博士論文

論文題目 Studies on the Bogomolov-Sommese vanishing theorem and Du Val del Pezzo surfaces in positive characteristic

(正標数の Bogomolov-Sommese 消滅定理と Du Val del Pezzo 曲面に関する研究)

氏　名　　河上　龍郎

Abstract

In this paper, we study the following.

(1) The Bogomolov-Sommese vanishing theorem for lc surfaces in positive characteristic.

(2) The Bogomolov-Sommese vanishing theorem for globally F-regular threefolds.

(3) Pathologies of Du Val del Pezzo surfaces in positive characteristic.

For (1), we show that the Bogomolov-Sommese vanishing theorem holds for a log canonical surface pair (X, B) with $\kappa(X, K_X + \lfloor B \rfloor) \neq 2$ in large characteristic. As an application, we prove that a surface pair (X, B) of a smooth projective surface X and a reduced simple normal crossing divisor B with $\kappa(X, K_X + B) \leqslant 0$ lifts to the ring of Witt vectors in large characteristic. Moreover, we give an explicit and optimal bound on the characteristic unless $\kappa(X, K_X + \lfloor B \rfloor) = 0$.

For (2), we show a weak version of the Bogomolov-Sommese vanishing theorem holds for globally F-regular threefolds. Indeed, we show that every invertible subsheaf of the cotangent bundle of a smooth globally F-regular threefold of characteristic $p > 3$ has Iitaka dimension less than or equal to one.

For (3), we study the relationship between pathological phenomena of Du Val del Pezzo surfaces and their non-liftability to the ring of Witt vectors. We investigate the following four conditions on Du Val del Pezzo surfaces:

- (NB) all the members of the anti-canonical linear system are singular,
- (ND) there does not exist Du Val del Pezzo surfaces over the field of complex numbers with the same Dynkin type, Picard rank, and anti-canonical degree,
- (NK) there exists an ample $\mathbb{Z}$-divisor that violates the Kodaira vanishing theorem, and
- (NL) the pair (Y, E) does not lift to the ring of Witt vectors, where Y is the minimal resolution and E is its reduced exceptional divisor.

We classify all the Du Val del Pezzo surfaces satisfying (NB) (resp. (ND),(NK),(NL)). Moreover, we see that none of these pathological conditions occur under the assumption of Frobenius splitting.

Acknowledgments

First, I would like to express my deep gratitude to my supervisor Shunsuke Takagi, who supported me by giving me many valuable suggestions and advice.

Moreover, I would like to thank Masaru Nagaoka and Shou Yoshikawa for their interesting conversations and helpful comments.

I am also grateful to Hiromu Tanaka, Kenta Sato, Fabio Bernasconi, Makoto Enokizono, Teppei Takamatsu, and Jakub Witaszek for useful discussion.

I was supported by JSPS KAKENHI Grant Number JP19J21085.

January, 2022
Tatsuro Kawakami

Contents

Chapter 1

Introduction

In this thesis, we study the Bogomolov-Sommese vanishing theorem and pathologies of Du Val del Pezzo surfaces in positive characteristic.

1.1 Bogomolov-Sommese vanishing theorem for lc surfaces in positive characteristic

This section is based on [58]. Vanishing theorems involving differential sheaves play a significant role in the analysis of algebraic varieties. The Bogomolov-Sommese vanishing theorem, originally proved in [15], is one of the most important tools of this kind and has been studied by many authors (see [34], [36], [37], [54], [94] for example).

Theorem 1.1.1 (Bogomolov-Sommese vanishing theorem [34, Corollary 1.3])**.** *Let (X, B) be a log canonical (lc, for short) projective pair over the field of complex numbers $\mathbb{C}$. Then*

$$H^0(X, (\Omega_X^{[i]}(\log \lfloor B \rfloor) \otimes \mathcal{O}_X(-D))^{**}) = 0$$

for every $\mathbb{Z}$-divisor D on X satisfying $\kappa(X, D) > i$.

In Theorem 1.1.1, $\kappa(X, D)$ denotes the Iitaka dimension of a $\mathbb{Z}$-divisor D (see Definition 2.3.1 for the definition), $(-)^{**}$ denotes reflexive hull, and $\Omega_X^{[i]}(\log \lfloor B \rfloor)$ denotes the sheaf of i-th logarithmic reflexive differential forms of the pair $(X, \lfloor B \rfloor)$, where $\lfloor B \rfloor$ is the round-down of B. Klt and lc singularities are important classes of singularities appearing in the minimal model program (MMP, for short), and it is very useful to generalize vanishing theorems to varieties with such singularities (see Definition 2.2.1 for their definition). The reader is also referred to Definition 2.2.4 for the definitions of varieties appearing in the MMP such as varieties of Fano type and of Calabi-Yau type.

We note that Theorem 1.1.1 is equivalent to saying that the sheaf $\Omega_X^{[i]}(\log \lfloor B \rfloor)$ does not contain any Weil divisorial sheaves with Iitaka dimension bigger than i. In particular, when dim $X = 2$, it suffices to check that the sheaf of the first logarithmic reflexive differential forms does not contain any big Weil divisorial sheaves.

The logarithmic extension theorem for $(n+1)$-dimensional lc pairs can be deduced from the Bogomolov-Sommese vanishing theorem for n-dimensional log Calabi-Yau pairs (see [35, Section 9]). Theorem 1.1.1 also can be applied to show the vanishing of the second cohomology of the tangent sheaf of lc projective surfaces with big anti-canonical divisors, so that they have no local-to-global obstructions (see [40, Proposition 3.1]). In this paper, we discuss an analog of Theorem 1.1.1 in positive characteristic. In the rest of this chapter, we work over an algebraically closed field k of positive characteristic $p > 0$.

Let X be a normal projective surface over k. It is well-known that the Bogomolov-Sommese vanishing theorem fails when K_X is big. For example, it is not difficult to see that the sheaf of the first differential forms of Raynaud's surface [92] contains an ample invertible sheaf. Moreover, Langer [74, Section 8] constructed a pair (S, F) of a smooth rational surface S and a disjoint union of smooth rational curves F such that $\Omega_S(\log F)$ contains a big invertible sheaf in every characteristic (see also [75, Section 11]). In other words, the Bogomolov-Sommese vanishing theorem fails even if X is a smooth rational surface. On the other hand, we can observe that the log canonical divisor $K_S + F$ is big except when the characteristic is equal to two (see Example 3.4.4 for the details). Therefore, it is natural to ask whether the Bogomolov-Sommese vanishing theorem holds when the log canonical divisor is not big and the characteristic is sufficiently large. We give an affirmative answer to this question.

Theorem 1.1.2. *There exists a positive integer p_0 with the following property. Let (X, B) be an lc projective surface pair over an algebraically closed field of characteristic $p > p_0$. If $\kappa(X, K_X + \lfloor B \rfloor) \neq 2$, then*

$$H^0(X, (\Omega_X^{[i]}(\log \lfloor B \rfloor) \otimes \mathcal{O}_X(-D))^{**}) = 0$$

for every $\mathbb{Z}$-divisor D on X satisfying $\kappa(X, D) > i$. Moreover, if $\kappa(X, K_X + \lfloor B \rfloor) = -\infty$ (resp. $\kappa(X, K_X + \lfloor B \rfloor) = 1$), then we can take $p_0 = 5$ (resp. $p_0 = 3$) as an optimal bound. If $\kappa(X, K_X + \lfloor B \rfloor) = 0$, then we can take p_0 as the maximum Gorenstein index of any klt Calabi-Yau surface over any algebraically closed field.

In Theorem 1.1.2, a klt Calabi-Yau surface means a klt projective surface whose canonical divisor is numerically trivial. If the base field is an algebraically closed field of characteristic zero, then the Gorenstein index of a klt Calabi-Yau surface is less than or equal to 21 by [14, Theorem C (a)]. In general, there exists a uniform bound on the Gorenstein index independent of the choice of the algebraically closed base field (see Lemma 3.1.9), but its explicit value is not known.

As an application of Theorem 1.1.2, we obtain a result on the liftability of log surfaces.

Theorem 1.1.3. *There exists a positive integer p_0 with the following property. Let X be a normal projective surface over an algebraically closed field k of characteristic*

$p > p_0$, B a reduced divisor on X, and $f\colon Y \longrightarrow X$ a log resolution of (X, B). Suppose that one of the followings holds:

(1) $\kappa(X, K_X + B) = -\infty$,

(2) $K_X + B \equiv 0$ *and* $B \neq 0$,

(3) $\kappa(X, K_X + B) = 0$.

Then $(Y, f_*^{-1}B + \mathrm{Exc}(f))$ lifts to the ring $W(k)$ of Witt vectors. Moreover, when the condition (1) or (2) holds, we can take $p_0 = 5$ as an optimal bound.

It is well-known that every smooth projective surface defined over an algebraically closed field of characteristic $p > 3$ with non-positive Kodaira dimension lifts to $W(k)$ (see [53, Proposition 2.6], [80, Section 11], and [88, Proposition 11.1]). Theorem 1.1.3 can be viewed as a log version of this fact.

In Theorem 1.1.3 (3), p_0 should be at least 19 by Example 3.4.3, but it is not clear whether we can take p_0 as the maximum Gorenstein index of klt Calabi-Yau surfaces.

In the proof of Theorem 1.1.3 (1) and (2), we apply Theorem 1.1.2 to obtain the vanishing of $H^2(Y, T_Y(-\log f_*^{-1}B + \mathrm{Exc}(f)))$, where the obstruction to the lifting lives. In the case of (3), such a vanishing does not always hold. Therefore, using an argument of Cascini-Tanaka-Witaszek [23], we show the boundedness of some ε-klt log Calabi-Yau surfaces, from which we deduce the desired liftability (see Lemma 3.1.12 and Proposition 3.1.13).

Using Theorems 1.1.2 and 1.1.3, we prove that the Kawamata-Viehweg vanishing theorem for $\mathbb{Z}$-divisors holds on normal projective surfaces whose canonical divisor is not big.

Theorem 1.1.4. *There exists a positive integer p_0 with the following property. Let X be a normal projective surface over an algebraically closed field of characteristic $p > p_0$ and D a nef and big $\mathbb{Z}$-divisor on X. Suppose that one of the followings holds:*

(1) $\kappa(X, K_X) \leqslant 0$,

(2) $\kappa(X, K_X) = 1$ *and X is lc.*

Then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$. Moreover, if $\kappa(X, K_X) = -\infty$ (resp. $\kappa(X, K_X) = 1$), then we can take $p_0 = 5$ (resp. $p_0 = 3$) as an optimal bound.

1.2 Bogomolov-Sommese type vanishing for globally F-regular threefolds

This section is based on [57]. We have studied the Bogomolov-Sommese vanishing theorem for surfaces in Section 1.1. In this section, we discuss the higher-dimensional case. Since the proof of Theorem 1.1.2 heavily depends on the classifications of lc surface singularities and klt del Pezzo surfaces of Picard rank one, a similar argument does not work for higher-dimensional varieties. On the other hand, the author proved in [59] that an analog of Theorem 1.1.1 holds when X is a smooth Fano threefold in characteristic $p > 0$, $B = 0$, and $i = 1$. In this section, we study a Bogomolov-Sommese type vanishing theorem for threefolds of Fano type, and we prove a weak version of the Bogomolov-Sommese vanishing theorem for globally F-regular threefolds, a special class of Frobenius split (F-split, for short) varieties that are of Fano type (see Definition 4.1.1 (2) for the precise definition).

Theorem 1.2.1 (Theorem 4.4.10)**.** *Let X be a smooth projective globally F-regular threefold over an algebraically closed field of characteristic $p > 3$. Then*

$$H^0(X, \Omega_X \otimes \mathcal{O}_X(-D)) = 0$$

for every $\mathbb{Z}$-divisor D on X satisfying $\kappa(X, D) > 1$. Furthermore, if $p > 7$, then the above vanishing holds for every $\mathbb{Z}$-divisor D satisfying $\kappa(X, D) > 0$.

We need the assumption "$p > 3$" only for running the MMP, which was recently established for threefolds of characteristic $p > 3$ (see [41] for the details). In the proof of Theorem 1.2.1, we run a K_X-MMP to reduce to the case where D is nef and big.

Theorem 1.2.2 (Theorem 4.4.5)**.** *Let X be a projective globally F-regular variety over an algebraically closed field of characteristic $p > 0$ and B a reduced $\mathbb{Z}$-divisor on X. Suppose that $\dim X \geqslant 2$ and the non-simple normal crossing locus of (X, B) has codimension at least three. Then*

$$H^0(X, (\Omega_X^{[1]}(\log B) \otimes \mathcal{O}_X(-D))^{**}) = 0$$

for every nef and big $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor D on X.

When X is smooth, Theorem 1.2.2 follows from the Cartier isomorphism and the F-splitting of X. However, even if we start from a smooth variety, the output of the MMP is not necessarily smooth. This is the reason why we have to consider singular varieties in Theorem 1.2.2. We use the global F-regularity of X to deal with singularities.

1.3 Pathologies of Du Val del Pezzo surfaces in positive characteristic

This section is based on [61] and [60], which are joint work with Masaru Nagaoka. Theorems 1.1.3 and 1.1.4 tells us that there exists a surface X over k with $\kappa(X, K_X) = -\infty$ that does not satisfy the Kodaira vanishing theorem and the liftability in Theorem 1.1.3 (see Example 3.4.1). In this section, we focus on del Pezzo surfaces with Du Val singularities and study their pathological phenomena systematically.

We say that X is a Du Val del Pezzo surface if X is a projective surface whose anti-canonical divisor is ample and that has at worst Du Val singularities, i.e., 2-dimensional canonical singularities. By *the Dynkin type of* X, we mean the Dynkin diagrams of singularities on X. For example, we say that X is of type $3A_1 + D_4$ if X has three A_1-singularities and one D_4-singularity. In this case, we also write $\mathrm{Dyn}(X) = 3A_1 + D_4$ and $X = X(3A_1 + D_4)$.

Definition 1.3.1 (cf. Definition 2.4.10)**.** Let X be a normal projective surface over an algebraically closed field k of characteristic $p > 0$. We say that X is *log liftable over* $W(k)$ if there exists a log resolution $f\colon Y \longrightarrow X$ such that the pair $(Y, \mathrm{Exc}(f))$ lifts to $W(k)$.

By Theorem 1.1.3, every Du Val del Pezzo surface over an algebraically closed field of characteristic $p > 5$ is log liftable over $W(k)$. We remark that all Du Val del Pezzo surfaces themselves are liftable over $W(k)$ (see Remark 5.1.2), but they are not necessarily log liftable. In this section, we study the following pathological conditions on Du Val del Pezzo surfaces.

Definition 1.3.2. For a Du Val del Pezzo surface X over an algebraically closed field k of characteristic $p > 0$, we say that X satisfies:

- (ND) if there does not exist any Du Val del Pezzo surface $X_{\mathbb{C}}$ over the field of complex numbers $\mathbb{C}$ with the same Dynkin type, the same Picard rank, and the same degree as X.
- (NB) if all members of the anti-canonical linear system of X are singular.
- (NK) if $H^1(X, \mathcal{O}_X(-A)) \neq 0$ for some ample $\mathbb{Z}$-divisor A on X.
- (NL) if X is not log liftable over $W(k)$.

For example, Keel-M$^{\mathrm{c}}$Kernan [62, end of Section 9] constructed a Du Val del Pezzo surface $X(7A_1)$ of Picard rank one and degree $K^2_{X(7A_1)} = 2$ in characteristic two. This surface satisfies (ND) (see [31, Theorem 2, Table (II)] or [9, Theorem 1.1]). Cascini-Tanaka pointed out in [21, Proposition 4.3 (iii)] and [22, Theorem 4.2 (6)] that it also satisfies (NB) and (NK). Since the anti-canonical linear system of $X(7A_1)$ is base point free, this surface is a counterexample to Bertini's theorem in positive characteristic. Furthermore, we easily see from Cascini-Tanaka's result [22, Theorem 4.2 (6)] that $X(7A_1)$ satisfies (NL).

Our main results consist of three theorems. The first one is the following, which shows the implications (NK) $\Rightarrow$ (NL) and (ND) $\Rightarrow$ (NL) $\Rightarrow$ (NB).

Theorem 1.3.3. *Let X be a Du Val del Pezzo surface over an algebraically closed field k of characteristic $p > 0$. Then the following hold.*

(1) *If a general member of anti-canonical linear system is smooth, then X is log liftable over $W(k)$.*

(2) *If X is log liftable over $W(k)$, then there exists a Du Val del Pezzo surface over $\mathbb{C}$ with the same Dynkin type, the same Picard rank, and the same degree as X.*

(3) *If X is log liftable over $W(k)$, then $H^1(X, \mathcal{O}_X(-A)) = 0$ for every ample $\mathbb{Z}$-divisor A.*

The second main theorem, Theorem 1.3.4, classifies Du Val del Pezzo surfaces satisfying (NB), the weakest condition among those listed in Definition 1.3.2.

Theorem 1.3.4. *Let X be a Du Val del Pezzo surface over an algebraically closed field k of characteristic $p > 0$. Suppose that X satisfies (NB). Then the following hold.*

(0) $K_X^2 \leqslant 2$ *and* $p = 2$ *or* 3.

(1) *When $K_X^2 = 1$ and $p = 2$ (resp. $p = 3$), the Dynkin type of X is E_8, D_8, $A_1 + E_7$, $2D_4$, $2A_1 + D_6$, $4A_1 + D_4$, or $8A_1$ (resp. E_8, $A_2 + E_6$, or $4A_2$). In particular, the Picard rank of X is equal to one.*

(2) *When $K_X^2 = 2$, the characteristic p has to be 2 and the Dynkin type of X is E_7, $A_1 + D_6$, $3A_1 + D_4$, or $7A_1$. In particular, the Picard rank of X is equal to one. Furthermore, the morphism $\varphi_{|-K_X|} \colon X \longrightarrow \mathbb{P}_k^2$ associated to the anti-canonical linear system is purely inseparable and therefore X is homeomorphic to $\mathbb{P}_k^2$.*

(3) *The isomorphism class of X is uniquely determined by its Dynkin type if and only if the Dynkin type is not $2D_4, 4A_1 + D_4$, or $8A_1$.*

Summarizing the above, we obtain Table 1.1.

Table 1.1

Degree	$K_X^2 = 1$				
Dynkin type	E_8	$A_2 + E_6$	$4A_2$	D_8	$A_1 + E_7$
Characteristic	$p = 2, 3$	$p = 3$		$p = 2$	
No. of isomorphism classes	1	1	1	1	1

$K_X^2 = 1$				$K_X^2 = 2$			
$2D_4$	$2A_1 + D_6$	$4A_1 + D_4$	$8A_1$	E_7	$A_1 + D_6$	$3A_1 + D_4$	$7A_1$
$p = 2$				$p = 2$			
∞	1	∞	∞	1	1	1	1

Remark 1.3.5. Combining Theorem 1.3.4 with Ito's results [50, 51], we obtain a complete classification of isomorphism classes of rational quasi-elliptic surfaces (see Corollary 5.5.24 for more details).

The last main theorem, Theorem 1.3.6, classifies Du Val del Pezzo surfaces satisfying (ND) (resp. (NK), (NL)). As a consequence, we have the implications (NK) $\Rightarrow$ (ND) $\Rightarrow$ (NL) $\Rightarrow$ (NB) among the conditions in Definition 1.3.2 and we see that none of the opposite implications hold.

Theorem 1.3.6. *Let X be a Du Val del Pezzo surface over an algebraically closed field k of characteristic $p > 0$. Then the following hold.*

(1) *X satisfies (NL) if and only if $(p, \mathrm{Dyn}(X)) = (3, 4A_2)$, $(2, 4A_1 + D_4)$, $(2, 8A_1)$, or $(2, 7A_1)$.*

(2) *X satisfies (ND) if and only if $(p, \mathrm{Dyn}(X)) = (2, 4A_1 + D_4)$, $(2, 8A_1)$, or $(2, 7A_1)$.*

(3) *X satisfies (NK) if and only if $(p, \mathrm{Dyn}(X)) = (2, 8A_1)$ or $(2, 7A_1)$.*

Remark 1.3.7. It follows from Theorem 1.3.6 that there exist Du Val del Pezzo surfaces over an algebraically closed field of characteristic $p = 2, 3$ on which the Bogomolov-Sommese vanishing theorem does not hold.

Finally, we show that F-split Du Val del Pezzo surfaces do not satisfy any conditions in Definition 1.3.2.

Theorem 1.3.8. *Let X be a Du Val del Pezzo surface over an algebraically closed field of characteristic $p > 0$. Suppose that X is F-split. Then a general member of the anti-canonical linear system is smooth. Moreover, if $p = 2$, then the general member is an ordinary elliptic curve.*

Remark 1.3.9. We cannot drop the assumption on the characteristic in the latter assertion of Theorem 1.3.8. Indeed, there exists an F-split Du Val del Pezzo surface over an algebraically closed field of characteristic $p = 3$ such that all smooth members of the anti-canonical linear system are supersingular elliptic curves. We refer to Remark 5.9.5 for the details.

Chapter 2

Preliminaries

2.1 Notation

A *variety* means an integral separated scheme of finite type over an algebraically closed field. A *curve* (resp. *surface*) means a variety of dimension one (resp. two). A *pair* (X, B) consists of a normal variety of X and an effective $\mathbb{Q}$-divisor B with coefficients in $[0, 1] \cap \mathbb{Q}$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Throughout this paper, we use the following notation:

- $\mathrm{Exc}(f)$: the reduced exceptional divisor of a birational morphism f.
- $\lfloor D \rfloor$ (resp. $\lceil D \rceil$): the *round-down* (resp. *round-up*) of a $\mathbb{Q}$-divisor D.
- $\mathcal{F}^*$: the dual of a coherent sheaf of $\mathcal{F}$.
- $\Omega_X^{[i]}(\log B)$: the *i-th logarithmic reflexive differential form* $j_*\Omega_U^i(\log B)$, where X is a normal variety, B is a reduced divisor on X, U is the snc locus of (X, B), and $j \colon U \hookrightarrow X$ is the natural inclusion morphism.
- $T_X(-\log B) := (\Omega_X^{[1]}(\log B))^*$: the *logarithmic tangent sheaf* of a normal variety X and a reduced divisor B on X.
- $W(k)$ (resp. $W_n(k)$): the ring of Witt vectors (resp. the ring of Witt vectors of length n), where k is an algebraically closed field of positive characteristic.

2.2 Singularities and varieties appearing in the MMP

In this section, we gather definitions of singularities and varieties appearing in the MMP.

Definition 2.2.1. Let (X, B) be a pair and $f \colon Y \longrightarrow X$ be a proper birational morphism from a normal variety Y and E a prime divisor on Y. Any such E is called a *divisor over* X and $f(E)$ is called *the center* of E. We take the canonical divisor K_Y of Y so that $f_*K_Y = K_X$. We call the coefficient $\mathrm{coeff}_E(K_Y - f^*(K_X+B))$ as *discrepancy* of E, and denote by $a(E, X, B)$. We say (X, B) is *klt (resp. ε-klt, lc)* if, for any prime divisor E over X, the discrepancy of E satisfies $a(E, X, B) > -1$ (resp. $a(E, X, B) > -1 + \varepsilon$, $a(E, X, B) \geqslant -1$).

Definition 2.2.2. Let Z be a Noetherian separated scheme. Let X be a smooth projective scheme over Z of relative dimension d and $B = \sum_{i=1}^{r} B_i$ a reduced divisor on X, where each B_i is an irreducible component. We say that B is a *simple normal crossing over* Z (snc over Z, for short) if, for any subset $J \subseteq \{1, \dots, r\}$ such that $\bigcap_{i \in J} B_i \neq \emptyset$, the scheme-theoretic intersection $\bigcap_{i \in J} B_i$ is smooth over Z of relative dimension $d - |J|$. When Z is a spectrum of an algebraically closed field, we just say that B is snc.

Let B' be a $\mathbb{Q}$-divisor on X. We say that (X, B') is log smooth over Z if $\mathrm{Supp}(B')$ is snc over Z. When Z is a spectrum of an algebraically closed field, we just say that (X, B') is log smooth.

Definition 2.2.3. Let (X, B) be a pair. We say (X, B) is *dlt* if there exists a closed subset $F \subset X$ such that

- (X, B) is log smooth outside F.
- for any divisor E over X whose center is contained in F, the discrepancy of E satisfies $a(E, X, B) > -1$.

Definition 2.2.4. We say a projective pair (X, B) is *log Fano* if (X, B) is klt and $-(K_X + B)$ is ample. We say X is *of Fano type* if there exists an effective $\mathbb{Q}$-divisor B such that (X, B) is log Fano. We say a projective pair (X, B) is *log Calabi-Yau* if (X, B) is lc and $K_X + B \equiv 0$. We say X is *of Calabi-Yau type* if there exists an effective $\mathbb{Q}$-divisor B such that (X, B) is log Calabi-Yau. If, in addition, the pair (X, B) is klt (resp. ε-klt), then we say that (X, B) is *klt (resp. ε-klt) log Calabi-Yau* and X is *klt (resp. ε-klt) of Calabi-Yau*

Definition 2.2.5. Let (X, B) be a pair and $f\colon X \longrightarrow Z$ a projective surjective morphism to a normal variety Z. We say $f\colon X \longrightarrow Z$ is a $(K_X + B)$-*Mori fiber space* if

- $-(K_X + B)$ is f-ample,
- $f_*\mathcal{O}_X = \mathcal{O}_Z$ and $\dim X > \dim Z$, and
- the relative Picard rank $\rho(X/Z) = 1$.

2.3 The Iitaka dimension for $\mathbb{Z}$-divisors

In this section, we recall the definition and basic properties of the Iitaka dimension of $\mathbb{Z}$-divisors.

Definition 2.3.1 ([37, Definition 2.18])**.** Let X be a normal projective variety and D a $\mathbb{Z}$-divisor on X. We define the *Iitaka dimension*$\in \{-\infty, 0, 1, \cdots, \dim X\}$ as follows. If $h^0(X, \mathcal{O}_X(mD)) = 0$ for all $m \in \mathbb{Z}_{>0}$, then we say D has *Iitaka dimension* $\kappa(X, D) := -\infty$. Otherwise, set

$$M := \{m \in \mathbb{Z}_{>0} \mid h^0(X, \mathcal{O}_X(mD)) > 0\},$$

and consider the natural rational mappings

$$\varphi_m : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(mD))^*) \quad \text{for each } m \in M.$$

Note that we can consider the rational map as above since $\mathcal{O}_X(mD)$ is invertible on the regular locus of X. The Iitaka dimension of D is then defined as

$$\kappa(X, D) := \max_{m \in M}\{\dim \overline{\varphi_m(X)}\}.$$

When D is a $\mathbb{Q}$-divisor, we define $\kappa(X, D)$ as $\kappa(X, mD)$, where m is any positive integer such that mD is a $\mathbb{Z}$-divisor. We say a $\mathbb{Q}$-divisor D is *big* if $\kappa(X, D) = \dim X$. Note that if D is $\mathbb{Q}$-Cartier, then the above definition coincides with the usual definition ([76, Definition 2.13]).

Remark 2.3.2. Let X be a normal projective variety. Suppose that a resolution $f: Z \longrightarrow X$ exists. We call $\kappa(Z, K_Z)$ as the *Kodaira dimension* of X. We note that the Kodaira dimension of X dose not depend on resolutions. In general, the Kodaira dimension of X is less than or equal to the Iitaka dimension $\kappa(X, K_X)$ of the canonical divisor K_X (see Lemma 3.2.1 (2)). In this paper, we mainly use the Iitaka dimension $\kappa(X, K_X)$ of the canonical divisor.

Definition 2.3.3. Let D be a $\mathbb{Q}$-divisor on a normal projective surface X. We say D is *nef* if $D \cdot C \geqslant 0$ for every curve C on X.

Let $f: Y \longrightarrow X$ be a projective birational morphism of normal surfaces. Let $\pi: \widetilde{Y} \longrightarrow Y$ be a resolution. Since $\mathrm{Supp}(\pi^* \mathrm{Exc}(f)) \subset \mathrm{Exc}(f \circ \pi)$, it follows that the intersection matrix of $\mathrm{Exc}(f)$ is negative definite. In particular, we can define the *Mumford pullback* ([19, 14.24]) for f.

Remark 2.3.4. We can see that the Mumford pullback preserves the Iitaka dimension by the projection formula. In addition, the Mumford pullback preserves nefness by definition.

In the rest of the paper, we just refer to the Mumford pullback as pullback.

2.4 Liftability of pairs to the ring of Witt vectors

In this section, we recall the fundamental facts about the liftability of pairs.

Definition 2.4.1. Let X be a normal projective variety over an algebraically closed field k and $B = \sum_{i=1}^{r} B_i$ a reduced divisor on X, where each B_i is an irreducible component. Let R be a Noetherian local ring with residue field k. When (X, B) is log smooth (resp. not log smooth), we say that the pair (X, B) *lifts to* R if there exist

- a projective flat morphism $\mathcal{X} \longrightarrow \operatorname{Spec} R$, and
- closed subschemes $\mathcal{B}_i$ $(i = 1, 2, \ldots, r)$ on $\mathcal{X}$

such that $(\mathcal{X}, \mathcal{B} := \sum_{i=1}^{r} \mathcal{B}_i)$ is log smooth (resp. $\mathcal{B}$ is flat) over R, $\mathcal{X} \otimes_R k = X$, and $\mathcal{B}_i \otimes_R k = B_i$.

In the setting of Definition 2.4.1, if we further assume that R is regular and (X, B) is log smooth, then a lifting $(\mathcal{X}, \mathcal{B})$ becomes automatically log smooth as follows.

Lemma 2.4.2. *Let X be a normal projective variety over an algebraically closed field k and $B = \sum_{i=1}^{r} B_i$ a reduced divisor on X, where each B_i is an irreducible component. Let R be a regular local ring with residue field k. Suppose that there exist*

- *a projective flat morphism $\mathcal{X} \longrightarrow \operatorname{Spec} R$, and*
- *closed subschemes $\mathcal{B}_i$ $(i = 1, 2, \ldots, r)$ on $\mathcal{X}$ flat over R*

such that $\mathcal{X} \otimes_R k = X$ and $\mathcal{B}_i \otimes_R k = B_i$. If (X, B) is log smooth, then $(\mathcal{X}, \mathcal{B})$ is log smooth over $\operatorname{Spec} R$.

Proof. Since $\mathcal{X}$ (resp. $\mathcal{B}_i$) is flat over $\operatorname{Spec} R$ by assumption, this is smooth of relative dimension d (resp. $d-1$) by [39, Théorème 12.2.4 (iii)]. In particular, each $\mathcal{B}_i$ is a Cartier divisor. We take a subset $J \subseteq \{1, \ldots, r\}$. Let us show that $\bigcap_{i \in J} \mathcal{B}_i$ is flat over R. We fix a closed point of $x \in \bigcap_{i \in J} \mathcal{B}_i$. Since each $\mathcal{B}_i$ is Cartier, we obtain

$$\begin{aligned} &\dim \mathcal{O}_{\mathcal{X},x} - |J| \leqslant \dim \mathcal{O}_{\bigcap_{i\in J} \mathcal{B}_i, x} \leqslant \dim \mathcal{O}_{\bigcap_{i\in J} B_i, x} + \dim R \\ =& \dim \mathcal{O}_{X,x} - |J| + \dim R = \dim \mathcal{O}_{\mathcal{X},x} - |J| \end{aligned}$$

and hence $\mathcal{O}_{\bigcap_{i\in J} \mathcal{B}_i, x}$ is Cohen-Macaulay and $\dim \mathcal{O}_{\bigcap_{i\in J} \mathcal{B}_i, x} = \dim \mathcal{O}_{\bigcap_{i\in J} B_i, x} + \dim R$. Then, by [84, Theorem 23.1], it follows that $\bigcap_{i\in J} \mathcal{B}_i$ is flat over R. Finally, by [39, Théorème 12.2.4 (iii)], the closed subscheme $\bigcap_{i\in J} \mathcal{B}_i$ is smooth over R and hence $(\mathcal{X}, \mathcal{B})$ is log smooth over R. $\square$

Lemma 2.4.3. *Let X be a smooth projective surface and B an snc divisor on X. Let $\mathrm{Bl}_x \colon Y \longrightarrow X$ a blow-up at a closed point $x \in X$. Suppose that (X, B) lifts to a complete regular local ring R. Then $(Y, (\mathrm{Bl}_x)_*^{-1} B + \mathrm{Exc}(\mathrm{Bl}_x))$ lifts to R.*

Proof. Let $(\mathcal{X}, \mathcal{B})$ be a lifting of (X, B) to R. Since R is regular, the pair $(\mathcal{X}, \mathcal{B})$ is log smooth over R by Lemma 2.4.2. Since R is henselian, [29, Propostion 2.8.13] shows that there exists a lifting $\widetilde{x}$ of x to R, which is compatible with the snc structure in the sense of [7, Theorem 2.7]. By [29, Theorem 2.5.8 (i)$\Rightarrow$ (ii)], there exists an open subset $\mathcal{U}$ of $\mathcal{X}$ containing x and an étale R-morphism $\widetilde{\varphi}\colon \mathcal{U} \longrightarrow \mathrm{Spec}\ R[T_1, T_2]$ such that

- $\mathcal{B}_i \cap \mathcal{U} = V(\widetilde{\varphi}^* T_i)$ for each irreducible component $\mathcal{B}_i$ of $\mathcal{B}$ satisfying $\mathcal{B}_i \cap \mathcal{U} \neq \emptyset$, and
- $\widetilde{x} = V(\widetilde{\varphi}^* T_1, \widetilde{\varphi}^* T_2)$.

We define an étale k-morphism $\varphi\colon U := \mathcal{U} \otimes_R k \longrightarrow \mathrm{Spec}\ k[T_1, T_2]$ as $\varphi := \widetilde{\varphi} \otimes_R k$. Then $x = V(\varphi^* T_1, \varphi^* T_2)$. Now, an argument of after Claim of [6, Lemma 4.4] shows that $(\mathcal{Y}, (\mathrm{Bl}_{\widetilde{x}})_*^{-1}\mathcal{B} + \mathrm{Exc}(\mathrm{Bl}_{\widetilde{x}}))$ is a lifting of $(Y, (\mathrm{Bl}_x)_*^{-1}B + \mathrm{Exc}(\mathrm{Bl}_x))$. $\square$

Lemma 2.4.4. *Let X be a normal projective surface and B a reduced divisor on X. Suppose that there exists a log resolution $f\colon Y \longrightarrow X$ of (X, B) such that $H^2(Y, \mathcal{O}_Y) = 0$ and $(Y, f_*^{-1}B + \mathrm{Exc}(f))$ lifts to a complete regular local ring R. Then, for every log resolution $g\colon Z \longrightarrow X$ of (X, B), the pair $(Z, g_*^{-1}B + \mathrm{Exc}(g))$ lifts to R.*

Proof. Let us take a log resolution $g\colon Z \longrightarrow X$ of (X, B) and show the liftability of $(Z, g_*^{-1}B + \mathrm{Exc}(g))$. We can take a log resolution $h\colon W \longrightarrow X$ of (X, B) such that both f and g factor through h. Since $W \longrightarrow Y$ is a composition of blow-ups at a smooth point, the pair $(W, h_*^{-1}B + \mathrm{Exc}(h))$ lifts to R by Lemma 2.4.3. Since $W \longrightarrow Z$ is also a composition of blow-ups at a smooth point, it follows from [2, Proposition 4.3 (1)] that $(Z, g_*^{-1}B + \mathrm{Exc}(g))$ formally lifts to R. By assumption, we have $H^2(Z, \mathcal{O}_Z) = H^2(Y, \mathcal{O}_Y) = 0$, and hence $(Z, g_*^{-1}B + \mathrm{Exc}(g))$ lifts to R as a scheme. $\square$

Theorem 2.4.5. *Let X be a smooth projective surface over an algebraically closed field k and B an snc divisor on X. Let (R, m) be a Noetherian complete local ring with residue field k. Suppose that $H^2(X, T_X(-\log B)) = 0$. Then (X, B) lifts to R as a formal scheme. In particular, (X, B) lifts to R/m^n for all $n \in \mathbb{Z}_{>0}$. Moreover, if we further assume that $H^2(X, \mathcal{O}_X) = 0$, then (X, B) lifts to R as a scheme.*

Proof. We denote R/m^n by R_n. Let (X^n, B^n) be a lifting of (X, B) over $\mathrm{Spec}\ R_n$. We first see that (X^n, B^n) is liftable to $\mathrm{Spec}\ R_{n+1}$. Since B^n is simple normal crossing over $\mathrm{Spec}\ R_n$, we can take an affine open covering $\{U_i\}$ of Y^n such that $(U_i, B|_{U_i})$ lifts to $\mathrm{Spec}\ R_{n+1}$. Then for each i and any open subset U of U_i, the set of equivalence classes of such liftings is a torsor under the action of $\mathcal{H}om(\Omega_U(\log B), m^{n-1}\mathcal{O}_U)$. We refer to the arguments of [26, Section 8] for the details. Then by a similar argument as in [27, Theorem 8.5.9 (b)], the obstruction for the lifting of (X^n, B^n) over $\mathrm{Spec}\ R_{n+1}$ is contained in $H^2(Y, T_X(-\log B)) \otimes m^n/m^{n+1}$. Thus the vanishing of $H^2(X, T_X(-\log B))$ gives a lifting of X and B_i over $\mathrm{Spec}\, R$ as formal schemes. We assume that $H^2(X, \mathcal{O}_X) = 0$ in addition. Then the formal liftings of X and

B_i are algebraizable and we get a projective scheme $\mathcal{X}$ over Spec R and a closed subscheme $\mathcal{B} := \sum_{i=1}^{r} \mathcal{B}_i$ on $\mathcal{X}$ such that $\mathcal{X} \otimes_R R_n = X^n$ and $\mathcal{B}_i \otimes_R R_n = B_i^n$ for each n and i by [27, Corollary 8.5.6 and Corollary 8.4.5]. We take a subset $J \subseteq \{1, \ldots, r\}$. Since $(\bigcap_{i \in J} \mathcal{B}_i) \otimes_R R_n = \bigcap_{i \in J} B_i^n$ is smooth over Spec R_n for all $n > 0$ and $\mathcal{X}$ is projective over Spec R, [38, Chapitre 0, Proposition (10.2.6)] and [39, Théorème 12.2.4 (iii)] show that $\bigcap_{i \in J} \mathcal{B}_i$ is smooth of relative dimension $\dim \mathcal{X}_\eta - |J|$ except when $\bigcap_{i \in J} \mathcal{B}_i = \varnothing$, where $\mathcal{X}_\eta$ is the generic fiber. Therefore $(\mathcal{X}, \mathcal{B} = \sum_{i=1}^{r} \mathcal{B}_i)$ is a lifting of (X, B) over Spec R. □

Hara [42, Corollary 3.8] showed the Akizuki-Nakano vanishing theorem for W_2-liftable pairs (X, B). In Theorem 2.4.6, we slightly generalize this theorem to the vanishing for nef and big divisors when dim $X = 2$.

Theorem 2.4.6 (cf. [42, Corollary 3.8])**.** *Let X be a smooth projective surface over an algebraically closed field k of characteristic $p > 0$ and B an snc divisor on X. Suppose that (X, B) lifts to $W_2(k)$. Let D be a nef and big $\mathbb{Q}$-divisor on X such that* $\mathrm{Supp}(\lceil D \rceil - D)$ *is contained in B. Then*

$$H^j(X, \Omega_X^i(\log B) \otimes \mathcal{O}_X(-\lceil D \rceil)) = 0$$

for $i, j \in \mathbb{Z}_{\geqslant 0}$ such that $i + j < 2$.

Remark 2.4.7. Langer [72, Example 1] showed that Theorem 2.4.6 does not hold when D is only big. In other words, the Bogomolov-Sommese vanishing theorem can fail on W_2-liftable surfaces.

Proof. By the Serre duality and the essentially same argument as in [42, Corollary 3.8], we can reduce the assertion to

$$H^j(X, \Omega_X^i(\log B)) \otimes \mathcal{O}_X(-B + \lceil p^e D \rceil) = 0$$

for all $i + j > 2$ and some $e > 0$. We remark that the assumption that $p >$ dim X in [42, Corollary 3.8] is relaxed to $p \geqslant$ dim X. Indeed, in the proof of [42, Corollary 3.8], the assumption that $p >$ dim X is only used for the quasi-isomorphism $\bigoplus_i \Omega_X^i(\log B)[-i] \cong F_* \Omega_X^\bullet(\log B)$, and this quasi-isomorphism holds even in $p =$ dim X by [26, 10.19 Proposition].

We take $m, n \in \mathbb{Z}_{>0}$ such that $p^m(p^n - 1)D$ is Cartier. Then we obtain

$$\begin{aligned} &H^j(X, \Omega_X^i(\log B)) \otimes \mathcal{O}_Y(-B + \lceil p^{m+ln} D \rceil) \\ =&H^j(X, \Omega_X^i(\log B) \otimes \mathcal{O}_Y(-B + \lceil p^m D \rceil + (\sum_{i=0}^{l-1} p^{ni}) p^m (p^n - 1) D)). \end{aligned}$$

When $j = 2$ (resp. $(i, j) = (2, 1)$), the last term vanishes for all sufficiently large $l \gg 0$ by [99, Proposition 2.3] (resp. [99, Theorem 2.6]), and we obtain the desired vanishing. □

Proposition 2.4.8. *Let X be a normal projective surface. Suppose that one of the following conditions folds.*

(1) $-K_X$ *is ample* $\mathbb{Q}$*-Cartier, the minimal resolution* $\pi\colon Y \longrightarrow X$ *is a log resolution, and there exists a log resolution* $f\colon Z \longrightarrow X$ *such that* (Z, E_f) *lifts to* $W_2(k)$.

(2) $H^2(X, T_X) = 0$ *and* $H^2(X, \mathcal{O}_X) = 0$.

Then, for every log resolution $f'\colon Z' \longrightarrow X$*, the pair* $(Z', E_{f'})$ *lifts to every Noetherian complete local ring with residue field* k.

Proof. We first show (1). We take a π-exceptional effective $\mathbb{Q}$-divisor F such that $\pi^*(-K_X) - F$ is ample and $\lceil \pi^*(-K_X) - F \rceil = \lceil \pi^*(-K_X) \rceil$. Since π is minimal, we have $-K_Y \geqslant \lceil \pi^*(-K_X) \rceil = \lceil \pi^*(-K_X) - F \rceil$. Let $f'\colon Z' \longrightarrow X$ be a log resolution. Then f' decomposes into $g\colon Z' \longrightarrow Y$ and the minimal resolution $\pi\colon Y \longrightarrow X$. We have the injective morphism

$$\begin{aligned} g_*(\Omega_{Z'}(\log\, E_{f'}) \otimes \mathcal{O}_{Z'}(K_{Z'})) \hookrightarrow & (g_*(\Omega_{Z'}(\log\, E_{f'}) \otimes \mathcal{O}_{Z'}(K_{Z'})))^{**} \\ = & \Omega_Y(\log\, E_\pi) \otimes \mathcal{O}_Y(K_Y) \end{aligned}$$

and then the Serre duality yields

$$\begin{aligned} H^2(Z', T_{Z'}(-\log\, E_{f'})) \cong & H^0(Z', \Omega_{Z'}(\log\, E_{f'}) \otimes \mathcal{O}_{Z'}(K_{Z'})) \\ \hookrightarrow & H^0(Y, \Omega_Y(\log\, E_\pi) \otimes \mathcal{O}_Y(K_Y)) \\ \hookrightarrow & H^0(Y, \Omega_Y(\log\, E_\pi) \otimes \mathcal{O}_Y(-\lceil \pi^*(-K_X) - F \rceil)). \end{aligned}$$

Since (Z, E_f) lifts to $W_2(k)$ by assumption, so does (Y, E_π) by [2, Proposition 4.3 (1)], and hence the last cohomology vanishes by Theorem 2.4.6. Together with

$$H^2(Z', \mathcal{O}_{Z'}) \cong H^0(Z', \mathcal{O}_{Z'}(K_{Z'})) \hookrightarrow H^0(X, \mathcal{O}_X(K_X)) = 0,$$

we obtain the liftability of $(Z', E_{f'})$ by Theorem 2.4.5.

In this case of (2), we have

$$\begin{aligned} H^2(Z', T_{Z'}(-\log\, E_{f'})) \cong & H^0(Z', \Omega_{Z'}(\log\, E_{f'}) \otimes \mathcal{O}_{Z'}(K_{Z'})) \\ \hookrightarrow & H^0(X, (\Omega_X \otimes \mathcal{O}_X(K_X))^{**}) \\ \cong & H^2(X, T_X) = 0, \end{aligned}$$

and the rest proof is similar to that of (1). □

Lemma 2.4.9. *Let* X *be a normal projective surface over an algebraically closed field* k *of positive characteristic and* D *a nef and big* $\mathbb{Z}$*-divisor on* X*. Suppose that there exists a log resolution* $\pi\colon Y \longrightarrow X$ *such that* $(Y, \mathrm{Exc}(\pi))$ *lifts to* $W_2(k)$*. Then* $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ *for all* $i > 0$.

Proof. By the Serre duality for Cohen-Macaulay sheaves ([66, Theorem 5.71]), it suffices to show that $H^i(X, \mathcal{O}_X(-D)) = 0$ for all $i < 2$. When $i = 0$, the vanishing follows from the bigness of D. Thus we assume that $i = 1$. By the spectral sequence

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{O}_Y(-\lceil \pi^* D \rceil)) \Rightarrow E^{p+q} = H^{p+q}(Y, \mathcal{O}_Y(-\lceil \pi^* D \rceil)),$$

we obtain an injective morphism

$$H^1(X, \pi_*\mathcal{O}_Y(-\lceil \pi^* D \rceil)) \hookrightarrow H^1(Y, \mathcal{O}_Y(-\lceil \pi^* D \rceil)).$$

By the projection formula, we have $\pi_*\mathcal{O}_Y(-\lceil \pi^* D \rceil) = \pi_*\mathcal{O}_Y(\lfloor -\pi^* D \rfloor) = \mathcal{O}_X(-D)$ and hence it suffices to show that $H^1(Y, \mathcal{O}_Y(-\lceil \pi^* D \rceil)) = 0$. Since $\mathrm{Supp}(\lceil \pi^* D \rceil - \pi^* D) \subset \mathrm{Exc}(\pi)$ and $\pi^* D$ is nef and big (see Remark 2.3.4), we obtain the desired vanishing by Theorem 2.4.6. $\square$

Finally, we define *log lifting*, which we will use in Section 5.

Definition 2.4.10. Let X be a normal projective surface. Fix a Noetherian irreducible scheme T and a morphism $\alpha\colon \operatorname{Spec} k \longrightarrow T$. We say that X is *log liftable over T via α* (or *log liftable over R via α* when $T = \operatorname{Spec} R$) if the pair (Z, E_f) lifts to T via α for some log resolution $f\colon Z \longrightarrow X$. When T is the spectrum of a local ring (R, m) and α is induced by $R/m \cong k$, We also say that X is *log liftable over R* for short.

Chapter 3

Bogomolov-Sommese vanishing and liftability for surface pairs in positive characteristic

In this chapter, we prove Theorems 1.1.2, 1.1.3, and 1.1.4.

3.1 Klt Calabi-Yau surfaces

In this section, we prove the liftability of a log resolution of a klt Calabi-Yau surface in large characteristic (Propositions 3.1.2 and 3.1.13). We also show that there exists a bound on the Gorenstein index for every klt Calabi-Yau surface over every algebraically closed field (Lemma 3.1.9).

Definition 3.1.1. Let X be a normal projective variety. We say that X is *canonical (resp. klt) Calabi-Yau* if X has only canonical (resp. klt) singularities and $K_X \equiv 0$. Moreover, if X is klt Calabi-Yau but not canonical Calabi-Yau, then we say that X is *strictly klt Calabi-Yau.*

First, we show the liftability of a log resolution of a canonical Calabi-Yau surface.

Proposition 3.1.2. *Let X be a canonical Calabi-Yau surface over an algebraically closed field k of characteristic $p > 19$. Then, for every log resolution $f\colon Z \longrightarrow X$ of X, the pair $(Z, \mathrm{Exc}(f))$ lifts to $W(k)$.*

Proof. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution. By Lemma 2.4.3, it suffices to show the liftability of $(Y, E := \mathrm{Exc}(\pi))$. Since $K_Y = \pi^*K_X = 0$, it follows that Y is one of an abelian surface, a hyperelliptic surface, a K3 surface, or an Enriques surface. If Y is an abelian surface, then $Y = X$ and Y lifts to $W(k)$ by [88, Proposition 11.1]. Next, we assume that Y is a hyperelliptic surface. In this case, $Y = X$ and Y is the quotient of the fiber product $C_1 \times C_2$ of elliptic curves by an action of some group scheme G. Let us recall that a smooth projective curve lifts to $W(k)$ with its automorphism if the degree of the automorphism is not divisible by p ([87, Theorem 1.5 and Remark 1.11]). Since $p \neq 2, 3$, comparing with the list of actions of G on $C_1 \times C_2$ in [19, List 10.27], we can take a $W(k)$-lifting $\mathcal{C}_i$ of C_i and $\mathcal{G}$ of G such that $\mathcal{G}$ acts on $\mathcal{C}_1 \times \mathcal{C}_2$ compatibly with the action of G on $C_1 \times C_2$. Then $\mathcal{C}_1 \times \mathcal{C}_2/\mathcal{G}$ gives a lifting of Y.

Next, we assume that Y is a K3 surface or an Enriques surface. Let us show that the determinant d of the intersection matrix of E is not divisible by p. For the sake of contradiction, we assume that d is divisible by p. Since the determinant of the intersection matrix of a rational double point of type A_n, (resp. D_n, E_6, E_7, E_8) is equal to $(-1)^n(n+1)$ (resp. $(-1)^n 4,\ 3, -2, 1$), it follows from the assumption of $p > 19$ that X has an A_{np-1}-singularity for some $n \in \mathbb{Z}_{>0}$. Hence we have $\rho(Y) \geqslant np \geqslant 23$, a contradiction since the Picard rank of a K3 surface (resp. an Enrique surface) is at most 22 (resp. 10) ([49, Chapter 17, 2.4] and [16, Section 3]). Thus d is not divisible by p and [35, Theorems 1.2 and 1.3] shows that $\pi_*\Omega_Y = \Omega_X^{[1]}$. Then we obtain

$$\begin{aligned} H^2(Y, T_Y(-\log E)) \hookrightarrow H^2(X, T_X) \cong & H^0(X, \Omega_X^{[1]} \otimes \mathcal{O}_X(K_X)) \\ = & H^0(Y, \Omega_Y \otimes \mathcal{O}_Y(K_Y)). \end{aligned}$$

For the first injection, we refer to Remark 3.2.2. Let us assume that Y is a K3 surface. Then we have $H^0(Y, \Omega_Y \otimes \mathcal{O}_Y(K_Y)) = H^0(Y, \Omega_Y) = 0$, and (Y, E) formally lifts to $W(k)$ by Theorem 2.4.5. Moreover, the formal lifting is algebraizable by [53, Proposition 2.6]. Finally, let us assume that Y is an Enriques surface. Then we have an étale morphism $\tau \colon \widetilde{Y} \longrightarrow Y$ from a K3 surface $\widetilde{Y}$ since $p \neq 2$. Thus we obtain $H^0(Y, \Omega_Y \otimes \mathcal{O}_Y(K_Y)) \hookrightarrow H^0(Y, \Omega_{\widetilde{Y}} \otimes \mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}})) = 0$. Moreover, since $p \neq 2$, we have $K_X \neq 0$, and in particular, $H^2(Y, \mathcal{O}_Y) \cong H^0(Y, \mathcal{O}_Y(K_Y)) = 0$. Therefore, the pair (Y, E) lifts to $W(k)$ by Theorem 2.4.5. □

Remark 3.1.3. In Proposition 3.1.2, we cannot drop the assumption $p > 19$ (see Example 3.4.3). On the other hand, when the minimal resolution Y is a K3 surface which is not supersingular, the pair (Y, E) lifts to $W(k)$ in any characteristic as follows.

First, by [79, Corollary 4.2], Y itself lifts to $W(k)$. Moreover, by [79, Lemma 2.3 and Corollary 4.2], each irreducible component of E lifts to $W(k)$. Then we obtain the desired liftability by Lemma 2.4.2.

From now, we focus on a strictly klt Calabi-Yau surface. We first prove the existence of the maximum number of the Gorenstein index of a klt Calabi-Yau surface.

Lemma 3.1.4. *Let X be a klt surface and $\pi \colon Y \longrightarrow X$ a resolution. Then the Cartier index of any $\mathbb{Z}$-divisor on X divides the determinant of the intersection matrix of* $\mathrm{Exc}(\pi)$.

Proof. Let d be the determinant of the intersection matrix of $\mathrm{Exc}(\pi)$. We take a $\mathbb{Z}$-divisor D on X and write $\pi^*D = \pi_*^{-1}D + \sum d_i E_i$ for some $d_i \in \mathbb{Q}$. Then it follows that $dd_i \in \mathbb{Z}$ for each i, and in particular, $\pi^* dD$ is Cartier. Now, we can conclude that dD is Cartier by [23, Lemma 2.1]. □

Lemma 3.1.5. *We fix a real number $\varepsilon \in (0, \frac{1}{\sqrt{3}})$. Then there exists $m := m(\varepsilon) \in Z_{>0}$ with the following property. For every ε-klt of Calabi-Yau type surface X over every algebraically closed field and every $\mathbb{Z}$-divisor D on X, the divisor mD is Cartier.*

Proof. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution and $\mathrm{Exc}(\pi) := \sum_i E_i$ the irreducible decomposition. Then Y is ε-klt of Calabi-Yau type. By [4, Lemma 1.2 and Theorem 1.8], we have $-\frac{2}{\varepsilon} \leqslant E_i^2 \leqslant -2$ and $\rho(Y) \leqslant \frac{128}{\varepsilon^5}$. In addition, we have $E_i \cdot E_j = 0$ or 1 for $i \neq j$ since X is klt. Thus there are only finitely many possibilities for the intersection matrix of $\mathrm{Exc}(\pi)$ when X moves surfaces as in the lemma. We take m as a product of all possible determinants of the intersection matrices of $\mathrm{Exc}(\pi)$. Now, Lemma 3.1.4 shows that m is the desired integer. □

Lemma 3.1.6. *We fix a real number $\varepsilon \in (0, \frac{1}{\sqrt{3}})$. When X moves every ε-klt of Calabi-Yau type surface over every algebraically closed field, there are only finitely many possibilities for K_X^2.*

Proof. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution. We can write $K_Y + \sum_i a_i E_i = \pi^* K_X$ for some $a_i \in \mathbb{Q}_{>0}$, where E_i is a π-exceptional prime divisor. Note that when we take a sum over the empty set, we define the sum as zero. As in the proof of Lemma 3.1.5, we have $\rho(Y) \leqslant \frac{128}{\varepsilon^5}$ and there are only finitely many possibilities for the intersection matrix of $\mathrm{Exc}(\pi)$ when X moves. We fix a positive integer $m := m(\varepsilon) \in \mathbb{Z}_{>0}$ as in Lemma 3.1.5. Then we have $a_i \in \{\frac{1}{m}, \cdots, \frac{m-1}{m}\}$ for each i.

If Y is rational, then Y is obtain from $\mathbb{P}^2_k$ or a Hirzebruch surface by at most $(\lfloor \frac{128}{\varepsilon^5} \rfloor - 1)$-times blow-ups, and in particular, $K_Y^2 \in \mathbb{Z} \cap (9 - \lfloor \frac{128}{\varepsilon^5} \rfloor, 9)$. If Y is not rational, then $K_Y^2 = 0$ by [4, Lemma 1.4]. Now, we can conclude that there are only finitely many possibilities for

$$K_X^2 = K_Y^2 + \sum_i a_i (K_Y \cdot E_i) = K_Y^2 + \sum_i a_i (-E_i^2 - 2)$$

and obtain the assertion. □

Lemma 3.1.7 (cf. [13, Proposition 11.7])**.** *Let $\Lambda \subset [0,1] \cap \mathbb{Q}$ be a DCC set. Then there exists a finite subset $\Gamma \subset \Lambda$ with the following property: for every projective morphism $X \longrightarrow Z$ over every algebraically closed field and every $\mathbb{Q}$-divisor B on X satisfying*

- *(X, B) is an lc surface,*
- *the coefficients of B are in Λ,*
- *$K_X + B$ is numerically trivial over Z,*
- $\dim X > \dim Z$,

all the π-horizontal coefficients of B are contained in Γ.

Proof. The assertion has been proved in [13, Proposition 11.7] when we fix the base field. We remark that the same proof works even when the base field moves every algebraically closed field. We note that, in Step 4 of the proof of [13, Proposition 11.7], we use [3, Theorem 6.9], which requires us to fix the base field. However, [3, Theorem 6.9] is applied to only show the boundedness of the Gorenstein index and the self-intersection number of the canonical divisor of an ε-klt del Pezzo surface, which do not depend on the base field (see also Lemmas 3.1.5 and 3.1.6). □

Lemma 3.1.8. *There exists a positive real number $\varepsilon \in \mathbb{R}_{>0}$ such that every klt Calabi-Yau surface over every algebraically closed field is ε-klt.*

Proof. First, we extract an exceptional divisor with minimum log discrepancy. We take a klt Calabi-Yau surface X as in the lemma. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution and write

$$K_Y + \sum_i a_{X,i}E_i = \pi^*K_X$$

for some $a_{X,i} \in \mathbb{Q}_{>0}$. We may assume that $a_{X,1} \geqslant a_{X,i}$ for all i. We run a $(K_Y + a_{X,1}E_1 + \sum_{i\geqslant 2} E_i)$-MMP over X to obtain a birational contraction $\varphi\colon Y \longrightarrow Y'$. Since $K_Y + a_{X,1}E_1 + \sum_{i\geqslant 2} E_i \equiv_X \sum_{i\geqslant 2}(1 - a_{X,i})E_i$, it follows that $\varphi_*E_1 \neq 0$ and $\sum_{i\geqslant 2}(1 - a_{X,i})\varphi_*E_i$ is nef over X. The negativity lemma shows that $\varphi_*E_i = 0$ for each $i \geqslant 2$ and hence

$$K_{Y'} + a_{X,1}\varphi_*E_1 \equiv \varphi_*(K_Y + \sum_i a_{X,i}E_i) \equiv 0.$$

Now, we prove the assertion. For the sake of contradiction, we assume that there exists a sequence of klt Calabi-Yau surfaces $\{X_m\}_{m\in\mathbb{Z}_{>0}}$ such that $\{a_{X_m,1}\}_{m\in\mathbb{Z}_{>0}}$ is a strictly increase sequence. Since $\{a_{X_m,1} | m \in \mathbb{Z}_{>0}\}$ is a DCC set, we can derive a contradiction by Lemma 3.1.7. □

Lemma 3.1.9. *There exists a minimum positive integer $n \in \mathbb{Z}_{>0}$ such that, for every klt Calabi-Yau surface X over every algebraically closed field, the Gorenstein index of X is less than or equal to n.*

Proof. The assertion follows from Lemmas 3.1.5 and 3.1.8. □

Remark 3.1.10. There exists a klt Calabi-Yau surface over $\mathbb{C}$ whose Gorenstein index is 19 by [14, Theorem C (a)]. Thus we have $n \geqslant 19$ in Lemma 3.1.9. Moreover, [14, Theorem C (a)] also shows that we can take $n = 21$ when the base field of X only moves algebraically closed fields of characteristic zero.

Lemma 3.1.11. *Let X be a strictly klt Calabi-Yau surface and n the Gorenstein index of X. Then n is a minimum positive integer such that $nK_X = 0$.*

Proof. By the abundance theorem ([98, Theorem 1.2]), we can take a minimum positive integer l such that $lK_X = 0$. By the definition, we have $n \leqslant l$. Let us show that $l \leqslant n$. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution of X. Then we have

$nK_Y + E = \pi^* nK_X \equiv 0$ for some effective Cartier divisor E. Since X is strictly klt Calabi-Yau, it follows from the proof of [4, Lemma 1.4] that Y is a rational surface. Thus numerically trivial Cartier divisors on Y are linearly trivial, and in particular, $nK_Y + E = 0$. Now we obtain $nK_X = \pi_*(nK_Y + E) = 0$ and hence $l \leqslant n$. $\square$

Lemmas 3.1.9 and 3.1.11 show that a global cyclic cover associated to the canonical divisor of a strictly klt Calabi-Yau surface is étale in codimension one in large characteristic.

Finally, we prove the liftability of a log resolution of a strictly klt Calabi-Yau surface in large characteristic.

Lemma 3.1.12. *We fix a finite set $I \subset [0,1) \cap \mathbb{Q}$ and a positive real number $\varepsilon \in (0, \frac{1}{\sqrt{3}})$. There exists a positive integer $p(I, \varepsilon) \in \mathbb{Z}_{>0}$ with the following property. Let (X, B) be an ε-klt log Calabi-Yau surface over an algebraically closed field k of characteristic bigger than $p(I, \varepsilon)$. Suppose that X admits a K_X-Mori fiber space structure $f\colon X \longrightarrow Z$ and all the coefficients of B are contained in I. Then, for every log resolution $g\colon W \longrightarrow X$ of (X, B), the pair $(W, g_*^{-1}(\mathrm{Supp}(B)) + \mathrm{Exc}(g))$ lifts to $W(k)$.*

Proof. First, we show the following claim.

Claim. There exists a flat family $(\mathcal{X}, \mathcal{B}) \longrightarrow T$ to a reduced quasi-projective scheme T over Spec $\mathbb{Z}$ such that every log Calabi-Yau surface (X, B) over every algebraically closed field of characteristic bigger than five satisfying

- (X, B) is ε-klt,
- X has a K_X-Mori fiber structure $f\colon X \longrightarrow Z$, and
- all the coefficients of B are contained in I,

is a geometric fiber of $(\mathcal{X}, \mathcal{B}) \longrightarrow T$.

Proof of Claim. By the proof of [23, Lemma 3.1], it suffices to show the following: there exists a positive integer $m \in \mathbb{Z}_{>0}$ not depending on X and a very ample divisor H_X on X such that

- mB is Cartier, and
- there are only finitely many possibilities for $\dim H^0(X, \mathcal{O}_X(H_X)), H_X^2, H_X \cdot K_X, H_X \cdot B, K_X \cdot B$, and B^2 when (X, B) moves log Calabi-Yau surfaces as in the claim.

We take a positive integer $m = m(\varepsilon)$ as in Lemma 3.1.5. Since all the coefficients of B are contained in a finite set I, we can assume that mB is Cartier when (X, B) moves, and the first assertion holds.

Let us show the latter assertion. Together with $B \equiv -K_X$ and Lemma 3.1.6, it suffices to check the values of $\dim H^0(X, \mathcal{O}_X(H_X)), H_X^2$, and $H_X \cdot K_X$. We first construct an ample Cartier divisor A_X on X such that there are only finitely many possibilities for $A_X^2, A_X \cdot K_X$ when X moves. If $\dim Z = 0$, then $-K_X$ is ample and we can take $A_X := -mK_X$. We next assume that $\dim Z = 1$. Let us show that $-K_X + (\lceil \frac{2}{\varepsilon} \rceil - 1)F$ is ample, where F is a fiber of $X \longrightarrow Z$. Let C be an irreducible curve whose numerical class spans an extremal ray of $\overline{NE}(X)$ that is not spanned by the numerical class of F. If $C^2 > 0$, then we have

$$(-K_X + (\lceil \frac{2}{\varepsilon} \rceil - 1)F) \cdot C = (B + (\lceil \frac{2}{\varepsilon} \rceil - 1)F) \cdot C \geqslant \lceil \frac{2}{\varepsilon} \rceil - 1 > 0,$$

and hence $-K_X + (\lceil \frac{2}{\varepsilon} \rceil - 1)F$ is ample by Kleiman's ampleness criterion. We next assume that $C^2 < 0$. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution. Then Y is an ε-klt of Calabi-Yau surface and hence [4, Lemma 1.2] shows that $-\frac{2}{\varepsilon} \leqslant (\pi_*^{-1}C)^2$. In particular, $-\frac{2}{\varepsilon} \leqslant C^2$. Now, we have

$$\begin{aligned}(-K_X + (\lceil \frac{2}{\varepsilon} \rceil - 1)F) \cdot C = (B + (\lceil \frac{2}{\varepsilon} \rceil - 1)F) \cdot C >&(1-\varepsilon)C^2 + \lceil \frac{2}{\varepsilon} \rceil - 1 \\ \geqslant& -\frac{2}{\varepsilon} + 2 + \lceil \frac{2}{\varepsilon} \rceil - 1 > 0,\end{aligned}$$

and hence $-K_X + (\lceil \frac{2}{\varepsilon} \rceil - 1)F$ is ample. Together with $F^2 = 0$, $K_X \cdot F = -2$, and Lemma 3.1.6, by taking $A_X := m(-K_X + (\lceil \frac{2}{\varepsilon} \rceil - 1)F)$, we can see that A_X is the desired ample Cartier divisor.

Now, by [102, Theorem 1.2], it follows that $13mK_X + 45mA_X$ is very ample. Moreover, we can see that $(13m-3)K_X + (45m-14)A_X$ is nef and hence $H^i(X, \mathcal{O}_X(13mK_X + 45mA_X)) = 0$ for all $i > 0$ by [102, Proposition 6.5]. We set $H_X := 13mK_X + 45mA_X$. Then there are only finitely many possibilities for H_X^2 and $H_X \cdot K_X$. Moreover, by the Riemann-Roch theorem, we have

$$\begin{aligned}&\dim H^0(X, \mathcal{O}_X(H_X)) = \mathcal{X}(\mathcal{O}_X(H_X)) = \mathcal{X}(\mathcal{O}_W(f^*H_X) \\ =&\frac{(f^*H_X)^2}{2} + \frac{f^*H_X \cdot (-K_W)}{2} + 1 = \frac{(H_X)^2}{2} + \frac{H_X \cdot (-K_X)}{2} + 1,\end{aligned}$$

where $f\colon W \longrightarrow X$ is a resolution and we used the fact that X has only rational singularities for the second equality. Therefore, there are only finitely many possibilities for $\dim H^0(X, \mathcal{O}_X(H_X))$, and we finish the proof of the claim.

By the claim and the proof of [23, Proposition 3.2], we can find a positive integer $p(I, \varepsilon)$ with the following property: let (X, B) be a log Calabi-Yau surface as in the lemma. Then, there exists a log resolution $g\colon W \longrightarrow X$ of (X, B) such that the pair $(W, g_*^{-1}(\mathrm{Supp}(B)) + \mathrm{Exc}(g))$ lifts to characteristic zero over a smooth base in the sense of [23, Definition 2.15]. Now, we obtain the desired liftability by [7, Proposition 2.5] and Lemma 2.4.4. $\square$

Proposition 3.1.13. *There exists a positive integer p_0 with the following property. Let X be a strictly klt Calabi-Yau surface over an algebraically closed field of characteristic $p > p_0$. Then, for every log resolution $g\colon W \longrightarrow X$, the pair $(W, \mathrm{Exc}(g))$ lifts to $W(k)$.*

Proof. By Lemma 3.1.8, there exists a positive real number $\varepsilon \in (0, \frac{1}{\sqrt{3}})$ such that every klt Calabi-Yau surface is ε-klt. We take $m = m(\varepsilon)$ as in Lemma 3.1.5 and define a finite set $I := \{\frac{1}{m}, \cdots, \frac{m-1}{m}\}$. Let us take $p_0 := p(I, \varepsilon)$ as in Lemma 3.1.12.

Let X be a strictly klt Calabi-Yau surface over an algebraically closed field of characteristic $p > p_0$. As in Lemma 3.1.8, we can take an extraction $f\colon Y \longrightarrow X$ of an exceptional prime divisor E_1 such that $a_1 := \mathrm{coeff}_{E_1}(f^*K_X - K_Y) \in I$. Since $K_Y \equiv -a_1E_1$ is not pseudo-effective, we can run a K_Y-MMP to obtain a birational contraction $\varphi\colon Y \longrightarrow Y'$ and a $K_{Y'}$-Mori fiber space Y'. Since $K_Y + a_1E_1 \equiv 0$, the negativity lemma shows that $K_Y + a_1E_1 = \varphi^*(K_{Y'} + a_1E_1')$ and hence (Y', a_1E_1') is ε-klt log Calabi-Yau, where $E_1' := \varphi_*E_1$. Then, by Lemma 3.1.12 and the definition of p_0, we can take a log resolution $\mu\colon Z \longrightarrow Y'$ of (Y', a_1E_1') such that φ factors through μ and $(Z, \mu_*^{-1}E_1' + \mathrm{Exc}(\mu))$ lifts to $W(k)$. We now have the following diagram:

$$\begin{tikzcd}
(Z, \mu_*^{-1}E_1' + \mathrm{Exc}(\mu)) \arrow[r, "h"'] \arrow[dr, "\mu"'] & (Y, E_1) \arrow[r, "f"'] \arrow[d, "\varphi"] & X \\
& (Y', E_1') &
\end{tikzcd}.$$

Since $\mathrm{Exc}(f \circ h) \subset \mu_*^{-1}E_1' + \mathrm{Exc}(\mu)$, the pair $(Z, \mathrm{Exc}(f \circ h))$ lifts to $W(k)$, and the assertion holds by Lemma 2.4.4. $\square$

3.2 The Bogomolov-Sommese vanishing theorem for lc surfaces

3.2.1 An extension type theorem for lc surfaces

In this subsection, we show an extension type theorem (Proposition 3.2.6), which plays an essential role in the proof of Theorem 1.1.2.

Lemma 3.2.1. *Let $f\colon Y \longrightarrow X$ be a projective birational morphism of normal surfaces. Let B_Y (resp. D_Y) be a reduced $\mathbb{Z}$-divisor (resp. $\mathbb{Z}$-divisor) on Y and $B := f_*B_Y$ (resp. $D := f_*D_Y$). Then the followings hold.*

(1) *The natural restriction morphism*

$$f_*(\Omega_Y^{[1]}(\log\, B_Y) \otimes \mathcal{O}_Y(-D_Y))^{**} \longrightarrow (\Omega_X^{[1]}(\log\, B) \otimes \mathcal{O}_X(-D))^{**}$$

is injective.

(2) *Suppose that X and Y are projective. Then $\kappa(Y, D_Y) \leqslant \kappa(X, D)$ holds.*

Proof. We first see the assertion (1). Since $f_*(\Omega_Y^{[1]}(\log\, D_Y) \otimes \mathcal{O}_Y(-B_Y))^{**}$ is torsion-free, we have an injective morphism

$$\begin{aligned} f_*(\Omega_Y^{[1]}(\log\, D_Y) \otimes \mathcal{O}_Y(-B_Y))^{**} \hookrightarrow& (f_*(\Omega_Y^{[1]}(\log\, D_Y) \otimes \mathcal{O}_Y(-B_Y))^{**})^{**} \\ \cong& (\Omega_X^{[1]}(\log\, D) \otimes \mathcal{O}_X(-B_X))^{**}. \end{aligned}$$

We note that, for every open subset U of X, the above morphism is the restriction of the sections of $H^0(f^{-1}(U), (\Omega_Y^{[1]}(\log D_Y) \otimes \mathcal{O}_Y(-B_Y)^{**})$ to $f^{-1}(U)\backslash \operatorname{Exc}(f) \cong U\backslash f(\operatorname{Exc}(f))$.

Similarly, we have the natural injective morphism

$$H^0(Y, \mathcal{O}_Y(mB_Y)) \hookrightarrow H^0(X, \mathcal{O}_X(mB_X))$$

for all $m \in \mathbb{Z}_{>0}$ and hence the assertion (2) holds. $\square$

Remark 3.2.2. In the setting of Lemma 3.2.1 (2), we have

$$\begin{aligned} H^2(Y, T_Y(-\log B_Y)) \cong & \operatorname{Hom}_{\mathcal{O}_Y}(T_Y(-\log B_Y), \mathcal{O}_Y(K_Y)) \\ \cong & H^0(Y, (\Omega_Y^{[1]}(\log B_Y) \otimes \mathcal{O}_Y(K_Y))^{**}) \end{aligned}$$

by the Serre duality. Then, by Lemma 3.2.1 (1), we obtain an injective morphism

$$H^2(Y, T_Y(-\log B_Y)) \hookrightarrow H^2(X, T_X(-\log B)).$$

We will use this fact in Section 3.3.

Definition 3.2.3. Let (X, B) be a dlt pair over an algebraically closed field of characteristic $p > 0$ such that B is reduced. We say that (X, B) is *tamely dlt* if the Cartier index of $K_X + B$ is not divisible by p.

Definition 3.2.4. Let X be a normal surface and B is a $\mathbb{Q}$-divisor with coefficients in $[0, 1]$. We say that a morphism $h\colon W \longrightarrow X$ is a *dlt blow-up of* (X, B) if

(1) h is a projective birational morphism,

(2) $(W, h_*^{-1}B + \operatorname{Exc}(h))$ is dlt, and

(3) $K_W + h_*^{-1}B + \operatorname{Exc}(h) + F = h^*(K_X + B)$ for some effective $\mathbb{Q}$-divisor F.

Lemma 3.2.5. *Let X be a normal surface and B is a $\mathbb{Q}$-divisor with coefficients in $[0, 1]$. Then the followings hold.*

(1) *Any log resolution $\pi\colon Y \longrightarrow X$ of (X, B) decomposes into a birational projective morphism $Y \longrightarrow W$ and a dlt blow-up $W \longrightarrow X$.*

(2) *$F = 0$ if and only if (X, B) is lc.*

Proof. We refer to [100, Theorem 4.7 and Remark 4.8] for the proof. Note that, by considering the Mumford pullback, [100, Remark 4.8 (1)] holds without the assumption that $K_X + B$ is $\mathbb{R}$-Cartier. $\square$

Proposition 3.2.6 (An extension type theorem for lc surfaces)**.** *Let (X, B) be an lc surface pair over an algebraically closed field of characteristic $p > 5$ and D a $\mathbb{Z}$-divisor on X. Let $f\colon Y \longrightarrow X$ be a projective birational morphism such that (Y, B_Y) is lc, where $B_Y := f_*^{-1}B + \operatorname{Exc}(f)$. Then the natural restriction morphism*

$$\Phi\colon f_*(\Omega_Y^{[1]}(\log \lfloor B_Y \rfloor) \otimes \mathcal{O}_Y(-\lceil f^*D \rceil))^{**} \longrightarrow (\Omega_X^{[1]}(\log \lfloor B \rfloor) \otimes \mathcal{O}_X(-D))^{**}$$

is isomorphic.

Remark 3.2.7. Proposition 3.2.6 is equivalent to saying that

$$f_*(\Omega_Y^{[1]}(\log \lfloor B_Y \rfloor) \otimes \mathcal{O}_Y(-\lceil f^*D \rceil))^{**}$$

is reflexive.

Remark 3.2.8. If we take $D = 0$ in the proposition, then this is nothing but Graf's extension theorem ([35, Theorem 1.2]). Let us see why we need to generalize Graf's extension theorem to Proposition 3.2.6 for the proof of Theorem 1.1.2.

We work in characteristic zero and follow the notation of Theorem 1.1.1. Let $\pi\colon Y \longrightarrow X$ be a log resolution and $B_Y := \pi_*^{-1}B + \mathrm{Exc}(\pi)$. Suppose that there exists a $\mathbb{Z}$-divisor D and an injective morphism $\mathcal{O}_X(D) \hookrightarrow \Omega_X^{[i]}(\log B)$. For simplicity, we assume that D is $\mathbb{Q}$-Cartier. Then, by applying the extension theorem in characteristic zero [37, Theorem 1.5], we can construct a $\mathbb{Z}$-divisor D_Y on Y such that there exists an injective morphism $\mathcal{O}_Y(D_Y) \hookrightarrow \Omega_Y^{[i]}(\log B_Y)$ and $\kappa(X, D) = \kappa(Y, D_Y)$. This means the Bogomolov-Sommese vanishing theorem can be reduced to the case of log smooth pairs by the extension theorem (see [37, 7.C. Proof of Theorem 7.2.] for the detailed argument). In the construction of D_Y, we use the fact that an index one cover of D is étale in codimension. However, when we work in characteristic $p > 0$ and the Cartier index of D is divisible by p, this fact is not always true. Therefore, we cannot apply Graf's extension theorem directly to reduce Theorem 1.1.2 to the case where (X, B) is log smooth.

Moreover, in positive characteristic, reducing to the case of log smooth surfaces is not enough because the Bogomolov-Sommese vanishing theorem is not known even for such pairs. Proposition 3.2.6 asserts that D_Y can be taken as $\lceil f^*D \rceil$, and this enables us to apply the Akizuki-Nakano vanishing theorem (Theorem 2.4.6) when D is ample.

Proof of Proposition 3.2.6. **Step 0.** Throughout the proof of this proposition, we denote $(\Omega_W^{[1]}(\log B_W) \otimes \mathcal{O}_W(-D_W))^{**}$ by $\Omega_W^{[1]}(\log B_W)(-D_W)$ for every surface pair (W, B_W) and $\mathbb{Z}$-divisor D_W. By Lemma 3.2.1 (1), Φ is injective. Since $(X, \lfloor B \rfloor)$ is lc (see [35, Proposition 7.2]), by replacing B with $\lfloor B \rfloor$, we may assume that B is reduced. Moreover, since the assertion of the proposition is local on X, we may assume that X is affine. Therefore, it suffices to show that

$$\Phi\colon H^0(Y, \Omega_Y^{[1]}(\log B_Y)(-\lceil f^*D \rceil)) \hookrightarrow H^0(X, \Omega_X^{[1]}(\log B)(-D))$$

is surjective.

Step 1. First, we prove the following claim.

Claim. Suppose that

- (Y, B_Y) is tamely dlt, and
- $-(K_Y + B_Y)$ is f-nef.

Then Φ is surjective.

Proof of Claim. We take $s \in H^0(X, \Omega_X^{[1]}(\log B)(-D))$. Let us construct a section in $H^0(Y, \Omega_Y^{[1]}(\log B_Y)(-\lceil \pi^* D \rceil))$ which maps to s by Φ. We may assume that s is non-zero and hence s is considered as an injective $\mathcal{O}_X$-module homomorphism $s\colon \mathcal{O}_X(D) \hookrightarrow \Omega_X^{[1]}(\log B)$. By [35, Theorem 6.1], the natural restriction morphism $f_*\Omega_Y^{[1]}(\log B_Y) \cong \Omega_X^{[1]}(\log B)$ is isomorphic. Then we have a generically injective $\mathcal{O}_Y$-module homomorphism

$$f^*\mathcal{O}_X(D) \xrightarrow{f^*s} f^*\Omega_X^{[1]}(\log B) \cong f^*f_*\Omega_Y^{[1]}(\log B_Y) \longrightarrow \Omega_Y^{[1]}(\log B_Y).$$

By taking double dual, we obtain an injective $\mathcal{O}_Y$-module homomorphism

$$s_Y\colon f^{[*]}\mathcal{O}_X(D) \hookrightarrow \Omega_Y^{[1]}(\log B_Y),$$

where $f^{[*]}\mathcal{O}_X(D) := (f^*\mathcal{O}_X(D))^{**}$. We take a $\mathbb{Z}$-divisor D_Y on Y such that $\mathcal{O}_Y(D_Y) = f^{[*]}\mathcal{O}_X(D)$. Since $\mathcal{O}_X(f_*D_Y) = (f_*\mathcal{O}_Y(D_Y))^{**} = (f_*f^{[*]}\mathcal{O}_X(D))^{**} = \mathcal{O}_X(D)$, it follows that f_*D_Y is linearly equivalent to D. By replacing D_Y with $D_Y + f^*(D - f_*D_Y)$, we may assume that $f_*D_Y = D$. In particular, $D_Y - f^*D$ is f-exceptional.

Now, we replace D_Y so that $D_Y - f^*D$ is effective. Let us assume that $D_Y - f^*D$ is not effective. By applying the negativity lemma to the negative coefficients part of $D_Y - f^*D$, we can take a prime f-exceptional divisor E_1 such that $\mathrm{mult}_{E_1}(D_Y - f^*D) < 0$ and $D_Y \cdot E_1 > 0$. Then we can show that s_Y factors though an injective $\mathcal{O}_Y$-module homomorphism $\mathcal{O}_Y(D_Y + E_1) \hookrightarrow \Omega_Y^{[1]}(\log B_Y)$. This follows from the essentially same argument as [35, Theorem 6.1], but we provide the proof here for the completeness.

Since (Y, B_Y) is tamely dlt, we have the following commutative diagram

$$\begin{tikzcd}
 & & \mathcal{O}_Y(D_Y) \arrow[dl, dotted] \arrow[d, "s_Y"] \arrow[dr, "t"] & & \\
0 \arrow[r] & \Omega_Y^{[1]}(\log B_Y - E_1) \arrow[r] & \Omega_Y^{[1]}(\log B_Y) \arrow[r, "\mathrm{res}_{E_1}"] & \mathcal{O}_{E_1} \arrow[r] & 0,
\end{tikzcd}$$

and a surjective morphism

$$\mathrm{res}_{E_1}^m\colon \mathrm{Sym}^{[m]}\,\Omega_Y^{[1]}(\log B_Y) := (\mathrm{Sym}^m\,\Omega_Y^{[1]}(\log B_Y))^{**} \longrightarrow \mathcal{O}_{E_1}$$

for each $m > 0$ which coincides with $\mathrm{Sym}^m(\mathrm{res}_{E_1})$ in the generic point of E_1 by [35, Theorem 1.4 (1.4.1)]. Let us show that t is the zero map. For the sake of contradiction, we assume that t is not zero. Since $\mathrm{Im}(t) \subset \mathcal{O}_{E_1}$ is a torsion-free $\mathcal{O}_{E_1}$-module, it follows that t is non-zero in the generic point of E_1 and so is $\mathrm{Sym}^m(t)\colon \mathcal{O}_Y(D_Y)^{\otimes m} \longrightarrow \mathcal{O}_{E_1}$. Since $\mathrm{Sym}^m(s_Y)$ (resp. $\mathrm{Sym}^m(\mathrm{res}_{E_1})$) coincides with $\mathrm{Sym}^{[m]}(s_Y) := (\mathrm{Sym}^m(s_Y))^{**}$ (resp. $\mathrm{res}_{E_1}^m$) in the generic point of E_1, the composition $\mathrm{res}_{E_1}^m \circ \mathrm{Sym}^{[m]}(s_Y)$ coincides with $\mathrm{Sym}^m(t) = \mathrm{Sym}^m(\mathrm{res}_{E_1}) \circ \mathrm{Sym}^m(s_Y)$, and in particular, is non-zero in the generic point of E_1. Now, we fix $m > 0$ such that mD_Y is Cartier. Note that Y is $\mathbb{Q}$-factorial since (Y, B_Y) is dlt. By restricting $\mathrm{res}_{E_1}^m \circ \mathrm{Sym}^{[m]}(s_Y)$ to E_1, we obtain an injective $\mathcal{O}_{E_1}$-module homomorphism

$\mathcal{O}_{E_1}(mD_Y) \hookrightarrow \mathcal{O}_{E_1}$ and hence $0 < mD_Y \cdot E_1 = \deg(\mathcal{O}_{E_1}(mD_Y)) \leqslant 0$, a contradiction. Therefore t is zero and the morphism s_Y factors through $\mathcal{O}_Y(D_Y) \longrightarrow \Omega_Y^{[1]}(\log\, B_Y - E_1)$. Then, by [35, Theorem 1.4 (1.4.2)], we obtain the following commutative diagram

$$\begin{tikzcd}
 & & \mathcal{O}_Y(D_Y) \arrow[dll, dotted] \arrow[d, "s_Y"] \arrow[dr, "v"] & & \\
0 \arrow[r] & \Omega_Y^{[1]}(\log\, B_Y)(-E_1) \arrow[r] & \Omega_Y^{[1]}(\log\, B_Y - E_1) \arrow[r, "\mathrm{restr}_{\mathrm{E}_1}"] & \omega_{E_1}(\lfloor E_1^c \rfloor) \arrow[r] & 0,
\end{tikzcd}$$

and a surjective morphism

$$\mathrm{restr}^m_{E_1} \colon \operatorname{Sym}^{[m]} \Omega_Y^{[1]}(\log\,(B_Y - E_1)) \longrightarrow \mathcal{O}_{E_1}(mK_{E_1} + \lfloor mE_1^c \rfloor)$$

which coincides with $\operatorname{Sym}^m(\mathrm{restr}_{\mathrm{E}_1})$ in the generic point E_1. Here, E_1^c denotes the different $\operatorname{Diff}_{E_1}(B_Y - E_1)$ (see [65, Definition 4.2] for the definition). Since $-(K_Y + B_Y)$ is f-nef, it follows that

$$\deg(\mathcal{O}_{E_1}(mK_{E_1} + \lfloor mE_1^c \rfloor)) \leqslant (mK_Y + mB_Y) \cdot E_1 \leqslant 0$$

for all $m > 0$ and hence an argument similar to above shows that $v = 0$ and s_Y factors through $\mathcal{O}_Y(D_Y) \hookrightarrow \Omega_Y^{[1]}(\log\, B_Y)(-E_1)$. In particular, we obtain an injective $\mathcal{O}_Y$-module homomorphism $\mathcal{O}_Y(D_Y + E_1) \hookrightarrow \Omega_Y^{[1]}(\log\, B_Y)$ which coincides with s_Y on $Y \setminus \operatorname{Exc}(f)$. By replacing D_Y with $D_Y + E_1$, and repeating the above procedure, we can assume that $D_Y - f^*D$ is effective.

Now, we obtain a $\mathbb{Z}$-divisor D_Y on Y such that $D_Y - f^*D \geqslant 0$ and a morphism $s_Y \in H^0(Y, \Omega_Y^{[1]}(\log\, B_Y)(-D_Y))$, which maps to s under the natural restriction morphism

$$\Phi' \colon H^0(Y, \Omega_Y^{[1]}(\log\, B_Y)(-D_Y)) \longrightarrow H^0(X, \Omega_X^{[1]}(\log\, B)(-D)).$$

Since $\lceil f^*D \rceil \leqslant \lceil D_Y \rceil = D_Y$, it follows that Φ' decomposes into the natural injective morphism

$$\Theta \colon H^0(Y, \Omega_Y^{[1]}(\log\, B_Y)(-D_Y)) \hookrightarrow H^0(Y, (\Omega_Y^{[1]}(\log\, B_Y)(-\lceil f^*D \rceil))$$

and the morphism

$$\Phi \colon H^0(Y, \Omega_Y^{[1]}(\log\, B_Y)(-\lceil f^*D \rceil)) \hookrightarrow H^0(X, \Omega_X^{[1]}(\log\, B)(-D)).$$

Now we have $\Phi(\Theta(s_Y)) = \Phi'(s_Y) = s$ and hence Φ is surjective. Thus we finish the proof of the claim.

Step 2. Next, let us show that we may assume that $f \colon Y \longrightarrow X$ is a log resolution of (X, B). Let $\pi \colon Z \longrightarrow Y$ be a log resolution of (Y, B_Y) and $\widetilde{f} := f \circ \pi$. Then $B_Z := \pi_*^{-1} B_Y + \operatorname{Exc}(\pi) = \widetilde{f}_*^{-1} B + \operatorname{Exc}(\widetilde{f})$, and in particular, $\widetilde{f}$ is a log resolution of (X, B). Suppose that the natural restriction morphisms

$$\Phi_{Z,X} \colon H^0(Z, \Omega_Z(\log\, B_Z)(-\lceil \widetilde{f}^*D \rceil)) \hookrightarrow H^0(X, \Omega_X^{[1]}(\log\, B)(-D)).$$

$$\Phi_{Z,Y} \colon H^0(Z, \Omega_Z(\log\, B_Z)(-\lceil \pi^*\lceil f^*D \rceil \rceil)) \hookrightarrow H^0(Y, (\Omega_Y^{[1]}(\log\, B_Y)(-\lceil f^*D \rceil))$$

are isomorphic. Since $\lceil \pi^*\lceil f^*D\rceil\rceil \geqslant \lceil \widetilde{f}^*D\rceil$, the isomorphism $\Phi_{Z,Y}$ factors through the natural restriction morphism

$$\Phi'_{Z,Y}\colon H^0(Z,\Omega_Z(\log\, B_Z)(-\lceil \widetilde{f}^*D\rceil)) \hookrightarrow H^0(X,\Omega_Y^{[1]}(\log\, B_Y)(-\lceil f^*D\rceil))$$

and hence $\Phi'_{Z,Y}$ is isomorphic. Now, we can conclude that $\Phi = \Phi_{Z,X}\circ(\Phi'_{Z,Y})^{-1}$ is an isomorphism. Thus we may assume that $f\colon Y \longrightarrow X$ is a log resolution of (X,B).

Step 3. Finally, let us show that the surjectivity of Φ and finish the proof of the proposition. By Lemma 3.2.5, a log resolution f decomposes into a birational morphism and a dlt blow-up. Then, by [35, 7.B. Proof of Theorem 1.2], we obtain a decomposition

$$f\colon Y_0 := Y \xrightarrow{f_1} Y_1 \longrightarrow \cdots \xrightarrow{f_m} Y_m := X$$

such that each f_i satisfies the assumption of the claim. Here, we use the assumption that $p>5$. By the claim, the natural restriction morphisms

$$\begin{aligned}\Phi_{m-1,m}\colon H^0(Y_{m-1},\Omega^{[1]}_{Y_{m-1}}(\log\, B_{Y_{m-1}})(-\lceil f_m^*D\rceil)) &\cong H^0(X,\Omega_X^{[1]}(\log\, B)(-D)),\\ \Phi_{m-2,m-1}\colon H^0(Y_{m-2},\Omega^{[1]}_{Y_{m-2}}(\log\, B_{Y_{m-2}})(-\lceil f_{m-1}^*\lceil f_m^*D\rceil\rceil))&\\ \cong H^0(Y_{m-1},\Omega^{[1]}_{Y_{m-1}}(\log\, B_{Y_{m-1}})(-\lceil f_m^*D\rceil))&\end{aligned}$$

are isomorphic. Then $\Phi_{m-1,m}\circ\Phi_{m-2,m-1}$ factors through the natural restriction morphism

$$\Phi_{m-2,m}\colon H^0(Y_{m-2},\Omega^{[1]}_{Y_{m-2}}(\log\, B_{Y_{m-2}})(-\lceil f_{m-1}^*f_m^*D\rceil)) \hookrightarrow H^0(X,\Omega_X^{[1]}(\log\, B)(-D)),$$

and hence $\Phi_{m-2,m}$ is isomorphic. By repeating this procedure, we can conclude that Φ is an isomorphism. □

3.2.2 Proof of Theorem 1.1.2

In this subsection, we prove Theorem 1.1.2. First, we show the Bogomolov-Sommese vanishing theorem on a surface admitting a fibration structure including a Mori fiber space and an lc trivial fibration.

Lemma 3.2.9. *Let X be a normal surface over an algebraically closed field k of characteristic $p>3$ and B a reduced divisor on X. Let $f\colon X\longrightarrow Z$ be a projective surjective morphism such that* $\dim Z = 1$*, $f_*\mathcal{O}_X=\mathcal{O}_Z$, and $-(K_X+B)$ is f-nef. Then*

$$f_*(\Omega_X^{[1]}(\log\, B)\otimes\mathcal{O}_X(-D))^{**}=0$$

for every $\mathbb{Z}$-divisor D satisfying $D\cdot F>0$ for a general fiber F of f.

Proof. Since $f_*(\Omega_X^{[1]}(\log\, B)\otimes\mathcal{O}_X(-D))^{**}$ is torsion-free, it suffices to show that the rank of the sheaf is zero, and in particular, we can shrink Z for the proof. First, we prove the following claim.

Claim. By shrinking Z, we may assume that (X,B) is log smooth over Z.

Proof of Claim. We note that the general fiber F is reduced and irreducible since $\dim Z = 1$ and $f_*\mathcal{O}_X = \mathcal{O}_Z$. By shrinking Z, we may assume that all irreducible components of B dominant Z. Let $n \in \mathbb{Z}_{\geqslant 0}$ be the number of the irreducible components of B. Then we have

$$\deg(K_F) = K_X \cdot F \leqslant K_X \cdot F + n \leqslant (K_X + B) \cdot F \leqslant 0$$

and hence $(\deg(K_F), n) = (0, 0), (-2, 0), (-2, 1)$, or $(-2, 2)$. If $(\deg(F), n) = (0, 0)$, then $B = 0$ and F is an elliptic curve since $p > 3$. Similarly, if $(\deg(F), n) = (-2, 0)$, then $B = 0$ and $F \cong \mathbb{P}^1_k$. Next, if $(\deg(F), n) = (-2, 1)$, then $F \cong \mathbb{P}^1_k$ and $B \cdot F = 1$ or 2. In the case where $B \cdot F = 1$, it follows that B and F intersect transversally. In the case where $B \cdot F = 2$, the restricted morphism $f|_B \colon B \longrightarrow Z$ is generically étale since $p \neq 2$. Finally, if $(\deg(F), n) = (-2, 2)$, then $B_1 \cdot F = B_2 \cdot F = 1$ and hence B_1 (resp. B_2) intersects transversally with F, where B_1 and B_2 are irreducible components of B.

Therefore, in each case, we can assume that (X, B) is log smooth over Z by shrinking Z and finish the proof of the claim.

Now, we show that the assertion of the lemma. We shrink Z so that Z is affine and (X, B) is log smooth over Z. Note that (X, B) is also log smooth over k in this case. For the sake of contradiction, we assume that

$$H^0(X, \Omega_X(\log B) \otimes \mathcal{O}_X(-D)) \neq 0$$

for some $\mathbb{Z}$-divisor D satisfying $D \cdot F > 0$. Then there exists an injective $\mathcal{O}_X$-module homomorphism $s \colon \mathcal{O}_X(D) \hookrightarrow \Omega_X(\log B)$. Since (X, B) is log smooth over Z, we have the following exact sequence.

$$\begin{tikzcd}
 & & \mathcal{O}_X(D) \arrow[dl, dotted] \arrow[d, "s"] \arrow[dr, "t"] & & \\
0 \arrow[r] & \mathcal{O}_X(f^*K_Z) \arrow[r] & \Omega_X(\log B) \arrow[r] & \Omega_{X/Z}(\log B) \arrow[r] & 0.
\end{tikzcd}$$

In the above diagram, when $B \neq 0$, we define $\Omega_X(\log B) \longrightarrow \Omega_{X/Z}(\log B)$ by $d(f^*z) \longmapsto 0, dx/x \longmapsto dx/x$, where z is a coordinate on Z and x is a local equation of B. Note that f^*z and x form coordinates on X since (X, B) is log smooth over Z. When $B = 0$, this is the usual relative differential sequence for f ([44, II Proposition 8.11]). Suppose that t is non-zero. Then, by restricting t to F, we have an injective $\mathcal{O}_F$-module homomorphism $t|_F \colon \mathcal{O}_F(D) \hookrightarrow \Omega_F(\log B|_F) = \mathcal{O}_F(K_F + B_F)$, where the injectivity follows from the generality of F. This shows that

$$0 < \deg(D|_F) \leqslant \deg(K_F + B|_F) = (K_X + B) \cdot F \leqslant 0,$$

a contradiction. Thus t is zero and the morphism s factors through $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(f^*K_Z)$. Then by considering the restriction to F, we obtain

$$0 < \deg(D|_F) \leqslant \deg(f^*K_Z|_F) = 0,$$

a contradiction. Hence we conclude that $H^0(X, \Omega_X(\log B) \otimes \mathcal{O}_X(-D)) = 0$. $\square$

Now, we prove Theorem 1.1.2.

Proof of Theorem 1.1.2. **Step 0.** By replacing B with $\lfloor B \rfloor$, we may assume that B is reduced. Since the assertion is obvious when $i = 0$ or 2, it suffices to show that

$$H^0(X, (\Omega_X^{[1]}(\log B) \otimes \mathcal{O}_X(-D))^{**}) = 0 \tag{a}$$

for every big $\mathbb{Z}$-divisor D. Let $h\colon (W, B_W := h_*^{-1}B + \mathrm{Exc}(h)) \longrightarrow (X, B)$ be a dlt blow-up. Then $\kappa(W, K_W + B_W) = \kappa(X, K_X + B)$ and the vanishing (a) is equivalent to saying that

$$H^0(W, (\Omega_W^{[1]}(\log B_W) \otimes \mathcal{O}_W(-\lceil h^*D \rceil))^{**}) = 0 \tag{b}$$

when $p > 5$ by Proposition 3.2.6. We set $D_W := \lceil h^*D \rceil$. By Remark 2.3.4, D_W is big.

Step 1. First, we assume that $\kappa(X, K_X + B) = -\infty$ and $p > 5$. Let us show the vanishing (b). In this case, $K_W + B_W$ is not pseudo-effective by the abundance theorem ([98, Theorem 1.2]). By Lemma 3.2.1 (1) and (2), we can replace W with an output of a $(K_W + B_W)$-MMP and assume that W has a $(K_W + B_W)$-Mori fiber space structure $f\colon W \longrightarrow Z$. If $\dim Z = 1$, then the assertion follows from Lemma 3.2.9. Thus we assume that $\dim Z = 0$. In this case, W is a klt del Pezzo surface of Picard rank one and D_W is an ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor. Let $\pi\colon Y \longrightarrow W$ be a log resolution of (W, B_W), $B' := \pi_*^{-1}B_W$, $E := \mathrm{Exc}(\pi)$, and $B_Y := B' + E$. Then by Proposition 3.2.6, it suffices to show that

$$H^0(Y, \Omega_Y(\log B_Y) \otimes \mathcal{O}_Y(-\lceil \pi^*D_W \rceil)) = 0.$$

For the sake of contradiction, we assume that there exists an injective $\mathcal{O}_Y$-module homomorphism $s\colon \mathcal{O}_Y(\lceil \pi^*D_W \rceil)) \hookrightarrow \Omega_Y(\log B_Y)$. Let us show that s factors through $s\colon \mathcal{O}_Y(\lceil \pi^*D_W \rceil)) \hookrightarrow \Omega_Y(\log E)$. Let B_1' be an irreducible component of B'. Since (Y, B_Y) is log smooth, we obtain the following diagram

$$\begin{tikzcd}
 & & \mathcal{O}_Y(\lceil \pi^*D_W \rceil) \arrow[dl, dotted] \arrow[d, "s"] \arrow[dr, "t"] & & \\
0 \arrow[r] & \Omega_Y(\log B_Y - B_1') \arrow[r] & \Omega_Y(\log B_Y) \arrow[r] & \mathcal{O}_{B_1'} \arrow[r] & 0.
\end{tikzcd}$$

Since B_1' is not π-exceptional and D_W is an ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor, it follows that

$$\lceil \pi^*D_W \rceil \cdot B_1' \geqslant \pi^*D_W \cdot B_1' = D_W \cdot \pi_*B_1' > 0,$$

and t is zero. Then s factors through $\mathcal{O}_Y(\lceil \pi^*D_W \rceil) \hookrightarrow \Omega_Y(\log B_Y - B_1')$. By repeating this procedure, we can show that s factors through $\mathcal{O}_Y(\lceil \pi^*D_W \rceil) \hookrightarrow \Omega_Y(\log E)$. By [67, Theorem 1.4] and Lemma 2.4.4 (1), it follows that (Y, E) lifts to $W(k)$. Now, since π^*D_W is a nef and big $\mathbb{Q}$-divisor whose support of the fractional part is contained in E, Theorem 2.4.6 shows that $0 \neq s \in H^0(Y, \Omega_Y(\log E)) \otimes \mathcal{O}_Y(-\lceil \pi^*D_W \rceil) = 0$, a contradiction.

Step 2. Next, we assume that $\kappa(X, K_X + B) = 0$ and prove the vanishing (b). We can replace (W, B_W) with the $(K_W + B_W)$-minimal model by Lemma 3.2.1 and hence assume that $K_W + B_W \equiv 0$.

Step 2-1. First, we assume that $B_W \neq 0$ and $p > 5$. In this case, K_W is not pseudo-effective and we can run a K_W-MMP to obtain a birational contraction $\varphi\colon W \longrightarrow W'$ and a $K_{W'}$-Mori fiber space $f\colon W' \longrightarrow Z$. Since $K_W + B_W \equiv 0$, the negativity lemma shows that $K_W + B_W = \varphi^*(K_{W'} + B_{W'})$, where $B_{W'} := \varphi_* B_W$. Thus $(W', B_{W'})$ is log Calabi-Yau and W' is klt. By Lemma 3.2.1, we can replace (W, B_W) with $(W', B_{W'})$. If $\dim Z = 1$, then the assertion follows from Lemma 3.2.9. Thus we may assume that $\dim Z = 0$. In this case, W is a klt del Pezzo surface of Picard rank one and D_W is an ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor. Let $\pi\colon Y \longrightarrow W$ be a log resolution of (W, B_W), $B' := \pi_*^{-1} B_W$, $E := \mathrm{Exc}(\pi)$, and $B_Y := B' + E$. As in Step 1, we derive a contradiction assuming there exists an injective $\mathcal{O}_Y$-module homomorphism $s\colon \mathcal{O}_Y(\lceil \pi^* D_W \rceil) \hookrightarrow \Omega_Y(\log\, B_Y)$. Since $B' \neq 0$, we can take an irreducible component B'_1 of B'. Since B'_1 is not π-exceptional and D_W is an ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor, an argument as in Step 1 shows that the morphism s factors through $\mathcal{O}_Y(\lceil \pi^* D_W \rceil) \longrightarrow \Omega_Y(\log\, B_Y - B'_1)$. Since $K_W + B_W \equiv 0$, we have $\kappa(Y, K_Y + B_Y - B'_1) = -\infty$. Now, we obtain a contradiction by Step 1.

Step 2-2. Next, we assume that $B_W = 0$. In this case, W is a klt Calabi-Yau surface. We take a positive integer n as in Lemma 3.1.9 and assume $p > n$. Let us show that we may assume that D_W is nef and big. Let $D_W \equiv P + N$ be the Zariski decomposition. Note that we can take the Zariski decomposition even when X is singular ([25, Theorem 3.1]). We take a rational number $0 < \varepsilon \ll 1$ such that $(W, \varepsilon N)$ is klt. Since K_W is torsion by the abundance theorem ([98, Theorem 1.2]) and N is negative definite, it follows that $\kappa(K_W + \varepsilon N) = \kappa(X, N) = 0$. We run a $(K_W + \varepsilon N)$-MMP to obtain a birational contraction $\varphi\colon W \longrightarrow W'$ to a $(K_W + \varepsilon N)$-minimal model W'. Then $K_{W'} = \varphi_* K_W \equiv 0$, and in particular, W' is klt Calabi-Yau. Moreover, $\varphi_* \varepsilon N \equiv K_{W'} + \varphi_* \varepsilon N \equiv 0$, and hence $\varphi_* D_W \equiv \varphi_* P$ is nef and big. By Lemma 3.2.1, we can replace W with W' and assume that D_W is nef and big.

We next reduce to the case where W is canonical Calabi-Yau. Let us assume that W is a strictly klt Calabi-Yau surface. By Lemma 3.1.11, the positive integer n is the minimum integer such that $nK_W = 0$. Then we can take a cyclic cover $\tau\colon \widetilde{W} \longrightarrow W$ associated to a non-zero global section of $nK_W = 0$. Since n is not divisible by p, it follows that τ is étale in codimension one, and hence we obtain an injective morphism

$$H^0(W, (\Omega_X^{[1]} \otimes \mathcal{O}_W(-D_W))^{**}) \hookrightarrow H^0(\widetilde{W}, (\Omega_{\widetilde{W}}^{[1]} \otimes \mathcal{O}_{\widetilde{W}}(-\tau^* D_W))^{**})$$

and $\tau^* D_W$ is nef and big. By replacing W with $\widetilde{W}$, we may assume that W has only canonical singularities.

Now, we show the vanishing (b). Let $\pi\colon Y \longrightarrow W$ be the minimal resolution and $E := \mathrm{Exc}(\pi)$. By Proposition 3.2.6, it suffices to show that

$$H^0(Y, \Omega_Y(\log\, E) \otimes \mathcal{O}_Y(-\lceil \pi^* D_W \rceil)) = 0.$$

Since $p > n \geqslant 19$ by Remark 3.1.10, the pair (Y, E) lifts to $W(k)$ by Proposition 3.1.2. Thus we conclude the desired vanishing by Theorem 2.4.6.

Step 3. Finally, we assume that $\kappa(X, K_X + B) = 1$ and $p > 3$. We prove the vanishing (a) directly. In this case, by replacing (X, B) with its $(K_X + B)$-minimal model, we may assume that $K_X + B$ is semiample and $\kappa(X, K_X + B) = 1$. Then there exists a projective morphism $f: X \longrightarrow Z$ such that $\dim Z = 1$, $f_*\mathcal{O}_X = \mathcal{O}_Z$, and $K_X + B$ is numerically trivial over Z. Now, by Lemma 3.2.9, we obtain the assertion.

We will check the sharpness of the explicit bounds on p_0 in Example 3.4.1. $\square$

Let us recall that the definition of a *globally sharply F-split pair*, which is a positive characteristic analog of a log Calabi-Yau pair in characteristic zero.

Definition 3.2.10 ([93, Definition 3.1])**.** Let (X, B) be a pair an algebraically closed field of characteristic $p > 0$. We say that (X, B) is *globally sharply F-split* if there exists a positive integer $e \in \mathbb{Z}_{>0}$ such that the composite map

$$\mathcal{O}_X \longrightarrow F_*^e\mathcal{O}_X \hookrightarrow F_*^e\mathcal{O}_X(\lceil (p^e - 1)B \rceil)$$

of the e-times iterated Frobenius morphism $\mathcal{O}_X \longrightarrow F_*^e\mathcal{O}_X$ and the natural inclusion $F_*^e\mathcal{O}_X \hookrightarrow F_*^e\mathcal{O}_X(\lceil (p^e - 1)B \rceil)$ splits as an $\mathcal{O}_X$-module homomorphism.

By a similar argument to Theorem 1.1.2, we can show the Bogomolov-Sommese vanishing theorem for a globally sharply F-split surface pair.

Proposition 3.2.11. *Let (X, B) be a globally sharply F-split surface pair over an algebraically closed field of characteristic $p > 5$. Then*

$$H^0(X, (\Omega_X^{[i]}(\log \lfloor B \rfloor) \otimes \mathcal{O}_X(-D))^{**}) = 0$$

for every $\mathbb{Z}$-divisor D on X satisfying $\kappa(X, D) > i$.

Proof. By [93, Theorem 4.4 (ii) and Theorem 4.3 (ii)], it follows that (X, B) is lc and $-(K_X + B)$ is effective. If $\kappa(X, K_X + \lfloor B \rfloor) = -\infty$, then the assertion follows from Theorem 1.1.2. Thus we may assume that $K_X + \lfloor B \rfloor \equiv 0$. First, we assume that $(X, \lfloor B \rfloor)$ is not klt. By Proposition 3.2.6, we can replace $(X, \lfloor B \rfloor)$ with its dlt blow-up. In this case, the boundary of the dlt pair is non-zero since $(X, \lfloor B \rfloor)$ is not klt. Then the assertion follows from Step 2-1 of the proof of Theorem 1.1.2.

Now, we assume that X is klt Calabi-Yau and $B = 0$. As in Step 2-2 of the proof of Theorem 1.1.2, by considering the Zariski decomposition, we can assume that D is nef and big. Note that the globally F-split property is preserved under a birational contraction ([17, 1.1.9 Lemma]). Next, a splitting morphism $F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ give a non-zero section of $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X((1-p)K_X))$, and together with $K_X \equiv 0$, we obtain $(1 - p)K_X = 0$. In particular, the minimum positive integer n such that $nK_X = 0$ is not divisible by p. Let us recall the globally F-split property is preserved under a finite cover which is étale in codimension one ([89, Lemma 11.1.]). Thus, by taking a cyclic cover associated to a non-zero global section of nK_X, we may assume that X is a canonical Calabi-Yau surface such that $K_X = 0$. If X is an abelian surface, then the same argument as Step 2-2 of the proof of Theorem 1.1.2 works. Thus we may assume that the minimal resolution Y of X is a K3 surface. Now, by [17, 1.3.13 Lemma] and [101, 5.1 Theorem], the K3 surface Y is not supersingular, and an argument of Step 2-2 of the proof of Theorem 1.1.2 and Remark 3.1.3 show the desired vanishing. $\square$

Remark 3.2.12. It is still open whether Proposition 3.2.6 holds for F-pure surface singularities in characteristic $p \leqslant 5$. This is the main reason why we need the assumption that $p > 5$ in Proposition 3.2.11.

3.3 Liftability of surface pairs

In this section, we prove Theorems 1.1.3 and 1.1.4. We also discuss deformations of an lc projective surface whose canonical divisor has negative Iitaka dimension (Proposition 3.3.6). First, we focus on the vanishing of the second cohomology of the logarithmic tangent sheaf.

Definition 3.3.1. Let X be a normal projective variety. We say X is *Q-abelian* if there exists a finite surjective morphism $\tau \colon \widetilde{X} \longrightarrow X$ such that $\widetilde{X}$ is an abelian variety and τ is étale in codimension one.

Proposition 3.3.2. *Let (X, B) be an lc projective surface pair over an algebraically closed field of characteristic $p > 0$ such that B is reduced. When $\kappa(X, K_X + B) = 0$, let (X', B') be the $(K_X + B)$-minimal model of (X, B), where B' is the pushforward of B. Suppose that one of the followings holds.*

(1) $\kappa(X, K_X + B) = -\infty$ *and* $p > 5$.

(2) $\kappa(X, K_X + B) = 0$ *and one of the followings holds.*

  (i) $B' \neq 0$ *and* $p > 5$,

  (ii) $B' = 0$, *X' is klt, the Gorenstein index of X' is not divisible by p, X' is not Q-abelian, and* $p > 19$.

Then $H^2(X, T_X(-\log B)) = 0$.

Remark 3.3.3. All the assumptions on p are sharp (see Examples 3.4.1, 3.4.2, and 3.4.3).

Proof. First, we assume that the condition (1) holds. We can reduce the desired vanishing to an output of a $(K_X + B)$-MMP by Remark 3.2.2, and hence assume that X admits a $(K_X + B)$-Mori fiber space structure $f \colon X \longrightarrow Z$. If $\dim Z = 1$, then the assertion follows from Lemma 3.2.9 since $-K_X$ is f-ample. We next assume that $\dim Z = 0$. Note that $-K_X$ is $\mathbb{Q}$-Cartier by [98, Theorem 5.4]. Then it follows from $\rho(X) = 1$ that $-K_X$ is an ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor, and the assertion follows from Theorem 1.1.2.

Next, we assume that the condition (2)-(i) holds. It suffices to show that $H^2(X', T_{X'}(-\log B')) = 0$. Since $K_{X'} + B' \equiv 0$ and $B' \neq 0$, it follows that $K_{X'}$ is not pseudo-effective. Then we can run a $K_{X'}$-MMP to obtain a birational contraction $\varphi\colon X' \longrightarrow \overline{X}$ to a $K_{\overline{X}}$-Mori fiber space $f\colon \overline{X} \longrightarrow Z$. It suffices to show that $H^2(\overline{X}, T_{\overline{X}}(-\log \overline{B})) = 0$. Since $K_{X'} + B' \equiv 0$, the negativity lemma shows that $K_{X'} + B' = \varphi^*(K_{\overline{X}} + \overline{B})$ and hence $(\overline{X}, \overline{B})$ is log Calabi-Yau, where $\overline{B} := \varphi_* B'$. If $\dim Z = 1$, then the assertion follows from Lemma 3.2.9 since $-K_{\overline{X}}$ is f-ample. If $\dim Z = 0$, then the assertion follows from Step 2-1 of the proof of Theorem 1.1.2 since $-K_{\overline{X}}$ is ample $\mathbb{Q}$-Cartier by [98, Theorem 5.4].

Finally, we assume that the condition (2)-(ii) holds. It suffices to show that $H^2(X', T_{X'}) = 0$. We first assume that X' is strictly klt Calabi-Yau. Let n be the minimum positive integer such that $nK_{X'} = 0$. Then n is equal to the Gorenstein index by Lemma 3.1.11, and hence n is not divisible by p by assumption. Let $\tau\colon \widetilde{X} \longrightarrow X'$ be a cyclic cover associated to a non-zero global section of $nK_{X'} = 0$. Since τ is étale in codimension one, we have

$$\begin{aligned} H^2(X', T_{X'}) \cong H^0(X', (\Omega^{[1]}_{X'} \otimes \mathcal{O}_{X'}(K_{X'}))^{**}) \hookrightarrow & H^0(\widetilde{X}, (\Omega^{[1]}_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}))^{**}) \\ = & H^2(\widetilde{X}, T_{\widetilde{X}}), \end{aligned}$$

Thus we may assume that X' is canonical Calabi-Yau. By the assumption that X' is not $\mathbb{Q}$-abelian, the minimal resolution of X' is a K3 surface or an Enriques surface. In these cases, we have already shown that $H^2(X', T_{X'}) = 0$ in the proof of Proposition 3.1.3. □

Now, we prove Theorems 1.1.3 and 1.1.4.

Proof of Theorem 1.1.3. Set $B_Y := \pi_*^{-1}B + \mathrm{Exc}(\pi)$. Suppose that the condition (1) holds and $p > 5$. Then $\kappa(Y, K_Y + B_Y) = -\infty$ by Lemma 3.2.1 (2), and hence (Y, B_Y) lifts to $W(k)$ by Proposition 3.3.2 (1) and Theorem 2.4.5.

Next, we assume that the condition (2) holds and $p > 5$. By Lemma 3.2.5, we can decompose $\pi\colon Y \longrightarrow X$ into a birational morphism $Y \longrightarrow W$ and a dlt blow-up $h\colon W \longrightarrow X$. Then there exists an effective $\mathbb{Q}$-divisor F such that $K_W + B_W + F \equiv h^*(K_X + B) \equiv 0$, where $B_W := h_*^{-1}B + \mathrm{Exc}(h)$. By assumption, we have $B_W \neq 0$ and hence $H^2(Y, T_Y(-\log B_Y)) \hookrightarrow H^2(W, T_W(-\log B_W)) = 0$ by Proposition 3.3.2 (1) and (2)-(i). Moreover, since $-K_X \equiv B$ is strictly effective, it follows that $H^2(Y, \mathcal{O}_Y) = 0$. Now, we conclude that (Y, B_Y) lifts to $W(k)$ by Theorem 2.4.5.

Finally, we assume that the condition (3) holds. In this case, $\kappa(Y, K_Y + B_Y) \leqslant 0$ by Lemma 3.2.1 (2). If $\kappa(Y, K_Y + B_Y) = -\infty$ and $p > 5$, then (Y, B_Y) lifts to $W(k)$ by (1). Thus we can assume that $\kappa(Y, K_Y + B_Y) = 0$. By Propositions 3.1.2 and 3.1.13, we can take a positive integer $p_0 > 19$ with the following property; for every klt Calabi-Yau surface Z over an algebraically closed field of characteristic bigger than p_0 and every log resolution $f\colon \widetilde{Z} \longrightarrow Z$, the pair $(\widetilde{Z}, \mathrm{Exc}(f))$ lifts to $W(k)$. We fix such a p_0 and assume that $p > p_0$. We run a $(K_Y + B_Y)$-MMP to obtain a birational contraction $\varphi\colon Y \longrightarrow Y'$ to the $(K_Y + B_Y)$-minimal model $(Y', B_{Y'} := \varphi_* B_Y)$, which is dlt and log Calabi-Yau. If $B_{Y'} \neq 0$, then (Y, B_Y) lifts to $W(k)$ by (2). If $B_{Y'} = 0$, then we obtain the desired liftability by the assumption of p_0.

We will check the sharpness of the explicit bound on p_0 in Examples 3.4.1 and 3.4.2. $\square$

Proof of Theorem 1.1.4. We take a positive integer p_0 as in Theorem 1.1.3. If $\kappa(X, K_X) = -\infty$ and $p > 5$ (resp. $\kappa(X, K_X) = 0$ and $p > p_0$), then the assertion follows from Theorem 1.1.3 and Lemma 2.4.9. Next, we assume that $\kappa(X, K_X) = 1$, X is lc, and $p > 3$. In this case, we have $H^0(X, (\Omega_X^{[1]} \otimes \mathcal{O}_X(-p^e D))^{**}) = 0$ for all $e \in \mathbb{Z}_{>0}$ by Theorem 1.1.2. Then, by the proof of [59, Lemma 2.5], we have the injective morphism $H^1(X, \mathcal{O}_X(-D)) \hookrightarrow H^1(X, \mathcal{O}_X(-p^e D))$ arising from the e-th iterated Frobenius morphism. Let $\pi\colon Y \longrightarrow X$ be a log resolution. By the proof of Lemma 2.4.9, it suffices to show that $H^1(Y, \mathcal{O}_Y(-\lceil p^e \pi^* D\rceil)) = 0$ for $e \gg 0$. We take $m, n \in \mathbb{Z}_{>0}$ such that $p^m(p^n - 1)\pi^* D$ is Cartier. Then we obtain

$$H^1(Y, \mathcal{O}_Y(-\lceil p^{m+nl}\pi^* D\rceil) = H^1(Y, \mathcal{O}_Y(-\lceil p^m \pi^* D\rceil + (\sum_{i=0}^{l-1} p^{ni}) p^m (p^n - 1)\pi^* D))$$

and the last cohomology vanishes for $l \gg 0$ by [99, Theorem 2.6].

We will check the sharpness of the explicit bounds on p_0 in Example 3.4.1. $\square$

Finally, we apply Proposition 3.3.2 to show the vanishing of local-to-global obstructions (see [78, Definition 4.11] for the definition). In particular, we prove Proposition 3.3.6 (3), which is a positive characteristic analog of [40, Proposition 3.1].

Definition 3.3.4. Let X be a normal projective variety. We say X admit a $\mathbb{Q}$-*Gorenstein smoothing* if there exists a flat projective morphism $\mathcal{X} \longrightarrow T$ from a normal $\mathbb{Q}$-Gorenstein scheme to a smooth curve T with a reference point $o \in T$ such that the fiber over o is isomorphic to X and $\mathcal{X} \longrightarrow T$ is smooth over $T - o$.

Definition 3.3.5. Let X be a normal variety over an algebraically closed field of positive characteristic. We say X is *F-pure* if the local ring $\mathcal{O}_{X,x}$ is F-split (i.e. the Frobenius morphism $\mathcal{O}_{X,x} \longrightarrow F_*\mathcal{O}_{X,x}$ splits as an $\mathcal{O}_{X,x}$-module homomorphism) for every closed point $x \in X$.

Proposition 3.3.6. *Let X be an lc projective surface over an algebraically closed field k of characteristic $p > 5$ with $\kappa(X, K_X) = -\infty$. Then X has no local-to-global obstructions. In particular, the followings hold.*

(1) *If X is F-pure, then X lifts to $W_2(k)$.*

(2) *If X is locally complete intersection (l.c.i. for short), then X lifts to $W(k)$.*

(3) *If every singular point of X is l.c.i or a T-singularity (see [78, Definition 3.4] for the definition), then X admits a $\mathbb{Q}$-Gorenstein smoothing.*

Proof. By taking $B = 0$ in Proposition 3.3.2 (1), we obtain $H^2(X, T_X) = 0$. Then we conclude that X has no local-to-global obstructions by [78, Theorem 4.13].

First, we show (1). Suppose that X is F-pure. Then X lifts to $W_2(k)$ locally by [72, Corollary 8]. Since X has no local-to-global obstructions, it follows that X lifts to $W_2(k)$. Next, we show (2). Suppose that X is locally complete intersection. Then X lifts to $W(k)$ locally by [45, Theorem 9.2]. Thus X formally lifts to $W(k)$ and this is algebraizable by $H^2(X, \mathcal{O}_X) = 0$. Finally, we show (3). Suppose that every singular point of X is l.c.i or a T-singularity. Then each singularity admit a smoothing by [78, Lemma 5.1 and Theorem 3.14]. Hence it follows from the proof of [78, Theorem 5.3] that X admits a smoothing. □

3.4 Sharpness of Theorems 1.1.2, 1.1.3, and 1.1.4

In this section, we observe the failure of Theorems 1.1.2, 1.1.3, and 1.1.4 in low characteristic or for surface pairs whose log canonical divisor is big. First, let us focus on the characteristic.

Example 3.4.1. By Theorem 1.3.6 (3), [10, Theorem 1.1], and [7, Proposition 5.1], we can take a klt del Pezzo surface X in each characteristic $p \in \{2, 3, 5\}$ with more than four singularities and an ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor D on X such that $H^1(X, \mathcal{O}_X(K_X + D)) \neq 0$. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution with $E :=$ $\mathrm{Exc}(\pi)$. Then $-K_Y$ is big and $\kappa(Y, K_Y + E) = -\infty$.

Firstly, (Y, E) does not lift to any Noetherian local domain with fractional field of characteristic zero because there are no klt del Pezzo surfaces with more than four singularities in characteristic zero by [9, Theorem 1.1]. In addition, (Y, E) dose not lift to $W_2(k)$ by Lemma 2.4.9, and it follows from Theorem 2.4.5 that $0 \neq H^2(Y, T_Y(-\log E)) \hookrightarrow H^2(X, T_X)$. Therefore, the explicit bound $p_0 = 5$ in Theorems 1.1.2, 1.1.3 (1), and 1.1.4 (1) is optimal. These examples also show that the sharpness of the assumption $p > 5$ in Proposition 3.3.2 (1).

By [92, Section 3,1], we can take a smooth projective surface X in each characteristic $p \in \{2, 3\}$ with $\kappa(X, K_X) = 1$ and an ample Cartier divisor D on X such that $H^1(X, \mathcal{O}_X(K_X + D)) \neq 0$. Then [59, Lemma 2.5] shows that there exists $n > 0$ such that $H^0(X, \Omega_X \otimes \mathcal{O}_X(-p^n D)) \neq 0$. Therefore, the explicit bound $p_0 = 3$ in Theorems 1.1.2, 1.1.4 (2) and the assumption that $p > 3$ in Lemma 3.2.9 are optimal.

Example 3.4.2. We show that there exists a klt del Pezzo surface X in each characteristic $p \in \{2, 3, 5\}$ and a non-zero reduced divisor B on X such that $K_X + B \equiv 0$, but $(Y, f_*^{-1}B + \mathrm{Exc}(f))$ does not lift to any Noetherian local domain with fractional field of characteristic zero for some log resolution $f\colon Y \longrightarrow X$ of (X, B).

We first assume $p = 5$. We take a del Pezzo surface X with two A_4-singularities and a cuspidal rational curve B in the smooth locus of X as in [67, Example 7.6]. Then we have $K_X + B \equiv 0$ by the adjunction formula. We take a three-times blow-up $f\colon Y \longrightarrow X$ at the cusp of B. Then there exists a contraction $\pi\colon Y \longrightarrow Z$ to a klt del Pezzo surface Z with five singularities and $\mathrm{Exc}(\pi) \subset f_*^{-1}B + \mathrm{Exc}(f)$ (see [67, Example 7.6] for the detail). Then $(Y, \mathrm{Exc}(\pi))$ does not lift to any Noetherian local domain with fractional field of characteristic zero by [9, Theorem 1.1] and neither does $(Y, f_*^{-1}B + \mathrm{Exc}(f))$. When $p = 3$, we can take $X = \mathbb{P}^2_k$ and a curve B as in [67, Example 7.5] to show the assertion. When $p = 2$, we can take a del Pezzo surface X as any one of Theorem 1.3.6 (3) and B is a general member of the anti-canonical linear system, which is integral.

Therefore, the explicit bound on p_0 in Theorem 1.1.3 (2) and the assumption $p > 5$ in Proposition 3.3.2 (2)-(i) are optimal.

Example 3.4.3 (cf. [11, Remark 3.4])**.** By [95, Corollary 1.2], there exists a canonical Calabi-Yau surface X in each characteristic $p \leqslant 19$ such that Y is a supersingular K3 surface and $E := \mathrm{Exc}(\pi)$ consists of 21 (-2)-curves, where $\pi\colon Y \longrightarrow X$ is the minimal resolution. Then (Y, E) does not lift to any Noetherian local domain R with fractional field K of characteristic zero. For the sake of contradiction, we assume that there exists a lifting $(\mathcal{Y}, \mathcal{E})$ of (Y, E) to such an R. Let $Y_{\overline{K}}$ (resp. $E_{\overline{K}}$) be the geometric generic fiber of $\mathcal{Y} \longrightarrow R$ (resp. $\mathcal{E} \longrightarrow R$).

Let us show that $Y_{\overline{K}}$ is a K3 surface. Since $H^1(Y, \mathcal{O}_Y) = 0$, a lifting of each invertible sheaf is unique by [27, Corollary 8.5.5]. Then $\omega_{\mathcal{Y}}|_Y = \omega_Y = \mathcal{O}_Y = \mathcal{O}_{\mathcal{Y}}|_Y$ shows that $\omega_{\mathcal{Y}} = \mathcal{O}_{\mathcal{Y}}$. Together with $\mathcal{X}(Y_{\overline{K}}, \mathcal{O}_{Y_{\overline{K}}}) = \mathcal{X}(Y, \mathcal{O}_Y) = 2$, we conclude that $Y_{\overline{K}}$ is a K3 surface. Since $Y_{\overline{K}}$ contains 21 (-2)-curves $E_{\overline{K}}$ which is negative definite, we obtain $\rho(Y_{\overline{K}}) \geqslant 22$, a contradiction with the fact that the Picard rank of a K3 surface in characteristic zero is at most 20 (see [49, Chapter 17, 1.1] for example). Finally, by the proof Proposition 3.1.2, we obtain $0 \neq H^2(Y, T_Y(-\log E)) \hookrightarrow H^2(X, T_X)$.

Therefore, p_0 in Theorem 1.1.3 (3) should be at least 19. Moreover, the assumption that $p > 19$ in Propositions 3.1.2 and 3.3.2 (2)-(ii) is sharp.

Finally, we close the paper by discussing the assumptions of Iitaka dimensions of (log) canonical divisors in Theorems 1.1.2, 1.1.3, and 1.1.4. By counterexamples ([92]) of the Kodaira vanishing theorem on smooth projective surfaces with big canonical divisor, we can see that Theorems 1.1.2, 1.1.3, and 1.1.4 do not hold for a surface with big canonical divisor in any characteristic. In the next example, we will see that Langer's surface pair [72] shows that Theorems 1.1.2 and 1.1.3 do not hold on a surface pair whose log canonical divisor is big even when the surface itself is rational.

Example 3.4.4. We first recall the construction of Langer's surface pair [72, Section 8]. Let $h\colon X \longrightarrow \mathbb{P}^2_k$ be the blow-up all the $\mathbb{F}_p$-rational points and $L_1, \ldots, L_{p^2+p+1}$ strict transforms of all the $\mathbb{F}_p$-lines. Then X is a smooth rational surface and $L_1, \ldots, L_{p^2+p+1}$ are pairwise disjoint smooth rational curves.

There exists a nef and big $\mathbb{Q}$-divisor D such that $H^1(X, \mathcal{O}_X(K_X + \lceil D \rceil)) \neq 0$ and $\mathrm{Supp}(\{D\}) = \sum_{i=1}^{p^2+p+1} L_i$ by [21, Theorem 3.1]. Thus $(X, \sum_{i=1}^{p^2+p+1} L_i)$ dose not lift to $W_2(k)$ by Theorem 2.4.6 and $H^2(X, T_X(-\log \sum_{i=1}^{p^2+p+1} L_i)) \neq 0$ by Theorem 2.4.5. Finally, there exists a big divisor M such that $\mathcal{O}_X(M)$ is contained in $\Omega_X(\log \sum_{i=1}^{p^2+p+1} L_i)$ by [75, Proposition 11.1].

Now, we check that $K_X + \sum_{i=1}^{p^2+p+1} L_i$ is big except when $p = 2$. Since $L_i^2 = -p$ for each i and $L_1, \ldots, L_{p^2+p+1}$ are pairwise disjoint, we can take the contraction $f\colon X \longrightarrow Z$ of $\sum_{i=1}^{p^2+p+1} L_i$. By the proof of [21, Lemma 2.4 (i)], we have $K_X + (1 - \frac{2}{p})(\sum_{i=1}^{p^2+p+1} L_i) = f^*K_Z$ and hence $K_X + \sum_{i=1}^{p^2+p+1} L_i = \lceil f^*K_Z \rceil$. If $p \neq 2$, then K_Z is ample by [21, Lemma 2.4 (iv)] and hence $K_X + \sum_{i=1}^{p^2+p+1} L_i$ is big. Note that if $p = 2$, then $\kappa(X, K_X + \sum_{i=1}^{p^2+p+1} L_i) = -\infty$ since $f_*(K_X + \sum_{i=1}^{p^2+p+1} L_i) = K_Z$ is anti-ample.

Therefore, Theorems 1.1.2, 1.1.3 and Proposition 3.3.2 do not hold on a surface pair whose log canonical divisor is big even when the surface itself is rational.

Remark 3.4.5. For a singular surface, it is often more useful to consider the liftability of a log resolution than that of itself (see [7], [23], and Section 5 for example). In Example 3.4.4, we constructed the pathological example from the log resolution of the pair consisting of $\mathbb{P}_k^2$ and all the $\mathbb{F}_p$-lines $\sum_{i=1}^{p^2+p+1} \overline{L}_i$. However, the pair $(\mathbb{P}_k^2, \sum_{i=1}^{p^2+p+1} \overline{L}_i)$ clearly lifts to $W(k)$. For this reason, when we discuss lifting of a non-log smooth pair, it is more suitable to consider the liftability of a log resolution of the pair to capture pathologies in positive characteristic.

Remark 3.4.6. Example 3.4.4 gives a slightly generalization of [21, Corollary 3.3]. Indeed, we can drop the assumption $p \geqslant 3$ and replace $\sum_{i=1}^{q^2+q+1} E_i + \sum_{i=1}^{q^2+q+1} L_i'$ with $\sum_{i=1}^{q^2+q+1} L_i'$ in [21, Corollary 3.3]. On the other hand, this fact also follows from [74, Proposition 4.1] and [75, Proposition 11.1].

Chapter 4

Bogomolov-Sommese type vanishing for globally F-regular threefolds

In this section, we prove Theorems 1.2.1 and 1.2.2.

4.1 F-split and globally F-regular varieties

In this section, we gather the results about F-split and globally F-regular varieties.

Definition 4.1.1 ([85], [97])**.** Let X be a normal variety.

(1) We say that X is *(globally) F-split* if the Frobenius map $\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-module homomorphism. We call $\sigma \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X((1-p)K_X))$ *a splitting section* if σ induces a splitting of $\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$. We often call the divisor $\Sigma \in |(1-p)K_X|$ corresponding to σ a splitting section.

(2) We say that X is *globally F-regular* if for every effective Weil divisor D on X, there exists an integer $e \geqslant 1$ such that the composite map

$$\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$$

of the e-times iterated Frobenius map $\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X$ and the natural inclusion $F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$ splits as an $\mathcal{O}_X$-module homomorphism.

Remark 4.1.2. (1) Let X be a globally F-regular variety and D an effective Weil divisor on X. Then the map $\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$ splits as an $\mathcal{O}_X$-module homomorphism for all sufficiently large e by [93, Proposition 3.8].

(2) Let $f\colon X \dashrightarrow Y$ be a small birational map or a projective surjective morphism satisfying $f_*\mathcal{O}_X = \mathcal{O}_Y$ of normal varieties. If X is F-split (resp. globally F-regular), then so is Y by [32, Lemma 1.5]. In particular, if we start the MMP

from an F-split (resp. globally F-regular) variety, then an output of the MMP is also F-split (resp. globally F-regular).

(3) Globally F-regular varieties are Cohen-Macaulay by [97, Proposition 4.1].

(4) Let $\pi\colon Y \longrightarrow X$ be a birational projective morphism between normal $\mathbb{Q}$-Gorenstein varieties such that $f^*K_X - K_Y$ is $\mathbb{Q}$-effective. If X is F-split (resp. globally F-regular), then so is Y by [33, Lemma 3.3]). In particular, the minimal resolution of a Du Val del Pezzo surface preserves the F-splitting property.

(5) Let C be an elliptic curve. Then C is F-split if and only if C is ordinary (see [17, 1.3.9 Remark (ii)]).

(6) Globally F regular projective varieties are of Fano type (see [93, Theorem 1.1]).

Theorem 4.1.3 ([32, Theorem 2.1])**.** *Let $f\colon X \longrightarrow Y$ be a projective surjective morphism of normal varieties satisfying $f_*\mathcal{O}_X = \mathcal{O}_Y$. If X is globally F-regular, then a general fiber of f is normal and globally F-regular.*

Theorem 4.1.4 (Proof of [32, Theorem 4.1])**.** *Let $f\colon X \longrightarrow Y$ be a projective surjective morphism from a terminal globally F-regular threefold to a normal variety over an algebraically closed field of characteristic $p > 0$ satisfying $f_*\mathcal{O}_X = \mathcal{O}_Y$. Suppose that $-K_X$ is f-ample and one of the following conditions holds.*

(1) $\dim Y = 2$.

(2) $p > 7$ *and* $\dim Y = 1$.

Then X is separably rationally connected.

We refer to [64, IV 3.2 Definition] for the definition of separably rationally connected varieties. Since the separably rationally connected property is preserved under birational maps, Theorem 4.1.4 states that if we start K_X-MMP from a smooth globally F-regular threefold and the MMP ends up with a Mori fiber space over a surface, or a curve and $p > 7$, then X is separably rationally connected. On the other hand, very little is known when the MMP ends up with a Fano variety. We refer to [32] for more details.

4.2 Logarithmic Cartier operators

In this section, we recall the logarithmic Cartier operator. Let X be a smooth variety and B a reduced divisor on X with snc support. The Frobenius push-forward of the logarithmic de Rham complex

$$F_*\Omega^\bullet_X(\log\ B)\colon F_*\mathcal{O}_X \xrightarrow{F_*d} F_*\Omega^1_X(\log\ B) \xrightarrow{F_*d} \cdots$$

is a complex of $\mathcal{O}_X$-module homomorphisms. For all $i \geqslant 0$, we define locally free $\mathcal{O}_X$-modules as follows.

$$\begin{aligned} B^i_X(\log B) &:= \mathrm{Im}(F_*d\colon F_*\Omega^{i-1}_X(\log B) \longrightarrow F_*\Omega^i_X(\log B)),\\ Z^i_X(\log B) &:= \mathrm{Ker}(F_*d\colon F_*\Omega^i_X(\log B) \longrightarrow F_*\Omega^{i+1}_X(\log B)). \end{aligned}$$

By definition, we have the following exact sequence

$$0 \longrightarrow Z^i_X(\log B) \longrightarrow F_*\Omega^i_X(\log B) \longrightarrow B^{i+1}_X(\log B) \longrightarrow 0$$

for $i \geqslant 0$.

In particular, by taking $i = 0$ in the above exact sequence, we have

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \xrightarrow{F_*d} B^1_X \longrightarrow 0.$$

We note that $B^1_X(\log B) = B^1_X$. We remark that the F-splitting (Definition 4.1.1 (1)) is nothing but to the splitting of this exact sequence. Moreover, we have the exact sequence arising from the logarithmic Cartier isomorphism

$$0 \longrightarrow B^i_X(\log B) \longrightarrow Z^i_X(\log B) \xrightarrow{C} \Omega^i_X(\log B) \longrightarrow 0.$$

We refer to [56, Theorem 7.2] for more details.

4.3 Bogomolov-Sommese type vanishing for several varieties

In this section, we prove a Bogomolov-Sommese type vanishing for varieties with special properties.

The following assertion states about a Bogomolov-Sommese type vanishing on separably uniruled varieties. We refer to [64, IV 1.1 Definition] for the definition of separably uniruled varieties.

Proposition 4.3.1 ([63, Lemma 7])**.** *Let X be a smooth projective separably uniruled variety and D a big Cartier divisor on X. Then $H^0(X, \Omega^i_X \otimes \mathcal{O}_X(-D)) = 0$ for all $i \geqslant 0$.*

Remark 4.3.2. Let (X, B) be a log smooth projective surface such that B is reduced. If $\kappa(X, K_X) = -\infty$ and $B = 0$, then the Bogomolov-Sommese vanishing holds by Proposition 4.3.1. On the other hand, as we have seen before in Example 3.4.4, this is not true if $B \neq 0$.

We will see that if X is F-split, then $\Omega_X(\log B)$ dose not contain a nef and big invertible sheaf in Proposition 4.4.6.

Next, we show a Bogomolov-Sommese type vanishing on separably rationally connected varieties. Let X be a smooth projective variety of $\dim X = n$. We recall that a rational curve $\varphi\colon \mathbb{P}^1_k \longrightarrow X$ is called *very free* if $\varphi^*\Omega_X = \mathcal{O}_{\mathbb{P}^1_k}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(-a_n)$ for $a_1, \cdots, a_n \in \mathbb{Z}_{>0}$.

Theorem 4.3.3. *Let X be a smooth projective variety. Then X is separably rationally connected if and only if there is a very free rational curve through a general point of X.*

Proof. We refer to [64, IV. Theorem 3.7] for the proof. □

The proof of the following proposition is essentially same as that of Proposition 4.3.1, but we include the proof for the convenience of the reader.

Proposition 4.3.4. *Let X be a smooth projective separably rationally connected variety and D a Cartier divisor on X. If $\kappa(X, D) \geqslant 0$, then $H^0(X, \Omega^i_X \otimes \mathcal{O}_X(-D)) = 0$ for all $i > 0$.*

Proof. We take a Cartier divisor D satisfying $\kappa(X, D) \geqslant 0$. Conversely, we assume that there exists a nonzero section $0 \neq s \in H^0(X, \Omega^i_X \otimes \mathcal{O}_X(-D))$ for some $i > 0$. We fix $m \in \mathbb{Z}_{>0}$ such that mD is linearly equivalent to an effective divisor. We take a very free rational curve $\varphi \colon \mathbb{P}^1_k \longrightarrow X$ through a general point of X. Then $\operatorname{Im} \varphi$ is not contained $\operatorname{Supp}(mD)$ and hence we have $\varphi^*\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^1_k}(b)$ for some $b \geqslant 0$. By the definition of a very free rational curve, it follows that $\varphi^*(\Omega^i_X \otimes \mathcal{O}_X(-D)) = \mathcal{O}_{\mathbb{P}^1_k}(-b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(-b_n)$ for some $b_1, \cdots, b_n > 0$.

On the other hand, since $\varphi \colon \mathbb{P}^1_k \longrightarrow X$ passes through a general point of X, it follows that $\operatorname{Im} \varphi$ is not contained in the zero locus of s and hence $s|_{\operatorname{Im}\varphi} \neq 0$.

Now, we obtain

$$\begin{aligned} 0 \neq s|_{\operatorname{Im}\varphi} \in \; & H^0(\operatorname{Im}\varphi, (\Omega^i_X \otimes \mathcal{O}_X(-D))|_{\operatorname{Im}\varphi}) \\ \hookrightarrow \; & H^0(\operatorname{Im}\varphi, ((\Omega^i_X \otimes \mathcal{O}_X(-D)) \otimes \varphi_*\mathcal{O}_{\mathbb{P}^1_k}) \\ = \; & H^0(\mathbb{P}^1_k, \varphi^*((\Omega^i_X \otimes \mathcal{O}_X(-D)) \\ = \; & H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(-b_n)) \\ = \; & 0, \end{aligned}$$

a contradiction. □

Remark 4.3.5. Let X be a smooth globally F-regular surface. Then X is rational by [32, Proposition 3.5]. Therefore if $\mathcal{O}_X(D) \subset \Omega^i_X$ is an invertible subsheaf for some $i > 0$, then $\kappa(X, D) = -\infty$ by Proposition 4.3.4.

Lemma 4.3.6. *Let $f \colon X \longrightarrow C$ be a minimal ruled surface. Then we have*

$$H^0(X, \Omega_X \otimes \mathcal{O}_X(-D)) = 0$$

for every Cartier divisor D satisfying $\kappa(X, D) \geqslant \min\{g := \dim_k H^1(X, \mathcal{O}_X), 2\}$.

Proof. If $g = 0$ or $g > 1$, then the assertion follows from Proposition 4.3.4 and Proposition 4.3.1, respectively. We assume that $g = 1$. We take an injective homomorphism $s \colon \mathcal{O}_X(D) \hookrightarrow \Omega_X$. Then we have the following commutative diagram

$$\begin{tikzcd} & & \mathcal{O}_X(D) \arrow[dl, dotted] \arrow[d, "s"] \arrow[dr, "t"] & & \\ 0 \arrow[r] & f^*\omega_C = \mathcal{O}_X \arrow[r] & \Omega^1_X \arrow[r] & \omega_X \arrow[r] & 0. \end{tikzcd}$$

Let F be a general fiber of f. By the generality of F, the restriction $t|_F \colon \mathcal{O}_F(D) \hookrightarrow \omega_X|_F$ is injective. Then we have $(D \cdot F) \leqslant (K_X \cdot F) = -2$ and hence $\kappa(X, D) = -\infty$ by the nefness of F. Now, we assume that t is zero. Then an injective homomorphism $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X$ is induced by the above diagram and hence $\kappa(X, D) = -\infty$ or $D = 0$. Therefore the assertion holds. $\square$

Next, we discuss the case where $\kappa(X, K_X) = 0$. The following proposition is an immediate consequence of [73, Corollary 3.3].

Proposition 4.3.7. *Let X be a smooth projective variety of* $\dim X = n$. *Suppose that $p \geqslant (n-1)(n-2)$, $K_X \equiv 0$, and X is not uniruled. If $\mathcal{O}_X(D) \subset \Omega^i_X$ is an invertible subsheaf for some $i \geqslant 0$, then $\kappa(X, D) \leqslant 0$.*

Proof. We may assume that k is an algebraically closed field. By [73, Corollary 3.3], it follows that Ω_X is strongly semistable with respect to any ample polarization H and so is Ω^i_X for each $i \geqslant 0$ by [91, Theorem 3.23]. We refer to [73] for the definition of the strongly semistability. We take an injective homomorphism $\mathcal{O}_X(D) \hookrightarrow \Omega^i_X$ for some $i \geqslant 0$. By the definition of the semistability, we have $(D \cdot H^{n-1}) \leqslant (-c_1(X) \cdot H^{n-1}) / \operatorname{rank} \Omega^i_X = 0$ and hence $\kappa(X, D) \leqslant 0$. Therefore we obtain the assertion. $\square$

Remark 4.3.8. A Calabi-Yau variety whose Artin-Mazur height is finite is not uniruled by [46, Theorem 1.3]. We note that the proof of [46, Theorem 1.3] works in any dimension.

Now, we show a Bogomolov-Sommese type vanishing for smooth projective F-split surfaces.

Theorem 4.3.9. *Let X be a smooth projective F-split surface. If $\mathcal{O}_X(D) \subset \Omega^i_X$ is an invertible subsheaf for some $i \geqslant 0$, then $\kappa(X, D) \leqslant 0$.*

Proof. When $i = 0$, the assertion is obvious. Since X is F-split, there exists a section $\sigma \in H^0(X, \mathcal{O}_X((1-p)K_X)) \cong \operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ which induces a splitting of the Frobenius map $\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$, and in particular the anti-canonical divisor $-K_X$ is effective. Then the assertion holds when $i = 2$. We assume that $i = 1$. By Lemma 3.2.1, we may assume that X is minimal.

- The case where $\kappa(X, K_X) = -\infty$. If $X \cong \mathbb{P}^2_k$, then the assertion follows from Proposition 4.3.4. Thus we assume that X has a ruled surface structure $f \colon X \longrightarrow C$. Since C is F-split by Remark 4.1.2 (2), it follows that $\dim H^1(C, \mathcal{O}_C) = 0$ or 1 and the assertion follows from Lemma 4.3.6.

- The case where $\kappa(X, K_X) = 0$. First, we assume that X is a K3 surface. Then the Artin-Mazur height of X is equal to one by [101, 5.1 Theorem] and hence we obtain the assertion by Proposition 4.3.7 and Remark 4.3.8.

  Next, we assume that X is an Enriques surface. We first assume that $p \neq 2$. Then there exists a finite étale morphism $f \colon Y \longrightarrow X$ from a K3 surface Y.

Since f is étale, it follows from [1, Lemma 2.5.5 (d)] that Y is F-split and in particular Y is not uniruled. The étaleness of f also shows that X is not uniruled and we obtain the assertion by Proposition 4.3.7. We next assume that $p = 2$. Since X is F-split, the Frobenius action on $H^1(X, \mathcal{O}_X)$ is bijective by [1, Lemma 2.5.5 (a)] and hence X is not supersingular. Moreover, since there exists a section $\sigma \in H^0(X, \mathcal{O}_X(-K_X))$ which induces a splitting of the Frobenius map, it follows that K_X is not torsion and hence X is not classical. Thus X is a singular Enriques surface and hence there exists a finite étale morphism $f\colon Y \longrightarrow X$ from a K3 surface Y by [16, Corollary in Section 3]. Now, the same argument as in the case where $p \neq 2$ shows the assertion.

If X is an Abelian surface, then the assertion follows immediately from Proposition 4.3.7. Finally, we assume that X is a (quasi-)hyperelliptic surface. Since X is F-split, a general fiber of the Albanese map is normal by [24, Proposition 7.2]. Thus X is a hyperelliptic surface and there exists a finite étale morphism $f\colon Y \longrightarrow X$ from an Abelian surface Y. Then X is not uniruled and we conclude the assertion by Proposition 4.3.7.

□

4.4 Bogomolov-Sommese type vanishing for globally F-regular threefolds

In this section, we prove a Bogomolov-Sommese type vanishing on globally F-regular threefolds.

Definition 4.4.1. Let X be a variety and $\mathcal{F}$ a coherent sheaf on X. We say that $\mathcal{F}$ satisfies *Serre's condition* S_n if $\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geqslant \min\{n, \dim \mathcal{O}_{X,x}\}$ holds for every (not necessary closed) point $x \in X$.

Lemma 4.4.2. *Let X be a projective variety and A an ample Cartier divisor on X. Let $\mathcal{F}$ be a coherent sheaf on X satisfying Serre's condition S_n. Then*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(-mA)) = 0$$

for all $i < l := \min\{n, \dim X\}$ and all sufficiently large m.

Proof. We may assume that X is the closed subscheme of $\mathbb{P}^N_k$ and $\mathcal{O}_X(A) = \mathcal{O}_X(1)$. We fix a closed point $x \in X$. Since $\mathcal{F}$ satisfies Serre's condition S_n, it follows that

$$\operatorname{pd}_{\mathcal{O}_{\mathbb{P}^N_k,x}}(\mathcal{F}_x) = N - \operatorname{depth}_{\mathcal{O}_{\mathbb{P}^N_k,x}}(\mathcal{F}_x) = N - \operatorname{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \leqslant N - l$$

and hence $\mathcal{E}xt^j_{\mathbb{P}^N_k}(\mathcal{F}, -) = 0$ for $j > N - l$. Now the Serre duality yields

$$\begin{aligned} H^i(X, \mathcal{F}(-m)) \cong& Ext^{N-i}(\mathcal{F}, \omega_{\mathbb{P}^N_k}(m)) & \\ \cong& H^0(\mathbb{P}^N_k, \mathcal{E}xt^{N-i}_{\mathbb{P}^N_k}(\mathcal{F}, \omega_{\mathbb{P}^N_k}(m))) & m \gg 0 \\ =& 0 & i < l \end{aligned}$$

and hence we obtain the assertion. □

Lemma 4.4.3. *Let X be a normal projective variety and D a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on X such that $\kappa(X, D) > 0$. Let $\mathcal{F}$ be a reflexive sheaf on X. Then $H^0(X, (\mathcal{F} \otimes \mathcal{O}_X(-p^e D))^{**}) = 0$ for all sufficiently large and divisible e.*

Proof. Since $\kappa(X, D) > 0$, there exists a rational map $\varphi := \varphi_{|p^m(p^n-1)D|} \colon X \dashrightarrow Y$ such that Y is a projective variety with $\dim Y > 0$ for some $m, n \in \mathbb{Z}_{\geqslant 0}$. We fix such m, n. Since $\mathcal{F}$ is reflexive, we can take an open subset U with $\operatorname{codim}_X(X - U) \geqslant 2$ such that $\mathcal{F}$ is locally free on U and $U \subset X_{\mathrm{reg}}$. By taking a resolution of indeterminacy of $\varphi|_U$, we have the following commutative diagram

$$\begin{tikzcd} V \arrow[d, "f"'] \arrow[dr, "g"] & \\ U \arrow[r, dotted, "\varphi|_U"'] & Y. \end{tikzcd}$$

We note that $p^m(p^n-1)f^*D|_U - g^*H \geqslant 0$ for some ample Cartier divisor H on Y by the construction of g. Then we have

$$\begin{aligned} & H^0(X, (\mathcal{F} \otimes \mathcal{O}_X(-p^{m+ln}D))^{**}) \\ = \ & H^0(U, (\mathcal{F}|_U \otimes \mathcal{O}_U(-p^m D)) \otimes \mathcal{O}_U(-p^m(p^{ln}-1)D)) \\ = \ & H^0(V, f^*(\mathcal{F}|_U \otimes \mathcal{O}_U(-p^m D)) \otimes \mathcal{O}_V(-p^m(p^{ln}-1)f^*D)) \\ \hookrightarrow \ & H^0(V, f^*(\mathcal{F}|_U \otimes \mathcal{O}_U(-p^m D)) \otimes \mathcal{O}_V(-g^*H_l)) \\ = \ & H^0(Y, g_*f^*(\mathcal{F}|_U \otimes \mathcal{O}_U(-p^m D)) \otimes \mathcal{O}_Y(-H_l)) \end{aligned}$$

for all $l \in \mathbb{Z}_{>0}$, where $H_l := (1 + p^n + \cdots + p^{(l-1)n})H$. Since $\mathcal{F}|_U$ and $\mathcal{O}_U(-p^m D)$ are locally free, it follows that $g_*f^*(\mathcal{F}|_U \otimes \mathcal{O}_U(-p^m D))$ is torsion-free. Therefore $H^0(Y, g_*f^*(\mathcal{F}|_U \otimes \mathcal{O}_U(-p^m D)) \otimes \mathcal{O}_Y(-H_l)) = 0$ for a sufficiently large integer l by Lemma 4.4.2. $\square$

Example 4.4.4. Let X be a normal projective variety which lifts to the ring of Witt vectors of length two $W_2(k)$ with its Frobenius morphism (see [20, Section 2] for more details). Then there exists a splitting injective map $\Omega_X^{[i]} \hookrightarrow F_*\Omega_X^{[i]}$ by [20, Theorem 2]. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on X. If $\kappa(X, D) > 0$, then it follows that

$$H^0(X, (\Omega_X^{[i]} \otimes \mathcal{O}_X(-D))^{**}) \hookrightarrow H^0(X, (\Omega_X^{[i]} \otimes \mathcal{O}_X(-p^e D))^{**}) = 0$$

for a sufficiently large and divisible integer e by Lemma 4.4.3. We remark that toric varieties lift to $W_2(k)$ with their Frobenius morphisms, but a stronger assertion than the above holds on them. We refer to [30, Theorem 2.22] for the detail.

Theorem 4.4.5. *Let X be a projective globally F-regular variety and B reduced divisor on X. Suppose that $\dim X \geqslant 2$ and $\operatorname{codim}_X((X, B)_{\mathrm{nsnc}}) \geqslant 3$. Then*

$$H^0(X, (\Omega_X^{[1]}(\log B) \otimes \mathcal{O}_X(-D))^{**}) = 0$$

for every nef and big $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor D on X.

Proof. First, we show the following claim.

Claim. $H^1(U, \mathcal{O}_U(-D)) = 0$ for every nef and big $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor D, where U denotes $(X, \Delta)_{\mathrm{snc}}$.

Proof of the Claim. Let D be a nef and big $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor. We fix $m, n \in \mathbb{Z}_{>0}$ such that $D' := p^m(p^n - 1)D$ is Cartier. The bigness of D shows that $p^m D$ is linearly equivalent to an effective $\mathbb{Z}$-divisor for all sufficiently large $m \gg 0$. By Remark 4.1.2 (1), there exists $l \gg 0$ such that

$$\mathcal{O}_X \hookrightarrow F_*^{m+ln}\mathcal{O}_X(p^m D)$$

splits. By restricting to U and tensoring $\mathcal{O}_U(-D)$, we have a splitting injective map

$$\begin{aligned} \mathcal{O}_U(-D) \hookrightarrow\ & F_*^{m+ln}\mathcal{O}_U(p^m D - p^{m+ln} D) \\ =\ & F_*^{m+ln}\mathcal{O}_U(-D'_l), \end{aligned}$$

where $D'_l := (1 + p^n + \cdots + p^{(l-1)n})D'$. By taking the cohomology, we have a splitting injection

$$H^1(U, \mathcal{O}_U(-D)) \hookrightarrow H^1(U, \mathcal{O}_U(-D'_l)),$$

and thus we may assume that D is Cartier. If $\dim X = 2$, then $U = X$ by the assumption that $\mathrm{codim}_X((X, \Delta)_{\mathrm{nsnc}}) \geqslant 3$, and the assertion of the claim follows from [97, Corollary 4.4]. Now we assume that $n := \dim X \geqslant 3$. Since X is globally F-regular, it follows that X is Cohen-Macaulay by Remark 4.1.2 (3) and the line bundle $\mathcal{O}_X(-D)$ satisfies Serre's condition S_n. By the assumption $\mathrm{codim}_X(Z) \geqslant 3$ and by [18, Proposition 1.2], we have $\mathcal{H}^j_Z(X, \mathcal{O}_X(-D)) = 0$ for all $j < 3$, where Z denotes $(X, \Delta)_{\mathrm{nsnc}}$. We consider the spectral sequence

$$E_2^{i,j} = H^i(X, \mathcal{H}^j_Z(X, \mathcal{O}_X(-D)) \longrightarrow H^{i+j} = H^{i+j}_Z(X, \mathcal{O}_X(-D)).$$

Since $E_2^{i,j} = H^i(X, \mathcal{H}^j_Z(X, \mathcal{O}_X(-D)) = 0$ for all $j < 3$, we have

$$H^i_Z(X, \mathcal{O}(-D)) = H^i = E_2^{i,0} = H^i(X, \mathcal{H}^0_Z(X, \mathcal{O}_X(-D)) = 0$$

for all $i < 3$. By the local cohomology exact sequence, we have the exact sequence

$$H^1(X, \mathcal{O}_X(-D)) \longrightarrow H^1(U, \mathcal{O}_X(-D) \longrightarrow H^2_Z(X, \mathcal{O}_X(-D)) = 0.$$

Therefore, it suffices to show that $H^1(X, \mathcal{O}_X(-D)) = 0$ and this follows from [97, Corollary 4.4]. $\square$

Now, we show the assertion of the theorem. Conversely, we assume that

$$H^0(X, (\Omega_X^{[1]}(\log B) \otimes \mathcal{O}_X(-D))^{**}) = H^0(U, \Omega_U(\log B) \otimes \mathcal{O}_U(-D)) \neq 0.$$

Then, by Lemma 4.4.3, there exists $l \in \mathbb{Z}_{\geqslant 0}$ such that

$$\begin{aligned} &H^0(X, (\Omega_X^{[1]}(\log B) \otimes \mathcal{O}_X(-p^l D))^{**}) \neq 0, \\ &H^0(X, (\Omega_X^{[1]}(\log B) \otimes \mathcal{O}_X(-p^{l+1} D))^{**}) = 0. \end{aligned}$$

We note that we can use the Cartier operator on $U = (X, \Delta)_{\mathrm{snc}}$. Since X is F-split, the exact sequence

$$0 \longrightarrow \mathcal{O}_U \longrightarrow F_*\mathcal{O}_U \longrightarrow B^1_U \longrightarrow 0$$

splits. By the claim and the splitting of the above exact sequence, we have

$$H^1(U, B^1_U \otimes \mathcal{O}_U(-p^l D)) = 0$$

for every nef and big $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor D. Since $Z^1_U(\log B) \subset F_*\Omega_U(\log B)$, we have

$$\begin{aligned} H^0(U, Z^1_U(\log B) \otimes \mathcal{O}_U(-p^l D)) &\hookrightarrow H^0(U, \Omega_U(\log B) \otimes \mathcal{O}_U(-p^{l+1}D)) \\ &= H^0(X, (\Omega^{[1]}_X(\log B) \otimes \mathcal{O}_X(-p^{l+1}D))^{**}) \\ &= 0. \end{aligned}$$

Now, since $H^0(U, Z^1_U(\log B) \otimes \mathcal{O}_U(-p^l D)) = H^1(U, B^1_U \otimes \mathcal{O}_U(-p^l D)) = 0$, the exact sequence

$$0 \longrightarrow B^1_U(\log B) = B^1_U \longrightarrow Z^1_U(\log B) \longrightarrow \Omega_U(\log B) \longrightarrow 0$$

shows

$$H^0(X, (\Omega^{[1]}_X(\log B) \otimes \mathcal{O}_X(-p^l D))^{**}) = H^0(U, \Omega_U(\log B) \otimes \mathcal{O}_U(-p^l D)) = 0,$$

a contradiction with the assumption of l. $\square$

If X is smooth in Theorem 4.4.5, then we can weaken the assumption that X is globally F-regular as follows.

Proposition 4.4.6. *Let (X, B) be a log smooth projective variety of $\dim X \geqslant 2$. Suppose that X is F-split. Then $H^0(X, \Omega_X(\log B) \otimes \mathcal{O}_X(-D)) = 0$ for every nef and big Cartier divisor D on X.*

Proof. We take a nef and big Cartier divisor D. Since X is F-split, we have a splitting injective map

$$H^1(X, \mathcal{O}_X(-D)) \hookrightarrow H^1(X, \mathcal{O}_X(-p^e D)).$$

By [71, Proposition 2.24], we have $H^1(X, \mathcal{O}_X(-p^e D)) = 0$ for a sufficiently large integer e and hence $H^1(X, \mathcal{O}_X(-D)) = 0$. Now the argument after the claim of Theorem 4.4.5 shows the assertion. $\square$

Globally F-regular surfaces have only F-regular singularities. We note that F-regular singularities are klt and in particular the minimal resolutions are log resolutions. We refer to [43] for more details.

Graf [35] shows that a surface with F-regular singularities satisfies the extension theorem for the logarithmic differential form.

Theorem 4.4.7 (cf. [35, Theorem 1.2])**.** *Let X be a normal surface with F-regular singularities and $\pi\colon Y \longrightarrow X$ the minimal resolution with the reduce π-exceptional divisor E. Then $\pi_*\Omega_Y(\log\, E) \cong \Omega_X^{[1]}$.*

Proof. We may assume that k is an algebraically closed field by [35, Proposition 7.4]. By [43, Theorem 1.1], the dual graph of π is one of the following.

1. The graphs of the singularity is a chain.
2. The graphs of the singularity is star-shaped and either
   (a) of type $(2, 2, d)$, $d \geqslant 2$, and $p \neq 2$,
   (b) of type $(2, 3, 3)$ or $(2, 3, 4)$, and $p > 3$,
   (c) of type $(2, 3, 5)$ and $p > 5$.

By applying [35, 7.B Proof of Theorem 1.2 (7.9.5)] (resp. [*ibid*, (7.9.6)], [*ibid*, (7.9.7)]) to (1) (resp. (2)(a), (2)(b) and (c)), we obtain the assertion. $\square$

Corollary 4.4.8. *Let X be a normal projective F-split surface with F-regular singularities. Then $H^0(X, \Omega_X^{[i]} \otimes \mathcal{O}_X(-D)) = 0$ for every $i \geqslant 0$ and every nef and big Cartier divisor D on X.*

Proof. When $i = 0$, we obtain the assertion by the bigness of D. Since X is F-split, it follows that $-K_X$ is effective and hence the assertion holds when $i = 2$. Now, we assume that $i = 1$. Conversely, we assume that there exists an injective homomorphism $\mathcal{O}_X(D) \hookrightarrow \Omega_X^{[1]}$ for some nef and big Cartier divisor D on X. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution with the reduced π-exceptional divisor E. Since π is crepant, it follows that Y is F-split by [17, 1.3.13 Lemma]. Now, by Theorem 4.4.7, we have an injective homomorphism $\mathcal{O}_Y(\pi^*D) \longrightarrow \pi^*\Omega_X^{[1]} \cong \pi^*\pi_*\Omega_Y(\log\, E) \longrightarrow \Omega_Y(\log E)$, a contradiction with Proposition 4.4.6. $\square$

Lemma 4.4.9. *Let $f\colon X \longrightarrow Y$ be a projective surjective morphism of normal varieties satisfying $f_*\mathcal{O}_X = \mathcal{O}_Y$. Suppose that a general fiber F of f is globally F-regular and* $\dim F = 1$ *or* 2*. In addition, assume that* $\operatorname{codim}_X(X_{\mathrm{sg}}) \geqslant 3$ *when* $\dim F = 2$*. Let D be an f-nef and f-big $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on X. Then $f_*(\Omega_X^{[i]} \otimes \mathcal{O}_X(-D))^{**} = 0$ for all $i \geqslant 0$.*

Proof. If $\dim Y = 0$, then X is a smooth rational curve or a smooth rational surface, and the assertion follows from Proposition 4.3.4. Thus we assume that $\dim Y > 0$. We may assume that Y is affine. Conversely, we assume that there exists an injective homomorphism $s\colon \mathcal{O}_X(D) \hookrightarrow \Omega_X^{[i]}$ for some $i \geqslant 0$. Since the closed point $y := f(F)$ is contained in Y_{reg}, we have $I_F/I_F^2 = f^*(\mathrm{m}_y/\mathrm{m}_y^2) = \mathcal{O}_F^{\oplus \dim Y}$, where I_F is the ideal sheaf of F. Now by the conormal exact sequence, we have

$$0 \to \mathcal{O}_{F_{\mathrm{reg}}}^{\oplus \dim Y} \to \Omega_X|_{F_{\mathrm{reg}}} \to \Omega_{F_{\mathrm{reg}}} \to 0.$$

By the generality of F, the restriction $s|_F\colon \mathcal{O}_F(D|_F) \hookrightarrow \Omega_X^{[i]}|_F = \bigwedge^i \Omega_X|_F$ is injective and we obtain an injective homomorphism $\mathcal{O}_F(D|_F) \hookrightarrow \Omega_F^{[j]} \otimes \bigwedge^{i-j} \mathcal{O}_F^{\oplus \dim Y}$ by [34, Lemma 3.14]. In particular, we have $\mathcal{O}_F(D|_F) \hookrightarrow \Omega_F^{[j]}$ for some $j \geqslant 0$. We first assume that $\dim F = 1$. Then $F \cong \mathbb{P}^1_k$ and $D|_F$ is a nef and big Cartier divisor. This is a contradiction. We next assume that $\dim F = 2$. Then $D|_F$ is a nef and big Cartier divisor by the assumption that $\operatorname{codim}_X(X_{\mathrm{sg}}) \geqslant 3$. Now we can derive a contradiction by Corollary 4.4.8. □

Now, we prove a Bogomolov-Sommese type vanishing for globally F-regular threefolds.

Theorem 4.4.10. *Let X be a smooth projective globally F-regular threefold and $\mathcal{O}_X(D) \subset \Omega_X$ an invertible subsheaf. If $p > 3$, then $\kappa(X, D) \leqslant 1$. Furthermore, if $p > 7$, then $\kappa(X, D) \leqslant 0$.*

Remark 4.4.11. In the above theorem, we need the assumption that $p > 3$ only for running K_X-MMP.

Proof. Let us prove the first assertion of the theorem. We assume that $p > 3$ and $\kappa(X, D) > 1$. Let us show that $H^0(X, \Omega_X \otimes \mathcal{O}_X(-D)) = 0$. Since X is globally F-regular, the anti-canonical divisor $-K_X$ is big by [93, Corollary 4.5]. Then by running K_X-MMP, we obtain a birational contraction $f\colon X \dashrightarrow X'$ and a Mori fiber space $g\colon X' \longrightarrow Y$ by [41, Theorem 1.2]. By Remark 4.1.2 (2), X' is a $\mathbb{Q}$-factorial terminal projective globally F-regular threefold. By Lemma 3.2.1, it suffices to show that $H^0(X', (\Omega^{[1]}_{X'} \otimes \mathcal{O}_{X'}(-D'))^{**}) = 0$. Moreover, we have $\kappa(X', D') \geqslant \kappa(X, D) > 1$.

First, we assume that $\dim Y = 0$. In this case, the divisor D' is ample since $\kappa(X', D') > 1$ and $\rho(X') = 1$. Since three-dimensional terminal singularities are isolated by [65, Corollary 2.13], we obtain $H^0(X', (\Omega^{[1]}_{X'} \otimes \mathcal{O}_{X'}(-D'))^{**}) = 0$ by Theorem 4.4.5.

Next, we assume that $\dim Y = 1$. Let G be a general fiber of g. Since $-K_{X'}$ and G form the basis of $N^1(X') \otimes_{\mathbb{Z}} \mathbb{Q}$, we can denote $D' \equiv a(-K_{X'}) + bG$ for some $a, b \in \mathbb{Q}$, where $N^1(X')$ is the quotient of $\mathrm{Pic}(X')$ by its subgroup consisting of all isomorphism classes numerically equivalent to zero. We denote by $\mathrm{Pic}^0(X')$ the subgroup of $\mathrm{Pic}(X')$ consisting of all isomorphism classes algebraically equivalent to zero and by $\mathrm{NS}(X')$ the quotient of $\mathrm{Pic}(X')$ by $\mathrm{Pic}^0(X')$. Since X' is globally F-regular, it follows from [97, Corollary 4.3] that $H^1(X', \mathcal{O}_{X'}) = 0$. Then by [27, Theorem 9.5.11], we obtain $\mathrm{Pic}^0(X') = 0$ and hence $\mathrm{Pic}(X') = \mathrm{NS}(X')$. Since the kernel of the natural map $\mathrm{NS}(X') \twoheadrightarrow N^1(X')$ is torsion by [76, Corollary 1.4.38], we obtain $\mathrm{Pic}(X') \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{NS}(X') \otimes_{\mathbb{Z}} \mathbb{Q} = N^1(X') \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular, D' is $\mathbb{Q}$-linearly equivalent to $a(-K_{X'}) + bG$. Since $\kappa(X', D') > 1 = \kappa(X', G)$, it follows that $a > 0$ and hence $D'|_G$ is ample. Now G is a globally F-regular surface by Theorem 4.1.3 and hence we obtain $H^0(X', (\Omega^{[1]}_{X'} \otimes \mathcal{O}_{X'}(-D'))^{**}) = 0$ by Lemma 4.4.9.

Finally, we assume that $\dim Y = 2$. In this case, X' is separably rationally connected by Theorem 4.1.4 (1) and hence so is X. Then we obtain $H^0(X, \Omega_X \otimes \mathcal{O}_X(-D)) = 0$ by Proposition 4.3.4.

Now, we show the latter assertion. We assume that $p > 7$ and $\kappa(X, D) > 0$. We take X', Y as above. When $\dim Y = 0$ or 2, we obtain the assertion by the essentially same argument as above. When dim $Y = 1$, Theorem 4.1.4 (2) shows that X' is separably rationally connected and hence so is X. Therefore we obtain $H^0(X, \Omega_X \otimes \mathcal{O}_X(-D)) = 0$ by Proposition 4.3.4. $\square$

Remark 4.4.12. Let X be a terminal projective globally F-regular threefold. Suppose that $p > 3$ and $\mathcal{O}_X(D) \subset \Omega_X^{[1]}$ is a Weil divisorial subsheaf. Then an argument similar to Theorem 4.4.10 shows that $\kappa(X, D) \leqslant 2$ as follows.

By taking a small $\mathbb{Q}$-factorialization and running K_X-MMP, the assertion is reduced to a Mori fiber space $g \colon X' \longrightarrow Y$. Let D' is the push-forward of D to X'. When dim $Y = 0$ or 1, we obtain the assertion by the proof of the first assertion of Theorem 4.4.10. On the other hand, when dim $Y = 2$, we need a different argument from Theorem 4.4.10 since Proposition 4.3.4 cannot be applied to singular varieties. In this case, since D' is big and $\rho(X'/Y) = 1$, it follows that $D'|_G$ is ample and the assertion follows from Lemma 4.4.9.

Chapter 5

Pathologies of Du Val del Pezzo surfaces in positive characteristic (joint work with Masaru Nagaoka)

In this chapter, we prove Theorem 1.3.3, 1.3.4, 1.3.6, and 1.3.8.

5.1 Du Val del Pezzo surfaces

In this section, we gather the basic results of Du Val del Pezzo surfaces.

Definition 5.1.1. Let X be a normal projective surface. We say that X is a *Du Val del Pezzo surface* if $-K_X$ is ample and X has only Du Val singularities. We write $\mathrm{Dyn}(X)$ for the Dynkin type of X. For D_n, E_n (resp. E_n), and in no other cases in $p = 2$ (resp. $p = 3$), there are more than one, finitely many, isomorphism classes of singularity sharing the same Dynkin type. They are classified and named as D_n^r and E_n^r by Artin [5], where r is called the *Artin coindex* of the Du Val singulairty. We write $\mathrm{Dyn}'(X)$ for the Dynkin type of X with Artin coindices.

Remark 5.1.2. Let X be a normal projective surface with only rational singularities with Iitaka dimension $\kappa(\widetilde{X}, K_{\widetilde{X}}) = -\infty$, where $\widetilde{X} \longrightarrow X$ is a resolution. Let us see that X lifts to every Noetherian complete local ring R with the residue field k.

First, we recall that $\widetilde{X}$ lifts to R (see [27, 8.5.26]). Then X is formally liftable to R by [2, Proposition 4.3(1)], and the formal lifting is algebraizable since $H^2(X, \mathcal{O}_X) = 0$. In particular, all Du Val del Pezzo surfaces lift to R.

Lemma 5.1.3. *Let X be a Du Val del Pezzo surface of degree $d := K_X^2$. Then the following hold.*

(1) $\dim |-K_X| = d$.

(2) $|-K_X|$ *has no fixed part.*

(3) *A general member of the anti-canonical linear system is a locally complete intersection curve with arithmetic genus one. Moreover, if $p > 3$, then a general member of the anti-canonical linear system is smooth.*

(4) *If $d \geqslant 3$, then $|-K_X|$ is very ample.*

(5) *If $d = 4$, then X is isomorphic to a complete intersection of two quadric hypersurfaces in $\mathbb{P}^4_k$.*

(6) *If $d = 3$, then X is isomorphic to a cubic hypersurface in $\mathbb{P}^3_k$.*

(7) *If $d = 2$, then $|-K_X|$ is base point free and X is isomorphic to a weighted hypersurface in $\mathbb{P}_k(1,1,1,2)$ of degree four.*

(8) *If $d = 1$, then $|-K_X|$ has the unique base point and X is isomorphic to a weighted hypersurface in $\mathbb{P}_k(1,1,2,3)$ of degree six.*

Proof. We refer to [12, Propositions 2.10, 2.12, and 2.14] and [59, Proposition 4.6] for the proof. □

5.2 Quasi-elliptic surfaces

In this subsection, we compile the results on rational quasi-elliptic surfaces by Ito [50, 51], which we will use in Sections 5.4 and 5.5.

Theorem 5.2.1 ([50, Theorems 3.1–3.3])**.** *Suppose $p = 3$. Then the following hold.*

(1) *The configurations of reducible fibers of rational quasi-elliptic surfaces and their Mordell-Weil groups are listed in Table 5.1, where we use Kodaira's notation.*

(2) *Rational quasi-elliptic surfaces of each type (1), (2), and (3) uniquely exist.*

(3) *Sections on rational quasi-elliptic surfaces are disjoint from each other. Moreover, the dual graphs of negative rational curves in rational quasi-elliptic surfaces are as in Figure 5.1 and Table 5.2, where black nodes (resp. white nodes) correspond to (-1)-curves (resp. (-2)-curves).*

Table 5.1

Type	Reducible fibers	$\mathrm{MW}(Z)$
(1)	II^*	$\{1\}$
(2)	$\mathrm{IV}^*, \mathrm{IV}$	$\mathbb{Z}/3\mathbb{Z}$
(3)	four IV	$(\mathbb{Z}/3\mathbb{Z})^2$

Theorem 5.2.2 ([51, §5])**.** *Suppose $p = 2$. Then the following hold.*

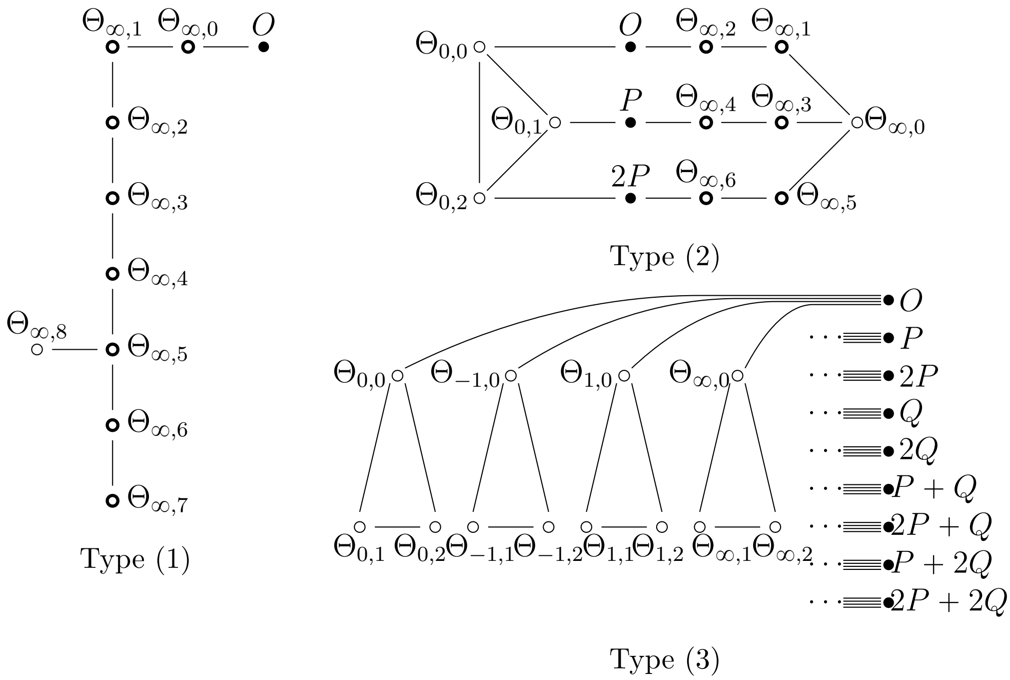

Figure 5.1: Dual graphs of negative rational curves in rational quasi-elliptic surfaces of types (1)–(3)

Table 5.2

	$\beta = 0$	$\beta = -1$	$\beta = 1$	$\beta = \infty$
Sections adjacent to $\Theta_{\beta,0}$	$O, 2P + Q$, $P + 2Q$	O, Q, $2Q$	O, P, $2P$	$O, P + Q$, $2P + 2Q$
Sections adjacent to $\Theta_{\beta,1}$	P, Q, $2P + 2Q$	$2P, 2P + Q$, $2P + 2Q$	$2Q, P + 2Q$, $2P + 2Q$	$P, 2Q$ $2P + Q$
Sections adjacent to $\Theta_{\beta,2}$	$2P, 2Q$, $P + Q$	$P, P + Q$, $P + 2Q$	$Q, P + Q$ $2P + Q$	$Q, 2P$, $P + 2Q$

(1) *The configurations of reducible fibers of rational quasi-elliptic surfaces and their Mordell-Weil groups are listed in Table 5.3, where we use Kodaira's notation.*

(2) *Rational quasi-elliptic surfaces of each type (a)–(c) and (e) uniquely exist.*

(3) *For each rational quasi-elliptic surface of one of the types (a)–(e), sections are disjoint from each other. Moreover, the dual graphs of negative rational curves in rational quasi-elliptic surfaces of types (a)–(e) are as in Figure 5.2, where black nodes (resp. white nodes) correspond to (-1)-curves (resp. (-2)-curves).*

(4) *For each rational quasi-elliptic surface of type (f), sections are disjoint from each other. There is an element $a \in k\backslash\{0\}$ such that the reducible fiber of type I_0^* lies over $t = 1$ and reducible fibers of type III lie over the points $t = 0, \infty, \alpha_1, \alpha_2$ of the base curve $\mathbb{P}^1_k$, where α_1 and α_2 are two solutions of the equation $t^2 + at + 1 = 0$. Moreover, Figure 5.3 and Table 5.4 describe the dual graph of the configuration of negative rational curves.*

(5) *For each rational quasi-elliptic surface of type (g), there are eight pairs of two sections intersecting with each other transversally and not intersecting with any other sections. There are no irreducible components of reducible fibers intersecting with two sections in a pair. Figure 5.4 describes the above situation.*

Table 5.3

Type	Reducible fibers	MW(Z)	Type	Reducible fibers	MW(Z)
(a)	II^*	$\{1\}$	(e)	$\mathrm{I}_2^*, \mathrm{III}, \mathrm{III}$	$(\mathbb{Z}/2\mathbb{Z})^2$
(b)	I_4^*	$\mathbb{Z}/2\mathbb{Z}$	(f)	I_0^* and four III	$(\mathbb{Z}/2\mathbb{Z})^3$
(c)	$\mathrm{III}^*, \mathrm{III}$	$\mathbb{Z}/2\mathbb{Z}$	(g)	eight III	$(\mathbb{Z}/2\mathbb{Z})^4$
(d)	$\mathrm{I}_0^*, \mathrm{I}_0^*$	$(\mathbb{Z}/2\mathbb{Z})^2$			

Remark 5.2.3. 1. Table 2 of [51] contains misprints. By substituting $t = 1$ to the equations of P_2, P_3, Q_1, and R_1 in the bottom of p. 246 of [*ibid*], we see at once that Q_1 and P_2 in the bottom table should be interchanged with each other. We also have to replace R_3 by R_2.

2. In Lemma 5.2.6, we will clarify that Figure 5.5 is the intersection matrix of negative rational curves in a rational quasi-elliptic surface of type (g).

3. In Corollary 5.5.24, we will give the parametrizing spaces of the isomorphism classes of rational quasi-elliptic surfaces of type (d), (f), or (g).

By [77, Theorem 3.1], each cuspidal cubic curve in $\mathbb{P}^2_{k,[x:y:z]}$ with an inflexion point is projectively equivalent to $C = \{x^3 + y^2 z = 0\}$. Moreover, since the automorphism $[x : y : z] \longmapsto [ax : y : a^3 z]$ of $\mathbb{P}^2_k$ with $a \in k^*$ fixes C, the pair of C and a point

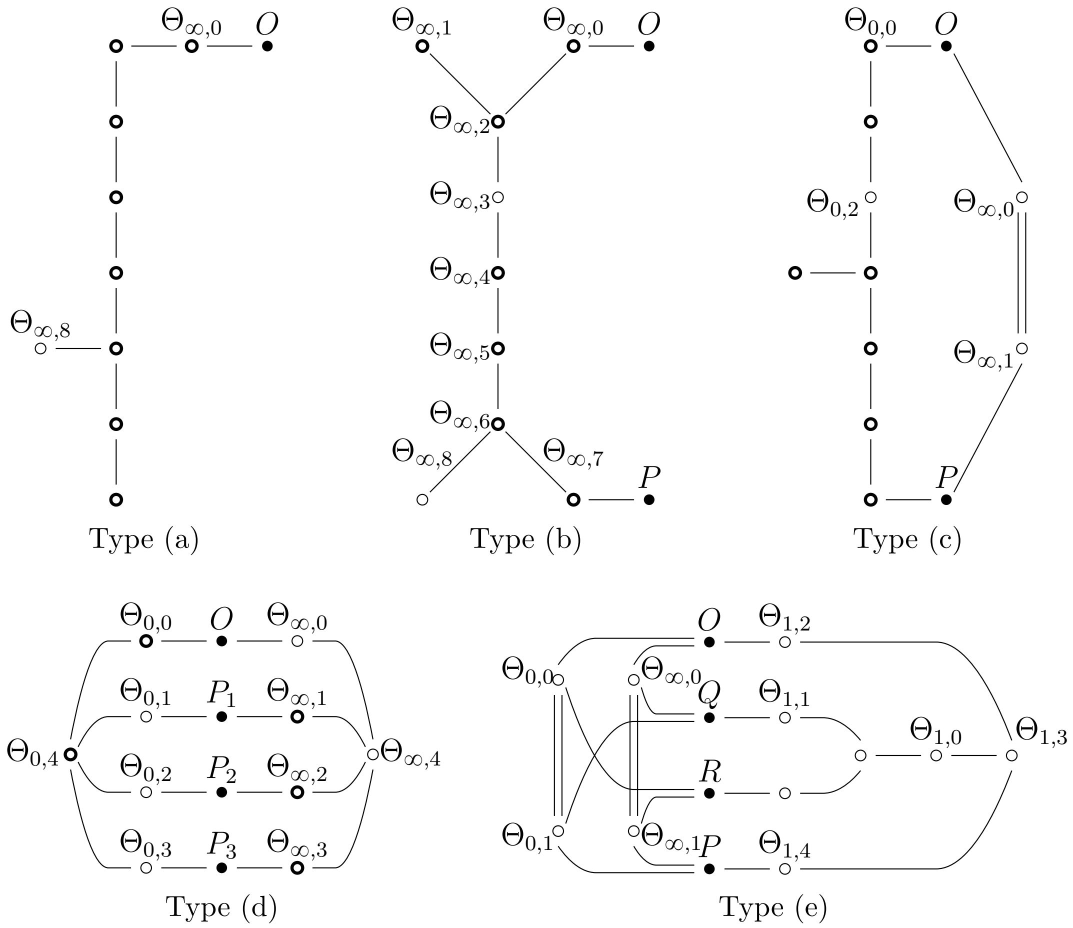

Figure 5.2: Dual graphs of negative rational curves in rational quasi-elliptic surfaces of types (a)–(e)

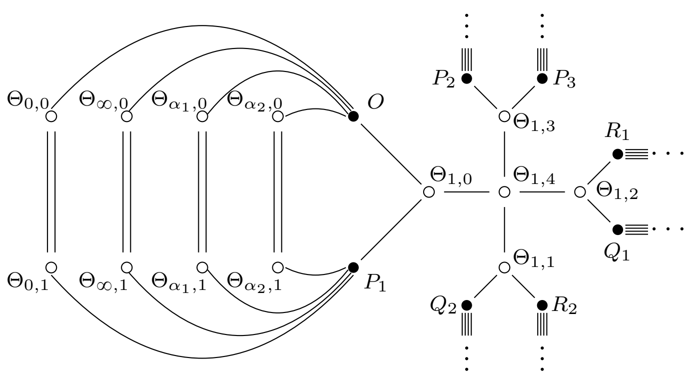

Figure 5.3: Dual graph of negative rational curves in a rational quasi-elliptic surface of type (f)

Table 5.4

	$\beta = 0$	$\beta = \infty$	$\beta = \alpha_1$	$\beta = \alpha_2$
Section intersecting with $\Theta_{\beta,0}$	O, R_2 R_1, P_3	O, Q_2 P_3, Q_1	O, Q_2 R_1, P_2	O, R_2 P_2, Q_1
Section intersecting with $\Theta_{\beta,1}$	P_1, Q_2 P_2, Q_1	P_1, R_2 R_1, P_2	P_1, R_2 P_3, Q_1	P_1, Q_2 R_1, P_3

	$\gamma = 1$
Section intersecting with $\Theta_{\gamma,0}$	O, P_1
Section intersecting with $\Theta_{\gamma,1}$	Q_2, R_2
Section intersecting with $\Theta_{\gamma,2}$	Q_1, R_1
Section intersecting with $\Theta_{\gamma,3}$	P_2, P_3

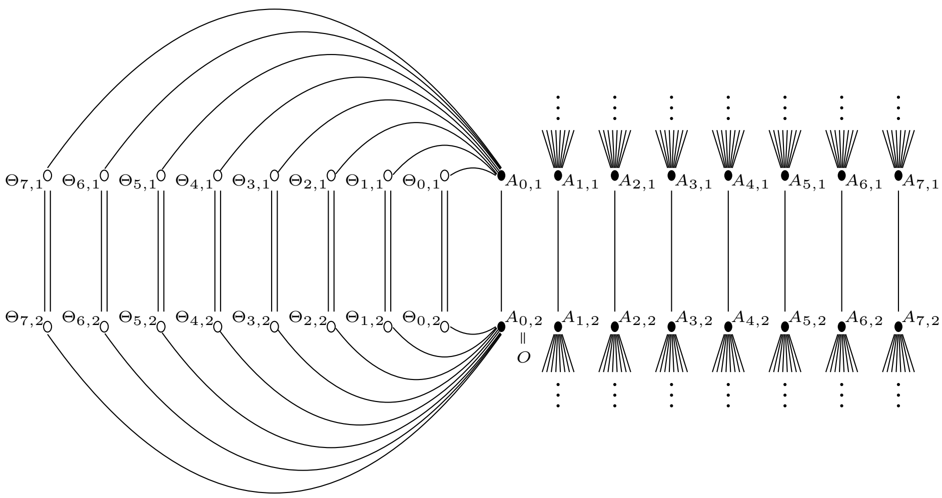

Figure 5.4: Dual graph of negative rational curves in a rational quasi-elliptic surface of type (g)

$p \in C$ is projectively equivalent to the pair of C and $[1 : 1 : -1]$ unless $p = [0 : 0 : 1]$ or $[0 : 1 : 0]$. From these facts, we can interpret [50, Example 3.8] and [51, Remark 4] as follows.

Lemma 5.2.4 ([50, Example 3.8], [51, Remark 4])**.** *Let Z be a quasi-elliptic surface of type one of (1)–(3) in characteristic three or one of (a)–(d) in characteristic two. When Z is of type (1), (a), or (b), we choose a general fiber F in addition. Then, contracting all curves corresponding to bold white node or black node in Figure 5.1 and Types (a)–(d) of Figure 5.2, we obtain a morphism $h\colon Z \longrightarrow \mathbb{P}^2_k$. Moreover, there are coordinates $[x : y : z]$ of $\mathbb{P}^2_k$ such that the images of F and negative rational curves by h are written as follows.*

If Z is of type (1), then

$$h(F) = \{x^3 + y^2 z = 0\}, \qquad h(\Theta_{\infty,8}) = \{z = 0\}, \qquad h(O) = [0 : 1 : 0].$$

If Z is of type (2), then

$$\begin{aligned} &h(\Theta_{\infty,0}) = \{x = 0\}, \quad h(\Theta_{0,0}) = \{y = 0\}, \quad h(\Theta_{0,1}) = \{y = z\}, \quad h(\Theta_{0,2}) = \{z = 0\}, \\ &h(O) = [0 : 0 : 1], \quad h(P) = [0 : 1 : 1], \quad h(2P) = [0 : 1 : 0]. \end{aligned}$$

If Z is of type (3), then

$$\begin{array}{lll} h(\Theta_{0,0}) = \{x = 0\}, & h(\Theta_{0,1}) = \{x = z\}, & h(\Theta_{0,2}) = \{x = -z\}, \\ h(\Theta_{-1,0}) = \{y = 0\}, & h(\Theta_{-1,1}) = \{y = z\}, & h(\Theta_{-1,2}) = \{y = -z\}, \\ h(\Theta_{1,0}) = \{x + y = 0\}, & h(\Theta_{1,1}) = \{x + y = -z\}, & h(\Theta_{1,2}) = \{x + y = z\}, \\ h(\Theta_{\infty,0}) = \{x - y = 0\}, & h(\Theta_{\infty,1}) = \{x - y = -z\}, & h(\Theta_{\infty,2}) = \{x - y = z\}, \\ h(O) = [0 : 0 : 1], & h(P) = [1 : -1 : 1], & h(2P) = [-1 : 1 : 1], \\ h(Q) = [1 : 0 : 1], & h(2Q) = [-1 : 0 : 1], & h(P + Q) = [-1 : -1 : 1], \\ h(2P + Q) = [0 : 1 : 1], & h(P + 2Q) = [0 : -1 : 1], & h(2P + 2Q) = [1 : 1 : 1]. \end{array}$$

If Z is of type (a), then

$$h(F) = \{x^3 + y^2 z = 0\}, \qquad h(\Theta_{\infty,8}) = \{z = 0\}, \qquad h(O) = [0 : 1 : 0].$$

If Z is of type (b), then

$$\begin{array}{lll} h(F) = \{x^3 + y^2 z = 0\}, & h(\Theta_{\infty,8}) = \{z = 0\}, & h(\Theta_{\infty,3}) = \{x + z = 0\}, \\ h(P) = [0 : 1 : 0], & h(O) = [1 : 1 : 1]. & \end{array}$$

If Z is of type (c), then

$$\begin{array}{lll} h(\Theta_{0,2}) = \{z = 0\}, & h(\Theta_{\infty,0}) = \{x = 0\}, & h(\Theta_{\infty,1}) = \{xz + y^2 = 0\}, \\ h(O) = [0 : 1 : 0], & h(P) = [1 : 0 : 0]. & \end{array}$$

If Z is of type (d), then

$$\begin{aligned} &h(\Theta_{\infty,4}) = \{x = 0\}, h(\Theta_{0,1}) = \{y = 0\}, h(\Theta_{0,2}) = \{y + z = 0\}, h(\Theta_{0,3}) = \{z = 0\}, \\ &h(\Theta_{0,4}) = [1 : 0 : 0], h(\Theta_{\infty,1}) = [0 : 0 : 1], h(\Theta_{\infty,2}) = [0 : 1 : 1], h(\Theta_{\infty,3}) = [0 : 1 : 0]. \end{aligned}$$

Rational quasi-elliptic surfaces are naturally endowed with the action of the Mordell-Weil groups. The next lemma shows that these surfaces may have other automorphisms.

Lemma 5.2.5. *A rational quasi-elliptic surface Z of type (d) has an involution which sends $\Theta_{0,i}$ in Type (d) of Figure 5.2 to $\Theta_{\infty,i}$ for $0 \leqslant i \leqslant 4$.*

Proof. Let $\varphi \colon Z \longrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$ be the contraction of O, P_1, P_2, P_3, $\Theta_{0,0}$, $\Theta_{0,1}$,$\Theta_{\infty,2}$, and $\Theta_{\infty,3}$. Then we can choose coordinates $([x : y], [s : t])$ of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ such that $\varphi(\Theta_{0,4}) = \{x = 0\}$ and $\varphi(\Theta_{\infty,4}) = \{y = 0\}$. Hence the involution $([x : y], [s : t]) \longmapsto ([y : x], [s : t])$ induces the desired involution on Z. $\square$

The next lemma clarifies the whole configuration of negative rational curves in a quasi-elliptic surface of type (g).

Lemma 5.2.6. *Figure 5.5 is the intersection matrix of negative rational curves on a rational quasi-elliptic surface of type (g).*

$$\left(\begin{array}{cccccccc|cccccccc|cccccccc|cccccccc}
-1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
&-1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1\\
&&-1&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&1&1&0&0&1&1\\
&&&-1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1\\
&&&&-1&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&0&1&1&0&0&1&1&1&1&0&0\\
&&&&&-1&0&0&0&0&0&0&0&1&0&0&1&0&0&1&1&0&0&1&0&1&1&0&0&1&1&0\\
&&&&&&-1&0&0&0&0&0&0&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&1&0&1&0\\
&&&&&&&-1&0&0&0&0&0&0&0&1&1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1\\
\hline
&&&&&&&&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
&&&&&&&&&-1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
&&&&&&&&&&-1&0&0&0&0&0&0&0&1&1&0&0&1&1&1&1&0&0&1&1&0&0\\
&&&&&&&&&&&-1&0&0&0&0&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0\\
&&&&&&&&&&&&-1&0&0&0&0&0&1&1&1&1&0&0&1&1&0&0&0&0&1&1\\
&&&&&&&&&&&&&-1&0&0&0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1\\
&&&&&&&&&&&&&&-1&0&0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1\\
&&&&&&&&&&&&&&&-1&0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0\\
\hline
&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\
&&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
&&&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0&0&2&0&0&0&0&0\\
&&&&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0&0&2&0&0&0&0\\
&&&&&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0&0&2&0&0&0\\
&&&&&&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0&0&2&0&0\\
&&&&&&&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0&0&2&0\\
&&&&&&&&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0&0&2\\
\hline
&&&&&&&&&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0&0\\
&&&&&&&&&&&&&&&&&&&&&&&&&-2&0&0&0&0&0&0\\
&&&&&&&&&&&&&&&&&&&&&&&&&&-2&0&0&0&0&0\\
&&&&&&&&&&&&&&&&&&&&&&&&&&&-2&0&0&0&0\\
&&&&&&&&&&&&&&&&&&&&&&&&&&&&-2&0&0&0\\
&&&&&&&&&&&&&&&&&&&&&&&&&&&&&-2&0&0\\
&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&-2&0\\
&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&-2
\end{array}\right)$$

Figure 5.5: The intersection matrix of $A_{0,1}, \ldots, A_{7,1}$, $A_{0,2}, \ldots, A_{7,2}$, $\Theta_{0,1}, \ldots, \Theta_{7,1}$, $\Theta_{0,2}, \ldots, \Theta_{7,2}$ in this order in a rational quasi-elliptic surface of type (g).

Proof. Let Z be a rational quasi-elliptic surface of type (g). Then there are exactly sixteen (-2)-curves $\{\Theta_{i,j}\}_{0 \leqslant i \leqslant 7, 1 \leqslant j \leqslant 2}$ on Z, which satisfies that

$$(\Theta_{i,j}, \Theta_{i',j'}) > 0 \iff (\Theta_{i,j}, \Theta_{i',j'}) = 2 \iff i = i' \text{ and } j \neq j'.$$

On the other hand, as we described in Theorem 5.2.2 (5), there is exactly sixteen sections $\{A_{k,l}\}_{0 \leqslant k \leqslant 7, 1 \leqslant l \leqslant 2}$ on Z, which satisfies that

$$(A_{k,l}, A_{k',l'}) > 0 \iff (A_{k,l}, A_{k',l'}) = 1 \iff k = k' \text{ and } l \neq l'.$$

By Theorem 5.2.2 (5), we may assume that

$$(\Theta_{i,j}, A_{0,2}) \neq 0 \iff (\Theta_{i,j}, A_{0,2}) = 1 \iff j = 2, \text{ and}$$
$$(\Theta_{0,2}, A_{k,l}) \neq 0 \iff (\Theta_{0,2}, A_{k,l}) = 1 \iff l = 2.$$

By contracting $A_{0,2}, \Theta_{0,2}$ and $A_{i,1}$ for $1 \leqslant i \leqslant 7$, we get a birational morphism $h\colon Z \longrightarrow \mathbb{P}^2_k$. Let $t = h(A_{0,2} \cup \Theta_{0,2})$, $t_i = h(A_{i,1})$, and $D_i = h_*\Theta_{i,1}$ for $1 \leqslant i \leqslant 7$. To show the assertion, we prepare some claims.

Claim. $h_*\Theta_{i,j} \sim \mathcal{O}_{\mathbb{P}^2_k}(j)$ for each $1 \leqslant i \leqslant 7$ and $1 \leqslant j \leqslant 2$.

Proof of Claim. We need only consider the case where $i = 1$ by symmetry and the case where $j = 1$ since $h_*(\Theta_{1,1} + \Theta_{1,2}) \sim h_*(-K_Y) \sim \mathcal{O}_{\mathbb{P}^2_k}(3)$. Suppose by contradiction that $h_*\Theta_{1,1} \sim \mathcal{O}_{\mathbb{P}^2_k}(2)$. Then exactly six of $A_{1,1}, A_{2,1}, \ldots, A_{7,1}$ intersect with $\Theta_{1,1}$ since $(h_*\Theta_{1,1})^2 - \Theta_{1,1}^2 = 6$. We may assume that $(A_{1,1}, \Theta_{1,1}) = 0$.

Assume that $h_*\Theta_{i,1} \sim \mathcal{O}_{\mathbb{P}^2_k}(2)$ for some $2 \leqslant i \leqslant 7$. Then $(h_*\Theta_{1,1}, h_*\Theta_{i,1}) = 4$. However, at least five of $A_{1,1}$, $A_{2,1}$, $\ldots$, $A_{7,1}$ intersect with both $\Theta_{1,1}$ and $\Theta_{i,1}$, which implies that $(h_*\Theta_{1,1}, h_*\Theta_{i,1}) \geqslant 5$, a contradiction. Hence $h_*\Theta_{i,1} \sim \mathcal{O}_{\mathbb{P}^2_k}(1)$ and $(h_*\Theta_{1,1}, h_*\Theta_{i,1}) = 2$ for each $2 \leqslant i \leqslant 7$. For such an i, exactly three of $A_{1,1}, A_{2,1}, \ldots, A_{7,1}$ intersect with $\Theta_{i,1}$ since $(h_*\Theta_{i,1})^2 - \Theta_{i,1}^2 = 3$. Moreover, $A_{1,1}$ intersects with $\Theta_{i,1}$ since otherwise we would obtain $(h_*\Theta_{1,1}, h_*\Theta_{i,1}) \geqslant 3$.

On the other hand, assume that $A_{k,1}$ intersects with both $\Theta_{i_1,1}$ and $\Theta_{i_2,1}$ for some $2 \leqslant k \leqslant 7$ and $2 \leqslant i_1 < i_2 \leqslant 7$. Then $(h_*\Theta_{i_1,1}, h_*\Theta_{i_2,1}) = 1$ since they are lines. However, $A_{1,1}$ also intersects with both $\Theta_{i_1,1}$ and $\Theta_{i_2,1}$, which implies that $(h_*\Theta_{i_1,1}, h_*\Theta_{i_2,1}) \geqslant 2$, a contradiction.

Hence we may assume that $\Theta_{i,1}$ intersects with $A_{1,1}, A_{2i-2,1}, A_{2i-1,1}$ for $2 \leqslant i \leqslant 4$. However, it implies that $(h_*\Theta_{5,1}, h_*\Theta_{i,1}) \geqslant 2$ for some $2 \leqslant i \leqslant 4$, a contradiction. Therefore $h_*(\Theta_{1,1}) \sim \mathcal{O}_{\mathbb{P}^2_k}(1)$. ■

Claim. There are coordinates of $\mathbb{P}^2_k$ such that $\{t_i\}_{1\leqslant i\leqslant 7}$ is the set of $\mathbb{F}_2$-rational points and $\{D_i\}_{1\leqslant i\leqslant 7}$ is the set of lines defined over $\mathbb{F}_2$.

Proof of Claim. By Claim 5.2, $\{D_i\}_{1\leqslant i\leqslant 7}$ is a set of lines passing through exactly three of $\{t_i\}_{1\leqslant i\leqslant 7}$. Hence the set $\Sigma := \{(i,j) \mid D_i \text{ passes through } t_j\}$ consists of 21 elements. On the other hand, distinct two lines cannot share two points. Combining this fact and $\sharp\Sigma = 21$, we conclude that $\{t_i\}_{1\leqslant i\leqslant 7}$ is a set of points contained in exactly three of $\{D_i\}_{1\leqslant i\leqslant 7}$.

Next, let us show that $\{t_i\}_{1\leqslant i\leqslant 7}$ contains four points in general position. Changing the indices of $\{D_i\}_{1\leqslant i\leqslant 7}$ and $\{t_i\}_{1\leqslant i\leqslant 7}$, we may assume that D_1 (resp. D_2) passes through t_1 and t_2 (resp. t_1 and t_3). Since three of $\{D_i\}_{1\leqslant i\leqslant 7}$ passes through t_2, it contains the line spanned by t_2 and t_3, say D_4. Then there is a unique point, say t_7, in $\{t_i\}_{1\leqslant i\leqslant 7}$ disjoint from $D_1 \cup D_2 \cup D_4$. Hence t_1, t_2, t_3, and t_7 are in general position.

Then there are coordinates of $\mathbb{P}^2_k$ such that $t_1 = [1:0:0]$, $t_2 = [0:1:0]$, $t_3 = [0:0:1]$ and $t_7 = [1:1:1]$. Then we may assume that $t_4 = [1:1:0], t_5 = [0:1:1]$ and $t_6 = [1:0:1]$. Since each of D_i is a span of two $\mathbb{F}_2$-rational point, it is also defined over $\mathbb{F}_2$. ■

Fix coordinates of $\mathbb{P}^2_k$ as above. By construction, t is not contained in D_i for $1 \leqslant i \leqslant 7$. Now define $\zeta \colon \{1, \dots, 7\} \longrightarrow I = \{(1,2,4), (1,3,6), (1,5,7), (2,3,5), (2,6,7), (3,4,7), (4,5,6)\}$ which maps 1, 2, 3, 4, 5, 6, and 7 to $(1,2,4)$, $(1,3,6)$, $(1,5,7)$, $(2,3,5)$, $(2,6,7)$, $(3,4,7)$, and $(4,5,6)$ respectively. By Claims 5.2 and 5.2, we may assume that D_i contains t_l for all $l \in \zeta(i)$ and $1 \leqslant i \leqslant 7$. Then the following hold for $1 \leqslant i \leqslant 7$.

- $\Theta_{i,1}$ is the strict transform by h of the line passing through t_l for all $l \in \zeta(i)$.
- $\Theta_{i,2}$ is the strict transform by h of the conic passing through t and t_l for all $l \in \{1, \dots, 7\} \backslash \zeta(i)$.
- $A_{i,1}$ is the exceptional divisor over t_i.
- $A_{i,2}$ is the strict transform by h of the line passing through t and t_i.
- $\Theta_{0,1}$ is the strict transform by h of the cubic passing through $t, t_1, \dots, t_7$ which has a cusp at t.
- The tangent line of $h(\Theta_{i,2})$ at t is independent of the choice of i, and $A_{0,1}$ is the strict transform of this line by h.
- Z is obtained by blowing up $\mathbb{P}^2_k$ at t_j once for $1 \leqslant j \leqslant 7$ and at t twice along $h(A_{0,1})$, and the h-exceptional divisor over t consists of $A_{0,2}$ and $\Theta_{0,2}$.

From these facts, it is easy to check that Figure 5.5 is the intersection matrix of $A_{0,1}, \dots, A_{7,1}$, $A_{0,2}, \dots, A_{7,2}$, $\Theta_{0,1}, \dots, \Theta_{7,1}$, $\Theta_{0,2}, \dots, \Theta_{7,2}$ in this order. ☐

5.3 Proof of Theorem 1.3.3

This section is devoted to proving Theorem 1.3.3. First, we show that (NL) $\Rightarrow$ (NB).

Proposition 5.3.1. *Let X be a Du Val del Pezzo surface whose general member of the anti-canonical linear system is smooth. Then X is log liftable over every Noetherian complete local ring with the residue field k.*

Proof. Let $\pi \colon Y \longrightarrow X$ be the minimal resolution. By Proposition 2.4.8 (2), it suffices to show that $H^2(X, T_X) = H^2(X, \mathcal{O}_X) = 0$. Since $-K_X$ is ample, it follows that $H^2(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X(K_X)) = 0$. Now we show that $H^2(X, T_X) = 0$. By the Serre duality, it follows that

$$\begin{aligned} H^2(X, T_X) \cong & \operatorname{Hom}_{\mathcal{O}_X}(T_X, \mathcal{O}_X(K_X)) \\ \cong & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-K_X), \Omega_X^{[1]}), \end{aligned}$$

where $\Omega_X^{[1]}$ denotes the double dual of Ω_X. Suppose by contradiction that there exists an injective $\mathcal{O}_X$-module homomorphism $s \colon \mathcal{O}_X(-K_X) \hookrightarrow \Omega_X^{[1]}$. Let $C \in |-K_X|$ be a general member and $s|_C \colon \mathcal{O}_C(-K_X) \to \Omega_X^{[1]}|_C$ be the restriction of s on C. By Lemma 5.1.3 (2), we may assume that C is not contained in the zero locus of s. In particular, $s|_C$ is injective. By assumption, we also may assume that C is a smooth Cartier divisor. In particular, X is smooth along C and hence $\Omega_X^{[1]}|_C = \Omega_X|_C$. Let $t \colon \mathcal{O}_C(-K_X) \longrightarrow \Omega_C$ be the composition of $s|_C \colon \mathcal{O}_C(-K_X) \hookrightarrow \Omega_X|_C$ and the canonical map $\Omega_X|_C \longrightarrow \Omega_C$. By the conormal exact sequence, we obtain the following diagram.

$$\begin{tikzcd}
 & & \mathcal{O}_C(-K_X) \arrow[dl, dotted] \arrow[d, "s|_C"] \arrow[dr, "t"] & & \\
0 \arrow[r] & \mathcal{O}_C(-C) \arrow[r] & \Omega_X|_C \arrow[r] & \Omega_C \arrow[r] & 0.
\end{tikzcd}$$

Then t is the zero map since $\mathcal{O}_C(-K_X)$ is ample and $\Omega_C = \mathcal{O}_C$. Hence the above diagram induces an injective $\mathcal{O}_C$-module homomorphism $\mathcal{O}_C(-K_X) \hookrightarrow \mathcal{O}_C(-C)$, but this is a contradiction because $\mathcal{O}_C(-C) = \mathcal{O}_C(K_X)$ is anti-ample. Therefore we obtain the assertion. $\square$

Next, we prove that (ND) $\Rightarrow$ (NL).

Proposition 5.3.2. *Let X be a Du Val del Pezzo surface. Let R be a Noetherian integral domain of characteristic zero with a surjective homomorphism $R \longrightarrow k$. If X is log liftable over R via the associated morphism $\alpha \colon \operatorname{Spec} k \longrightarrow \operatorname{Spec} R$, then there exists a Du Val del Pezzo surface over $\mathbb{C}$ which has the same Dynkin type, the same Picard rank, and the same degree as X.*

Proof. Let m be the kernel of the homomorphism $R \longrightarrow k$. Replacing R with the completion of R_m, we may assume that R is a Noetherian complete local ring with residue field k. Thus, by Lemma 2.4.4, the pair (Y, E_π) lifts to R, where $\pi \colon Y \longrightarrow X$ is the minimal resolution. We denote by $E_\pi := \sum_{i=1}^r E_i$ the irreducible decomposition. Let $(\mathcal{Y}, \mathcal{E} := \sum_{i=1}^r \mathcal{E}_i)$ be an R-lifting of (Y, E_π). We take a subfield K of the field of fractions of R such that K is of finite transcendence degree over $\mathbb{Q}$, and the generic fiber of $\mathcal{Y}$ and that of each $\mathcal{E}_i$ are defined over K. Fix an inclusion $K \subset \mathbb{C}$ and take $\overline{K} \subset \mathbb{C}$ as the algebraic closure of K. For a field extension $K \subset F$, we use the notation $Y_F := \mathcal{Y} \otimes_R F$ and $E_{i,F} := \mathcal{E}_i \otimes_R F$ for each i. Since the geometrical connectedness are open property by [39, Théorème 12.2.4 (viii)], $Y_{\mathbb{C}}$ and $E_{i,\mathbb{C}}$ are smooth varieties. Since $E_{\mathbb{C}} := \sum_{i=1}^r E_{i,\mathbb{C}}$ has the same intersection matrix as E_π, we have a contraction $\pi_{\mathbb{C}} \colon Y_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$ of $E_{\mathbb{C}}$ and $X_{\mathbb{C}}$ has the same Dynkin type as X. By the crepantness of π and $\pi_{\mathbb{C}}$, we obtain $K_X^2 = K_Y^2 = K_{Y_{\mathbb{C}}}^2 = K_{X_{\mathbb{C}}}^2$.

Next, we prove that $X_{\mathbb{C}}$ is a Du Val del Pezzo surface. For the sake of contradiction, we assume that $-K_{X_{\mathbb{C}}}$ is not ample. Since $K_{X_{\overline{K}}}^2 = K_{X_{\mathbb{C}}}^2 > 0$, there exists an integral curve $C_0 \subset Y_{\mathbb{C}}$ defined over $\overline{K}$ such that C_0 is not contained in $E_{\mathbb{C}}$ and $(-K_{Y_{\mathbb{C}}} \cdot C_0) \leqslant 0$. We take a finite Galois extension field L of K such that C_0 is defined over L and write $C := \sum_{\sigma \in \mathrm{Gal}(L/K)} \sigma(C_0)$, which is defined over K. By the choice of C_0, there are no components contained in both C and E_K. We denote by $\overline{C}$ the closure of C in $\mathcal{Y}$ and define an effective divisor $C_k := \overline{C} \otimes_R k$.

Now, assume that $\operatorname{Supp} C_k \subset E_\pi$. Then we can write $C_k = \sum_{i=1}^r a_i E_i$ for some $a_i \geqslant 0$. Since C and E_K have no common components, we have $C_k^2 = (C \cdot \sum_{i=1}^r a_i E_{i,K}) \geqslant 0$. By the negative definiteness of E_π, we obtain $a_i = 0$ for $1 \leqslant i \leqslant r$, a contradiction. Thus there exists an integral curve $C'_k \subset C_k$ such that C'_k is not contained in E_π. Since $-K_Y$ is nef, we have $0 \leqslant (-K_Y \cdot C'_k) \leqslant (-K_Y \cdot C_k) = (-K_{Y_K} \cdot C) = |\mathrm{Gal}(L/K)|(-K_{Y_\mathbb{C}} \cdot C_0) \leqslant 0$. Hence $(-K_Y \cdot C'_k) = (-K_X \cdot \pi_*(C'_k)) = 0$, a contradiction with the ampleness of $-K_X$. Therefore, $X_\mathbb{C}$ is a Du Val del Pezzo surface.

Finally, we show that $\rho(X) = \rho(X_\mathbb{C})$. Since Y and $Y_\mathbb{C}$ are smooth rational surfaces, we have $\rho(Y_\mathbb{C}) = 10 - K_{Y_\mathbb{C}}^2 = 10 - K_Y^2 = \rho(Y)$. Then we obtain $\rho(X) = \rho(X_\mathbb{C})$ because $\pi_\mathbb{C}$ contracts the same number of (-2)-curves as π. $\square$

Finally, we prove that (NK) $\Rightarrow$ (NL).

Lemma 5.3.3. *Let $f\colon Z \longrightarrow X$ be a birational morphism of normal projective klt surfaces and A an ample $\mathbb{Z}$-divisor on X. Suppose that $(Z, \lceil f^*A \rceil - f^*A)$ is klt. Then $H^i(X, \mathcal{O}_X(-A)) = H^i(Z, \mathcal{O}_Z(-\lceil f^*A \rceil))$ for $i \geqslant 0$.*

Proof. By [99, Theorem 2.12], it follows that $R^i f_* \mathcal{O}_Z(K_Z + \lceil f^*A \rceil)) = 0$ for $i \geqslant 1$. Then the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{O}_Z(K_Z + \lceil f^*A \rceil)) \Rightarrow E^{p+q} = H^{p+q}(Z, \mathcal{O}_Z(K_Z + \lceil f^*A \rceil))$$

gives $H^i(X, f_*\mathcal{O}_Z(K_Z + \lceil f^*A \rceil)) \cong H^i(Z, \mathcal{O}_Z(K_Z + \lceil f^*A \rceil))$. Since X is klt, we obtain

$$\begin{aligned} K_Z + \lceil f^*A \rceil &= \lceil K_Z - f^*K_X + f^*(K_X + A) \rceil \\ &= \lfloor f^*(K_X + A) \rfloor + F \end{aligned}$$

for some effective f-exceptional $\mathbb{Z}$-divisor F. Then $H^i(X, f_*\mathcal{O}_Z(K_Z + \lceil f^*A \rceil)) = H^i(X, \mathcal{O}_X(K_X + A))$ by the projection formula. Hence the assertion follows from the Serre duality for Cohen-Macaulay sheaves [66, Theorem 5.71]. $\square$

Proposition 5.3.4. *Let X be a normal projective surface and A an ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor. Suppose that there exists a log resolution $f\colon Z \longrightarrow X$ such that (Z, E_f) lifts to $W_2(k)$. Then $H^1(X, \mathcal{O}_X(-A)) = 0$.*

Proof. By Lemma 5.3.3, it follows that $H^1(X, \mathcal{O}_X(-A)) = H^1(Z, \mathcal{O}_Z(-\lceil f^*A \rceil))$. Take an f-exceptional effective $\mathbb{Q}$-divisor F such that $\lceil f^*A - F \rceil = \lceil f^*A \rceil$ and $f^*A - F$ is ample. Since $\operatorname{Supp}(\lceil f^*A - F \rceil - (f^*A - F))$ is contained in E_f, Theorem 2.4.6 shows that $H^1(Z, \mathcal{O}_Z(-\lceil f^*A - F \rceil)) = 0$. Hence we get the assertion. $\square$

Now we can prove Theorem 1.3.3.

Proof of Theorem 1.3.3. The assertions (1), (2), and (3) follow from Propositions 5.3.1, 5.3.2, and 5.3.4 respectively. $\square$

5.4 Dynkin types

In this section, we determine the Dynkin types of Du Val del Pezzo surfaces satisfying (NB). By Lemma 5.1.3 (3),(4), such a del Pezzo surface is of degree at most two, and $p = 2$ or 3. First, we treat the case where the degree is one.

Proposition 5.4.1. *Let X be a Du Val del Pezzo surface with $K_X^2 = 1$ and $\pi\colon Y \longrightarrow X$ the minimal resolution. Take $g\colon Z \longrightarrow Y$ as the blow-up at the base point of $|-K_Y|$ and $f\colon Z \longrightarrow \mathbb{P}^1_k$ the genus one fibration defined by $|-K_Z|$.*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[drr, "f"'] & Y \arrow[r, "\pi"] & X \arrow[d, dotted, "\varphi_{|-K_X|}"] \\ & & \mathbb{P}^1_k \end{tikzcd}$$

Then the following hold.

(1) *X satisfies (NB) if and only if f is a quasi-elliptic fibration.*

(2) *For another Du Val del Pezzo surface X' of degree one, take $\pi'\colon Y' \longrightarrow X'$ and $g'\colon Z' \longrightarrow Y'$ as above. Then $X \cong X'$ if and only if $Z \cong Z'$.*

(3) *Suppose that $p = 3$ and Z is a rational quasi-elliptic surface. Then Z is of type (1) (resp. (2), (3)) if and only if* $\mathrm{Dyn}(X) = E_8$ *(resp. $A_2 + E_6, 4A_2$).*

(4) *Suppose that $p = 2$ and Z is a rational quasi-elliptic surface. Then Z is of type (a) (resp. (b), (c), (d), (e), (f), (g)) if and only if* $\mathrm{Dyn}(X) = E_8$ *(resp. $D_8, A_1 + E_7, 2D_4, 2A_1 + D_6, 4A_1 + D_4, 8A_1$)*

Proof. (1): A general member of $|-K_Y|$ is isomorphic to its image by π since it is disjoint from the exceptional divisor E_π of π. On the other hand, the base locus of $|-K_Y|$ consists of one point, say y. Since any two members of $|-K_Y|$ intersect transversely with each other at y, each f-fiber is isomorphic to its image on Y. Hence a general f-fiber is isomorphic to its image on X, and the assertion holds.
(2): Take $f'\colon Z' \longrightarrow \mathbb{P}^1_k$ as the morphism given by $|-K_{Z'}|$. Suppose that there is an isomorphism $X \cong X'$. Then it ascends to an isomorphism $Z \cong Z'$ since the construction of π, π', g, and g' are canonical. On the other hand, suppose that there is an isomorphism $\sigma\colon Z \cong Z'$. Then this isomorphism is compatible with the genus one fibration structures since $-K_Z \sim \sigma^*(-K_Z')$. In particular, σ maps each f-section to an f'-section. Since E_g (resp. $E_{g'}$) is an f-section (resp. an f'-section) and the Mordell-Weil group $\mathrm{MW}(Z)$ acts on the set of f-sections transitively, we may assume that σ maps E_g to $E_{g'}$. Hence it descends to an isomorphism $\sigma_Y\colon Y \cong Y'$. Since both π (resp. π') is the contraction of all the (-2)-curves on Y (resp. Y'), σ_Y also descends to the desired isomorphism $\sigma_X\colon X \cong X'$.
(3) and (4): Since $\mathrm{MW}(Z)$ acts on the set of f-sections transitively, we may assume that E_g is the section O in Figures 5.1–5.4. Hence the assertions follows from Figures 5.1–5.5 and Tables 5.2 and 5.4. $\square$

We will use the following proposition in Section 5.6.

Proposition 5.4.2. *Let X be a Du Val del Pezzo surface with* $\mathrm{Dyn}(X) = 4A_2$. *Then X satisfies (NB) if and only if $p = 3$.*

Proof. The only if part follows from Proposition 5.4.1. To show the other direction, we suppose that $p = 3$. Let us take Y and Z as in Proposition 5.4.1. Suppose by contradiction that a general member of the anti-canonical linear system of X is smooth. Then Z is an extremal rational elliptic surface with four singular fibers. By [69, Theorem 2.1], its singular fibers are $(\mathrm{I}_8, \mathrm{I}_2, \mathrm{I}_1, \mathrm{I}_1)$, $(\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1)$, or $(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$. However, this implies that $\mathrm{Dyn}(X) = A_1+A_7, 2A_4$, or $2A_1+2A_3$, a contradiction. □

Next, we treat the case where the degree is two. The following proposition claims that the double covering associated to the anti-canonical linear system must be purely inseparable.

Proposition 5.4.3. *Let X be a Du Val del Pezzo surface with $K_X^2 = 2$. Suppose that the double covering $\varphi_{|-K_X|}\colon X \longrightarrow \mathbb{P}^2_k$ associated to the anti-canonical linear system is separable. Then a general member of the anti-canonical linear system is smooth.*

Proof. Take the minimal resolution $\pi\colon Y \longrightarrow X$. Let $t \in \mathbb{P}^2_k$ be a general point and $V \subset |-K_Y|$ the pullback of the pencil of lines in $\mathbb{P}^2_k$ passing through t. Then the base locus of V consists of two points, say y_1 and y_2, such that there are no (-1)-curves passing through y_1 or y_2 because t is general and there exist only finitely many (-1)-curves on Y. Let $g\colon Z \longrightarrow Y$ be the blow-up at y_1 and y_2, and E_i the g-exceptional divisor over y_i for $i \in \{1, 2\}$. Then g gives a resolution $f\colon Z \longrightarrow \mathbb{P}^1_k$ of the indeterminacy of the pencil $\varphi_V\colon Y \dashrightarrow \mathbb{P}^1_k$. Since any two members of V intersect transversely at y_1 and y_2, a general f-fiber is isomorphic to its image on Y.

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[drr, "f"'] & Y \arrow[r, "\pi"] & X \arrow[r, "\varphi_{|-K_X|}"] \arrow[d, dashed, "\varphi_V"] & \mathbb{P}^2_k \\
 & & \mathbb{P}^1_k &
\end{tikzcd}$$

Now let us show that a general member of $|-K_X|$ is smooth. Suppose by contradiction that members of $|-K_X|$ are all singular. Then Z is a quasi-elliptic surface, and E_1 and E_2 are f-sections by the same arguments as in Proposition 5.4.1. Since there are no (-1)-curves on Y which pass through y_1 or y_2, each (-2)-curves in Z either intersects with both E_1 and E_2 or is disjoint from both E_1 and E_2. However, there is no such a choice of two sections by Figures 5.1–5.5 and Tables 5.2 and 5.4, a contradiction. □

Proposition 5.4.4. *Let X be a Du Val del Pezzo surface with $K_X^2 = 2$ satisfying (NB). Then $p = 2$ and* $\mathrm{Dyn}(X) = E_7$, $A_1 + D_6$, $3A_1 + D_4$, *or* $7A_1$.

Proof. By Proposition 5.4.3, the double covering $\varphi_{|-K_X|}\colon X \longrightarrow \mathbb{P}^2_k$ associated to the anti-canonical linear system is purely inseparable. In particular, we have $p = 2$. Take π, t and V as in Proposition 5.4.3. By the generality of t, the base locus of V consists of one point, say y, and no (-1)-curves pass through y. For general two members C_1 and C_2 of V, they intersect with each other at y with multiplicity two since $\varphi_{|-K_X|}$ is a homeomorphism. Moreover, one of them is smooth at y since otherwise $2 = K_Y^2 = (C_1 \cdot C_2) \geqslant 4$. Thus general members of V are smooth at y, and have the same tangent direction at y. Hence there is a point y' infinitely near y such that the blow-up $g\colon Z \longrightarrow Y$ at y and y' gives a resolution $f\colon Z \longrightarrow \mathbb{P}^1_k$ of indeterminacy of the pencil $\varphi_V\colon Y \dashrightarrow \mathbb{P}^1_k$. Since a general member of V is smooth at y, a general f-fiber is isomorphic to its image on X. In particular, Z is a quasi-elliptic surface. By construction, E_g consists of a (-1)-curve E_1 and a (-2)-curve E_2. In particular, E_1 is an f-section and E_2 is contained in a reducible f-fiber.

Suppose that the f-fiber containing E_2 has simple normal crossing support. Then there is another (-2)-curve C intersecting with E_2. Since C and E_2 are contained in the same f-fiber, E_1 is disjoint from C. This implies, however, g_*C is a (-1)-curve passing through y, a contradiction with the choice of y. Hence E_2 is contained in a reducible f-fiber whose support is not simple normal crossing. Theorem 5.2.2 now shows that E_2 is contained in a reducible f-fiber of type III, where we use Kodaira's notation, and Y is one of the types (c), (e), (f), and (g) in Table 5.3. By Figures 5.2–5.5 and Table 5.4, we conclude that $\mathrm{Dyn}(X) = E_7$, $A_1 + D_6$, $3A_1 + D_4$, or $7A_1$. □

Finally, let us show that there are several constructions of Du Val del Pezzo surfaces of degree two satisfying (NB).

Lemma 5.4.5. *Let X be a del Pezzo surface satisfying (NB) such that* $\mathrm{Dyn}(X) = E_7, A_1 + D_6, 3A_1 + D_4,$ *or* $7A_1$. *Let Y be the minimal resolution of X. Then the following hold.*

(0) $K_X^2 = 2$ *and* $p = 2$.

(1) *For each point $t \in Y$ not contained in any negative rational curves, there is a rational quasi-elliptic surface Z, an irreducible component T of reducible fiber of type III, and a section S of Z intersecting with T such that Y is given from Z by contracting $S \cup T$ to t.*

(2) *If* $\mathrm{Dyn}(X) = E_7$ *(resp.* $A_1 + D_6, 3A_1 + D_4, 7A_1$*), then Z as in the assertion (1) is of type (c) (resp. (e), (f), (g)).*

(3) *If* $\mathrm{Dyn}(X) = E_7, A_1 + D_6,$ *or* $3A_1 + D_4$, *then the union of the negative rational curves on Y is a simple normal crossing divisor. Moreover, Figure 5.6 is the dual graph of the configuration of the negative rational curve, where black nodes (resp. white nodes) corresponds to a (-1)-curve (resp. a (-2)-curve).*

(4) *If* $\mathrm{Dyn}(X) = 7A_1$, *then there are exactly seven (-1)-curves and seven (-2)-curves whose intersection matrix is as in Figure 5.7.*

(5) *If* $\mathrm{Dyn}(X) = E_7$*, then* Y *is also obtained from the rational quasi-elliptic surface of type (a) by blowing down* O *and* $\Theta_{\infty,0}$ *in Type (a) of Figure 5.2.*

(6) *If* $\mathrm{Dyn}(X) = A_1 + D_6$*, then* Y *is also obtained from the rational quasi-ellptic surface of type (b) (resp. (c)) by blowing down* O *and* $\Theta_{\infty,0}$ *(resp.* O *and* $\Theta_{0,0}$*) in Type (b) (resp. (c)) of Figure 5.2.*

(7) *If* $\mathrm{Dyn}(X) = 3A_1 + D_4$*, then* Y *is also obtained from a rational quasi-ellptic surface of type (d) (resp. (e)) by blowing down* O *and* $\Theta_{0,0}$ *(resp.* O *and* $\Theta_{1,2}$*) in Type (d) (resp. (e)) of Figure 5.2.*

(8) *If* $\mathrm{Dyn}(X) = 7A_1$*, then* Y *is also obtained from a rational quasi-ellptic surface of type (f) by blowing down* O *and* $\Theta_{1,0}$ *in Figure 5.3.*

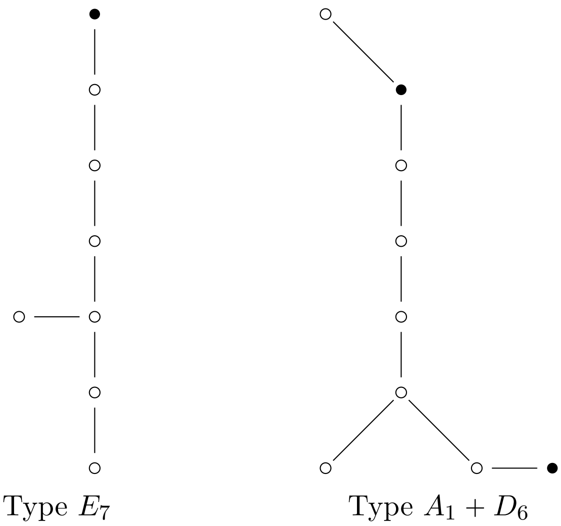

Figure 5.6: Dual graphs of negative rational curves in a Du Val del Pezzo surface of type $E_7, A_1 + D_6$, or $3A_1 + D_4$ satisfying (NB)

$$\left(\begin{array}{ccccccc|ccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & & & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & & & & -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & -2 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & -2 & 0 & 0 & 0 \\
 & & & & & & & & & & & -2 & 0 & 0 \\
 & & & & & & & & & & & & -2 & 0 \\
 & & & & & & & & & & & & & -2
\end{array}\right)$$

Figure 5.7: The intersection matrix of negative rational curves in a Du Val del Pezzo surface of type $7A_1$

Proof. The assertion (0) follows from Lemma Propositions 5.4.1 and 5.4.4. The essentially same proof as in that of Proposition 5.4.4 shows the assertions (1) and (2). We see at once that the contraction of O and $\Theta_{\infty,0}$ in Types (c) and (e) of Figure 5.2 and Figure 5.3 gives the dual graph as in Figure 5.6, and the assertion (3) holds.

Suppose that $\mathrm{Dyn}(X) = 7A_1$ and we follow the notation of the proof of Lemma 5.2.6. By contracting $A_{0,2}$ and $\Theta_{0,2}$ in Figure 5.4, $A_{i,1}$, $A_{i,2}$, $\Theta_{i,1}$, $\Theta_{i,2}$ $A_{0,1}$, and $\Theta_{0,1}$ become a (-1)-curve, a (0)-curve, a (-2)-curve, a (0)-curve, a (1)-curve, and a cuspidal curve of self intersection number two respectively for $1 \leqslant i \leqslant 7$. Hence the assertion (4) holds.

Finally, let us show the assertions (5)–(8). Let E be a (-1)-curve in Y and $t \in E$ a point not contained in any (-2)-curve. Then the blow-up Y_t of Y at t is a weak del Pezzo surface whose all members of anti-canonical linear system are singular. Hence Y_t is the blow-down of a section in a rational quasi-elliptic surface Z_t.

Now suppose that $\mathrm{Dyn}(X) = E_7$ and let E correspond the black node in Type E_7 of Figure 5.6. Then Y_t contains eight (-2)-curves whose configuration is the Dynkin diagram E_8. By Proposition 5.4.1 (4), Z_t is of type (a), and hence the assertion (5) holds.

Similarly, if $\mathrm{Dyn}(X) = A_1 + D_6$ (resp. $3A_1 + D_4$), then by Type $A_1 + D_6$ (resp. $3A_1 + D_4$) of Figure 5.6, there are two possibility of the number of (-2)-curves intersecting with E, and Y_t contains eight (-2)-curves whose configuration is the Dynkin diagram D_8 or $A_1 + E_7$ (resp. $2D_4$ or $2A_1 + D_6$). On the other hand, if $\mathrm{Dyn}(X) = 7A_1$, then Figure 5.7 shows that E is unique up to symmetry, and Y_t contains eight (-2)-curves whose configuration is the Dynkin diagram $D_4 + 4A_1$. Therefore Proposition 5.4.1 (4) shows assertions (6)–(8). □

5.5 Isomorphism classes

In this section, we determine the isomorphism classes and the automorphism groups of Du Val del Pezzo surfaces satisfying (NB).

5.5.1 Characteristic three

In this subsection, we treat the case where $p = 3$.

Proposition 5.5.1. *Let X be a Du Val del Pezzo surface satisfying (NB) in $p = 3$ and $\pi\colon Y \longrightarrow X$ be the minimal resolution. Then the following hold.*

(1) $K_X^2 = 1$.

(2) $\mathrm{Dyn}(X) = E_8$, $A_2 + E_6$, *or* $4A_2$. *Moreover, X is uniquely determined up to isomorphism by* $\mathrm{Dyn}(X)$.

(3) *If* $\mathrm{Dyn}(X) = E_8$, *then Y is constructed from $\mathbb{P}^2_{k,[x:y:z]}$ by blowing up at* $[0:1:0]$ *eight times along $\{x^3 + y^2 z = 0\}$. Moreover, each negative rational curve is either exceptional over $\mathbb{P}^2_k$ or the strict transform of $\{z = 0\}$.*

(4) *If* $\mathrm{Dyn}(X) = A_2 + E_6$*, then* Y *is constructed from* $\mathbb{P}^2_{k,[x:y:z]}$ *by blowing up at* $[0:0:1]$ *twice along* $\{y = 0\}$*, at* $[0:1:1]$ *three times along* $\{y = z\}$*, and at* $[0:1:0]$ *three times along* $\{z = 0\}$*. Moreover, each negative rational curve is either exceptional over* $\mathbb{P}^2_k$ *or the strict transform of* $\{y = 0\}$*,* $\{y = z\}$*,* $\{z = 0\}$*, or* $\{x = 0\}$*.*

(5) *If* $\mathrm{Dyn}(X) = 4A_2$*, then* Y *is constructed from* $\mathbb{P}^2_{k,[x:y:z]}$ *by blowing up all the* $\mathbb{F}_3$*-rational points on* $\{z \neq 0\}$ *except* $[0:0:1]$*. Moreover, each negative rational curve on* Y *is either exceptional over* $\mathbb{P}^2_k$ *or the strict transform of lines passing through two of the eight points as above.*

(6) Y *and each negative rational curve on* Y *are defined over* $\mathbb{F}_3$*.*

Proof. (1): The assertion follows from Lemma 5.1.3 (3) and Proposition 5.4.4.
(2): The assertion follows from Proposition 5.4.1 and Theorem 5.2.1.
(3)–(5): Let $g\colon Z \longrightarrow Y$ be the blow-up at the base point of $|-K_Y|$. By Proposition 5.4.1 (3), Z is a rational quasi-elliptic surface of type (1), (2), and (3) when $\mathrm{Dyn}(X) = E_8, A_2 + E_6$, and $4A_2$ respectively. Since the $\mathrm{MW}(Z)$-action on the set of sections of Z is transitive, we may assume that g is the contraction of the section O in Figure 5.1. Take $h\colon Z \longrightarrow \mathbb{P}^2_k$ as in Lemma 5.2.4. Then the assertion follows from the description of the induced morphism $h'\colon Y \longrightarrow \mathbb{P}^2_k$ and the image of negative rational curves on Z via h.
(6): The assertions directly follow from the assertions (3)–(5). $\square$

Corollary 5.5.2. *Let* X *be a Du Val del Pezzo surface satisfying (NB) in* $p = 3$*. When* $\mathrm{Dyn}(X) = E_8$ *(resp.* $A_2 + E_6$*,* $4A_2$*),* $\mathrm{Aut}\, X$ *is isomorphic to*

$$\left\{ \begin{pmatrix} a & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & a^3 \end{pmatrix} \in \mathrm{PGL}(3,k) \,\middle|\, a \in k^*, c \in k \right\} \ \textit{(resp. } k^* \times \mathbb{Z}/2\mathbb{Z}, \mathrm{GL}(2,\mathbb{F}_3)\textit{)}.$$

Proof. We follow the notation of the proof of Proposition 5.5.1.

Since every morphism from Y to X factors through the minimal resolution π, we have a canonical homomorphism $\varphi\colon \mathrm{Aut}\, X \longrightarrow \mathrm{Aut}\, Y$ such that $\sigma \circ \pi = \pi \circ \varphi(\sigma)$ for all $\sigma \in \mathrm{Aut}\, X$. On the other hand, π is the contraction of all the (-2)-curves on Y. Since each automorphism of Y fixes the union of (-2)-curves, we also have a canonical homomorphism $\psi\colon \mathrm{Aut}\, Y \longrightarrow \mathrm{Aut}\, X$, which is the inverse of φ. Hence $\mathrm{Aut}\, X \cong \mathrm{Aut}\, Y$.

First, suppose that $\mathrm{Dyn}(X) = E_8$. By Type (1) of Figure 5.1, each negative rational curve on Y is $g(\Theta_{\infty,i})$ for some $0 \leqslant i \leqslant 8$. The $\mathrm{Aut}\, Y$-action on Y fixes the unique (-1)-curve $g(\Theta_{\infty,0})$. It also fixes $g(\Theta_{\infty,1})$, which is the unique (-2)-curve intersecting with $g(\Theta_{\infty,0})$. By a similar argument, it fixes each negative rational curve. Hence the $\mathrm{Aut}\, Y$-action descends to $\mathbb{P}^2_k$ via h'. In particular, $\mathrm{Aut}\, Y$ is contained in the subgroup G of $\mathrm{PGL}(3,k) \cong \mathrm{Aut}\, \mathbb{P}^2_k$ fixing $h'_*|-K_Y|$. On the other hand, since $h_*|-K_Z| = h'_*|-K_Y|$ and $|-K_Z|$ are base point free, Z is the minimal resolution of indeterminacy of $h'_*|-K_Y|$. In particular, the G-action on $\mathbb{P}^2_k$ ascends to Z. Since Z has a unique section, it descends to Y. Therefore

$\operatorname{Aut} Y \cong G$. Since $h(F) = \{x^3 + y^2 z = 0\}$ and $h(\Theta_{\infty,8}) = \{z = 0\}$, we have $h'_*|-K_Y| = \{sz^3 + t(x^3 + y^2 z) = 0 \mid [s : t] \in \mathbb{P}^1_k\}$. Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

be an element of G. Since $\operatorname{Aut} Y$ fixes $h'(g(\Theta_{\infty,8})) = h(\Theta_{\infty,8}) = \{z = 0\}$, we have $g = h = 0$ and $i \neq 0$. On the other hand, we have

$$A \cdot (x^3 + y^2 z) = (a^3 x^3 + b^3 y^3 + c^3 z^3) + (dx + ey + fz)^2 (iz) \in h'_*|-K_Y|.$$

Since the coefficients of y^3, $x^2 z$, and yz^2 must be zero, we have $b = 0$, $d = 0$, $e \neq 0$, and $f = 0$. Since the coefficient of x^3 must coincide with that of $y^2 z$, we have $a^3 = e^2 i$. Fixing $e = 1$, we obtain the assertion.

Next, suppose that $\operatorname{Dyn}(X) = A_2 + E_6$. By Type (2) of Figure 5.1, each negative rational curve on Y is $g(P)$, $g(2P)$, $g(\Theta_{0,i})$ for some $0 \leqslant i \leqslant 2$, or $g(\Theta_{\infty,i})$ for some $0 \leqslant i \leqslant 6$. The $\operatorname{Aut} Y$-action on Y fixes $g(\Theta_{0,0})$, which is the unique (-1)-curve intersecting with one (-1)-curve and two (-2)-curves. Then it also fixes $g(\Theta_{\infty,2})$, $g(\Theta_{\infty,1})$, and $g(\Theta_{\infty,0})$. On the other hand, there are exactly two (-1)-curves on Y intersecting with no other (-1)-curves, which are $g(P)$ and $g(2P)$. Then the $\operatorname{Aut} Y$-action on Y fixes $g(P) \cup g(2P)$. Similarly, it fixes $g(\Theta_{\infty,4}) \cup g(\Theta_{\infty,6})$, $g(\Theta_{\infty,3}) \cup g(\Theta_{\infty,5})$, and $g(\Theta_{0,1}) \cup g(\Theta_{0,2})$. Hence the $\operatorname{Aut} Y$-action descends to $\mathbb{P}^2_k$ via h'. In particular, the $\operatorname{Aut} Y$-action on $\mathbb{P}^2_k$ fixes $h(O) = [0 : 0 : 1]$, $h(\Theta_{\infty,0}) = \{x = 0\}$, and $h(\Theta_{0,1}) \cup h(\Theta_{0,2}) = \{z(y + z) = 0\}$. In particular, it fixes $h(\Theta_{0,1}) \cap h(\Theta_{0,2}) = [1 : 0 : 0]$ and $h(\Theta_{\infty,0}) \cap (h(\Theta_{0,1}) \cup h(\Theta_{0,2})) = \{[0 : 1 : 0], [0 : 1 : 1]\}$. On the other hand, by construction, every automorphism on $\mathbb{P}^2_k$ ascends to Y via h' if they fix $[0 : 0 : 1]$, $[1 : 0 : 0]$, and $\{[0 : 1 : 0], [0 : 1 : 1]\}$. Hence $\operatorname{Aut} Y$ is isomorphic to

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & h & i \end{pmatrix} \in \mathrm{PGL}(3, k) \,\middle|\, a \in k^*, (h, i) = (0, 1) \text{ or } (1, -1) \right\} \cong k^* \times \mathbb{Z}/2\mathbb{Z}.$$

Finally, suppose that $\operatorname{Dyn}(X) = 4A_2$. By Type (3) of Figure 5.1, each (-1)-curve on Y is either $g(\Theta_{i,0})$ for some $i = 0, -1, 1, \infty$, or the image of a section. The former intersects with another (-1)-curve at $g(O)$ and the latter intersects with no other (-1)-curves. Hence the $\operatorname{Aut} Y$-action on Y fixes $g(O)$ and $E_{h'}$. In particular, it descends to $\mathbb{P}^2_k$ via h' and fixes $h(O) = [0 : 0 : 1]$ and $h(E_h)$, which are $\mathbb{F}_3$-rational points not contained in $\{z = 0\}$. On the other hand, by construction, every automorphism on $\mathbb{P}^2_k$ ascends to Y via h' if they fix $[0 : 0 : 1]$ and $h(E_h)$. Hence $\operatorname{Aut} Y$ is isomorphic to the subgroup of $\mathrm{PGL}(3, \mathbb{F}_3)$ which fixes $\{z = 0\}$ and $[0 : 0 : 1]$, which is $\mathrm{GL}(2, \mathbb{F}_3)$.

Combining these arguments, we complete the proof. □

5.5.2 Characteristic two

In this subsection, we always assume that $p = 2$.

First let us show that, when the degrees are two, Dynkin types determine the isomorphism classes of Du Val del Pezzo surfaces satisfying (NB).

Proposition 5.5.3. *The minimal resolution of each del Pezzo surface of type E_7 satisfying (NB) is constructed from $\mathbb{P}^2_{k,[x:y:z]}$ by blowing up at $[0:1:0]$ seven times along $\{x^3 + y^2 z = 0\}$. In particular, there is a unique del Pezzo surface of type E_7 satisfying (NB).*

Proof. We follow the notation of Type (a) of Figure 5.2. Let Z be the rational quasi-elliptic surface of type (a) and F a general fiber. Let $g\colon Z \longrightarrow Y$ be the contraction of O and $\Theta_{\infty,0}$ and $\pi\colon Y \longrightarrow X$ the contraction of all (-2)-curves. Then the desired del Pezzo surface must be X by Lemma 5.4.5 (5). Take $h\colon Z \longrightarrow \mathbb{P}^2_k$ and coordinates of $\mathbb{P}^2_k$ as in Lemma 5.2.4. Let $h'\colon Y \longrightarrow \mathbb{P}^2_k$ be the morphism induced by h. Then h' is the blow-up at $h(O) = [0:1:0]$ seven times along $h(F) = \{x^3 + y^2 z = 0\}$. Hence it suffices to show that X satisfies (NB).

Since π and h' is an isomorphism around a general member of $|-K_X|$, we are reduced to proving that $h'_*|-K_Y|$ has only a singular member. By construction, $h'_*|-K_Y|$ consists of cubic curves intersecting with $h(F) = \{x^3 + y^2 z = 0\}$ at $h(O) = [0:1:0]$ with multiplicity seven. Then it is generated by $\{x^3 + y^2 z = 0\}, \{z^3 = 0\}$, and $\{xz^2 = 0\}$. The Jacobian criterion now shows that $h'_*|-K_{Y'}|$ has only a singular member, and the assertion holds. $\square$

Corollary 5.5.4. *Let X be the del Pezzo surface of type E_7 satisfying (NB) and $\pi\colon Y \longrightarrow X$ the minimal resolution. Then the following hold.*

(1) *Y and each negative rational curve on Y are defined over $\mathbb{F}_2$.*

(2) $\operatorname{Aut} X$ *is isomorphic to*

$$\left\{ \begin{pmatrix} a & 0 & d^2 a \\ d & 1 & f \\ 0 & 0 & a^3 \end{pmatrix} \in \mathrm{PGL}(3,k) \,\middle|\, a \in k^*, d \in k, f \in k \right\}.$$

Proof. We follow the notation of the proof of Proposition 5.5.3.
(1): By the construction of h', Y and each irreducible component of the exceptional divisor $E_{h'}$ of h' are defined over $\mathbb{F}_2$. Since Z is of type (a), Lemma 5.2.4 shows that a negative rational curve on Y is either a component of $E_{h'}$ or the strict transform of $h(\Theta_{\infty,8}) = \{z = 0\}$. Hence the assertion holds.
(2): As in the proof of Corollary 5.5.2, we have $\operatorname{Aut} X \cong \operatorname{Aut} Y$. By Type E_7 of Figure 5.6, the $\operatorname{Aut} Y$-action on Y fixes the (-1)-curve and each (-2)-curve. In particular, the $\operatorname{Aut} Y$-action naturally descends to $\mathbb{P}^2_k$ via h'. Hence $\operatorname{Aut} Y$ is contained the subgroup G of $\mathrm{PGL}(3,k)$ which fixes the net $h'_*|-K_Y| = \{sz^3 + t(xz^2) + u(x^3 + y^2 z) = 0 \mid [s:t:u] \in \mathbb{P}^2_k\}$.

On the other hand, $|-K_Y|$ is base point free by Lemma 5.1.3 (7). Since (-1)-curves on Y are of $(-K_Y)$-degree one, every blow-down of Y collapses the base point freeness of $|-K_Y|$. Hence Y is the minimal resolution of indeterminacy of $h'_*|-K_Y|$. In particular, we obtain $G \subset \operatorname{Aut} Y$.

Therefore $\operatorname{Aut} Y \cong G$. Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

be an element of G. Then

$$\begin{aligned}&A\cdot z^3\\ =&i^3z^3+gi^2xz^2+(g^3x^3+h^2iy^2z)+h^3y^3+g^2hx^2y+g^2ix^2z+h^2gxy^2+hi^2yz^2.\end{aligned}$$

Since the coefficient of y^3 must be zero, we have $h=0$. Now the coefficient of x^3 also must be zero. Hence $g=0$ and $i\neq 0$. Similarly,

$$A\cdot xz^2=ci^2z^3+ai^2xz^2+bi^2yz^2$$

implies that $b=0$ and

$$A\cdot(x^3+y^2z)=(c^3+f^2i)z^3+ac^2xz^2+(a^3x^3+e^2iy^2z)+(a^2c+d^2i)x^2z$$

implies that $a^3=e^2i$ and $a^2c=d^2i$. Fixing $e=1$, we obtain the assertion. □

Proposition 5.5.5. *The minimal resolution of each del Pezzo surface of type A_1+D_6 satisfying (NB) is constructed from $\mathbb{P}^2_{k,[x:y:z]}$ by blowing up at $[0:1:0]$ five times along $\{x^3+y^2z=0\}$ and at $[1:1:1]$ twice along $\{x^3+y^2z=0\}$. In particular, there is a unique del Pezzo surface of type A_1+D_6 satisfying (NB).*

Proof. We follow the notation of Type (b) of Figure 5.2. Let Z be the rational quasi-elliptic surface of type (b) and F a general fiber. Let $g\colon Z\longrightarrow Y$ be the contraction of O and $\Theta_{\infty,0}$ and $\pi\colon Y\longrightarrow X$ the contraction of all (-2)-curves. Then the desired del Pezzo surface must be X by Lemma 5.4.5 (6). Take $h\colon Z\longrightarrow\mathbb{P}^2_k$ and coordinates of $\mathbb{P}^2_k$ as in Lemma 5.2.4. Let $h'\colon Y\longrightarrow\mathbb{P}^2_k$ be the morphism induced by h. Then h' is the composition of the blow-ups at $h(P)=[0:1:0]$ five times along $h(F)=\{x^3+y^2z=0\}$ and at $h(O)=[1:1:1]$ twice along $\{x^3+y^2z=0\}$. Hence it suffices to show that X satisfies (NB).

Since π and h' is an isomorphism around a general member of $|-K_X|$, it suffices to show that $h'_*|-K_Y|$ has only a singular member. By construction, $h'_*|-K_Y|$ consists of cubic curves intersecting with $h(F)=\{x^3+y^2z=0\}$ at $h(P)=[0:1:0]$ five times and at $h(O)=[1:1:1]$ twice. Then it is generated by $\{x^3+y^2z=0\},\{(x+z)z^2=0\}$, and $\{(x+z)^2z=0\}$. The Jacobian criterion now shows that $h'_*|-K_Y|$ has only a singular member, and the assertion holds. □

Corollary 5.5.6. *Let X be the del Pezzo surface of type A_1+D_6 satisfying (NB) and Y the minimal resolution of X. Then the following hold.*

(1) *Y and each negative rational curve on Y are defined over $\mathbb{F}_2$.*

(2) $\operatorname{Aut}X$ *is isomorphic to*

$$\left\{\left(\begin{smallmatrix}a&0&a^3+a\\ d&1&a^3+d+1\\ 0&0&a^3\end{smallmatrix}\right)\in\mathrm{PGL}(3,k)\,\middle|\,a\in k^*,d\in k\right\}.$$

(3) *There is a birational morphism $h'_1\colon Y\longrightarrow\mathbb{P}^1_{k,[x:y]}\times\mathbb{P}^1_{k,[s:t]}$ such that each negative rational curve on Y is either h'_1-exceptional or the strict transform of $\{x=0\}$, $\{y=0\}$, or $\{s=0\}$. Moreover, h'_1 is decomposed into six blow-ups at $\mathbb{F}_2$-rational points.*

Proof. We follow the notation of the proof of Proposition 5.5.5.
(1): By the construction of h', Y and each irreducible component of $E_{h'}$ are defined over $\mathbb{F}_2$. Since Z is of type (b), Lemma 5.2.4 shows that a negative rational curve on Y is either a component of $E_{h'}$ or the strict transform of $h(\Theta_{\infty,8}) = \{z = 0\}$ or $h(\Theta_{\infty,3}) = \{x + z = 0\}$. Hence the assertion holds.
(2): Analysis similar to that in the proof of Corollary 5.5.4 shows that $\mathrm{Aut}\, X \cong \mathrm{Aut}\, Y$ is the subgroup of $\mathrm{PGL}(3, k)$ which fixes $h'_*| - K_Y| = \{s((x + z)z^2) + t((x + z)^2 z) + u(x^3 + y^2 z) = 0 \mid [s : t : u] \in \mathbb{P}^2_k\}$, and the assertion holds.
(3): Take $h_1 \colon Z \longrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$ as the contraction of O, P, and $\Theta_{\infty,i}$ for $i = 0, 2, 3, 5, 6$, and 7. The induced morphism $h'_1 \colon Y \longrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$ satisfies the first assertion. The second assertion follows from (1). □

Proposition 5.5.7. *The minimal resolution of each del Pezzo surface of type $3A_1 + D_4$ satisfying (NB) is constructed from $\mathbb{P}^2_{k,[x:y:z]}$ by blowing up at $[1 : 0 : 0]$ once, at $[0 : 0 : 1]$ twice along $\{y = 0\}$, at $[0 : 1 : 1]$ twice along $\{y + z = 0\}$, and at $[0 : 1 : 0]$ twice along $\{z = 0\}$. In particular, there is a unique del Pezzo surface of type $3A_1 + D_4$ satisfying (NB).*

Proof. We follow the notation of Type (d) of Figure 5.2. Let Z be a rational quasi-elliptic surface of type (d). Let $g \colon Z \longrightarrow Y$ be the contraction of O and $\Theta_{0,0}$ and $\pi \colon Y \longrightarrow X$ the contraction of all (-2)-curves. Then the desired del Pezzo surface must be X by suitable choice of Z by Lemma 5.4.5 (7). Hence it suffices to show that X is independent of the choice of Z and satisfies (NB). Take $h \colon Z \longrightarrow \mathbb{P}^2_k$ and coordinates of $\mathbb{P}^2_k$ as in Lemma 5.2.4. Let $h' \colon Y \longrightarrow \mathbb{P}^2_k$ be the morphism induced by h. Then h' is the composition of the blow-ups at $h(\Theta_{0,4}) = [1 : 0 : 0]$ once, at $h(\Theta_{\infty,1}) = [0 : 0 : 1]$ twice along $h(\Theta_{0,1}) = \{y = 0\}$, at $h(\Theta_{\infty,2}) = [0 : 1 : 1]$ twice along $h(\Theta_{0,2}) = \{y + z = 0\}$, and at $h(\Theta_{\infty,3}) = [0 : 1 : 0]$ twice along $h(\Theta_{0,3}) = \{z = 0\}$. Hence it suffices to show that X satisfies (NB).

Since π and h' is an isomorphism around a general member of $| - K_X|$, we are reduced to proving that $h'_*|-K_Y|$ has only a singular member. By construction, $h'_*|-K_Y|$ consists of cubic curves intersecting with $h(\Theta_{0,i})$ at $h(\Theta_{\infty,i})$ with multiplicity two for $1 \leqslant i \leqslant 3$ and passing through $h(\Theta_{0,4})$. Then it is generated by $\{x^2 y = 0\}$, $\{x^2 z = 0\}$, and $\{yz(y + z) = 0\}$. The Jacobian criterion now shows that $h'_*| - K_Y|$ has only a singular member, and the assertion holds. □

Corollary 5.5.8. *Let X be the del Pezzo surface of type $3A_1 + D_4$ satisfying (NB) and Y the minimal resolution of X. Then the following hold.*

(1) *Y and each negative rational curve on Y are defined over $\mathbb{F}_2$.*

(2) $\mathrm{Aut}\, X \cong k^* \times \mathrm{PGL}(2, \mathbb{F}_2)$.

Proof. We follow the notation of the proof of Proposition 5.5.7.
(1): By the construction of h', Y and every irreducible component of $E_{h'}$ are defined over $\mathbb{F}_2$. Since Z is of type (d), Lemma 5.2.4 shows that a negative rational curve on Y is either h'-exceptional or the strict transform of one of $\{y = 0\}$, $\{y + z = 0\}$, $\{z = 0\}$, or $h(\Theta_{\infty,4}) = \{x = 0\}$. Hence the assertion holds.

(2): By the symmetry of Type $3A_1 + D_4$ of Figure 5.6, the $\mathrm{Aut}\, X \cong \mathrm{Aut}\, Y$-action on Y naturally descends to $\mathbb{P}^2_k$ via h'. Hence $\mathrm{Aut}\, Y$ is isomorphic to the subgroup of $\mathrm{Aut}\, \mathbb{P}^2_k$ generated by automorphisms fixing $\{[0:1:0], [0:1:1], [0:0:1]\}$ and $[1:0:0]$, which is

$$\left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{smallmatrix} \right) \in \mathrm{PGL}(3,k) \middle| a \in k^*, \left(\begin{smallmatrix} e & f \\ h & i \end{smallmatrix} \right) \in \mathrm{PGL}(2, \mathbb{F}_2) \right\} \cong k^* \times \mathrm{PGL}(2, \mathbb{F}_2).$$

□

Proposition 5.5.9. *The minimal resolution of each del Pezzo surface of type* $7A_1$ *is constructed from* $\mathbb{P}^2_{k,[x:y:z]}$ *by blowing up all the* $\mathbb{F}_2$*-rational points. In particular, there is a unique del Pezzo surface of type* $7A_1$.

Proof. We follow the notation of the proof of Lemma 5.2.6. Since [103] shows that the desired surface satisfies (ND), it also satisfies (NB) by Theorem 1.3.3.

Let Z be a rational quasi-elliptic surface of type (g). Let $g\colon Z \longrightarrow Y$ be the contraction of $A_{0,2}$ and $\Theta_{0,2}$ and $\pi\colon Y \longrightarrow X$ the contraction of all (-2)-curves. Then the desired del Pezzo surface must be X by a suitable choice of Z by Lemma 5.4.5 (1) and (2). Claim 5.2 in the proof of Lemma 5.2.6 now shows that the morphism $h'\colon Y \longrightarrow \mathbb{P}^2_k$ induced by $h\colon Z \longrightarrow \mathbb{P}^2_k$ is the blow-up of all the points in $\mathbb{P}^2_k$ defined over $\mathbb{F}_2$. □

Remark 5.5.10. Cascini-Tanaka [21, Proposition 6.4] proved that some del Pezzo surfaces constructed by Keel-McKernan [62, end of section 9] are isomorphic to the del Pezzo surface constructed by Langer [74, Example 8.2]. Proposition 5.5.9 gives another proof of this fact. Moreover, Proposition 5.5.9 says that this surface is also isomorphic to a counterexample to the Akizuki-Nakano vanishing theorem in [35, Proposition 11.1 (1)] with $p = n = 2$.

Corollary 5.5.11. *Let* X *be the del Pezzo surface of type* $7A_1$ *and* Y *the minimal resolution of* X. *Let* $h'\colon Y \longrightarrow \mathbb{P}^2_k$ *be the blow-up of all the* $\mathbb{F}_2$*-rational points. Then the following hold.*

(1) (-1)*-curves (resp.* (-2)*-curves) on* Y *are* h'*-exceptional (resp. the strict transform of lines in* $\mathbb{P}^2_k$ *are defined over* $\mathbb{F}_2$*). In particular,* Y *and every negative rational curve on* Y *are defined over* $\mathbb{F}_2$.

(2) *The class divisor group of* Y *is generated by the seven* (-1)*-curves and any one of* (-2)*-curves.*

(3) $\mathrm{Aut}\, X \cong \mathrm{Aut}\, Y \cong \mathrm{PGL}(3, \mathbb{F}_2)$.

(4) $\mathrm{Aut}\, Y$ *acts on both the set of* (-1)*-curves on* Y *and that of* (-2)*-curves transitively.*

(5) *For each* (-1)*-curve* E *on* Y*, the stabilizer subgroup of* $\mathrm{Aut}\, Y$ *with respect to* E *is isomorphic to* $\mathbb{F}_2^2 \rtimes \mathrm{PGL}(2, \mathbb{F}_2)$. *The first (resp. second) factor acts on* E *trivially (resp. as* $\mathrm{Aut}\, \mathbb{P}^1_{\mathbb{F}_2}$*).*

Proof. (1): There are seven h'-exceptional curves and the strict transform of lines in $\mathbb{P}^2_k$ defined over $\mathbb{F}_2$, which are (-1)-curves and (-2)-curves respectively. On the other hand, Lemma 5.4.5 (4) shows that Y contains exactly seven (-1)-curves and seven (-2)-curves. Hence the assertion holds.
(2): The assertion is obvious from the assertion (1).
(3): By the assertion (1), the $\operatorname{Aut} Y$-action on Y fixes $E_{h'}$ and descends to $\mathbb{P}^2_k$ via h'. Hence $\operatorname{Aut} Y$ equals the stabilizer subgroup of $\mathrm{PGL}(3,k)$ with respect to the set of $\mathbb{F}_2$-rational points on $\mathbb{P}^2_k$, which is $\mathrm{PGL}(3,\mathbb{F}_2)$.
(4): The assertion is obvious from the assertion (3).
(5): Fix coordinates $[x:y:z]$ of $\mathbb{P}^2_k$. By the assertion (4), we may assume that E is the strict transform of $\{x=0\} \subset \mathbb{P}^2_k$. Then the stabilizer subgroup of $\operatorname{Aut} Y$ with respect to E is

$$\begin{aligned}\left\{\begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathrm{PGL}(3,\mathbb{F}_2)\right\} &\cong \left\{\begin{pmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ g & 0 & 1 \end{pmatrix} \in \mathrm{PGL}(3,\mathbb{F}_2)\right\} \rtimes \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \in \mathrm{PGL}(3,\mathbb{F}_2)\right\} \\ &\cong \mathbb{F}_2^2 \rtimes \mathrm{PGL}(2,\mathbb{F}_2),\end{aligned}$$

and the assertion holds. □

Next, we treat the case where the degree is one.

Proposition 5.5.12. *Let X be a Du Val del Pezzo surface satisfying (NB). Suppose that $p=2$ and $\mathrm{Dyn}(X) = E_8, D_8, A_1+E_7$, or $2A_1+D_6$. Then the isomorphism class of X is uniquely determined by $\mathrm{Dyn}(X)$.*

Proof. By Proposition 5.4.1 (4), the minimal resolution of X is obtained from the rational quasi-elliptic surface Z of type (a), (b), (c), or (e) by contracting a section. Since Z is unique up to isomorphism for each types by Theorem 5.2.2, the assertion follows from Proposition 5.4.1 (2). □

Lemma 5.5.13. *Let X be a Du Val del Pezzo surface satisfying (NB) and $\pi\colon Y \longrightarrow X$ be the minimal resolution. Suppose that $p=2$ and $\mathrm{Dyn}(X) = E_8$, D_8, or A_1+E_7 in addition. Then the following hold.*

(1) *Y and every negative rational curve on Y are defined over $\mathbb{F}_2$.*

(2) *$\operatorname{Aut} X$ is isomorphic to*

$$\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & a^3 \end{pmatrix} \in \mathrm{PGL}(3,k) \middle| a \in k^*, f \in k\right\}$$

when $\mathrm{Dyn}(X) = E_8$,

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ d & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{PGL}(3,k) \middle| d \in k\right\} \cong k$$

when $\mathrm{Dyn}(X) = D_8$, *and*

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{pmatrix} \in \mathrm{PGL}(3,k) \middle| e \in k^*\right\} \cong k^*$$

when $\mathrm{Dyn}(X) = A_1+E_7$.

Proof. Let $g\colon Z \longrightarrow Y$ be the blow-up at the base point of $|-K_Y|$. When $\mathrm{Dyn}(X) = E_8$ (resp. D_8, $A_1 + E_7$), Z is the rational quasi-elliptic surface of type (a) (resp. (b), (c)) by Proposition 5.4.1 (4). We may assume that g is the contraction of O in Types (a)–(c) of Figure 5.2 by virtue of the $\mathrm{MW}(Z)$-action on Y. From now on, we follow the notation of Lemma 5.2.4. Then $h\colon Z \longrightarrow \mathbb{P}^2_k$ induces a morphism $h'\colon Y \longrightarrow \mathbb{P}^2_k$.
(1): First, suppose that $\mathrm{Dyn}(X) = E_8$. Then h' is the blow-up of $\mathbb{P}^2_k$ at $h(O) = [0:1:0]$ eight times along $h(F) = \{x^3 + y^2z = 0\}$. Hence Y and each irreducible component of $E_{h'}$ are defined over $\mathbb{F}_2$. Since each negative rational curve on Y is either a component of E_h or the strict transform of $h(\Theta_{\infty,8}) = \{z = 0\}$, the assertion holds.

Next, suppose that $\mathrm{Dyn}(X) = D_8$. Then h' is the composition of the blow-up of $\mathbb{P}^2_k$ at $h(P) = [0:1:0]$ five times along $h(F) = \{x^3 + y^2z = 0\}$ and the blow-up at $h(O) = [1:1:1]$ three times along $\{x^3 + y^2z = 0\}$. Hence Y and each irreducible component of $E_{h'}$ are defined over $\mathbb{F}_2$. Since each negative rational curve on Y is either a component of $E_{h'}$ or the strict transform of $h(\Theta_{\infty,8}) = \{z = 0\}$ or $h(\Theta_{\infty,3}) = \{x + z = 0\}$, the assertion holds.

Finally, suppose that $\mathrm{Dyn}(X) = A_1 + E_7$. Then h' is the composition of the blow-up of $\mathbb{P}^2_k$ at $h(P) = [1:0:0]$ six times along $h(\Theta_{\infty,1}) = \{xz + y^2 = 0\}$ and the blow-up at $h(O) = [0:1:0]$ twice along $h(\Theta_{\infty,0}) = \{x = 0\}$. Hence Y and each irreducible component of $E_{h'}$ are defined over $\mathbb{F}_2$. Since each negative rational curve on Y is either a component of $E_{h'}$ or the strict transform of $\{xz + y^2 = 0\}$, $\{x = 0\}$, or $h(\Theta_{0,2}) = \{z = 0\}$, the assertion holds.
(2): From Types (a)–(c) of Figure 5.2, it is easily seen that the $\mathrm{Aut}\,Y$-action on Y fixes each negative rational curve. In particular, the $\mathrm{Aut}\,Y$-action naturally descends to $\mathbb{P}^2_k$ via h'. Hence $\mathrm{Aut}\,Y$ is contained in the subgroup G of $\mathrm{PGL}(3,k)$ which fixes the net $h'_*|-K_Y|$.

On the other hand, we have $h'_*|-K_Y| = h_*|-K_Z|$. Since $|-K_Z|$ is base point free, Z is the minimal resolution of indeterminacy of $h'_*|-K_Y|$. Hence the G-action on $\mathbb{P}^2_k$ ascends to Z. When $\mathrm{Dyn}(X) = E_8$, it descends to Y since there is a unique section on Z. On the other hand, when $\mathrm{Dyn}(X) = D_8$ or $A_1 + E_7$, it also descends to Y by the asymmetry of E_h. Therefore $\mathrm{Aut}\,Y \cong G$. By the choice of coordinates $[x:y:z]$ of $\mathbb{P}^2_k$, $h'_*|-K_Y|$ is generated by $\{x^3 + y^2z = 0\}$ and $\{z^3 = 0\}$ (resp. $\{x^3 + y^2z = 0\}$ and $\{(x+z)^2z = 0\}$, $\{(xz + y^2)x = 0\}$ and $\{z^3 = 0\}$) when $\mathrm{Dyn}(X) = E_8$ (resp. D_8, $A_1 + E_7$). Hence an easy computation as in the proof of Corollary 5.5.4 gives the assertion. $\square$

Lemma 5.5.14. *Let X be the Du Val del Pezzo surface of type $2A_1 + D_6$ satisfying (NB) and $\pi\colon Y \longrightarrow X$ be the minimal resolution. Then the following hold.*

(1) *Y and every negative rational curve on Y are defined over $\mathbb{F}_2$.*

(2) $\mathrm{Aut}\,X$ *is isomorphic to*

$$\left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix} \in \mathrm{PGL}(3,k) \right\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. Since π is the minimal resolution, we have $\mathrm{Aut}\, Y \cong \mathrm{Aut}\, X$. Let $g\colon Z \longrightarrow Y$ be the blow-up at the base point of $|-K_Y|$. Then Z be the rational quasi-elliptic surface of type (e) by Proposition 5.4.1 (4). In what follows, we use the notation of Type (e) of Figure 5.2. Then we may assume that g is the contraction of O. By the shape of Type (e) of Figure 5.2, the $\mathrm{Aut}\, Y$-action on Y fixes $g(\Theta_{1,2})$. By Lemma 5.4.5 (7), the contraction of $g(\Theta_{1,2})$ gives a morphism $h\colon Y \longrightarrow W$ to the minimal resolution of the Du Val del Pezzo surface of type $3A_1 + D_4$ satisfying (NB). Hence $\mathrm{Aut}\, Y$ is isomorphic to the stabilizer subgroup of $\mathrm{Aut}\, W \cong k^* \times \mathrm{PGL}(2, \mathbb{F}_2)$ with respect to $t = h \circ g(\Theta_{1,2})$.
(2): By Type (e) of Figure 5.2, $E = h \circ g(\Theta_{1,3})$ is the unique negative rational curve containing t. Hence the $\mathrm{Aut}\, Y$-action on W fixes E. Moreover E is a (-1)-curve intersecting with exactly two (-2)-curves, which are $E_1 = h \circ g(\Theta_{1,0})$ and $E_2 = h \circ g(\Theta_{1,4})$. We have seen in the proof of Corollary 5.5.8 that the first factor (resp. the second factor) of $\mathrm{Aut}\, W \cong k^* \times \mathrm{PGL}(2, \mathbb{F}_2)$ acts on $E \backslash (E \cap (E_1 \cup E_2)) \cong k^*$ freely and transitively (resp. acts as a permutation of the third nodes from the top in Type $3A_1 + D_4$ of Figure 5.6). Hence the assertion holds.
(1): By Corollary 5.5.8, W and each negative rational curve on W are defined over $\mathbb{F}_2$. By virtue of the k^*-action on W, we may assume that t is an $\mathbb{F}_2$-rational point. Hence Y and each negative rational curve on Y except $g(\Theta_{0,0})$ and $g(\Theta_{\infty,0})$ are defined over $\mathbb{F}_2$. On the other hand, $g(\Theta_{0,0})$ and $g(\Theta_{\infty,0})$ are defined over $\mathbb{F}_{2^m}$ for some $m > 0$ since Y is defined over $\overline{\mathbb{F}_2}$, and are the unique (-1)-curves on Y intersecting with $g(\Theta_{0,1})$ and $g(\Theta_{\infty,1})$ twice respectively. Since the field extension $\mathbb{F}_{2^m}/\mathbb{F}_2$ is Galois, they are also defined over $\mathbb{F}_2$. Hence the assertion holds. □

To determine the isomorphism classes of Du Val del Pezzo surfaces of one of the types $2D_4$, $4A_1 + D_4$, and $8A_1$ satisfying (NB), we need the following notation and auxiliary lemmas.

Definition 5.5.15. For coordinates of $\mathbb{P}^n_k$, let $\mathcal{D}_n \subset \mathbb{P}^n_k$ denote the complement of all the hyperplane sections defined over $\mathbb{F}_2$. Note that $\mathrm{PGL}(n+1, \mathbb{F}_2)$ naturally acts on $\mathcal{D}_n$.

Lemma 5.5.16. *Let Σ_t be the stabilizer subgroup of $\mathrm{PGL}(2, \mathbb{F}_2)$ with respect to $t \in \mathcal{D}_1$. Then the following hold.*

(1) *Σ_t is trivial unless t is an $\mathbb{F}_4$-rational point.*

(2) *The $\mathrm{PGL}(2, \mathbb{F}_2)$-action on the set $\mathcal{D}_1(\mathbb{F}_4)$ of $\mathbb{F}_4$-rational points on $\mathcal{D}_1$ is transitive.*

(3) *$\Sigma_t = \mathbb{Z}/3\mathbb{Z}$ if t is an $\mathbb{F}_4$-rational point.*

(4) *$\mathcal{D}_1/\mathrm{PGL}(2, \mathbb{F}_2) \cong \mathbb{A}^1_k$ with a distinct point which corresponds to $\mathcal{D}_1(\mathbb{F}_4)$.*

Proof. (1): Suppose that there is a non-trivial element $A \in \Sigma_t$. Since $\mathrm{PGL}(2, \mathbb{F}_2)$ is isomorphic to the symmetric group of three letters, it has exactly three conjugacy classes. Since $A_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ are non-trivial and have different minimal polynomials, A is conjugate to A_i for some i. Then A_i also fixes some point in $\mathcal{D}_1$.

In $\mathbb{P}^1_{k,[x:y]}$, the fixed point locus of A_1 (resp. A_2) equals $\{[1:1]\}$ (resp. $\{[1:s] \mid s^2+s+1=0\}$). Hence $i=2$ and $t \in \mathcal{D}_1(\mathbb{F}_4)$.

(2): Since A_1 interchanges two points in $\mathcal{D}_1(\mathbb{F}_4)$ with each other, the assertion holds.

(3): Since the order of Σ_t equals $|\operatorname{PGL}(2,\mathbb{F}_2)|/|\mathcal{D}_1(\mathbb{F}_4)| = 3$, we obtain $\Sigma_t = \mathbb{Z}/3\mathbb{Z}$.

(4): $\mathcal{D}_1/\operatorname{PGL}(2,\mathbb{F}_2)$ is naturally embedded into $\mathbb{P}^1_k/\operatorname{PGL}(2,\mathbb{F}_2) \cong \mathbb{P}^1_k$. The complement is a point since $\operatorname{PGL}(2,\mathbb{F}_2)$ acts on $\mathbb{P}^1_k(\mathbb{F}_2)$ transitively. □

Lemma 5.5.17. *Let Σ_t be the stabilizer subgroup of $\operatorname{PGL}(3,\mathbb{F}_2)$ with respect to $t \in \mathcal{D}_2$. Then the following hold.*

1. *Σ_t is trivial unless t is an $\mathbb{F}_8$-rational point.*
2. *The $\operatorname{PGL}(3,\mathbb{F}_2)$-action on the set $\mathcal{D}_2(\mathbb{F}_8)$ of $\mathbb{F}_8$-rational points on $\mathcal{D}_2$ is transitive.*
3. *$\Sigma_t = \mathbb{Z}/7\mathbb{Z}$ if t is an $\mathbb{F}_8$-rational point.*
4. *$\mathcal{D}_2/\operatorname{PGL}(3,\mathbb{F}_2)$ is a surface with a unique singular point, which corresponds to $\mathcal{D}_2(\mathbb{F}_8)$.*

Proof. (1): Suppose that there is a non-trivial element $A \in \Sigma_t$. By [55, 27.1 Lemma], $\operatorname{PGL}(3,\mathbb{F}_2) \cong \operatorname{PSL}(2,\mathbb{F}_7)$ has exactly six conjugacy classes. Since

$$A_1 = \begin{pmatrix} 1&1&0\\0&1&0\\0&0&1 \end{pmatrix}, A_2 = \begin{pmatrix} 1&1&0\\0&1&1\\0&0&1 \end{pmatrix}, A_3 = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&1 \end{pmatrix}, A_4 = \begin{pmatrix} 0&1&0\\0&0&1\\1&1&0 \end{pmatrix}, \text{ and } A_5 = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&1 \end{pmatrix}$$

are non-trivial and have different minimal polynomials to each other, A is conjugate to A_i for some $1 \leqslant i \leqslant 5$. Then A_i also fixes some point in $\mathcal{D}_2$.

In $\mathbb{P}^2_{k,[x:y:z]}$, the fixed point locus of A_i equals

$$\begin{cases} \{y=0\} & (i=1) \\ \{[1:0:0]\} & (i=2) \\ \{[1:0:0]\} \cup \{[0:1:s] \mid s^2+s+1=0\} & (i=3) \\ \{[1:s:s^2] \mid s^3+s+1=0\} & (i=4) \\ \{[1:s:s^2] \mid s^3+s^2+1=0\} & (i=5) \end{cases}$$

Hence $i=4$ or 5, and $t \in \mathcal{D}_2(\mathbb{F}_8)$. We have proved more, namely that A is of order seven and fixes exactly three points in $\mathcal{D}_2(\mathbb{F}_8)$. By [55, 27.1 Lemma], the size of its conjugacy class is 24.

(2): $\mathbb{P}^1_k(\mathbb{F}_8)$ (resp. $\mathbb{P}^2_k(\mathbb{F}_8)$) consists of nine (resp. 73) points. Since $\mathbb{P}^2_k \backslash \mathcal{D}_2$ is the union of seven $\mathbb{P}^1_k$'s passing through three $\mathbb{F}_2$-rational points, $\mathcal{D}_2(\mathbb{F}_8)$ consists of $73 - 7\cdot 9 + (3-1)\cdot 7 = 24$ points. The Burnside lemma now shows that the number of the $\operatorname{PGL}(3,\mathbb{F}_2)$-orbits is

$$|\mathcal{D}_2(\mathbb{F}_8)/\operatorname{PGL}(3,\mathbb{F}_2)| = \frac{1}{168}(24\cdot 3 + 24\cdot 3 + 1\cdot 24 + (168-24-24-1)\cdot 0) = 1.$$

Hence $\mathrm{PGL}(3, \mathbb{F}_2)$ acts on $\mathcal{D}_2(\mathbb{F}_8)$ transitively.
(3): Since the order of Σ_t equals $|\mathrm{PGL}(3, \mathbb{F}_2)|/|\mathcal{D}_2(\mathbb{F}_8)| = 7$, we obtain $\Sigma_t = \mathbb{Z}/7\mathbb{Z}$.
(4): The quotient morphism $\mathcal{D}_2 \longrightarrow \mathcal{D}_2/\mathrm{PGL}(3, \mathbb{F}_2)$ is étale outside the image of $\mathcal{D}_2(\mathbb{F}_8)$. Hence the assertion holds. □

Finally, let us investigate Du Val del Pezzo surfaces of one of types $2D_4$, $4A_1+D_4$, and $8A_1$ satisfying (NB).

Proposition 5.5.18. *Let W be the minimal resolution of the Du Val del Pezzo surface of type $3A_1 + D_4$ satisfying (NB) and E the (-1)-curve intersecting with three (-2)-curves. Note that E is unique by Lemma 5.4.5 (3) and W and E are defined over $\mathbb{F}_2$ by Corollary 5.5.8. Then the following holds.*

(1) *The minimal resolution of each Du Val del Pezzo surface of type $2D_4$ satisfying (NB) is obtained from W by blowing up a point in $E\backslash E(\mathbb{F}_2) \cong \mathcal{D}_1$.*

(2) *Let $h_t \colon Y_t \longrightarrow W$ be the blow-up at $t \in E\backslash E(\mathbb{F}_2)$. Then Y_t is the minimal resolution of a Du Val del Pezzo surface of type $2D_4$ satisfying (NB). Moreover, for $t' \in E\backslash E(\mathbb{F}_2)$, $Y_t \cong Y_{t'}$ if and only if t' is contained in the* $\mathrm{PGL}(2, \mathbb{F}_2)$*-orbit of t.*

(2) *The isomorphism classes of del Pezzo surfaces of type $2D_4$ satisfying (NB) corresponds to the closed points of $\mathcal{D}_1/\mathrm{PGL}(2, \mathbb{F}_2)$.*

Proof. We follow the notation of Type (d) of Figure 5.2.
(1): Let Z be a rational quasi-elliptic surface of type (d). Let $g \colon Z \longrightarrow Y$ be the contraction of O. Then the minimal resolution of each del Pezzo surface of type $2D_4$ satisfying (NB) is isomorphic to Y by suitable choice of Z by Proposition 5.4.1 (4). On the other hand, by Lemma 5.4.5 (7), the contraction of $g(\Theta_{0,0})$ gives a morphism $h \colon Y \longrightarrow W$. We check at once that $E = h \circ g(\Theta_{0,4})$ and it contains the point $t = h \circ g(\Theta_{0,0})$, which is not contained in any (-2)-curves. By Corollary 5.5.8, the set of $\mathbb{F}_2$-rational points on E is the intersection of E and all (-2)-curves. Therefore Y is the blow-up of W at $t \in E\backslash E(\mathbb{F}_2)$.
(2): By Lemma 5.4.5 (7), Y_t is obtained from a rational quasi-elliptic surface of type (d) by contracting a section. Hence the former assertion follows from Proposition 5.4.1 (4).

We have seen in the proof of Corollary 5.5.8 that first (resp. second) factor of $\mathrm{Aut}\, W = k^* \times \mathrm{PGL}(2, \mathbb{F}_2)$ acts on $h(\Theta_{\infty,4})$ trivially (resp. as $\mathrm{Aut}\, \mathbb{P}^1_{\mathbb{F}_2}$). The same conclusion can be drawn for E by the choice of its coordinates. In particular, $t' \in E\backslash E(\mathbb{F}_2)$ is contained in the $\mathrm{PGL}(2, \mathbb{F}_2)$-orbit of t if and only if it is contained in the $\mathrm{Aut}\, W$-orbit of t in W. On the other hand, $Y_t \cong Y_{t'}$ if t' is contained in the $\mathrm{Aut}\, W$-orbit of t. Hence it remains to prove that t' is contained in the $\mathrm{Aut}\, W$-orbit of t if $Y_t \cong Y_{t'}$.

Suppose that there is an isomorphism $\sigma \colon Y_t \cong Y_{t'}$. Since the involution as in Lemma 5.2.5 fixes the section O, it descends to an involution $\tau \in \mathrm{Aut}\, Y_t$. Then, replacing σ with $\sigma \circ \tau$ if necessary, we may assume that $\sigma(E_{h_t}) = E_{h_{t'}}$. Hence σ

descends to an isomorphism $\overline{\sigma} \in \operatorname{Aut} W$ such that $\overline{\sigma}(t) = t'$, and the latter assertion holds.

(3): This is an immediate consequence of (1) and (2). $\square$

Corollary 5.5.19. *Let X_t be the contraction of all (-2)-curves in Y_t as in Proposition 5.5.18. Then $\operatorname{Aut} X_t \cong \operatorname{Aut} Y_t \cong (k^* \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$ when t is an $\mathbb{F}_4$-rational point of $\mathcal{D}_1$ and $k^* \rtimes \mathbb{Z}/2\mathbb{Z}$ otherwise. In particular, there is a unique Du Val del Pezzo surface $X(2D_4)$ satisfying (NB) such that $\operatorname{Aut} X \cong (k^* \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Proof. We follow the notation of Proposition 5.5.18. Let Σ be the stabilizer subgroup of $\operatorname{Aut} W$ with respect to t. Then $\Sigma = k^* \times \Sigma'$ for some $\Sigma' \subset \operatorname{PGL}(2, \mathbb{F}_2)$ since k^* acts on E trivially. By Lemma 5.5.16, $\Sigma = k^* \times \mathbb{Z}/3\mathbb{Z}$ if $t \in \mathcal{D}_1(\mathbb{F}_4)$ and $\Sigma = k^*$ otherwise. On the other hand, we can identify Σ with the stabilizer subgroup of $\operatorname{Aut} Y_t$ with respect to E_{h_t}. For $\eta \in \operatorname{Aut} Y_t$, either η or $\eta \circ \tau$ belongs to Σ. Hence $\operatorname{Aut} Y_t \cong \Sigma \rtimes \mathbb{Z}/2\mathbb{Z}$, where the last factor is generated by τ, and the first assertion holds. Since $\operatorname{PGL}(2, \mathbb{F}_2)$ acts on $\mathcal{D}_1(\mathbb{F}_4)$ transitively, the second assertion follows from Proposition 5.5.18 (2). $\square$

Proposition 5.5.20. *Let W be the minimal resolution of the Du Val del Pezzo surface of type $7A_1$ and E a (-1)-curve. Note that W and E are defined over $\mathbb{F}_2$ and E is unique up to the $\operatorname{Aut} W$-action on W by Corollary 5.5.11 (1) and (4). Then the following hold.*

(1) *The minimal resolution of each Du Val del Pezzo surface of type $4A_1 + D_4$ is obtained from W by blowing up a point in $E \backslash E(\mathbb{F}_2) \cong \mathcal{D}_1$.*

(2) *Let $h_t \colon Y_t \longrightarrow W$ be the blow-up at $t \in E \backslash E(\mathbb{F}_2)$. Then Y_t is the minimal resolution of a Du Val del Pezzo surface of type $4A_1 + D_4$. Moreover, for $t' \in E \backslash E(\mathbb{F}_2)$, $Y_t \cong Y_{t'}$ if and only if t' is contained in the $\operatorname{PGL}(2, \mathbb{F}_2)$-orbit of t.*

As a result, there is a one-to-one correspondence between the isomorphism classes of del Pezzo surfaces of type $4A_1 + D_4$ and the closed points of $\mathcal{D}_1 / \operatorname{PGL}(2, \mathbb{F}_2)$.

Proof. We follow the notation of Figure 5.3. Note that, since [103] shows that $4A_1 + D_4$ is not feasible over $\mathbb{C}$, every Du Val del Pezzo surface of type $4A_1 + D_4$ satisfies (NB) by Theorem 1.3.3.

(1): Let Z be a rational quasi-elliptic surface of type (f). Let $g \colon Z \longrightarrow Y$ be the contraction of O. Then the minimal resolution of each del Pezzo surface of type $4A_1 + D_4$ is isomorphic to Y by suitable choice of Z by Proposition 5.4.1 (4). On the other hand, by Lemma 5.4.5 (8), the contraction of $g(\Theta_{1,0})$ gives a morphism $h \colon Y \longrightarrow W$. We may assume that $E = h \circ g(\Theta_{1,4})$. Then E contains the point $t = h \circ g(\Theta_{1,0})$, which is not contained in any (-2)-curves. By Corollary 5.5.11 (1), the set of $\mathbb{F}_2$-rational points on E is the intersection of E and all (-2)-curves. Therefore Y is the blow-up of W at $t \in E \backslash E(\mathbb{F}_2)$.

(2): By Lemma 5.4.5 (8), Y_t is obtained from a rational quasi-elliptic surface of type (f) by contracting a section. Hence the former assertion follows from Proposition 5.4.1 (4).

Let C_t be the strict transform of E in Y_t, which is a (-2)-curve. By Figure 5.7, C_t intersects with three (-2)-curves in Y_t. Hence C_t is the central curve of the Dynkin diagram D_4. In particular, every automorphism of Y_t fixes C_t.

By Corollary 5.5.11 (5), $t' \in E \backslash E(\mathbb{F}_2)$ is contained in the $\mathrm{PGL}(2, \mathbb{F}_2)$-orbit of t if and only if it is contained in the $\mathrm{Aut}\, W = \mathbb{F}_2^2 \rtimes \mathrm{PGL}(2, \mathbb{F}_2)$-orbit of t in W. On the other hand, $Y_t \cong Y_{t'}$ if t' is contained in the $\mathrm{Aut}\, W$-orbit of t. Hence it remains to prove that t' is contained in the $\mathrm{Aut}\, W$-orbit of t if $Y_t \cong Y_{t'}$.

Suppose that there is an isomorphism $\sigma \colon Y_t \cong Y_{t'}$. Then $\sigma(C_t) = C_{t'}$. By Figure 5.3, E_{h_t} is the unique (-1)-curve intersecting with C_t. Hence $\sigma(E_{h_t}) = E_{h_{t'}}$ and σ descends to an isomorphism $\overline{\sigma} \in \mathrm{Aut}\, W$ such that $\overline{\sigma}(t) = t'$, and the latter assertion holds. $\square$

Corollary 5.5.21. *Let X_t be the contraction of all (-2)-curves in Y_t as in Proposition 5.5.20. Then $\mathrm{Aut}\, X_t \cong \mathrm{Aut}\, Y_t \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/3\mathbb{Z}$ when t is an $\mathbb{F}_4$-rational point and $(\mathbb{Z}/2\mathbb{Z})^2$ otherwise. In particular, there is a unique Du Val del Pezzo surface $X(4A_1 + D_4)$ such that $\mathrm{Aut}\, X \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/3\mathbb{Z}$.*

Proof. We follow the notation of Proposition 5.5.20. Since each automorphism of Y_t fixes E_{h_t}, the group $\mathrm{Aut}\, Y_t$ equals the stabilizer subgroup Σ of $\mathrm{Aut}\, W = \mathbb{F}_2^2 \rtimes \mathrm{PGL}(2, \mathbb{F}_2)$ with respect to t. Then $(\mathbb{Z}/2\mathbb{Z})^2 \cong \mathbb{F}_2^2 \subset \mathrm{Aut}\, Y_t$ since $\mathbb{F}_2^2$ acts on E trivially. The rest of the proof runs as in Corollary 5.5.19. $\square$

Proposition 5.5.22. *Let W be the minimal resolution of the Du Val del Pezzo surface of type $7A_1$ and B the union of all negative rational curves on W. Note that W and B are defined over $\mathbb{F}_2$ and $W \backslash B \cong \mathcal{D}_2$ by Corollary 5.5.11 (1). Then the following hold.*

(1) *The minimal resolution of each Du Val del Pezzo surface of type $8A_1$ is obtained from W by blowing up a point in $W \backslash B$.*

(2) *Let $h_t \colon Y_t \longrightarrow W$ be the blow-up at $t \in W \backslash B$. Then Y_t is the minimal resolution of a Du Val del Pezzo surface of type $8A_1$.*

(3) *For $t \in W \backslash B$, Figure 5.8 is the intersection matrix of negative rational curves on Y_t. Moreover, there is a (-2)-curve C_t such that E_{h_t} is a unique (-1)-curve intersecting with C_t twice.*

(4) *For $t \in W \backslash B$, $\mathrm{Aut}\, Y_t$ is contained in the affine linear group $\mathbb{F}_2^3 \rtimes \mathrm{GL}(3, \mathbb{F}_2)$ and contains its normal subgroup $\mathbb{F}_2^3$, which acts on the set of (-2)-curves transitively.*

(5) *For t and $t' \in W \backslash B$, $Y_t \cong Y_{t'}$ if and only if t' is contained in the $\mathrm{Aut}\, W \cong \mathrm{PGL}(3, \mathbb{F}_2)$-orbit of t.*

As a result, there is a one-to-one correspondence between the isomorphism classes of del Pezzo surfaces of type $8A_1$ and the closed points of $\mathcal{D}_2 / \mathrm{PGL}(3, \mathbb{F}_2)$.

$$
\left(\begin{array}{ccccccc|ccccccc|cccccccc|cccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
 & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
 & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
 & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
 & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
\hline
 & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & & & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & & & & & & & & & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & & & & & & & & -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & & & & & & & & & & & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
 & & & & & & & & & & & & & & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\hline
 & & & & & & & & & & & & & & & & & & & & & & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & & & & & & & & & & & & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & & & & & & & & & & & & & -1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & & & & & & & & & & & & & & -1 & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & & & & & & & & & & & & & & & -1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & & & & & & & & & & & & & & & & -1 & 1 & 1 \\
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & -1 & 1 \\
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & -1
\end{array}\right).
$$

Figure 5.8: The intersection matrix of the (-1)-curves and (-2)-curves in a del Pezzo surface of type $8A_1$

Proof. We follow the notation of the proof of Lemma 5.2.6. Note that, since [103] shows that $8A_1$ is not feasible over $\mathbb{C}$, every Du Val del Pezzo surface of type $8A_1$ satisfies (NB) by Theorem 1.3.3.

(1): Let Z be a rational quasi-elliptic surface of type (g). Let $g\colon Z \longrightarrow Y$ be the contraction of $A_{0,2}$. Then the minimal resolution of each del Pezzo surface of type $8A_1$ is isomorphic to Y by suitable choice of Z by Proposition 5.4.1 (4). On the other hand, by Lemma 5.4.5 (1) and (2), the contraction of $g(\Theta_{0,2})$ gives a morphism $h\colon Y \longrightarrow W$ such that $t = h \circ g(\Theta_{0,2}) \in W \backslash B$. Therefore Y is the blow-up of W at $t \in W \backslash B$.

(2): By Lemma 5.4.5 (1) and (2), Y_t is obtained from a rational quasi-elliptic surface Z_t of type (g) by contracting a section. Hence the assertion follows from Proposition 5.4.1 (4).

(3): By Lemma 5.2.6, Z_t has exactly sixteen (-1)-curves $A_{0,1}, \ldots, A_{7,1}, A_{0,2}, \ldots, A_{7,2}$ and exactly sixteen (-2)-curves $\Theta_{0,1}, \ldots, \Theta_{7,1}, \Theta_{0,2}, \ldots, \Theta_{7,2}$, whose intersection matrix is Figure 5.5. We may assume that the contraction of $A_{0,2}$ gives a morphism $g_t\colon Z_t \longrightarrow Y_t$ and $E_{h_t} = g_t(\Theta_{0,2})$. Then $g_t(A_{0,1})$ is a (0)-curve and $A'_{i,j} := g_t(A_{i,j})$ is a (-1)-curve for $1 \leqslant i \leqslant 7$ and $j = 1, 2$. Moreover, $\Theta'_{i,1} := g_t(\Theta_{i,1})$ and $\Theta'_{i,2} := g_t(\Theta_{i,2})$ is a (-2)-curve and a (-1)-curve respectively for $0 \leqslant i \leqslant 7$. Hence Figure 5.8 is the intersection matrix of $A'_{1,1}, \ldots, A'_{7,1}$, $A'_{1,2}, \ldots, A'_{7,2}$, $\Theta'_{0,1}, \ldots, \Theta'_{7,1}$, $\Theta'_{0,2}, \ldots, \Theta'_{7,2}$ in this order. Moreover, $E_{h_t} = \Theta'_{0,2}$ is a unique (-1)-curve intersecting with $C_t = \Theta'_{0,1}$ twice.

(4): Suppose that an automorphism of Y_t fixes each (-1)-curve and each (-2)-curve. Then it fixes $A'_{1,1}, \ldots, A'_{7,1}$, and $\Theta'_{0,2}$. By Claim 5.2 in the proof of Lemma 5.2.6, it descends to an automorphism of $\mathbb{P}^2_k$ fixing all the $\mathbb{F}_2$-rational points, which is the identity. Thus an automorphism of Y_t is determined by the image of all (-1)-curves

$$\left(\begin{array}{cccc|cccc|cccccc|cc}
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
 & & & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
 & & & & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
 & & & & & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
 & & & & & & -1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & & & & & & & -1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & & & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & -1 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & -1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & -1 & 1 & 0 & 0 \\
 & & & & & & & & & & & & & -1 & 0 & 0 \\
\hline
 & & & & & & & & & & & & & & -1 & 1 \\
 & & & & & & & & & & & & & & & -1
\end{array}\right).$$

Figure 5.9: The intersection matrix of the (-1)-curves in $\overline{Y}$

and (-2)-curves.

Let S_8 be the permutation group of $\{0, 1, \dots, 7\}$. By Figure 5.8, the images of all (-1)-curves are determined by those of (-2)-curves $\Theta'_{0,1}, \dots, \Theta'_{7,1}$. Hence there is an injection $\iota\colon \operatorname{Aut} Y_t \longrightarrow S_8$ which sends $\eta \in \operatorname{Aut} Y_t$ to $\sigma \in S_8$ such that $\eta(\Theta'_{i,1}) = \Theta'_{\sigma(i),1}$ for $0 \leqslant i \leqslant 7$. Moreover, (0^8), (1^8), and fourteen rows in the $(1, 3)$ block or $(2, 3)$ block of Figure 5.8 form the $[8, 4, 4]$ extended Hamming code, which is also the Reed-Muller code $R(1, 3)$. Hence ι factors through the automorphism group of $R(1, 3)$, which is the affine linear group $\mathbb{F}_2^3 \rtimes \mathrm{GL}(3, \mathbb{F}_2) \subset S_8$ by [81, Chapter 13, §9, Theorem 24]. Since the normal group $\mathbb{F}_2^3 \subset S_8$ is generated by $(01)(23)(45)(67)$, $(02)(13)(46)(57)$ and $(04)(15)(26)(37)$, it acts on $\{0, 1, \dots, 7\}$ transitively. Hence it suffices to show that $\mathbb{F}_2^3 \subset \operatorname{Aut} Y_t$. We show only the existence of $\eta \in \operatorname{Aut} Y_t$ such that $\iota(\eta) = (01)(23)(45)(67)$; the same proof works for $(02)(13)(46)(57)$ and $(04)(15)(26)(37)$.

Let $\varphi\colon Y_t \longrightarrow \overline{Y}$ be the contraction of $A'_{1,1}$, $A'_{2,1}$, and $A'_{4,1}$. Set $s_i = \varphi(A'_{i,1})$ for $i = 1, 2$, and 4. Then $\overline{Y}$ is a smooth del Pezzo surface of degree four since each (-2)-curve in Y_t intersects with $A'_{1,1}$, $A'_{2,1}$, or $A'_{4,1}$. Generally speaking, a smooth del Pezzo surface of degree four contains sixteen (-1)-curves, and each (-1)-curve intersects with five (-1)-curves. In the present case, $\overline{A}_{i,1} = \varphi(A'_{i,1})$, $\overline{A}_{i,2} = \varphi(A'_{i,2})$, $\overline{\Theta}_{j,1} = \varphi(\Theta'_{j,1})$, and $\overline{\Theta}_{k,1} = \varphi(\Theta'_{k,1})$ are (-1)-curves on Y for $i = 3, 5, 6$, or 7, $2 \leqslant j \leqslant 7$, and $k = 0, 1$. Figure 5.9 is the intersection matrix of $\overline{A}_{3,1}, \dots, \overline{A}_{7,1}, \overline{A}_{3,2}, \dots, \overline{A}_{7,2}, \overline{\Theta}_{2,1}, \dots, \overline{\Theta}_{7,1}, \overline{\Theta}_{0,2}$, and $\overline{\Theta}_{1,2}$ in this order. In particular, $\mathcal{M} = (\overline{\Theta}_{1,2}, \overline{A}_{3,2}, \overline{A}_{5,2}, \overline{A}_{6,2}, \overline{A}_{7,2})$ and $\mathcal{M}' = (\overline{\Theta}_{0,2}, \overline{A}_{3,1}, \overline{A}_{5,1}, \overline{A}_{6,1}, \overline{A}_{7,1})$ are the 5-tuples of (-1)-curves intersecting with $\overline{\Theta}_{0,2}$ and $\overline{\Theta}_{1,2}$ respectively.

Since $\mathcal{M}$ and $\mathcal{M}'$ satisfy the condition (1) of [48, Theorem 2.1], there is an automorphism $\overline{\eta}$ of $\overline{Y}$ which interchanges $\mathcal{M}$ with $\mathcal{M}'$. By Figure 5.9, $\overline{\eta}$ also interchanges $\overline{\Theta}_{2,1}$ with $\overline{\Theta}_{3,1}$, $\overline{\Theta}_{4,1}$ with $\overline{\Theta}_{5,1}$, and $\overline{\Theta}_{6,1}$ with $\overline{\Theta}_{7,1}$. Then $\overline{\eta}$ fixes $s_1 = \overline{\Theta}_{2,1} \cap \overline{\Theta}_{3,1}$, $s_2 = \overline{\Theta}_{4,1} \cap \overline{\Theta}_{5,1}$, and $s_4 = \overline{\Theta}_{6,1} \cap \overline{\Theta}_{7,1}$. Hence $\overline{\eta}$ induces an automorphism $\eta \in \operatorname{Aut} Y_t$, which interchanges $\Theta'_{0,2}$ with $\Theta'_{1,2}$, $\Theta'_{2,1}$ with $\Theta'_{3,1}$, $\Theta'_{4,1}$ with $\Theta'_{5,1}$, and $\Theta'_{6,1}$ with $\Theta'_{7,1}$. Since $\Theta'_{0,1}$ (resp. $\Theta'_{1,1}$) is the unique (-2)-curve which intersects with $\Theta'_{0,2}$ (resp. $\Theta'_{1,2}$) twice, η also interchanges $\Theta'_{0,1}$ with $\Theta'_{1,1}$. Hence $\iota(\eta) = (01)(23)(45)(67)$, and the assertion holds.

(5): If some automorphism of W sends t to t', then it ascends to an isomorphism

$Y_t \cong Y_{t'}$. On the other hand, suppose that there is an isomorphism $Y_t \cong Y_{t'}$. By the assertion (4), we may assume that this isomorphism sends C_t to $C_{t'}$. Then by the assertion (3), it also sends E_{h_t} to $E_{h_{t'}}$. Hence it descends to an isomorphism of W, which sends t to t'. □

Corollary 5.5.23. *Let X_t be the contraction of all (-2)-curves in Y_t as in Proposition 5.5.22. Then $\operatorname{Aut} X_t \cong \operatorname{Aut} Y_t \cong (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathbb{Z}/7\mathbb{Z}$ when t is an $\mathbb{F}_8$-rational point and $(\mathbb{Z}/2\mathbb{Z})^3$ otherwise. In particular, there is a unique Du Val del Pezzo surface $X(8A_1)$ such that $\operatorname{Aut} X \cong (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathbb{Z}/7\mathbb{Z}$.*

Proof. We follow the notation of Proposition 5.5.22. Let $\Sigma \subset \operatorname{Aut} Y_t$ be the stabilizer subgroup with respect to C_t. Since $\mathbb{F}_2^3 \cong (\mathbb{Z}/2\mathbb{Z})^3$ is a normal subgroup of $\operatorname{Aut} Y_t$ which acts on the set of (-2)-curves in Y_t transitively, we obtain $\operatorname{Aut} Y_t \cong (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \Sigma$. By Proposition 5.5.22 (3), Σ is the same as the stabilizer subgroup of $\mathrm{PGL}(3, \mathbb{F}_2)$ with respect to $t \in \mathcal{D}_2$. Now the first assertion follows from Lemma 5.5.17. Since $\mathrm{PGL}(3, \mathbb{F}_2)$ acts on $\mathcal{D}_2(\mathbb{F}_8)$ transitively, the second assertion follows from Proposition 5.5.22 (5). □

Corollary 5.5.24. *There are one-to-one correspondences between the isomorphism classes of rational quasi-elliptic surfaces of type (d), (f), and (g), and the closed points of $\mathcal{D}_1/\mathrm{PGL}(2, \mathbb{F}_2)$, $\mathcal{D}_1/\mathrm{PGL}(2, \mathbb{F}_2)$, and $\mathcal{D}_2/\mathrm{PGL}(3, \mathbb{F}_2)$ respectively.*

Proof. By Proposition 5.4.1, there is one-to-one correspondence between isomorphism classes of del Pezzo surfaces of type $2D_4$ satisfying (NB) (resp. type $4A_1 + D_4$, type $8A_1$) and those of rational quasi-elliptic surfaces of type (d) (resp. (f), (g)). Hence the assertion follows from Propositions 5.5.18, 5.5.20 and 5.5.22. □

Now we can prove Theorem 1.3.4.

Proof of Theorem 1.3.4. The assertions (0), (1), and (2) follow from Lemma 5.1.3 (3),(4), Proposition 5.4.1, and Propositions 5.4.3 and 5.4.4 respectively. The assertion (3) follows from Propositions 5.5.1, 5.5.3, 5.5.5, 5.5.7, 5.5.9, 5.5.12, 5.5.18, 5.5.20, and 5.5.22. □

5.5.3 List of automorphism groups

As a consequence, we obtain the list of automorphisms of Du Val del Pezzo surfaces satisfying (NB) and rational quasi-elliptic surfaces as follows.

Theorem 5.5.25. *Let X be a Du Val del Pezzo surface satisfying (NB). Then $\operatorname{Aut} X$ is described in Table 5.5. Furthermore, suppose that $p = 2$. Then for each of types $2D_4$, $4A_1 + D_4$, and $8A_1$, there is a unique del Pezzo surface of the given type such that the group G in Table 5.5 is non-trivial.*

Proof. The assertion follows from Corollaries 5.5.2, 5.5.4, 5.5.6, 5.5.8, 5.5.11, Lemmas 5.5.13, 5.5.14, Corollaries 5.5.19, 5.5.21, and 5.5.23. □

Table 5.5

$\mathrm{Dyn}(X)$	Characteristic	Automorphism groups
E_8	$p=2$	$\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & a^3 \end{pmatrix} \in \mathrm{PGL}(3,k) \mid a \in k^*, f \in k\right\}$
	$p=3$	$\left\{\begin{pmatrix} a & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & a^3 \end{pmatrix} \in \mathrm{PGL}(3,k) \mid a \in k^*, c \in k\right\}$
A_2+E_6		$k^* \times \mathbb{Z}/2\mathbb{Z}$
$4A_2$		$\mathrm{GL}(2,\mathbb{F}_3)$
D_8	$p=2$	k
A_1+E_7		k^*
$2D_4$		$(k^* \times G) \rtimes \mathbb{Z}/2\mathbb{Z}$ with $G=\{1\}$ or $\mathbb{Z}/3\mathbb{Z}$
$2A_1+D_6$		$\mathbb{Z}/2\mathbb{Z}$
$4A_1+D_4$		$(\mathbb{Z}/2\mathbb{Z})^2 \rtimes G$ with $G=\{1\}$ or $\mathbb{Z}/3\mathbb{Z}$
$8A_1$		$(\mathbb{Z}/2\mathbb{Z})^3 \rtimes G$ with $G=\{1\}$ or $\mathbb{Z}/7\mathbb{Z}$
E_7		$\left\{\begin{pmatrix} a & 0 & d^2a \\ d & 1 & f \\ 0 & 0 & a^3 \end{pmatrix} \in \mathrm{PGL}(3,k) \mid a \in k^*, d \in k, f \in k\right\}$
A_1+D_6		$\left\{\begin{pmatrix} a & 0 & a^3+a \\ d & 1 & a^3+d+1 \\ 0 & 0 & a^3 \end{pmatrix} \in \mathrm{PGL}(3,k) \mid a \in k^*, d \in k\right\}$
$3A_1+D_4$		$k^* \times \mathrm{PGL}(2,\mathbb{F}_2)$
$7A_1$		$\mathrm{PGL}(3,\mathbb{F}_2)$

Corollary 5.5.26. *Let Z be a rational quasi-elliptic surface and $O \subset Z$ a section. Take $g\colon Z \longrightarrow Y$ as the contraction of O and $\pi\colon Y \longrightarrow X$ the contraction of all the (-2)-curves. Then $\operatorname{Aut} Z \cong \mathrm{MW}(Z) \cdot \operatorname{Aut} X$. In particular, $\operatorname{Aut} Z \cong (\mathbb{Z}/p\mathbb{Z})^n \cdot H$ for some $0 \leqslant n \leqslant 4$ and for some group H listed in Table 5.5.*

Proof. Note that X is a Du Val del Pezzo surface satisfying (NB) by Proposition 5.4.1 (1) and $\operatorname{Aut} Y \cong \operatorname{Aut} X$ since π is the minimal resolution. Since $h(O)$ is the base point of $|-K_Y|$, h induces an isomorphism between $\operatorname{Aut} Y$ and the stabilizer subgroup of $\operatorname{Aut} Z$ with respect to O. Hence the first assertion follows from the transitivity of the $\mathrm{MW}(Z)$-action on the set of sections on Z. The second assertion follows from Theorems 5.2.1, 5.2.2, and 5.5.25. □

Remark 5.5.27. We follow the notation in Corollary 5.5.26. We have described the reduced scheme structure of $\operatorname{Aut} Y$ and $\operatorname{Aut} Z$. We can also describe the scheme structure of them by virtue of [82, Main Theorem], which calculates the identity component of $\operatorname{Aut} Y$ as a scheme.

On the other hand, what is still lacking is the determination of the scheme structure of $\operatorname{Aut} X$ since the contraction of (-2)-curves may thicken the scheme structures of the automorphism groups. For example, smooth K3 surfaces in characteristic $p > 0$ admit no non-trivial μ_p-actions but RDP K3 surfaces may admit such actions (see [83, Remark 2.3]).

5.6 Log liftability

In this section, we determine all the Du Val del Pezzo surfaces which are not log liftable over $W(k)$. Note that by Theorem 1.3.3 (1), it suffices to consider Du Val del Pezzo surfaces satisfying (NB).

Proposition 5.6.1. *Let X be a Du Val del Pezzo surface satisfying (NB) and $\pi\colon Y \longrightarrow X$ the minimal resolution. Suppose that $p = 3$ and $\mathrm{Dyn}(X) = E_8$ or $A_2 + E_6$. Then the pair of (Y, E_π) lifts to $\operatorname{Spec}\mathbb{Z}$ via $\operatorname{Spec}\mathbb{F}_3 \longrightarrow \operatorname{Spec}\mathbb{Z}$. As a result, X is log liftable both over $\mathbb{Z}$ via $\operatorname{Spec}\mathbb{F}_3 \longrightarrow \operatorname{Spec}\mathbb{Z}$ and over $W(k)$.*

Proof. Note that Y and each (-2)-curve on Y are defined over $\mathbb{F}_3$ by Proposition 5.5.1 (6). Suppose that $\mathrm{Dyn}(X) = E_8$. Take a birational morphism $h'_{\mathbb{Z}}\colon Y_{\mathbb{Z}} \longrightarrow \mathbb{P}^2_{\mathbb{Z}}$ as the blow-up at $[0:1:0]$ eight times along $\{x^3 + y^2z = 0\}$. By Proposition 5.5.1 (3), we have $Y \cong Y_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_3$ and each negative rational curve on Y is the specialization of either an $h'_{\mathbb{Z}}$-exceptional curve or the strict transform of $\{z = 0\} \subset \mathbb{P}^2_{\mathbb{Z}}$ via $h'_{\mathbb{Z}}$. Hence we obtain the desired lift. The proof for the case where $\mathrm{Dyn}(X) = A_2 + E_6$ is similar by virtue of Proposition 5.5.1 (4). □

Proposition 5.6.2. *Let X be a Du Val del Pezzo surface satisfying (NB) and $\pi\colon Y \longrightarrow X$ the minimal resolution. Suppose that $p = 2$. Then the following hold.*

(1) *Suppose that $\mathrm{Dyn}(X) = E_7$, $A_1 + D_6$, or $3A_1 + D_4$. Then the log smooth pair of Y and the union B of negative rational curves lifts to $\operatorname{Spec}\mathbb{Z}$ via $\operatorname{Spec}\mathbb{F}_2 \longrightarrow \operatorname{Spec}\mathbb{Z}$.*

(2) *Suppose that $\mathrm{Dyn}(X) = E_8$, D_8, $A_1 + E_7$, or $2A_1 + D_6$. Then the pair (Y, E_π) lifts to $\operatorname{Spec}\mathbb{Z}$ via $\operatorname{Spec}\mathbb{F}_2 \longrightarrow \operatorname{Spec}\mathbb{Z}$.*

As a result, X is log liftable both over $\mathbb{Z}$ via $\operatorname{Spec}\mathbb{F}_2 \longrightarrow \operatorname{Spec}\mathbb{Z}$ and over $W(k)$.

Proof. By Theorem 1.3.4, X is uniquely determined up to isomorphism by $\mathrm{Dyn}(X)$. Moreover, we have shown in §5.5 that Y and each negative rational curve on Y are defined over $\mathbb{F}_2$.
(1): By Lemma 5.4.5 (3), the pair (Y, B) is log smooth. Now suppose that $\mathrm{Dyn}(X) = E_7$. Take a birational morphism $h'_{\mathbb{Z}}\colon Y_{\mathbb{Z}} \longrightarrow \mathbb{P}^2_{\mathbb{Z}}$ as the blow-up at $[0:1:0]$ seven times along $\{x^3 + y^2z = 0\}$. By Proposition 5.5.3, we have $Y \cong Y_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_2$ and each negative rational curve on Y is the specialization of either an $h'_{\mathbb{Z}}$-exceptional curve or the strict transform of $\{z = 0\} \subset \mathbb{P}^2_{\mathbb{Z}}$ via $h'_{\mathbb{Z}}$. Hence we obtain the desired lift. The proof for the cases where $\mathrm{Dyn}(X) = A_1 + D_6$ and $3A_1 + D_4$ is similar by virtue of Corollary 5.5.6 (3) and Proposition 5.5.7 respectively.
(2): By Proposition 5.4.1 (4) and Lemma 5.4.5 (5)–(7), for some Du Val del Pezzo surface of type E_7, $A_1 + D_6$, or $3A_1 + D_4$ satisfying (NB) and its minimal resolution W, there exist a (-1)-curve $E \subset W$ and an $\mathbb{F}_2$-rational point $t \in E$ not contained in any (-2)-curves such that Y is the blow-up of W at t. Hence the assertion follows from the assertion (1) and [7, Proposition 2.9]. □

By Proposition 5.5.18, there are infinitely many Du Val del Pezzo surfaces of type $2D_4$ satisfying (NB). In particular, they are not defined over $\mathbb{F}_2$ in general. On the other hand, we can show their log liftability over $W(k)$ as follows.

Proposition 5.6.3. *Let X be a Du Val del Pezzo surface of type $2D_4$ satisfying (NB) in $p = 2$. Take R as a Noetherian irreducible ring with surjective ring homomorphism $f\colon R \to k$. Then X is log liftable over R via the induced morphism* $\operatorname{Spec} k \longrightarrow \operatorname{Spec} R$.

Proof. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution. By Proposition 5.5.18 (1), on the minimal resolution W of the Du Val del Pezzo surface of type $3A_1 + D_4$ satisfying (NB), there are the (-1)-curve $E \subset W$ intersecting with exactly three (-2)-curves and a closed point $t \in E$ not contained in any (-2)-curves such that Y is the blow-up of W at t.

Let D be the union of the (-2)-curves in W. By Proposition 5.6.2 (1), the log smooth pair $(W, D \cup E)$ lifts to $\operatorname{Spec} \mathbb{Z}$ via $\operatorname{Spec} \mathbb{F}_2 \longrightarrow \operatorname{Spec} \mathbb{Z}$. Take $(\mathcal{W}, \mathcal{D} \cup \mathcal{E})$ as the base change of such a lifting by the natural homomorphism $\mathbb{Z} \longrightarrow R$.

Fix coordinates $[x : y]$ of $\mathcal{E} \cong \mathbb{P}^1_R$ and choose $a, b \in k$ so that $t = [a : b] \in \mathbb{P}^1_{k,[x:y]}$. Since $f\colon R \longrightarrow k$ is surjective, we can take a lifting $\widetilde{a}$ (resp. $\widetilde{b}$) of a (resp. b). Then $\widetilde{t} = [\widetilde{a} : \widetilde{b}] \in \mathcal{E} \cong \mathbb{P}^1_R$ is a lifting of t. Let $\Phi\colon \mathcal{Y} \longrightarrow \mathcal{W}$ be the blow-up along $\widetilde{t}$. Then $(\mathcal{Y}, \Phi_*^{-1}(\mathcal{D} \cup \mathcal{E}))$ is the desired lift. □

Proposition 5.6.4. *Let X be a Du Val del Pezzo surface with* $\operatorname{Dyn}(X) = 4A_1 + D_4$, *$8A_1$, or $7A_1$. Then X is not log liftable over any Noetherian integral domain R of characteristic zero via any morphism* $\operatorname{Spec} k \longrightarrow \operatorname{Spec} R$ *induced by a surjective homomorphism $R \longrightarrow k$.*

Proof. By [103, Theorem 1.2], the surface X satisfies (ND). Hence the assertion follows from Proposition 5.3.2. □

Proposition 5.6.5. *Let X be a Du Val del Pezzo surface of type $4A_2$ in $p = 3$. Then X is not log liftable over $W(k)$.*

Proof. We note that X satisfies (NB) by Proposition 5.4.2. Suppose by contradiction that X is log liftable over $W(k)$. Take $\pi\colon Y \longrightarrow X$ as the minimal resolution and $(\mathcal{Y}, \mathcal{E})$ as a $W(k)$-lifting of (Y, E_π). We follow the notation used in the proof of Proposition 5.3.2. Then the blow-up $Z_K \longrightarrow Y_K$ at the base point of $|-K_{Y_K}|$ gives the morphism $f_K\colon Z_K \longrightarrow \mathbb{P}^1_K$ associated to the anti-canonical linear system. Let G be the strict transform of $E_K = \sum_{i=1}^8 E_{i,K}$ in Z_K. Then $f_K(G)$ consists of four K-rational points. We fix coordinates of $\mathbb{P}^1_K$ such that $f(G) = \{0, 1, \infty, \alpha\}$ for some $\alpha \in \mathbb{P}^1_K \backslash \{0, 1, \infty\}$.

On the other hand, by Proposition 5.3.2, $X_{\mathbb{C}}$ is the del Pezzo surface of degree one of type $4A_2$. By [103, Table 4.1], the blow-up $Z_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$ at the base point of $|-K_{Y_{\mathbb{C}}}|$ gives an elliptic fibration $f_{\mathbb{C}}\colon Z_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ with four singular fibers of type I_3. Since $f_K(G) \subset \mathbb{P}^1_{\mathbb{C}}$ is the singular fiber locus of $f_{\mathbb{C}}$, [8, Théorème] now yields the existence of $\sigma \in \operatorname{Aut} \mathbb{P}^1_{\mathbb{C}}$ which sends $f_K(G)$ to $\{1, \omega, \omega^2, \infty\}$, where ω is a primitive cube root of unity. An easy computation shows that $\alpha = -\omega$ and hence $\omega \in K$. However, by the Eisenstein criterion and the Gauss lemma, the cyclotomic polynomial $t^2 + t + 1$ is irreducible in $K[t]$, a contradiction. Therefore (Y, E_π) does not lift to $W(k)$. $\square$

Remark 5.6.6. One question still unanswered is whether $X(4A_2)$ in $p = 3$ is log liftable over any Noetherian integral domain of characteristic zero.

Remark 5.6.7. As we saw in the proof of Propositions 5.5.9, 5.5.20, and 5.5.22 (resp. Proposition 5.5.1 (4)), the surfaces as in Proposition 5.6.4 (resp. Proposition 5.6.5) are obtained from the configuration of all the lines in $\mathbb{P}^2_k$ defined over $\mathbb{F}_p$, which is not realizable in $\mathbb{P}^2_{\mathbb{C}}$ by the Hirzebruch inequality for line arrangements (see [47] and [86, Example 3.2.2]). This is the reason why we cannot apply the proof of Proposition 5.6.1 for such surfaces.

5.7 Kodaira type vanishing theorem

In this section, we determine all the Du Val del Pezzo surfaces which violate the Kodaira vanishing theorem for ample $\mathbb{Z}$-divisors. Note that by Theorem 1.3.3 (3), it suffices to consider Du Val del Pezzo surfaces satisfying (NL).

Lemma 5.7.1 (cf. [59, Theorem 4.8]). *Let X be a Du Val del Pezzo surface and A an ample $\mathbb{Z}$-divisor on X. If $H^1(X, \mathcal{O}_X(-A)) \neq 0$, then $p = 2$ and $(-K_X \cdot A) = 1$.*

Proof. We refer to the proof of [59, Theorem 4.8] for the details. $\square$

Proposition 5.7.2. *Let X be a del Pezzo surface of type $8A_1$. Then there is an ample $\mathbb{Z}$-divisor A such that $H^1(X, \mathcal{O}_X(-A)) \neq 0$.*

Proof. We follow the notation of the proof of Proposition 5.5.22. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution and $A := \pi_*(A'_{1,1} + A'_{2,1} - A'_{4,1})$. Then A is ample since $\rho(X) = 1$ and $(-K_X \cdot A) = 1$. By Figure 5.8, we have

$$\begin{aligned}
&\lceil \pi^* A \rceil \\
=&\lceil A'_{1,1} + \frac{1}{2}(\Theta'_{0,1} + \Theta'_{1,1} + \Theta'_{2,1} + \Theta'_{3,1}) + A'_{2,1} + \frac{1}{2}(\Theta'_{0,1} + \Theta'_{1,1} + \Theta'_{4,1} + \Theta'_{5,1}) \\
&- A'_{4,1} - \frac{1}{2}(\Theta'_{0,1} + \Theta'_{1,1} + \Theta'_{6,1} + \Theta'_{7,1}) \rceil \\
=&A'_{1,1} + A'_{2,1} - A'_{4,1} + \Theta'_{0,1} + \Theta'_{1,1} + \Theta'_{2,1} + \Theta'_{3,1} + \Theta'_{4,1} + \Theta'_{5,1}
\end{aligned}$$

In particular, $\lceil \pi^* A \rceil^2 = -3$ and $(-K_Y \cdot \lceil \pi^* A \rceil) = 1$. Lemma 5.3.3 now yields $H^i(Y, \mathcal{O}_Y(-\lceil \pi^* A \rceil)) = H^i(X, \mathcal{O}_X(-A))$ for $i \geqslant 0$. Since $\lceil \pi^* A \rceil$ is big, we have $H^0(Y, \mathcal{O}_Y(-\lceil \pi^* A \rceil)) = 0$.

Next assume that $H^2(Y, \mathcal{O}_Y(-\lceil \pi^* A \rceil)) \neq 0$. Then there is an effective divisor $C \sim K_Y + \lceil \pi^* A \rceil$ by the Serre duality. Since $(-K_Y \cdot C) = 0$, the curve C is a sum of (-2)-curves. Since (-2)-curves in Y are disjoint from each other, we have $(C \cdot \Theta'_{0,1}) \in 2\mathbb{Z}$. However, $(C \cdot \Theta'_{0,1}) = (\lceil \pi^* A \rceil \cdot \Theta'_{0,1}) = -1$ by Figure 5.8, a contradiction.

Combining these results and the Riemann-Roch theorem, we conclude that

$$\begin{aligned}\dim_k H^1(X, \mathcal{O}_X(-A)) &= \dim_k H^1(Y, \mathcal{O}_Y(-\lceil \pi^* A \rceil)) \\ &= -\chi(Y, \mathcal{O}_Y(-\lceil \pi^* A \rceil)) \\ &= -(\chi(Y, \mathcal{O}_Y) + \frac{1}{2}((-\lceil \pi^* A \rceil)^2 + (-K_Y \cdot -\lceil \pi^* A \rceil))) = 1.\end{aligned}$$

Therefore $H^1(X, \mathcal{O}_X(-A)) \neq 0$. □

Proposition 5.7.3. *Let X be a del Pezzo surface of type $4A_1 + D_4$. Then*

$$H^1(X, \mathcal{O}_X(-A)) = 0$$

for any ample $\mathbb{Z}$-divisor A.

Proof. Let $\pi\colon Y \longrightarrow X$ be the minimal resolution. By Proposition 5.4.1 (4), there exist a rational quasi-elliptic surface Z of type (f) and a section O such that the contraction of O gives a birational morphism $g\colon Z \longrightarrow Y$. In what follows, we use the notation of Figure 5.3. For a birational morphism $Z \longrightarrow S$ and a curve $C \subset Z$, we denote $(C)_S$ the strict transform of C in S.

By Lemma 5.4.5 (8), the contraction of $(\Theta_{1,0})_Y$ gives a morphism $h\colon Y \longrightarrow W$ to the minimal resolution of the Du Val del Pezzo surface V of type $7A_1$. Let $\xi\colon W \longrightarrow V$ be the contraction of all the (-2)-curves and $\nu = \xi \circ h$. By Corollary 5.5.11 (2), the class divisor group of W is generated by $(\Theta_{1,4})_W$, $(R_2)_W$, $(Q_2)_W$, $(R_1)_W$, $(Q_1)_W$, $(P_3)_W$, $(P_2)_W$, and any one of (-2)-curves. Since the point $h((\Theta_{1,0})_Y)$ lies on $(\Theta_{1,4})_W$ and π contracts all the (-2)-curves, the class divisor group of X is generated by $(R_2)_X$, $(Q_2)_X$, $(R_1)_X$, $(Q_1)_X$, $(P_3)_X$, $(P_2)_X$, and $(\Theta_{1,0})_X$, whose anti-canonical degrees are one. Then an easy computation shows the following.

$$\begin{aligned}
\pi^*(\Theta_{1,0})_X &= (\Theta_{1,0})_Y + (\Theta_{1,1})_Y + (\Theta_{1,2})_Y + (\Theta_{1,3})_Y + 2(\Theta_{1,4})_Y \\
&= 2((\Theta_{1,0})_Y + \frac{1}{2}(\Theta_{1,1})_Y + \frac{1}{2}(\Theta_{1,2})_Y + \frac{1}{2}(\Theta_{1,3})_Y + (\Theta_{1,4})_Y) - (\Theta_{1,0})_Y \\
&= 2\nu^*(\Theta_{1,4})_V - (\Theta_{1,0})_Y, \\
\pi^*(Q_1)_X &= (Q_1)_Y + \frac{1}{2}(\Theta_{0,1})_Y + \frac{1}{2}(\Theta_{\alpha_1,1})_Y \\
&\quad + \frac{1}{2}(\Theta_{1,1})_Y + (\Theta_{1,2})_Y + \frac{1}{2}(\Theta_{1,3})_Y + (\Theta_{1,4})_Y \\
&= ((Q_1)_Y + \frac{1}{2}(\Theta_{0,1})_Y + \frac{1}{2}(\Theta_{\alpha_1,1})_Y + \frac{1}{2}(\Theta_{1,2})_Y) \\
&\quad + ((\Theta_{1,0})_Y + \frac{1}{2}(\Theta_{1,1})_Y + \frac{1}{2}(\Theta_{1,2})_Y + \frac{1}{2}(\Theta_{1,3})_Y + (\Theta_{1,4})_Y) - (\Theta_{1,0})_Y \\
&= \nu^*(Q_1)_V + \nu^*(\Theta_{1,4})_V - (\Theta_{1,0})_Y, \\
\pi^*(R_1)_X &= \nu^*(R_1)_V + \nu^*(\Theta_{1,4})_V - (\Theta_{1,0})_Y, \\
\pi^*(Q_2)_X &= \nu^*(Q_2)_V + \nu^*(\Theta_{1,4})_V - (\Theta_{1,0})_Y, \\
\pi^*(R_2)_X &= \nu^*(R_2)_V + \nu^*(\Theta_{1,4})_V - (\Theta_{1,0})_Y, \\
\pi^*(P_2)_X &= \nu^*(P_2)_V + \nu^*(\Theta_{1,4})_V - (\Theta_{1,0})_Y, \\
\pi^*(P_3)_X &= \nu^*(P_3)_V + \nu^*(\Theta_{1,4})_V - (\Theta_{1,0})_Y.
\end{aligned}$$

Now let us show the assertion. Let A be an ample $\mathbb{Z}$-divisor on X. By Lemma 5.3.3, we only have to show that $H^1(Y, \mathcal{O}_Y(-\lceil \pi^*A \rceil)) = 0$. By Lemma 5.7.1, we may assume that $(-K_X \cdot A) = 1$. Then $A \sim n_1(R_2)_X + n_2(Q_2)_X + n_3(R_1)_X + n_4(Q_1)_X + n_5(P_3)_X + n_6(P_2)_X + n_7(\Theta_{1,0})_X$ with $n_1 + \cdots + n_7 = 1$. Set $B = n_1(R_2)_V + n_2(Q_2)_V + n_3(R_1)_V + n_4(Q_1)_V + n_5(P_3)_V + n_6(P_2)_V + n_7(\Theta_{1,4})_V$. Then we obtain $\pi^*A = \nu^*(B + (\Theta_{1,4})_V) - (\Theta_{1,0})_Y$. Since ν sends $E_h = (\Theta_{1,0})_Y$ to a smooth point of V, the support of $\lceil \nu^*(B + (\Theta_{1,4})_V) \rceil - \nu^*(B + (\Theta_{1,4})_V)$ is contained in E_ξ. Since E_h is disjoint from E_ξ in Y, we obtain $(E_h \cdot \lceil \pi^*A \rceil) = (E_h \cdot \nu^*(B + (\Theta_{1,4})_V) - E_h) = 1$. Hence we have an exact sequence

$$0 \to \mathcal{O}_Y(-\lceil \nu^*(B + (\Theta_{1,4})_V) \rceil) \to \mathcal{O}_Y(-\lceil \pi^*A \rceil) \to \mathcal{O}_{E_h}(-1) \to 0.$$

Thus $H^1(Y, \mathcal{O}_Y(-\lceil \pi^*A \rceil)) \cong H^1(Y, \mathcal{O}_Y(-\lceil \nu^*(B + (\Theta_{1,4})_V) \rceil))$. Since (Y, E_ν) is a log smooth pair, Lemma 5.3.3 yields $H^1(Y, \mathcal{O}_Y(-\lceil \nu^*(B+(\Theta_{1,4})_V) \rceil)) \cong H^1(V, \mathcal{O}_V(-(B + (\Theta_{1,4})_V)))$. Since $(-K_V \cdot B + (\Theta_{1,4})_V) = 2$, Lemma 5.7.1 yields $H^1(V, \mathcal{O}_V(-(B + (\Theta_{1,4})_V))) \cong 0$. Hence the assertion holds. $\square$

Now we can prove Theorem 1.3.6.

Proof of Theorem 1.3.6. By Theorem 1.3.3, it suffices to show the assertions when X satisfies (NB), i.e., X is listed in Table 1.1. Then the assertions (1) and (2) follow from Propositions 5.6.1–5.6.5 and [31, Theorem 2, Table (II)] respectively. Finally, we show the assertion (3). Suppose that X satisfies (NK). Then $p = 2$ and X satisfies (NL) by Lemma 5.7.1 and Theorem 1.3.3 (3) respectively. The assertion

(1) now shows that $\mathrm{Dyn}(X) = 7A_1$, $8A_1$, or $4A_1 + D_4$. If $\mathrm{Dyn}(X) = 7A_1$, then X satisfies (NK) by [22, Theorem 4.2 (6)] with $(d, q_1, q_2) = (3, 1, 2)$. If $\mathrm{Dyn}(X) = 8A_1$, then X satisfies (NK) by Proposition 5.7.2. If $\mathrm{Dyn}(X) = 4A_1 + D_4$, then X does not satisfy (NK) by Proposition 5.7.3. Hence we get the assertion (3). □

5.8 Classification of Du Val del Pezzo surface of rank one

In this section, we prove the following theorem.

Theorem 5.8.1. *Let X be a Du Val del Pezzo surface over an algebraically closed field of characteristic $p > 0$. Suppose that X is singular and the Picard rank of X is one. Then the following holds.*

(1) *The Dynkin types of X and the number of the isomorphism classes of the del Pezzo surfaces of the given Dynkin type are listed in Table 5.6.*

(2) *When $p = 2$ (resp. $p = 3$), X is uniquely determined up to isomorphism by its Dynkin type with Artin coindices except when its Dynkin type is D_8, $2D_4$, $4A_1 + D_4$, or $8A_1$ (resp. $2D_4$).*

Table 5.6

Dynkin type			E_8		D_8	
Characteristic			$p = 2, 3$	$p > 3$	$p = 2$	$p > 2$
No. of isomorphism classes			3	2	∞	1
A_8	$A_1 + A_7$	$2A_4$	$A_1 + A_2 + A_5$	$A_3 + D_5$	$4A_2$	$2A_1 + D_6$
$p > 0$						
1	1	1	1	1	1	1
$A_2 + E_6$	$A_1 + E_7$	$2D_4$	$2A_1 + 2A_3$	$4A_1 + D_4$	$8A_1$	A_7
$p > 0$			$p > 2$	$p = 2$		$p > 0$
2	2	∞	1	∞	∞	1
E_7			$A_1 + D_6$		$A_2 + A_5$	$3A_1 + D_4$
$p = 2$	$p = 3$	$p > 3$	$p = 2$	$p > 2$	$p > 0$	
3	2	1	2	1	1	1
$A_1 + 2A_3$	$7A_1$	E_6		$A_1 + A_5$	$3A_2$	$2A_1 + A_3$
$p > 0$	$p = 2$	$p = 2, 3$	$p > 3$	$p > 0$		$p > 0$
1	1	2	1	1	1	1
D_5		A_4	$A_1 + A_2$	A_1		
$p = 2$	$p > 2$	$p > 0$	$p > 0$	$p > 0$		
2	1	1	1	1		

When $p > 3$, the assertion (1) has already proven by [67, Theorem B.7]. For this reason, we assume that $p = 2$ or 3 in this section. The proof is similar in spirit to [103]. However, we have to follow Ye's method carefully because the classification of extremal rational elliptic surfaces in $p = 2$ or 3 is quite different from that in $p > 3$ and rational quasi-elliptic surfaces appear in $p = 2$ or 3. We also investigate $\mathrm{Dyn}'(X)$ of some Du Val del Pezzo surfaces X to get the assertion (2).

5.8.1 Defining equations of Du Val del Pezzo surfaces

In this section, we calculate defining equations satisfying (NL).

Proposition 5.8.2. *There are coordinates $[x : y : z : w]$ of $\mathbb{P}_k(1, 1, 1, 2)$ such that the defining equation of the Du Val del Pezzo surface $X(7A_1)$ is $w^2 + xyz(x + y + z) = 0$.*

Proof. Let Y be the minimal resolution of X. Fix coordinates $[s : t : u]$ of $\mathbb{P}^2_k$. Then there is the blow-down $h \colon Y \longrightarrow \mathbb{P}^2_k$ such that $h(E_h) \subset \mathbb{P}^2_k$ is the set of closed points defined over $\mathbb{F}_2$ by Proposition 5.5.9. Set $x := st(s + t), y := tu(t + u), z := us(u + s)$, and $w := stu(s + t)(t + u)(u + s)$. Then $x, y, z \in H^0(\mathbb{P}^2_k, h_*\mathcal{O}_Y(-K_Y))$ and $w \in H^0(\mathbb{P}^2_k, h_*\mathcal{O}_Y(-2K_Y))$ because $H^0(\mathbb{P}^2_k, h_*\mathcal{O}_Y(-nK_Y)) \subset H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(3n))$ consists of elements which have zero of order at least n at each points in $h(E_h)$ for $n \geqslant 1$. Moreover, it is easy to check that $\{x^2, y^2, z^2, xy, yz, zx, w\}$ is a basis of $H^0(\mathbb{P}^2_k, h_*\mathcal{O}_Y(-2K_Y))$. Hence X is the closure of the image of the map

$$\begin{aligned} \Phi \colon \mathbb{P}^2_k &\longrightarrow \mathbb{P}_k(1, 1, 1, 2) \\ [s : t : u] &\longmapsto [x : y : z : w]. \end{aligned}$$

Now an easy computation gives the desired equation. $\square$

Proposition 5.8.3. *Let X be a Du Val del Pezzo surface of degree one which is not log liftable. When $p = 2$, set $\mathcal{D}_n \subset \mathbb{P}^n_k$ as the complement of the union of all the hyperplane sections defined over $\mathbb{F}_2$ for $n = 1, 2$ in addition. Then the defining equation of X in $\mathbb{P}_k(1, 1, 2, 3)$ is listed as in Table 5.7.*

Proof. We give the proof only for the case $\mathrm{Dyn}(X) = 8A_1$; the other cases are left to the reader. Let Y be the minimal resolution of X. Fix coordinates $[s : t : u]$ of $\mathbb{P}^2_k$. Then there is $t = [a : b : c] \in \mathcal{D}_2$ and the blow-down $h \colon Y \longrightarrow \mathbb{P}^2_k$ such that $t \in h(E_h)$ and $h(E_h) \backslash \{t\} \subset \mathbb{P}^2_k$ is the set of closed points defined over $\mathbb{F}_2$ by Proposition 5.5.22. Set

$$\begin{aligned} x :=& c(a + b)st(s + t) + a(b + c)tu(t + u) + b(c + a)us(u + s), \\ y :=& c^2 st(s + t) + a^2 tu(t + u) + b^2 us(u + s), \\ z :=& ((b + c)s + (c + a)t + (a + b)u)^2 stu(s + t + u), \text{ and} \\ w :=& ((b + c)s + a(t + u))((c + a)t + b(u + s))((a + b)u + c(s + t)) \\ & \times stu(s + t)(t + u)(u + s). \end{aligned}$$

Table 5.7

characteristic	Dynkin type	defining equation of $X \subset \mathbb{P}_k(1,1,2,3)_{[x:y:z:w]}$
$p=3$	$4A_2$	$w^2+z^3-x^2y^2(x+y)^2=0$
$p=2$	$4A_1+D_4$	$w^2+z^3+abx^2z^2$ $+y^4z+(a^2+ab+b^2)x^2y^2z$ $+ab(a+b)x^3yz=0$ for some $[a:b]\in\mathcal{D}_1$
	$8A_1$	w^2+abcz^3 $+((ab+bc+ca)^2+abc(a+b+c))y^2z^2$ $+(a+b+c)(a+b)(b+c)(c+a)xyz^2$ $+(ab+bc+ca)^2x^2z^2$ $+(a+b+c)^2(a+b)(b+c)(c+a)xy^3z$ $+(a+b+c)^2((a+b+c)^3+abc)x^2y^2z$ $+(a+b+c)^2(a+b)(b+c)(c+a)x^3yz$ $+(a+b+c)^2abcx^4z$ $+(a+b)^2(b+c)^2(c+a)^2y^6$ $+((a+b+c)^3+abc)^2x^2y^4$ $+(a+b)^2(b+c)^2(c+a)^2x^4y^2$ $+a^2b^2c^2x^6=0$ for some $[a:b:c]\in\mathcal{D}_2$

Then we can see that $x,y\in H^0(\mathbb{P}^2_k,h_*\mathcal{O}_Y(-K_Y))$, $z\in H^0(\mathbb{P}^2_k,h_*\mathcal{O}_Y(-2K_Y))$ and $w\in H^0(\mathbb{P}^2_k,h_*\mathcal{O}_Y(-3K_Y))$ because $H^0(\mathbb{P}^2_k,h_*\mathcal{O}_Y(-nK_Y))\subset H^0(\mathbb{P}^2_k,\mathcal{O}_{\mathbb{P}^2_k}(3n))$ consists of functions which has zero of order at least n at each points in $h(E_h)$ for $n\geqslant 1$. Moreover, it is easy to check that $\{x^3,x^2y,xy^2,y^3,xz,yz,w\}$ is a basis of $H^0(\mathbb{P}^2_k,h_*\mathcal{O}_Y(-3K_Y))$. Hence X is the closure of the image of the map

$$\begin{aligned}\Phi\colon \mathbb{P}^2_k &\longrightarrow \mathbb{P}_k(1,1,2,3)\\ [s:t:u] &\longmapsto [x:y:z:w].\end{aligned}$$

Now an easy computation gives the desired equation. □

5.8.2 Rational genus one fibrations

In this section, we compile the results on rational extremal elliptic surfaces by Lang [69, 70] and rational quasi-elliptic surfaces by Ito [50, 51], which we will use in Section 5.8.

Theorem 5.8.4 ([69, 70, 52]). *When $p=2$ (resp. $p=3$), the configurations of singular fibers of extremal rational elliptic surfaces and the order of their Mordell-Weil groups are listed in Table 5.8 (resp. Table 5.9) using Kodaira's notation.*

Moreover, there is a unique extremal rational elliptic surface with each configuration of singular fibers in the table except for the type I (resp. type VI). In this case, there are infinitely many isomorphism classes of extremal rational elliptic surfaces with that configuration of singular fibers.

Table 5.8

No.	Singular fibers	$\lvert\mathrm{MW}(X)\rvert$	No.	Singular fibers	$\lvert\mathrm{MW}(X)\rvert$
I	I_4^*	2	VII	$\mathrm{IV}, \mathrm{IV}^*$	3
II	II^*	1	VIII	$\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6$	6
III	$\mathrm{III}, \mathrm{I}_8$	4	IX	$\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3$	3
IV	$\mathrm{I}_1^*, \mathrm{I}_4$	4	SI	$\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1$	3
V	$\mathrm{III}^*, \mathrm{I}_2$	2	SII	$\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1$	5
VI	$\mathrm{II}^*, \mathrm{I}_1$	1	SIII	$\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3$	9

Table 5.9

No.	Singular fibers	$\lvert\mathrm{MW}(X)\rvert$	No.	Singular fibers	$\lvert\mathrm{MW}(X)\rvert$
I	II^*	1	VIII	$\mathrm{I}_1^*, \mathrm{I}_1, \mathrm{I}_4$	4
II	$\mathrm{II}, \mathrm{I}_9$	3	IX	$\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2$	4
III	$\mathrm{IV}^*, \mathrm{I}_3$	3	X	$\mathrm{I}_4^*, \mathrm{I}_1, \mathrm{I}_1$	2
IV	$\mathrm{II}^*, \mathrm{I}_1$	1	XI	$\mathrm{III}^*, \mathrm{I}_1, \mathrm{I}_2$	2
V	$\mathrm{III}^*, \mathrm{III}$	2	SI	$\mathrm{I}_8, \mathrm{I}_2, \mathrm{I}_1, \mathrm{I}_1$	4
VI	$\mathrm{I}_0^*, \mathrm{I}_0^*$	4	SII	$\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1$	5
VII	$\mathrm{III}, \mathrm{I}_3, \mathrm{I}_6$	6	SIII	$\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2$	8

Definition 5.8.5. For a smooth weak del Pezzo surface Y and the union D of all the (-2)-curves in Y, a curve $E \subset Y$ is called *a nice exceptional curve* (NEC for short) if E is a (-1)-curve such that $(E \cdot D) = 1$.

Lemma 5.8.6. *Let X be a Du Val del Pezzo surface with $\rho(X) = 1$ and $d = K_X^2 \leqslant 7$. Let $Y \longrightarrow X$ be the minimal resolution. Then there are an extremal rational elliptic surface or a rational quasi-elliptic surface Y_0 and blow-downs $\{f_i \colon Y_{i-1} \longrightarrow Y_i\}_{1 \leqslant i \leqslant d}$ of (-1)-curves such that $Y_d = Y$ and E_{f_i} is an NEC for $2 \leqslant i \leqslant d$.*

Proof. Since a general member of the anti-canonical linear system is a curve with arithmetic genus one by Theorem 5.1.3 (3), the same proof as in [67, Theorem B.6] remains valid for this case after admitting for Y_0 to be a rational quasi-elliptic surface. □

In the remaining of this section, we follow the notation in Tables 5.8–5.1. In addition, we use the following notation.

Definition 5.8.7. We denote by $Y_n^m(l)$ the successive blow-down of a section and NECs from an extremal rational elliptic surface or a rational quasi-elliptic surface of type m such that the anti-canonical degree is n and the configuration of (-2)-curves is the Dynkin diagram l.

5.8.3 Characteristic two

In this subsection, we treat the case where $p = 2$. Let X be a singular Du Val del Pezzo surface with Picard rank one. By the same proof as in [103, pp.14–15], we obtain that $K_X^2 \neq 7, 9$, and X is the quadric cone in $\mathbb{P}_k^3$ when $K_X^2 = 8$. For this reason, we assume that $K_X^2 \leqslant 6$. We follow the notation of Lemma 5.8.6.

We start with the case where $K_X^2 = 1$. The pairs of Y_0 and $Y = Y_1$ are listed as in Table 5.10, where n is the number of the NECs on Y_1.

Table 5.10

Type of Y_0	Y_1	n	Type of Y_0	Y_1	n
I	$Y_1^{\mathrm{I}}(D_8)$	2	SII	$Y_1^{\mathrm{SII}}(2A_4)$	0
II	$Y_1^{\mathrm{II}}(E_8)$	1	SIII	$Y_1^{\mathrm{SIII}}(4A_2)$	0
III	$Y_1^{\mathrm{III}}(A_1 + A_7)$	1	(a)	$Y_1^{(\mathrm{a})}(E_8)$	1
IV	$Y_1^{\mathrm{IV}}(A_3 + D_5)$	1	(b)	$Y_1^{(\mathrm{b})}(D_8)$	2
V	$Y_1^{\mathrm{V}}(A_1 + E_7)$	1	(c)	$Y_1^{(\mathrm{c})}(A_1 + E_7)$	1
VI	$Y_1^{\mathrm{VI}}(E_8)$	1	(d)	$Y_1^{(\mathrm{d})}(2D_4)$	2
VII	$Y_1^{\mathrm{VII}}(A_2 + E_6)$	1	(e)	$Y_1^{(\mathrm{e})}(2A_1 + D_6)$	1
VIII	$Y_1^{\mathrm{VIII}}(A_1 + A_2 + A_5)$	0	(f)	$Y_1^{(\mathrm{f})}(4A_1 + D_4)$	1
IX	$Y_1^{\mathrm{IX}}(A_2 + E_6)$	1	(g)	$Y_1^{(\mathrm{g})}(8A_1)$	0
SI	$Y_1^{\mathrm{SI}}(A_8)$	2			

The isomorphism class of Y_1 is independent of the choice of E_{f_1} by virtue of the $\mathrm{MW}(Y_0)$-action. Hence there is one-to-one correspondence between the isomorphism classes of Y_0 and those of Y_1. Theorems 5.8.4 and 5.2.2 now show the assertion (1) of Theorem 5.8.1 in the case where $K_X^2 = 1$ and $p = 2$.

Next, we consider the case where $K_X^2 = 2$. Then $Y = Y_2$, which is the blow-down of an NEC in one of Y_1 listed in Table 5.10. The pairs of Y_1 and Y_2 are listed as in Table 5.11, where n is the number of the NECs on Y_2.

Table 5.11

Y_1	Y_2	n	Y_1	Y_2	n
$Y_1^{\mathrm{I}}(D_8)$	$Y_2^{\mathrm{I}}(A_7)$	2	$Y_1^{\mathrm{SI}}(A_8)$	$Y_2^{\mathrm{SI}}(A_2 + A_5)$	1
$Y_1^{\mathrm{I}}(D_8)$	$Y_2^{\mathrm{I}}(A_1 + D_6)$	1	$Y_1^{(\mathrm{a})}(E_8)$	$Y_2^{(\mathrm{a})}(E_7)$	1
$Y_1^{\mathrm{II}}(E_8)$	$Y_2^{\mathrm{II}}(E_7)$	1	$Y_1^{(\mathrm{b})}(D_8)$	$Y_2^{(\mathrm{b})}(A_7)$	2
$Y_1^{\mathrm{III}}(A_1 + A_7)$	$Y_2^{\mathrm{III}}(A_1 + 2A_3)$	0	$Y_1^{(\mathrm{b})}(D_8)$	$Y_2^{(\mathrm{b})}(A_1 + D_6)$	1
$Y_1^{\mathrm{IV}}(A_3 + D_5)$	$Y_2^{\mathrm{IV}}(A_1 + 2A_3)$	0	$Y_1^{(\mathrm{c})}(A_1 + E_7)$	$Y_2^{(\mathrm{c})}(A_1 + D_6)$	1
$Y_1^{\mathrm{V}}(A_1 + E_7)$	$Y_2^{\mathrm{V}}(A_1 + D_6)$	1	$Y_1^{(\mathrm{d})}(2D_4)$	$Y_2^{(\mathrm{d})}(3A_1 + D_4)$	0
$Y_1^{\mathrm{VI}}(E_8)$	$Y_2^{\mathrm{VI}}(E_7)$	1	$Y_1^{(\mathrm{e})}(2A_1 + D_6)$	$Y_2^{(\mathrm{e})}(3A_1 + D_4)$	0
$Y_1^{\mathrm{VII}}(A_2 + E_6)$	$Y_2^{\mathrm{VII}}(A_2 + A_5)$	1	$Y_1^{(\mathrm{f})}(4A_1 + D_4)$	$Y_2^{(\mathrm{f})}(7A_1)$	0
$Y_1^{\mathrm{IX}}(A_2 + E_6)$	$Y_2^{\mathrm{IX}}(A_2 + A_5)$	1			

Remark 5.8.8. The isomorphism class of $Y_2^{\mathrm{SI}}(A_2 + A_5)$ is independent of the choice of E_{f_2} because $\mathrm{MW}(Y_0^{\mathrm{SI}}) \cong \mathbb{Z}/3\mathbb{Z}$ and two NECs in $Y_1^{\mathrm{SI}}(A_8)$ are the images of the sections in Y_0^{SI}. The isomorphism class of $Y_2^{(\mathrm{d})}(3A_1+D_4)$ is also independent of the choice of E_{f_2} because it maps to the other NEC in $Y_1^{(\mathrm{d})}(2D_4)$ by the involution τ on $Y_1^{(\mathrm{d})}(2D_4)$ constructed as in the proof of Proposition 5.5.18.

Lemma 5.8.9. *We have the following isomorphisms:*

(1) $Y_2^{\mathrm{I}}(A_7) \cong Y_2^{(\mathrm{b})}(A_7)$.

(2) $Y_2^{\mathrm{I}}(A_1+D_6) \cong Y_2^{\mathrm{V}}(A_1+D_6)$.

(3) $Y_2^{\mathrm{III}}(A_1+2A_3) \cong Y_2^{\mathrm{IV}}(A_1+2A_3)$.

(4) $Y_2^{\mathrm{VII}}(A_2+A_5) \cong Y_2^{\mathrm{IX}}(A_2+A_5) \cong Y_2^{\mathrm{SI}}(A_2+A_5)$.

(5) $Y_2^{(\mathrm{b})}(A_1+D_6) \cong Y_2^{(\mathrm{c})}(A_1+D_6)$.

(6) $Y_2^{(\mathrm{d})}(3A_1+D_4) \cong Y_2^{(\mathrm{e})}(3A_1+D_4)$.

Proof. We give the proof only for the assertions (1), (2), (5), and (6): the proof of the assertions (3) and (4) run as in [103, Claim 4.5 (4) and (2)] respectively.
(1): Let $Z \longrightarrow Y_2^{\mathrm{I}}(A_7)$ be the contraction of two sections of an extremal rational elliptic surface of type I. Then there is one to one correspondence between the isomorphism classes of $Y_2^{\mathrm{I}}(A_7)$ and those of Z, which are uniquely determined by those j-invariants $\alpha \in k^*$ by [70, p.432].

On the other hand, $Y' := Y_2^{(\mathrm{b})}(A_7)$ is obtained from the rational quasi-elliptic surface of type (b) by contracting two sections. Fix coordinates $[x : y : z]$ of $\mathbb{P}_k^2$ and take $C := \{x^3 + y^2z = 0\}$. By [51, Remark 4], Y' is also obtained from $\mathbb{P}_k^2$ by blowing up four points on C infinitely near $[0 : 1 : 0]$ and three points on C infinitely near $[1 : 1 : 1]$. Moreover, the push forward of $|-K_{Y'}|$ to $\mathbb{P}_k^2$ is generated by $x^3 + y^2z, (x+z)^2z$ and $(x+z)(y+z)z$. Now take C_α as the strict transform of $\{x^3 + y^2z + \alpha^{\frac{1}{8}}(x+z)(y+z)z = 0\} \subset \mathbb{P}_k^2$ in Y'. Then C_α is a smooth member of $|-K_{Y'}|$ whose j-invariant equals α. By blowing up Y' at the intersection of C_α and two NECs on Y', we obtain the extremal rational elliptic surface of type I whose j-invariant is α, which is isomorphic to Z. Therefore $Y_2^{\mathrm{I}}(A_7) \cong Y_2^{(\mathrm{b})}(A_7)$.
(2): Let $Z \longrightarrow Y_2^{\mathrm{I}}(A_1+D_6)$ be the blow-up at a general point of the unique NEC. Then $|-K_Z|$ has a smooth member and Z contains eight (-2)-curves whose configuration is the Dynkin diagram A_1+E_7. Since members of the anti-canonical linear system of $Y_1^{(\mathrm{c})}(A_1+E_7)$ are all singular, Table 5.10 now shows that $Z \cong Y_1^{\mathrm{V}}(A_1+E_7)$. Hence $Y_2^{\mathrm{I}}(A_1+D_6) \cong Y_2^{\mathrm{V}}(A_1+D_6)$.
(5): It follows from Proposition 5.5.5.
(6): It follows from Proposition 5.5.7. □

Lemma 5.8.10. *We have the following:*

(1) $Y_2^{\mathrm{I}}(A_1+D_6) \not\cong Y_2^{(\mathrm{b})}(A_1+D_6)$.

(2) $Y_2^{(\mathrm{a})}(E_7) \not\cong Y_2^{\mathrm{II}}(E_7)$ *and* $Y_2^{(\mathrm{a})}(E_7) \not\cong Y_2^{\mathrm{VI}}(E_7)$.

Proof. A general member of the anti-canonical linear system of $Y_2^{\mathrm{I}}(A_1 + D_6)$ is smooth and that of $Y_2^{\mathrm{(b)}}(A_1 + D_6)$ is singular by Proposition 5.5.5. Hence we have the assertion (1). Similarly, Proposition 5.5.3 gives the assertion (2). □

We will show that $Y_2^{\mathrm{II}}(E_7) \not\cong Y_2^{\mathrm{VI}}(E_7)$ in Corollary 5.8.14. In conclusion, there are 10 isomorphism classes of X with $K_X^2 = 2$.

Next, we deal with the case where $K_X^2 = 3$. Then $Y = Y_3$, which is the blow-down of an NEC in one of Y_2 listed in Table 5.11. The pairs of Y_2 and Y_3 are listed as in Table 5.12, where n is the number of the NECs on Y_3.

Table 5.12

Y_2	Y_3	n	Y_2	Y_3	n
$Y_2^{\mathrm{I}}(A_7)$	$Y_3^{\mathrm{I}}(A_1 + A_5)$	1	$Y_2^{\mathrm{VII}}(A_2 + A_5)$	$Y_3^{\mathrm{VII}}(3A_2)$	0
$Y_2^{\mathrm{II}}(E_7)$	$Y_3^{\mathrm{II}}(E_6)$	1	$Y_2^{\mathrm{(a)}}(E_7)$	$Y_3^{\mathrm{(a)}}(E_6)$	1
$Y_2^{\mathrm{V}}(A_1 + D_6)$	$Y_3^{\mathrm{V}}(A_1 + A_5)$	1	$Y_2^{\mathrm{(b)}}(A_1 + D_6)$	$Y_3^{\mathrm{(b)}}(A_1 + A_5)$	1
$Y_2^{\mathrm{VI}}(E_7)$	$Y_3^{\mathrm{VI}}(E_6)$	1			

Remark 5.8.11. The isomorphism class of $Y_3^{\mathrm{I}}(A_1 + A_5)$ is independent of the choice of E_{f_3} because the $\mathrm{MW}(Y_0)$-action naturally descends to $Y_2^{\mathrm{I}}(A_7)$, which sends one NEC to the other.

Lemma 5.8.12. *We have the isomorphisms* $Y_3^{\mathrm{V}}(A_1{+}A_5) \cong Y_3^{\mathrm{I}}(A_1{+}A_5) \cong Y_3^{\mathrm{(b)}}(A_1{+}A_5)$.

Proof. It follows from Lemma 5.8.9 (1) and (2). □

Lemma 5.8.13. *It holds that* $Y_3^{\mathrm{(a)}}(E_6) \cong Y_3^{\mathrm{II}}(E_6)$ *and* $Y_3^{\mathrm{(a)}}(E_6) \not\cong Y_3^{\mathrm{VI}}(E_6)$.

Proof. Fix coordinates $[x : y : z]$ of $\mathbb{P}_k^2$ and let $C := \{x^3 + y^2 z = 0\}$. By Proposition 5.5.3, $Y_3^{\mathrm{(a)}}(E_6)$ is obtained by blowing up six points on C infinitely near $t := [0 : 1 : 0]$. The anti-canonical linear system of $Y_3^{\mathrm{(a)}}(E_6)$ corresponds to the linear system Λ of cubic curves intersecting with C at t with multiplicity at least six. Then $\Lambda = \{a(x^3 + y^2 z) + bz^3 + cxz^2 + dyz^2 | [a : b : c : d] \in \mathbb{P}_k^3\}$. It is easy to check that $[0 : 1 : 0 : 0]$ corresponds to the member $3\{z = 0\}$ and the locus of singular members of Λ is $\{ad = 0\}$. In particular, a pencil in Λ passing through $[0 : 1 : 0 : 0]$ either consists of singular members or contains exactly one singular member, which is $3\{z = 0\}$.

On the other hand, we recall that the configuration of singular fibers of the extremal rational elliptic surface of type II (resp. VI) is (II^*) (resp. $(\mathrm{II}^*, \mathrm{I}_1)$), where we use Kodaira's notation. Since an elimination of a general pencil in Λ passing through $[0 : 1 : 0 : 0]$ gives the extremal rational elliptic surface of type II, we conclude that $Y_3^{\mathrm{(a)}}(E_6) \cong Y_3^{\mathrm{II}}(E_6)$. On the other hand, there is no pencil in Λ passing through $[0 : 1 : 0 : 0]$ which contains exactly two singular members. Hence $Y_3^{\mathrm{(a)}}(E_6) \not\cong Y_3^{\mathrm{VI}}(E_6)$. □

Corollary 5.8.14. *It holds that* $Y_2^{\mathrm{II}}(E_7) \not\cong Y_2^{\mathrm{VI}}(E_7)$

Proof. Suppose the assertion of the lemma is false. Then $Y_3^{\mathrm{II}}(E_6) \cong Y_3^{\mathrm{VI}}(E_6)$, a contradiction to Lemma 5.8.13. □

Therefore there are four isomorphism classes of X with $K_X^2 = 3$.

Next, we deal with the case where $K_X^2 = 4$. Then $Y = Y_4$, which is the blow-down of an NEC in one of Y_3 listed in Table 5.12. The pairs of Y_3 and Y_4 are listed as in Table 5.13, where n is the number of the NECs on Y_4.

Table 5.13

Y_3	Y_4	n
$Y_3^{\mathrm{I}}(A_1 + A_5)$	$Y_4^{\mathrm{I}}(2A_1 + A_3)$	0
$Y_3^{\mathrm{VI}}(E_6)$	$Y_4^{\mathrm{VI}}(D_5)$	1
$Y_3^{(\mathrm{a})}(E_6)$	$Y_4^{(\mathrm{a})}(D_5)$	1

Lemma 5.8.15. *It holds that* $Y_4^{(\mathrm{a})}(D_5) \not\cong Y_4^{\mathrm{VI}}(D_5)$.

Proof. We follow the notation of the proof of Lemma 5.8.13. Then $Y_4^{(\mathrm{a})}(D_5)$ is obtained by blowing up five points on $C \subset \mathbb{P}_k^2$ infinitely near t and the anti-canonical linear system of $Y_4^{(\mathrm{a})}(D_5)$ corresponds to the linear system $\Lambda' = \{a(x^3 + y^2z) + bz^3 + cxz^2 + dyz^2 + ex^2z | [a : b : c : d : e] \in \mathbb{P}_k^4\}$. It is easy to check that $[0 : 1 : 0 : 0 : 0]$ corresponds to the member $3\{z = 0\}$ and the locus of singular members of Λ' is $\{ad = 0\}$. In particular, there is no pencil in Λ' passing through $[0 : 1 : 0 : 0 : 0]$ which contains exactly two singular members. Hence $Y_4^{(\mathrm{a})}(D_5) \not\cong Y_4^{\mathrm{VI}}(D_5)$. □

Therefore there are three isomorphism classes of X with $K_X^2 = 4$.

Finally, we deal with the case where $K_X^2 = 5$ or 6. When $K_X^2 = 5$, the surface Y is isomorphic to either $Y_5^{(\mathrm{a})}(A_4)$ or $Y_5^{\mathrm{VI}}(A_4)$ by Table 5.13.

Lemma 5.8.16. *It holds that* $Y_5^{(\mathrm{a})}(A_4) \cong Y_5^{\mathrm{VI}}(A_4)$.

Proof. We follow the notation of the proof of Lemma 5.8.13. Then $Y_5^{(\mathrm{a})}(A_4)$ is obtained by blowing up four points on $C \subset \mathbb{P}_k^2$ infinitely near t and the anti-canonical linear system of $Y_5^{(\mathrm{a})}(A_4)$ corresponds to the linear system $\Lambda'' = \{a(x^3 + y^2z) + bz^3 + cxz^2 + dyz^2 + ex^2z + fxyz | [a : b : c : d : e : f] \in \mathbb{P}_k^5\}$. Since $\{x^3 + y^2z + xyz = 0\}$ is a nodal cubic, an elimination of the pencil $\langle z^3, x^3 + y^2z + xyz\rangle \subset \Lambda''$ gives the extremal rational elliptic surface of type VI. Hence $Y_5^{(\mathrm{a})}(A_4) \cong Y_5^{\mathrm{VI}}(A_4)$. □

By blowing down the unique NEC in $Y_5^{(\mathrm{a})}(A_4)$, we obtain $Y_6^{(\mathrm{a})}(A_1 + A_2)$.

Now we can prove Theorem 5.8.1 in the case where $p = 2$.

Proof of Theorem 5.8.1 in the case where $p = 2$. The assertion (1) follows from the above arguments in this subsection. The assertion (2) will be proved once we prove the proposition below. □

Proposition 5.8.17. *Let X be a Du Val del Pezzo surface with $\rho(X) = 1$ and $p = 2$. If $\mathrm{Dyn}(X) \neq D_8, 2D_4, 4A_1 + D_4$, or $8A_1$, then its isomorphism class is uniquely determined by its Dynkin type with Artin coindices.*

Proof. Let Y be the minimal resolution of X. By Theorem 5.8.1 (1), we may assume that $\mathrm{Dyn}(X) = D_5, E_6, E_7, A_1 + D_6, E_8, A_1 + E_7$, or $A_2 + E_6$.

When $\mathrm{Dyn}(X) \neq A_1 + E_7$ or $A_2 + E_6$ in addition, we calculate the defining equation of X as in the proof of Propositions 5.8.2 and 5.8.3; firstly, we choose a suitable blow-down $h\colon Y \longrightarrow \mathbb{P}^2_k$ and calculate a basis of $\Lambda := h_*|-nK_Y|$ with $n = 1$ (resp. $= 2, = 3$) when $K_Y^2 \geqslant 3$ (resp. $= 2, = 1$). Then X is the closure of the image of the map from $\mathbb{P}^2_k$ to $\mathbb{P} := \mathbb{P}^d_k$ with $d = K_Y^2$ (resp. $\mathbb{P}_k(1,1,1,2)$, $\mathbb{P}_k(1,1,2,3)$) defined by Λ when $K_Y^2 \geqslant 3$ (resp. $= 2, = 1$). Finally we compute the defining equation of X in $\mathbb{P}$ to determine $\mathrm{Dyn}'(X)$. As a result, we get the defining equation of X as in Table 5.14, where $[x_0 : \cdots : x_4]$ stands for coordinates of $\mathbb{P}^4_k$ and $[x : y : z : w]$ stands for coordinates of $\mathbb{P}^3_k, \mathbb{P}_k(1,1,1,2)$, or $\mathbb{P}_k(1,1,2,3)$.

Table 5.14

K_Y^2	Y	defining equation of $X \subset \mathbb{P}$	$\mathrm{Dyn}'(X)$
4	$Y_4^{(\mathrm{a})}(D_5)$	$x_2^2 + x_1x_4, x_0x_1 + x_2x_4 + x_3^2$	D_5^0
	$Y_4^{\mathrm{VI}}(D_5)$	$x_2^2 + x_1x_4, x_0x_1 + x_2x_4 + x_3^2 + x_2x_3$	D_5^1
3	$Y_3^{(\mathrm{a})}(E_6)$	$wz^2 + x^3 + y^2z$	E_6^0
	$Y_3^{\mathrm{VI}}(E_6)$	$wz^2 + x^3 + y^2z + xyz$	E_6^1
2	$Y_2^{(\mathrm{a})}(E_7)$	$w^2 + yz^3 + xy^3$	E_7^0
	$Y_2^{\mathrm{II}}(E_7)$	$w^2 + yz^3 + xy^3 + y^2w$	E_7^2
	$Y_2^{\mathrm{VI}}(E_7)$	$w^2 + yz^3 + xy^3 + yzw$	E_7^3
	$Y_2^{(\mathrm{b})}(A_1 + D_6)$	$w^2 + xyz^2 + y^3z$	$A_1 + D_6^0$
	$Y_2^{\mathrm{I}}(A_1 + D_6)$	$w^2 + xyz^2 + y^3z + yzw$	$A_1 + D_6^2$
1	$Y_1^{(\mathrm{a})}(E_8)$	$w^2 + z^3 + xy^5$	E_8^0
	$Y_1^{\mathrm{II}}(E_8)$	$w^2 + z^3 + xy^5 + y^3w$	E_8^3
	$Y_1^{\mathrm{VI}}(E_8)$	$w^2 + z^3 + xy^5 + yzw$	E_8^4

Finally, suppose that $\mathrm{Dyn}(X) = A_1 + E_7$ or $A_2 + E_6$. Then a suitable choice of blow-down $Y \longrightarrow Y'$ gives the minimal resolution Y' of a del Pezzo surface of Picard rank one of type E_7 or E_6 as in Table 5.15.

Table 5.15

Y	Y'	$\mathrm{Dyn}'(X)$
$Y_1^{(\mathrm{c})}(A_1 + E_7)$	$Y_2^{(\mathrm{a})}(E_7)$	$A_1 + E_7^0$
$Y_1^{\mathrm{V}}(A_1 + E_7)$	$Y_2^{\mathrm{VI}}(E_7)$	$A_1 + E_7^3$
$Y_1^{\mathrm{VII}}(A_2 + E_6)$	$Y_3^{(\mathrm{a})}(E_6)$	$A_2 + E_6^0$
$Y_1^{\mathrm{XI}}(A_2 + E_6)$	$Y_3^{\mathrm{VI}}(E_6)$	$A_2 + E_6^1$

Combining these results, we get the assertion. $\square$

5.8.4 Characteristic three

In this subsection, we treat the case where $p = 3$. Let X be a singular Du Val del Pezzo surface with Picard rank one. As in the case where $p = 2$, we may assume that $K_X^2 \leqslant 6$. We follow the notation of Lemma 5.8.6.

We start with the case where $K_X^2 = 1$. The pairs of Y_0 and $Y = Y_1$ are listed as in Table 5.16, where n is the number of the NECs on Y_1.

Table 5.16

Type of Y_0	Y_1	n	Type of Y_0	Y_1	n
I	$Y_1^{\mathrm{I}}(E_8)$	1	X	$Y_1^{\mathrm{X}}(D_8)$	2
II	$Y_1^{\mathrm{II}}(A_8)$	2	XI	$Y_1^{\mathrm{XI}}(A_1 + E_7)$	1
III	$Y_1^{\mathrm{III}}(A_2 + E_6)$	1	SI	$Y_1^{\mathrm{SI}}(A_1 + A_7)$	1
IV	$Y_1^{\mathrm{IV}}(E_8)$	1	SII	$Y_1^{\mathrm{SII}}(2A_4)$	0
V	$Y_1^{\mathrm{V}}(A_1 + E_7)$	1	SIII	$Y_1^{\mathrm{SIII}}(2A_1 + 2A_3)$	0
VI	$Y_1^{\mathrm{VI}}(2D_4)$	2	(1)	$Y_1^{(1)}(E_8)$	1
VII	$Y_1^{\mathrm{VII}}(A_1 + A_2 + A_5)$	0	(2)	$Y_1^{(2)}(A_2 + E_6)$	1
VIII	$Y_1^{\mathrm{VIII}}(A_3 + D_5)$	1	(3)	$Y_1^{(3)}(4A_2)$	0
IX	$Y_1^{\mathrm{IX}}(2A_1 + D_6)$	1			

By virtue of the $\mathrm{MW}(Y_0)$-action, there is one to one correspondence between the isomorphism classes of Y_0 and those of Y_1. Theorems 5.8.4 and 5.2.1 now show the assertion (1) of Theorem 5.8.1 in the case where $K_X^2 = 1$ and $p = 3$.

Next, we consider the case where $K_X^2 = 2$. Then $Y = Y_2$, which is the blow-down of an NEC in one of Y_1 listed in Table 5.16. The pairs of Y_1 and Y_2 are listed as in Table 5.17, where n is the number of the NECs on Y_2.

Table 5.17

Y_1	Y_2	n	Y_1	Y_2	n
$Y_1^{\mathrm{I}}(E_8)$	$Y_2^{\mathrm{I}}(E_7)$	1	$Y_1^{\mathrm{IX}}(2A_1 + D_6)$	$Y_2^{\mathrm{IX}}(3A_1 + D_4)$	0
$Y_1^{\mathrm{II}}(A_8)$	$Y_2^{\mathrm{II}}(A_2 + A_5)$	1	$Y_1^{\mathrm{X}}(D_8)$	$Y_2^{\mathrm{X}}(A_7)$	2
$Y_1^{\mathrm{III}}(A_2 + E_6)$	$Y_2^{\mathrm{III}}(A_2 + A_5)$	1	$Y_1^{\mathrm{X}}(D_8)$	$Y_2^{\mathrm{X}}(A_1 + D_6)$	1
$Y_1^{\mathrm{IV}}(E_8)$	$Y_2^{\mathrm{IV}}(E_7)$	1	$Y_1^{\mathrm{XI}}(A_1 + E_7)$	$Y_2^{\mathrm{XI}}(A_1 + D_6)$	1
$Y_1^{\mathrm{V}}(A_1 + E_7)$	$Y_2^{\mathrm{V}}(A_1 + D_6)$	1	$Y_1^{\mathrm{SI}}(A_1 + A_7)$	$Y_2^{\mathrm{SI}}(A_1 + 2A_3)$	0
$Y_1^{\mathrm{VI}}(2D_4)$	$Y_2^{\mathrm{VI}}(3A_1 + D_4)$	0	$Y_1^{(1)}(E_8)$	$Y_2^{(1)}(E_7)$	1
$Y_1^{\mathrm{VIII}}(A_3 + D_5)$	$Y_2^{\mathrm{VIII}}(A_1 + 2A_3)$	0	$Y_1^{(2)}(A_2 + E_6)$	$Y_2^{(2)}(A_2 + A_5)$	1

Analysis similar to that in Remark 5.8.8 shows that $Y_2^{\mathrm{II}}(A_2 + A_5)$ and $Y_2^{\mathrm{VI}}(3A_1 + D_4)$ are unique up to isomorphism.

Lemma 5.8.18. *We have the following isomorphisms:*

(1) $Y_2^{\mathrm{III}}(A_2 + A_5) \cong Y_2^{\mathrm{II}}(A_2 + A_5) \cong Y_2^{(2)}(A_2 + A_5)$.

(2) $Y_2^{\mathrm{V}}(A_1 + D_6) \cong Y_2^{\mathrm{X}}(A_1 + D_6) \cong Y_2^{\mathrm{XI}}(A_1 + D_6)$.

(3) $Y_2^{\mathrm{VI}}(3A_1 + D_4) \cong Y_2^{\mathrm{IX}}(3A_1 + D_4)$.

(4) $Y_2^{\mathrm{VIII}}(A_1 + 2A_3) \cong Y_2^{\mathrm{SI}}(A_1 + 2A_3)$.

Proof. We give the proof only for the isomorphism $Y_2^{\mathrm{II}}(A_2 + A_5) \cong Y_2^{(2)}(A_2 + A_5)$: the proof of the other assertions run as in [103, Claim 4.5].

Let $Z \longrightarrow Y_2^{(2)}(A_2 + A_5)$ be the blow-up at a general point of a (-1)-curve which is not an NEC. Then Z contains eight (-2)-curves whose configuration is the Dynkin diagram A_8. Hence $Z = Y_1^{\mathrm{II}}(A_8)$ and $Y_2^{\mathrm{II}}(A_2 + A_5) \cong Y_2^{(2)}(A_2 + A_5)$. □

Lemma 5.8.19. *It holds that $Y_2^{(1)}(E_7) \cong Y_2^{\mathrm{I}}(E_7)$ and $Y_2^{(1)}(E_7) \not\cong Y_2^{\mathrm{IV}}(E_7)$.*

Proof. First we recall the construction of $Y_2^{(1)}(E_7)$. Fix coordinates $[x : y : z]$ of $\mathbb{P}_k^2$ and let $C := \{x^3 + y^2 z = 0\}$. By [50, Example 3.8], an elimination of the pencil $\langle x^3 + y^2 z, z^3\rangle$ is the rational quasi-elliptic surface of type (1). Thus $Y_2^{(1)}(E_7)$ is obtained by blowing up seven points on C infinitely near $t := [0 : 1 : 0]$ and the anti-canonical linear system of $Y_2^{(1)}(E_7)$ corresponds to the linear system $\Lambda = \{a(x^3 + y^2 z) + bz^3 + cxz^2 | [a : b : c] \in \mathbb{P}_k^2\}$. It is easy to check that $[0 : 1 : 0]$ corresponds to the member $3\{z = 0\}$ and the locus of singular members of Λ is $\{ac = 0\}$. In particular, a pencil in Λ passing through $[0 : 1 : 0]$ either consists of singular members or contains exactly one singular member, which is $3\{z = 0\}$.

On the other hand, we recall that the configuration of singular fibers of the extremal rational elliptic surface of type I (resp. IV) is (II^*) (resp. $(\mathrm{II}^*, \mathrm{I}_1)$), where we use Kodaira's notation. Since an elimination of a general pencil in Λ passing through $[0 : 1 : 0]$ gives the extremal rational elliptic surface of type I, we conclude that $Y_2^{(1)}(E_7) \cong Y_2^{\mathrm{I}}(E_7)$. On the other hand, there is no pencil in Λ passing through $[0 : 1 : 0]$ which contains exactly two singular members. Hence $Y_2^{(1)}(E_7) \not\cong Y_2^{\mathrm{IV}}(E_7)$ □

In conclusion, there are seven isomorphism classes of X with $K_X^2 = 2$.

Next, we deal with the case where $K_X^2 = 3$. Then $Y = Y_3$, which is the blow-down of an NEC in one of Y_2 listed in Table 5.17. The pairs of Y_2 and Y_3 are listed as in Table 5.18, where n is the number of the NECs on Y_3.

Table 5.18

Y_2	Y_3	n	Y_2	Y_3	n
$Y_2^{\mathrm{II}}(A_2 + A_5)$	$Y_3^{\mathrm{II}}(3A_2)$	0	$Y_2^{\mathrm{X}}(A_7)$	$Y_3^{\mathrm{X}}(A_1 + A_5)$	1
$Y_2^{\mathrm{IV}}(E_7)$	$Y_3^{\mathrm{IV}}(E_6)$	1	$Y_2^{(1)}(E_7)$	$Y_3^{(1)}(E_6)$	1
$Y_2^{\mathrm{V}}(A_1 + D_6)$	$Y_3^{\mathrm{V}}(A_1 + A_5)$	1			

Analysis similar to that in Remark 5.8.11 shows that $Y_3^{\mathrm{X}}(A_1 + A_5)$ is unique up to isomorphism.

Lemma 5.8.20. *We have the isomorphism $Y_3^{\mathrm{V}}(A_1 + A_5) \cong Y_3^{\mathrm{X}}(A_1 + A_5)$.*

Proof. The assertion follows from Lemma 5.8.18 (2). $\square$

Lemma 5.8.21. *It holds that* $Y_3^{(1)}(E_6) \ncong Y_3^{\mathrm{IV}}(E_6)$.

Proof. We follow the notation of the proof of Lemma 5.8.19. Then $Y_3^{(1)}(E_6)$ is obtained by blowing up six points on $C \subset \mathbb{P}_k^2$ infinitely near t and the anti-canonical linear system of $Y_3^{(1)}(E_6)$ corresponds to the linear system $\Lambda' = \{a(x^3+y^2z)+bz^3+cxz^2+dyz^2 | [a:b:c:d] \in \mathbb{P}_k^3\}$. It is easy to check that $[0:1:0:0]$ corresponds to the member $3\{z=0\}$ and the locus of singular members of Λ' is $\{ac=0\}$. In particular, there is no pencil in Λ' passing through $[0:1:0:0]$ which contains exactly two singular members. Hence $Y_3^{(1)}(E_6) \ncong Y_3^{\mathrm{IV}}(E_6)$. $\square$

Therefore there are four isomorphism classes of X with $K_X^2 = 3$.

Finally, we deal with the case where $4 \leqslant K_X^2 \leqslant 6$. When $K_X^2 = 4$, the pairs of Y_3 and $Y = Y_4$ are listed as in Table 5.19, where n is the number of the NECs on Y_4.

Table 5.19

Y_3	Y_4	n
$Y_3^{\mathrm{IV}}(E_6)$	$Y_4^{\mathrm{IV}}(D_5)$	1
$Y_3^{\mathrm{V}}(A_1+A_5)$	$Y_4^{\mathrm{V}}(2A_1+A_3)$	0
$Y_3^{(1)}(E_6)$	$Y_4^{(1)}(D_5)$	1

Lemma 5.8.22. *It holds that* $Y_4^{(1)}(D_5) \cong Y_4^{\mathrm{IV}}(D_5)$.

Proof. We follow the notation of the proof of Lemma 5.8.19. Then $Y_4^{(1)}(D_5)$ is obtained by blowing up five points on $C \subset \mathbb{P}_k^2$ infinitely near t and the anti-canonical linear system of $Y_4^{(1)}(D_5)$ corresponds to the linear system $\Lambda'' = \{a(x^3+y^2z)+bz^3+cxz^2+dyz^2+ex^2z | [a:b:c:d:e] \in \mathbb{P}_k^4\}$. Since $\{x^3+y^2z+x^2z=0\}$ is a nodal cubic, an elimination of the pencil $\langle z^3, x^3+y^2z+x^2z\rangle \subset \Lambda''$ gives the extremal rational elliptic surface of type IV. Hence $Y_4^{(1)}(D_5) \cong Y_4^{\mathrm{IV}}(D_5)$. $\square$

Therefore there are two isomorphism classes of X with $K_X^2 = 4$. By blowing down the unique NEC on $Y_4^{(1)}(D_5)$, we obtain $Y_5^{(1)}(A_4)$. Then it also contains a unique NEC, and the blow-down of this NEC gives $Y_6^{(1)}(A_1+A_2)$.

Now we can prove Theorem 5.8.1 in the case where $p=3$.

Proof of Theorem 5.8.1 in the case where $p=3$. The assertion (1) follows from the above arguments in this subsection. The assertion (2) will be proved once we prove the proposition below. $\square$

Proposition 5.8.23. *Let* X *be a Du Val del Pezzo surface with* $\rho(X)=1$ *and* $p=3$. *If* $\mathrm{Dyn}(X) \neq 2D_4$, *then its isomorphism class is uniquely determined by its Dynkin type with Artin coindices.*

Proof. We follow the notation as in Proposition 5.8.17. By Theorem 5.8.1 (1), we may assume that $\mathrm{Dyn}(X) = E_6, E_7, E_8, A_1 + E_7$, or $A_2 + E_6$. When $\mathrm{Dyn}(X) \neq A_1 + E_7$ or $A_2 + E_6$ in addition, analysis similar to that in the proof of Proposition 5.8.17 gives the defining equation of X in $\mathbb{P}$ as in Table 5.20.

Table 5.20

K_Y^2	Y	defining equation of $X \subset \mathbb{P}$	$\mathrm{Dyn}'(X)$
3	$Y_3^{(1)}(E_6)$	$wz^2 + x^3 + y^2z$	E_6^0
	$Y_3^{\mathrm{IV}}(E_6)$	$wz^2 + x^3 + y^2z + x^2z$	E_6^1
2	$Y_2^{(1)}(E_7)$	$w^2 + yz^3 + xy^3$	E_7^0
	$Y_2^{\mathrm{IV}}(E_7)$	$w^2 + yz^3 + xy^3 + y^2z^2$	E_7^1
1	$Y_1^{(1)}(E_8)$	$w^2 + z^3 + xy^5$	E_8^0
	$Y_1^{\mathrm{I}}(E_8)$	$w^2 + z^3 + xy^5 + y^4z$	E_8^1
	$Y_1^{\mathrm{IV}}(E_8)$	$w^2 + z^3 + xy^5 + y^2z^2$	E_8^2

Finally, suppose that $\mathrm{Dyn}(X) = A_1 + E_7$ or $A_2 + E_6$. Then a suitable choice of blow-down $Y \longrightarrow Y'$ gives the minimal resolution Y' of a del Pezzo surface of Picard rank one of type E_7 or E_6 as in Table 5.21.

Table 5.21

Y	Y'	$\mathrm{Dyn}'(X)$
$Y_1^{\mathrm{V}}(A_1 + E_7)$	$Y_2^{(1)}(E_7)$	$A_1 + E_7^0$
$Y_1^{\mathrm{XI}}(A_1 + E_7)$	$Y_2^{\mathrm{IV}}(E_7)$	$A_1 + E_7^1$
$Y_1^{(2)}(A_2 + E_6)$	$Y_3^{(1)}(E_6)$	$A_2 + E_6^0$
$Y_1^{\mathrm{III}}(A_2 + E_6)$	$Y_3^{\mathrm{IV}}(E_6)$	$A_2 + E_6^1$

Combining these results, we get the assertion. □

5.9 Proof of Theorem 1.3.8

In this subsection, we prove Theorem 1.3.8. We also give a corollary of this theorem.

The following lemma is an immediate consequence of Fedder's criterion [28, Proposition 1.7].

Lemma 5.9.1. *Fix coordinates $[x : y : z : w]$ of $\mathbb{P}_k(1, 1, 2, 3)$ (resp. $\mathbb{P}_k(1, 1, 1, 2), \mathbb{P}_k^3$) and let $f(x, y, z, w) = 0$ be the defining equation of a Du Val del Pezzo surface X of degree one (resp. two, three). Then X is F-split if and only if $f^{p-1} \notin (x^p, y^p, z^p, w^p)$.*

Proof. Let $R := k[x, y, z, w]/(f)$. By Fedder's criterion [28, Proposition 1.7], Spec R is F-split if and only if $f^{p-1} \notin (x^p, y^p, z^p, w^p)$. Since $R \cong \bigoplus_{m\geqslant 0} H^0(X, \mathcal{O}_X(-mK_X))$ and $-K_X$ is ample, it follows that X is F-split by [96, Proposition 4.10]. □

Lemma 5.9.2 ([68, Proposition 2.1]). *Let X be a normal F-split variety and Z a smooth closed subscheme of codimension d which is contained in the smooth locus of X. Let $\Sigma \in |(1-p)K_X|$ be a splitting section. If Σ passes through Z with multiplicity at least $(d-1)(p-1)$, then the blow-up of X along Z is F-split.*

Proof. We refer to the proof of [68, Proposition 2.1] for the details. □

Proposition 5.9.3. *Let X be a Du Val del Pezzo surface. Suppose that $p = 2$ and X is F-split. Then the set of ordinary elliptic curves in $|-K_X|$ is dense.*

Proof. Conversely, suppose that the closure H of the set of ordinary elliptic curves in $|-K_X|$ is the proper closed subset. Choose $C \in |-K_X|\backslash H$ such that X is smooth along C. Let Σ be the splitting section of X and V the pencil generated by C and Σ. Then a general member of V is not an ordinary elliptic curve. When $K_X^2 \neq 2$, we may assume that C is smooth along Σ by Lemma 5.1.3 (4). In particular, a general member of V is also smooth along Σ. Next, suppose that $K_X^2 = 2$. For general two members C_1 and C_2 of V, both $C_1 \cap \Sigma$ and $C_2 \cap \Sigma$ coincides with $C_1 \cap C_2$. Moreover, one of them is smooth along $C_1 \cap C_2$ since otherwise $2 = K_Y^2 = (C_1 \cdot C_2) \geqslant 4$. As a result, a general member of V is smooth along Σ without the assumption on K_X^2.

Let $\varphi\colon Z \longrightarrow \mathbb{P}^1_k$ be an elimination of a rational map associated to the pencil V. Then Z is normal and F-split by Lemma 5.9.2. Thus a general φ-fiber is an ordinary elliptic curve by [90, Corollary 2.3] and Remark 4.1.2 (5). Since a general member of V is smooth along Σ, a general fiber of φ is isomorphic to its image on X. Therefore a general member of V is an ordinary elliptic curve, a contradiction. □

Remark 5.9.4. Let $f\colon V \longrightarrow W$ be a smooth projective morphism between varieties. Yoshikawa [104, Proposition 2.11 (2)] showed that the subset $\{w \in W | V_{\overline{w}} \text{ is } F\text{-split}\}$ is constructible, where $V_{\overline{w}}$ denotes the geometric fiber over w. Making use of this, we can show that a general member of $|-K_X|$ as in Proposition 5.9.3 is an ordinary elliptic curve as follows.

Let $\pi\colon Y \longrightarrow X$ be the minimal resolution. It suffices to show that a general member of $|-K_Y|$ is an ordinary elliptic curve. Note that a member of $|-K_Y|$ is corresponding a fiber of the projection $\mathrm{pr}_2\colon \mathcal{H} \longrightarrow |-K_Y|$, where $\mathcal{H} := \{(y, D) \in Y \times_k |-K_Y| \mid y \in D\} \subset Y \times_k |-K_Y|$. Since $\mathcal{H}$ is Cohen-Macaulay, $|-K_Y|$ is smooth, and pr_2 has equi-dimensional fibers, it follows that pr_2 is flat. Proposition 5.9.3 shows there exists a dense subset $U \subset |-K_Y|$ such that $\mathcal{H}_{\overline{s}}$ is an ordinary elliptic curve for all $s \in U$. By shrinking $|-K_Y|$, we may assume pr_2 is smooth. Hence $\{s \in |-K_Y| | \mathcal{H}_{\overline{s}} \text{ is } F\text{-split}\}$ is constructible by [*ibid.*], and U contains a non-empty open subset.

Remark 5.9.5. Proposition 5.9.3 does not hold without the assumption of characteristic. For example, assuming $p = 3$, consider a Du Val del Pezzo surface $\{w^2 + z^3 + x^2y^2z - x^4z + x^6 = 0\} \subset \mathbb{P}_k(1,1,2,3)_{[x:y:z:w]}$. Then we can see that it is F-split, but a general member of the anti-canonical linear system is $\{w^2 + z^3 + a^2x^4z - x^4z + x^6 = 0\}$ with $a \in k\backslash\{\pm 1\}$, which is a supersingular elliptic curve.

Now we prove Theorem 1.3.8.

Proof of Theorem 1.3.8. By Lemma 5.1.3 (3), we may assume that $p = 2$ or 3. When $p = 2$, the assertion follows from Proposition 5.9.3.

Now suppose that $p = 3$. Let Y be the minimal resolution of X. If $Y \cong Y_1^{(1)}(E_8)$ (resp. $Y_1^{(3)}(4A_2)$) as in Table 5.16, then X is not F-split by Table 5.20 (resp. Table 5.7) and Lemma 5.9.1, a contradiction. Hence it suffices to show that Y is not isomorphic to $Y_1^{(2)}(A_2 + E_6)$ by Theorem 1.3.4 and Proposition 5.4.1. On one hand, Table 5.20 and Lemma 5.9.1 show that $Y_3^{(1)}(E_6)$ is not F-split. Combining Table 5.21 and Remark 4.1.2 (2), we conclude that $Y_1^{(2)}(A_2 + E_6)$ is also not F-split. On the other hand, Y is F-split by Remark 4.1.2 (4). Hence $Y \not\cong Y_1^{(2)}(A_2 + E_6)$. □

At the end of this paper, let us state a corollary of Theorem 1.3.8. The second cohomology of the tangent bundle is important because this contains local-to-global obstructions to deformations (cf. [78, Theorem 4.13]). In characteristic $p = 2$ or 3, there exists a Du Val del Pezzo surface X such that $H^2(X, T_X) \neq 0$ by Theorem 1.3.6 (1). On the other hand, if X is F-split, then Theorem 1.3.8 shows the following.

Corollary 5.9.6. *Let S be a normal projective surface with only Du Val singularities such that $\kappa(S, K_S) = -\infty$. Suppose that S is F-split. Then $H^2(S, T_S) = 0$.*

Proof. By running a K_S-MMP, we obtain a birational contraction $\varphi\colon S \longrightarrow S'$ and a Mori fiber space $S' \longrightarrow B$. Note that K_S is not pseudo-effective. Since

$$\varphi_*(\Omega_S^{[1]} \otimes \mathcal{O}_S(K_S)) \hookrightarrow (\varphi_*(\Omega_S^{[1]} \otimes \mathcal{O}_S(K_S)))^{**} = \Omega_{S'}^{[1]} \otimes \mathcal{O}_{S'}(K_{S'}),$$

the Serre duality yields

$$H^2(S, T_S) \cong H^0(S, \Omega_S^{[1]} \otimes \mathcal{O}_S(K_S)) \subset H^0(S, \Omega_{S'}^{[1]} \otimes \mathcal{O}_{S'}(K_{S'})),$$

where $\Omega_S^{[1]}$ denotes the reflexive hull of Ω_S. When dim $B = 1$, then the assertion follows from [59, Theorem 5.3 (1)]. Now we assume that $\dim B = 0$. Since S' is F-split, it follows that a general member of $|-K_{S'}|$ is smooth by Theorem 1.3.8. Hence the proof similar to Theorem 5.3.1 shows that $H^0(S, \Omega_S'^{[1]} \otimes \mathcal{O}_{S'}(K_{S'})) = 0$. □

Bibliography

[1] Piotr Achinger, Jakub Witaszek, and Maciej Zdanowicz. Liftability of the frobenius morphism and images of toric varieties. *arXiv preprint arXiv:1708.03777*, 2017.

[2] Piotr Achinger and Maciej Zdanowicz. Some elementary examples of non-liftable varieties. *Proc. Amer. Math. Soc.*, 145(11):4717–4729, 2017.

[3] Valery Alexeev. Boundedness and K^2 for log surfaces. *Internat. J. Math.*, 5(6):779–810, 1994.

[4] Valery Alexeev and Shigefumi Mori. Bounding singular surfaces of general type. In *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000)*, pages 143–174. Springer, Berlin, 2004.

[5] M. Artin. Coverings of the rational double points in characteristic p. In *Complex analysis and algebraic geometry*, pages 11–22. 1977.

[6] Emelie Arvidsson. On the Kodaira vanishing theorem for log del Pezzo surfaces in positive characteristic. *Mathematische Zeitschrift*, pages 1–16, 2021.

[7] Emelie Arvidsson, Fabio Bernasconi, and Justin Lacini. On the Kawamata-Viehweg vanishing theorem for log del Pezzo surfaces in positive characteristic. *arXiv preprint arXiv:2006.03571*, 2020.

[8] Arnaud Beauville. Les familles stables de courbes elliptiques sur $\mathbf{P}^1$ admettant quatre fibres singulières. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(19):657–660, 1982.

[9] Grigory Belousov. The maximal number of singular points on log del Pezzo surfaces. *J. Math. Sci. Univ. Tokyo*, 16(2):231–238, 2009.

[10] Fabio Bernasconi. Kawamata-Viehweg vanishing fails for log del pezzo surfaces in characteristic 3. *arXiv preprint arXiv:1709.09238*, 2017.

[11] Fabio Bernasconi, Iacopo Brivio, Tatsuro Kawakami, and Jakub Witaszek. On the log liftability of globally F-split surfaces. In preparation.

[12] Fabio Bernasconi and Hiromu Tanaka. ON DEL PEZZO FIBRATIONS IN POSITIVE CHARACTERISTIC. *J. Inst. Math. Jussieu*, 21(1):197–239, 2022.

[13] Caucher Birkar. Existence of flips and minimal models for 3-folds in char p. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(1):169–212, 2016.

[14] Raimund Blache. The structure of l.c. surfaces of Kodaira dimension zero. I. *J. Algebraic Geom.*, 4(1):137–179, 1995.

[15] F. A. Bogomolov. Holomorphic tensors and vector bundles on projective manifolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(6):1227–1287, 1439, 1978.

[16] E. Bombieri and D. Mumford. Enriques' classification of surfaces in char. p. III. *Invent. Math.*, 35:197–232, 1976.

[17] Michel Brion and Shrawan Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2005.

[18] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[19] Lucian Bădescu. *Algebraic surfaces*. Universitext. Springer-Verlag, New York, 2001. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.

[20] Anders Buch, Jesper F. Thomsen, Niels Lauritzen, and Vikram Mehta. The Frobenius morphism on a toric variety. *Tohoku Math. J. (2)*, 49(3):355–366, 1997.

[21] Paolo Cascini and Hiromu Tanaka. Smooth rational surfaces violating Kawamata-Viehweg vanishing. *Eur. J. Math.*, 4(1):162–176, 2018.

[22] Paolo Cascini and Hiromu Tanaka. Purely log terminal threefolds with non-normal centres in characteristic two. *Amer. J. Math.*, 141(4):941–979, 2019.

[23] Paolo Cascini, Hiromu Tanaka, and Jakub Witaszek. On log del Pezzo surfaces in large characteristic. *Compos. Math.*, 153(4):820–850, 2017.

[24] Sho Ejiri. When is the Albanese morphism an algebraic fiber space in positive characteristic? *Manuscripta Math.*, 160(1-2):239–264, 2019.

[25] Makoto Enokizono. An integral version of zariski decompositions on normal surfaces. *arXiv preprint arXiv:2007.06519*, 2020.

[26] Hélène Esnault and Eckart Viehweg. *Lectures on vanishing theorems*, volume 20 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1992.

[27] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[28] Richard Fedder. F-purity and rational singularity. *Trans. Amer. Math. Soc.*, 278(2):461–480, 1983.

[29] Lei Fu. *Etale cohomology theory*, volume 13 of *Nankai Tracts in Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[30] Osamu Fujino. Minimal model theory for log surfaces. *Publ. Res. Inst. Math. Sci.*, 48(2):339–371, 2012.

[31] Mikio Furushima. Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space $\mathbf{C}^3$. *Nagoya Math. J.*, 104:1–28, 1986.

[32] Yoshinori Gongyo, Zhiyuan Li, Zsolt Patakfalvi, Karl Schwede, Hiromu Tanaka, and Runhong Zong. On rational connectedness of globally F-regular threefolds. *Adv. Math.*, 280:47–78, 2015.

[33] Yoshinori Gongyo and Shunsuke Takagi. Surfaces of globally F-regular and F-split type. *Math. Ann.*, 364(3-4):841–855, 2016.

[34] Patrick Graf. Bogomolov-Sommese vanishing on log canonical pairs. *J. Reine Angew. Math.*, 702:109–142, 2015.

[35] Patrick Graf. Differential forms on log canonical spaces in positive characteristic. *arXiv preprint arXiv:1905.01968*, 2019.

[36] Daniel Greb, Stefan Kebekus, and Sándor J. Kovács. Extension theorems for differential forms and Bogomolov-Sommese vanishing on log canonical varieties. *Compos. Math.*, 146(1):193–219, 2010.

[37] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell. Differential forms on log canonical spaces. *Publ. Math. Inst. Hautes Études Sci.*, (114):87–169, 2011.

[38] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

[39] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.

[40] Paul Hacking and Yuri Prokhorov. Smoothable del Pezzo surfaces with quotient singularities. *Compos. Math.*, 146(1):169–192, 2010.

[41] Christopher Hacon and Jakub Witaszek. The minimal model program for threefolds in characteristic five. *arXiv preprint arXiv:1911.12895*, 2019.

[42] Nobuo Hara. A characterization of rational singularities in terms of injectivity of Frobenius maps. *Amer. J. Math.*, 120(5):981–996, 1998.

[43] Nobuo Hara. Classification of two-dimensional F-regular and F-pure singularities. *Adv. Math.*, 133(1):33–53, 1998.

[44] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[45] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[46] Masayuki Hirokado. A non-liftable Calabi-Yau threefold in characteristic 3. *Tohoku Math. J. (2)*, 51(4):479–487, 1999.

[47] F. Hirzebruch. Arrangements of lines and algebraic surfaces. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 113–140. Birkhäuser, Boston, Mass., 1983.

[48] Toshio Hosoh. Automorphism groups of quartic del Pezzo surfaces. *J. Algebra*, 185(2):374–389, 1996.

[49] Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158. Cambridge University Press, 2016.

[50] Hiroyuki Ito. The Mordell-Weil groups of unirational quasi-elliptic surfaces in characteristic 3. *Math. Z.*, 211(1):1–39, 1992.

[51] Hiroyuki Ito. The Mordell-Weil groups of unirational quasi-elliptic surfaces in characteristic 2. *Tohoku Math. J. (2)*, 46(2):221–251, 1994.

[52] Hiroyuki Ito. On extremal elliptic surfaces in characteristic 2 and 3. *Hiroshima Math. J.*, 32(2):179–188, 2002.

[53] Kazuhiro Ito. Finiteness of Brauer groups of $K3$ surfaces in characteristic 2. *Int. J. Number Theory*, 14(6):1813–1825, 2018.

[54] Kelly Jabbusch and Stefan Kebekus. Families over special base manifolds and a conjecture of Campana. *Math. Z.*, 269(3-4):847–878, 2011.

[55] Gordon James and Martin Liebeck. *Representations and characters of groups.* Cambridge University Press, New York, second edition, 2001.

[56] Nicholas M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, (39):175–232, 1970.

[57] Tatsuro Kawakami. Bogomolov-Sommese type vanishing for globally F-regular threefolds. *Math. Z.*, 299(3-4):1821–1835, 2021.

[58] Tatsuro Kawakami. Bogomolov-sommese vanishing and liftability for surface pairs in positive characteristic. *arXiv preprint arXiv:2108.03768*, 2021.

[59] Tatsuro Kawakami. On Kawamata-Viehweg type vanishing for three dimensional Mori fiber spaces in positive characteristic. *Trans. Amer. Math. Soc.*, 374(8):5697–5717, 2021.

[60] Tatsuro Kawakami and Masaru Nagaoka. Classification of Du Val del Pezzo surfaces of Picard rank one in positive characteristic. *arXiv preprint arXiv:2012.09405*, 2020.

[61] Tatsuro Kawakami and Masaru Nagaoka. Pathologies and liftability of Du Val del Pezzo surfaces in positive characteristic. *arXiv preprint arXiv:2008.07700*, 2020, to appear in Math. Z.

[62] Seán Keel and James McKernan. Rational curves on quasi-projective surfaces. *Mem. Amer. Math. Soc.*, 140(669):viii+153, 1999.

[63] János Kollár. Nonrational hypersurfaces. *J. Amer. Math. Soc.*, 8(1):241–249, 1995.

[64] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[65] János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

[66] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[67] Justin Lacini. On rank one log del Pezzo surfaces in characteristic different from two and three. *arXiv preprint arXiv:2005.14544*, 2020.

[68] V. Lakshmibai, V. B. Mehta, and A. J. Parameswaran. Frobenius splittings and blow-ups. *J. Algebra*, 208(1):101–128, 1998.

[69] William E. Lang. Extremal rational elliptic surfaces in characteristic p. I. Beauville surfaces. *Math. Z.*, 207(3):429–437, 1991.

[70] William E. Lang. Extremal rational elliptic surfaces in characteristic p. II. Surfaces with three or fewer singular fibres. *Ark. Mat.*, 32(2):423–448, 1994.

[71] Adrian Langer. Moduli spaces of sheaves and principal G-bundles. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 273–308. Amer. Math. Soc., Providence, RI, 2009.

[72] Adrian Langer. Bogomolov's inequality for Higgs sheaves in positive characteristic. *Invent. Math.*, 199(3):889–920, 2015.

[73] Adrian Langer. Generic positivity and foliations in positive characteristic. *Adv. Math.*, 277:1–23, 2015.

[74] Adrian Langer. The Bogomolov-Miyaoka-Yau inequality for logarithmic surfaces in positive characteristic. *Duke Math. J.*, 165(14):2737–2769, 2016.

[75] Adrian Langer. Birational geometry of compactifications of Drinfeld half-spaces over a finite field. *Adv. Math.*, 345:861–908, 2019.

[76] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[77] Wanseok Lee, Euisung Park, and Peter Schenzel. On the classification of non-normal cubic hypersurfaces. *J. Pure Appl. Algebra*, 215(8):2034–2042, 2011.

[78] Yongnam Lee and Noboru Nakayama. Simply connected surfaces of general type in positive characteristic via deformation theory. *Proc. Lond. Math. Soc. (3)*, 106(2):225–286, 2013.

[79] Max Lieblich and Davesh Maulik. A note on the cone conjecture for K3 surfaces in positive characteristic. *Math. Res. Lett.*, 25(6):1879–1891, 2018.

[80] Christian Liedtke. Algebraic surfaces in positive characteristic. In *Birational geometry, rational curves, and arithmetic*, pages 229–292. Springer, 2013.

[81] F. J. MacWilliams and N. J. A. Sloane. *The theory of error-correcting codes. II.* North-Holland Mathematical Library, Vol. 16. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[82] Gebhard Martin and Claudia Stadlmayr. Weak del Pezzo surfaces with global vector fields. *arXiv preprint arXiv:2007.03665*, 2020.

[83] Yuya Matsumoto. μ_n-actions on K3 surfaces in positive characteristic. *arXiv preprint arXiv:1710.07158*, 2017.

[84] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[85] V. B. Mehta and A. Ramanathan. Frobenius splitting and cohomology vanishing for Schubert varieties. *Ann. of Math. (2)*, 122(1):27–40, 1985.

[86] Yoichi Miyaoka. Themes and variations—on inequalities of Chern classes. *Sūgaku*, 41(3):193–207, 1989.

[87] Andrew Obus. Lifting of curves with automorphisms. In *Open problems in arithmetic algebraic geometry*, volume 46 of *Adv. Lect. Math. (ALM)*, pages 9–59. Int. Press, Somerville, MA, [2019] ©2019.

[88] Frans Oort. Lifting algebraic curves, abelian varieties, and their endomorphisms to characteristic zero. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 165–195. Amer. Math. Soc., Providence, RI, 1987.

[89] Zsolt Patakfalvi and Maciej Zdanowicz. On the Beauville–Bogomolov decomposition in characteristic $p \geqslant 0$. *arXiv preprint arXiv:1912.12742*, 2019.

[90] Zsolt Patakfalvi and Maciej Zdanowicz. Ordinary varieties with trivial canonical bundle are not uniruled. *arXiv preprint arXiv:2006.04692*, 2020.

[91] S. Ramanan and A. Ramanathan. Some remarks on the instability flag. *Tohoku Math. J. (2)*, 36(2):269–291, 1984.

[92] M. Raynaud. Contre-exemple au "vanishing theorem" en caractéristique $p > 0$. In *C. P. Ramanujam—a tribute*, volume 8 of *Tata Inst. Fund. Res. Studies in Math.*, pages 273–278. Springer, Berlin-New York, 1978.

[93] Karl Schwede and Karen E. Smith. Globally F-regular and log Fano varieties. *Adv. Math.*, 224(3):863–894, 2010.

[94] Bernard Shiffman and Andrew John Sommese. *Vanishing theorems on complex manifolds*, volume 56 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.

[95] Ichiro Shimada. Rational double points on supersingular $K3$ surfaces. *Math. Comp.*, 73(248):1989–2017, 2004.

[96] Karen E. Smith. Vanishing, singularities and effective bounds via prime characteristic local algebra. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 289–325. Amer. Math. Soc., Providence, RI, 1997.

[97] Karen E. Smith. Globally F-regular varieties: applications to vanishing theorems for quotients of Fano varieties. *Michigan Math. J.*, 48:553–572, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[98] Hiromu Tanaka. Minimal models and abundance for positive characteristic log surfaces. *Nagoya Math. J.*, 216:1–70, 2014.

[99] Hiromu Tanaka. The X-method for klt surfaces in positive characteristic. *J. Algebraic Geom.*, 24(4):605–628, 2015.

[100] Hiromu Tanaka. Minimal model program for excellent surfaces. *Ann. Inst. Fourier (Grenoble)*, 68(1):345–376, 2018.

[101] G. van der Geer and T. Katsura. On a stratification of the moduli of $K3$ surfaces. *J. Eur. Math. Soc. (JEMS)*, 2(3):259–290, 2000.

[102] Jakub Witaszek. Effective bounds on singular surfaces in positive characteristic. *Michigan Math. J.*, 66(2):367–388, 2017.

[103] Qiang Ye. On Gorenstein log del Pezzo surfaces. *Japan. J. Math. (N.S.)*, 28(1):87–136, 2002.

[104] Shou Yoshikawa. Liftability of the Frobenius morphism versus the existence of an int-amplified endmorphism. In preparation.