博士論文

On the Helmholtz decomposition of vector fields with bounded mean oscillation in various domains
（諸領域における有界平均振動ベクトル場のヘルムホルツ分解）

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Chapter 1

Introduction

1.1 Purpose of the thesis

The purpose of this thesis is to study the Helmholtz decomposition of vector fields with bounded mean oscillation in various domains other than the whole space. Specifically speaking, the Helmholtz decompositions of vector fields with bounded mean oscillation are established in the cases where the domain is a half space, a bounded C^3 domain and a perturbed C^3 half space with small perturbation.

The study of Helmholtz decomposition investigates the standard question whether a space of vector fields, which is defined in some domain, can be decomposed into the direct sum of a solenoidal subspace and a subspace that is exactly a gradient field. This decomposition plays a fundamental role in the mathematical theory of the Navier-Stokes equations, see e.g. [9]. This is the reason why we are interested in such problems. For vector fields of L^p spaces over domains with $1 < p < \infty$, such decompositions are widely studied. It is well-known that by the Hilbert space method, the Helmholtz decomposition of the L^2 vector fields holds for any arbitrary domain. In the case where p is not equal to 2, whether the Helmholtz decomposition of the L^p vector fields holds or not actually depends on the domain. For bounded domains, the most general result on this decomposition was given by Fujiwara and Morimoto [6]. Their proof was based on the general theory for elliptic partial differential equations by Lions and Magenes [16], [17]. Simader and Sohr [22] generalized this result to both bounded and exterior domains by a variational approach. On the other hand, Bogovskiĭ [5] showed that there exists an unbounded domain in which the Helmholtz decomposition does not hold. However, if one considers the $\tilde{L}^p$ vector fields where $\tilde{L}^p$ is defined to be $L^2 \cap L^p$ for $2 \leq p < \infty$ and $L^2 + L^p$ for $1 < p < 2$, then the Helmholtz decomposition holds for arbitrary uniformly C^2 domain, this is the result due to Farwig, Kozono and Sohr [7]. Their proof was also a variational approach based on duality. In the case when p equals infinity, the Helmholtz decomposition does not hold even in the whole space. The projection mapping to the gradient field in this case is a kind of Riesz operator, which is unbounded in L^∞. Hence, we consider vector fields with bounded mean oscillation as an alternate choice for the L^∞ vector fields.

In the case of the whole space, the Helmholtz decomposition of the space of vector fields with bounded mean oscillation was established by Miyakawa [19]. In his work, the Helmholtz projection was explicitly presented to prove its boundedness in the space of vector fields with bounded mean oscillation. In the case of the half space, we make use of this projection to construct the Helmholtz projection in the half space case explicitly through extending a vector field, defined in the half space, to the whole space by the trick of even and odd extensions. In the cases of a bounded C^3 domain and a perturbed C^3 half space with small perturbation, we establish the Helmholtz decomposition by directly constructing the volume potential. The ideas in this thesis to establish the Helmholtz decomposition are more of potential-theoretical approaches. Although there is a chance that variational approaches through duality might also be possible to establish the Helmholtz decomposition, that would require a thorough understanding for the predual space of space of vector fields with bounded mean oscillation in domain, i.e., we need to have the theory for spaces of vector fields in real Hardy space in domains in advanced. At this moment, we are not ready to consider a variational approach to establish the Helmholtz decomposition for vector fields with bounded mean oscillation in domains. This would be our future target.

1.2 Introduction to Chapter 2

Chapter 2 is devoted to consider the Helmholtz decompositions for vector fields with bounded mean oscillation and vector fields in real Hardy spaces over the half space. We define the space of vector fields with bounded mean oscillation or in real Hardy spaces over the half space in a way such that the even extension of the tangential component and the odd extension of the normal component of a vector field are of bounded mean oscillation or in real Hardy spaces.

By making use of the Helmholtz projection constructed in the whole space case [19], we construct the Helmholtz projection in the half space case directly by considering even and odd extensions and restriction. We show that this projection constructed is bounded linear in both spaces of vector fields with bounded mean oscillation and in real Hardy spaces over the half space. The famous John-Nirenberg inequality, see e.g. [14, Theorem 3.1.6], says that functions of bounded mean oscillation are indeed locally L^2. Hence, for the space of vector fields with bounded mean oscillation over the half space, the trace can be taken in the sense of distributions. By finally invoking the De Rahm's theorem, see e.g. [9, Lemma III.1.1], we show that our projection that is directly constructed indeed induces the correct Helmholtz decomposition for vector fields with bounded mean oscillation over the half space. On the other hand, we do not know how to take the trace properly for vector fields in real Hardy spaces over the half space, therefore we only obtain a partial decomposition for vector fields in real Hardy spaces over the half space in this chapter.

Moreover, by considering the restrictions of atoms defined in the theory of real Hardy spaces in the whole space, we establish the atomic decomposition theorem for the space of vector fields in real Hardy spaces over the half space defined in this chapter. Following the duality argument due to Fefferman and Stein [8], we prove that the space of vector fields with bounded mean oscillation over the half space defined in this chapter is indeed the dual space of the space of vector fields in real Hardy spaces over the half space defined in this chapter. We develop two sets of theories of real Hardy spaces and spaces of bounded mean oscillation defined in the half space which are compatible with the theory of Miyachi [18], where he established the theory of real Hardy spaces defined in domains.

Chapter 2 is based on the joint work [10] with Professor Yoshikazu Giga.

1.3 Introduction to Chapter 3

In Chapter 3, we introduce local bounded mean oscillation spaces in domains. The local bounded mean oscillation space defined in the whole space consists of functions of bounded mean oscillation that are uniformly locally L^1 in the whole space. We define different types of local bounded mean oscillation spaces in a domain by allowing functions to be uniformly locally L^1 only in the δ-neighborhood of the boundary in that domain for $0 < \delta \leq \infty$. We give a classification to these different types of spaces according to different values of δ. We then define a local bounded mean oscillation space of vector fields which admits some boundary control on the normal component of every vector field. We call the boundary control as the b^ν estimate. This b^ν estimate was introduced in the previous works [1], [2], [3] and [4].

Due to Jones [20], we see that the bounded mean oscillation space defined in a domain can be extended linearly continuously to the bounded mean oscillation space defined in the whole space if and only if the domain is a uniform domain. Following Jones' argument, we show that if the domain is a uniform domain, then the local bounded mean oscillation space defined in this domain can be extended linearly continuously to the local bounded mean oscillation space defined in the whole space in a way such that the support of every extended function is contained in a small neighborhood of this domain. Since the local bounded mean oscillation space is the dual space of the local real Hardy space and multiplication by a Hölder function is bounded linear in the local real Hardy space, see e.g. [21, Chapter 3], by our extension theorem for the local bounded mean oscillation space defined in a uniform domain, we deduce that the multiplication by a Hölder function is bounded linear in the local bounded mean oscillation space defined in a uniform domain. This means that we can do cut-off to functions of local bounded mean oscillation defined in a uniform domain.

If the domain is the half space. For a vector field of local bounded mean oscillation with boundary control on its normal component, by the formula of integration by parts, we give an estimate on the L^∞ norm of the normal component of the vector field on the boundary by the local bounded mean oscillation norm of the vector field in the domain, the b^ν estimate of the normal component of the vector field on the boundary and the uniformly locally L^n norm of the divergence of the vector field in the δ-neighborhood of the boundary. This can be done as for a L^1 function defined on the boundary, there exists a bounded linear lifting operator that maps the L^1 function to a function that belongs to the Triebel-Lizorkin space $F^1_{1,2}$, see e.g. [24, Section 4.4.3]. Since the gradient of a function in $F^1_{1,2}$ is indeed in the local Hardy space, we can apply the duality relation. We can then generalize this result to any uniformly $C^{2+\beta}$ domain with $0 < \beta < 1$ by localizing the problem to small neighborhoods of points on the boundary and then flatten the boundary by invoking the normal coordinate change in each of these small neighborhoods. When the boundary is flattened, the problem locally reduces to the half space case. We therefore obtain a trace theorem that holds for any uniformly $C^{2+\beta}$ domain.

Chapter 3 is based on the joint work [11] with Professor Yoshikazu Giga.

1.4 Introduction to Chapter 4

Chapter 4 is devoted to the Helmholtz decomposition of the space of vector fields of bounded mean oscillation defined in a bounded C^3 domain that requires the normal component of every vector field to be b^ν bounded. As we have shown in Chapter 3, in the case of a bounded C^2 domain, the space of vector fields of bounded mean oscillation that requires the normal component of every vector field to be b^ν bounded is indeed L^1. Hence, we do not need to assume the space of vector fields to be of local bounded mean oscillation. In the case of a bounded C^3 domain, multiplication by a Hölder function is bounded linear in the space of vector fields of bounded mean oscillation that implements the b^ν condition on the normal component of every vector field, i.e., we can do cut-off to vector fields with bounded mean oscillation whose normal components are controlled on the boundary.

Our strategy to establish the Helmholtz decomposition is a potential-theoretic approach. Simply speaking, we construct the volume potential corresponding to the divergence of a vector field directly and then solve a Neumann problem with bounded data. The idea of constructing the volume potential is simply applying the minus Laplacian to the divergence of a vector field. However, if we apply the minus Laplacian directly to the divergence of a vector field, we would get a volume potential whose gradient has normal component that is not necessarily b^ν bounded on the boundary. We construct the volume potential in a delicate way. We do cut-off to split a vector field into the sum of a vector field supported away from the boundary and a vector field supported in a small neighborhood of the boundary. For the vector field supported away from the boundary, we construct the corresponding volume potential by applying the minus Laplacian to the divergence of the vector field directly. We can estimate the L^∞ norm of the gradient of this volume potential in a small neighborhood of the boundary, thus this gradient certainly has b^ν bounded normal component. For the vector field supported in a small neighborhood of the boundary, we extend this vector field in a way such that the tangential component of the extended vector field is even with respect to the boundary whereas the normal component of the boundary is odd with respect to the boundary. Then we consider a finite partition of unity to localize the extended vector field to finitely many compact small neighborhoods of points on the boundary. In each of these compact small neighborhoods, we consider the normal coordinate change so that the boundary becomes flattened. Thus locally the problem can be viewed as in the half space. Applying the minus Laplacian in normal coordinate to the localized extended vector field, we construct the corresponding volume potential by Neumann series. Our parity setting for the extended vector field ensures that the gradient of the volume potential constructed from each of the compact small neighborhoods has b^ν bounded normal component. Adding up all volume potentials constructed from each of the compact small neighborhoods together with the volume potential constructed from the vector field supported away from the boundary, we obtain our desired volume potential.

Finally, we solve the Neumann problem with bounded data. Since the domain is a bounded C^3 domain, we recall the Green's function from [13]. For a bounded data defined on the boundary, the unique solution (up to an additive constant) to the Neumann problem is given by the convolution of the Green's function with bounded data on the boundary. In the case of a bounded domain, the Green's function contains two parts, the first part is the usual Newton potential $E(x-y)$, the second part $h(x,y)$ has gradient L^1 with respect to the y variable on the boundary for any point x in the bounded domain (see [13, Lemma 3.1]). The gradient of the convolution of this second part with boundary data on the boundary is thus estimated directly by the L^∞ norm of the boundary data on the boundary. It is sufficient to consider only the Newton potential part. The BMO estimate for the Newton potential part follows from the standard $L^\infty - BMO$ estimate, see e.g. [14, Theorem 4.2.7]. By a direct calculation, we show that the normal derivative of the Newton potential is L^1 with respect to the y variable on the boundary. Hence, the normal derivative of the convolution of the Newton potential with bounded data on the boundary is uniformly bounded by the L^∞ norm of the boundary data on the boundary. The b^ν estimate of the normal component of our solution to the Neumann problem follows naturally. Therefore, we solve our Neumann problem.

Chapter 4 is based on the joint work [12] with Professor Yoshikazu Giga.

1.5 Introduction to Chapter 5

In Chapter 5, we generalize the extension result in Chapter 3 for local bounded mean oscillation functions defined in a domain. In Chapter 3, we follow the idea of Jones [20] and establish an extension theorem for local bounded mean oscillation functions defined in a uniform domain. In this chapter, we invoke the extension introduced in Chapter 4 which extends a function supported in a small neighborhood of the boundary evenly with respect to the boundary, in order to establish an extension theorem for local bounded mean oscillation functions defined in arbitrary uniformly C^2 domain. Although we requires the boundary to be uniformly C^2, our extension theorem extends Jones' result [20] in the sense that local bounded mean oscillation functions defined in a non-uniform domain can also be extended linearly continuously.

Our strategy is to firstly decompose a function into the sum of a function supported away from the boundary and a function supported in a small neighborhood of the boundary in our domain. We can achieve this by multiplying a cut-off test function supported in our domain. At this stage, multiplication is not necessarily bounded linear as the domain is not necessarily uniform. However, we can still uniformly estimate the mean oscillation of the function supported in a small neighborhood of the boundary over all balls in the domain with sufficiently small radius. This is because for a small ball close to the boundary, we may find a bounded C^2 subdomain that is contained in our domain such that the small ball is contained in this bounded C^2 subdomain, hence we can perform the multiplication rule for local bounded mean oscillation functions inside this bounded C^2 subdomain. For a small ball lying sufficiently away from the boundary, then the function supported in a small neighborhood of the boundary is actually zero in this small ball. We then even extend this function with respect to the boundary, the idea of even extension with respect to the boundary is introduced in Chapter 4. Since the extended function in this case is supported in a small neighborhood of the boundary in the whole space, by invoking the normal coordinate we can also uniformly estimate the mean oscillation of the extended function over all balls in the whole space with sufficiently small radius. As we have shown in Chapter 3, if a function is uniformly locally L^1, then being able to uniformly estimate the mean oscillation of the function over all balls with sufficiently small radius is equivalent to prove that the function is of bounded mean oscillation. Hence, the extended function is of local bounded mean oscillation in the whole space. For the function supported away from the boundary, we extend it to the whole space by simply considering its zero extension. Since we can also uniformly estimate the mean oscillation of its zero extension over all balls in the whole space with sufficiently small radius, this zero extended function is of local bounded mean oscillation in the whole space. Add up these two extended function together, we extend our original function to the whole space.

By our extension theorem, we further deduce that the multiplication by any Hölder function is bounded linear in the local bounded mean oscillation space defined in a uniformly C^2 domain. Moreover, we also obtain several uniform estimates regarding to a uniformly C^2 domain. We show that for each point on the boundary, the gradient of the normal coordinate change in a small neighborhood of that point with fixed size is uniformly controlled by a constant depending only on the size of the small neighborhood of that point. We also obtain a locally finite partition of unity for a small neighborhood of the boundary such that the C^1 norm of each partition function is uniformly controlled. These uniform estimates will also be used in the next chapter.

1.6 Introduction to Chapter 6

Chapter 6 is devoted to the Helmholtz decomposition of a space of vector fields with bounded mean oscillation defined in a perturbed C^3 half space with small perturbation. A perturbed C^3 half space is the region above a compactly supported C^3 function. By small perturbation we require the C^2 norm of the boundary function to be small and the support of the boundary function to be not too big. In Chapter 4, since we consider the Helmholtz decomposition for vector fields defined in a bounded C^3 domain, the space of vector fields of bounded mean oscillation, in which boundary control is implemented on the normal component of each vector field, is indeed L^1 in the bounded domain. We do not need to assume the vector fields to be of local bounded mean oscillation in order to allow cut-off by multiplication. In the case of a perturbed C^3 half space, in order to allow cut-off by multiplication we need some extra integrability, other than requiring the vector fields to be of bounded mean oscillation and to have b^ν bounded normal components. We consider the space of L^2 vector fields that is of bounded mean oscillation having bounded b^ν normal components. This is in some sense compatible with the result of Farwig, Kozono and Sohr [7] where the Helmholtz decomposition of the $L^p \cap L^2$ vector fields $(2 \leq p < \infty)$ was established.

Our strategy follows from the potential theoretical approach introduced in Chapter 4. We firstly construct the volume potential and then solve a Neumann problem. In the case of a bounded C^3 domain, the construction of the volume potential works as the boundary is compact. There exist finitely many points on the boundary such that small neighborhoods of these points provide an open cover of a small neighborhood of the boundary. Thus, the gradient of the normal coordinate change in each of these small neighborhoods is uniformly controlled. In addition, since we have a finite partition of unity for a small neighborhood of the boundary, the C^1 norm of each partition function is uniformly controlled. In Chapter 5, we see that in the case of a uniformly C^2 domain, although the boundary is not compact, the gradient of the normal coordinate change in a small neighborhood of every point on the boundary is uniformly controlled regardless of where the point is. Moreover, in the case of a uniformly C^2 domain, there exist countably many points on the boundary such that small neighborhoods of these points provide a locally finite open cover of a small neighborhood of the boundary. By considering the normal coordinate change in each of these small neighborhood in this locally finite open cover, we can construct a partition of unity for a small neighborhood of the boundary such that the C^1 norm of each partition function is uniformly controlled. Hence by following the argument of constructing volume potential in Chapter 4, we can generalize the volume potential construction to arbitrary uniformly C^3 domain instead of just to a perturbed C^3 half space.

At the end of Chapter 4, we solve the Neumann problem with bounded data. In this case of a perturbed C^3 half space, since we consider the L^2 vector fields that are of bounded mean oscillation, the normal trace is actually $L^\infty \cap H^{-\frac{1}{2}}$ on the boundary. In this chapter, our target is to solve the Neumann problem under $L^\infty \cap H^{-\frac{1}{2}}$ data. For a $L^\infty \cap H^{-\frac{1}{2}}$ boundary data, we consider its double layer potential on the boundary. By separating the boundary into the straight part and the curved part and then viewing the straight part as part of the half space boundary and the curved part as part of the boundary of a bounded C^2 domain, the trace of this double layer potential on the boundary is indeed an bounded linear operator in L^∞ of the form $(\frac{1}{2}I - S)$ acting on the boundary data. By considering Neumann series, we can construct the inverse to the operator $(I-2S)$. The Neumann series converges if the operator norm of S is small enough. That is why we need the perturbation to be small. The unique solution (up to an additive constant) to our Neumann problem is then given by the single layer potential of $2(I - 2S)^{-1}$ acting on the boundary data. Similar as in the case of a bounded domain, the BMO estimate of the gradient of our solution follows from the standard $L^\infty - BMO$ estimate, see e.g. [14, Theorem 4.2.7]. The L^∞ norm of the normal component of the gradient of our solution in a small neighborhood of the boundary is estimated by the $L^\infty \cap H^{-\frac{1}{2}}$ norm of the boundary data. In the case of a half space, the solution to the Neumann problem is explicitly given by twice of the single layer potential of the boundary data. The standard theory says that the L^2 estimate of the gradient of this solution in the half space is estimated by the $H^{-\frac{1}{2}}$ norm of the boundary data, see e.g. [23, Remark 1.2 and Remark 1.3], [15, Section 1.7]. In our problem, we again separate the boundary into the straight part and the curved part. We view the straight part as part of the boundary of the half space, hence we can invoke the standard theory of the half space case to estimate the contribution of the straight part in the L^2 estimate. The contribution of the curved part in the L^2 estimate can be calculated directly as the curved part is compact. Therefore, we have our desired L^2 estimate for the gradient of our solution to the Neumann problem.

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Chapter 2

On the Helmholtz decompositions of vector fields of bounded mean oscillation and in real Hardy spaces over the half space

This chapter is concerned with the Helmholtz decompositions of vector fields of bounded mean oscillation over the half space and vector fields in real Hardy spaces over the half space. It proves the Helmholtz decomposition for vector fields of bounded mean oscillation over the half space whereas a partial Helmholtz decomposition for vector fields in real Hardy spaces over the half space. Meanwhile, it also establishes two sets of theories of real Hardy spaces over the half space which are compatible with the theory of Miyachi (1990).

2.1 Introduction

In this chapter, we investigate the Helmholtz decompositions of vector fields of bounded mean oscillation over the half space and vector fields in real Hardy spaces over the half space. The subject of studying Helmholtz decompositions asks the standard question whether a vector field, in some specific function spaces over some specific domains, can be decomposed into the direct sum of a solenoidal subspace and a subspace which is exactly a gradient field. The reason why we are interested in this subject is due to the well known fact that Helmholtz decomposition plays an important role in constructing mild solutions of the Navier-Stokes equations.

Helmholtz decompositions are widely studied for vector fields of L^p spaces over many kinds of different domains when $1 < p < \infty$. For example, we have the result that for every open domain $\Omega \subset \mathbb{R}^n$ the Helmholtz decomposition holds for vector fields of $L^2(\Omega)$. When p does not equal to 2, we also know that the Helmholtz decompositions of vector fields of L^p spaces hold for some domains while there exists other domains where the Helmholtz decompositions of vector fields of L^p spaces fail to hold, e.g. see [4]. Although problems when p does not equal to 2 are much more difficult than the case when p equals to 2, we still had various results. However, this subject is poorly studied for vector fields of other function spaces. In the case for vector fields of bounded mean oscillation and vector fields in real Hardy spaces, we only have a single piece of result, obtained by Miyakawa [8], states that the Helmholtz decompositions of vector fields of bounded mean oscillation over $\mathbb{R}^n$ and vector fields in real Hardy spaces over $\mathbb{R}^n$ hold. This lack of study is due to the fact that the theories of real Hardy spaces and BMO spaces over domains other than $\mathbb{R}^n$ are harder to deal with and moreover, the proper definitions of the space of vector fields of bounded mean oscillation and the space of vector fields in real Hardy spaces over other domains are not known perfectly. The purpose of this chapter seeks to extend the result of Miyakawa [8] from $\mathbb{R}^n$ to $\mathbb{R}^n_+ = \{(\mathbf{x}', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} | x_n > 0\}$. In the meantime, we show that our definitions of the space of vector fields of bounded mean oscillation over $\mathbb{R}^n_+$ and the space of vector fields in real Hardy spaces over $\mathbb{R}^n_+$ are valid, in the sense that they admit a duality relation.

In order to define the space of vector fields of bounded mean oscillation over $\mathbb{R}^n_+$, we need to define two types of BMO spaces over $\mathbb{R}^n_+$ firstly, one corresponds to the function space for the tangent direction while the other one corresponds to the function space for the normal direction. The BMO space over $\mathbb{R}^n_+$ for the tangent direction we define is the space $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$. In Section 2.5, we prove that $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ is equivalent to $BMO(\mathbb{R}^n_+) := r_{\mathbb{R}^n_+} BMO$, the restriction of functions of BMO to $\mathbb{R}^n_+$. The BMO space over $\mathbb{R}^n_+$ for the normal direction we define is the space $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$. In [1], it is proved that $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$ is equivalent to $BMO_M(\mathbb{R}^n_+)$ where $BMO_M(\mathbb{R}^n_+)$ is the BMO space defined by Miyachi in [7]. Therefore the space of vector fields of bounded mean oscillation over $\mathbb{R}^n_+$, denoted by $\mathbf{X}$, can be defined as $\mathbf{X} := (BMO(\mathbb{R}^n_+))^{n-1} \times BMO_M(\mathbb{R}^n_+)$. The first main theorem of this chapter reads as follows. Let $\mathbf{n}$ be the exterior unit normal of the boundary of $\mathbb{R}^n_+$, i.e., $\mathbf{n} = (0, 0, -1)$ so that the inner product $\mathbf{v} \cdot \mathbf{n}$ denotes the normal trace to $\partial\mathbb{R}^n_+$ of a vector field $\mathbf{v}$ on $\mathbb{R}^n_+$.

Theorem 2.1.1. *Let $\mathbf{X}$ be the space of vector fields of bounded mean oscillation over the half space $\mathbb{R}^n_+$, then $\mathbf{X}$ admits the Helmholtz decomposition*

$$\mathbf{X} = \mathbf{X}_\sigma \oplus \mathbf{X}_\pi$$

with the Helmholtz projection $\mathbb{P}_{\mathbb{R}^n_+}$ where

$$\begin{aligned} \mathbf{X}_\sigma &= \{\, \mathbf{v} \in \mathbf{X} \mid \operatorname{div} \mathbf{v} = 0 \ \ \text{in } \mathbb{R}^n_+ \ \ \& \ \ \mathbf{v} \cdot \mathbf{n} = 0 \ \ \text{on } \partial\mathbb{R}^n_+ \,\}, \\ \mathbf{X}_\pi &= \{\, \nabla p \in \mathbf{X} \mid p \in L^1_{loc}(\overline{\mathbb{R}^n_+}) \,\}. \end{aligned}$$

The key idea of the proof of Theorem 2.1.1 is to consider extension and restriction. When Miyakawa [8] established the Helmholtz decomposition of vector fields of bounded mean oscillation over $\mathbb{R}^n$ and vector fields in real Hardy spaces over $\mathbb{R}^n$, he considered the Helmholtz projection $\mathbb{P}$ where $\mathbb{P}_{i,j} := \delta_{i,j} + R_i R_j$ and R_i is the i-th Riesz transform for $1 \le i, j \le n$. Here we make use of this idea. We define our projection by $\mathbb{P}_{\mathbb{R}^n_+} := r_{\mathbb{R}^n_+} \mathbb{P} E$ where E is the extension operator which extends vectors in $\mathbf{X}$ to vectors in BMO and $r_{\mathbb{R}^n_+}$ is the restriction operator which restricts vectors in BMO back to vectors in $\mathbf{X}$. Then we prove that our projection $\mathbb{P}_{\mathbb{R}^n_+}$ is actually a bounded linear map from $\mathbf{X}$ to $\mathbf{X}$. Hence through this projection we have a natural decomposition of our space $\mathbf{X}$ of the form

$$\mathbf{X} = \mathbb{P}_{\mathbb{R}^n_+} \mathbf{X} \oplus (I - \mathbb{P}_{\mathbb{R}^n_+}) \mathbf{X}.$$

Then we prove that the subspace $\mathbb{P}_{\mathbb{R}^n_+} \mathbf{X}$ is actually the solenoidal part and the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+}) \mathbf{X}$ is actually the gradient part. As for the trace problem, we can make use of the theory of Temam [10] since $\mathbf{X} \subset \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+})$. Notice that the space $\mathbf{X}$ is not a proper Banach space due to the fact that the BMO-type norm is just a seminorm. Therefore, in order to avoid any ambiguity, we mean the Helmholtz decomposition not for $\mathbf{X}$ in the usual sense but for the quotient space $\mathbf{X}/(\mathbb{R}^{n-1} \times \{0\})$. Here we direct the readers to Section 2.2 for the precise definitions of the extension E, the restriction $r_{\mathbb{R}^n_+}$, the space $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ and the space $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$.

By similar ideas as above, we need to define two types of real Hardy spaces over $\mathbb{R}^n_+$ in order to define the space of vector fields in real Hardy spaces over $\mathbb{R}^n_+$. For the real Hardy space over $\mathbb{R}^n_+$ in the tangent direction, denoted by $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$, is defined to be the restriction of all even functions in the real Hardy space over $\mathbb{R}^n$ to the half space $\mathbb{R}^n_+$. For the real Hardy space over $\mathbb{R}^n_+$ in the normal direction, denoted by $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, is defined to be the restriction of all odd functions in the real Hardy space over $\mathbb{R}^n$ to the half space $\mathbb{R}^n_+$. In Section 2.5, we also prove that $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ is equivalent to $\mathscr{H}^1_{M}(\mathbb{R}^n_+)$ where $\mathscr{H}^1_{M}(\mathbb{R}^n_+)$ is the real Hardy space defined by Miyachi in [7]. Hence the space of vector fields in real Hardy spaces over $\mathbb{R}^n_+$, denoted by $\mathbf{Y}$, can be defined as $\mathbf{Y} := (\mathscr{H}^1_{even}(\mathbb{R}^n_+))^{n-1} \times \mathscr{H}^1_{M}(\mathbb{R}^n_+)$. Let $\mathbf{Y}_\sigma = \{\, \mathbf{v} \in \mathbf{Y} \mid \operatorname{div} \mathbf{v} = 0 \ \text{ in } \mathbb{R}^n_+ \ \ \& \ \ \mathbf{v} \cdot \mathbf{n} = 0 \ \text{ on } \partial\mathbb{R}^n_+ \,\}$, the second main theorem in this chapter reads as follows.

Theorem 2.1.2. *Let $\mathbf{Y}$ be the vector field in real Hardy spaces over the half space $\mathbb{R}^n_+$, then $\mathbf{Y}$ admits a decomposition of the form*

$$\mathbf{Y} = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} \oplus \mathbf{Y}_\pi$$

with a bounded linear projection $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{Y} \to \mathbf{Y}$ where

$$\begin{aligned} \mathbf{Y}_\sigma \subset \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} &\subset \{\, \mathbf{v} \in \mathbf{Y} \mid \operatorname{div} \mathbf{v} = 0 \ \ \textit{in} \ \ \mathbb{R}^n_+ \,\}, \\ \mathbf{Y}_\pi &= \{\, \nabla p \in \mathbf{Y} \mid p \in L^1_{loc}(\overline{\mathbb{R}^n_+}) \,\}. \end{aligned}$$

Similar to the proof of Theorem 2.1.1, we consider the same projection $\mathbb{P}_{\mathbb{R}^n_+} := r_{\mathbb{R}^n_+}\mathbb{P}E$ and we prove that $\mathbb{P}_{\mathbb{R}^n_+}$ is also a bounded linear map from $\mathbf{Y}$ to $\mathbf{Y}$. Using the same idea, we can see that $\mathbf{Y}$ also admits a natural decomposition of the form

$$\mathbf{Y} = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} \oplus (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}.$$

Although the later theory is basically the same as the previous case for vector fields of bounded mean oscillation, in this case we do not know how to solve the trace problem. Hence for the subspace $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y}$ we can only say that it is divergence free, we cannot say that it is the right solenoidal part in the Helmholtz decomposition. We have no problems in characterizing the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$. Indeed, $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$ is the right gradient part, just like the previous case. For the precise definitions of the spaces $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, we direct the readers to Section 2.2. Notice that if we can solve the trace problem, then this decomposition turns into the full Helmholtz decomposition immediately. Hence for this decomposition, we call it a partial Helmholtz decomposition.

By the standard theory of real Hardy spaces, we can see that the space of vector fields of bounded mean oscillation over $\mathbb{R}^n$ is exactly the dual space of the space of vector fields in real Hardy spaces $\mathscr{H}^1(\mathbb{R}^n)$. In order to make the theory over $\mathbb{R}^n_+$ to be compatible with the theory over $\mathbb{R}^n$, it is necessary to consider the relation between the spaces $\mathbf{X}$ and $\mathbf{Y}$. Fortunately, we have a positive answer to this question.

Theorem 2.1.3. *Suppose $\mathbf{v} \in \mathbf{X}$. Then the linear functional l defined on $\mathbf{Y}$ by*

$$l(\mathbf{u}) = \int_{\mathbb{R}^n_+} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}$$

for $\mathbf{u} \in \mathbf{Y}$ *is a bounded linear functional which satisfies* $||l|| \leq c \cdot ||\mathbf{v}||_{\mathbf{X}}$ *with some constant* c*. Conversely, every bounded linear functional on* $\mathbf{Y}$ *can be written in the form of*

$$l(\mathbf{u}) = \int_{\mathbb{R}^n_+} \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} \quad \textit{for all } \mathbf{u} \in \mathbf{Y}$$

with $\mathbf{v} \in \mathbf{X}$ *and* $||\mathbf{v}||_{\mathbf{X}} \leq c \cdot ||l||$ *with some constant* c*. Here* $\|l\|$ *means the norm of* l *as a bounded linear functional on* $\mathbf{Y}$*.*

In short, the above theorem states the simple fact that $\mathbf{X}$ is the dual space of $\mathbf{Y}$. To prove the above theorem, we prove that $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ is the dual space of $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$ is the dual space of $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. The key idea in showing these two duality relations is again to consider extensions and restrictions. By the theories in the previous part, we see that the even extension of elements in $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ produce elements in $\mathscr{H}^1(\mathbb{R}^n)$ and the odd extension of elements in $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ also produce elements in $\mathscr{H}^1(\mathbb{R}^n)$. Since elements in $\mathscr{H}^1(\mathbb{R}^n)$ admit atomic decompositions, by taking the restrictions we can get the half space version of atomic decompositions of elements in $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Then by similar arguments of Fefferman and Stein [3] in proving that BMO is the dual space of $\mathscr{H}^1(\mathbb{R}^n)$, we can prove the two duality relations concerning $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. The proof of Theorem 2.1.3 establishes two sets of complete theories for our two types of real Hardy spaces over $\mathbb{R}^n_+$. These two sets of theories are indeed compatible with the theory of Miyachi [7] where he established the theory of real Hardy spaces over arbitrary open subsets of $\mathbb{R}^n$. As a result, Theorem 2.1.3 verifies the validity of the definitions of $\mathbf{X}$ and $\mathbf{Y}$.

In the work of Miyakawa [8], he also found the fact that the dual operator of the whole space Helmholtz projection $\mathbb{P}$ is indeed $\mathbb{P}$ itself. In this chapter we also investigate the dual operator of our half space Helmholtz projection $\mathbb{P}_{\mathbb{R}^n_+}$ and we obtain the following result.

Theorem 2.1.4. *The dual operator of* $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{Y} \to \mathbf{Y}$ *is* $\mathbb{P}_{\mathbb{R}^n_+}$ *itself as a map from* $\mathbf{X}$ *to* $\mathbf{X}$*, i.e.,* ${\mathbb{P}_{\mathbb{R}^n_+}}^* = \mathbb{P}_{\mathbb{R}^n_+}$ *as a map from* $\mathbf{X}$ *to* $\mathbf{X}$*.*

The key idea lies in the proof of Theorem 2.1.3. This theorem can be easily deduced by simply considering the dual operators of E, $\mathbb{P}$ and $r_{\mathbb{R}^n_+}$. By making use of this theorem, we can further deduce the following important corollary.

Corollary 2.1.5. ${\mathbf{X}_\sigma} = {\mathbf{Y}_\pi}^\perp$ *and* $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} = {\mathbf{X}_\pi}^\perp$*.*

Notice that here because we do not know how to take the trace of elements in $\mathbf{Y}$ properly, we can only say that $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y}$ is the annihilator of $\mathbf{X}_\pi$. If the trace problem is settled, this relation turns into $\mathbf{Y}_\sigma = {\mathbf{X}_\pi}^\perp$ immediately.

This chapter is organized as follow. In section 2.2, we give out the basic definitions. In section 2.3, we investigate the Helmholtz decomposition of $\mathbf{X}$. In section 2.4, we investigate the Helmholtz decomposition of $\mathbf{Y}$. In section 2.5, we study the duality relationship between $\mathbf{X}$ and $\mathbf{Y}$. In section 2.6, we study the dual operator of our Helmholtz projection $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{Y} \to \mathbf{Y}$.

2.2 Definitions and notations

Let $\mathbb{R}^{\mathrm{n}}_+ := \{\mathbf{x} \in \mathbb{R}^n | x_n > 0\}$ be the half space where x_n here is the n-th component of $\mathbf{x}$ and let $\partial\mathbb{R}^{\mathrm{n}}_+ := \{\mathbf{x} \in \mathbb{R}^n | x_n = 0\}$ be the boundary of the half space $\mathbb{R}^{\mathrm{n}}_+$. The space $L^1_{loc}(\mathbb{R}^{\mathrm{n}}_+)$ is defined in the usual way as the set

$$\{f : \mathbb{R}^n_+ \to \mathbb{R} \text{ measurable} \mid \|f\|_{L^1(\Omega)} < \infty \text{ for any open subsets } \Omega \subset\subset \mathbb{R}^n_+\}$$

and $\mathbf{L}^1_{loc}(\mathbb{R}^n_+) := (L^1_{loc}(\mathbb{R}^n_+))^n$.

Definition 2.2.1. Let $f \in L^1_{loc}(\mathbb{R}^n_+)$ and $B_r(\mathbf{x})$ be the open ball of radius r centered at $\mathbf{x}$, we define three types of BMO-type seminorms as the following:

- $[f]_{BMO^\infty(\mathbb{R}^n_+)} := \sup\limits_{B\subset\mathbb{R}^n_+} \frac{1}{|B|}\int_B |f(\mathbf{y}) - f_B|\, \mathrm{d}\mathbf{y}$

  where $f_B := \frac{1}{|B|}\int_B f(\mathbf{y})\, \mathrm{d}\mathbf{y}$ and B is an open ball.

- $[f]_{b^\infty(\mathbb{R}^n_+)} := \sup\limits_{\substack{r>0 \\ \mathbf{x}\in\partial\mathbb{R}^n_+}} \frac{1}{|B_r(\mathbf{x})\cap\mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y})|\, \mathrm{d}\mathbf{y}$.

- $[f]_{ba^\infty(\mathbb{R}^n_+)} := \sup\limits_{\substack{r>0 \\ \mathbf{x}\in\partial\mathbb{R}^n_+}} \frac{1}{|B_r(\mathbf{x})\cap\mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y}) - f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+}|\, \mathrm{d}\mathbf{y}$

  where $f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} := \frac{1}{|B_r(\mathbf{x})\cap\mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} f(\mathbf{y})\, \mathrm{d}\mathbf{y}$.

The seminorm $[\,\cdot\,]_{b^\infty(\mathbb{R}^n_+)}$ is already introduced in [1] with a more general form. In [1], the definition of this seminorm is of the form $[\,\cdot\,]_{b^\nu p(\Omega)}$ where ν could be any real number including ∞ and $p \in [1,\infty)$. In our case, when ν is equal to ∞ and $p = 1$, an easy check quickly shows that this seminorm is indeed a norm. Therefore it is unambiguous to replace $[\,\cdot\,]_{b^\infty(\mathbb{R}^n_+)}$ by $\|\cdot\|_{b^\infty(\mathbb{R}^n_+)}$.

Definition 2.2.2. We define two types of BMO spaces over the half space $\mathbb{R}^n_+$ in the following way:

- $BMO^{\infty,\infty}_b(\mathbb{R}^n_+) := \{f \in L^1_{loc}(\mathbb{R}^n_+) \mid \|f\|_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)} < \infty\}$

  where $\|f\|_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)} := [f]_{BMO^\infty(\mathbb{R}^n_+)} + \|f\|_{b^\infty(\mathbb{R}^n_+)}$.

- $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+) := \{f \in L^1_{loc}(\mathbb{R}^n_+) \mid [f]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)} < \infty\}$

  where $[f]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)} := [f]_{BMO^\infty(\mathbb{R}^n_+)} + [f]_{ba^\infty(\mathbb{R}^n_+)}$.

Since $\|\cdot\|_{b^\infty(\mathbb{R}^n_+)}$ is indeed a norm, $\|\cdot\|_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)}$ is also a norm. However, $[\,\cdot\,]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)}$ is simply a seminorm.

Definition 2.2.3. The space of vector fields of bounded mean oscillation over the half space $\mathbb{R}^n_+$ is defined in the following way:

$$\mathbf{X}(\mathbb{R}^n_+, \mathbb{R}^n) := \{(\mathbf{v}', v^n) \mid \mathbf{v}' \in (BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+))^{n-1}, v^n \in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)\}$$

where $\mathbf{v}' := (v^1, \ldots, v^{n-1})$ and $\mathbf{v} := (v^1, \ldots, v^{n-1}, v^n)$. We define the seminorm $[\,\cdot\,]_{\mathbf{X}}$ on the space of vector fields $\mathbf{X}(\mathbb{R}^n_+, \mathbb{R}^n)$ as follow:

$$[\mathbf{v}]_{\mathbf{X}} := \sum_{i=1}^{n-1} [v^i]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)} + \|v^n\|_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)}.$$

From now on, without any ambiguity, we shall denote $(\mathbf{X}, [\,\cdot\,]_{\mathbf{X}})$ simply by $\mathbf{X}$ for abbreviation.

Next we would like to define two extension operators which extend functions over the half space $\mathbb{R}^n_+$ to functions over the whole space $\mathbb{R}^n$.

Definition 2.2.4. Let $f : \mathbb{R}^n_+ \to \mathbb{R}$, we say that $E_{odd}\, f : \mathbb{R}^n \to \mathbb{R}$ is the odd extension of f if

$$E_{odd}\, f(\mathbf{x}', x_n) = \begin{cases} f(\mathbf{x}', x_n) & \text{if } x_n > 0, \\ -f(\mathbf{x}', -x_n) & \text{if } x_n < 0. \end{cases}$$

a.e. (almost everywhere).

Definition 2.2.5. Let $f : \mathbb{R}^n_+ \to \mathbb{R}$, we say that $E_{even}\, f : \mathbb{R}^n \to \mathbb{R}$ is the even extension of f if

$$E_{even}\, f(\mathbf{x}', x_n) = \begin{cases} f(\mathbf{x}', x_n) & \text{if } x_n > 0, \\ f(\mathbf{x}', -x_n) & \text{if } x_n < 0. \end{cases}$$

a.e. (almost everywhere).

Based on these two definitions of extension, we are able to define an extension operator for vector fields of functions over the half space $\mathbb{R}^n_+$.

Definition 2.2.6. Let $f^i : \mathbb{R}^n_+ \to \mathbb{R}$ for $1 \le i \le n$ and let $\mathbf{f} = (f^1, \dots, f^{n-1}, f^n)$, we define the extension of f by

$$E\mathbf{f} = \begin{cases} (E\mathbf{f})^i := E_{even}\, f^i & \text{for } 1 \le i \le n-1, \\ (E\mathbf{f})^n := E_{odd}\, f^n. \end{cases}$$

After we defined the extension operator, we shall now define the restriction operator, for functions and vector fields.

Definition 2.2.7. The restriction operator is defined as follow in two cases:

- Let $f : \mathbb{R}^n \to \mathbb{R}$, we define the restriction $r_{\mathbb{R}^n_+} f$ by $r_{\mathbb{R}^n_+} f := f\,|_{\mathbb{R}^n_+} : \mathbb{R}^n_+ \to \mathbb{R}^n$.
- Let $\mathbf{f} = (f^1, \dots, f^{n-1}, f^n)$ and $f^i : \mathbb{R}^n \to \mathbb{R}$ with $1 \le i \le n$, we define the i-th component of the restriction $r_{\mathbb{R}^n_+} \mathbf{f}$ by $(r_{\mathbb{R}^n_+} \mathbf{f})^i := r_{\mathbb{R}^n_+} f^i$.

Now we have done enough preparations for defining our vector field of real Hardy space $\mathscr{H}^1$ over $\mathbb{R}^n_+$.

Definition 2.2.8. We define two types of real Hardy space $\mathscr{H}^1$ over the half space $\mathbb{R}^n_+$ in the following way:

- $\mathscr{H}^1_{odd}(\mathbb{R}^n_+) := \{ f \in L^1(\mathbb{R}^n_+) \mid \|f\|_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} < \infty \}$

  where $\|f\|_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} := \big\| \sup_{t>0} |r_{\mathbb{R}^n_+}\, e^{t\Delta}\, E_{odd}\, f|\,(\mathbf{x}) \big\|_{L^1(\mathbb{R}^n_+)}$.

- $\mathscr{H}^1_{even}(\mathbb{R}^n_+) := \{ f \in L^1(\mathbb{R}^n_+) \mid \|f\|_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} < \infty \}$

  where $\|f\|_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} := \big\| \sup_{t>0} |r_{\mathbb{R}^n_+}\, e^{t\Delta}\, E_{even}\, f|\,(\mathbf{x}) \big\|_{L^1(\mathbb{R}^n_+)}$.

Here $e^{t\Delta}$ is the heat semigroup. In other words, $(e^{t\Delta}f)(\mathbf{x}) = \int_{\mathbb{R}^n} G_t(\mathbf{x}-\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y}$ where $G_t(\mathbf{x}) = \frac{1}{(4\pi t)^n}e^{-\frac{|\mathbf{x}|^2}{4t}}$ denotes the heat kernel. We also write as $(G_t * f)(\mathbf{x})$ by using the notation of convolution.

Definition 2.2.9. The space of vector fields in real Hardy spaces over the half space $\mathbb{R}^n_+$ is defined in the following way:

$$\mathbf{Y}(\mathbb{R}^n_+, \mathbb{R}^n) := \{(\mathbf{u}', u^n) \mid \mathbf{u}' \in (\mathscr{H}^1_{even}(\mathbb{R}^n_+))^{n-1}, u^n \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)\}$$

where $\mathbf{u}' := (u^1, \ldots, u^{n-1})$ and $\mathbf{u} := (u^1, \ldots, u^{n-1}, u^n)$. We define the norm $\|\cdot\|_{\mathbf{Y}}$ on $\mathbf{Y}$ by

$$\|\mathbf{u}\|_{\mathbf{Y}} := \sum_{i=1}^{n-1} \|u^i\|_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} + \|u^n\|_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}.$$

From now on, without any ambiguity, we shall denote $(\mathbf{Y}, \|\cdot\|_{\mathbf{Y}})$ simply by $\mathbf{Y}$ for abbreviation.

Definition 2.2.10. We define $\mathbb{P}$ by $(\mathbb{P})_{ij} := \delta_{ij} + R_iR_j$ with $1 \leq i, j \leq n$ where R_i is the i-th Riesz transform.

Here $\mathbb{P}$ is an $n \times n$ matrix whose entries are transforms. This $\mathbb{P}$ is exactly the Helmholtz projection established by Miyakawa in [8].

Definition 2.2.11. We define the half space projection operator $\mathbb{P}_{\mathbb{R}^n_+}$ by $\mathbb{P}_{\mathbb{R}^n_+} := r_{\mathbb{R}^n_+}\,\mathbb{P}\,E$, that means for $\mathbf{v} \in \mathbf{X}$ (or $\mathbf{Y}$) we have that $\mathbb{P}_{\mathbb{R}^n_+}\,\mathbf{v} := r_{\mathbb{R}^n_+}\,\mathbb{P}\,E\,\mathbf{v}$.

Before we end this section, let us recall the real Hardy space and the BMO space defined by Miyachi in [7] when the domain $\Omega = \mathbb{R}^n_+$ and $p = 1$. Let $\varphi \in C_0^\infty(B(0,1))$ such that $\int_{\mathbb{R}^n} \varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$. For $\mathbf{x} \in \mathbb{R}^n_+$, let $d_{\mathbb{R}^n_+}(\mathbf{x}) := \mathrm{dist}(\mathbf{x}, (\mathbb{R}^n_+)^c)$.

Definition 2.2.12. We denote by $\mathscr{H}^1_M(\mathbb{R}^n_+)$ the set of those $f \in L^1(\mathbb{R}^n_+)$ such that

$$\left\| \sup_{0<t<d_{\mathbb{R}^n_+}(\mathbf{x})} |\varphi_t * f|(\mathbf{x}) \right\|_{L^1(\mathbb{R}^n_+)} < \infty.$$

Definition 2.2.13. Let $f \in L^1_{loc}(\mathbb{R}^n_+)$, we say $f \in BMO_M(\mathbb{R}^n_+)$ if

$$\|f\|_{BMO_M(\mathbb{R}^n_+)} := [f]_{BMO(\mathbb{R}^n_+)} + [f]_{b(\mathbb{R}^n_+)} < \infty$$

where

$$[f]_{BMO(\mathbb{R}^n_+)} := \sup\left\{ \frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |f - f_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \mid B_{2r}(\mathbf{x}) \subset \mathbb{R}^n_+ \right\},$$

$$[f]_{b(\mathbb{R}^n_+)} := \sup\left\{ \frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |f|\,\mathrm{d}\mathbf{y} \mid B_{2r}(\mathbf{x}) \subset \mathbb{R}^n_+ \text{ and } B_{5r}(\mathbf{x}) \cap (\mathbb{R}^n_+)^c \neq \emptyset \right\}.$$

2.3 Helmholtz decomposition of vector fields of bounded mean oscillation over the half space

2.3.1 Boundedness of projection $\mathbb{P}_{\mathbb{R}^n_+}$ from $\mathbf{X}$ to $\mathbf{X}$

Let $\mathbf{v} \in \mathbf{X}$ and $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} := r_{\mathbb{R}^n_+}\,\mathbb{P}\,E\,\mathbf{v}$.

Lemma 2.3.1. *Let $f \in BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$, then we have that $E_{odd}f \in BMO(\mathbb{R}^n,\mathbb{R})$ and there exists a constant C which only depends on n such that*

$$[E_{odd}f]_{BMO} \leq C \cdot ||f||_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)}.$$

Proof. This lemma has already been established in [1, Lemma 7]. □

Lemma 2.3.2. *Let $f \in BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)$, then we have that $E_{even}f \in BMO(\mathbb{R}^n,\mathbb{R})$ and there exists a constant C which only depends on n such that*

$$[E_{even}f]_{BMO} \leq C \cdot [f]_{BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)}.$$

Proof. For simplicity let us denote $E_{even}f$ by $\tilde{f}$, let $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. If $B_r(\mathbf{x}) \subset \mathbb{R}^n_+$ or $B_r(\mathbf{x}) \subset (\mathbb{R}^n_+)^c$, we can easily verify that

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \leq [f]_{BMO^\infty(\mathbb{R}^n_+)}.$$

(1). If $B_r(\mathbf{x}) \cap \partial\mathbb{R}^n_+ \neq \varnothing$ and $\mathbf{x} \in \partial\mathbb{R}^n_+$, then due to the fact that $\tilde{f}$ is even with respect to x_n, we have

$$\begin{aligned}\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} &\leq \frac{2}{|B_r(\mathbf{x}) \cap \mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \\ &\leq \frac{2}{|B_r(\mathbf{x}) \cap \mathbb{R}^n_+|}\Big(\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y}) - f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+}|\,\mathrm{d}\mathbf{y} \\ &\quad + \int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y}\Big) \cdots\cdots (*1).\end{aligned}$$

Here $f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} := \frac{1}{|B_r(\mathbf{x})\cap\mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} f(\mathbf{y})\,\mathrm{d}\mathbf{y}$. By simple check we can further notice that

$$\tilde{f}_{B_r(\mathbf{x})} = \frac{1}{|B_r(\mathbf{x}) \cap \mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} f(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Therefore $f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} = \tilde{f}_{B_r(\mathbf{x})}$ if $\mathbf{x} \in \partial\mathbb{R}^n_+$ and hence

$$\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} = 0.$$

By continuing the calculation we can deduce that

$$(*1) = \frac{2}{|B_r(\mathbf{x}) \cap \mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y}) - f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+}|\,\mathrm{d}\mathbf{y} \leq 2 \cdot [f]_{ba^\infty(\mathbb{R}^n_+)}.$$

Thus if $\mathbf{x} \in \partial\mathbb{R}^n_+$, then for any $r > 0$ we have that

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \le 2\cdot [f]_{ba^\infty(\mathbb{R}^n_+)}.$$

(2). If $B_r(\mathbf{x})\cap\partial\mathbb{R}^n_+ \ne \varnothing$ and $\mathbf{x}\notin\partial\mathbb{R}^n_+$, then $\exists\,\mathbf{x}^*\in B_r(\mathbf{x})\cap\partial\mathbb{R}^n_+$ and $B_r(\mathbf{x})\subset B_{2r}(\mathbf{x}^*)$. Notice that

$$\begin{aligned}\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y} &\le \frac{|B_{2r}(\mathbf{x}^*)|}{|B_r(\mathbf{x})|}\cdot\frac{1}{|B_{2r}(\mathbf{x}^*)|}\cdot\int_{B_{2r}(\mathbf{x}^*)} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y}\\ &\le \frac{|B_{2r}(\mathbf{x}^*)|}{|B_r(\mathbf{x})|}\cdot 2\cdot [f]_{ba^\infty(\mathbb{R}^n_+)}\\ &= 2^{n+1}\cdot [f]_{ba^\infty(\mathbb{R}^n_+)}.\end{aligned}$$

The second inequality here holds because of (1). Notice that

$$\begin{aligned}\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \le \Big(&\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y}\\ &+\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}_{B_{2r}(\mathbf{x}^*)} - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y}\Big)\cdots\cdots(*2).\end{aligned}$$

and

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}_{B_{2r}(\mathbf{x}^*)} - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \le \frac{1}{|B_r(\mathbf{x})|}\cdot\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y}.$$

Therefore

$$(*2) \le \frac{2}{|B_r(\mathbf{x})|}\cdot\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y} \le 2^{n+2}\cdot [f]_{ba^\infty(\mathbb{R}^n_+)}.$$

As a result, for any $\mathbf{x}\in\mathbb{R}^n_+$ and $r > 0$, we have that

$$\begin{aligned}\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} &\le ([f]_{BMO^\infty(\mathbb{R}^n_+)} + 2^{n+2}\cdot [f]_{ba^\infty(\mathbb{R}^n_+)})\\ &= 2^{n+2}\cdot [f]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)}\end{aligned}$$

by (1) and (2). Therefore it is true that

$$[\tilde{f}]_{BMO} \le 2^{n+2}\cdot [f]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)}.$$

□

Lemma 2.3.3. *Let $f\in BMO(\mathbb{R}^n,\mathbb{R})$ and f be odd with respect to x_n, i.e., $f(\mathbf{x}', x_n) = -f(\mathbf{x}', -x_n)$, then we have that $r_{\mathbb{R}^n_+}f\in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$ and there exists a universal constant C such that*

$$||r_{\mathbb{R}^n_+}f||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)} \le C\cdot [f]_{BMO}.$$

Proof. (1). Notice that

$$[r_{\mathbb{R}^n_+}f]_{BMO^\infty(\mathbb{R}^n_+)} \le \sup_{\substack{\mathbf{x}\in\mathbb{R}^n\\ r>0}}\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |f(\mathbf{y}) - f_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} = [f]_{BMO}.$$

(2). Let $\mathbf{x} \in \partial\mathbb{R}^n_+$ and $r > 0$. Let $B_r^+(\mathbf{x}) := B_r(\mathbf{x}) \cap \mathbb{R}^n_+$ and $B_r^-(\mathbf{x}) := B_r(\mathbf{x}) \cap (\mathbb{R}^n_+)^c$. We have that

$$f_{B_r(\mathbf{x})} = \frac{1}{|B_r(\mathbf{x})|}\Big(\int_{B_r^+(\mathbf{x})} f(\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{B_r^-(\mathbf{x})} f(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big).$$

Notice that by change of variables we can easily deduce that

$$\int_{B_r^-(\mathbf{x})} f(\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{B_r^+(\mathbf{x})} f(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Hence

$$f_{B_r(\mathbf{x})} = \frac{1}{|B_r(\mathbf{x})|}\cdot\Big(\int_{B_r^+(\mathbf{x})} f(\mathbf{y})\,\mathrm{d}\mathbf{y} - \int_{B_r^+(\mathbf{x})} f(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big) = 0.$$

Therefore in this case, we have that

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |f(\mathbf{y}) - f_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} = \frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |f(\mathbf{y})|\,\mathrm{d}\mathbf{y}.$$

By taking the supremum, we can deduce that

$$\begin{aligned}\sup_{\substack{r>0\\ \mathbf{x}\in\partial\mathbb{R}^n_+}} r^{-n}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y})|\,\mathrm{d}\mathbf{y} &\leq \sup_{\substack{r>0\\ \mathbf{x}\in\partial\mathbb{R}^n_+}} \frac{C}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |f(\mathbf{y}) - f_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y}\\ &\leq C\cdot[f]_{BMO}.\end{aligned}$$

Thus

$$||r_{\mathbb{R}^n_+} f||_{b^\infty(\mathbb{R}^n_+)} \leq C\cdot[f]_{BMO}.$$

Therefore by (1) and (2), we have that

$$||r_{\mathbb{R}^n_+} f||_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)} \leq C\cdot[f]_{BMO}.$$

□

Lemma 2.3.4. *Let $f \in BMO(\mathbb{R}^n_+, \mathbb{R})$, then we have that $r_{\mathbb{R}^n_+} f \in BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)$ and there exists a universal constant C such that*

$$[r_{\mathbb{R}^n_+} f]_{BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)} \leq C\cdot[f]_{BMO}.$$

Proof. Firstly let us recall the fact that in defining the BMO-seminorm it is equivalent to consider the supremum over all balls and all squares. Here we make use of this idea. Let $f \in BMO(\mathbb{R}^n_+, \mathbb{R})$, $\mathbf{x} \in \partial\mathbb{R}^n_+$ and $r > 0$, let $B_r^+(\mathbf{x})$ be the intersection of the ball $B_r(\mathbf{x})$ and the half space $\mathbb{R}^n_+$. Let $\tilde{Q}_c$ be the set of squares whose centers are on the boundary $\partial\mathbb{R}^n_+$ with sides parallel to the coordinate system. Notice that a simple triangle inequality would give us the fact that if for each half ball $B_r^+(\mathbf{x})$ there exists a constant $c_{B_r^+(\mathbf{x})}$ such that

$$\sup_{\substack{\mathbf{x}\in\partial\mathbb{R}^n_+\\ r>0}} \frac{1}{|B_r^+(\mathbf{x})|}\int_{B_r^+(\mathbf{x})} |f(\mathbf{y}) - c_{B_r^+(\mathbf{x})}|\,\mathrm{d}\mathbf{y} < \infty, \tag{2.3.1}$$

then $[f]_{ba^\infty} < \infty$. Now we let $Q_* \in \tilde{Q}_c$ be the smallest square that contains $B_r(\mathbf{x})$, then we can easily deduce that

$$\begin{aligned}\frac{1}{|B_r^+(\mathbf{x})|}\int_{B_r^+(\mathbf{x})}|f(\mathbf{y})-f_{Q_*^+}|\,\mathrm{d}\mathbf{y} &\leq \frac{|Q_*^+|}{|B_r^+(\mathbf{x})|}\cdot\frac{1}{|Q_*^+|}\int_{Q_*^+}|f(\mathbf{y})-f_{Q_*^+}|\,\mathrm{d}\mathbf{y}\\ &\leq c\cdot\sup_{Q\in\tilde{Q}_c}\frac{1}{|Q^+|}\int_{Q^+}|f(\mathbf{y})-f_{Q^+}|\,\mathrm{d}\mathbf{y}\end{aligned}$$

where c is a constant independent of the radius r and Q^+ is the intersection of Q and $\mathbb{R}^n_+$. Hence by (2.3.1) there exists a constant c such that

$$[f]_{ba^\infty(\mathbb{R}^n_+)} \leq c\cdot\sup_{Q\in\tilde{Q}_c}\frac{1}{|Q^+|}\int_{Q^+}|f(\mathbf{y})-f_{Q^+}|\,\mathrm{d}\mathbf{y}.$$

For the opposite direction let $Q^* \in \tilde{Q}_c$ be the largest square that is contained in the ball $B_r(\mathbf{x})$, then we have

$$\frac{1}{|Q^{*+}|}\int_{Q^{*+}}|f(\mathbf{y})-f_{B_r^+(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \leq \frac{|B_r^+(\mathbf{x})|}{|Q^{*+}|}\cdot\frac{1}{|B_r^+(\mathbf{x})|}\int_{B_r^+(\mathbf{x})}|f(\mathbf{y})-f_{B_r^+(\mathbf{x})}|\,\mathrm{d}\mathbf{y}.$$

By similar arguments as proving (2.3.1), if we take the supremum over all squares, we have that

$$\sup_{Q\in\tilde{Q}_c}\frac{1}{|Q^+|}\int_{Q^+}|f(\mathbf{y})-f_{Q^+}|\,\mathrm{d}\mathbf{y} \leq c\cdot[f]_{ba^\infty(\mathbb{R}^n_+)}.$$

Therefore the seminorm $[f]_{ba^\infty(\mathbb{R}^n_+)}$ is equivalent to the seminorm $\sup_{Q\in\tilde{Q}_c}\frac{1}{|Q^+|}\int_{Q^+}|f(\mathbf{y})-f_{Q^+}|\,\mathrm{d}\mathbf{y}$. To prove Lemma 2.3.4, we only need to check that the seminorm $\sup_{Q\in\tilde{Q}_c}\frac{1}{|Q^+|}\int_{Q^+}|f(\mathbf{y})-f_{Q^+}|\,\mathrm{d}\mathbf{y}$ is less than infinity. This is indeed since we always have that

$$\begin{aligned}\frac{1}{|Q^+|}\int_{Q^+}|f(\mathbf{y})-f_Q|\,\mathrm{d}\mathbf{y} &\leq \frac{|Q|}{|Q^+|}\cdot\frac{1}{|Q|}\cdot\int_Q|f(\mathbf{y})-f_Q|\,\mathrm{d}\mathbf{y}\\ &= c\cdot[f]_{BMO}\\ &< \infty.\end{aligned}$$

By applying the argument of the square version of (2.3.1) again, we can deduce that

$$\frac{1}{|Q^+|}\int_{Q^+}|f(\mathbf{y})-f_{Q^+}|\,\mathrm{d}\mathbf{y} \leq c\cdot[f]_{BMO} < \infty.$$

Therefore by taking the supremum, we are done. □

Now we are ready to prove the main lemma in this subsection.

Lemma 2.3.5. $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{X} \to \mathbf{X}$ *is a bounded linear operator.*

Proof. (1). Let $\mathbf{v} \in \mathbf{X}$, by Lemma 2.3.1 and Lemma 2.3.2, we can deduce that there exists a constant C such that

$$\begin{aligned}[E\mathbf{v}]_{BMO} &= \sum_{i=1}^{n-1}[E_{even}\, v^i]_{BMO} + [E_{odd}\, v^n]_{BMO} \\ &\leq C \cdot (\sum_{i=1}^{n-1}[v^i]_{BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)} + ||v^n||_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)}) \\ &\leq C \cdot [\mathbf{v}]_{\mathbf{X}}.\end{aligned}$$

Therefore $E : \mathbf{X} \to BMO(\mathbb{R}^n, \mathbb{R}^n)$ is a bounded linear operator.

(2). Since the Riesz transform R_i is a bounded linear operator from $BMO(\mathbb{R}^n, \mathbb{R}^n)$ to $BMO(\mathbb{R}^n, \mathbb{R}^n)$ for each i, we can easily deduce that the projection $\mathbb{P} := I + R \otimes R$ is also a bounded linear operator from $BMO(\mathbb{R}^n, \mathbb{R}^n)$ to $BMO(\mathbb{R}^n, \mathbb{R}^n)$. As for the boundedness of Riesz transforms from BMO to BMO, please refer to Fefferman and Stein [3].

(3). Notice the fact that $(\mathbb{P}E\mathbf{v})^i$ is even with respect to x_n for i such that $1 \leq i \leq n-1$ whereas $(\mathbb{P}E\mathbf{v})^n$ is odd with respect to x_n. This fact will be proved in subsection 2.3.3. Then by Lemma 2.3.3 and Lemma 2.3.4, we can deduce that there exists a constant C such that

$$[\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v}]_{\mathbf{X}} \leq C \cdot [\mathbf{v}]_{\mathbf{X}}.$$

□

2.3.2 Trace problem

Let $\mathbf{u} \in \mathbf{X}$, then by Lemma 2.3.1 and Lemma 2.3.2 we know that $E\mathbf{u} \in BMO(\mathbb{R}^n, \mathbb{R}^n)$. Let $\mathbf{L}^2_{loc}(\Omega) := (L^2_{loc}(\Omega))^n$ where $\Omega \subseteq \mathbb{R}^n$.

Lemma 2.3.6. *Let $\mathbf{u} \in \mathbf{X}$, then we have that $\mathbf{u} \in \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+})$.*

Proof. Let $u \in L^1_{loc}(\mathbb{R}^n_+)$ and $Eu \in L^1_{loc}(\mathbb{R}^n)$ be an extension of u.

(1). $Eu \in BMO$ implies that $Eu \in L^2_{loc}(\mathbb{R}^n)$. This is indeed true since if we let B be any open ball in $\mathbb{R}^n$, by the John-Nirenberg inequality we have that

$$\begin{aligned}||Eu||^2_{L^2(B)} &= 2 \cdot \int_0^\infty \alpha\mu(\{\mathbf{x} \in B \mid |Eu(\mathbf{x}) - Eu_B| > \alpha\})\, \mathrm{d}\alpha \\ &\leq C_1 \cdot |B| \cdot \int_0^\infty \alpha \cdot \exp(-\frac{C_2\alpha}{[Eu]_{BMO}})\, \mathrm{d}\alpha \\ &< \infty.\end{aligned}$$

The first equality above is due to $||f||^p_{L^p} = p\int_0^\infty \alpha^{p-1} d_f(\alpha)\, \mathrm{d}\alpha$ where $d_f(\alpha)$ is the distribution function of f, for this fact please refer to L.Grafakos [5].

(2). Let $K \subset\subset \overline{\mathbb{R}^n_+}$, it is certainly that $K \subset B_r(\mathbf{x}) \cap \mathbb{R}^n_+$ for some $\mathbf{x} \in \partial\mathbb{R}^n_+$ and $r > 0$, then we have that

$$||u||_{L^2(K)} \leq ||u||_{L^2(B_r(\mathbf{x})\cap\mathbb{R}^n_+)} \leq ||Eu||_{L^2(B_r(\mathbf{x}))} < \infty.$$

Therefore $u \in L^2(K)$ for any $K \subset\subset \overline{\mathbb{R}^n_+}$, that means $u \in L^2_{loc}(\overline{\mathbb{R}^n_+})$.

For $\mathbf{u} \in \mathbf{X}$, we have that $E_{even}\, u^i \in BMO$ for $1 \leq i \leq n-1$ and $E_{odd}\, u^n \in BMO$, hence by (1) and (2) $u^i \in L^2_{loc}(\overline{\mathbb{R}^n_+})$ for $1 \leq i \leq n$. □

Since we have proved that $\mathbf{u} \in \mathbf{X}$ implies that $\mathbf{u} \in \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+})$, we are able to make use of the theory of R.Temam [10] to define the trace.

Definition 2.3.7. We define the space $E_{loc}(\overline{\mathbb{R}^n_+})$ in the following way :

- $E_{loc}(\overline{\mathbb{R}^n_+}) := \{\mathbf{u} \in \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+}) \mid \operatorname{div}\mathbf{u} \in L^2_{loc}(\overline{\mathbb{R}^n_+})\}$.

  Here $\operatorname{div}\mathbf{u}$ means the divergence of $\mathbf{u}$, i.e., $\operatorname{div}\mathbf{u} := \sum_{i=1}^{n} \partial_{x_i} u^i$.

- Let $\mathbf{u} \in E_{loc}(\overline{\mathbb{R}^n_+})$, we define a family of seminorms $\|\cdot\|_{E(\Omega_i)}$ for all $i \in \mathbb{N}$ on $E_{loc}(\overline{\mathbb{R}^n_+})$ by

$$||\mathbf{u}||^2_{E(\Omega_i)} := \int_{\Omega_i} |\operatorname{div}\mathbf{u}|^2 + |\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}$$

  where Ω_i is an open domain in $\mathbb{R}^n_+$ with C^2 boundary $\partial\Omega_i$ for each $i \in \mathbb{N}$, moreover we require that $B_i(0)' \subset \partial\Omega_i$ for all $i \in \mathbb{N}$ where $B_i(0)' := \{\mathbf{x} \in B_i(0) \mid x_n = 0\}$ and $\Omega_i \uparrow \mathbb{R}^n_+$ as $i \to \infty$.

Definition 2.3.8. (Trace space)

- We denote the interior of the region $\overline{\Omega_i} \cap \partial\mathbb{R}^n_+$ in $\mathbb{R}^{n-1}$ by Ω_i'.
- $\Gamma(\mathbb{R}^{n-1}) := \{\, T \in \mathscr{D}'(\mathbb{R}^{n-1}) \mid \; |< T, \phi >| \le C_i \cdot ||\,\phi\,||_{H^{\frac{1}{2}}(\Omega_i')}$ for any $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$ *with* $\operatorname{supp}\phi \subset \Omega_i' \,\}$
- We define a family of seminorms $\{\, ||\cdot||_{\Omega_i'} \mid i \in \mathbb{N}\,\}$ on $\Gamma(\mathbb{R}^{n-1})$ by:

$$||\,T\,||_{\Omega_i'} := \sup_{\substack{\phi \in \mathscr{D}(\mathbb{R}^{n-1}),\\ \operatorname{supp}\phi \subset \Omega_i',\\ ||\,\phi\,||_{H^{\frac{1}{2}}(\Omega_i')} = 1.}} |< T, \phi >|.$$

It is not hard to verify the fact that these two spaces $E_{loc}(\overline{\mathbb{R}^n_+})$ and $\Gamma(\mathbb{R}^{n-1})$ are indeed Frechet spaces, thus we omit the details here and proceed directly to define the trace.

Lemma 2.3.9. *Let* $\gamma : E_{loc}(\overline{\mathbb{R}^n_+}) \to \Gamma(\mathbb{R}^{n-1})$ *by* $\mathbf{u} \mapsto \gamma_{\mathbf{u}}$*, where for* $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$ *with* $\operatorname{supp}\phi \subset \Omega_i'$ *we have the map*

$$\gamma_{\mathbf{u}}(\phi) := \int_{\Omega_i} \operatorname{div}\mathbf{u}\cdot\omega + \mathbf{u}\cdot\nabla\omega \,\mathrm{d}\mathbf{x}.$$

Here we choose $\omega \in H^1(\Omega_i)$ *with the trace operator* $\gamma_0 : H^1(\Omega_i) \to H^{\frac{1}{2}}(\partial\Omega_i)$ *such that the trace of* ω *is* ϕ*. Then we have that the map* γ *is a bounded linear operator.*

Proof. Here we make use of the theory of R.Temam [10]. Notice that for each $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$ with $\operatorname{supp}\phi \subset \Omega_i'$, we can actually find an $\omega \in H^1(\Omega_i)$ such that its trace $\gamma_0\,\omega = \phi$. Let $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$ with $\operatorname{supp}\phi \subset \Omega_i'$, notice that by definition we have that $\Omega_i' \subset \Omega_i$. We define a function g on $\partial\Omega_i$ by

$$g(\mathbf{x}) := \begin{cases} \phi(\mathbf{x}') & \text{if} \quad x_n = 0, \\ 0 & \text{else.} \end{cases}$$

Since $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$, an easy check quickly tells us that this function $g \in H^{\frac{1}{2}}(\partial\Omega_i)$ and $||g||_{H^{\frac{1}{2}}(\partial\Omega_i)} = ||\phi||_{H^{\frac{1}{2}}(\Omega_i')}$. Then by R.Temam [10], there exists an $\omega \in H^1(\Omega_i)$ such that its trace $\gamma_0\, \omega = g$. Therefore by the definition of our $\gamma_{\mathbf{u}}$, we have that

$$\begin{aligned} |\, \gamma_{\mathbf{u}}(\phi)\, | &\le ||\, \mathrm{div}\, \mathbf{u}\, ||_{L^2(\Omega_i)} \cdot ||\, \omega\, ||_{L^2(\Omega_i)} + ||\, \mathbf{u}\, ||_{\mathbf{L}^2(\Omega_i)} \cdot ||\, \nabla\omega\, ||_{\mathbf{L}^2(\Omega_i)} \\ &\le C \cdot (\, ||\, \mathrm{div}\, \mathbf{u}\, ||_{L^2(\Omega_i)} + ||\, \mathbf{u}\, ||_{\mathbf{L}^2(\Omega_i)}\,) \cdot ||\, \omega\, ||_{H^1(\Omega_i)} \\ &\le C \cdot ||\, \mathbf{u}\, ||_{E(\Omega_i)} \cdot ||\, \omega\, ||_{H^1(\Omega_i)} \end{aligned}$$

by the triangle inequality and Hölder's inequality. Since by R.Temam [10], there exists $l_{\Omega_i} \in \mathcal{L}\,(\, H^{1/2}(\partial\Omega_i),\, H^1(\Omega_i)\,)$ where l_{Ω_i} is the lifting operator such that $l_{\Omega_i} g = \omega$, hence by above we have that

$$\begin{aligned} |\, \gamma_{\mathbf{u}}(\phi)\, | &\le C \cdot ||\, \mathbf{u}\, ||_{E(\Omega_i)} \cdot ||\, l_{\Omega_i} g\, ||_{H^1(\Omega_i)} \\ &\le C_i \, \cdot ||\, \mathbf{u}\, ||_{E(\Omega_i)} \cdot ||\, g\, ||_{H^{1/2}(\partial\Omega_i)} \\ &= C_i \, \cdot ||\, \mathbf{u}\, ||_{E(\Omega_i)} \cdot ||\, \phi\, ||_{H^{1/2}(\Omega_i')}. \end{aligned}$$

The last equality holds since $g(\mathbf{x}) = 0$ for $\mathbf{x} \notin \Omega_i'$. Therefore, we can deduce that

$$||\, \gamma_{\mathbf{u}}\, ||_{\Omega_i'} \le C_i \, \cdot ||\, \mathbf{u}\, ||_{E(\Omega_i)}$$

where C_i is simply a constant which depends on i. As a result, we see that

$$\gamma : E_{loc}(\overline{\mathbb{R}^n_+}) \to \Gamma(\mathbb{R}^{n-1})$$

is indeed a bounded linear operator in the sense of Frechet spaces. □

By Lemma 2.3.6 we know that $\mathbf{X} \subset \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+})$ and by Lemma 2.3.9 there exists a bounded linear operator γ which maps $E_{loc}(\overline{\mathbb{R}^n_+})$ to $\Gamma(\mathbb{R}^{n-1})$. For the subspace $\{\, \mathbf{u} \in \mathbf{X} \mid \mathrm{div}\ \mathbf{u} \in L^2_{loc}(\overline{\mathbb{R}^n_+})\, \} \subset \mathbf{X}$, it is trivial to see that the map γ is also a bounded linear operator from $\{\, \mathbf{u} \in \mathbf{X} \mid \mathrm{div}\ \mathbf{u} \in L^2_{loc}(\overline{\mathbb{R}^n_+})\, \}$ to $\Gamma(\mathbb{R}^{n-1})$. This is how we take the trace for elements in $\mathbf{X}$.

2.3.3 Validity of $\mathbb{P}_{\mathbb{R}^n_+}$ as the Helmholtz projection

Lemma 2.3.10. *Let* $\mathbf{v} \in \mathbf{X}$*, then* $\mathrm{div}\ \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} = 0$ *in* $\mathbb{R}^n_+$ *in the sense of distributions.*

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^n_+)$. By the definition of distributions, we have that

$$\int_{\mathbb{R}^n_+} \mathrm{div}\ \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \phi \, \mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^n_+} \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \nabla\phi \, \mathrm{d}\mathbf{x}.$$

Since $\mathrm{supp}\ \phi \subset\subset \mathbb{R}^n_+$, we can easily deduce that $\mathrm{supp}\ \partial_{x_i}\phi \subset\subset \mathbb{R}^n_+$ for any $1 \le i \le n$, therefore

$$\int_{\mathbb{R}^n_+} \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \nabla\phi \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \mathbb{P}E\mathbf{v} \cdot \nabla\phi \, \mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^n} \mathrm{div}\ (\mathbb{P}E\mathbf{v}) \cdot \phi \, \mathrm{d}\mathbf{x}.$$

Because $\mathrm{div}\ (\mathbb{P}E\mathbf{v}) = 0$ in the sense of distributions, we have that

$$\int_{\mathbb{R}^n} \mathrm{div}\ (\mathbb{P}E\mathbf{v}) \cdot \phi \, \mathrm{d}\mathbf{x} = 0.$$

Thus

$$\int_{\mathbb{R}^n_+} \text{div } \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \phi \, \mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^n_+} \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \nabla\phi \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \text{div } (\mathbb{P}E\mathbf{v}) \cdot \phi \, \mathrm{d}\mathbf{x} = 0.$$

Notice that the above equality holds for any $\phi \in C_0^\infty(\mathbb{R}^n_+)$, hence

$$\text{div } \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} = 0 \quad \text{in} \quad \mathbb{R}^n_+$$

in the sense of distributions. As for the reason why div $\mathbb{P}E\mathbf{v} = 0$ in the sense of distributions, by considering Fourier transforms we can quickly prove it through simple calculations. □

Let us recall some facts about Riesz transforms. Notice that the j-th Riesz transform R_j is defined as

$$R_j(f)(\mathbf{x}) := \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|\,\mathbf{x} - \mathbf{y}\,|^{n+1}} \cdot f(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

By [9, p.232], we have that $R_j(f)$ is well-defined for any $f \in \mathscr{H}^1(\mathbb{R}^n)$ and $1 \leq j \leq n$. By [3], we have that for $f \in BMO$ and $1 \leq j \leq n$, $R_j(f) \in \mathscr{H}^1(\mathbb{R}^n)^*$. Hence by the fact that $BMO = \mathscr{H}^1(\mathbb{R}^n)^*$, there exists $h \in BMO$ such that $R_j(f) = h$ in the sense of bounded linear functionals on $\mathscr{H}^1(\mathbb{R}^n)$. Therefore for any $f \in BMO$ and $1 \leq j \leq n$, $R_j(f)$ is defined by its corresponding h. Based on these facts, we have the next lemma which proves an interesting property about Riesz transforms.

Lemma 2.3.11. *Let f belongs to BMO or $\mathscr{H}^1(\mathbb{R}^n)$,*
(1). If f is even with respect to x_n, then

$$\begin{cases} R_j(f) \textit{ is even with respect to } x_n \textit{ for } j \textit{ satisfying } 1 \leq j \leq n-1, \\ R_n(f) \textit{ is odd with respect to } x_n. \end{cases}$$

(2). If f is odd with respect to x_n, then

$$\begin{cases} R_j(f) \textit{ is odd with respect to } x_n \textit{ for } j \textit{ satisfying } 1 \leq j \leq n-1, \\ R_n(f) \textit{ is even with respect to } x_n. \end{cases}$$

Proof. For $f \in \mathscr{H}^1(\mathbb{R}^n)$, since $R_j(f)$ is well-defined for each $1 \leq j \leq n$, we can prove this lemma directly through change of variables. Let $g \in BMO$ be odd with respect to x_n and $1 \leq j \leq n-1$, let $w \in BMO$ such that $R_j(g) = w$. Let $\tilde{w}(\mathbf{x}', x_n) := w(\mathbf{x}', -x_n)$ and $f \in \mathscr{H}^1(\mathbb{R}^n)$, then by change of variables we have that

$$< \tilde{w}, f > = < w, \tilde{f} > = - < g, R_j(\tilde{f}) > .$$

Notice that the second equality above holds since $\tilde{f} \in \mathscr{H}^1(\mathbb{R}^n)$ if $f \in \mathscr{H}^1(\mathbb{R}^n)$. Again by change of variables, we can further deduce that

$$R_j(\tilde{f})(\mathbf{x}', x_n) = R_j(f)(\mathbf{x}', -x_n).$$

Then,

$$\begin{aligned}
- < g, R_j(\tilde{f}) > &= - \int_{\mathbb{R}^n} g \cdot R_j(\tilde{f}) \,\mathrm{d}\mathbf{x} \\
&= - \int_{\mathbb{R}^n} g(\mathbf{x}', x_n) \cdot R_j(f)(\mathbf{x}', -x_n) \,\mathrm{d}\mathbf{x} \\
&= - \int_{\mathbb{R}^n} g(\mathbf{x}', -x_n) \cdot R_j(f)(\mathbf{x}', x_n) \,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^n} g(\mathbf{x}', x_n) \cdot R_j(f)(\mathbf{x}', x_n) \,\mathrm{d}\mathbf{x} \\
&=< g, R_j(f) > \\
&= - < w, f > .
\end{aligned}$$

Hence $< \tilde{w} + w, f >= 0$ for any $f \in \mathscr{H}^1(\mathbb{R}^n)$ and thus w is odd with respect to x_n. The other three cases can be proved by similar arguments. □

Lemma 2.3.12. *Let $\mathbf{v} \in \mathbf{X}$, then we have that*

$$\begin{cases} (\mathbb{P}E\mathbf{v})^i \text{ is even with respect to } x_n \text{ for } i \text{ satisfying } 1 \le i \le n-1, \\ (\mathbb{P}E\mathbf{v})^n \text{ is odd with respect to } x_n. \end{cases}$$

Proof. This is a direct application of Lemma 2.3.11. □

Lemma 2.3.13. *Let $\mathbf{v} \in \mathbf{X}$, then the trace $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\mathbb{R}^n_+$ in the sense of distributions.*

Proof. Let B_R be the ball $B_R(0)$. Let $B_R^+ := B_R \cap \mathbb{R}^n_+$ and $B_R^- := B_R \cap (\mathbb{R}^n_+)^c$. Let $\mathbf{v} \in \mathbf{X}$ and let $\mathbf{u} := \mathbb{P}E\mathbf{v}$. By the above lemma we can see that u^n is odd with respect to x_n. Let

$$\mathbf{u}_1(\mathbf{x}', x_n) := \begin{cases} \mathbf{u}(\mathbf{x}', x_n) & \text{if } x_n > 0, \\ 0 & \text{if } x_n < 0. \end{cases}$$

and

$$\mathbf{u}_2(\mathbf{x}', x_n) := \begin{cases} 0 & \text{if } x_n > 0, \\ \mathbf{u}(\mathbf{x}', x_n) & \text{if } x_n < 0. \end{cases}$$

Let $\phi \in C_0^\infty(B_R)$, then we have that

$$\begin{aligned}
< \mathrm{div}\ \mathbf{u}_1, \phi > &:= - < \mathbf{u}_1, \nabla\phi > \\
&= \int_{B_R^+} \mathrm{div}\ \mathbf{u}_1 \cdot \phi \,\mathrm{d}\mathbf{x} + \int_{\{x_n=0\}\cap B_R} (\mathbf{u}_1 \cdot \mathbf{n}_1)\phi \ \mathrm{d}\mathscr{H}^{n-1}
\end{aligned}$$

where $\mathbf{n}_1$ is the normal vector on $\partial\mathbb{R}^n_+$ which points outward B_R^+. In the mean time, we also have that

$$\begin{aligned}
< \mathrm{div}\ \mathbf{u}_2, \phi > &:= - < \mathbf{u}_2, \nabla\phi > \\
&= \int_{B_R^-} \mathrm{div}\ \mathbf{u}_2 \cdot \phi \,\mathrm{d}\mathbf{x} + \int_{\{x_n=0\}\cap B_R} (\mathbf{u}_2 \cdot \mathbf{n}_2)\phi \ \mathrm{d}\mathscr{H}^{n-1}
\end{aligned}$$

where $\mathbf{n}_2$ is the normal vector on $\partial\mathbb{R}^n_+$ which points outward B_R^-. By similar arguments as in the proof of Lemma 2.3.10, we can see that div $\mathbf{u} = 0$ in B_R, div $\mathbf{u}_1 = 0$ in B_R^+ and div $\mathbf{u}_2 = 0$ in B_R^-. Therefore

$$\begin{aligned} 0 &= < \text{div } \mathbf{u}_1, \phi > + < \text{div } \mathbf{u}_2, \phi > \\ &= \int_{B_R^+} \text{div } \mathbf{u}_1 \cdot \phi \, \mathrm{d}\mathbf{x} + \int_{B_R^-} \text{div } \mathbf{u}_2 \cdot \phi \, \mathrm{d}\mathbf{x} + \int_{\{x_n=0\}\cap B_R} \big(\mathbf{u}_1 \cdot \mathbf{n}_1 + \mathbf{u}_2 \cdot \mathbf{n}_2\big) \, \phi \, \mathrm{d}\mathscr{H}^{n-1} \\ &= \int_{\{x_n=0\}\cap B_R} \big(\mathbf{u}_1 \cdot \mathbf{n}_1 - \mathbf{u}_2 \cdot \mathbf{n}_1\big) \, \phi \, \mathrm{d}\mathscr{H}^{n-1}. \end{aligned}$$

Thus we see that on $\{x_n = 0\} \cap B_R$, $\big(\mathbf{u}_1 \cdot \mathbf{n}_1 - \mathbf{u}_2 \cdot \mathbf{n}_1\big) = 0$ in the sense of distributions. Notice that if $x_n < 0$, then

$$u_2^n\,(\mathbf{x}', x_n) = -u_1^n\,(\mathbf{x}', -x_n).$$

At $\{x_n = 0\} \cap B_R$, we have that

$$\mathbf{u}_1 \cdot \mathbf{n}_1 = u_1^n\,(\mathbf{x}', 0) \quad \text{and} \quad \mathbf{u}_2 \cdot \mathbf{n}_2 = -u_1^n\,(\mathbf{x}', 0).$$

and thus $u_1^n\,(\mathbf{x}', 0) = 0$ in the sense of distributions. Notice that

$$u_1^n\,(\mathbf{x}', 0) = \mathbb{P}_{\mathbb{R}^n_+} \mathbf{v} \cdot \mathbf{n} \,|_{\{x_n=0\}\cap B_R} .$$

Since $\{x_n = 0\} \cap B_R \uparrow \partial\mathbb{R}^n_+$ as $R \to \infty$, we can easily deduce that the trace

$$\mathbb{P}_{\mathbb{R}^n_+} \mathbf{v} \cdot \mathbf{n} \,|_{\partial\mathbb{R}^n_+} = 0$$

in the sense of distributions. □

Lemma 2.3.14. *Let* $\mathbf{v} \in \mathbf{X}$ *such that*

$$\begin{cases} \text{div } \mathbf{v} = 0 & \text{in } \mathbb{R}^n_+, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Then we have that $\mathbf{v} \in \mathbb{P}_{\mathbb{R}^n_+} \mathbf{X}$. *Notice that both equalities above hold in the sense of distributions.*

Proof. Let $\mathbf{v} \in \mathbf{X}$ such that

$$\begin{cases} \text{div } \mathbf{v} = 0 & \text{in } \mathbb{R}^n_+, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

in the sense of distributions and let E be our extension operator. Throughout the proof of this lemma we mean equal to 0 in the sense of distributions.

(1). Here we prove that div $E\mathbf{v} = 0$ in $\mathbb{R}^n$. Let B_R be the ball $B_R(0)$. Let $B_R^+ := B_R \cap \mathbb{R}^n_+$ and $B_R^- := B_R \cap (\mathbb{R}^n_+)^c$. If $x_n > 0$, then $E\mathbf{v}\,(\mathbf{x}', x_n) = \mathbf{v}\,(\mathbf{x}', x_n)$ and div $E\mathbf{v} =$ div $\mathbf{v} = 0$ in $\mathbb{R}^n_+$ by our assumptions. If $x_n < 0$, then $E\mathbf{v}\,(\mathbf{x}', x_n) = (\mathbf{v}'(\mathbf{x}', -x_n), -v^n(\mathbf{x}', -x_n))$ and

$$\text{div } E\mathbf{v} = \sum_{i=1}^{n-1} \partial_{x_i} v^i(\mathbf{x}', -x_n) + \partial_{-x_n} v^n(\mathbf{x}', -x_n) = 0$$

since div $\mathbf{v} = 0$ in $\mathbb{R}^n_+$. Let $\phi \in C_0^\infty(B_R)$, then

$$\begin{aligned} < \text{div } E\mathbf{v}, \phi > &:= - < E\mathbf{v}, \nabla\phi > \\ &= \int_{B_R^+} \text{div } E\mathbf{v} \cdot \phi \, \mathrm{d}\mathbf{x} + \int_{B_R^-} \text{div } E\mathbf{v} \cdot \phi \, \mathrm{d}\mathbf{x} \\ &- \int_{B_R \cap \{x_n = 0\}} \{((E\mathbf{v})_+ - (E\mathbf{v})_-) \cdot \mathbf{n}_+\} \phi \, \mathrm{d}\mathscr{H}^{n-1}. \end{aligned}$$

The first two terms in the last equality equal to 0 since div $E\mathbf{v} = 0$ in both B_R^+ and B_R^-. The third term equals to 0 since $(E\mathbf{v})_+ \cdot \mathbf{n}_+ = v^n(\mathbf{x}', 0)$, $(E\mathbf{v})_- \cdot \mathbf{n}_+ = -v^n(\mathbf{x}', 0)$ and $v^n(\mathbf{x}', 0) = 0$ by our assumptions. Hence div $E\mathbf{v} = 0$ in $\mathbb{R}^n$.

(2). Notice that by simply considering Fourier transforms it is easy to verify that $R_i \sum_j R_j u^j = 0$ for any $1 \leq i \leq n$ if div $\mathbf{u} = 0$ in $\mathbb{R}^n$. Therefore if div $\mathbf{u} = 0$ in $\mathbb{R}^n$, then $(\mathbb{P}\mathbf{u})^i = u^i$ for any $1 \leq i \leq n$.

Now let $\mathbf{u} := E\mathbf{v}$, by (1) and (2) we have that $\mathbb{P}\mathbf{u} = \mathbf{u}$. Then by applying the restriction on both sides of this equality, we get that $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} = \mathbf{v}$. □

Definition 2.3.15. We define the solenoidal subspace $\mathbf{X}_\sigma$ of $\mathbf{X}$ by

$$\mathbf{X}_\sigma := \{ \mathbf{v} \in \mathbf{X} \mid \text{div } \mathbf{v} = 0 \text{ in } \mathbb{R}^n_+ \ \& \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\mathbb{R}^n_+ \}.$$

Here the two equalities hold in the sense of distributions.

By Lemma 2.3.10 and Lemma 2.3.13 we can see that $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{X} \subseteq \mathbf{X}_\sigma$. And by Lemma 2.3.14 we can see that $\mathbf{X}_\sigma \subseteq \mathbb{P}_{\mathbb{R}^n_+}\mathbf{X}$. Therefore $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{X} = \mathbf{X}_\sigma$. This fact justifies the validity of $\mathbb{P}_{\mathbb{R}^n_+}$ as the Helmholtz projection.

2.3.4 Characterization of the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$

Lemma 2.3.16. *Let $\mathbf{v} \in \mathbf{X}$, then there exists $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ such that $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{v} = \nabla p$.*

Proof. We seek to make use of De Rham's theorem [4] here. In order to make use of De Rham's theorem, it is sufficient to show that

$$< (I - \mathbb{P})E\mathbf{v}, \phi >= 0 \quad \forall \phi \in C^\infty_{0,\sigma}(\mathbb{R}^n).$$

Let $\phi \in C^\infty_{0,\sigma}(\mathbb{R}^n)$ and $\mathbf{u} := E\mathbf{v}$, notice that

$$\{(I - \mathbb{P})\mathbf{u}\}^i = -R_i \sum_j R_j u^j.$$

Therefore by substitution $< (I - \mathbb{P})\mathbf{u}, \phi >= \sum_i < -R_i \sum_j R_j u^j, \phi^i >$. Let $f := \sum_j R_j u^j$, notice that

$$< -R_i(f), \phi^i > =< f, R_i(\phi^i) > .$$

Therefore

$$\begin{aligned} < (I - \mathbb{P})\mathbf{u}, \phi > &= \sum_i < \sum_j R_j u^j, R_i \phi^i > \\ &=< \sum_j R_j u^j, \sum_i R_i \phi^i > . \end{aligned}$$

By div $\phi = 0$ we can easily deduce that $\sum_i R_i \phi^i = 0$ by considering Fourier transforms. Thus

$$< (I - \mathbb{P})\mathbf{u}, \phi >= 0 \quad \forall\, \phi \in C^\infty_{0,\sigma}(\mathbb{R}^n).$$

Therefore by De Rham [4], there exists $p \in L^1_{loc}(\mathbb{R}^n)$ such that $(I - \mathbb{P})\mathbf{u} = \nabla p$. By applying the restriction operator we have that

$$r_{\mathbb{R}^n_+}\,(I - \mathbb{P})\,E\mathbf{v} = (I - \mathbb{P}_{\mathbb{R}^n_+})\,\mathbf{v} = r_{\mathbb{R}^n_+}\,\nabla p.$$

Notice that we can further deduce that $r_{\mathbb{R}^n_+}\nabla p = \nabla\,(r_{\mathbb{R}^n_+} p)$. Indeed since for any $\phi \in C^\infty_0(\mathbb{R}^n_+)$ we have that

$$\begin{aligned} < r_{\mathbb{R}^n_+}\nabla p, \phi > \;&:=\; \int_{\mathbb{R}^n_+} r_{\mathbb{R}^n_+}\nabla p \cdot \phi\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \nabla p \cdot \phi\, \mathrm{d}\mathbf{x} \\ &= -\int_{\mathbb{R}^n} p \cdot \mathrm{div}\,\phi\, \mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^n_+} p \cdot \mathrm{div}\,\phi\, \mathrm{d}\mathbf{x} \\ &= -\int_{\mathbb{R}^n} (r_{\mathbb{R}^n_+} p) \cdot \mathrm{div}\,\phi\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n_+} \nabla(r_{\mathbb{R}^n_+} p) \cdot \phi\, \mathrm{d}\mathbf{x} \\ &= < \nabla(r_{\mathbb{R}^n_+} p), \phi > . \end{aligned}$$

Therefore we have that $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{v} = \nabla(r_{\mathbb{R}^n_+} p)$. Since $p \in L^1_{loc}(\mathbb{R}^n)$, it is easy to deduce that $r_{\mathbb{R}^n_+} p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$. □

Lemma 2.3.17. *Let $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ such that $\nabla p \in \mathbf{X}$, then $\nabla p \in (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$.*

Proof. Let $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ such that $\nabla p \in \mathbf{X}$, it is sufficient to prove that $\mathbb{P}_{\mathbb{R}^n_+}\nabla p = 0$. Then by this fact we can see that

$$(I - \mathbb{P}_{\mathbb{R}^n_+})\nabla p = \nabla p - \mathbb{P}_{\mathbb{R}^n_+}\nabla p = \nabla p.$$

and thus $\nabla p \in (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$. Let q be defined as follow:

$$q(\mathbf{x}', x_n) := \begin{cases} p(\mathbf{x}', x_n) & \text{if } \; x_n > 0, \\ p(\mathbf{x}', -x_n) & \text{if } \; x_n < 0. \end{cases}$$

Since q is the even extension of p, $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ would imply $q \in L^1_{loc}(\mathbb{R}^n)$. Moreover, simple calculations would tell us $\nabla q = E\,\nabla p$. This is indeed since for $x_n < 0$ we have that

$$\frac{\partial}{\partial x_n}\, q(\mathbf{x}', x_n) = \frac{\partial}{\partial x_n}\, p(\mathbf{x}', -x_n) = -\frac{\partial}{\partial(-x_n)}\, p(\mathbf{x}', -x_n) = -\frac{\partial}{\partial z_n}\, p(\mathbf{x}', z_n)$$

where $z_n > 0$. Again by considering Fourier transforms, it is easy to verify that $(\mathbb{P}\nabla q)^i = 0$ for any $1 \le i \le n$. As a result,

$$\mathbb{P}_{\mathbb{R}^n_+}\nabla p = r_{\mathbb{R}^n_+}\mathbb{P}E\nabla p = r_{\mathbb{R}^n_+}\mathbb{P}\nabla q = 0.$$

Hence $\nabla p = (I - \mathbb{P}_{\mathbb{R}^n_+})\nabla p$ and we are done. □

Definition 2.3.18. We define the subspace $\mathbf{X}_\pi$ of $\mathbf{X}$ by

$$\mathbf{X}_\pi := \{\, \nabla p \in \mathbf{X} \mid \; p \in L^1_{loc}(\overline{\mathbb{R}^n_+})\,\}.$$

By Lemma 2.3.16 we can see that $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X} \subseteq \mathbf{X}_\pi$ and by Lemma 2.3.17 we can see that $\mathbf{X}_\pi \subseteq (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$. Therefore $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X} = \mathbf{X}_\pi$. This fact gives the characterisation of the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$.

2.3.5 Proof of Theorem 2.1.1

Proof. By Lemma 2.3.5 we see that $\mathbb{P}_{\mathbb{R}^n_+}$ is a bounded linear operator which maps $\mathbf{X}$ to $\mathbf{X}$. By this bounded linear map we can easily see that the vector field $\mathbf{X}$ admits a natural decomposition

$$\mathbf{X} = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{X} \oplus (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$$

where both $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{X}$ and $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$ are linear subspaces of $\mathbf{X}$. Since this natural decomposition is induced by the projection $\mathbb{P}_{\mathbb{R}^n_+}$, this decomposition is certainly unique. Moreover, we have already proved that

$$\mathbb{P}_{\mathbb{R}^n_+}\mathbf{X} = \mathbf{X}_\sigma$$

and

$$(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X} = \mathbf{X}_\pi.$$

As a result, Theorem 2.1.1 holds and we are done. □

Remark 2.3.19. Although the Helmholtz decomposition we established for $\mathbf{X}$ is true, due to the fact that $[\,\cdot\,]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)}$ is a seminorm, it is inevitable to think about the question where constant vectors are mapped to under this Helmholtz projection $\mathbb{P}_{\mathbb{R}^n_+}$. Unfortunately, this question is not answered in this research, in order to avoid this ambiguity, we shall consider our Helmholtz decomposition not for the space $\mathbf{X}$ but for the quotient space $\mathbf{X}/(\mathbb{R}^{n-1} \times \{0\})$. From now on, without causing any ambiguity, we shall denote $\mathbf{X}/(\mathbb{R}^{n-1} \times \{0\})$ simply by $\mathbf{X}$.

2.4 Partial Helmholtz decomposition of vector fields in real Hardy spaces over the half space

2.4.1 Boundedness of projection $\mathbb{P}_{\mathbb{R}^n_+}$ from $\mathbf{Y}$ to $\mathbf{Y}$

Let $\mathbf{v} \in \mathbf{Y}$ and $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} := r_{\mathbb{R}^n_+}\mathbb{P}E\mathbf{v}$.

Lemma 2.4.1. *Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, then we have that $E_{odd}f \in \mathscr{H}^1(\mathbb{R}^n)$ and*

$$||E_{odd}f||_{\mathscr{H}^1} = 2 \cdot ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}.$$

Proof. For simplicity we denote $E_{odd}f$ by $\bar{f}$. Let G_t be the heat kernel on $\mathbb{R}^n$ so that $(e^{t\Delta}g)(\mathbf{x}) = (G_t * g)(\mathbf{x})$ for a function g on $\mathbb{R}^n$. By Definition 2.2.8, we have that

$$\begin{aligned} ||\bar{f}||_{\mathscr{H}^1} &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, G_t * \bar{f}\,|(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n_-} \sup_{t>0} |\, G_t * \bar{f}\,|(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= (1) + (2). \end{aligned}$$

(1). For $\mathbf{x} \in \mathbb{R}^n_+$ and $t > 0$, we have that $(G_t * \bar{f})(\mathbf{x}, t) = (r_{\mathbb{R}^n_+}(G_t * \bar{f}))(\mathbf{x}, t)$. Since this is true for all $t > 0$, by taking the supremum over all $t > 0$, we have that

$$\sup_{t>0} |\, G_t * \bar{f}\,|(\mathbf{x}) = \sup_{t>0} |\, r_{\mathbb{R}^n_+}(G_t * \bar{f})\,|(\mathbf{x}).$$

Since the above equality holds for all $\mathbf{x} \in \mathbb{R}^n_+$, we can see that

$$\begin{aligned}(1) &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+} (G_t * \bar{f}) \,| (\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+} e^{t\Delta} \, \bar{f} \,| (\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ &= ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}.\end{aligned}$$

(2). Notice that $(G_t * \bar{f})(\mathbf{x}, t)$ is actually odd with respect to x_n since $\bar{f}$ is odd with respect to x_n, hence

$$|\, G_t * \bar{f} \,| (\mathbf{x}', x_n, t) = |\, -(G_t * \bar{f})(\mathbf{x}', -x_n, t) \,| = |\, G_t * \bar{f} \,| (\mathbf{x}', -x_n, t).$$

Let $\bar{f}^+_{G_t}(\mathbf{x}) := \sup_{t>0} |\, G_t * \bar{f} \,| (\mathbf{x})$, $\bar{f}^+_{G_t}$ is even with respect to x_n. Hence,

$$(2) = \int_{\mathbb{R}^n_+} \bar{f}^+_{G_t}(\mathbf{z}', -z_n) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}z_n = \int_{\mathbb{R}^n_+} \bar{f}^+_{G_t}(\mathbf{z}', z_n) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}z_n = (1).$$

□

Lemma 2.4.2. *Let $f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$, then we have that $E_{even} f \in \mathscr{H}^1(\mathbb{R}^n)$ and*

$$||E_{even} f||_{\mathscr{H}^1(\mathbb{R}^n)} = 2 \cdot ||f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)}.$$

Proof. For simplicity we denote $E_{even} f$ by $\tilde{f}$. Let G_t be the heat kernel. By Definition 2.2.8, we have that

$$\begin{aligned}||\tilde{f}||_{\mathscr{H}^1} &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, G_t * \tilde{f} \,| (\mathbf{x}) \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n_-} \sup_{t>0} |\, G_t * \tilde{f} \,| (\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ &= (1) + (2).\end{aligned}$$

(1). For $\mathbf{x} \in \mathbb{R}^n_+$ and $t > 0$, we have that $(G_t * \tilde{f})(\mathbf{x}, t) = (r_{\mathbb{R}^n_+} (G_t * \tilde{f}))(\mathbf{x}, t)$. Since this is true for all $t > 0$, by taking the supremum over all $t > 0$, we have that

$$\sup_{t>0} |\, G_t * \tilde{f} \,| (\mathbf{x}) = \sup_{t>0} |\, r_{\mathbb{R}^n_+} (G_t * \tilde{f}) \,| (\mathbf{x}).$$

Since the above equality holds for all $\mathbf{x} \in \mathbb{R}^n_+$, we can see that

$$(1) = ||f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)}.$$

(2). Notice that $(G_t * \tilde{f})(\mathbf{x}, t)$ is even with respect to x_n since $\tilde{f}$ is even with respect to x_n. We have that $\tilde{f}^+_{G_t}(\mathbf{x}) := \sup_{t>0} |\, G_t * \tilde{f} \,| (\mathbf{x})$ is even with respect to x_n. Therefore,

$$(2) = \int_{\mathbb{R}^n_+} \tilde{f}^+_{G_t}(\mathbf{z}', -z_n) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}z_n = \int_{\mathbb{R}^n_+} \tilde{f}^+_{G_t}(\mathbf{z}', z_n) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}z_n = (1).$$

□

Lemma 2.4.3. *Let $f \in \mathscr{H}^1(\mathbb{R}^n)$ and f be odd with respect to x_n, i.e., $f(\mathbf{x}', x_n) = -f(\mathbf{x}', -x_n)$, then we have that $r_{\mathbb{R}^n_+} f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and*

$$||r_{\mathbb{R}^n_+} f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} \leq ||f||_{\mathscr{H}^1}.$$

Proof. Let $f \in \mathscr{H}^1(\mathbb{R}^n)$ such that f is odd with respect to x_n, then

$$\begin{aligned} ||r_{\mathbb{R}^n_+} f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} &:= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}\, e^{t\Delta}\, E_{odd}\, r_{\mathbb{R}^n_+} f \,|\, (\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}\, e^{t\Delta}\, f \,|\, (\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &\leq \int_{\mathbb{R}^n} \sup_{t>0} |\, e^{t\Delta}\, f \,|\, (\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= ||f||_{\mathscr{H}^1(\mathbb{R}^n)}. \end{aligned}$$

□

Lemma 2.4.4. *Let $f \in \mathscr{H}^1(\mathbb{R}^n)$ and f be even with respect to x_n, i.e., $f(\mathbf{x}', x_n) = f(\mathbf{x}', -x_n)$, then we have that $r_{\mathbb{R}^n_+} f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and*

$$||r_{\mathbb{R}^n_+} f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} \leq ||f||_{\mathscr{H}^1}.$$

Proof. Let $f \in \mathscr{H}^1(\mathbb{R}^n)$ such that f is even with respect to x_n, then

$$\begin{aligned} ||r_{\mathbb{R}^n_+} f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} &:= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}\, e^{t\Delta}\, E_{even}\, r_{\mathbb{R}^n_+} f \,|\, (\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}\, e^{t\Delta}\, f \,|\, (\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &\leq \int_{\mathbb{R}^n} \sup_{t>0} |\, e^{t\Delta}\, f \,|\, (\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= ||f||_{\mathscr{H}^1(\mathbb{R}^n)}. \end{aligned}$$

□

Lemma 2.4.5. $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{Y} \to \mathbf{Y}$ *is a bounded linear operator.*

Proof. The proof is basically identical to the proof of Lemma 2.3.5. □

2.4.2 Properties of projection $\mathbb{P}_{\mathbb{R}^n_+}$

Except some places due to the fact that we cannot take the trace properly, the theory in this subsection is completely identical to the theory in subsection 2.3.3. This is due to the fact that all properties hold not because of the space where $\mathbf{v}$ belongs to, but the properties of projection $\mathbb{P}$ itself has.

Lemma 2.4.6. *Let $\mathbf{v} \in \mathbf{Y}$, then* $\operatorname{div} \mathbb{P}_{\mathbb{R}^n_+} \mathbf{v} = 0$ *in $\mathbb{R}^n_+$ in the sense of distributions.*

Proof. The proof is completely identical to the proof of Lemma 2.3.10. □

Lemma 2.4.7. *Let $\mathbf{v} \in \mathbf{Y}$ such that*

$$\begin{cases} \operatorname{div} \mathbf{v} = 0 & \text{in } \mathbb{R}^n_+, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Then we have that $\mathbf{v} \in \mathbb{P}_{\mathbb{R}^n_+} \mathbf{Y}$. Notice that both equalities above hold in the sense of distributions.

Proof. The proof is completely identical to the proof of Lemma 2.3.14. □

Definition 2.4.8. We define the subspace $\mathbf{Y}_\sigma$ of $\mathbf{Y}$ by

$$\mathbf{Y}_\sigma := \{\, \mathbf{v} \in \mathbf{Y} \mid \text{div } \mathbf{v} = 0 \text{ in } \mathbb{R}^n_+ \;\; \& \;\; \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\mathbb{R}^n_+ \,\}.$$

Lemma 2.4.9. *In the case for the space* $\mathbf{Y}$*, we have that*

$$\mathbf{Y}_\sigma \subset \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} \subset \{\, \mathbf{v} \in \mathbf{Y} \mid \text{div } \mathbf{v} = 0 \;\; in \;\; \mathbb{R}^n_+ \,\}.$$

Proof. By Lemma 2.4.6 and Lemma 2.4.7, we are done. □

2.4.3 Characterization of the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$

Due to the fact that the theory in this section depends only on the properties of projection $\mathbb{P}_{\mathbb{R}^n_+}$ and the trace problem which we do not know how to deal with is not involved in any sense, it is completely identical to the theory in subsection 2.3.4.

Lemma 2.4.10. *Let* $\mathbf{v} \in \mathbf{Y}$*, then there exists* $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ *such that* $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{v} = \nabla p$.

Proof. The proof is completely identical to the proof of Lemma 2.3.16. □

Lemma 2.4.11. *Let* $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ *such that* $\nabla p \in \mathbf{Y}$*, then* $\nabla p \in (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$.

Proof. The proof is completely identical to the proof of Lemma 2.3.17. □

Definition 2.4.12. We define the subspace $\mathbf{Y}_\pi$ of $\mathbf{Y}$ by

$$\mathbf{Y}_\pi := \{\, \nabla p \in \mathbf{Y} \mid p \in L^1_{loc}(\overline{\mathbb{R}^n_+}) \,\}.$$

Lemma 2.4.13. $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y} = \mathbf{Y}_\pi$.

Proof. By Lemma 2.4.10 and Lemma 2.4.11, we are done. □

2.4.4 Proof of Theorem 2.1.2

Proof. By Lemma 2.4.5 we see that $\mathbb{P}_{\mathbb{R}^n_+}$ is a bounded linear operator which maps $\mathbf{Y}$ to $\mathbf{Y}$. By this bounded linear map we can easily see that the vector field $\mathbf{Y}$ admits a natural decomposition

$$\mathbf{Y} = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} \oplus (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$$

where both $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y}$ and $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$ are linear subspaces of $\mathbf{Y}$. Since this natural decomposition is induced by the projection $\mathbb{P}_{\mathbb{R}^n_+}$, this decomposition is certainly unique. Moreover, we have already proved that

$$\mathbf{Y}_\sigma \subset \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} \subset \{\, \mathbf{v} \in \mathbf{Y} \mid \text{div } \mathbf{v} = 0 \text{ in } \mathbb{R}^n_+ \,\}$$

and

$$(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y} = \mathbf{Y}_\pi.$$

As a result, Theorem 2.1.2 holds and we are done. □

2.5 Duality theorem

Before we start this section we would like to recall the definition that a function $h \in \mathscr{H}^1(\mathbb{R}^n)$ is called a 2-atom if $\operatorname{supp} h \subset B$, $||h||_{L^2(\mathbb{R}^n)} \leq |B|^{-1/2}$ and $\int_B h \, \mathrm{d}\mathbf{x} = 0$. Here $B \subset \mathbb{R}^n$ is an open ball.

2.5.1 Duality theorem for the case of odd extension

Throughout this subsection, we denote the odd extension operator E_{odd} by E.

Definition 2.5.1. We define the set of symmetric 2-atoms by the set

$$\{Er_{\mathbb{R}^n_+}\alpha \mid \alpha \text{ is a 2-atom s.t. } \operatorname{supp}\alpha \subset B \text{ and } B \cap \partial\mathbb{R}^n_+ \neq \varnothing\}$$
$$\bigcup \quad \{Er_{\mathbb{R}^n_+}\beta \mid \beta \text{ is a 2-atom s.t. } \operatorname{supp}\beta \subset B \subset \mathbb{R}^n_+\}.$$

Let $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+) := \{E\mathbf{v} \mid \mathbf{v} \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)\}$. Then $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+) \subset \mathscr{H}^1(\mathbb{R}^n)$ is a linear subspace.

Lemma 2.5.2. *The norm*

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \textit{all symmetric 2-atomic decompositions}\}$$

is equivalent to the norm $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ *on the subspace* $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

Proof. Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, then $Ef \in \mathscr{H}^1(\mathbb{R}^n)$.

(1). By the atomic decompositions of functions of the real Hardy space $\mathscr{H}^1(\mathbb{R}^n)$, we see that Ef admits 2-atomic decompositions. Let

$$Ef = \sum_i \lambda_i \alpha_i + \sum_j \mu_j \beta_j$$

be a 2-atomic decomposition of Ef. Apply $r_{\mathbb{R}^n_+}$ firstly and then E secondly on both sides of this 2-atomic decomposition, we can deduce that

$$Ef = Er_{\mathbb{R}^n_+}Ef = \sum_i \lambda_i Er_{\mathbb{R}^n_+}\alpha_i + \sum_j \mu_j Er_{\mathbb{R}^n_+}\beta_j.$$

This is a symmetric 2-atomic decomposition of Ef with exactly the same coefficients just as the original 2-atomic decomposition. Hence we see that every 2-atomic decomposition of Ef gives rise to a symmetric 2-atomic decomposition of Ef with exactly the same coefficients. Therefore,

$$\begin{aligned} ||Ef||_{\mathscr{H}^1(\mathbb{R}^n)} &= \inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all 2-atomic decompositions}\} \\ &\geq \inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\}. \end{aligned}$$

(2). Let $Ef = \sum_i \lambda_i Er_{\mathbb{R}^n_+}\alpha_i + \sum_j \mu_j Er_{\mathbb{R}^n_+}\beta_j$ be a symmetric 2-atomic decomposition. Pick an i, suppose that $\operatorname{supp}\alpha_i \subset B_i$ where B_i is a ball in $\mathbb{R}^n$ such that $B_i \cap \partial\mathbb{R}^n_+ \neq \varnothing$.

Then there exists $\mathbf{x}^* \in B_i \cap \partial\mathbb{R}^n_+$ such that $\operatorname{supp} Er_{\mathbb{R}^n_+}\alpha_i \subset B_{2i}(\mathbf{x}^*)$. Moreover, we have that

$$||Er_{\mathbb{R}^n_+}\alpha_i||_{L^2(\mathbb{R}^n)} \leq 2 \cdot ||\alpha_i||_{L^2(\mathbb{R}^n)} = 2^{\frac{n}{2}+1} \cdot |B_{2i}(\mathbf{x}^*)|^{-1/2}.$$

Since E is the odd extension, we certainly have that

$$\int_{B_{2i}(\mathbf{x}^*)} Er_{\mathbb{R}^n_+}\alpha_i \,\mathrm{d}\mathbf{x} = 0.$$

Therefore, $\frac{1}{2^{\frac{n}{2}+1}} \cdot Er_{\mathbb{R}^n_+}\alpha_i$ is a 2-atom in $\mathscr{H}^1(\mathbb{R}^n)$ for any i. In addition, since $\operatorname{supp}\beta_j \subset B_j \subset \mathbb{R}^n_+$ for some ball B_j, for any j we can decompose $Er_{\mathbb{R}^n_+}\beta_j$ into the form $\beta_j + \beta_j^-$ where β_j^- is a 2-atom which is contained in $(\mathbb{R}^n_+)^c$. Hence we can rewrite the symmetric 2-atomic decomposition in the following way:

$$Ef = \sum_i (\lambda_i 2^{\frac{n}{2}+1}) \cdot (\frac{1}{2^{\frac{n}{2}+1}} Er_{\mathbb{R}^n_+}\alpha_i) + \sum_j \mu_j \cdot \beta_j + \sum_j \mu_j \cdot \beta_j^-.$$

Here $(\frac{1}{2^{\frac{n}{2}+1}} Er_{\mathbb{R}^n_+}\alpha_i)$, β_j and β_j^- are all 2-atoms for any i, j. Therefore we can get a 2-atomic decomposition of Ef from each symmetric 2-atomic decomposition of Ef with coefficients $\{\lambda_i'\}_{i=1}^\infty$ and $\{\mu_j'\}_{j=1}^\infty$ where $\lambda_i' = \lambda_i \cdot 2^{\frac{n}{2}+1}$ for all i and $\mu_j' = 2 \cdot \mu_j$ for all j. Notice that

$$\begin{aligned}\sum_i |\lambda_i| + \sum_j |\mu_j| &\geq \frac{1}{2^{\frac{n}{2}+1}} \cdot \Big(\sum_i (|\lambda_i| \cdot 2^{\frac{n}{2}+1}) + \sum_j 2 \cdot |\mu_j|\Big) \\ &= \frac{1}{2^{\frac{n}{2}+1}} \cdot \Big(\sum_i |\lambda_i'| + \sum_j |\mu_j'|\Big).\end{aligned}$$

Therefore we have that

$$\begin{aligned} &\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\} \\ \geq\ & \frac{1}{2^{\frac{n}{2}+1}} \cdot \inf\{\sum_i |\lambda_i'| + \sum_j |\mu_j'| \mid \text{all 2-atomic decompositions}\}.\end{aligned}$$

Since the norm $\inf\{\sum_i |\lambda_i'| + \sum_j |\mu_j'| \mid \text{all 2-atomic decompositions}\}$ is equivalent to the norm $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ by the standard theory of real Hardy spaces, we can deduce that

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\} \geq C||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$$

for some constant C. □

By making use of Lemma 2.5.2 we can deduce the half space atomic decomposition for elements of $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

Theorem 2.5.3. *Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, then there exists sequences of non-negative numbers $\{\lambda_i\}_{i=1}^\infty$ & $\{\mu_j\}_{j=1}^\infty$, a sequence of 2-atoms $\{\alpha_i\}_{i=1}^\infty$ where for each i $\operatorname{supp}\alpha_i \subset B_i$ for some* *ball B_i and $B_i \cap \partial\mathbb{R}^n_+ \neq \varnothing$ and a sequence of 2-atoms $\{\beta_j\}_{j=1}^\infty$ where for each j $\operatorname{supp}\beta_j \subset B_j \subset \mathbb{R}^n_+$ for some ball B_j such that*

$$f = \sum_i \lambda_i \cdot \alpha_i \,|_{r_{\mathbb{R}^n_+}} + \sum_j \mu_j \cdot \beta_j.$$

We refer such a decomposition of f as a half space atomic decomposition of f and moreover, the norm

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \textit{all half space atomic decompositions}\}$$

is equivalent to the norm $||\cdot||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}$ on $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

Proof. By Lemma 2.5.2, we have that

$$\begin{aligned} f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+). &\implies Ef \in \mathscr{H}^1(\mathbb{R}^n). \\ &\implies Ef \text{ admits 2-atomic decompositions.} \\ &\implies Ef \text{ admits symmetric 2-atomic decompositions.} \\ &\implies f \text{ admits half space atomic decompositions by taking} \\ &\qquad \text{restrictions of symmetric 2-atomic decompositions.} \end{aligned}$$

By Lemma 2.4.1 and Lemma 2.4.3, there exists constants C_1 and C_2 such that

$$C_1 \cdot ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} \leq ||Ef||_{\mathscr{H}^1(\mathbb{R}^n)} \leq C_2 \cdot ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}.$$

Moreover, the norm $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ is equivalent to the norm

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\}$$

on $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ by Lemma 2.5.2. Since each of the half space atomic decomposition of f gives rise naturally to a symmetric 2-atomic decomposition of Ef with exactly the same coefficients by odd extension, we have that

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all half space atomic decompositions}\} \approx ||\cdot||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}$$

on $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. □

Definition 2.5.4. We denote the set of all finite linear combinations of symmetric 2-atoms by $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$.

Notice that $\mathscr{H}^1_{0,s}(\mathbb{R}^n) \subset \mathscr{H}^1_0(\mathbb{R}^n) \cap E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ where $\mathscr{H}^1_0(\mathbb{R}^n)$ is the set of all finite linear combinations of 2-atoms.

Lemma 2.5.5. *$E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ is a closed subspace of $\mathscr{H}^1(\mathbb{R}^n)$.*

Proof. Let $F \in \overline{E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}^{||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}} \setminus E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, then there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ such that $Eu_n \to F$ in $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ as $n \to \infty$. Since $\mathscr{H}^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, we have that

$$||Eu_n - F||_{L^1(\mathbb{R}^n)} \leq ||Eu_n - F||_{\mathscr{H}^1(\mathbb{R}^n)} \to 0.$$

This means that $Eu_n(\mathbf{x}) \to F(\mathbf{x})$ *a.e.*. Notice that for $\mathbf{x} \in \mathbb{R}^n$,

$$F(\mathbf{x}', x_n) \leftarrow Eu_n(\mathbf{x}', x_n) = -Eu_n(\mathbf{x}', -x_n) \to -F(\mathbf{x}', -x_n).$$

Therefore, F is odd with respect to x_n *a.e.* and $F \in E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. □

Lemma 2.5.6. *$\mathscr{H}^1_{0,s}(\mathbb{R}^n)$ is dense in $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.*

Proof. Through the proof of Lemma 2.5.2 we know that every element of $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ admits symmetric 2-atomic decompositions and by Lemma 2.5.5 we see that $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ is closed in $\mathscr{H}^1(\mathbb{R}^n)$. We are done. □

Theorem 2.5.7. *Suppose $g \in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$. Then the linear functional l defined on $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ by*

$$l(f) = \int_{\mathbb{R}^n_+} f \cdot g \,\mathrm{d}\mathbf{x}$$

for $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ is a bounded linear functional which satisfies $||l|| \leq c \cdot ||g||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)}$ with some constant c. Conversely, every bounded linear functional l on $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ can be written in the form of

$$l(f) = \int_{\mathbb{R}^n_+} f \cdot g \,\mathrm{d}\mathbf{x} \text{ for all } f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$$

with $g \in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$ and $||g||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)} \leq c \cdot ||l||$ with some constant c. Here $||l||$ means the norm of l as a bounded linear functional on $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

Proof. (1). Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and $g \in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$. Then we have the estimates

$$\begin{aligned}
|\int_{\mathbb{R}^n_+} f \cdot g \,\mathrm{d}\mathbf{x}| &= \frac{1}{2} \cdot |\int_{\mathbb{R}^n} Ef \cdot Eg \,\mathrm{d}\mathbf{x}| \\
&\leq \frac{1}{2} \cdot ||Ef||_{\mathscr{H}^1(\mathbb{R}^n)} \cdot ||Eg||_{BMO} \\
&\leq c \cdot ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} \cdot ||g||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)}.
\end{aligned}$$

Therefore, $l : f \mapsto \displaystyle\int_{\mathbb{R}^n_+} f \cdot g \,\mathrm{d}\mathbf{x} \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$ and the above inequalities imply that $||l|| \leq c \cdot ||g||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)}$ with some constant c.

(2). Let $l \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$. We define $\tilde{l}(Ef) := 2 \cdot l(f)$ for all $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Fix a ball $B \subset \mathbb{R}^n_+$, let $L^2_0(B)$ be the subspace $\{f \in L^2(B) \mid \int_B f \,\mathrm{d}\mathbf{x} = 0\}$, notice that $L^2_0(B) \subset \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Let $u \in L^2_0(B)$ be a 2-atom, i.e., we require that $\operatorname{supp} u \subset B \subset \mathbb{R}^n_+$ for some ball B, $\int_B u \,\mathrm{d}\mathbf{x} = 0$ and $||u||_{L^2(B)} \leq |B|^{-1/2}$. We then have that

$$\begin{aligned}
|\tilde{l}(Eu)| &:= 2 \cdot |l(u)| \leq c \cdot ||u||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} \\
&\leq c \cdot ||Eu||_{\mathscr{H}^1(\mathbb{R}^n)} = c \cdot ||u^+ + u^-||_{\mathscr{H}^1(\mathbb{R}^n)} \\
&\leq c \cdot (||u^+||_{\mathscr{H}^1(\mathbb{R}^n)} + ||u^-||_{\mathscr{H}^1(\mathbb{R}^n)}) \leq c \cdot |B|^{1/2} \cdot ||u||_{L^2_0(B)} \\
&\leq c \cdot |B|^{1/2} \cdot ||Eu||_{EL^2_0(B)}.
\end{aligned}$$

Here $||\cdot||_{L_0^2(B)} := (\int_B |\cdot|^2 \,\mathrm{d}\mathbf{x})^{\frac{1}{2}}$ and $||\cdot||_{EL_0^2(B)} := (\int_{B\cup B^-} |\cdot|^2 \,\mathrm{d}\mathbf{x})^{\frac{1}{2}}$ with $B^- := \{(\mathbf{x}', -x_n) \mid (\mathbf{x}', x_n) \in B\}$. For general $w \in L_0^2(B)$, we have that $w = \lambda \cdot u$ where $u \in L_0^2(B)$ is a 2-atom, then

$$|\tilde{l}(Ew)| := 2\cdot |l(w)| = 2\cdot |\lambda| \cdot |l(u)| \le c\cdot |B|^{1/2} \cdot ||Ew||_{EL_0^2(B)}.$$

Thus $\tilde{l}\,|_{EL_0^2(B)}$ is a bounded linear functional on $EL_0^2(B)$.

Claim 1 : $EL_0^2(B)^* = EL_0^2(B)$.

Proof of *Claim* 1 : Let $\tilde{T} \in EL_0^2(B)^*$, by definition we have that $|\tilde{T}(Eu)| \le c\cdot ||Eu||_{EL_0^2(B)}$. Let's define $T(u)$ for each $u \in L_0^2(B)$ by $T(u) = \frac{1}{2}\cdot \tilde{T}(Eu)$, thus

$$|T(u)| = \frac{1}{2}\cdot |\tilde{T}(Eu)| \le c\cdot ||Eu||_{EL_0^2(B)} \le c\cdot ||u||_{L_0^2(B)}.$$

Hence $T \in L_0^2(B)^*$. By the Riesz representation theorem for the Hilbert space $L_0^2(B)$, we deduce that there exists $g^B \in L_0^2(B)$ such that

$$T(u) = \int_B u\cdot g^B \,\mathrm{d}\mathbf{x} \;\text{ for all } u \in L_0^2(B).$$

Notice that

$$\tilde{T}(Eu) = 2\cdot T(u) = 2\cdot \int_B u\cdot g^B \,\mathrm{d}\mathbf{x} = \int_{B\cup B^-} Eu\cdot Eg^B \,\mathrm{d}\mathbf{x}$$

and $Eg^B \in EL_0^2(B)$, hence $EL_0^2(B)^* = EL_0^2(B)$ and the proof of *Claim* 1 is finished.

By *Claim* 1, $\tilde{l}\,|_{EL_0^2(B)} \in EL_0^2(B)^* = EL_0^2(B)$ implies that there exists $g^B \in L_0^2(B)$ such that $\tilde{l}\,|_{EL_0^2(B)} = Eg^B$ as a bounded linear functional on $EL_0^2(B)$, i.e.,

$$\tilde{l}(Eu) = \int_{B\cup B^-} Eu\cdot Eg^B \,\mathrm{d}\mathbf{x} \;\text{ for all } Eu \in EL_0^2(B).$$

Since B is any ball in $\mathbb{R}^n_+$, we can find Eg^B for any $B \subset \mathbb{R}^n_+$. If $B_1 \subset B_2 \subset \mathbb{R}^n_+$, then we can easily see that $Eg^{B_2} - Eg^{B_1}$ is a constant on $B_1 \cup B_1^-$.

Consider the ball $B_r(\mathbf{x})$ where $\mathbf{x} \in \partial\mathbb{R}^n_+$ and $r > 0$. Let $B_r^+(\mathbf{x}) := B_r(\mathbf{x}) \cap \mathbb{R}^n_+$. For simplicity, we denote $B_r(\mathbf{x})$ by B_r. Let $u \in B_r^+$, notice that $Eu \in L^2(B_r)$ and $\int_{B_r} Eu \,\mathrm{d}\mathbf{x} = 0$ as E is the odd extension. Since $Eu \in EL_0^2(B_r)$ and Eu is odd with respect to x_n, we have that $L^2(B_r^+) \subset \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. By similar arguments as above, we see that $\tilde{l}\,|_{EL^2(B_r^+)}$ is a bounded linear functional on $EL^2(B_r^+)$. By the same proof of *Claim* 1, we have that $EL^2(B_r^+)^* = EL^2(B_r^+)$. Hence $\tilde{l}\,|_{EL^2(B_r^+)} \in EL^2(B_r^+)^* = EL^2(B_r^+)$ implies that $\tilde{l}\,|_{EL^2(B_r^+)} = Eg^{B_r^+} \in EL^2(B_r^+)$ as a bounded linear functional on $EL^2(B_r^+)$ for some $g^{B_r^+} \in L^2(B_r^+)$. For any ball $B_r(\mathbf{x})$ where $\mathbf{x} \in \partial\mathbb{R}^n_+$, we can find $Eg^{B_r^+}$. If $B_{r_1} \subset B_{r_2}$, then $Eg^{B_{r_2}^+} - Eg^{B_{r_1}^+}$ is a constant on B_{r_1}.

Now we seek to find a uniform $Eg(\mathbf{x})$ defined on $\mathbb{R}^n$. We define that

$$Eg(\mathbf{x}) := Eg^{B_r^+(0)} - \frac{1}{|B_1(0)|}\cdot \int_{B_1(0)} Eg^{B_r^+(0)} \,\mathrm{d}\mathbf{x} = Eg^{B_r^+(0)}.$$

The last equality holds as $\underset{B_1(0)}{Avg}\, Eg^{B_r^+(0)} = 0$. For $B \subset \mathbb{R}^n_+$, we have $Eg^B(\mathbf{x})$ defined on B, then there exists $B_R(0)$ for some R large enough such that $B \subset B_R^+(0)$. Hence

$$\begin{aligned} Eg^B(\mathbf{x}) &= Eg^B(\mathbf{x}) - Eg^{B_R^+(0)}(\mathbf{x}) + Eg^{B_R^+(0)}(\mathbf{x}) \\ &= c_B + Eg(\mathbf{x}) \end{aligned}$$

where $c_B := Eg^B(\mathbf{x}) - Eg^{B_R^+(0)}(\mathbf{x})$ is a constant which depends on B.

Next we prove that the function $g(\mathbf{x})$ defined by $g(\mathbf{x}) := r_{\mathbb{R}^n_+} Eg(\mathbf{x})$ belongs to the space $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$.

1*. If $B \subset \mathbb{R}^n_+$, we have that

$$\begin{aligned} \frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} &= \frac{1}{|B|}\int_B |Eg^B(\mathbf{x})|\,\mathrm{d}\mathbf{x} \\ &\le \frac{1}{|B|}\Big(\int_B |Eg^B|^2\,\mathrm{d}\mathbf{x}\Big)^{\frac{1}{2}} \cdot |B|^{\frac{1}{2}} \\ &= |B|^{-\frac{1}{2}} \cdot ||Eg^B||_{EL_0^2(B)} \end{aligned}$$

where the second inequality above is by the Hölder inequality. Since

$$\Big|\int_{B\cup B^-} Eg^B \cdot Eu\,\mathrm{d}\mathbf{x}\Big| = |\tilde{l}(Eu)| \le c\cdot |B|^{\frac{1}{2}} \cdot ||Eu||_{EL_0^2(B)},$$

we can deduce that

$$||Eg^B||_{EL_0^2(B)} = ||\tilde{l}|| \le c\cdot|B|^{\frac{1}{2}}$$

where $||\tilde{l}||$ is the operator norm of $\tilde{l}$. Therefore we have that

$$\frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \le |B|^{-\frac{1}{2}}\cdot c\cdot |B|^{\frac{1}{2}} = c.$$

By taking the supremum over all balls in $\mathbb{R}^n_+$, we can deduce that

$$\sup_{B\subset\mathbb{R}^n_+}\frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \le c.$$

Then by the triangle inequality, we can easily get that

$$[g]_{BMO^\infty(\mathbb{R}^n_+)} \le 2\cdot \sup_{B\subset\mathbb{R}^n_+}\frac{1}{|B|}\int_B |g(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \le 2\cdot c.$$

2*. For balls of the form $B_r(\mathbf{x})$ where $\mathbf{x} \in \partial\mathbb{R}^n_+$, we have that

$$Eg(\mathbf{x}) = Eg^{B_r^+(\mathbf{x})} - c_{B_r}.$$

Now we integrate this equality over the ball $B_r(\mathbf{x})$, we have that

$$\int_{B_r(\mathbf{x})} Eg(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{B_r(\mathbf{x})} Eg^{B_r^+(\mathbf{x})}\,\mathrm{d}\mathbf{y} - \int_{B_r(\mathbf{x})} c_{B_r}\,\mathrm{d}\mathbf{y}.$$

Notice that Eg and $Eg^{B_r^+(\mathbf{x})}$ are both odd with respect to x_n, we certainly have

$$\int_{B_r(\mathbf{x})} Eg(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{B_r(\mathbf{x})} Eg^{B_r^+(\mathbf{x})}\,\mathrm{d}\mathbf{y} = 0.$$

Hence c_{B_r} must equal 0. By making use of this fact and similar arguments as the previous part, we also have that

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |Eg(\mathbf{y}) - (-c_{B_r})|\,\mathrm{d}\mathbf{y} \le c.$$

Therefore,

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |Eg|\,\mathrm{d}\mathbf{y} = \frac{1}{|B_r^+(\mathbf{x})|}\int_{B_r^+(\mathbf{x})} |Eg|\,\mathrm{d}\mathbf{y} \le c.$$

As $Eg(\mathbf{y}) = g(\mathbf{y})$ in $B_r^+(\mathbf{x})$, we have that

$$\frac{1}{|B_r^+(\mathbf{x})|}\int_{B_r^+(\mathbf{x})} |g(\mathbf{y})|\,\mathrm{d}\mathbf{y} \le c.$$

By taking the supremum over all balls centered at $\partial\mathbb{R}^n_+$, we can easily deduce that

$$||g||_{b^\infty(\mathbb{R}^n_+)} = \sup_{\substack{r>0\\ \mathbf{x}\in\partial\mathbb{R}^n_+}} \frac{1}{|B_r^+(\mathbf{x})|}\int_{B_r^+(\mathbf{x})} |g(\mathbf{y})|\,\mathrm{d}\mathbf{y} \le c < \infty.$$

Hence by 1^* and 2^*, $g \in BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$.

Let Eu be a (2, s)-atom, we have that

$$\int_{\mathbb{R}^n_+} g\cdot u\,\mathrm{d}\mathbf{x} = \frac{1}{2}\cdot\int_{\mathbb{R}^n} Eg\cdot Eu\,\mathrm{d}\mathbf{x} = \frac{1}{2}\cdot\tilde{l}(Eu) = l(u).$$

Since this representation has been established for the subspace $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$ and $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$ is dense in $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, therefore $Eg = \tilde{l} \in E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$ and thus $g = l \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$. □

Notice that in the proof of Theorem 2.5.7, there is a step where we proved that for $B \subset \mathbb{R}^n_+$ and $u \in L^2_0(B)$ we have that

$$|\tilde{l}(Eu)| \le c\cdot|B|^{\frac{1}{2}}\cdot||Eu||_{EL^2_0(B)}.$$

For the ball $B_r(\mathbf{x})$ with $\mathbf{x}\in\partial\mathbb{R}^n_+$ we also have the same estimates. By L.Grafakos [6], the constant c depends only on the dimension n and it is independent of the ball B or $B_r(\mathbf{x})$, hence the later arguments in the proof are valid.

2.5.2 Duality theorem for the case of even extension

Throughout this subsection, we denote the even extension operator E_{even} by E.

Definition 2.5.8. We define the set of symmetric 2-atoms by

$$\{Er_{\mathbb{R}^n_+}\alpha \mid \alpha \text{ is a 2-atom such that } \operatorname{supp}\alpha\subset B \;\&\; B\cap\partial\mathbb{R}^n_+\neq\varnothing$$
$$\& \int_{\mathbb{R}^n_+}\alpha\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n_-}\alpha\,\mathrm{d}\mathbf{x} = 0\}$$
$$\cup\,\{Er_{\mathbb{R}^n_+}\beta \mid \beta \text{ is a 2-atom such that } \operatorname{supp}\beta\subset B\subset\mathbb{R}^n_+\}.$$

Let $E\mathscr{H}^1_{even}(\mathbb{R}^n_+) := \{E\mathbf{v} \mid \mathbf{v} \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)\}$. Then $E\mathscr{H}^1_{even}(\mathbb{R}^n_+) \subset \mathscr{H}^1(\mathbb{R}^n)$ is a linear subspace.

Lemma 2.5.9. *The norm*

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \textit{all symmetric 2-atomic decompositions}\}$$

is equivalent to the norm $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ *on the subspace* $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$.

Proof. Let $f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$, then $Ef \in \mathscr{H}^1(\mathbb{R}^n)$.

(1). By the atomic decompositions of functions of the real Hardy space $\mathscr{H}^1(\mathbb{R}^n)$, we see that Ef admits 2-atomic decompositions. Let

$$Ef = \sum_i \lambda_i \alpha_i + \sum_j \mu_j \beta_j$$

be a 2-atomic decomposition of Ef. Notice that

$$f = r_{\mathbb{R}^n_+} Ef = \sum_i \lambda_i r_{\mathbb{R}^n_+} \alpha_i + \sum_j \mu_j r_{\mathbb{R}^n_+} \beta_j.$$

Without loss of generality, assume that $\operatorname{supp} \alpha_i \subset B_i$ for some ball B_i and $B_i \cap \partial\mathbb{R}^n_+ \neq \varnothing$, assume further that $\operatorname{supp} \beta_j \subset B_j \subset \mathbb{R}^n_+$ or $\mathbb{R}^n_-$. Therefore we have that

$$f = \sum_i \lambda_i r_{\mathbb{R}^n_+} \alpha_i + \sum_j \mu_j \beta_j.$$

Let $B_i^+ := B_i \cap \mathbb{R}^n_+$ and $B_i^- := B_i \cap \mathbb{R}^n_-$. Since α_i can be any 2-atom, we know that $\displaystyle\int_{B_i} \alpha_i \,\mathrm{d}\mathbf{x} = 0$ but $\displaystyle\int_{B_i^+} \alpha_i \,\mathrm{d}\mathbf{x}$ and $\displaystyle\int_{B_i^-} \alpha_i \,\mathrm{d}\mathbf{x}$ are not necessarily zero. Here we need to do some tricks to $\displaystyle\int_{B_i^+} \alpha_i \,\mathrm{d}\mathbf{x}$ and $\displaystyle\int_{B_i^-} \alpha_i \,\mathrm{d}\mathbf{x}$. Since E is the even extension, except

$$Ef = Er_{\mathbb{R}^n_+} Ef = \sum_i \lambda_i Er_{\mathbb{R}^n_+} \alpha_i + \sum_j \mu_j Er_{\mathbb{R}^n_+} \beta_j$$

we also have that

$$Ef = Er_{\mathbb{R}^n_-} Ef = \sum_i \lambda_i Er_{\mathbb{R}^n_-} \alpha_i + \sum_j \mu_j Er_{\mathbb{R}^n_-} \beta_j.$$

Therefore,

$$\begin{aligned} 2Ef &= Er_{\mathbb{R}^n_+} Ef + Er_{\mathbb{R}^n_-} Ef \\ &= \sum_i \lambda_i \cdot (Er_{\mathbb{R}^n_+} \alpha_i + Er_{\mathbb{R}^n_-} \alpha_i) + \sum_j \mu_j \cdot (Er_{\mathbb{R}^n_+} \beta_j + Er_{\mathbb{R}^n_-} \beta_j). \end{aligned}$$

Suppose that $\operatorname{supp} \alpha_i \subset B_i(\mathbf{x})$ and $B_i(\mathbf{x}) \cap \partial\mathbb{R}^n_+ \neq \varnothing$, there exists $\mathbf{x}^* \in B_i(\mathbf{x}) \cap \partial\mathbb{R}^n_+$ such that $\operatorname{supp} Er_{\mathbb{R}^n_+} \alpha_i \subset B_{2r_i}(\mathbf{x}^*)$ and $\operatorname{supp} Er_{\mathbb{R}^n_-} \alpha_i \subset B_{2r_i}(\mathbf{x}^*)$. Therefore we have that $\operatorname{supp}(Er_{\mathbb{R}^n_+} \alpha_i + Er_{\mathbb{R}^n_-} \alpha_i) \subset B_{2r_i}(\mathbf{x}^*)$. Notice that $Er_{\mathbb{R}^n_+} \alpha_i + Er_{\mathbb{R}^n_-} \alpha_i$ is also even with respect to x_n. Let's consider $r_{\mathbb{R}^n_+}(Er_{\mathbb{R}^n_+}\alpha_i + Er_{\mathbb{R}^n_-}\alpha_i) = r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i$. There is no doubt that supp $(r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i) \subset B_{2r_i}(\mathbf{x}^*) \cap \mathbb{R}^n_+$ and

$$\begin{aligned}\int_{\mathbb{R}^n_+} r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i \, \mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^n_+} r_{\mathbb{R}^n_+}\alpha_i \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n_-} Er_{\mathbb{R}^n_-}\alpha_i \, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n} \alpha_i \, \mathrm{d}\mathbf{x} \\ &= 0.\end{aligned}$$

Let $\alpha_i^* := r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i$, notice that

$$\begin{aligned}||\alpha_i^*||_{L^2(\mathbb{R}^n)} &= ||r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i||_{L^2(\mathbb{R}^n)} \\ &\le ||\alpha_i||_{L^2(\mathbb{R}^n)} + ||Er_{\mathbb{R}^n_-}\alpha_i||_{L^2(\mathbb{R}^n)} \\ &\le ||\alpha_i||_{L^2(\mathbb{R}^n)} + 2 \cdot ||\alpha_i||_{L^2(\mathbb{R}^n)} \\ &\le 3 \cdot 2^{\frac{n}{2}} \cdot |B_{2r_i}(\mathbf{x}^*)|^{-1/2}.\end{aligned}$$

Let $c_{3,2} := 3 \cdot 2^{\frac{n}{2}}$. Therefore $c_{3,2}^{-1} \cdot \alpha_i^*$ is a 2-atom and more importantly, we have that

$$\int_{\mathbb{R}^n_-} c_{3,2}^{-1} \cdot \alpha_i^* \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n_+} c_{3,2}^{-1} \cdot \alpha_i^* \, \mathrm{d}\mathbf{x} = 0.$$

Hence $E(c_{3,2}^{-1} \cdot \alpha_i^*) = c_{3,2}^{-1} \cdot E\alpha_i^*$ is a symmetric 2-atom. We have that

$$2 \cdot Ef = \sum_i \lambda_i' \cdot (c_{3,2}^{-1} \cdot E\alpha_i^*) + \sum_j \mu_j \cdot Er_{\mathbb{R}^n_+}\beta_j + \sum_j \mu_j \cdot Er_{\mathbb{R}^n_-}\beta_j$$

where $\lambda_i' := \lambda_i \cdot c_{3,2}$. Therefore from a 2-atomic decomposition of Ef we can get a symmetric 2-atomic decomposition of Ef. In addition, for a 2-atomic decomposition $Ef = \sum_i \lambda_i\alpha_i + \sum_j \mu_j\beta_j$ such that $\sum_i |\lambda_i| + \sum_j |\mu_j| < \infty$, the corresponding symmetric 2-atomic decomposition of this 2-atomic decomposition is $Ef = \sum_i \frac{\lambda_i'}{2} \cdot (c_{3,2}^{-1} \cdot E\alpha_i^*) + \sum_j \frac{\mu_j}{2} \cdot Er_{\mathbb{R}^n_+}\beta_j + \sum_j \frac{\mu_j}{2} \cdot Er_{\mathbb{R}^n_-}\beta_j$. In this case we have that

$$\sum_i \frac{|\lambda_i'|}{2} + \sum_j \frac{|\mu_j|}{2} + \sum_j \frac{|\mu_j|}{2} \le 3 \cdot 2^{\frac{n}{2}-1} \cdot (\sum_i |\lambda_i| + \sum_j |\mu_j|) < \infty.$$

Therefore,

$$\sum_i |\lambda_i| + \sum_j |\mu_j| \ge \frac{1}{3 \cdot 2^{\frac{n}{2}-1}} \cdot (\sum_i |\lambda_i''| + \sum_j |\mu_j''|)$$

where $\lambda_i'' := \frac{\lambda_i'}{2}$ for all i and $\mu_j'' := \frac{\mu_j}{2}$ for all j. λ_i'' and μ_j'' are the coefficients of the corresponding symmetric 2-atomic decomposition induced by the original 2-atomic decomposition. As a result, we have that

$$\begin{aligned}&\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all 2-atomic decompositions}\} \\ \ge\; & C_1 \cdot \inf\{\sum_i |\lambda_i''| + \sum_j |\mu_j''| \mid \text{all symmetric 2-atomic decompositions}\}\end{aligned}$$

where $C_1 := \frac{1}{3 \cdot 2^{\frac{n}{2}-1}}$.

(2). Let $Ef = \sum_i \lambda_i \cdot Er_{\mathbb{R}^n_+} \alpha_i + \sum_j \mu_j \cdot Er_{\mathbb{R}^n_+} \beta_j$ be a symmetric 2-atomic decomposition. Since α_i is a 2-atom, we have that

$$||Er_{\mathbb{R}^n_+} \alpha_i||_{L^2(\mathbb{R}^n)} \leq 2^{\frac{n}{2}+1} \cdot |B_{2r_i}(\mathbf{x}^*)|^{-1/2}.$$

Therefore

$$Ef = \sum_i (\lambda_i \cdot 2^{\frac{n}{2}+1}) \cdot (\frac{1}{2^{\frac{n}{2}+1}} \cdot Er_{\mathbb{R}^n_+} \alpha_i) + \sum_j \mu_j \beta_j^+ + \sum_j \mu_j \beta_j^-$$

is a 2-atomic decomposition of Ef. Thus every symmetric 2-atomic decomposition of Ef gives rise to a 2-atomic decomposition. For this symmetric 2-atomic decomposition of Ef where $\sum_i |\lambda_i| + \sum_j |\mu_j| < \infty$, the coefficients of the corresponding 2-atomic decomposition of Ef satisfies

$$\sum_i (|\lambda_i| \cdot 2^{\frac{n}{2}+1}) + \sum_j 2 \cdot |\mu_j| \leq 2^{\frac{n}{2}+1} \cdot (\sum_i |\lambda_i| + \sum_j |\mu_j|).$$

Therefore,

$$\begin{aligned} & \inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\} \\ \geq \quad & C_2 \cdot \inf\{\sum_i |\lambda_i^{'}| + \sum_j |\mu_j^{'}| \mid \text{all 2-atomic decompositions}\} \end{aligned}$$

where $C_2 := \frac{1}{2^{\frac{n}{2}+1}}$. □

Theorem 2.5.10. *Let $f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$, then there exists sequences of non-negative numbers $\{\lambda_i\}_{i=1}^\infty$ and $\{\mu_j\}_{j=1}^\infty$, a sequence of 2-atoms $\{\alpha_i\}_{i=1}^\infty$ where for each i $\operatorname{supp} \alpha_i \subset B_i$ & $B_i \cap \partial\mathbb{R}^n_+ \neq \varnothing$ & $\int_{\mathbb{R}^n_+} \alpha_i \, \mathrm{d}\mathbf{x} = 0$ for some ball B_i and a sequence of 2-atoms $\{\beta_j\}_{j=1}^\infty$ where for each j $\operatorname{supp} \beta_j \subset B_j \subset \mathbb{R}^n_+$ for some ball B_j such that*

$$f = \sum_i \lambda_i \cdot \alpha_i \mid_{\mathbb{R}^n_+} + \sum_j \mu_j \cdot \beta_j.$$

We refer such a decomposition of f as a half space atomic decomposition of f and moreover, the norm

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \textit{all half space atomic decompositions}\}$$

is equivalent to the norm $|| \cdot ||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)}$ on $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$.

Proof. By Lemma 2.5.9 we are done. □

Definition 2.5.11. We denote the set of all finite linear combinations of symmetric 2-atoms by $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$.

By similar arguments as in the previous subsection, we can easily deduce that $\mathscr{H}^1_{0,s}(\mathbb{R}^n) \subset \mathscr{H}^1_0(\mathbb{R}^n) \cap E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$, $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ is a closed subspace of $\mathscr{H}^1(\mathbb{R}^n)$ and $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$ is dense in $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$. Then by making use of these facts, we can prove our duality theorem for the case of even extension.

Theorem 2.5.12. *Suppose $g \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$. Then the linear functional l defined on $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ by*

$$l(f) = \int_{\mathbb{R}^n_+} f \cdot g \,\mathrm{d}\mathbf{x}$$

for $f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$ is a bounded linear functional which satisfies $||l|| \leq c \cdot [g]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)}$ with some constant c. Conversely, every bounded linear functional l on $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ can be written in the form of

$$l(f) = \int_{\mathbb{R}^n_+} f \cdot g \,\mathrm{d}\mathbf{x} \quad \text{for all } f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$$

with $g \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ and $[g]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)} \leq c \cdot ||l||$ with some constant c. Here $||l||$ means the norm of l as a bounded linear functional on $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$.

Proof. The only difference from the proof of Theorem 2.5.7 is the last part where here we prove that the unified function $g(\mathbf{x}) \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ instead of $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$. For the rest of the details, please refer to the proof of Theorem 2.5.7.

We define the unified function $Eg(\mathbf{x})$ on $\mathbb{R}^n$ by

$$\begin{aligned} Eg(\mathbf{x}) &:= Eg^{B_r^+(0)} - \frac{1}{|B_1(0)|}\int_{B_1(0)} Eg^{B_r^+(0)}\,\mathrm{d}\mathbf{x} \\ &= Eg^{B_r^+(0)} - \underset{B_1(0)}{Avg}\, Eg^{B_r^+(0)}. \end{aligned}$$

For $B \subset \mathbb{R}^n_+$ we have $Eg^B(\mathbf{x})$ defined on the ball B, then there exists $B_r(0)$ for some r large enough such that $B \subset B_r(0)$. We can rewrite $Eg^B(\mathbf{x})$ as

$$Eg^B(\mathbf{x}) = Eg^B(\mathbf{x}) - Eg^{B_r^+(0)}(\mathbf{x}) + Eg^{B_r^+(0)}(\mathbf{x}) - \underset{B_1(0)}{Avg}\, Eg^{B_r^+(0)} + \underset{B_1(0)}{Avg}\, Eg^{B_r^+(0)}.$$

Notice that $Eg^B(\mathbf{x}) - Eg^{B_r^+(0)}(\mathbf{x})$ and $\underset{B_1(0)}{Avg}\, Eg^{B_r^+(0)}$ are both constants which depend on B, hence let $c_B := Eg^B(\mathbf{x}) - Eg^{B_r^+(0)}(\mathbf{x}) + \underset{B_1(0)}{Avg}\, Eg^{B_r^+(0)}$, we have that $Eg^B(\mathbf{x}) = c_B + Eg(\mathbf{x})$. Next we prove that the function $g(\mathbf{x})$ defined by $g(\mathbf{x}) := r_{\mathbb{R}^n_+} Eg(\mathbf{x}) \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$.

$*1$. If $B \subset \mathbb{R}^n_+$, we have that

$$\frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \leq c \cdot |B|^{-1/2} \cdot ||Eg^B||_{EL^2_0(B)}$$

by the Hölder inequality. Since

$$|\int_{B\cup B^-} Eg^B \cdot Eu \,\mathrm{d}\mathbf{x} = |\tilde{l}(Eu)| \leq c \cdot |B|^{-1/2} \cdot ||Eu||_{EL^2_0(B)},$$

we have that

$$||Eg^B||_{EL^2_0(B)} = ||\tilde{l}|| \leq c \cdot |B|^{1/2}.$$

Therefore we can deduce that

$$\frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)| \, \mathrm{d}\mathbf{x} \leq c.$$

Notice that the c here is just a number which is independent of B. Therefore by taking the supremum over all balls contained in $\mathbb{R}^n_+$, we can see that

$$\sup_{B \subset \mathbb{R}^n_+} \frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)| \, \mathrm{d}\mathbf{x} \leq c.$$

and thus,

$$[r_{\mathbb{R}^n_+} Eg]_{BMO^\infty(\mathbb{R}^n_+)} = [g]_{BMO^\infty(\mathbb{R}^n_+)} \leq 2 \cdot c.$$

$*2$. If $B_r(\mathbf{x})$ is a ball where $\mathbf{x} \in \partial\mathbb{R}^n_+$ and $r > 0$, we have that $Eg(\mathbf{x}) = Eg^{B_r^+}(\mathbf{x}) - c_{B_r}$. Therefore we have the following calculations:

$$\begin{aligned} 2 \cdot \int_{B_r^+} g(\mathbf{x}) \, \mathrm{d}\mathbf{x} &= \int_{B_r} Eg(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ &= \int_{B_r} Eg^{B_r^+}(\mathbf{x}) \, \mathrm{d}\mathbf{x} - \int_{B_r} c_{B_r} \, \mathrm{d}\mathbf{x} \\ &= 0 - c_{B_r} \cdot |B_r|. \end{aligned}$$

Hence $c_{B_r} = -g_{B_r^+}$ and we have that

$$\begin{aligned} \frac{1}{|B_r|}\int_{B_r} |Eg(\mathbf{x}) - (-c_{B_r})| \, \mathrm{d}\mathbf{x} &= \frac{1}{|B_r|}\int_{B_r} |Eg(\mathbf{x}) - g_{B_r^+}| \, \mathrm{d}\mathbf{x} \\ &= \frac{1}{|B_r^+|}\int_{B_r^+} |g(\mathbf{x}) - g_{B_r^+}| \, \mathrm{d}\mathbf{x} \leq c. \end{aligned}$$

Take the supremum over all balls centered on $\mathbb{R}^n_+$, we have that

$$[g]_{ba^\infty(\mathbb{R}^n_+)} = \sup_{\substack{r>0 \\ \mathbf{x} \in \partial\mathbb{R}^n_+}} \frac{1}{|B_r^+|}\int_{B_r^+} |g(\mathbf{x}) - g_{B_r^+}| \, \mathrm{d}\mathbf{x} \leq c$$

and hence $g \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$. □

2.5.3 Proof of Theorem 2.1.3

Proof. By Theorem 2.5.7 and Theorem 2.5.12, we are done. □

2.5.4 Comments

Remark 2.5.13. If we look at the proof of Lemma 2.4.1 and Lemma 2.4.2, we can see that it is completely all right for us to replace the heat kernel $e^{t\Delta}$ in the definition of $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ by any radial symmetric function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi \, d\mathbf{x} = 1$. Therefore, the definitions of the norms $|| \cdot ||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)}$ and $|| \cdot ||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}$ are independent of the choice of φ if φ is radial symmetric with integral over $\mathbb{R}^n$ equals 1.

Remark 2.5.14. When we established the half space atomic decompositions for $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, we made use of the 2-atomic decomposition of $\mathscr{H}^1(\mathbb{R}^n)$ in order to carry out the arguments of Fefferman and Stein [3] to prove the duality theorem. However, if we carry out the arguments using the p-atomic decomposition of $\mathscr{H}^1(\mathbb{R}^n)$ instead where $p \geq 1$, then we get the half space atomic decompositions for $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ in the form of symmetric p-atomic decompositions.

In [1], it is proved that $BMO_M(\mathbb{R}^n_+)$ and $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$ are actually the same space. Since $BMO_M(\mathbb{R}^n_+)$ is the dual space of $\mathscr{H}^1_M(\mathbb{R}^n_+)$ and $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$ is the dual space of $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, it is natural to ask the question about the relation between $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_M(\mathbb{R}^n_+)$. Here we give an answer to this question.

Lemma 2.5.15. $\mathscr{H}^1_{odd}(\mathbb{R}^n_+) = \mathscr{H}^1_M(\mathbb{R}^n_+)$.

Proof. (1). By the theory of Miyachi [7], $f \in \mathscr{H}^1_M(\mathbb{R}^n_+)$ implies that f admits the half space atomic decomposition of the form

$$f = \sum_i \lambda_i \alpha_i + \sum_j \mu_j \beta_j$$

where $\{\beta_j\}_{j=1}^{\infty}$ is a sequence of 1-atom such that β_j is supported on some ball B_j with $2B_j \subset \mathbb{R}^n_+$ for each j and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence of $(1, \mathbb{R}^n_+)$-atom such that α_i is supported on some ball B_i with $2B_i \subset \mathbb{R}^n_+$ but $5B_i \cap (\mathbb{R}^n_+)^c \neq \varnothing$ for each i. Let $B_i = B_r(\mathbf{x}_i)$ and $\mathbf{x}^* := (\mathbf{x}_i^{'}, 0)$. Since $2B_i \subset \mathbb{R}^n_+$ but $5B_i \cap (\mathbb{R}^n_+)^c \neq \varnothing$, we can easily deduce that $B_i \subset B_{6r}(\mathbf{x}^*)$. Notice that $\alpha_i = r_{\mathbb{R}^n_+} E_{odd}\alpha_i$ and $\int_{B_{6r}(\mathbf{x}^*)} E_{odd}\alpha_i \, d\mathbf{x} = 0$, therefore we have that

$$E_{odd} f = \sum_i (\lambda_i \cdot 6^n) \cdot (\frac{1}{6^n} \cdot E_{odd}\alpha_i) + \sum_j \mu_j E_{odd}\beta_j. \tag{2.5.1}$$

Here $\frac{1}{6^n} \cdot E_{odd}\alpha_i$ is a 1-atom for any i, hence by (2.5.1) we see that $E_{odd} f \in \mathscr{H}^1(\mathbb{R}^n)$ and thus by Remark 2.5.14 $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

(2). Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, let η be the standard mollifier. For $\mathbf{x} \in \mathbb{R}^n_+$ and $0 < t < dist(\mathbf{x}, \partial\mathbb{R}^n_+)$, we have that $(\eta_t * f)(\mathbf{x}) = (\eta_t * E_{odd} f)(\mathbf{x})$ since $\operatorname{supp} \eta_t \subset B_t(0)$. Hence for $\mathbf{x} \in \mathbb{R}^n_+$,

$$\begin{aligned} \sup_{0<t<dist(\mathbf{x},\partial\mathbb{R}^n_+)} |\eta_t * f|(\mathbf{x}) &= \sup_{0<t<dist(\mathbf{x},\partial\mathbb{R}^n_+)} |\eta_t * E_{odd} f|(\mathbf{x}) \\ &\leq \sup_{t>0} |\eta_t * E_{odd} f|(\mathbf{x}). \end{aligned}$$

Thus

$$\begin{aligned}||f||_{\mathscr{H}^1_M(\mathbb{R}^n_+)} &:= \int_{\mathbb{R}^n_+} \sup_{0<t<dist(\mathbf{x},\partial\mathbb{R}^n_+)} |\eta_t * f|(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &\leq \int_{\mathbb{R}^n_+} \sup_{t>0} |\eta_t * E_{odd} f|(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}\end{aligned}$$

and therefore $f \in \mathscr{H}^1_M(\mathbb{R}^n_+)$. □

Remark 2.5.16. Let us consider a function $f \in L^2(B^+_r(0))$ with integral over $B^+_r(0)$ not equals to 0. Notice that although $\int_{B^+_r(0)} f\,\mathrm{d}\mathbf{x} \neq 0$, the odd extension $E_{odd}f$ has integral zero over the ball $B_r(0)$. Hence we have that $E_{odd}f \in L^2(B_r(0))$, $\int_{B_r(0)} E_{odd}f\,\mathrm{d}\mathbf{x} = 0$ and thus $E_{odd}f \in \mathscr{H}^1(\mathbb{R}^n)$. Then $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. However, $\int_{B^+_r(0)} f\,\mathrm{d}\mathbf{x} \neq 0$ implies that $\int_{B_r(0)} E_{even}f\,\mathrm{d}\mathbf{x} \neq 0$ and thus $E_{even}f \notin \mathscr{H}^1(\mathbb{R}^n)$. Hence $f \notin \mathscr{H}^1_{even}(\mathbb{R}^n_+)$. Therefore $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ are two different spaces.

Remark 2.5.17. Let us consider the function $\log|\mathbf{x}|$, by the standard theory of BMO spaces we see that $\log|\mathbf{x}| \in BMO$. Then $\log|\mathbf{x}|\;|_{\mathbb{R}^n_+} \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$. However, $\log|\mathbf{x}|\;|_{\mathbb{R}^n_+} \notin BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$ since the integral

$$\frac{1}{B^+_r(0)} \int_{B^+_r(0)} |\log|\mathbf{x}||\,\mathrm{d}\mathbf{x} \to \infty \quad \text{as } r \to \infty.$$

Therefore $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$ and $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ are also two different spaces.

Remark 2.5.18. Notice that by Theorem 2.5.3 we can easily see that $\mathscr{H}^1_{odd}(\mathbb{R}^n_+) = \mathscr{H}^1(\mathbb{R}^n_+)$ where $\mathscr{H}^1(\mathbb{R}^n_+) := \{r_{\mathbb{R}^n_+} f \,|\, f \in \mathscr{H}^1(\mathbb{R}^n)\}$. Moreover, by Lemma 2.3.2 and Lemma 2.3.4, we can also see that $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+) = BMO(\mathbb{R}^n_+)$ where $BMO(\mathbb{R}^n_+) := \{r_{\mathbb{R}^n_+} f \,|\, f \in BMO(\mathbb{R}^n)\}$. As a result, we can clarify the relationship between various function spaces in this chapter as follow:

$$\begin{array}{ccccc} BMO(\mathbb{R}^n_+) & = & BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+) & =^* & \mathscr{H}^1_{even}(\mathbb{R}^n_+) \\ & & \cup & & \cap \\ & & BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+) & =^* & \mathscr{H}^1_{odd}(\mathbb{R}^n_+) = \mathscr{H}^1(\mathbb{R}^n_+) \\ & & \| & & \| \\ & & BMO_M(\mathbb{R}^n_+) & =^* & \mathscr{H}^1_M(\mathbb{R}^n_+). \end{array}$$

Here $A =^* B$ means that A is the dual space of B.

2.6 Dual operator of the Helmholtz projection

2.6.1 Dual operators of E_{odd} and $r_{\mathbb{R}^n_+}$

In this subsection, for simplicity, we shall denote the odd extension operator E_{odd} by E. Since $E : \mathscr{H}^1_{odd}(\mathbb{R}^n_+) \to E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, we have that $E^* : E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)^* \to \mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$. By the theories in section 2.5 we have that $E^* : EBMO^{\infty,\infty}_{b}(\mathbb{R}^n_+) \to BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$.

Lemma 2.6.1. *The dual operator of E is indeed $2 \cdot r_{\mathbb{R}^n_+}$, i.e., $E^* = 2 \cdot r_{\mathbb{R}^n_+}$.*

Proof. Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and $g \in BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$, by the definition of dual operator, we can deduce that

$$< E^* Eg, f > := < Eg, Ef > = 2 < g, f > .$$

Therefore, we have that

$$< E^* Eg - 2g, f > = 0 \;\text{ for all } f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+).$$

Let $B_r(0)$ be the ball centered at 0 with radius r and $B_r^+(0) := B_r(0) \cap \mathbb{R}^n_+$. For simplicity, we denote $B_r^+(0)$ by B_r^+. Notice that from the previous chapter, we see that $L^2(B_r^+) \subset \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Hence fix $r > 0$, we have that

$$< E^* Eg - 2g, f > = 0 \;\text{ for all } f \in L^2(B_r^+).$$

Since $C_0^\infty(B_r^+) \subset L^2(B_r^+)$, by the fundamental lemma of variational calculus, we see that

$$E^* Eg - 2g = 0 \quad \text{a.e. in } B_r^+.$$

This means $E^* = 2 \cdot r_{\mathbb{R}^n_+}$ and we are done. □

By similar arguments as above, we can also deduce that ${r_{\mathbb{R}^n_+}}^* : BMO_b^{\infty,\infty}(\mathbb{R}^n_+) \to EBMO_b^{\infty,\infty}(\mathbb{R}^n_+)$ and the dual operator of $r_{\mathbb{R}^n_+}$, where $r_{\mathbb{R}^n_+}$ corresponds to the restriction of $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, is indeed $\frac{1}{2} \cdot E$.

2.6.2 Dual operators of E_{even} and $r_{\mathbb{R}^n_+}$

We denote the even extension operator E_{even} by E. By similar arguments as in the previous subsection, we have that the dual operator of E is indeed $2 \cdot r_{\mathbb{R}^n_+}$ and the dual operator of $r_{\mathbb{R}^n_+}$, which corresponds to the restriction of $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$, is indeed $\frac{1}{2} \cdot E$.

2.6.3 Proof of Theorem 2.1.4

Proof. Since $\mathbb{P}_{\mathbb{R}^n_+}$ is a bounded linear operator from $\mathbf{Y}$ to $\mathbf{Y}$ and $\mathbf{X}$ is the dual space of $\mathbf{Y}$, we have that

$${\mathbb{P}_{\mathbb{R}^n_+}}^* : \mathbf{X} \to \mathbf{X}.$$

Then let $\mathbf{v} \in \mathbf{X}$ and $\mathbf{u} \in \mathbf{Y}$, we have that

$$< {\mathbb{P}_{\mathbb{R}^n_+}}^* \mathbf{v}, \mathbf{u} > = \sum_{i=1}^{n-1} < v^i, r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{u})^i > + < v^n, r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{u})^n > .$$

Notice that $(\mathbb{P}E\mathbf{u})^i$ is even with respect to x_n for $1 \le i \le n-1$ and $(\mathbb{P}E\mathbf{u})^n$ is odd with respect to x_n. Hence for $1 \le i \le n-1$, the $r_{\mathbb{R}^n_+}$ in $r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{u})^i$ corresponds to the restriction of $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ whereas for $i = n$, the $r_{\mathbb{R}^n_+}$ in $r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{u})^n$ corresponds to the restriction of $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Therefore,

$$< {\mathbb{P}_{\mathbb{R}^n_+}}^* \mathbf{v}, \mathbf{u} > = \frac{1}{2} < E\mathbf{v}, \mathbb{P}E\mathbf{u} > \cdots\cdots (*).$$

By [8], we see that the dual operator of $\mathbb{P} : \mathscr{H}^1(\mathbb{R}^n) \to \mathscr{H}^1(\mathbb{R}^n)$ is itself as a map from BMO to BMO. Therefore

$$\begin{aligned}
(*) &= \frac{1}{2} < \mathbb{P}E\mathbf{v}, E\mathbf{u} > \\
&= \frac{1}{2}\Big(\sum_{i=1}^{n-1} < (\mathbb{P}E\mathbf{v})^i, E_{even}u^i > + < (\mathbb{P}E\mathbf{v})^n, E_{odd}u^n > \Big) \\
&= \frac{1}{2}\Big(\sum_{i=1}^{n-1} < 2r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{v})^i, u^i > + < 2r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{v})^n, u^n > \Big) \\
&=< \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v}, \mathbf{u} > .
\end{aligned}$$

□

Remark 2.6.2. When we are considering the dual operator of $\mathbb{P}_{\mathbb{R}^n_+}$, notice that the space $\mathbf{X}$ must be viewed as $\mathbf{X}/(\mathbb{R}^{n-1} \times \{0\})$!

2.6.4 Proof of Corollary 2.1.5

Proof. By [2, Th 2.19] and Theorem 2.1.4 in this chapter, we are done. □

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Chapter 3

Normal trace for a vector field of bounded mean oscillation

We introduce various spaces of vector fields of bounded mean oscillation (BMO) defined in a domain so that normal trace of a vector field on the boundary is bounded when its divergence is well controlled. The behavior of "normal" component and "tangential" component may be different for our BMO vector fields. As a result the zero extension of the normal component stays in BMO although such property may not hold for tangential components.

3.1 Introduction

One of basic questions on vector fields defined on a domain Ω in $\mathbf{R}^n$ ($n \geq 2$) is whether the normal trace is well controlled without estimating all partial derivatives when the divergence is well controlled. Such a type of estimates is well known when a vector field is L^p ($1 < p < \infty$) or L^∞. Here are examples. Let Ω be a bounded domain with smooth boundary Γ. Let $\mathbf{n}$ denotes its exterior unit normal vector field on Γ. For simplicity, we assume that a vector field v satisfies $\operatorname{div} v = 0$. Then there is a constant C independent of v such that

$$\|v \cdot \mathbf{n}\|_{W^{-1/p,p}(\Gamma)} \leq C\|v\|_{L^p(\Omega)} \tag{3.1.1}$$

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C\|v\|_{L^\infty(\Omega)}. \tag{3.1.2}$$

Here $W^{s,p}$ denotes the Sobolev space which is actually a Besov space $B^s_{p,p}$ for non-integer s. The first estimate is a key to establish the Helmholtz decomposition of an L^p vector field; see e.g. [6]. The second estimate is important to study, for example, a total variation flow; see e.g. [1, Appendix C1]. These estimates (3.1.1), (3.1.2) hold for various domains including the case that Ω is a half space $\mathbf{R}^n_+$, i.e.,

$$\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_n > 0\}.$$

Our goal in this chapter is to extend (3.1.2) by replacing $\|v\|_{L^\infty(\Omega)}$ by some BMO type norm. However, it turns out that the normal trace on $\Gamma = \partial\mathbf{R}^n_+$ of divergence free BMO vector fields in $\mathbf{R}^n$ may not be bounded. Indeed, consider

$$v = (v^1, v^2), \quad v^1(x) = v^2(x) = \log|x_1 - x_2|. \quad x = (x_1, x_2) \in \mathbf{R}^2.$$

This vector field is in $BMO(\mathbf{R}^2)$ and it is divergence free in distribution sense. Indeed,

$$\int_{\mathbf{R}^2} v \cdot \nabla\varphi \, dx = \frac{1}{2} \int_{\mathbf{R}^2} \log|\zeta| \big((\partial_\zeta - \partial_\zeta)\tilde{\varphi} + (\partial_\eta + \partial_\eta)\tilde{\varphi} \big) \, d\zeta d\eta = 0,$$

$$\zeta = x_1 - x_2, \quad \eta = x_1 + x_2$$

for all compactly supported smooth function φ, i.e., $\varphi \in C_c^\infty(\mathbf{R}^2)$. Here, $\tilde{\varphi}(\zeta, \eta) = \varphi\big((\zeta + \eta)/2, (\eta - \zeta)/2\big)$. However, if we consider $\Omega = \mathbf{R}^2_+$ and $\Gamma = \{x_2 = 0\}$, then $v \cdot \mathbf{n} = -v_2$ on Γ is clearly unbounded. This example indicates that we need some control near the boundary. Such a control is introduced in [?BG], [?BGS], [?BGMST], [?BGST]. More precisely, for $f \in L^1_{\mathrm{loc}}(\Omega)$ and $\nu \in (0, \infty]$, they introduced a seminorm

$$[f]_{b^\nu} := \sup \left\{ r^{-n} \int_{\Omega \cap B_r(x)} |f(y)| \, dy \;\middle|\; x \in \Gamma, \ 0 < r < \nu \right\},$$

where $B_r(x)$ denotes the closed ball of radius r centered at x. For $\mu \in (0, \infty]$, they define

$$[f]_{BMO^\mu} := \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| f - f_{B_r(x)} \right| dy \;\middle|\; B_r(x) \subset \Omega, \ r < \mu \right\},$$

where $f_B = \dfrac{1}{|B|} \displaystyle\int_B f(y) \, dy$, the average over B; here $|B|$ denotes the Lebesgue measure of B. The BMO type space $BMO_b^{\mu,\nu}$ introduced in these papers is the space of $f \in L^1_{\mathrm{loc}}(\Omega)$ having finite

$$\|f\|_{BMO_b^{\mu,\nu}} := [f]_{BMO^\mu} + [f]_{b^\nu}.$$

This space is very convenient to study the Stokes semigroup in [2], [4], [3], [5] as well as the heat semigroup [5]. One of our main results (Theorem 3.4.7) yields

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C \|v\|_{BMO_b^{\mu,\nu}} \tag{3.1.3}$$

for any $\mu, \nu \in (0, \infty]$ for any uniformly $C^{1+\beta}$ domain with $\beta \in (0, 1)$.

However, for applications, especially to establish the Helmholtz decomposition, requiring all components to be $BMO_b^{\mu,\nu}$ bounded is too strong so we would like to estimate by a weaker norm. We only use b^ν seminorm for normal component of a vector field v. To decompose the vector field, let $d_\Omega(x)$ be the distance of $x \in \Omega$ from the boundary Γ, i.e.,

$$d_\Omega(x) := \inf\big\{ |x - y| \bigm| y \in \Gamma \big\}.$$

If Ω is uniformly C^2, then d_Ω is C^2 in a δ-tubular neighborhood Γ_δ of Γ for some $\delta < R_*$, where R_* is the reach of Γ [10, Chapter 14, Appendix], [11, §4.4]; here

$$\Gamma_\delta := \big\{ x \in \Omega \bigm| d_\Omega(x) < \delta \big\}.$$

Instead of (3.1.3), our main results (Theorem 3.4.2, 3.4.3) together with Theorem 3.2.9 read as

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C \big([v]_{BMO^\mu} + [\nabla d_\Omega \cdot v]_{b^\nu} \big) \tag{3.1.4}$$

for $\nu \leq \delta$, $\mu \in (0, \infty]$ provided that Ω is a bounded $C^{2+\beta}$ domain with $\beta \in (0, 1)$. The quantity $\nabla d_\Omega \cdot v$ is a kind of a normal component of v.

Our main strategy is to use the formula

$$\int_\Gamma (v \cdot \mathbf{n})\psi \, d\mathcal{H}^{n-1} = \int_\Omega (\operatorname{div} v)\varphi \, dx - \int_\Omega v \cdot \nabla\varphi \, dx$$

for any $\varphi \in C_c^\infty(\overline{\Omega})$ with $\varphi|_\Gamma = \psi$, where $d\mathcal{H}^{n-1}$ denotes the surface element. This formula is obtained by integration by parts. If $\operatorname{div} v = 0$, then it reads

$$\int_\Gamma (v \cdot \mathbf{n})\psi \, d\mathcal{H}^{n-1} = -\int_\Omega v \cdot \nabla\varphi \, dx. \tag{3.1.5}$$

Our estimate (3.1.4) follows from localization, flattening the boundary and duality argument. To get the flavor, we explain the case when Ω is the half space $\mathbf{R}^n_+$. For $\psi \in L^1(\Gamma)$ it is known that there is $\varphi \in F^1_{1,2}(\mathbf{R}^n)$ such that its trace to the boundary equals to ψ; see e.g. [19, Section 4.4.3]. Here $F^1_{1,2}$ denotes the Triebel-Lizorkin space which means that $\nabla\varphi \in h^1$, a local Hardy space. We may assume that φ is even in x_n. We extend $v = (v', v_n)$ even in x_n for tangential part v' and odd in x_n for the normal part $v_n = \nabla d_\Omega \cdot v$. Although extended v' is still in $BMO^\infty(\mathbf{R}^n)$, the extended v_n may not be in $BMO^\infty(\mathbf{R}^n)$ unless we assume $[v_n]_{b^\nu} < \infty$. Here we invoke $[\nabla d_\Omega \cdot v]_{b^\nu} < \infty$. By these extensions, our (3.1.5) yields

$$\int_\Gamma (v \cdot \mathbf{n})\psi \, d\mathcal{H}^{n-1} = -\frac{1}{2}\int_{\mathbf{R}^n} v \cdot \nabla\varphi \, dx, \tag{3.1.6}$$

where v denotes the extended vector field. We apply h^1-bmo duality [16, Theorem 3.22] for (3.1.6) to get

$$\left| \int_\Gamma (v \cdot \mathbf{n})\psi \, d\mathcal{H}^{n-1} \right| \leq C\|v\|_{bmo}\|\varphi\|_{F^1_{1,2}},$$

where $bmo = BMO \cap L^1_{\mathrm{ul}}$ a localized BMO space. Here L^1_{ul} denotes a uniformly local L^1 space; see Section 3.2 for details. Since $\|\varphi\|_{F^1_{1,2}} \leq C\|\psi\|_{L^1}$, this implies

$$\begin{aligned} \|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} &\leq C\|v\|_{bmo(\mathbf{R}^n)} \\ &\leq C\left([v]_{BMO^\infty(\Omega)} + [v]_{L^1_{\mathrm{ul}}(\Omega)} + [\nabla d_\Omega \cdot v]_{b^\infty}\right). \end{aligned} \tag{3.1.7}$$

Here and hereafter C denotes a constant independent of v and its numerical value may be different line by line.

In the case of a curved domain we need localization and flattening procedure by using a normal (principal) coordinate system. The localized space $bmo^\mu_\delta = BMO^\mu \cap L^1_{\mathrm{ul}}(\Gamma_\delta)$ is convenient for this purpose. Again we have to handle normal component $\nabla d_\Omega \cdot v$ separately. If the domain has a compact boundary, we are able to remove L^1_{ul} term in (3.1.7) and we deduce the estimate (3.1.4). Note that in this trace estimate only the behavior of v near Γ is important so one may use finite exponents in BMO^μ and b^ν.

As a byproduct we notice the extension problem of BMO functions. In general, zero extension of $v \in BMO^\mu(\Omega)$ may not belong to $BMO^\mu(\mathbf{R}^n)$ but if v is in $BMO^{\mu,\nu}_b$, as noticed in [5], its zero extension belongs to $BMO^\mu(\mathbf{R}^n)$ for $\nu \geq 2\mu$. We also note that it is possible to extend general $bmo^\mu_\delta(\Omega)$ to BMO^μ whose support is near $\overline{\Omega}$. We develop such a theory to explain the role of b^ν.

This chapter is organized as follows. In Section 3.2 we introduce several localized BMO spaces and compared these spaces. Some of them are discussed in [5]. We introduce a new space $vbmo^{\mu,\nu}_\delta$ which requires that the b^ν seminorm of the normal component is bounded in

$(bmo_\delta^\mu)^n$. A key observation is that if the boundary of the domain is compact, i.e., either a bounded or an exterior domain, the requirement in $L^1_{\mathrm{ul}}(\Gamma_\delta)$ is redundant in the definition of $vbmo_\delta^{\mu,\nu}$. In Section 3.3 we discuss extension problem as well as localization problem. In Section 3.4 we shall prove our main results. In Appendix we discuss coordinate change of vector fields by normal coordinates for the reader's convenience.

3.2 Spaces

In this section we fix notation of important function spaces. Let $L^1_{\mathrm{ul}}(\mathbf{R}^n)$ be a uniformly L^1 space, i.e., for a fixed $r_0 > 0$

$$L^1_{\mathrm{ul}}(\mathbf{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) \;\middle|\; \|f\|_{L^1_{\mathrm{ul}}} := \sup_{x \in \mathbf{R}^n} \int_{B_{r_0}(x)} \bigl|f(y)\bigr|\, dy < \infty \right\}.$$

The space is independent of the choice of r_0. For a domain Ω, the space L^1_{ul} is the space of all L^1_{loc} functions f in Ω whose zero extension belongs to $L^1_{\mathrm{ul}}(\mathbf{R}^n)$. In other words,

$$L^1_{\mathrm{ul}}(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; \|f\|_{L^1_{\mathrm{ul}}(\Omega)} := \sup_{x \in \mathbf{R}^n} \int_{B_{r_0}(x) \cap \Omega} \bigl|f(y)\bigr|\, dy < \infty \right\}.$$

As in [2], we set

$$BMO^\mu(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; [f]_{BMO^\mu} < \infty \right\}.$$

For $\delta \in (0, \infty]$, we set

$$bmo_\delta^\mu(\Omega) := BMO^\mu(\Omega) \cap L^1_{\mathrm{ul}}(\Gamma_\delta) = \left\{ f \in BMO^\mu(\Omega) \;\middle|\; \text{restriction of } f \text{ on } \Gamma_\delta \text{ is in } L^1_{\mathrm{ul}}(\Gamma_\delta) \right\}.$$

This is a Banach space equipped with the norm

$$\|f\|_{bmo_\delta^\mu} := [f]_{BMO^\mu(\Omega)} + [f]_{\Gamma_\delta}, \quad [f]_{\Gamma_\delta} := \|f\|_{L^1_{\mathrm{ul}}(\Gamma_\delta)},$$

where the restriction of f on Γ_δ is still denoted by f. If there is no boundary, we set

$$bmo(\mathbf{R}^n) := BMO^\infty(\mathbf{R}^n) \cap L^1_{\mathrm{ul}}(\mathbf{R}^n)$$

which is a local BMO space and it agrees with the Triebel-Lizorkin space $F^0_{\infty,2}$; see e.g. [19, Section 1.7.1], [16, Theorem 3.26].

For vector-valued function spaces, we still write BMO^μ instead of $(BMO^\mu)^n$. For example, for vector field v, by $v \in bmo_\delta^\mu(\Omega)$ we mean that

$$v = (v_1, \dots, v_n), \quad v_i \in bmo_\delta^\mu(\Omega), \quad 1 \le i \le n.$$

We next introduce the space of vector fields whose normal component has finite b^ν of the form

$$vbmo_\delta^{\mu,\nu}(\Omega) := \left\{ v \in bmo_\delta^\mu(\Omega) \;\middle|\; [\nabla d_\Omega \cdot v]_{b^\nu} < \infty \right\}$$

for $\nu \in (0, \infty]$. This space is a Banach space equipped with the norm

$$\|v\|_{vbmo_\delta^{\mu,\nu}} := \|v\|_{bmo_\delta^\mu} + [\nabla d_\Omega \cdot v]_{b^\nu}.$$

Similarly, we introduce another space

$$vBMO^{\mu,\nu}(\Omega) := \left\{ v \in BMO^\mu(\Omega) \;\middle|\; [\nabla d_\Omega \cdot v]_{b^\nu} < \infty \right\}$$

equipped with a seminorm

$$[v]_{vBMO^{\mu,\nu}} := [v]_{BMO^\mu} + [\nabla d_\Omega \cdot v]_{b^\nu}.$$

Of course, this is strictly larger than the Banach space

$$BMO_b^{\mu,\nu}(\Omega) := \left\{ v \in BMO^\mu(\Omega) \;\middle|\; [v]_{b^\nu} < \infty \right\}$$

equipped with the norm

$$\|v\|_{BMO_b^{\mu,\nu}} := [v]_{BMO^\mu} + [v]_{b^\nu}$$

introduced essentially in [2]. Indeed, in the case when Ω is the half space $\mathbf{R}^n_+$,

$$vBMO^{\mu,\nu}(\mathbf{R}^n_+) = \left(BMO^\mu(\mathbf{R}^n_+)\right)^{n-1} \times BMO_b^{\mu,\nu}(\mathbf{R}^n_+), \tag{3.2.1}$$

where in the right-hand side the each space denotes the space of scalar functions not of vector fields. This shows that $vBMO^{\mu,\nu}(\mathbf{R}^n_+)$ is strictly larger than $BMO_b^{\mu,\nu}(\mathbf{R}^n_+)$ for $n \geq 2$.

Although there are many exponents, the spaces may be the same for different exponents. By definition, for $0 < \mu_1 \leq \mu_2 \leq \infty$, $0 < \nu_1 \leq \nu_2 \leq \infty$, $0 < \delta_1 \leq \delta_2 \leq \infty$,

$$[f]_{BMO^{\mu_1}} \leq [f]_{BMO^{\mu_2}}, \quad [f]_{b^{\nu_1}} \leq [f]_{b^{\nu_2}}, \quad [f]_{\Gamma_{\delta_1}} \leq [f]_{\Gamma_{\delta_2}}.$$

Proposition 3.2.1. *Let Ω be an arbitrary domain in $\mathbf{R}^n$.*

(i) Let $0 < \mu_1 < \mu_2 < \infty$. Then seminorms $[\cdot]_{BMO^{\mu_1}}$ and $[\cdot]_{BMO^{\mu_2}}$ are equivalent. If Ω is bounded, one may take $\mu_2 = \infty$.

(ii) Let $0 < \delta_1 < \delta_2 < \infty$ and $\mu \in (0, \infty]$. Then there exists a constant $C > 0$ depending only on n, μ, δ_1, δ_2 and Ω such that

$$[f]_{\Gamma_{\delta_2}} \leq C\left([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\right).$$

In particular, the norms $\|\cdot\|_{bmo^\mu_{\delta_1}}$ and $\|\cdot\|_{bmo^\mu_{\delta_2}}$ are equivalent. If Ω is bounded, one may take $\delta_2 = \infty$.

Proof. (i) This is [5, Theorem 4] which follows from [5, Theorem 3].

(ii) Since the space $L^1_{\mathrm{ul}}(\Gamma_\delta)$ is independent of the radius r_0 in its definition, without loss of generality, we may assume that $r_0 > \delta_1$. Let us firstly consider the case where the dimension $n > 1$. Let k be the smallest integer such that $2^{-k} < \frac{\delta_1}{\sqrt{n}}$ and $x \in \mathbf{R}^n$. Notice that

$$\int_{B_{r_0}(x)\cap\Gamma_{\delta_2}} |f|\, dy = \int_{B_{r_0}(x)\cap\Gamma_{\delta_1}} |f|\, dy + \int_{B_{r_0}(x)\cap(\Gamma_{\delta_2}\setminus\Gamma_{\delta_1})} |f|\, dy,$$

we can estimate $\|f\|_{L^1(B_{r_0}(x)\cap\Gamma_{\delta_1})}$ directly by $[f]_{\Gamma_{\delta_1}}$. Assume that $\Gamma_{\delta_2} \setminus \Gamma_{\delta_1} \neq \emptyset$. Let $D_k(x)$ be the set of dyadic cubes of side length 2^{-k} that intersect with $B_{r_0}(x) \cap (\Gamma_{\delta_2} \setminus \Gamma_{\delta_1})$. For a dyadic cube $Q_j \in D_k(x)$, we define B_j to be the ball which has radius $\frac{\sqrt{n}}{2} \cdot 2^{-k}$ and shares the same center with Q_j. Let $C_k(x) := \{B_j \,|\, Q_j \in D_k(x)\}$ and $\Sigma := \{x \in \Omega \,|\, d_\Omega(x) = \delta_1\}$.

For $Q_j \in D_k(x)$ that intersects Σ, we seek to estimate $\|f\|_{L^1(B_j)}$. Let c_j be a point on $\Sigma \cap Q_j$, we have that $B_{\delta_1}(c_j) \subset \Omega$. Indeed as otherwise, there exists $z \in B_{\delta_1}(c_j) \cap \Omega^c$. Then the line segment joining c_j and z must intersect Γ at some point, say z^*. Then $|z^* - c_j| \le |z - c_j| < \delta_1$. This contradicts the fact that $d_\Omega(c_j) = \delta_1$. For $y \in B_j$, $|y - c_j| < \sqrt{n} \cdot \ell(Q_j) = \sqrt{n} \cdot 2^{-k} < \delta_1$. So $B_j \subset B_{\delta_1}(c_j)$. Let $d_j \in \Gamma$ be a point such that $|c_j - d_j| = \delta_1$, then on the line segment joining c_j and d_j, we can find a point o_j such that $|o_j - d_j| = \frac{\sqrt{n}}{2} \cdot 2^{-k}$. For $y \in B_{\frac{\sqrt{n}}{2} \cdot 2^{-k}}(o_j)$, we have that $|d_\Omega(y) - d_\Omega(o_j)| \le |y - o_j|$. Hence $d_\Omega(y) \le d_\Omega(o_j) + |y - o_j| < \sqrt{n} \cdot 2^{-k} < \delta_1$. This means that $B_{\frac{\sqrt{n}}{2} \cdot 2^{-k}}(o_j) \subset \Gamma_{\delta_1}$. Moreover,

$$|c_j - y| \le |c_j - o_j| + |o_j - y| \le \delta_1 - \frac{\sqrt{n}}{2} \cdot 2^{-k} + \frac{\sqrt{n}}{2} \cdot 2^{-k} = \delta_1.$$

Thus $B_{\frac{\sqrt{n}}{2} \cdot 2^{-k}}(o_j) \subset B_{\delta_1}(c_j)$. Denote $B_{\frac{\sqrt{n}}{2} \cdot 2^{-k}}(o_j)$ by B_j^*. We have that

$$\int_{B_j} |f| \, dy \le \int_{B_{\delta_1}(c_j)} |f - f_{B_{\delta_1}(c_j)}| \, dy + \int_{B_{\delta_1}(c_j)} |f_{B_{\delta_1}(c_j)} - f_{B_j^*}| \, dy + \int_{B_{\delta_1}(c_j)} |f_{B_j^*}| \, dy.$$

Notice that

$$\begin{aligned}
\int_{B_{\delta_1}(c_j)} |f - f_{B_{\delta_1}(c_j)}| \, dy &\le C_n \cdot \delta_1^n \cdot [f]_{BMO^\mu}, \\
\int_{B_{\delta_1}(c_j)} |f_{B_{\delta_1}(c_j)} - f_{B_j^*}| \, dy &\le \frac{|B_{\delta_1}(c_j)|^2}{|B_j^*|} \cdot [f]_{BMO^\mu}, \\
\int_{B_{\delta_1}(c_j)} |f_{B_j^*}| \, dy &\le \frac{|B_{\delta_1}(c_j)|}{|B_j^*|} \cdot [f]_{\delta_1}.
\end{aligned}$$

Since $|B_{\delta_1}(c_j)| = C_n \cdot \delta_1^n$ and $\frac{|B_{\delta_1}(c_j)|}{|B_j^*|} = \frac{C_n \cdot \delta_1^n}{(\frac{\sqrt{n}}{2} \cdot 2^{-k})^n} \le \frac{C_n \cdot \delta_1^n}{(\frac{\delta_1}{4})^n} = C_n$, $\|f\|_{L^1(B_j)}$ is therefore controlled by $C_{\delta_1,n} \cdot \Big([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\Big)$.

Next we consider $Q_j' \in D_k(x)$ that does not intersect Σ. Suppose that $Q_j \in D_k(x)$ has a touching edge with Q_j'. There exists a ball B_i^j of radius $\frac{\sqrt{n}-1}{2} \cdot 2^{-k}$ which is contained in $B_j \cap B_j'$ where B_j, B_j' are the smallest balls that contain Q_j, Q_j' respectively. Similar to above, as $B_i^j \subset B_j$,

$$\begin{aligned}
\int_{B_j'} |f| \, dy &\le \int_{B_j'} |f - f_{B_j'}| \, dy + \int_{B_j'} |f_{B_j'} - f_{B_i^j}| \, dy + \int_{B_j'} |f_{B_i^j}| \, dy \\
&\le |B_j'| \cdot [f]_{BMO^\mu} + \frac{|B_j'|^2}{|B_i^j|} \cdot [f]_{BMO^\mu} + \frac{|B_j'|}{|B_i^j|} \cdot \int_{B_j} |f| \, dy.
\end{aligned}$$

Therefore if $\|f\|_{L^1(B_j)}$ is controlled by $C_{\delta_1,n} \cdot \Big([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\Big)$, $\|f\|_{L^1(B_j')}$ is also controlled by $C_{\delta_1,n} \cdot \Big([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\Big)$.

Since $B_{r_0}(x) \cap (\Gamma_{\delta_2} \backslash \Gamma_{\delta_1})$ is connected, we can estimate $\|f\|_{L^1(B_j)}$ for every $Q_j \in D_k(x)$ where B_j is the smallest ball that contains Q_j. For each $Q_j \in D_k(x)$, there exists $y \in$

$Q_j \cap B_{r_0}(x)$, so for any $z \in B_j$, $|z-x| \le |z-y| + |y-x| < \sqrt{n} \cdot 2^{-k} + r_0 < r_0 + \delta_1$. Thus $\bigcup_{Q_j \in D_k(x)} B_j \subset B_{r_0+\delta_1}(x)$. Let $N(D_k(x))$ be the number of cubes in $D_k(x)$, we have that

$$N(D_k(x)) \le \frac{|B_{r_0+\delta_1}(x)|}{2^{-kn}} \le C_n \cdot \left(\frac{r_0+\delta_1}{\delta_1}\right)^n.$$

Therefore,

$$\begin{aligned}\int_{B_{r_0}(x)\cap(\Gamma_{\delta_2}\backslash\Gamma_{\delta_1})} |f|\,dy &\le \sum_{B_j \in C_{r_0}(x)} \int_{B_j} |f|\,dy \\ &\le N(D_k(x)) \cdot C_{\delta_1,n} \cdot \left([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\right) \\ &\le C_{n,\delta_1,r_0} \cdot \left([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\right).\end{aligned}$$

For the case where the dimension $n=1$, we let k to be the smallest integer such that $2^{-k} < \frac{\delta_1}{2}$ and D_k to be the set of dyadic cubes of side length 2^{-k} that intersects $\Gamma_{\delta_2}\backslash\Gamma_{\delta_1}$. Notice that the region $\Gamma_{\delta_2}\backslash\Gamma_{\delta_1}$ is indeed a union of intervals. Without loss of generality, we can assume Ω to be $(0,\infty)$ and take $\mu=\infty$ by part (i) of this proposition. Thus in this case $\Gamma_{\delta_2}\backslash\Gamma_{\delta_1} = (\delta_1,\delta_2)$. For $Q_0 \in D_k$ such that $\delta_1 \in Q_0$,

$$\begin{aligned}\int_{Q_0} |f|\,dy \le \int_{2Q_0} |f|\,dy \le \int_{2Q_0} |f - f_{2Q_0}|\,dy + \int_{2Q_0} |f_{2Q_0} - f_{Q_0^*}|\,dy + \int_{2Q_0} |f_{Q_0^*}|\,dy \\ \le C \cdot \left([f]_{BMO^\infty} + [f]_{\Gamma_{\delta_1}}\right),\end{aligned}$$

where $Q_0^* = 2Q_0 \backslash \left(Q_0 \cup [\delta_1,\infty)\right)$ and $\ell(Q_0^*) = \frac{1}{2}\ell(Q_0) = 2^{-(k+1)}$.

We then put an ordering on the elements of D_k in the following way. For $j \in \mathbf{N}$, suppose that we have ordered intervals $Q_0, Q_1, ..., Q_{j-1}$, we pick $Q_j \in D_k\backslash\{Q_0, Q_1, ..., Q_{j-1}\}$ such that Q_j has a touching edge with Q_{j-1}. For $Q_j \in D_k$, similarly we have that

$$\begin{aligned}\int_{Q_j} |f|\,dy \le \int_{2Q_j} |f|\,dy \le \int_{2Q_j} |f - f_{2Q_j}|\,dy + \int_{2Q_j} |f_{2Q_j} - f_{Q_j^*}|\,dy + \int_{2Q_j} |f_{Q_j^*}|\,dy \\ \le C \cdot \left([f]_{BMO^\infty} + [f]_{\Gamma_{\delta_1}}\right),\end{aligned}$$

where $Q_j^* = 2Q_{j-1} \cap 2Q_j$ and $\ell(Q_j^*) = \ell(Q_j) = 2^{-k}$.

Let $N(D_k)$ be the number of elements of D_k, we have that

$$N(D_k) \le \frac{\delta_2 - \delta_1}{2^{-k}} + 2 \le \frac{4(\delta_2-\delta_1)}{\delta_1} + 2$$

and therefore

$$\int_{\Gamma_{\delta_2}\backslash\Gamma_{\delta_1}} |f|\,dy \le C_{\delta_2,\delta_1} \cdot \left([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\right).$$

The proof is now complete. □

By this observation, when we discuss the space bmo_δ^μ, there are only four types of spaces

$$bmo_\delta^\mu, \quad bmo_\delta^\infty, \quad bmo_\infty^\mu, \quad bmo_\infty^\infty$$

for finite $\mu, \delta > 0$. If Ω is bounded, it is clear that these four spaces agree with each other. However, if Ω is unbounded, these four spaces may be different because they requires different growth at infinity. Indeed, if $\Omega = (0, \infty)$

$$bmo^\infty_\infty \subsetneq bmo^\infty_\delta$$

since $\log(x+1) \in bmo^\infty_\delta$ while it does not belong to bmo^∞_∞. Moreover, since $x \in bmo^\mu_\delta$ but it does not belong to neither bmo^μ_∞ nor bmo^∞_δ, we see that

$$bmo^\infty_\delta \subsetneq bmo^\mu_\delta, \quad bmo^\mu_\infty \subsetneq bmo^\mu_\delta.$$

It is possible to prove that $bmo^\infty_\infty = bmo^\mu_\infty$. Indeed, $bmo^\infty_\infty(\Omega) \subset bmo^\mu_\infty(\Omega)$ is simply by the definition of the BMO seminorm. It is sufficient to show the contrary, i.e., $[f]_{BMO^\infty} \leq C \cdot ([f]_{BMO^\mu} + [f]_{\Gamma_\infty})$. Without loss of generality, in defining the seminorm $[\,\cdot\,]_{L^1_{\mathrm{ul}}(\Gamma_\infty)}$, we set the radius of the ball to be $\frac{\sqrt{n}}{2}$. For $B_r(x) \subset \Omega$ with $r < \mu$,

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}|\, dy \leq [f]_{BMO^\mu}.$$

For $B_r(x) \subset \Omega$ with $r \geq \mu$, if $r \leq \frac{\sqrt{n}}{2}$, then

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}|\, dy \leq \frac{2}{|B_r(x)|} \int_{B_{\frac{\sqrt{n}}{2}}(x) \cap \Omega} |f|\, dy \leq C_{\mu,n} \cdot [f]_{\Gamma_\infty}.$$

If $r > \frac{\sqrt{n}}{2}$, $B_r(x)$ is contained in the cube Q_r with center x and side length $2([r]+1)$, here $[r]$ is the largest integer less than or equal to r. By dividing each side length of Q_r equally into $2([r]+1)$ parts, we can divide the cube Q_r into $(2[r]+2)^n$ subcubes of side length 1. Let S_{Q_r} be the set of these $(2[r]+2)^n$ subcubes of Q_r. For $Q^i_r \in S_{Q_r}$, let B^i_r be the smallest ball that contains Q^i_r. Let $C_{Q_r} := \{B^i_r \mid Q^i_r \in S_{Q_r}\}$. We have that

$$\int_{B_r(x)} |f|\, dy \leq \sum_{i=1}^{(2[r]+2)^n} \int_{B^i_r \cap \Omega} |f|\, dy \leq (2[r]+2)^n \cdot [f]_{\Gamma_\infty}.$$

Since $r \geq \mu$,

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}|\, dy \leq \frac{2}{|B_r(x)|} \int_{B_r(x)} |f|\, dy \leq C_{\mu,n} \cdot [f]_{\Gamma_\infty}.$$

Therefore $bmo^\infty_\infty = bmo^\mu_\infty$ and thus $bmo^\mu_\infty \subsetneq bmo^\infty_\mu$.

We summarize these equivalences.

Theorem 3.2.2. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Then*

$$bmo^\infty_\infty(\Omega) = bmo^\mu_\infty(\Omega) \subset bmo^\infty_\delta(\Omega) \subset bmo^\mu_\delta(\Omega)$$

for finite $\delta, \mu > 0$. The inclusions can be strict when Ω is unbounded. If Ω is bounded, all four spaces are the same.

As a simple application of Proposition 3.2.1, we conclude that the space $BMO^{\mu,\nu}_b$ is included in bmo^μ_ν since $[f]_\nu \leq c[f]_{b^\nu}$ $(\nu < \infty)$ with $c > 0$ depending only on ν and n.

Theorem 3.2.3. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. For $\mu \in (0,\infty]$ the inclusion*

$$BMO_b^{\mu,\nu}(\Omega) \subset bmo_\nu^\mu(\Omega)$$

holds for $\nu \in (0,\infty)$.

Since b^ν-seminorm controls boundary growth stronger than L^1 sense, this inclusion is in general strict even when Ω is bounded. Here is a simple example when $\Omega = (0,1)$. The b^ν-seminorm of $f(x) = \log x$ is infinite but $\|f\|_{L^1(\Omega)}$ is finite.

We next discuss the space $vbmo_\delta^{\mu,\nu}$.

Remark 3.2.4. As proved in [5, Theorem 9], if Ω is a bounded Lipschitz domain, the space $BMO_b^{\mu,\nu}$ ($\mu,\nu \in (0,\infty]$) agrees with the Miyachi BMO space [14] defined by

$$\begin{aligned}
BMO^M(\Omega) &= \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; \|f\|_{BMO^M} < \infty \right\}, \\
\|f\|_{BMO^M} &:= [f]_{BMO^M} + [f]_{b^M}, \\
[f]_{BMO^M} &:= \sup\left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}|\, dy \;\middle|\; B_{2r}(x) \subset \Omega \right\}, \\
[f]_{b^M} &:= \sup\left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f|\, dy \;\middle|\; B_{2r}(x) \subset \Omega \text{ and } B_{5r}(x) \cap \Omega^c \neq \emptyset \right\}.
\end{aligned}$$

Proposition 3.2.5. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Let $0 < \nu_1 \le \nu_2 \le \delta \le \infty$. Then there exists a constant $c > 0$ depending only on n, ν_1, ν_2, δ such that*

$$[\nabla d_\Omega \cdot v]_{b^{\nu_2}} \le [\nabla d_\Omega \cdot v]_{b^{\nu_1}} + c[v]_{\Gamma_\delta}$$

for all $v \in L^1_{\mathrm{ul}}(\Gamma_\delta)$.

Proof. We may assume that $\nu_1 < \infty$. Let $Q_r(x)$ denote a cube centered at x with side length $2r$, Since $|\nabla d| = 1$ and $B_r(x) \subset Q_r(x)$, we see that

$$\begin{aligned}
[\nabla d_\Omega \cdot v]_{b^{\nu_2}} - [\nabla d_\Omega \cdot v]_{b^{\nu_1}} &\le \sup\left\{ \frac{1}{r^n} \int_{B_r(x)\cap\Omega} |\nabla d_\Omega \cdot v|\, dy \;\middle|\; x \in \partial\Omega,\ \nu_1 \le r < \nu_2 \right\} \\
&\le \sup\left\{ \frac{1}{r^n} \int_{Q_r(x)} |\tilde{v}|\, dy \;\middle|\; x \in \partial\Omega,\ \nu_1 \le r \le \nu_2 \right\}
\end{aligned}$$

where $\tilde{v}$ denotes the zero extension of v to $\mathbf{R}^n$. Since $\nu_2 \le \delta$ so that $Q_r(x) \cap \Omega \subset \Gamma_\delta$, we see that

$$\sup_{x\in\partial\Omega} \int_{Q_r(x)} |\tilde{v}|\, dy \le \|v\|_{L^1_{\mathrm{ul}}(\Gamma_\delta)} \quad \text{for} \quad \nu_1 \le r \le \nu_2$$

provided that ν_2 is finite by taking an equivalent norm of L^1_{ul}; in fact, we take $r_0 = \sqrt{n}\,\nu_2$. This implies that

$$[\nabla d_\Omega \cdot v]_{b^{\nu_2}} - [\nabla d_\Omega \cdot v]_{b^{\nu_1}} \le \frac{1}{\nu_1^n}[v]_{\Gamma_\delta}.$$

If $\nu_2 = \delta = \infty$, we may assume $r = 2^\ell \nu_1$. We divide $Q_r(x)$ into subcube Q_j, $j = 1,\dots,2^{\ell n}$ of side length $2\nu_1$. Then

$$\frac{1}{|Q_r(x)|} \int_{Q_r(x)} |\tilde{v}|\, dy \le \frac{1}{2^{\ell n}(2\nu_1)^n} \sum_{j=1}^{2^{\ell n}} \int_{Q_j} |\tilde{v}|\, dy \le \frac{2^{\ell n}}{2^{\ell n}(2\nu_1)^n} \|\tilde{v}\|_{L^1_{\mathrm{ul}}} \le \frac{1}{(2\nu_1)^n} \|\tilde{v}\|_{L^1_{\mathrm{ul}}}$$

where r_0 in L^1_{ul} norm is taken as $\sqrt{n}\,\nu_1$. We thus observe that

$$[\nabla d \cdot v]_{b^{\nu_2}} - [\nabla d \cdot v]_{b^{\nu_1}} \le c[v]_{\Gamma_\delta}.$$

□

By Proposition 3.2.1 and 3.2.5, we do not need to care about ν. More precisely,

Theorem 3.2.6. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Assume that $\mu \in (0,\infty]$ and that $\delta \in (0,\infty]$. Then norms $\|\cdot\|_{vbmo^{\mu,\nu_1}_\delta}$ and $\|\cdot\|_{vbmo^{\mu,\nu_2}_\delta}$ are equivalent provided that $0<\nu_1<\nu_2<\infty$. In the case $\delta=\infty$, we may take $\nu_2=\infty$.*

In general, different from Theorem 3.2.3, the space $vBMO^{\mu,\nu}$ may not be included in bmo^μ_ν even for finite μ by the decomposition (3.2.1) and the fact that BMO^μ is not contained in $L^1_{\mathrm{ul}}(\Gamma_\delta)$ for any δ. However, if each connected component of the boundary Γ of Ω has a curved part, we are able to compare these spaces.

Definition 3.2.7. Let Ω be a uniformly C^1 domain in $\mathbf{R}^n$ and Γ^0 be a connected component of the boundary Γ of Ω. We say that Γ^0 has a *fully curved part* if the set of all normals of Γ^0 spans $\mathbf{R}^n$. In other words, the set $\left\{\mathbf{n}(x) \in \mathbf{R}^n \mid x \in \Gamma^0\right\}$ contains n linearly independent vectors, when $\mathbf{n}$ denotes the unit exterior normal of Γ^0.

We introduce $b^\nu(\Gamma^0)$-seminorm for convenience. Let us decompose Γ into its connected component Γ^j so that $\Gamma = \bigcup_{j=1}^m \Gamma^j$. We set

$$[f]_{b^\nu(\Gamma^j)} := \sup\left\{ r^{-n} \int_{\Omega\cap B_r(x)} \left|f(y)\right| dy \;\middle|\; x \in \Gamma^j,\ 0<r<\nu \right\}.$$

Evidently, $[f]_{b^\nu} = \max_{1\le j\le m}[f]_{b^\nu(\Gamma^j)}$ at least for small $\nu>0$.

The existence of a fully curved part implies "non-degeneracy" of the seminorm $[\nabla d\cdot f]_{b^\nu}$.

Lemma 3.2.8. *Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$. Let Γ^j be a connected component of the boundary Γ of Ω. If $c \in \mathbf{R}^n$ satisfies*

$$[\nabla d_\Omega \cdot c]_{b^\nu(\Gamma^j)} = 0,$$

for some $\nu>0$, then $c=0$ provided that Γ^j has a fully curved part.

Proof. If Ω is uniformly C^2, then d_Ω is C^2 in $(\Gamma^j)_\delta$ for sufficiently small $\delta>0$. Since $-\nabla d_\Omega(x)$ at $x\in\Gamma^j$ equals $\mathbf{n}(x)$, we see that

$$\frac{1}{r^n}\int_{B_r(x)\cap\Omega} \nabla d_\Omega(y)\,dy \to c_0\mathbf{n}(x) \quad \text{as} \quad r\to 0$$

with scalar constant c_0. Our assumption now implies that $c\cdot\mathbf{n}(x)=0$ for $x\in\Gamma^j$. If Γ^j has a curved part, then by definition this implies that $c=0$. □

Here is a few comments on examples of such domains. All connected components of the boundary of a bounded domain, exterior domain has a fully curved part. A perturbed half space

$$\mathbf{R}^n_\psi = \left\{(x',x_n)\in\mathbf{R}^n \;\middle|\; x_n>\psi(x'),\ x'=(x_1,\ldots,x_{n-1})\in\mathbf{R}^{n-1}\right\}$$

with $\psi \in C^1_c(\mathbf{R}^{n-1})$, $\psi \not\equiv 0$ is another example. However, a half-space $\mathbf{R}^n_+$, cylindrical domain $G\times\mathbf{R}^{n-k}$ with $k\ge 1$, $G\subset\mathbf{R}^k$ does not have a boundary having a fully curved part. Our goal is to show that for a domain with boundary components having a fully curved part the space $vBMO^{\mu,\nu}$ is comparable with $vbmo^{\mu,\nu}_\delta$ space if the boundary is compact.

Theorem 3.2.9. *Let Ω be a C^2 bounded or exterior domain in $\mathbf{R}^n$ so that each component of the boundary has a fully curved part. For $\mu \in (0,\infty]$ and $\nu \in (0,R_*)$ the identity holds:*

$$vBMO^{\mu,\nu}(\Omega) = vbmo^{\mu,\nu}_{\nu}.$$

Proof. Let Γ^j be a j-th connected component of the boundary $\Gamma = \partial\Omega$ such that $\Gamma = \bigcup_{j=1}^m \Gamma^j$. Since Γ^j is C^2 and compact, there is a number $r_0 \in (0,\nu/2)$ such that

$$(\Gamma^j)_\nu = \bigcup_{x\in\Lambda} \operatorname{int} B_{r_0}(x), \quad \Lambda \subset (\Gamma^j)_\nu,$$

where Γ^j is a connected component of Γ and $(\Gamma^j)_\nu$ denotes its ν-neighborhood. The next lemma shows that

$$vBMO^{\nu,\nu}(\Omega) \subset L^1_{\mathrm{ul}}(\Gamma_\nu)$$

which yields the desired result. Note that we may assume $\nu \le \mu$ by Proposition 3.2.1. □

Lemma 3.2.10. *Under the same assumption of Theorem 3.2.9 with $\mu \le \nu$ assume that $r_0 < \nu/2 < R_*/2$ is taken so that*

$$(\Gamma^i)_\nu = \bigcup_{x\in\Lambda} \operatorname{int} B_{r_0}(x)$$

with some $\Lambda \subset (\Gamma^j)_\nu$. Then there exists $C > 0$ depending only on r_0, n, Γ^j, ν such that

$$\sup_{x\in\Lambda} \frac{1}{|B_{r_0}(x)|} \int_{B_{r_0}(x)} |f(y)|\,dy \le C\Big([f]_{BMO^\mu((\Gamma^j)_\nu)} + [\nabla d_\Omega \cdot f]_{b^\nu(\Gamma^j)}\Big).$$

Proof. We shall suppress r_0 dependence since it is fixed. We shall prove the average $f_{B(x)} = \dfrac{1}{|B(x)|}\displaystyle\int_{B(x)} f\,dy$ has an estimate

$$\sup_{x\in\Lambda} \left|f_{B(x)}\right| \le C\Big([f]_{BMO^\mu((\Gamma^j)_\nu)} + [\nabla d \cdot f]_{b^\nu(\Gamma^j)}\Big). \tag{3.2.2}$$

If this is proved, applying the triangle inequality

$$(|f|)_{B(x)} \le \frac{1}{|B(x)|} \int_{B(x)} \left|f - f_{B(x)}\right| dy + \left|f_{B(x)}\right|$$

yields the desired result.

We shall prove the key inequality (3.2.2) by contradiction argument. Assume the inequality (3.2.2) were false. Then, there would exist a sequence $\{f^k\}_{k=1}^\infty$ such that

$$1 = \sup_{x\in\Lambda} \left|f^k_{B(x)}\right| \ge k\left(\left[f^k\right]_{BMO^\mu} + \left[\nabla d_\Omega \cdot f^k\right]_{b^\nu}\right).$$

Here we suppress $(\Gamma^i)_\nu$ and Γ^j in the right-hand side. Since

$$\sup_{x\in\Lambda} \left|c^k(x)\right| = 1 \quad \text{with} \quad c^k(x) = f^k_{B(x)} \in \mathbf{R}^n,$$

there is a sequence $\{x_k\}_{k=1}^\infty$ in Λ with the property

$$1 \ge \left|c^k(x_k)\right| \ge 1/2.$$

By taking a subsequence, we may assume that x_k converges to some $\hat{x} \in (\Gamma^j)_\nu$ since Γ^j is compact and $d\left(x_k, \partial(\Gamma^j)_\nu\right) \geq r_0$, where $d(x_k, A)$ denotes the distance from a point x_k to a set A. Since Γ^j is connected, there is an increasing sequence $\{K_\ell\}_{\ell=1}^\infty$ of connected compact sets in $(\Gamma^j)_\nu$ such that $\operatorname{int} K_\ell \ni \hat{x}$ for $\ell \geq 1$ and $(\Gamma^j)_\nu = \bigcup_{\ell=1}^\infty K_\ell$. By compactness, there is a finite subset Λ_ℓ of Λ with the property that

$$K_\ell \subset \bigcup_{x \in \Lambda_\ell} \operatorname{int} B(x), \quad \Lambda_\ell \subset \Lambda_{\ell+1}$$

and the right-hand side is connected. By taking a further subsequence, we may assume that $c^k(x) \to c(x)$ for $x \in \Lambda_\ell$. However, since $\left[f^k\right]_{BMO^\mu} \to 0$ so that

$$\int_{B(x)} \left| f^k - c^k \right| dx \to 0$$

as $k \to \infty$, we see that $c(x) = c(y)$ if $\operatorname{int} B(x) \cap \operatorname{int} B(y) \neq \emptyset$. Since

$$\bigcup_{x \in \Lambda_\ell} \operatorname{int} B(x)$$

is connected, $c(x)$ is independent of $x \in \Lambda_\ell$, say $c = c_\ell$. By taking a further subsequence of $\left\{f^k\right\}$, we may assume that $c^k(x) \to c_\ell$ in Λ_ℓ. By a diagonal argument, there is a subsequence of $\left\{f^k\right\}$ such that

$$c^k(x) \to c \quad \text{for} \quad x \in \bigcup_{\ell=1}^\infty \Lambda_\ell =: \Lambda_\infty \subset \Lambda.$$

We thus observe that

$$\int_{B(x)} \left| f^k(y) - c \right| dy \to 0 \quad \text{for} \quad x \in \Lambda_\infty \quad \text{as} \quad k \to \infty.$$

If we take $B(x)$ such that $\hat{x} \in \operatorname{int} B(x)$, c should not be equal to zero since $\left|c^k(x_k)\right| \geq 1/2$ and $x_k \to \hat{x}$ as $k \to \infty$. We now invoke the property that

$$\left[\nabla d_\Omega \cdot f^k\right]_{b^\nu} \to 0.$$

Since

$$(\Gamma^j)_\nu = \bigcup_{x \in \Lambda_\infty} B(x),$$

we observe that $f^k \to c$ in $L^1_{\mathrm{loc}}\left((\Gamma^j)_\nu\right)$. By taking a subsequence we may assume that $f^k(x) \to c$ for a.e. $x \in (\Gamma^j)_\nu$ so that $\nabla d_\Omega \cdot f^k \to \nabla d_\Omega \cdot c$, a.e. By lower semicontinuity of integrals (Fatou's lemma) and supremum operation, the seminorm b^ν is lower semicontinuous under this convergence. We thus conclude that

$$[\nabla d_\Omega \cdot c]_{b^\nu} \leq \varliminf_{k \to \infty} \left[\nabla d_\Omega \cdot f^k\right]_{b^\nu} = 0.$$

By Lemma 3.2.8, this c must be zero which leads to a contradiction. We thus proved the key estimate (3.2.2). This completes the proof of Lemma 3.2.10. □

3.3 A variant of Jones' extension theorem

Different from L^∞ functions, it is in general impossible to extend BMO function by setting zero outside the domain. Indeed, the zero-extension of $\log \min(x,1) \in bmo^\infty_\infty(\mathbf{R}^1_+)$ does not belong to $BMO^\infty(\mathbf{R})$. The goal in this section is to give a linear, extention operator of BMO type function so that the support of extended function is contained in an ε-neighborhood of the original domain, of a function.

For this purpose we recall an extension given by P. W. Jones [15]. Since we modify the way of construction, we will give a sketch of this construction. We first recall a dyadic Whitney decomposition of a set A in $\mathbf{R}^n$. Let $\mathcal{A} = \{Q_j\}_{j\in\mathbf{N}}$ be a set of dyadic closed cubes with side length $\ell(Q_j)$ contained in A satisfying following four conditions.

(i) $A = \cup_j Q_j$,

(ii) $\operatorname{int} Q_j \cap \operatorname{int} Q_k = \emptyset$ if $j = k$,

(iii) $\sqrt{n} \leq d(Q_j, \mathbf{R}^n \backslash A)/\ell(Q_j) \leq 4\sqrt{n}$ for all $j \in \mathbf{N}$,

(iv) $1/4 \leq \ell(Q_k)/\ell(Q_j) \leq 4$ if $Q_j \cap Q_k \neq \emptyset$.

We say that $\mathcal{A}$ is called a dyadic Whitney decomposition of A. Such a decomposition exists for any open sets; see [18, Chapter VI, Theorem 1]. Here $d(B,C)$ for sets B, C in $\mathbf{R}^n$ is defined as

$$d(B,C) = \inf\left\{|x-y| \;\middle|\; x \in B,\ y \in C\right\}.$$

If B is a point x, we write $d(x,C)$ instead of $d(\{x\},C)$.

There are at least two important distance functions on $\mathcal{A}$. For $Q_j, Q_k \in \mathcal{A}$, a family $\{Q(\ell)\}_{\ell=0}^m \subset \mathcal{A}$ is called a Whitney chain of length m if $Q(0) = Q_j$ and $Q(m) = Q_k$ such that $Q(\ell) \cap Q(\ell+1) \neq \emptyset$ for ℓ with $0 \leq \ell \leq m-1$. Then the length of the shortest Whitney chain connecting Q_j and Q_k gives a distance on $\mathcal{A}$, which is denoted by $d_1(Q_j, Q_k)$. The second distance for $Q_j, Q_k \in \mathcal{A}$ is defined as

$$d_2(Q_j, Q_k) := \log\left|\frac{\ell(Q_j)}{\ell(Q_k)}\right| + \log\left|\frac{\ell(Q_j, Q_k)}{\ell(Q_j) + \ell(Q_k)} + 1\right|.$$

Note that d_1 and d_2 are invariant under dilation as well as translation and rotation. P. W. Jones [15] gives a necessary and sufficient condition for a domain such that there exists a linear extension operator. A domain Ω is called a uniform domain if there exist constants $a, b > 0$ such that for all $x, y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ of length $s(\gamma) \leq a|x-y|$ with $\min\{s(\gamma(x,z)),\ s(\gamma(y,z))\} \leq bd(z, \partial\Omega)$, where $\gamma(x,z)$ denotes the part of γ between x and z on the curve; see e.g. [8]. It is equivalent to saying that there is a constant $K > 0$ such that

$$d_1(Q_j, Q_k) \leq K d_2(Q_j, Q_k) \tag{3.3.1}$$

for all $Q_j, Q_k \in \mathcal{A}$ and some dyadic Whitney decomposition $\mathcal{A}$ of Ω.

Theorem 3.3.1. *Let $A \subset \mathbf{R}^n$ be a uniform domain. Then there is a constant $C(K)$ depending only on K in* (3.3.1) *such that for each $f \in BMO^\infty(A)$ there is an extension $\overline{f} \in BMO^\infty(\mathbf{R}^n)$ satisfying*

$$[\overline{f}]_{BMO^\infty(\mathbf{R}^n)} \leq C(K)[f]_{BMO^\infty(A)}.$$

The operator $f \mapsto \overline{f}$ is a bounded linear operator. Conversely, if there exists such an extension, then A is a uniform domain.

A bounded Lipschitz domain is a typical example of a uniform domain. The constant K in (3.3.1) depends only on the Lipschitz regularity of the domain. A Lipschitz half space $\mathbf{R}^n_\psi$ is another example of a uniform domain; here ψ is a Lipschitz function on $\mathbf{R}^{n-1}$.

We next note that if we modify the construction by P. W. Jones, the support of the extension $\overline{f}$ is contained in an ε-neighborhood of $\overline{\Omega}$ if f is also in L^1_{ul} type space.

Theorem 3.3.2. *Let $\Omega \subset \mathbf{R}^n$ be a uniform domain. For each $\varepsilon > 0$ there is a constant $C = C(K, \varepsilon)$ with K in (3.3.1) such that for each $f \in bmo^\infty_\infty(\Omega)$ there is an extension $\overline{f} \in bmo^\infty_\infty(\Omega_{2\varepsilon})$ such that*

$$[\overline{f}]_{bmo^\infty_\infty(\Omega_{2\varepsilon})} \leq C[f]_{bmo^\infty_\infty(\Omega)}$$

and $\operatorname{supp} \overline{f} \subset \overline{\Omega_\varepsilon}$, *where*

$$\Omega_\varepsilon := \left\{ x \in \mathbf{R}^n \mid d(x, \overline{\Omega}) < \varepsilon \right\}.$$

The operator $f \mapsto \overline{f}$ is a bounded linear operator.

This can be proved almost along the same way as in [15]. We shall give an explicit proof.

Proof. Let k_ε be the smallest integer such that $2^{-k_\varepsilon} < \frac{\varepsilon}{5\sqrt{n}}$. So $2^{-k_\varepsilon} \geq \frac{\varepsilon}{10\sqrt{n}}$. Let $E = \{Q_j\}$ be the Whitney decomposition of Ω and $E' = \{Q'_j\}$ be the Whitney decomposition of Ω^c. Let E_* be the set of Whitney cubes in E whose side length is strictly greater than 2^{-k_ε}. For each $Q_m \in E_*$, we define a function g_m on Ω by

$$g_m(x) := \begin{cases} f_{Q_m}, & \text{if } x \in Q_m \\ 0, & \text{else} \end{cases}$$

and we further define a function g on Ω by

$$g := \sum_{Q_m \in E_*} g_m.$$

Here $f_{Q_m} = \frac{1}{|Q_m|} \int_{Q_m} f(y)dy$ for each $Q_m \in E_*$. Let $\widetilde{g}$ be the zero extension of g from Ω to $\mathbf{R}^n$.

Without loss of generality, we assume that the radius r_0 of the ball equals 1 in defining the space $L^1_{\mathrm{ul}}(\Omega)$. Notice that

$$\|g_m\|_{L^\infty(\Omega)} \leq \frac{1}{|Q_m|} \cdot \int_{Q_m} |f| \, dy.$$

Let k_0 be the smallest integer such that $2^{-k_0} < \frac{2}{\sqrt{n}}$. If $\ell(Q_m) \leq 2^{-k_0}$, then $\|f\|_{L^1(Q_m)} \leq [f]_{\Gamma_\infty}$. In this case, as $\ell(Q_m) > 2^{-k_\varepsilon}$,

$$\|g_m\|_{L^\infty(\Omega)} \leq \frac{1}{|Q_m|} \cdot \int_{Q_m} |f| \, dy \leq (\frac{10\sqrt{n}}{\varepsilon})^n \cdot [f]_{\Gamma_\infty}.$$

If $\ell(Q_m) > 2^{-k_0}$, we divide Q_m into $(\frac{\ell(Q_m)}{2^{-k_0}})^n$ small subcubes of side length 2^{-k_0}. Hence,

$$\int_{Q_m} |f| \, dy = \sum_{i=1}^{(\ell(Q_m)/2^{-k_0})^n} \int_{Q^i_m} |f| \, dy \leq (\frac{\ell(Q_m)}{2^{-k_0}})^n \cdot [f]_{\Gamma_\infty} \leq |Q_m| \cdot n^{\frac{n}{2}} \cdot [f]_{\Gamma_\infty},$$

in this case $\|g_m\|_{L^\infty(\Omega)} \leq n^{\frac{n}{2}} \cdot [f]_{\Gamma_\infty}$. Therefore,

$$\|g\|_{L^\infty(\Omega)} \leq C_{n,\varepsilon} \cdot [f]_{\Gamma_\infty}$$

and we deduce that $g \in bmo^\infty_\infty(\Omega)$ as $L^\infty(\Omega) \subset bmo^\infty_\infty(\Omega)$.

Let $f^* := f - g \in bmo^\infty_\infty(\Omega)$. We do Jones extension to f^*. If Ω is unbounded, for each $Q_j^{'} \in E^{'}$, we find a nearset $Q_j \in E$ satisfying $\ell(Q_j) \geq \ell(Q_j^{'})$. We define that $\widetilde{f^*} = f^*$ on Ω and $\widetilde{f^*}(x) = f^*_{Q_j}$ for $x \in Q_j^{'}$. If Ω is bounded, we pick $Q_0 \in E$ such that $\ell(Q_0) = \sup_{Q_j \in E} \ell(Q_j)$. We define that $\widetilde{f^*} = f^*$ on Ω, $\widetilde{f^*}(x) = f^*_{Q_j}$ for $x \in Q_j^{'}$ where $\ell(Q_j^{'}) \leq \ell(Q_0)$ and $\widetilde{f^*}(x) = f^*_{Q_0}$ for $x \in Q_j^{'}$ where $\ell(Q_j) > \ell(Q_0)$. By Jones [?PJ], $\widetilde{f^*} \in BMO$ and $[\widetilde{f^*}]_{BMO} \leq C_K \cdot [f^*]_{BMO^\infty(\Omega)}$. By this extension, for $\widetilde{f^*}(x) \neq 0$, either $x \in \Omega$ or $x \in Q_j^{'}$ such that $\ell(Q_j^{'}) \leq 2^{-k_\varepsilon}$. Since $d(Q_j^{'}, \Omega) \leq 4\sqrt{n} \cdot \ell(Q_j^{'})$, pick $x \in \overline{Q_j^{'}}$ and $z \in \Gamma$ such that $|x - z| = d(Q_j^{'}, \Omega)$. For any $y \in Q_j^{'}$, $|y - z| \leq |y - x| + |x - z| \leq 5\sqrt{n} \cdot \ell(Q_j^{'})$. So $\operatorname{int} Q_j^{'} \subset B_{5\sqrt{n}\cdot\ell(Q_j^{'})}(z)$ for some $z \in \Gamma$. Since $5\sqrt{n} \cdot \ell(Q_j^{'}) \leq 5\sqrt{n} \cdot 2^{-k_\varepsilon} < \varepsilon$, $\operatorname{int} Q_j^{'} \subset \Omega_\varepsilon$. Let $\widetilde{f} := \widetilde{f^*} + \widetilde{g}$ and $\overline{f} = \widetilde{f}|_{\Omega_{2\varepsilon}}$, we have that $\operatorname{supp} \overline{f} \subset \overline{\Omega_\varepsilon}$ and by previous calculation,

$$\begin{aligned}
[\overline{f}]_{BMO^\infty(\Omega_{2\varepsilon})} &\leq [\widetilde{f}]_{BMO} \leq [\widetilde{f^*}]_{BMO} + [\widetilde{g}]_{BMO} \leq C_K \cdot [f^*]_{BMO^\infty(\Omega)} + 2\|g\|_\infty \\
&\leq C_{K,n,\varepsilon} \cdot ([f]_{BMO^\infty(\Omega)} + [f]_{\Gamma_\infty}).
\end{aligned}$$

Let $B(x)$ denotes the ball of radius 1 centered at x and $\Gamma^\varepsilon := \{x \in \Omega^c \,|\, d_\Omega(x) < \varepsilon\}$. For $B(x) \cap \Omega_\varepsilon \neq \emptyset$,

$$\int_{B(x)\cap\Omega_\varepsilon} |\overline{f}|\, dy = \int_{B(x)\cap\Omega} |f|\, dy + \int_{B(x)\cap\Gamma^\varepsilon} |\overline{f}|\, dy.$$

The first integral on the right hand side is directly estimated by $[f]_\infty$, so we only need to consider the second integral. Let $Q_*^{'}$ be a largest Whitney cube in $E^{'}$ that intersects $B(x) \cap \Gamma^\varepsilon$. For $Q_j^{'} \in E^{'}$, [15, Lemma 2.10] says that if $Q_j \in E$ is a nearest Whitney cube satisfying $\ell(Q_j) \geq \ell(Q_j^{'})$, then $d(Q_j, Q_j^{'}) \leq 65K^2 \cdot \ell(Q_j^{'})$. Consider $Q_j^{'} \in E^{'}$ such that $Q_j^{'} \cap B(x) \cap \Gamma^\varepsilon \neq \emptyset$, let $x_j \in Q_j$ where Q_j is a nearest Whitney cube satisfying $\ell(Q_j) \geq \ell(Q_j^{'})$, let $x_j^{'} \in Q_j^{'} \cap B(x) \cap \Gamma^\varepsilon$ and $x_*^{'} \in Q_*^{'} \cap B(x) \cap \Gamma^\varepsilon$. By choosing K large such that $K^2 \geq 2\sqrt{n}$, we have that

$$|x_j - x_*^{'}| \leq |x_*^{'} - x_j^{'}| + |x_j^{'} - x_j| \leq 2 + 2\sqrt{n} \cdot \ell(Q_j^{'}) + 65K^2 \cdot \ell(Q_j^{'}) \leq 2 + 66K^2 \cdot \ell(Q_j).$$

Since $\ell(Q_j) \leq 2\ell(Q_j^{'}) \leq 2\ell(Q_*^{'}) \leq 2\ell(Q_*)$ where $Q_* \in E$ is a nearest cube satisfying $\ell(Q_*) \geq \ell(Q_*^{'})$, $|x_j - x_*^{'}| \leq 2 + 132K^2 \cdot \ell(Q_*)$.

If $B(x) \cap \Gamma \neq \emptyset$, then $\sqrt{n} \cdot \ell(Q_*^{'}) \leq d(Q_*^{'}, \Omega) \leq 2$. Hence $\ell(Q_*) \leq 2\ell(Q_*^{'}) \leq \frac{4}{\sqrt{n}}$, for any $x_j \in Q_j$, $|x_j - x_*^{'}| < 2 + 133K^2 \cdot \frac{4}{\sqrt{n}}$. Consider the cube $\widetilde{Q_*^{'}}$ with center $x_*^{'}$ and side length $4 + \frac{1064K^2}{\sqrt{n}}$. For each $Q_j^{'} \in E^{'}$ such that $Q_j^{'} \cap B(x) \cap \Gamma^\varepsilon \neq \emptyset$, the corresponding nearest $Q_j \in E$ such that $\ell(Q_j) \geq \ell(Q_j^{'})$ we choose to define $\widetilde{f^*}$ is contained in $\widetilde{Q_*^{'}}$, i.e., $Q_j \subset \widetilde{Q_*^{'}}$. Hence,

$$\int_{B(x)\cap\Gamma^\varepsilon} |\overline{f}|\, dy = \sum_{\substack{Q_j^{'} \in E^{'},\\ Q_j^{'} \cap B(x)\cap\Gamma^\varepsilon \neq \emptyset}} \int_{Q_j^{'} \cap B(x)\cap\Gamma^\varepsilon} |f^*_{Q_j}|\, dy \leq \int_{\widetilde{Q_*^{'}} \cap \Omega} |f^*|\, dy.$$

Let p be the largest integer such that $2^{-p} > 4 + \frac{1064K^2}{\sqrt{n}}$, so $2^{-p} \leq 8 + \frac{2128K^2}{\sqrt{n}}$. Let $\widetilde{Q'_*}$ be contained in a larger cube $\widetilde{Q}$ where $\widetilde{Q}$ has center x'_* and side length 2^{-p}. We can divide $\widetilde{Q}$ into $(\frac{2^{-p}}{2^{-k_0}})^n$ subcubes of side length 2^{-k_0}, thus

$$\int_{\widetilde{Q'_*}\cap\Omega} |f^*|\, dy \leq \sum_{i=1}^{(2^{-p}/2^{-k_0})^n} \int_{\widetilde{Q_i}\cap\Omega} |f^*|\, dy \leq (\frac{2^{-p}}{2^{-k_0}})^n \cdot [f^*]_{\Gamma_\infty} \leq C_{K,n} \cdot [f^*]_{\Gamma_\infty}.$$

If $B(x) \cap \Gamma = \emptyset$, i.e., $B(x) \subset \overline{\Omega}^c$. Let $E'_1 := \{Q'_j \in E' \mid Q'_j \cap B(x) \neq \emptyset\}$. Let $\ell_m := \inf_{Q'_j \in E'_1} \ell(Q'_j)$ and Q'_* be a largest $Q'_j \in E'_1$. If $\ell_m = 0$, then there exists $z \in \Gamma \cap \partial B(x)$. In this case, $\sqrt{n} \cdot \ell(Q'_*) \leq d(Q'_*, \Omega) \leq 2$. Therefore same argument as in the case where $B(x)\cap\Gamma \neq \emptyset$ gives that $\|\overline{f}\|_{L^1(B(x)\cap\Gamma^\varepsilon)} \leq C_{K,n}\cdot[f^*]_\infty$. If $0 < \ell_m \leq 2$, then pick $Q'_m \in E'_1$ such that $\ell(Q'_m) = \ell_m$. Since $\sqrt{n}\cdot\ell(Q'_*) \leq d(Q'_*, \Omega) \leq 2+\sqrt{n}\cdot\ell(Q'_m)+d(Q'_m, \Omega) \leq 2+10\sqrt{n}$, we have that $\ell(Q_*) \leq \frac{4}{\sqrt{n}}+20$. Hence $|x_j - x'_*| \leq 2+133K^2\cdot(\frac{4}{\sqrt{n}}+20)$. Following the argument as in the case where $B(x) \cap \Gamma \neq \emptyset$, we can deduce that $\|\overline{f}\|_{L^1(B(x)\cap\Gamma^\varepsilon)} \leq C_{K,n} \cdot [f^*]_{\Gamma_\infty}$. If $\ell_m > 2$, then $B(x)$ intersects at most 2^n Whitney cubes in E'. Without loss of generality, assume that E'_1 has 2^n elements. Then

$$\int_{B(x)\cap\Gamma^\varepsilon} |\overline{f}|\, dy \leq \sum_{Q'_i \in E'_1} \int_{B(x)\cap Q'_i} |f^*_{Q_i}|\, dy \leq \sum_{Q'_i \in E'_1} \frac{|B(x)\cap Q'_i|}{|Q_i|} \cdot \int_{Q_i} |f^*|\, dy.$$

Divide Q_i into $\left(\frac{\ell(Q_i)}{2^{-k_0}}\right)^n$ subcubes of side length 2^{-k_0}, we have that

$$\int_{Q_i} |f^*|\, dy \leq \left(\frac{\ell(Q_i)}{2^{-k_0}}\right)^n \cdot [f^*]_{\Gamma_\infty} \leq |Q_i| \cdot n^{\frac{n}{2}} \cdot [f^*]_{\Gamma_\infty}.$$

Therefore,

$$\int_{B(x)\cap\Gamma^\varepsilon} |\overline{f}|\, dy \leq \Big(\sum_{Q'_i \in E'_1} |B(x)\cap Q'_i| \Big) \cdot n^{\frac{n}{2}} \cdot [f^*]_{\Gamma_\infty} \leq C_n \cdot [f^*]_{\Gamma_\infty}.$$

Since $[f^*]_{\Gamma_\infty} \leq [f]_{\Gamma_\infty} + [g]_{\Gamma_\infty}$ and $[g]_{\Gamma_\infty}$ is estimated by $C_{n,\varepsilon} \cdot [f]_{\Gamma_\infty}$, we are done. □

As an application we give an estimate for the product of a Hölder function and a function in bmo^∞_∞. We first recall properties of point multipliers. It is known that for a local hardy space $h^1 = F^0_{1,2}$ [16, Theorem 3.18], there is a constant C such that

$$\|\varphi g\|_{F^0_{1,2}} \leq C\|\varphi\|_{C^\gamma}\|g\|_{F^0_{1,2}} \quad g \in F^0_{1,2} \tag{3.3.2}$$

for $\varphi \in C^\gamma(\mathbf{R}^n)$, $\gamma \in (0,1)$, where

$$\|\varphi\|_{C^\gamma} = \sup_{x\in\mathbf{R}^n} |\varphi(x)| + \sup_{\substack{x,y\in\mathbf{R}^n \\ x\neq y}} |\varphi(x) - \varphi(y)| \,/\, |x-y|^\gamma;$$

see e.g. [16, Remark 4.4]. Since

$$bmo = BMO^\infty(\mathbf{R}^n) \cap L^1_{\mathrm{ul}}(\mathbf{R}^n)$$

equals to $F^0_{\infty,2}$ [16, Theorem 3.26], it is a dual space of $h^1 = F^0_{1,2}$ [16, Theorem 3.22]. Thus

$$\|\varphi f\|_{bmo} \leq C\|\varphi\|_{C^\gamma}\|f\|_{bmo}. \tag{3.3.3}$$

Theorem 3.3.3. *Let $\Omega \subset \mathbf{R}^n$ be a uniform domain. Let $\varphi \in C^\gamma(\Omega)$, $\gamma \in (0,1)$. For each $f \in bmo^\infty_\infty(\Omega)$, the function $\varphi f \in bmo^\infty_\infty(\Omega)$ satisfies*

$$\|\varphi f\|_{bmo^\infty_\infty(\Omega)} \leq C \|\varphi\|_{C^\gamma(\Omega)} \|f\|_{bmo^\infty_\infty(\Omega)}$$

with C independent of φ and f.

Proof. By [13], there exists $\overline{\varphi} \in C^\gamma(\mathbf{R}^n)$ such that $\overline{\varphi}\,|_\Omega = \varphi$ and

$$\|\overline{\varphi}\|_{C^\gamma(\mathbf{R}^n)} \leq \|\varphi\|_{C^\gamma(\Omega)}.$$

For our current purpose it suffices to set $\overline{\varphi} = \max\{\min\{\varphi_*, \|\varphi\|_\infty\}, -\|\varphi\|_\infty\}$ with

$$\varphi_*(x) = \inf_{y \in \Omega} \left\{ \varphi(y) + [\varphi]_{C^\gamma} \cdot |x-y|^\gamma \right\},$$

where $\|\varphi\|_{C^\gamma(\Omega)} = \|\varphi\|_{L^\infty(\Omega)} + [\varphi]_{C^\gamma(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} |\varphi(x)|$ and $[\varphi]_{C^\gamma(\Omega)} = \sup_{x,y\in\Omega} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\gamma}$; we often suppress Ω. By definition $\varphi_*(x) \leq \varphi(x)$. Moreover, since $\varphi(x) \leq \varphi(y) + [\varphi]_{C^\gamma} \cdot |x-y|^\gamma$ for $x, y \in \Omega$, we see that $\varphi(x) \leq \varphi_*(x)$ which implies $\varphi = \varphi_*$ on Ω. For any $x \in \mathbf{R}^n$ and $\varepsilon > 0$ there is $y_\varepsilon \in \Omega$ such that

$$\varphi(y_\varepsilon) + [\varphi]_{C^\gamma} \cdot |x - y_\varepsilon|^\gamma \leq \varphi_*(x) + \varepsilon.$$

For $x_1 \in \mathbf{R}^n$ we observe that

$$\varphi_*(x_1) - \varphi_*(x) \leq \varphi(y_\varepsilon) + [\varphi]_{C^\gamma} \cdot |x_1 - y_\varepsilon|^\gamma - \{\varphi(y_\varepsilon) + [\varphi]_{C^\gamma} \cdot |x - y_\varepsilon|^\gamma\} + \varepsilon \leq [\varphi]_{C^\gamma} \cdot |x - x_1|^\gamma + \varepsilon.$$

Since ε is arbitrary, we see that $\varphi_*(x_1) - \varphi_*(x) \leq [\varphi]_{C^\gamma} \cdot |x - x_1|^\gamma$. Interchanging the role of x_1 and x, we conclude that

$$[\varphi_*]_{C^\gamma(\mathbf{R}^n)} \leq [\varphi]_{C^\gamma(\Omega)}.$$

Since $\|\varphi\|_\infty < \infty$, $\overline{\varphi} = \varphi$ on Ω and $\overline{\varphi}$ is still Hölder. More precisely, $[\overline{\varphi}]_{C^\gamma} \leq [\varphi_*]_{C^\gamma}$. By definition $\|\overline{\varphi}\|_\infty \leq \|\varphi\|_\infty$ so we conclude that $\|\overline{\varphi}\|_{C^\gamma} \leq \|\varphi\|_{C^\gamma}$.

Extending $f \in bmo^\infty_\infty(\Omega)$ to $\overline{f} \in bmo$ by Theorem 3.3.2, we conclude from multiplication estimate (3.3.3) that

$$\begin{aligned} \|\varphi f\|_{bmo^\infty_\infty(\Omega)} &\leq \|\overline{\varphi}\overline{f}\|_{bmo} \\ &\leq C \cdot \|\overline{\varphi}\|_{C^\gamma(\mathbf{R}^n)} \cdot \|\overline{f}\|_{bmo} \\ &\leq C \cdot \|\varphi\|_{C^\gamma(\Omega)} \cdot \|f\|_{bmo^\infty_\infty(\Omega)}. \end{aligned}$$

□

Remark 3.3.4. If we prove that the extension $f \mapsto \overline{f}$ constructed in Theorem 3.3.1 is bounded from bmo^∞_∞ to $bmo = BMO \cap L^1_{\mathrm{ul}}$, then the support condition will follow by taking $\varphi \in C^\gamma(\mathbf{R}^n)$ in Theorem 3.3.3 as a cut off function of Ω, i.e., $\varphi \equiv 1$ on Ω with $\operatorname{supp} \varphi \subset \Omega_\varepsilon$. In other words, we consider $f \mapsto \varphi \overline{f}$. However, the proof that $\overline{f} \in L^1_{\mathrm{ul}}$ needs some argument so we give a direct proof of Theorem 3.3.2.

For $BMO^{\mu,\infty}_b$ function in Ω it is easy to see that its zero extension is in BMO space; see e.g. [5, Lemma 4].

Theorem 3.3.5. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Assume that $\mu \in (0,\infty]$. For $f \in BMO_b^{\mu,\nu}(\Omega)$ with $\nu \geq 2\mu$, let f_0 be the zero extension to $\mathbf{R}^n$, i.e., $f_0(x) = 0$ for $x \in \Omega^c$ and $f_0(x) = f(x)$ for $x \in \Omega$. Then $f_0 \in BMO^\mu(\mathbf{R}^n)$ and $[f_0]_{BMO^\mu} \leq C[f]_{BMO_b^{\mu,\nu}}$ with C independent of f.*

Proof. If the ball B of radius $\leq \mu$ is in Ω, then

$$\frac{1}{|B|}\int_B |f_0 - f_{0B}|\,dy \leq [f]_{BMO^\mu}.$$

If B is in Ω^c, then $\int_B |f_0 - f_{0B}|\,dy = 0$. It remains to estimate the integral if B has nonempty intersection with the boundary $\Gamma = \partial\Omega$. For each $B_r(x) \cap \Gamma \neq \emptyset$, $r < \mu$, we take $x_0 \in B_r(x) \cap \Gamma$. Then, $B_r(x) \subset B_{2r}(x_0)$ and thus

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f_0 - f_{0B_r(x)}|\,dy \leq \frac{2}{|B_r(x)|}\int_{B_{2r}(x_0)} |f_0|\,dy \leq \frac{2^{n+1}}{\omega_n}\cdot [f]_{b^{2\mu}},$$

where ω_n is the volume of an n-dimensional ball. □

Remark 3.3.6. In [5, Lemma 4], it is assumed that $\Omega = \Omega' \times \mathbf{R}^{n-k}$ where Ω' is a bounded Lipschitz domain in $\mathbf{R}^k$. However, from the proof above it is clear that we do not need this requirement. Thus we give a full proof here.

As an application of boundedness of multiplication, we give invariance of function spaces under coordinate changes. We say that Ψ is a global $C^{k+\beta}$ (resp. C^k)-diffeomorphism if $C^{k+\beta}$ (resp. C^k)-norms of Ψ and Ψ^{-1} are bounded in $\mathbf{R}^n$, where $k \in \mathbf{N}$ and $\beta \in (0,1)$.

Proposition 3.3.7. *The space bmo is invariant under bi-Lipschitz coordinate change and the space h^1 is invariant under global $C^{1+\beta}$-diffeomorphism.*

Proof. For $f \in bmo$, by a simple change of variables on the equivalent definition of the seminorm $[f]_{BMO}$ where

$$[f]_{BMO} = \sup_{B \subset \mathbf{R}^n} \inf_{c \in \mathbf{R}} \int_B |f(y) - c|\,dy,$$

see e.g. [9, Proposition 3.1.2], we can easily deduce that *bmo* is invariant under bi-Lipschitz coordinate change.

Let $g \in h^1(\mathbf{R}^n)$ and Ψ be a global $C^{1+\beta}$-diffeomorphism. We have that

$$\|g \circ \Psi\|_{h^1} = \sup_{\|f\|_{bmo} \leq 1} \left| \int_{\mathbf{R}^n} f \cdot g \circ \Psi\,dy \right|.$$

By change of variable we have that

$$\left| \int_{\mathbf{R}^n} f(y) \cdot g \circ \Psi(y)\,dy \right| = \left| \int_{\mathbf{R}^n} f \circ \Psi^{-1}(x) \cdot g(x) \cdot J_{\Psi^{-1}}(x)\,dx \right|,$$

where $J_{\Psi^{-1}}$ is the Jacobian which is of regularity C^β. Then by the $bmo - h^1$ duality [16, Theorem 3.22] and multiplication estimate (3.3.2), we deduce that

$$\left| \int_{\mathbf{R}^n} f \circ \Psi^{-1} \cdot g \cdot J_{\Psi^{-1}}\,dx \right| \leq \|f \circ \Psi^{-1}\|_{bmo} \cdot \|gJ_{\Psi^{-1}}\|_{h^1} \leq \|f \circ \Psi^{-1}\|_{bmo} \cdot \|J_{\Psi^{-1}}\|_{C^\beta} \cdot \|g\|_{h^1}.$$

Since *bmo* is independent of bi-Lipschitz coordinate change, we have that

$$\|g \circ \Psi\|_{h^1} \leq C \cdot \|\nabla \Psi^{-1}\|_{L^\infty} \cdot \|J_{\Psi^{-1}}\|_{C^\beta} \cdot \|g\|_{h^1}$$

for some constant C independent of g and Ψ. □

Proposition 3.3.8. *The space $F^1_{1,2}(\mathbf{R}^n)$ is invariant under global $C^{1+\beta}$-diffeomorphism.*

Proof. Let $g \in F^1_{1,2}$ and Ψ be a global $C^{1+\beta}$-diffeomorphism. By multiplication estimate (3.3.2) and Proposition 3.3.7, we have that

$$\|\nabla(g\circ\Psi)\|_{F^0_{1,2}} \leq C\cdot\|\nabla\Psi\|_{C^\beta}\cdot\|(\nabla g)\circ\Psi\|_{F^0_{1,2}} \leq C\cdot\|\nabla\Psi\|_{C^\beta}\cdot\|\nabla\Psi^{-1}\|_{L^\infty}\cdot\|J_{\Psi^{-1}}\|_{C^\beta}\cdot\|\nabla g\|_{F^0_{1,2}},$$

where $J_{\Psi^{-1}}$ is the Jacobian for Ψ^{-1} and C is a constant independent of g and Ψ. Hence $\nabla(g\circ\Psi) \in F^0_{1,2}$ by $\|\nabla g\|_{F^0_{1,2}} \leq C\|g\|_{F^1_{1,2}}$ since the differentiation mapping is bounded from $F^s_{p,q}$ to $F^{s-1}_{p,q}$ for $p \in (0,\infty)$, $q \in (0,\infty]$ and $s \in \mathbf{R}$, see e.g. [16, Theorem 2.12]. Since $\nabla(g\circ\Psi) \in F^0_{1,2}$, we also get $\Delta(g\circ\Psi) \in F^{-1}_{1,2}$. Since $F^1_{1,2} \hookrightarrow F^0_{1,2}$, Proposition 3.3.7 tells us that $g\circ\Psi \in F^0_{1,2} \subset F^{-1}_{1,2}$. Therefore, $(I-\Delta)(g\circ\Psi) \in F^{-1}_{1,2}$. Notice that [16, Theorem 2.12] also tells us that for $\sigma \in \mathbf{R}$, $(I-\Delta)^\sigma$ is an isomorphism from $F^s_{p,q}$ to $F^{s-2\sigma}_{p,q}$. Hence by letting $\sigma = -1$, we deduce that

$$\begin{aligned}
\|g\circ\Psi\|_{F^1_{1,2}} &= \|(I-\Delta)^{-1}(I-\Delta)(g\circ\Psi)\|_{F^1_{1,2}} \\
&\leq C\cdot\|(I-\Delta)(g\circ\Psi)\|_{F^{-1}_{1,2}} \\
&\leq C\cdot\left(\|g\circ\Psi\|_{F^0_{1,2}} + \|\nabla(g\circ\Psi)\|_{F^0_{1,2}}\right) \\
&\leq C\cdot(1+\|\nabla\Psi\|_{C^\beta})\cdot\|\nabla\Psi^{-1}\|_{L^\infty}\cdot\|J_{\Psi^{-1}}\|_{C^\beta}\cdot\|g\|_{F^1_{1,2}},
\end{aligned}$$

where C is a constant independent of g and Ψ. □

Remark 3.3.9. The proof of Proposition 3.3.8 also says that $F^1_{1,2} = \{f \in F^0_{1,2} \mid \nabla f \in (F^0_{1,2})^n\}$.

3.4 Trace problems

In this section we show that the normal trace of a vector field in $vbmo^{\mu,\nu}_\delta$ is in $L^\infty(\Gamma)$ if its divergence is well controlled. We begin with the case that Ω is the half space $\mathbf{R}^n_+$.

We first recall that the trace operator $(\mathrm{Tr} f)(x') = f(x',0)$ for $f \in F^1_{1,2}(\mathbf{R}^n)$ gives a surjective bounded linear operator from $F^1_{1,2}(\mathbf{R}^n)$ to $L^1(\mathbf{R}^{n-1})$; see [19, Section 4.4.3].

Proposition 3.4.1 ([19])**.** *The operator* Tr *from $F^1_{1,2}$ to $L^1(\mathbf{R}^{n-1})$ is surjective for $n \geq 2$. Actually, surjectivity holds for a smaller space $B^1_{1,1}$. There exists an inverse operator called the extension which is a bounded linear operator.*

For a C^2 domain Ω a normal trace $v\cdot\mathbf{n}$ on $\Gamma = \partial\Omega$ of v is well-defined as an element of $W^{-1/p,p}_{\mathrm{loc}}(\Gamma)$ if v and $\operatorname{div} v$ is in L^p_{loc}; see e.g. [6] or [7]. If $v \in vbmo^{\mu,\nu}_\delta(\Omega)$ so that $v \in L^1_{\mathrm{loc}}$, then by an interpolation inequality (see e.g. [5, Theorem 11]) v is in L^p_{loc} for any $p \geq 1$. Thus if $\operatorname{div} v$ is in L^p_{loc}, $v\cdot\mathbf{n}$ is well-defined. We derive L^∞ estimate for $v\cdot\mathbf{n}$ when Ω is the half space.

Theorem 3.4.2. *Let μ,ν,δ be in $(0,\infty]$ and $n \geq 2$. Then there is a constant $C = C(\mu,\nu,\delta,n)$ such that*

$$\|v\cdot\mathbf{n}\|_{L^\infty(\mathbf{R}^{n-1})} \leq C\left(\|v\|_{vbmo^{\mu,\nu}_\delta(\mathbf{R}^n_+)} + \|\operatorname{div} v\|_{L^n_{\mathrm{ul}}(\Gamma_\delta)}\right)$$

for all $v \in vbmo^{\mu,\nu}_\delta(\mathbf{R}^n_+)$.

Proof. Let $v \in vbmo_\delta^{\mu,\nu}(\mathbf{R}^n_+)$. By definition, the n-th component v_n of $v = (v', v_n)$ belongs to $BMO_b^{\mu,\nu}(\mathbf{R}^n_+)$. For $x_0' \in \mathbf{R}^{n-1}$, we consider the region $U = B_1(x_0') \times (-\delta, \delta)$ where $B_1(x_0')$ denotes the ball in $\mathbf{R}^{n-1}$ centered at x_0' with radius 1. Let v_{re} denotes the restriction of v on $U \cap \mathbf{R}^n_+$, i.e., $v_{\mathrm{re}} = v\,|_{U \cap \mathbf{R}^n_+}$. We have that $v_{\mathrm{re}} \in bmo^\infty_\infty(U \cap \mathbf{R}^n_+)$ and

$$\sup_{\substack{x' \in B_1(x_0') \\ r < \nu}} \frac{1}{|B_r((x',0))|} \int_{B_r((x',0))} |(v_{\mathrm{re}})_n|\,dy < \infty.$$

Let $\overline{(v_{\mathrm{re}})_n}$ be the zero extension of $(v_{\mathrm{re}})_n$ to U. By Theorem 3.3.5, $\overline{(v_{\mathrm{re}})_n}$ is in $BMO^\infty(U)$. Let $\overline{v_{\mathrm{re}}'}$ be the even extension of v_{re}' to U of the form

$$\overline{v_{\mathrm{re}}'}(x', x_n) = \begin{cases} v_{\mathrm{re}}'(x', x_n), & x' \in B_1(x_0') \text{ and } x_n > 0 \\ v_{\mathrm{re}}'(x', -x_n), & x' \in B_1(x_0') \text{ and } x_n < 0 \end{cases} \tag{3.4.1}$$

and set $\widetilde{v} = (\overline{v_{\mathrm{re}}'}, \overline{(v_{\mathrm{re}})_n})$. We have that $\widetilde{v} \in bmo^\infty_\infty(U)$. By Theorem 3.3.2 its Jones' extension v_U belongs to $bmo^\infty_\infty(\mathbf{R}^n)$.

Integration by parts formally yields

$$\int_{\mathbf{R}^{n-1}} v_U \cdot \mathbf{n}\rho\,dx' = \int_{\mathbf{R}^n_+} (\operatorname{div} v_U)\rho\,dx - \int_{\mathbf{R}^n_+} v_U \cdot \nabla\rho\,dx. \tag{3.4.2}$$

By Proposition 3.4.1 there is an extension operator $\mathrm{Ext} : L^1(\mathbf{R}^{n-1}) \to F^1_{1,2}(\mathbf{R}^n)$ such that $\mathrm{Tr} \circ \mathrm{Ext}$ is the identity operator on L^1. For $\varphi \in C_c^\infty\left(B_{\frac{1}{2}}(x_0')\right)$ we set $\sigma = \mathrm{Ext}\,\varphi$. Multiplying a cut off function $\theta \in C_c^\infty(U)$ such that $\theta \equiv 1$ in $\frac{1}{2}U$ and considering $\rho = \theta\sigma$, we still find $\rho \in F^1_{1,2}(\mathbf{R}^n)$ by a multiplier theorem [16, Theorem 3.18], [19, Section 4.2.2]. We estimate (3.4.2) to get

$$\begin{aligned} \left| \int_{\mathbf{R}^{n-1}} v_U \cdot \mathbf{n}\rho\,dx' \right| &\le \left| \int_U (\operatorname{div} v_U)\rho\,dx \right| + \left| \int_{\mathbf{R}^n_+} v_U' \cdot \nabla'\rho\,dx \right| \\ &\quad + \left| \int_{\mathbf{R}^n} v_{U^n} \frac{\partial\rho}{\partial x_n} dx \right| = I + I\!I + I\!I\!I. \end{aligned}$$

We may assume that ρ is even in x_n by taking $(\rho(x', x_n) + \rho(x', -x_n))/2$ so that the second term is estimated by bmo-h^1 duality $(h^1)^* = (F^0_{1,2})^* = F^0_{\infty,2} = bmo$ as follows

$$\begin{aligned} I\!I = \left| \int_{\mathbf{R}^n_+} v_U' \cdot \nabla'\rho\,dx \right| &= \frac{1}{2} \left| \int_{\mathbf{R}^n} v_U' \cdot \nabla'\rho\,dx \right| \\ &\le C\|v_U'\|_{bmo} \|\nabla'\rho\|_{h^1}. \end{aligned}$$

The third term is estimated as

$$I\!I\!I \le C\|v_{U^n}\|_{bmo} \left\| \frac{\partial\rho}{\partial x_n} \right\|_{h^1}.$$

The first term is estimated by

$$\begin{aligned} I &\le \|\operatorname{div} v_U\|_{L^n(U)} \|\rho\|_{L^{n/(n-1)}(U)} \\ &\le C\|\operatorname{div} v\|_{L^n_{\mathrm{ul}}(\Gamma_\delta)} \|\nabla\rho\|_{L^1(U)} \end{aligned}$$

by the Sobolev inequality. Since $\|\nabla\rho\|_{L^1} \leq \|\nabla\rho\|_{h^1}$ and $\|\nabla\rho\|_{h^1} \leq \|\rho\|_{F^1_{1,2}} \leq C\|\varphi\|_{L^1\left(B_{\frac{1}{2}}(x_0')\right)}$, collecting these estimates yields

$$\left|\int_{B_{\frac{1}{2}}(x_0')} v\cdot\mathbf{n}\varphi\, dx'\right| \leq C\|\varphi\|_{L^1\left(B_{\frac{1}{2}}(x_0')\right)}\left(\|v\|_{vbmo^{\mu,\nu}_\delta(\mathbf{R}^n_+)} + \|\operatorname{div} v\|_{L^n_{\mathrm{ul}}(\Gamma_\delta)}\right).$$

This yields the desired estimate since $C_c^\infty\left(B_{\frac{1}{2}}(x_0')\right)$ is dense in $L^1\left(B_{\frac{1}{2}}(x_0')\right)$ and C in the right-hand side is independent of $x_0' \in \mathbf{R}^{n-1}$. □

We now consider a curved domain. Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$ so that the reach R_* of Γ is positive and $\beta \in (0,1)$.

Theorem 3.4.3. *Let Ω be a uniformly $C^{2+\beta}$ domain in $\mathbf{R}^n$ with $n \geq 2$. Let μ,ν,δ be in $(0,\infty]$. Then there is a constant $C = C(\mu,\nu,\delta,\Omega)$ such that*

$$\|v\cdot\mathbf{n}\|_{L^\infty(\Gamma)} \leq C\left(\|v\|_{vbmo^{\mu,\nu}_\delta(\Omega)} + \|\operatorname{div} v\|_{L^n_{\mathrm{ul}}(\Gamma_\delta)}\right)$$

for all $v \in vbmo^{\mu,\nu}_\delta(\Omega)$.

We shall prove this result by localizing the problems near the boundary and by using a normal coordinate system. Let Ω be a uniformly $C^{2+\beta}$ domain. In other words, there exist $r_*, \delta_* > 0$ such that for each $z_0 \in \Gamma$, up to translation and rotation, there exists a function $h_{z_0} \in C^{2+\beta}(B_{r_*}(0'))$ with

$$\begin{aligned}&|(\nabla')^k h_{z_0}| \leq L \ \text{ in } \ B_{r_*}(0') \ \text{ for } \ k = 0,1,2,\\ &[(\nabla')^2 h_{z_0}]_{C^\beta(B_{r_*}(0'))} < \infty,\ \nabla' h_{z_0}(0') = 0',\ h_{z_0}(0') = 0\end{aligned}$$

such that the neighborhood

$$U_{r_*,\delta_*,h_{z_0}}(z_0) := \{(x', x_n) \in \mathbf{R}^n \mid h_{z_0}(x') - \delta_* < x_n < h_{z_0}(x') + \delta_*,\ |x'| < r_*\}$$

satisfies

$$\Omega \cap U_{r_*,\delta_*,h_{z_0}}(z_0) = \{(x', x_n) \in \mathbf{R}^n \mid h_{z_0}(x') < x_n < h_{z_0}(x') + \delta_*,\ |x'| < r_*\}$$

and

$$\partial\Omega \cap U_{r_*,\delta_*,h_{z_0}}(z_0) = \{(x', x_n) \in \mathbf{R}^n \mid x_n = h_{z_0}(x'),\ |x'| < r_*\}.$$

For $x \in \Omega$, let πx be a point on Γ such that $|x - \pi x| = d_\Omega(x)$. If x is within the reach of Γ, then this πx is unique. There exist $r < r_*$ and $\delta < \delta_*$ such that

$$U(z_0) = \{x \in \mathbf{R}^n \mid (\pi x)' \in B_r(0'),\ d_\Gamma(x) < \delta\}$$

is contained in $U_{r_*,\delta_*,h_{z_0}}(z_0)$. Since d_Ω is $C^{2+\beta}$ in $\overline{\Gamma}_\sigma$ for $\sigma < R_*$ [10, Chap. 14, Appendix] [11, §4.4], we may take δ smaller (independent of z_0) so that d_Ω is $C^{2+\beta}$ in $\overline{U(z_0)\cap\Omega}$.

We next consider a normal coordinate system in $U(z_0)$

$$\left\{\begin{aligned} x' &= y' + y_n\nabla' d_\Omega\left(y', h_{z_0}(y')\right)\\ x_n &= h_{z_0}(y') + y_n\partial_{x_n} d_\Omega\left(y', h_{z_0}(y')\right)\end{aligned}\right. \tag{3.4.3}$$

or shortly

$$x = \pi x - d_\Omega(x)\mathbf{n}(\pi x).$$

Let this coordinate change be denoted by $x = \psi(y)$, $\psi \in C^{1+\beta}(V)$, where V is a neighborhood defined below. Notice that $\nabla\psi(0) = I$. If we consider r and δ small, this coordinate change is indeed a local C^1-diffeomorphism which maps $U(z_0)$ to V where $V := B_r(0^{'}) \times (-\delta, \delta)$. Moreover, by [12], we extend ψ to a global C^1-diffeomorphism $\widetilde{\psi}$ such that $\widetilde{\psi}|_V = \psi$ and $\|\nabla\widetilde{\psi}\|_{L^\infty(\mathbf{R}^n)} < 2$. Let the inverse of ψ in V be denoted by ϕ, i.e., $\phi = \psi^{-1}$.

Lemma 3.4.4. *Let W be a vector field with measurable coefficient in Γ_σ, $\sigma < R_*$ of the form*

$$W = \sum_{i=1}^{n} w_i \frac{\partial}{\partial x_i}.$$

Let y be the normal coordinate such that $y_n = d_\Omega(x)$. Let $\tilde{W}$ be W in y coordinate of the form $\overline{W} = \sum_{j=1}^{n} \tilde{w}_j(y)\partial/\partial y_j$. Then

$$\tilde{w}_n(y) = \nabla d_\Omega\left(x(y)\right) \cdot w\left(x(y)\right).$$

We shall prove this lemma in Appendix which follows from a simple linear algebra.

Proof of Theorem 3.4.3. We first observe that the restriction of v on $U(z_0) \cap \Omega$ belongs to $bmo_\infty^\infty\left(U(z_0) \cap \Omega\right)$. By considering the following equivalent definition of the seminorm $[f]_{BMO^\infty(D)}$ where

$$[f]_{BMO^\infty(D)} = \sup_{B_r(x)\subset D} \inf_{c\in\mathbf{R}} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - c|\, dy,$$

(see [9, Proposition 3.1.2]), we can deduce that the space bmo_∞^∞ on a bounded domain is independent of bi-Lipschitz coordinate change. We introduce normal coordinate for a vector field $v = \sum_{i=1}^{n} v_i\partial/\partial x_i$ with $v_i \in bmo_\infty^\infty\left(U(z_0) \cap \Omega\right)$. Let w be the transformed vector field under the normal coordinate y. By Lemma 3.4.4, w_n of $w = \sum_{i=1}^{n} w_i\partial/\partial y_i$ fulfills $w_n = \nabla d_\Omega\left(x(y)\right) \cdot v\left(x(y)\right)$. Since $v \in vbmo_\delta^{\mu,\nu}(\Omega)$, this implies that $w \in bmo_\infty^\infty(V \cap \mathbf{R}_+^n)$ and moreover,

$$\sup_{\ell<\delta,\, B_\ell(x)\subset V} \ell^{-n} \int_{B_\ell(x)\cap\mathbf{R}_+^n} |w_n|\, dy < \infty.$$

Thus, as in the proof of Theorem 3.4.2, the zero extension of w_n for $y_n < 0$ is in $bmo_\infty^\infty(V)$, we still denote this extension by w_n. Let $J = J(y)$ denote the Jacobian of the mapping $y \mapsto x$ in V. For tangential part $w^{'}$ of $w = (w^{'}, w_n)$, we take an even extension with weight J of the form

$$\hat{w}'(y', y_n) = \begin{cases} w'(y', y_n), & y_n > 0 \\ w'(y', -y_n)J(y', -y_n)/J(y', y_n), & y_n < 0 \end{cases} \tag{3.4.4}$$

and set $\tilde{w}(y', y_n) = (\hat{w}', w_n)$. Let $\overline{w}^{'}$ denote the unweighted even extension of $w^{'}$ to V, thus $w \in bmo_\infty^\infty(V \cap \mathbf{R}_+^n)$ implies that $\overline{w}^{'} \in bmo_\infty^\infty(V)$. Let f be the function defined on V such that $f \equiv 1$ for $y_n \geq 0$ and $f = J(y', -y_n)/J(y', y_n)$ for $y_n < 0$. Since $J(y)^{-1} = |\det D\psi(y)|^{-1} = |\det D\phi(\psi(y))|$ for $y \in V$, we have that $f \in C^\beta(V)$. Notice that $\hat{w}'(y) = \overline{w}'(y)f(y)$, therefore by Theorem 3.3.3, we can deduce that $\tilde{w}$ belongs to $bmo_\infty^\infty(V)$. By

Theorem 3.3.2, the Jones' extension w_U of $\tilde{w}$ belongs to $bmo^\infty_\infty(\mathbf{R}^n)$. Its expression in x coordinate is v_U which is only defined near Γ.

If the support of ρ is in $U(z_0)$, then integration by parts implies that

$$\int_\Gamma v_U \cdot \mathbf{n}\rho \, d\mathcal{H}^{n-1} = \int_\Omega (\operatorname{div} v_U)\rho \, dx - \int_\Omega v_U \cdot \nabla\rho \, dx. \tag{3.4.5}$$

We shall estimate the left-hand side as in the case of $\mathbf{R}^n_+$. The first integral in the right-hand side can be estimated similarly as in the proof of Theorem 3.4.2. It is sufficient to only consider the second integral. Let $\Psi : B_r(0') \to \Gamma \cap U(z_0)$ by $(y', 0) \mapsto (y', h_{z_0}(y'))$. Extend $h_{z_0} \in C^2(B_r(0'))$ to $\tilde{h} \in C^2_c(\mathbf{R}^{n-1})$ such that $\tilde{h}\,|_{B_r(0')} = h_{z_0}$. Define $\widetilde{\Psi} : \mathbf{R}^{n-1} \to \tilde{h}(\mathbf{R}^{n-1})$ by $(y', 0) \mapsto (y', \tilde{h}(y'))$. Hence $\widetilde{\Psi}\,|_{B_r(0')} = \Psi$. Extend further $\widetilde{\Psi}$ to $\widetilde{\Psi}^* : \mathbf{R}^n \to \mathbf{R}^n$ by $(y', d) \mapsto \widetilde{\Psi}(y', 0) + (0', d)$. Notice that this $\widetilde{\Psi}^*$ is a global C^2-diffeomorphism whose derivatives are bounded in $\mathbf{R}^n$ up to second-order. We may assume $z_0 = 0$ by translation. Let $\zeta > 0$ be a constant to be determined later. For $\varphi \in C^1_c(\Gamma \cap \zeta U(z_0))$, we observe that $\varphi \circ \Psi \in C^1_c(B_{\zeta r}(0'))$. Let $\widetilde{\sigma} = \operatorname{Ext}(\varphi \circ \Psi)$ as in the proof of Theorem 3.4.2 and let $\sigma = \widetilde{\sigma} \circ (\widetilde{\Psi}^*)^{-1}$. With this choice of σ, we observe that for $(y', h_{z_0}(y')) \in \Gamma \cap \zeta U(z_0)$,

$$\sigma(y', h_{z_0}(y')) = \widetilde{\sigma} \circ (\widetilde{\Psi}^*)^{-1}(y', h_{z_0}(y')) = \widetilde{\sigma}(y', 0) = \varphi \circ \Psi(y', 0) = \varphi(y', h_{z_0}(y')).$$

Thus φ is an extension of σ. Since $(\widetilde{\Psi}^*)^{-1}$ is a global C^2-diffeomorphism and $\widetilde{\sigma} \in F^1_{1,2}(\mathbf{R}^n)$, we observe that $\sigma \in F^1_{1,2}(\mathbf{R}^n)$, see e.g. see Proposition 3.3.8 or [19, Section 4.3.1].

For each $z_0 \in \Gamma$, there exists $\epsilon_{z_0} > 0$ such that we can find a cut off function $\theta_{z_0} \in C^\infty_c(U(z_0))$ for which $\theta_{z_0} \equiv 1$ within $\epsilon_{z_0} U(z_0)$ and

$$\sum_{|\alpha| \le 2} \|D^\alpha \theta_{z_0}\|_{L^\infty(\mathbf{R}^n)} \le M$$

for some fixed universal constant $M > 1$ independent of z_0. By multiplying this cut off function θ_{z_0}, we have that $\rho = \theta_{z_0}\sigma \in F^1_{1,2}(\mathbf{R}^n)$ and $\|\rho\|_{F^1_{1,2}(\mathbf{R}^n)} \le M \cdot \|\sigma\|_{F^1_{1,2}(\mathbf{R}^n)}$. Hence we take the constant ζ above to be ϵ_{z_0}.

By coodinate change, we observe that

$$\int_\Omega v_U \cdot \nabla\rho \, dx = \int_{U(z_0)\cap\Omega} \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\rho \, dx = \int_{V\cap\mathbf{R}^n_+} \sum_{j=1}^n w_{U_j}(y) J(y) \frac{\partial}{\partial y_j}\big(\rho \circ \psi(y)\big) dy.$$

The n-th component equals

$$\int_{V\cap\mathbf{R}^n_+} w_{U_n}(y) J(y) \frac{\partial}{\partial y_n}\big(\rho \circ \psi(y)\big) dy = \int_V w_{U_n}(y) J(y) \frac{\partial}{\partial y_n}\big(\rho \circ \psi(y)\big) dy$$

since w_{U_n} equals zero for $y_n < 0$. Considering extensions of Hölder functions [13] and local diffeomorphism [12], by the $F^0_{1,2} - F^0_{\infty,2}$ duality [16, Theorem 3.22] and Proposition 3.3.7, we conclude that

$$\begin{aligned}
\left| \int_V w_{U_n}(y) J(y) \frac{\partial}{\partial y_n}\big(\rho \circ \psi(y)\big) dy \right| &\le C \cdot \sum_{i=1}^n \|w_{U_n}\|_{bmo} \cdot \|J\|_{C^\beta(V)} \cdot \|\partial_{y_n}\psi\|_{C^\beta(V)} \cdot \|\nabla\rho \circ \widetilde{\psi}\|_{h^1} \\
&\le C \cdot \|w_{U_n}\|_{bmo} \cdot \|\nabla\rho\|_{h^1}.
\end{aligned}$$

For tangential part we may assume that

$$(\rho \circ \psi)(y', y_n) = (\rho \circ \psi)(y', -y_n) \quad \text{for } \; y_n < 0. \tag{3.4.6}$$

In fact, for a given ρ we take

$$g(y', y_n) = \big(\rho \circ \psi(y', y_n) + \rho \circ \psi(y', -y_n)\big)/2$$

which satisfies evenness $g(y', y_n) = g(y', -y_n)$ and

$$g(y', 0) = \theta \circ \psi(y', 0) \cdot \sigma \circ \psi(y', 0) = \theta(y', h_{z_0}(y')) \cdot \varphi(y', h_{z_0}(y')).$$

It suffices to take ρ such that $\rho \circ \psi(y) = g(y)$. Thus, we may assume that $\rho \circ \psi$ is even in y_n so that $\partial_{y_j}(\rho \circ \psi)$ is also even in y_n for $j = 1, 2, \dots, n-1$. Since $w_{U_j} J$ is even in y_n for y in V, we observe that

$$\int_{V \cap \mathbf{R}^n_+} w_{U_j}(y) J(y) \frac{\partial}{\partial y_j}(\rho \circ \psi)\, dy = \frac{1}{2} \int_V w_{U_j}(y) J(y) \frac{\partial}{\partial y_j}(\rho \circ \psi)\, dy$$

for $1 \leq j \leq n-1$. Similar to the case for the n-th component, we thus conclude that

$$\left| \int_V w_{U_j}(y) J(y) \frac{\partial}{\partial y_j}(\rho \circ \psi)\, dy \right| \leq C \cdot \|w_{U_j}\|_{bmo} \cdot \|\nabla \rho\|_{h^1}.$$

Collecting these estimates, we conclude that

$$\begin{aligned} \left| \int_{\Gamma \cap \epsilon_{z_0} U(z_0)} v \cdot \mathbf{n}\, \varphi \, dx^{n-1} \right| &\leq C \|w_U\|_{bmo} \|\nabla \rho\|_{h^1} \\ &\leq C \|v\|_{vbmo^{\mu,\nu}_\delta(\Omega)} \|\varphi\|_{L^1(\Gamma \cap \epsilon_{z_0} U(z_0))}. \end{aligned}$$

Thus $\|v \cdot \mathbf{n}\|_{L^\infty} \leq C \|v\|_{vbmo^{\mu,\nu}_\delta(\Omega)}$. □

Remark 3.4.5. (i) Since $BMO^{\mu,\nu}_b \subset vbmo^{\mu,\nu}_\delta$ for $\delta < \infty$, the estimate in Theorem 3.4.3 holds if we replace $vbmo^{\mu,\nu}_\delta$ by $BMO^{\mu,\nu}_b$. Moreover, since we are able to use zero extension in this case. We can follow the proof of Theorem 3.4.3 directly without the necessity to invoke normal coordinates. We shall state a version of Theorem 3.4.3 for $BMO^{\mu,\nu}_b$ in the end of this section.

(ii) By Theorem 3.2.9 we may replace $vbmo^{\mu,\nu}_\delta$ by $vBMO^{\mu,\nu}(\Omega)$ in the estimate in Theorem 3.4.3 since we may always take $\delta \leq \nu < R_*$ provided that Ω is a bounded or an exterior domain.

Remark 3.4.6. If we assume that the vector field v is continuous in $\overline{\Omega}$, then by Lebesgue differentiation theorem we have the natural estimate $\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C[\nabla d_\Omega \cdot v]_{b^\nu}$ for some constant C independent of v. Therefore, if we replace the space $vbmo^{\mu,\nu}_\delta(\Omega)$ by the $vbmo^{\mu,\nu}_\delta$ closure of $C^\infty_c(\overline{\Omega})$, then Theorem 3.4.2 and 3.4.3 trivially hold. However, the $vbmo^{\mu,\nu}_\delta$ closure of $C^\infty_c(\overline{\Omega})$ seems to be strictly smaller than the space $vbmo^{\mu,\nu}_\delta(\Omega)$ since it is known that a similar space VMO, the BMO closure of $C^\infty_c(\mathbf{R}^n)$, is a proper subspace of BMO [17]. Thus, our trace theorems stay non-trivial. Generally speaking, we cannot directly estimate the L^∞ norm on the boundary by the b^ν-seminorm. Here is an example. In dimension 1, for any $m \in \mathbf{N} \cap \{0\}$ we define f in $(0,1)$ by

$$f(x) = \begin{cases} m+1, & \text{if} \quad x \in (\frac{1}{2^m+1}, \frac{1}{2^m}), \\ 0, & \text{otherwise}. \end{cases}$$

A simple calculation tells us that $[f]_{b^\nu} \leq 2 \cdot \sum_{i=1}^\infty \frac{i}{2^i} < \infty$ but for any $M > 0$ there exists $\delta_M > 0$ such that there exists a subset $S \subset (0, \delta_M)$ with Lebesgue measure $|S| > 0$ and $f(x) > M$ for any $x \in S$.

Theorem 3.4.7. *Let Ω be a uniformly $C^{1+\beta}$ domain in $\mathbf{R}^n$ with $n \geq 2$. Let μ, ν, δ be in $(0, \infty]$. Then there is a constant $C = C(\mu, \nu, \delta, \Omega)$ such that*

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C(\|v\|_{BMO_b^{\mu,\nu}(\Omega)} + \|\operatorname{div} v\|_{L^n_{\mathrm{ul}}(\Gamma_\delta)})$$

for all $v \in BMO_b^{\mu,\nu}(\Omega)$.

Proof. For $z_0 \in \Gamma$, let $U(z_0) = U_{r_*,\delta_*,h_{z_0}}(z_0)$ with $\delta_* \leq R_*$. We then follow the proof of Theorem 3.4.3 without invoking the normal coordinates. For $v \in BMO_b^{\mu,\nu}(\Omega)$, let v_0 be the zero extension of v. We have that $v_0 \in bmo^\infty_\infty(U(z_0))$. Let v_U be the Jones' extension of $r_{U(z_0)}v_0$ by Theorem 3.3.2 where $r_{U(z_0)}v_0$ denotes the restriction of v_0 on $U(z_0)$. For $\varphi \in C_c^1(\Gamma \cap \frac{1}{2}U(z_0))$, we construct the function σ in the same way as in the proof of Theorem 3.4.3. Since the boundary Γ is uniformly $C^{1+\beta}$, $\widetilde{\Psi}^*$ is a global $C^{1+\beta}$-diffeomorphism. By Proposition 3.3.8, we have that $\sigma = \widetilde{\sigma} \circ (\widetilde{\Psi}^*)^{-1} \in F^1_{1,2}(\mathbf{R}^n)$. Pick θ in $C_c^\infty(U(z_0))$ such that $\theta \equiv 1$ within $\frac{1}{2}U(z_0)$ and let $\rho = \theta\sigma$, we deduce that $\rho \in F^1_{1,2}(\mathbf{R}^n)$ and

$$\left| \int_\Omega v_U \cdot \nabla\rho \, dx \right| \leq C \cdot \|v_U\|_{bmo} \cdot \|\nabla\rho\|_{h^1} \leq C \cdot \|v\|_{BMO_b^{\mu,\nu}(\Omega)} \cdot \|\nabla\rho\|_{h^1}.$$

Therefore,

$$\left| \int_{\Gamma \cap \frac{1}{2}U(z_0)} v \cdot \mathbf{n} \, \varphi \, dx^{n-1} \right| \leq C \cdot \|v\|_{BMO_b^{\mu,\nu}(\Omega)} \cdot \|\varphi\|_{L^1(\Gamma \cap \frac{1}{2}U(z_0))}.$$

The proof is therefore complete. □

3.5 Appendix

We shall prove Lemma 3.4.4. We first recall a simple property of a matrix.

Proposition 3.5.1. *Let A be an invertible matrix*

$$A = (\vec{a}_1, \ldots, \vec{a}_n)$$

when $\vec{a}_j = {}^t\, (a_{ij})_{1 \leq i \leq n}$ is a column vector. Assume that $\vec{a}_n$ is a unit vector and orthogonal to $\vec{a}_j$ with $1 \leq j \leq n-1$. Then n-row vector of A^{-1} equals ${}^t\vec{a}_n$. In other words, if one writes $A^{-1} = (b_{ij})_{1 \leq i,j \leq n}$, then $b_{nj} = a_{jn}$ for $1 \leq j \leq n$.

Proof. By definition the row vector $\vec{b} = (b_{nj})_{1 \leq j \leq n}$ must satisfies $\vec{b} \cdot \vec{a}_j = 0$ $(j = 1, \ldots, n-1)$, $\vec{b} \cdot \vec{a}_n = 1$. Since $\{\vec{a}_j\}_{j=1}^{n-1}$ spans $\mathbf{R}^{n-1}$ orthogonal to $\vec{a}_n$, first identities imply that $\vec{b}$ is parallel to $\vec{a}_n$. We thus conclude that $\vec{b} = \vec{a}_n$ since $\vec{b} \cdot \vec{a}_n = 1$ and $|\vec{a}_n| = 1$. □

Proof of Lemma 3.4.4. We recall the explicit representation (4.4.1) of the normal coordinate system. The Jacobi matrix from $y \longmapsto x$ is of the form

$$A = (\vec{a}_1, \ldots, \vec{a}_n)$$

with $\vec{a}_j = {}^t \left(\delta_{ij} - y_n \partial_j \mathbf{n}_i\, (y', \psi(y'))\,,\ \partial_j\psi(y') - y_n \partial_j \mathbf{n}_n\, (y', \psi(y'))\right)_{1 \leq i \leq n-1}$, $1 \leq j \leq n-1$,

$$\vec{a}_n = -{}^t\mathbf{n}\left(y', \psi(y')\right) \quad \text{where} \quad \mathbf{n} = -\nabla d_\Omega.$$

Note that the vector $(\delta_{ij}, \partial_j\psi(y'))_{1\le i\le n-1}$ is a tangential vector to Γ. Moreover, the vector $(\partial_j \mathbf{n}_1, \ldots, \partial_j \mathbf{n}_n)$ is also tangential since $\partial_j \mathbf{n}\cdot\mathbf{n} = \partial_j|\mathbf{n}|^2/2 = 0$. Thus $\vec{a}_j$ is orthogonal to $\vec{a}_n$ for $1 \le j \le n-1$. The invertibility of A is guaranteed if $y_n < R_*$.

By a chain rule we have

$$\begin{aligned}\overline{w} &= \sum_{j=1}^{n} \tilde{w}_j(y)(\partial/\partial y_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{w}_j \frac{\partial x_i}{\partial y_j}\frac{\partial}{\partial x_i}\end{aligned}$$

so that

$$w_i\,(x(y)) = \sum_{j=1}^{n} \tilde{w}_j(y)\frac{\partial x_i}{\partial y_j} \quad \text{i.e.,} \quad w = A\tilde{w},$$

where $A = (\partial x_i/\partial y_j)_{1\le i,j\le n}$, $\tilde{w} =^t (\tilde{w}_i, \ldots, \tilde{w}_n)$, $w =^t (w_1, \ldots, w_n)$. Thus

$$\tilde{w} = A^{-1}w.$$

By Proposition 3.5.1, the last row of A^{-1} equals ∇d_Ω.

We thus conclude that $\tilde{w}_n = \nabla d_\Omega \cdot w$. This is what we would like to prove. □

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Chapter 4

The Helmholtz decomposition of a space of vector fields with bounded mean oscillation in a bounded domain

We introduce a space of vector fields with bounded mean oscillation whose "tangential" and "normal" components to the boundary behave differently. We establish its Helmholtz decomposition when the domain is bounded. This substantially extends the authors' earlier result for a half space.

4.1 Introduction

The Helmholtz decomposition of a vector field is a fundamental tool to analyze the Stokes and the Navier-Stokes equations. It is formally a decomposition of a vector field $v = (v^1, \ldots, v^n)$ in a domain Ω of $\mathbf{R}^n$ into

$$v = v_0 + \nabla q; \tag{4.1.1}$$

here v_0 is a divergence free vector field satisfying supplemental conditions like boundary condition and ∇q denotes the gradient of a function (scalar field) q. If v is in L^p $(1 < p < \infty)$ in Ω, such a decomposition is well-studied. For example, a topological direct sum decomposition

$$(L^p(\Omega))^n = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

holds for various domains including $\Omega = \mathbf{R}^n$, a half space $\mathbf{R}^n_+$, a bounded smooth domain [8]; see e.g. G. P. Galdi [9]. Here, $L^p_\sigma(\Omega)$ denotes the L^p-closure of the space of all div-free vector fields compactly supported in Ω and $G^p(\Omega)$ denotes the totality of L^p gradient fields. It is impossible to extend this Helmholtz decomposition to L^∞ even if $\Omega = \mathbf{R}^n$ since the projection $v \mapsto \nabla q$ is a composite of the Riesz operators which is not bounded in L^∞. We have to replace L^∞ with a class of functions of bounded mean oscillation. However, if the vector field is of bounded mean oscillation (BMO for short), such a problem is only studied when Ω is a half space $\mathbf{R}^n_+$ [10], where the boundary is flat.

Our goal is to establish the Helmholtz decomposition of BMO vector fields in a smooth bounded domain in $\mathbf{R}^n$, which is a typical example of a domain with curved boundary.

Although the space of BMO functions in $\mathbf{R}^n$ is well studied, the situation is less clear when one considers such a space in a domain, because there are several possible definitions. One should be careful about the behavior of a function near the boundary $\Gamma = \partial\Omega$. In this chapter we study a space of BMO vector fields introduced in [11] and establish its Helmholtz decomposition when Ω is a bounded C^3 domain.

Let us recall the space $vBMO(\Omega)$ introduced in [11]. We first recall the BMO seminorm for $\mu \in (0, \infty]$. For a locally integrable function f, i.e., $f \in L^1_{\mathrm{loc}}(\Omega)$ we define

$$[f]_{BMO^\mu(\Omega)} := \sup\left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} \left|f(y) - f_{B_r(x)}\right| dy \;\middle|\; B_r(x) \subset \Omega,\ r < \mu \right\},$$

where f_B denotes the average over B, i.e.,

$$f_B := \frac{1}{|B|} \int_B f(y)\, dy$$

and $B_r(x)$ denotes the closed ball of radius r centered at x and $|B|$ denotes the Lebesgue measure of B. The space $BMO^\mu(\Omega)$ is defined as

$$BMO^\mu(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; [f]_{BMO^\mu} < \infty \right\}.$$

This space may not agree with the space of restrictions $r_\Omega f$ of $f \in BMO^\mu(\mathbf{R}^n)$. As in [1], [2], [3], [4] we introduce a seminorm controlling the boundary behavior. For $\nu \in (0, \infty]$, we set

$$[f]_{b^\nu} := \sup\left\{ r^{-n} \int_{\Omega \cap B_r(x)} |f(y)|\, dy \;\middle|\; x \in \Gamma,\ 0 < r < \nu \right\}.$$

In these papers, the space

$$BMO_b^{\mu,\nu}(\Omega) := \left\{ f \in BMO^\mu(\Omega) \;\middle|\; [f]_{b^\nu} < \infty \right\}$$

is considered. Note that this space $BMO_b^{\infty,\infty}(\Omega)$ is identified with Miyachi's BMO introduced by [19] if Ω is a bounded Lipschitz domain or a Lipschitz half space as proved in [4]. However, unfortunately, it turns out such a boundary control for whole components of vector fields is too strict to have the Helmholtz decomposition. We separate tangential and normal components. Let $d_\Gamma(x)$ denote the distance from the boundary Γ, i.e.,

$$d_\Gamma(x) := \inf\left\{ |x - y|,\ y \in \Gamma \right\}.$$

For vector fields, we consider

$$vBMO^{\mu,\nu}(\Omega) := \left\{ v \in (BMO^\mu(\Omega))^n \;\middle|\; [\nabla d_\Gamma \cdot v]_{b^\nu} < \infty \right\},$$

where $\cdot$ denotes the standard inner product in $\mathbf{R}^n$. The quantity $(\nabla d_\Gamma \cdot v)\nabla d_\Gamma$ on Γ is the component of v normal to the boundary Γ. We set

$$[v]_{vBMO^{\mu,\nu}(\Omega)} := [v]_{BMO^\mu(\Omega)} + [\nabla d_\Gamma \cdot v]_{b^\nu}.$$

If Ω is the half space, this is not a norm but a seminorm. However, if it has a fully curved part in the sense of [11, Definition 7], then this becomes a norm [11, Lemma 8]. In particular, when Ω is a bounded C^2 domain, this is a norm. Roughly speaking, the boundary behavior of a vector field v is controlled for only normal part of v if $v \in vBMO^{\mu,\nu}(\Omega)$. For a bounded domain, this norm is equivalent no matter how μ and ν are taken; in other words, $vBMO^{\mu,\nu}(\Omega) = vBMO^{\infty,\infty}(\Omega)$. This is because $vBMO^{\mu,\nu}(\Omega) \subset L^1(\Omega)$ when Ω is bounded, which follows from the characterization of $vBMO^{\mu,\nu}(\Omega)$ in [11, Theorem 9]. We shall simply write $vBMO^{\mu,\nu}(\Omega)$ as $vBMO(\Omega)$. We are now in a position to state our main result.

Theorem 4.1.1. *Let Ω be a bounded C^3 domain in $\mathbf{R}^n$. Then the topological direct sum decomposition*

$$vBMO(\Omega) = vBMO_\sigma(\Omega) \oplus GvBMO(\Omega) \tag{4.1.2}$$

holds with

$$\begin{aligned} vBMO_\sigma(\Omega) &:= \left\{ v \in vBMO(\Omega) \mid \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}, \\ GvBMO(\Omega) &:= \left\{ \nabla q \in vBMO(\Omega) \mid q \in L^1_{\mathrm{loc}}(\Omega) \right\}, \end{aligned}$$

where $\mathbf{n}$ denotes the exterior unit normal vector field. In other words, for $v \in vBMO(\Omega)$, there is unique $v_0 \in vBMO_\sigma(\Omega)$ and $\nabla q \in GvBMO(\Omega)$ satisfying $v = v_0 + \nabla q$. Moreover, the mapping $v \mapsto v_0$, $v \mapsto \nabla q$ is bounded in $vBMO(\Omega)$.

As shown in [11], the norm trace $v \cdot \mathbf{n}$ is well defined as an element of $L^\infty(\Gamma)$ for $v \in vBMO(\Omega)$ with $\operatorname{div} v = 0$. So far, the Helmholtz decomposition BMO type space in a domain is only known for $vBMO^{\infty,\infty}$ when Ω is the half space

$$\mathbf{R}^n_+ = \left\{ x = (x_1, \dots, x_n) \in \mathbf{R}^n \mid x_n > 0 \right\}$$

as shown in [10], where the normal trace is taken in locally $H^{-1/2}$ sense.

Here is our strategy to show Theorem 6.1.1. For a vector field v, we construct a linear map $v \longmapsto q_1$ such that q_1 satisfies

$$-\Delta q_1 = \operatorname{div} v \quad \text{in} \quad \Omega,$$

where the divergence is taken in the sense of distribution. There are many ways to construct such a map because there is no boundary condition. A naive way is to extend v in a suitable way to a function $\overline{v}$ on $\mathbf{R}^n$ so that $v \longmapsto \overline{v}$ is linear. We next consider the volume potential of $\operatorname{div} \overline{v}$, i.e.,

$$q_0(x) := \int_{\mathbf{R}^n} E(x-y) \operatorname{div} \overline{v}(y)\, dy = E * \operatorname{div} \overline{v},$$

where E is the fundamental solution of $-\Delta$ in $\mathbf{R}^n$, i.e.,

$$E(x) := \begin{cases} -\log|x|/2\pi & (n=2) \\ |x|^{2-n}/\left(n(n-2)\alpha(n)\right) & (n \geq 3), \end{cases}$$

where $\alpha(n)$ denotes the volume of the unit ball $B_1(0)$ of $\mathbf{R}^n$. By the famous BMO-BMO estimate due to Fefferman and Stein [7], we have

$$[\nabla q_0]_{BMO^\infty(\mathbf{R}^n)} \leq C_0 [\overline{v}]_{BMO^\infty(\mathbf{R}^n)}$$

with $C_0 > 0$ independent of $\overline{v}$. However, it is difficult to control $[\nabla d_\Gamma \cdot \nabla q_0]_{b^\nu}$ so we construct another function q_1 instead of q_0.

Although BMO space does not allow the standard cut-off procedure, our space is in L^1, so we are able to decompose v into two parts $v = v_1 + v_2$ such that the support of v_2 is close to Γ while the support of v_1 is away from Γ; see Proposition 6.2.4. For v_1 we just set

$$q_1^1 = E * \operatorname{div} v_1$$

by extending v_1 as zero outside its support. Then, the L^∞ bound for ∇q_1^1 is well controlled near Γ, which yields a bound for b^ν semi-norm. To estimate v_2, we use a normal coordinate system near Γ and reduce the problem to the half space. Let d denotes the signed distance function where $d = d_\Gamma$ in Ω and $d = -d_\Gamma$ outside Ω. We extend v_2 to $\mathbf{R}^n$ so that the normal part $(\nabla d \cdot \overline{v_2})\nabla d$ is odd and the tangential part $\overline{v_2} - (\nabla d \cdot \overline{v_2})\nabla d$ is even in the direction of ∇d with respect to Γ. In such type of coordinate system, the minus Laplacian can be transformed as

$$L = A - B + \text{lower order terms}, \ A = -\Delta_\eta, \ B = \sum_{1 \leq i,j \leq n-1} \partial_{\eta_i} b_{ij} \partial_{\eta_j},$$

where η_n is the normal direction to the boundary so that $\{\eta_n > 0\}$ is the half space. By choosing a suitable coordinate system to represent Γ locally, we are able to arrange $b_{ij} = 0$ at one point of the boundary of the local coordinate system. We use a freezing coefficient method to construct volume potential q_1^2 and q_1^3, which corresponds to the contribution from the tangential part $\overline{v_2}^{\tan}$ and the normal part $\overline{v_2}^{\text{nor}}$ respectively. Since the leading term of $\operatorname{div} \overline{v_2}^{\text{nor}}$ in normal coordinate consists of the differential of η_n only, if we extend the coefficient b_{ij} even in η_n, q_1^3 is constructed so that the leading term of $\nabla d \cdot \nabla q_1^3$ is odd in the direction of ∇d. On the other hand, as the leading term of $\operatorname{div} \overline{v_2}^{\tan}$ in normal coordinate consists of the differential of $\eta' = (\eta_1, ..., \eta_{n-1})$ only, the even extension of b_{ij} in η_n gives rise to q_1^2 so that the leading term of $\nabla d \cdot \nabla q_1^2$ is also odd in the direction of ∇d. Disregarding lower order terms and localization procedure, we set q_1^2 and q_1^3 of the form

$$\begin{aligned} q_1^2 &= -L^{-1} \operatorname{div} \overline{v}_2^{\tan} = -A^{-1}(I - BA^{-1})^{-1} \operatorname{div} \overline{v}_2^{\tan}, \\ q_1^3 &= -L^{-1} \operatorname{div} \overline{v}_2^{\text{nor}} = -A^{-1}(I - BA^{-1})^{-1} \operatorname{div} \overline{v}_2^{\text{nor}}. \end{aligned}$$

One is able to arrange BA^{-1} small by taking a small neighborhood of a boundary point. Then $(I - BA^{-1})^{-1}$ is given as the Neumann series $\sum_{m=0}^\infty (BA^{-1})^m$. We are able to establish BMO-BMO estimate for ∇q_1^2 and ∇q_1^3, i.e.

$$\left[\nabla q_1^2\right]_{BMO(\mathbf{R}^n)} \leq C_0' \left[\overline{v}_2^{\tan}\right]_{BMO(\mathbf{R}^n)}, \ \left[\nabla q_1^3\right]_{BMO(\mathbf{R}^n)} \leq C_0' \left[\overline{v}_2^{\text{nor}}\right]_{BMO(\mathbf{R}^n)}$$

with some constant C_0' independent of $\overline{v_2}$. Since the leading term of $\nabla d \cdot (\nabla q_1^2 + \nabla q_1^3)$ is odd in the direction of ∇d with respect to Γ, the BMO bound implies b^ν bound. Note that $[\overline{v_2}^{\text{nor}}]_{BMO(\mathbf{R}^n)}$ is controlled by $[v_2]_{b^\nu}$ and $[v_2]_{BMO(\Omega)}$ since $\overline{v_2}^{\text{nor}}$ is odd in the direction of ∇d with respect to Γ. By the procedure sketched above, we are able to construct a suitable operator by setting $q_1 = q_1^1 + q_1^2 + q_1^3$.

Theorem 4.1.2 (Construction of a suitable volume potential)**.** *Let Ω be a bounded C^3 domain in $\mathbf{R}^n$. Then, there exists a linear operator $v \longmapsto q_1$ from $vBMO(\Omega)$ to $L^\infty(\Omega)$ such that*

$$-\Delta q_1 = \operatorname{div} v \quad in \quad \Omega$$

and that there exists a constant $C_1 = C_1(\Omega)$ satisfying

$$\|\nabla q_1\|_{vBMO(\Omega)} \leq C_1 \|v\|_{vBMO(\Omega)}.$$

In particular, the operator $v \longmapsto \nabla q_1$ is a bounded linear operator in $vBMO(\Omega)$.

By this operator, we observe that $w = v - \nabla q_1$ is divergence free in Ω. Unfortunately, this w may not fulfill the trace condition $w \cdot \mathbf{n} = 0$ on the boundary Γ. We construct another potential q_2 by solving the Neumann problem

$$\begin{aligned} \Delta q_2 &= 0 && \text{in} \quad \Omega \\ \frac{\partial q_2}{\partial n} &= w \cdot \mathbf{n} && \text{on} \quad \Gamma. \end{aligned}$$

We then set $q = q_1 + q_2$. Since $\partial q_2 / \partial \mathbf{n} = \nabla q_2 \cdot \mathbf{n}$, $v_0 = v - \nabla q$ gives the Helmholtz decomposition (6.1.1). To complete the proof of Theorem 6.1.1, it suffices to prove that $\|\nabla q_2\|_{vBMO(\Omega)}$ is bounded by a constant multiply of $\|v\|_{vBMO(\Omega)}$.

Lemma 4.1.3 (Estimate of the normal trace)**.** *Let Ω be a bounded $C^{2+\kappa}$ domain in $\mathbf{R}^n$ with $\kappa \in (0,1)$. Then there is a constant $C_2 = C_2(\Omega)$ such that*

$$\|w \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C_2 \|w\|_{vBMO(\Omega)}$$

for all $w \in vBMO(\Omega)$ with $\operatorname{div} w = 0$.

This is a special case of the trace theorem established in [11]. We finally need the estimate for the Neumann problem.

Lemma 4.1.4 (Estimate for the Neumann problem)**.** *Let Ω be a bounded C^2 domain. For $g \in L^\infty(\Gamma)$ satisfying $\int_\Gamma g \, d\mathcal{H}^{n-1} = 0$, there exists a unique (up to constant) solution u to the Neumann problem*

$$\begin{aligned} \Delta u &= 0 && \textit{in} \quad \Omega \\ \frac{\partial u}{\partial \mathbf{n}} &= g && \textit{on} \quad \Gamma \end{aligned} \tag{4.1.3}$$

such that the operator $g \longmapsto u$ is linear and that there exists a constant $C_3 = C_3(\Omega)$ such that

$$\|\nabla u\|_{vBMO(\Omega)} \leq C_3 \|g\|_{L^\infty(\Gamma)}.$$

Combining these two lemmas, Theorem 6.1.2 yields

$$\begin{aligned} \|\nabla q_2\|_{vBMO(\Omega)} &\leq C_3 C_2 \|v - \nabla q_1\|_{vBMO(\Omega)} \\ &\leq C_3 C_2 (1 + C_1) \|v\|_{vBMO(\Omega)}. \end{aligned}$$

Setting $q = q_1 + q_2$ and $v_0 = v - \nabla q$, we now observe that the projections $v \longmapsto v_0$, $v \longmapsto \nabla q$ are bounded in $vBMO(\Omega)$, which yields (6.1.3) in Theorem 6.1.1.

To show Lemma 6.1.4 let $N(x,y)$ be the Neumann Green function. Then a solution of (6.1.4) is given by $\int_\Gamma N(x,y) g(y) \, \mathrm{d}\mathcal{H}^{n-1}$. It is well-known (see e.g. [12, Appendix]) that leading part of N is $E(x-y)$. We have to estimate

$$\|\nabla E * (\delta_\Gamma \otimes g)\|_{vBMO^{\infty,\nu}(\Omega)} \,.$$

Here δ_Γ denotes the delta function supported on Γ, i.e.,

$$\delta_\Gamma : \psi \mapsto \int_\Gamma \psi \, \mathrm{d}\mathcal{H}^{n-1}$$

for $\psi \in C_c^\infty(\mathbb{R}^n)$. We take a C^2 cutoff function $\theta \geq 0$ such that $\theta(\sigma) = 1$ for $\sigma \leq 1$, $\theta(\sigma) = 0$ for $\sigma \geq 2$. We take δ small so that 2δ is smaller than the reach of Γ. By this choice, $\theta_d = \theta(d/\delta)$ is C^2 in $\mathbf{R}^n$, where d denotes the signed distance function from Γ so that $\nabla d = -\mathbf{n}$ on Γ. For $g \in L^\infty(\Gamma)$, we extend g so that $\nabla d \cdot g = 0$ near the 2δ-neighborhood of Γ. Let g_e denotes this extension and set $g_{e,c} = \theta_d g_e$. A key observation is that

$$\delta_\Gamma \otimes g = (\nabla 1_\Omega \cdot \nabla d) g_{e,c} = \operatorname{div}(g_{e,c} 1_\Omega \nabla d) - 1_\Omega \operatorname{div}(g_{e,c} \nabla d)$$
$$\operatorname{div}(g_{e,c} \nabla d) = g_{e,c} \Delta d + \nabla d \cdot \nabla g_{e,c} = g_{e,c} \Delta d + \frac{\theta'(d/\delta)}{\delta} g_e,$$

where 1_Ω is the characteristic function of Ω. The leading (singular) part of $\nabla E * (\delta_\Gamma \otimes g)$ is the term involving $\operatorname{div}(g_{e,c} 1_\Omega \nabla d)$. The famous L^∞-BMO estimate for the singular integral operator $\nabla E * \operatorname{div}$ yields

$$\|\nabla E * \operatorname{div}(g_{e,c} 1_\Omega \nabla d)\|_{BMO(\mathbf{R}^n)} \leq C \|g_{e,c} \nabla d\|_{L^\infty(\Omega)} \leq C' \|g\|_{L^\infty(\Gamma)}$$

with C and C' independent of g. All other terms can be estimated easily since the integral kernel is integrable. A direct calculation gives an L^∞ estimate near Γ for $\nabla d \cdot \nabla E * (\delta_\Gamma \otimes g)$ which yields

$$[\nabla d \cdot \nabla E * (\delta_\Gamma \otimes g)]_{b^\nu} \leq C_4 \|g\|_{L^\infty(\Gamma)}$$

with C_4 independent of g, but it is impossible to estimate b^ν-seminorm of the tangential part. This is the main reason why we use $vBMO$ instead of BMO_b-type space where b^ν-boundedness of ALL components of vector fields is imposed; see the end of Section 6.3.2.

To extend our results to a more general domain it seems to be reasonable to consider $vBMO \cap L^2$. This is because $L^p \cap L^2$ $(p > 2)$ admits the Helmholtz decomposition for arbitrary uniformly C^2 domains as proved in [5], [6].

Our approach in this chapter is to derive the boundedness of the operator $v \mapsto \nabla q$ by a potential-theoretic approach. In L^p setting there is a variational approach based on duality introduced by [21]; see also [5]. The key estimate is

$$\|\nabla q\|_{L^p(\Omega)} \leq C_5 \sup \left\{ \int_\Omega \nabla q \cdot \nabla \varphi \, dx \;\middle|\; \|\nabla \varphi\|_{L^{p'}(\Omega)} \leq 1 \right\}$$

with C_5 independent of q, where $1/p + 1/p' = 1$, $1 < p < \infty$. Formally, this estimate yields the desired bound $\|\nabla q\|_{L^p(\Omega)} \leq C_5 \|v\|_{L^p(\Omega)}$ since

$$\int_\Omega \nabla q \cdot \nabla \varphi \, dx = \int_\Omega v \cdot \nabla \varphi \, dx.$$

At this moment, it is not clear that similar estimate holds if one replaces $L^p(\Omega)$ by $vBMO$ since the predual space of $vBMO$ is not clear.

For BMO_b type solution, it is known that the Stokes semigroup is analytic [1], [3]. However, it is nontrivial to extend to the space $vBMO$ since in the half space the Stokes operator with Dirichlet boundary condition does not generate a semigroup because $[u(t)]_{vBMO}$ for the solution $u(t)$ may be non-zero for $t > 0$ for initial data u_0 with $[u_0]_{vBMO} = 0$ so that $u_0^{\tan}$ may be a non-zero constant [1, Example 6.5].

This chapter is organized as follows. In Section 4.2, to construct a volume potential of $\operatorname{div} v$, we localize the problem and reduce the problem to small neighborhoods of points on the boundary. In Section 4.3, we construct a leading part of the volume potential by a perturbation method called the freezing coefficient method. In these two sections, we complete the proof of Theorem 6.1.2. In Section 4.4, we prove Lemma 6.1.4 by estimating the single layer potential.

4.2 Construction of volume potential

For $v \in vBMO(\Omega)$, we shall construct a suitable potential q_1 so that $v \longmapsto \nabla q_1$ is a bounded linear operator in $vBMO$ as stated in Theorem 6.1.2. In this section, as a preliminary, we reduce the problem to the case that the support of v is contained in a small neighborhood of a point on the boundary and it consists of only normal part.

4.2.1 Localization procedure

Let Ω be a uniformly C^k domain in $\mathbf{R}^n$ ($k \geq 1$). In other words, there exists $r_*, \delta_* > 0$ such that for each $z_0 \in \Gamma$, up to translation and rotation, there exists a function h_{z_0} which is C^k in a closed ball $B_{r_*}(0')$ of radius r_* centered at the origin $0'$ of $\mathbf{R}^{n-1}$ satisfying following properties:

(i) $K_\Gamma := \sup_{B_{r_*}(0')} |(\nabla')^s h_{z_0}| < \infty$ for $s = 0, 1, 2, \ldots, k$, where ∇' denotes the gradient in $x' \in \mathbf{R}^{n-1}$; $\nabla' h(0') = 0$, $h(0') = 0$,

(ii) $\Omega \cap U_{r_*,\delta_*,h_{z_0}}(z_0) = \left\{ (x', x_n) \in \mathbf{R}^n \;\middle|\; h_{z_0}(x') < x_n < h_{z_0}(x') + \delta_*,\ |x'| < r_* \right\}$ for

$$U_{r_*,\delta_*,h_{z_0}}(z_0) := \left\{ (x', x_n) \in \mathbf{R}^n \;\middle|\; h_{z_0}(x') - \delta_* < x_n < h_{z_0}(x') + \delta_*,\ |x'| < r_* \right\},$$

(iii) $\Gamma \cap U_{r_*,\delta_*,h_{z_0}}(z_0) = \left\{ (x', x_n) \in \mathbf{R}^n \;\middle|\; x_n = h_{z_0}(x'),\ |x'| < r_* \right\}$.

A bounded C^k domain is, of course, a uniformly C^k domain.

Let d denote the signed distance function from Γ which is defined by

$$d(x) = \begin{cases} \inf_{y \in \Gamma} |x - y| & \text{for} \quad x \in \Omega, \\ -\inf_{y \in \Gamma} |x - y| & \text{for} \quad x \notin \Omega \end{cases} \tag{4.2.1}$$

so that $d(x) = d_\Gamma(x)$ for $x \in \Omega$. If Ω is a bounded C^2 domain, then there is $R_* > 0$ such that if $|d(x)| < R_*$, there is unique point πx such that $|x - \pi x| = |d(x)|$. The supremum of such R_* is called the reach of Ω and Ω^c. Moreover, d is C^2 in the R_*-neighborhood of Γ, i.e., $d \in C^2\left(\Gamma_{R_*}^{\mathbf{R}^n}\right)$ with

$$\Gamma_{R_*}^{\mathbf{R}^n} := \left\{ x \in \mathbf{R}^n \;\middle|\; |d(x)| < R_* \right\};$$

see [13, Chap. 14, Appendix], [17, §4.4]. Note that R_* satisfies

$$R_* = \min\left(R_*^\Omega, R_*^{\Omega^c}\right),$$

where R_*^Ω is the reach of Γ in Ω while $R_*^{\Omega^c}$ is the reach of Γ in the complement Ω^c of Ω. Let $K_\Gamma^* := \max\{K_\Gamma, 1\}$. There exists $0 < \rho_0 < \min(r_*, \delta_*, \frac{R_*}{2}, \frac{1}{2nK_\Gamma^*})$ such that

$$U_\rho(z_0) := \left\{ x \in \mathbf{R}^n \;\middle|\; (\pi x)' \in \operatorname{int} B_\rho(0'),\ |d(x)| < \rho \right\}$$

is contained in the coordinate chart $U_{r_*,\delta_*,h_{z_0}}(z_0)$ for any $\rho \leq \rho_0$.

We always take $\rho < \rho_0$. Since Ω is bounded and

$$\bigcup_{z \in \Gamma} U_\rho(z)$$

covers the compact set $K = \mathrm{cl}\left(\Gamma^{\mathbf{R}^n}_{\rho/2}\right)$, there exists a finite subcover $\{U_\rho(z_j)\}_{j=1}^m$ of K, where the number m depends on ρ. For $\sigma > 0$, we denote that

$$\Omega^\sigma = \Omega \backslash \Gamma^{\mathbf{R}^n}_\sigma, \ U_{\sigma,j} := U_\sigma(z_j).$$

Observe that

$$\overline{\Omega} \subset \bigcup_{j=1}^m U_{\rho,j} \cup \Omega^{\rho/2}.$$

Let $\{\varphi_j\}_{j=0}^m$ be a partition of the unity associated with $\{U_{\rho,j}\} \cup \{\Omega^{\rho/2}\}$ in the sense that

$$\begin{aligned} &\varphi_j \in C_c^\infty(U_{\rho,j} \cap \overline{\Omega}), \ 0 \leq \varphi_j \leq 1 \quad \text{for} \quad j = 1, \dots, m, \\ &\varphi_0 \in C_c^\infty(\Omega^{\rho/2}), \ 0 \leq \varphi_0 \leq 1, \ \varphi_0 = 1 \quad \text{in} \quad \Omega^\rho \end{aligned}$$

and

$$\sum_{j=0}^m \varphi_j = 1 \quad \text{in} \quad \overline{\Omega}.$$

Here $C_c^\infty(W)$ denotes the space of all smooth function in W whose support is compact in W.

Throughout this chapter, unless otherwise specified, the symbol C in an inequality represents a positive constant independent of quantities that appeared in the inequality. For a fixed $\rho > 0$, C_ρ represents a constant depending only on ρ. C_n represents a constant depending only on n and $C_{\Omega,n}$ represents a constant depending only on Ω and n.

4.2.2 Cut-off and extension

In general, multiplication by a smooth function to BMO is not bounded in BMO. Fortunately, our space is closed by multiplication.

Proposition 4.2.1 (Multiplication)**.** *Let Ω be a bounded C^2 domain in $\mathbf{R}^n$. Let $\varphi \in C^\gamma(\Omega)$, $\gamma \in (0,1)$. For each $v \in vBMO(\Omega)$, the function $\varphi v \in vBMO(\Omega)$ satisfies*

$$\|\varphi v\|_{vBMO(\Omega)} \leq C \|\varphi\|_{C^\gamma(\Omega)} \|v\|_{vBMO(\Omega)}$$

with C independent of φ and v.

Proof. Since

$$[\nabla d \cdot \varphi v]_{b^\nu} \leq \|\varphi\|_{L^\infty(\Omega)} [\nabla d \cdot v]_{b^\nu},$$

it suffices to establish the estimate

$$[\varphi v]_{BMO(\Omega)} \leq c_0 \|\varphi\|_{C^\gamma(\Omega)} \|v\|_{vBMO(\Omega)} \tag{4.2.2}$$

with c_0 independent of φ and v. Since a bounded Lipschitz domain is a uniform domain, we are able to apply [11, Theorem 13] to get

$$[\varphi v]_{BMO(\Omega)} \leq c_1 \|\varphi\|_{C^\gamma(\Omega)} ([v]_{BMO(\Omega)} + \|v\|_{L^1(\Omega)}).$$

This is based on the product estimate of a Hölder function and a function in $bmo(\mathbf{R}^n) := BMO(\mathbf{R}^n) \cap L^1_{\mathrm{ul}}(\mathbf{R}^n)$ where

$$L^1_{\mathrm{ul}}(\mathbf{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) \ \middle|\ \|f\|_{L^1_{\mathrm{ul}}(\mathbf{R}^n)} := \sup_{x \in \mathbf{R}^n} \int_{B_1(x)} |f(y)|\, dy < \infty \right\}.$$

The space $bmo(\mathbf{R}^n)$ is equipped with the norm

$$\|f\|_{bmo(\mathbf{R}^n)} := [f]_{BMO(\mathbf{R}^n)} + \|f\|_{L^1_{\mathrm{ul}}(\mathbf{R}^n)}$$

for $f \in bmo(\mathbf{R}^n)$. The product estimate for bmo follows from a similar result for a local Hardy space $h^1 = F^0_{1,2}$ [20, Remark 4.4] and duality $bmo = (h^1)'$ [20, Theorem 3.26]. To handle a function in Ω, we need an extension to conclude [11, Theorem 13]. Fortunately, by the characterization of $vBMO$ for a bounded C^2 domain [11, Theorem 9],

$$\|v\|_{L^1(\Omega)} \leq c_2 \|v\|_{vBMO(\Omega)}.$$

Here c_j denotes a constant independent of v and φ for $j = 1, 2$. Combining these two estimates, we obtain (4.2.2) with $c_0 = c_1(1 + c_2)$. This yields Proposition 6.2.4. □

For a bounded C^3 domain, we next consider an extension based on the normal coordinate in $U_\rho(z_0)$ for $\rho \leq \rho_0$ of the form

$$\left\{ \begin{array}{rcl} x' & = & \eta' + \eta_n \nabla' d(\eta', h_{z_0}(\eta')); \\ x_n & = & h_{z_0}(\eta') + \eta_n \partial_{x_n} d(\eta', h_{z_0}(\eta')). \end{array} \right. \tag{4.2.3}$$

Let $V_\sigma := B_\sigma(0') \times (-\sigma, \sigma)$ for $\sigma \in (0, \rho_0)$. We shall write this coordinate change by $x = \psi(\eta)$ with $\psi \in C^2(V_{\rho_0})$ and

$$x = \pi x - d(x)\mathbf{n}(\pi x), \quad \mathbf{n}(\pi x) = -\nabla d(\pi x).$$

We consider the projection to the direction to ∇d. For $x \in \Gamma^{\mathbf{R}^n}_{\rho_0}$, we set

$$P(x) = \nabla d(\pi x) \otimes \nabla d(\pi x) = \mathbf{n}(\pi x) \otimes \mathbf{n}(\pi x).$$

For later convenience, we set $Q(x) = I - P(x)$ which is the tangential projection for $x \in \Gamma^{\mathbf{R}^n}_{\rho_0}$. For a function f in $\Gamma^{\mathbf{R}^n}_\rho \cap \overline{\Omega}$, let f_{even} (resp. f_{odd}) denote its even (odd) extension to $\Gamma^{\mathbf{R}^n}_\rho$ defined by

$$\begin{array}{lll} f_{\mathrm{even}}\left(\pi x + d(x)\mathbf{n}(\pi x)\right) = f\left(\pi x - d(x)\mathbf{n}(\pi x)\right) & \text{for} & x \in \Gamma^{\mathbf{R}^n}_\rho \backslash \overline{\Omega}, \\ f_{\mathrm{odd}}\left(\pi x + d(x)\mathbf{n}(\pi x)\right) = -f\left(\pi x - d(x)\mathbf{n}(\pi x)\right) & \text{for} & x \in \Gamma^{\mathbf{R}^n}_\rho \backslash \overline{\Omega}. \end{array}$$

We denote r_W to be the restriction in W for any subset $W \subset \mathbf{R}^n$. Let f be a function (or a vector field) defined in V_σ for some $\sigma \in (0, \infty]$. We set $E_{\mathrm{even}} f$ to be the even extension of f in $V_\sigma \cap \mathbf{R}^n_+$ to V_σ with respect to the n-th variable, i.e.,

$$E_{\mathrm{even}} f(\eta', -\eta_n) = f(\eta', \eta_n)$$

for any $(\eta', \eta_n) \in V_\sigma \cap \mathbf{R}^n_+$.

For $v \in vBMO(\Omega)$ with supp $v \subset U_\rho(z_0) \cap \overline{\Omega}$, let $\overline{v}$ be its extension of the form

$$\overline{v}(x) := (P v_{\mathrm{odd}})(x) + (Q v_{\mathrm{even}})(x) \tag{4.2.4}$$

for $x \in U_\rho(z_0)$. Notice that supp $\overline{v} \subset U_\rho(z_0)$, $\overline{v}$ is indeed defined in $\mathbf{R}^n$ with $\overline{v}(x) = 0$ for any $x \in U_\rho(z_0)^{\mathrm{c}}$. Define

$$L_* := \sup_{z_0 \in \Gamma,\, \rho \leq \rho_0} \max \{ \|\nabla \psi\|_{L^\infty(V_\rho)} + \|\nabla \psi^{-1}\|_{L^\infty(U_\rho(z_0))}, 1 \}.$$

Since the boundary Γ is uniformly C^3, L_* is finite that depends on Ω only. We set $\rho_{0,*} = \rho_0 / 12 L_*$.

Proposition 4.2.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded C^2 domain, $z_0 \in \Gamma$ and $\rho \in (0, \rho_{0,*})$. There exists a constant C_ρ, which depends on ρ only, such that*

$$[\overline{v}]_{BMO(\mathbf{R}^n)} \leq C_\rho \|v\|_{vBMO(\Omega)},$$
$$[\nabla d \cdot \overline{v}]_{b^\nu(\Gamma)} \leq C_\rho \|v\|_{vBMO(\Omega)}$$

for all $v \in vBMO(\Omega)$ with $\operatorname{supp} v \subset U_\rho(z_0) \cap \overline{\Omega}$ *and $\nu > 0$.*

In the normal coordinate, $P\overline{v} = Pv_{\mathrm{odd}}$ is odd in η_n and $Q\overline{v} = Qv_{\mathrm{even}}$ is even in η_n. The key idea of proving this proposition is to reduce the problem to the case where the boundary is locally flat by invoking the normal coordinate.

Proof. Since $vBMO(\Omega) \subset L^1(\Omega)$, see e.g. [11, Theorem 9], by considering the normal coordinate change $y = \psi(\eta)$ in $U_\rho(z_0)$ we can deduce that $v_{\mathrm{even}}, v_{\mathrm{odd}} \in L^1(\mathbf{R}^n)$ satisfying

$$\|v_{\mathrm{even}}\|_{L^1(\mathbf{R}^n)} = \|v_{\mathrm{odd}}\|_{L^1(\mathbf{R}^n)} \leq 2L_*^2 \|v\|_{L^1(\Omega)}.$$

Hence $\overline{v} \in L^1(\mathbf{R}^n)$ satisfies the estimate $\|\overline{v}\|_{L^1(\mathbf{R}^n)} \leq C_{\Omega,n} \|v\|_{L^1(\Omega)}$. Since Ω is a uniform domain, by [16, Theorem 1] there exists $v_J \in BMO(\mathbf{R}^n)$ with $r_\Omega v_J = v$ and

$$[v_J]_{BMO(\mathbf{R}^n)} \leq C_{\Omega,n} [v]_{BMO^\infty(\Omega)}.$$

Suppose that $B_r(\zeta) \subset V^+_{4\rho L_*} := V_{4\rho L_*} \cap \mathbf{R}^n_+$. The mean value theorem implies that $\psi(B_r(\zeta)) \subset B_{L_* r}(x)$ with $x = \psi(\zeta)$. By change of variables $y = \psi(\eta)$ in $U_{4\rho L_*}(z_0)$, we see that

$$\begin{aligned}\frac{1}{|B_r(\zeta)|} \int_{B_r(\zeta)} |v \circ \psi(\eta) - c| \, d\eta &\leq L_* \cdot \frac{1}{|B_r(\zeta)|} \int_{\psi(B_r(\zeta))} |v(y) - c| \, dy \\ &\leq C_n L_*^{n+1} \cdot \frac{1}{|B_{L_* r}(x)|} \int_{B_{L_* r}(x)} |v_J(y) - c| \, dy\end{aligned}$$

for any constant vector $c \in \mathbf{R}^n$. By considering an equivalent definition of the BMO-seminorm, see e.g. [14, Proposition 3.1.2], we deduce that

$$[v \circ \psi]_{BMO^\infty(V^+_{4\rho L_*})} \leq C_{\Omega,n} [v]_{BMO^\infty(\Omega)}.$$

By recalling the results concerning the even extension of BMO functions in the half space, see [10, Lemma 3.2] and [10, Lemma 3.4], we can deduce that

$$[v_{\mathrm{even}} \circ \psi]_{BMO^\infty(V_{4\rho L_*})} \leq C_{\Omega,n} [v]_{BMO^\infty(\Omega)}. \tag{4.2.5}$$

Next, we shall estimate the BMO-seminorm of v_{even}. Let $B_r(x)$ be a ball with radius $r \leq \frac{\rho}{2}$. If either $B_r(x) \cap U_\rho(z_0) = \emptyset$ or $B_r(x) \subset \Omega$, there is nothing to prove. It is sufficient to consider $B_r(x)$ that intersects both $U_\rho(z_0)$ and Ω^{c}. In this case we can find $x_0 \in B_r(x) \cap U_\rho(z_0)$. Since $B_r(x) \subset B_{2r}(x_0) \subset B_{4\rho}(z_0) \subset U_{8\rho}(z_0)$, by considering change of variables $y = \psi(\eta)$ in $U_{8\rho}(x_0)$, we have that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |v_{\mathrm{even}}(y) - c| \, dy \leq \frac{L_*}{|B_r(x)|} \int_{\psi^{-1}(B_r(x))} |v_{\mathrm{even}} \circ \psi(\eta) - c| \, d\eta.$$

For any $y \in B_r(x)$, we have that $|y - z_0| < 4\rho$. Hence $\psi^{-1}(B_r(x)) \subset B_{L_* r}(\zeta) \subset B_{4\rho L_*}(0) \subset V_{4\rho L_*}$. By (5.4.1), we deduce that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |v_{\text{even}}(y) - (v_{\text{even}})_{B_r(x)}| \, dy \leq C_{\Omega,n} [v]_{BMO^\infty(\Omega)}.$$

Thus, we obtain that

$$[v_{\text{even}}]_{BMO^{\frac{\rho}{2}}(\mathbf{R}^n)} \leq C_{\Omega,n} [v]_{BMO^\infty(\Omega)}.$$

For a ball B with radius $r(B) > \frac{\rho}{2}$, a simple triangle inequality implies that

$$\frac{1}{|B|} \int_B |v_{\text{even}}(y) - (v_{\text{even}})_B| \, dy \leq \frac{2}{|B|} \int_B |v_{\text{even}}(y)| \, dy \leq \frac{C_n}{\rho^n} \|v_{\text{even}}\|_{L^1(\mathbf{R}^n)}.$$

Therefore, we obtain the BMO estimate for v_{even}, i.e.,

$$[v_{\text{even}}]_{BMO(\mathbf{R}^n)} \leq \frac{C_{\Omega,n}}{\rho^n} \|v\|_{vBMO(\Omega)}.$$

We shall then give the BMO estimate for Pv_{odd}. Since $\nabla d \in C^1(\Gamma^{\mathbf{R}^n}_{\rho_0})$, there exists $D_e \in C^1(\mathbf{R}^n)$ such that $\|D_e\|_{C^1(\mathbf{R}^n)} \leq \|\nabla d\|_{C^1(\Gamma^{\mathbf{R}^n}_{\rho_0})}$ and $r_{\Gamma^{\mathbf{R}^n}_{\rho_0}} D_e = \nabla d$, see the proof of [11, Theorem 13]. By the multiplication rule for bmo functions, we have that $(Pv)_E := (D_e \cdot v_{\text{even}}) D_e \in bmo(\mathbf{R}^n)$, see also [11, Theorem 13]. Consider the normal coordinate change in $U_{4\rho L_*}(z_0)$. Since $(Pv)_E = Pv$ in $U_{4\rho L_*}(z_0) \cap \Omega$, same argument in the second paragraph implies that

$$[Pv \circ \psi]_{BMO^\infty(V^+_{4\rho L_*})} \leq C_{\Omega,n} \|(Pv)_E\|_{bmo(\mathbb{R}^n)} \leq \frac{C_{\Omega,n}}{\rho^n} \|v\|_{vBMO(\Omega)}.$$

Let $\zeta \in V_{12\rho L_*} = \psi^{-1}(U_{12\rho L_*}(z_0))$ with $\zeta_n = 0$. Let $B_r(\zeta) \subset V_{12\rho L_*}$ and $x = \psi(\zeta)$. Since $F(B_r(\zeta) \cap V^+_{12\rho L_*}) \subset B_{L_* r}(x) \cap \Omega$, by considering change of variables $y = \psi(\eta)$ in $U_{12\rho L_*}(z_0)$, we can deduce that

$$\frac{1}{|B_r(\zeta)|} \int_{B_r(\zeta) \cap V^+_{12\rho L_*}} |Pv_{\text{odd}} \circ \psi(\eta)| \, d\eta \leq L_*^{n+1} [\nabla d \cdot v]_{b^\nu}. \tag{4.2.6}$$

Recall the results concerning the odd extension of BMO functions in the half space, see [10, Lemma 3.1], we have the estimate

$$[Pv_{\text{odd}} \circ \psi]_{BMO^\infty(V_{4\rho L_*})} \leq \frac{C_{\Omega,n}}{\rho^n} \|v\|_{vBMO(\Omega)}. \tag{4.2.7}$$

By considering (4.2.7) and the fact that $Pv_{\text{odd}} = (Pv)_E$ in Ω, same argument in the third paragraph implies the BMO estimate for Pv_{odd}, i.e.,

$$[Pv_{\text{odd}}]_{BMO(\mathbf{R}^n)} \leq \frac{C_{\Omega,n}}{\rho^n} \|v\|_{vBMO(\Omega)}.$$

Combining the BMO estimates for v_{even} and Pv_{odd}, we have that

$$[\overline{v}]_{BMO(\mathbf{R}^n)} \leq \frac{C_{\Omega,n}}{\rho^n} \|v\|_{vBMO(\Omega)}.$$

Notice that $\nabla d \cdot \overline{v} = v_{\text{odd}} \cdot \nabla d$ in $\mathbf{R}^n$. Let $x \in \Gamma$ and $r \leq \frac{\rho}{L_*}$. If $B_r(x) \cap U_\rho(z_0) = \emptyset$, then $v_{\text{odd}} = 0$ in $B_r(x)$. Suppose that $B_r(x) \cap U_\rho(z_0) \neq \emptyset$. Then we can find $x_0 \in B_r(x) \cap U_\rho(z_0) \cap \Gamma$. Let $\zeta_0 = \psi^{-1}(x_0)$, we have that $\psi^{-1}(B_r(x)) \subset B_{2L_* r}(\zeta_0) \subset V_{12\rho L_*}$. Hence,

$$
\begin{aligned}
r^{-n} \int_{B_r(x)} |v_{\text{odd}} \cdot \nabla d| \, dy &\leq \frac{2L_*}{r^n} \int_{B_{2L_* r}(\zeta_0) \cap V^+_{12\rho L_*}} |(v \cdot \nabla d) \circ \psi| \, d\eta \\
&\leq \frac{2L_*^2}{r^n} \int_{B_{2L_*^2 r}(x_0) \cap \Omega} |\nabla d \cdot v| \, dy \leq C_{\Omega,n} [\nabla d \cdot v]_{b^\nu}.
\end{aligned}
$$

For $r > \frac{\rho}{L_*}$, we simply have that

$$
r^{-n} \int_{B_r(x)} |v_{\text{odd}} \cdot \nabla d| \, dy \leq \frac{C_{\Omega,n}}{\rho^n} \|v_{\text{odd}}\|_{L^1(\mathbf{R}^n)} \leq \frac{C_{\Omega,n}}{\rho^n} \|v\|_{vBMO(\Omega)}.
$$

□

4.2.3 Volume potentials

To construct mapping $v \mapsto q_1$ in Theorem 6.1.2, for some ρ_* to be determined later in the next section, we localize v by using the partition of the unity $\{\varphi_j\}_{j=0}^m$ associated with the covering

$$
\{U_{\rho,j}\}_{j=1}^m \cup \Omega^{\rho/2}
$$

as in Section 6.2.1, where ρ is always assumed to satisfy $\rho \leq \rho_*/2$. Here and hereafter we always assumed that Ω is a bounded C^3 domain in $\mathbf{R}^n$.

Proposition 4.2.3. *There exists a constant C_ρ, which depends on ρ only, such that*

$$
\begin{aligned}
[\nabla q_1^1]_{BMO^\infty(\mathbf{R}^n)} &\leq C_\rho \|v\|_{vBMO(\Omega)}, \\
\|\nabla q_1^1(x)\|_{L^\infty(\Gamma^{\mathbf{R}^n}_{\rho/4})} &\leq C_\rho \|v\|_{vBMO(\Omega)}
\end{aligned}
$$

for $q_1^1 = E * \operatorname{div}(\varphi_0 v)$ and $v \in vBMO(\Omega)$. In particular,

$$
\left[\nabla q_1^1\right]_{b^\nu(\Gamma)} \leq C_\rho \|v\|_{vBMO(\Omega)}
$$

for $\nu < \rho/4$.

Proof. By the BMO-BMO estimate [7], we have the estimate

$$
\left[\nabla q_1^1\right]_{BMO(\mathbf{R}^n)} \leq C[\varphi_0 v]_{BMO(\mathbf{R}^n)}.
$$

Consider $x \in \Gamma^{\mathrm{R}^n}_{\rho/4}$. Since ∇q_1^1 is harmonic in $\Gamma^{\mathrm{R}^n}_{\rho/2}$ and $B_{\frac{\rho}{4}}(x) \subset \Gamma^{\mathrm{R}^n}_{\rho/2}$, the mean value property for harmonic functions implies that

$$
\nabla q_1^1(x) = \frac{C_n}{\rho^n} \int_{B_{\frac{\rho}{4}}(x)} \nabla q_1^1(y) \, dy.
$$

By Hölder's inequality, we can estimate $|\nabla q_1^1(x)|$ by $\frac{C_n}{\rho^{n/2}} \|\nabla q_1^1\|_{L^2(\mathbf{R}^n)}$. Since the convolution with $\nabla^2 E$ is bounded in L^p for any $1 < p < \infty$, see e.g. [14, Theorem 5.2.7 and Theorem 5.2.10], an interpolation inequality (cf. [4, Lemma 5]) implies that

$$
\|\nabla q_1^1\|_{L^2(\mathbf{R}^n)} \leq C \|\varphi_0 v\|_{L^2(\mathbf{R}^n)} \leq C \|\varphi_0 v\|^{\frac{1}{2}}_{L^1(\mathbf{R}^n)} [\varphi_0 v]^{\frac{1}{2}}_{BMO(\mathbf{R}^n)}.
$$

View $\varphi_0 v$ as the extension of $\varphi_0 v$ from Ω to $\mathbf{R}^n$. By the extension theorem for *bmo* functions [11, Theorem 12], we estimate $[\varphi_0 v]_{BMO(\mathbf{R}^n)}$ by $C_\rho[\varphi_0 v]_{BMO^\infty(\Omega)}$. Since $vBMO(\Omega) \subset L^1(\Omega)$, see [11, Theorem 9], Proposition 6.2.4 implies that

$$|\nabla q_1^1(x)| \leq C_\rho \|v\|_{vBMO(\Omega)}$$

for any $x \in \Gamma^{\mathrm{R}^n}_{\rho/4}$. □

We next set $v_1 := \varphi_0 v$ and $v_2 := 1 - v_1$. For each $\varphi_j v_2$ $(j = 1, ., m)$, we extend as in Proposition 6.2.5 to get $\overline{\varphi_j v_2}$ and set

$$\overline{v_2} := \sum_{j=1}^{m} \overline{\varphi_j v_2}.$$

Indeed, this extension is independent of the choice φ_j's but we do not use this fact. We next set

$$\overline{v_2}^{\tan} := Q\,\overline{v_2} = \sum_{j=1}^{m} Q\,(\varphi_j v_2)_{\mathrm{even}}.$$

For $1 \leq j \leq m$, $\varphi_j \in C_{\mathrm{c}}^\infty(U_{\rho,j} \cap \overline{\Omega})$ implies that the even extension of φ_j in $U_{\rho,j}$ with respect to Γ is Hölder continuous in the sense that $(\varphi_j)_{\mathrm{even}} \in C^{0,1}(U_{\rho,j})$. Moreover, we have that $(\varphi_j)_{\mathrm{even}} \in C^{0,1}(\mathbf{R}^n)$ satisifes

$$\|(\varphi_j)_{\mathrm{even}}\|_{C^{0,1}(\mathbf{R}^n)} \leq C_\rho \|(\varphi_j)_{\mathrm{even}}\|_{C^{0,1}(U_{\rho,j})}.$$

For simplicity of notations, we denote $Q\,(\varphi_j v_2)_{\mathrm{even}}$ by $w_j^{\tan}$ for every $1 \leq j \leq m$. Now, we are ready to construct the suitable potential corresponding to $\overline{v_2}^{\tan}$.

Proposition 4.2.4. *There exists $\rho_* > 0$ such that if $\rho < \rho_*/2$, then for every $1 \leq j \leq m$, there exists a linear operator $v \longmapsto p_j^{\tan}$ from $vBMO(\Omega)$ to $L^\infty(\mathbf{R}^n)$ such that*

$$-\Delta p_j^{\tan} = \operatorname{div} w_j^{\tan} \ \ \text{in} \ \ U_{2\rho,j} \cap \Omega$$

and that there exists a constant C_ρ, independent of v, such that

$$[\nabla p_j^{\tan}]_{BMO(\mathbf{R}^n)} \leq C_\rho \|v\|_{vBMO(\Omega)},$$

$$\sup_{x\in\Gamma, r<\rho} \frac{1}{r^n} \int_{B_r(x)} |\nabla d \cdot \nabla p_j^{\tan}|\, dy \leq C_\rho \|v\|_{vBMO(\Omega)}.$$

Having the estimate for the volume potential near the boundary regarding its tangential component, we are left to handle the contribution from $\overline{v}_2^{\mathrm{nor}} := \overline{v}_2 - \overline{v}_2^{\tan}$. We recall its decomposition

$$\overline{v}_2^{\mathrm{nor}} = \sum_{j=1}^{m} P\,(\varphi_j v_2)_{\mathrm{odd}}.$$

For simplicity of notations, we denote $P\,(\varphi_j v_2)_{\mathrm{odd}}$ by w_j^{nor} for every $1 \leq j \leq m$. With a similar idea of proof, we can establish the suitable potential corresponding to $\overline{v}_2^{\mathrm{nor}}$.

Proposition 4.2.5. *There exists $\rho_* > 0$ such that if $\rho < \rho_*/2$, then for every $1 \leq j \leq m$, there exists a linear operator $v \longmapsto p_j^{\mathrm{nor}}$ from $vBMO(\Omega)$ to $L^\infty(\mathbf{R}^n)$ such that*

$$-\Delta p_j^{\mathrm{nor}} = \operatorname{div} w_j^{\mathrm{nor}} \quad \text{in } U_{2\rho,j} \cap \Omega$$

and that there exists a constant C_ρ, independent of v, such that

$$[\nabla p_j^{\mathrm{nor}}]_{BMO(\mathbf{R}^n)} \leq C_\rho \|v\|_{vBMO(\Omega)},$$
$$\sup_{x\in\Gamma, r<\rho} \frac{1}{r^n} \int_{B_r(x)} |\nabla d \cdot \nabla p_j^{\mathrm{nor}}| \, dy \leq C_\rho \|v\|_{vBMO(\Omega)}.$$

Once these two propositions are proved, we are able to prove Theorem 6.1.2.

Theorem 6.1.2 admitting Proposition 6.2.7 and 6.2.8. Fix $1 \leq j \leq m$. Let us first consider the contribution from the tangential part. We take a cut-off function $\theta_j \in C_c^\infty(U_{2\rho,j})$ such that $\theta_j = 1$ on $U_{\rho,j}$ and $0 \leq \theta_j \leq 1$. We next set

$$q_{1,j}^{\tan} := \theta_j p_j^{\tan} + E * \left(p_j^{\tan} \Delta\theta_j + 2\nabla\theta_j \cdot \nabla p_j^{\tan}\right).$$

By definition, Proposition 6.2.7 says that

$$\begin{aligned} -\Delta q_{1,j}^{\tan} &= -\Delta(\theta_j p_j^{\tan}) + p_j^{\tan}\Delta\theta_j + 2\nabla\theta_j \cdot \nabla p_j^{\tan} \\ &= \theta_j \operatorname{div} w_j^{\tan} = \operatorname{div} w_j^{\tan} \end{aligned}$$

in Ω as $\operatorname{supp} w_j^{\tan} \subset U_{\rho,j}$. By interpolation as in the proof of Proposition 6.2.7, we observe that $\|p_j^{\tan}\|_{L^\infty(\mathbf{R}^n)}$, $\|\nabla p_j^{\tan}\|_{L^p(\mathbf{R}^n)}$ are controlled by $\|v\|_{BMO(\Omega)}$. Since ∇E is in $L^{p'}(B_R)$ for $p' < n/(n-1)$ where $R = \operatorname{diam}\Omega + 4\rho$, it follows that

$$\sup_{\mathbf{R}^n} |\nabla E * (p_j^{\tan}\Delta\theta_j + 2\nabla\theta_j \cdot \nabla p_j^{\tan})| \leq C_\rho \|v\|_{vBMO(\Omega)}.$$

Thus, by Proposition 6.2.7, we conclude that the restriction of $q_{1,j}^{\tan}$ on Ω, which is still denoted by $q_{1,j}^{\tan}$, fulfills

$$\|\nabla q_{1,j}^{\tan}\|_{vBMO(\Omega)} \leq C_\rho \|v\|_{vBMO(\Omega)}. \tag{4.2.8}$$

By Proposition 6.2.8, a similar argument yields an estimate of type (4.2.8) for

$$q_{1,j}^{\mathrm{nor}} := \theta_j p_j^{\mathrm{nor}} + E * (p_j^{\mathrm{nor}}\Delta\theta_j + 2\nabla\theta_j \cdot \nabla p_j^{\mathrm{nor}}).$$

Set

$$q_1^2 = \sum_{j=1}^m q_{1,j}^{\tan}, \; q_1^3 = \sum_{j=1}^m q_{1,j}^{\mathrm{nor}}, \; q_1 = q_1^1 + q_1^2 + q_1^3.$$

Observe that q_1^2 and q_1^3 satisfy the desired estimates in Theorem 6.1.2. Moreover, by construction we have that

$$\begin{aligned} -\Delta q_1 &= -\Delta q_1^1 - \Delta q_1^2 - \Delta q_1^3 \\ &= \operatorname{div} v_1 + \sum_{j=1}^m \operatorname{div} w_j^{\tan} + \sum_{j=1}^m \operatorname{div} w_j^{\mathrm{nor}} \\ &= \operatorname{div}(v_1 + v_2) = \operatorname{div} v \end{aligned}$$

in Ω. □

4.3 Volume potentials based on normal coordinates

Our goal in this section is to prove Proposition 6.2.7 and Proposition 6.2.8. We write the Laplace operator by a normal coordinate system and construct a volume potential keeping the parity of functions with respect to the boundary. For this purpose, we adjust a perturbation method called a freezing coefficient method which is often used to construct a fundamental solution to an operator with variable coefficients.

4.3.1 A perturbation method keeping parity

We consider an elliptic operator of the form

$$L_0 = A - B, \quad A = -\Delta_\eta, \quad B = \sum_{1\le i,j\le n-1} \partial_{\eta_i} b_{ij}(\eta)\partial_{\eta_j}$$

in a cylinder $V_{4\rho}$. We assume that

(B1) (Regularity) $b_{ij} \in \mathrm{Lip}(V_{4\rho})$ $(1 \le i, j \le n-1)$,

(B2) (Parity) b_{ij} is even in η_n, i.e., $b_{ij}(\eta', \eta_n) = b_{ij}(\eta', -\eta_n)$ for $\eta \in V_{4\rho}$,

(B3) (Smallness) $b_{ij}(0) = 0$ $(1 \le i, j \le n-1)$.

For $\rho > 0$, let Y_ρ denotes the space

$$\left\{ g \in BMO(\mathbf{R}^n) \cap L^2(\mathbf{R}^n) \;\middle|\; \operatorname{supp} g \subset V_\rho,\ g(\eta', \eta_n) = g(\eta', -\eta_n) \text{ for } \eta \in V_\rho \right\},$$

whereas Z_ρ denotes the space

$$\left\{ f \in BMO(\mathbf{R}^n) \;\middle|\; \operatorname{supp} f \subset V_\rho,\ f(\eta', \eta_n) = -f(\eta', -\eta_n) \text{ for } \eta \in V_\rho \right\}.$$

The oddness condition in Z_ρ guarantees that

$$\frac{1}{r^n} \int_{B_r(\eta',0)} f \, d\eta = 0$$

for any $r > 0$ and $\eta' \in \mathbf{R}^{n-1}$, which implies that

$$\frac{1}{r^n} \int_{B_r(\eta',0)} |f| \, d\eta \le [f]_{BMO(\mathbf{R}^n)}$$

for any $r > 0$ and $\eta' \in \mathbf{R}^{n-1}$. Hence f is L^1 in $\mathbf{R}^n$.

Lemma 4.3.1. *Assume that (B1) – (B3). Then, there exists $\rho_* > 0$ depending only on n and b such that the following property holds provided that $\rho \in (0, \rho_*)$. There exists a bounded linear operator $f \mapsto q_o$ from Z_ρ to $L^\infty(\mathbf{R}^n)$ such that*

(i)

$$[\nabla_\eta q_o]_{BMO(\mathbf{R}^n)} \le C[f]_{BMO(\mathbf{R}^n)} \quad \textit{for all} \quad f \in Z_\rho$$

with some C independent of f;

(ii)

$$L_0 q_o = \partial_{\eta_n} f \quad \textit{in} \quad V_{2\rho};$$

(iii) q_o *is even in* $\mathbf{R}^n$ *with respect to* η_n, *i.e.* $q_o(\eta', \eta_n) = q_o(\eta', -\eta_n)\ \forall\, \eta \in \mathbf{R}^n$;

(iv)

$$\sup\left\{ \frac{1}{r^n} \int_{B_r(\eta',0)} |\partial_{\eta_n} q_o| \, d\eta \;\middle|\; 0 < r < \infty,\, \eta' \in \mathbf{R}^{n-1} \right\} \le C[f]_{BMO(\mathbf{R}^n)}.$$

Proof. By (B3) and (B1), we observe that

$$\overline{\lim_{\rho\downarrow 0}}\, \|b_{ij}\|_{C^\gamma(V_{4\rho})} \,/\, \rho^{1-\gamma} < \infty$$

for any $\gamma \in (0,1)$ and $1 \le i,j \le n-1$. Indeed, for $1 \le i,j \le n-1$, (B1) and (B3) imply that

$$\begin{aligned} \|b_{ij}\|_{L^\infty(V_{4\rho})} &\le 8L\rho, \\ [b_{ij}]_{C^\gamma(V_{4\rho})} &:= \sup\left\{ |b_{ij}(\eta) - b_{ij}(\zeta)| \,/\, |\eta - \zeta|^\gamma \;\middle|\; \eta, \zeta \in V_{4\rho} \right\} \\ &\le L(16\rho)^{1-\gamma}, \end{aligned}$$

where L is the maximum of Lipschitz bound for b_{ij} for all $1 \le i,j \le n-1$. We next take a cut-off function. We take $\theta \in C_c^\infty(V_4)$ such that $\theta = 1$ on V_2 and $0 \le \theta \le 1$ in V_4, we may assume θ is radial so that θ is even in η_n. We rescale θ by setting

$$\theta_\rho(\eta) = \theta(\eta/\rho)$$

so that $\theta_\rho = 1$ on $V_{2\rho}$. Since $\|\nabla\theta_\rho\|_\infty \rho$ is bounded as $\rho \to 0$, we see that

$$\overline{\lim_{\rho\downarrow 0}}\, [\theta_\rho]_{C^\gamma(V_{4\rho})} \rho^\gamma < \infty.$$

Hence, the estimate

$$[\theta_\rho b_{ij}]_{C^\gamma(V_{4\rho})} \le [\theta_\rho]_{C^\gamma(V_{4\rho})} \|b_{ij}\|_{L^\infty(V_{4\rho})} + [b_{ij}]_{C^\gamma(V_{4\rho})} \|\theta_\rho\|_{L^\infty(V_{4\rho})}$$

implies that

$$\overline{\lim_{\rho\downarrow 0}}\, \|\theta_\rho b_{ij}\|_{C^\gamma(V_{4\rho})} \,/\, \rho^{1-\gamma} < \infty.$$

We then set

$$L_1 = A - B_1, \quad B_1 = \sum_{1\le i,j\le n-1} \partial_{\eta_i} b_{ij}^\rho \partial_{\eta_j}, \quad b_{ij}^\rho = b_{ij}\theta_\rho.$$

For $1 \le i,j \le n-1$, notice that b_{ij}^ρ satisfies the same property of b_{ij} in (B1) – (B3). Moreover,

$$\operatorname{supp} b_{ij}^\rho \subset V_{4\rho} \quad \text{and} \quad \left\| b_{ij}^\rho \right\|_{C^\gamma(V_{4\rho})} \le c_b \rho^{1-\gamma}, \ \rho > 0$$

with some c_b independent of ρ. Since $\operatorname{supp} b_{ij}^\rho \subset V_{4\rho}$, we actually have that $b_{ij}^\rho \in C^\gamma(\mathbf{R}^n)$ together with the estimate

$$\|b_{ij}^\rho\|_{C^\gamma(\mathbf{R}^n)} \le \|b_{ij}^\rho\|_{C^\gamma(V_{4\rho})}.$$

For a given $f \in Z_\rho$, we define q_o by

$$q_o := \sum_{k=0}^\infty A^{-1} \left(B_1 A^{-1}\right)^k \partial_{\eta_n} f,$$

where formally for a function h we mean $A^{-1}h$ by $E * h$. The parity condition (iii) is clear once q_o is well defined as a function. Since

$$L_1 q_o = \sum_{k=0}^{\infty} \left(B_1 A^{-1}\right)^k \partial_{\eta_n} f - \sum_{k=1}^{\infty} \left(B_1 A^{-1}\right)^k \partial_{\eta_n} f = \partial_{\eta_n} f$$

in $\mathbf{R}^n$, the property (ii) then follows since $L_1 = L_0$ in $V_{2\rho}$.

It remains to prove the convergence of q_o as well as (i). For this purpose, we reinterpret q_o in a different way. We rewrite

$$B_1 = \operatorname{div}' \cdot \nabla'_B \quad \text{with} \quad \nabla'_B = \left(\sum_{j=1}^{n-1} b^\rho_{ij} \partial_{\eta_j} \right)_{1 \le i \le n-1}$$

and observe that

$$\begin{aligned} q_o &= \sum_{k=0}^{\infty} A^{-1} \operatorname{div}' \cdot G^k \cdot \nabla'_B A^{-1} \partial_{\eta_n} f + A^{-1} \partial_{\eta_n} f, \\ G &:= \nabla'_B A^{-1} \operatorname{div}'. \end{aligned}$$

Denote

$$b^\rho := \left(b^\rho_{ij} \right)_{1 \le i,j \le n-1}.$$

Since $\partial_{\eta_\alpha} A^{-1} \partial_{\eta_\beta}$ is bounded in BMO [7] and also in L^p $(1 < p < \infty)$ for all $\alpha, \beta = 1, \dots, n$, see e.g. [14, Theorem 5.2.7 and Theorem 5.2.10], by a multiplication theorem we can deduce the estimates

$$\|Gh\|_{L^p(\mathbf{R}^n)} \le C_p \|b^\rho\|_{L^\infty(\mathbf{R}^n)} \|h\|_{L^p(\mathbf{R}^n)}, \tag{4.3.1}$$

$$[Gh]_{BMO(\mathbf{R}^n)} \le C'_\infty \|b^\rho\|_{C^\gamma(\mathbf{R}^n)} \left([h]_{BMO(\mathbf{R}^n)} + \|h\|_{L^1(\mathbf{R}^n)} \right) \tag{4.3.2}$$

provided that $\operatorname{supp} h \subset V_{4\rho}$ and $\rho < 1$. Here C_p and C'_∞ are independent of ρ and h. Similar estimate holds for $\nabla'_B A^{-1} \partial_{\eta_n}$. Since $\|f\|_{L^1(\mathbf{R}^n)} \le C_\rho [f]_{BMO(\mathbf{R}^n)}$ for $f \in Z_\rho$, by an interpolation (cf. [4, Lemma 5]) we see that the L^p norm of f is also controlled, i.e., $\|f\|_{L^p(\mathbf{R}^n)} \le C_\rho [f]_{BMO(\mathbf{R}^n)}$ for any $1 \le p < \infty$. By the support condition, $A^{-1} \operatorname{div}'$ and $A^{-1} \partial_{\eta_n}$ is bounded from $L^p \to L^\infty$ for $p > n$ with bound K, we see that

$$\begin{aligned} \|q_o\|_{L^\infty(\mathbf{R}^n)} &\le K \left(\left\| \sum_{k=0}^{\infty} G^k \nabla'_B A^{-1} \partial_{\eta_n} f \right\|_{L^p(\mathbf{R}^n)} + \|f\|_{L^p(\mathbf{R}^n)} \right) \\ &\le K \left(\sum_{k=0}^{\infty} C_p^{k+1} \|b^\rho\|^{k+1}_{L^\infty(\mathbf{R}^n)} \|f\|_{L^p(\mathbf{R}^n)} + \|f\|_{L^p(\mathbf{R}^n)} \right), \quad p > n. \end{aligned}$$

If ρ is taken small so that

$$\sum_{k=0}^{\infty} (C_p \cdot 8L\rho)^{k+1} < \infty,$$

then q_o converges uniformly in $\mathbf{R}^n$ and $\|q_o\|_{L^\infty(\mathbf{R}^n)} \le C_\rho [f]_{BMO(\mathbf{R}^n)}$ with some C_ρ independent of f.

Set

$$\|h\|_{BMOL^p(\mathbf{R}^n)} := [h]_{BMO(\mathbf{R}^n)} + \|h\|_{L^p(\mathbf{R}^n)}.$$

By estimates (4.3.1) and (4.3.2), we observe that

$$\|Gh\|_{BMOL^p(\mathbf{R}^n)} \leq C_* \|b^\rho\|_{C^\gamma(\mathbf{R}^n)} \|h\|_{BMOL^p(\mathbf{R}^n)}, \quad 1 < p < \infty,$$

where $C_* = C_p + C'_\infty \cdot C_n$ with C_n independent of ρ and h. We next estimate ∇q_o. By the similar estimate for $\nabla'_B A^{-1} \operatorname{div}'$ and $\nabla'_B A^{-1} \partial_{\eta_n}$, we have that

$$\|\nabla q_o\|_{BMOL^p(\mathbf{R}^n)} \leq \left(\sum_{k=0}^{\infty} C_*^{k+1} \|b^\rho\|_{C^\gamma(\mathbf{R}^n)}^{k+1} + C_* \|b^\rho\|_{C^\gamma(\mathbf{R}^n)} \right) \|f\|_{BMOL^p(\mathbf{R}^n)}.$$

We fix $p > n$ and take $\rho < \frac{1}{8LC_p}$ sufficiently small so that

$$\sum_{k=0}^{\infty} \left(C_* \cdot c_b \rho^{1-\gamma} \right)^{k+1} < \infty.$$

Then we get our desired estimate

$$\|\nabla q_o\|_{BMOL^p(\mathbf{R}^n)} \leq C_\rho \|f\|_{BMOL^p(\mathbf{R}^n)} \leq C_\rho [f]_{BMO(\mathbf{R}^n)}$$

for $f \in Z_\rho$. This completes the proof of (i).

Since $\partial_{\eta_n} q_o$ is odd in η_n so that

$$\frac{1}{r^n} \int_{B_r(\eta',0)} \partial_{\eta_n} q_o \, d\eta = 0$$

for any $\eta' \in \mathbf{R}^{n-1}$, the left-hand side of (iv) is estimated by a constant multiple of $[\partial_{\eta_n} q_o]_{BMO(\mathbf{R}^n)}$, which is estimated by a constant multiple of $[f]_{BMO(\mathbf{R}^n)}$. The proof of (iv) is now complete. □

Similarly, we are able to establish the following which corresponds to a version of Lemma 4.3.1 for the space Y_ρ.

Lemma 4.3.2. *Assume that (B1) – (B3). Then, there exists $\rho_* > 0$ depending only on n and b such that the following property holds provided that $\rho \in (0, \rho_*)$. For each $1 \leq i \leq n-1$, there exists a bounded linear operator $g \mapsto q_{e,i}$ from Y_ρ to $L^\infty(\mathbf{R}^n)$ such that*

(i)

$$[\nabla q_{e,i}]_{BMO(\mathbf{R}^n)} \leq C \|g\|_{BMOL^2(\mathbf{R}^n)} \quad \text{for all} \quad g \in Y_\rho$$

with some C independent of f;

(ii)

$$L_0 q_{e,i} = \partial_{\eta_i} g \quad \text{in} \quad V_{2\rho};$$

(iii) $q_{e,i}$ is even in $\mathbf{R}^n$ with respect to η_n, i.e. $q_{e,i}(\eta', \eta_n) = q_{e,i}(\eta', -\eta_n) \; \forall \, \eta \in \mathbf{R}^n$;

(iv)

$$\sup \left\{ \frac{1}{r^n} \int_{B_r(\eta',0)} |\partial_{\eta_n} q_{e,i}| \, d\eta \;\middle|\; 0 < r < \infty, \, \eta' \in \mathbf{R}^{n-1} \right\} \leq C \|g\|_{BMOL^2(\mathbf{R}^n)}.$$

Proof. Fix $1 \le i \le n-1$. Since g is even in $\mathbf{R}^n$ with respect to η_n, $\partial_{\eta_i} g$ is also even in $\mathbf{R}^n$ with respect to η_n. This means that $\partial_{\eta_i} g$ has the same parity with $\partial_{\eta_n} f$ in Lemma 4.3.1. By considering

$$q_{e,i} := \sum_{k=0}^{\infty} A^{-1}(B_1 A^{-1})^k \partial_{\eta_i} g,$$

exactly the same arguments of the proof of Lemma 4.3.1 finish the rest of the work. □

We take ρ_* in Lemma 4.3.1 and Lemma 4.3.2 to be

$$\rho_* := \min \left\{ \rho_{0,*},\ \frac{1}{8LC_p},\ \left(\frac{1}{C_* \cdot c_b}\right)^{\frac{1}{1-\gamma}} \right\}.$$

4.3.2 Laplacian in a normal coordinate system

Take $z_0 \in \Gamma$. Let us recall the normal coordinate system (4.2.3) introduced in Section 6.2.1, i.e.,

$$\begin{cases} x' &= \eta' + \eta_n \nabla' d(\eta', h_{z_0}(\eta')); \\ x_n &= h_{z_0}(\eta') + \eta_n \partial_{\eta_n} d(\eta', h_{z_0}(\eta')) \end{cases}$$

in $U_{\rho_0}(z_0)$ with $\nabla' h_{z_0}(0') = 0$, $h_{z_0}(0') = 0$ up to translation and rotation such that $z_0 = 0$ and

$$-\mathbf{n}\left(\eta', h_{z_0}(\eta')\right) = \left(-\nabla' h_{z_0}(\eta'), 1\right) \Big/ \left(1 + \left|\nabla'_{z_0} h(\eta')\right|^2\right)^{1/2}, \quad \eta' \in B_{\rho_0}.$$

Since Γ is C^3, the mapping $x = \psi(\eta) \in C^2(V_{\rho_0})$ in $U_{\rho_0}(z_0)$, it is a local C^2-diffeomorphism. Moreover, its Jacobi matrix $D\psi$ is the identity at 0, i.e.,

$$\nabla\psi(0) = I = \nabla\psi^{-1}(0).$$

A direct calculation shows that in $U_{\rho_0}(z_0) \cap \Omega$,

$$-\Delta_x = -\Delta_\eta - \left\{ \sum_{\substack{1 \le i,j \le n-1 \\ i \ne j}} \gamma_{ij} \partial_{\eta_i} \partial_{\eta_j} + \sum_{j=1}^{n-1} (\gamma_{jj} - 1) \partial^2_{\eta_j} \right\}$$
$$- \sum_{1 \le i,j \le n} \frac{\partial^2 \eta_j}{\partial x_i^2} \partial_{\eta_j},\ \gamma_{ij} = \sum_{k=1}^{n} \frac{\partial \eta_j}{\partial x_k} \frac{\partial \eta_i}{\partial x_k}.$$

Note that $\gamma_{jj}(0) = 1$ while $\gamma_{ij}(0) = 0$ if $i \ne j$. Changing order of multiplication and differentiation, we conclude that

$$\begin{aligned} &-\Delta_x = \tilde{L}_0 + \tilde{M}, \\ &\tilde{L}_0 := A - \tilde{B}, \quad A := -\Delta_\eta, \quad \tilde{B} := \sum_{1 \le i,j \le n-1} \partial_{\eta_i} \tilde{b}_{ij}(\eta) \partial_{\eta_j}, \\ &\tilde{M} := \sum_{j=1}^{n} \tilde{c}_j(\eta) \partial_{\eta_j} \end{aligned}$$

with $\tilde{b}_{ij} = \gamma_{ij} - \delta_{ij}$, $\tilde{c}_j = -\sum_{i=1}^n \frac{\partial^2 \eta_j}{\partial x_i^2} + \sum_{i=1}^n \partial_{\eta_i} \gamma_{ij}$. Note that if $\Gamma = \partial\Omega$ is C^3, $\tilde{b}_{ij} \in C^1(V_{\rho_0})$ and $\tilde{c}_j \in C(V_{\rho_0})$. We restrict $\tilde{b}_{ij}$, $\tilde{c}_j$ in $V_{\rho_0} \cap \mathbf{R}^n_+$ and extend to V_{ρ_0} so that the extended function b_{ij}, c_j's are even in V_{ρ_0} with respect to η_n, i.e., we set $b_{ij} = E_{\mathrm{even}}\ r_{V_{\rho_0}\cap\mathbf{R}^n_+}\ \tilde{b_{ij}}$ and $c_j = E_{\mathrm{even}}\ r_{V_{\rho_0}\cap\mathbf{R}^n_+}\ \tilde{c_j}$. By this extension, b_{ij} may not be in C^1 but still Lipschitz and $c_j \in C(V_{\rho_0})$. We set

$$B := \sum_{1\le i,j\le n-1} \partial_{\eta_i} b_{ij}(\eta)\partial_{\eta_j},$$
$$M := \sum_{j=1}^{n} c_j(\eta)\partial_{\eta_j}$$

and

$$L := L_0 + M, \quad L_0 = A - B.$$

The operator L may not agree with $-\Delta_x$ outside $U_{\rho_0}(z_0)\cap\Omega$. We summarize what we observe so far.

Proposition 4.3.3. *Let $\Gamma = \partial\Omega$ be C^3 and ρ_0 be chosen as in Section 6.2.1. For $z_0 \in \Gamma$, L_0 satisfies (B1) – (B3). Moreover, $-\Delta_x = L$ in $U_{\rho_0}(z_0)\cap\Omega$ and the coefficient of M is in $C(V_{\rho_0})$.*

Although we do not use the explicit formula of Δ in normal coordinates, we give it for $n = 2$ when we take the arc length parameter s to represent Γ. The coordinate transform is of the form

$$x_1 = \phi_1(x) + r\phi_2'(s)$$
$$x_2 = \phi_2(x) - r\phi_1'(s)$$

with $\phi_1'^2 + \phi_2'^2 = 1$ and $r = d(x)$. A direct calculation yields

$$-\Delta_x = -\Delta_{s,r} - \partial_s\left(\frac{1}{J^2} - 1\right)\partial_s - \frac{\partial_s J}{J^3}\partial_s - \frac{1}{r}\left(1 - \frac{1}{J}\right)\partial_r,$$

where $J = 1 + r\kappa$ and κ is the curvature. We see that that the even extension of coefficient does not agree with $-\Delta_x$ outside Ω.

4.3.3 *bmo* invariant under local C^1-diffeomorphism

Before we give the proofs to Proposition 6.2.7 and 6.2.8, we shall first establish the fact that the *bmo* estimate of a compactly supported function is preserved under a local C^1-diffeomorphism. Let $V, U \subset \mathbf{R}^n$ be two domains, we consider a local C^1-diffeomorphism $\psi : V \mapsto U$. Suppose that

$$\|\nabla_\eta\psi\|_{L^\infty(V)} + \|\nabla_x\psi^{-1}\|_{L^\infty(U)} < \infty.$$

Let $\rho > 0$. Assume that there exist two bounded subdomains $V_\rho \subset V, U_\rho \subset U$ such that $\psi : V_\rho \mapsto U_\rho$ is also a local C^1-diffeomorphism. Set

$$K_* := \max\left\{1, \|\nabla_\eta\psi\|_{L^\infty(V)} + \|\nabla_x\psi^{-1}\|_{L^\infty(U)}\right\}.$$

We assume further that there exists a constant c_0 such that for some $\eta_0 \in V_\rho$,

$$V_\rho \subset B_{c_0\rho}(\eta_0) \subset B_{K_*(c_0+3)\rho}(\eta_0) \subset V, \;\; U_\rho \subset B_{c_0\rho}(x_0) \subset B_{K_*(c_0+3)\rho}(x_0) \subset U$$

where $x_0 = \psi(\eta_0)$.

Proposition 4.3.4. *Let $f \in bmo(\mathbf{R}^n)$ with* $\operatorname{supp} f \subset V_\rho$*, then $f \circ \psi^{-1} \in bmo(\mathbf{R}^n)$ satisfies*

$$\|f \circ \psi^{-1}\|_{bmo(\mathbf{R}^n)} \leq C_\rho \|f\|_{bmo(\mathbf{R}^n)}.$$

Proof. Since $\operatorname{supp} f \circ \psi^{-1} \subset U_\rho$, we can treat $f \circ \psi^{-1}$ as a function in $\mathbf{R}^n$ with value zero outside U_ρ. The compactness of V_ρ in $\mathbf{R}^n$ implies that $\|f\|_{bmo(\mathbf{R}^n)} = \|f\|_{BMOL^1(\mathbf{R}^n)}$. Thus, the L^1 estimate

$$\|f \circ \psi^{-1}\|_{L^1(\mathbf{R}^n)} \leq C\|f\|_{L^1(\mathbf{R}^n)}$$

is obvious. Since $\psi \in C^1(V_\rho)$, an equivalent definition of the BMO-seminorm (cf. [15, Proposition 3.1.2]) implies that

$$[f \circ \psi^{-1}]_{BMO^\infty(B_{(c_0+1)\rho}(x_0))} \leq \|\nabla_x \psi^{-1}\|^n_{L^\infty(U)} \cdot \|\nabla_\eta \psi\|_{L^\infty(V)} \cdot [f]_{BMO(\mathbf{R}^n)}.$$

As $U_\rho \subset B_{c_0\rho}(x_0)$, by the extension theorem of *bmo* functions [11, Theorem 12], we obtain that

$$\|f \circ \psi^{-1}\|_{bmo(\mathbf{R}^n)} \leq C_\rho \|f \circ \psi^{-1}\|_{bmo^\infty_\infty(B_{(c_0+1)\rho}(x_0))} \leq C_\rho \|f\|_{bmo(\mathbf{R}^n)}.$$

□

Similarly, if $g \in bmo(\mathbf{R}^n)$ with $\operatorname{supp} g \subset U_\rho$, then we have that $g \circ \psi \in bmo(\mathbf{R}^n)$ satisfying

$$\|g \circ \psi\|_{bmo(\mathbf{R}^n)} \leq C_\rho \|g\|_{bmo(\mathbf{R}^n)}.$$

Even if we are considering vector fields instead of scalar functions, similar results hold.

Proposition 4.3.5. *Let $\nabla_\eta f \in bmo(\mathbf{R}^n)$ with* $\operatorname{supp} \nabla_\eta f \subset V_\rho$*, then $\nabla_x(f \circ \psi^{-1}) \in bmo(\mathbf{R}^n)$ satisfying*

$$\|\nabla_x(f \circ \psi^{-1})\|_{bmo(\mathbf{R}^n)} \leq C_\rho \|\nabla_\eta f\|_{bmo(\mathbf{R}^n)}.$$

Proof. Since $\nabla_\eta f$ is compactly supported, the L^1 estimate

$$\|\nabla_x(f \circ \psi^{-1})\|_{L^1(\mathbf{R}^n)} \leq C\|\nabla_\eta f\|_{L^1(\mathbf{R}^n)}$$

is obvious. Pick a cut-off function $\theta_{*,\rho} \in C_c^\infty(B_{K_*(c_0+3)\rho}(\eta_0))$ such that $\theta_{*,\rho} = 1$ in $B_{K_*(c_0+2)\rho}(\eta_0)$. Consider $B_r(x) \subset B_{(c_0+1)\rho}(x_0)$ with $r < \rho$. Let $\eta = \psi^{-1}(x)$. Since $\psi^{-1}(B_r(x)) \subset B_{K_*(c_0+2)\rho}(\eta_0)$, we have that

$$\frac{1}{r^n} \int_{B_r(x)} |\partial_{x_i}(f \circ \psi^{-1}) - c| \, dy \leq \frac{K_*}{r^n} \int_{\psi^{-1}(B_r(x))} \left| \sum_{1 \leq l \leq n} \theta_{*,\rho} \left(\frac{\partial \eta_l}{\partial x_i} \right)_\psi \partial_{\eta_l} f - c \right| d\eta$$

for any $c \in \mathbf{R}^n$, $1 \leq i \leq n$. By considering an equivalent definition of the BMO-seminorm, see e.g. [15, Proposition 3.1.2], we deduce that

$$\begin{aligned} [\nabla_x(f \circ \psi^{-1})]_{BMO^\infty(B_{(c_0+1)\rho}(x_0))} &\leq K_*^{n+1} \left[\sum_{1 \leq i,l \leq n} \theta_{*,\rho} \left(\frac{\partial \eta_l}{\partial x_i} \right)_\psi \partial_{\eta_l} f \right]_{BMO(\mathbf{R}^n)} \\ &\leq C_\rho \|\nabla_\eta f\|_{bmo(\mathbf{R}^n)}. \end{aligned}$$

As $U_\rho \subset B_{c_0\rho}(x_0)$, by the extension theorem of *bmo* functions [11, Theorem 12], we obtain that

$$\|\nabla_x(f \circ \psi^{-1})\|_{bmo(\mathbf{R}^n)} \leq C_\rho \|\nabla_x(f \circ \psi^{-1})\|_{bmo^\infty_\infty(B_{(c_0+1)\rho}(x_0))} \leq C_\rho \|\nabla_\eta f\|_{bmo(\mathbf{R}^n)}.$$

□

If $\nabla_x g \in bmo(\mathbf{R}^n)$ with $\operatorname{supp} \nabla_x g \subset U_\rho$, same proof of Proposition 4.3.5 shows that $\nabla_\eta(g \circ \psi) \in bmo(\mathbf{R}^n)$ satisfying

$$\|\nabla_\eta(g \circ \psi)\|_{bmo(\mathbf{R}^n)} \leq C_\rho \|\nabla_x g\|_{bmo(\mathbf{R}^n)}.$$

Let h be either a scalar function or a vector field which is compactly supported in U_ρ, for simplicity of notations we denote $h_\psi := h \circ \psi$. If h is a vector field, we denote $h_{\psi,i} := h_i \circ \psi$ for $1 \leq i \leq n$.

4.3.4 Volume potential for tangential component

Let $\rho \in (0, \rho_*/2)$ and fix $1 \leq j \leq m$. Since $\varphi_j v_2 \in vBMO(\Omega)$ with $\operatorname{supp} \varphi_j v_2 \subset U_{\rho,j} \cap \overline{\Omega}$, Proposition 6.2.5 implies that $(\varphi_j v_2)_{\text{even}} \in BMOL^1(\mathbf{R}^n)$. By the product estimate for *bmo* functions [11, Theorem 13], we see that $w_j^{\tan} = Q(\varphi_j v_2) \in BMOL^1(\mathbf{R}^n)$ with $\operatorname{supp} w_j^{\tan} \subset U_{\rho,j}$. For simplicity of notations, we set $v_{2,j} := (\varphi_j v_2)_{\text{even}}$.

Let $\psi : V_{4\rho} \mapsto U_{4\rho,j}$ be the normal coordinate change defined by (4.2.3) in Section 6.2.1. Since $\rho < \rho_*/2$, we have that

$$V_{4\rho} \subset B_{12\rho}(0) \subset B_{24L_*\rho}(0) \subset V_{\rho_0}, \quad U_{4\rho,j} \subset B_{12\rho}(z_j) \subset B_{24L_*\rho}(z_j) \subset U_{\rho_0,j}.$$

By Proposition 4.3.4 and 4.3.5, we see that ψ, in this case, is a local C^2-diffeomorphism that preserves *bmo* estimates for functions or vector fields compactly supported in $V_{4\rho}$. As a result, $(v_{2,j})_\psi \in BMOL^1(\mathbf{R}^n)$ satisfies the estimate

$$\|(v_{2,j})_\psi\|_{BMOL^1(\mathbf{R}^n)} \leq C_\rho \|v_{2,j}\|_{BMOL^1(\mathbf{R}^n)}.$$

Note that similar conclusions hold if we consider $\psi^{-1} : U_{4\rho,j} \mapsto V_{4\rho}$ instead.

Proposition 6.2.7. For $1 \leq i \leq n$ and $1 \leq k \leq n-1$, we define

$$\left(\frac{\partial \eta_k}{\partial x_i}\right)_* := E_{\text{even}}\, r_{V_{4\rho} \cap \mathbf{R}^n_+} \left(\frac{\partial \eta_k}{\partial x_i}\right)_\psi \quad \text{and} \quad g_{i,k} := \left(\frac{\partial \eta_k}{\partial x_i}\right)_* \cdot (v_{2,j})_{\psi,i}.$$

We consider

$$(\operatorname{div}_x w_j^{\tan})_{\psi,*} := \sum_{\substack{1 \leq i \leq n, \\ 1 \leq k \leq n-1}} \left\{ \partial_{\eta_k} g_{i,k} - \partial_{\eta_k} \left(\frac{\partial \eta_k}{\partial x_i}\right)_\psi \cdot (v_{2,j})_{\psi,i} \right\}$$
$$- \sum_{\substack{1 \leq i \leq n, \\ 1 \leq k \leq n-1}} \left(\frac{\partial \eta_k}{\partial x_i}\right)_\psi \cdot \left(\sum_{1 \leq l \leq n} (v_{2,j})_{\psi,l} \cdot \left(\frac{\partial \eta_n}{\partial x_l}\right)_\psi \right) \cdot \frac{\partial^2 x_i}{\partial \eta_k \partial \eta_n}$$

in $V_{4\rho} = \psi^{-1}(U_{4\rho,j})$. Let $L = L_0 + M$ be the operator in Proposition 4.3.3 and L_0^{-1} be the operator in Lemma 4.3.2. Let $1 \leq i \leq n$ and $1 \leq k \leq n-1$. We set

$$q_{j,1,\psi}^{i,k} := -\theta_\rho L_0^{-1} \partial_{\eta_k} g_{i,k}$$

where θ_ρ is the cut-off function defined in the proof of Lemma 4.3.1. There exists $\overline{(\frac{\partial \eta_k}{\partial x_i})_*} \in C^{0,1}(\mathbf{R}^n)$, see e.g. [11, Theorem 13], such that the restriction of $\overline{(\frac{\partial \eta_k}{\partial x_i})_*}$ in $V_{4\rho}$ equals $(\frac{\partial \eta_k}{\partial x_i})_*$ and $\|\overline{(\frac{\partial \eta_k}{\partial x_i})_*}\|_{C^{0,1}(\mathbf{R}^n)} \leq \|(\frac{\partial \eta_k}{\partial x_i})_*\|_{C^{0,1}(V_{4\rho})}$. By viewing $g_{i,k}$ as $\overline{(\frac{\partial \eta_k}{\partial x_i})_*} \cdot (v_{2,j})_{\psi,i}$, we see that

$g_{i,k} \in BMOL^1(\mathbf{R}^n)$. Hence, $q^{i,k}_{j,1,\psi} \in L^\infty(\mathbf{R}^n)$ is well-defined which satisfies all conditions in Lemma 4.3.2. Let $f^{i,k}_{j,1,\psi} := M\theta_\rho L_0^{-1}\partial_{\eta_k} g_{i,k}$. We can define

$$q^{i,k}_{j,1} := q^{i,k}_{j,1,\psi} \circ \psi^{-1}, f^{i,k}_{j,1} := f^{i,k}_{j,1,\psi} \circ \psi^{-1}$$

in $U_{\rho_0,j}$. Notice that $\operatorname{supp} q^{i,k}_{j,1}, \operatorname{supp} f^{i,k}_{j,1} \subset U_{4\rho,j}$, we can indeed treat $q^{i,k}_{j,1}, f^{i,k}_{j,1}$ as functions defined in $\mathbf{R}^n$ where their values outside $U_{4\rho,j}$ equal zero. Proposition 4.3.5 shows that $\nabla_x q^{i,k}_{j,1} \in BMO(\mathbf{R}^n)$ satisfies the estimate

$$[\nabla_x q^{i,k}_{j,1}]_{BMO(\mathbf{R}^n)} \leq C_\rho \|\nabla_\eta q^{i,k}_{j,1,\psi}\|_{BMOL^2(\mathbf{R}^n)} \leq C_\rho \|g_{i,k}\|_{BMOL^2(\mathbf{R}^n)}.$$

Let $p^{i,k}_{j,1} := E * f^{i,k}_{j,1}$. By Lemma 4.3.2 again, we can prove that

$$\|p^{i,k}_{j,1}\|_{L^\infty(\mathbf{R}^n)} + \|\nabla_x p^{i,k}_{j,1}\|_{L^\infty(\mathbf{R}^n)} \leq C_\rho \|f^{i,k}_{j,1,\psi}\|_{L^p(V_{2\rho})} \leq C_\rho \|g_{i,k}\|_{L^p(\mathbf{R}^n)}$$

with some $p > n$. Thus, $p^{i,k}_{j,1}$ is well-defined. By Proposition 6.2.5, we have that

$$\|g_{i,k}\|_{BMOL^1(\mathbf{R}^n)} \leq C_\rho \|v_{2,j}\|_{BMOL^1(\mathbf{R}^n)} \leq C_\rho \|v\|_{vBMO(\Omega)}.$$

Hence, by an interpolation (cf. [4, Lemma 5]),

$$\|g_{i,k}\|_{L^p(\mathbf{R}^n)} \leq C_\rho \|v\|_{vBMO(\Omega)}$$

for any $1 < p < \infty$.

For lower order term $q^{i,k}_{j,2,\psi} := \partial_{\eta_k}(\frac{\partial \eta_k}{\partial x_i})_\psi \cdot (v_{2,j})_{\psi,i}$, we set $q^{i,k}_{j,2} := q^{i,k}_{j,2,\psi} \circ \psi^{-1}$ in $U_{\rho_0,j}$. Similar as $q^{i,k}_{j,1}$, we can treat $q^{i,k}_{j,2}$ as a function in $\mathbf{R}^n$ with value zero outside $U_{\rho,j}$ since $\operatorname{supp} q^{i,k}_{j,2} \subset U_{\rho,j}$. Define $p^{i,k}_{j,2} := E * q^{i,k}_{j,2}$. Since E and $\nabla_x E$ are locally integrable, we have that

$$\|p^{i,k}_{j,2}\|_{L^\infty(\mathbf{R}^n)} + \|\nabla_x p^{i,k}_{j,2}\|_{L^\infty(\mathbf{R}^n)} \leq C_\rho \|q^{i,k}_{j,2,\psi}\|_{L^p(V_\rho)} \leq C_\rho \|v_{2,j}\|_{L^p(U_{\rho,j})}$$

for some $p > n$. By an interpolation (cf. [4, Lemma 5]) again, we deduce that

$$\|p^{i,k}_{j,2}\|_{L^\infty(\mathbf{R}^n)} + \|\nabla_x p^{i,k}_{j,2}\|_{L^\infty(\mathbf{R}^n)} \leq C_\rho \|v\|_{vBMO(\Omega)}.$$

This argument also holds for lower order term

$$q^{i,k}_{j,3,\psi} := \left(\frac{\partial \eta_k}{\partial x_i}\right)_\psi \cdot \left(\sum_{1\leq l\leq n} (v_{2,j})_{\psi,l} \cdot \left(\frac{\partial \eta_n}{\partial x_l}\right)_\psi\right) \cdot \frac{\partial^2 x_i}{\partial \eta_k \partial \eta_n}.$$

By letting $q^{i,k}_{j,3} := q^{i,k}_{j,3,\psi} \circ \psi^{-1}$ in $U_{\rho_0,j}$ and $p^{i,k}_{j,3} := E * q^{i,k}_{j,3}$, we can show that

$$\|p^{i,k}_{j,3}\|_{L^\infty(\mathbf{R}^n)} + \|\nabla_x p^{i,k}_{j,3}\|_{L^\infty(\mathbf{R}^n)} \leq C_\rho \|v\|_{vBMO(\Omega)}.$$

Set

$$p^{\tan}_j := \sum_{\substack{1\leq i\leq n,\\ 1\leq k\leq n-1}} \left(q^{i,k}_{j,1} + p^{i,k}_{j,1} + p^{i,k}_{j,2} + p^{i,k}_{j,3}\right).$$

Since a direct calculation implies that

$$(\mathrm{div}_x\, w_j^{\tan})_\psi = \sum_{\substack{1\le i\le n,\\ 1\le k\le n-1}} \left(\frac{\partial \eta_k}{\partial x_i}\right)_\psi \cdot \partial_{\eta_k}(v_{2,j})_{\psi,i} - \sum_{\substack{1\le i\le n,\\ 1\le k\le n-1}} \left(\frac{\partial \eta_k}{\partial x_i}\right)_\psi \cdot \left(\sum_{1\le l\le n} (v_{2,j})_{\psi,l}\cdot \left(\frac{\partial \eta_n}{\partial x_l}\right)_\psi\right)\cdot \frac{\partial^2 x_i}{\partial \eta_k \partial \eta_n}$$

in normal coordinate in $V_{4\rho} = \psi^{-1}(U_{4\rho,j})$, it is easy to see that

$$-\Delta_x p_j^{\tan} = \mathrm{div}\, w_j^{\tan}$$

in $U_{2\rho,j} \cap \Omega$. Calculations above ensures that

$$[\nabla_x p_j^{\tan}]_{BMO(\mathbf{R}^n)} \le C_\rho \|v\|_{vBMO(\Omega)}.$$

Since $\mathrm{supp}\, q_{j,1}^{i,k} \subset U_{4\rho,j}$, we consider $x \in \Gamma$ and $r < \rho$ such that $B_r(x) \cap U_{4\rho,j} \ne \emptyset$. By change of variables $y = \psi(\eta)$ in $U_{4\rho,j}$, we deduce that

$$\int_{B_r(x)\cap U_{4\rho,j}} |\nabla_y q_{j,1}^{i,k} \cdot \nabla_y d|\, dy \le C \int_{B_{L_* r}(\zeta)} |\partial_{\eta_n} q_{j,1,\psi}^{i,k}|\, d\eta$$

where $\zeta = \psi^{-1}(x)$ and $\zeta_n = 0$. By Lemma 4.3.2, we see that

$$\int_{B_{L_* r}(\zeta)} |\partial_{\eta_n} q_{j,1,\psi}^{i,k}|\, d\eta \le r^n C_\rho \|v\|_{vBMO(\Omega)}.$$

Since $\nabla_x p_{j,l}^{i,k} \in L^\infty(\mathbf{R}^n)$ for $l = 1, 2, 3$, we finally obtain that

$$\frac{1}{r^n} \int_{B_r(x)} |\nabla_y p_j^{\tan} \cdot \nabla_y d|\, dy \le C_\rho \|v\|_{vBMO(\Omega)}.$$

□

4.3.5 Volume potential for normal component

Consider $\rho \in (0, \rho_*/2)$ and $1 \le j \le m$. We let $g_j := \nabla d \cdot (\varphi_j v_2)_{\mathrm{odd}}$. Since $\varphi_j v_2 \in vBMO(\Omega)$ with $\mathrm{supp}\, \varphi_j v_2 \subset U_{\rho,j} \cap \overline{\Omega}$, by Proposition 6.2.5 we see that $g_j \in BMO(\mathbf{R}^n) \cap b^\nu(\Gamma)$. In particular, we have the estimate

$$[g_j]_{BMO(\mathbf{R}^n)} + [g_j]_{b^\nu(\Gamma)} \le C_\rho \|v\|_{vBMO(\Omega)}.$$

Considering the normal coordinate in $U_{4\rho,j}$, g_j is odd in η_n. Note that $w_j^{\mathrm{nor}} = g_j \nabla d$.

Proposition 6.2.8. Since $\nabla d \in C^1(U_{\rho_0,j})$, by Proposition 6.2.5 we have that

$$[w_j^{\mathrm{nor}}]_{BMO(\mathbf{R}^n)} \le C \|\nabla d\|_{C^\gamma(U_{\rho_0,j})} \|g_j\|_{BMOL^1(\mathbf{R}^n)} \le C_\rho \|v\|_{vBMO(\Omega)}.$$

We note that

$$\mathrm{div}_x\, w_j^{\mathrm{nor}} = \nabla_x g_j \cdot \nabla_x d + g_j \Delta_x d.$$

Let $g_{j,\psi} := g_j \circ \psi$ in $U_{\rho_0,j}$. We may treat $g_{j,\psi}$ as a function in $\mathbf{R}^n$ with value zero outside V_ρ. By Proposition 4.3.4, we have that

$$[g_{j,\psi}]_{BMO(\mathbf{R}^n)} \leq C_\rho \|g_j\|_{BMOL^1(\mathbf{R}^n)}.$$

In normal coordinate, $\nabla_x g_j \cdot \nabla_x d = \partial_{\eta_n} g_{j,\psi}$. We introduce the operator $L = L_0 + M$ in Proposition 4.3.3. Since $g_{j,\psi} \in Z_\rho$, we set

$$p_{1,j,\psi} := \theta_\rho L_0^{-1} \partial_{\eta_n} g_{j,\psi}$$

where θ_ρ is the cut-off function of $V_{2\rho}$ in the proof of Lemma 4.3.1. $p_{1,j,\psi}$ satisfies all conditions in Lemma 4.3.1. Set $f_{j,\psi} := -M\theta_\rho L_0^{-1}\partial_{\eta_n} g_{j,\psi}$. We define

$$p_{1,j} := p_{1,j,\psi} \circ \psi^{-1}, f_j := f_{j,\psi} \circ \psi^{-1}$$

in $U_{\rho_0,j}$. Notice that $p_{1,j} \in L^\infty(\mathbf{R}^n)$ and $f_j \in L^p(\mathbf{R}^n)$ with some $p > n$. By Proposition 4.3.5,

$$[\nabla_x p_{1,j}]_{BMO(\mathbf{R}^n)} \leq C_\rho [\nabla_\eta p_{1,j,\psi}]_{BMO(\mathbf{R}^n)} \leq C_\rho [g_{j,\psi}]_{BMO(\mathbf{R}^n)}.$$

Set

$$p_j^{\mathrm{nor}} = p_{1,j} + p_{2,j} + p_{3,j}$$

with $p_{2,j} = E * f_j$ and $p_{3,j} = E * (g_j \Delta_x d)$. This p_j^{nor} satisfies all desired properties required. For lower order terms $p_{2,j}$ and $p_{3,j}$, we have that

$$\|p_2\|_{L^\infty(\mathbf{R}^n)} + \|\nabla p_2\|_{L^\infty(\mathbf{R}^n)} + \|\nabla p_3\|_{L^\infty(\mathbf{R}^n)} + \|p_3\|_{L^\infty(\mathbf{R}^n)} \leq C_\rho \|g_j\|_{L^p(\mathbf{R}^n)}$$

as E and $\nabla_x E$ are both locally integrable. By an interpolation (cf. [4, Lemma 5]), we obtain that

$$[\nabla_x p_j^{\mathrm{nor}}]_{BMO(\mathbf{R}^n)} \leq C_\rho \|g_j\|_{BMOL^1(\mathbf{R}^n)} \leq C_\rho \|v\|_{vBMO(\Omega)}.$$

Since $\operatorname{supp} p_{1,j} \subset U_{\rho,j}$, we consider $x \in \Gamma$ and $r < \rho$ such that $B_r(x) \cap U_{\rho,j} \neq \emptyset$. Set $\zeta = \psi^{-1}(x)$ with $\zeta_n = 0$. Consider change of variable $y = \psi(\eta)$ in $U_{4\rho,j}$, by Lemma 4.3.1 we see that

$$\int_{B_r(x)\cap U_{\rho,j}} |\nabla_y d \cdot \nabla_y p_{1,j}|\, dy \leq C \int_{B_{L*r}(\zeta)} |\partial_{\eta_n} p_{1,j,\psi}|\, d\eta \leq C_\rho [g_{j,\psi}]_{BMO(\mathbf{R}^n)}.$$

By the L^∞-estimates of $\nabla_y p_2$ and $\nabla_y p_3$, we get that

$$\frac{1}{r^n} \int_{B_r(x)} |\nabla_y d \cdot \nabla_y p_j^{\mathrm{nor}}|\, dy \leq C_\rho \|v\|_{vBMO(\Omega)}.$$

Finally, a simple substitution shows that

$$-\Delta_x p_j^{\mathrm{nor}} = \nabla_x d \cdot \nabla_x g_j - f_j + f_j + g_j \Delta_x d = \operatorname{div}_x w_j^{\mathrm{nor}}$$

in $U_{2\rho}(z_0) \cap \Omega$. □

4.4 Neumann problem with bounded data

We consider the Neumann problem for the Laplace equation problem (6.1.4) for the Laplace equation. If Ω is a smooth bounded domain, as well-known, for $g \in H^{-1/2}(\Gamma)$, there is a unique (up to constant) weak solution $u \in H^1(\Omega)$ provided that g fulfills the compatibility condition

$$\int_\Gamma g \, d\mathcal{H}^{n-1} = 0; \tag{4.4.1}$$

see e.g. [18]. The main goal of this section is to prove that ∇u belongs to $vBMO^{\infty,\infty}(\Omega)$ provided that $g \in L^\infty(\Gamma)$. In other words, we prove Lemma 6.1.4.

To prove Lemma 6.1.4, we represent the solution by using the Neumann-Green function. Let $N(x,y)$ be the Green function, i.e., a solution v of

$$\begin{aligned} -\Delta_x v &= \delta(x-y) - |\Omega|^{-1} && \text{in} \quad \Omega \\ \frac{\partial v}{\partial \mathbf{n}_x} &= 0 && \text{on} \quad \partial\Omega \end{aligned}$$

for $y \in \Omega$. It is easy to see that the solution u of (6.1.4) satisfying $\int_\Omega u \, dx = 0$ is given as

$$u(x) = \int_\Gamma N(x,y) g(y) \, d\mathcal{H}^{n-1}(y).$$

The function N is decomposed as

$$N(x,y) = E(x-y) + h(x,y),$$

where $h \in C^\infty(\Omega \times \Omega)$ is a milder part. We recall $h(x,y) = h(y,x)$ and

$$\sup_{x\in\Omega} \int_\Omega \left| \nabla_y^k h(x,y) \right|^{1+\delta} dy < \infty$$

for $k = 0, 1, 2$ with some $\delta > 0$; see [12, Lemma 3.1]. In particular, by applying the standard L^p estimate for the Neumann problem in the proof of [12, Lemma 3.1] to $\nabla_y h(\cdot, y)$, we can deduce that

$$\sup_{x\in\Omega} \int_\Omega \left| \nabla_x \nabla_y h(x,y) \right|^{1+\delta} dy < \infty.$$

Hence, we see that $\nabla_x h(x,\cdot) \in W^{1,1+\delta}(\Omega_y)$. By the trace theorem for Sobolev space $W^{1,1+\delta}(\Omega_y)$, this yields

$$M_0 := \sup_{x\in\Omega} \int_\Gamma \left| \nabla_x h(x,y) \right|^{1+\delta} d\mathcal{H}^{n-1}(y) < \infty. \tag{4.4.2}$$

We decompose u as

$$u(x) = E * (\delta_\Gamma \otimes g) + \int_\Gamma h(x,y) g(y) \, d\mathcal{H}^{n-1}(y) = I + \mathit{II}.$$

The estimate (4.4.2) yields

$$\|\nabla \mathit{II}\|_{L^\infty(\Omega)} \leq M_0 \|g\|_{L^\infty(\Gamma)},$$

so to prove Lemma 6.1.4 it suffices to estimate ∇I. In other words, Lemma 6.1.4 follows from the next lemma.

Lemma 4.4.1. *Let Ω be a bounded domain in $\mathbf{R}^n$ with C^2 boundary $\Gamma = \partial\Omega$.*

(i) (BMO estimate) There exists a constant C_1 such that

$$[\nabla (E * (\delta_\Gamma \otimes g))]_{BMO(\mathbf{R}^n)} \leq C_1 \|g\|_{L^\infty(\Gamma)} \tag{4.4.3}$$

for all $g \in L^\infty(\Gamma)$.

(ii) (L^∞ estimate for normal component) There exists a constant C_2 such that

$$\|\nabla d \cdot \nabla (E * (\delta_\Gamma \otimes g))\|_{L^\infty(\Gamma_{\rho_0}^{\mathbf{R}^n} \cap \Omega)} \leq C_2 \|g\|_{L^\infty(\Gamma)} \tag{4.4.4}$$

for all $g \in L^\infty(\Gamma)$.

Here $E * (\delta_\Gamma \otimes g)$ is defined as $E * (\delta_\Gamma \otimes g)(x) := \int_\Gamma E(x-y) g(y) \, d\mathcal{H}^{n-1}(y)$ for a function g on Γ. We shall prove Lemma 6.3.3 in following subsections.

4.4.1 *BMO* estimate

To see the idea, we shall prove (4.4.3) in the case where Γ is flat. Let $\Gamma = \partial\mathbf{R}^n_+$ and $\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_n > 0\}$. In this case,

$$\nabla (E * (\delta_\Gamma \otimes g)) = \nabla \partial_{x_n} E * 1_{\mathbf{R}^n_+} \widetilde{g}$$

where $\widetilde{g} \in L^\infty(\mathbf{R}^n)$ is defined by $\widetilde{g}(x', x_n) := g(x', 0)$ for any $x \in \mathbf{R}^n$. By the L^∞-BMO estimate for the singular integral operator [15, Theorem 4.2.7], we obtain (4.4.3) when $\Gamma = \partial\mathbf{R}^n_+$.

Lemma 6.3.3 (i). Note that the signed distance function d is C^2 in $\Gamma_{\rho_0}^{\mathbf{R}^n}$, see [13, Section 14.6]. Let $\delta \in \rho_0/2$. We take a C^2 cut-off function $\theta \geq 0$ such that $\theta(\sigma) = 1$ for $\sigma \leq 1$ and $\theta(\sigma) = 0$ for $\sigma \geq 2$. By the choice of δ, we see that $\theta_d = \theta(d/\delta)$ is C^2 in $\mathbf{R}^n$. We extend $g \in L^\infty(\Gamma)$ to $g_e \in L^\infty(\Gamma_{2\delta}^{\mathbf{R}^n})$ by setting

$$g_e(x) := g(\pi x)$$

for any $x \in \Gamma_{2\delta}^{\mathbf{R}^n}$ with πx denoting the projection of x on Γ. For $x \in \Gamma_{2\delta}^{\mathbf{R}^n}$, by considering the normal coordinate $x = \psi(\eta)$ in $U_{2\delta}(\pi x)$, we have that

$$(\nabla_x d)_\psi \cdot (\nabla_x g_e)_\psi = \partial_{\eta_n} (g_e)_\psi = 0$$

as $(g_e)_\psi(\eta', \alpha) = (g_e)_\psi(\eta', \beta)$ for any $|\eta'| < 2\delta$ and $\alpha, \beta \in (-2\delta, 2\delta)$. Hence, we see that $\nabla d \cdot \nabla g_e = 0$ in $\Gamma_{2\delta}^{\mathbf{R}^n}$.

Let us consider $g_{e,c} := \theta_d g_e$. A key observation is that

$$\begin{aligned}
\delta_\Gamma \otimes g &= (\nabla 1_\Omega \cdot \nabla d) g_{e,c} \\
&= \operatorname{div}(g_{e,c} 1_\Omega \nabla d) - 1_\Omega \operatorname{div}(g_{e,c} \nabla d), \\
\operatorname{div}(g_{e,c} \nabla d) &= g_{e,c} \Delta d + \nabla d \cdot \nabla g_{e,c} = g_{e,c} \Delta d + \frac{\theta'(d/\delta)}{\delta} g_e.
\end{aligned}$$

Thus

$$\nabla E * (\delta_\Gamma \otimes g) = \nabla \operatorname{div} (E * (g_{e,c} 1_\Omega \nabla d)) - \nabla E * (1_\Omega g_e f_{\theta,\delta}) = I_1 + I_2$$

where $f_{\theta,\delta} := \theta_d \Delta d + \frac{\theta'(d/\delta)}{\delta}$. By the L^∞-BMO estimate for the singular integral operator [15, Theorem 4.2.7], the first term is estimated as

$$[I_1]_{BMO(\mathbf{R}^n)} \leq C\|g_{e,c}\nabla d\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\Gamma)}.$$

Since

$$A = \sup_{x\in\mathbf{R}^n\setminus\{0\}} |x|^{n-1} |\nabla E(x)| < \infty,$$

for $x \in \mathbf{R}^n$ with $d(x,\Omega) = \inf_{y\in\Omega} |x-y| < 1$ we have that

$$|I_2(x)| \leq A \int_\Omega \frac{1}{|x-y|^{n-1}}\, dy \|f_{\theta,\delta}\|_{L^\infty(\Gamma^{\mathbf{R}^n}_{2\delta})} \|g_{e,c}\|_{L^\infty(\Gamma^{\mathbf{R}^n}_{2\delta})} \leq C_{\Omega,\delta}\|g\|_{L^\infty(\Gamma)}$$

with $C_{\Omega,\delta}$ depending only on Ω and δ. For $x \in \mathbf{R}^n$ with $d(x,\Omega) = \inf_{y\in\Omega} |x-y| \geq 1$, the above estimate is trivial as $|x-y|^{-(n-1)} \leq 1$ for any $y \in \Omega$. The proof of (i) is now complete. □

4.4.2 Estimate for normal derivative

We shall estimate normal derivative of E.

Lemma 4.4.2. *Let Ω be a bounded domain in $\mathbf{R}^n$ with C^2 boundary Γ. Then*

(i)

$$\int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y)\, d\mathcal{H}^{n-1}(y) = -1 \quad \text{for} \quad x \in \Omega,$$

(ii)

$$\sup_{x\in\Omega} \int_\Gamma \left| \frac{\partial E}{\partial \mathbf{n}_y}(x-y) \right| d\mathcal{H}^{n-1}(y) < \infty.$$

Proof. (i) This follows from the Gauss divergence theorem. We observe that

$$\int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y)\, d\mathcal{H}^{n-1}(y) = \int_\Omega \Delta_y E(x-y)\, dy.$$

Since $\Delta_y E(x-y) = -\delta(x-y)$, we obtain

$$\int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y)\, d\mathcal{H}^{n-1}(y) = -1$$

for $x \in \Omega$.

(ii) We recall our local coordinate patches $\{U_i\}_{i=1}^m$ with $U_i = U_{\rho,i}$ as in Section 6.2.1. For $x \in \Omega^\rho$ and $y \in \Gamma$, obviously $|\nabla E(x-y)| \leq C\rho^{-(n-1)}$. Let $x \in \Gamma^{\mathbf{R}^n}_\rho \cap \Omega$. If $d(x, U_i \cap \Gamma) \geq \rho$, similarly $|\nabla E(x-y)| \leq C\rho^{-(n-1)}$ for $y \in U_i \cap \Gamma$. Hence, it is sufficient to consider U_i such that $d(x, U_i \cap \Gamma) < \rho$, i.e., it suffices to prove

$$\int_{U_i\cap\Gamma} \left| \frac{\partial E}{\partial \mathbf{n}_y}(x-y) \right| d\mathcal{H}^{n-1}(y) < \infty.$$

for U_i such that $d(x, U_i \cap \Gamma) < \rho$. Since $-\partial E/\partial \mathbf{n}_y(x-y)$ is invariant under translations and rotations, we can write $-\partial E/\partial \mathbf{n}_y(x-y)$ in the local coordinate. Let U_i be such that $d(x, U_i \cap \Gamma) < \rho$ and denote h_{z_i} by h_i for simplicity. Let us observe that

$$-\mathbf{n}\left(y', h_i(y')\right) = \left(-\nabla' h_i(y'), 1\right)/\omega_i(y')$$

with $\omega_i(y') = \left(1 + |\nabla' h_i(y')|^2\right)^{1/2}$, where ∇' is the gradient in y' variables. This implies that

$$-n\alpha(n)\frac{\partial E}{\partial \mathbf{n}_y}(x-y) = \frac{\sigma_i(y')}{\omega_i(y')\left(|x'-y'|^2 + (x_n - h_i(y'))^2\right)^{n/2}}$$

for $y \in \Gamma_i$ with

$$\sigma_i(y') := -\nabla' h_i(y) \cdot (x'-y') + \left(x_n - h_i(y')\right) \quad \text{where } \ x_n > h_i(x'), \ x' \in B_{3\rho}(0').$$

We set

$$K_i(x', y', x_n) = \frac{\sigma_i(y')}{\left(|x'-y'|^2 + (x_n - h_i(y'))^2\right)^{n/2}}.$$

By the Taylor expansion

$$h_i(x') = h_i(y') + \nabla' h_i(y') \cdot (x'-y') + r_i(x', y')$$

with

$$r_i(x', y') = (x'-y')^{\mathrm{T}} \cdot \int_0^1 (1-\theta)\nabla'^2 h_i\left(\theta x' + (1-\theta)y'\right) d\theta \cdot (x'-y'),$$

we obtain

$$\sigma_i(y') = x_n - h_i(x') + r_i(x', y')$$

with an estimate

$$\left|r_i(x', y')\right| \leq \|\nabla'^2 h_i\|_{L^\infty(B_{3\rho}(0'))}|x'-y'|^2. \tag{4.4.5}$$

We decompose K_i into a leading term and a remainder term

$$K_i(x', y', x_n) = K_0^i(x', y', x_n) + R_i(x', y', x_n)$$

with

$$K_0^i(x', y', x_n) := \frac{x_n - h_i(x')}{\left(|x'-y'|^2 + (x_n - h_i(y'))^2\right)^{n/2}}$$

$$R_i(x', y', x_n) := \frac{r_i(x, y)}{\left(|x'-y'|^2 + (x_n - h_i(y'))^2\right)^{n/2}}.$$

The term K_0^i is very singular but it is positive. The term R_i is estimated as

$$\left|R_i(x', y', x_n)\right| \leq \|\nabla'^2 h_i\|_{L^\infty(B_{3\rho}(0'))}|x'-y'|^{2-n}$$

by the estimate (6.3.5). Hence,

$$\int_{\Gamma\cap U_i} \left| \frac{R_i(x', y', x_n)}{\omega_i(y')} \right| d\mathcal{H}^{n-1}(y) \le C \int_{B_\rho(0')} \frac{1}{|x'-y'|^{n-2}}\, dy' \le C\rho$$

with C independent of ρ and i. By (i), we observe that

$$\begin{aligned} n\alpha(n) = &\sum_{i:d(x,U_i\cap\Gamma)<\rho} \int_{B_\rho(0')} \frac{K_i(x', y', x_n)}{\omega_i(y')}\, dy' \\ &- n\alpha(n) \sum_{j:d(x,U_j\cap\Gamma)\ge\rho} \int_{U_j\cap\Gamma} \frac{\partial E}{\partial \mathbf{n}_y}(x-y)\, d\mathcal{H}^{n-1}(y). \end{aligned}$$

Since K_0^i is positive for any i such that $d(x, U_i\cap\Gamma) < \rho$,

$$\sum_{i:d(x,U_i\cap\Gamma)<\rho} \int_{B_\rho(0')} \frac{K_0^i(x', y', x_n)}{\omega_i(y')}\, dy' \le n\alpha(n)\cdot(1 + \frac{m\cdot C\cdot S(\Gamma)}{\rho^{n-1}}) + m\cdot C\cdot\rho$$

where $S(\Gamma)$ denotes the surface area of Γ, which is bounded. Thus, the estimate

$$\int_{U_i\cap\Gamma} \left| \frac{\partial E}{\partial \mathbf{n}_y}(x-y) \right| d\mathcal{H}^{n-1}(y) \le \frac{1}{n\alpha(n)} \int_{B_\rho(0')} \frac{K_0^i + |R_i|}{\omega_i(y')}\, dy' < \infty$$

holds for any U_i such that $d(x, U_i\cap\Gamma) < \rho$. The proof of (ii) is now complete.

□

Based on Lemma 6.3.4, we are able to prove Lemma 6.3.3 (ii).

Lemma 6.3.3 (ii). We decompose

$$\begin{aligned} \nabla d(x)\cdot\nabla\left(E*(\delta_\Gamma\otimes g)\right)(x) = &\int_\Gamma (\nabla d(x) - \nabla d(y))\cdot\nabla E(x-y) g(y)\, d\mathcal{H}^{n-1}(y) \\ &+ \int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y) g(y)\, d\mathcal{H}^{n-1}(y) = I_1 + I_2. \end{aligned}$$

Let $x \in \Gamma_{\rho_0}^{\mathbf{R}^n}$ and πx be the projection of x on Γ. For $y \in U_{\rho_0}(\pi x)$, there exists a constant L', independent of x and y, such that

$$|\nabla d(x) - \nabla d(y)| \le L'|x-y|.$$

For $y \in \Gamma_{\rho_0}^{\mathbf{R}^n} \setminus U_{\rho_0}(\pi x)$, we have that $|x-y| \ge \frac{\rho_0}{2}$. Since $\overline{\Gamma_{\rho_0/2}^{\mathbf{R}^n}}$ is compact in $\mathbf{R}^n$, by considering a finite subcover of $\cup_{z\in\Gamma} U_{\rho_0}(z)$ we are able to show that there exists $M > 0$ such that the estimate

$$|\nabla d(x) - \nabla d(y)| \le M|x-y|$$

holds for any $x, y \in \Gamma_{\rho_0}^{\mathbf{R}^n}$. Thus,

$$H(x,y) = (\nabla d(x) - \nabla d(y))\cdot\nabla E(x-y)$$

is estimated as

$$|H(x,y)| \le \frac{M}{|x-y|^{n-2}}$$

in $\Gamma^{\mathbf{R}^n}_{\rho_0} \times \Gamma^{\mathbf{R}^n}_{\rho_0}$. We observe that

$$\begin{aligned}\sup_{x\in\Gamma^{\mathbf{R}^n}_{\rho_0}\cap\Omega} |I_1(x)| &\leq \sup_{x\in\Gamma^{\mathbf{R}^n}_{\rho_0}\cap\Omega} \int_\Gamma H(x,y)\, d\mathcal{H}^{n-1}(y) \|g\|_{L^\infty(\Gamma)} \\ &\leq M \sup_{x\in\Gamma^{\mathbf{R}^n}_{\rho_0}\cap\Omega} \int_\Gamma \frac{d\mathcal{H}^{n-1}(y)}{|x-y|^{n-2}} \|g\|_{L^\infty(\Gamma)}.\end{aligned}$$

Since

$$\sup_{x\in\Gamma^{\mathbf{R}^n}_{\rho_0}\cap\Omega} |I_2(x)| \leq \sup_{x\in\Gamma^{\mathbf{R}^n}_{\rho_0}\cap\Omega} \int_\Gamma \left| \frac{\partial E}{\partial \mathbf{n}_y}(x-y) \right| d\mathcal{H}^{n-1}(y) \|g\|_{L^\infty(\Gamma)},$$

Lemma 6.3.4 (ii) now yields (4.4.4). The proof is now complete. □

We wonder whether the tangential component of $\nabla E*(\delta_\Gamma \otimes g)$ satisfies the same estimate. Unfortunately, the estimate

$$\|\nabla\left(E*(\delta_\Gamma\otimes g)\right)\|_{L^\infty(\Gamma^{\mathbf{R}^n}_{\rho_0}\cap\Omega)} \leq C\|g\|_{L^\infty(\Gamma)}$$

should not hold even if Γ is flat. Even weaker estimate

$$\left[\nabla\left(E*(\delta_\Gamma\otimes g)\right)\right]_{b^\nu(\Gamma)} \leq C\|g\|_{L^\infty(\Gamma)}$$

should not hold in general.

To illustrate the problem, we consider the case that Γ is flat. We may assume $\Gamma = \partial\mathbf{R}^n_+$, $\mathbf{R}^n_+ = \{x_n > 0\}$.

Lemma 4.4.3. *The estimate*

$$\|\partial_{x_n}\left(E*(\delta_\Gamma\otimes g)\right)\|_{L^\infty(\mathbf{R}^n_+)} \leq \frac{1}{2}\|g\|_{L^\infty(\mathbf{R}^{n-1})}$$

holds for $g \in L^\infty(\mathbf{R}^{n-1})$.

Proof. This is because $-\partial_{x_n}\left(E*(\delta_\Gamma\otimes g)\right)$ is the half of the Poisson integral, i.e.,

$$-\partial_{x_n}\left(E*(\delta_\Gamma\otimes g)\right)(x) = \frac{1}{2}\int_{\mathbf{R}^{n-1}} P_{x_n}(x'-y')g(y')dy',$$

where P_{x_n} denotes the Poisson kernel. Thus the desired L^∞ estimate follows from the maximum principle of the Dirichlet problem for the Laplacian or from the property that $\int_{\mathbf{R}^{n-1}} P_{x_n}(x')dx' = 1$ and $P_{x_n} \geq 0$. □

Theorem 4.4.4. *There is a bounded sequence of smooth functions* $\{g_\ell\}_{\ell\in\mathbf{N}} \subset L^\infty(\mathbf{R}^{n-1})$ *such that*

$$\lim_{\ell\to\infty} \left[\partial_{x'}\left(E*(\delta_\Gamma\otimes g_\ell)\right)\right]_{b^\nu} = \infty$$

for any $\nu > 0$.

Proof. If g is smooth, then $E*(\delta_\Gamma\otimes g)$ is smooth up to the boundary. In this case, if $\left[\partial_{x'}\left(E*(\delta_\Gamma\otimes g)\right)\right]_{b^\nu}$ is bounded by $C\|g\|_{L^\infty(\mathbf{R}^{n-1})}$, $\|\partial_{x'}\left(E*(\delta_\Gamma\otimes g)\right)\|_{L^\infty(\Gamma)}$ is also bounded by $c_0C\|g\|_{L^\infty(\mathbf{R}^{n-1})}$ with a constant c_0 depending only on n since the mean value over r-ball around x converges to its value at x as $r \to 0$.

We consider the Neumann problem

$$\Delta u = 0 \quad \text{in} \quad \mathbf{R}^n_+,$$
$$\frac{\partial u}{\partial \mathbf{n}} = g \quad \text{on} \quad \Gamma = \partial \mathbf{R}^n_+.$$

By using the tangential Fourier transform, we see that

$$u(x,t) = \Lambda^{-1} \exp(-x_n \Lambda) g$$

where $\Lambda = (-\Delta')^{1/2}$. If $\|\nabla' u\|_{L^\infty(\Gamma)} \leq C\|g\|_{L^\infty(\mathbf{R}^{n-1})}$ were true, sending $x_n > 0$ to zero would imply L^∞ boundedness of the Riesz operator $\nabla'\Lambda^{-1}$, which is absurd.

The operator $E*(\delta_\Gamma \otimes g)$ is the half of the solution operator of the Neumann problem, so L^∞ bound for $\nabla' E*(\delta_\Gamma \otimes g)$ should not hold even if it is restricted to smooth functions. □

Corollary 4.4.5. *Assume that $\Omega = \mathbf{R}^n_+$. Let $v \mapsto \nabla q$ be the Helmholtz projection to a gradient field. Then, this projection is unbounded from $(L^\infty(\Omega))^n$ to $\left(BMO^{\mu,\nu}_b(\Omega)\right)^n$ for any $\mu, \nu > 0$.*

Proof. We consider

$$v = \big(0, \ldots, 0, v_n(x')\big)$$

with $v_n \in L^\infty(\mathbf{R}^{n-1})$. This evidently solves $\operatorname{div} v = 0$. The normal trace equals $-v_n(x')$. If

$$[\nabla q]_{b^\nu} \leq C\|v_n\|_{L^\infty(\mathbf{R}^{n-1})}$$

for all $v_n \in L^\infty(\mathbf{R}^{n-1})$ with C independent of v, then this would contradict Theorem 4.4.4. □

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Chapter 5

Extension theorem for *bmo* in a domain

In this chapter, we establish an extension theorem for functions defined in an arbitrary uniformly C^2 domain in the local BMO space. This extension theorem results in a product estimate for the local BMO space in an arbitrary uniformly C^2 domain.

5.1 Introduction

For a function space defined in an open domain $\Omega \subset \mathbf{R}^n$, it is natural to consider the problem if functions of this space can be continuously extended from Ω to $\mathbf{R}^n$. For example, if f is in $L^p(\Omega)$ with $1 \leq p \leq \infty$, its zero extension $f^{ze} = f \cdot 1_\Omega$ naturally belongs to $L^p(\mathbf{R}^n)$ where 1_Ω denotes the characteristic function for domain Ω. Although such extension problem is trivial for L^p, the story completely changes when it comes to the space of bounded mean oscillation (BMO for short). In the case for BMO, $f \in BMO^\infty(\Omega)$ is not sufficient to have that $f^{ze} \in BMO(\mathbf{R}^n)$. In fact, there exist domains Ω where bounded linear extension operator from $BMO^\infty(\Omega)$ to $BMO(\mathbf{R}^n)$ does not exist. P. W. Jones [11] gives a necessary and sufficient condition for a domain such that there exists a bounded linear extension operator.

An open connected subset $D \subset \mathbf{R}^n$ is called a uniform domain if there exists constants $a, b > 0$ such that for all $x, y \in D$ there exists a rectifiable curve $\gamma \subset D$ of length $s(\gamma) \leq a|x-y|$ with $\min\{s(\gamma(x,z)), s(\gamma(y,z))\} \leq bd(z, \partial D)$, where $\gamma(x,z)$ denotes the part of γ between x and z on the curve and $d(z,\partial D) = \inf_{w\in\partial D}|z-w|$ denotes the distance from z to the boundary ∂D; see e.g. [6]. Let $D \subset \mathbf{R}^n$ be a uniform domain. Jones' extension theorem guarantees that there is a constant C_J such that for each $f \in BMO^\infty(D)$, there is an extension $\overline{f} \in BMO(\mathbf{R}^n)$ satisfying

$$[\overline{f}]_{BMO(\mathbf{R}^n)} \leq C_J [f]_{BMO^\infty(D)}$$

with C_J independent of f. The operator $f \mapsto \overline{f}$ is a bounded linear operator. Conversely, if there exists such an extension, then D is a uniform domain.

In [8], a small modification was made to Jones' extension theorem so that we obtained an extension theorem regarding the local BMO space $bmo^\infty_\infty(D) := BMO^\infty(D) \cap L^1_{\mathrm{ul}}(D)$ where

$$L^1_{\mathrm{ul}}(D) := \left\{ f \in L^1_{\mathrm{loc}}(D) \;\middle|\; \|f\|_{L^1_{\mathrm{ul}}(D)} := \sup_{x\in\mathbf{R}^n} \int_{B_1(x)\cap D} |f(y)|\, dy < \infty \right\}.$$

If D is a uniform domain, the modified Jones' extension theorem says that for $f \in bmo^\infty_\infty(D)$ there exists $\overline{f} \in bmo := BMO \cap L^1_{\mathrm{ul}}(\mathbf{R}^n)$ satisfies

$$\|\overline{f}\|_{bmo(\mathbf{R}^n)} \leq C_J \|f\|_{bmo^\infty_\infty(D)} \tag{5.1.1}$$

with C_J independent of f. Moreover, the support of $\overline{f}$ is contained in a small neighborhood of $\overline{D}$. The reason why we are interested in such local BMO spaces (bmo) is that multiplication by a Hölder function in such spaces is bounded, i.e., for $\varphi \in C^\gamma(D)$ with $\gamma \in (0,1)$, we have that $\varphi f \in bmo^\infty_\infty(D)$ satisfies the product estimate

$$\|\varphi f\|_{bmo^\infty_\infty(D)} \leq C_J \|\varphi\|_{C^\gamma(D)} \|f\|_{bmo^\infty_\infty(D)} \tag{5.1.2}$$

with C_J independent of φ and f. Because of this multiplication principle, cut-off becomes possible in the space $bmo^\infty_\infty(D)$. The product estimate for bmo follows from the fact that such estimate holds for the local Hardy space h^1 and bmo is the dual space of h^1, see e.g. [13, Section 3].

Since the extension theorem and the product estimate for $bmo^\infty_\infty(D)$ relies heavily on the original extension theorem by Jones, we don't know if these results hold or not in the case where D is not a uniform domain. For instance, an aperture domain is an example for a non-uniform domain which is of special interests in fluid mechanics.

Our goal in this chapter is to establish the extension theorem for $bmo^\infty_\infty(\Omega)$ in the case where Ω is any arbitrary uniformly C^2 domain. We would like to clarify several relevant concepts before we state our main theorem. Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain with $n \geq 2$. Let $\Gamma := \partial\Omega$ denotes the boundary of Ω. Let R_0 be the reach of the boundary $\Gamma = \partial\Omega$. By considering R_0 sufficiently small, we may assume that R_0 is not only the reach of Γ in Ω but also the reach of Γ in Ω^c. Let d denote the signed distance function from Γ which is defined by

$$d(x) = \begin{cases} \inf_{y\in\Gamma} |x-y| & \text{for} \quad x \in \Omega, \\ -\inf_{y\in\Gamma} |x-y| & \text{for} \quad x \notin \Omega \end{cases}$$

so that $d(x) = d_\Gamma(x)$ for $x \in \Omega$. For $0 < \rho < R_0$, let Γ_ρ be the ρ-neighborhood of Γ in Ω, i.e.,

$$\Gamma_\rho = \{x \in \Omega \mid d_\Gamma(x) < \rho\}$$

and Γ^ρ be the ρ-neighborhood of Γ in $\mathbf{R}^n$, i.e.,

$$\Gamma^\rho = \{x \in \mathbf{R}^n \mid |d(x)| < \rho\}.$$

We recall the BMO^μ-seminorm for $\mu \in (0,\infty]$ which was defined in [1], [2], [3], [4]. For $f \in L^1_{\mathrm{loc}}(\Omega)$, we define

$$[f]_{BMO^\mu(\Omega)} := \sup\left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} \left|f(y) - f_{B_r(x)}\right| dy \,\middle|\, B_r(x) \subset \Omega,\ r < \mu \right\},$$

where f_B denotes the average over B, i.e.,

$$f_B := \frac{1}{|B|} \int_B f(y)\, dy$$

and $B_r(x)$ denotes the closed ball of radius r centered at x and $|B|$ denotes the Lebesgue measure of B. The space $BMO^\mu(\Omega)$ is defined as

$$BMO^\mu(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \,\middle|\, [f]_{BMO^\mu} < \infty \right\}.$$

As in [8], for $\delta \in (0, \infty]$ we set

$$bmo^\mu_\delta(\Omega) := BMO^\mu(\Omega) \cap L^1_{\mathrm{ul}}(\Gamma_\delta)$$

with the norm

$$\|v\|_{bmo^\mu_\delta} := [v]_{BMO^\mu(\Omega)} + [v]_{L^1_{\mathrm{ul}}(\Gamma_\delta)}.$$

We are now in a position to state our main result.

Theorem 5.1.1. *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain with $n \geq 2$. There exists $c^*_\Omega > 0$ such that for any $\rho \in (0, c^*_\Omega)$ and $v \in bmo^\infty_\infty(\Omega)$, there is an extension $\widetilde{v} \in bmo(\mathbf{R}^n)$ such that*

$$\|\widetilde{v}\|_{bmo(\mathbf{R}^n)} \leq \frac{C}{\rho^n} \|v\|_{bmo^\infty_\infty(\Omega)}$$

with C independent of v and ρ. Moreover, $\operatorname{supp} \widetilde{v} \subset \overline{\Omega_{2\rho}}$ where

$$\Omega_{2\rho} := \{x \in \mathbf{R}^n \mid d(x, \overline{\Omega}) < 2\rho\}.$$

The operator $v \mapsto \widetilde{v}$ is a bounded linear operator.

Different from the construction by Jones which delicately deals with the Whitney decomposition of both Ω and Ω^c, our strategy firstly decomposes v into the sum of v_1 and v_2 such that the support of v_1 is close to Γ whereas the support of v_2 is away from Γ. Such decomposition of v is achieved by the multiplication of v with a cut-off function θ_ρ supported in a small neighborhood of Γ, i.e., $v_1 := \theta_\rho v$. Since Ω is not necessarily uniform, at this moment we cannot apply the product estimate that was established for the case of uniform domains to v_1 directly. Instead, we apply a localization argument so that we can estimate the BMO^ρ-seminorm of v_1 in Ω. The key idea of the localization argument is as follow. If a ball B of radius $r(B) \leq \rho$ in Ω is away from the boundary, then v_1 vanishes in this ball. If B is close to the boundary, then we can find a bounded Lipschitz domain W_ρ such that the boundary of W_ρ coincides with Γ for a small part and $B \subset W_\rho$. Since Γ is uniformly C^2, by considering the normal coordinate change in Γ^{R_0}, we are able to show that the Lipschitz regularity of ∂W_ρ can be uniformly controlled. As $r_{W_\rho} v_1 \in bmo^\infty_\infty(W_\rho)$, we can apply the product estimate to $r_{W_\rho} v_1$ in W_ρ. Since a bounded Lipschitz domain is a typical example of a uniform domain and the constant C_J in (5.1.1) and (5.1.2) depends only on the Lipschitz regularity of the domain, we obtain a uniform estimate for $[v_1]_{BMO^\rho(\Omega)}$.

Next, we recall the extension introduced in [9] for functions supported in a small neighborhood of Γ. We extend v_1 to v^e_1 in $\mathbf{R}^n$ so that v^e_1 is even in the direction of ∇d with respect to Γ. By considering the normal coordinate change, we then reduce the problem to the half space and prove that $v^e_1 \in bmo^\rho_\infty(\mathbf{R}^n)$. Since the BMO^∞-seminorm can be estimated by the bmo^ρ_∞-norm, we thus deduce that $v^e_1 \in bmo(\mathbf{R}^n)$. For v_2, we simply zero extend it. By a similar argument, it is not hard to show that its zero extension $v^{ze}_2 \in bmo(\mathbf{R}^n)$. Setting $\widetilde{v} = v^e_1 + v^{ze}_2$ gives us Theorem 5.1.1.

Since there exists a bounded linear extension operator from $C^\gamma(\Omega)$ to $C^\gamma(\mathbf{R}^n)$ for arbitrary domain Ω, the product estimate for $bmo^\infty_\infty(\Omega)$ follows naturally from Theorem 5.1.1.

Theorem 5.1.2. *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain with $n \geq 2$. Let $\varphi \in C^\gamma(\Omega)$ with $\gamma \in (0, 1)$. For each $v \in bmo^\infty_\infty(\Omega)$, the function $\varphi v \in bmo^\infty_\infty(\Omega)$ satisfies*

$$\|\varphi v\|_{bmo^\infty_\infty(\Omega)} \leq C \|\varphi\|_{C^\gamma(\Omega)} \|v\|_{bmo^\infty_\infty(\Omega)}$$

with C independent of φ and v.

This chapter is organized as follow. In Section 5.2, we establish several uniform estimates which are essential for our localization argument. In Section 5.3, we perform the localization argument to do the cut-off to v and get v_1. In Section 5.4, we extend v_1 from Ω to $\mathbf{R}^n$ and prove Theorem 5.1.1 and Theorem 5.1.2. Besides, we apply a similar argument to further obtain an extension theorem for $bmo^\mu_\delta(\Omega)$ in the case where $\delta, \mu < \infty$. In Section 5.5, we give a simple application of our main extension theorem to construct an example regarding the space $BMO^{\infty,\infty}_b(\Omega)$. In Section 5.6, we update an extension result that is essential in establishing the Helmholtz decomposition of vector fields of BMO in a domain.

5.2 Uniform estimates

We denote $x' := (x_1, x_2, ..., x_{n-1})$ for $x \in \mathbf{R}^n$ and $\nabla' := (\partial_1, \partial_2, ..., \partial_{n-1})$. Since Ω is a uniformly C^2 domain, there exists $r_*, \delta_*, L_\Gamma > 0$ such that for each $w_0 \in \Gamma$, up to translation and rotation, there exists a function $\psi_{w_0} \in C^2(B_{r_*}(0'))$ with

$$\begin{aligned} &|\nabla^k \psi_{w_0}| \leq L_\Gamma \ \text{ in } \ B_{r_*}(0') \ \text{ for } \ k = 0, 1, 2, \\ &\nabla' \psi_{w_0}(0') = 0', \ \psi_{w_0}(0') = 0 \end{aligned} \tag{5.2.1}$$

such that the neighborhood

$$U_{r_*,\delta_*,\psi_{w_0}}(w_0) := \{(x', x_n) \in \mathbf{R}^n \mid \psi_{w_0}(x') - \delta_* < x_n < \psi_{w_0}(x') + \delta_*, \ |x'| < r_*\}$$

satisfies

$$\Omega \cap U_{r_*,\delta_*,\psi_{w_0}}(w_0) = \{(x', x_n) \in \mathbf{R}^n \mid \psi_{w_0}(x') < x_n < \psi_{w_0}(x') + \delta_*, \ |x'| < r_*\}$$

and

$$\partial\Omega \cap U_{r_*,\delta_*,\psi_{w_0}}(w_0) = \{(x', x_n) \in \mathbf{R}^n \mid x_n = \psi_{w_0}(x'), \ |x'| < r_*\}.$$

For simplicity of explanation, we say that Ω is of type $(r_*, \delta_*, L_\Gamma)$. For $x \in \Omega$, let πx be a point on Γ such that $|x - \pi x| = d_\Gamma(x)$. If x is within the reach of Γ, then this πx is unique. There exists $0 < \rho_0 < \min\{r_*, \delta_*, R_0, 1\}$ such that for any $w_0 \in \Gamma$,

$$U_{\rho_0}(w_0) := \{x \in U_{r_*,\delta_*,\psi_{w_0}}(w_0) \mid (\pi x)' \in B_{\rho_0}(0'), \ |d(x)| < \rho_0\} \tag{5.2.2}$$

is contained in $U_{r_*,\delta_*,\psi_{w_0}}(w_0)$.

We next consider the normal coordinate in $U_{\rho_0}(w_0)$, i.e.,

$$x = F(\eta) = \left\{ \begin{array}{l} \eta' + \eta_n \nabla' d(\eta', \psi_{w_0}(\eta')); \\ \psi_{w_0}(\eta') + \eta_n \partial_{x_n} d(\eta', \psi_{w_0}(\eta')) \end{array} \right. \tag{5.2.3}$$

or shortly

$$x = \pi x - d(x)\mathbf{n}(\pi x).$$

For each $w_0 \in \Gamma$, F is indeed a local C^1-diffeomorphism which maps V_{ρ_0} to $U_{\rho_0}(w_0)$ where $V_{\rho_0} := B_{\rho_0}(0') \times (-\rho_0, \rho_0)$. We indeed have that $F \in C^1(V_{\rho_0})$ and $(\nabla_\eta F)(0) = I$. Our first uniform control is for the gradient of F with respect to different $w_0 \in \Gamma$.

Proposition 5.2.1. *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain with $n \geq 2$, $\varepsilon \in (0,1)$. Then there exists a constant $c^\varepsilon_\Omega > 0$, depending on Ω, n and ε only, such that for any $\rho \in (0, c^\varepsilon_\Omega]$ and $w_0 \in \Gamma$,*

$$\begin{aligned} \|\nabla F - I\|_{L^\infty(V_\rho)} &< \varepsilon, \\ \|\nabla F^{-1} - I\|_{L^\infty(U_\rho(w_0))} &< \varepsilon \end{aligned}$$

hold simultaneously.

Proof. Let $0 < \varepsilon < 1$ and fix $w_0 \in \Gamma$, $\rho < \rho_0$. By the mean value theorem together with the upper bound of second order derivatives of ψ_{w_0} in (5.2.1), we deduce that

$$|\nabla' \psi_{w_0}(\eta')| = |\nabla' \psi_{w_0}(\eta') - \nabla' \psi_{w_0}(0')| \leq \|\nabla^2 \psi_{w_0}\|_{L^\infty(B_\rho(0'))} \cdot |\eta'| \leq L_\Gamma \cdot \rho \tag{5.2.4}$$

for any $|\eta'| < \rho$. Since

$$\begin{aligned} \partial_{\eta_j} x_i &= \delta_{i,j} + \eta_n \cdot \partial_{\eta_j}(\partial_{x_i} d) \\ &= \delta_{i,j} - \eta_n \cdot \frac{\partial_{\eta_j} \partial_{\eta_i} \psi_{w_0}}{(1 + |\nabla' \psi_{w_0}|^2)^{\frac{1}{2}}} + \eta_n \cdot \frac{\sum_{k=1}^{n-1} \partial_{\eta_i} \psi_{w_0} \cdot \partial_{\eta_k} \psi_{w_0} \cdot \partial_{\eta_j} \partial_{\eta_k} \psi_{w_0}}{(1 + |\nabla' \psi_{w_0}|^2)^{\frac{3}{2}}} \end{aligned}$$

in V_ρ for $1 \leq i, j \leq n-1$, by estimates (5.2.1) and (5.2.4) we have that

$$|\partial_{\eta_j} x_i(\eta) - \delta_{i,j}| \leq L_\Gamma \rho + (n-1) \cdot (L_\Gamma \rho)^3$$

for any $\eta \in V_\rho$. By similar calculations, for $\eta \in V_\rho$ we can also deduce that

$$|\partial_{\eta_n} x_i(\eta)| \leq L_\Gamma \rho$$

for each $1 \leq i \leq n-1$ and

$$\begin{aligned} |\partial_{\eta_j} x_n(\eta)| &\leq L_\Gamma \rho + (n-1) \cdot (L_\Gamma \rho)^2 \quad \text{for} \quad 1 \leq j \leq n-1, \\ |\partial_{\eta_n} x_n(\eta) - 1| &\leq (n-1) \cdot (L_\Gamma \rho)^2. \end{aligned}$$

Notice that for an invertible matrix A, we have that $A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A)$ where $\mathrm{adj}(A)$ denotes the adjugate of matrix A. Since we have obtained estimates for each entry of ∇F, by considering the inverse of ∇F we can deduce similar estimates for entries of ∇F^{-1}. Denote $c_{L_\Gamma \rho} := L_\Gamma \rho + (n-1) \cdot (L_\Gamma \rho)^2$. Assume that $L_\Gamma \rho << 1$, then for any $\eta \in V_\rho$,

$$(1 - c_{L_\Gamma \rho})^n - n! \cdot c_{L_\Gamma \rho}^2 \cdot (1 + c_{L \rho})^{n-2} \leq |\det(\nabla F)(\eta)| \leq (1 + c_{L_\Gamma \rho})^n + n! \cdot c_{L_\Gamma \rho}^2 \cdot (1 + c_{L_\Gamma \rho})^{n-2}.$$

By considering the adjugate of ∇F, in V_ρ we also have that

$$(1 - c_{L_\Gamma \rho})^{n-1} - (n-1)! \cdot c_{L_\Gamma \rho}^2 \cdot (1 + c_{L_\Gamma \rho})^{n-3} \leq |\partial_{x_i} \eta_i(\eta)| \leq (1 + c_{L_\Gamma \rho})^{n-1} + (n-1)! \cdot c_{L_\Gamma \rho}^2 \cdot (1 + c_{L_\Gamma \rho})^{n-3}$$

for every $1 \leq i \leq n$ and

$$|\partial_{x_j} \eta_i(\eta)| \leq (n-1)! \cdot c_{L_\Gamma \rho} \cdot (1 + c_{L_\Gamma \rho})^{n-2}$$

for every $1 \leq i, j \leq n$ with $i \neq j$. Therefore, if ρ is chosen to be sufficiently small, then for each $w_0 \in \Gamma$ we can have

$$\begin{aligned} \|\nabla F - I\|_{L^\infty(V_\rho)} &< \varepsilon, \\ \|\nabla F^{-1} - I\|_{L^\infty(U_\rho(w_0))} &< \varepsilon \end{aligned}$$

simultaneously.

Next we determine how small for ρ is enough. It is easy to see that if $\rho < \min\{\frac{\varepsilon}{2L_\Gamma}, \frac{1}{(n-1)L_\Gamma}\}$, we have that $\|\nabla F - I\|_{L^\infty(V_\rho)} < \varepsilon$. Suppose further that $c_{L_\Gamma \rho} < 2L_\Gamma \rho << 1$, then in V_ρ we have that

$$1 - c_{L_\Gamma \rho} \cdot (n+1)! \cdot 2^n < |\det(\nabla F)(\eta)| < 1 + c_{L_\Gamma \rho} \cdot (n+1)! \cdot 2^n.$$

Hence if $2L_\Gamma\rho < \frac{1}{(n+1)!\cdot 2^{n+1}}$, then

$$1 - c_{L_\Gamma\rho} \cdot (n+1)! \cdot 2^{n+1} < \frac{1}{|\det(\nabla F)(\eta)|} < 1 + c_{L_\Gamma\rho} \cdot (n+1)! \cdot 2^{n+1}$$

in V_ρ. Since

$$1 - c_{L_\Gamma\rho} \cdot n! \cdot 2^n < |\partial_{x_i}\eta_i(\eta)| \leq 1 + c_{L_\Gamma\rho} \cdot n! \cdot 2^n,$$

we deduce that

$$\left| \frac{1}{|\det(\nabla F)(\eta)|} \cdot \partial_{x_i}\eta_i(\eta) - 1 \right| < c_{L_\Gamma\rho} \cdot ((n+1)!)^2 \cdot 2^{2n+3}$$

for every $1 \leq i \leq n$ in V_ρ. Similar calculations enable us to also obtain that

$$\left| \frac{1}{|\det(\nabla F)(\eta)|} \cdot \partial_{x_j}\eta_i(\eta) \right| < c_{L_\Gamma\rho} \cdot n! \cdot 2^{n+1}$$

for every $1 \leq i, j \leq n$ with $i \neq j$ in V_ρ.

Therefore, if $\rho \leq c_\Omega^\varepsilon := \min\{\frac{\varepsilon}{L_\Gamma \cdot ((n+1)!)^2 \cdot 2^{2n+5}}, \frac{\rho_0}{2}\}$, we indeed have

$$\begin{aligned} \|\nabla F - I\|_{L^\infty(V_\rho)} &< \varepsilon, \\ \|\nabla F^{-1} - I\|_{L^\infty(U_\rho(w_0))} &< \varepsilon \end{aligned}$$

simultaneously. □

We would like to give a uniform estimate, regardless of $w_0 \in \Gamma$, on the size of the ball centered at w_0 that is contained in $U_\rho(w_0)$.

Proposition 5.2.2. *Let $\varepsilon \in (0,1)$. If $\rho < \min\{\frac{\varepsilon}{8L_\Gamma}, \rho_0\}$, then*

$$B_{\rho(1-\frac{\varepsilon}{2})}(w_0) \subset U_\rho(w_0)$$

for any $w_0 \in \Gamma$.

Proof. For a ball $B_r(w_0)$ to be contained in $U_\rho(w_0)$, we must have $r \leq \rho$. If $B_r(w_0)$ intersects $U_\rho(w_0)^c$ with some $r \leq \rho$, we can find $x \in B_r(w_0)$ of the form $(\eta', h_{w_0}(\eta')) + \tau\nabla d(\eta', h_{w_0}(\eta'))$ with $|\eta'| = \rho$ and $|\tau| \in [0, \rho)$. Notice that

$$|x - w_0|^2 = |(\eta', h_{w_0}(\eta'))|^2 + \tau^2 + 2\tau(\eta', h_{w_0}(\eta')) \cdot \nabla d(\eta', h_{w_0}(\eta')).$$

By the mean value theorem, we can estimate $|\partial_{\eta_i} h_{w_0}(\eta')|$ by ρL_Γ and $|h_{w_0}(\eta')|$ by $\rho^2 L_\Gamma$ for any $|\eta'| \leq \rho$ and $1 \leq i \leq n-1$. Thus, we deduce that

$$|(\eta', h_{w_0}(\eta'))|^2 + \tau^2 + 2\tau(\eta', h_{w_0}(\eta')) \cdot \nabla d(\eta', h_{w_0}(\eta')) \geq \rho^2 + \tau^2 - 4\tau\rho^2 L_\Gamma$$

for any $|\eta'| < \rho$. Since $\rho L_\Gamma < \frac{\varepsilon}{8}$, we have that

$$|x - w_0| \geq \rho\sqrt{1 - \frac{\varepsilon}{2}} > \rho(1 - \frac{\varepsilon}{2})$$

for any x of the form $(\eta', h_{w_0}(\eta')) + \tau\nabla d(\eta', h_{w_0}(\eta'))$ with $|\eta'| = \rho$ and $|\tau| \in [0, \rho)$. Hence for any $w_0 \in \Gamma$, we have that $B_{\rho(1-\frac{\varepsilon}{2})}(w_0) \subset U_\rho(w_0)$. □

Next, we establish a partition of unity for a small neighborhood of the boundary Γ in which not only partition functions but also their gradients are uniformly controlled.

Proposition 5.2.3. *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain with $n \geq 2$, $\rho \in (0, \frac{\rho_0}{2})$. There exist a countable family of points in Γ, say $S := \{x_i \in \Gamma \mid i \in \mathbf{N}\}$, and a natural number $N_* \in \mathbf{N}$ such that*

$$\Gamma^{2\rho} = \bigcup_{x_i \in S} U_{2\rho}(x_i)$$

and for any $x_j \in S$, there exist at most N_* points in S, say $\{x_{j_1}, ..., x_{j_{N_*}}\} \subset S$, with

$$U_{2\rho}(x_j) \cap U_{2\rho}(x_{j_l}) \neq \emptyset$$

for each $1 \leq l \leq N_*$.

Proof. Let $k_* \in \mathbf{N}$ be the smallest integer such that $2^{-k_*} \leq \frac{\rho}{\sqrt{n}}$. Let $\mathscr{D}$ be the collection of all dyadic cubes of the form

$$\{(y_1, ..., y_n) \in \mathbf{R}^n \mid m_j 2^{-k_*} \leq y_j < (m_j + 1)2^{-k_*}\},$$

where $m_j \in \mathbf{Z}$. Since $\mathscr{D}$ covers the whole space $\mathbf{R}^n$, we can pick out the set of dyadic cubes in $\mathscr{D}$ that intersect the boundary Γ. Let this subset be denoted by $G = \{Q_i \in \mathscr{D} \mid i \in \mathbf{N}\}$ and we have that

$$\Gamma \subset \bigcup_{i \in \mathbf{N}} Q_i.$$

We choose $x_i \in Q_i \cap \Gamma$ for each $i \in \mathbf{N}$ and set S to be the set of these points.

This is indeed the set of points we are seeking. For $y \in \Gamma^{2\rho}$, there exists $y_0 \in \Gamma$ such that $d(y) = |y - y_0|$. As G covers the boundary Γ, we have that $y_0 \in Q_j$ for some $j \in \mathbf{N}$. Hence $y \in U_{2\rho}(x_j)$. We have that

$$\Gamma^{2\rho} = \bigcup_{x_i \in S} U_{2\rho}(x_i).$$

By the mean value theorem, we can deduce that

$$\sup_{y \in U_{2\rho}(x)} |y - x| < 5\rho$$

for every $x \in \Gamma$. We fix $x_i \in S$. For $Q_j \in G$ with $d(Q_j, Q_i) > 10\rho$, by the triangle inequality we obviously have that

$$U_{2\rho}(x_j) \cap U_{2\rho}(x_i) = \emptyset.$$

This means that if $U_{2\rho}(x_j)$ intersects $U_{2\rho}(x_i)$, we must have that $d(Q_j, Q_i) \leq 10\rho$. If $d(Q_j, Q_i) \leq 10\rho$, then

$$\sup_{y \in Q_j,\, x \in Q_i} |y - x| < 12\rho.$$

Denote x_{i_c} to be the center of the cube Q_i. If $U_{2\rho}(x_j)$ intersects $U_{2\rho}(x_i)$, we have that $Q_j \subset Q_i^*$ where Q_i^* is the cube of side-length 24ρ with center x_{i_c}. Since elements of S belong to cubes that do not intersect, we can choose N_* to be $24^n \cdot n^{\frac{n}{2}}$. □

Based on $\{U_{c^{\varepsilon}_{\Omega}}(x_i) \mid x_i \in S\}$, a locally finite open cover of $\Gamma^{c^{\varepsilon}_{\Omega}}$, our desired partition of unity for $\Gamma^{c^{\varepsilon}_{\Omega}}$ can be constructed as follow.

Proposition 5.2.4. *There exist $\varphi_i \in C^1(\Gamma^{c^\varepsilon_\Omega})$ for each $i \in \mathbf{N}$ and a constant C_U such that properties*

$$\begin{aligned} &0 \le \varphi_i \le 1 \quad \text{for any} \quad i \in \mathbf{N}, \\ &\operatorname{supp} \varphi_i \subset \overline{U_{c^\varepsilon_\Omega}(x_i)} \quad \text{for any} \quad i \in \mathbf{N}, \\ &\sum_{i=1}^\infty \varphi_i(x) \equiv 1 \quad \text{for any} \quad x \in \Gamma^{c^\varepsilon_\Omega}, \\ &\sup_{i \in \mathbf{N}} \|\nabla \varphi_i\|_{L^\infty(\Gamma^{c^\varepsilon_\Omega})} \le C_U \end{aligned} \tag{5.2.5}$$

hold.

Similar proposition appears in [5]. For the completeness of the theory, we shall provide a proof here.

Proof of Proposition 6.2.3. Let us recall an empirical cutoff function that is widely used in various contents, e.g. see [12, Lemma 2.20 and Lemma 2.21]. We consider

$$f(t) = \begin{cases} \exp(-\frac{1}{t}) & t > 0, \\ 0 & t \le 0 \end{cases}$$

and

$$\theta(t) := \frac{f(2-t)}{f(t-1) + f(2-t)}$$

for $t \in \mathbf{R}$. A simple calculation tells us that $\theta \in C^\infty_c(\mathbf{R})$ with $\theta(t) = 1$ for $|t| \le 1$ and $\theta(t) = 0$ for $|t| \ge 2$. For $i \in \mathbf{N}$, we define that

$$\phi_i(x) := \theta\big(2|(F^{-1}(x))'|/c^\varepsilon_\Omega\big)$$

for $x \in U_{c^\varepsilon_\Omega}(x_i)$ where F in this case is the normal coordinate change between $V_{c^\varepsilon_\Omega}$ and $U_{c^\varepsilon_\Omega}(x_i)$. By Proposition 6.2.2, there exists $S_i := \{x_{i_1}, x_{i_2}, ..., x_{i_m}\} \subset S$ with $m \le N_*$ and $U_{c^\varepsilon_\Omega}(x_{i_l}) \cap U_{c^\varepsilon_\Omega}(x_i) \neq \emptyset$ for any $1 \le l \le m$. Without loss of generality, we assume that $i_l \neq i$ for each $1 \le l \le m$. Then we define φ_i in $\Gamma^{c^\varepsilon_\Omega}$ by

$$\varphi_i(x) := \begin{cases} \frac{\phi_i(x)}{\phi_i(x) + \sum_{l=1}^m \phi_{i_l}(x)} & x \in U_{c^\varepsilon_\Omega}(x_i), \\ 0 & x \in \Gamma^{c^\varepsilon_\Omega} \setminus U_{c^\varepsilon_\Omega}(x_i). \end{cases}$$

It is trivial to see that $0 \le \varphi_i \le 1$ for any $i \in \mathbf{N}$ and

$$\sum_{i=1}^\infty \varphi_i(x) \equiv 1 \quad \text{in} \quad \Gamma^{c^\varepsilon_\Omega}.$$

It is sufficient to estimate the gradient of φ_i. Note that

$$\partial_j \varphi_i = \frac{\partial_j \phi_i}{\phi_i + \sum_{l=1}^m \phi_{i_l}} - \frac{\phi \cdot (\partial_j \phi_i + \sum_{l=1}^m \partial_j \phi_{i_l})}{(\phi + \sum_{l=1}^m \phi_{i_l})^2}.$$

Let $x \in U_{c^\varepsilon_\Omega}(x_i)$ and πx be the projection of x in Γ. By the construction of the set S in the proof of Proposition 6.2.2, there exists $x_{i_k} \in S_i$ such that $|\pi x - x_{i_k}| < \frac{c^\varepsilon_\Omega}{2}$. This means that $|(F^{-1}(x))'| < \frac{c^\varepsilon_\Omega}{2}$, i.e., we have that $\phi_{i_k}(x) = 1$. Hence, we deduce that

$$\phi_i + \sum_{l=1}^m \phi_{i_l} \ge 1 \quad \text{in} \quad U_{c^\varepsilon_\Omega}(x_i).$$

As a result, we have the estimate

$$|\partial_j \varphi_i| \leq 2 \cdot |\partial_j \phi_i| + \sum_{l=1}^{m} |\partial_j \phi_{i_l}|.$$

For any $k \in \mathbf{N}$, we have that

$$\|\nabla \phi_k\|_{L^\infty(U_{c_\Omega^\varepsilon}(x_k))} \leq \frac{C_n}{\rho} \cdot \|\theta'\|_{L^\infty(\mathbf{R})} \cdot \|\nabla F^{-1}\|_{L^\infty(U_{c_\Omega^\varepsilon}(x_k))}.$$

By Proposition 6.2.1, we have a uniform estimate for $\|\nabla F^{-1}\|_{L^\infty(U_{c_\Omega^\varepsilon}(x_k))}$. Therefore, combining all estimates together, we finally obtain that

$$\sup_{i \in \mathbf{N}} \|\nabla \varphi_i\|_{L^\infty(\Gamma^{c_\Omega^\varepsilon})} \leq \frac{C_{n,N_*}}{\rho} \|\theta'\|_{L^\infty(\mathbf{R})}.$$

□

5.3 Cut-off

We consider $v \in bmo_\infty^\infty(\Omega)$. Let $0 < \rho < c_\Omega^\varepsilon/32$ be sufficiently small for which the smallness of ρ will be determined later. For $x \in \Gamma_{\rho_0}^{\mathbf{R}^n}$, we set $\theta_\rho(x) := \theta(d(x)/\rho)$ where θ is defined in the proof of Proposition 6.2.3. Note that $\theta_\rho \in C^2(\mathbf{R}^n)$. We then consider $v_1 := \theta_\rho v$.

Lemma 5.3.1. *$v_1 \in bmo_\infty^\rho(\Omega)$ satisfies the estimate*

$$\|v_1\|_{bmo_\infty^\rho(\Omega)} \leq \frac{C}{\rho} \|v\|_{bmo_\infty^\infty(\Omega)}$$

with C independent of v and ρ.

Since the domain Ω is not assumed to be a Jones domain, this lemma cannot be derived by applying the product estimate to *bmo* functions directly. To establish Lemma 5.3.1, we consider a localization argument in which we apply the product estimate to *bmo* functions locally. For $w_0 \in \Gamma$, we invoke the normal coordinate change $x = F(\eta)$ in $U_{32\rho}(w_0)$. There exists a bounded C^2 domain W such that $V_{16} \cap \mathbf{R}^n_+ \subset W \subset V_{32} \cap \mathbf{R}^n_+$ and $\partial W \cap \mathbf{R}^{n-1} \times \{0\} = B_{16}(0') \times \{0\}$. Without loss of generality, we assume that W is of type $(\alpha, \beta, L_{\partial W})$ with some constant $L_{\partial W}$. Let $W_\rho := \{\rho x \mid x \in W\}$. A simple check tells us that W_ρ is of type $(\alpha\rho, \beta\rho, L_{\partial W}/\rho)$.

Proposition 5.3.2. *$F(W_\rho)$ is a bounded Lipschitz domain with Lipschitz constant depending on $L_{\partial W}$ only. Moreover, we have that $U_{16\rho}(w_0) \cap \Omega \subset F(W_\rho) \subset U_{32\rho}(w_0) \cap \Omega$ and $\partial F(W_\rho) \cap \Gamma = U_{16\rho}(w_0) \cap \Gamma$.*

Proof. Since the normal coordinate change F is a C^1-diffeomorphism, we see that $F(W_\rho)$ is a bounded domain which satisfies $F(\partial W_\rho) = \partial F(W_\rho)$. Let $\tau_0 \in \partial W_\rho$ and $\delta < \min\{\alpha\rho, \beta\rho, \rho\}$. Without loss of generality we may assume that $\delta = c_0\rho$ for some sufficiently small universal constant c_0. Since ∂W_ρ is uniformly C^2, there exist a rotation R_{τ_0} and $h_{\tau_0} \in C^2(B_\delta(0'))$ such that $\widetilde{\eta_0} := R_{\tau_0}(\eta_0 - \tau_0)$ satisfies

$$(\widetilde{\eta_0})_n = h_{\tau_0}(\widetilde{\eta_0}')$$

for any $\eta_0 \in \partial W_\rho$ with $|\widetilde{\eta_0}| < \delta$. Let $y_0 := F(\tau_0)$ and e_{τ_0} to be the unit normal through τ_0 with respect to boundary ∂W_ρ. We set $\tau_n := \tau_0 + \delta e_{\tau_0}$ and $y_n := F(\tau_n)$. There exists another rotation matrix R_{y_0} such that $R_{y_0}(y_n - y_0) = \delta e_n$ where $e_n = (0', 1)$. Let $\zeta_0 \in \partial W_\rho$ such that $|\widetilde{\zeta_0}| < \delta$ where $\widetilde{\zeta_0} := R_{\tau_0}(\zeta_0 - \tau_0)$. We set $x_0 := F(\zeta_0)$ and $z_0 := F(\eta_0)$. In the coordinate system centered at y_0 with y_n lying on the n-axis in the positive direction, the coordinate of x_0 becomes $\widetilde{x_0} := R_{y_0}(x_0 - y_0)$ whereas the coordinate of z_0 becomes $\widetilde{z_0} := R_{y_0}(z_0 - y_0)$. By applying the mean value theorem, we have that

$$(\widetilde{x_0})_n - (\widetilde{z_0})_n = R_{y_0,n} \cdot \int_0^1 (\nabla F)(\eta_0 + t(\zeta_0 - \eta_0))\, dt \cdot R_{\tau_0}^{-1} \cdot (\widetilde{\zeta_0} - \widetilde{\eta_0})$$

with $R_{y_0,n}$ denoting the n-th row of rotation matrix R_{y_0}. Since $(\widetilde{\zeta_0})_n - (\widetilde{\eta_0})_n = h_{\tau_0}(\widetilde{\zeta_0}') - h_{\tau_0}(\widetilde{\eta_0}')$, we deduce that

$$|(\widetilde{x_0})_n - (\widetilde{z_0})_n| \le \|\nabla F\|_{L^\infty(V_{16\rho})} \cdot (1 + \|h_{\tau_0}\|_{L^\infty(B_\delta(0'))}) \cdot |\widetilde{\zeta_0}' - \widetilde{\eta_0}'|. \tag{5.3.1}$$

Applying the mean value theorem again to rewrite $\widetilde{\zeta_0} - \widetilde{\eta_0}$ back to $\widetilde{x_0} - \widetilde{z_0}$, for $1 \le i \le n-1$ we have that

$$(\widetilde{\zeta_0})_i - (\widetilde{\eta_0})_i = R_{\tau_0,i} \cdot \int_0^1 (\nabla F^{-1})(z_0 + t(x_0 - z_0))\, dt \cdot R_{y_0}^{-1} \cdot (\widetilde{x_0} - \widetilde{z_0}) \tag{5.3.2}$$

with $R_{\tau_0,i}$ denoting the i-th row of rotation matrix R_{τ_0}.

Fix $1 \le i \le n-1$. By deducting the identity matrix I from ∇F^{-1} in (5.3.2) and then adding I back, we have that

$$|(\widetilde{\zeta_0})_i - (\widetilde{\eta_0})_i| \le \|\nabla F^{-1} - I\|_{L^\infty(U_{32\rho}(w_0))} \cdot |\widetilde{x_0} - \widetilde{z_0}| + |R_{\tau_0,i} \cdot R_{y_0}^{-1} \cdot (\widetilde{x_0} - \widetilde{z_0})|.$$

In the coordinate system centered at τ_0, there exists $\eta_i \in V_{32\rho}$ such that $R_{\tau_0}(\eta_i - \tau_0) = \delta e_i$ where e_i denotes the vector whose j-th entry equals $\delta_{i,j}$ for each $1 \le j \le n$. Hence, $R_{\tau_0,i} = \frac{1}{\delta}(\eta_i - \tau_0)$. Similarly, in the coordinate system centered at y_0, we can find $y_i \in U_{32\rho}(w_0)$ such that $R_{y_0,i} = \frac{1}{\delta}(y_i - y_0)$ where $R_{y_0.j}$ denotes the j-th row of R_{y_0} for any $1 \le j \le n$. Since $R_{y_0}^{-1} = R_{y_0}^{\mathrm{T}}$, we see that

$$R_{\tau_0,i} \cdot R_{y_0}^{-1} \cdot (\widetilde{x_0} - \widetilde{z_0}) = (R_{\tau_0,i} - R_{y_0,i}) \cdot R_{y_0}^{\mathrm{T}} \cdot (\widetilde{x_0} - \widetilde{z_0}) + (\widetilde{x_0})_i - (\widetilde{z_0})_i.$$

Focus on the term that involves $(\widetilde{x_0})_n - (\widetilde{z_0})_n$, characterizations of rows of R_{τ_0} and R_{y_0} say that

$$\big((\widetilde{x_0})_n - (\widetilde{z_0})_n\big)\big((R_{\tau_0,i} - R_{y_0,i}) \cdot R_{y_0,n}\big) = \frac{\big((\widetilde{x_0})_n - (\widetilde{z_0})_n\big)}{\delta^2}\big((\eta_i - y_i) - (\tau_0 - y_0)\big) \cdot (y_n - y_0).$$

For $\zeta \in V_{32\rho}$,

$$F(\zeta) - \zeta = (0', \psi_{w_0}(\zeta') - \zeta_n) + \zeta_n \cdot (\nabla d)(\zeta', \psi_{w_0}(\zeta')).$$

An easy check gives that

$$|\zeta_n \cdot (\partial_{x_j} d)(\zeta', \psi_{w_0}(\zeta'))| \le |\zeta_n \cdot (\partial_{\zeta_j} \psi_{w_0})(\zeta')| \le C_{L_\Gamma} \rho^2.$$

for $1 \le j \le n-1$ and

$$|\psi_{w_0}(\zeta')| + |\zeta_n| \cdot |((\partial_{x_n} d)(\zeta', \psi_{w_0}(\zeta')) - 1)| \le C_{L_\Gamma, n} \rho^2.$$

Hence, for any $\zeta \in V_{32\rho}$, we have the estimate

$$|F(\zeta) - \zeta| \leq \frac{C_{L_\Gamma,n}}{c_0^2}\delta^2.$$

By the mean value theorem, we see that

$$|(y_0 - \tau_0) \cdot (y_n - y_0)| \leq |F(\tau_0) - \tau_0| \cdot |F(\tau_n) - F(\tau_0)| \leq \frac{C_{L_\Gamma,n}}{c_0^2} \cdot \|\nabla F\|_{L^\infty(V_{32\rho})} \cdot \delta^3. \quad (5.3.3)$$

On the other hand,

$$|(\eta_i - y_i) \cdot (y_n - y_0)| \leq |(\eta_i - \tau_0) \cdot (y_n - y_0)| + |(\tau_0 - y_0) \cdot (y_n - y_0)| + |(y_0 - y_i) \cdot (y_n - y_0)|.$$

By decomposing $y_n - y_0$ into $(y_n - \tau_n) + (\tau_n - \tau_0) + (\tau_0 - y_0)$ and applying the estimate (5.3.3), we deduce that

$$\begin{aligned}|(\eta_i - y_i) \cdot (y_n - y_0)| &\leq |(\eta_i - \tau_0) \cdot (y_n - \tau_n)| + |(\eta_i - \tau_0) \cdot (\tau_0 - y_0)| + |(\tau_0 - y_0) \cdot (y_n - y_0)| \\ &\leq \frac{C_{L_\Gamma,n}}{c_0^2} \cdot (2 + \|\nabla F\|_{L^\infty(V_{32\rho})}) \cdot \delta^3.\end{aligned}$$

Therefore,

$$\left|\big((\widetilde{x_0})_n - (\widetilde{z_0})_n\big)\big((R_{\tau_0,i} - R_{y_0,i}) \cdot R_{y_0,n}\big)\right| \leq \frac{C_{L_\Gamma,n}}{c_0^2} \cdot (1 + \|\nabla F\|_{L^\infty(V_{32\rho})}) \cdot \delta \cdot |(\widetilde{x_0})_n - (\widetilde{z_0})_n|.$$

If $\rho < c_\Omega^\varepsilon/32$, by Proposition 6.2.1 we see that

$$|R_{\tau_0,i} \cdot R_{y_0}^{-1} \cdot (\widetilde{x_0} - \widetilde{z_0})| \leq (n+1) \cdot |(\widetilde{x_0})' - (\widetilde{z_0})'| + \frac{C_{L_\Gamma,n}}{c_0^2} \cdot \delta \cdot |(\widetilde{x_0})_n - (\widetilde{z_0})_n|.$$

Hence,

$$|(\widetilde{\zeta_0})_i - (\widetilde{\eta_0})_i| \leq (n+2) \cdot |(\widetilde{x_0})' - (\widetilde{z_0})'| + \big(\frac{C_{L_\Gamma,n}}{c_0^2} \cdot \delta + \varepsilon\big) \cdot |(\widetilde{x_0})_n - (\widetilde{z_0})_n|.$$

Substitute this estimate back to the inequality (5.3.1), we obtain that

$$|(\widetilde{x_0})_n - (\widetilde{z_0})_n| \leq C_{n,L_{\partial W}}|(\widetilde{x_0})' - (\widetilde{z_0})'| + 2n(1 + L_{\partial W}) \cdot \big(\frac{C_{L_\Gamma,n}}{c_0^2} \cdot \delta + \varepsilon\big) \cdot |(\widetilde{x_0})_n - (\widetilde{z_0})_n|.$$

Therefore, if we take $\varepsilon < \frac{1}{8n(1+L_{\partial W})}$ and $\rho < \min\{\frac{c_0^2}{8n(1+L_{\partial W}) \cdot C_{L_\Gamma,n}}, \frac{c_\Omega^\varepsilon}{32}\}$, then we have that

$$|(\widetilde{x_0})_n - (\widetilde{z_0})_n| \leq 2C_{n,L_{\partial W}}|(\widetilde{x_0})' - (\widetilde{z_0})'|.$$

□

Based on this proposition, we have the tool to localize the problem and then to apply the product estimate for *bmo* functions in a bounded domain.

Proof of Lemma 5.3.1. Obviously, the estimate $\|v_1\|_{L^1(B_1(x)\cap\Omega)} \leq \|v\|_{L^1(B_1(x)\cap\Omega)}$ holds for any $x \in \mathbf{R}^n$. It is sufficient to estimate the BMO^ρ-seminorm for v_1. Let $r \leq \rho$. For $x \in \Omega$ such that $d(x) \geq 3\rho$, $v_1 \equiv 0$ in $B_r(x)$ as $B_r(x) \subset \Omega \setminus \overline{\Gamma^{\mathbf{R}^n}_{2\rho}}$, there is nothing to prove in this case. We then consider $x \in \Omega$ with $d(x) < 3\rho$ and $B_r(x) \subset \Omega$. Let πx be the projection of x on Γ, i.e., $d(x) = |x - \pi x|$. We have that $B_r(x) \subset U_{8\rho}(\pi x) \cap \Omega$. By Proposition 5.3.2, we see that $B_r(x) \subset F(W_\rho) \subset U_{32\rho}(\pi x) \cap \Omega$ where F in this case is the normal coordinate change between $U_{32\rho}(\pi x)$ and $V_{32\rho}$. Since a bounded Lipschitz domain is a uniform (Jones) domain, we can apply the product estimate for *bmo* functions [8, Theorem 13] in $F(W_\rho)$, i.e., we have that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |v_1(y) - (v_1)_{B_r(x)}|\, dy \leq \|v_1\|_{bmo^\infty_\infty(F(W_\rho))} \leq C_0\|\theta_\rho\|_{C^1(F(W_\rho))}\|v\|_{bmo^\infty_\infty(F(W_\rho))}$$

where C_0 depends only on the Lipschitz constant of $\partial F(W_\rho)$, which is universal by Proposition 5.3.2. Therefore, we obtain that

$$[v_1]_{BMO^\rho(\Omega)} \leq \frac{C_0}{\rho}\|v\|_{bmo^\infty_\infty(\Omega)}.$$

□

Next, let us consider further cut-offs induced by the partition of unity for $\Gamma^{2\rho}$. For $i \in \mathbf{N}$, we set $v_{1,i} := \varphi_i v_1$ where φ_i is the cut-off function defined in Proposition 6.2.3.

Lemma 5.3.3. *$v_{1,i} \in bmo^\rho_\infty(\Omega)$ satisfies the estimate*

$$\|v_{1,i}\|_{bmo^\rho_\infty(\Omega)} \leq \frac{C}{\rho}\|v\|_{bmo^\infty_\infty(\Omega)}$$

with C independent of v and ρ.

Proof. The estimate $\|v_{1,i}\|_{L^1(B_1(x)\cap\Omega)} \leq \|v\|_{L^1(B_1(x)\cap\Omega)}$ is trivial for any $x \in \mathbf{R}^n$. Let $r \leq \rho$. We only need to consider $x \in \Omega$ such that $d(x) < 3\rho$, $B_r(x) \subset \Omega$ and $B_r(x) \cap U_{2\rho}(x_i) \neq \emptyset$. Proposition 5.2.2 ensures that if $\varepsilon < \frac{2}{3}$ and $\rho < \frac{1}{4L_\Gamma}$, then $B_r(x) \subset B_{7\rho}(x_i) \cap \Omega \subset U_{16\rho}(x_i) \cap \Omega \subset F(W_\rho)$ where F in this case is the normal coordinate change that maps $V_{32\rho}$ to $U_{32\rho}(x_i)$. Again, by applying the product estimate for *bmo* functions [8, Theorem 13] in $F(W_\rho)$, we have that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |v_{1,i}(y) - (v_{1,i})_{B_r(x)}|\, dy \leq \|v_{1,i}\|_{bmo^\infty_\infty(F(W_\rho))} \leq C_1\|\varphi_i\|_{C^1(F(W_\rho))}\|v_1\|_{bmo^\infty_\infty(F(W_\rho))}$$

with C_1 depending only on the Lipschitz constant of $\partial F(W_\rho)$. Note that $bmo^\infty_\infty(F(W_\rho)) = bmo^\rho_\infty(F(W_\rho))$. Since $F(W_\rho) \subset U_{32\rho}(x_i) \cap \Omega \subset \Gamma^{c^\varepsilon_\Omega}$, by Proposition 6.2.3 and Proposition 5.3.2 we can deduce that

$$[v_{1,i}]_{BMO^\rho(\Omega)} \leq \frac{C_1(1 + C_U)(1 + C_0)}{\rho}\|v\|_{bmo^\infty_\infty(\Omega)}.$$

□

5.4 Extension

5.4.1 Extension to a neighborhood of Γ

We are now in a position to extend $v_{1,i}$ with respect to the boundary Γ for $i \in \mathbf{N}$. Let us recall the extension introduced in [9]. For a function h defined in $\Gamma^{\rho_0} \cap \overline{\Omega}$, let h^e denote the even extension of h with respect to Γ to Γ^{ρ_0} defined by

$$h^e\left(\pi x + d(x)\mathbf{n}(\pi x)\right) = h\left(\pi x - d(x)\mathbf{n}(\pi x)\right) \quad \text{for} \quad x \in \Gamma^{\rho_0} \setminus \overline{\Omega}.$$

Let h^o denote the odd extension of h with respect to Γ to Γ^{ρ_0} defined by

$$h^o\left(\pi x + d(x)\mathbf{n}(\pi x)\right) = -h\left(\pi x - d(x)\mathbf{n}(\pi x)\right) \quad \text{for} \quad x \in \Gamma^{\rho_0} \setminus \overline{\Omega}.$$

Lemma 5.4.1. *Let $\rho < \frac{c_\Omega^\varepsilon}{32}$. There exists a constant C, independent of v and ρ, such that the estimate*

$$[v_{1,i}^e]_{bmo(\mathbf{R}^n)} \leq \frac{C}{\rho^n} \|v\|_{bmo_\infty^\infty(\Omega)}$$

holds for any $i \in \mathbf{N}$.

Proof. It is trivial to see that

$$\int_{U_{2\rho}(x_i)} |v_{1,i}^e|\, dy \leq 2\|\nabla F\|_{L^\infty(V_{2\rho})} \cdot \|\nabla F^{-1}\|_{L^\infty(U_{2\rho}(x_i))} \cdot \int_{U_{2\rho}(x_i)\cap\Omega} |v_{1,i}|\, dy.$$

Since $\operatorname{supp} v_{1,i} \subset U_{2\rho}(x_i)$, $\rho < \frac{c_\Omega^\varepsilon}{32}$ implies that

$$\|v_{1,i}^e\|_{L^1(\mathbf{R}^n)} \leq 8\|v_{1,i}\|_{L^1(B_1(x_i)\cap\Omega)} \leq 8\|v\|_{bmo_\infty^\infty(\Omega)}.$$

Since $F(W_\rho)$ is a bounded Lipschitz domain and $v_{1,i} \in bmo_\infty^\infty(F(W_\rho))$, by the extension theorem for BMO functions [11], there exists $v_{1,i}^J \in BMO(\mathbf{R}^n)$ satisfying $r_{F(W_\rho)} v_{1,i}^J = v_{1,i}$ and

$$[v_{1,i}^J]_{BMO(\mathbf{R}^n)} \leq C[v_{1,i}]_{BMO^\infty(F(W_\rho))}$$

where by Proposition 5.3.2 the constant C depends on $L_{\partial W}$ only. Let $c \in \mathbb{R}^n$ be a constant vector. For $B_r(\zeta) \subset V_{16\rho}^+$, by change of variable $\eta = F^{-1}(y)$ in $V_{16\rho} = F^{-1}(U_{16\rho}(x_i))$, we see that

$$\frac{1}{|B_r(\zeta)|} \int_{B_r(\zeta)} |v_{1,i} \circ F(\eta) - c|\, d\eta \leq \|\nabla F^{-1}\|_{L^\infty(U_{16\rho}(x_i))} \cdot \frac{1}{|B_r(\zeta)|} \int_{F(B_r(\zeta))} |v_{1,i}(y) - c|\, dy.$$

Let $x = F(\zeta)$. By Proposition 6.2.1, $\rho < \frac{c_\Omega^\varepsilon}{32}$ implies that $\|\nabla F^{-1}\|_{L^\infty(U_{16\rho}(x_i))} < 2$ and $F(B_r(\zeta)) \subset B_{2r}(x)$. Thus,

$$\frac{1}{|B_r(\zeta)|} \int_{F(B_r(\zeta))} |v_{1,i}(y) - c|\, dy \leq 2^n \cdot \frac{1}{|B_{2r}(x)|} \int_{B_{2r}(x)} |v_{1,i}^J(y) - c|\, dy.$$

By considering an equivalent definition of the BMO-seminorm, see e.g. [10, Proposition 3.1.2], we deduce that

$$[v_{1,i} \circ F]_{BMO^\infty(V_{16\rho}^+)} \leq C_n [v_{1,i}]_{BMO^\infty(F(W_\rho))} \leq \frac{C_n}{\rho} \|v\|_{bmo_\infty^\infty(\Omega)}.$$

By recalling the results concerning the even extension of BMO functions in the half space, see [7, Lemma 3.2] and [7, Lemma 3.4], we can deduce that

$$[v^e_{i,n} \circ F]_{BMO^\infty(V_{8\rho})} \leq \frac{C_n}{\rho} \|v\|_{bmo^\infty_\infty(\Omega)}. \tag{5.4.1}$$

Let $B_r(x)$ be a ball with radius $r \leq \rho$. If $B_r(x) \cap U_{2\rho}(x_i) = \emptyset$, there is nothing to prove. It is sufficient to consider $B_r(x)$ that intersects $U_{2\rho}(x_i)$. Proposition 5.2.2 ensures that if $\varepsilon < \frac{1}{4}$, then $B_r(x) \subset B_{7\rho}(x_i) \subset U_{8\rho}(x_i)$. By change of variable $y = F(\eta)$ in $U_{16\rho}(x_i)$, we have that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |v^e_{1,i}(y) - c| \, dy \leq \|\nabla F\|_{L^\infty(V_{16\rho})} \cdot \frac{1}{|B_r(x)|} \int_{F^{-1}(B_r(x))} |v^e_{1,i} \circ F(\eta) - c| \, d\eta.$$

Since $F^{-1}(B_r(x)) \subset B_{2r}(\zeta) \subset B_{8\rho}(0) \subset V_{8\rho}$, by (5.4.1) we deduce that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |v^e_{1,i}(y) - (v^e_{1,i})_{B_r(x)}| \, dy \leq \frac{C_n}{\rho} \|v\|_{bmo^\infty_\infty(\Omega)}.$$

Thus, we obtain that

$$[v^e_{1,i}]_{BMO^\rho(\mathbf{R}^n)} \leq \frac{C_n}{\rho} \|v\|_{bmo^\infty_\infty(\Omega)}.$$

For a ball B with radius $r(B) > \rho$, a simple triangle inequality implies that

$$\frac{1}{|B|} \int_B |v^e_{1,i}(y) - (v^e_{1,i})_B| \, dy \leq \frac{2}{|B|} \int_B |v^e_{1,i}(y)| \, dy \leq \frac{C_n}{\rho^n} \|v^e_{1,i}\|_{L^1(\mathbf{R}^n)}.$$

Therefore, we obtain the BMO estimate for $v^e_{1,i}$, i.e.,

$$[v^e_{1,i}]_{BMO(\mathbf{R}^n)} \leq \frac{C_n}{\rho^n} \|v\|_{bmo^\infty_\infty(\Omega)}.$$

□

Since $\{U_{2\rho}(x_i) \mid x_i \in S\}$ is a locally finite open cover of $\Gamma^{2\rho}$, we are able to estimate the *bmo* norm for v^e_1.

Lemma 5.4.2. *$v^e_1 \in bmo(\mathbf{R}^n)$ satisfies the estimate*

$$\|v^e_1\|_{bmo(\mathbf{R}^n)} \leq \frac{C}{\rho^n} \|v\|_{bmo^\infty_\infty(\Omega)}$$

with C independent of v and ρ.

Proof. Let $r < \rho$ and consider $B_r(x)$ that intersects $\Gamma^{2\rho}$. By the construction of S in Proposition 6.2.2, there exists $x_{i_0} \in S$ such that $|\pi x - x_{i_0}| < \rho$. Thus, by Proposition 5.2.2 we have that $B_r(x) \subset B_{5\rho}(x_{i_0}) \subset U_{6\rho}(x_{i_0})$ as $\varepsilon < \frac{1}{3}$. If $x_j \in S$ such that $U_{2\rho}(x_j) \cap B_r(x) \neq \emptyset$, then $U_{6\rho}(x_j) \cap U_{6\rho}(x_{i_0}) \neq \emptyset$. This means that the number of $x_j \in S$ such that $U_{2\rho}(x_j) \cap B_r(x) \neq \emptyset$ is smaller than the number of $x_j \in S$ such that $U_{6\rho}(x_j) \cap U_{6\rho}(x_{i_0}) \neq \emptyset$. Same proof of Proposition 6.2.2 also shows that for any $x_k \in S$, the number of $x_j \in S$ such that $U_{6\rho}(x_j) \cap U_{6\rho}(x_k) \neq \emptyset$ is smaller than some $N_{*,0} \in \mathbf{N}$ independent of x_k. Hence, we can find at most $N_{*,0}$ points in S, say $\{x_{j_1}, ..., x_{j_{N_{*,0}}}\} \subset S$, such that $U_{2\rho}(x_{j_l}) \cap B_r(x) \neq \emptyset$ for each $1 \leq l \leq N_{*,0}$.

The L^1 norm of v_1^e in $B_r(x)$ is estimated as

$$\|v_1^e\|_{L^1(B_r(x))} \leq \sum_{l=1}^{N_{*,0}} \|v_{1,j_l}^e\|_{L^1(B_r(x)\cap U_{2\rho}(x_{j_l}))} \leq 8N_{*,0}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

Since this estimate holds regardless of $x \in \mathbf{R}^n$, we obtain that

$$\|v_1^e\|_{L^1_{\mathrm{ul}}(\mathbf{R}^n)} \leq 8N_{*,0}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

Since

$$r_{B_r(x)}v_1^e = \sum_{l=1}^{N_{*,0}} r_{B_r(x)}v_{1,j_l}^e,$$

we have that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |v_1^e(y) - (v_1^e)_{B_r(x)}|\, dy \leq \sum_{l=1}^{N_{*,0}} \frac{1}{|B_r(x)|}\int_{B_r(x)} |v_{1,j_l}^e(y) - (v_{1,j_l}^e)_{B_r(x)}|\, dy.$$

By Lemma 5.4.1, we deduce that

$$[v_1^e]_{BMO^\rho(\mathbf{R}^n)} \leq \frac{N_{*,0}C_n}{\rho}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

Let B be a ball in $\mathbf{R}^n$ with radius $r(B) > \rho$. By the triangle inequality,

$$\frac{1}{|B|}\int_B |v_1^e(y) - (v_1^e)_B|\, dy \leq \frac{2}{|B|}\int_B |v_1^e(y)|\, dy$$

Let $M \in \mathbb{N}$ be the largest integer such that $M\rho \leq r(B)$. By definition we have that $(M+1)\rho > r(B)$. Note that the ball B is contained in a cube Q of side length $(M+1)\rho$ which shares the same center as B. Separating each side of Q equally into $M+1$ parts, we can divide Q equally into $(M+1)^n$ subcubes of side length ρ. Hence, we have that

$$\int_B |v_1^e(y)|\, dy \leq \int_Q |v_1^e(y)|\, dy \leq C_n(M+1)^n \cdot \|v_1^e\|_{L^1_{\mathrm{ul}}(\mathbf{R}^n)}.$$

Since $r(B) \geq M\rho$, we deduce that

$$\frac{2}{|B|}\int_B |v_1^e|\, dy \leq \frac{C_n}{\rho^n} \cdot \|v_1^e\|_{L^1_{\mathrm{ul}}(\mathbf{R}^n)}.$$

Therefore, we finally obtain the estimate

$$[v_1^e]_{bmo(\mathbf{R}^n)} \leq \frac{N_{*,0}C_n}{\rho^n}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

□

5.4.2 Extension to $\mathbf{R}^n$

Let $v_2 := v - v_1$. Note that $\operatorname{supp} v_2 \subset \Omega \setminus \Gamma_\rho$. Let v_2^{ze} denote the zero extension of v_2 to $\mathbf{R}^n$, i.e.,

$$v_2^{ze}(x) = \begin{cases} v_2(x) & \text{for} \quad x \in \Omega, \\ 0 & \text{for} \quad x \notin \Omega. \end{cases}$$

We next estimate the *bmo* norm of v_2^{ze}.

Lemma 5.4.3. *$v_2^{ze} \in bmo(\mathbf{R}^n)$ satisfies the estimate*

$$\|v_2^{ze}\|_{bmo(\mathbf{R}^n)} \leq \frac{C}{\rho^n}\|v\|_{bmo_\infty^\infty(\Omega)}$$

with C independent of v and ρ.

Proof. Since $r_\Omega v_1^e = v_1$, Lemma 5.4.2 implies that $v_1 \in bmo_\infty^\infty(\Omega)$ with the estimate

$$\|v_1\|_{bmo_\infty^\infty(\Omega)} \leq \frac{C}{\rho^n}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

Hence, $v_2 = v - v_1 \in bmo_\infty^\infty(\Omega)$ satisfies the estimate

$$\|v_2\|_{bmo_\infty^\infty(\Omega)} \leq \frac{C}{\rho^n}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

Since v_2^{ze} is the zero extension of v_2, the estimate $\|v_2^{ze}\|_{L^1_{\mathrm{ul}}(\mathbf{R}^n)} \leq \|v_2\|_{L^1_{\mathrm{ul}}(\Omega)}$ is trivial. Let $B \subset \mathbf{R}^n$ be a ball with radius $r(B) \leq \rho/2$. If B intersects $\overline{\Omega \setminus \Gamma_\rho}$, then $B \subset \Omega$. In this case, we naturally have that

$$\frac{1}{|B|}\int_B |v_2^{ze}(y) - (v_2^{ze})_B|\, dy \leq [v_2]_{BMO^\infty(\Omega)}.$$

If $B \cap \overline{\Omega \setminus \Gamma_\rho} = \emptyset$, then $v_2^{ze} = 0$ in B, there is nothing to prove in this case. Hence, we have the estimate

$$[v_2^{ze}]_{BMO^{\rho/2}(\mathbf{R}^n)} \leq [v_2]_{BMO^\infty(\Omega)} \leq \frac{C}{\rho^n}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

Let $B \subset \mathbf{R}^n$ be a ball with radius $r(B) > \rho/2$. By same argument in the proof of Lemma 5.4.2 that decomposes the smallest cube Q containing B into small subcubes of side-length $\rho/2$, we deduce that

$$\frac{1}{|B|}\int_B |v_2^{ze}(y) - (v_2^{ze})_B|\, dy \leq \frac{2}{|B|}\int_B |v_2^{ze}(y)|\, dy \leq \frac{C}{\rho^n}\|v_2^{ze}\|_{L^1_{\mathrm{ul}}(\mathbf{R}^n)}.$$

Therefore, we finally obtain that

$$\|v_2^{ze}\|_{bmo(\mathbf{R}^n)} \leq \frac{C}{\rho^n}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

□

Up till here, we have gathered enough results to prove our main theorem.

Proof of Theorem 5.1.1. Let

$$\begin{aligned}\varepsilon &< \frac{1}{8n(1+L_{\partial W})},\\ c_\Omega^\varepsilon &= \min\{\frac{\varepsilon}{L_\Gamma \cdot ((n+1)!)^2 \cdot 2^{2n+4}},\ \rho_0\},\\ c_\Omega^* &:= \min\{\frac{c_0^2}{16n(1+L_{\partial W})\cdot C_{L_\Gamma,n}},\ \frac{c_\Omega^\varepsilon}{64}\}.\end{aligned}$$

We set $\widetilde{v} := v_1^e + v_2^{ze}$ and let $\rho < c_\Omega^*$. An easy check ensures that $\operatorname{supp}\widetilde{v} \subset \overline{\Omega_{2\rho}}$ and $r_\Omega \widetilde{v} = v$. By Lemma 5.4.2 and Lemma 5.4.3, we see that $\widetilde{v} = v_1^e + v_2^{ze} \in bmo(\mathbf{R}^n)$ satisfies the estimate

$$\|\widetilde{v}\|_{bmo(\mathbf{R}^n)} \leq \frac{C}{\rho^n}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

□

The product estimate for $v \in bmo_\infty^\infty(\Omega)$ follows directly from the extension theorem.

Proof of Theorem 5.1.2. Let $\gamma \in (0,1)$. By [8, Theorem 13], we see that for $\varphi \in C^\gamma(\Omega)$, there exists $\widetilde{\varphi} \in C^\gamma(\mathbf{R}^n)$ such that $r_\Omega \widetilde{\varphi} = \varphi$ and

$$\|\widetilde{\varphi}\|_{C^\gamma(\mathbf{R}^n)} \leq \|\varphi\|_{C^\gamma(\Omega)}.$$

Extending $v \in bmo_\infty^\infty(\Omega)$ to $\widetilde{v} \in bmo(\mathbf{R}^n)$ by Theorem 5.1.1, we naturally have that

$$\|\varphi v\|_{bmo_\infty^\infty(\Omega)} \leq \|\widetilde{\varphi}\widetilde{v}\|_{bmo(\mathbf{R}^n)} \leq C\|\widetilde{\varphi}\|_{C^\gamma(\mathbf{R}^n)}\|\widetilde{v}\|_{bmo(\mathbf{R}^n)} \leq C\|\varphi\|_{C^\gamma(\Omega)}\|v\|_{bmo_\infty^\infty(\Omega)}.$$

□

By almost the same proof of Theorem 5.1.1, we are able to further establish an extension theorem for $bmo_\delta^\mu(\Omega)$ with $\delta, \mu < \infty$. We recall that $bmo_\infty^\infty(\Omega) \subset bmo_\delta^\mu(\Omega)$ for arbitrary domain Ω and $\delta, \mu < \infty$ [8, Theorem 2].

Theorem 5.4.4. *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain with $n \geq 2$ and $\mu, \delta \in (0,\infty)$. There exists $c_\Omega^* > 0$ such that for any $\rho \in (0, c_\Omega^*)$ and $v \in bmo_\delta^\mu(\Omega)$, there is an extension $\widetilde{v} \in BMO^\mu(\mathbf{R}^n) \cap L_{\mathrm{ul}}^1(\Gamma^\delta)$ such that*

$$[\widetilde{v}]_{BMO^\mu(\mathbf{R}^n)} + [\widetilde{v}]_{L_{\mathrm{ul}}^1(\Gamma^\delta)} \leq \frac{C}{\rho}\|v\|_{bmo_\delta^\mu(\Omega)}$$

with C independent of v and ρ. Moreover, $\operatorname{supp}\widetilde{v} \subset \overline{\Omega_{2\rho}}$ where

$$\Omega_{2\rho} := \{x \in \mathbf{R}^n \mid d(x, \overline{\Omega}) < 2\rho\}.$$

The operator $v \mapsto \widetilde{v}$ is a bounded linear operator.

Proof. By [8, Proposition 1], we see that the space $bmo_{\delta_1}^{\mu_1}(\Omega)$ and the space $bmo_{\delta_2}^{\mu_2}(\Omega)$ are equivalent for any $0 < \delta_1, \delta_2, \mu_1, \mu_2 < \infty$. Without loss of generality, we may assume that $\mu, \delta > c_\Omega^*$ where c_Ω^* is defined in the proof of Theorem 5.1.1. Let $\rho \in (0, c_\Omega^*)$. Follow the proofs of Lemma 5.3.1, Lemma 5.3.3, Lemma 5.4.1 and Lemma 5.4.2, we can deduce that $v_1^e \in BMO^\rho(\mathbf{R}^n) \cap L_{\mathrm{ul}}^1(\Gamma^\delta)$ satisfies the estimate

$$[v_1^e]_{BMO^\rho(\mathbf{R}^n)} + [v_1^e]_{L_{\mathrm{ul}}^1(\Gamma^\delta)} \leq \frac{C}{\rho}\|v\|_{bmo_\delta^\mu(\Omega)}.$$

Moreover, in this case it is trivial that $v_2^{ze} \in BMO^\rho(\mathbf{R}^n) \cap L^1_{\mathrm{ul}}(\Gamma^\delta)$. Still by setting $\widetilde{v} = v_1^e + v_2^{ze}$, we finally obtain that $\widetilde{v} \in BMO^\rho(\mathbf{R}^n) \cap L^1_{\mathrm{ul}}(\Gamma^\delta)$ satisfies the estimate

$$[\widetilde{v}]_{BMO^\rho(\mathbf{R}^n)} + [\widetilde{v}]_{L^1_{\mathrm{ul}}(\Gamma^\delta)} \leq \frac{C}{\rho} \|v\|_{bmo^\mu_\delta(\Omega)}$$

with C independent of v and ρ. □

5.5 Application of the extension theorem

As defined in [1], [2], [3], [4], we recall a seminorm that controls the boundary behavior. For $\nu \in (0, \infty]$, we set

$$[f]_{b^\nu} := \sup \left\{ r^{-n} \int_{\Omega \cap B_r(x)} |f(y)| \, dy \;\middle|\; x \in \Gamma, \; 0 < r < \nu \right\}.$$

We define the space

$$BMO_b^{\mu,\nu}(\Omega) := \left\{ f \in BMO^\mu(\Omega) \;\middle|\; [f]_{b^\nu} < \infty \right\}$$

with

$$\|f\|_{BMO_b^{\mu,\nu}(\Omega)} := [f]_{BMO^\mu(\Omega)} + [f]_{b^\nu}.$$

Let $\mu_0, \nu_0 < \infty$. In [4, Example 1], we see that there exist examples in $BMO_b^{\mu_0,\nu_0}$, $BMO_b^{\mu_0,\infty}$ and BMO_b^{∞,ν_0}. By making use of the extension theorem and the product estimate established in this chapter, we shall give an example of a function that belongs to $BMO_b^{\infty,\infty}$ but does not belong to L^∞.

We consider the case where the domain Ω is the half space $\mathbf{R}^2_+$. Let $f = \log x_2$ defined in the layer domain $D_L := \{0 < x_2 < 1\}$. For a cube $Q = [a, a+1] \times [b, b+1]$ that intersects D_L, we have that

$$\int_{Q \cap D_L} |\log x_2| \, dx = -\int_0^{b+1} \log x_2 \, dx_2 \leq 1.$$

Hence, we see that $f \in bmo^\infty_\infty(D_L)$. By Theorem 5.1.1, we can find $\widetilde{f} \in bmo(\mathbf{R}^2)$ such that $r_{D_L} \widetilde{f} = f$ and $\operatorname{supp} \widetilde{f} \subset \{-1 < x_2 < 2\}$. Set $\widetilde{g}(x_1, x_2) := \widetilde{f}(x_1, x_2 - 2)$ for any $x = (x_1, x_2) \in \mathbf{R}^2$ and $g := r_{\mathbf{R}^2_+} \widetilde{g}$. Note that $\operatorname{supp} g \subset \{1 < x_2 < 4\}$.

Proposition 5.5.1. *$g \in BMO_b^{\infty,\infty}(\mathbf{R}^2_+)$ but $g \notin L^\infty(\mathbf{R}^2_+)$.*

Proof. It is trivial to see that $g \in BMO^\infty(\mathbf{R}^2_+)$ and $g \notin L^\infty(\mathbf{R}^2_+)$. We only need to estimate the b^∞-norm for g. Since $\operatorname{supp} g \subset \{1 < x_2 < 4\}$, it is sufficient to estimate

$$\frac{2}{|Q_r(x)|} \int_{Q_r(x) \cap \mathbf{R}^2_+} |g| \, dy$$

for $r \geq 1$ and $x = (x_1, 0) \in \partial \mathbf{R}^2_+$ where $Q_r(x)$ denotes the square with center x of side-length $2r$. Without loss of generality, we may assume that g is only a function of x_2. Hence, a direct calculation shows that

$$\frac{2}{|Q_r(x)|} \int_{Q_r(x) \cap \mathbf{R}^2_+} |g| \, dy = \frac{1}{2r^2} \int_{x_1 - r}^{x_1 + r} \int_1^r |g| \, dy_2 \, dy_1 \leq 2 \int_0^1 |\log z_2| \, dz_2 \leq 2.$$

□

Remark 5.5.2. Let $\phi \in C_c^\infty(B_8(0))$ with $\phi \equiv 1$ in $B_6(0)$, by Proposition 5.5.1 we see that $\phi g \in BMO_b^{\infty,\infty}(\mathbf{R}^2_+) \cap L^2(\mathbf{R}^2_+)$ but $\phi g \notin L^\infty(\mathbf{R}^2_+)$.

5.6 Extension of vector fields in *bmo* in a domain

Note that Lemma 5.4.1 basically coincide with [9, Proposition 2] in the statement. However, the proof of Lemma 5.4.1 involves the localization argument in this chapter, which actually improves [9, Proposition 2] in the sense that [9, Proposition 2] holds for any uniformly C^2 domain instead of just for bounded domain. Here we provide an update of [9, Proposition 2].

We consider the space

$$vbmo(\Omega) := \{u \in bmo^{\infty}_{\infty}(\Omega) \mid [\nabla d \cdot u]_{b^\nu} < \infty\}$$

equipped with the norm

$$\|u\|_{vbmo(\Omega)} := \|u\|_{bmo^{\infty}_{\infty}(\Omega)} + [\nabla d \cdot u]_{b^\nu}.$$

This space is independent of $\nu \in (0, \infty]$. Let $u \in vbmo(\Omega)$. We set $u_1 = \theta_\rho u$, $u_{1,i} = \varphi_i u_1$. Let $Pu^o_{1,i} := (\nabla d \cdot u^o_{1,i})\nabla d$ denotes the normal component of $u^o_{1,i}$ whereas $Qu^e_{1,i} := u^e_{1,i} - (\nabla d \cdot u^e_{1,i})\nabla d$ denotes the tangential component of $u^e_{1,i}$.

Lemma 5.6.1. *Let $\rho < \frac{c^\varepsilon_\Omega}{48}$. There exists a constant C, independent of v and ρ, such that the estimates*

$$[Pu^o_{1,i}]_{bmo(\mathbf{R}^n)} \leq \frac{C}{\rho^n}\|u\|_{vbmo(\Omega)},$$
$$[\nabla d \cdot Pu^o_{1,i}]_{b^\infty(\Gamma)} \leq \frac{C}{\rho^n}\|u\|_{vbmo(\Omega)}$$

hold for any $i \in \mathbf{N}$ and $\nu \in (0, \infty]$.

Proof. Follow the proofs of [9, Proposition 2] and Lemma 5.4.1, we are done. □

Lemma 5.6.2. *$Pu^o_1 \in bmo(\mathbf{R}^n)$ satisfies the estimates*

$$\|Pu^o_1\|_{bmo(\mathbf{R}^n)} \leq \frac{C}{\rho^n}\|u\|_{vbmo(\Omega)},$$
$$[\nabla d \cdot Pu^o_1]_{b^\infty(\Gamma)} \leq \frac{C}{\rho^n}\|u\|_{vbmo(\Omega)}$$

with C independent of u and ρ.

Proof. Follow the proof of Lemma 5.4.2, we are done. □

Similar as in [9, Proposition 2], we set

$$\overline{u_1} := Pu^o_1 + Qu^e_1.$$

By Lemma 5.4.2, we have that $\overline{u_1} \in bmo(\mathbf{R}^n)$. Let $u_2 := u - u_1$ and u^{ze}_2 be the zero extension of u_2 to $\mathbf{R}^n$. Since $\overline{u_1}$ coincide with u_1 in Ω, following the proof of Lemma 5.4.3 we can show that $u^{ze}_2 \in bmo(\mathbf{R}^n)$ satisfying

$$\|u^{ze}_2\|_{bmo(\mathbf{R}^n)} \leq \frac{C}{\rho^n}\|u\|_{vbmo(\Omega)}$$

with C independent of u and ρ. Therefore, by setting $\overline{u} := \overline{u_1} + u^{ze}_2$, we obtain an extension of u whose normal component in a small neighborhood of Γ is odd with respect to Γ whereas the tangential component in a small neighborhood of Γ is even with respect to Γ. We summarize the extension theorem for a vector field of *bmo* in a domain as follow.

Theorem 5.6.3. *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain with $n \geq 2$. There exists $c_\Omega^{**} > 0$ such that for any $\rho \in (0, c_\Omega^{**})$ and $u \in vbmo(\Omega)$, there is an extension $\overline{u} \in bmo(\mathbf{R}^n)$ such that*

$$\|\overline{u}\|_{bmo(\mathbf{R}^n)} + [\nabla d \cdot \overline{u}]_{b^\infty(\Gamma)} \leq \frac{C}{\rho^n} \|u\|_{vbmo(\Omega)}$$

with C independent of u and ρ. Moreover, $\operatorname{supp} \overline{u} \subset \overline{\Omega_{2\rho}}$ *where*

$$\Omega_{2\rho} := \{x \in \mathbf{R}^n \mid d(x, \overline{\Omega}) < 2\rho\}.$$

The operator $u \mapsto \overline{u}$ is a bounded linear operator.

The constant c_Ω^{**} can be taken as

$$c_\Omega^{**} := \min\{\frac{c_0^2}{16n(1 + L_{\partial W}) \cdot C_{L_\Gamma, n}}, \frac{c_\Omega^\varepsilon}{96}\}.$$

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Chapter 6

The Helmholtz decomposition of a space of vector fields with bounded mean oscillation in a perturbed half space with small perturbation

We introduce a space of vector fields with bounded mean oscillation whose "tangential" and "normal" components to the boundary behave differently. We establish its Helmholtz decomposition when the domain is a perturbed half space with small perturbation. This substantially extends the authors' earlier results for a half space and a bounded domain.

6.1 Introduction

The Helmholtz decomposition of a vector field is a fundamental tool to analyze the Stokes and the Navier-Stokes equations. It is formally a decomposition of a vector field $v = (v^1, \dots, v^n)$ in a domain Ω of $\mathbf{R}^n$ into

$$v = v_0 + \nabla q; \tag{6.1.1}$$

here v_0 is a divergence free vector field satisfying supplemental conditions like boundary condition and ∇q denotes the gradient of a function (scalar field) q. If v is in L^p $(1 < p < \infty)$ in Ω, such a decomposition is well-studied. For example, a topological direct sum decomposition

$$(L^p(\Omega))^n = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

holds for various domains including $\Omega = \mathbf{R}^n$, a half space $\mathbf{R}^n_+$, a bounded smooth domain [8]; see e.g. G. P. Galdi [9]. Here, $L^p_\sigma(\Omega)$ denotes the L^p-closure of the space of all div-free vector fields compactly supported in Ω and $G^p(\Omega)$ denotes the totality of L^p gradient fields. It is impossible to extend this Helmholtz decomposition to L^∞ even if $\Omega = \mathbf{R}^n$ since the projection $v \mapsto \nabla q$ is a composite of the Riesz operators which is not bounded in L^∞. We have to replace L^∞ with a class of functions of bounded mean oscillation. If the vector field is of bounded mean oscillation (BMO for short), such a problem is studied in the cases when Ω is a half space $\mathbf{R}^n_+$ [10] and a bounded C^3 domain [12]. Our goal in this chapter is to establish the Helmholtz decomposition of BMO vector fields in a perturbed C^3 half space with small perturbation in $\mathbf{R}^n$, which is an example of a domain with curved and non-compact boundary.

Let us recall the BMO space of vector fields introduced in [11] and [12]. We first recall the BMO seminorm for $\mu \in (0, \infty]$. For a locally integrable function f, i.e., $f \in L^1_{\mathrm{loc}}(\Omega)$ we define

$$[f]_{BMO^\mu(\Omega)} := \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| f(y) - f_{B_r(x)} \right| dy \;\middle|\; B_r(x) \subset \Omega,\ r < \mu \right\},$$

where f_B denotes the average over B, i.e.,

$$f_B := \frac{1}{|B|} \int_B f(y)\, dy$$

and $B_r(x)$ denotes the closed ball of radius r centered at x and $|B|$ denotes the Lebesgue measure of B. The space $BMO^\mu(\Omega)$ is defined as

$$BMO^\mu(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; [f]_{BMO^\mu} < \infty \right\}.$$

This space may not agree with the space of restrictions $r_\Omega f$ of $f \in BMO^\mu(\mathbf{R}^n)$. As in [2], [3], [4], [5] we introduce a seminorm controlling the boundary behavior. For $\nu \in (0, \infty]$, we set

$$[f]_{b^\nu} := \sup \left\{ r^{-n} \int_{\Omega \cap B_r(x)} |f(y)|\, dy \;\middle|\; x \in \Gamma,\ 0 < r < \nu \right\}.$$

In these papers, the space

$$BMO_b^{\mu,\nu}(\Omega) := \left\{ f \in BMO^\mu(\Omega) \;\middle|\; [f]_{b^\nu} < \infty \right\}$$

is considered. Note that this space $BMO_b^{\infty,\infty}(\Omega)$ is identified with Miyachi's BMO introduced by [22] if Ω is a bounded Lipschitz domain or a Lipschitz half space as proved in [5]. Unfortunately, it turns out such a boundary control for all components of vector fields is too strict to have the Helmholtz decomposition. We separate tangential and normal components. Let d denote the signed distance function from Γ which is defined by

$$d(x) = \begin{cases} \inf_{y \in \Gamma} |x - y| & \text{for} \quad x \in \Omega, \\ -\inf_{y \in \Gamma} |x - y| & \text{for} \quad x \notin \Omega \end{cases}$$

so that $d(x) = d_\Gamma(x)$ for $x \in \Omega$.

For vector fields of bounded mean oscillation, we consider

$$vBMO^{\mu,\nu}(\Omega) := \left\{ v \in (BMO^\mu(\Omega))^n \;\middle|\; [\nabla d \cdot v]_{b^\nu} < \infty \right\},$$

where $\cdot$ denotes the standard inner product in $\mathbf{R}^n$. We call the quantity $(\nabla d \cdot v)\nabla d$ on Γ to be the component of v normal to the boundary Γ. We set

$$[v]_{vBMO^{\mu,\nu}(\Omega)} := [v]_{BMO^\mu(\Omega)} + [\nabla d_\Gamma \cdot v]_{b^\nu}.$$

In the case where Ω is the half space $\mathbf{R}^n_+$, $[\cdot]_{vBMO^{\mu,\nu}(\Omega)}$ is not a norm but a seminorm if either μ or ν is finite. However, if the boundary Γ has a fully curved part in the sense of [11, Definition 7], then this becomes a norm [11, Lemma 8]. In particular, when Ω is a bounded C^2 domain, this is a norm. Roughly speaking, the boundary behavior of a vector field v is controlled for only normal part of v if $v \in vBMO^{\mu,\nu}(\Omega)$. If Ω is a bounded domain, this norm is equivalent no matter how μ and ν are taken; in other words, $vBMO^{\mu,\nu}(\Omega) = vBMO^{\infty,\infty}(\Omega)$. This is because $vBMO^{\mu,\nu}(\Omega) \subset L^1(\Omega)$ when Ω is bounded, which follows from the characterization of $vBMO^{\mu,\nu}(\Omega)$ in [11, Theorem 9]. Without loss of generality, we can simply write $vBMO^{\mu,\nu}(\Omega)$ as $vBMO(\Omega)$ in this case. However, if Ω is an unbounded space, then Γ does not necessarily have a fully curved part. Hence in this case, $[\cdot]_{vBMO^{\mu,\nu}(\Omega)}$ is not necessarily a norm. Moreover, the space $vBMO^{\mu,\nu}(\Omega)$ depends on the value of μ and ν. As a result, instead of working with $vBMO^{\mu,\nu}(\Omega)$ directly, we consider its intersection with the $(L^2(\Omega))^n$, i.e., we consider the space

$$vBMOL^2(\Omega) := vBMO^{\mu,\nu}(\Omega) \cap (L^2(\Omega))^n$$

with

$$\|v\|_{vBMOL^2(\Omega)} := [v]_{vBMO^{\mu,\nu}(\Omega)} + \|v\|_{(L^2(\Omega))^n}.$$

Note that this space $vBMOL^2(\Omega)$ is a Banach space which is independent of $\mu, \nu \in (0, \infty]$.

We denote $x' := (x_1, x_2, ..., x_{n-1})$ for $x \in \mathbf{R}^n$. Let $h \in C_0^3(\mathbf{R}^{n-1})$. We define the perturbed half space $\mathbf{R}^n_h$ to be the space

$$\mathbf{R}^n_h := \{x = (x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}.$$

Without loss of generality, we may assume that $\operatorname{supp} h \subset B_{R_h}(0')$ for some $R_h > 0$ where $B_{R_h}(0')$ denotes the ball in $\mathbf{R}^{n-1}$ with center $0'$ and radius R_h. Let $C_* > 0$ be a fixed constant that are going to be determined later in this chapter. We say the perturbed C^3 half space $\mathbf{R}^n_h$ is of small perturbation if the condition

$$(R_h^{n-1} + 1)\|h\|_{C^2(\mathbf{R}^{n-1})} < \frac{1}{2C_*} \tag{6.1.2}$$

holds. Now we are ready to state our main theorem.

Theorem 6.1.1. *Let Ω be a perturbed C^3 half space in $\mathbf{R}^n$ with small perturbation. Then the topological direct sum decomposition*

$$vBMOL^2(\Omega) = vBMOL^2_\sigma(\Omega) \oplus GvBMOL^2(\Omega) \tag{6.1.3}$$

holds with

$$\begin{aligned} vBMOL^2_\sigma(\Omega) &:= \left\{v \in vBMOL^2(\Omega) \mid \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot \mathbf{n} = 0 \text{ on } \Gamma\right\}, \\ GvBMOL^2(\Omega) &:= \left\{\nabla q \in vBMOL^2(\Omega) \mid q \in L^1_{\mathrm{loc}}(\Omega)\right\}, \end{aligned}$$

where $\mathbf{n}$ denotes the exterior unit normal vector field. In other words, for $v \in vBMOL^2(\Omega)$, there is unique $v_0 \in vBMOL^2_\sigma(\Omega)$ and $\nabla q \in GvBMOL^2(\Omega)$ satisfying $v = v_0 + \nabla q$. Moreover, the mappings $v \mapsto v_0$, $v \mapsto \nabla q$ are bounded in $vBMOL^2(\Omega)$.

Our strategy to prove Theorem 6.1.1 follows from the strategy we used to establish the Helmholtz decomposition in a bounded C^3 domain [12]. Let E be the fundamental solution of $-\Delta$ in $\mathbf{R}^n$, i.e.,

$$E(x) := \begin{cases} -\log|x|/2\pi & (n = 2) \\ |x|^{2-n}/\left(n(n-2)\alpha(n)\right) & (n \geq 3), \end{cases}$$

where $\alpha(n)$ denotes the volume of the unit ball $B_1(0)$ of $\mathbf{R}^n$. By [Theorem 5.1.2, Chapter 5], we see that as long as the regularity of Γ is of uniformly C^2, the space $BMO^\infty(\Omega) \cap L^2(\Omega)$ allows the standard cut-off, i.e., we are able to decompose v into two parts $v = v_1 + v_2$ with $v_1 = \varphi v$ and $v_2 = v - v_1$ with some $\varphi \in C_0^\infty(\Omega)$. Then, the support of v_2 lies in a small neighborhood of Γ whereas the support of v_1 is away from Γ. For v_1 we just set

$$q_1^1 = E * \operatorname{div} v_1$$

by extending v_1 as zero outside its support. Then, the L^∞ bound for ∇q_1^1 is well controlled near Γ, which yields a bound for b^ν semi-norm. To estimate v_2, we use a normal coordinate system near Γ and reduce the problem to the half space. Let d denotes the signed distance function where $d = d_\Gamma$ in Ω and $d = -d_\Gamma$ outside Ω. We extend v_2 to $\mathbf{R}^n$ so that the normal part $(\nabla d \cdot \overline{v_2})\nabla d$ is odd and the tangential part $\overline{v_2} - (\nabla d \cdot \overline{v_2})\nabla d$ is even in the direction of ∇d with respect to Γ. In such type of coordinate system, the minus Laplacian can be transformed as

$$L = A - B + \text{lower order terms}, \;\; A = -\Delta_\eta, \;\; B = \sum_{1 \leq i,j \leq n-1} \partial_{\eta_i} b_{ij} \partial_{\eta_j},$$

where η_n is the normal direction to the boundary so that $\{\eta_n > 0\}$ is the half space. By choosing a suitable coordinate system to represent Γ locally, we are able to arrange $b_{ij} = 0$ at one point of the boundary of the local coordinate system. We use a freezing coefficient method to construct volume potential q_1^{tan} and q_1^{nor}, which corresponds to the contribution from the tangential part $\overline{v_2}^{\mathrm{tan}}$ and the normal part $\overline{v_2}^{\mathrm{nor}}$ respectively. Since the leading term of $\operatorname{div} \overline{v_2}^{\mathrm{nor}}$ in normal coordinate consists of the differential of η_n only, if we extend the coefficient b_{ij} even in η_n, q_1^{nor} is constructed so that the leading term of $\nabla d \cdot \nabla q_1^{\mathrm{nor}}$ is odd in the direction of ∇d. On the other hand, as the leading term of $\operatorname{div} \overline{v_2}^{\mathrm{tan}}$ in normal coordinate consists of the differential of $\eta' = (\eta_1, ..., \eta_{n-1})$ only, the even extension of b_{ij} in η_n gives rise to q_1^{tan} so that the leading term of $\nabla d \cdot \nabla q_1^{\mathrm{tan}}$ is also odd in the direction of ∇d. Disregarding lower order terms and localization procedure, we set q_1^{tan} and q_1^{nor} of the form

$$\begin{aligned} q_1^{\mathrm{tan}} &= -L^{-1} \operatorname{div} \overline{v}_2^{\mathrm{tan}} = -A^{-1}(I - BA^{-1})^{-1} \operatorname{div} \overline{v}_2^{\mathrm{tan}}, \\ q_1^{\mathrm{nor}} &= -L^{-1} \operatorname{div} \overline{v}_2^{\mathrm{nor}} = -A^{-1}(I - BA^{-1})^{-1} \operatorname{div} \overline{v}_2^{\mathrm{nor}}. \end{aligned}$$

One is able to arrange BA^{-1} small by taking a small neighborhood of a boundary point. Then $(I - BA^{-1})^{-1}$ is given as the Neumann series $\sum_{m=0}^\infty (BA^{-1})^m$. We are able to establish BMO-BMO estimate for $\nabla q_1^{\mathrm{tan}}$ and $\nabla q_1^{\mathrm{nor}}$, i.e.

$$\left[\nabla q_1^{\mathrm{tan}}\right]_{BMO(\mathbf{R}^n)} \leq C_0' \left[\overline{v}_2^{\mathrm{tan}}\right]_{BMO(\mathbf{R}^n)}, \; \left[\nabla q_1^{\mathrm{nor}}\right]_{BMO(\mathbf{R}^n)} \leq C_0' \left[\overline{v}_2^{\mathrm{nor}}\right]_{BMO(\mathbf{R}^n)}$$

with some constant C_0' independent of $\overline{v_2}$. Since the leading term of $\nabla d \cdot (\nabla q_1^{\mathrm{tan}} + \nabla q_1^{\mathrm{nor}})$ is odd in the direction of ∇d with respect to Γ, the BMO bound implies b^ν bound. Note that $[\overline{v_2}^{\mathrm{nor}}]_{BMO(\mathbf{R}^n)}$ is controlled by $[v_2]_{b^\nu}$ and $[v_2]_{BMO^\infty(\Omega)}$ since $\overline{v_2}^{\mathrm{nor}}$ is odd in the direction of ∇d with respect to Γ. By the procedure sketched above, we are able to construct a suitable operator by setting $q_1 = q_1^1 + q_1^{\mathrm{tan}} + q_1^{\mathrm{nor}}$. Since many steps in the construction of volume potential q_1 in this case follows exactly from the theory in [12, Section 3] and Chapter 5, for these parts we provide necessary results directly without giving their proofs.

Theorem 6.1.2 (Construction of a suitable volume potential)**.** *Let* Ω *be a uniformly* C^3 *domain in* $\mathbf{R}^n$*. Then, there exists a linear operator* $v \longmapsto q_1$ *from* $vBMOL^2(\Omega)$ *to* $L^\infty(\Omega)$ *such that*

$$-\Delta q_1 = \operatorname{div} v \quad \text{in} \quad \Omega$$

and that there exists a constant $C_1 = C_1(\Omega)$ *satisfying*

$$\|\nabla q_1\|_{vBMOL^2(\Omega)} \leq C_1 \|v\|_{vBMOL^2(\Omega)}.$$

In particular, the operator $v \longmapsto \nabla q_1$ *is a bounded linear operator in* $vBMOL^2(\Omega)$*.*

By this operator, we observe that $w = v - \nabla q_1$ is divergence free in Ω. Unfortunately, this w may not fulfill the trace condition $w \cdot \mathbf{n} = 0$ on the boundary Γ. We construct another potential q_2 by solving the Neumann problem

$$\begin{aligned} \Delta q_2 &= 0 && \text{in} \quad \Omega \\ \frac{\partial q_2}{\partial n} &= w \cdot \mathbf{n} && \text{on} \quad \Gamma. \end{aligned}$$

We then set $q = q_1 + q_2$. Since $\partial q_2/\partial \mathbf{n} = \nabla q_2 \cdot \mathbf{n}$, $v_0 = v - \nabla q$ gives the Helmholtz decomposition (6.1.1). To complete the proof of Theorem 6.1.1, it suffices to prove that $\|\nabla q_2\|_{vBMOL^2(\Omega)}$ is bounded by a constant multiply of $\|v\|_{vBMOL^2(\Omega)}$.

Lemma 6.1.3 (Estimate of the normal trace)**.** *Let* Ω *be a uniformly* $C^{2+\kappa}$ *domain in* $\mathbf{R}^n$ *with* $\kappa \in (0,1)$*. Then there is a constant* $C_2 = C_2(\Omega)$ *such that*

$$\|w \cdot \mathbf{n}\|_{L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)} \leq C_2 \|w\|_{vBMOL^2(\Omega)}$$

for all $w \in vBMOL^2(\Omega)$ *with* $\operatorname{div} w = 0$*.*

The L^∞ estimate of $w \cdot \mathbf{n}$ follows from the trace theorem established in [11]. For the $H^{-\frac{1}{2}}$ estimate for $w \cdot \mathbf{n}$, we split the boundary into the straight part and the curved part. Since we have the L^∞ estimate for $w \cdot \mathbf{n}$ and the curved part is compact, the contribution in the $H^{-\frac{1}{2}}$ estimate for $w \cdot \mathbf{n}$ that comes from the curved part can be estimated by the L^∞ norm of $w \cdot \mathbf{n}$ directly. For the contribution in the $H^{-\frac{1}{2}}$ estimate of $w \cdot \mathbf{n}$ that comes from the straight part, we invoke the $H^{-\frac{1}{2}}$ estimate of $w \cdot \mathbf{n}$ in the case of the half space. Hence, we finally need the estimate for the Neumann problem.

Lemma 6.1.4 (Estimate for the Neumann problem)**.** *Let* $\Omega \subset \mathbf{R}^n$ *be a perturbed* C^2 *half space with small perturbation. For* $g \in L^\infty(\Gamma)$ *satisfying* $\int_\Gamma g \, d\mathcal{H}^{n-1} = 0$*, there exists a unique (up to constant) solution* u *to the Neumann problem*

$$\begin{aligned} \Delta u &= 0 && \text{in} \quad \Omega \\ \frac{\partial u}{\partial \mathbf{n}} &= g && \text{on} \quad \Gamma \end{aligned} \tag{6.1.4}$$

such that the operator $g \longmapsto u$ *is linear and that there exists a constant* $C_3 = C_3(\Omega)$ *such that*

$$\|\nabla u\|_{vBMOL^2(\Omega)} \leq C_3 \|g\|_{L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)}.$$

Combining these two lemmas with Theorem 6.1.2 yields

$$\begin{aligned}\|\nabla q_2\|_{vBMOL^2(\Omega)} &\le C_3C_2\|v-\nabla q_1\|_{vBMOL^2(\Omega)}\\ &\le C_3C_2(1+C_1)\|v\|_{vBMOL^2(\Omega)}.\end{aligned}$$

Setting $q = q_1+q_2$ and $v_0 = v-\nabla q$, we now observe that the projections $v \longmapsto v_0$, $v \longmapsto \nabla q$ are bounded in $vBMOL^2(\Omega)$, which yields (6.1.3) in Theorem 6.1.1.

For $g \in L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)$, we consider the single layer potential

$$E * (\delta_\Gamma \otimes g)(x) := \int_\Gamma E(x-y)g(y)\, d\mathcal{H}^{n-1}(y)$$

for $x \in \mathbf{R}^n$. To show Lemma 6.1.4, we firstly estimate

$$\|\nabla E * (\delta_\Gamma \otimes g)\|_{vBMO^{\infty,\nu}(\Omega)}\,.$$

For the BMO estimate, we set $g_1(y', h(y')) := 1_{B_{2R_h}(0')}(y')g(y', h(y'))$ for $(y', h(y')) \in \Gamma$ and $g_2 := g - g_1$. By setting $\overline{g_2}(y', 0) = 0$ for $|y'| < 2R_h$ and $\overline{g_2}(y', 0) = g_2(y', 0)$ for $|y'| \ge 2R_h$, we see that the equality

$$E * (\delta_\Gamma \otimes g_2)(x) = E * (\delta_{\partial\mathbf{R}^n_+} \otimes \overline{g_2})(x)$$

holds for any $x \in \mathbf{R}^n$. Thus, the BMO estimate of $\nabla E * (\delta_\Gamma \otimes g_2)$ follows from the BMO estimate of $E * (\delta_{\partial\mathbf{R}^n_+} \otimes \overline{g_2})$. Since $g_1(\cdot, h(\cdot))$ is compactly supported in $\mathbf{R}^{n-1}$, the BMO estimate for $\nabla E * (\delta_\Gamma \otimes g_1)$ follows directly from [12, Lemma 5], which contains a similar estimate that is established in the case of a compact boundary. It is very subtle but by a direct calculation, we may deduce the estimate

$$\sup_{x\in\Gamma_\nu} \int_\Gamma \left| \frac{\partial E}{\partial \mathbf{n}_y}(x-y) \right| d\mathcal{H}^{n-1}(y) < \infty,$$

where $\Gamma_\nu := \{x \in \Omega \mid d(x) < \nu\}$ denotes a small neighborhood of Γ in Ω. Let $x \in \Gamma_\nu$. By making use of this estimate, we can show that

$$\left|\nabla d(x) \cdot \nabla \left(E * (\delta_\Gamma \otimes g_1)\right)(x)\right| \le C\|g\|_{L^\infty(\Gamma)}.$$

Since the kernel $|\cdot'|^{1-n}$ is integrable in $L^2(B_M(0')^{\mathrm{c}})$ for any $M > 0$, we are able to prove that

$$\left|\nabla d(x) \cdot \nabla \big(E * (\delta_\Gamma \otimes g_2)\big)(x)\right| \le C\|g\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Hence, we obtain an estimate for $\|\nabla d \cdot \nabla E * (\delta_\Gamma \otimes g)\|_{L^\infty(\Gamma_\nu)}$ by $\|g\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)}$. The b^ν estimate therefore follows.

Let $g \in L^\infty(\Gamma)$. The trace of the double layer potential

$$(Pg)(x) = \int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y)g(y)\, d\mathcal{H}^{n-1}(y), \quad x \in \Gamma_\nu$$

is of the form

$$\big(\gamma(Pg)\big)(x', h(x')) = \frac{1}{2}g(x', h(x')) - \big(Sg\big)(x', h(x')),$$

where S is a bounded linear operator on $L^\infty(\Gamma)$ satisfying

$$\|S\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)} \le C_* R_h^{n-1}\|h\|_{C^2(\mathbf{R}^{n-1})}$$

for some constant C_* independent of h. Moreover, we have that $Sg \in L^2(\Gamma)$ satisfies the estimate

$$\|Sg\|_{L^2(\Gamma)} \leq C^* R_h^{\frac{n-1}{2}} \|g\|_{L^\infty(\Gamma)}$$

with some constant C^* independent of h and g. Therefore, if Ω is a perturbed half space of small perturbation, the inverse of $I - 2S$ is well-defined as a bounded linear map from $L^\infty(\Gamma)$ to $L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)$ by the Neumann series

$$(I - 2S)^{-1} = \sum_{i=0}^{\infty} (2S)^i.$$

Since Pg is harmonic in Ω, the solution to the Neumann problem (6.1.4) is formally given by

$$u(x) = E * \big(\delta_\Gamma \otimes (2(I - 2S)^{-1} g)\big)(x), \quad x \in \Omega.$$

If we can estimate the L^2 norm of ∇u in Ω, then we are done. Fortunately, we indeed have this estimate. In the case of a half space, if $g \in H^{-\frac{1}{2}}(\partial \mathbf{R}^n_+)$ satisfies

$$\int_{\partial \mathbf{R}^n_+} g(y)\, d\mathcal{H}^{n-1}(y) = 0,$$

then the estimate

$$\|\nabla E * (\delta_{\partial \mathbf{R}^n_+} \otimes g)\|_{(L^2(\mathbf{R}^n_+))^n} \leq C \|g\|_{H^{-\frac{1}{2}}(\partial \mathbf{R}^n_+)}$$

holds with some constant C independent of g. This estimate holds for the reason that the single layer potential $E * (\delta_{\partial \mathbf{R}^n_+} \otimes g)$ is exactly half of the solution to the Neumann problem in the half space, and the Neumann problem in the half space admits a unique weak solution (up to an additive constant) $u \in H^1(\mathbf{R}^n_+)$ which satisfies

$$\|\nabla u\|_{(L^2(\mathbf{R}^n_+))^n} \leq C \|g\|_{H^{-\frac{1}{2}}(\partial \mathbf{R}^n_+)}$$

with C independent of g, see e.g. [26, Remark 1.2 and Remark 1.3], [21, Section 1.7]. In the case that Ω is a perturbed C^2 half space, for $g \in L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)$, we consider g as a sum of g_1 and g_2. If the integral of g on Γ equals zero, then there exists a constant $I_c \in \mathbf{R}$ such that

$$\int_{\partial \mathbf{R}^n_+} \overline{g_2}(y) + f_s(y)\, d\mathcal{H}^{n-1}(y) = 0$$

with

$$f_s(x', 0) = \begin{cases} 0 & \text{for} \quad |x'| \geq 2M_h \\ \frac{I_c}{|B_{2M_h}(0')|} & \text{for} \quad |x'| < 2M_h. \end{cases}$$

Since $\overline{g_2} + f_s \in H^{-\frac{1}{2}}(\partial \mathbf{R}^n_+)$, by the L^2 estimate in the half space case, we deduce that

$$\|\nabla E * \big(\delta_{\partial \mathbf{R}^n_+} \otimes \overline{g_2}\big)\|_{(L^2(\mathbf{R}^n_+))^n} \leq \|\nabla E * \big(\delta_{\partial \mathbf{R}^n_+} \otimes f_s\big)\|_{(L^2(\mathbf{R}^n_+))^n} + C \|g\|_{L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)}.$$

In addition, if the support of $g(\cdot', h(\cdot'))$ is contained in $\overline{B_{2R_h}(0')}$, we apply the idea in [12, Lemma 5] which extends $g \in L^\infty(\Gamma)$ to $g_e \in L^\infty(\Gamma^{2\delta})$ by letting $g_e(x) := g(\pi x)$ for any $x \in \Gamma^{2\delta}$ with πx denoting the projection of x on Γ. By multiplying a cutoff function θ_d to g_e where $\theta_d(x) = 1$ for $|d(x)| \leq \delta$ and $\theta_d(x) = 0$ for $|d(x)| \geq 2$, we see that $g_{e,c} := \theta_d g_e \in L^\infty(\mathbf{R}^n)$ is of compact support. Since

$$\begin{aligned}\delta_\Gamma \otimes g &= (\nabla 1_\Omega \cdot \nabla d) g_{e,c} \\ &= \operatorname{div}(g_{e,c} 1_\Omega \nabla d) - 1_\Omega \operatorname{div}(g_{e,c} \nabla d), \\ \operatorname{div}(g_{e,c}\nabla d) &= g_{e,c}\Delta d + \nabla d \cdot \nabla g_{e,c} = g_{e,c}\Delta d + (\nabla d \cdot \nabla \theta_d) g_e,\end{aligned}$$

we have that

$$\nabla E * (\delta_\Gamma \otimes g_1) = \nabla \operatorname{div}\left(E * (g_{e,c} 1_\Omega \nabla d)\right) - \nabla E * (1_\Omega g_e f_\theta) = I_1 + I_2$$

where $f_\theta := \theta_d \Delta d + \nabla d \cdot \nabla \theta_d$. Since $\nabla \operatorname{div} E$ is L^p for any $1 < p < \infty$, see e.g. [15, Theorem 5.2.7 and Theorem 5.2.10], I_1 can be estimated as

$$\|I_1\|_{(L^2(\mathbf{R}^n))^n} \leq C \|g_{e,c} 1_\Omega \nabla\|_{(L^2(\mathbf{R}^n))^n} \leq C \|g\|_{L^\infty(\Gamma)}.$$

Since $\nabla E \sim |\cdot|^{1-n}$, the famous Hardy-Littlewood-Sobolev inequality [1, Theorem 1.7] implies that

$$\|I_2\|_{(L^2(\mathbf{R}^n))^n} \leq C \|1_\Omega g_e f_\theta\|_{(L^2(\mathbf{R}^n))^n} \leq C \|g\|_{L^\infty(\Gamma)}.$$

Hence, it can be deduced that

$$\|\nabla E * \left(\delta_{\partial \mathbf{R}^n_+} \otimes f_s\right)\|_{(L^2(\mathbf{R}^n_+))^n} + \|\nabla E * \left(\delta_\Gamma \otimes g_1\right)\|_{(L^2(\Omega))^n} \leq C \|g\|_{L^\infty(\Gamma)}.$$

Since $\nabla E * \left(\delta_{\partial \mathbf{R}^n_+} \otimes \overline{g_2}\right) = \nabla E * \left(\delta_\Gamma \otimes g_2\right)$, we finally obtain our desired L^2 estimate

$$\|\nabla E * \left(\delta_\Gamma \otimes g\right)\|_{(L^2(\Omega))^n} \leq C \|g\|_{L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)}.$$

If $g \in L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma) \cap L^1(\Gamma)$, then without assuming the integral of g on Γ to be zero, we can deduce the L^2 estimate

$$\|\nabla E * \left(\delta_\Gamma \otimes g\right)\|_{(L^2(\Omega))^n} \leq C \|g\|_{L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma) \cap L^1(\Gamma)}$$

in a similar way. Since we also have $Sg \in L^1(\Gamma)$ for $g \in L^\infty(\Gamma)$, the series $\sum_{i=1}^\infty (2S)^i g$ is well-defined in $L^1(\Gamma)$ as long as the smallness condition

$$(R_h^{n-1} + 1) \|h\|_{C^2(\mathbf{R}^{n-1})} < \frac{1}{2C_*}$$

is satisfied. Therefore, the L^2 estimate for ∇u holds. We obtain Lemma 6.1.4.

Our approach in this chapter is to derive the boundedness of the operator $v \mapsto \nabla q$ by a potential-theoretic approach. In L^p setting there is a variational approach based on duality introduced by [23]; see also [6]. The key estimate is

$$\|\nabla q\|_{L^p(\Omega)} \leq C_5 \sup \left\{ \int_\Omega \nabla q \cdot \nabla \varphi \, dx \;\middle|\; \|\nabla \varphi\|_{L^{p'}(\Omega)} \leq 1 \right\}$$

with C_5 independent of q, where $1/p + 1/p' = 1$, $1 < p < \infty$. Formally, this estimate yields the desired bound $\|\nabla q\|_{L^p(\Omega)} \leq C_5 \|v\|_{L^p(\Omega)}$ since

$$\int_\Omega \nabla q \cdot \nabla \varphi \, dx = \int_\Omega v \cdot \nabla \varphi \, dx.$$

At this moment, it is not clear that similar estimate holds if one replaces $L^p(\Omega)$ by $vBMO$ since the predual space of $vBMO$ is not clear.

This chapter is organized as follows. In Section 6.2, we construct a volume potential corresponding to $\operatorname{div} v$. We localize the problem and reduce the problem to small neighborhoods of points on the boundary. Here we invoke the theory established in [12] and Chapter 5 to give a proof to Theorem 6.1.2. In Section 6.3, we establish Lemma 6.1.4 by estimating the single layer potential.

Throughout this chapter, unless otherwise specified, the symbol C in an inequality represents a positive constant independent of quantities that appeared in the inequality. For a fixed $\rho > 0$, C_ρ represents a constant depending only on ρ. C_n represents a constant depending only on n and $C_{\Omega,n}$ represents a constant depending only on Ω and n.

6.2 Volume potential construction in a uniformly C^3 domain

For $v \in vBMOL^2(\Omega)$, we shall construct a suitable potential q_1 so that $v \longmapsto \nabla q_1$ is a bounded linear operator in $vBMOL^2$ as stated in Theorem 6.1.2. The construction in the case where Ω is a uniformly C^3 domain basically follows from the theory in [12], in which Ω is assumed to be a bounded C^3 domain.

6.2.1 Localization tools

Let us recall the uniform estimates established in Chapter 5. We denote $x' = (x_1, x_2, ..., x_{n-1})$ for $x \in \mathbf{R}^n$ and $\nabla' := (\partial_1, \partial_2, ..., \partial_{n-1})$. Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$. In other words, there exists $r_*, \delta_* > 0$ such that for each $z_0 \in \Gamma$, up to translation and rotation, there exists a function h_{z_0} which is C^2 in a closed ball $B_{r_*}(0')$ of radius r_* centered at the origin $0'$ of $\mathbf{R}^{n-1}$ satisfying following properties:

(i) $K_\Gamma := \sup_{B_{r_*}(0')} |(\nabla')^s h_{z_0}| < \infty$ for $s = 0, 1, 2$; $\nabla' h(0') = 0$, $h(0') = 0$,

(ii) $\Omega \cap U_{r_*,\delta_*,h_{z_0}}(z_0) = \left\{(x', x_n) \in \mathbf{R}^n \;\middle|\; h_{z_0}(x') < x_n < h_{z_0}(x') + \delta_*,\ |x'| < r_*\right\}$ for

$$U_{r_*,\delta_*,h_{z_0}}(z_0) := \left\{(x', x_n) \in \mathbf{R}^n \;\middle|\; h_{z_0}(x') - \delta_* < x_n < h_{z_0}(x') + \delta_*,\ |x'| < r_*\right\},$$

(iii) $\Gamma \cap U_{r_*,\delta_*,h_{z_0}}(z_0) = \left\{(x', x_n) \in \mathbf{R}^n \;\middle|\; x_n = h_{z_0}(x'),\ |x'| < r_*\right\}$.

We say that Ω is of type $(r_*, \delta_*, K_\Gamma)$.

Since Ω is a uniformly C^2 domain, there is $R_* > 0$ such that if $|d(x)| < R_*$, there is unique point πx such that $|x - \pi x| = |d(x)|$. The supremum of such R_* is called the reach of Ω and Ω^c. For $\delta \in (0, R_*]$, we set that

$$\Gamma^\delta := \left\{x \in \mathbf{R}^n \;\middle|\; |d(x)| < \delta\right\}$$

and

$$\Gamma_\delta := \left\{x \in \Omega \;\middle|\; d_\Gamma(x) < \delta\right\}.$$

Moreover, d is C^2 in the R_*-neighborhood of Γ, i.e., $d \in C^2(\Gamma^{R_*})$; see [14, Chap. 14, Appendix], [20, §4.4]. Note that R_* satisfies

$$R_* = \min\left(R_*^\Omega, R_*^{\Omega^c}\right),$$

where R_*^Ω is the reach of Γ in Ω while $R_*^{\Omega^c}$ is the reach of Γ in the complement Ω^c of Ω. Let $K_\Gamma^* := \max\{K_\Gamma, 1\}$. There exists $0 < \rho_0 < \min(r_*, \delta_*, \frac{R_*}{2}, \frac{1}{2nK_\Gamma^*})$ such that

$$U_\rho(z_0) := \{x \in \mathbf{R}^n \mid (\pi x)' \in \operatorname{int} B_\rho(0'),\ |d(x)| < \rho\}$$

is contained in the coordinate chart $U_{r_*,\delta_*,h_{z_0}}(z_0)$ for any $\rho \le \rho_0$.

We next consider the normal coordinate in $U_{\rho_0}(z_0)$

$$x = \psi(\eta) = \begin{cases} \eta' + \eta_n \nabla' d(\eta', h_{z_0}(\eta')); \\ h_{z_0}(\eta') + \eta_n \partial_{x_n} d(\eta', h_{z_0}(\eta')) \end{cases} \tag{6.2.1}$$

or shortly

$$x = \pi x - d(x)\mathbf{n}(\pi x), \quad \mathbf{n}(\pi x) = -\nabla d(\pi x).$$

For each $z_0 \in \Gamma$, ψ is indeed a local C^1-diffeomorphism which maps V_{ρ_0} to $U_{\rho_0}(z_0)$ where $V_{\rho_0} := B_{\rho_0}(0') \times (-\rho_0, \rho_0)$. We indeed have that $\psi \in C^1(V_{\rho_0})$ with $(\nabla_\eta \psi)(0) = I$. Let $\varepsilon \in (0,1)$ and $c_\Omega^\varepsilon := \min\{\frac{\varepsilon}{K_\Gamma^* \cdot ((n+1)!)^2 \cdot 2^{2n+5}}, \frac{\rho_0}{2}\}$. Regardless of $z_0 \in \Gamma$, we can uniformly estimate the gradient of ψ and ψ^{-1} simultaneously.

Proposition 6.2.1 (Chapter 5). *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain with $n \ge 2$. Then for any $\rho \in (0, c_\Omega^\varepsilon]$ and $z_0 \in \Gamma$, the estimates*

$$\|\nabla F - I\|_{L^\infty(V_\rho)} < \varepsilon,$$
$$\|\nabla F^{-1} - I\|_{L^\infty(U_\rho(z_0))} < \varepsilon$$

hold simultaneously.

For $\rho \in (0, \rho_0/2)$, there exists a locally finite open cover of Γ^ρ, i.e., we have that

Proposition 6.2.2 (Chapter 5). *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain with $n \ge 2$. There exist a countable family of points in Γ, say $S := \{x_i \in \Gamma \mid i \in \mathbf{N}\}$, and a natural number $N_* \in \mathbf{N}$ such that*

$$\Gamma^\rho = \bigcup_{x_i \in S} U_\rho(x_i)$$

and for any $x_j \in S$, there exist at most N_* points in S, say $\{x_{j_1}, ..., x_{j_{N_*}}\} \subset S$, with

$$U_\rho(x_j) \cap U_\rho(x_{j_l}) \neq \emptyset$$

for each $1 \le l \le N_*$.

Based on this open cover of Γ^ρ, a partition of unity for Γ^ρ can be constructed as follow.

Proposition 6.2.3 (Chapter 5). *There exist $\varphi_i \in C^1(\Gamma^\rho)$ for each $i \in \mathbf{N}$ and a constant C_U such that properties*

$$\begin{aligned} &0 \le \varphi_i \le 1 \quad \text{for any} \quad i \in \mathbf{N}, \\ &\operatorname{supp} \varphi_i \subset \overline{U_\rho(x_i)} \quad \text{for any} \quad i \in \mathbf{N}, \\ &\operatorname{supp} \varphi_i \circ \psi \subset B_\rho(0') \times [-\rho, \rho], \\ &\sum_{i=1}^\infty \varphi_i(x) \equiv 1 \quad \text{for any} \quad x \in \Gamma^\rho, \\ &\sup_{i \in \mathbf{N}} \|\nabla \varphi_i\|_{L^\infty(\Gamma^\rho)} \le C_U \end{aligned} \tag{6.2.2}$$

hold.

6.2.2 Cut-off and extension

In general, multiplication by a smooth function to BMO is not bounded in BMO. However, such multiplication is bounded in bmo.

Proposition 6.2.4 (Multiplication)**.** *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain, $n \geq 2$. Let $\varphi \in C^\gamma(\Omega)$ with $\gamma \in (0,1)$. For each $v \in vBMOL^2(\Omega)$, the function $\varphi v \in vBMOL^2(\Omega)$ satisfies*

$$\|\varphi v\|_{vBMOL^2(\Omega)} \leq C\|\varphi\|_{C^\gamma(\Omega)}\|v\|_{vBMOL^2(\Omega)}$$

with C independent of φ and v.

Proof. Since

$$[\nabla d \cdot \varphi v]_{b^\nu} \leq \|\varphi\|_{L^\infty(\Omega)} [\nabla d \cdot v]_{b^\nu},$$

this proposition follows trivially from the product estimate [Theorem 5.1.2, Chapter 5], by which $\|\varphi v\|_{bmo^\infty_\infty(\Omega)}$ is estimated by the product of $\|\varphi\|_{C^\gamma(\Omega)}$ and $\|v\|_{bmo^\infty_\infty(\Omega)}$ with a constant C independent of φ and v. □

We consider the projection to the direction to ∇d. For $x \in \Gamma^{\rho_0}$, we set

$$P(x) = \nabla d(\pi x) \otimes \nabla d(\pi x) = \mathbf{n}(\pi x) \otimes \mathbf{n}(\pi x).$$

For later convenience, we set $Q(x) = I - P(x)$ which is the tangential projection for $x \in \Gamma^{\rho_0}$. For a function f in $\Gamma^{\rho_0} \cap \overline{\Omega}$, let f_{even} (resp. f_{odd}) denote its even (odd) extension to Γ^{ρ_0} defined by

$$\begin{aligned} f_{\mathrm{even}}\left(\pi x + d(x)\mathbf{n}(\pi x)\right) &= f\left(\pi x - d(x)\mathbf{n}(\pi x)\right) && \text{for} \quad x \in \Gamma^{\rho_0}\backslash\overline{\Omega}, \\ f_{\mathrm{odd}}\left(\pi x + d(x)\mathbf{n}(\pi x)\right) &= -f\left(\pi x - d(x)\mathbf{n}(\pi x)\right) && \text{for} \quad x \in \Gamma^{\rho_0}\backslash\overline{\Omega}. \end{aligned}$$

We denote r_W to be the restriction in W for any subset $W \subset \mathbf{R}^n$. Let f be a function (or a vector field) defined in V_σ for some $\sigma \in (0,\infty]$. We set $E_{\mathrm{even}} f$ to be the even extension of f in $V_\sigma \cap \mathbf{R}^n_+$ to V_σ with respect to the n-th variable, i.e.,

$$E_{\mathrm{even}} f(\eta', -\eta_n) = f(\eta', \eta_n)$$

for any $(\eta', \eta_n) \in V_\sigma \cap \mathbf{R}^n_+$.

For $v \in vBMOL^2(\Omega)$ with $\operatorname{supp} v \subset U_\rho(z_0) \cap \overline{\Omega}$, let $\overline{v}$ be its extension of the form

$$\overline{v}(x) := (Pv_{\mathrm{odd}})(x) + (Qv_{\mathrm{even}})(x) \tag{6.2.3}$$

for $x \in U_\rho(z_0)$. Notice that $\operatorname{supp} \overline{v} \subset U_\rho(z_0)$, $\overline{v}$ is indeed defined in $\mathbf{R}^n$ with $\overline{v}(x) = 0$ for any $x \in U_\rho(z_0)^{\mathrm{c}}$. Let $c^{**}_\Omega < c^\varepsilon_\Omega/96$ be the constant defined in Chapter 5.

Proposition 6.2.5. *Let $\Omega \subset \mathbf{R}^n$ be a uniformly C^2 domain, $z_0 \in \Gamma$ and $\rho \in (0, c^{**}_\Omega)$. There exists a constant C, independent of v and ρ, such that*

$$\begin{aligned} [\overline{v}]_{BMOL^2(\mathbf{R}^n)} &\leq \frac{C}{\rho^n}\|v\|_{vBMOL^2(\Omega)}, \\ [\nabla d \cdot \overline{v}]_{b^\nu(\Gamma)} &\leq \frac{C}{\rho^n}\|v\|_{vBMOL^2(\Omega)} \end{aligned}$$

for all $v \in vBMOL^2(\Omega)$ with supp $v \subset U_\rho(z_0) \cap \overline{\Omega}$ *and $\nu > 0$.*

This proposition is simply the $vBMOL^2$ version of [12, Proposition 2].

Proof. By considering the normal coordinate change $y = \psi(\eta)$ in $U_\rho(z_0)$, we can deduce that $v_{\mathrm{even}}, v_{\mathrm{odd}} \in L^2(\mathbf{R}^n)$ satisfying

$$\|v_{\mathrm{even}}\|_{L^2(\mathbf{R}^n)} = \|v_{\mathrm{odd}}\|_{L^2(\mathbf{R}^n)} \leq 4\|v\|_{L^2(\Omega)}.$$

Hence $\overline{v} \in L^2(\mathbf{R}^n)$ satisfies the estimate $\|\overline{v}\|_{L^2(\mathbf{R}^n)} \leq C_n\|v\|_{L^2(\Omega)}$. Hence, this proposition follows from the estimate

$$\|\overline{u}\|_{BMO(\mathbf{R}^n)} + [\nabla d \cdot \overline{u}]_{b^\infty(\Gamma)} \leq \frac{C}{\rho^n}\|u\|_{vBMOL^2(\Omega)},$$

which is guaranteed by [Theorem 5.6.3, Chapter 5]. □

6.2.3 Volume potentials

In this subsection, we always assume that Ω is a uniformly C^3 domain in $\mathbf{R}^n$. Let $\rho \in (0, \rho_*/2)$ for some sufficiently small ρ_* that is to be determined later. We consider a cut off function $\theta \in C_c^\infty(\mathbf{R})$ such that $0 \leq \theta \leq 1$, $\theta(t) = 1$ for any $0 < |t| \leq 1/2$ and $\theta(t) = 0$ for any $|t| \geq 1$. Set $\theta_\rho := \theta(d(x)/\rho)$. Note that $\theta_\rho \in C^3(\mathbf{R}^n)$. We then let $v_2 := \theta_\rho v$ and $v_1 := (1-\theta_\rho)v$. Same proof of [Theorem 5.1.1, Chapter 5] implies that $v_1 \in BMOL^2(\mathbf{R}^n)$ and $v_2 \in BMOL^2(\mathbf{R}^n) \cap b^\nu(\Gamma)$.

To construct the mapping $v \mapsto q_1$ in Theorem 6.1.2, we localize v_2 by using the partition of the unity $\{\varphi_i\}_{i=1}^\infty$ associated with the covering $\{U_{\rho,i}\}_{i=1}^\infty$ of Γ^ρ. Here for each $i \in \mathbf{N}$, $U_{\rho,i}$ denote $U_\rho(x_i)$ in Proposition 6.2.3. The corresponding volume potential to v_1 can be estimated directly.

Proposition 6.2.6. *There exists a constant C_ρ, which depends on ρ only, such that*

$$\|\nabla q_1^1\|_{BMOL^2(\mathbf{R}^n)} \leq C_\rho\|v\|_{vBMOL^2(\Omega)},$$
$$\|\nabla q_1^1\|_{L^\infty(\Gamma^{\mathbf{R}^n}_{\rho/4})} \leq C_\rho\|v\|_{vBMOL^2(\Omega)}$$

for $q_1^1 = E * \operatorname{div} v_1$ and $v \in vBMOL^2(\Omega)$. In particular,

$$\left[\nabla q_1^1\right]_{b^\nu(\Gamma)} \leq C_\rho\|v\|_{vBMOL^2(\Omega)}$$

for $\nu < \rho/4$.

Proof. By the *BMO*-*BMO* estimate [7] and Proposition 6.2.4, we have the estimate

$$\left[\nabla q_1^1\right]_{BMO(\mathbf{R}^n)} \leq C[v_1]_{BMO(\mathbf{R}^n)} \leq C_\rho\|v\|_{vBMOL^2(\Omega)}.$$

Consider $x \in \Gamma^{\mathrm{R}^n}_{\rho/4}$. Since ∇q_1^1 is harmonic in $\Gamma^{\mathrm{R}^n}_{\rho/2}$ and $B_{\frac{\rho}{4}}(x) \subset \Gamma^{\mathrm{R}^n}_{\rho/2}$, the mean value property for harmonic functions implies that

$$\nabla q_1^1(x) = \frac{C_n}{\rho^n}\int_{B_{\frac{\rho}{4}}(x)} \nabla q_1^1(y)\,dy.$$

By Hölder's inequality, we can estimate $|\nabla q_1^1(x)|$ by $\frac{C_n}{\rho^{n/2}}\|\nabla q_1^1\|_{L^2(\mathbf{R}^n)}$. Since the convolution with $\nabla^2 E$ is bounded in L^p for any $1 < p < \infty$, see e.g. [15, Theorem 5.2.7 and Theorem 5.2.10], we have that

$$\|\nabla q_1^1\|_{L^2(\mathbf{R}^n)} \leq C\|v_1\|_{L^2(\mathbf{R}^n)} \leq C\|v\|_{L^2(\mathbf{R}^n)}.$$

Therefore, the estimate

$$|\nabla q_1^1(x)| \leq C_\rho \|v\|_{vBMOL^2(\Omega)}$$

holds for any $x \in \Gamma^{\mathrm{R}^n}_{\rho/4}$. □

For $i \in \mathbf{N}$, we extend $(r_\Omega \varphi_i)v_2$ as in Proposition 6.2.5 to get $\overline{(r_\Omega \varphi_i)v_2}$ and set

$$\overline{v_2} := \sum_{i=1}^{\infty} \overline{(r_\Omega \varphi_i)v_2}.$$

Indeed, this extension is independent of the choice of $\{\varphi_i\}_{i=1}^\infty$ as long as $\{\varphi_i\}_{i=1}^\infty$ is a partition of unity for Γ^ρ. We next set

$$\overline{v_2}^{\tan} := Q\,\overline{v_2} = \sum_{i=1}^{\infty} Q\ \left(\varphi_i (v_2)_{\mathrm{even}}\right).$$

For $i \in \mathbf{N}$, we have that $\varphi_i \in C^2(U_{\rho,i})$. For simplicity of notation, we denote $\varphi_i(v_2)_{\mathrm{even}}$ by $v_{2,i}$. Proposition 6.2.3 and the construction of v_2 imply that $v_{2,i} \in BMOL^2(\mathbf{R}^n)$ with $\operatorname{supp} v_{2,i} \subset U_{\rho,i}$. In addition, we denote $Q\, v_{2,i}$ by $w_i^{\tan}$. Now, we are ready to construct the suitable potential corresponding to

$$\overline{v_2}^{\tan} = \sum_{i=1}^{\infty} Q\, v_{2,i}.$$

Proposition 6.2.7 ([12]). *There exists $\rho_* > 0$ such that if $\rho < \rho_*/2$, then for every $i \in \mathbf{N}$, there exist bounded linear operators $v \longmapsto p_{i,1}^{\tan}$ and $v \longmapsto p_{i,2}^{\tan}$ from $vBMOL^2(\Omega)$ to $L^\infty(\mathbf{R}^n)$ such that*

$$-\Delta p_i^{\tan} = \operatorname{div} w_i^{\tan} \ \ \text{in} \ \ U_{2\rho,i} \cap \Omega$$

with $p_i^{\tan} := p_{i,1}^{\tan} + p_{i,2}^{\tan}$, $\operatorname{supp} p_{i,1}^{\tan} \subset U_{2\rho,i}$. Moreover, there exists a constant C_ρ, independent of v, such that

$$\|\nabla p_{i,1}^{\tan}\|_{BMOL^2(\mathbf{R}^n)} \leq C_\rho \|v_{2,i}\|_{BMOL^2(\mathbf{R}^n)},$$
$$\|\nabla p_{i,2}^{\tan}\|_{L^\infty(\mathbf{R}^n)} \leq C_\rho \|v\|_{vBMOL^2(\Omega)},$$
$$\sup_{x \in \Gamma, r<\rho} \frac{1}{r^n} \int_{B_r(x)} |\nabla d \cdot \nabla p_i^{\tan}|\, dy \leq C_\rho \|v\|_{vBMOL^2(\Omega)}.$$

This proposition is simply a rewrite of [12, Proposition 4]. Having the estimate for the volume potential near the boundary regarding its tangential component, we are left to handle the contribution from $\overline{v}_2^{\mathrm{nor}} := \overline{v}_2 - \overline{v}_2^{\tan}$. We recall its decomposition

$$\overline{v}_2^{\mathrm{nor}} = \sum_{i=1}^{\infty} P\ \left(\varphi_i (v_2)_{\mathrm{odd}}\right).$$

For simplicity of notations, we denote $\nabla d \cdot (\varphi_i (v_2)_{\mathrm{odd}})$ by g_i and $P\ (\varphi_i(v_2)_{\mathrm{odd}})$ by w_i^{nor} for every $i \in \mathbf{N}$. By this notation $w_i^{\mathrm{nor}} = g_i \nabla d$. With a similar idea of proof, we can establish the suitable potential corresponding to $\overline{v}_2^{\mathrm{nor}}$.

Proposition 6.2.8 ([12]). *There exists $\rho_* > 0$ such that if $\rho < \rho_*/2$, then for every $i \in \mathbf{N}$, there exist bounded linear operators $v \longmapsto p_{i,1}^{\mathrm{nor}}$ and $v \longmapsto p_{i,2}^{\mathrm{nor}}$ from $vBMOL^2(\Omega)$ to $L^\infty(\mathbf{R}^n)$ such that*

$$-\Delta p_i^{\mathrm{nor}} = \operatorname{div} w_i^{\mathrm{nor}} \quad \text{in } U_{2\rho,i} \cap \Omega$$

with $p_i^{\mathrm{nor}} := p_{i,1}^{\mathrm{nor}} + p_{i,2}^{\mathrm{nor}}$, $\operatorname{supp} p_{i,1}^{\mathrm{nor}} \subset U_{2\rho,i}$. Moreover, there exists a constant C_ρ, independent of v, such that

$$\begin{aligned} \|\nabla p_{i,1}^{\mathrm{nor}}\|_{BMOL^2(\mathbf{R}^n)} &\le C_\rho \|g_i\|_{BMOL^2(\mathbf{R}^n)}, \\ \|\nabla p_{i,2}^{\mathrm{nor}}\|_{L^\infty(\mathbf{R}^n)} &\le C_\rho \|v\|_{vBMOL^2(\Omega)}, \\ \sup_{x\in\Gamma, r<\rho} \frac{1}{r^n} \int_{B_r(x)} |\nabla d \cdot \nabla p_i^{\mathrm{nor}}|\, dy &\le C_\rho \|v\|_{vBMOL^2(\Omega)}. \end{aligned}$$

Similarly, this proposition is just a rewrite of [12, Proposition 5]. By these two propositions, we are now ready to prove Theorem 6.1.2.

Proof of Theorem 6.1.2 admitting Proposition 6.2.7 and 6.2.8. Let $i \in \mathbf{N}$. We first consider the contribution from the tangential part. We take a cut-off function $\theta_i \in C_c^\infty(U_{2\rho,i})$ such that $\theta_i = 1$ on $U_{\rho,i}$ and $0 \le \theta_i \le 1$. We next set

$$q_{1,i}^{\tan} := \theta_i p_i^{\tan} + E * \left(p_i^{\tan} \Delta\theta_i + 2\nabla\theta_i \cdot \nabla p_i^{\tan}\right).$$

By definition, Proposition 6.2.7 says that

$$\begin{aligned} -\Delta q_{1,i}^{\tan} &= -\Delta(\theta_i p_i^{\tan}) + p_i^{\tan}\Delta\theta_i + 2\nabla\theta_i \cdot \nabla p_i^{\tan} \\ &= \theta_i \operatorname{div} w_i^{\tan} = \operatorname{div} w_i^{\tan} \end{aligned}$$

in Ω as $\operatorname{supp} w_i^{\tan} \subset U_{\rho,i}$. We then set

$$q_1^{\tan} := \sum_{i=1}^\infty q_{1,i}^{\tan}.$$

Since $\operatorname{supp} p_{i,1}^{\tan} \subset U_{2\rho,i}$, by Proposition 6.2.7 we see that

$$\sum_{i=1}^\infty \|\nabla(\theta_i p_{i,1}^{\tan})\|_{L^2(\mathbf{R}^n)} \le C_\rho \sum_{i=1}^\infty \|v_{2,i}\|_{L^2(\mathbf{R}^n)}.$$

Since our partition of unity for Γ^ρ is locally finite according to Proposition 6.2.2 and 6.2.3, we can deduce that

$$\sum_{i=1}^\infty \|v_{2,i}\|_{L^2(\mathbf{R}^n)} \le 8N_* \|v_2\|_{L^2(\Omega)} \le 8N_* \|v\|_{L^2(\Omega)}$$

with the constant N_* defined in Proposition 6.2.2. Suppose that B is a ball of radius $r(B) < \rho$. If B does not intersect $\Gamma^{2\rho}$, then

$$\frac{1}{|B|} \int_B |\nabla(\theta_i p_{i,1}^{\tan}) - \left(\nabla(\theta_i p_{i,1}^{\tan})\right)_B|\, dy = 0$$

for each $i \in \mathbf{N}$. If B intersects $\Gamma^{2\rho}$, then by the proof of [Lemma 5.4.2, Chapter 5], we see that B intersects at most N_* neighborhoods of $\{U_{2\rho}(x_i) \mid x_i \in S\}$. Hence in this case, we have that

$$\begin{aligned}\frac{1}{|B|}\int_B \left| \sum_{i=1}^\infty \nabla(\theta_i p_{i,1}^{\tan}) - \left(\sum_{i=1}^\infty \nabla(\theta_i p_{i,1}^{\tan})\right)_B \right| dy &\le \sum_{l=1}^{N_*} [\nabla(\theta_{i_l} p_{i_l,1}^{\tan})]_{BMO(\mathbf{R}^n)} \\ &\le C_\rho N_* \|v\|_{vBMOL^2(\Omega)}.\end{aligned}$$

Thus, we deduce that

$$\left\| \sum_{i=1}^\infty \nabla(\theta_i p_{i,1}^{\tan}) \right\|_{BMOL^2(\mathbf{R}^n)} \le C_\rho N_* \|v\|_{vBMOL^2(\Omega)}.$$

Since $\operatorname{supp} \theta_i p_{i,2}^{\tan} \subset U_{2\rho,i}$, by Proposition 6.2.7 and 6.2.2 we have that

$$\left[\sum_{i=1}^\infty \nabla(\theta_i p_{i,2}^{\tan}) \right]_{BMO(\mathbf{R}^n)} \le 2 \left\| \sum_{i=1}^\infty \nabla(\theta_i p_{i,2}^{\tan}) \right\|_{L^\infty(\mathbf{R}^n)} \le C_\rho N_* \|v\|_{vBMOL^2(\Omega)}.$$

In addition, as

$$\|\nabla(\theta_i p_{i,2}^{\tan})\|_{L^2(\mathbf{R}^n)} \le |U_{2\rho,i}|^{1/2} \|\nabla(\theta_i p_{i,2}^{\tan})\|_{L^\infty(\mathbf{R}^n)} \le C_\rho \|v_{2,i}\|_{L^2(\mathbf{R}^n)},$$

similar argument as above implies that

$$\left\| \sum_{i=1}^\infty \nabla(\theta_i p_{i,2}^{\tan}) \right\|_{L^2(\mathbf{R}^n)} \le \sum_{i=1}^\infty \|\nabla(\theta_i p_{i,2}^{\tan})\|_{L^2(\mathbf{R}^n)} \le C_\rho \sum_{i=1}^\infty \|v_{2,i}\|_{L^2(\mathbf{R}^n)} \le C_\rho N_* \|v\|_{L^2(\Omega)}.$$

Hence, we obtain that

$$\left\| \sum_{i=1}^\infty \nabla(\theta_i p_i^{\tan}) \right\|_{BMOL^2(\mathbf{R}^n)} \le C_\rho N_* \|v\|_{vBMOL^2(\Omega)}. \tag{6.2.4}$$

Let $f_i^{\tan} = p_i^{\tan} \Delta\theta_i + 2\nabla\theta_i \cdot \nabla p_i^{\tan}$. Since $\operatorname{supp} f_i^{\tan} \subset U_{2\rho,i}$, we actually have that

$$\|f_i^{\tan}\|_{L^1(U_{2\rho,i})} \le |U_{2\rho,i}|^{1/2} \cdot \|f_i^{\tan}\|_{L^2(U_{2\rho,i})}.$$

By the same argument in the above paragraph which proves the estimate (6.2.4), we can show that

$$\left[\sum_{i=1}^\infty f_i^{\tan} \right]_{BMO(\mathbf{R}^n)} + \sum_{i=1}^\infty \|f_i^{\tan}\|_{L^1(\mathbf{R}^n)} \le C_\rho N_* \|v\|_{vBMOL^2(\Omega)}.$$

By an interpolation (cf. [5, Lemma 5], [19, Theorem 2.2], [18, Theorem 1 and Remark 1]), we see that the estimate

$$\begin{aligned}\left\| \sum_{i=1}^\infty f_i^{\tan} \right\|_{L^s(\mathbf{R}^n)} &\le C_n \left\| \sum_{i=1}^\infty f_i^{\tan} \right\|_{L^1(\mathbf{R}^n)}^{\frac{1}{s}} \left[\sum_{i=1}^\infty f_i^{\tan} \right]_{BMO(\mathbf{R}^n)}^{1-\frac{1}{s}} \\ &\le C_n \left\| \sum_{i=1}^\infty f_i^{\tan} \right\|_{BMOL^1(\mathbf{R}^n)}\end{aligned} \tag{6.2.5}$$

holds for any $1 < s < \infty$. Since ∇E is in $L^{p'}(B_{6\rho}(0))$ for $1 < p' < n/(n-1)$, it follows that

$$\sup_{\mathbf{R}^n} \left| \nabla E * \left(\sum_{i=1}^{\infty} f_i^{\tan} \right) \right| \leq C_\rho \left\| \sum_{i=1}^{\infty} f_i^{\tan} \right\|_{L^p(\mathbf{R}^n)}.$$

Thus, we deduce that

$$\sup_{\mathbf{R}^n} \left| \nabla E * \left(\sum_{i=1}^{\infty} f_i^{\tan} \right) \right| \leq C_\rho N_* \|v\|_{vBMOL^2(\Omega)}.$$

By the well-known Hardy-Littlewood-Sobolev inequality, see e.g. [1, Theorem 1.7], the estimate

$$\left\| \nabla E * \left(\sum_{i=1}^{\infty} f_i^{\tan} \right) \right\|_{L^2(\mathbf{R}^n)} \leq C \left\| \sum_{i=1}^{\infty} f_i^{\tan} \right\|_{L^r(\mathbf{R}^n)}$$

holds with $r = \frac{2n}{n+2}$. Hence by (6.2.5), we get that

$$\left\| \nabla E * \left(\sum_{i=1}^{\infty} f_i^{\tan} \right) \right\|_{BMOL^2(\mathbf{R}^n)} \leq C_\rho N_* \|v\|_{vBMOL^2(\Omega)}.$$

Combine with (6.2.4), we finally obtain that

$$\|\nabla q_1^{\tan}\|_{BMOL^2(\Omega)} \leq C_\rho N_* \|v\|_{vBMOL^2(\Omega)}.$$

By Proposition 6.2.7, the control on the boundary with respect to $q_1^{\tan}$ is estimated by

$$\sup_{x \in \Gamma, r < \rho} \frac{1}{r^n} \int_{B_r(x)} |\nabla d \cdot \nabla q_1^{\tan}| \, dy \leq C_\rho N_* \|v\|_{vBMOL^2(\Omega)}$$

as the partition $\{U_\rho(x_i) \mid x_i \in S\}$ is a locally finite open cover of Γ^ρ according to Proposition 6.2.2.

Set

$$q_{1,i}^{\mathrm{nor}} := \theta_i p_i^{\mathrm{nor}} + E * (p_i^{\mathrm{nor}} \Delta\theta_i + 2\nabla\theta_i \cdot \nabla p_i^{\mathrm{nor}})$$

and

$$q_1^{\mathrm{nor}} := \sum_{i=1}^{\infty} q_{1,i}^{\mathrm{nor}}.$$

By making use of Proposition 6.2.8 and repeating the whole argument above that treats the case for $q_1^{\tan}$, we can prove that

$$\|\nabla q_1^{\mathrm{nor}}\|_{vBMOL^2(\Omega)} \leq C_\rho N_* \|v\|_{vBMOL^2(\Omega)}$$

in the same way. Then we set

$$q_1 := q_1^1 + q_1^{\tan} + q_1^{\mathrm{nor}}.$$

By our construction we have that

$$\begin{aligned} -\Delta q_1 &= -\Delta q_1^1 - \Delta q_1^{\tan} - \Delta q_1^{\mathrm{nor}} \\ &= \operatorname{div} v_1 + \sum_{i=1}^{\infty} \operatorname{div} w_i^{\tan} + \sum_{i=1}^{\infty} \operatorname{div} w_i^{\mathrm{nor}} \\ &= \operatorname{div}(v_1 + v_2) = \operatorname{div} v \end{aligned}$$

in Ω. We are done. □

6.3 Neumann problem with bounded data in a perturbed C^2 half space with small perturbation

We consider the Neumann problem for the Laplace equation in a perturbed C^2 half space in $\mathbf{R}^n$ with L^∞-initial data. We shall begin with the half space. Let E be the fundamental solution of $-\Delta$ in $\mathbf{R}^n$. A solution of the Neumann problem

$$\begin{aligned} \Delta u = 0 \quad &\text{in} \quad \mathbf{R}^n_+ \\ \frac{\partial u}{\partial \mathbf{n}} = g \quad &\text{on} \quad \partial \mathbf{R}^n_+ \end{aligned} \tag{6.3.1}$$

is formally given by

$$u(x) = \int_{\partial \mathbf{R}^n_+} N(x,y) g(y) \, d\mathcal{H}^{n-1}, \tag{6.3.2}$$

where N denotes the Neumann-Green function. In the case of a half space, it is well-known that

$$N(x,y) = E(x-y) + E(x'-y', x_n + y_n).$$

Its exterior normal derivative $\partial N / \partial \mathbf{n}_x$ for $y_n = 0$ is nothing but the Poisson kernel with the parameter x_n. By symmetry we observe that

$$-\frac{\partial}{\partial x_n} \int_{\mathbf{R}^{n-1}} E(x'-y', x_n) g(y') \, dy' \to \frac{1}{2} g(x')$$

as $x_n > 0$ tends to zero. Thus u gives a solution of (6.3.1) formally. The function

$$E * (\delta_{\partial \mathbf{R}^n_+} \otimes g) := \int_{\partial \mathbf{R}^n_+} E(x'-y', x_n) g(y') \, dy'$$

is called the single layer potential of g. For $g \in L^\infty(\partial \mathbf{R}^n_+)$, we let $\widetilde{g}(x', x_n) := g(x', 0)$ for any $x \in \mathbf{R}^n$. Natrually, $\widetilde{g} \in L^\infty(\mathbf{R}^n)$. Let $1_{\mathbf{R}^n_+}$ be the characteristic function associated with the half space $\mathbf{R}^n_+$. In this case, we have that

$$\nabla E * (\delta_{\partial \mathbf{R}^n_+} \otimes g) = \nabla \partial_{x_n} E * 1_{\mathbf{R}^n_+} \widetilde{g}$$

Hence by the L^∞-BMO estimate for the singular integral operator [16, Theorem 4.2.7], we recall the following.

Proposition 6.3.1 ([12])**.** *There exists a constant C, independent of g, such that*

$$[\nabla (E * (\delta_{\partial \mathbf{R}^n_+} \otimes g))]_{BMO(\mathbf{R}^n)} \le C \|g\|_{L^\infty(\partial \mathbf{R}^n_+)}.$$

Since $-\partial_{x_n}(E * (\delta_\Gamma \otimes g))$ is the half of the Poisson integral, i.e.,

$$-\partial_{x_n}(E * (\delta_{\partial \mathbf{R}^n_+} \otimes g)) = \frac{1}{2} \int_{\mathbf{R}^{n-1}} P_{x_n}(x'-y') g(y') \, dy',$$

we also have the following.

Proposition 6.3.2 ([12])**.** *The estimate*

$$\left\| \partial_{x_n} \big(E * (\delta_{\partial \mathbf{R}^n_+} \otimes g) \big) \right\|_{L^\infty(\mathbf{R}^n_+)} \le \frac{1}{2} \|g\|_{L^\infty(\partial \mathbf{R}^n_+)}$$

holds for $g \in L^\infty(\partial \mathbf{R}^n_+)$.

We then seek to establish these estimates for the case where Ω is a perturbed C^2 half space with small perturbation. Here and hereafter, we set

$$\Omega = \mathbf{R}^n_h = \{(x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}$$

with $h \in C^2_c(\mathbf{R}^{n-1})$ satisfying smallness condition (6.1.2) and $\Gamma = \partial\mathbf{R}^n_h$. Let $1'_{B_{2R_h}(0')}$ be the characteristic function associated with $B_{2R_h}(0')$ in $\mathbf{R}^{n-1}$. We define $g_1, g_2 \in L^\infty(\Gamma)$ by setting $g_1(x', h(x')) := 1'_{B_{2R_h}(0')}(x')g(x', h(x'))$ and $g_2(x', h(x')) := g(x', h(x')) - g_1(x', h(x'))$ for any $x' \in \mathbf{R}^{n-1}$.

Lemma 6.3.3. *Let $\Omega = \mathbf{R}^n_h$ be a perturbed C^2 half space in $\mathbf{R}^n$ with boundary $\Gamma = \partial\mathbf{R}^n_h$.*

(i) (BMO estimate) There exists a constant C_1 such that

$$[\nabla (E * (\delta_\Gamma \otimes g))]_{BMO(\mathbf{R}^n)} \leq C_1 \|g\|_{L^\infty(\Gamma)} \tag{6.3.3}$$

for all $g \in L^\infty(\Gamma)$.

(ii) (L^∞ estimate for normal component) There exists a constant C_2 such that

$$\|\nabla d \cdot \nabla (E * (\delta_\Gamma \otimes g))\|_{L^\infty(\Gamma^{\rho_0} \cap \Omega)} \leq C_2 \|g\|_{L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)} \tag{6.3.4}$$

for all $g \in L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)$.

Here $E * (\delta_\Gamma \otimes g)$ is defined as $E * (\delta_\Gamma \otimes g)(x) := \int_\Gamma E(x-y)g(y)\, d\mathcal{H}^{n-1}(y)$ for a function g on Γ. We shall prove Lemma 6.3.3 in following subsections.

6.3.1 BMO estimate

Lemma 6.3.3 (i). We define $\overline{g_2} \in L^\infty(\partial\mathbf{R}^n_+)$ by setting

$$\overline{g_2}(x', 0) = \begin{cases} g_2(x', 0) & \text{for} \quad |x'| \geq 2R_h, \\ 0 & \text{for} \quad |x'| < 2R_h. \end{cases}$$

By Proposition 6.3.1, the estimate

$$[\nabla(E * (\delta_{\partial\mathbf{R}^n_+} \otimes \overline{g_2}))]_{BMO(\mathbf{R}^n)} \leq C\|\overline{g_2}\|_{L^\infty(\partial\mathbf{R}^n_+)} \leq C\|g\|_{L^\infty(\Gamma)}$$

holds with C independent of g.

Note that the signed distance function d is C^2 in Γ^{ρ_0}, see [14, Section 14.6]. Let $\delta < \rho_0/2$. We take a C^2 cut-off function $\theta \geq 0$ such that $\theta(\sigma) = 1$ for $\sigma \leq 1$ and $\theta(\sigma) = 0$ for $\sigma \geq 2$. By the choice of δ, we see that $\theta_d = \theta(d/\delta)$ is C^2 in $\mathbf{R}^n$. We extend $g_1 \in L^\infty(\Gamma)$ to $g_1^e \in L^\infty(\Gamma^{2\delta})$ by setting

$$g_1^e(x) := g_1(\pi x)$$

for any $x \in \Gamma^{2\delta}$ with πx denoting the projection of x on Γ. For $x \in \Gamma^{2\delta}$, by considering the normal coordinate $x = \psi(\eta)$ in $U_{2\delta}(\pi x)$, we have that

$$(\nabla_x d)_\psi \cdot (\nabla_x g_1^e)_\psi = \partial_{\eta_n}(g_1^e)_\psi = 0$$

as $(g_1^e)_\psi(\eta', \alpha) = (g_1^e)_\psi(\eta', \beta)$ for any $|\eta'| < 2\delta$ and $\alpha, \beta \in (-2\delta, 2\delta)$. Hence, we see that $\nabla d \cdot \nabla g_1^e = 0$ in $\Gamma^{2\delta}$.

Let us consider $g^e_{1,c} := \theta_d g^e_1$. A key observation is that

$$\begin{aligned} \delta_\Gamma \otimes g_1 &= (\nabla 1_\Omega \cdot \nabla d) g^e_{1,c} \\ &= \operatorname{div}(g^e_{1,c} 1_\Omega \nabla d) - 1_\Omega \operatorname{div}(g^e_{1,c} \nabla d), \\ \operatorname{div}(g^e_{1,c} \nabla d) &= g^e_{1,c} \Delta d + \nabla d \cdot \nabla g^e_{1,c} = g^e_{1,c} \Delta d + \frac{\theta'(d/\delta)}{\delta} g^e_1. \end{aligned}$$

Thus

$$\nabla E * (\delta_\Gamma \otimes g_1) = \nabla \operatorname{div} \left(E * (g^e_{1,c} 1_\Omega \nabla d) \right) - \nabla E * (1_\Omega g^e_1 f_{\theta,\delta}) = I_1 + I_2$$

where $f_{\theta,\delta} := \theta_d \Delta d + \frac{\theta'(d/\delta)}{\delta}$. By the L^∞-BMO estimate for the singular integral operator [16, Theorem 4.2.7], the first term is estimated as

$$[I_1]_{BMO(\mathbf{R}^n)} \leq C \| g^e_{1,c} \nabla d \|_{L^\infty(\Omega)} \leq C \| g \|_{L^\infty(\Gamma)}.$$

Let $U_\mathrm{c} := \{ x \in \Gamma^{2\delta} \mid |(\pi x)'| < 2R_h \}$. Since

$$A = \sup_{x \in \mathbf{R}^n \setminus \{0\}} |x|^{n-1} |\nabla E(x)| < \infty,$$

for $x \in \mathbf{R}^n$ with $d(x, \Omega) = \inf_{y \in \Omega} |x - y| < 1$ we have that

$$|I_2(x)| \leq A \int_{U_\mathrm{c}} \frac{1}{|x-y|^{n-1}} \, dy \| f_{\theta,\delta} \|_{L^\infty(U_\mathrm{c})} \| g^e_1 \|_{L^\infty(U_\mathrm{c})} \leq C_{R_h,\delta} \| g \|_{L^\infty(\Gamma)}$$

with $C_{R_h,\delta}$ depending only on R_h and δ. For $x \in \mathbf{R}^n$ with $d(x, U_\mathrm{c}) = \inf_{y \in U_\mathrm{c}} |x - y| \geq 1$, the above estimate holds trivially as $|x-y|^{-(n-1)} \leq 1$ for any $y \in U_\mathrm{c}$. The proof of the first part of Lemma 6.3.3 is now complete. □

6.3.2 Estimate for normal derivative

We shall estimate normal derivative of E.

Lemma 6.3.4. *Let Ω be a perturbed C^2 half space in $\mathbf{R}^n$ with $\Gamma = \partial\Omega$, $\nu < \rho_0$. Then*

(i)

$$\int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y) \, d\mathcal{H}^{n-1}(y) = -\frac{1}{2} \quad \text{for} \quad x \in \Gamma_\nu,$$

(ii)

$$\sup_{x \in \Gamma_\nu} \int_\Gamma \left| \frac{\partial E}{\partial \mathbf{n}_y}(x-y) \right| \, d\mathcal{H}^{n-1}(y) < \infty.$$

Proof. (i) This follows from the Gauss divergence theorem. For a bounded smooth domain D, we have that

$$\int_{\partial D} \frac{\partial E}{\partial \mathbf{n}_y}(x-y) \, d\mathcal{H}^{n-1}(y) = \int_D \Delta_y E(x-y) \, dy.$$

Since $\Delta_y E(x-y) = -\delta(x-y)$, we obtain that

$$\int_{\partial D} \frac{\partial E}{\partial \mathbf{n}_y}(x-y) \, d\mathcal{H}^{n-1}(y) = -1$$

for $x \in D$. We take the domain D_R as

$$D_R := \mathbf{R}^n_h \cap \{(y, y_n) \,|\, y_n < \|h\|_{L^\infty(\mathbf{R}^{n-1})} + \epsilon\} \cap \{|y'| < R\}$$

with $\epsilon > 0$. Suppose that $R > R_h$. By applying the Gauss divergence theorem in D_R, we deduce that

$$-1 = \int_{\substack{y_n = \|h\|_\infty + \epsilon, \\ |y'| < R}} \frac{\partial E}{\partial \mathbf{n}_y}(x - y)\, d\mathcal{H}^{n-1}(y) + \int_{\substack{y \in \Gamma, \\ |y'| < R}} \frac{\partial E}{\partial \mathbf{n}_y}(x - y)\, d\mathcal{H}^{n-1}(y) + \int_{\substack{0 < y_n < \|h\|_\infty + \epsilon, \\ |y'| = R}} \frac{\partial E}{\partial \mathbf{n}_y}(x - y)\, d\mathcal{H}^{n-1}(y).$$

The last term tends to zero naturally as $R \to \infty$. In the first term, since $\mathbf{n}_y$ is pointing upward but x is located below $\{(y', y_n) \,|\, y_n = \|h\|_\infty + \epsilon\}$, the kernel is exactly the half of the Poisson kernel. Hence, the first term tends to $-\frac{1}{2}$ as $R \to \infty$. We obtain (i).

(ii) Let us observe that

$$-\mathbf{n}\left(y', h(y')\right) = \left(-\nabla' h(y'), 1\right)/\omega(y')$$

with $\omega(y') = \left(1 + |\nabla' h(y')|^2\right)^{1/2}$, where ∇' is the gradient in y' variables. This implies that

$$-n\alpha(n) \frac{\partial E}{\partial \mathbf{n}_y}(x - y) = \frac{\sigma(y')}{\omega(y')\left(|x' - y'|^2 + (x_n - h(y'))^2\right)^{n/2}}$$

for $y \in \Gamma$ with

$$\sigma(y') := -\nabla' h(y') \cdot (x' - y') + \left(x_n - h(y')\right) \quad \text{where } \; x_n > h(x'),\; x', y' \in \mathbf{R}^{n-1}.$$

We set

$$K(x', y', x_n) = \frac{\sigma(y')}{\left(|x' - y'|^2 + (x_n - h(y'))^2\right)^{n/2}}.$$

By the Taylor expansion, for $|x' - y'| < 1$ we have that

$$h(x') = h(y') + \nabla' h(y') \cdot (x' - y') + r(x', y')$$

with

$$r(x', y') = (x' - y')^{\mathrm{T}} \cdot \int_0^1 (1 - \theta) \nabla'^2 h\left(\theta x' + (1 - \theta) y'\right) d\theta \cdot (x' - y').$$

We obtain that

$$\sigma(y') = x_n - h(x') + r(x', y')$$

with an estimate

$$\left|r(x', y')\right| \le \|\nabla'^2 h\|_{L^\infty(B_1(x'))} |x' - y'|^2. \tag{6.3.5}$$

We decompose K into a leading term and a remainder term

$$K(x', y', x_n) = K_0(x', y', x_n) + R(x', y', x_n)$$

with

$$K_0(x', y', x_n) := \frac{x_n - h(x')}{\Big(|x' - y'|^2 + (x_n - h(y'))^2\Big)^{n/2}}$$

$$R(x', y', x_n) := \frac{r(x, y)}{\Big(|x' - y'|^2 + (x_n - h(y'))^2\Big)^{n/2}}.$$

The term R is estimated as

$$\big|R(x', y', x_n)\big| \leq \|\nabla'^2 h\|_{L^\infty(B_1(x'))} |x' - y'|^{2-n}$$

for $|x' - y'| < 1$ by (6.3.5). Hence,

$$\int_{\substack{y\in\Gamma,\\ |x'-y'|<1}} \left|\frac{R(x', y', x_n)}{\omega(y')}\right| d\mathcal{H}^{n-1}(y) \leq C_n \|\nabla'^2 h\|_{L^\infty(\mathbf{R}^{n-1})}.$$

Since

$$|\sigma(y')| \leq |\nabla' h(y')| \cdot |x' - y'| + |x_n| + |h(y')|$$

for any $y' \in \mathbf{R}^{n-1}$, we have that

$$\int_{\substack{y\in\Gamma,\\ |y'-x'|\geq 1}} \left|\frac{K(x', y', x_n)}{\omega(y')}\right| d\mathcal{H}^{n-1}(y) \leq \int_{|y'-x'|\geq 1} |\nabla' h(y')|\, dy' + \int_{|y'-x'|\geq 1} |h(y')|\, dy' + \int_{|y'-x'|\geq 1} \frac{|x_n|}{|x' - y'|^n}\, dy'. \tag{6.3.6}$$

Since the support of h is contained in $\overline{B_{R_h}(0')}$, the first two terms of (6.3.6) can be estimated by $C_{R_h,n}\|h\|_{C^1(\mathbf{R}^{n-1})}$. Note that there exists a constant C, independent of $x \in \Gamma_\nu$, such that the estimate

$$|x_n - h(x')| \leq C\nu$$

holds for any $x \in \Gamma_\nu$. The third term of (6.3.6) is estimated by $C(\nu + \|h\|_{L^\infty(\mathbf{R}^{n-1})})$. By (i), we observe that

$$\frac{n\alpha(n)}{2} = \int_{\substack{y\in\Gamma,\\ |y'-x'|\geq 1}} \frac{K(x', y', x_n)}{\omega(y')}\, d\mathcal{H}^{n-1}(y) + \int_{\substack{y\in\Gamma,\\ |y'-x'|<1}} \frac{K_0(x', y', x_n)}{\omega(y')}\, d\mathcal{H}^{n-1}(y) + \int_{\substack{y\in\Gamma,\\ |y'-x'|<1}} \frac{R(x', y', x_n)}{\omega(y')}\, d\mathcal{H}^{n-1}(y).$$

The term K_0 is very singular but it is positive for $x \in \Gamma_\nu$. Hence, we have that

$$\int_{\substack{y\in\Gamma,\\ |y'-x'|<1}} \frac{K_0(x', y', x_n)}{\omega(y')}\, d\mathcal{H}^{n-1}(y) \leq \frac{n\alpha(n)}{2} + C_{R_h,n} \cdot \big(\|h\|_{C^2(\mathbf{R}^{n-1})} + \nu\big).$$

Therefore, we finally obtain the estimate

$$\int_\Gamma \left|\frac{\partial E}{\partial \mathbf{n}_y}(x - y)\right| d\mathcal{H}^{n-1}(y) \leq \frac{n\alpha(n)}{2} + C_{R_h,n} \cdot \big(\|h\|_{C^2(\mathbf{R}^{n-1})} + \nu\big),$$

which holds for any $x \in \Gamma_\nu$. The proof of (ii) is now complete.

□

6.3.3 Review of boundary integral equation

For $g \in L^\infty(\Gamma)$, we define the double layer potential

$$(Pg)(x) = \int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y)g(y)\,d\mathcal{H}^{n-1}(y), \quad x \in \Gamma_\nu,$$

where $\partial/\partial \mathbf{n}_y$ denotes the exterior normal derivative with respect to y-variable.

Theorem 6.3.5. *Assume that $\nu < \rho_0$.*

(i) There exists a constant C_h, depending only on h, such that

$$\|Pg\|_{L^\infty(\Gamma_\nu)} \le C_h \|g\|_{L^\infty(\Gamma)}.$$

(ii) The boundary trace is of the form

$$\big(\gamma(Pg)\big)(x', h(x')) = \frac{1}{2} g(x', h(x')) - \big(Sg\big)(x', h(x'))$$

for $g \in L^\infty(\Gamma)$, where S is a bounded linear operator on $L^\infty(\Gamma)$ satisfying

$$\|S\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)} \le C_*(R_h^{n-1}+1)\|h\|_{C^2(\mathbf{R}^{n-1})}$$

with some constant C_* independent of h and g.

Proof. (i) This follows from the second part of Lemma 6.3.4 directly.

(ii) Suppose that $x \in \Gamma_\nu$ with $|x'| \ge 2R_h$. By decomposing g into a straight part and a curved part, we see that

$$\begin{aligned}\big(Pg\big)(x) &= \int_{\{y\in\Gamma \,|\, |y'|\ge 2R_h\}} \frac{\partial E}{\partial \mathbf{n}_y}(x-y)g_2(y)\,d\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\{y\in\Gamma \,|\, |y'|< 2R_h\}} \frac{\partial E}{\partial \mathbf{n}_y}(x-y)g_1(y)\,d\mathcal{H}^{n-1}(y) = I_1(x) + I_2(x).\end{aligned}$$

Note that

$$I_1(x) = -\int_{|y'|\ge 2R_h} P_{x_n}(x'-y')g_2(y')\,dy' = -\int_{\mathbf{R}^{n-1}} P_{x_n}(x'-y')\overline{g_2}(y')\,dy'.$$

Let x tends x_0 on the boundary, in this case we have that I_1 tends to $\frac{1}{2}\overline{g_2}(x_0)$, which is indeed $\frac{1}{2}g(x_0)$. Recall the proof of the second part of Lemma 6.3.4, if $|x_0'| \ge 2R_h$ then we have that

$$|I_2(x_0)| \le C_n R_h^{n-1} \|h\|_{C^2(\mathbf{R}^{n-1})} \|g\|_{L^\infty(B_{2R_h}(0'))}.$$

For $x_0 \in \Gamma$ with $|x_0'| \ge 2R_h$, by setting

$$\big(T_s g\big)(x_0) = \int_{\{y\in\Gamma \,|\, |y'|< 2R_h\}} \frac{\partial E}{\partial \mathbf{n}_y}(x_0-y)g(y)\,d\mathcal{H}^{n-1}(y),$$

we get that

$$\big(\gamma(Pg)\big)(x_0) = \frac{1}{2}g(x_0) + \big(T_s g\big)(x_0)$$

with

$$\|T_s\|_{\mathrm{op}} \leq C_n R_h^{n-1} \|h\|_{C^2(\mathbf{R}^{n-1})}.$$

Suppose that $x \in \Gamma_\nu$ with $|x'| < 2R_h$. There exists a bounded C^2 domain $D_c \subset \Omega$ such that $\partial D_c \cap \Gamma = \{y \in \Gamma \mid |y'| < 2R_h\}$. Let us recall a standard result concerning the double layer potential, see e.g. [17, Lemma 6.17]. Let $f \in L^\infty(\partial D_c)$, then the boundary trace of the double layer potential

$$(Qf)(z) = \int_{\partial D_c} \frac{\partial E}{\partial \mathbf{n}_y}(z-y) f(y)\, d\mathcal{H}^{n-1}(y), \quad z \in D_c$$

is of the form

$$\big(\gamma(Qf)\big)(w) = \frac{1}{2} f(w) + \int_{\partial D_c} \frac{\partial E}{\partial \mathbf{n}_y}(w-y) f(y)\, d\mathcal{H}^{n-1}(y)$$

for $w \in \partial D_c$. We define $g_c \in L^\infty(\partial D_c)$ by letting

$$g_c(w) = \begin{cases} g_1(w) & \text{for} \quad w \in \partial D_c \cap \Gamma, \\ 0 & \text{for} \quad w \in \partial D_c \setminus \Gamma. \end{cases}$$

Without loss of generality, we may assume that $\{x \in \Gamma_\nu \mid |x'| < 2R_h\} \subset D_c$. Thus, for $x \in \Gamma_\nu$ with $|x'| < 2R_h$, we have that

$$(Qg_c)(x) = (Pg_1)(x).$$

Let x tends to x_0 on the boundary, we see that

$$\big(\gamma(Pg_1)\big)(x_0) = \big(\gamma(Qg_c)\big)(x_0) = \frac{1}{2} g(x_0) + \big(T_c g\big)(x_0).$$

where $\big(T_c g\big)(x_0)$ is defined as

$$\big(T_c g\big)(x_0) = \int_{\{y \in \Gamma \mid |y'| < 2R_h\}} \frac{\partial E}{\partial \mathbf{n}_y}(x_0 - y) g(y)\, d\mathcal{H}^{n-1}(y).$$

Again, the proof of the second part of Lemma 6.3.4 tells us that

$$|T_c g(x_0)| \leq C_n R_h^{n-1} \|h\|_{C^2(\mathbf{R}^{n-1})} \|g\|_{L^\infty(B_{2R_h}(0'))}.$$

Note that in this case,

$$\big(\gamma(Pg_2)\big)(x_0) = -\int_{|y'| \geq 2R_h} P_{h(x_0')}(x_0' - y') g_2(y')\, dy'.$$

By the argument in proof of Lemma 6.3.4 (ii), we can deduce that

$$\begin{aligned} \big|\big(\gamma(Pg_2)\big)(x_0)\big| &\leq \int_{|y'-x_0'|<1} \frac{\|\nabla'^2 h\|_{L^\infty(\mathbf{R}^{n-1})}}{|x_0' - y'|^{n-2}}\, dy' + \int_{|y'-x_0'|\geq 1} \frac{\|h\|_{L^\infty(\mathbf{R}^{n-1})}}{|x_0' - y'|^{n}}\, dy' \\ &\leq C_n \|h\|_{C^2(\mathbf{R}^{n-1})} \|g\|_{L^\infty(\Gamma)}. \end{aligned}$$

Therefore, by setting

$$\big(Sg\big)(x_0) = -\int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x_0 - y) g(y)\, d\mathcal{H}^{n-1}(y)$$

for $x_0 \in \Gamma$ with $|x_0'| < 2R_h$ and

$$\big(Sg\big)(x_0) = -\int_{\{y \in \Gamma \mid |y'| < 2R_h\}} \frac{\partial E}{\partial \mathbf{n}_y}(x_0 - y) g(y)\, d\mathcal{H}^{n-1}(y)$$

for $x_0 \in \Gamma$ with $|x_0'| \geq 2R_h$, we obtain the second part of Theorem 6.3.5.

□

6.3.4 Solution to the Neumann Problem

We would like to give an essential tool for proving Lemma 6.3.3 (ii). Let us recall that for $f \in H^{\frac{1}{2}}(\Gamma)$ we mean that the norm

$$\|f\|_{H^{\frac{1}{2}}(\Gamma)} := \left(\|f\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|f(x) - f(y)|^2}{|x-y|^n} \, d\mathcal{H}^{n-1}(x) \, d\mathcal{H}^{n-1}(y) \right)^{\frac{1}{2}}$$

is finite, see e.g. [24, Section I.3.6]. Our essential tool is a similar result to [13, Lemma 3.2].

Proposition 6.3.6. *Let $n \geq 3$. Suppose that $f \in C^1(\mathbf{R}^{n-1})$ satisfies*

$$\operatorname{supp} f \subset B_1(0')^{\mathrm{c}}, \quad |f(x')| \cdot |x'|^{n-1} \leq c_1, \quad |\nabla' f(x')| \cdot |x'|^n \leq c_2$$

with some constants c_1 and c_2 independent of $x' \in \mathbf{R}^{n-1}$. Then the quantity

$$\|f\|^2_{L^2(\mathbf{R}^{n-1})} + \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^{n-1}} \frac{|f(x') - f(y')|^2}{|x'-y'|^n} \, dx' \, dy'$$

is finite which depends on n, c_1 and c_2 only.

Proof. By a direct calculation, we see that

$$\|f\|^2_{L^2(\mathbf{R}^{n-1})} \leq c_1 \int_{B_1(0')^{\mathrm{c}}} \frac{1}{|y'|^{2n-2}} \, dy' \leq C_n c_1.$$

It is sufficient to estimate

$$I = \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^{n-1}} \frac{|f(x') - f(y')|^2}{|x'-y'|^n} \, dx' \, dy'.$$

We follow the argument that proves [13, Lemma 3.2].

Assume that $|x'| \leq |y'|$ and connect x' and y' by a geodesic curve in $B_{|x'|}(0')^{\mathrm{c}}$. Since the curve length is less than $(\pi/2)|x'-y'|$, by a fundamental theorem of calculus, we observe that

$$\begin{aligned} \big|f(x') - f(y')\big| &\leq (\pi/2)|x'-y'| \cdot \sup\big\{ \big|\nabla' f(z')\big| \;\big|\; z' \in B_{|x'|}(0')^{\mathrm{c}} \big\} \\ &\leq (\pi/2) c_2 |x'-y'| \cdot |x'|^{-n}. \end{aligned}$$

Since the integrand of I is symmetric with respect to x' and y', we now estimate

$$\frac{I}{2} = \iint_{D_1} + \iint_{D_2} \frac{|f(x') - f(y')|^2}{|x'-y'|^n} \, dx' \, dy' = I_1 + I_2$$

with

$$\begin{aligned} D_1 &= \big\{ (x', y') \;\big|\; 1 \leq |x'| \leq |y'|, \; |x'-y'| \leq |x'| \big\}, \\ D_2 &= \big\{ (x', y') \;\big|\; 1 \leq |x'| \leq |y'|, \; |x'-y'| \geq |x'| \big\}. \end{aligned}$$

To estimate I_1, we observe that

$$\begin{aligned} \frac{|f(x') - f(y')|^2}{|x'-y'|^n} &\leq (\pi/2) c_2 |x'|^{-2n} |x'-y'|^{-(n-2)} \\ &\leq (\pi/2) c_2 |x'|^{-2n+1+\delta} |x'-y'|^{-(n-2)-1-\delta} \end{aligned}$$

for $0 < \delta < 1$ since $|x' - y'| \le |x'|$. Thus,

$$\begin{aligned} I_1 \le& (\pi/2)c_2 \int_{B_1(0')^c} \int_{B_1(x')} |y' - x'|^{-(n-2)}\, dy' |x'|^{-2n}\, dx' \\ &+ (\pi/2)c_2 \int_{B_1(0')^c} \int_{B_1(x')^c} |y' - x'|^{-(n-2)-1-\delta}\, dy' |x'|^{-2n+1+\delta}\, dx' < \infty. \end{aligned}$$

To estimate I_2, we observe that

$$\frac{|f(x') - f(y')|^2}{|x' - y'|^n} \le 2\frac{|f(x')|^2 + |f(y')|^2}{|x' - y'|^n} \le 4c_1 |x' - y'|^{-n} |x'|^{-(2n-2)}$$

since $|x'| \le |y'|$. Since $|x' - y'| \ge |x'|$ in this case, we have that

$$|x' - y'|^{-n} |x'|^{-(2n-2)} \le |x' - y'|^{-(n-2)} |x'|^{-2n}.$$

and

$$|x' - y'|^{-n} |x'|^{-(2n-2)} \le |x' - y'|^{-(n-\delta)} |x'|^{-(2n-2)-\delta}$$

for $0 < \delta < 1$. Hence,

$$\begin{aligned} I_2 \le& 4c_1 \int_{B_1(0')^c} \int_{B_1(x')} |y' - x'|^{-(n-2)}\, dy' |x'|^{-2n}\, dx' \\ &+ 4c_1 \int_{B_1(0')^c} \int_{B_1(x')^c} |y' - x'|^{-(n-\delta)}\, dy' |x'|^{-(2n-2)-\delta}\, dx' < \infty. \end{aligned}$$

□

Now we are ready to give a proof to Lemma 6.3.3 (ii).

Proof of Lemma 6.3.3 (ii). Let $x \in \Gamma_{\rho_0}$. Suppose that $|x'| \ge R_h$. In this case, we decompose

$$\begin{aligned} \nabla d(x) \cdot \nabla\big(E * (\delta_\Gamma \otimes g)\big)(x) &= \partial_{x_n}\big(E * (\delta_\Gamma \otimes g)\big)(x) \\ &= \partial_{x_n}\big(E * (\delta_{\partial \mathbf{R}^n_+} \otimes \overline{g_2})\big)(x) + \int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y) g_1(y)\, d\mathcal{H}^{n-1}(y). \end{aligned}$$

By Proposition 6.3.2, we see that

$$|\partial_{x_n}\big(E * (\delta_{\partial \mathbf{R}^n_+} \otimes \overline{g_2})\big)(x)| \le \frac{1}{2} \|\overline{g_2}\|_{L^\infty(\partial \mathbf{R}^n_+)} \le \frac{1}{2} \|g\|_{L^\infty(\Gamma)}.$$

By Lemma 6.3.4 (ii), we have that

$$\left| \int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y) g_1(y)\, d\mathcal{H}^{n-1}(y) \right| \le C \|g\|_{L^\infty(\Gamma)}.$$

Thus for $x \in \Gamma_{\rho_0}$ with $|x'| \ge R_h$, we show that

$$|\nabla d(x) \cdot \nabla\big(E * (\delta_\Gamma \otimes g)\big)(x)| \le C \|g\|_{L^\infty(\Gamma)}$$

with C independent of g.

Suppose that $|x'| < R_h$. We decompose

$$\nabla d(x) \cdot \nabla \left(E * (\delta_\Gamma \otimes g_1)\right)(x) = \int_\Gamma (\nabla d(x) - \nabla d(y)) \cdot \nabla E(x-y) g_1(y)\, d\mathcal{H}^{n-1}(y) + \int_\Gamma \frac{\partial E}{\partial \mathbf{n}_y}(x-y) g_1(y)\, d\mathcal{H}^{n-1}(y) = I_1 + I_2.$$

For $|y'| < 2R_h$, there exists a constant M, independent of x and y, such that the estimate

$$|\nabla d(x) - \nabla d(y)| \leq M|x-y|$$

holds. In this case, we have that

$$\left| \int_\Gamma (\nabla d(x) - \nabla d(y)) \cdot \nabla E(x-y) g_1(y)\, d\mathcal{H}^{n-1}(y) \right| \leq C_n M \int_{|y'|<2R_h} \frac{1}{|x'-y'|^{n-2}}\, dy' \|g\|_{L^\infty(\Gamma)}.$$

Thus, $|I_1(x)|$ is estimated by $C_n M R_h \|g\|_{L^\infty(\Gamma)}$. By Lemma 6.3.4 (ii), $|I_2(x)|$ is estimated by $C\|g\|_{L^\infty(\Gamma)}$.

We define that

$$H_x(y') := \Big(\nabla d(x) \cdot \nabla E\big((y', h(y')) - x\big) \Big) 1_{B_{R_h}(x')^c}(y')$$

for $y' \in \mathbf{R}^{n-1}$. With this notation, we have that

$$\big|\nabla d(x) \cdot \nabla \big(E * (\delta_\Gamma \otimes g_2)\big)(x)\big| \leq \int_{\mathbf{R}^{n-1}} \big|H_x(y') g(y', h(y'))\big|\, dy'.$$

Note that $H_x(\cdot') \in C^1(\mathbf{R}^{n-1})$ satisfies

$$\operatorname{supp} H_x(\cdot') \subset B_{R_h}(x')^c, \quad |H_x(y')| \cdot |x'-y'|^{n-1} \leq c_1, \quad |\nabla'_{y'} H_x(y')| \cdot |x'-y'|^n \leq c_2$$

with some constant c_1 and c_2 independent of $x', y' \in \mathbf{R}^{n-1}$. By Proposition 6.3.6, we deduce that $H_x(\cdot') \in H^{\frac{1}{2}}(\Gamma)$. By the duality relation, we see that

$$\big|\nabla d(x) \cdot \nabla \big(E * (\delta_\Gamma \otimes g_2)\big)(x)\big| \leq \|H_x(\cdot')\|_{H^{\frac{1}{2}}(\Gamma)} \|g\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C_n \|g\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Combining all estimates above regarding different $x' \in \mathbf{R}^{n-1}$, we are done. □

Finally, if we can show that $Sg \in H^{-\frac{1}{2}}(\Gamma)$ for $g \in L^\infty(\Gamma)$ with the operator norm

$$\|S\|_{L^\infty(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)}$$

sufficiently small, then we solve Neumann problem (6.1.4). Fortunately, we have an affirmative answer to this question.

Lemma 6.3.7. *For $g \in L^\infty(\Gamma)$, we have that $Sg \in L^2(\Gamma)$ satisfies the estimate*

$$\|Sg\|_{L^2(\Gamma)} \leq C^* (\|h\|_{C^2(\mathbf{R}^{n-1})} + 1)(R_h^{n-1} + 1) R_h^{\frac{n-1}{2}} \|g\|_{L^\infty(\Gamma)}$$

with some constant C^* independent of h and g.

Proof. Let $x \in \Gamma$ with $|x'| < 3R_h$. Since

$$\|Sg\|_{L^\infty(\Gamma)} \le C_*(R_h^{n-1}+1)\|h\|_{C^2(\mathbf{R}^{n-1})}\|g\|_{L^\infty(\Gamma)}$$

by Theorem 6.3.5 (ii), we have that

$$\left(\int_{\{x\in\Gamma \,|\, |x'|<3R_h\}} |Sg(z)|^2\, d\mathcal{H}^{n-1}(z)\right)^{\frac{1}{2}} \le C_{*,h}\|h\|_{C^2(\mathbf{R}^{n-1})}\|g\|_{L^\infty(\Gamma)}\mu\bigl(\{x\in\Gamma \,|\, |x'|<3R_h\}\bigr)^{\frac{1}{2}}$$

where $\mu\bigl(\{x\in\Gamma \,|\, |x'|<3R_h\}\bigr)$ denotes the surface area of $\{x\in\Gamma \,|\, |x'|<3R_h\}$ and $C_{*,h} := C_*(R_h^{n-1}+1)$. Thus,

$$\left(\int_{\{x\in\Gamma \,|\, |x'|<3R_h\}} |Sg(z)|^2\, d\mathcal{H}^{n-1}(z)\right)^{\frac{1}{2}} \le C_{*,h}R_h^{\frac{n-1}{2}}\|h\|_{C^2(\mathbf{R}^{n-1})}\|g\|_{L^\infty(\Gamma)}.$$

Suppose that $x \in \Gamma$ with $|x'| \ge 3R_h$. For $y \in \Gamma$ with $|y'| < 2R_h$, the triangle inequality implies that $|x'-y'| \ge |x'| - 2R_h$. In this case, we have that

$$|Sg(x)| \le \int_{\{|y'|<2R_h\}} \frac{1}{|x'-y'|^{n-1}}\, dy'\|g\|_{L^\infty(\Gamma)} \le \frac{|B_{2R_h}(0')|\cdot\|g\|_{L^\infty(\Gamma)}}{(|x'|-2R_h)^{n-1}}.$$

Hence,

$$\int_{\{x\in\Gamma \,|\, |x'|\ge 3R_h\}} |Sg(x)|^2\, d\mathcal{H}^{n-1}(x) \le CR_h^{2(n-1)}\|g\|^2_{L^\infty(\Gamma)}\cdot\int_{\{|x'|\ge 3R_h\}} \frac{1}{(|x'|-2R_h)^{2(n-1)}}\, dx'.$$

Assume that $R_h < 1$. We have that

$$\int_{\{|x'|\ge 3R_h\}} \frac{1}{(|x'|-2R_h)^{2(n-1)}}\, dx' \le C\int_{R_h}^\infty \frac{(r+2R_h)^{n-2}}{r^{2n-2}}\, dr \le C\sum_{i=0}^{n-2} R_h^{n-2-i}\int_{R_h}^\infty \frac{r^i}{r^{2n-2}}\, dr.$$

Therefore, we obtain that

$$\left(\int_{\{x\in\Gamma \,|\, |x'|\ge 3R_h\}} |Sg(x)|^2\, d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{2}} \le CR_h^{\frac{n-1}{2}}\|g\|_{L^\infty(\Gamma)}.$$

□

Since $L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$ is a natural embedding, for $g \in L^\infty(\Gamma)$ we have that $Sg \in H^{-\frac{1}{2}}(\Gamma)$ satisfies the estimate

$$\|Sg\|_{H^{-\frac{1}{2}}(\Gamma)} \le C^*(\|h\|_{C^2(\mathbf{R}^{n-1})}+1)(R_h^{n-1}+1)R_h^{\frac{n-1}{2}}\|g\|_{L^\infty(\Gamma)}$$

with some constant C^* independent of h and g.

Proof of Lemma 6.1.4. For $i \in \mathbf{N}$, we have that

$$\begin{aligned}\|(2S)^i g\|_{H^{-\frac{1}{2}}(\Gamma)} &\le 2C^*(\|h\|_{C^2(\mathbf{R}^{n-1})}+1)(R_h^{n-1}+1)R_h^{\frac{n-1}{2}}\|(2S)^{i-1}g\|_{L^\infty(\Gamma)}\\ &\le \bigl(2C^*(\|h\|_{C^2(\mathbf{R}^{n-1})}+1)R_h^{\frac{n-1}{2}}\bigr)2^{i-1}C_*^{i-1}(R_h^{n-1}+1)^i\|h\|^{i-1}_{C^2(\mathbf{R}^{n-1})}\|g\|_{L^\infty(\Gamma)}\end{aligned}$$

and

$$\|(2S)^i g\|_{L^\infty(\Gamma)} \leq 2^i C_*^i (R_h^{n-1}+1)^i \|h\|^i_{C^2(\mathbf{R}^{n-1})} \|g\|_{L^\infty(\Gamma)}.$$

Let the perturbation h be sufficiently small so that

$$(R_h^{n-1}+1)\|h\|_{C^2(\mathbf{R}^{n-1})} < \frac{1}{2C_*}$$

with C_* defined in Theorem 6.3.5 (ii). Then the operator $I-2S$, which is bounded linear from $L^\infty(\Gamma)$ to $L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)$, has a bounded inverse by a standard Neumann series argument. The inverse of $I-2S$ can be constructed as

$$(I-2S)^{-1} = \sum_{i=0}^\infty (2S)^i,$$

which is well-defined as a bounded linear map from $L^\infty(\Gamma)$ to $L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)$ since the operator norm

$$\|2S\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)} < 1.$$

Therefore, for $g\in L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)$, the solution to the Neumann problem (6.1.4) is formally given by

$$u(x) = E * \big(\delta_\Gamma \otimes (2(I-2S)^{-1}g)\big)(x), \quad x\in\Omega$$

since Pg is harmonic in Ω.

If the L^2 estimate

$$\|\nabla E * \big(\delta_\Gamma \otimes (2(I-2S)^{-1}g)\big)\|_{(L^2(\Omega))^n} \leq C\|2(I-2S)^{-1}g\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)}$$

holds, then we are done. Fortunately, we indeed have this L^2 estimate, we shall give a proof to this estimate in the next subsection. Combine this estimate with Lemma 6.3.3, we obtain our desired estimate

$$\|\nabla u\|_{vBMOL^2(\Omega)} \leq C\|g\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)}$$

with a constant C independent of g. □

6.3.5 L^2 estimate to the solution of the Neumann problem

Firstly, we start with the half space case.

Proposition 6.3.8. *For $g\in H^{-\frac{1}{2}}(\partial\mathbf{R}^n_+)$ satisfying*

$$\int_{\partial\mathbf{R}^n_+} g(y)\,d\mathcal{H}^{n-1}(y) = 0,$$

the estimate

$$\|\nabla E * (\delta_{\partial\mathbf{R}^n_+}\otimes g)\|_{(L^2(\mathbf{R}^n_+))^n} \leq C\|g\|_{H^{-\frac{1}{2}}(\partial\mathbf{R}^n_+)}$$

holds with some constant C independent of g.

Proof. Since

$$E * (\delta_{\partial \mathbf{R}^n_+} \otimes g) = \int_{\mathbf{R}^{n-1}} E(x'-y', x_n) g(y')\, dy',$$

the single layer potential $E * (\delta_{\partial \mathbf{R}^n_+} \otimes g)$ is exactly half of the solution to the Neumann problem (6.3.1). Since $g \in H^{-\frac{1}{2}}(\partial \mathbf{R}^n_+)$ satisfying

$$\int_{\partial \mathbf{R}^n_+} g(y)\, d\mathcal{H}^{n-1}(y) = 0,$$

there exists a unique weak solution (up to an additive constant) $u_* \in H^1(\mathbf{R}^n_+)$ to the Neumann problem (6.3.1) which satisfies

$$\|\nabla u_*\|_{(L^2(\mathbf{R}^n_+))^n} \leq C \|g\|_{H^{-\frac{1}{2}}(\partial \mathbf{R}^n_+)}$$

with C independent of g, see e.g. [26, Remark 1.2 and Remark 1.3], [21, Section 1.7]. Therefore, the single layer potential $2E * (\delta_{\partial \mathbf{R}^n_+} \otimes g)$ indeed differs from u_* by an additive constant. We do have that

$$\|\nabla E * (\delta_{\partial \mathbf{R}^n_+} \otimes g)\|_{(L^2(\mathbf{R}^n_+))^n} = \frac{1}{2} \|\nabla u_*\|_{(L^2(\mathbf{R}^n_+))^n} \leq C \|g\|_{H^{-\frac{1}{2}}(\partial \mathbf{R}^n_+)}.$$

□

We then generalize this result to any perturbed half space $\mathbf{R}^n_h$.

Lemma 6.3.9. *Let $\Omega = \mathbf{R}^n_h$ be a perturbed C^2 half space with $\Gamma = \partial\Omega$. For any $g \in L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)$ that satisfies*

$$\int_\Gamma g(y)\, d\mathcal{H}^{n-1}(y) = 0,$$

the estimate

$$\|\nabla E * (\delta_\Gamma \otimes g)\|_{(L^2(\Omega))^n} \leq C \|g\|_{L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)}$$

holds with some constant C independent of g.

Proof. Without loss of generality, we may assume that $\operatorname{supp} h \subset B_{M_h}(0')$ for some $M_h > 0$. We set

$$g_1(y', h(y')) := 1_{B_{2M_h}(0')}(y') g(y', h(y')), \quad g_2(y', h(y')) := g(y', h(y')) - g_1(y', h(y'))$$

for any $y' \in \mathbf{R}^{n-1}$. Since $g \in L^\infty(\Gamma)$, it is trivial to see that $g_1 \in L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$. Hence, we deduce that $g_1, g_2 \in L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)$. Since $\operatorname{supp} g_1(\cdot', h(\cdot')) \subset B_{2M_h}(0')$, there exists a constant $I_c \in \mathbf{R}$ such that

$$I_c = \int_\Gamma g_1(y)\, d\mathcal{H}^{n-1}(y) = -\int_\Gamma g_2(y)\, d\mathcal{H}^{n-1}.$$

Let $\overline{g_2} \in L^\infty(\partial \mathbf{R}^n_+)$ be defined by

$$\overline{g_2}(x', 0) = \begin{cases} g_2(x', 0) & \text{for} \quad |x'| \geq 2M_h \\ 0 & \text{for} \quad |x'| < 2M_h. \end{cases}$$

Note that $T : \eta \mapsto \zeta$ with $\zeta(y', h(y')) = \eta(y', 0)$ for any $y' \in \mathbf{R}^{n-1}$ is an isomorphism between $H^{\frac{1}{2}}(\partial\mathbf{R}^n_+)$ and $H^{\frac{1}{2}}(\Gamma)$. For any $\eta \in H^{\frac{1}{2}}(\partial\mathbf{R}^n_+)$, we have that

$$\int_{\partial\mathbf{R}^n_+} \overline{g_2}(y)\eta(y)\, d\mathcal{H}^{n-1}(y) = \int_\Gamma g_2(y)\zeta(y)\, d\mathcal{H}^{n-1}(y).$$

By considering

$$\|\overline{g_2}\|_{H^{-\frac{1}{2}}(\partial\mathbf{R}^n_+)} = \sup_{\|\eta\|_{H^{\frac{1}{2}}(\partial\mathbf{R}^n_+)} \le 1} \left| \int_{\partial\mathbf{R}^n_+} \overline{g_2}(y)\eta(y)\, d\mathcal{H}^{n-1}(y) \right|,$$

we can deduce that $\overline{g_2} \in H^{-\frac{1}{2}}(\partial\mathbf{R}^n_+)$ satisfies the estimate

$$\|\overline{g_2}\|_{H^{-\frac{1}{2}}(\partial\mathbf{R}^n_+)} \le C\|g_2\|_{H^{-\frac{1}{2}}(\Gamma)} \le C\|g\|_{H^{-\frac{1}{2}}(\Gamma)}$$

with C independent of g. Let $f_s \in L^\infty(\partial\mathbf{R}^n_+) \cap H^{-\frac{1}{2}}(\partial\mathbf{R}^n_+)$ be defined by

$$f_s(x', 0) = \begin{cases} 0 & \text{for} \quad |x'| \ge 2M_h \\ \frac{I_c}{|B_{2M_h}(0')|} & \text{for} \quad |x'| < 2M_h \end{cases}$$

where $|B_{2M_h}(0')|$ denotes the size of the ball $B_{2M_h}(0')$ in $\mathbf{R}^{n-1}$. Note that

$$\int_{\partial\mathbf{R}^n_+} \overline{g_2}(y) + f_s(y)\, d\mathcal{H}^{n-1}(y) = 0.$$

Let us recall an argument from the proof of Lemma 6.3.3 (i). Let $\delta < \rho_0/2 < R_*/2$. We take a C^2 cut-off function $\theta \ge 0$ such that $\theta(\sigma) = 1$ for $\sigma \le 1$ and $\theta(\sigma) = 0$ for $\sigma \ge 2$. By the choice of δ, we see that $\theta_d = \theta(d/\delta)$ is C^2 in $\mathbf{R}^n$. We extend $g_1 \in L^\infty(\Gamma)$ to $g_1^e \in L^\infty(\Gamma^{2\delta})$ by setting

$$g_1^e(x) := g_1(\pi x)$$

for any $x \in \Gamma^{2\delta}$ with πx denoting the projection of x on Γ. As explained in the proof of Lemma 6.3.3 (i), we have that $\nabla d \cdot \nabla g_1^e = 0$ in $\Gamma^{2\delta}$. Set $g_{1,c}^e := \theta_d g_1^e$. Since

$$\begin{aligned} \delta_\Gamma \otimes g_1 &= (\nabla 1_\Omega \cdot \nabla d) g_{1,c}^e \\ &= \operatorname{div}(g_{1,c}^e 1_\Omega \nabla d) - 1_\Omega \operatorname{div}(g_{1,c}^e \nabla d), \\ \operatorname{div}(g_{1,c}^e \nabla d) &= g_{1,c}^e \Delta d + \nabla d \cdot \nabla g_{1,c}^e = g_{1,c}^e \Delta d + \frac{\theta'(d/\delta)}{\delta} g_1^e, \end{aligned}$$

for any $x \in \mathbf{R}^n$ we have that

$$\nabla E * (\delta_\Gamma \otimes g_1)(x) = \nabla \operatorname{div}\left(E * (g_{1,c}^e 1_\Omega \nabla d)\right)(x) - \nabla E * \left(1_\Omega g_1^e f_{\theta,\delta}\right)(x) = I_1(x) + I_2(x)$$

where $f_{\theta,\delta} := \theta_d \Delta d + \frac{\theta'(d/\delta)}{\delta}$. Since $\nabla \operatorname{div} E$ is bounded in L^p for $1 < p < \infty$, see e.g. [15, Theorem 5.2.7 and Theorem 5.2.10], we deduce that

$$\|I_1\|_{(L^2(\mathbf{R}^n)^n} \le C\|g_{1,c}^e 1_\Omega \nabla d\|_{(L^2(\mathbf{R}^n))^n} \le C\|g\|_{L^\infty(\Gamma)}$$

as $\operatorname{supp} g_{1,c}^e \subset \{x \in \mathbf{R}^n \mid |d(x)| < 2\delta, |(\pi x)'| < 2M_h\}$. Since $\nabla E \sim |\cdot|^{1-n}$, by the famous Hardy-Littlewood-Sobolev inequality, see e.g. [1, Theorem 1.7], we have that

$$\|I_2\|_{(L^2(\mathbf{R}^n))^n} \le C\|1_\Omega g_1^e f_{\theta,\delta}\|_{L^r(\mathbf{R}^n)}$$

where $r = \frac{2n}{n+2}$. As $\operatorname{supp} g_1^e f_{\theta,\delta} \subset \{x \in \mathbf{R}^n \mid |d(x)| < 2\delta, |(\pi x)'| < 2M_h\}$, the estimate

$$\|1_\Omega g_1^e f_{\theta,\delta}\|_{L^r(\mathbf{R}^n)} \leq C\|g\|_{L^\infty(\Gamma)}$$

holds. Hence, we obtain the L^2 estimate for g_1, i.e., it holds that

$$\|\nabla E * (\delta_\Gamma \otimes g_1)\|_{(L^2(\Omega))^n} \leq C\|g\|_{L^\infty(\Gamma)}.$$

By Proposition 6.3.8, we have the estimate

$$\|\nabla E * \big(\delta_{\partial\mathbf{R}^n_+} \otimes (\overline{g_2} + f_s)\big)\|_{(L^2(\mathbf{R}^n_+))^n} \leq C\big(\|\overline{g_2}\|_{H^{-\frac{1}{2}}(\partial\mathbf{R}^n_+)} + \|f_s\|_{H^{-\frac{1}{2}}(\partial\mathbf{R}^n_+)}\big)$$

with some C independent of g. Since $f_s \in L^2(\partial\mathbf{R}^n_+)$, we have that

$$\|f_s\|_{H^{-\frac{1}{2}}(\partial\mathbf{R}^n_+)} \leq \frac{C}{M_h^{\frac{n-1}{2}}} \cdot |I_c| \leq \frac{C}{M_h^{\frac{n-1}{2}}} \cdot S_c \cdot \|g\|_{L^\infty(\Gamma)},$$

where $S_c := \mu(\{y \in \Gamma \mid |y'| < 2M_h\})$ denotes the surface area of the curved part $\{y \in \Gamma \mid |y'| < 2M_h\}$. Hence, we get that

$$\|\nabla E * \big(\delta_{\partial\mathbf{R}^n_+} \otimes \overline{g_2}\big)\|_{(L^2(\mathbf{R}^n_+))^n} \leq \|\nabla E * \big(\delta_{\partial\mathbf{R}^n_+} \otimes f_s\big)\|_{(L^2(\mathbf{R}^n_+))^n} + C\|g\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)}.$$

Since $f_s \in L^\infty(\partial\mathbf{R}^n_+)$ and $\operatorname{supp} f_s \subset B_{2M_h}(0')$, by the argument in the above paragraph, we can deduce that

$$\|\nabla E * (\delta_{\partial\mathbf{R}^n} \otimes f_s)\|_{(L^2(\mathbf{R}^n_+))^n} \leq C\|f_s\|_{L^\infty(\partial\mathbf{R}^n_+)} \leq \frac{C}{M_h^{n-1}} \cdot |I_c|.$$

Note that for any $x \in \Omega$, it holds that

$$\nabla E * (\delta_\Gamma \otimes g_2)(x) = \nabla E * (\delta_{\partial\mathbf{R}^n_+} \otimes \overline{g_2})(x).$$

In addition, for $x = (x', x_n) \in \mathbf{R}^n_+$, we have that

$$\big|\nabla E * (\delta_{\partial\mathbf{R}^n_+} \otimes \overline{g_2})(x', -x_n)\big| = \big|\nabla E * (\delta_{\partial\mathbf{R}^n_+} \otimes \overline{g_2})(x', x_n)\big|.$$

Hence, we deduce that

$$\|\nabla E * (\delta_\Gamma \otimes g_2)\|_{(L^2(\Omega))^n} \leq 2\|\nabla E * \big(\delta_{\partial\mathbf{R}^n_+} \otimes \overline{g_2}\big)\|_{(L^2(\mathbf{R}^n_+))^n} \leq C\|g\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)}.$$

Combine with the L^2 estimate for $\nabla E * (\delta_\Gamma \otimes g_1)$, we finally obtain our desired L^2 estimate

$$\|\nabla E * (\delta_\Gamma \otimes g)\|_{(L^2(\Omega))^n} \leq C\|g\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)}$$

with some constant C independent of g. □

If $g \in L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma) \cap L^1(\Gamma)$, then we obtain a similar lemma to Lemma 6.3.9 which does not need to require the integral of g on Γ to be zero.

Lemma 6.3.10. *Let $\Omega = \mathbf{R}^n_h$ be a perturbed C^2 half space with $\Gamma = \partial\Omega$. For any $g \in L^\infty(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma) \cap L^1(\Gamma)$, the estimate*

$$\|\nabla E * (\delta_\Gamma \otimes g)\|_{(L^2(\Omega))^n} \leq C\|g\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)\cap L^1(\Gamma)}$$

holds with some constant C independent of g.

Proof. Since $\|g\|_{L^1(\Gamma)}$ is finite, there exists a constant $I_c \in \mathbf{R}$ such that

$$\int_\Gamma g_2(y)\, d\mathcal{H}^{n-1}(y) = I_c$$

where $g_2(y', h(y')) := 1_{B_{2M_h}(0')^c}(y')g(y', h(y'))$. Since we can estimate $|I_c|$ by $\|g\|_{L^1(\Gamma)}$ directly, following the proof of Lemma 6.3.9 gives us Lemma 6.3.10. □

For $g \in L^\infty(\Gamma)$, by Lemma 6.3.7 we see that $Sg \in L^2(\Gamma)$. Actually, we can also estimate the L^1 norm of Sg.

Lemma 6.3.11. *For $g \in L^\infty(\Gamma)$, we have that $Sg \in L^1(\Gamma)$ satisfying the estimate*

$$\|Sg\|_{L^1(\Gamma)} \le C\big((R_h^{n-1}+1)\|h\|_{C^2(\mathbf{R}^{n-1})}+1\big)R_h^{n-1}\|g\|_{L^\infty(\Gamma)}$$

with some constant C independent of g and h.

Proof. Let $x \in \Gamma$ with $|x'| < 3R_h$. Since

$$\|Sg\|_{L^\infty(\Gamma)} \le C_*(R_h^{n-1}+1)\|h\|_{C^2(\mathbf{R}^{n-1})}\|g\|_{L^\infty(\Gamma)}$$

by Theorem 6.3.5 (ii), we have that

$$\int_{\{x\in\Gamma\,|\,|x'|<3R_h\}} |Sg(z)|\, d\mathcal{H}^{n-1}(z) \le C_{*,h}\|h\|_{C^2(\mathbf{R}^{n-1})}\|g\|_{L^\infty(\Gamma)} \cdot \mu\big(\{x\in\Gamma\,|\,|x'|<3R_h\}\big)$$

where $\mu\big(\{x\in\Gamma\,|\,|x'|<3R_h\}\big)$ denotes the surface area of $\{x\in\Gamma\,|\,|x'|<3R_h\}$ and $C_{*,h} := C_*(R_h^{n-1}+1)$. Thus,

$$\int_{\{x\in\Gamma\,|\,|x'|<3R_h\}} |Sg(z)|\, d\mathcal{H}^{n-1}(z) \le C_{*,h}R_h^{n-1}\|h\|_{C^2(\mathbf{R}^{n-1})}\|g\|_{L^\infty(\Gamma)}.$$

Suppose that $x \in \Gamma$ with $|x'| \ge 3R_h$. Then we have that

$$\int_{|x'|\ge 3R_h} |Sg(x',0)|\, dx' \le \int_{\{y\in\Gamma\,|\,|y'|<2R_h\}} |g_1(y',h(y'))| \cdot \left(\int_{|x'|\ge 3R_h} \frac{1}{|x'-y'|^{n-1}}\, dx'\right) dy'.$$

By Hölder's inequality, we deduce that

$$\int_{|x'|\ge 3R_h} |Sg(x',0)|\, dx' \le A_c^{1/2}\|g\|_{L^\infty(\Gamma)} \cdot \left(\int_{|y'|<2R_h}\left(\int_{|x'|\ge 3R_h} \frac{1}{|x'-y'|^{n-1}}\, dx'\right)^2 dy'\right)^{1/2}$$

where $A_c := \mu\big(\{y\in\Gamma\,|\,|y'|<3R_h\}\big)$ denotes the surface area of the curved part $\{y \in \Gamma\,|\,|y'|<2R_h\}$. By Minkowski's inequality for integrals, see e.g. [25, Appendices A.1], we see that

$$\begin{aligned}&\left(\int_{|y'|<2R_h}\left(\int_{|x'|\ge 3R_h} \frac{1}{|x'-y'|^{n-1}}\, dx'\right)^2 dy'\right)^{1/2}\\ &\le \int_{|y'|<2R_h}\left(\int_{|x'|\ge 3R_h} \frac{1}{|x'-y'|^{2(n-1)}}\, dx'\right)^{1/2} dy'\end{aligned}$$

For $y \in \Gamma$ with $|y'| < 2R_h$, the triangle inequality implies that $|x'-y'| \geq |x'| - 2R_h$. In this case,

$$\int_{|x'|\geq 3R_h} \frac{1}{|x'-y'|^{2(n-1)}}\, dx' \leq \int_{|x'|\geq 3R_h} \frac{1}{(|x'|-2R_h)^{2(n-1)}}\, dx'.$$

Recall the calculation in the proof of Lemma 6.3.7, it can be deduced that

$$\int_{|x'|\geq 3R_h} \frac{1}{(|x'|-2R_h)^{2(n-1)}}\, dx' \leq \frac{C}{R_h^{n-1}}.$$

Therefore,

$$\int_{|x'|\geq 3R_h} |Sg(x',0)|\, dx' \leq A_c^{1/2} C R_h^{\frac{n-1}{2}} \|g\|_{L^\infty(\Gamma)}.$$

Combine with the L^1 estimate of $|Sg|$ on $\{x \in \Gamma \mid |x'| < 3R_h\}$, we are done. □

Proof of Lemma 6.1.4 (Continued). If

$$(R_h^{n-1}+1)\|h\|_{C^2(\mathbf{R}^{n-1})} < \frac{1}{2C_*},$$

then Theorem 6.3.5, Lemma 6.3.7 and Lemma 6.3.11 together imply that

$$\left\| \sum_{i=1}^{\infty} (2S)^i g \right\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)\cap L^1(\Gamma)} \leq C\|g\|_{L^\infty(\Gamma)}$$

with some constant C independent of g. Let us view $\nabla E * \big(\delta_\Gamma \otimes (2(I-2S)^{-1}g\big)$ as

$$\nabla E * (\delta_\Gamma \otimes 2g) + \nabla E * \left(\delta_\Gamma \otimes \left(2\sum_{i=1}^{\infty}(2S)^i g\right)\right). \tag{6.3.7}$$

Since the integral of g on Γ is zero, by applying Lemma 6.3.9 to the first term of (6.3.7), we obtain that

$$\|\nabla E * (\delta_\Gamma \otimes 2g)\|_{(L^2(\Omega))^n} \leq C\|g\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)}.$$

Since

$$\sum_{i=1}^{\infty}(2S)^i g \in L^1(\Gamma)\cap L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma),$$

by applying Lemma 6.3.10 to the second term of (6.3.7) and estimating the L^1 norm of $\sum_{i=1}^{\infty}(2S)^i g$ by $\|g\|_{L^\infty(\Gamma)}$, we get that

$$\left\|\nabla E * \left(\delta_\Gamma \otimes \left(2\sum_{i=1}^{\infty}(2S)^i g\right)\right)\right\|_{(L^2(\Omega))^n} \leq C\|g\|_{L^\infty(\Gamma)\cap H^{-\frac{1}{2}}(\Gamma)}.$$

This completes the proof of Lemma 6.1.4. □

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