2.2. The Gaussian weight is also used in the context of the resonance theory to justify the contour deformation in the oscillatory integral (see Galkowski-Zworski [16, Appendix B.1]).

Remark 2.3. The second part of Lemma 2.1 is used in the appendix to this section.

This lemma implies that for $\rho \in C_b^\infty(\mathbb{R}^d;\mathbb{R})$, the semiclassical principal symbol of $\widetilde{H}_\rho(h)$ is

$$\sigma_h(\widetilde{H}_\rho(h)) = p(\xi - i\partial\rho(x), x).$$

In the proof of the Agmon estimate, the weight $\rho \in C^\infty(\mathbb{R}^d;\mathbb{R})$ may be only lower semibounded and $\partial\rho \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Take $\nu(t) \in C^\infty(\mathbb{R};\mathbb{R})$ with $0 \le \nu'(t) \le 1$ and $\nu''(t) \le 0$ such that $\nu(t) = t$ for $t < 0.9$ and $\nu(t) = 1$ for $t > 1.1$. We then set $\rho_M(x) = M\nu(\rho(x)/M)$. We note that $\rho_M(x) \nearrow \rho(x)$ when $M \to \infty$ since $\nu''(t) \le 0$.

By the proof of Lemma 2.1, we see that the first statement in Lemma 2.1 with ρ replaced by ρ_M is valid uniformly for $M > 1$ since $\partial\rho_M \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$ uniformly for $M > 1$. The second statement with ρ replaced by ρ_M is also valid uniformly for $M > 1$ if we add the assumption that $\rho(x) \gtrsim |x|$ for $|x| \gg 1$ to ensure that (1) with ρ replaced by ρ_M is valid uniformly for $M > 1$.

We set

$$\widetilde{H}_M(h) = e^{\rho_M(hD_\xi)}\widetilde{H}(h)e^{-\rho_M(hD_\xi)}.$$

It may be possible to prove that $\widetilde{H}_\rho(h) = \lim_{M\to\infty}\widetilde{H}_M(h) \in \mathrm{Op}S$ and that $\sigma_h(\widetilde{H}_\rho(h)) = p(\xi - i\partial\rho(x), x)$ even in this case. In fact, in the proof of the Agmon estimate, we do not use these and we take the limit $M \to \infty$ in a later step of the proof.

2.3.2 The Agmon-Finsler metric

We recall that

$$p_0(\xi) = \sum_{j=1}^d (2 - 2\cos\xi_j) = 4\sum_{j=1}^d \sin^2\frac{\xi_j}{2}.$$

We will find a condition which ensures that

$$\mathrm{Re}\,\big(p_0(\xi - i\partial\rho(x)) + V(x) - E\big)$$

is positive away from $\mathcal{G}_E = \{x \in \mathbb{R}^d |\, V(x) \le E\}$. We note that

$$\begin{aligned}4\sin^2\frac{\xi + i\lambda}{2} &= 4(\sin\frac{\xi}{2}\cos\frac{i\lambda}{2} + \cos\frac{\xi}{2}\sin\frac{i\lambda}{2})^2 \\ &= 4(\sin\frac{\xi}{2}\cosh\frac{\lambda}{2} + i\cos\frac{\xi}{2}\sinh\frac{\lambda}{2})^2.\end{aligned}$$

Then we have

$$\mathrm{Re}\,(4\sin^2\frac{\xi + i\lambda}{2}) \ge -4\sinh^2\frac{\lambda}{2}.$$

This implies that

$$\mathrm{Re}\,\big(p_0(\xi - i\partial\rho(x)) + V(x) - E\big) \ge V(x) - E - 4\sum_{j=1}^d \sinh^2\frac{\partial_j\rho(x)}{2}. \qquad (2)$$

We set

$$K_x = \{\xi \in \mathbb{R}^d \mid \sum_{j=1}^{d} \sinh^2 \frac{\xi_j}{2} \leq \frac{(V(x)-E)_+}{4}\},$$

which is interpreted as a convex subset of $T_x^*\mathbb{R}^d$.

We present a construction of a function $d(x)$ such that

$$\partial d(x) \in K_x$$

for (almost all) $x \in \mathbb{R}^d$, which is valid for more general K_x. We introduce a Finsler metric given by the supporting function of K_x;

$$L(x,v) = \sup_{\xi \in K_x} \langle \xi, v \rangle,$$

which gives the length of $v \in T_x\mathbb{R}^d = \mathbb{R}^d$ in this metric.

Remark 2.4. We note that K_x for x with $V(x) > E$ is a strictly convex compact set such that ∂K_x is smooth and has non-vanishing Gaussian curvature. This implies that $\left(\frac{1}{2}\partial_{v_i}\partial_{v_j} L(x,v)^2\right)_{ij}$ is positive definite for $v \neq 0$ and x with $V(x) > E$. Thus $L(x,v)$ satisfies the conditions of the definition of the Finsler metric (for instance, [2, Section 1.1]) on $\mathcal{G}_E^c = \{x \in \mathbb{R}^d \mid V(x) > E\}$.

We set

$$d_E(x,y) = \inf_{x(\cdot)} \int_0^1 L(x(t), x'(t))dt,$$

where $x(\cdot) : [0,1] \to \mathbb{R}^d$ ranges over C^1 curves such that $x(0) = x$ and $x(1) = y$. We note that $d_E(x,y) = d_E(y,x)$ since $L(x,v) = L(x,-v)$. Take any closed set $\mathcal{G}$ in $\mathbb{R}^d$ and set

$$d_{\mathcal{G}}(x) = d_{E,\mathcal{G}}(x) = \inf_{y \in \mathcal{G}} d_E(x,y).$$

Note that $d_{\mathcal{G}}$ is a Lipschitz continuous function and thus is differentiable at almost all $x \in \mathbb{R}^d$. We have the following.

Lemma 2.5. *For almost all* $x \in \mathbb{R}^d$,

$$\partial d_{\mathcal{G}}(x) \in K_x.$$

Proof. Take x such that $d_{\mathcal{G}}(x)$ is differentiable at x. Take any $v \in T_x\mathbb{R}^d$. The triangle inequality implies

$$|d_{\mathcal{G}}(x) - d_{\mathcal{G}}(x+tv)|/t \leq d_E(x, x+tv)/t.$$

Taking limit $t \to 0$, we learn

$$|\langle \partial d_{\mathcal{G}}(x), v \rangle| \leq L(x,v).$$

Since the compact convex set K_x is recovered from its supporting function as

$$K_x = \{\xi \in \mathbb{R}^d \mid \langle \xi, v \rangle \leq L(x,v) \text{ for any } v \in \mathbb{R}^d\}$$

(see [19, subsection 4.3]), this implies $\partial d_{\mathcal{G}}(x) \in K_x$. □

We call $L(x,v)$ or $L : T\mathbb{R}^d \to [0,\infty)$ with respect to our K_x the Agmon-Finsler metric for discrete Schrödinger operators. Then the exponential decay of the eigenfunctions of $H(h)$ is stated in terms of

$$d_E(x) = d_{E,\mathcal{G}_E}(x).$$

The inequality (2) and Lemma 2.5 imply that

$$\mathrm{Re}\left(p_0(\xi - i\partial d_E(x)) + V(x) - E\right) \geq 0.$$

outside $\mathcal{G}_E$.

2.3.3 Proof of Theorem 2

Proof of theorem 2. We should modify $d_E(x)$ as follows. For a given $\varepsilon > 0$, we take a sufficiently small $\delta > 0$. We fix $\psi_\delta \in C_c^\infty(\mathbb{R};\mathbb{R}_{\geq 0})$ such that $\mathrm{supp}\,\psi_\delta \subset \{x \in \mathbb{R}^d |\, |x| < \delta/30\}$ and $\int_{\mathbb{R}^d} \psi_\delta(x)dx = 1$. Set $\chi = \mathbb{1}_{\mathcal{G}^c_{E,\frac{3}{4}\delta}} * \psi_\delta$, $\chi_1 = \mathbb{1}_{\mathcal{G}^c_{E,\frac{1}{2}\delta}} * \psi_\delta$ and $\tilde{\chi} = \mathbb{1}_{\mathcal{G}_{E,\frac{7}{8}\delta} \setminus \mathcal{G}_{E,\frac{5}{8}\delta}} * \psi_\delta$. Here $\mathbb{1}$ denotes the indicator function of a set. Then $\chi, \chi_1, \tilde{\chi} \in C_b^\infty(\mathbb{R}^d;[0,1])$, $\chi\chi_1 = \chi$, $\tilde{\chi}\partial\chi = \partial\chi$ and

$$\mathrm{supp}\,(1-\chi) \subset \mathcal{G}_{E,\delta}, \quad \mathrm{supp}\,\tilde{\chi} \subset \mathcal{G}_{E,\delta} \setminus \mathcal{G}_{E,\delta/2}.$$

By mollifying $(1-\varepsilon)d_{E,\mathcal{G}_{E,\delta}}$, we obtain $\rho \in C^\infty(\mathbb{R}^d;\mathbb{R}_{\geq 0})$ satisfying $|(1-\varepsilon)d_E(x) - \varepsilon| \leq \varepsilon$ for $x \in \mathbb{R}^d$ and $\partial\rho(x) \in (1-\varepsilon/2)K_x$ on $\mathrm{supp}\,\chi_1$. Moreover, $\mathrm{dist}(\mathrm{supp}\,\rho, \mathrm{supp}\,\partial\chi) > \delta/10$.

Define ρ_M and $\widetilde{H}_M(h)$ from this ρ as in subsubsection 2.3.1. Then $\partial\rho_M(x) \in (1-\varepsilon/2)K_x$ on $\mathrm{supp}\,\chi_1$.

Take any $z \in [E - C_0, E + C_0h] + i[-C_0, C_0]$ for a given C_0. By Lemma 2.1 we have

$$\chi_1(hD_\xi)(\widetilde{H}_M(h) - z)^*(\widetilde{H}_M(h) - z)\chi_1(hD_\xi) - \gamma^2\chi_1(hD_\xi)^2$$

belongs to $\mathrm{Op}S$ uniformly for $M > 1$ and its principal symbol is

$$\chi_1(x)^2|p(\xi - i\partial\rho_M(x), x) - z|^2 - \gamma^2\chi_1(x)^2.$$

The inequality (2) and the estimate for $\partial\rho_M(x)$ above imply that this is non-negative for small $\gamma > 0$. Here we replace z with $z - C_0h$ if $E \leq \mathrm{Re}\,z \leq E + C_0h$.

Then the Gårding inequality implies that there exists $h_0 > 0$ such that

$$\|(\widetilde{H}_M(h) - z)\chi_1(hD_\xi)\hat{u}\|_{L^2(\mathbb{T}^d)} \geq \gamma\|\chi_1(hD_\xi)\hat{u}\|_{L^2(\mathbb{T}^d)} - \frac{\gamma}{2}\|\hat{u}\|_{L^2(\mathbb{T}^d)}$$

for any $\hat{u} \in L^2(\mathbb{T}^d)$ and any $0 < h < h_0$. Here h_0 is independent of $M > 1$ by the uniform estimate of the symbol. Replacing $\hat{u}$ with $\chi(hD_\xi)\hat{u}$, we have

$$\|e^{\rho_M(x)/h}(H(h) - z)e^{-\rho_M(x)/h}\chi u\|_{\ell^2} \geq \frac{\gamma}{2}\|\chi u\|_{\ell^2}$$

for $u \in \ell^2(h\mathbb{Z}^d)$ and $0 < h < h_0$. We replace u with $e^{\rho_M(x)/h}u$ and obtain

$$\|e^{\rho_M(x)/h}(H(h)-z)\chi u\|_{\ell^2} \geq \frac{\gamma}{2}\|e^{\rho_M(x)/h}\chi u\|_{\ell^2}$$

for $u \in \ell^2(h\mathbb{Z}^d)$ and $0 < h < h_0$. Taking the limit $M \to \infty$, we have this estimate with $\rho_M(x)$ replaced by $\rho(x)$. Then we have

$$\begin{aligned}
\|\chi e^{\rho(x)/h}u\|_{\ell^2} &\leq C\|e^{\rho(x)/h}(H(h)-z)\chi u\|_{\ell^2} \\
&\leq C\|\chi e^{\rho(x)/h}(H(h)-z)u\|_{\ell^2} + C\|e^{\rho(x)/h}[H(h),\chi]u\|_{\ell^2} \\
&\leq C\|\chi e^{\rho(x)/h}(H(h)-z)u\|_{\ell^2} + C\|\tilde{\chi}u\|_{\ell^2}.
\end{aligned}$$

In the last inequality, we used the facts that $\rho = 0$ near $\operatorname{supp}\partial\chi$ and that $\tilde{\chi} = 1$ near $\operatorname{supp}\partial\chi$. □

2.4 WKB solutions near a potential minimum

In this subsection, we set $q(x,\xi) = 4\sum_{j=1}^d \sinh^2\frac{\xi_j}{2} - V(x)$ and give a proof of Theorem 3.

2.4.1 Solution to the eikonal equation

After some orthogonal transformation, we have

$$q(y,\eta) = \eta^2 - \sum_{j=1}^{d}\lambda_j^2 y_j^2 + \mathcal{O}((y,\eta)^3).$$

Thus there exists a real valued smooth function ϕ defined near $x = 0$ such that the local unstable and stable manifolds at $(0,0)$ of $H_q = \frac{\partial q}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial q}{\partial x}\frac{\partial}{\partial \xi}$ are given as $\Lambda_\pm = \{(x,\xi) | \, \xi = \pm\partial\phi(x)\}$. Moreover, $\phi = \sum_{j=1}^d \frac{\lambda_j}{2}y_j^2 + \mathcal{O}(|y|^3)$ in the above coordinate y and the phase ϕ satisfies the eikonal equation $q(x,\partial\phi(x)) = 0$. These facts are proved by the same proof as in [11, Section 3].

We recall that $d(x)$ is the Agmon-Finsler distance to $0 \in \mathbb{R}^d$ at energy 0.

Lemma 2.6. *Under the above notation, $\phi(x) = d(x)$ near $x = 0$.*

Proof. We follow the strategy of Proposition A.1 in [11, Section 6]. We should be careful since we work with a Finsler metric. Take a small neighborhood $\widetilde{\Omega}$ of $0 \in \mathbb{R}^d$ where ϕ is defined. We also take a small neighborhood $\Omega \subset \widetilde{\Omega}$ of $0 \in \mathbb{R}^d$. Then for $x, \tilde{x} \in \Omega$, the Agmon-Finsler distance $d(x,\tilde{x})$ is computed by the C^1 curves in $\widetilde{\Omega}$ joining them.

Suppose that $x(t)$ is a C^1 curves in $\widetilde{\Omega}$ such that $x(0) = x$, $x(1) = \tilde{x}$. Since $(\partial\phi)(x(t)) \in \partial K_{x(t)}$ by the eikonal equation, the definition of L implies that

$$|\phi(x) - \phi(\tilde{x})| = |\int_0^1 \langle(\partial\phi)(x(t)), x'(t)\rangle dt| \leq \int_0^1 L(x(t), x'(t))dt.$$

Taking the infimum over $x(t)$, we have $|\phi(x) - \phi(\tilde{x})| \leq d(x, \tilde{x})$. In particular, we have $0 \leq \phi(x) \leq d(x)$ by setting $\tilde{x} = 0$.

We next take $x \in \Omega$ and set $\exp(-tH_q)(x, \partial\phi(x)) = (x(t), \xi(t))$, where $\exp(tH_q)$ is the flow generated by H_q. Since ϕ generates the local unstable manifold, we see that $\lim_{t\to\infty} x(t) = 0$ and $\xi(t) = (\partial\phi)(x(t)) \in \partial K_{x(t)}$. Then we have

$$\phi(x) - \phi(x(t)) = \int_0^t \langle (\partial\phi)(x(s)), -x'(s)\rangle ds = \int_0^t \langle \xi(s), -x'(s)\rangle ds.$$

We have $-x'(s) = \frac{\partial q}{\partial \xi}(x(s), \xi(s))$ by the Hamilton equation. Thus the supremum in $L(x(s), x'(s)) = L(x(s), -x'(s)) = \sup_{\xi \in K_{x(s)}} \langle \xi, -x'(s)\rangle$ is achieved at $\xi(s)$. Thus we have

$$\phi(x) - \phi(x(t)) = \int_0^t L(x(s), x'(s))ds \geq d(x, x(t)).$$

Taking the limit $t \to \infty$, we have $\phi(x) \geq d(x)$. □

2.4.2 Transport equation

We next calculate $e^{\phi(x)/h} H(h) e^{-\phi(x)/h}$. Difference operators such as $H(h)$ act on functions both on $\mathbb{R}^d$ and $h\mathbb{Z}^d$.

Proposition 2.7. *Under the above notation*

$$e^{\phi(x)/h} H(h) e^{-\phi(x)/h} a = h(\mathcal{L}a)(x) + h^2 \Phi(x, h; a)$$

for $a \in C^\infty(\mathbb{R}^d)$ near $x = 0$, where

$$(\mathcal{L}a)(x) = 2\sum_{j=1}^d (\sinh \partial_j\phi(x))\partial_j a(x) + \sum_{j=1}^d (\cosh \partial_j \phi(x))(\partial_j^2 \phi(x)) a(x)$$

and

$$\Phi(x, h; a) \sim \sum_{n=0}^\infty h^n (\Phi_n a)(x).$$

Here Φ_n is a $(n+2)$th order differential operator defined in terms of ϕ.

Proof. We have

$$e^{\phi(x)/h} H(h) e^{-\phi(x)/h} a = -\sum_{|y-x|=h} (a(y) e^{(\phi(x)-\phi(y))/h} - a(x)) + V(x)a(x).$$

Since ϕ satisfies the eikonal equation, we have

$$V(x) = \sum_{j,\pm} (e^{\mp \partial_j \phi(x)} - 1).$$

This and the Taylor expansions of $\phi(y)$ and $a(y)$ around x imply that

$$\begin{aligned}
&e^{\phi(x)/h}H(h)e^{-\phi(x)/h}a \\
&\sim -\sum_{j,\pm}\big(\big(\sum_{n=0}^{\infty}(\pm h)^n(\partial_j^n a(x)\big)/n!\big)e^{\mp\sum_{n=1}^{\infty}(\pm h)^{n-1}(\partial_j^n\phi(x))/n!}-e^{\mp\partial_j\phi(x)}a(x)\big) \\
&\sim -\sum_{j,\pm}\big(\pm h\partial_j a(x)e^{\mp\partial_j\phi(x)}-ha(x)e^{\mp\partial_j\phi(x)}\frac{1}{2}\partial_j^2\phi(x)\big)+h^2\Phi(x,h;a) \\
&= h(\mathcal{L}a)(x)+h^2\Phi(x,h;a)
\end{aligned}$$

for some $\Phi(x,h;a)\sim\sum_{n=0}^{\infty}h^n(\Phi_n a)(x)$. This is justified as an asymptotic expansion if we expand

$$e^{\mp\sum_{n=1}^{\infty}(\pm h)^{n-1}(\partial_j^n\phi(x))/n!}=e^{\mp\partial_j\phi(x)}e^{\mp\sum_{n=2}^{\infty}(\pm h)^{n-1}(\partial_j^n\phi(x))/n!}$$

using $e^z=\sum_{m=0}^{\infty}z^m/m!$. □

See [21, Section 8] for the case of $d=1$. This proposition implies that in order to solve

$$e^{\phi(x)/h}(H(h)-hE(h))(a(x)e^{-\phi(x)/h})=\mathcal{O}(h^{\infty}),$$

it is enough to solve the following transport equations

$$(\mathcal{L}-E_0)a_0=0,\ (\mathcal{L}-E_0)a_n=\sum_{m=0}^{n-1}(E_{n-m}-\Phi_{n-m-1})a_m,\ (n\geq 1).$$

2.4.3 Solution to the transport equation

Proof of Theorem 3. We recall that $\phi(y)=\sum_{j=1}^{d}\frac{\lambda_j}{2}y_j^2+\mathcal{O}(|y|^3)$ after some orthogonal transformation. Thus $\mathcal{L}=2\sum_{j=1}^{d}(\lambda_j y_j+\mathcal{O}(|y|^2))\partial_{y_j}+\sum_j\lambda_j+\mathcal{O}(|y|)$. Then the same arguments as in [11, Section 3] implies the existence part of Theorem 3.

We next prove the uniqueness of E_j and the essential uniqueness of a_j. Suppose that $\tilde{E}_j$ and $\tilde{a}_j$ are other solutions. Recall that for any $g\in C^{\infty}(\mathbb{R}^d)$, the equation $(\mathcal{L}-E_0)f=g-\lambda a_0$ near $x=0$ has a solution $f\in C^{\infty}$ for precisely one $\lambda\in\mathbb{C}$ and the solution is unique modulo $\mathbb{C}a_0$ ([11, Proposition 3.4, 3.5]). Considering the first nonzero $\tilde{a}_j$, we may assume that $\tilde{a}_0=a_0$. We prove the uniqueness by induction. We assume that $\tilde{E}_j=E_j$ and $\tilde{a}_j=a_j+\sum_{\ell=1}^{j}c_\ell a_{j-\ell}$ for some $c_\ell\in\mathbb{C}$ up to $j-1$. Then

$$\begin{aligned}
(\mathcal{L}-E_0)\tilde{a}_j+(E_j-\tilde{E}_j)a_0&=\sum_{m=0}^{j-1}(E_{j-m}-\Phi_{j-m-1})\tilde{a}_m \\
&=\sum_{m=0}^{j-1}(E_{j-m}-\Phi_{j-m-1})(a_m+\sum_{\ell=1}^{m}c_\ell a_{m-\ell})
\end{aligned}$$

by the inductive hypothesis and the transport equation for $\tilde{a}_j$. This is equal to

$$(\mathcal{L}-E_0)a_j+\sum_{\ell=1}^{j-1}c_\ell\sum_{m=\ell}^{j-1}(E_{j-m}-\Phi_{j-m-1})a_{m-\ell}$$
$$=(\mathcal{L}-E_0)a_j+\sum_{\ell=1}^{j-1}c_\ell\sum_{m=0}^{j-1-\ell}(E_{j-m-\ell}-\Phi_{j-m-\ell-1})a_m$$

by the transport equation for a_j. The transport equation for $a_{j-\ell}$ implies that

$$(\mathcal{L}-E_0)\tilde{a}_j+(E_j-\tilde{E}_j)a_0=(\mathcal{L}-E_0)a_j+\sum_{\ell=1}^{j-1}c_\ell(\mathcal{L}-E_0)a_{j-\ell}.$$

Since this has a solution $\tilde{a}_j$, we have $\tilde{E}_j=E_j$ and $\tilde{a}_j=a_j+\sum_{\ell=1}^{j}c_\ell a_{j-\ell}$ for some $c_j\in\mathbb{C}$. This shows the uniqueness part of Theorem 3. $\square$

2.5 Appendix to Section 2: The Agmon estimate and the exponential decay of eigenfunctions for discrete Schrödinger operators

2.5.1 The Agmon estimate

In this appendix to Section 2, we prove the Agmon estimate for

$$Hu(x)=-\sum_{|x-y|=1}(u(y)-u(x))+V(x)u(x),$$

where $x,y\in\mathbb{Z}^d$. The proof is similar to that of Theorem 2. While we used the semiclassical Gårding inequality in the proof of Theorem 2, we employ the non-semiclassical sharp Gårding inequality in this appendix.

Assumption 3. The potential $V:\mathbb{Z}^d\to\mathbb{R}$ has a smooth extension $\widetilde{V}:\mathbb{R}^d\to\mathbb{R}$ such that

$$|\partial^\alpha\widetilde{V}(x)|\le C_\alpha(1+|x|)^{-\theta|\alpha|}\quad\text{for any }\ \alpha\in\mathbb{Z}^d_{\ge 0}\tag{3}$$

for some $0<\theta\le 1$ and $\underline{\lim}_{|x|\to\infty}\widetilde{V}(x)\ge 0$.

We note that any $V\in\ell^\infty_{\mathrm{comp}}(\mathbb{Z}^d)$ satisfies the Assumption 3. We write $\widetilde{V}=V$ without confusion. We fix $E<0$.

Remark 2.8. Note that a necessary and sufficient condition for the existence of an extension $V:\mathbb{R}^d\to\mathbb{R}$ of $V:\mathbb{Z}^d\to\mathbb{R}$ satisfying (3) is given in Nakamura [39, Lemma 2.1]. Although the case of $\theta=1$ is discussed in [39], the case of $0<\theta<1$ is similar.

We set $q(\xi)=4\sum_{j=1}^d\sinh^2\frac{\xi_j}{2}$ in this appendix. We set $K^E=\{\xi\in\mathbb{R}^d\,|\,q(\xi)\le|E|\}$ and

$$\rho_E(x)=\sup_{\xi\in K^E}\langle x,\xi\rangle.$$

We define the Gauss map $G_E : \partial K^E \to \mathbb{S}^{d-1}$ by $G_E(x) = \partial q(\xi)/|\partial q(\xi)|$ for $\xi \in \partial K^E$. This is bijective since K^E is convex and the Gaussian curvature of ∂K^E does not vanish. Then we have

$$\rho_E(x) = x \cdot G_E^{-1}(x/|x|).$$

Theorem 4. *Under Assumption 3 and the above notation, for any $C_0 > 0$ and $\varepsilon > 0$ there exist $C > 0$ and $1 - \chi$, $\tilde{\chi} \in \ell^\infty_{\mathrm{comp}}(\mathbb{Z}^d)$ such that*

$$\|\chi e^{(1-\varepsilon)\rho_E(x)} u\|_{\ell^2} \leq C\|\tilde{\chi} u\|_{\ell^2} + C\|\chi e^{(1-\varepsilon)\rho_E(x)}(H - z)u\|_{\ell^2}$$

for any $u \in \ell^2(\mathbb{Z}^d)$ and $z \in [E - C_0, E] + i[-C_0, C_0]$.

Corollary 2.9. *Under Assumption 3 and the above notation, if $(H - E)u = 0$ and $u \in \ell^2(\mathbb{Z}^d)$, then for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|u(x)| \leq C_\varepsilon e^{-(1-\varepsilon)\rho_E(x)}$$

for any $x \in \mathbb{Z}^d$.

Remark 2.10. We note that $\rho_E(x)$ coincides with the length of the line segment joining 0 and x with respect to the Agmon-Finsler metric $L(x, v)$ at energy E for $V \equiv 0$. The geodesics with respect to the Agmon-Finsler metric in this case are the straight lines since $L(x, v)$ is independent of x. Thus $\rho_E(x)$ coincides with $d_E(x, 0)$ for $V \equiv 0$ (see [2, subsection 5.3, 6.6]).

2.5.2 Proof of Theorem 4

We first note that $\rho_E(x)$ satisfies the eikonal equation.

Lemma 2.11. *For any $x \in \mathbb{R}^d \setminus \{0\}$,*

$$q(\partial \rho_E(x)) = |E|.$$

Proof. The definition of G_E implies

$$q(G_E^{-1}(x/|x|)) = |E| \ \text{ and } \ (\partial q)(G_E^{-1}(x/|x|)) = x/|x|.$$

Differentiating the first equality and using the second, we learn

$$x \cdot \partial_{x_j}(G_E^{-1}(x/|x|)) = 0.$$

This implies that

$$\partial \rho_E(x) = \partial \left(x \cdot G_E^{-1}(x/|x|)\right) = G_E^{-1}(x/|x|).$$

Then we have

$$q(\partial \rho_E(x)) = q(G_E^{-1}(x/|x|)) = |E|.$$

□

Remark 2.12. Set $\Lambda_0 = T_0^*\mathbb{R}^d \cap \{q(\xi) = |E|\}$, which is a $(d-1)$-dimensional isotropic submanifold of $T^*\mathbb{R}^d$ with the standard symplectic structure. Then the solution in Lemma 2.11 corresponds to the Lagrangian submanifold $\Lambda = \bigcup_{t>0} \Lambda_t$, where Λ_t is the image of Λ_0 under the time t map of the Hamilton flow generated by the Hamilton vector field of $q(\xi)$.

Proof of Theorem 4. Take a smooth modification $\tilde{\rho}_E(x)$ of $\rho_E(x)$ such that $\tilde{\rho}_E(x) = \rho_E(x)$ for $|x| > 1$. We have $|\partial^\alpha \tilde{\rho}_E(x)| \le C_\alpha (1+|x|)^{1-|\alpha|}$ for any $\alpha \in \mathbb{Z}^d_{\ge 0}$. We also note that $\tilde{\rho}_E(x) \gtrsim |x|$ for large $|x|$. For a given small $\varepsilon > 0$, we define ρ_M and $\widetilde{H}_M = \widetilde{H}_M(1)$ from $(1-\varepsilon)\tilde{\rho}_E$ as in subsubsection 2.3.1.

We take any $z \in [E - C_0, E] + i[-C_0, C_0]$ for a fixed C_0. We also take $\chi_1 \in C^\infty(\mathbb{R}^d; [0,1])$ such that $\operatorname{supp}\chi_1 \subset \{x \in \mathbb{R}^d | \, |x| > R-2\}$ and $\chi_1(x) = 1$ for $|x| > R-1$. Lemma 2.1 implies that

$$\chi_1(D_\xi)(\widetilde{H}_M - z)^*(\widetilde{H}_M - z)\chi(D_\xi) - \gamma^2 \chi_1(D_\xi)^2$$

belongs to $\mathrm{Op}S^0_{\theta,0}$ uniformly for $M > 1$ and its symbol is

$$\chi_1(x)^2 |p(\xi - i\partial\rho_M(x), x) - z|^2 - \gamma^2 \chi_1(x)^2$$

modulo $S^{-\theta}_{\theta,0}$, where $0 < \theta \le 1$ is that in Assumption 3. If $R > 2$ is sufficiently large and $\gamma > 0$ is sufficiently small, this is everywhere nonnegative for any $M > 1$ by Assumption 3, Lemma 2.11 and the construction of $\tilde{\rho}_E$.

Then the sharp Gårding inequality implies

$$\|(\widetilde{H}_M - z)\chi_1(D_\xi)\hat{u}\|^2_{L^2} - \gamma^2 \|\chi_1(D_\xi)\hat{u}\|^2_{L^2} \ge -C\|\hat{u}\|^2_{H^{-\theta/2}}$$

for any $\hat{u} \in L^2(\mathbb{T}^d)$. Here $H^{-\theta/2}$ is the Sobolev space on $\mathbb{T}^d$. We replace $\hat{u}$ with $\chi(D_\xi)\hat{u}$, where $\chi \in C^\infty(\mathbb{R}^d; [0,1])$ satisfies $\operatorname{supp}\chi \subset \{x \in \mathbb{R}^d | \, |x| > R\}$ and $\chi(x) = 1$ for $|x| > R+1$. Then we obtain

$$\|(\widetilde{H}_M - z)\chi(D_\xi)\hat{u}\|^2_{L^2} - \gamma^2 \|\chi(D_\xi)\hat{u}\|^2_{L^2} \ge -C\|\chi(D_\xi)\hat{u}\|^2_{H^{-\theta/2}}.$$

Taking $R > 1$ large enough, we have

$$C\|\chi(D_\xi)\hat{u}\|^2_{H^{-\theta/2}} \le \frac{\gamma^2}{2}\|\chi(D_\xi)\hat{u}\|^2_{L^2}.$$

Here C and thus R are independent of $M > 1$ by the uniform estimate of the symbol. This shows that

$$\|e^{\rho_M(x)}(H - z)e^{-\rho_M(x)}\chi(x)u\|_{\ell^2} \ge \frac{\gamma}{2}\|\chi(x)u\|_{\ell^2}$$

for any $u \in \ell^2(\mathbb{Z}^d)$. We then have

$$\|e^{\rho_M(x)}(H - z)\chi(x)u\|_{\ell^2} \ge \frac{\gamma}{2}\|e^{\rho_M(x)}\chi(x)u\|_{\ell^2}$$

for any $u \in \ell^2(\mathbb{Z}^d)$. Taking the limit $M \to \infty$, we obtain

$$\|e^{(1-\varepsilon)\rho_E(x)}(H-z)\chi(x)u\|_{\ell^2} \geq \frac{\gamma}{2}\|e^{(1-\varepsilon)\rho_E(x)}\chi(x)u\|_{\ell^2}$$

for any $u \in \ell^2(\mathbb{Z})$. We then calculate the commutator as in the proof of Theorem 2 and take $\tilde{\chi} \in \ell^\infty_{\mathrm{comp}}$ which is 1 on $\{x \in \mathbb{Z}^d |\, R-1 < |x| < R+2\}$. Then the proof of Theorem 4 is finished. □

2.5.3 The optimality of Theorem 4

We prove that the exponential decay of eigenfunctions in Theorem 4 is optimal for a concrete discrete Schrödinger operator. Fix any $E < 0$. We define $u_E \in \ell^2(\mathbb{Z}^d)$ by

$$u_E(x) = (2\pi)^{-d}\int_{\mathbb{T}^d}\big(4\sum_{j=1}^d \sin^2\frac{\xi_j}{2} + |E|\big)^{-1}e^{-i\langle x,\xi\rangle}d\xi.$$

Then $(H_0+|E|)u_E(x) = \delta_0(x)$. Here H_0 is the free discrete Schrödinger operator and δ_0 is the delta function supported on $0 \in \mathbb{Z}^d$. We note that $u_E(0) > 0$. Thus we have $(H_0+V)u_E(x) = Eu_E(x)$ if we set $V(x) = -u_E(0)^{-1}\delta_0(x)$. We study the exponential decay of this eigenfunction u_E. We note that Corollary 2.9 for u_E is also proved by the deformation of the integral in the definition of u_E.

Take a bounded domain $0 \in \Omega \subset \mathbb{R}^d$ and set

$$\rho_\Omega(x) = \sup_{\xi\in\Omega}\langle x,\xi\rangle.$$

Recall that $K^E = \{\xi \in \mathbb{R}^d |\, 4\sum_{j=1}^d \sinh^2\frac{\xi_j}{2} \leq |E|\}$. The following proposition shows the optimality of Theorem 4.

Proposition 2.13. *Under the above notation, assume that*

$$|u_E(x)| \leq Ce^{-\rho_\Omega(x)}$$

for some $C > 0$ and any $x \in \mathbb{Z}^d$. Then $\Omega \subset K^E$.

Proof. By the Fourier inversion formula, we have

$$(4\sum_{j=1}^d \sin^2\frac{\xi_j}{2} + |E|\big)^{-1} = \sum_{x\in\mathbb{Z}^d} u_E(x)e^{i\langle x,\xi\rangle}.$$

The assumption on u_E implies

$$|u_E(x)e^{i\langle x,\xi\rangle}| \leq Ce^{-\rho_\Omega(x)}e^{-\langle \operatorname{Im}\xi,x\rangle}.$$

We then see that $(4\sum_{j=1}^d \sin^2\frac{\xi_j}{2} + |E|\big)^{-1}$ has an analytic continuation to $\{\xi \in \mathbb{C}^d/2\pi\mathbb{Z}^d |\, -\operatorname{Im}\xi \in \Omega\}$. Since $4\sin^2\xi_j/2 = -4\sinh^2\operatorname{Im}\xi_j/2$ for $\operatorname{Re}\xi_j = 0$, we conclude that $\Omega \subset K^E$. □

Remark 2.14. For $d = 1$, we have $u_E(x) = (|E|(4 + |E|))^{-1/2}e^{-\rho_E(x)}$ ([29, Theorem 2.2]). Ito-Jensen [29, Theorem 2.1, 2.4] showed that u_E is expressed by a hypergeometric function of several variables for $d \geq 2$ and by a generalized hypergeometric function of one variable for $d = 2$. The precise pointwise asymptotics of $u_E(x)$ when $|x| \to \infty$ does not seem to be immediate from these expressions.

3 Resonances and viscosity limit for Wigner-von Neumann-type Hamiltonians

As mentioned in Section 1, this section is based on the joint work [31] with Shu Nakamura.

3.1 Introduction to Section 3

In this section, we consider the one-dimensional Schrödinger operator

$$P = -\frac{d^2}{dx^2} + V(x) \quad \text{on} \quad L^2(\mathbb{R}),$$

where $V(x)$ is an oscillatory and slowly decaying potential. A typical example is

$$P = -\frac{d^2}{dx^2} + a\frac{\sin 2x}{x} \quad \text{on} \quad L^2(\mathbb{R}),$$

where $a \in \mathbb{R}$.

We note that P is not dilation analytic in this case since the potential is exponentially growing in the complex direction. More generally, we consider the following class of potentials.

Assumption 4. The potential $V(x)$ has the following form:

$$V(x) = \sum_{j=1}^{J} s_j(x)W_j(x)$$

for some $J \in \mathbb{N}$, where $s_j \in C(\mathbb{R};\mathbb{R})$ is a periodic function with period π whose Fourier series converges absolutely, and $W_j \in C^\infty(\mathbb{R};\mathbb{R})$ has an analytic continuation to the region $\{z = x + iy \mid |x| > R_0, |y| < K|x|\}$ for some $R_0 > 0$ and $K > 0$ with the bound $|W_j(z)| \leq C|z|^{-\mu}$ for some $\mu > 0$ in this region.

We note that $V(x) = a\frac{\sin 2x}{x}$ satisfies Assumption 4 for any $K > 0$ with $J = 2$. We also note that dilation analytic potentials satisfy Assumption 4 by setting $s_j(x) = 1$. Our first aim is to show that resonances can be defined for this class of potentials. We write the set of thresholds by $\mathcal{T} = \{n^2 \mid n \in \mathbb{N}\cup\{0\}\}$ (see Remark 3.6 for the reason for which we introduce $\mathcal{T}$). We denote the resolvent on the upper half plane by $R_+(z) = (z - P)^{-1}$, $\operatorname{Im} z > 0$.

Theorem 5. *Under Assumption 4, there exists a complex neighborhood $\Omega \subset \mathbb{C}$ of $[0,\infty) \setminus \mathcal{T}$ such that the following holds. For any $f, g \in L^2_{\mathrm{comp}}(\mathbb{R})$, the matrix element of the resolvent $(f, R_+(z)g)$ has a meromorphic continuation to Ω.*

Remark 3.1. Unfortunately, the original Wigner-von Neumann potential ([41], see also [45, Section XIII.13])

$$V(x) = (1+g(x)^2)^{-2}(-32\sin x)(g(x)^3\cos x - 3g(x)^2\sin^3 x + g(x)\cos x + \sin^3 x)$$

does not seem to satisfy Assumption 4, where $g(x) = 2x - \sin 2x$. The argument principle implies that if $\nu > 1/2$ and $\ell \gg 1$ with $\ell \in \mathbb{Z}$, $g(z) \pm i$ have two zeros in the region $\{z \in \mathbb{C} \mid (\ell - 1/2)\pi \le \mathrm{Re}\, z \le (\ell + 1/2)\pi, -\nu \log \ell \le \mathrm{Im}\, z \le \nu \log \ell\}$. Thus we need another method to study the complex resonances for the original Wigner-von Neumann Hamiltonian.

We define the complex resonances using this meromorphic continuation.

Definition 3.2. A complex number $z \in \Omega$ is called a resonance if z is a pole of $(f, R_+(z)g)$ for some $f, g \in L^2_{\mathrm{comp}}(\mathbb{R})$. The multiplicity of resonance m_z at z is defined as the maximal number m such that there exist $f_1, \dots, f_m, g_1, \dots, g_m \in L^2_{\mathrm{comp}}(\mathbb{R})$ with $\det\big(\frac{1}{2\pi i}\oint_{C(z)}(f_i, R_+(\zeta)g_j)d\zeta\big)_{i,j=1}^m \neq 0$, where $C(z)$ is a small circle around z. The set of resonances is denoted by $\mathrm{Res}(P)$ including multiplicities.

Remark 3.3. $\mathrm{Res}(P)$ is discrete in Ω and $m_z < \infty$ for any $z \in \Omega$ (see Remark 3.7).

We prove Theorem 5 by introducing the periodic complex distortion in the Fourier space. See subsection 3.2 for the definition and the underlying idea.

We next add a complex absorbing potential as follows:

$$P_\varepsilon = -\frac{d^2}{dx^2} + V(x) - i\varepsilon x^2, \quad \varepsilon > 0.$$

It is not difficult to see that P_ε, $\varepsilon > 0$, has purely discrete spectrum on $L^2(\mathbb{R})$. Zworski [62] proved that the resonances can be characterized as limit points of the eigenvalues of P_ε as $\varepsilon \to 0$, namely $\lim_{\varepsilon\to 0}\sigma_d(P_\varepsilon) = \mathrm{Res}(P)$ for compactly supported potentials. The proof employed the dilation analytic method. Zworski [62] also proposed a problem of finding a potential $V(x)$ such that the limit set of $\sigma_d(P_\varepsilon)$ when $\varepsilon \to 0$ is not discrete. He suggested $V(x) = \frac{\sin x}{x}$ as a candidate for such $V(x)$. Our next result disproves this conjecture (away from the thresholds). We set $B(z,\rho) = \{w \in \mathbb{C} \mid |w - z| \le \rho\}$.

Theorem 6. *Under Assumption 4, there exists a complex neighborhood $\Omega \subset \mathbb{C}$ of $[0,\infty) \setminus \mathcal{T}$ such that $\lim_{\varepsilon\to 0}\sigma_d(P_\varepsilon) = \mathrm{Res}(P)$ in Ω including multiplicities. More precisely, for any $z \in \Omega$ there exists $\rho_0 > 0$ such that for any $0 < \rho < \rho_0$ there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$,*

$$\#\big(\sigma_d(P_\varepsilon) \cap B(z,\rho)\big) = m_z.$$

In particular, $\lim_{\varepsilon\to 0}\sigma_d(P_\varepsilon)$ is discrete in Ω.

Remark 3.4. Ω in Theorem 5 and Theorem 6 are given explicitly in subsection 3.2 and subsection 3.3.

Wigner-von Neumann-type Hamiltonians have been investigated by many authors. See for instance [4], [5], [6], [9], [15], [18], [34], [35], [46], [47] and references therein. To our knowledge, the definition of the complex resonances based on the complex distortion for Schrödinger operators with oscillatory and slowly decaying potentials is new. The complex distortion in the Fourier space was studied by Cycon [7] and Sigal [50] for radially symmetric dilation analytic, or sufficiently smooth exponentially decaying potentials. This method was extended to the not necessarily radially symmetric case in [37]. See the references in [37] for earlier works on the complex distortion.

Complex absorbing potential method was introduced in physical chemistry ([48], [49]). Theorem 6 was proved by Zworski [62] for the compactly supported potentials, which gave the mathematical justification of this method. This was extended to several settings (see [31], [58], [59], [60]). Analogous results were proved for Pollicott-Ruelle resonances by Dyatlov-Zworski [13] (see also [8], [12]), and for 0th order pseudodifferential operators by Galkowski-Zworski [16]. Stefanov [53] studied the approximation of resonances in the semiclassical limit by a fixed complex absorbing potential. Similar methods were also used in Nonnenmacher-Zworski [42], [43] and Vasy [55].

This section is organized as follows. In subsection 3.2, we prove the theorems for the model case $V(x) = a\frac{\sin 2x}{x}$. It contains essential ideas for the general case. In subsection 3.3, we present technical arguments which complete the proofs of the theorems for the general case.

3.2 The proofs for the model case

In this subsection, we present the general ideas for the proofs and give the proofs for the model case $V(x) = a\frac{\sin 2x}{x}$, $a \in \mathbb{R}$.

3.2.1 Periodic distortion in the Fourier space

The idea of the proof of Theorem 5 is as follows. While the standard dilation analytic method for the resonances does not apply to our potentials, it is known that we can construct a Mourre theory with the conjugate operator A', where

$$A' = \frac{1}{2}(x \cdot D' + D' \cdot x), \quad D'u(x) = \frac{1}{2\pi}(u(x+\pi) - u(x-\pi))$$

(see [38]). In the Fourier space, A' is a differential operator

$$\widetilde{A'} = (i\partial_\xi) \cdot \sin(\pi\xi) + \sin(\pi\xi) \cdot (i\partial_\xi).$$

We may use $e^{-i\theta A'}$ to define the resonances for our model. Although the flow of the vector field $\sin(\pi\xi)$ is calculated explicitly, it is complicated.

Thus we use the Hunziker-type distortion from the vector field $\sin(\pi\xi)$ (see [37] for Hunziker-type local distortion in the Fourier space). Namely, we set

$$\Phi_\theta(\xi) = \xi + \theta\sin(\pi\xi), \quad U_\theta f(\xi) = \Phi'_\theta(\xi)^{\frac{1}{2}} f(\Phi_\theta(\xi)),$$

where $\theta \in (-\pi^{-1}, \pi^{-1})$. This is our periodic distortion in the Fourier space. In the Fourier space, P has the form $\widetilde{P} = \xi^2 + \widetilde{V}$, where $\widetilde{V} = (2\pi)^{-1/2}\hat{V}*$ is a convolution operator and $\hat{V}$ is the Fourier transform of V with the convention $\hat{V}(\xi) = (2\pi)^{-1/2}\int V(x)e^{-ix\xi}dx$. Hence we have

$$\widetilde{P}_\theta := U_\theta \widetilde{P} U_\theta^{-1} = (\xi + \theta\sin(\pi\xi))^2 + \widetilde{V}_\theta, \quad \widetilde{V}_\theta = U_\theta \widetilde{V} U_\theta^{-1}.$$

We next prove the analyticity of $\widetilde{V}_\theta$ for the model case.

Lemma 3.5. *Let $V(x) = a\frac{\sin 2x}{x}$ for $a \in \mathbb{R}$. Then $\widetilde{V}_\theta = (\Phi_\theta')^{\frac{1}{2}}\widetilde{V}(\Phi_\theta')^{\frac{1}{2}}$, where $(\Phi_\theta')^{\frac{1}{2}}$ is a multiplication operator by $\Phi_\theta'(\xi)^{\frac{1}{2}}$, and $\widetilde{V} = \frac{a}{2}\mathbb{1}_{[-2,2]}*$. Here $\mathbb{1}_{[-2,2]}$ denotes the indicator function of $[-2,2]$. In particular, $\widetilde{V}_\theta$ is analytic with respect to θ and ξ^2-compact, where θ ranges over $\mathbb{C} \setminus \big((-\infty, -\pi^{-1}] \cup [\pi^{-1}, \infty)\big)$.*

Proof. By a simple computation, we immediately have $\widetilde{V} = \frac{a}{2}\mathbb{1}_{[-2,2]}*$. Thus we have

$$\begin{aligned}
\widetilde{V}_\theta f(\xi) &= U_\theta \widetilde{V} U_\theta^{-1} f(\xi) \\
&= \int_{\mathbb{R}} \Phi_\theta'(\xi)^{\frac{1}{2}} \frac{a}{2} \mathbb{1}_{[-2,2]}(\Phi_\theta(\xi) - \eta)(\Phi_\theta^{-1})'(\eta)^{\frac{1}{2}} f(\Phi_\theta^{-1}(\eta))d\eta \\
&= \int_{\mathbb{R}} \Phi_\theta'(\xi)^{\frac{1}{2}} \frac{a}{2} \mathbb{1}_{[-2,2]}(\Phi_\theta(\xi) - \Phi_\theta(\eta))\Phi_\theta'(\eta)^{\frac{1}{2}} f(\eta)d\eta
\end{aligned}$$

for $\theta \in (-\pi^{-1}, \pi^{-1})$. To simplify this expression, we note that

$$\frac{d}{d\xi}\left(\Phi_\theta(\xi) - \Phi_\theta(\eta)\right) = 1 + \theta\pi\cos(\pi\xi) > 0$$

for $\theta \in (-\pi^{-1}, \pi^{-1})$ and that

$$\Phi_\theta(\eta \pm 2) - \Phi_\theta(\eta) = \pm 2 + \theta(\sin(\pi(\eta \pm 2)) - \sin(\pi\eta)) = \pm 2.$$

These imply that $-2 \le \Phi_\theta(\xi) - \Phi_\theta(\eta) \le 2$ if and only if $-2 \le \xi - \eta \le 2$. Then we see that

$$\begin{aligned}
\widetilde{V}_\theta f(\xi) &= \int_{\mathbb{R}} \Phi_\theta'(\xi)^{\frac{1}{2}} \frac{a}{2} \mathbb{1}_{[-2,2]}(\xi - \eta)\Phi_\theta'(\eta)^{\frac{1}{2}} f(\eta)d\eta \\
&= (\Phi_\theta')^{\frac{1}{2}}\widetilde{V}(\Phi_\theta')^{\frac{1}{2}} f(\xi).
\end{aligned}$$

The first part of the Lemma 3.5 implies the second part. Note that $(\Phi_\theta')^{\frac{1}{2}}$ is well-defined for $\theta \in \mathbb{C} \setminus \big((-\infty, -\pi^{-1}] \cup [\pi^{-1}, \infty)\big)$ since $\mathbb{C} \setminus \big((-\infty, -\pi^{-1}] \cup [\pi^{-1}, \infty)\big)$ is simply connected and $\Phi_\theta'(\xi) = 1 + \theta\pi\cos(\pi\xi) \neq 0$ for such θ. □

3.2.2 Definition of resonances

In subsubsection 3.2.2 and subsubsection 3.2.3, we set $V(x) = a\frac{\sin 2x}{x}$ for $a \in \mathbb{R}$. The modifications needed for the general case are explained in subsection 3.3.

By Lemma 3.5, we see that $\widetilde{P}_\theta$ is analytic with respect to θ in the sense of Kato and the essential spectrum of $\widetilde{P}_\theta$ is given by

$$\sigma_{\mathrm{ess}}(\widetilde{P}_\theta) = \left\{(\xi + \theta\sin(\pi\xi))^2 \;\middle|\; \xi \in \mathbb{R}\right\}.$$

Remark 3.6. We note that

$$\sigma_{\rm ess}(\widetilde{P}_\theta)\cap[0,\infty)=\{n^2\mid n\in\mathbb{N}\cup\{0\}\}$$

for complex θ. Thus $\mathcal{T}=\{n^2\mid n\in\mathbb{N}\cup\{0\}\}\subset[0,\infty)$ is the set of thresholds with respect to our periodic complex distortion in the Fourier space. This is analogous to the set of threshold $\{0\}\subset[0,\infty)$ in the case of the usual complex dilation. In addition to the usual threshold 0, our $\mathcal{T}$ contains energy n^2, $n\in\mathbb{N}$, at which the corresponding wave $e^{\pm inx}$ has a half-multiple wavenumber of that of the oscillating part of the potential.

We fix $n\in\mathbb{N}$ and consider the energy interval $((n-1)^2,n^2)$. We take $\theta=(-1)^n i\delta=\pm i\delta$. We easily see that for $0<\delta<\pi^{-1}$ the essential spectrum of $\widetilde{P}_{\pm i\delta}$ is the graph of a function $\kappa_{\pm\delta}:[0,\infty)\to\mathbb{R}$ in $\mathbb{R}^2\cong\mathbb{C}$. Namely, we may define $\kappa_{\pm\delta}(x)$, $x=\operatorname{Re}z\geq 0$, by the relation

$$\sigma_{\rm ess}(\widetilde{P}_{\pm i\delta})=\bigl\{z\in\mathbb{C}\bigm|\operatorname{Im}z=\kappa_{\pm\delta}(\operatorname{Re}z),\operatorname{Re}z\geq 0\bigr\}.$$

If $x=\xi^2-\delta^2\sin^2(\pi\xi)$ for $\xi\in\mathbb{R}$, then $\kappa_{\pm\delta}(x)=\pm 2\delta\xi\sin(\pi\xi)$. An important fact is that $\kappa_{(-1)^n\delta}(x)<0$ for $x\in((n-1)^2,n^2)$.

We take any $0<\delta<\delta_0$, where $\delta_0=\pi^{-1}$. We set

$$\Omega_{n,\delta}=\bigl\{z=x+iy\bigm|(n-1)^2<x<n^2,y>\kappa_{(-1)^n\delta}(x)\bigr\}.$$

We note that $\Omega_{n,\delta}\subset\Omega_{n,\delta'}$ if $0<\delta<\delta'<\delta_0$.

Proof of Theorem 5 for the model case. We fix $n\in\mathbb{N}$ and $\delta>0$ as above. We denote $\mathcal{A}=L^2_{\rm comp}(\mathbb{R})$. We then see that $U_\theta\hat{f}$, $f\in\mathcal{A}$, has an analytic continuation for complex θ. We denote the resolvent $R_+(z)$ on the Fourier space by $\widetilde{R_+}(z)$. For $f,g\in\mathcal{A}$, we see

$$(\hat{f},\widetilde{R_+}(z)\hat{g})=(U_\theta\hat{f},U_\theta\widetilde{R_+}(z)U_\theta^{-1}U_\theta\hat{g})=(U_{\overline{\theta}}\hat{f},(z-\widetilde{P_\theta})^{-1}U_\theta\hat{g})$$

where $\theta\in\mathbb{R}$ and $\operatorname{Im}z>0$. The right hand side is analytic with respect to θ by Lemma 3.5, where θ ranges over a complex neighborhood of $\{(-1)^n i\delta\mid 0\leq\delta<\delta_0\}$. This implies that the left hand side has a meromorphic continuation to Ω_{n,δ_0} with respect to z. Then Theorem 5 is proved with $\Omega=\bigcup_{n\in\mathbb{N}}\Omega_{n,\delta_0}$. □

Remark 3.7. We set the spectral projection for $\widetilde{P_\theta}$ as $\Pi_z^\theta=\frac{1}{2\pi i}\oint_{C(z)}(\zeta-\widetilde{P_\theta})^{-1}d\zeta$. Then we have

$$\frac{1}{2\pi i}\oint_{C(z)}(f,R_+(\zeta)g)d\zeta=\frac{1}{2\pi i}\oint_{C(z)}(U_{\overline{\theta}}\hat{f},(\zeta-\widetilde{P_\theta})^{-1}U_\theta\hat{g})d\zeta=(U_{\overline{\theta}}\hat{f},\Pi_z^\theta U_\theta\hat{g}).$$

We note that $\{U_\theta\hat{f}\mid f\in\mathcal{A}\}$ is dense in L^2. This is proved by an argument similar to [27, Theorem 3]. Then we see that $m_z=\operatorname{rank}\Pi_z^\theta$. Namely, the resonances coincide with the discrete eigenvalues of $\widetilde{P_\theta}$ including multiplicities. This implies that $\operatorname{Res}(P)$ is discrete in Ω and $m_z<\infty$ for any $z\in\Omega$.

3.2.3 Viscosity limit

As in [62], the essential part of the proof of Theorem 6 is the resolvent estimate for the free distorted operator which is uniform with respect to ε. We prove this by the semiclassical analysis in the Fourier space with the semiclassical parameter $h = \sqrt{\varepsilon}$. Since we work in the Fourier space, the term $-i\varepsilon x^2 = i\varepsilon\partial_\xi^2$ is the usual viscosity term (multiplied by i). The viscosity limit corresponds to the semiclassical limit.

For simplicity, we set $P_0 = P$, $\widetilde{P}_0 = \widetilde{P}$ and $\widetilde{P}_{0,\theta} = \widetilde{P}_\theta$. In the Fourier space, P_ε, $\varepsilon \geq 0$, has the following form:

$$\widetilde{P}_\varepsilon = \xi^2 + \widetilde{V} + i\varepsilon\partial_\xi^2.$$

Thus the distorted operator $\widetilde{P}_{\varepsilon,\theta} = U_\theta \widetilde{P}_\varepsilon U_\theta^{-1}$ is given by

$$\widetilde{P}_{\varepsilon,\theta} = (\xi + \theta\sin(\pi\xi))^2 + \widetilde{V}_\theta - i\varepsilon D_\xi(1 + \pi\theta\cos(\pi\xi))^{-2}D_\xi - i\varepsilon r_\theta(\xi),$$

where $r_\theta(\xi) = -\Phi_\theta'(\xi)^{-\frac{1}{2}}\partial_\xi(\Phi_\theta'(\xi)^{-1}\partial_\xi(\Phi_\theta'(\xi)^{-\frac{1}{2}}))$ is a function which is analytic with respect to θ and bounded with respect to ξ. Since the resolvent of $\widetilde{P}_{\varepsilon,\theta}$ is compact , $\widetilde{P}_{\varepsilon,\theta}$, $\varepsilon > 0$, has purely discrete spectrum. Moreover, for fixed $\varepsilon > 0$, $\widetilde{P}_{\varepsilon,\theta}$ is analytic with respect to θ in the sense of Kato. These imply that the eigenvalues of $\widetilde{P}_{\varepsilon,\theta}$ coincide with those of $\widetilde{P}_\varepsilon$ including multiplicities by the same argument as in Remark 3.7. Then it is enough to show that the eigenvalues of $\widetilde{P}_{\varepsilon,\theta}$ converge to those of $\widetilde{P}_\theta$ as $\varepsilon \to 0+$.

Proof of Theorem 6 for the model case. We set the distorted free Hamiltonian as follows:

$$\widetilde{Q}_{\varepsilon,\theta} = (\xi + \theta\sin(\pi\xi))^2 - i\varepsilon D_\xi(1 + \pi\theta\cos(\pi\xi))^{-2}D_\xi - i\varepsilon r_\theta(\xi), \quad \varepsilon \geq 0.$$

We first prove the resolvent estimate (4) below for this operator. In this proof, we fix $n \in \mathbb{N}$, set $\theta = (-1)^n i\delta = \pm i\delta$, $0 < \delta < \delta_0$ as in subsubsection 3.2.2.

We set $h = \sqrt{\varepsilon}$ and regard $\widetilde{Q}_{\varepsilon,\theta}$ as an h-pseudodifferential operator in the Fourier space. Recall the definition of $\Omega_{n,\delta}$ in subsubsection 3.2.2. We easily see that the range

$$\left\{(\xi + \theta\sin(\pi\xi))^2 - i(1 + \pi\theta\cos(\pi\xi))^{-2}x^2 \;\middle|\; x, \xi \in \mathbb{R}\right\}$$

of the h-principal symbol of $\widetilde{Q}_{\varepsilon,\theta}$ is disjoint from $\Omega_{n,\delta}$ for small $\delta > 0$. This is true for $0 < \delta \leq \delta_1$, where $\delta_1 = (\sqrt{2} - 1)\pi^{-1}$. The constant δ_1 comes from the requirement that $\sup_{x\geq 0}|\frac{d}{dx}\kappa_{\pm\delta}(x)| = |\frac{d}{dx}\kappa_{\pm\delta}(0)| = \frac{2\pi\delta}{1-\pi^2\delta^2}$ is less than or equal to the minimal value $\frac{1}{2}\left(\frac{1}{\pi\delta} - \pi\delta\right)$ with respect to $\xi \in \mathbb{R}$ of the absolute value of the slope of the half line $\left\{-i(1 \pm \pi\delta i\cos(\pi\xi))^{-2}x^2 \;\middle|\; x \in \mathbb{R}\right\}$ in the complex plane. We consider $0 < \delta < \delta_1$ and do not pursue the optimal δ. We fix $0 < \delta < \delta_1$ and $z \in \Omega_{n,\delta}$. Then there exists $\rho_0 > 0$ such that there is no resonance in $B(z, \rho_0) \Subset \Omega_{n,\delta}$ possibly expect for z, where $B(z, \rho)$ denotes the disk of radius ρ with the center z. In the following, we fix $0 < \rho < \rho_0$, and consider $w \in B_z = B(z,\rho)$. By the standard semiclassical calculus we see that $(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}$ exists and

$$\|(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}\|_{L^2\to L^2} \le C \tag{4}$$

for $w \in B_z$ and $0 < \varepsilon \ll 1$. We note that it also holds for $\varepsilon = 0$.

We next employ the perturbation argument. By the existence of $(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}$, we have

$$\widetilde{P}_{\varepsilon,\theta} - w = (1 + \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1})(\widetilde{Q}_{\varepsilon,\theta} - w).$$

By Lemma 3.5 and the boundedness of $(\xi^2+i)(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}$, we see that $\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}$ is compact for $0 \le \varepsilon \ll 1$. Then we may apply the analytic Fredholm theory. We have

$$\begin{aligned}(w - \widetilde{P}_{\varepsilon,\theta})^{-1} &= \big(\partial_w(\widetilde{P}_{\varepsilon,\theta} - w)\big)(\widetilde{P}_{\varepsilon,\theta} - w)^{-1}\\ &= (\partial_w \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1})(1 + \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1})^{-1}\\ &\quad + (1 + \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1})(w - \widetilde{Q}_{\varepsilon,\theta})^{-1}(1 + \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1})^{-1}.\end{aligned}$$

The Gohberg-Sigal factorization theorem ([17, Theorem 3.1]) applied to $1 + \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}$, Cauchy's theorem and the cyclicity of the trace imply that

$$\operatorname{tr} \oint_{\partial B_z} (1 + \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1})(w - \widetilde{Q}_{\varepsilon,\theta})^{-1}(1 + \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1})^{-1} dw = 0.$$

Thus the number of the eigenvalues of $P_{\varepsilon,\theta}$, $0 \le \varepsilon \ll 1$, in B_z is given by

$$\operatorname{tr} \oint_{\partial B_z} (w - \widetilde{P}_{\varepsilon,\theta})^{-1} dw = \operatorname{tr} \oint_{\partial B_z} (\partial_w \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1})(1 + \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1})^{-1} dw.$$

We note that the right hand side of this equality is the number of zeros of $1 + \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}$ in B_z in the sense of Gohberg-Sigal ([17, Theorem 2.1]). Thus operator-valued Rouché's theorem ([17, Theorem 2.2]) implies that to prove Theorem 6, it is enough to show that

$$\big\|((1+\widetilde{V}_\theta(\widetilde{Q}_{0,\theta}-w)^{-1})-(1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}))(1+\widetilde{V}_\theta(\widetilde{Q}_{0,\theta}-w)^{-1})^{-1}\big\|_{L^2\to L^2} < 1$$

for $w \in \partial B_z$ and small $\varepsilon > 0$. We note that $(1 + \widetilde{V}_\theta(\widetilde{Q}_{0,\theta} - w)^{-1})^{-1}$ exists and is independent of $\varepsilon > 0$ for $w \in \partial B_z$ since we assumed that $w \in \partial B_z$ is not a resonance. Then the above estimate hold if we show

$$\lim_{\varepsilon\to 0} \|\widetilde{V}_\theta(\widetilde{Q}_{0,\theta} - w)^{-1} - \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}\|_{L^2\to L^2} = 0 \tag{5}$$

uniformly for $w \in \partial B_z$.

Take any $\gamma > 0$. We claim that there exists a decomposition $\widetilde{V}_\theta = \widetilde{V}_{\theta,1} + \widetilde{V}_{\theta,2}$, where $\widetilde{V}_{\theta,1}$ is a smoothing pseudodifferential operator in the Fourier space and $\|\widetilde{V}_{\theta,2}\|_{L^2\to L^2} < \gamma$. To see this, we write

$$\widetilde{V}_\theta = (\Phi'_\theta)^{\frac12}\widetilde{V}(\Phi'_\theta)^{\frac12} = (\Phi'_\theta)^{\frac12}\widetilde{V}_{1,R}(\Phi'_\theta)^{\frac12} + (\Phi'_\theta)^{\frac12}\widetilde{V}_{2,R}(\Phi'_\theta)^{\frac12} = \widetilde{V}_{\theta,1} + \widetilde{V}_{\theta,2}$$

for large $R > 0$, where $\widetilde{V}_{j,R}$ is the Fourier multiplier on the Fourier space by $V_{j,R}$. Here we took $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi = 1$ near $x = 0$ and decomposed $a\frac{\sin 2x}{x} = a\frac{\sin 2x}{x}\chi(x/R) + a\frac{\sin 2x}{x}(1-\chi(x/R)) = V_{1,R} + V_{2,R}$. The claimed properties are easily verified.

Since $\|(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}\|_{L^2 \to L^2} \le C$ for $0 \le \varepsilon \ll 1$ and $w \in B_z$, we see that

$$\|\widetilde{V}_{\theta,2}(\widetilde{Q}_{0,\theta} - w)^{-1} - \widetilde{V}_{\theta,2}(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}\|_{L^2 \to L^2} \le 2C\gamma,$$

where C is independent of γ. By the resolvent equation, we have

$$\begin{aligned}&\widetilde{V}_{\theta,1}(\widetilde{Q}_{0,\theta} - w)^{-1} - \widetilde{V}_{\theta,1}(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}\\&= -i\varepsilon\widetilde{V}_{\theta,1}(\widetilde{Q}_{0,\theta} - w)^{-1}(D_\xi(1+\pi\theta\cos(\pi\xi))^{-2}D_\xi + r_\theta(\xi))(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}.\end{aligned}$$

Since $\widetilde{V}_{\theta,1}$ is smoothing and $(\widetilde{Q}_{0,\theta} - w)^{-1}$ preserves the Sobolev space on the Fourier space, we see that $\widetilde{V}_{\theta,1}(\widetilde{Q}_{0,\theta} - w)^{-1}D_\xi^2$ is L^2-bounded. Thus we have

$$\|\widetilde{V}_{\theta,1}(\widetilde{Q}_{0,\theta} - w)^{-1} - \widetilde{V}_{\theta,1}(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}\|_{L^2 \to L^2} \le C_\gamma\varepsilon$$

with some constant $C_\gamma > 0$. We take small ε such that $\varepsilon \le (C/C_\gamma)\gamma$. Then we have

$$\|\widetilde{V}_\theta(\widetilde{Q}_{0,\theta} - w)^{-1} - \widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta} - w)^{-1}\|_{L^2 \to L^2} \le 2C\gamma + C_\gamma\varepsilon \le 3C\gamma$$

and thus (5) is proved since $\gamma > 0$ is arbitrary small. Then Theorem 6 is proved with $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_{n,\delta_1}$. □

3.3 The proofs for the general case

3.3.1 The analyticity of $\widetilde{V}_\theta$

Recall that $\widetilde{V}_\theta$ was defined in subsubsection 3.2.1. We also recall the constant K in Assumption 4.

Lemma 3.8. *Under Assumption 4, the distorted operator $\widetilde{V}_\theta$ is analytic with respect to θ and ξ^2-compact, where θ ranges over some complex neighborhood of $\{i\delta \mid -K\pi^{-1} < \delta < K\pi^{-1}\}$.*

Proof of Lemma 3.8. For real θ, the integral kernel of $\widetilde{V}_\theta$ is given by

$$\widetilde{V}_\theta(\xi,\eta) = \Phi_\theta'(\xi)^{\frac{1}{2}}\hat{V}(\Phi_\theta(\xi) - \Phi_\theta(\eta))\Phi_\theta'(\eta)^{\frac{1}{2}}, \quad \xi,\eta \in \mathbb{R}.$$

We first consider the case of $V \in C_c^\infty(\mathbb{R};\mathbb{R})$. Then $\widetilde{V}_\theta(\xi,\eta)$ is analytic with respect to $\theta \in \mathbb{C}$ and has the off-diagonal decay bounds

$$|\partial_\xi^\alpha\partial_\eta^\beta\widetilde{V}_\theta(\xi,\eta)| \le C_{\alpha,\beta,N}\langle\xi-\eta\rangle^{-N}, \quad \xi,\eta \in \mathbb{R}$$

for any α, β and N. Here $C_{\alpha,\beta,N}$ is independent of θ when $\theta \in \mathbb{C}$ ranges over a bounded set. We also recall the formula ([61, subsection 8.1])

$$\widetilde{V}_\theta = b^{\mathrm{w}}(\xi, D_\xi;\theta), \quad b(\xi,x;\theta) = \int_{\mathbb{R}} \widetilde{V}_\theta\Big(\xi + \frac{\eta}{2}, \xi - \frac{\eta}{2}\Big)e^{-i\langle\eta,x\rangle}d\eta,$$

where b^{w} denotes the Weyl quantization

$$b^{\mathrm{w}}(\xi, D_\xi; \theta)f(\xi) = (2\pi)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} b\Big(\frac{\xi+\eta}{2}, x; \theta\Big) e^{i\langle \xi-\eta, x\rangle} f(\eta) d\eta dx.$$

These imply that $\widetilde{V}_\theta$ is a pseudodifferential operator in the Fourier space whose symbol is rapidly decaying with respect to x, that is,

$$|\partial_\xi^\alpha \partial_x^\beta b(\xi, x; \theta)| \le C_{\alpha,\beta,N} \langle x \rangle^{-N}, \quad \xi, x \in \mathbb{R}$$

for any α, β and N. Here $C_{\alpha,\beta,N}$ is independent of θ when $\theta \in \mathbb{C}$ ranges over a bounded set. Moreover, $b(\xi, x; \theta)$ is analytic with respect to θ. Thus the Lemma 3.8 is proved in this case.

We next consider the case of $V(x) = s(x)W(x)$, where $s(x)$ and $W(x)$ satisfy the condition in Assumption 4. We first estimate the Fourier transform of $W(x)$. By the deformation of the integral, we have

$$\hat{W}(\xi) = (2\pi)^{-1/2} \int_{C_{\pm,\tau}} W(z) e^{-iz\xi} dz, \quad \pm\xi > 0,$$

where

$$C_{\pm,\tau} = (e^{\pm i\tau}(-\infty, 0] - 2R_0) \cup [-2R_0, 2R_0] \cup (2R_0 + e^{\mp i\tau}[0, \infty)).$$

Here $0 < \tau < \arctan K$ and R_0 is that in Assumption 4. By this expression we see that $\hat{W}(\xi)$ has an analytic continuation to

$$S_\tau = \{z \in \mathbb{C}^* | -\tau < \arg z < \tau\} \cup \{z \in \mathbb{C}^* | -\tau < \arg z - \pi < \tau\}.$$

We see that $\hat{W}(\xi)$ decays rapidly in S_τ when $|\xi| \to \infty$ by the smoothness of W. We claim that we have $|\hat{W}(\xi)| \le C|\xi|^{-\frac{1}{1+\mu}}$ for small $\xi \in S_\tau$, where $\mu > 0$ is the constant in Assumption 4. To see this, we take $C_{\pm,\tau'}$ for $0 < \tau < \tau' < \arctan K$ and estimate

$$|\hat{W}(\xi)| \le C \int_0^\infty e^{-cx|\xi|} \langle x \rangle^{-\mu} dx = C|\xi|^{-1} \int_0^\infty e^{-c|x|} \langle x/|\xi| \rangle^{-\mu} dx.$$

We divide the integral into $\int_0^\varepsilon + \int_\varepsilon^\infty$, which shows the bound $\frac{\varepsilon}{|\xi|} + \frac{1}{|\xi|}\langle \varepsilon/|\xi| \rangle^{-\mu}$. We take $\varepsilon = |\xi|^{\frac{\mu}{1+\mu}}$ and see that $|\hat{W}(\xi)| \le C|\xi|^{-\frac{1}{1+\mu}}$.

We next show that the Fourier transform $\hat{V}(\xi)$ has an analytic continuation to the region $T_\tau = \bigcup_{k\in\mathbb{Z}} T_{\tau,k}$, where

$$T_{\tau,k} = \{z \in \mathbb{C} \setminus \{0, 2\} | -\tau < \arg z < \tau, -\tau < \arg(2 - z) < \tau\} + 2k,$$

and the estimate

$$\sum_{k\in\mathbb{Z}} \sup_{\xi \in T_{\tau,k}} |\xi - 2k|^{\frac{1}{1+\mu}} |\xi - 2k - 2|^{\frac{1}{1+\mu}} |\hat{V}(\xi)| < \infty \tag{6}$$

holds. To see this, we first write the Fourier transform of s as

$$\hat{s}(\xi) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} a_k \delta(\xi - 2k).$$

Then we have

$$\hat{V}(\xi) = \sum_{k\in\mathbb{Z}} a_k \hat{W}(\xi - 2k).$$

We have $\sum_{k\in\mathbb{Z}} |a_k| < \infty$ by Assumption 4. The estimates on $\hat{W}(\xi)$ above show

$$\sum_{k\in\mathbb{Z}} \sup_{\xi\in T_{\tau,k}} |\xi - 2k|^{\frac{1}{1+\mu}} |\xi - 2k - 2|^{\frac{1}{1+\mu}} |\hat{W}(\xi)| < \infty.$$

Young's inequality in $\ell^1(\mathbb{Z})$ applied to sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{\sup_{\xi\in T_{\tau,k}} |\xi - 2k|^{\frac{1}{1+\mu}} |\xi - 2k - 2|^{\frac{1}{1+\mu}} |\hat{W}(\xi)|\}_{k\in\mathbb{Z}}$ then implies the estimate (6).

By (6), we have $|\widetilde{V}_\theta(\xi,\eta)| \le g(\xi - \eta)$ for some integrable function g. This is also true for the derivatives of $\widetilde{V}_\theta(\xi,\eta)$ with respect to θ by Cauchy's formula. Thus Young's inequality implies that the operator $\widetilde{V}_\theta$ with integral kernel $\widetilde{V}_\theta(\xi,\eta)$ is L^2-bounded and analytic with respect to θ. We note that if $\theta \in i\mathbb{R}$, we have

$$|\mathrm{Im}\,(\Phi_\theta(\xi) - \Phi_\theta(\eta))| \le \pi|\theta||\mathrm{Re}\,(\Phi_\theta(\xi) - \Phi_\theta(\eta)) - 2k|,$$

for any $k \in \mathbb{Z}$, in particular, for k with $|\xi-\eta-2k| \le 1$. Thus θ can be taken from a complex neighborhood of $\{i\delta | -\tan\tau < \delta < \tan\tau\}$. Since $0 < \tau < \arctan K$ is arbitrary, $\widetilde{V}_\theta$ is analytic for θ, where θ ranges as claimed in Lemma 3.8.

To show ξ^2-compactness, we approximate V by C_c^∞ functions. Take $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi = 1$ near $x = 0$. We decompose $V(x) = V_{1,R} + V_{2,R}$, where $R \gg 1$,

$$V_{1,R} = \chi(x/R)W(x) \sum_{|k|\le R} a_k e^{2ikx}$$

and

$$V_{2,R} = W(x) \sum_{|k|>R} a_k e^{2ikx} + (1 - \chi(x/R))W(x) \sum_{|k|\le R} a_k e^{2ikx}.$$

We denote the corresponding distorted operator on the Fourier space by $\widetilde{V}_{\theta,1,R}$ and $\widetilde{V}_{\theta,2,R}$. Since $V_{1,R} \in C_c^\infty$, we see that $\widetilde{V}_{\theta,1,R}$ is ξ^2-compact. We also see that $\lim_{R\to\infty} \|\widetilde{V}_{\theta,2,R}\|_{L^2\to L^2} = 0$ by the estimate for $V = s(x)W(x)$ discussed above. This completes the proof of Lemma 3.8 □

3.3.2 Modifications of the proofs for the general case

While we set $\delta_0 = \pi^{-1}$ for the model case in subsection 3.2, we set $\delta_0 = \min\{\pi^{-1}, K\pi^{-1}\}$ for the general case in this subsubsection in view of Lemma 3.8. Similarly we set $\delta_1 = \min\{(\sqrt{2}-1)\pi^{-1}, K\pi^{-1}\}$ in this subsubsection. Then all the statements in subsubsection 3.2.2 and subsubsection 3.2.3 remain valid for these δ_0 and δ_1.

Proof of Theorem 5 for the general case. We replace Lemma 3.5 by Lemma 3.8. Then the proof is exactly the same as that for the model case discussed in subsection 3.2. □

Proof of Theorem 6 for the general case. The proof is almost the same as that for the model case discussed in subsection 3.2. In the claim that we can decompose $\widetilde{V}_\theta = \widetilde{V}_{\theta,1} + \widetilde{V}_{\theta,2}$, where $\widetilde{V}_{\theta,1}$ is a smoothing pseudodifferential operator in the Fourier space and $\|\widetilde{V}_{\theta,2}\|_{L^2\to L^2} < \gamma$, the proof for the model case used the special form of $\widetilde{V}_\theta$ in Lemma 3.5. For the general case, we set $\widetilde{V}_{\theta,1} = \widetilde{V}_{\theta,1,R}$ and $\widetilde{V}_{\theta,2} = \widetilde{V}_{\theta,2,R}$ for large $R \gg 1$, where $\widetilde{V}_{\theta,j,R}$ was defined in the ξ^2-compactness part of the proof of Lemma 3.8. This is the only necessary modification for the general case. □

Remark 3.9. In the case of $V = a\frac{\sin 2x}{x} + V_0$ with $a \in \mathbb{R}$ and $V_0 \in C_c^\infty(\mathbb{R};\mathbb{R})$, Lemma 3.8 holds for $\theta \in \mathbb{C} \setminus \big((-\infty, -\pi^{-1}] \cup [\pi^{-1}, \infty)\big)$ by Lemma 3.5 and the proof of Lemma 3.8. Then the set of resonances $\mathrm{Res}_n(P)$ may be defined in $\mathbb{C} \setminus (0,\infty)$ for any $n \in \mathbb{N}$ including multiplicities by the meromorphic continuation of $(f, R_+(z)g)$ from $\{z \mid 0 < \arg z < \pi\}$ to

$$\{z \mid 0 < \arg z < \pi\} \cup \{z \mid \arg z = 0, (n-1)^2 < |z| < n^2\} \cup \{z \mid -2\pi < \arg z < 0\}.$$

At this time we do not know whether $\mathrm{Res}_n(P) \neq \mathrm{Res}_{n'}(P)$ when $n \neq n'$.

4 Complex absorbing potential method for the Stark resonances

4.1 Introduction to Section 4

In this section, we discuss the complex absorbing potential method for the Stark resonances. We study the Stark Hamiltonian

$$P = -\Delta + x_1 + V(x),$$

where $V(x) \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is a potential.

We write $x' = (x_2, \dots, x_n)$ and set the cone $C(K,\rho) = \{x \in \mathbb{R}^n \mid |x'| \le K(x_1 + \rho)\}$ for $K > 0$ and $\rho \in \mathbb{R}$. Its complement is denoted by $C(K,\rho)^c$. The set of smooth functions which is bounded with all its derivatives is denoted by C_b^∞.

We first recall the definition of resonances following [30].

Assumption 5. The potential V has a decomposition $V = V_1 + V_{\mathrm{sing}}$, where V_1 and V_{sing} satisfy the following.
(1). The smooth part $V_1(x) \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ has an analytic continuation to the region $\{x \in \mathbb{C}^n \mid \mathrm{Re}\, x \in C(K_0,\rho_0)^c, |\mathrm{Im}\, x| < \delta_0\}$ for some $\rho_0 \in \mathbb{R}, K_0 > 0$ and $\delta_0 > 0$. Moreover $\partial V(x) \to 0$ when $|\mathrm{Re}\, x| \to \infty$ in this region.
(2). The compactly supported singular part $V_{\mathrm{sing}} \in L^2_{\mathrm{comp}}(\mathbb{R}^n;\mathbb{R})$ is $-\Delta$-bounded with relative bound 0.

The resolvent on the upper half plane is denoted by $R_+(z) = (z-P)^{-1}$ for $\operatorname{Im} z > 0$.

Theorem 7. *Under Assumption 1, for any $\chi_1, \chi_2 \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ such that $\chi_j = 1$ near $\operatorname{supp} V_{\mathrm{sing}}$, the cutoff resolvent $\chi_1 R_+(z)\chi_2$, $(\operatorname{Im} z > 0)$, has a meromorphic continuation with finite rank poles to $\{z | \operatorname{Im} z > -\delta_0\}$. The pole z of $\chi_1 R_+(z)\chi_2$ is called a resonance and its multiplicity is defined by*

$$m_z = \operatorname{rank} \frac{1}{2\pi i} \oint_z \chi_1 R_+(z)\chi_2 dz.$$

The set of resonances $\operatorname{Res}(P)$ is independent of the choices of χ_1, χ_2 including multiplicities.

We set $m_z = 0$ if $z \notin \operatorname{Res}(P)$. We give a proof of Theorem 7 based on the complex distortion outside a cone introduced in [30]. In [30], Theorem 7 was proved when $V_{\mathrm{sing}} = 0$ based on this method. The modification for the case of $V_{\mathrm{sing}} \neq 0$ is presented in the appendix to this section. Our proof of Theorem 8 below is also based on the complex distortion outside a cone. We recall that Theorem 7 can be proved based on the complex distortion on a half space ([25, Chapter 23]). Nevertheless, it seems to the author that the proof of Theorem 8 would be more difficult if it were based on the half space distortion.

The main purpose of this section is the complex absorbing potential method.

Assumption 6. In addition to Assumption 5, the following hold.
(1). The smooth part satisfies $\lim_{|x|\to\infty} V_1(x) = 0$.
(2). The compactly supported singular part V_{sing} is $-\Delta$-compact.

We note that Assumption 6.(1) for $\operatorname{Re} x \in C(K_0, \rho_0)^c$, $|\operatorname{Im} x| < \delta_0$ follows from that for $x \in \mathbb{R}^n$ by Assumption 5.(1).

For $\varepsilon > 0$, we set

$$P_\varepsilon = P - i\varepsilon x^2.$$

Then P_ε for $\varepsilon > 0$ with the domain $D(P_\varepsilon) = D(-\Delta) \cap D(x^2)$ has purely discrete spectrum. Then our main result is the following. We set $B(z, \gamma) = \{w \in \mathbb{C} | \, |w - z| \leq \gamma\}$.

Theorem 8. *Under Assumption 6,*

$$\lim_{\varepsilon\to 0+} \sigma_d(P_\varepsilon) = \operatorname{Res}(P).$$

More precisely, for any $z \in \{z \in \mathbb{C} | \operatorname{Im} z > -\delta_0\}$ there exists $\gamma_0 > 0$ such that for any $0 < \gamma < \gamma_0$ there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$,

$$\#\big(\sigma_d(P_\varepsilon) \cap B(z, \gamma)\big) = m_z.$$

Example 4.1. The Coulomb potential $V(x) = \sum_{i=1}^N \frac{e_i}{|x-R_i|}$ on $\mathbb{R}^n$, $(n \geq 3)$, satisfies the Assumption 6 with arbitrary large δ_0 if we take ρ_0 and K_0 large.

Note that $(|\mathrm{Re}\, x|^2 - |\mathrm{Im}\, x|^2 + 2i \mathrm{Re}\, x \cdot \mathrm{Im}\, x)^{-1/2}$ is well-defined for $|\mathrm{Re}\, x| > |\mathrm{Im}\, x|$. Thus we have

$$\mathrm{Res}(-\Delta + x_1 + \sum_{i=1}^{N} \frac{e_i}{|x - R_i|}) = \lim_{\varepsilon \to 0+} \sigma_d(-\Delta + x_1 + \sum_{i=1}^{N} \frac{e_i}{|x - R_i|} - i\varepsilon x^2)$$

on the whole complex plane.

The resonances for Stark Hamiltonians were investigated by many authors (for instance, [22], [23], [24], [51], [52], [56], [57]). See subsection 3.1 for the history of the complex absorbing potential method.

This section is organized as follows. In subsection 4.2, we recall the complex distortion outside a cone and discuss the properties of its generalization to the Stark Hamiltonian with a complex absorbing potential. In subsection 4.3, we prove Theorem 8. The main ingredient of the proof is the construction of an approximate resolvent of the free Stark distorted operator. In subsection 4.4, we prove two technical lemmas used in the proof of Theorem 8. In the appendix to this section, we give the modifications for including local singularities of the potential. The proof of Theorem 7 is given there.

4.2 Complex distortion and complex absorbing potential

We recall the complex distortion outside a cone introduced in [30]. Take $K > K_0$ and sufficiently large $\rho > 0$. Take a convex set $\widetilde{C}(K, \rho)$ such that its boundary $\partial \widetilde{C}(K, \rho)$ is smooth, $\widetilde{C}(K, \rho)$ is rotationally symmetric with respect to x' and $\widetilde{C}(K, \rho) = C(K, \rho)$ in $x_1 > -\rho + 1$. We set $F = -(1 + K^{-2})^{\frac{1}{2}} \mathrm{dist}\left(\bullet, \widetilde{C}(K, \rho)\right) * \phi$. Here $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfies $0 \le \phi \le 1$ and $\int \phi(x) dx = 1$. We set $v(x) = (v_1(x), \ldots, v_n(x)) = \partial F(x) \in C_b^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and set $\Phi_\theta(x) = x + \theta v(x)$, which is a diffeomorphism for real θ with small $|\theta|$. We note that $v_1(x) \ge 1$ for $\mathrm{dist}(x, \widetilde{C}(K, \rho)) \gg 1$. We also write $x_\theta = \Phi_\theta(x)$. We define the distortion outside a cone $U_\theta f(x) = (\det \Phi'_\theta(x))^{\frac{1}{2}} f(\Phi_\theta(x))$, which is unitary on $L^2(\mathbb{R}^n)$. We then define the distorted operator $P_\theta = U_\theta P U_\theta^{-1}$. Then P_θ has the following form:

$$P_\theta = -\sum_{i,j} \partial_i g_\theta^{ij} \partial_j + r_\theta(x) + x_1 + \theta v_1 + V(x_\theta),$$

where $(g_\theta^{ij}) = (\Phi'_\theta)^{-2}$, $g_\theta = \det(\Phi'_\theta)^2$ and $r_\theta = -\sum_{i,j} g_\theta^{-\frac{1}{4}} (\partial_i (g_\theta^{\frac{1}{2}} g_\theta^{ij} \partial_j g_\theta^{-\frac{1}{4}}))$. If $\mathrm{Im}\, \theta < 0$, we have $\mathrm{Im}\, (-\sum_{i,j} \partial_i g_\theta^{ij} \partial_j) \le 0$ in the sense of quadratic form ([30, Lemma 2.1]). The function $r_\theta(x) \in C_b^\infty(\mathbb{R}^n)$ satisfies $|r_\theta(x)| \le C\langle x \rangle^{-1}$ when $\mathrm{dist}(x, \partial \widetilde{C}(K, \rho)) \gg 1$ since we have $|\partial^\alpha v_j(x)| \le C_\alpha \langle x \rangle^{-1}$ when $|\alpha| \ge 1$ and $\mathrm{dist}(x, \partial \widetilde{C}(K, \rho)) \gg 1$. The distorted operator P_θ is an analytic family of closed operators for θ with $|\mathrm{Im}\, \theta| < \delta_0 (1 + K^{-2})^{-\frac{1}{2}}$ and $|\mathrm{Re}\, \theta|$ small. We have $P_\theta^* = P_{\bar{\theta}}$. For $\mathrm{Im}\, \theta < 0$, the spectrum of P_θ is discrete in $\{\mathrm{Im}\, z > \mathrm{Im}\, \theta\}$ and resonances for P coincide with discrete eigenvalues of P_θ in this region. These facts were proved in [30, Section 2] for $V_{\rm sing} = 0$ and is proved in the appendix to this section for $V_{\rm sing} \neq 0$.

We recall the notation

$$S(m) = \{a(x,\xi) \in C^\infty(\mathbb{R}^{2n}) \,|\, |\partial^\alpha_{x,\xi} a(x,\xi)| \le C_\alpha m(x,\xi)\}.$$

Recall $P_\varepsilon = P - i\varepsilon x^2$ and set $P_{\varepsilon,\theta} = U_\theta P_\varepsilon U_\theta^{-1}$. By the ellipticity in the symbol class $S(1+x^2+\xi^2)$, we see analogous properties for $P_{\varepsilon,\theta}$ with $\varepsilon > 0$. Namely, $P_{\varepsilon,\theta}$ is closed on the domain $D(P_{\varepsilon,\theta}) = D(-\Delta) \cap D(x^2)$ and is an analytic family of type (A) for θ with $|\mathrm{Im}\,\theta| < \delta_0(1+K^{-2})^{-\frac{1}{2}}$ and $|\mathrm{Re}\,\theta|$ small. We have $P^*_{\varepsilon,\theta} = P_{-\varepsilon,\bar\theta}$. The spectrum of $P_{\varepsilon,\theta}$ with $\varepsilon > 0$ is purely discrete on the whole complex plane and the eigenvalues are independent of θ including multiplicities.

Lemma 4.2.

$$x \cdot v(x) \le 0 \;\; \textit{for any} \;\; x \in \mathbb{R}^n.$$

Proof. By the rotational symmetry with respect to x', it is enough to consider the case of $n = 2$ and $x' = x_2 \ge 0$. We first assume that $x_1 > 0$. We then have $v_2(x) = -K^{-1}v_1(x)$ by the construction of $v(x)$. If $x_2 > Kx_1$, we thus have $x \cdot v(x) = v_1(x)(x_1 - K^{-1}x_2) \le 0$ since $v_1(x) \ge 0$ everywhere. If $x_2 \le Kx_1$, then $v(x) = 0$ by the construction of v and thus $x \cdot v(x) = 0$. We next assume that $x_1 \le 0$. Then we have $v_2(x) \le 0$ by the construction of v. Since $v_1(x) \ge 0$, we see that $x \cdot v(x) \le 0$, which completes the proof. □

The full symbol of a Stark Hamiltonian is not globally elliptic in the sense that there is no z such that $||\xi|^2 + x_1 + V(x) - z| \gtrsim (|\xi|^2 + |x_1| + 1)$ while the natural symbol class for $|\xi|^2 + x_1 + V(x)$ is $S(|\xi|^2 + |x_1| + 1)$. This is due to $\lim_{x_1 \to -\infty}(V(x) + x_1) = -\infty$ and is one of main difficulties of the analysis of Stark Hamiltonians. Lemma 4.2 shows that

$$\mathrm{Re}\,(-i\varepsilon x^2_{-i\delta}) = -2\delta\varepsilon x \cdot v(x) \ge 0$$

for $\varepsilon, \delta > 0$. Hence the distorted complex absorbing potential does not cause an additional difficulty on the lack of the global ellipticity of $\mathrm{Re}\,P_{\varepsilon,\theta}$. Thus Lemma 4.2 simplifies the proof of Theorem 8.

4.3 The proof of Theorem 8

4.3.1 Free distorted resolvent estimate with complex absorbing potential

We write $P_0 = P$ and $P_{0,\theta} = P_\theta$ for simplicity. Take any $\Omega \Subset \{z | \,\mathrm{Im}\, z > -\delta_0\}$ and fix $\theta = -i\delta$ with $0 < \delta < \delta_0$ such that $\Omega \Subset \{z | \,\mathrm{Im}\, z > -\delta\}$. Then $P_{\varepsilon,\theta}$ is defined for this θ if we take $K > 0$ large enough in the definition of the complex distortion outside a cone. Stark resonances coincide with eigenvalues of $P_{0,\theta}$ in Ω. We denote $P_{\varepsilon,\theta}$ with $V \equiv 0$ by $Q_{\varepsilon,\theta}$. The following Proposition and its proof is crucial in the proof of Theorem 8.

Proposition 4.3. *There exists $C > 0$ such that*

$$\|(Q_{\varepsilon,\theta} - z)^{-1}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C$$

for small $\varepsilon > 0$ and $z \in \Omega$.

Remark 4.4. We remark the relation with the harmonic oscillator. We have

$$Q_\varepsilon = -\Delta + x_1 - i\varepsilon x^2 = -\Delta - i\varepsilon(x_1 + \frac{i}{2\varepsilon})^2 - i\varepsilon x'^2 - \frac{i}{4\varepsilon}.$$

Then the eigenfunctions for this operator are obtained by a suitable complex coordinate transform of those for the harmonic oscillator. Then we see that

$$\sigma(Q_\varepsilon) = \{\varepsilon^{1/2}e^{-\pi i/4}(2|\alpha| + n) - \frac{i}{4\varepsilon} \mid \alpha \in \mathbb{Z}^n_{\ge 0}\}$$

including multiplicities. Note that this diverges to infinity when $\varepsilon \to 0+$. Since $\sigma(Q_\varepsilon) = \sigma(Q_{\varepsilon,\theta})$, we see that $(Q_{\varepsilon,\theta} - z)^{-1}$ exists for small $\varepsilon > 0$ and $z \in \Omega$. Proposition 4.3 claims the stronger uniform resolvent estimate. In [62], the complex absorbing potential method was justified in the region $\{z| -\pi/4 < \arg z \le 0\}$. This is related to the fact that $\sigma(-\Delta - i\varepsilon x^2) = \{\varepsilon^{1/2}e^{-\pi i/4}(2|\alpha| + n)| \,\alpha \in \mathbb{Z}^n_{\ge 0}\}$. There is no such restriction in the Stark Hamiltonian case.

We first construct an approximation of $(Q_{\varepsilon,\theta} - z)^{-1}$ as in [62, Section 3]. We note that $(Q_{0,\theta} - z)^{-1}$ exists for $z \in \Omega$ by the non-existence of the free Stark resonances ([30, Corollary 2.2]). Thus $(Q_{0,\theta} - z)^{-1}$ approximate $(Q_{\varepsilon,\theta} - z)^{-1}$ on a bounded set in $\mathbb{R}^n$ if $0 < \varepsilon \ll 1$. We next construct an approximation of $(Q_{\varepsilon,\theta} - z)^{-1}$ near the infinity. Take $\chi \in C_c^\infty(\mathbb{R}^n; [0, 1])$ such that $\chi = 1$ near $x = 0$. We set

$$Q^R_{\varepsilon,\theta} = Q_{\varepsilon,\theta} - iR\chi(x/R), \quad R \gg 1.$$

Lemma 4.5. *There exist $R_0 > 1$, $\tilde{\varepsilon} > 0$ and $C > 0$ such that*

$$\|(Q^R_{\varepsilon,\theta} - z)^{-1}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C$$

for $R > R_0$, $0 < \varepsilon \le \tilde{\varepsilon}$ and $z \in \Omega$.

We prove Lemma 4.5 in subsubsection 4.4.1. Although Lemma 4.5 is also true for $\varepsilon = 0$ by the same proof, we do not use this since $(Q_{0,\theta} - z)^{-1}$ exists. We fix $R > R_0$. We set $\chi^M(x) = \chi(x/M)$, where χ is as above. Then we introduce an approximate resolvent of $Q_{\varepsilon,\theta}$ defined by

$$T^M_{\varepsilon,z} = \chi^M(Q_{0,\theta} - z)^{-1} + (1 - \chi^M)(Q^R_{\varepsilon,\theta} - z)^{-1}.$$

Lemma 4.5 shows that $T^M_{\varepsilon,z}$ is uniformly bounded for $0 < \varepsilon \le \tilde{\varepsilon}$, $M > 1$ and $z \in \Omega$. We set $(Q_{\varepsilon,\theta} - z)T^M_{\varepsilon,z} = 1 + E^M_{\varepsilon,z}$ and estimate $E^M_{\varepsilon,z}$. A simple calculation shows that

$$E^M_{\varepsilon,z} = [Q_{\varepsilon,\theta}, \chi^M](Q_{0,\theta} - z)^{-1} - [Q_{\varepsilon,\theta}, \chi^M](Q^R_{\varepsilon,\theta} - z)^{-1} - \chi^M i\varepsilon x^2_\theta(Q_{0,\theta} - z)^{-1}$$

for $M \gg R$. We set $\widetilde{Q}_\varepsilon = Q^R_{\varepsilon,\theta}$ for $0 < \varepsilon \le \tilde{\varepsilon}$ and $\widetilde{Q}_0 = Q_{0,\theta}$. For any $\tilde{\chi} \in C_b^\infty(\mathbb{R})$, we write $\tilde{\chi}^M(x) = \tilde{\chi}(x_1/M)$.

Lemma 4.6. *For any $\tilde{\chi} \in C_b^\infty(\mathbb{R})$ such that $\tilde{\chi}(x_1) = 0$ for $-x_1 \gg 1$, there exists $C > 0$ such that*

$$\|\tilde{\chi}^M (\widetilde{Q}_\varepsilon - z)^{-1}\|_{L^2(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \leq CM^{k/2}$$

for $0 \leq k \leq 2$, $M > 1$, $z \in \Omega$ and $0 \leq \varepsilon \leq \tilde{\varepsilon}$.

We prove Lemma 4.6 in subsubsection 4.4.2. Since we obtain M^{-1} from the commutator, Lemma 4.6 with $k = 1$ shows that

$$\|E^M_{\varepsilon,z}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \mathcal{O}(M^{-1/2}) + \mathcal{O}_M(\varepsilon).$$

If we take M large and then take $\varepsilon > 0$ small, we learn $\|E^M_{\varepsilon,z}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < 1/2$ for $z \in \Omega$. Then the Neumann series argument implies that $T^M_{\varepsilon,z}(1+E^M_{\varepsilon,z})^{-1}$ is a right inverse of $Q_{\varepsilon,\theta}-z$. The same argument shows that the adjoint $Q_{-\varepsilon,\bar{\theta}}-\bar{z}$ also has a right inverse. We then conclude that $(Q_{\varepsilon,\theta} - z)^{-1}$ exists on $L^2(\mathbb{R}^n)$ and is equal to $T^M_{\varepsilon,z}(1+E^M_{\varepsilon,z})^{-1}$. Then Proposition 4.3 follows from the uniform boundedness of $T^M_{\varepsilon,z}$.

4.3.2 Convergence to Stark resonances

Proof of Theorem 8. We follow the strategy of [62, Section 5] (see also [31, subsection 2.3]). By Proposition 4.3, we have

$$P_{\varepsilon,\theta} - z = (1 + V_\theta(Q_{\varepsilon,\theta} - z)^{-1})(Q_{\varepsilon,\theta} - z),$$

where $V_\theta(x) = V(x_\theta)$. Note that $\lim_{x\to\infty} V_\theta(x) = 0$ and that V_θ is $-\Delta$-compact by Assumption 6. An approximation of V_θ by compactly supported $-\Delta$-compact functions and Lemma 4.6 with $k = 2$ imply that $V_\theta(Q_{\varepsilon,\theta} - z)^{-1}$ is a compact operator. Thus $1 + V_\theta(Q_{\varepsilon,\theta} - z)^{-1}$ is a Fredholm operator. Then by the same arguments as in [62], [31, subsection 2.3] based on the analytic Fredholm theory and the Gohberg-Sigal theory, Theorem 8 follows if we prove

$$\lim_{\varepsilon\to 0+} \|V_\theta(Q_{\varepsilon,\theta} - z)^{-1} - V_\theta(Q_{0,\theta} - z)^{-1}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = 0 \qquad (7)$$

uniformly for $z \in \Omega$. While [62], [31] employ the resolvent equation, it requires the estimate of the distorted Stark resolvent with the weight x_θ^2 in this case, which does not seem to be easy. We instead use the construction of approximate resolvent in subsubsection 4.3.1. By approximating V_θ by compactly supported functions $\widetilde{V_\theta}$ and using Proposition 4.3, it is enough to prove (7) with V_θ replaced by some $\widetilde{V_\theta}$ such that $\widetilde{V_\theta} \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$ and $\widetilde{V_\theta}$ is $-\Delta$-compact. By subsubsection 4.3.1, we see that

$$\begin{aligned}\widetilde{V_\theta}(Q_{\varepsilon,\theta} - z)^{-1} - \widetilde{V_\theta}(Q_{0,\theta} - z)^{-1} &= \widetilde{V_\theta}(T^M_{\varepsilon,z}(1 + E^M_{\varepsilon,z})^{-1} - (Q_{0,\theta} - z)^{-1}) \\ &= \widetilde{V_\theta}(Q_{0,\theta} - z)^{-1}((1 + E^M_{\varepsilon,z})^{-1} - 1)\end{aligned}$$

for $M \gg 1$. We used fact that $\widetilde{V_\theta}T^M_{\varepsilon,z} = \widetilde{V_\theta}(Q_{0,\theta} - z)^{-1}$ for $0 < \varepsilon \leq \tilde{\varepsilon}$ since $\widetilde{V_\theta}(1 - \chi^M) = 0$. Note that $\widetilde{V_\theta}(Q_{0,\theta} - z)^{-1}$ is independent of $\varepsilon > 0$ and $M > 1$.

It is a bounded operator on $L^2(\mathbb{R}^n)$ by the $-\Delta$-boundedness of $\widetilde{V_\theta}$, Lemma 4.6 with $k=2$ and the compactness of $\operatorname{supp}\widetilde{V_\theta}$. Then we obtain

$$\begin{aligned}
&\|\widetilde{V_\theta}(Q_{\varepsilon,\theta}-z)^{-1}-\widetilde{V_\theta}(Q_{0,\theta}-z)^{-1}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\
&\lesssim \|(1+E^M_{\varepsilon,z})^{-1}-1\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}.
\end{aligned}$$

By the estimate on $\|E^M_{\varepsilon,z}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}$, the operator $(1+E^M_{\varepsilon,z})^{-1}$ is close to the identity operator in the operator norm if we take large $M>1$ and then take small $\varepsilon>0$. This completes the proof of Theorem 8. □

4.4 Proofs of technical lemmas

In this subsection, we present proofs of two lemmas in subsection 4.3. The notation is the same as in subsection 4.3. Take $w\in C^\infty(\mathbb{R}^n;\mathbb{R}_{\geq 1})$ depending only on x_1 and $w=|x_1|$ for $x_1\leq -2$ and $w=1$ for $x_1\geq -1$.

4.4.1 Proof of Lemma 4.5

We take sufficiently small $c_0>0$ and $\tilde\chi_1,\tilde\chi_2,\tilde\chi_3,\tilde\chi_4\in C^\infty(\mathbb{R};[0,1])$ such that $\tilde\chi_1(x_1)=1$ for $x_1>5c_0$, $\tilde\chi_1(x_1)=0$ for $x_1<4c_0$, $\tilde\chi_2(x_1)=1$ for $x_1<5c_0$, $\tilde\chi_2(x_1)=0$ for $x_1>6c_0$, $\tilde\chi_3(x_1)=1$ for $|x_1-5c_0|<c_0$, $\tilde\chi_3(x_1)=0$ for $|x_1-5c_0|>2c_0$, $\tilde\chi_4(x_1)=1$ for $x_1<c_0$ and $\tilde\chi_4(x_1)=0$ for $x_1>2c_0$. We then set $\tilde\chi_j^R(x)=\tilde\chi_j(x_1/R)$ for $j=1,2,3,4$.

We take any $u\in C_c^\infty(\mathbb{R}^n)$. We have $\|u\|\leq\|\tilde\chi_1^R u\|+\|\tilde\chi_2^R u\|$. We first estimate these by quadratic form arguments. We have

$$\operatorname{Re}(\tilde\chi_1^R u,(Q^R_{\varepsilon,\theta}-z)\tilde\chi_1^R u)\gtrsim R\|\tilde\chi_1^R u\|^2$$

by the Stark potential. We also have

$$\operatorname{Im}(\tilde\chi_2^R u,(Q^R_{\varepsilon,\theta}-z)\tilde\chi_2^R u)\lesssim -\|\tilde\chi_2^R u\|^2$$

by the complex distortion outside a cone and the $-iR\chi(x/R)$ term in $Q^R_{\varepsilon,\theta}$. Here we assumed that c_0 is sufficiently small. We also assumed that $0<\varepsilon\leq\tilde\varepsilon$, where $\tilde\varepsilon>0$ is sufficiently small, to estimate the term $i\varepsilon\delta^2 v(x)^2$ in $-i\varepsilon x^2_{-i\delta}$. Then we have

$$\begin{aligned}
\|u\| &\lesssim \sum_{j=1}^2\|(Q^R_{\varepsilon,\theta}-z)\tilde\chi_j^R u\| \\
&\lesssim \|(Q^R_{\varepsilon,\theta}-z)u\|+\sum_{j=1}^2\|[Q^R_{\varepsilon,\theta},\tilde\chi_j^R]u\| \\
&\lesssim \|(Q^R_{\varepsilon,\theta}-z)u\|+R^{-1}\|\tilde\chi_3^R u\|_{H^1}.
\end{aligned}$$

To estimate $\|\tilde\chi_3^R u\|_{H^1}$, we set

$$\widehat{Q}=\widehat{Q}^R_{\varepsilon,z}=Q^R_{\varepsilon,\theta}-z+\tilde\chi_4^R w+R\tilde\chi_4^R.$$

The estimates below are uniform with respect to $0 < \varepsilon \le \tilde{\varepsilon}$ and $z \in \Omega$. We define the symbol $\widehat{q}(x,\xi;R)$ by $\widehat{Q}^R_{\varepsilon,z} = \widehat{q}(x,D;R)$. We claim that $\widehat{Q}^{-1} \in \mathrm{Op}S(\langle\xi\rangle^{-2})$ uniformly for $R \gg 1$. By Lemma 4.2 and $x_1 + \tilde{\chi}_4^R w + R\tilde{\chi}_4^R \gtrsim R$, we see that

$$\left|\frac{1+\xi^2+\varepsilon x^2}{\widehat{q}(x,\xi;R)}\right|, \left|\frac{\partial_x\widehat{q}(x,\xi;R)}{\widehat{q}(x,\xi;R)}\right| \lesssim \frac{1+\xi^2+\varepsilon x^2}{|\xi^2+R-i\varepsilon x^2|} \lesssim 1$$

and

$$\left|\frac{\partial_\xi\widehat{q}(x,\xi;R)}{\widehat{q}(x,\xi;R)}\right| \lesssim \frac{|\xi|}{|\xi^2+R-i\varepsilon x^2|}.$$

We have

$$\frac{\langle\xi\rangle}{|\xi^2+R-i\varepsilon x^2|} \lesssim R^{-1/2},$$

which follows from estimating it separately for $|\xi|/R^{1/2} \gg 1$ and $|\xi|/R^{1/2} \lesssim 1$. These imply that $\widehat{q}^{-1}(x,\xi;R) = \mathcal{O}(1)$ in $S((1+\xi^2+\varepsilon x^2)^{-1}) \subset S(\langle\xi\rangle^{-2})$ for $R > 1$, $\partial_x\widehat{q}^{-1}(x,\xi;R) = \mathcal{O}(R^{-1/2})$ in $S(\langle\xi\rangle^{-1})$ and $\partial_\xi\widehat{q}^{-1}(x,\xi;R) = \mathcal{O}(R^{-1/2})$ in $S((1+\xi^2+\varepsilon x^2)^{-1})$. Thus the symbols of $\widehat{q}(x,D;R)\widehat{q}^{-1}(x,D;R) - 1$ and $\widehat{q}^{-1}(x,D;R)\widehat{q}(x,D;R) - 1$ are $\mathcal{O}(R^{-1/2})$ in $S(1)$ since $\partial_x\widehat{q}(x,\xi;R) \in S(1+\xi^2+\varepsilon x^2)$ and $\partial_\xi\widehat{q}(x,\xi;R) \in S(\langle\xi\rangle)$ by employing a standard argument of pseudodifferential operators. This estimate based on the pseudodifferential calculus in the symbol classes $S((1+\xi^2+\varepsilon x^2)^{\pm1})$ is uniform with respect to $0 < \varepsilon \le 1$ since we have $|(1+\xi^2+\varepsilon x^2)|/|(1+\eta^2+\varepsilon y^2)| \le C(1+|x-y|+|\xi-\eta|)^N$ uniformly for $0 < \varepsilon \le 1$ (see [61, Chapter 5]). In these arguments, we may replace $1+\xi^2+\varepsilon x^2$ by $1+\xi^2+\varepsilon|x|$ if we use $|\partial^\alpha(\varepsilon x_\theta^2)| \le C_\alpha\varepsilon\langle x\rangle$. In fact, we only need $|\partial^\alpha(\varepsilon x_\theta^2)| \le C_\alpha\varepsilon\langle x\rangle^2$. Then by the Neumann series argument and the Beals's theorem we conclude that $\widehat{Q}^{-1} \in \mathrm{Op}S(\langle\xi\rangle^{-2})$ uniformly for $R \gg 1$. In particular, we see that $\widehat{Q}^{-1} : H^{-1}(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ is uniformly bounded for $R \gg 1$.

Thus we have

$$\|\tilde{\chi}_3^R u\|_{H^1} \lesssim \|\widehat{Q}\tilde{\chi}_3^R u\|_{H^{-1}}.$$

This is equal to $\|(Q^R_{\varepsilon,\theta} - z)\tilde{\chi}_3^R u\|_{H^{-1}}$ since $\operatorname{supp}\tilde{\chi}_3^R \cap \operatorname{supp}\tilde{\chi}_4^R = \emptyset$. This is bounded by

$$\|(Q^R_{\varepsilon,\theta} - z)u\| + \|[Q^R_{\varepsilon,\theta}, \tilde{\chi}_3^R]u\|_{H^{-1}} \lesssim \|(Q^R_{\varepsilon,\theta} - z)u\| + R^{-1}\|u\|.$$

Thus we conclude that $\|u\| \lesssim \|(Q^R_{\varepsilon,\theta} - z)u\|$ for $R \gg 1$. Since $Q^R_{\varepsilon,\theta}$ is the closure of its restriction to $C_c^\infty(\mathbb{R}^n)$, this is true for any u in the domain of $Q^R_{\varepsilon,\theta}$. Since the adjoint $(Q^R_{\varepsilon,\theta} - z)^* = Q_{-\varepsilon,\bar{\theta}} - \bar{z} + iR\chi(x/R)$ has the same estimate, we see that $(Q^R_{\varepsilon,\theta} - z)^{-1}$ exists on $L^2(\mathbb{R}^n)$ and $\|(Q^R_{\varepsilon,\theta} - z)^{-1}\|_{L^2\to L^2} \le C$. This completes the proof of Lemma 4.5.

4.4.2 Proof of Lemma 4.6

Lemma 4.6 for $M > 1$ follows from that for $M \gg 1$. By Lemma 4.5 and the existence of $(Q_{0,\theta} - z)^{-1}$, Lemma 4.6 is valid for $k = 0$ (without $\tilde{\chi}^M$). Thus it is enough to prove the case of $k = 2$ by the interpolation theorem.

We take $\tilde{\chi}_5.\tilde{\chi}_6 \in C_b^\infty(\mathbb{R};[0,1])$ such that $\tilde{\chi}_5 = 1$ near $\operatorname{supp}\tilde{\chi}$ and $\tilde{\chi}_6 = 1$ near $\operatorname{supp}\tilde{\chi}_5$. We set $\tilde{\chi}_j^M(x) = \tilde{\chi}_j(x_1/M)$. We fix sufficiently large $C_1 > 0$ and set

$$A = A_{\varepsilon,z}^M = \widetilde{Q}_\varepsilon + (1-\tilde{\chi}_6^M)w + C_1 M - z.$$

The estimates below are uniform with respect to $0 \le \varepsilon \le \tilde{\varepsilon}$ and $z \in \Omega$. We define the symbol $a(x,\xi;M)$ by $A_{\varepsilon,z}^M = a(x,D;M)$. We claim that $A^{-1} \in \operatorname{Op}S(\langle\xi\rangle^{-2})$ uniformly for $M \gg 1$. We see that

$$\left|\frac{1+\xi^2+\varepsilon x^2}{a(x,\xi;R)}\right|, \left|\frac{\partial_x a(x,\xi;M)}{a(x,\xi;M)}\right| \lesssim \frac{1+\xi^2+\varepsilon x^2}{|\xi^2+M-i\varepsilon x^2|} \lesssim 1$$

and

$$\left|\frac{\partial_\xi a(x,\xi;M)}{a(x,\xi;M)}\right| \lesssim \frac{|\xi|}{|\xi^2+M-i\varepsilon x^2|}$$

by Lemma 4.2 and the fact that $\xi^2 + x_1 + (1-\tilde{\chi}_6^M)w + C_1 M \gtrsim \xi^2 + M$ for $C_1 \gg 1$.

We argue as in the proof of Lemma 4.5 with R replaced by M and q replaced by a and conclude that $A^{-1} \in \operatorname{Op}S(\langle\xi\rangle^{-2})$ uniformly for $M \gg 1$. In particular, $A^{-1} : H^k(\mathbb{R}^n) \to H^{k+2}(\mathbb{R}^n)$ is uniformly bounded for any $k \in \mathbb{R}$ for $M \gg 1$.

We now estimate $\|\tilde{\chi}^M(\widetilde{Q}_\varepsilon - z)^{-1}\|_{L^2\to H^2}$. For this, we decompose

$$\tilde{\chi}^M(\widetilde{Q}_\varepsilon - z)^{-1} = \tilde{\chi}^M A^{-1}\tilde{\chi}_5^M A(\widetilde{Q}_\varepsilon - z)^{-1} + \tilde{\chi}^M A^{-1}(1-\tilde{\chi}_5^M)A(\widetilde{Q}_\varepsilon - z)^{-1}.$$

Since $\tilde{\chi}_5^M(1-\tilde{\chi}_6^M) = 0$, we have

$$\begin{aligned}
&\|\tilde{\chi}^M A^{-1}\tilde{\chi}_5^M A(\widetilde{Q}_\varepsilon - z)^{-1}\|_{L^2\to H^2} \\
&\lesssim \|A^{-1}\|_{L^2\to H^2}\cdot\|(\widetilde{Q}_\varepsilon + C_1 M - z)(\widetilde{Q}_\varepsilon - z)^{-1}\|_{L^2\to L^2} \\
&\lesssim M,
\end{aligned}$$

where we also used Lemma 4.5. Since $\tilde{\chi}^M(1-\tilde{\chi}_5^M) = 0$, we also estimate

$$\begin{aligned}
&\|\tilde{\chi}^M A^{-1}(1-\tilde{\chi}_5^M)A(\widetilde{Q}_\varepsilon - z)^{-1}\|_{L^2\to H^2} \\
&= \|A^{-1}[\tilde{\chi}^M, A]A^{-1}(1-\tilde{\chi}_5^M)A(\widetilde{Q}_\varepsilon - z)^{-1}\|_{L^2\to H^2} \\
&= \|A^{-1}[\tilde{\chi}^M, A]A^{-1}[\tilde{\chi}_5^M, A](\widetilde{Q}_\varepsilon - z)^{-1}\|_{L^2\to H^2} \\
&\lesssim \|A^{-1}\|_{L^2\to H^2}\cdot\|[\tilde{\chi}^M, A]\|_{H^1\to L^2}\cdot\|A^{-1}\|_{H^{-1}\to H^1}\cdot\|[\tilde{\chi}_5^M, A]\|_{L^2\to H^{-1}} \\
&\lesssim M^{-2}.
\end{aligned}$$

Here we used the fact that $[\tilde{\chi}^M, A](1-\tilde{\chi}_5^M) = 0$. We also used the fact that $[\tilde{\chi}^M, A]$ and $[\tilde{\chi}_5^M, A]$ are first order differential operators with $\mathcal{O}(M^{-1})$ coefficients. From these, we have $\|\tilde{\chi}^M(\widetilde{Q}_\varepsilon - z)^{-1}\|_{L^2\to H^2} \lesssim M$ for $M \gg 1$, which completes the proof of Lemma 4.6.

Remark 4.7. In fact, we can prove Theorem 8 without relying on the property of the vector field $v(x)$ in Lemma 4.2. If we do not use Lemma 4.2, we replace $\tilde{\chi}_j$ with cutoffs near $C(K_j, \rho_j)$, $C(K_j, \rho_j)^c$ or $\partial C(K_j, \rho_j)$ for suitable $K_j > 0$ and $\rho_j \in \mathbb{R}$ in the proofs of Lemma 4.5 and Lemma 4.6. We also replace $w(x_1)$ with $\langle x \rangle$ in the proofs. Then Lemma 4.5 and Lemma 4.6 are proved with $\tilde{\chi}$ in Lemma 4.6 replaced by a cutoff near $C(\tilde{K}, \tilde{\rho})$ for any $\tilde{K} > 0$ and $\tilde{\rho} > 0$.

4.5 Appendix to Section 4: Local singularities

In this appendix to Section 4, we give modifications to include local singularities of the potential. In particular, we prove that Stark resonances for the Coulomb potential are defined on the whole complex plane based on our complex distortion outside a cone.

We set $P = -\Delta + x_1 + V(x)$ and assume Assumption 5 throughout this appendix. We define the distorted operator P_θ from P as in subsection 4.2. The distortion is performed outside $\operatorname{supp} V_{\mathrm{sing}}$.

Lemma 4.8. *The singular part V_{sing} is P_θ-bounded with relative bound* 0.

Proof. Set $P_{1,\theta} = P_\theta - V_{\mathrm{sing}}$. We take a cutoff function $\chi \in C_c^\infty(\mathbb{R}^n)$ near $\operatorname{supp} V_{\mathrm{sing}}$ which is supported away from the distortion region. Assumption 5 implies that for any small $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|V_{\mathrm{sing}} u\| = \|V_{\mathrm{sing}} \chi u\| \le \varepsilon \| -\Delta \chi u\| + C_\varepsilon \|\chi u\|$$

for any $u \in C_c^\infty(\mathbb{R}^n)$. Since $P_{1,\theta} = -\Delta + x_1 + V_1(x)$ near $\operatorname{supp} \chi$ and $x_1 + V_1(x)$ is bounded there,

$$\begin{aligned}
\|V_{\mathrm{sing}} u\| &\le \varepsilon \|P_{1,\theta} \chi u\| + C_\varepsilon \|u\| \\
&\le \varepsilon \|P_{1,\theta} u\| + C_\varepsilon \|u\| + \varepsilon \|[-\Delta, \chi] u\| \\
&\le 2\varepsilon \|P_{1,\theta} u\| + C_\varepsilon \|u\| \\
&\le 2\varepsilon \|P_\theta u\| + 2\varepsilon \|V_{\mathrm{sing}} u\| + C_\varepsilon \|u\|,
\end{aligned}$$

where the third inequality follows from the elliptic estimate. We subtract $2\varepsilon \|V_{\mathrm{sing}} u\|$ for small $\varepsilon > 0$ from both sides, which completes the proof. □

Proof of Theorem 7. We first prove [30, Proposition 2.1] in this case, namely, P_θ is an analytic family of type (A) and $P_\theta^* = P_{\bar{\theta}}$. As in [30, subsection 2.1], it is enough to prove that

$$\|(P_\theta - P_{\theta'}) u\| \le C|\theta - \theta'| \|P_\theta u\| + C_{\theta, \theta'} \|u\| \tag{A.1}$$

for any $u \in C_c^\infty(\mathbb{R}^n)$. By the case where $V_{\mathrm{sing}} = 0$, we see

$$\begin{aligned}
\|(P_\theta - P_{\theta'}) u\| &= \|(P_{1,\theta} - P_{1,\theta'}) u\| \\
&\le C|\theta - \theta'| \|P_{1,\theta} u\| + C_{\theta,\theta'} \|u\| \\
&\le C|\theta - \theta'| \|P_\theta u\| + C|\theta - \theta'| \|V_{\mathrm{sing}} u\| + C_{\theta,\theta'} \|u\|.
\end{aligned}$$

Then the inequality (A.1) follows from Lemma 4.8.

We next prove [30, Proposition 2.2] in this case, namely, P_θ with $\operatorname{Im}\theta < 0$ has purely discrete spectrum in $\{z | \operatorname{Im} z > \operatorname{Im}\theta\}$. As in [30, subsection 2.2], it is enough to prove that we have $\|(\widetilde{P_\theta} - z)u\| \geq c\|u\|$ for any $u \in C_c^\infty(\mathbb{R}^n)$ if $\operatorname{Im} z > \operatorname{Im}\theta$ and $M \gg 1$, where

$$\widetilde{P_\theta} = P_\theta - iM\phi(x/M)\phi(D/M)^2\phi(x/M).$$

Here we took $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ near $\{|x| \leq 1/3\}$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$. We fix small $\varepsilon_1 > 0$ and set $\chi_{j,M} = \tau_j(G(x)/M)$, where $\tau_0 \in C_b^\infty(\mathbb{R})$ is a cutoff near $(-\infty, \varepsilon_1]$, $\tau_1 \in C_b^\infty(\mathbb{R})$ is a cutoff near $[\frac{1}{2}\varepsilon_1, \frac{3}{2}\varepsilon_1]$, $\tau_2 \in C_b^\infty(\mathbb{R})$ is a cutoff near $[2\varepsilon_1, \infty)$ and $G(x) = (1+K^{-2})^{\frac{1}{2}} \operatorname{dist}\left(\bullet, \widetilde{C}(K,0)\right) * \phi$. The function ϕ is the same as above. We fix z such that $\operatorname{Im} z > \operatorname{Im}\theta$. We set $Q = \widetilde{P_\theta} - V_{\text{sing}} - z + \chi_{2,M} w - iM\chi_{2,M}$ and define the symbol q by $Q = q(x, D; M)$. Here w is the same as in subsection 4.4. We denote the seminorms in $S(\langle\xi\rangle^k)$ by $|a|_{k,\alpha} = \sup_{x,\xi} |\partial_{x,\xi}^\alpha a|/\langle\xi\rangle^k$. We claim that

$$\left|\frac{\langle\xi\rangle^{2-k}}{q(x,\xi;R)}\right| \leq CM^{-k/2}$$

for $0 \leq k \leq 2$. This is proved if we estimate them separately for $|x| < M/3$, $x \in C(K, 3\varepsilon_1 M)^c$ and $x_1/M \gtrsim 1$ for small $\varepsilon_1 > 0$. For each case, we also estimate them separately for $|\xi|/M^{1/2} \lesssim 1$ and $|\xi|/M^{1/2} \gg 1$ as in [30, Proposition 2.2]. Then we have

$$\left|\frac{\partial_x q(x,\xi;M)}{q(x,\xi;M)}\right| \lesssim \left|\frac{\langle\xi\rangle^2}{q(x,\xi;M)}\right| \lesssim 1$$

and

$$\left|\frac{\partial_\xi q(x,\xi;M)}{q(x,\xi;M)}\right| \lesssim \left|\frac{\langle\xi\rangle}{q(x,\xi;M)}\right| \lesssim M^{-1/2}.$$

Thus $|\partial_\xi q^{-1}|_{-2,\alpha} = \mathcal{O}(M^{-1/2})$ and $|q^{-1}|_{k-2,\alpha} = \mathcal{O}(M^{-k/2})$ for $0 \leq k \leq 2$ (see Remark 4.9 below). We note that $\partial_x q \in S(\langle\xi\rangle^2)$ and $\partial_\xi q \in S(\langle\xi\rangle)$. Thus the estimates $|\partial_\xi q^{-1}|_{-2,\alpha}$, $|\partial_x q^{-1}|_{-1,\alpha} = \mathcal{O}(M^{-1/2})$ imply that the symbols of $q^{-1}(x,D;M)q(x,D;M) - 1$ and $q(x,D;M)q^{-1}(x,D;M) - 1$ are $\mathcal{O}(M^{-1/2})$ in $S(1)$. By the Neumann series argument and Beals's theorem, we conclude that $Q^{-1} = \mathcal{O}(M^{-k/2})$ in $\operatorname{Op}S(\langle\xi\rangle^{k-2})$ for $M \gg 1$ and $0 \leq k \leq 2$.

Then we have

$$\begin{aligned}
M\|\chi_{0,M}u\| &\leq C\|Q\chi_{0,M}u\| = C\|(\widetilde{P_\theta} - V_{\text{sing}} - z)\chi_{0,M}u\| \\
&\leq C\|(\widetilde{P_\theta} - z)\chi_{0,M}u\| + C\|V_{\text{sing}}\chi_{0,M}u\| \\
&\leq C\|(\widetilde{P_\theta} - z)\chi_{0,M}u\| + \varepsilon C\|(P_\theta - z)\chi_{0,M}u\| + C_\varepsilon\|\chi_{0,M}u\|,
\end{aligned}$$

where we used Lemma 4.8 for the last inequality. We take $\varepsilon < \frac{1}{4C}$ and then take $M > 2C_\varepsilon$. Subtracting $C_\varepsilon \|\chi_{0,M}u\| \leq \frac{M}{2}\|\chi_{0,M}u\|$ from both sides, we have

$$\begin{aligned}\|\chi_{0,M}u\| &\leq C\|(\widetilde{P_\theta} - z)\chi_{0,M}u\| + \frac{2\varepsilon C}{M}\|(P_\theta - z)\chi_{0,M}u\| \\ &\leq C\|(\widetilde{P_\theta} - z)\chi_{0,M}u\| + \frac{1}{2M}\|iM\phi(x/M)\phi(D/M)^2\phi(x/M)\chi_{0,M}u\| \\ &\leq C\|(\widetilde{P_\theta} - z)\chi_{0,M}u\| + \frac{1}{2}\|\chi_{0,M}u\|.\end{aligned}$$

Subtracting $\frac{1}{2}\|\chi_{0,M}u\|$, we see that $\|\chi_{0,M}u\| \leq C\|(\widetilde{P_\theta} - z)\chi_{0,M}u\|$.

The remaining part of the proof of [30, Proposition 2.2] for $V_{\mathrm{sing}} \not\equiv 0$ is similar to that in [30, subsection 2.1] with minor modifications as follows. We set $\widetilde{\chi}_{0,M} = 1 - \chi_{0,M}$. We note that $V_{\mathrm{sing}} = 0$ near $\operatorname{supp}\chi_{1,M}$, $\operatorname{supp}\chi_{2,M}$ and $\operatorname{supp}\widetilde{\chi}_{0,M}$. We have

$$-\mathrm{Im}\,(\widetilde{\chi}_{0,M}u, (\widetilde{P_\theta} - z)\widetilde{\chi}_{0,M}u) \geq c\|\widetilde{\chi}_{0,M}u\|^2$$

for $M \gg 1$ by the complex distortion outside a cone. Thus we see that

$$\begin{aligned}\|u\| &\leq \|\chi_{0,M}u\| + \|\widetilde{\chi}_{0,M}u\| \\ &\leq C\|(\widetilde{P_\theta} - z)\chi_{0,M}u\| + C\|(\widetilde{P_\theta} - z)\widetilde{\chi}_{0,M}u\| \\ &\leq C\|(\widetilde{P_\theta} - z)u\| + C\|[\widetilde{P_\theta}, \chi_{0,M}]u\|.\end{aligned}$$

We note that

$$\|[\widetilde{P_\theta}, \chi_{0,M}]u\| \leq CM^{-1}\|\chi_{1,M}u\|_{H^1} + \mathcal{O}(M^{-\infty})\|u\|.$$

Since $\operatorname{supp}\chi_{1,M} \cap \operatorname{supp}V_{\mathrm{sing}} = \emptyset$, we obtain

$$\begin{aligned}\|\chi_{1,M}u\|_{H^1} &\leq C\|Q\chi_{1,M}u\|_{H^{-1}} \\ &= C\|(\widetilde{P_\theta} - z)\chi_{1,M}u\|_{H^{-1}} \\ &\leq C\|(\widetilde{P_\theta} - z)u\| + C\|[\widetilde{P_\theta}, \chi_{1,M}]u\|_{H^{-1}} \\ &\leq C\|(\widetilde{P_\theta} - z)u\| + CM^{-1}\|u\|.\end{aligned}$$

Summing up these inequalities, we have

$$\|u\| \leq C\|(\widetilde{P_\theta} - z)u\| + CM^{-2}\|u\|.$$

By subtracting $CM^{-2}\|u\|$ from both sides, we conclude that $\|(\widetilde{P_\theta} - z)u\| \geq c\|u\|$ for large $M > 1$.

Once Proposition 2.1 and Proposition 2.2 in [30] are proved, we completes the proof of Theorem 7 by the same proof as that of [30, Theorem 1]. □

Remark 4.9. The statement that $|q^{-1}|_{-2,\alpha} = \mathcal{O}(M^{-|\alpha|/2})$ in the proof of [30, Proposition 2.2] is too strong. The necessary argument for the modification is straightforward and contained in the above proof.

Resonances for P coincide with discrete eigenvalues of P_θ in the region $\{z | \operatorname{Im} z > \operatorname{Im} \theta\}$ for $\operatorname{Im} \theta < 0$ including multiplicities, which is proved by the same proof as in [30, Section 2].

Remark 4.10. The other results in [30, Section 2] are also valid in the almost same form under Assumption 5 by the same proofs as in [30]. For instance, we can replace L^p_{comp} by $L^p_{\mathrm{cone}} = \{f \in L^p | \operatorname{supp} f \subset C(K, \rho) \text{ for some } K, \rho\}$ in Theorem 7 in this section. There is some modifications related to the unique continuation argument. Namely, we assumed $\chi_j = 1$ near $\operatorname{supp} V_{\mathrm{sing}}$ in Theorem 7 in this section and should assume $U \supset \operatorname{supp} V_{\mathrm{sing}}$ in [30, Proposition 2.3]. These modifications are not needed if we moreover assume that there is a closed set $S \subset \mathbb{R}^n$ of Lebesgue measure zero such that $\mathbb{R}^n \setminus S$ is connected and V_{sing} is bounded on any compact subset of $\mathbb{R}^n \setminus S$.

For [30, Theorem 3], the same proof shows the existence of a one-to-one correspondence between eigenvalues of a reference operator P^{int} and the shape resonances of P such that their distances are bounded by $e^{-S/\hbar}$ for some $S > 0$. Thus [30, Theorem 3] is true even when $V_{\mathrm{sing}} \not\equiv 0$ if the Weyl law for P^{int} is true.

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