博士論文

論文題目　Multidimensional continued fractions and Fujiki-Oka resolutions of cyclic quotient singularities
(多次元連分数と巡回商特異点に対する藤木岡特異点解消)

氏　　名　佐藤　悠介

Abstract: This thesis shows some relationships between multi-dimensional continued fractions and Fujiki-Oka resolutions of cyclic quotient singularities. First, we will show a necessary and sufficient condition for the Fujiki-Oka resolutions of Gorenstein abelian quotient singularities to be crepant in all dimensions. This result is obtained in joint work with Kohei Sato. Second, we introduce n-dimensional complete coprime cyclic quotient singularities. It has a good resolution which is obtained by subdivision using only points of Hilbert basis. Moreover, there is one-to-one correspondence between exceptional divisors of this resolution and the multidimensional continued fraction.

Contents

Acknowledgements: First, the author would like to thank his supervisor Yukari Ito for advice and support over the past five years. She has provided a great deal of encouragement and inspiration for the author's study. He couldn't have made this thesis without them. He would also like to thank Kohei Sato for discussing various problems together as a collaborative researcher. Thanks to him, the author was able to proceed with his research.

The author is grateful to Tadashi Ashikaga for useful advice about multidimensional continued fractions. His advice stimulated research motivation. The author would like to thank those who commented and advised on his presentation at conferences and seminars, especially the author thanks to Akira Ishii, Alastair Craw, Masataka Tomari, and Miles Reid. The author wants to thank Yusuke Nakajima, Ryo Yamagishi, and Takuro Matsumoto for their advice and comments as a member of the "Singularity Seminar".

The author would like to express his gratitude to his grandmother for her financial support and encouragement. Her words made him believe in himself and keep studying. Finally, the author wants to thank his family for agreeing to go to graduate school and for continuing to support him.

1 Introduction

1.1 Background

In this thesis, we study the resolution of cyclic quotient singularities which obtained by multi-dimensional continued fractions. Especially, we pay special attention to crepant resolutions and Hilbert basis resolutions.

Let G be a finite subgroup of $\mathrm{GL}(n,\mathbb{C})$, and let $(\mathbb{C}^n/G,[0])$ be the n-dimensional quotient singularity. In the case $n=2$, the cyclic quotient singularity $(\mathbb{C}^2/G,[0])$ has a unique minimal resolution, and the self-intersection number of each exceptional divisor of the minimal resolution corresponds to a coefficient of the Hirzebruch-Jung continued fraction related to the group action of G (see Section 3). As a generalization of the Hirzebruch-Jung continued fraction, the multi-dimensional continued fraction was introduced by Ashikaga to control the Fujiki-Oka resolution of cyclic quotient singularities of type $\frac{1}{r}(1,a_1,\ldots,a_n)$ (cf. [1]). The Fujiki-Oka resolution is a certain resolution of cyclic quotient singularities proposed by Fujiki [14], and represented by Oka [34] as a toric resolution. Especially, the Fujiki-Oka resolution coincides with the minimal resolution in dimension two.

On the other hands, the minimal resolution of quotient singularities is studied for a McKay correspondence. If G is a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, then the quotient singularity is Gorenstein, and the dual of the weighted graph obtained from the exceptional divisors of the minimal resolution corresponds to the Dynkin diagram obtained from the non-trivial irreducible representations of G. This correspondence remarked by J. McKay [31] is called the *McKay correspondence.* This correspondence has not been shown in the general dimension. It is because minimal resolutions do not necessarily exist for cyclic quotient singularities in the case of $n\geq 3$. The McKay correspondence has been generalized to the case of $n=3$ by Batyrev and Dais [3] and by Ito and Reid [22] as the following;

$$\{\text{Conjugacy classes of } G \text{ of age } i\} \longleftrightarrow \{\text{A basis of } H^{2i}(\widetilde{\mathbb{C}^3/G},\mathbb{Q})\}$$

where $\widetilde{\mathbb{C}^3/G}$ is a *crepant resolution* of $\mathbb{C}^3/G$, i.e., a resolution whose canonical divisor of $\widetilde{\mathbb{C}^3/G}$ is trivial.

Therefore the existence of crepant resolutions is a necessary condition to construct the above correspondences, but, unfortunately, there does not necessarily exist a crepant resolution of arbitrary Gorenstein quotient singularity. Hence, "The Existence Problem of Crepant Resolutions" is a basic question to hold the McKay correspondence in a higher dimension.

Existence Problem of Crepant Resolution: Do there exist crepant resolutions of $\mathbb{C}^n/G$ for $G\subset \mathrm{SL}(n,\mathbb{C})$ with $n\geq 4$?

As for known results with respect to the existence problem, when $n=2$, the minimal resolutions are always crepant. In the case of $n=3$, all Gorenstein quotient singularities possess at least one crepant resolution. This result was proved by Ito [18, 19], Markushevich [27, 28, 29] and Roan [36] case by case based on the classification of finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ given by Yau and Yu [41]. In the case of $n \geq 4$, the Gorenstein quotient singularities do not necessarily have a crepant resolution. On the other hand, Dais, Henk, Ziegler, and others have proved that all complete intersection Gorenstein quotient singularities possess at least one crepant resolution and have constructed some infinite series of Gorenstein quotient singularities which possess a crepant resolution [7, 8, 9, 11]. Moreover, some infinite series of Gorenstein quotient singularities that possess a crepant resolution was constructed by others [15, 37].

In this thesis we introduce a necessary and sufficient condition for the Fujiki-Oka resolutions to be crepant. And we proposed a crepant resolution of three-dimensional abelian quotient singularities as a Fujiki-Oka resolution.

The generalization of crepant resolution of toric quotient singularities is a Hilbert basis resolution. The Hilbert basis resolution was introduced as a **Hilb**-*desingularization* in [6, 7], and also *G-désingularization* in [2]. If a toric quotient singularity has a crepant resolution, then it is a Hilbert basis resolution [7]. We will consider Hilbert basis resolutions instead of crepant resolutions for quotient singularities. In this thesis, we give a condition for the Fujiki-Oka resolution that coincides with Hilbert basis resolution. In addition, we give series of cyclic quotient singularities which holds weakly McKay correspondence on the Hilbert basis resolution.

1.2 Statement of the results

In this subsection, we will introduce a summary of our results.

Definition 1.1. *Let n be an integer greater than or equal to 1. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $r \in \mathbb{Z}$ such that $0 \leq a_i \leq r-1$ for $1 \leq i \leq n$. We call the symbol*

$$\frac{\boldsymbol{a}}{r} = \frac{(a_1, \ldots, a_n)}{r}$$

an n-dimensional proper fraction.

We will denote by $\overline{\mathbb{Q}_n^{\mathrm{prop}}}$ the union set of n-dimensional proper fractions and $\{\infty\}$. For n-dimensional proper fractions, Ashikaga defined the ***i*-th remainder map** $R_i : \overline{\mathbb{Q}_n^{\mathrm{prop}}} \to \overline{\mathbb{Q}_n^{\mathrm{prop}}}$ and the remainder polynomial [1]. The i-th remainder map is defined by the following. If $a_i \neq 0$, then

$$R_i\left(\frac{(a_1, \ldots, a_n)}{r}\right) = \frac{(\overline{a_1}^{a_i},\ \ldots,\ \overline{a_{i-1}}^{a_i},\ \overline{-r}^{a_i},\ \overline{a_{i+1}}^{a_i},\ \ldots, \overline{a_n}^{a_i})}{a_i}$$

where $\overline{x}^{a_i} \equiv x \pmod{a_i}$ with $0 \leq \overline{x}^{a_i} \leq a_i - 1$. If $a_i = 0$, then $R_i\left(\frac{(a_1, \ldots, a_n)}{r}\right) = \infty$.

The remainder map describes the remainder of the division for one component of the n-dimensional proper fraction. The remainder polynomial is defined by repeatedly acting this map. Let $\frac{\mathbf{a}}{r}$ be an n-dimensional proper fraction, and let $\mathbf{I} = \{1, 2, \dots, n\}$ signify the index set of the variables.

Definition 1.2. The **remainder polynomial** $\mathcal{R}_*\left(\frac{\mathbf{a}}{r}\right) \in \overline{\mathbb{Q}_n^{\mathrm{prop}}}\langle x_1, \dots, x_n\rangle$ is defined by

$$\mathcal{R}_*\left(\frac{\mathbf{a}}{r}\right) = \frac{\mathbf{a}}{r} + \sum_{\substack{(i_1, i_2, \dots, i_l) \in \mathbf{I}^l \\ l \geq 1}} (R_{i_l} \cdots R_{i_2} R_{i_1}) \left(\frac{\mathbf{a}}{r}\right) x_{i_1} \cdots x_{i_l}$$

where we exclude terms with coefficient $\frac{(0,\dots,0)}{1}$ and ∞.

Multidimensional continued fractions consist of two polynomials. Another polynomial which is called **the round down polynomial** is introduced Section 3. The remainder polynomial (resp. The round down polynomial) indicates the types of the quotient singularities (resp. the $\mathbb{Z}^{n-1}$-weight) which appear in each step of the *Fujiki-Oka resolution* [14, 34]. The definition of the Fujiki-Oka resolution is also introduced in Section 3.

Our first main result is a necessary and sufficient condition for the Fujiki-Oka resolutions of Gorenstein abelian quotient singularities to be crepant in all dimensions. This is a joint work with Kohei Sato[38]. We shall show that this condition can be expressed by the coefficients of the remainder polynomials as follows.

Theorem 1.3. (Theorem 4.1.) *For a cyclic quotient singularity of $\frac{1}{r}(1, a_2, \dots, a_n)$-type, the Fujiki-Oka resolution is crepant if and only if the ages of all the coefficients of the corresponding remainder polynomial $\mathcal{R}_*\left(\frac{(1, a_2, \dots, a_n)}{r}\right)$ are* 1.

In Section 4.4, we introduce an extension of the Fujiki-Oka resolutions to the abelian case, which is named the *iterated Fujiki-Oka resolutions*. By using them, we shall generalize this theorem to the abelian case.

Proposition 1.4. (Theorem 4.14.) *Let $\widetilde{Y_{H_1}}, \widetilde{Y_{H_2}}, \dots, \widetilde{Y_{H_k}} = \widetilde{Y_G}$ be a sequence of iterated Fujiki-Oka resolutions for an n-dimensional Gorenstein abelian quotient singularity $\mathbb{C}^n/G$. The iterated Fujiki-Oka resolution $\widetilde{Y_G}$ is a crepant resolution of $\mathbb{C}^n/G$ if and only if the ages of all the coefficients in the remainder polynomials associated with every $\widetilde{Y_{H_i}}$ $(i = 1, \dots, k)$ are* 1.

As a corollary of this proposition, we have the following result of the existence of crepant resolutions of three-dimensional Gorenstein abelian quotient singularities.

Corollary 1.5. (Corollary 4.15.) *Any three-dimensional Gorenstein abelian quotient singularity possesses a crepant iterated Fujiki-Oka resolution.*

The proof of this corollary is an alternative proof of the existence of crepant resolutions for the three-dimensional Gorenstein abelian quotient singularities, and that needs only simple computations compared with known results.

Our second result is the property of complete coprime Fujiki-Oka resolutions.

Definition 1.6. The remainder polynomial $\mathcal{R}_*$ is complete coprime if arbitrary coefficients $\frac{(b_1,\dots,b_n)}{b_0}$ of $\mathcal{R}_*$ satisfy $\mathrm{GCD}(b_i, b_j) = 1$ for all $i \neq j$. Moreover, the Fujiki-Oka resolution is complete coprime if it is obtained by a complete coprime remainder polynomial $\mathcal{R}_*$.

In Section 5, we classify the type of cyclic quotient singularity which has a complete coprime Fujiki-Oka resolution. In this case, the Fujiki-Oka resolution is a Hilbert basis resolution, and there is one to one correspondence between exceptional divisors of this resolution and the multidimensional continued fraction. Let G (resp. G') be a cyclic group of type $\frac{1}{r}(a,b)$ (resp. $\frac{1}{r}(a,b,1\dots,1)$). We denote by Y_G and $Y_{G'}$ the minimal resolution of $\mathbb{C}^2/G$ and the Fujiki-Oka resolution of $\mathbb{C}^n/G'$, respectively. Then the following holds.

Theorem 1.7. (Theorem 5.11.) *Under the above assumption, further assume that the Fujiki-Oka resolution of $\mathbb{C}^n/G'$ is complete coprime. Then the Fujiki-Oka resolution is a Hilbert basis resolution. Moreover, there is a one-to-one correspondence between exceptional divisors of Y_G and exceptional divisors of $Y_{G'}$.*

In addition, Section 5.3 shows several examples of complete coprime cyclic quotient singularities which satisfy the Euler number of the Fujiki-Oka resolution equal to the order of G. This is a kind of the generalized McKay correspondence.

Theorem 1.8. (Theorem 5.14.) *Let H be of type $\frac{1}{r}(1, r-n+1)$ with $r = (n-1)k+1$ where r, n, k are some positive integers. For $h = \frac{1}{r}(a,b) \in H$, we have a two dimensional proper fraction $(a,b)/r$. If $\mathcal{R}_*((a,b)/r)$ is complete coprime, then the Euler number of the Fujiki-Oka resolution of $\mathbb{C}^n/G$ is the order of G, where G is of type $\frac{1}{r}(a,b,1^{n-2})$.*

Our last main result is characterize binary trees which gives the Fujiki-Oka resolution of the above two series of cyclic quotient singularities $\frac{1}{r}(1,a,r-a-1)$ and $\frac{1}{r}(1,a,r-a)$.

For two-dimensional proper fractions, the remainder polynomial can be represented by a binary tree (see Section 6). $\mathbf{T}_{\frac{(a,b)}{r}}$ denotes the binary tree which is obtained from the remainder polynomial $\mathcal{R}_*\left(\frac{(a,b)}{r}\right)$. We call the tree $\mathbf{T}_{\frac{(a,b)}{r}}$ terminal (resp. Gorenstein canonical) if $a+b=r$ (resp. $a+b+1=r$). This terminology comes from the fact that the quotient singularity of type $\frac{1}{r}(1,a,b)$ with $a+b=r$ (resp. $a+b+1=r$) is terminal (resp. Gorenstein canonical). In the following theorem, we will denote by $\mathbf{T}_x$ the binary tree whose topmost node is x. Our result is:

Theorem 1.9. (Theorem 6.12) *Let* $\mathbf{T}$ *be a full binary tree. Let* x_1 *be an arbitrary node which has a parent node* x*, a sibling node* x_2 *and a nephew node* y*. Then* $\mathbf{T}$ *is terminal if and only if* $\mathbf{T}$ *satisfies the following conditions.*

(i) *A sibling node of a leaf is a leaf.*

(ii) *If* $\mid \mathbf{T}_{x_1} \mid = \mid \mathbf{T}_{x_2} \mid$*, then* $\mathbf{T}_{x_1} = \mathbf{T}_{x_2} = \mathbf{T}_{\frac{(0,0)}{1}}$.

(iii) *If* $\mid \mathbf{T}_{x_1} \mid < \mid \mathbf{T}_{x_2} \mid$*, then* $\mathbf{T}_{x_1} = \mathbf{T}_y$.

This theorem characterizes the shape of a terminal tree. Section 6 shows the Gorenstein canonical tree version of this theorem. In addition, this result gives a condition for combining two terminal trees. Let Σ_1 (resp. Σ_2) denote the fan which gives the Fujiki-Oka resolution of the cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$ (resp. $\frac{1}{3}(1,2,1)$).

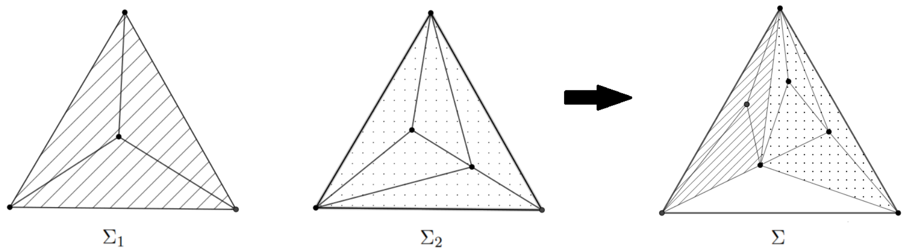
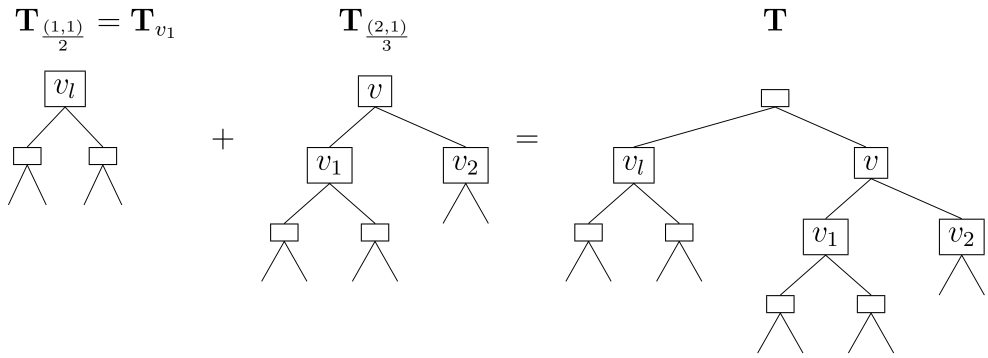


Fig. 1: Fujiki-Oka resolutions for type $\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,2,1)$ and $\frac{1}{5}(1,2,3)$.

By combining two fans as in Figure 1, we obtain a new fan Σ which gives the Fujiki-Oka resolution of the quotient singularity of type $\frac{1}{5}(1,2,3)$. In general, we can't determine the type of this quotient singularity. However, since the binary tree $\mathbf{T}$ which is obtained by combining $\mathbf{T}_{\frac{(1,1)}{2}}$ and $\mathbf{T}_{\frac{(2,1)}{3}}$ satisfies the conditions in Theorem 1.9, this tree is terminal. In this case, the type of the quotient singularity is determined by denominators 2 and 3, that is, it is the type of $\frac{1}{2+3}(1,2,3)$ (the first component is always one). By combining two terminal trees, we get a new economic resolution from two economic resolutions for terminal quotient singularities.



At the end of this section, we will introduce how this paper is organized. In Section 2, we recall some definitions and properties of cyclic quotient singularities as toric varieties. In particular, we introduce a crepant resolution and a Hilbert basis resolution. Section 3 explains multi-dimensional continued fractions and Fujiki-Oka resolutions. The definitions of the proper fraction, the remainder map, and the remainder polynomial are introduced in this section. In addition, we summarize the Fujiki-Oka resolution. In Section 4, we will show the our first main result which is the condition for the Fujiki-Oka resolution to be crepant. This section contains some application of our first result. We will define the iterated Fujiki-Oka resolution and prove any three-dimensional Gorenstein abelian quotient singularity possesses a crepant Fujiki-Oka resolution. Section 5 defines the complete coprime quotient singularities and shows that the Fujiki-Oka resolution of this singularity coincides with Hilbert basis resolution. In section 6, we will characterize binary trees which gives the Fujiki-Oka resolution of the above two series of cyclic quotient singularities $\frac{1}{r}(1, a, r-a-1)$ and $\frac{1}{r}(1, a, r-a)$.

2 Quotient singularities

In this section, we construct certain crepant resolutions of the quotient singularity $\mathbb{C}^n/G$, for G is an abelian group using the methods of toric variety.

Most of the necessary fact in toric geometry can be found Oda [33], and the facts about Hilbert basis resolution based on Bouvier and Gonzalez-Sprinberg[2].

2.1 Notations from Toric Geometry

The purpose of this section is to introduce some basic notions of toric geometry. Let G be a finite abelian subgroup of $GL(n, \mathbb{C})$ of order r. Then all elements in G are simultaneously diagonalizable. Therefore, any element in G can be written as the form $g = \mathrm{diag}(e^{\frac{2a_1\pi\sqrt{-1}}{r}}, \ldots, e^{\frac{2a_n\pi\sqrt{-1}}{r}})$ where $1 \leq i \leq n$ and $0 \leq a_i < r$. For simplicity, the matrix $\mathrm{diag}(e^{\frac{2a_1\pi\sqrt{-1}}{r}}, \ldots, e^{\frac{2a_n\pi\sqrt{-1}}{r}})$ is denoted by $\frac{1}{r}(a_1, \ldots, a_n)$.

Let N be a free $\mathbb{Z}$-module of rank n and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be a fixed basis of N. If the convex hull $\mathrm{Conv}\{\boldsymbol{0}, \boldsymbol{n}\}$ contains no elements in N except $\boldsymbol{0}$ and $\boldsymbol{n}$, the element $\boldsymbol{n} \in N$ is said to be *primitive*. For $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_k \in N$, the subset $\tau = \mathbb{R}_{\geq 0}\boldsymbol{n}_1 + \cdots + \mathbb{R}_{\geq 0}\boldsymbol{n}_k \subset N_{\mathbb{R}}$ satisfying $\tau \cap (-\tau) = \boldsymbol{0}$ is called a *rational strongly convex polyhedral cone* where $\mathbb{R}_{\geq 0}$ is the set of all non negative elements in $\mathbb{R}$. For simplicity, τ also signifies the finite fan consists of all faces of τ. The *dimension* of a cone τ is defined as the dimension of $\mathbb{R} \cdot \tau$ as vector space over $\mathbb{R}$. If the dimension of a cone τ is n, then the cone is said to be *maximal*. Let $\sigma = \mathbb{R}_{\geq 0}\boldsymbol{e}_1 + \cdots + \mathbb{R}_{\geq 0}\boldsymbol{e}_n \subset N_{\mathbb{R}}$. The toric variety $X(N, \sigma)$ determined by N and the finite fan σ is isomorphic to $\mathbb{C}^n$. There exists a morphism of toric varieties $\phi_T : X(N, \sigma) \to X(N_G, \sigma)$ corresponding to the quotient map $\phi : \mathbb{C}^n \to \mathbb{C}^n/G$ where N_G is the free $\mathbb{Z}$-module of rank n satisfying $N \subset N_G$ and $N_G/N \cong G$ as groups. Therefore, there is an element $\bar{g} = \frac{1}{r}(a_1, \ldots, a_n) \in N_G$ for each $g \in G$. We set N_G as the following:

$$N_G = N + \sum_{\frac{1}{r}(a_1,\ldots,a_n)\in G} \frac{1}{r}(a_1, \ldots, a_n)\mathbb{Z},$$

and $\bar{g} = \frac{1}{r}(a_1, \ldots, a_n) \in N_G$ maps to $g = \frac{1}{r}(a_1, \ldots, a_n) \in G$ by the composition of the quotient map and the isomorphism from N_G to G. We note that $N_{G,\mathbb{R}}$ satisfies $N_{G,\mathbb{R}} = N_G \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.1. Define the *age* of an element $g = \frac{1}{r}(a_1, \ldots, a_n) \in G$ to be

$$\mathrm{age}(g) = \frac{1}{r}\sum_{i=1}^{n} a_i.$$

Similarly, we define the *age* of an element $\bar{g} = \frac{1}{r}(a_1, \ldots, a_n) \in N_G$ to be

$$\mathrm{age}(\bar{g}) = \frac{1}{r}\sum_{i=1}^{n} a_i.$$

Definition 2.2. Let $g \in G$ and I_G be the unit of G. Then, the rank

$$\mathrm{rank}(g - I_G)$$

is called the *height* of g and denoted by $\mathrm{ht}(g)$.

Proposition 2.3. ([3, Prop. 5.2.]) Let $g \in G$. The following formula holds.

$$\mathrm{ht}(g) = \mathrm{ht}(g^{-1}) = \mathrm{age}(g) + \mathrm{age}(g^{-1}).$$

We shall recall the definition of a crepant resolution in matters of toric geometry. If a fan Σ subdivides the fan σ, then we have a birational map $f : X(N_G, \Sigma) \to X(N_G, \sigma)$, and the following relation holds between the canonical divisors:

$$K_{X(N_G,\Sigma)} = f^*(K_{X(N_G,\sigma)}) + \sum_{\tau\in\Sigma(1)} a_\tau D_\tau,$$

where D_τ is an exceptional divisor corresponding to the one dimensional cone $\tau \in \Sigma(1)$ in Σ and $a_\tau = \mathrm{age}(A_\tau) - 1$, where A_τ is the primitive element in τ. The rational number a_τ is called the *discrepancy* of D_τ.

Remark 2.4. Let Σ be a subdivision of σ by using only lattice points whose ages are 1. If the toric variety $X(N_G, \Sigma)$ is smooth, then $X(N_G, \Sigma)$ is a crepant resolution of $\mathbb{C}^n/G$.

The convex hull $\mathfrak{s}_G \subset N_{G,\mathbb{R}}$ spanned by $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ is called the *junior simplex*. An element in the junior simplex is called a *junior element*. By Remark 2.4, a crepant resolution $X(N_G, \Sigma)$ can be identified with a basic triangulation of $\mathfrak{s}_G$ by using points in N_G. As for this fact, a necessary condition for the Gorenstein abelian quotient singularities to admit a crepant resolution via Hilbert basis is known as follows.

Definition 2.5. ([8, p.11]) Let $\mathrm{Hlb}_{N_G}(\sigma)$ be as follows:

$$\mathrm{Hlb}_{N_G}(\sigma) = \left\{ n \in \sigma \cap (N_G \backslash \{0\}) \;\middle|\; \begin{array}{c} n \text{ can not be expresed as} \\ \text{the sum of two other vectors} \\ \text{belonging to } \sigma \cap (N_G \backslash \{0\}) \end{array} \right\}.$$

The set $\mathrm{Hlb}_{N_G}(\sigma)$ is called the *Hilbert basis* of σ with reference to the lattice N_G.

Theorem 2.6. ([8, pp. 30–31]) *Let $\mathbb{C}^r/G$ be a Gorenstein abelian quotient singularity. If $\mathfrak{s}_G$ has a basic triangulation, then*

$$\mathrm{Hlb}_{N_G}(\sigma) = \mathfrak{s}_G \cap N_G,$$

i.e., each of the members of the Hilbert basis of σ has to be either a junior element or a vertex of $\mathfrak{s}_G$.

2.2 Hilbert basis resolution

The exceptional divisors corresponding to the Hilbert basis play a very important role in the toric version of the "Nash problem", and it is known that those divisors are *essential divisors* over X (see [2, 20]). An *essential divisor* over X is an exceptional divisor of which the center on Y is an irreducible component of $f^{-1}(\mathrm{Sing}X)$ for every resolution $f : Y \to X$. If $\rho \in \Delta(1)$ is a ray, then there exists a primitive vector $n(\rho) \in N_G \cap \rho$ with $\rho = \mathbb{R}_{\geq 0} n(\rho)$. The set of minimal generators of $\sigma \in \Delta$ is defined by

$$\mathrm{Gen}(\sigma) := \{n(\rho) \mid \rho \in \Delta(1),\ \rho \prec \sigma\}.$$

For Δ, we define analogously $\mathrm{Gen}(\Delta) := \bigcup_{\sigma \in \Delta} \mathrm{Gen}(\sigma)$.

Definition 2.7. *The subdivision Δ of σ is called a Hilbert basis resolution of σ if Δ satisfies the following conditions:*

- Δ *is smooth.*
- $\mathrm{Gen}(\Delta) = \mathbf{Hilb}_{N_G}(\sigma)$.

The Hilbert basis resolution is a resolution of which all exceptional divisors are essential, and it is called **Hilb**-*desingularization* in [6, 7], and also *G-désingularization* in [2]. The previous works on Hilbert basis resolutions related to this paper are as follows.

- For two-dimensional toric singularity (i.e. cyclic quotient singularity $\mathbb{C}^2/G$), the minimal resolution is a Hilbert basis resolution.
- In dimension three, Bouvier and Gonzalez-Sprinberg shows existence of Hilbert basis resolutions [2]. However, it is not necessarily unique.
- They give an example of singularity in dimension four which has no Hilbert basis resolutions [2].
- If a toric quotient singularity in any dimension has a toric crepant resolution, then it is an Hilbert basis resolution [7].
- For three-dimensional terminal quotient singularities, Danilov [12] and Reid [35] introduce the economic resolution which is obtained by a sequence of weighted blow-ups. It coincides with an Hilbert basis resolution. We will introduce in Section 6.

2.3 Weighted blow-ups

In this subsection, we introduce the weighted blow-ups for cyclic quotient singularities. We fix a primitive lattice point $v = \frac{1}{r}(a_1, \dots, a_n)$. Assume that v ganerates N_G/N. We consider the subdivision of $\sigma = \mathrm{Cone}(e_1, \dots, e_n)$ at v. Namely, let σ_k denote a n-dimensional cone $\mathrm{Cone}(e_1, \dots, \hat{e_k}, v, \dots, e_n)$ for $k = 1, \dots, n$, and Σ denote the fan consisting of these cones and their all faces. The subdivision $\Sigma \to \sigma$ is called the star subdivision. In addition, we call the induced toric morphism $f : X(N_G, \Sigma) \to X(N_G, \sigma)$ "**the weighted blow-up**" of $X(N_G, \sigma)$ with weight $\frac{1}{r}(a_1, \dots, a_n)$.

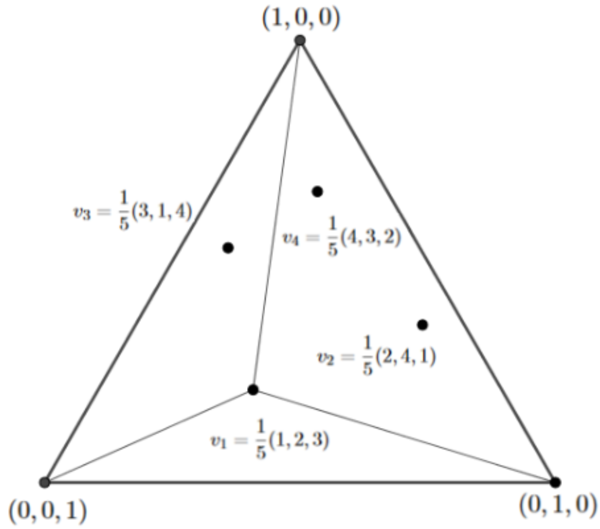
Example 2.8. Let G be the cyclic group of type $\frac{1}{5}(1, 2, 3)$, and $N_G = \mathbb{Z}^3 + \frac{1}{5}(1, 2, 3)\mathbb{Z}$. Then the fan Σ_{v_1} obtained by weighted blow-up with weight $v_1 = \frac{1}{5}(1, 2, 3)$ is the following.

Let $N_{G,k}$ be the sublattice of N_G which is generated by $e_1, \dots, \hat{e_k}, v, e_{k+1}, \dots, e_n$ for $k = 1, \dots, n$. Then the dual lattice $M_{G,k} := \mathrm{Hom}(N_{G,k}, \mathbb{Z})$ has dual basis $\{\xi_1, \dots, \xi_n\}$ which satisfy:

$$\xi_j = \begin{cases} x_j x_k^{-\frac{a_i}{a_k}} & \text{if } j \neq k, \\ x_k^{\frac{r}{a_k}} & \text{if } j = k. \end{cases}$$

The affine toric variety $X(N_G, \sigma_k)$ has a cyclic quotient singularity of type $\frac{1}{a_k}(a_1, \dots, a_{k-1}, -r, a_{k-1}, \dots, a_n)$ with coordinates $\{\xi_1, \dots, \xi_n\}$. Note that this singularity type is obtained by the image of the k-th remainder map $R_k\left(\frac{1}{r}(a_1, \dots, a_n)\right)$. If the lattice point $v' = R_k\left(\frac{1}{r}(a_1, \dots, a_n)\right)$ of $N_{G,k}$ generates $N_{G,k}/N$, then we can repeat the above operation.

The Fujiki-Oka resolution introduced in the next section is the obtained by repeating this weighted blow-up.



3 Multidimensional Continued fractions

3.1 Hirzebruch-Jung continued fractions

In this section, we shall mention relations between the minimal resolution of an A_n singularity and the Hirzebruch-Jung Continued Fraction obtained from the type of quotient singularities. Let $\mathbb{C}^2/G$ be a quotient singularity of $\frac{1}{r}(1,a)$-type where $a \in \mathbb{Z}$ and $r \in \mathbb{N}$ are coprime. The *Hirzebruch-Jung continued fraction* of $\frac{r}{a}$ is defined as follows:

$$\frac{r}{a} = x_1 - \frac{1}{x_2 - \frac{1}{x_3 - \cdots \frac{1}{x_s}}} = [x_1, \ldots, x_s]$$

where $x_1, \ldots, x_s \in \mathbb{Z}_{>0}$.

Let $\mathbb{C}^2/G \cong X(N_G, \sigma)$. We set that $\sigma = \mathbb{R}_{\geq 0}\boldsymbol{e}_1 + \mathbb{R}_{\geq 0}\boldsymbol{e}_2$, $N_G = \mathbb{Z}^2 + \frac{1}{r}(1,a)\mathbb{Z}$, $v_0 = \boldsymbol{e}_2$ and $v_{s+1} = \boldsymbol{e}_1$. The *Newton polygon* L is given as the convex hull of lattice points $(N_G \cap \sigma) \setminus \{(0,0)\}$ (see Fig. 2).

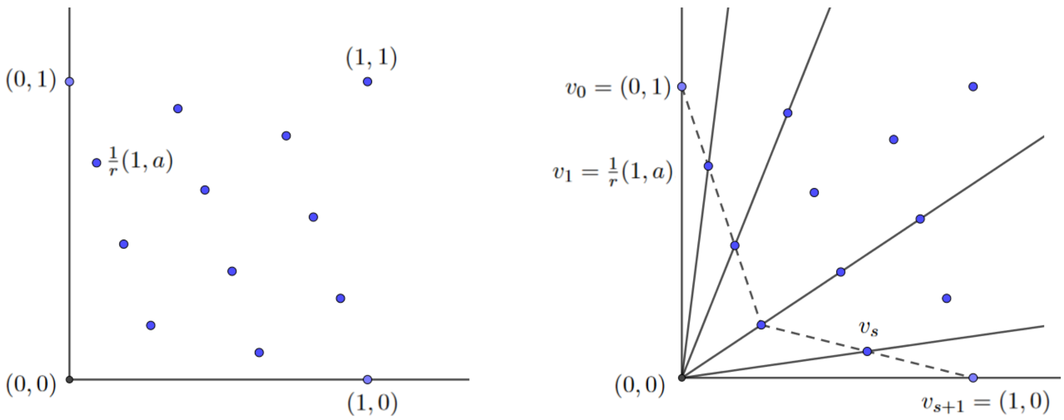


Fig. 2: The minimal resolution of $\mathbb{C}^2/G$ and the Newton Polygon

Let $X(N_G, \Sigma)$ be the minimal resolution of $\mathbb{C}^2/G$. The fan Σ is the subdivision of σ by the half lines from $(0, 0)$ to the primitive elements $v_0 = \boldsymbol{e}_2, v_1 = \frac{1}{r}(1, a), \ldots, v_{s+1} = \boldsymbol{e}_1$ in N_G. These elements are on the edge of L. Moreover, it is known that the following formula holds for the coordinates of these primitive elements and coefficients $x_1, \ldots, x_s$ which appear in the Hirzebruch-Jung continued fraction:

$$v_{i+1} + v_{i-1} = x_i v_i \quad (\ i = 1, \ldots, s).$$

Therefore, the coordinates of $v_1, \ldots, v_s$ can be computed from the Hirzebruch-Jung continued fractions concretely. Every exceptional divisor E_i of the minimal resolution corresponds to the primitive element v_i, and its self-intersection number is $-x_i$.

Example 3.1. If $a = 8$ and $r = 11$, then the Hirzebruch-Jung continued fraction is as follows:

$$\frac{11}{8} = 2 - \cfrac{1}{2 - \frac{1}{3 - \frac{1}{2}}} = [2, 2, 3, 2].$$

The following list is on the exceptional divisors of the minimal resolution of $X(N_G, \Sigma)$.

Exceptional Divisors	Primitive Elements in N_G	Self-Intersection Number
E_1	$v_1 = \frac{1}{11}(1, 8)$	-2
E_2	$v_2 = \frac{1}{11}(2, 5)$	-2
E_3	$v_3 = \frac{1}{11}(3, 2)$	-3
E_4	$v_4 = \frac{1}{11}(7, 1)$	-2

Conversely, type of a quotient singularity is given by series of coefficients of continued fraction. In the case of $[3, 2, 2]$, the given quotient singularity is of $\frac{1}{7}(1, 3)$-type.

3.2 Ashikaga continued fractions

We shall introduce a generalization of Hirzebruch-Jung continued fractions by Ashikaga [1]. This generalized continued fractions summarizes information of the *Fujiki-Oka resolution* (see [14, 34]) for *semi-isolated* quotient singularities (i.e., cyclic quotient singularities of $\frac{1}{r}(1, a_2, \ldots, a_n)$-type). The Fujiki-Oka resolution is a canonical resolution of any semi-isolated quotient singularity. We call this continued fraction *Ashikaga's continued fraction.*

Definition 3.2. Let n be a positive integer. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $r \in \mathbb{N}$ which satisfies $0 \leq a_i \leq r-1$ for $1 \leq i \leq n$. We call the symbol

$$\frac{\mathbf{a}}{r} = \frac{(a_1, \ldots, a_n)}{r} = (a_1, \ldots, a_n)/r$$

an *n-dimensional proper fraction.*

Definition 3.3. Define the *age* of an n-dimensional proper fraction $\frac{\mathbf{a}}{r} = \frac{(a_1,\ldots,a_n)}{r}$ to be

$$\operatorname{age}\left(\frac{\mathbf{a}}{r}\right) = \frac{1}{r}\sum_{i=1}^{n} a_i.$$

In the following, the symbol $\mathbb{Q}_n^{prop}$ (resp. $\overline{\mathbb{Q}_n^{prop}}$) means the set of n-dimensional proper fractions (resp. the set $\mathbb{Q}_n^{prop} \cup \{\infty\}$). Similarly, $\overline{\mathbb{Z}^n} = \mathbb{Z}^n \cup \{\infty\}$. Moreover, $\overline{\mathbb{Q}_n^{prop}}[x_2, \ldots, x_n]$ (resp. $\overline{\mathbb{Z}^n}[x_2, \ldots, x_n]$) denotes the set consisting of all noncommutative polynomials with $n-1$ variables over $\overline{\mathbb{Q}_n^{prop}}$ (resp. $\overline{\mathbb{Z}^n}$), and $\mathbf{I} = \{2, \ldots, n\}$ signifies the index set of the variables $x_2, \ldots, x_n$.

Ashikaga's continued fraction consists of a *round down polynomial* and a *remainder polynomial*, and these polynomials are obtained via *round down maps* and *remainder maps* for a *semi-unimodular proper fraction* (i.e., a proper fraction such that at least one component of $\mathbf{a}$ is 1). Roughly speaking, these maps are division for just one component of the vector $\mathbf{a}$ by r. In the following, we may assume that the first component of a semi-unimodular proper fraction is always 1 by changing coordinates.

Definition 3.4. ([1, Def 3.1]) Let $\frac{(1,a_2,\ldots,a_n)}{r}$ be a semi-unimodular proper fraction.

(*i*) For $2 \leq i \leq n$, the *i-th remainder map* $R_i : \overline{\mathbb{Q}_n^{prop}} \to \overline{\mathbb{Q}_n^{prop}}$ is defined by

$$R_i\left(\frac{(1, a_2, \ldots, a_n)}{r}\right) = \begin{cases} \frac{\left(\overline{1}^{a_i},\ \overline{a_2}^{a_i}, \ldots,\ \overline{a_{i-1}}^{a_i},\ \overline{-r}^{a_i},\ \overline{a_{i+1}}^{a_i}, \ldots,\ \overline{a_n}^{a_i}\right)}{a_i} & \text{if } a_i \neq 0 \\ \infty & \text{if } a_i = 0 \end{cases}$$

and $R_i(\infty) = \infty$ where $\overline{a_j}^{a_i}$ is an integer satisfying $0 \leq \overline{a_j}^{a_i} < a_i$ and $\overline{a_j}^{a_i} \equiv a_j$ modulo a_i.

(*ii*) For $2 \le i \le n$, the *i-th round down map* $Z_i : \overline{\mathbb{Q}_n^{prop}} \to \overline{\mathbb{Z}^n}$ is defined by

$$Z_i\left(\frac{(1,a_2,\ldots,a_n)}{r}\right) = \begin{cases} \left(\lfloor\frac{1}{a_i}\rfloor, \lfloor\frac{a_2}{a_i}\rfloor, \ldots, \lfloor\frac{a_{i-1}}{a_i}\rfloor, \lfloor\frac{-r}{a_i}\rfloor, \lfloor\frac{a_{i+1}}{a_i}\rfloor, \ldots, \lfloor\frac{a_n}{a_i}\rfloor\right) & \text{if } a_i \neq 0 \\ \infty & \text{if } a_i = 0 \end{cases}$$

and $Z_i(\infty) = \infty$ where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

Example 3.5. If $v = \frac{(1,2,5,7)}{8}$, then

$$\begin{aligned} Z_2(v) &= (0,-4,2,3), \\ Z_3(v) &= (0,0,-2,1), \\ R_2(v) &= \frac{(1,0,1,1)}{2} \text{ and} \\ R_3(v) &= \frac{(1,2,2,2)}{5}. \end{aligned}$$

Definition 3.6. [1, Def 3.2] Let $\frac{\mathbf{a}}{r}$ be an n-dimensional semi-unimodular proper fraction.

(i) The *remainder polynomial* $\mathcal{R}_*\left(\frac{\mathbf{a}}{r}\right) \in \overline{\mathbb{Q}_n^{prop}}[x_2,\ldots,x_n]$ is defined by

$$\mathcal{R}_*\left(\frac{\mathbf{a}}{r}\right) = \frac{\mathbf{a}}{r} + \sum_{(i_1,i_2,\ldots,i_l)\in\mathbf{I}^l,\, l\ge 1} (R_{i_l}\cdots R_{i_2}R_{i_1})\left(\frac{\mathbf{a}}{r}\right)\cdot x_{i_1}x_{i_2}\cdots x_{i_l}$$

where we exclude terms with coefficients ∞ or $\frac{(0,0,\ldots,0)}{1}$.

(ii) The *round down polynomial* $\mathcal{Z}_* \in \overline{\mathbb{Z}^n}[x_2,\ldots,x_n]$ is defined by

$$\mathcal{Z}_*\left(\frac{\mathbf{a}}{r}\right) = \sum_{j=2}^{n} Z_j\left(\frac{\mathbf{a}}{r}\right)x_j + \sum_{j=2}^{n}\sum_{(i_1,i_2,\ldots,i_l)\in\mathbf{I}^l,\, l\ge 1}(Z_jR_{i_l}\cdots R_{i_2}R_{i_1})\left(\frac{\mathbf{a}}{r}\right)\cdot x_{i_1}x_{i_2}\cdots x_{i_l}x_j.$$

Remark 3.7. In the case $n=2$, the series of the coefficients of $\mathcal{Z}_*\left(\frac{\mathbf{a}}{r}\right)$ coincides with the series of the coefficients of Hirzebruch-Jung continued fraction.

Example 3.8. Let $v = \frac{(1,2,8)}{11}$, then the remainder polynomial is

$$\begin{aligned} \mathcal{R}_*\left(\frac{(1,2,8)}{11}\right) = \frac{(1,2,8)}{11} &+ \frac{(1,1,0)}{2}x_2 + \frac{(1,2,5)}{8}x_3 \\ &+ \frac{(1,0,1)}{2}x_3x_2 + \frac{(1,2,2)}{5}x_3x_3 \\ &+ \frac{(1,1,0)}{2}x_3x_3x_2 + \frac{(1,0,1)}{2}x_3x_3x_3. \end{aligned}$$

The round down polynomial is

$$\begin{aligned} \mathcal{Z}_*\left(\frac{(1,2,8)}{11}\right) &= (0,-6,4)x_2 + (0,0,-2)x_3 \\ &+ (1,-4,2)x_3x_2 + (0,0,-2)x_3x_3 \\ &+ (0,-3,1)x_3x_3x_2 + (0,1,-3)x_3x_3x_3. \end{aligned}$$

3.3 Fujiki-Oka resolution

The remainder polynomial consists of datum of blow-up centers of a Fujiki-Oka resolution. In this subsection, we shall summarize Fujiki-Oka resolutions. For the details of Definition 3.9 and Lemma 3.10, see the articles written by Ashikaga [1] and Oka [34] respectively.

Definition 3.9. Let $P_1, \dots, P_n$ be primitive elements in N_G. If an n-dimensional cone $\tau = \mathbb{R}_{\geq 0}P_1 + \cdots + \mathbb{R}_{\geq 0}P_n$ in $N_{G,\mathbb{R}}$ has a smooth facet $\mathbb{R}_{\geq 0}P_1 + \cdots + \mathbb{R}_{\geq 0}P_{n-1}$, then we call the cone *semi-unimodular* over the vertex P_n.

If a cone τ is semi-unimodular over all vertices $P_1, \dots, P_n \in N_G$, then the toric variety $X(N_G, \tau)$ has an isolated singularity or no singularities. If the toric variety $X(N_G, \sigma)$ has a quotient singularity of $\frac{1}{r}(a_1, \dots, a_n)$-type satisfying $\mathrm{GCD}(r, a_i) = 1$, then σ is semi-unimodular over $\boldsymbol{e}_i$, and $X(N_G, \sigma)$ has a semi-isolated singularity.

Lemma 3.10. *Let an n-dimensional cone $\tau = \mathbb{R}_{\geq 0}P_1 + \cdots + \mathbb{R}_{\geq 0}P_n \subset N_{G,\mathbb{R}}$ be semi-unimodular over P_1 and $r = |\det(P_1, P_2, \cdots, P_n)|$. If $C \in \mathbb{Z}^n$ is a primitive element such that the n-dimensional cone $\mathbb{R}_{\geq 0}C + \mathbb{R}_{\geq 0}P_2 + \cdots + \mathbb{R}_{\geq 0}P_n$ is smooth, then there exist integers $0 \leq a_2, \dots, a_n \leq r-1$ such that*

$$C = \frac{P_1 + \sum_{i=2}^{n} a_i P_i}{r}.$$

This element $C \in N_G$ is called a *Oka center* of τ over P_1. The Oka center exists uniquely for a cone which is semi-unimodular over an element.

Lemma 3.11. ([14, Lemma 3]) *Suppose $(X, [0])$ is a cyclic quotient singularity of $\frac{1}{r}(a_1, a_2, \dots, a_n)$-type, where $\mathrm{GCD}(r, a_1, \dots, a_n) = 1$ and $a_1 a_2 \cdots a_n \neq 0$. Then there exist a variety $\widetilde{X}$, a finite affine open covering $\mathcal{U} = \{U_1, \dots, U_l\}$ of $\widetilde{X}$ for an integer $1 \leq l \leq n$, and a proper birational morphism $f : \widetilde{X} \to X$ such that U_i is the quotient singularity of $\frac{1}{r_i}(a_{i1}, a_{i2}, \dots, a_{in})$-type for each i, where the integers r_i and a_{ij} $(1 \leq j \leq n)$ are determined by the following formula:*

$$\begin{cases} r_i = a_i/d \ \textit{where}\ d = \mathrm{GCD}(a_1, \dots, a_n), \\ a_{ij} \equiv r_j \ \textit{modulo}\ r_i \ \textit{and}\ 0 \leq a_{ij} < r_i \ (j \neq i), \\ a_{ij} + r \equiv 0 \ \textit{modulo}\ r_i \ (j = i). \end{cases}$$

The proper fraction $\frac{\mathbf{a}}{r} = \frac{(1, a_2, \dots, a_n)}{r}$ obtained from Lemma 3.10 and Lemma 3.11 is called the *proper fraction of τ over P_1*.

Lemma 3.12. ([1]) *If a cone $\tau = \mathbb{R}_{\geq 0}P_1 + \cdots + \mathbb{R}_{\geq 0}P_n \subset N_{G,\mathbb{R}}$ contains a primitive element $C \in N_G$ in Lemma 3.10, then the toric variety $X(N_G, \tau)$ has a quotient singularity of $\frac{1}{r}(1, a_2, \dots, a_n)$-type.*

By Lemmas 3.10, 3.11 and 3.12, any semi-isolated quotient singularity is resolved by blow-ups with the Oka center repeatedly, and these toric resolutions are called *Fujiki-Oka resolutions.* Each coefficient which appears in a remainder polynomial coincides with the type of quotient singularities appearing in each step of the Fujiki-Oka resolution.

Lemma 3.13. ([1]) *Let* $\tau = \mathbb{R}_{\geq 0}P_1 + \cdots + \mathbb{R}_{\geq 0}P_n \subset N_{G,\mathbb{R}}$ *be a semi-unimodular cone over* P_1 *and* C *be the Oka center of* τ*. Then the cone*

$$\tau_i = \mathbb{R}_{\geq 0}P_1 + \cdots + \mathbb{R}_{\geq 0}P_{i-1} + \mathbb{R}_{\geq 0}c_i + \mathbb{R}_{\geq 0}P_{i+1} + \cdots + \mathbb{R}_{\geq 0}P_n$$

is semi-unimodular over P_1 *and its Oka center is*

$$c_i = \frac{\sum_{j \neq i,n} \overline{a_j}^{a_i} P_j + \overline{-d}^{a_i} P_i}{a_i}.$$

By Lemma 3.13, each remainder polynomial can be understood as a overview of a Fujiki-Oka resolution of a semi-isolated quotient singularity.

Assume that $G = \frac{1}{r}(1, a_2, \ldots, a_n)$. We consider the consecutive star subdivision, which starts with subdivision at v. Then the type of quotient singularities appearing at each stage of star subdivisions is obtained from the remainder polynomial $\mathcal{R}_*\left(\frac{1}{r}(1, a_2, \ldots, a_n)\right)$. The induced toric morphism is called the Fujiki-Oka resolution (see [1] for more detail).

Example 3.14. Let $X(N_G, \sigma)$ have a quotient singularity of $\frac{1}{11}(1, 2, 8)$-type, i.e., $N_G = \mathbb{Z}^3 + \frac{1}{11}(1, 2, 8)\mathbb{Z}$ and $\sigma = \mathbb{R}_{\geq 0}\boldsymbol{e}_1 + \mathbb{R}_{\geq 0}\boldsymbol{e}_2 + \mathbb{R}_{\geq 0}\boldsymbol{e}_3$. Then, the cone σ is semi-unimodular over $\boldsymbol{e}_1$, and the Oka center is $c = \frac{1}{11}(1, 2, 8)$, and the remainder polynomial of the proper fraction $\frac{(1,2,8)}{11}$ is as follows:

$$\begin{aligned}
\mathcal{R}_*\left(\frac{(1,2,8)}{11}\right) = \frac{(1,2,8)}{11} \quad &+ \quad \frac{(1,1,0)}{2}x_2 + \frac{(1,2,5)}{8}x_3 \\
&+ \quad \frac{(1,0,1)}{2}x_3x_2 + \frac{(1,2,2)}{5}x_3x_3 \\
&+ \quad \frac{(1,1,0)}{2}x_3x_3x_2 + \frac{(1,0,1)}{2}x_3x_3x_3.
\end{aligned}$$

This expanding of Ashikaga's continued fraction indicates that the toric variety after the blow-up with the Oka center $\frac{1}{11}(1, 2, 8)$ has two semi-isolated quotient singularities of $\frac{1}{2}(1, 1, 0)$-type and $\frac{1}{8}(1, 2, 5)$-type. For these quotient singularities, the corresponding cones which appear in σ after the subdivision by $\frac{1}{11}(1, 2, 8) \in N_G$ are $\sigma_2 = \mathbb{R}_{\geq 0}\boldsymbol{e}_1 + \mathbb{R}_{\geq 0}c + \mathbb{R}_{\geq 0}\boldsymbol{e}_3$ and $\sigma_3 = \mathbb{R}_{\geq 0}\boldsymbol{e}_1 + \mathbb{R}_{\geq 0}\boldsymbol{e}_2 + \mathbb{R}_{\geq 0}c$ respectively. $\frac{1}{2}(1, 1, 0)$ and $\frac{1}{8}(1, 2, 5)$ are the Oka center of semi-unimodular cones σ_2, σ_3 over $\boldsymbol{e}_1$ respectively. Therefore, we can take blow-ups with the Oka centers again. The blow-up with Oka centers of $X(N_G, \sigma_3)$ consists a smooth toric variety and quotient singularities of $\frac{1}{2}(1, 0, 1)$-type and $\frac{1}{5}(1, 2, 2)$-type respectively. By repeating blow-ups with Oka centers, we have the smooth toric variety (see Fig. 3).

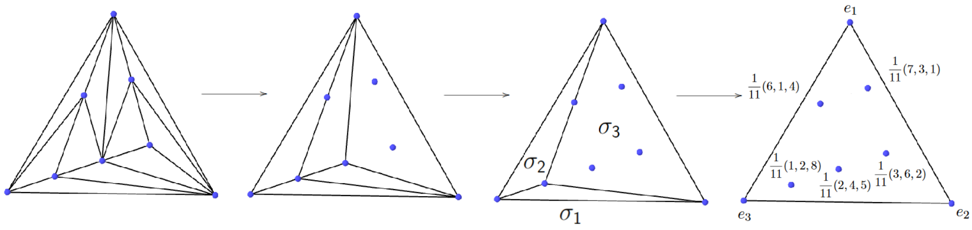


Fig. 3: The basic triangulation of $\mathfrak{s}_G$ by Fujiki-Oka resolution

Let $X(N_G, \sigma)$ have a semi-isolated quotient singularity. The cone $\sigma \subset N_{G,\mathbb{R}}$ can be semi-unimodular over $\boldsymbol{e}_1$ by exchanging basis of N_G. For a semi-unimodular cone σ over $\boldsymbol{e}_1$, we call the terminal smooth fan obtained from its Fujiki-Oka resolution the *continued fraction fan*, and that fan is denoted as $\mathrm{CFF}_{\boldsymbol{e}_1}(\sigma)$ or, more simply, $\mathrm{CFF}(\sigma)$. Clearly, there exists at least one $\mathrm{CFF}(\sigma)$ for a semi-unimodular cone σ.

As seen above, the remainder polynomial controls the Fujiki-Oka resolution. On the other hand, the round down polynomial gives the $\mathbb{Z}^{n-1}$-weight of $(n-1)$-dimensional cone. For simplicity, we treat only $\mathbb{Z}^2$-weight in this paper.

Definition 3.15. Let $\sigma_1 = \mathrm{Cone}(v, v', v_1)$ and $\sigma_2 = (v, v', v_2)$ be three-dimensional smooth cone. Then the two-dimensional common face $\tau = \mathrm{Cone}(v, v')$ has a $\mathbb{Z}^2$-weight $(\alpha, \beta) \in \mathbb{Z}^2$ which satisfies

$$\alpha \cdot v + \beta \cdot v' + v_1 + v_2 = (0, 0, 0).$$

Note that a $\mathbb{Z}^2$-weight gives a self intersection number of a curve $X(N, \tau)$ on $X(N, \mathrm{Cone}(v))$ and $X(N, \mathrm{Cone}(v'))$.

Proposition 3.16. ([1, Lemma5.1]) *Let $\sigma = \mathrm{Cone}(v_1, v_2, v_3)$ and $v = \frac{1}{r}(v_1, a \cdot v_2, b \cdot v_3)$. After a star subdivision at v, we have two-dimensional cones $\tau_2 = \mathrm{Cone}(v, v_2)$ and $\tau_3 = \mathrm{Cone}(v, v_3)$. The $\mathbb{Z}^2$-weighting of τ_i coincides with the image of the i-th round-down map $Z_i\left(\frac{1}{r}(a, b)\right)$ for $i = 2, 3$.*

4 Crepant property of Fujiki-Oka resolution

4.1 Sufficient condition of crepant resolution

The purpose of this subsection is to show a sufficient condition of existence of a crepant resolution of semi-isolated cyclic quotient singularities. In particular, all isolated cyclic quotient singularities are included in this case.

Theorem 4.1. *For a cyclic quotient singularity of $\frac{1}{r}(1, a_2, \ldots, a_n)$-type, the Fujiki-Oka resolution is crepant if and only if the ages of all the coefficients of the corresponding remainder polynomial $\mathcal{R}_*\left(\frac{(1,a_2,\ldots,a_n)}{r}\right)$ are 1.*

Let $G = \left\langle \frac{1}{r}(1, a_2, \dots, a_n) \right\rangle = \langle g \rangle$, we will denote by G_i the cyclic group which is generated by g_i, where g_i is determined by the image of the i-th remainder map

$$\mathcal{R}_i \left(\frac{(1, a_2, \dots, a_n)}{r} \right) = \frac{(1, \overline{a_2}^{a_i}, \dots, \overline{a_{i-1}}^{a_i}, \overline{-r}^{a_i}, \overline{a_{i+1}}^{a_i}, \dots, \overline{a_n}^{a_i})}{a_i},$$

i.e., $\mathbb{C}^n/G_i$ is the cyclic quotient singularity of $\frac{1}{a_i}(1, \overline{a_2}^{a_i}, \dots, \overline{a_{i-1}}^{a_i}, \overline{-r}^{a_i}, \overline{a_{i+1}}^{a_i}, \dots, \overline{a_n}^{a_i})$-type. We first show that the generator $g = \frac{1}{r}(1, a_2, \dots, a_n)$ satisfies $\mathrm{age}(g) = 1$ if $\mathbb{C}^n/G$ has a crepant resolution.

Proposition 4.2. *Assume that $1 + a_2 + \cdots + a_n \geq 2r$ for $G = \left\langle \frac{1}{r}(1, a_2, \dots, a_n) \right\rangle$. Then, $\mathbb{C}^n/G$ has no toric crepant resolutions.*

Proof. Assume that $\mathbb{C}^n/G \cong X(\sigma, N_G)$ has a toric crepant resolution $X(\Sigma, N_G)$. Since G has the generator of which the first component is $\frac{1}{r}$, we see that there are no lattice points on $\mathfrak{s}_G \cap \tau_1$ where τ_1 is an $n-1$ dimensional cone with vertices $\boldsymbol{e}_2, \dots, \boldsymbol{e}_n$. Therefore, there is a lattice point $\boldsymbol{q} \in N_G$ such that $\mathrm{age}(\boldsymbol{q}) = 1$ and $\mathrm{Cone}(\boldsymbol{q}, \boldsymbol{e}_2, \boldsymbol{e}_3, \dots, \boldsymbol{e}_n) \in \Sigma$. Since G is generated by $\frac{1}{r}(1, a_2, \dots, a_n)$ with $1 + a_2 + \cdots + a_n \geq 2r$, we can write $\boldsymbol{q} = \frac{1}{r}(i, \overline{a_2 i}^r, \overline{a_3 i}^r, \dots, \overline{a_n i}^r)$ where $i \neq 1$. Thus, we have $\{\boldsymbol{q}, \boldsymbol{e}_2, \boldsymbol{e}_3, \dots, \boldsymbol{e}_n\}$ as $\mathbb{Z}$-basis of N_G, so there exist integers $k_1, \dots, k_n \in \mathbb{Z}$ such that

$$\boldsymbol{p} = \frac{1}{r}(1, a_2, \dots, a_n) = k_1 \boldsymbol{q} + \sum_{j=2}^{n} k_j \boldsymbol{e}_j.$$

We now turn to the first component. This formula gives $\frac{1}{r} = \frac{k_1 i}{r}$, but it contradicts $i \neq 1$. Therefore, if $1 + a_2 + \cdots + a_n \geq 2r$, then $\mathbb{C}^n/G$ has no crepant resolutions. $\square$

Proposition 4.3. *Let $G = \left\langle \frac{1}{r}(1, a_2, \dots, a_n) \right\rangle$ with $1 + a_2 + \cdots + a_n = r$. If $\mathbb{C}^n/G_i$ have a crepant resolution of all $i = 2, \dots, n$, then $\mathbb{C}^n/G$ have a crepant resolution.*

Proof. Let $X(\Sigma_i, N_i)$ be a toric crepant resolution of $\mathbb{C}^n/G_i$ where $N_i = \mathbb{Z}^n + g_i\mathbb{Z}$ with canonical basis $\overline{\boldsymbol{e}_1} \dots, \overline{\boldsymbol{e}_n}$. For simplicity of notation, we write $N_{i\mathbb{R}}$ insteads of $N_i \otimes_{\mathbb{Z}} \mathbb{R}$. Fix a smooth cone in Σ_i, and write this cone $\sigma = \mathrm{Cone}(\overline{\boldsymbol{v}_1}, \dots, \overline{\boldsymbol{v}_n})$. For $\boldsymbol{x} = (x_1, \dots, x_n) \in N_{i\mathbb{R}}$, the map $\phi_i : N_{i\mathbb{R}} \hookrightarrow N_{G,\mathbb{R}}$ is defined as follows:

$$\phi_i(\boldsymbol{x}) = \left(x_1 + \frac{1}{r}x_i, x_2 + \frac{a_2}{r}x_i, \dots, x_{i-1} + \frac{a_{i-1}}{r}x_i, \frac{a_i}{r}x_i, x_{i+1} + \frac{a_{i+1}}{r}x_i, \dots, x_n + \frac{a_n}{r}x_i \right).$$

The proof will be divided into two steps. The first step is to check $\phi_i(\boldsymbol{x})$ satisfies $\mathrm{age}(\phi_i(\boldsymbol{x})) = 1$ for a point $\boldsymbol{x} \in N_{i\mathbb{R}}$ with $\mathrm{age}(\boldsymbol{x}) = 1$, the second step is to prove $\phi_i(\sigma) \subset N_{G,\mathbb{R}}$ is also smooth.

(i) If $\boldsymbol{x} = (x_1, \dots, x_n) \in N_i$ satisfies $\mathrm{age}(\boldsymbol{x}) = 1$, then $\mathrm{age}(\phi_i(\boldsymbol{x})) = x_1 + x_2 + \cdots + \hat{x}_i + \cdots + x_n + \frac{1}{r}(1 + a_2 + \cdots + a_n)x_i$. By assumption, we have $x_1 + \cdots + x_n = 1$ and $1 + a_2 + \cdots + a_n = r$. These formulae give $\mathrm{age}(\phi_i(\boldsymbol{x})) = x_1 + \cdots + x_n = 1$.

(ii) Since σ is smooth on N_i, $\{\overline{\boldsymbol{v}_1}, \ldots, \overline{\boldsymbol{v}_n}\}$ is a $\mathbb{Z}$-basis of N_i, namely genarates the canonical basis $\overline{\boldsymbol{e}_1}, \ldots, \overline{\boldsymbol{e}_n}$ of N_i and $\overline{\boldsymbol{q}} = \frac{1}{a_i}(1, \overline{a_2}^{a_i}, \ldots, \overline{-r}^{a_i}, \ldots, \overline{a_n}^{a_i})$. Let $\boldsymbol{v}_j$ denote $\phi_i(\overline{\boldsymbol{v}_j})$, then we have $\phi_i(\sigma) = \mathrm{Cone}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$. It is easy to see that $\phi_i(\boldsymbol{e}_j)$ and $\phi_i(\overline{\boldsymbol{q}})$ are generated by $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, where $j \in \{1, \ldots, n\} \backslash \{i\}$. To show V is a $\mathbb{Z}$-basis of N_G, it is sufficient to prove that $\boldsymbol{e}_i$ is generated by V. Let us denote by Q_z the quotient of z devided by a_i. We have

$$\boldsymbol{q} = \phi_i(\overline{\boldsymbol{q}}) = \frac{1}{r}(-Q_{-r}, Q_{a_2}r - a_2 Q_{-r}, \ldots, (-r) - a_i Q_{-r}, \ldots, Q_{a_n}r - a_n Q_{-r}),$$

and we get the formula

$$\boldsymbol{q} + Q_{-r}\boldsymbol{p} = \frac{1}{r}(0, Q_{a_2}r, \ldots, Q_{a_{i-1}}r, -r, Q_{a_{i+1}}r, \ldots, Q_{a_n}r).$$

Therefore, the following equation holds

$$\boldsymbol{e}_i = \boldsymbol{q} + Q_{-r}\boldsymbol{p} - \sum_{j \in \{1, \ldots, n\} \backslash \{i\}} Q_{a_j} \boldsymbol{e}_j.$$

This implies that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ generates $\boldsymbol{e}_i$. Thus, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a $\mathbb{Z}$-basis of N_G.

From (i) and (ii), we see that if $\mathbb{C}^n/G_i$ has a crepant resolution $X(\Sigma_i, N_i)$, then we have a fan on N_G corresponding to a crepant resolution of $\mathbb{C}^n/G$ by taking the union of all $\phi_i(\Sigma_i)$. □

Proof of Theorem 4.1: Assume that the ages of all coefficients of the remainder polynomial of $\frac{(1, a_2, \ldots, a_n)}{r}$ are equal to 1. By Proposition 4.3, whether $\mathbb{C}^n/G$ has a crepant resolution depends on whether $\mathbb{C}^n/G_i$ has a crepant resolution of all i. It is obvious that the order of G_i is less than the order of G. The repeated application of the remainder map enables us to get $G_{i_1 i_2 \cdots i_j} = \frac{1}{k}(1, c_2, c_3, \ldots, c_n)$ with $c_j \in \{0, 1\}$ for all j. Since $\mathrm{age}\left(\frac{1}{k}(1, c_2, c_3, \ldots, c_n)\right) = 1$, the Fujiki-Oka resolution of $\mathbb{C}^n/G_{i_1 i_2 \cdots i_j}$ is crepant. By the proof of Proposition 4.3, the Fujiki-Oka resolution of $\mathbb{C}^n/G$ is crepant. Conversely, if the Fujiki-Oka resolution of $\mathbb{C}^n/G$ is crepant, then $\mathrm{age}\left(\frac{1}{r}(1, a_2, a_3, \ldots, a_n)\right) = 1$ and $\mathrm{age}(g_i) = 1$ for $i = 2, \ldots, n$. Therefore, the ages of all the coefficients of the remainder polynomial are 1, which completes the proof.

The Gorenstein property of $\mathbb{C}^n/G_i$ comes from the property of the cyclic group G. We have the following lemma.

Lemma 4.4. *Assume that* $1 + a_2 + a_3 + \cdots + a_n = r$ *for* $G = \left\langle \frac{1}{r}(1, a_2, \ldots, a_n) \right\rangle$. *Then* $\mathrm{age}\left(\mathcal{R}_i\left(\frac{(1, a_2, \ldots, a_n)}{r}\right)\right)$ *is an integer.*

Proof. It is enough to prove in the case of $i = 2$. We have the equation $\mathcal{R}_2\left(\frac{(1,a_2,\ldots,a_n)}{r}\right) = \frac{(1,\overline{-r}^{a_2},\overline{a_3}^{a_2},\ldots,\overline{a_n}^{a_2})}{a_2}$. We claim that $1 + \overline{-r}^{a_2} + \overline{a_3}^{a_2} + \cdots + \overline{a_n}^{a_2}$ is divided by a_2. It is sufficient to show that $1 + (-r) + a_3 + a_4 + \cdots + a_n$ is divided by a_2. By the assumption $1 + a_2 + a_3 + \cdots + a_n = r$, we have $1 + (-r) + a_3 + a_4 + \cdots + a_n = a_2$. Therefore, age$\left(\mathcal{R}_2\left(\frac{(1,a_2,\ldots,a_n)}{r}\right)\right)$ is an integer. □

Lemma 4.4 and Theorem 4.1 lead to the following corollary.

Corollary 4.5. *For all three dimensional semi-isolated Gorenstein quotient singularities, the Fujiki-Oka resolutions are crepant.*

Proof. Let $G = \left\langle \frac{1}{r}(1,a,b)\right\rangle$ where $1+a+b = r$. we have $\mathcal{R}_2\left(\frac{(1,a,b)}{r}\right) = \frac{(1,\overline{-r}^a,\overline{b}^a)}{a}$, and the age of $\mathcal{R}_2\left(\frac{(1,a,b)}{r}\right)$ is an integer by Lemma 4.4. Clearly, $1 + \overline{-r}^a + \overline{b}^a < 2a$. So, the age of $\mathcal{R}_2\left(\frac{(1,a,b)}{r}\right)$ is equal to 1. Thus, the ages of all coefficients of $\mathcal{R}_*\left(\frac{(1,a,b)}{r}\right)$ are equal to 1. By Theorem 4.1, the Fujiki-Oka resolution $X(N_G, \mathrm{CFF}(\sigma))$ is crepant. □

4.2 First Existence Criterion via Continued Fractions

We will give the continued fraction version of Theorem 2.6.

Definition 4.6. The term with the variable $x_i \cdots x_i$ in a remainder polynomial is called *iterated* where $1 \leq i \leq n$, and the lattice point in N_G corresponding to the coefficient of iterated terms is also called to be *iterated*.

Every iterated point can be written as $\phi_i^{-1}(\frac{\mathbf{a}}{r}) \in N_G$ for the coefficient $\frac{\mathbf{a}}{r}$ of an iterated term.

We shall consider a relationship between iterated points and Hilbert basis, and apply the relationship to Theorem 2.6.

In the following, for a cyclic group $\left\langle \frac{1}{r}(1,a_2,\ldots,a_n)\right\rangle \subseteq \mathrm{SL}(n,\mathbb{C})$ satisfying $1 + a_2 + \cdots + a_n = r$, the symbol A_i denotes the cyclic subgroup $\left\langle \frac{1}{r}(1,a_i)\right\rangle \subset \mathrm{GL}(2,\mathbb{C})$ for $a_i \neq 0$, and $\boldsymbol{v}_{i_1},\ldots,\boldsymbol{v}_{i_s}$ denote the lattice points in $N_{A_i} = \mathbb{Z}^2 + \frac{1}{r}(1,a_i)\mathbb{Z}$ such that

$$\boldsymbol{v}_{i_{j-1}} + \boldsymbol{v}_{i_{j+1}} = \alpha_{i_j}\boldsymbol{v}_{i_j} \text{ for } j = 1,\ldots,s$$

where the integers $\alpha_{i_1},\ldots,\alpha_{i_s}$ are the entries of the Hirzbruch-Jung continued fraction $\frac{r}{a_i} = [\alpha_{i_1},\ldots,\alpha_{i_s}]$ and $\boldsymbol{v}_{i_0} = (0,1), \boldsymbol{v}_{i_{s+1}} = (1,0)$. The lattice point $\boldsymbol{v}_{i_j}$ can be written as $\boldsymbol{v}_{i_j} = \frac{1}{r}(k_{i_j}, \overline{a_i \cdot k_{i_j}}^r)$ for some positive integer k_{i_j}.

Definition 4.7. Let r, a_i and k_{i_j} be as above. We define an *i-th minimal point* $\boldsymbol{u}_{ij} \in N_G$ as follows:

$$\boldsymbol{u}_{ij} = \frac{1}{r}(k_{i_j}, \overline{a_2 \cdot k_{i_j}}^r, \ldots, \overline{a_n \cdot k_{i_j}}^r).$$

We note that $\boldsymbol{v}_{i_0}, \ldots, \boldsymbol{v}_{i_{s+1}}$ are elements in $\mathrm{Hlb}_{N_{A_i}}(\sigma_{A_i})$, where $\sigma_{A_i} = \mathrm{Cone}((1,0),(0,1)) \subset N_{A_i} \otimes \mathbb{R}$. One of the good properties of minimal points is that they are in Hilbert basis as shown in the next lemma.

Lemma 4.8. *All minimal points are in* $\mathrm{Hlb}_{N_G}(\sigma)$.

Proof. Let $\boldsymbol{u} = (u_1, \ldots, u_n) \in N_G$ be an i-th minimal point and $\boldsymbol{v} = (u_1, u_i) \in N_{A_i}$ be the element corresponding to $\boldsymbol{u}$. If $\boldsymbol{u} \notin \mathrm{Hlb}_{N_G}(\sigma)$, then there exists $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ in N_G such that $\boldsymbol{u} = X + Y$. By focusing on the first and i-th components, the following equations hold:

$$\begin{aligned} u_1 &= x_1 + y_1, \\ u_i &= x_i + y_i. \end{aligned}$$

Let $X_i = (x_1, x_i)$, $Y_i = (y_1, y_i) \in N_{A_i}$, then we have $\boldsymbol{v} = X_i + Y_i$ by the above formula. This contradicts the fact that $\boldsymbol{v} \in \mathrm{Hlb}_{N_{A_i}}(\sigma_{A_i})$. Therefore, we get $\boldsymbol{u} \in \mathrm{Hlb}_{N_G}(\sigma)$. □

An iterated point is either a minimal point or a sum of canonical basis and a minimal point. An iterated point is minimal if and only if it satisfies the conditions of the proper fractions. See Definition 3.2.

Proposition 4.9. *Let $\mathbb{C}^n/G$ be a quotient singularity of $\frac{1}{r}(1, a_2, \ldots, a_n)$-type satisfying $1 + a_2 + \cdots + a_n = r$. If the remainder polynomial $\mathcal{R}_*\left(\frac{(1,a_2,\ldots,a_n)}{r}\right)$ contains an iterated term of which the age of the coefficient is equal to or larger than 2, then $\mathbb{C}^n/G$ has no toric crepant resolutions.*

The problem with Proposition 4.9 is that if G has some representation $\frac{1}{r}(1, a_2, \ldots, a_n)$, $\frac{1}{r}(b_1, 1, b_3, \ldots, b_n)$ and so on, their remainder polynomials are different from each other, so if necessary, we have to calculate iterated points for all representations. Moreover, in higher dimension, there are many groups which fulfil Theorem 2.6 and possess no crepant resolutions. For example, $G = \left\langle \frac{1}{39}(1, 5, 8, 25) \right\rangle$.

4.3 Two parameter Gorenstein cyclic quotient singularities

D. I. Dais, U. U. Haus and M. Henk have proposed a condition for $\mathbb{C}^n/A$ where $A = \frac{1}{r}(a, b, 1, \ldots, 1)$ with $r = a + b + (n-2)$ to possess a crepant resolution of all dimension [10]. We call this "two-parameter Gorenstein cyclic quotient singularities". After that, a new criterion for these quotient singularities to admit a crepant resolution is introduced by S. Davis, T. Logvinenko and M. Reid [13]. In this subsection, we will give the remainder polynomial version of their results. What's better than their results is that if a crepant resolution exists, it can be concretely constructed as Fujiki-Oka resolution. This is the first application of Theorem 4.1 and Proposition 4.9.

We consider two-parameter cyclic quotient singularities $\mathbb{C}^n/A$ where A denote a cyclic group generated by $\frac{1}{r}(a, b, 1, \ldots, 1)$. These singularities have the following three cases:

(1) $\mathrm{GCD}(r,a,b) = d > 1$,

(2) $\mathrm{GCD}(r,a,b) = 1$, $\mathrm{GCD}(r,a) = d_1 > 1$ and $\mathrm{GCD}(r,b) = d_2 > 1$,

(3) $\mathrm{GCD}(r,a) = 1$ or $\mathrm{GCD}(r,b) = 1$.

If A satisfies (1), it is easily seen that $\mathbb{C}^n/A$ has a crepant resolution (see [13]). In the case of (2), $\mathbb{C}^n/A$ has a crepant resolution if and only if lattice points $\frac{1}{r}(0,k_1,r_1,\ldots,r_1)$ and $\frac{1}{r}(k_2,0,r_2.\ldots,r_2)$ are on the junior simplex with $r = r_i \cdot d_i$ and $r = k_i + r_i(n-2)$ for $i = 1,2$.

From now on, we assume that $\mathrm{GCD}(r,a) = 1$. In other words, we treat only the case $A = \frac{1}{r}(1,d,c,\ldots,c)$ with $r = 1+d+(n-2)c$.

Applying Theorem 4.1 and Proposition 4.9 to the cyclic group $A = \frac{1}{r}(1,d,c,\ldots,c)$ gives the conditions to the existence of crepant resolutions.

Lemma 4.10. *If $\mathcal{R}_*\left(\frac{(1,d,c,\ldots,c)}{r}\right)$ with $1+d+(n-2)c = r$ does not satisfy the condition (i) , then $\mathcal{R}_*\left(\frac{(1,d,c,\ldots,c)}{r}\right)$ satisfies the condition (ii), where the condition (i) and (ii) are the followings;*

(i) The remainder polynomial $\mathcal{R}_*\left(\frac{(1,a_2,\ldots,a_n)}{r}\right)$ contains an iterated term of which the age of the coefficient is equal to or bigger than 2.

(ii) The ages of all coefficients of $\mathcal{R}_*\left(\frac{(1,a_2,\ldots,a_n)}{r}\right)$ are 1.

Proof. It is easily to check that the age of $R_i\left(\frac{(1,d,c,\ldots,c)}{r}\right) = \frac{(1,\overline{d}^c,0,\ldots,0,\overline{-r}^c,0,\ldots,0)}{c}$ is equal to 1 for $i = 3,\ldots,n$. By the proof of Corollary 4.5, $\mathcal{R}_*\left(R_i\left(\frac{(1,d,c,\ldots,c)}{r}\right)\right)$ satisfies the condition (ii). On the other hand, by assumption, the image of the remainder map $(R_2\cdots R_2)\left(\frac{(1,d,c,\ldots,c)}{r}\right)$ is $\frac{1}{r'}(1,d',c',\ldots,c')$ for some positive integer r',d',c' with $1+d'+(n-2)c' = r'$. Thus, $\mathcal{R}_*\left(R_i\left(\frac{(1,d',c',\ldots,c')}{r'}\right)\right)$ satisfies the condition (ii) for $i = 3,\ldots,n$. By induction, it follows that $\mathcal{R}_*(\frac{(1,d,c,\ldots,c)}{r})$ satisfies the condition (ii). □

Lemma 4.10 and Theorem 4.1 lead to the following theorem.

Theorem 4.11. *Let $\mathbb{C}^n/G$ be a quotient singularity of $\frac{1}{r}(1,d,c,\ldots,c)$-type. $\mathbb{C}^n/G$ has a crepant resolution if and only if the ages of all coefficients of the remainder polynomial $\mathcal{R}_*\left(\frac{(1,d,c,\ldots,c)}{r}\right)$ are 1.*

4.4 Iterated Fujiki-Oka resolutions

In this section, we give a way to construct Fujiki-Oka resolutions for Gorenstein abelian quotient singularities by using Ashikaga's continued fractions repeatedly. As the goal of this section, we prove that iterated Fujiki-Oka resolutions for three dimensional Gorenstein abelian quotient singularities are crepant.

4.5 Basic Generating Systems of G

Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite abelian subgroup. Since all the elements in G are simultaneously diagonalizable, there exists a conjugacy class of G which is generated by diagonal matrices. Therefore, we may assume that G is generated by diagonal matrices. By Proposition 2.3, if $G \subset \mathrm{SL}(3, \mathbb{C})$, then it is possible to take elements in G of which age is one as the generators of G. In higher dimensional case, we assume that the ages of all generators of G are one, because it is clear that $\mathbb{C}^n/G$ has no crepant resolutions if the ages of a generator g and the inverse g^{-1} in G are more than one by Proposition 4.2. Therefore, we assume the ages of the generators of G are one. By the fundamental theorem of finite abelian groups and the Chinese remainder theorem, there exists a generating system of G as follows:

$$\left\{ \frac{1}{r_1}(a_{11}, a_{12}, \ldots, a_{1n}), \frac{1}{r_2}(0, a_{22}, \ldots, a_{2n}), \ldots, \frac{1}{r_{n-1}}(0, \ldots, 0, a_{n-1\ n-1}, a_{n-1\ n}) \right\}$$

where r_i, a_{ij} $(1 \leq i \leq n-1,\ i \leq j \leq n)$ are positive integers satisfying $\mathrm{LCM}(r_1, \ldots, r_{n-1}) = |G|$ and the following conditions:

(i) if $a_{ii} = 0$, then $a_{ij} = 0$ for $i \leq j \leq n$,

(ii) if $a_{ii} \neq 0$, then $a_{ii} = 1$ and $\displaystyle\sum_{j=i}^{n} a_{ij} = r_i$.

In this paper, we call a generating system of G satisfying the above conditions a *basic generating system* of G. Additionally, G can be decomposed to the cyclic components as follows:

$$G \cong \left\langle \frac{1}{r_1}(a_{11}, a_{12}, \ldots, a_{1n}) \right\rangle \times \cdots \times \left\langle \frac{1}{r_{n-1}}(0, \ldots, 0, a_{n-1\ n-1}, a_{n-1\ n}) \right\rangle.$$

Clearly, every cyclic component can be decomposed to the product of p-Sylow subgroups.

4.6 Iterated Fujiki-Oka resolutions

We shall introduce the *iterated Fujiki-Oka resolutions* in general dimension. Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite abelian subgroup and H be a component of a decomposition by cyclic subgroups of G. If the singularity $\mathbb{C}^n/H$ is semi-isolated, then we have the Fujiki-Oka resolution $(\widetilde{Y_H}, \mathrm{FO}_1)$ and the toric partial resolution (Y_G, ϕ) satisfying the following diagram:

$$\begin{tikzcd}
 & \mathbb{C}^n \arrow[d, "\pi_H"] \\
\widetilde{Y_H} \arrow[r, "\mathrm{FO}_1", "\text{Fujiki-Oka Resolution}"'] \arrow[d, "\pi_{G/H}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C}^n/H \arrow[d, "\pi_{G/H}"] \\
\widetilde{Y_H}\Big/(G/H) = Y_G \arrow[r, "\phi", "\text{Toric Partial Resolution}"'] & \mathbb{C}^n/G
\end{tikzcd}$$

where π_H (resp. $\pi_{G/H}$) is the quotient map by H (resp. G/H). Let $X(N_G, \Sigma_\phi) = Y_G$. If all maximal cones in Σ_ϕ are semi-unimodular with respect to N_G, then we have the Fujiki-Oka resolution $(\widetilde{Y_G}, \mathrm{FO}_2)$ for the quotient singularities corresponding to the maximal cones in Σ_ϕ.

$$\widetilde{Y_G} \xrightarrow[\text{Fujiki-Oka Resolution}]{\mathrm{FO}_2} Y_G$$

We note that every singularity in Y_G corresponding to a maximal cone in Σ_ϕ is at worst a Gorenstein cyclic quotient singularity which is canonical but not terminal because of the construction.

Definition 4.12. We call the resolution $(\widetilde{Y_G}, \mathrm{FO}_2 \circ \phi)$ in the above diagrams an *iterated Fujiki-Oka resolution* of $\mathbb{C}^n/G$.

Let G' be a finite abelian subgroup which acts on $\widetilde{Y_G}$ equivariant with the torus action and G be a component of a decomposition by cyclic subgroups of G'. Let $Y_{G'} = X(N_{G'}, \Sigma_{\phi'})$. If all maximal cones in Σ_ϕ are again semi-unimodular with respect to $N_{G'}$, then we have a new iterated Fujiki-Oka resolution by extending the above diagram.

$$\begin{tikzcd}
 & & & \mathbb{C}^n \arrow[d, "\pi_H"] \\
 & & \widetilde{Y_H} \arrow[r, "\mathrm{FO}_1"] \arrow[d, "\pi_{G/H}"'] & \mathbb{C}^n/H \arrow[d, "\pi_{G/H}"] \\
 & \widetilde{Y_G} \arrow[r, "\mathrm{FO}_2"] \arrow[d, "\pi_{G'/G}"'] & Y_G \arrow[r, "\phi", "\text{T.P.R.}"'] & \mathbb{C}^n/G \arrow[d, "\pi_{G'/G}"] \\
\widetilde{Y_{G'}} \arrow[r, "\mathrm{FO}_3"] & Y_{G'} \arrow[rr, "\phi'", "\text{T.P.R.}"'] & & \mathbb{C}^n/G'
\end{tikzcd}$$

As $(\widetilde{Y_{G'}}, \mathrm{FO}_3 \circ \phi')$ in the above, iterated Fujiki-Oka resolutions can be extended under the suitable conditions. We also call these resolutions and the ordinary Fujiki-Oka resolutions *iterated Fujiki-Oka resolutions.*

Lemma 4.13. *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite abelian subgroup. There exist at least one iterated Fujiki-Oka resolution of $\mathbb{C}^n/G$.*

Proof. Let $\left\{ \frac{1}{r_1}(a_{11}, a_{12}, \ldots, a_{1n}), \ldots, \frac{1}{r_{n-1}}(0, \ldots, 0, a_{n-1\ n-1}, a_{n-1\ n}) \right\}$ be a basic generating system of G. We set

$$H_1 = \left\langle \frac{1}{r_{n-1}}(0, \ldots, 0, a_{n-1\ n-1}, a_{n\ n}) \right\rangle.$$

Then, we have the Fujiki-Oka resolution $X(N_1, \Sigma_1)$ of the singularity $\mathbb{C}^n/H_1$ such that the maximal cones in Σ_1 are obtained from subdividing the two dimensional junior simplex $\mathfrak{s}_2$ spanned by $\boldsymbol{e}_{n-1}$ and $\boldsymbol{e}_n$ into r_{n-1} equal sections. Let E_i be the edge of which endpoints are $\frac{i-1}{r_{n-1}}\boldsymbol{e}_{n-1} + \frac{r_{n-1}-i+1}{r_{n-1}}\boldsymbol{e}_n$ and $\frac{i}{r_{n-1}}\boldsymbol{e}_{n-1} + \frac{r_{n-1}-i}{r_{n-1}}\boldsymbol{e}_n$ for $i = 1, \ldots, r_{n-1}$ on $\mathfrak{s}_2$.

As the next step, we set

$$H_2 = \left\langle \frac{1}{r_{n-1}}(0, \ldots, 0, a_{n-2\ n-2}, a_{n-2\ n-1}, a_{n-2\ n}) \right\rangle \times \left\langle \frac{1}{r_{n-1}}(0, \ldots, 0, a_{n-1\ n-1}, a_{n-1\ n}) \right\rangle.$$

We have the quotient map $\pi_{H_2/H_1} : \mathbb{C}^n/H_1 \to \mathbb{C}^n/H_2 = X(N_2, \Sigma_1)$. Focus the three dimensional junior simplex $\mathfrak{s}_3$ spanned by $\boldsymbol{e}_{n-2}, \boldsymbol{e}_{n-1}$ and $\boldsymbol{e}_n$. By the definition of the basic generating system, there are no lattice points on the edges $E_i \subset \mathfrak{s}_2 \subset \mathfrak{s}_3$ for all i. Therefore, every maximal cone in Σ_1 is semi-unimodular, and we have an iterated Fujiki-Oka resolution $X(N_2, \Sigma_2)$.

By repeating similar operation to the above for the subgroup sequence:

$$H_1 \subset H_2 \subset \cdots \subset H_{n-1} = G,$$

we have the sequence of iterated Fujiki-Oka resolutions:

$$\widetilde{Y_{H_1}} = X(N_1, \Sigma_1), \widetilde{Y_{H_2}} = X(N_2, \Sigma_2), \ldots, \widetilde{Y_G} = X(N_{n-1}, \Sigma_{n-1}).$$

$\square$

By applying Theorem 4.1, Proposition 4.3 and Lemma 4.13 to the iterated Fujiki-Oka resolutions, we have the following theorem.

Theorem 4.14. *Let $\widetilde{Y_{H_1}}, \widetilde{Y_{H_2}}, \ldots, \widetilde{Y_{H_k}} = \widetilde{Y_G}$ be the sequence of iterated Fujiki-Oka resolutions for an n-dimensional Gorenstein abelian quotient singularity $\mathbb{C}^n/G$. If the ages of all the coefficients in the remainder polynomials associated with every $\widetilde{Y_{H_i}}$ $(i = 1, \ldots, k)$ are 1, then the corresponding iterated Fujiki-Oka resolution $\widetilde{Y_G}$ for $\mathbb{C}^n/G$ is crepant.*

Theorem 4.14 and Corollary 4.5 lead to the following corollary.

Corollary 4.15. *Assume that G is a finite abelian subgroup of* $\mathrm{SL}(3, \mathbb{C})$. *Then a crepant iterated Fujiki-Oka resolution exists for $\mathbb{C}^3/G$.*

4.7 Examples of Iterated Fujiki-Oka resolutions

At first, we shall see an example of iterated Fujiki-Oka resolutions in three dimension.

Example 4.16. Let $G = \left\langle \frac{1}{4}(1,3,0), \frac{1}{4}(1,0,3) \right\rangle$, then $X = \mathbb{C}^3/G$ has a Gorenstein hypersurface singularity defined by $xyz - w^4 = 0$. In this case, we have the set $\{\frac{1}{4}(1,2,1), \frac{1}{4}(0,3,1)\}$ as a basic generating system of G. According to Lemma 4.13, we set $H = \left\langle \frac{1}{4}(0,3,1) \right\rangle$. Then the junior simplex of the iterated Fujiki-Oka resolution is transformed as Fig. 6.

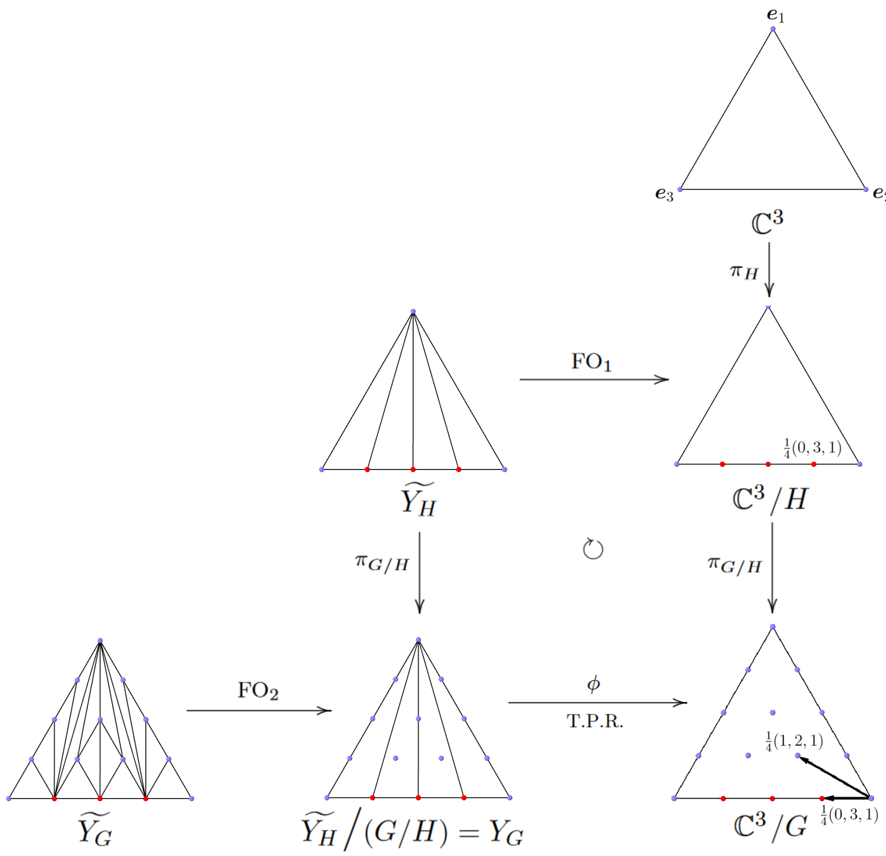


Fig. 4: The iterated Fujiki-Oka resolution of $\left\langle \frac{1}{4}(1,3,0), \frac{1}{4}(1,0,3) \right\rangle$

On the other hand, if we choose $\frac{1}{4}(1,2,1)$ as a generator instead of $\frac{1}{4}(0,3,1)$, then we obtain an iterated Fujiki-Oka resolution via a subgroup $H' = \left\langle \frac{1}{4}(1,2,1) \right\rangle$ (see Fig. 7). In general, the iterated Fujiki-Oka resolution is not unique, and it depends on the choice of the generator.

The next example is in four dimensional case.

Example 4.17. Let $G = \langle \frac{1}{2}(1,1,0,0), \frac{1}{2}(1,0,1,0), \frac{1}{2}(1,0,0,1) \rangle$, then $X = \mathbb{C}^4/G$ has a Gorenstein canonical hypersurface singularity. It is known that X has crepant resolutions. However, G-Hilb$(\mathbb{C}^4)$ is not a crepant resolution, it is a blow-up of certain crepant resolutions.

We can obtain a crepant resolution of X by iterated Fujiki-Oka resolutions. Let $H = \left\langle \frac{1}{2}(1,1,0,0) \right\rangle \subset G$. In addition, this crepant resolution is not blow-down of G-Hilb$(\mathbb{C}^4)$. In general, the iterated Fujiki-Oka resolution and G-Hilb give different fans.

5 Complete coprime cyclic quotient singularities

5.1 Complete coprime remainder polynomials

This section deals with a complete coprime Fujiki-Oka resolution of $\mathbb{C}^n/G$ where G is a cyclic group of $\mathrm{GL}(n, \mathbb{C})$. This resolution is one of Hilbert basis resolutions.

Definition 5.1. The remainder polynomial $\mathcal{R}_*$ is complete coprime if arbitrary coefficients $(b_1, \ldots, b_n)/b_0$ of $\mathcal{R}_*$ satisfies $\mathrm{GCD}(b_i, b_j) = 1$ for all $i \neq j$. Moreover, the Fujiki-Oka resolution is complete coprime if it is obtained by a complete coprime remainder polynomial $\mathcal{R}_*$.

Example 5.2. Let G be the following type. Then the Fujiki-Oka resolution of $\mathbb{C}^n/G$ is complete coprime.

(i) $G = \frac{1}{r}(1, a) \subset \mathrm{GL}(2, \mathbb{C})$ with $\mathrm{GCD}(r, a) = 1$.

(ii) $G = \frac{1}{r}(1, a, r-a) \subset \mathrm{GL}(3, \mathbb{C})$ with $\mathrm{GCD}(r, a) = 1$.

Note that a Fujiki-Oka resolution of the case (i) is a minimal resolution. In the case (ii), $\mathbb{C}^3/G$ has a terminal singularity. A Fujiki-Oka resolution coincides with a Hilbert basis resolution which is called an economic resolution.

Remark 5.3. Let Σ_G denote the fan corresponding to the Fujiki-Oka resolution of $\mathbb{C}^n/G$. Suppose that the Fujiki-Oka resolution is complete coprime. Then $\mathrm{Cone}(e_1, v_i)$ is in Σ_G for any one dimensional cone $\tau_i = \mathrm{Cone}(v_i)$ which is element of $\Sigma_G(1)\backslash\sigma(1)$. Moreover, the equation ${}_{\#}\{\Sigma_G(n)\} = (n-1)_{\#}\{\Sigma(1)\} + 1$ holds.

Lemma 5.4. *If a remainder polynomial $\mathcal{R}_*$ is complete coprime, then any coefficients $(a_1, \ldots, a_n)/a_0$ in $\mathcal{R}_*$ satisfy $a_i + a_j \leq r$ for all $i \neq j$.*

Proof. It is sufficient to prove that $\mathcal{R}_*\left((a, b)/r\right)$ is not complete coprime when $a+b > r$ and $r > b > a$. Let c denote the positive integers which satisfies $a+b+c = r$. Suppose that $\mathcal{R}_*\left((a, b)/r\right)$ is complete coprime. The image of second remainder map $R_2\left((a, b)/r\right)$ is $(a, \overline{-r}^b)/b$. Since $2b > r > b$, we have $r - \overline{r}^b = b$. It follows that $a + \overline{-r}^b = a + b - \overline{r}^b = b - c$. By our assumption, we have $a, \overline{-r}^b \neq 0$ and $\mathrm{GCD}(a, \overline{-r}^b) = 1$. Then the above discussion can be repeated, which contradicts that the term of the remainder polynomial is finite. $\square$

Lemma 5.5. *Let a, b and r be positive integers with $r > b > a$ and $r - b > a$. Assume that $v_1 = (a, b)/r$ and $v_2 = (a, r-b)/r$ is complete coprime. Then $\overline{b}^a$ and $\overline{r-b}^a$ is an even number.*

Proof. Since b or $r-b$ is greater then $\frac{r}{2}$, there is no loss of generality in assuming $r-b>\frac{r}{2}$. The images of the remainder map for v_1 and v_2 are as follows. All of the following two-dimensional fraction are complete coprime.

$$\begin{tikzcd}
& v_1=\frac{(a,b)}{r} \arrow[dl, "R_1"'] \arrow[dr, "R_2"] & \\
\frac{(\overline{-r}^a,\overline{b}^a)}{a} & & \frac{(a,\overline{-r}^b)}{b} \arrow[dl, "R_1"] \\
& \frac{(\overline{-b}^a,\overline{\overline{-r}^b}^a)}{a} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& v_2=\frac{(a,r-b)}{r} \arrow[dl, "R_1"'] \arrow[dr, "R_2"] & \\
\frac{(\overline{-r}^a,\overline{r-b}^a)}{a} & & \frac{(a,r-2b)}{r-b} \arrow[dl, "R_1"] \\
& \frac{(\overline{-(r-b)}^a,\overline{r-2b}^a)}{a} &
\end{tikzcd}$$

If $\overline{b}^a>\overline{r-b}^a$, then we have

$$\begin{aligned}
\overline{-(r-b)}^a+\overline{r-2b}^a &= a-\overline{(r-b)}^a+\overline{r-b}^a-\overline{b}^a+a \\
&= 2a-\overline{b}^a>a.
\end{aligned}$$

We apply Lemma 5.4 to $R_1R_2(v_2)=(\overline{-(r-b)}^a,\overline{r-2b}^a)/a$, then it contradicts complete coprime. We thus get $\overline{r-b}^a>\overline{b}^a$, and then $\overline{r-2b}^a=\overline{r-b}^a-\overline{b}^a$ holds.
We show that assuming $\overline{b}^a$ is an odd number contradicts complete coprime. If $\overline{r-b}^a$ is even, then we see that $\overline{r}^a$ is odd and $\overline{-r}^a$ is even. It follows that $R_1(v_2)=(\overline{-r}^a,\overline{r-b}^a)/a$ is not complete coprime. On the other hands, if $\overline{r-b}^a$ is odd, then $\overline{r-b}^a$ and $\overline{r-2b}^a$ is even. It contradicts to $R_1R_2(v_2)$ is complete coprime. Therefore, we conclude $\overline{b}^a$ is an even number. Since $R_1(v_1)=(\overline{-r}^a,\overline{b}^a)/a$ is complete coprime, we have $\overline{-r}^a$ is odd. It follows that $\overline{r-b}^a$ is also an even number.

$\square$

Proposition 5.6. *Let v be $(a,b,c)/r$ with $1<a<b<c$. Then $\mathcal{R}_*(v)$ is not complete coprime.*

Proof. We can assume that one of numerators of $R_i(v)$ equals to 1 for $i=1,2,3$. If not, we should consider $\mathcal{R}_*(R_i(v))$ instead of $\mathcal{R}_*(v)$ for some i. Since this assumption and $R_3(v)=(a,b,\overline{-r}^c)/c$, we have $\overline{-r}^c=1$.

$$
\begin{tikzcd}
 & v = \frac{(a,b,c)}{r} \arrow[dl, "R_1"'] \arrow[d, "R_2"'] \arrow[dr, "R_3"] & \\
\frac{(\overline{-r}^a,\overline{b}^a,\overline{c}^a)}{a} & \frac{(a,\overline{-r}^b,\overline{c}^b)}{b} \arrow[dl, "R_1"'] & \frac{(a,b,1)}{c} \arrow[dl, "R_1"'] \arrow[d, "R_2"] \\
\frac{(\overline{-b}^a,\overline{\overline{-r}^b}^a,\overline{\overline{c}^b}^a)}{a} & \frac{(\overline{-c}^a,\overline{b}^a,\overline{\overline{-r}^c}^a)}{a} & \frac{(a,\overline{-c}^b,1)}{b} \arrow[dl, "R_1"'] \\
 & \frac{(\overline{-b}^a,\overline{\overline{-c}^b}^a,1)}{a} &
\end{tikzcd}
$$

If $\overline{c}^b = 1$, then $\overline{-c}^b = b - 1$ and $a + b - 1 > b$. This leads to $R_2R_3(v)$ is not complete coprime by Proposition 5.4. We thus get $\overline{-r}^b = 1$. Similar arguments apply to the case $\overline{c}^a = 1$, we have $\overline{-r}^a = 1$.

On the other hand, either $\overline{b}^a$ or $\overline{-b}^a$ is an even number, and either $\overline{c}^a$ or $\overline{-c}^a$ is also an even number. Since $R_1(v)$ and $R_1R_3(v)$ are complete coprime, $\overline{b}^a$ is odd number. Thus $\overline{-b}^a$ is an even number. For $R_2(v) = (a, 1, \overline{c}^b)/b$ and $R_2R_3(v)$, we have $\overline{c}^b + a < b$ and $\overline{-c}^b + a > b$. By Lemma 5.5, $\overline{\overline{c}^b}^a$ and $\overline{\overline{-c}^b}^a$ is an even number. Therefore, $R_1R_2R_3(v) = (\overline{-b}^a, \overline{\overline{-c}^b}^a, 1)/a$ is not complete coprime. $\square$

As a corollary of Proposition5.6, the following theorem holds.

Theorem 5.7. *Let G be a cyclic group of type $\frac{1}{r}(a_1, \ldots, a_n)$. If the remainder polynomial $\mathcal{R}_*((a_1, \ldots, a_n)/r)$ is complete coprime, then G is isomorphic to a cyclic group of type $\frac{1}{r}(a, b, 1, \ldots, 1)$.*

5.2 The resolution of complete coprime quotient singularities

From now on, G and G' denote the cyclic group of type $\frac{1}{r}(a, b)$ and $\frac{1}{r}(a, b, 1^{n-2})$, respectively. In our case, the lattice $N_G := \mathbb{Z}^2 + \frac{1}{r}(a, b)\mathbb{Z}$.

Assume that the remainder polynomial $\mathcal{R}_*((a, b)/r)$ is complete coprime. We propose the resolution of $\mathbb{C}^2/G$ which is obtained by the remainder polynomial $\mathcal{R}_*((a, b)/r)$. By the assumption, $v = \frac{1}{r}(a, b) \in N_G$ generate $N_G/\mathbb{Z}^2$. It follows that remainder maps indicate the types of quotient singularities corresponding each three dimensional cone after star subdivision at v. We now apply this argument again, with $G = \frac{1}{r}(a, b)$ replaced by $G_1 = \frac{1}{a}(\overline{-r}^a, \overline{b}^a)$ (resp. $G_2 = \frac{1}{b}(\overline{a}^b, \overline{-r}^b)$) and $N_{G_1} = \mathbb{Z}^2 + \frac{1}{a}(\overline{-r}^a, \overline{b}^a)\mathbb{Z}$ (resp. $N_{G_2} = \mathbb{Z}^2 + \frac{1}{b}(\overline{a}^b, \overline{-r}^b)\mathbb{Z}$) until the fan which is obtained by consecutive subdivisions is smooth. Then we have the resolution which is called a continued fractional resolution of $\mathbb{C}^2/G$.

Definition 5.8. Let $\mathrm{COEF}\{\mathcal{R}_*((a,b)/r)\}$ be the set of all coefficients in $\mathcal{R}_*((a,b)/r)$. Then there is a natural map $\phi_G : \mathrm{COEF}\{\mathcal{R}_*((a,b)/r)\} \to N_G$.

We will denote by ψ_i the natural injective morphism $\psi_i : \mathrm{Im}(\phi_{N_{G_i}}) \to \mathrm{Im}(\phi_{N_G})$ for $i = 1, 2$.

Example 5.9. Let G be of type $\frac{1}{5}(2,3)$. Then the remainder polynomial is

$$\begin{aligned}\mathcal{R}_*\left(\frac{(2,3)}{5}\right) &= \frac{(2,3)}{5} + \frac{(1,1)}{2}x_2 \\ &+ \frac{(2,1)}{3}x_3 + \frac{(1,1)}{2}x_3x_2.\end{aligned}$$

The image of ϕ_G, ϕ_{G_1} and ϕ_{G_2} are

$$\begin{aligned}\mathrm{Im}(\phi_G) &= \left\{\frac{1}{5}(1,4), \frac{1}{5}(2,3), \frac{1}{5}(3,2), \frac{1}{5}(4,1)\right\}, \\ \mathrm{Im}(\phi_{G_1}) &= \left\{\frac{1}{2}(1,1)\right\}, \text{and} \\ \mathrm{Im}(\phi_{G_2}) &= \left\{\frac{1}{3}(1,2), \frac{1}{3}(2,1)\right\}.\end{aligned}$$

We have $\psi_1\left(\frac{1}{2}(1,1)\right) = \frac{1}{5}(1,4)$, $\psi_2\left(\frac{1}{3}(1,2)\right) = \frac{1}{5}(3,2)$ and $\psi_2\left(\frac{1}{3}(2,1)\right) = \frac{1}{5}(4,1)$.

Proposition 5.10. *Let $G = \frac{1}{r}(a,b)$. Then $\mathrm{Im}(\phi_G)$ coincides with $\mathrm{Hilb}_{N_G}(\sigma)\backslash\{(1,0),(0,1)\}$.*

Proof. We give proof by induction on the order of G. It is easily seen that the statement holds for $r = 2, 3$. We show that if the statement holds for $r \le k-1$, then it holds for $r = k$.
Assume that $v = \frac{1}{r}(a,b)$ is not in $\mathrm{Hilb}_{N_G}(\sigma)$. We will denote by $v_0, v_1, \ldots, v_s, v_{s+1}$ the elements of $\mathrm{Hilb}_{N_G}(\sigma)$ in order of the smallest x coordinates, where $v_0 = e_2$, $v_{s+1} = e_1$. By assumption, there exists a integer t $(1 \le t \le s)$ such that $v = v_t + v_{t+1}$.
Let G_1 be a cyclic group of type $R_1(v) = \frac{1}{a}(\overline{-r}^a, \overline{b}^a)$ and G_2 be of type $R_2(v) = \frac{1}{b}(a, \overline{-r}^b)$. The coordinate of $\psi_1^{-1}(v_t)$ in N_{G_1} is $\frac{1}{a}(i,1)$ for some integer i. Since v_{t-1}, v_t, v are not on the same straight line, we get $2i > a$. Similarly, the coordinate of $\psi_2^{-1}(v_{t+1})$ is $\frac{1}{a}(1,j)$, and we have $2j > a$.
On the other hand, the coordinate of v_t is $\frac{1}{r}\left(i, \frac{bi+r}{a}\right)$ and v_{t+1} is $\frac{1}{r}\left(\frac{\overline{-r}^b+r}{b} + j, \frac{\overline{-r}^b+bj}{a}\right)$ in N_G. Since $v = v_t + v_{t+1}$, we have $ab = bi + bj + \overline{-r}^b + r$. It follows that $b(a-i-j) = \overline{-r}^b + r > 0$. However, leads to $a - i - j < 0$, a contradiction. Therefore, we conclude that $\mathrm{Im}(\phi_G)$ coincides with $\mathrm{Hilb}_{N_G}(\sigma)\backslash\{(1,0),(0,1)\}$. $\square$

Let Y_G and $Y_{G'}$ denote the minimal resolution of $\mathbb{C}^2/G$ and the Fujiki-Oka resolution of $\mathbb{C}^n/G'$, respectively. It is clear that the remainder polynomial $R_*((a,b,1^{n-2})/r)$ and $R_*((a,b)/r)$ have the same number of terms. By Proposition 5.10, the following holds.

Theorem 5.11. *Under the above assumption, further assume that the Fujiki-Oka resolution of $\mathbb{C}^n/G'$ is complete coprime. Then the Fujiki-Oka resolution is a Hilbert basis resolution. In addition, there is one-to-one correspondence between exceptional divisors of Y_G and exceptional divisors of $Y_{G'}$.*

Note that minimal resolution (it coincides with Hilbert basis resolution) of a toric surface quotient singularity has no (-1)-curves. A complete coprime Fujiki-Oka resolution of three-dimensional cyclic quotient singularities has the same properties.

Proposition 5.12. *Let $G = \frac{1}{r}(a, b, 1)$. If a Fujiki-Oka resolution of $\mathbb{C}^3/G$ is complete coprime, then there is no exceptional $(-1, -1)$-curves.*

Proof.
It suffices to show that all $\mathbb{Z}^2$-weight of $\tau \in \Sigma_G(2)$ is not equal $(-1, -1)$. By Proposition3.16, if τ has not e_3 as a generator, then $\mathbb{Z}^2$-weight of τ is obtained by the round down polynomial $\mathcal{Z}_*\left((a, b)/r\right)$. Clearly, the image of the round down map is not equal $(-1, -1)$.

Assume that the $\mathbb{Z}^2$-weight of $\tau = \mathrm{Cone}(e_3, v)$ is $(-1, -1)$ where $v = (x_1, x_2, x_3)$. Let $\Sigma_{\min}$ denote the fan corresponding a minimal resolution of the quotient singularity of type $\frac{1}{r}(a, b)$, and let $v' = (x_1, x_2)$ in $N_{G'}$. Then Σ_{min} has a one dimensional cone $\tau' = \mathrm{Cone}(v')$ which corresponding to an exceptional (-1)-curve, which contradicts the minimal resolution has no (-1)-curves. Therefore, all $\mathbb{Z}^2$-weight of two dimensional cone in Σ_G is not equal $(-1, -1)$. □

5.3 McKay correspondence of Fujiki-Oka resolutions

We show several examples of Fujiki-Oka resolutions which satisfies the Euler number equal to the order of G. Since the number of conjugacy classes of G is just the order of G for a cyclic group, this can be considered a kind of generalized McKay correspondence. In toric geometry, the following fact is well known.

Fact 5.13. *Let X_Σ denote a toric variety associated with a fan Σ. Then the Euler number of X_Σ equals the number of cones of maximal dimension in Σ.*

Theorem 5.14. *Let H be of type $\frac{1}{r}(1, r-n+1)$ with $r = (n-1)k + 1$ where r, n, k are some positive integers. For $h = \frac{1}{r}(a, b) \in H$, we have a two dimensional proper fraction $(a, b)/r$. If $\mathcal{R}_*((a, b)/r)$ is complete coprime, then the Euler number of the Fujiki-Oka resolution of $\mathbb{C}^n/G$ is the order of G, where G is of type $\frac{1}{r}(a, b, 1^{n-2})$.*

Proof. Let χ_G denote the Eular number of the Fujiki-Oka resolution of $\mathbb{C}^n/G$. Σ_G (resp. Σ_H) denote the fan corresponding to the Fujiki-Oka resolution of $\mathbb{C}^n/G$ (resp. $\mathbb{C}^2/G$). By remark 5.3, we have $\chi_G = (n-1)_\#\{\Sigma_G(1)\} + 1$.

On the other hand, ${}_{\#}\{\Sigma_G(1)\}$ is the number of terms in the remainder polynomial $\mathcal{R}_*((a,b,1,\ldots,1)/r)$. Theorem5.14 now leads to

$$ {}_{\#}\left\{\mathcal{R}_*\left((a,b,1,\ldots,1)/r\right)\right\} = {}_{\#}\left\{\mathcal{R}_*\left((1,r-n+1)/r\right)\right\} = k $$

where ${}_{\#}\{\mathcal{R}_*\}$ denote the number of terms in the remainder polynomial. It follows that $\chi_G = (n-1)k+1 = r$. □

There are at least two elements of H that satisfy Theorem5.14. Actually, if we choose $\frac{1}{r}(1,r-n-1)$ or $\frac{1}{r}(k,1)$ in H, remainder polynomials $\mathcal{R}_*$ are complete coprime.

Example 5.15. Let us consider the case of $n=3$ and $r=11$, that is $H=\frac{1}{11}(1,9)$. Remainder polynomials $\mathcal{R}_*((1,9)/11)$ and $\mathcal{R}_*((4,3)/11)$ are complete coprime. The following figure shows the cross section of each fans corresponding to a Fujiki-Oka resolution of $\mathbb{C}^3/G_i$, where $G_1 = \frac{1}{11}(1,9,1)$ and $G_4 = \frac{1}{11}(4,3,1)$. Since there are eleven three-dimensional cones, the Eular number of Fujiki-Oka resolutions is 11.

Example 5.16. Let G be following type. Then the Fujiki-Oka resolution of $\mathbb{C}^n/G$ has the Euler number equal to the order of G.

- $\frac{1}{6k+1}(1,3,6k-5)$
- $\frac{1}{6k-1}(1,3,3k-2)$

6 Three dimensinal quotient singularities and binary trees

In this section, we construct a binary tree by using remainder polynomial, and we characterize binary tree which gives the Fujiki-Oka resolution for two series of cyclic quotient singularities. Originally, the remainder polynomial has no terms with coefficient $\frac{[0,\ldots,0]}{1}$ and ∞. In this section, we allow a remainder polynomial to have these terms to define full binary tree.

For n-dimensional proper fractions, we defined the **extended i-th remainder map** $\overline{R_i} : \overline{\mathbb{Q}_n^{\mathrm{prop}}} \cup \{-\infty\} \to \overline{\mathbb{Q}_n^{\mathrm{prop}}} \cup \{-\infty\}$ and the remainder polynomial [1]. The i-th remainder map is defined by the following. If $a_i \neq 0$, then

$$ \overline{R_i}\left(\frac{(a_1,\ldots,a_n)}{r}\right) = \frac{(\overline{a_1}^{a_i},\ \ldots,\ \overline{a_{i-1}}^{a_i},\ \overline{-r}^{a_i},\ \overline{a_{i+1}}^{a_i},\ \ldots,\overline{a_n}^{a_i})}{a_i} $$

where $\overline{x}^{a_i} \equiv x \pmod{a_i}$ with $0 \leq \overline{x}^{a_i} \leq a_i - 1$. If $a_i = 0$, then $\overline{R_i}\left(\frac{(a_1,\ldots,a_n)}{r}\right) = \infty$. In addition, we define $\overline{R_i}(\infty) = -\infty$ and $\overline{R_i}(-\infty) = -\infty$.

In the original paper [1], both ∞ and $-\infty$ are written as ∞. Since we will represent the continued fraction as full binary trees, the two symbols are distinguished in this section.

Definition 6.1. The **extended remainder polynomial** $\overline{\mathcal{R}_*}\left(\frac{\mathbf{a}}{r}\right) \in \overline{\mathbb{Q}_n^{\mathrm{prop}}}[x_1, \ldots, x_n]$ is defined by

$$\overline{\mathcal{R}_*}\left(\frac{\mathbf{a}}{r}\right) = \frac{\mathbf{a}}{r} + \sum_{\substack{(i_1,i_2,\ldots,i_l)\in\mathbf{I}^l \\ l\geq 1}} (\overline{R_{i_l}} \cdots \overline{R_{i_2} R_{i_1}}) \left(\frac{\mathbf{a}}{r}\right) x_{i_1} \cdots x_{i_l}$$

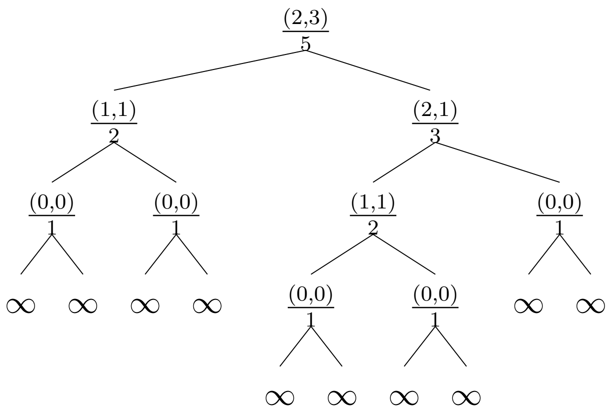
where we exclude terms with coefficient $-\infty$.

Example 6.2. Let $v = \frac{(2,3)}{5}$. Then the extended remainder polynomial is

$$\begin{aligned}
\overline{\mathcal{R}_*}\left(\frac{(2,3)}{5}\right) &= \frac{(2,3)}{5} + \frac{(1,1)}{2}x_1 + \frac{(2,1)}{3}x_2 \\
&+ \frac{(0,0)}{1}x_1x_1 + \frac{(0,0)}{1}x_1x_2 + \frac{(1,1)}{2}x_2x_1 + \frac{(0,0)}{1}x_2x_2 \\
&+ \infty x_1x_1x_1 + \infty x_1x_1x_2 + \infty x_1x_2x_1 + \infty x_1x_2x_2 \\
&+ \frac{(0,0)}{1}x_2x_1x_1 + \frac{(0,0)}{1}x_2x_1x_2 + \infty x_2x_2x_1 + \infty x_2x_2x_2 \\
&+ \infty x_2x_1x_1x_1 + \infty x_2x_1x_1x_2 + \infty x_2x_1x_2x_1 + \infty x_2x_1x_2x_2.
\end{aligned}$$

6.1 Binary tree and continued fraction

We note that remainder polynomials can be represented by trees and proper fractions as follows. Each term of a remainder polynomial corresponds to node of tree, and we connect two nodes if these nodes correspond to the terms with variable $x_{i_1} \cdots x_{i_l}$ and $x_{i_1} \cdots x_{i_l} \cdot x_j$ for some l and j. In the above example, the binary tree obtained by two-dimensional proper fraction is the following.

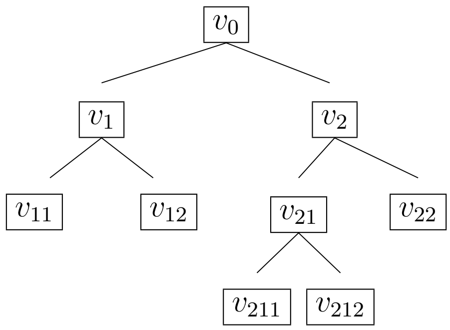


We will introduce the definition of binary trees.

Definition 6.3. The topmost node of a tree is called **the root**. Every node is the root, or is connected by a directed edge from one node which is called **a parent node**. On the other hand, every node connects to some nodes which are called **child nodes**. A node with no children is called **a leaf**, and node with the same parent is called **a sibling**.

Definition 6.4. A **binary tree** is a tree whose elements have at most 2 children. In addition, the tree is a **full** binary tree if each node has exactly zero or two children. Since each node can have only two children, we name them **a left child** and **a right child**.

We will denote by $\mathbf{T}_v$ the binary tree whose root is the node v. For example, v_{11} and v_{12} are children of v_1 and v_2 is sibling of v_1 in the following tree $\mathbf{T}_{v_0}$. A subtree $\mathbf{T}_x$ of a tree $\mathbf{T}$ is a tree consisting of a node x in $\mathbf{T}$ and all of its descendants in $\mathbf{T}$.



In this paper, we define a **nephew node** as follows.

Definition 6.5. Let v be an arbitrary node which is a left (resp. right) child. If there exists a left (resp. right) child of a sibling node of v, then we call this node a nephew of v.

In the above figure, v_{21} is a nephew of v_1 and v_{212} is a nephew of v_{22}. We will denote by $\mathbf{T}_{\frac{(a,b)}{r}}$ the binary tree obtained by two-dimensional proper fraction $\frac{(a,b)}{r}$, and we call this tree **the continued fractional tree**. For convenience, we define the tree which consists only one node is also the continued fractional tree. We call this tree trivial.

Definition 6.6. The size of tree $\mathbf{T}$, denoted by $|\mathbf{T}|$, is defined to be the number of leaves.

6.2 Terminal trees

We define a terminal tree and show some properties of this one.

Definition 6.7. A two-dimensional proper fraction $\frac{(a,b)}{r}$ is terminal if $a + b = r$ and $\mathrm{GCD}(r, a) = \mathrm{GCD}(r, b) = 1$. In addition, $\mathbf{T}$ is terminal if it is obtained by terminal fraction, or it is trivial or $\frac{(0,0)}{1}$.

Since $a + b = r$, we can write terminal fraction as $\frac{(a,r-a)}{r}$. The multidimensional continued fraction for $\frac{(a,r-a)}{r}$ gives the economic resolution of the quotient singularity of type $\frac{1}{r}(1, a, r - a)$. This quotient singularity is terminal, so we call this fraction terminal.

Proposition 6.8. *For given two 2-dimensional terminal proper fractions* $\frac{(a_1,b_1)}{r_1}$ *and* $\frac{(a_2,b_2)}{r_2}$*, if the lifting* $\frac{(a_0,b_0)}{r_0}$ *which satisfies* $\left(\frac{(a_0,b_0)}{r_0}\right) = \frac{(a_1,b_1)}{r_1}$ *and* $\overline{R_2}\left(\frac{(a_0,b_0)}{r_0}\right) = \frac{(a_2,b_2)}{r_2}$ *exists and it is terminal, then it is uniquely determined.*

Proof. By Definition 6.1, the lifting $\frac{(a_0,b_0)}{r_0}$ satisfies $a_0 = r_1$ and $b_0 = r_2$. Since the lifting is terminal, it follows that $r_0 = r_1 + r_2$. $\square$

This proposition says that the proper fraction corresponding to the parent node is uniquely determined from the denominator of two child nodes in terminal tree.

Remark 6.9. A Gorenstein canonical proper fraction (see Definition 6.14) also has this property.

Proposition 6.10. *If* $\mathbf{T}_{\frac{(a,b)}{r}}$ *is terminal, then all subtrees are also terminal.*

Proof. We claim that the image of remainder map $\overline{R_i}(\frac{(a,b)}{r})$ is also a terminal two-dimensional proper fraction for $i = 1, 2$. Since $\overline{R_1}(\frac{(a,b)}{r}) = \frac{(\overline{-r}^a, \overline{b}^a)}{a}$ and $-r + b = -a$, we have $\overline{-r}^a + \overline{b}^a \equiv 0 \pmod a$. By assumption, $0 \le \overline{-r}^a < a$ and $0 \le \overline{b}^a < a$, it follows $\overline{-r}^a + \overline{b}^a = a$. $\square$

Corollary 6.11. *Let* $\mathbf{T}_{\frac{(a,b)}{r}}$ *be a terminal tree, then the sibling node of a leaf is a leaf. Especially,* $|\mathbf{T}_{\frac{(a,b)}{r}}| = 2r$*.*

Proof. The leaves correspond to ∞ as coefficient of the remainder polynomial. We claim that there are no nodes which correspond to $\frac{(\alpha,0)}{r}$ or $\frac{(0,\alpha)}{r}$ where $0 < \alpha < r$. If this node exists, then we have $\alpha + 0 = r$ by Proposition 6.10. This contradicts $\mathrm{GCD}(r, a) = 1$. $\square$

Theorem 6.12. (Theorem 1.9) *Let* $\mathbf{T}$ *be a full binary tree. Let* x_1 *be an arbitrary node which has a parent node* x*, a sibling node* x_2 *and a nephew node* y*. Then* $\mathbf{T}$ *is terminal if and only if* $\mathbf{T}$ *satisfies the following conditions.*

(i) *A sibling node of a leaf is a leaf.*

(ii) *If* $|\mathbf{T}_{x_1}| = |\mathbf{T}_{x_2}|$*, then* $\mathbf{T}_{x_1} = \mathbf{T}_{x_2} = \mathbf{T}_{\frac{(0,0)}{1}}$*.*

(iii) *If* $|\mathbf{T}_{x_1}| < |\mathbf{T}_{x_2}|$*, then* $\mathbf{T}_{x_1} = \mathbf{T}_y$*.*

Proof. First, we show the sufficient condition. Let x be the internal node of a terminal tree $\mathbf{T}$, then x corresponds to the two-dimensional proper fraction $\frac{(a,b)}{r}$, where $a + b = r$. By definition of the remainder map, the nodes x_1 and x_2 correspond to $\frac{(\overline{-r}^a, \overline{b}^a)}{a}$ and $\frac{(\overline{a}^b, \overline{-r}^b)}{b}$, respectively. If x_1 is a leaf, then x_2 is also a leaf by Corollary 6.11. If x_1 is not a leaf , then there exists a nephew node of x_1. The node y denotes this nephew node. If $|\mathbf{T}_{x_1}| = |\mathbf{T}_{x_2}|$, then $a = b$ and $\mathbf{T}_{x_1} = \mathbf{T}_{x_2} = \mathbf{T}_{\frac{(0,0)}{1}}$ by Corollary 6.11. Hence Theorem 6.12 holds in this case.

If $|\ \mathbf{T}_{x_1}\ |<|\ \mathbf{T}_{x_2}\ |$, then the nephew node y corresponds to $\frac{(\overline{-b}^a, \overline{-\overline{r}^b}^a)}{a}$. Since $\frac{(a,b)}{r}$ is terminal type, we have $\overline{a+b}^a = \overline{r}^a$. This gives $\overline{b}^a = \overline{r}^a - \overline{a}^a = \overline{r}^a$, and so $\overline{-b}^a = \overline{-r}^a$. Therefore, $\frac{(\overline{-r}^a, \overline{b}^a)}{a}$ is equal to $\frac{(\overline{-b}^a, \overline{-\overline{r}^b}^a)}{a}$ by Proposition 6.10.

Next we will show the converse. Assume that a tree $\mathbf{T}$ satisfies conditions (i) $\sim$ (iii). Let x be the internal node which has a left child node x_1 and a right child node x_2. Clearly, the followings hold.

($*$) If x_1 or x_2 is a leaf , then $\mathbf{T}_x = \mathbf{T}_{\frac{(0,0)}{1}}$ is a terminal tree by condition (i).

($**$) If $|\mathbf{T}_{x_1}| = 2$ and $|\mathbf{T}_{x_2}| = 2b \geq 2$, then $\mathbf{T}_x$ corresponds to $\mathbf{T}_{\frac{(1,b)}{b+1}}$ by condition (ii).

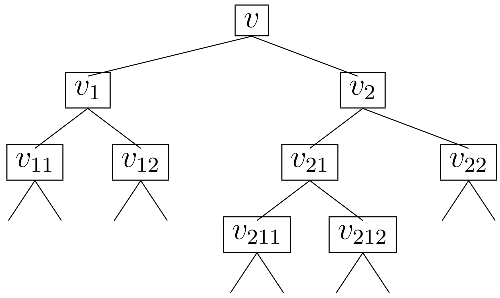
We need only consider the case where $|\mathbf{T}_{x_1}| = 2a \geq 2$ and $|\mathbf{T}_{x_2}| = 2b \geq 2$ hold. We claim that if $\mathbf{T}_{x_1} = \mathbf{T}_{\frac{(a_1,a_2)}{a}}$ and $\mathbf{T}_{x_2} = \mathbf{T}_{\frac{(b_1,b_2)}{b}}$ are terminal trees, then $\mathbf{T}_x$ is equal to the terminal tree $\mathbf{T}_{\frac{(a,b)}{a+b}}$. It is sufficient to show equations

$$\begin{aligned} a_1 &= \overline{-(a+b)}^a = \overline{-b}^a, \\ a_2 &= \overline{b}^a, \\ b_1 &= a, \\ \text{and} \qquad b_2 &= \overline{-(a+b)}^b = \overline{-a}^b. \end{aligned}$$

Since $\mathbf{T}_{x_1}$ and $\mathbf{T}_{x_2}$ are terminal, we have $a_1 + a_2 = a$ and $b_1 + b_2 = b$. We can assume $b > a$, and then there exists the nephew node of x_1. By condition (iii), $\mathbf{T}_y$ is equal to $\mathbf{T}_{x_1} = \mathbf{T}_{\frac{(a_1,a_2)}{a}}$ where y is the nephew node of x_1. On the other hand, $\mathbf{T}_y = \mathbf{T}_{\frac{(\overline{-b}^{b_1}, \overline{b_2}^{b_1})}{b_1}}$ holds by the definition of the remainder map. It follows that $a = b_1$, $a_1 = \overline{-b}^{b_1} = \overline{-b}^a$ and $a_2 = \overline{b_2}^{b_1} = \overline{b_2}^a$. By the assumption $b > a$ and $b_1 + b_2 = b$, we have $b_2 = b - b_1 = b - a = \overline{-a}^b$ and $a_2 = \overline{b-a}^a = \overline{b}^a$.

Therefore, whether $\mathbf{T}_x$ is terminal depends on whether $\mathbf{T}_{x_1}$ and $\mathbf{T}_{x_2}$ are terminal. Since the orders of subtrees $\mathbf{T}_{x_1}$ and $\mathbf{T}_{x_2}$ are strictly smaller than that of $\mathbf{T}_x$, we need only consider the cases of ($*$) and ($**$). Thus the binary tree $\mathbf{T}$ which satisfies conditions (i) $\sim$ (iii) is terminal. $\square$

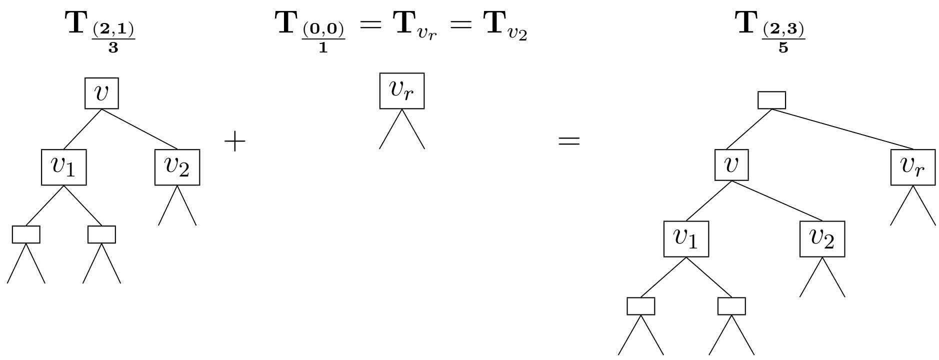
Example 6.13. Let us show an example. By Theorem 6.12, the following tree $\mathbf{T}_v$ is a terminal tree.

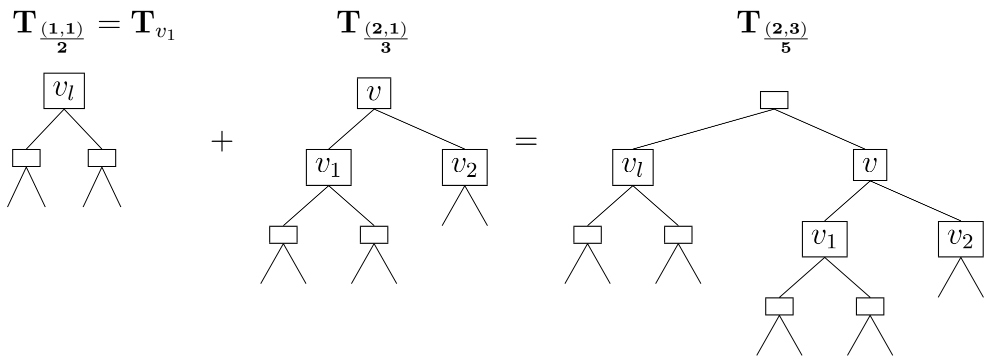


Clearly, this tree satisfies the condition (i). Since the subtrees $\mathbf{T}_{v_{11}}$, $\mathbf{T}_{v_{12}}$, $\mathbf{T}_{v_{211}}$ and $\mathbf{T}_{v_{212}}$ are equal to $\mathbf{T}_{\frac{(0,0)}{1}}$, so condition (ii) holds. The node v_1 has a nephew node v_{21}, and we have $|\mathbf{T}_{v_1}| < |\mathbf{T}_{v_2}|$ and $\mathbf{T}_{v_1} = \mathbf{T}_{v_{21}}$. This means condition (iii) holds for v_1. Similarly, condition (iii) holds for v_{22}. The other nodes have no nephew or satisfy $|\mathbf{T}_{x_1}| > |\mathbf{T}_{x_2}|$, so all internal nodes satisfy the condition (i),(ii) and (iii). Actually, the terminal tree $\mathbf{T}_{\frac{(2,3)}{5}}$ coincides with the above tree.

In general, if $\mathbf{T}_v$ satisfies the conditions (i),(ii),(iii) , $|\mathbf{T}_{v_1}| = 2a$ and $|\mathbf{T}_{v_2}| = 2b$, then $\mathbf{T}_v = \mathbf{T}_{\frac{(a,b)}{a+b}}$, where v_1 and v_2 are children of v.

As an application of this theorem, the Fujiki-Oka resolution of a new terminal quotient singularity can be constructed by combining two binary trees. Let $\mathbf{T}_v$ be a terminal tree which has a left child v_1 and a right child v_2. We will denote by $\mathbf{T}_l$ (resp. $\mathbf{T}_r$) the terminal tree which coincides with $\mathbf{T}_{v_1}$ (resp. $\mathbf{T}_{v_2}$). Then we can combine $\mathbf{T}_l$ with $\mathbf{T}_v$ from left, or $\mathbf{T}_r$ with $\mathbf{T}_v$ from right. This tree is a terminal tree by Theorem 6.12. The following shows the example of $\mathbf{T}_v = \mathbf{T}_{\frac{(2,1)}{3}}$.





6.3 Gorenstein canonical trees

We charactrize the shape of Gorenstein canonical trees as in the previous section. Let G be a cyclic subgroup of $\mathrm{SL}(3, \mathbb{C})$. Then $\mathbb{C}^3/G$ has a Gorenstein canonical singularity ([35], [40]). To consider the Fujiki-Oka resolution, we assume $\mathbb{C}^3/G$ has a semi-isolated singularity (see Section 2.2). In other words, G is generated by $\frac{1}{r}(1, a, r-a-1)$. Hence we treat the two-dimensional proper fraction $\frac{(a, r-a-1)}{r}$ in this section.

Definition 6.14. *A two-dimensional proper fraction $\frac{(a,b)}{r}$ is Gorenstein canonical if $a+b+1 = r$. In addition, $\mathbf{T}_{\frac{(a,b)}{r}}$ is Gorenstein canonical tree if it is obtained by Gorenstein canonical fraction, or it is a trivial tree.*

The following proposition is proved almost in the same way as Proposition 6.10.

Proposition 6.15. *If $\mathbf{T}_{\frac{(a,b)}{r}}$ is a Gorenstein canonical tree. Then all subtrees are also Gorenstein canonical.*

Corollary 6.16. *Let $\mathbf{T}_{\frac{(a,b)}{r}}$ be a Gorenstein canonical tree. Then $|\mathbf{T}_{\frac{(a,b)}{r}}| = r+1$.*

Proof. We can easily confirm that the claim holds for $\mathbf{T}_{\frac{(r-1,0)}{r}}$, $\mathbf{T}_{\frac{(0,r-1)}{r}}$ and $\mathbf{T}_{\frac{(0,0)}{1}}$. We assume the claim holds for $r \leq k-1$. If $r = k$, that is $\mathbf{T}_{\frac{(a,b)}{k}}$ with $a, b \neq 0$, we have the equation

$$|\mathbf{T}_{\frac{(a,b)}{k}}| = |\mathbf{T}_{\frac{(\overline{-k}^a, \overline{b}^a)}{a}}| + |\mathbf{T}_{\frac{(\overline{a}^b, \overline{-k}^b)}{b}}|.$$

Since $a, b \leq k-1$, $|\mathbf{T}_{\frac{(a,b)}{k}}| = a+1+b+1 = k+1$ by assumption. Therefore, the statement holds by induction. $\square$

We will give the Gorenstein canonical tree version of Theorem 6.12. This theorem can be proved in the same way as Theorem 6.12. In the following theorem, a and b denote positive integers.

Theorem 6.17. *Let* $\mathbf{T}$ *be a full binary tree. Let* x_1 *be an arbitrary node which has a parent node* x*, a sibling node* x_2 *and a nephew node* y*. Then* $\mathbf{T}$ *is Gorenstein canonical if and only if* $\mathbf{T}$ *satisfies the following conditions.*

(i) *If* $|\mathbf{T}_{x_1}| = |\mathbf{T}_{x_2}|$*, then* $\mathbf{T}_{x_1} = \mathbf{T}_{\frac{(a-1,0)}{a}}$ *and* $\mathbf{T}_{x_2} = \mathbf{T}_{\frac{(0,a-1)}{a}}$.

(ii) *If* $|\mathbf{T}_{x_1}| < |\mathbf{T}_{x_2}|$ *and* x_1 *is a leaf, then* y *is also a leaf.*

(iii) *If* $|\mathbf{T}_{x_1}| < |\mathbf{T}_{x_2}|$ *and* x_1 *has two children* y_1 *and* y_2 *with* $|\mathbf{T}_{y_1}| = \alpha, |\mathbf{T}_{y_2}| = 1$*, then* y *has two children* z_1, z_2 *such that* $|\mathbf{T}_{z_1}| = 1, |\mathbf{T}_{z_2}| = \alpha$*.*

(iv) *If* $|\mathbf{T}_{x_1}| < |\mathbf{T}_{x_2}|$ *and* x_1 *has two children* y_1 *and* y_2 *with* $|\mathbf{T}_{y_1}| = \alpha, |\mathbf{T}_{y_2}| = \beta > 1$*, then* y *has two children* z_1, z_2 *such that* $|\mathbf{T}_{z_1}| = \alpha + 1, |\mathbf{T}_{z_2}| = \beta - 1$*.*

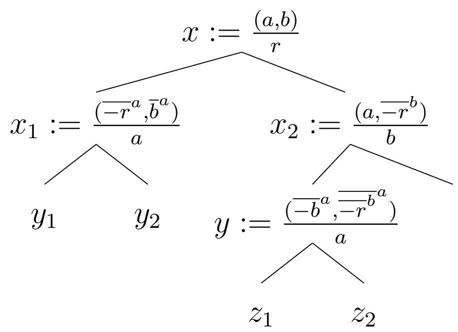
Proof. First, assume that $\mathbf{T}$ is a Gorenstein canonical tree. We check the case (i). If $|\mathbf{T}_{x_1}| = |\mathbf{T}_{x_2}|$, then

$$\mathbf{T}_{x_1} = \mathbf{T}_{\frac{(b,a-b-1)}{a}} \text{ and } \mathbf{T}_{x_2} = \mathbf{T}_{\frac{(c,a-c-1)}{a}}$$

for some positive integers a, b and c. This leads to $\mathbf{T}_x = \mathbf{T}_{\frac{(a,a)}{2a+1}}$. Thus we see that $b = a - 1$ and $c = 0$.

In the case (ii), we have $\mathbf{T}_x = \mathbf{T}_{\frac{(0,a-1)}{a}}$. It follows that $\mathbf{T}_{x_2} = \mathbf{T}_{\frac{(0,a-2)}{a-1}}$ and the nephew node y is a leaf.

Next, we check the cases (iii) and (iv) . Let $\mathbf{T}_x = \mathbf{T}_{\frac{(a,b)}{r}}$ with $a + b + 1 = r$, $a < b$. Then the two-dimensional proper fractions corresponding to $\mathbf{T}_{x_1}, \mathbf{T}_{x_2}$ and $\mathbf{T}_y$ are $\frac{(\overline{-r}^a, \overline{b}^a)}{a}$, $\frac{(a, \overline{-r}^b)}{b}$ and $\frac{(\overline{-b}^a, \overline{\overline{-r}^b}^a)}{a}$, respectively.



If $|\mathbf{T}_{y_1}| = \alpha, |\mathbf{T}_{y_2}| = 1$, then $\mathbf{T}_{x_1} = \mathbf{T}_{\frac{(\alpha-1,0)}{\alpha}}$. It implies $\overline{b}^a = \overline{-b}^a = 0$. Thus (iii) holds. In the case of $|\mathbf{T}_{y_1}| = \alpha$ and $|\mathbf{T}_{y_2}| = \beta > 1$, we have $\overline{b}^a \neq 0$ and $\overline{-b}^a = a - \overline{b}^a$. Since $\mathbf{T}_y$ is a Gorenstein canonical tree, $\overline{-b}^a + \overline{\overline{-r}^b}^a + 1 = a$. It follows that $\overline{\overline{-r}^b}^a = \overline{b}^a - 1$. We conclude from Corollary 6.16 that $\beta = |\mathbf{T}_{y_2}| = \overline{b}^a + 1$ and $|\mathbf{T}_{z_2}| = \overline{\overline{-r}^b}^a + 1 = \overline{b}^a = \beta - 1$.

On the other hand, we have $\alpha = |\mathbf{T}_{y_1}| = \overline{-r}^a + 1 = a - \overline{b}^a$ since $\mathbf{T}_{x_1}$ is a Gorenstein canonical tree. Hence $|\mathbf{T}_{z_1}| = \overline{-b}^a + 1 = a - \overline{b}^a + 1 = \alpha + 1$.

It remains to prove that if a full binary tree $\mathbf{T}$ satisfies conditions (i) to (iv) then $\mathbf{T}$ is a Gorenstein canonical tree. Let x' denote the root of $\mathbf{T}$, and x'_1, x'_2 the children of x'. In the same way as in the proof of Theorem 6.12, if $\mathbf{T}_{x'_1} = \mathbf{T}_{\frac{(a_1,a_2)}{a}}$ and $\mathbf{T}_{x'_2} = \mathbf{T}_{\frac{(b_1,b_2)}{b}}$ are Gorenstein canonical trees where $a, b > 0$, then $\mathbf{T}_{x'}$ is equal to the Gorenstein canonical tree $\mathbf{T}_{\frac{(a,b)}{a+b+1}}$. We need only consider the case where x'_1 or x'_2 is a leaf. If x'_1 and x'_2 are leaves, then $\mathbf{T} = \mathbf{T}_{\frac{(0,0)}{1}}$. Suppose that $\mathbf{T}_{x'_1} = \mathbf{T}_{\frac{(a_1,a_2)}{a}}$ and x'_2 is a leaf. By the condition (ii), $\mathbf{T}_{x'} = \mathbf{T}_{\frac{(a,0)}{a+1}}$ and it is Gorenstein canonical tree. Therefore we obtain the latter assertion by the similar inductive arguments. $\square$

Let us explain how conditions (i)$\sim$(iv) characterize the shape of a binary tree. First, the condition (i) means that subtree $\mathbf{T}_x$ of a Gorenstein canonical tree coincides with the following tree (Fig.5) if $|\mathbf{T}_{x_1}| = |\mathbf{T}_{x_2}|$. The condition (ii) means that a nephew node of a leaf is also a leaf if it exists in the Gorenstein canonical tree. Figure 6 shows the

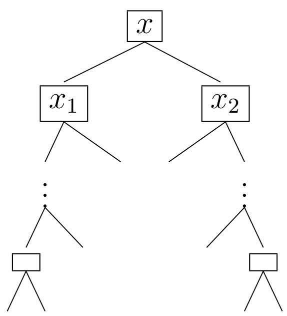


Fig. 5: The tree which satisfies condition (i).

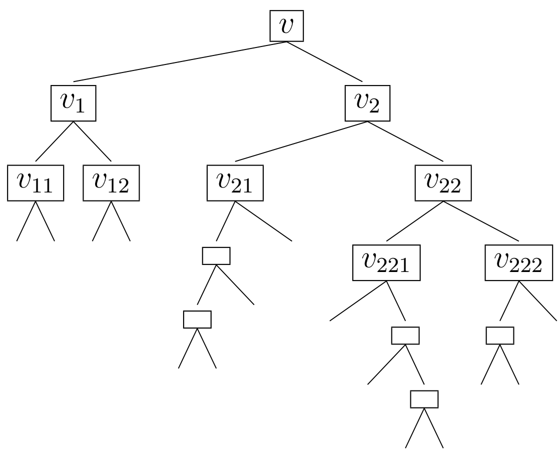


Fig. 6: The Gorenstein canonical tree $\mathbf{T}_{\frac{(3,10)}{14}}$.

example of a Gorenstein canonical tree obtained by $\frac{(3,10)}{14}$. In this case, let us focus on the node v_1. Since $|\ \mathbf{T}_{v_1}\ | < |\ \mathbf{T}_{v_2}\ |$ and v_1 has two children v_{11} and v_{12} with $|\mathbf{T}_{v_{11}}| = 2$ and $|\mathbf{T}_{v_{12}}| = 2 > 1$, this is the case where the condition (iv) is applied. Actually, v_{21} has children of size three and one respectively. Similarly, the relationship between v_{21} and v_{221} is obtained by the condition (iii). In general, the proper fractions of the nodes which are nephew nodes each other in the canonical Gorenstein tree have the following properties. Namely, they have the same denominator and the numerators systematically change as

$$\frac{(a,b)}{r} \to \frac{(a+1,b-1)}{r} \to \cdots \to \frac{(a+b-1,1)}{r} \to \frac{(a+b,0)}{r} \to \frac{(0,a+b)}{r} \to \frac{(1,a+b-1)}{r} \to \cdots.$$

In this example, the nodes v_1, v_{21}, v_{221} are nephew nodes each other such that their proper fractions are $\frac{(1,1)}{3}$, $\frac{(2,0)}{3}$ and $\frac{(0,2)}{3}$ respectively.

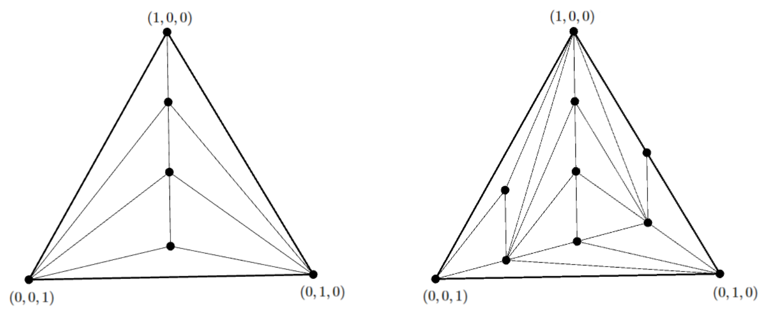


Fig. 7: Fujiki-Oka resolutions of type $\frac{1}{7}(1,3,3)$ and $\frac{1}{14}(1,3,10)$.

Next, let us see the subdivision in the case (i) and (iii). Figure 6 shows the fans which are subdivided by the Fijiki-Oka resolutions for the quotient singularity of type $\frac{1}{7}(1,3,3)$ and $\frac{1}{14}(1,3,10)$. In these cases, the subdivision processes occur at the common Oka centers of the common faces of both sides of semi-unimodular cones. Generally, the subtree which satisfies the condition (i) or (iii) induces the above subdivision.

Similarly as for a terminal tree, we can get a new Gorenstein canonical tree by connecting two Gorenstein canonical trees which satisfy the conditions of Theorem 6.17. In other words, we can construct a crepant resolution for the quotient singularity of a higher order cyclic group. Thus, if we can extend this result to the case of general dimensions (we need the n-ary tree instead of the binary tree), we can construct many examples of cyclic quotient singularities which possess a crepant resolution in higher dimension.

References

[1] T. Ashikaga, *Multidimensional continued fractions for cyclic quotient singularities and Dedekind sums*, Kyoto J. Math. **59** (2019), no. 4, 993–1039.

[2] C. Bouvier and G.Gonzalez-Sprinberg, *Système générateur minimal, diviseurs essentiels et G-désingularisations de variétés toriques*, Tohoku Math. J. (2) **47** (1995), no.1, 125–149.

[3] V. V. Batyrev and D. I. Dais, *Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry*, Topology **35** (1996), no. 4, 901–929.

[4] T. Bridgeland, A. King and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554.

[5] A. Craw and M. Reid, *How to calculate* A-Hilb$(\mathbb{C}^3)$, Geometry of toric varieties, 129–154, Sémin. Congr., **6**, Soc. Math. France, Paris, 2002.

[6] D. I. Dais, *Resolving* 3*-dimensional toric singularities*, Geometry of toric varieties, 155–186, Sémin. Congr., **6**, Soc. Math. France, Paris, 2002.

[7] D. I. Dais, M. Henk, and G. M. Ziegler, *All abelian quotient c.i.-singularities admit projective crepant resolutions in all dimensions*, Adv. Math. **139** (1998), no. 2, 194–239.

[8] D. I. Dais, M. Henk, and G. M. Ziegler, *On the existence of crepant resolutions of Gorenstein Abelian quotient singularities in dimensions* ≥ 4, Algebraic and geometric combinatorics, 125–193, Contemp. Math., **423**, Amer. Math. Soc., Providence, RI, 2006.

[9] D. I. Dais, C. Haase and G. M. Ziegler, *All toric local complete intersection singularities admit projective crepant resolutions*, Tohoku Math. J. (2) **53** (2001), no. 1, 95–107.

[10] D. I. Dais, U. U. Haus and M. Henk, *On crepant resolutions of* 2*-parameter series of Gorenstein cyclic quotient singularities*, Results Math. **33** (1998), no.3-4, 208–265.

[11] D. I. Dais and M. Henk, *On a series of Gorenstein cyclic quotient singularities admitting a unique projective crepant resolution*, preprint, `math.AG/9803094`.

[12] V. I. Danilov, *Birational geometry of three-dimensional toric varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 971–982, 1135.

[13] S. Davis, T. Logvinenko and M. Reid, *How to calculate* A-Hilb$(\mathbb{C}^n)$ *for* $\frac{1}{r}(a, b, 1, \ldots, 1)$, preprint, available at: `https://www.cantab.net/users/t.logvinenko/Traps/Traps2MR1.pdf`.

[14] A. Fujiki, *On resolutions of cyclic quotient singularities*, Publ. Res. Inst. Math. Sci. **10** (1974/75), no. 1, 293–328.

[15] T. Hayashi, Y. Ito and Y. Sekiya, *Existence of crepant resolutions*, Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata's sixtieth birthday, 185–202, Adv. Stud. Pure Math., **74**, Math. Soc. Japan, Tokyo, 2017.

[16] A. Ishii, *On the McKay correspondence for a finite small subgroup of* $GL(2;\mathbb{C})$, J. Reine Angew. Math. **549** (2002), 221–233.

[17] A. Ishii, Y. Ito and A. Nola de Celis, *On* G/N-Hilb *of* N-Hilb, Kyoto J. Math. **53**, (2013), no. 1, 91–130.

[18] Y. Ito, *Crepant resolution of trihedral singularities and the orbifold Euler characteristic*, Internat. J. Math. **6** (1995), no. 1, 33–43.

[19] Y. Ito, *Gorenstein quotient singularities of monomial type in dimension three*, J. Math. Sci. Univ. Tokyo **2** (1995), no. 2, 419–440.

[20] S. Ishii and J. Kollár, *The Nash problem on arc families of singularities*, Duke Math. J. **120** (2003), no. 3, 601-–620.

[21] Y. Ito and I. Nakamura, *Hilbert schemes and simple singularities*, New trends in algebraic geometry (Warwick, 1996), 151–233, London Math. Soc. Lecture Note Ser., **264**, Cambridge Univ. Press, Cambridge, 1999.

[22] Y. Ito and M. Reid, *The McKay correspondence for finite subgroups of* $\mathrm{SL}(3, \mathbb{C})$, Higher dimensional Complex Varieties, Higher-dimensional complex varieties (Trento, 1994), 221–240, de Gruyter, Berlin, 1996.

[23] S. J. Jung, *Terminal quotient singularities in dimension three via Variation of GIT*, J. Algebra **468** (2016), 354–394.

[24] R. Kidoh, *Hilbert schemes and cyclic quotient singularities*, Hokkaido Math. J. **30** (2001), no. 1, 91–103.

[25] O. Kedzierski, *Cohomology of the G-Hilbert scheme for* $\frac{1}{r}(1,1,r-1)$, Serdica Math. J. **30** (2004), no.2-3, 293–302.

[26] O. Kedzierski, *Danilov's resolution and representations of the McKay quiver*, Tohoku Math. J. (2) **66** (2014), no. 3, 355–375.

[27] D. G. Markushevich, *Resolution of* $\mathbb{C}^3/H_{168}$, Math. Ann. **308** (1997), no. 2, 279–289.

[28] D. G. Markushevich, *Resolution of singularities (Toric Method)*: Appendix in the article: D. G. Markushevich, M. A. Olshanetsky and A. M. Perelomov, *Description of a class of superstring compactifications related to semi-simple Lie algebras*, Commun. in Math. Phys. **111** (1987) 247–274.

[29] D. G. Markushevich, M. A. Olshanetsky and A. M. Perelomov, *Description of a class of superstring compactifications related to semi-simple Lie algebras*, Commun. in Math. Phys. **111** (1987) 247–274.

[30] D. Matsushita, *On smooth* 4*-fold flops*, Saitama Math. J. **15** (1997), 47–54.

[31] J. McKay, *Graphs, singularities and finite groups*, The Santa Cruz Conference on Finite Groups (Univ. California, SantaCruz, Calif., 1979), pp. 183–186, Proc. Sympos. Pure Math., **37**, Amer. Math. Soc., Providence, R.I., 1980.

[32] I. Nakamura, *Hilbert schemes of abelian group orbits*, J Algebraic Geom. **10** (2001), no. 4, 757–759.

[33] T. Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties.*, Translated from the Japanese. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **15**. Springer-Verlag, Berlin, 1988.

[34] M. Oka, *On the resolution of hypersurface singularities*, Complex analytic singularities, 405–436, Adv. Stud. Pure Math., **8**, North-Holland, Amsterdam, 1987.

[35] M. Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[36] S. S. Roan, *Minimal resolutions of Gorenstein orbifolds in dimension three*, Topology **35** (1996), no. 2, 489–508.

[37] K. Sato, *Existence of crepant resolution for abelian quotient singularities by order p elements in dimension* 4, Saitama Math. J. **27** (2010), 9–23.

[38] K. Sato and Y. Sato, *Crepant Property of Fujiki-Oka Resolutions for Gorenstein Abelian Quotient Singularities*, Nihonkai Math. J. **32** (2021), 41–69.

[39] Y. Sato, *Fujiki-Oka resolution for three-dimensional cyclic quotient singularities via binary trees*, Tokyo Journal of Mathematics (in press).

[40] K. Watanabe, Certain invariant subrings are Gorenstein I, Osaka J. Math. **11** (1974), 1–8.

[41] S. S. Yau, Y. Yu, *Gorenstein quotient singularities in dimension three*, Mem. Amer. Math. Soc. **105** (1993), no. 505.

Graduate School Of Mathematical Sciences, University Of Tokyo 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.
E-mail:yusuke.sato@ipmu.jp