博士論文

論文題目　Inverse problems for hyperbolic partial differential equations with time-dependent coefficients

(時間依存する係数を持つ双曲型偏微分方程式の逆問題)

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Preface

The author considers several inverse problems for hyperbolic partial differential equations with time-dependent coefficients. He focuses on first-order hyperbolic equations in part I and second-order hyperbolic equations in part II. Both chapters consist of three chapters. He mentions contents of both parts below.

Part I

The author considers inverse source and coefficient problems for non-degenerate first-order hyperbolic equations in chapter 1, first-order symmetric hyperbolic systems in chapter 2, and degenerate first-order hyperbolic equations in chapter 3. He mentions abstracts of three chapters.

Chapter 1. We prove global Lipschitz stability for inverse source and coefficient problems for the first-order linear hyperbolic equation

$$A^0(x,t)\frac{\partial}{\partial t}u + A(x,t)\cdot\nabla u + p(x,t)u = R(x,t)f(x),$$

the coefficients of which depend on both space and time. We use a global Carleman estimate, and a crucial point, introduced in this chapter, is the choice of the length of integral curves of a vector field generated by the principal part of the hyperbolic operator to construct a weight function for the Carleman estimate. To define the weight function using integral curves, we define dissipativeness for vector-valued functions. These integral curves coincides with the characteristic curves in the case when the vector field is independent of time. This chapter is based on Floridia and Takase [23].

Chapter 2. We introduce geometric analysis to inverse problems for the strongly coupled symmetric hyperbolic system

$$\sum_{\mu=0}^{n} A^\mu(t,x)\frac{\partial}{\partial x^\mu}u + p(t,x)u = R(t,x)f(x),$$

where A^μ are matrix-valued functions. Although one can easily obtain Carleman estimates for weakly coupled systems, few results are known for strongly coupled systems. In this chapter, we establish Carleman estimates under assumptions that suitable weight functions for the systems exist on compact manifolds and prove global Lipschitz stability for an inverse source problem of determining the spatially varying function f by applying the Carleman estimates. We provide a unifying approach to hyperbolic inverse problems through geometric analysis. This chapter is partly based on Floridia, Takase, and Yamamoto [25].

Chapter 3. We prove an observability inequality for a degenerate transport equation with time-dependent coefficients

$$\frac{\partial}{\partial t}u + H(t)\cdot\nabla u = 0,$$

where $H(0) = 0$. First we introduce a local in time Carleman estimate for the degenerate equation assuming $|H'(t)|$ is uniformly positive, then we apply it to obtain a global in time observability inequality by using also an energy estimate. We apply the methodology without partitions to non-degenerate equations as well. This chapter is based on Floridia and Takase [24].

Part II

The author considers inverse problems for systems of wave equations on Lorentzian manifolds in chapter 4 and one-dimensional Saint-Venant equations in chapter 5. In chapter 6, the author constructs infinitely many examples which violate unique continuation properties concerning two-dimensional wave and Schrödinger equations.

Chapter 4. A system of wave equations on a Lorentzian manifold, the coefficients of which depend on time relates to the Einstein equation in general relativity. We consider inverse source problem for the system

$$\Box_g h + a(t,x)h = S(t,x)f(x),$$

where $\Box_g$ denotes the Laplace–Beltrami operator with respect to a Lorentzian metric g. Having established the Carleman estimate with a second large parameter for the operator $\Box_g$ on a Lorentzian manifold under the assumption independent of a choice of local coordinates on a suitable weight function, we consider its application to the inverse source problem for the system and prove local Hölder stability. This chapter is based on Takase [64].

Chapter 5. The one-dimensional Saint-Venant equation describes unsteady water flow in channels and is derived from the one-dimensional Euler equation by imposing several physical assumptions. In this chapter, we consider the linearized and simplified equation in the one-dimensional case featuring a mixed derivative term

$$\frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u + a\frac{\partial^2}{\partial x\partial t}u = R(x,t)f(x),$$

where $a > 0$ is a constant, and prove the global Lipschitz stability of the inverse source problem via a global Carleman estimate. This chapter is based on Takase [63].

Chapter 6. In 1963, Kumano-go presented one non-uniqueness example for the two-dimensional wave equation with a time-dependent potential. We construct infinitely many non-uniqueness examples for Cauchy problems of the two-dimensional wave and Schrödinger equations

$$Lu + V(x,t)u = 0,$$

where $L = \partial_t^2 - \Delta$ or $L = -i\partial_t - \Delta$, as a generalization of the construction by Kumano-go. The main tool is asymptotic analysis for the Bessel functions. This chapter is based on Takase [65].

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Contents

Part I

First-order hyperbolic equations

CHAPTER 1

Non-degenerate hyperbolic equations

1. Introduction

Let $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $T > 0$, and $Q := \Omega \times (0, T)$. For $a, b \in \mathbb{R}^d$, we denote by $a \cdot b$ the inner product on $\mathbb{R}^d$. We define the first-order partial differential operator P such that

$$Pu := A^0(x,t)\partial_t u + A(x,t) \cdot \nabla u,$$

where $A^0 \in C^1(\overline{Q}) \cap L^\infty(\Omega \times (0,\infty))$ is a positive function, and $A = (A^1, \cdots, A^d)^\mathsf{T} \in C^2(\overline{Q}; \mathbb{R}^d)$ is a vector-valued function on $\overline{Q}$. In this chapter, we obtain global Lipschitz stability results for three inverse problems for equations with the principal part of type P.

The arguments of this chapter are based on the Carleman estimates, which were introduced by Carleman in [12] to prove unique continuation properties for elliptic partial differential equations with not necessarily analytic coefficients, and the Bukhgeim–Klibanov method introduced in [8]. The methodology using the Carleman estimates is widely applicable to not only inverse problems and unique continuation (e.g., [6], [33], [35], [42], [43], [49], and [69]), but also control theory (e.g., [11], [14], [27], [28], [50], and [56]) for various partial differential equations.

Now, the author describes some results concerned with the operator P. For the radiative transport equation having the principal part of type

$$\partial_t u(x,v,t) + v \cdot \nabla u(x,v,t), \quad (x,v,t) \in \Omega \times \mathbb{S}^{d-1} \times (0,T),$$

where $\mathbb{S}^{d-1} := \{v \in \mathbb{R}^d \mid |v| = 1\}$ is a set of a velocity field, Klibanov and Pamyatnykh [44] and [45] proved the Carleman estimates and global uniqueness theorem for inverse coefficient problem of determining a zeroth-order coefficient. In [44] and [45], the weight function for the Carleman estimate was independent of the principal parts:

$$\varphi(x,t) = |x - x_0|^2 - \beta t^2,$$

where $x_0 \in \mathbb{R}^d$ and $\beta > 0$ were fixed. For the same weight function used for transport equations with space-dependent first-order coefficients, see also Gaitan and Ouzzane [29]. Machida and Yamamoto [53] and [54] also proved global Lipschitz stability for inverse coefficient problems, where they took a linear function as the weight function for the Carleman estimate:

$$\varphi(x,t) = \gamma \cdot x - \beta t,$$

where $\gamma \in \mathbb{R}^d$ and $\beta > 0$ were fixed. Recently, Lai and Li [48] proved Lipschitz stability for inverse source and coefficient problems of determining a zeroth-order coefficient under the assumption that there existed a suitable weight function for the Carleman estimate.

For first-order hyperbolic operators of type P with a variable principal part, Gölgeleyen and Yamamoto [31] proved Lipschitz stability and conditional Hölder stability for inverse source and inverse coefficient problems, where they assumed the existence of a suitable weight function $\varphi = \varphi(x,t)$ for the Carleman estimate satisfying

$$\min_{(x,t)\in\overline{Q}} P\varphi(x,t) > 0$$

when $A^0 \equiv 1$ and $A = A(x)$. In the same time-independent case, Cannarsa, Floridia, Gölgeleyen, and Yamamoto [9] proved local Hölder stability for inverse coefficient problems of determining the principal part and a zeroth-order coefficient, where they took a function

$$\varphi(x,t) = A(x)\cdot x - \beta t$$

as the weight function for the Carleman estimate, and determined the coefficients up to a local domain, depending on the weight function, from local boundary data. In the same time-independent case, we also mention that Gaitan and Ouzzane [29] proved global Lipschitz stability for inverse coefficient problem of determining a zeroth-order coefficient via the Carleman estimate.

In these results mentioned above, in general, one must impose some assumptions on the principal parts and weight functions to guarantee the Carleman estimates that is not needed in this chapter. Moreover, the author must note that these results were all for first-order equations with coefficients independent of time t. However, equations with time-dependent principal parts of type P often appear in mathematical physics, for example, the conservation law of mass in time-dependent velocity fields, and the mathematical analysis for such equations is needed (e.g., Taylor [68, Section 17.1] and Evans [20, Section 11.1]). In regard to first-order hyperbolic equations having time-dependent principal parts, although the theory about direct problems for the above equations is quite complete, there are some open questions for inverse problems due to the major difficulties in dealing with time-dependent coefficients. About inverse problems and time-dependent principal parts, we mention Cannarsa, Floridia, and Yamamoto [10] that proved an observability inequality for a non-degenerate case. Floridia and Takase [24] proved the observability inequality for a degenerate case, which was motivated by applications to inverse problems. In both papers, they dealt with the case $A^0 \equiv 1$ and $A = A(t)$. For more references regarding inverse problems and controllability for conservation laws with time-dependent coefficients, see [32], [40], [41], and [46]. Regarding inverse problems for nonlinear first-order equations, readers are referred to Esteve and Zuazua [19], which studies Hamilton–Jacobi equations (see also Porretta and Zuazua [56]).

For the second-order hyperbolic equations with time-dependent coefficients, the literature about inverse problems is more extensive. In this context, Jiang, Liu, and Yamamoto [37], and Yu, Liu, and Yamamoto [70] proved the local Hölder stability for inverse source and coefficient problems in the Euclidean space assuming the Carleman estimates existed. Takase [64] proved local Hölder stability for the wave equation and obtained some sufficient conditions for the Carleman estimate by using geometric analysis on Lorentzian manifolds, which is presented in part II.

Finally, the author notes that, on the well-posedness by the method of characteristics of first-order hyperbolic equations with principal parts of type P, readers are referred to John [39, Chapter 1], Rauch [57, Chapter 1], Evans [20, Chapter

3], and Bressan [7]. In addition to that, for symmetric hyperbolic systems, readers are referred to Rauch [57, Chapter 2], Ringström [58, Chapter 7], and Taylor [68, Section 16.2].

Although a large number of studies have been made on inverse problems for first-order equations, as already mentioned, what seems to be lacking is analysis for equations with time-dependent coefficients. In this chapter the author investigates equations with coefficients depending on both space and time. The important point the author wants to make is the decisive way to choose the weight function in the Carleman estimate for applications to inverse problems. Indeed, the weight function of the Carleman estimate (see Proposition 1.14 and Lemma 1.15) is linear in t, which is similar to Machida–Yamamoto [53], Gölgeleyen–Yamamoto [31], and Cannarsa–Floridia–Yamamoto [10]. However, the novelty is that the spatial term of the weight function in the Carleman estimate is the length of integral curves of the vector-valued function $A(\cdot, 0)$, which is different from the ones in all the above results ([10], [24], [29], [31], [44], [45], and [53]) and a new attempt. Owing to the choice, we need not assume any assumptions on A to guarantee the Carleman estimates like in [31] and [9], but assume only the finiteness of the length of integral curves (see Definition 1.4 and (1.2)). The author remarks that these integral curves correspond to the characteristic curves in the case $A^0 \equiv 1$ and $A = A(x)$. In addition, the author notes that thanks to the above linearity with respect to t, we do not need to extend the solution to $(-T, 0)$, which enables us to apply the Carleman estimate to inverse problems for wider functional spaces of time-dependent coefficients A^0 and A.

A structure of this chapter is following. The main results in this chapter are global Lipschitz stability for the inverse source problem (Theorem 1.11), inverse coefficient problem to determine the zeroth-order coefficient (Theorem 1.12), and inverse coefficient problem to determine the time-independent principal part (Theorem 1.13). After describing some settings, we present them in section 2. In section 3, we establish the global Carleman estimate (Proposition 1.14), which is the main tool to prove the main results, under the assumption that a suitable weight function exists. After that, we prove the existence of such a weight function by taking the length of integral curves generated by the vector-valued function $A(\cdot, 0)$ (Lemma 1.15). In addition, in section 3, we introduce energy estimates needed to prove the main results. In section 3, we show the proofs of the main results. In section 5, we give the proofs of auxiliary and original results.

2. Preliminary and statements of main results

Before showing main results, we describe some definitions and settings needed to present them.

Definition 1.1. *For a vector-valued function $X \in C^2(\overline{\Omega}; \mathbb{R}^d)$ and $x \in \overline{\Omega}$, a C^2 curve $c : [-\eta_1, \eta_2] \to \overline{\Omega}$ for some $\eta_1 \geq 0$ and $\eta_2 \geq 0$ with $\eta_1 + \eta_2 > 0$ is called an integral curve of X through x if it solves the following initial problem for ordinary differential equations*

$$\begin{cases} c'(\sigma) := \dfrac{dc}{d\sigma}(\sigma) = X(c(\sigma)), \quad \sigma \in [-\eta_1, \eta_2], \\ c(0) = x. \end{cases}$$

REMARK 1.2. *If c_x denotes the integral curve of X through x, then $c_x(\sigma)$ is C^2 with respect to $x \in \overline{\Omega}$.*

DEFINITION 1.3. *Let $a, b \in \mathbb{R}$ with $a < b$. An integral curve $c : [a, b] \to \overline{\Omega}$ is called maximal if it cannot be extended in $\overline{\Omega}$ to a segment $[a - \eta_1, b + \eta_2]$ for some $\eta_1 \geq 0$ and $\eta_2 \geq 0$ with $\eta_1 + \eta_2 > 0$.*

DEFINITION 1.4. *A vector-valued function $X \in C^2(\overline{\Omega}; \mathbb{R}^d)$ is called dissipative if the maximal integral curve c_x of X through x is defined on a finite segment $[\sigma_-(x), \sigma_+(x)]$ and $\sigma_- \in C(\overline{\Omega}) \cap H^2(\Omega)$.*

REMARK 1.5. *If $X \in C^2(\overline{\Omega}; \mathbb{R}^d)$ is dissipative, then $c_x(\sigma_-(x))$, $c_x(\sigma_+(x)) \in \partial\Omega$, where c_x is the maximal integral curve of X through x.*

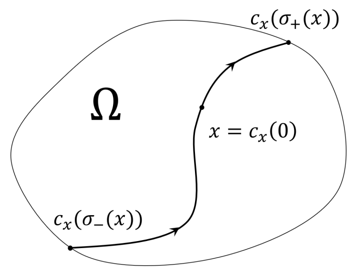

FIGURE 1. c_x is the maximal integral curve of X through x.

The terminology dissipative for vector fields seems not to be widely-used. However, the authors use this terminology on the analogy of CDRM (compact dissipative Riemannian manifold) used in a setting of integral geometry problems for tensor fields. In this subject, CDRM is equivalent to the absence of a geodesic of infinite length in a compact Riemannian manifold with strictly convex boundary (e.g., [62, Chapter 4]).

We assume the followings on the vector-valued function $A \in C^2(\overline{Q}; \mathbb{R}^d)$:

$$\exists \rho > 0 \text{ s.t. } \min_{(x,t)\in\overline{Q}} |A(x,t)| \geq \rho \; ; \tag{1.1}$$

$$\exists t_* \in [0, T) \text{ s.t. } A(\cdot, t_*) \text{ is dissipative.}$$

Without loss of generality, we assume $t_* = 0$ in the above, i.e.,

$$A(\cdot, 0) \text{ is dissipative} \tag{1.2}$$

because it suffices to consider the change of variables $\tilde{t} := t - t_*$ and $\tilde{A}(\cdot, \tilde{t}) := A(\cdot, \tilde{t} + t_*)$.

REMARK 1.6. *In the case $A^0 \equiv 1$ and $A = A(x)$,* (1.2) *means that any maximal characteristic curves have finite length.*

EXAMPLE 1.7. *Let $d = 2$ and $B_r := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < r^2\}$ for $r > 0$. Then, $X(x, y) := \begin{pmatrix} -x \\ -1 \end{pmatrix}$ on $\Omega = B_r \cap \{y > 0\}$ is dissipative because we see σ_- is smooth on $\overline{\Omega}$. However, $Y(x, y) := \begin{pmatrix} -y \\ x \end{pmatrix}$ on $B_r \setminus \overline{B_{\frac{r}{2}}}$ is not dissipative because we can not define σ_-.*

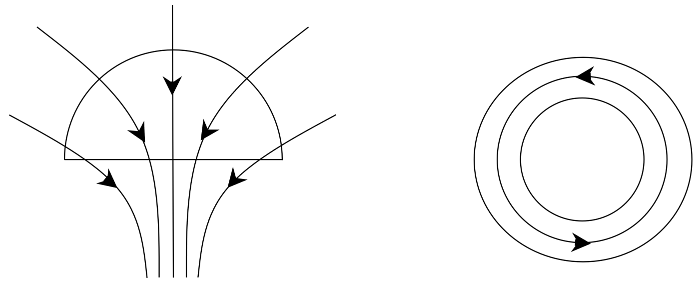

FIGURE 2. Pictures of X (left) and Y (right).

Under the assumption (1.2), we can give the following notations. For a fixed $x \in \overline{\Omega}$, let $c_x : [\sigma_-(x), \sigma_+(x)] \to \overline{\Omega}$ be the maximal integral curve of $A(\cdot, 0)$ through x, i.e., c_x satisfies

$$\begin{cases} c_x'(\sigma) = A(c_x(\sigma), 0), & \sigma \in [\sigma_-(x), \sigma_+(x)], \\ c_x(0) = x. \end{cases}$$

Since c_x is a rectifiable curve by (1.2), we can define the function φ_0 on $\overline{\Omega}$ as the length of the arc of the maximal integral curves defined on $[\sigma_-(x), 0]$:

$$\varphi_0(x) := \int_{\sigma_-(x)}^{0} |c_x'(\sigma)| d\sigma, \tag{1.3}$$

the integral of which is independent of a choice of parameters.

LEMMA 1.8. *Let $A \in C^2(\overline{Q}; \mathbb{R}^d)$ be a vector-valued function. Assume* (1.1) *and* (1.2). *Then, the function φ_0 defined by* (1.3) *is in the class $C(\overline{\Omega}) \cap H^2(\Omega)$.*

PROOF. It follows from Definition 1.4 and Remark 1.2. □

To prove the global Lipschitz stability for inverse problems for the hyperbolic equations, the observation time should be given large enough for the solutions to reach the boundaries owing to the finite propagation speeds (see Bardos, Lebeau, and Rauch [3]). Then, we define the following quantities to describe this situation mathematically.

For the positive function $A^0 \in C^1(\overline{\Omega}) \cap L^\infty(\Omega \times (0, \infty))$ and φ_0 defined by (1.3), we define the positive number

$$T_0 := \frac{\left(\sup_{x \in \Omega, t > 0} A^0(x,t)\right)\left(\max_{x \in \overline{\Omega}} \varphi_0(x)\right)}{\rho}. \tag{1.4}$$

Moreover, considering inverse problems for the hyperbolic equation with time-dependent principal part, we will assume

$$\exists C > 0 \text{ s.t. } \forall \xi \in \mathbb{R}^d, \ \forall (x,t) \in \overline{Q}, \quad |\partial_t A(x,t) \cdot \xi| \le C |A(x,t) \cdot \xi|. \tag{1.5}$$

The condition (1.5) will be decisive in the the energy estimate given in Lemma 1.16 and in the proofs of Theorem 1.11 and Theorem 1.12.

REMARK 1.9. *When $d = 1$,* (1.1) *implies* (1.5).

If a non-vanishing vector valued function A satisfies (1.5), then A has the following structure.

Proposition 1.10. *If a vector-valued function* $A \in C^2(\overline{Q}; \mathbb{R}^d)$ *satisfies* (1.1) *and* (1.5)*, then* A *can be represented by*

$$A(x,t) = A(x,0)e^{\int_0^t \phi(x,s)ds}, \quad (x,t) \in \overline{Q}$$

for some function $\phi \in C^1(\overline{Q})$.

The proof of Proposition 1.10 is presented in section 5. Proposition 1.10 is decisive in the realization of a weight function for the Carleman estimate, which will be given in Lemma 1.15.

Now, we define some notations. Set

$$\Sigma_+ := \{(x,t) \in \partial\Omega \times (0,T) \mid A(x,t) \cdot \nu(x) > 0\},$$

where we recall ν is the outer unit normal to $\partial\Omega$. Moreover, we set $\Sigma_- := (\Sigma_+)^c = (\partial\Omega \times (0,T)) \setminus \Sigma_+$.

We use the notations $H^0(\Omega) := L^2(\Omega)$, $H^0(0,T;H^1(\Omega)) := L^2(0,T;H^1(\Omega))$, and $\partial_t^0 w = w$ for a function w throughout this chapter to avoid notational complexity.

2.1. Inverse source problems. We consider the initial boundary value problem

$$\begin{cases} Pu + p(x,t)u = R(x,t)f(x) & \text{in } Q, \\ u = 0 & \text{on } \Sigma_-, \\ u(\cdot,0) = 0 & \text{on } \Omega, \end{cases} \tag{1.6}$$

where $p \in W^{1,\infty}(0,T;L^\infty(\Omega))$, $R \in H^1(0,T;L^\infty(\Omega))$, and $f \in L^2(\Omega)$. Given A^0, A, p, and R, we consider the inverse source problem to determine the source term f in Ω by observation data u on Σ_+.

Theorem 1.11. *Let* $A^0 \in C^1(\overline{Q}) \cap L^\infty(\Omega \times (0,\infty))$ *satisfying* $\min_{(x,t)\in\overline{Q}} A^0(x,t) > 0$*, and* $A \in C^2(\overline{Q}; \mathbb{R}^d)$ *satisfying* (1.1), (1.2)*, and* (1.5)*. Let* $p \in W^{1,\infty}(0,T;L^\infty(\Omega))$, $R \in H^1(0,T;L^\infty(\Omega))$*, and* $f \in L^2(\Omega)$ *satisfying*

$$\exists m_0 > 0 \text{ s.t. } |R(x,0)| \geq m_0 \quad \text{a.e. } x \in \Omega. \tag{1.7}$$

Assume

$$T_0 < T, \tag{1.8}$$

where T_0 *is defined by* (1.4)*, and there exists a function* u *satisfying* (1.6) *in the class*

$$u \in \bigcap_{k=1}^{2} H^k(0,T;H^{2-k}(\Omega)).$$

Then, there exists a constant $C > 0$ *independent of* f *and* u *such that*

$$\|f\|_{L^2(\Omega)} \leq C \sum_{k=0}^{1} \|\partial_t^k u\|_{L^2(\Sigma_+)}.$$

2.2. Inverse coefficient problems. We consider the initial boundary value problem

$$
\begin{cases} Pu + p(x,t)u = 0 & \text{in } Q, \\ u = g & \text{on } \Sigma_-, \\ u(\cdot,0) = \alpha & \text{on } \Omega, \end{cases} \tag{1.9}
$$

where $p \in W^{1,\infty}(0,T;L^\infty(\Omega))$, $g \in H^1(0,T;H^{\frac{1}{2}}(\partial\Omega))$, and $\alpha \in H^1(\Omega)$ satisfying the compatibility conditions. The author presents two nonlinear inverse coefficient problems.

2.2.1. *Zeroth-order coefficient.* Given A^0, A, g, and α, we consider the inverse coefficient problem to determine the time-independent zeroth-order coefficient $p = p(x)$ in Ω by observation data on Σ_+.

For a fixed $M > 0$, define the conditional set

$$D(M) := \{p \in L^\infty(\Omega) \mid \|p\|_{L^\infty(\Omega)} \le M\}.$$

THEOREM 1.12. *Let $M > 0$ be fixed, $A^0 \in C^1(\overline{Q}) \cap L^\infty(\Omega \times (0,\infty))$ satisfying $\min_{(x,t)\in\overline{Q}} A^0(x,t) > 0$, and $A \in C^2(\overline{Q};\mathbb{R}^d)$ satisfying* (1.1)*,* (1.2)*, and* (1.5)*. Let $p_i \in D(M)$ for $i = 1,2$, $g \in H^1(0,T;H^{\frac{1}{2}}(\partial\Omega))$, and $\alpha \in H^1(\Omega)$ satisfying*

$$\exists m_0 > 0 \text{ s.t. } |\alpha(x)| \ge m_0 \quad \text{a.e. } x \in \Omega. \tag{1.10}$$

Assume $T_0 < T$, where T_0 is defined by (1.4)*, and for $i = 1,2$ there exist functions u_i satisfying* (1.9) *with $p = p_i$ in the class*

$$u_i \in \bigcap_{k=1}^{2} H^k(0,T;H^{2-k}(\Omega))$$

such that

$$u_2 \in H^1(0,T;L^\infty(\Omega)) \text{ and } \|u_2\|_{H^1(0,T;L^\infty(\Omega))} \le M.$$

Then, there exists a constant $C > 0$ independent of $p_i \in D(M)$ for $i = 1,2$ such that

$$\|p_1 - p_2\|_{L^2(\Omega)} \le C \sum_{k=0}^{1} \|\partial_t^k u_1 - \partial_t^k u_2\|_{L^2(\Sigma_+)}.$$

2.2.2. *First-order coefficients.* We consider (1.9) with the time-independent principal coefficients A^0 and A, more precisely, with $A^0 \in C^1(\overline{\Omega})$ and $A \in C^2(\overline{\Omega};\mathbb{R}^d)$. Given p, finitely many initial values α, and boundary values g, we consider the inverse coefficient problem to determine the time-independent coefficients A^0 and A simultaneously by finitely many observation data on Σ_+.

Let $\rho > 0$ be fixed. We will assume that the unknown coefficients A^0 and A satisfy the following condition:

$$\frac{\left(\max_{x\in\overline{\Omega}} A^0(x)\right)\left(\max_{x\in\overline{\Omega}} \varphi_0(x)\right)}{\rho} < T, \tag{1.11}$$

where φ_0 is defined by (1.3).

For $A \in C^2(\overline{\Omega};\mathbb{R}^d)$, set

$$\Gamma_{+,A} := \{x \in \partial\Omega \mid A(x)\cdot\nu(x) > 0\}$$

and $\Gamma_{-,A} := \partial\Omega \setminus \Gamma_{+,A}$.

For fixed $M > 0$, $\rho > 0$, and a subset $\Gamma \subset \partial\Omega$, define the conditional set

$$D(M, \rho, \Gamma) := \left\{ (A^0, A) \in C^1(\overline{\Omega}) \times C^2(\overline{\Omega}; \mathbb{R}^d) \,\middle|\, \begin{cases} \|A^0\|_{C^1(\overline{\Omega})} + \|A\|_{C^2(\overline{\Omega};\mathbb{R}^d)} \leq M, \\ \min_{x\in\overline{\Omega}} A^0(x) \geq \rho, \ \min_{x\in\overline{\Omega}} |A(x)| \geq \rho, \\ (1.2),\ (1.11),\ \text{and } \Gamma_{+,A} \subset \Gamma \text{ hold.} \end{cases} \right\}.$$

THEOREM 1.13. *Let $M > 0$, $\rho > 0$, $\Gamma \subset \partial\Omega$ be a subset, and $(A_i^0, A_i) \in D(M, \rho, \Gamma)$ for $i = 1, 2$. Let $p \in W^{1,\infty}(0, T; L^\infty(\Omega))$, $g_m \in H^1(0, T; H^{\frac{1}{2}}(\partial\Omega))$, and $\alpha_m \in W^{1,\infty}(\Omega)$ for $m = 1, \dots, d + 1$ satisfying*

$$\exists m_0 > 0 \text{ s.t. } |p(x, 0)| \left| \det \begin{pmatrix} \alpha_1(x) & \cdots & \alpha_{d+1}(x) \\ \nabla\alpha_1(x) & \cdots & \nabla\alpha_{d+1}(x) \end{pmatrix} \right| \geq m_0 \quad a.e.\ x \in \Omega. \tag{1.12}$$

Assume that for $i = 1, 2$ and $m = 1, \dots, d + 1$ there exist functions $u_{i,m}$ satisfying (1.9) with $P = P_i := A_i^0\partial_t + A_i \cdot \nabla$, $g = g_m$, and $\alpha = \alpha_m$ in the class

$$u_{i,m} \in \bigcap_{k=1}^{2} H^k(0, T; W^{2-k,\infty}(\Omega))$$

such that for all $m = 1, \dots, d + 1$,

$$\sum_{k=1}^{2} \|u_{2,m}\|_{H^k(0,T;W^{2-k,\infty}(\Omega))} \leq M.$$

Then, there exists a constant $C > 0$ independent of $(A_i^0, A_i) \in D(M, \rho, \Gamma)$ for $i = 1, 2$ such that

$$\sum_{\mu=0}^{d} \|A_1^\mu - A_2^\mu\|_{L^2(\Omega)} \leq C \sum_{m=1}^{d+1} \|u_{1,m} - u_{2,m}\|_{H^1(0,T;L^2(\Gamma))}.$$

3. Carleman estimate and energy estimates

In this section, the author introduces the Carleman estimate and energy estimates needed to prove the main results.

3.1. Carleman estimate. In this subsection, we prove the global Carleman estimate for the operator $P + p(x, t)\cdot$, where $p \in L^\infty(Q)$. In section 3.1.1, we present the general statement for the Carleman estimate assuming the existence of a suitable weight function φ satisfying some sufficient conditions. In section 3.1.2, we construct such a weight function satisfying the sufficient conditions using φ_0 defined by (1.3).

3.1.1. *General statements.* To obtain the local in time Carleman estimate, we first assume the existence of a function $\varphi \in H^2(Q)$ satisfying

$$\exists \delta > 0 \text{ s.t. } P\varphi(x, t) \geq \delta \quad \text{a.e. } (x, t) \in Q. \tag{1.13}$$

PROPOSITION 1.14. *Let $A^0 \in C^1(\overline{Q})$ satisfying $\min_{(x,t)\in\overline{Q}} A^0(x, t) > 0$, $A \in C^1(\overline{Q}; \mathbb{R}^d)$, and $p \in L^\infty(Q)$. Assume that there exists a function $\varphi \in H^2(Q)$*

satisfying (1.13)*. Then, there exist constants* $s_* > 0$ *and* $C > 0$ *such that*

$$s^2 \int_Q e^{2s\varphi}|u|^2 dxdt + s\int_\Omega e^{2s\varphi(x,0)}|u(x,0)|^2 dx \tag{1.14}$$
$$\le C\int_Q e^{2s\varphi}|(P+p(x,t))u|^2 dxdt + Cs\int_{\Sigma_+} e^{2s\varphi}|u|^2 dSdt$$
$$+ Cs\int_\Omega e^{2s\varphi(x,T)}|u(x,T)|^2 dx$$

holds for all $s > s_*$ *and* $u \in \bigcap_{k=0}^{1} H^k(0,T;H^{1-k}(\Omega))$*, where* dS *denotes the area element of* $\partial\Omega$*.*

Proof. It suffices to prove Proposition 1.14 when $p \equiv 0$ due to the sufficiently large parameter s. Let $z := e^{s\varphi}u$ and $P_s z := e^{s\varphi}P(e^{-s\varphi}z)$ for $s > 0$. Then, we obtain

$$P_s z = Pz - sP\varphi z,$$

which implies

$$\begin{aligned}
\|P_s z\|^2_{L^2(Q)} &= \|Pz\|^2_{L^2(Q)} + 2(Pz, -sP\varphi z)_{L^2(Q)} + \|sP\varphi z\|^2_{L^2(Q)}\\
&\ge \|sP\varphi z\|^2_{L^2(Q)} + 2(Pz, -sP\varphi z)_{L^2(Q)}\\
&= s^2\int_Q |P\varphi|^2|z|^2 dxdt - s\int_Q P\varphi\Big(A^0\partial_t(|z|^2) + A\cdot\nabla(|z|^2)\Big)dxdt\\
&\ge s^2\int_Q \delta^2|z|^2 dxdt + s\int_Q \Big[\partial_t((P\varphi)A^0) + \nabla\cdot((P\varphi)A)\Big]|z|^2 dxdt - \mathcal{B},
\end{aligned}$$

by our assumption (1.13), where

$$\mathcal{B} := s\int_\Omega \Big[(P\varphi)A^0|z|^2\Big]_{t=0}^{t=T} dx + s\int_{\partial\Omega\times(0,T)} P\varphi(A(x,t)\cdot\nu(x))|z|^2 dSdt.$$

Therefore, there exists $C > 0$ such that

$$C\int_Q s^2\left[1 + O\left(\frac{1}{s}\right)\right]|z|^2 dxdt \le \|P_s z\|^2_{L^2(Q)} + \mathcal{B}$$

as $s \to +\infty$. By choosing $s > 0$ large enough, we complete the proof. □

3.1.2. *Realization of weight functions.* We construct the weight function $\varphi \in C(\overline{Q}) \cap H^2(Q)$ depending on the vector field generated by the coefficients A, and satisfying (1.13).

Lemma 1.15. *Let* $A^0 \in C(\overline{Q}) \cap L^\infty(\Omega\times(0,\infty))$ *satisfying* $\min_{(x,t)\in\overline{Q}} A^0(x,t) > 0$*, and* $A \in C^2(\overline{Q};\mathbb{R}^d)$ *be given functions satisfying* (1.1)*,* (1.2)*, and* (1.5)*. Then, for an arbitrary real number* $\beta > 0$ *independent of* T *satisfying*

$$0 < \beta < \frac{\rho}{\sup_{x\in\Omega, t>0} A^0(x,t)}, \tag{1.15}$$

the function φ *defined by*

$$\varphi(x,t) := \varphi_0(x) - \beta t, \quad (x,t) \in \overline{Q}, \tag{1.16}$$

with φ_0 *defined by* (1.3)*, is in the class* $\varphi \in C(\overline{Q}) \cap H^2(Q)$ *and satisfies* (1.13)*.*

PROOF. It is obvious that $\varphi \in C(\overline{Q}) \cap H^2(Q)$ by Lemma 1.8. We prove that φ defined by (1.16) satisfies (1.13). It follows that

$$\begin{aligned} P\varphi(x,t) &= A(x,t)\cdot\nabla\varphi_0(x) - \beta A^0(x,t) \\ &\geq A(x,t)\cdot\nabla\varphi_0(x) - \beta \sup_{x\in\Omega, t>0} A^0(x,t). \end{aligned} \tag{1.17}$$

For a fixed $x \in \Omega$, let $c_x : [\sigma_-(x), \sigma_+(x)] \to \overline{\Omega}$ be the maximal integral curve with $c_x(0) = x$ of $A(\cdot, 0)$. For a sufficiently small $\eta \in [\sigma_-(x), \sigma_+(x)]$, we set $x_\eta := c_x(\eta)$. Because we can verify

$$\begin{cases} \dfrac{d}{d\sigma}\big(c_x(\sigma+\eta)\big) = c_x'(\sigma+\eta) = A(c_x(\sigma+\eta), 0), \\ c_x(0+\eta) = x_\eta, \end{cases}$$

we have $c_{x_\eta}(\sigma) = c_x(\sigma+\eta)$ by the uniqueness of the solution to the initial problem of the ordinary differential equation. Hence, $\sigma_-(x_\eta) = \sigma_-(x) - \eta$ holds. Therefore, we obtain

$$\varphi_0(c_x(\eta)) = \varphi_0(x_\eta) = \int_{\sigma_-(x_\eta)}^0 |c_{x_\eta}'(\sigma)|d\sigma = \int_{\sigma_-(x)-\eta}^0 |c_x'(\sigma+\eta)|d\sigma = \int_{\sigma_-(x)}^\eta |c_x'(\sigma)|d\sigma.$$

Differentiating both sides with respect to η and substituting $\eta = 0$ yield

$$c_x'(0)\cdot\nabla\varphi_0(c_x(0)) = |c_x'(0)| = |A(x,0)|.$$

Therefore, by (1.5), Proposition 1.10, and (1.1), we obtain

$$\begin{aligned} A(x,t)\cdot\nabla\varphi_0(x) &= A(x,0)\cdot\nabla\varphi_0(x)e^{\int_0^t \phi(x,s)ds} \\ &= c_x'(0)\cdot\nabla\varphi_0(c_x(0))e^{\int_0^t \phi(x,s)ds} \\ &= |A(x,0)|e^{\int_0^t \phi(x,s)ds} \\ &= |A(x,t)| \geq \rho. \end{aligned} \tag{1.18}$$

Applying (1.18) to (1.17) yields

$$P\varphi(x,t) \geq \rho - \beta \sup_{x\in\Omega, t>0} A^0(x,t) > 0$$

for almost all $(x,t) \in Q$.

□

3.2. Energy estimates. The following Lemma 1.16 is the energy estimate for the first-order hyperbolic equations with the time-dependent principal part needed to prove Theorem 1.11 and Theorem 1.12. Moreover, we describe Lemma 1.17, which is the energy estimate for first-order hyperbolic equations with time-independent principal part needed to prove Theorem 1.13. Their proofs are presented in section 5.

For a positive function $A^0 \in C^1(\overline{Q})$ and $u \in \bigcap_{k=1}^{2} H^k(0,T; H^{2-k}(\Omega))$, we define the quantity

$$E(t) := \int_\Omega \left(A^0(x,t)|\partial_t u(x,t)|^2 + |u(x,t)|^2\right)dx, \quad t \in [0,T].$$

LEMMA 1.16. *Let* $A^0 \in C^1(\overline{Q})$ *satisfying* $\min_{(x,t)\in\overline{Q}} A^0(x,t) > 0$, $A \in C^1(\overline{Q};\mathbb{R}^d)$, $p \in W^{1,\infty}(0,T;L^\infty(\Omega))$, $R \in H^1(0,T;L^\infty(\Omega))$, *and* $f \in L^2(\Omega)$. *Then, there exists a constant* $C > 0$ *independent of* u *and* f *such that*

$$E(t) \le C\Big(\|\partial_t A \cdot \nabla u\|^2_{L^2(Q)} + \|f\|^2_{L^2(\Omega)}\Big) \tag{1.19}$$

holds for all $t \in [0,T]$ *and* $u \in \bigcap_{k=1}^{2} H^k(0,T;H^{2-k}(\Omega))$ *satisfying* (1.6).

Moreover, if we assume (1.5), *then there exists a constant* $C > 0$ *independent of* u *and* f *such that*

$$E(t) \le C\|f\|^2_{L^2(\Omega)} \tag{1.20}$$

holds for all $t \in [0,T]$ *and* $u \in \bigcap_{k=1}^{2} H^k(0,T;H^{2-k}(\Omega))$ *satisfying* (1.6).

LEMMA 1.17. *Let* $\ell \in \mathbb{N}$ *be a fixed number,* $A^0 \in C^1(\overline{\Omega})$ *satisfying* $\min_{x\in\overline{\Omega}} A^0(x) > 0$, $A \in C^1(\overline{\Omega};\mathbb{R}^d)$, $p \in W^{1,\infty}(0,T;L^\infty(\Omega))$, $R \in H^1(0,T;L^\infty(\Omega;\mathbb{R}^\ell))$, *and* $F \in L^2(\Omega;\mathbb{R}^\ell)$. *Let us consider the initial boundary value problem*

$$\begin{cases} A^0(x)\partial_t u + A(x)\cdot\nabla u + p(x,t)u = R(x,t)\cdot F(x) & \text{in } Q,\\ u = 0 & \text{on } \Gamma_{-,A}\times(0,T),\\ u(\cdot,0) = 0 & \text{on } \Omega. \end{cases} \tag{1.21}$$

Then, there exists a constant $C > 0$ *independent of* u *and* F *such that*

$$E(t) \le C\|F\|^2_{L^2(\Omega;\mathbb{R}^\ell))} \tag{1.22}$$

holds for all $t \in [0,T]$ *and* $u \in \bigcap_{k=1}^{2} H^k(0,T;H^{2-k}(\Omega))$ *satisfying* (1.21).

4. Proofs of main results

Using several estimates introduced in section 3, we prove the three main theorems in the subsequently sections.

4.1. Proof of Theorem 1.11.

PROOF OF THEOREM 1.11. By our assumption (1.8), we can take $0 < \beta < \dfrac{\rho}{\sup_{x\in\Omega,t>0} A^0(x,t)}$ independent of T satisfying

$$(T_0 <)\frac{\max_{x\in\overline{\Omega}}\varphi_0(x)}{\beta} < T.$$

Then, there exists $\kappa > 0$ such that

$$\max_{x\in\overline{\Omega}}\varphi_0(x) - \beta T < -\kappa. \tag{1.23}$$

Henceforth, by $C > 0$ we denote a generic constant independent of u which may change from line to line, unless specified otherwise. Applying the Carleman estimate (1.14) of Proposition 1.14 to $\partial_t u \in \bigcap_{k=0}^{1} H^k(0,T;H^{1-k}(\Omega))$ yields

$$\begin{aligned}(1.24)\qquad & s^2\int_Q e^{2s\varphi}|\partial_t u|^2 dxdt + s\int_\Omega e^{2s\varphi(x,0)}|R(x,0)f(x)|^2dx \\ &\le C\int_Q e^{2s\varphi}|(P+p(x,t))\partial_t u|^2dxdt + Cs\int_{\Sigma_+} e^{2s\varphi}|\partial_t u|^2 dSdt \\ &\quad + Cs\int_\Omega e^{2s\varphi(x,T)}|\partial_t u(x,T)|^2dx.\end{aligned}$$

Since we obtain

$$\begin{aligned}(P+p(x,t))\partial_t u &= \partial_t\Big(A^0(x,t)\partial_t u + A(x,t)\cdot\nabla u + p(x,t)u\Big) \\ &\quad - \partial_t A^0(x,t)\partial_t u - \partial_t A(x,t)\cdot\nabla u - \partial_t p(x,t)u \\ &= \partial_t R(x,t)f(x) - \partial_t A^0(x,t)\partial_t u - \partial_t A(x,t)\cdot\nabla u - \partial_t p(x,t)u,\end{aligned}$$

we have

$$\begin{aligned}(1.25)\qquad |(P+p(x,t))\partial_t u|^2 &\le C\Big(|\partial_t Rf|^2 + |\partial_t u|^2 + |\partial_t A(x,t)\cdot\nabla u|^2 + |u|^2\Big) \\ &\le C\Big(|\partial_t Rf|^2 + |\partial_t u|^2 + |A(x,t)\cdot\nabla u|^2 + |u|^2\Big),\end{aligned}$$

where we used the assumption (1.5) to obtain the second inequality. Therefore, applying the equation in (1.6) to the above estimate (1.25) yields

$$(1.26)\qquad |(P+p(x,t))\partial_t u|^2 \le C\Big(|\partial_t Rf|^2 + |Rf|^2 + |\partial_t u|^2 + |u|^2\Big).$$

Furthermore, applying (1.23) and the energy estimate (1.20) of Lemma 1.16 yields

$$\begin{aligned}(1.27)\qquad s\int_\Omega e^{2s\varphi(x,T)}|\partial_t u(x,T)|^2dx &\le Cse^{-2\kappa s}\int_\Omega A^0(x,T)|\partial_t u(x,T)|^2dx \\ &\le Cse^{-2\kappa s}\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

Applying (1.26) and (1.27) to (1.24) and choosing $s > s_*$ large enough yield

$$\begin{aligned}(1.28)\qquad & s^2\int_Q e^{2s\varphi}|\partial_t u|^2dxdt + s\int_\Omega e^{2s\varphi(x,0)}|R(x,0)f(x)|^2dx \\ &\le C\int_Q e^{2s\varphi}\Big(\sum_{k=0}^{1}|\partial_t^k R|^2\Big)|f|^2dxdt + C\int_Q e^{2s\varphi}|u|^2dxdt \\ &\quad + Cs\int_{\Sigma_+} e^{2s\varphi}|\partial_t u|^2dSdt + Cse^{-2\kappa s}\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

In regard to the left-hand side of (1.28), using (1.7), for some $C > 0$ we obtain

$$(1.29)\qquad s^2\int_Q e^{2s\varphi}|\partial_t u|^2dxdt + s\int_\Omega e^{2s\varphi(x,0)}|R(x,0)f(x)|^2dx \ge Cs\|e^{s\varphi_0}f\|^2_{L^2(\Omega)}.$$

In regard to right-hand side of (1.28), applying the Carleman estimate (1.14) of Proposition 1.14 to $u \in \bigcap_{k=1}^{2} H^k(0,T;H^{2-k}(\Omega))$ and then using (1.23) and the energy estimate (1.20) yield

$$
\begin{aligned}
(1.30)\qquad & \int_Q e^{2s\varphi}|u|^2 dxdt \\
& \le \frac{C}{s^2}\int_Q e^{2s\varphi}|Rf|^2 dxdt + \frac{C}{s}\int_{\Sigma_+} e^{2s\varphi}|u|^2 dSdt \\
& \quad + \frac{C}{s}\int_\Omega e^{2s\varphi(x,T)}|u(x,T)|^2 dx \\
& \le \frac{C}{s^2}\int_Q e^{2s\varphi}|Rf|^2 dxdt + \frac{C}{s}\int_{\Sigma_+} e^{2s\varphi}|u|^2 dSdt + \frac{C}{s}e^{-2\kappa s}\|f\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Applying (1.29) and (1.30) to (1.28) and choosing sufficiently large $s > s_*$ yield

$$
\begin{aligned}
& s\|e^{s\varphi_0} f\|^2_{L^2(\Omega)} \\
& \le C\int_Q e^{2s\varphi}\Big(\sum_{k=0}^{1}|\partial_t^k R|^2\Big)|f|^2 dxdt + \frac{C}{s}\int_{\Sigma_+} e^{2s\varphi}|u|^2 dSdt \\
& \quad + Cse^{Cs}\|\partial_t u\|^2_{L^2(\Sigma_+)} + Cse^{-2\kappa s}\|f\|^2_{L^2(\Omega)} \\
& \le C\int_Q e^{2s\varphi}\Big(\sum_{k=0}^{1}|\partial_t^k R|^2\Big)|f|^2 dxdt + Cse^{Cs}\sum_{k=0}^{1}\|\partial_t^k u\|^2_{L^2(\Sigma_+)} \\
& \quad + Cse^{-2\kappa s}\|f\|^2_{L^2(\Omega)} \\
& = C\int_\Omega \left(\int_0^T e^{-2s(\varphi_0(x)-\varphi(x,t))}\Big(\sum_{k=0}^{1}\|\partial_t^k R(\cdot,t)\|^2_{L^\infty(\Omega)}\Big)dt\right) e^{2s\varphi_0}|f|^2 dx \\
& \quad + Cse^{Cs}\sum_{k=0}^{1}\|\partial_t^k u\|^2_{L^2(\Sigma_+)} + Cse^{-2\kappa s}\|f\|^2_{L^2(\Omega)} \\
& = C\int_\Omega \left(\int_0^T e^{-2\beta ts}\Big(\sum_{k=0}^{1}\|\partial_t^k R(\cdot,t)\|^2_{L^\infty(\Omega)}\Big)dt\right) e^{2s\varphi_0}|f|^2 dx \\
& \quad + Cse^{Cs}\sum_{k=0}^{1}\|\partial_t^k u\|^2_{L^2(\Sigma_+)} + Cse^{-2\kappa s}\|f\|^2_{L^2(\Omega)} \\
& \le o(1)\|e^{s\varphi_0} f\|^2_{L^2(\Omega)} + Cse^{Cs}\sum_{k=0}^{1}\|\partial_t^k u\|^2_{L^2(\Sigma_+)} + Cse^{-2\kappa s}\|e^{s\varphi_0} f\|^2_{L^2(\Omega)} \\
& = o(1)\|e^{s\varphi_0} f\|^2_{L^2(\Omega)} + Cse^{Cs}\sum_{k=0}^{1}\|\partial_t^k u\|^2_{L^2(\Sigma_+)}
\end{aligned}
$$

as $s \to +\infty$ by the Lebesgue dominated convergence theorem. Choosing $s > s_*$ large enough yields

$$
\|e^{s\varphi_0} f\|_{L^2(\Omega)} \le Ce^{Cs}\sum_{k=0}^{1}\|\partial_t^k u\|_{L^2(\Sigma_+)}.
$$

Since $\varphi_0(x) \ge 0$ for all $x \in \overline{\Omega}$, $\|e^{s\varphi_0} f\|_{L^2(\Omega)} \ge \|f\|_{L^2(\Omega)}$ holds. Then, we complete the proof. $\square$

4.2. Proof of Theorem 1.12.

PROOF OF THEOREM 1.12. We show that Theorem 1.12 comes down to Theorem 1.11. Setting

$$v := u_1 - u_2, \quad R := -u_2, \quad f := p_1 - p_2,$$

we obtain

$$\begin{cases} Pv + p_1(x)v = R(x,t)f(x) & \text{in } Q, \\ v = 0 & \text{on } \Sigma_-, \\ v(\cdot, 0) = 0 & \text{on } \Omega, \end{cases}$$

and (1.7) is satisfied due to the assumption (1.10). Therefore, by Theorem 1.11, the proof is completed. □

4.3. Proof of Theorem 1.13.

PROOF OF THEOREM 1.13. By our assumption (1.11), we can take $0 < \beta < \dfrac{\rho}{\max_{x\in\overline{\Omega}} A_1^0(x)}$ independent of T satisfying

$$\frac{\left(\max_{x\in\overline{\Omega}} A_1^0(x)\right)\left(\max_{x\in\overline{\Omega}} \varphi_0(x)\right)}{\rho} < \frac{\max_{x\in\overline{\Omega}} \varphi_0(x)}{\beta} < T.$$

Then, there exists $\kappa > 0$ such that

$$\max_{x\in\overline{\Omega}} \varphi_0(x) - \beta T < -\kappa. \tag{1.31}$$

Henceforth, by $C > 0$ we denote a generic constant independent of u which may change from line to line, unless specified otherwise. For $m = 1, \ldots, d+1$, setting

$$v_m := u_{1,m} - u_{2,m}, \quad f_1 := A_1^0 - A_2^0, \quad f_2 := A_1 - A_2,$$

and

$$F := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2(\Omega; \mathbb{R}^{d+1}),$$

$$R_m := \begin{pmatrix} -\partial_t u_{2,m} & -\partial_{x^1} u_{2,m} & \cdots & -\partial_{x^d} u_{2,m} \end{pmatrix} \in H^1(0,T; L^\infty(\Omega; \mathbb{R}^{d+1})).$$

Thus, we obtain

$$\begin{cases} P_1 v_m + p(x,t)v_m = R_m(x,t)F(x) & \text{in } Q, \\ v_m = 0 & \text{on } \Sigma_-, \\ v_m(\cdot, 0) = 0 & \text{on } \Omega, \end{cases}$$

where the product in the right-hand side of the equation is a product of matrices. Applying the Carleman estimate (1.14) of Proposition 1.14 with $P = P_1$ to

$$\partial_t v_m \in \bigcap_{k=0}^{1} H^k(0,T; W^{1-k,\infty}(\Omega)) \subset \bigcap_{k=0}^{1} H^k(0,T; H^{1-k}(\Omega))$$

yields

$$
\begin{aligned}
&s^2\int_Q e^{2s\varphi}|\partial_t v_m|^2 dxdt + s\int_\Omega e^{2s\varphi(x,0)}|R_m(x,0)F(x)|^2dx\\
&\le C\int_Q e^{2s\varphi}|(P_1+p(x,t))\partial_t v_m|^2dxdt + Cs\int_{\Gamma_{+,A_1}\times(0,T)} e^{2s\varphi}|\partial_t v_m|^2 dSdt\\
&\quad + Cs\int_\Omega e^{2s\varphi(x,T)}|\partial_t v_m(x,T)|^2dx.
\end{aligned}
$$

Summing up with respect to $m=1,\dots,d+1$ yields

$$
\begin{aligned}
&s^2\int_Q e^{2s\varphi}|\partial_t v|^2 dxdt + s\int_\Omega e^{2s\varphi(x,0)}|R(x,0)F(x)|^2dx\\
&\le C\int_Q e^{2s\varphi}|(P_1+p(x,t))\partial_t v|^2dxdt + Cs\int_{\Gamma_{+,A_1}\times(0,T)} e^{2s\varphi}|\partial_t v|^2 dSdt\\
&\quad + Cs\int_\Omega e^{2s\varphi(x,T)}|\partial_t v(x,T)|^2dx,
\end{aligned}
\tag{1.32}
$$

where we define

$$
v:=\begin{pmatrix} v_1\\ \vdots\\ v_{d+1}\end{pmatrix},\quad R:=\begin{pmatrix} R_1\\ \vdots\\ R_{d+1}\end{pmatrix},\quad (P_1+p(x,t))\partial_t v:=\begin{pmatrix} (P_1+p(x,t))\partial_t v_1\\ \vdots\\ (P_1+p(x,t))\partial_t v_{d+1}\end{pmatrix}.
$$

Since we obtain

$$
\begin{aligned}
(P_1+p(x,t))\partial_t v_m &= \partial_t\Big(A_1^0(x)\partial_t v_m + A_1(x)\cdot\nabla v_m + p(x,t)v_m\Big) - \partial_t p(x,t)v_m\\
&= \partial_t(R_mF) - \partial_t p(x,t)v_m
\end{aligned}
$$

for each $m=1,\dots,d+1$, we have

$$
|(P_1+p(x,t))\partial_t v|^2 \le C\Big(|\partial_t RF|^2 + |v|^2\Big). \tag{1.33}
$$

Furthermore, applying (1.31) and the energy estimate (1.22) of Lemma 1.17 for $m=1,\dots,d+1$ yields

$$
\begin{aligned}
s\int_\Omega e^{2s\varphi(x,T)}|\partial_t v_m(x,T)|^2dx &\le Cse^{-2\kappa s}\int_\Omega A_1^0(x,T)|\partial_t v_m(x,T)|^2dx\\
&\le Cse^{-2\kappa s}\|F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})},
\end{aligned}
$$

which implies

$$
s\int_\Omega e^{2s\varphi(x,T)}|\partial_t v(x,T)|^2dx \le Cse^{-2\kappa s}\|F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}. \tag{1.34}
$$

Applying (1.33) and (1.34) to (1.32) and choosing $s>s_*$ large enough yield

$$
\begin{aligned}
&s^2\int_Q e^{2s\varphi}|\partial_t v|^2 dxdt + s\int_\Omega e^{2s\varphi(x,0)}|R(x,0)F(x)|^2dx\\
&\le C\int_Q e^{2s\varphi}|\partial_t RF|^2dxdt + C\int_Q e^{2s\varphi}|v|^2dxdt\\
&\quad + Cs\int_{\Gamma\times(0,T)} e^{2s\varphi}|\partial_t v|^2dSdt + Cse^{-2\kappa s}\|F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}.
\end{aligned}
\tag{1.35}
$$

In regard to the left-hand side of (1.35), we obtain

$$
\begin{aligned}
& s^2 \int_Q e^{2s\varphi} |\partial_t v|^2 dxdt + s \int_\Omega e^{2s\varphi(x,0)} |R(x,0)F(x)|^2 dx \\
& \qquad \geq Cs \| e^{s\varphi_0} F \|^2_{L^2(\Omega;\mathbb{R}^{d+1})}
\end{aligned} \tag{1.36}
$$

for some $C > 0$ by (1.12). Indeed, by $\min_{x\in\overline{\Omega}} A_2^0(x) \geq \rho > 0$, it follows that

$$
\begin{aligned}
|\det R(x,0)| &= \left| \det \begin{pmatrix} \partial_t u_{2,1}(x,0) & \cdots & \partial_t u_{2,d+1}(x,0) \\ \nabla u_{2,1}(x,0) & \cdots & \nabla u_{2,d+1}(x,0) \end{pmatrix} \right| \\
&\geq C \left| \det \begin{pmatrix} A_2 \cdot \nabla \alpha_1 + p(x,0)\alpha_1 & \cdots & A_2 \cdot \nabla \alpha_{d+1} + p(x,0)\alpha_{d+1} \\ \nabla \alpha_1 & \cdots & \nabla \alpha_{d+1} \end{pmatrix} \right| \\
&= C \left| \det \begin{pmatrix} p(x,0)\alpha_1 & \cdots & p(x,0)\alpha_{d+1} \\ \nabla \alpha_1 & \cdots & \nabla \alpha_{d+1} \end{pmatrix} \right| \\
&= C |p(x,0)| \left| \det \begin{pmatrix} \alpha_1(x) & \cdots & \alpha_{d+1}(x) \\ \nabla \alpha_1(x) & \cdots & \nabla \alpha_{d+1}(x) \end{pmatrix} \right| \geq m_0 \quad \text{a.e. } x \in \Omega.
\end{aligned}
$$

In regard to the right-hand side of (1.35), applying the Carleman estimate (1.14) of Proposition 1.14 to $v_m \in \bigcap_{k=1}^{2} H^k(0,T; W^{2-k,\infty}(\Omega))$ for each $m = 1, \ldots, d+1$ and then using (1.31) and the energy estimate (1.22) of Lemma 1.17 yield

$$
\begin{aligned}
& \int_Q e^{2s\varphi} |v|^2 dxdt \\
& \leq \frac{C}{s^2} \int_Q e^{2s\varphi} |Rf|^2 dxdt + \frac{C}{s} \int_{\Gamma_{+,A_1} \times (0,T)} e^{2s\varphi} |v|^2 dSdt \\
& \quad + \frac{C}{s} \int_\Omega e^{2s\varphi(x,T)} |v(x,T)|^2 dx \\
& \leq \frac{C}{s^2} \int_Q e^{2s\varphi} |Rf|^2 dxdt + \frac{C}{s} \int_{\Gamma \times (0,T)} e^{2s\varphi} |v|^2 dSdt \\
& \quad + \frac{C}{s} e^{-2\kappa s} \|F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}.
\end{aligned} \tag{1.37}
$$

Applying (1.36) and (1.37) to (1.35) and choosing sufficiently large $s > s_*$ yield

$$
\begin{aligned}
&s\|e^{s\varphi_0}F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}\\
&\le C\int_Q e^{2s\varphi}|\partial_t RF|^2dxdt + \frac{C}{s^2}\int_Q e^{2s\varphi}|Rf|^2dxdt + \frac{C}{s}\int_{\Gamma\times(0,T)} e^{2s\varphi}|v|^2dSdt\\
&\quad + Cs\int_{\Gamma\times(0,T)} e^{2s\varphi}|\partial_t v|^2dSdt + Cse^{-2\kappa s}\|F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}\\
&\le C\int_Q e^{2s\varphi}\Big(\sum_{k=0}^{1}|\partial_t^k RF|^2\Big)dxdt + Cse^{Cs}\|v\|^2_{H^1(0,T;L^2(\Gamma;\mathbb{R}^{d+1}))}\\
&\quad + Cse^{-2\kappa s}\|F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}\\
&= C\int_\Omega\left(\int_0^T e^{-2\beta ts}\Big(\sum_{k=0}^{1}\|\partial_t^k R(\cdot,t)\|^2_{L^\infty(\Omega;\mathbb{R}^{(d+1)\times(d+1)})}\Big)dt\right)e^{2s\varphi_0}|F|^2dx\\
&\quad + Cse^{Cs}\|v\|^2_{H^1(0,T;L^2(\Gamma;\mathbb{R}^{d+1}))} + Cse^{-2\kappa s}\|F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}\\
&\le o(1)\|e^{s\varphi_0}F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})} + Cse^{Cs}\|v\|^2_{H^1(0,T;L^2(\Gamma;\mathbb{R}^{d+1}))} + Cse^{-2\kappa s}\|e^{s\varphi_0}F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}\\
&= o(1)\|e^{s\varphi_0}F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})} + Cse^{Cs}\|v\|^2_{H^1(0,T;L^2(\Gamma;\mathbb{R}^{d+1}))}
\end{aligned}
$$

as $s \to +\infty$ by the Lebesgue dominated convergence theorem. Choosing $s > s_*$ large enough yields

$$\|e^{s\varphi_0}F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})} \le Ce^{Cs}\|v\|^2_{H^1(0,T;L^2(\Gamma;\mathbb{R}^{d+1}))}$$

Since $\varphi_0(x) \ge 0$ for all $x \in \overline{\Omega}$, $\|e^{s\varphi_0}F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})} \ge \|F\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}$ holds. Then, we complete the proof. □

5. Proofs of auxiliary results

In this section, we prove Proposition 1.10, Lemma 1.16, and Lemma 1.17.

5.1. Proof of Proposition 1.10.

Proof of Proposition 1.10. When $d \ge 2$, we note that there exists a vector-valued function $A_\perp(x,t) \ne 0$ for each $(x,t) \in \overline{Q}$ such that

$$A(x,t)\cdot A_\perp(x,t) = 0$$

due to (1.1). Applying (1.5) to $\xi = A_\perp(x,t)$ yields

$$\forall(x,t)\in\overline{Q},\ \partial_t A(x,t)\cdot A_\perp(x,t) = 0,$$

which implies that there exists a function $\phi \in C^1(\overline{Q})$ such that

$$\forall(x,t)\in\overline{Q},\quad \partial_t A(x,t) = \phi(x,t)A(x,t).$$

Therefore, $A(x,t)$ is represented by

$$A(x,t) = A(x,0)e^{\int_0^t \phi(x,s)ds}.$$

When $d = 1$, noting Remark 1.9, setting

$$\phi(x,t) := \frac{\partial_t A(x,t)}{A(x,t)}$$

completes the proof. □

5.2. Proof of Lemma 1.16.

Proof of Lemma 1.16. Differentiating the equation in (1.6) with respect to t yields

$$\begin{aligned} &A^0(x,t)\partial_t^2 u + \partial_t A^0(x,t)\partial_t u + A(x,t)\cdot\nabla\partial_t u \\ &\quad +\partial_t A(x,t)\cdot\nabla u + p(x,t)\partial_t u + \partial_t p(x,t) u = \partial_t R(x,t) f(x). \end{aligned}$$

Multiplying $2\partial_t u$ to the above equality and integrating over Ω yield

$$\begin{aligned} &\int_\Omega A^0(x,t)\partial_t(|\partial_t u|^2)dx + \int_\Omega 2\partial_t A^0(x,t)|\partial_t u|^2 dx + \int_\Omega A(x,t)\cdot\nabla(|\partial_t u|^2)dx \\ &\quad + \int_\Omega 2\partial_t u(\partial_t A(x,t)\cdot\nabla u)dx + \int_\Omega 2p(x,t)|\partial_t u|^2 dx + \int_\Omega 2\partial_t p(x,t) u\partial_t u dx \\ &= \int_\Omega 2\partial_t u\partial_t R(x,t) f(x) dx. \end{aligned}$$

Integration by parts yields

$$\begin{aligned} &\frac{d}{dt}\int_\Omega A^0(x,t)|\partial_t u|^2 dx \\ &= -\int_\Omega (\partial_t A^0(x,t) + 2p(x,t))|\partial_t u|^2 dx + \int_\Omega (\nabla\cdot A(x,t))|\partial_t u|^2 dx \\ &\quad - \int_\Omega 2\partial_t u(\partial_t A(x,t)\cdot\nabla u)dx - \int_\Omega 2\partial_t p(x,t) u\partial_t u dx + \int_\Omega 2\partial_t u\partial_t R f dx \\ &\quad - \int_{\partial\Omega} (A(x,t)\cdot\nu)|\partial_t u|^2 dS \\ &\le C\left(\int_\Omega A^0(x,t)|\partial_t u|^2 dx + \int_\Omega |u|^2 dx + \int_\Omega |\partial_t A(x,t)\cdot\nabla u|^2 dx + \int_\Omega |\partial_t R f|^2 dx\right) \\ &\quad - \int_{\partial\Omega} (A(x,t)\cdot\nu)|\partial_t u|^2 dS. \end{aligned}$$

Adding $\frac{d}{dt}\int_\Omega |u|^2 dx$ to the both sides of the above estimate, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega A^0(x,t)|\partial_t u|^2 dx + \int_\Omega |u|^2 dx\right) \\ &\le C\Big(\int_\Omega A^0(x,t)|\partial_t u|^2 dx + \int_\Omega |u|^2 dx + \int_\Omega |\partial_t A(x,t)\cdot\nabla u|^2 dx \\ &\quad + \int_\Omega |\partial_t R f|^2 dx\Big) + \int_\Omega 2|u||\partial_t u| dx - \int_{\partial\Omega} (A(x,t)\cdot\nu)|\partial_t u|^2 dS \\ &\le C\Big(\int_\Omega A^0(x,t)|\partial_t u|^2 dx + \int_\Omega |u|^2 dx + \int_\Omega |\partial_t A(x,t)\cdot\nabla u|^2 dx \\ &\quad + \int_\Omega |\partial_t R f|^2 dx\Big) - \int_{\partial\Omega} (A(x,t)\cdot\nu)|\partial_t u|^2 dS, \end{aligned} \tag{1.38}$$

which implies

$$\begin{aligned}&\frac{d}{dt}\left(e^{-Ct}\int_\Omega\left(A^0(x,t)|\partial_t u|^2+|u|^2\right)dx\right)\\&\le e^{-Ct}\left(C\int_\Omega\left(|\partial_t A(x,t)\cdot\nabla u|^2+|\partial_t Rf|^2\right)dx-\int_{\partial\Omega}(A(x,t)\cdot\nu)|\partial_t u|^2dS\right).\end{aligned}$$

Integrating over $(0,t)$ for $t\le T$ yields

$$E(t)\le C\left(E(0)+\int_Q|\partial_t A(x,t)\cdot\nabla u|^2dxdt+\int_\Omega|f|^2dx\right).$$

Since, using the equation (1.6), we obtain

$$E(0)\le C\int_\Omega|f|^2dx, \tag{1.39}$$

we prove (1.19).

Moreover, if we assume the assumption (1.5), then there exists $C>0$ such that for all $(x,t)\in\overline{Q}$,

$$|\partial_t A(x,t)\cdot\nabla u|^2\le C|A(x,t)\cdot\nabla u|^2.$$

Therefore, applying the above inequality to (1.38) and using the equation in (1.6) yield

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega A^0(x,t)|\partial_t u|^2dx+\int_\Omega|u|^2dx\right)\\&\le C\left(\int_\Omega A^0(x,t)|\partial_t u|^2dx+\int_\Omega|u|^2dx+\int_\Omega|\partial_t Rf|^2dx+\int_\Omega|Rf|^2dx\right)\\&\quad-\int_{\partial\Omega}(A(x,t)\cdot\nu)|\partial_t u|^2dS,\end{aligned}$$

which implies

$$\begin{aligned}&\frac{d}{dt}\left(e^{-Ct}\int_\Omega\left(A^0(x,t)|\partial_t u|^2+|u|^2\right)dx\right)\\&\le e^{-Ct}\left(C\int_\Omega\left(\sum_{k=0}^1|\partial_t^k R|^2\right)|f|^2dx-\int_{\partial\Omega}(A(x,t)\cdot\nu)|\partial_t u|^2dS\right).\end{aligned}$$

Integrating over $(0,t)$ for $t\le T$ yields

$$E(t)\le C\left(E(0)+\int_\Omega|f|^2dx\right).$$

By (1.39), we complete the proof. □

5.3. Proof of Lemma 1.17.

PROOF OF LEMMA 1.17. Differentiating the equation with respect to t yields

$$A^0(x)\partial_t^2u+A(x)\cdot\nabla\partial_t u+p(x,t)\partial_t u+\partial_t p(x,t)u=\partial_t R(x,t)\cdot F(x).$$

Multiplying $2\partial_t u$ to the above equation and integrating over Ω yield

$$\begin{aligned}&\int_\Omega A^0(x)\partial_t(|\partial_t u|^2)dx+\int_\Omega A(x)\cdot\nabla(|\partial_t u|^2)dx\\&\quad+\int_\Omega 2p(x,t)|\partial_t u|^2dx+\int_\Omega 2\partial_t p(x,t)u\partial_t udx=\int_\Omega 2\partial_t u\partial_t R(x,t)\cdot F(x)dx.\end{aligned}$$

Integration by parts yields

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega A^0(x)|\partial_t u|^2 dx \\
&= \int_\Omega (\nabla\cdot A(x) - 2p(x,t))|\partial_t u|^2 dx - \int_\Omega 2\partial_t p(x,t) u\partial_t u dx + \int_\Omega 2\partial_t u \partial_t R\cdot F dx \\
&\quad - \int_{\partial\Omega} (A(x)\cdot\nu)|\partial_t u|^2 dS \\
&\le C\left(\int_\Omega A^0(x)|\partial_t u|^2 dx + \int_\Omega |u|^2 dx + \int_\Omega |\partial_t R\cdot F|^2 dx\right) - \int_{\partial\Omega}(A(x)\cdot\nu)|\partial_t u|^2 dS.
\end{aligned}
$$

Adding $\frac{d}{dt}\int_\Omega |u|^2 dx$ to the both sides of the above estimate, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega A^0(x)|\partial_t u|^2 dx + \int_\Omega |u|^2 dx\right) \\
&\le C\left(\int_\Omega A^0(x)|\partial_t u|^2 dx + \int_\Omega |u|^2 dx + \int_\Omega |\partial_t R\cdot F|^2 dx\right) + \int_\Omega 2|u||\partial_t u| dx \\
&\quad - \int_{\partial\Omega}(A(x)\cdot\nu)|\partial_t u|^2 dS \\
&\le C\left(\int_\Omega A^0(x)|\partial_t u|^2 dx + \int_\Omega |u|^2 dx + \int_\Omega |\partial_t R\cdot F|^2 dx\right) - \int_{\partial\Omega}(A(x)\cdot\nu)|\partial_t u|^2 dS,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{d}{dt}\left(e^{-Ct}\int_\Omega \left(A^0(x)|\partial_t u|^2 + |u|^2\right) dx\right) \\
&\le e^{-Ct}\left(C\int_\Omega |\partial_t R\cdot F|^2 dx - \int_{\partial\Omega}(A(x)\cdot\nu)|\partial_t u|^2 dS\right).
\end{aligned}
$$

Integrating over $(0,t)$ for $t \le T$ yields

$$
E(t) \le C\left(E(0) + \int_\Omega |F|^2 dx\right).
$$

Since, using the equation in (1.21), we obtain

$$
E(0) \le C\int_\Omega |F|^2 dx,
$$

we prove (1.22). □

CHAPTER 2

Symmetric hyperbolic systems

1. Introduction and main result

Let $n, \ell \in \mathbb{N}$, (M, g) be a compact oriented n-dimensional smooth Riemannian manifold with boundary ∂M, $T > 0$, and $L := [0, T] \times M$. For $a, b \in \mathbb{R}^\ell$, $a \cdot b$ denotes the inner product on $\mathbb{R}^\ell$, $\mathbb{M}^{\ell \times \ell}$ denotes the space of real square matrices of order ℓ, and Sym_ℓ denotes the subspace of real symmetric $\ell \times \ell$ matrices, i.e.,

$$\mathrm{Sym}_\ell := \{X \in \mathbb{M}^{\ell \times \ell} \mid X^\mathsf{T} = X\},$$

where $\cdot^\mathsf{T}$ denotes the transposition of a matrix. Let $u = (u^1, \dots, u^\ell)^\mathsf{T} : L \to \mathbb{R}^\ell$ be a vector-valued function and

$$A := A^\mu(t, x) \frac{\partial}{\partial x^\mu} := \sum_{\mu=0}^{n} A^\mu(t, x) \frac{\partial}{\partial x^\mu}, \quad (t = x^0, x^1, \dots, x^n) \in L$$

be a matrix-valued vector field on L, where $A^\mu = (A^\mu_{ij})_{1 \le i,j \le \ell} \in C^1(L; \mathrm{Sym}_\ell)$ are matrix-valued functions for $\mu = 0, 1, \dots, n$. We use the Einstein summation convention throughout this chapter and note that summations with respect to Greek indices range from 0 to n, whereas those for Roman indices range from 1 to n.

For a vector-valued function u, define the symmetric hyperbolic operator P as

$$Pu := Au = A^\mu(t, x) \partial_\mu u,$$

where $\partial_\mu = \partial_{x^\mu}$. This type of symmetric hyperbolic operator P has been considered in describing some equations in mathematical physics such as the Maxwell's equations and the elasticity equations, and readers are referred to [26] and [55] for a theory of well-posedness concerning the first-order symmetric hyperbolic equations. Some of these equations can be described via the wave equations and it is meaningful to consider the equivalent first-order systems which form a more general class of hyperbolic equations.

The main focus of this chapter is the inverse source problem for the symmetric hyperbolic system concerning the operator P as an application of a Carleman estimate by the Bukhgeim–Klibanov method [8]. The Carleman estimate is one of the most important tools when we study inverse problems and a large number of studies have been made on the Carleman estimates and their applications to inverse problems (e.g., [6], [33], [43], and references therein). If A^μ are all diagonal for $\mu = 0, \dots, n$, then we can easily obtain the Carleman estimate, where the system is called a weakly coupled system. However, to the best of author's knowledge, little is known about the Carleman estimates for strongly coupled systems where principal parts are coupled except for Floridia–Takase–Yamamoto [25] because it is not easy to establish the estimates directly. For other approaches to inverse problems for the first-order hyperbolic systems, readers are referred to [18] and [36]. In [18], they consider the system with an integral term of a convolution type and determine the convolution kernel by means of the Volterra integral equation. In [36], they proved the existence and uniqueness to determine time-independent potential by the method of characteristics.

We show, as a first approach to strongly coupled systems, the way to prove the Carleman estimate and apply them to the inverse problem for the symmetric hyperbolic system. Furthermore, the crucial point of this chapter is geometric analysis on manifolds. Regarding the symmetric hyperbolic operator P as a matrix-valued vector field gives a comprehensive viewpoint for the hyperbolic inverse problems. To achieve this, the analysis independent of local coordinate systems is essential. We introduce some notations needed to describe the geometric analysis.

Let $\tau : L \to [0,T]$ and $\pi : L \to M$ be the projections, and $\langle \cdot, \cdot \rangle$ denotes a pairing between 1-forms and vector fields. We assume that all eigenvalues $\lambda_i(t,x)$ of $\langle d\tau, A\rangle = P\tau \in C^1(L; \mathrm{Sym}_\ell)$ are positive and uniformly bounded, i.e.,

$$\begin{cases} \forall i = 1, \dots, \ell, \ \lambda_i(t,x) > 0, \\ \lambda := \sup\limits_{t>0, x\in M, i=1,\dots,\ell} \lambda_i(t,x) < \infty, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \exists \alpha = \alpha_k dx^k \in C(T^*M) \text{ s.t. } \exists \varphi_0 \in C^1(M) \text{ with } \alpha = d\varphi_0 \text{ and} \\ \langle \alpha, \pi_* A\rangle = P\varphi_0 \in C(L; \mathrm{Sym}_\ell) \text{ is uniformly positive definite, i.e.,} \\ \exists \delta > 0 \text{ s.t. } \inf\limits_{|v|=1, t>0, x\in M} \partial_k \varphi_0(x) A^k(t,x) v\cdot v \geq \delta, \end{cases} \tag{2.2}$$

where $C(T^*M)$ denotes the space of continuous sections of T^*M.

Remark 2.1. *If $P\tau = I$, which is the identity matrix of order ℓ,* (2.1) *is obviously satisfied. Moreover, regarding* (2.2)*, if the exterior derivative of α vanishes in M, i.e., $d\alpha = 0$, the existence of φ_0 satisfying $\alpha = d\varphi_0$ follows in some cases. This is related to the Poincaré's lemma presented in section 4.*

In addition, what seems to be lacking is analysis for hyperbolic inverse problems with time-dependent principal parts. Although there are several works related to this kind of hyperbolic equations (e.g., [23], [37], and [64]), it is not enough for the first-order hyperbolic systems. We investigate the equations with coefficients depending on both space variable x and time variable t. Let $N := -\nabla\tau$ be the future directed timelike vector field, where ∇ denotes the Levi-Civita connection with respect to the metric $-dt\otimes dt + g$. To deal with the time dependence, we set the following ansatz:

$$\exists C > 0 \text{ s.t. } \forall \xi \in C(T^*M; \mathbb{R}^\ell), \ |\langle \xi, \pi_*(\nabla_N A)\rangle| \leq C|\langle \xi, \pi_* A\rangle|, \tag{2.3}$$

where $C(T^*L; \mathbb{R}^\ell)$ denotes the space of $\mathbb{R}^\ell$-valued continuous sections of T^*L.

Remark 2.2. *It follows that $\nabla_N A = \partial_t A^\mu \frac{\partial}{\partial x^\mu}$ by the local coordinate $(t, x^1, \dots, x^n)$. Therefore,* (2.3) *implies that there exists a constant $C > 0$ such that for all $\xi = \xi_k dx^k \in C(T^*M; \mathbb{R}^\ell)$ and $(t,x) \in L$,*

$$|\partial_t A^k(t,x)\xi_k(x)| \leq C|A^k(t,x)\xi_k(x)|.$$

Clearly, if A^k is independent of t for all $k = 1, \dots, n$, then (2.3) *is satisfied.*

Define the set the elements in which satisfy the above three conditions (2.1), (2.2), and (2.3):

$$\mathcal{A} := \left\{ A = A^\mu(t,x)\frac{\partial}{\partial x^\mu} \in C^1(TL; \mathrm{Sym}_\ell) \mid (2.1),\ (2.2),\ (2.3) \right\}.$$

To describe our main result, we define the Sobolev space on manifolds, which should be defined so as not to depend on a choice of coordinate systems.

Definition 2.3. *Let M be a compact oriented n-dimensional smooth manifold, and $\{(U_i, x_i)\}_i$ be a coordinate system. Assume $\{\chi_i\}_i$ is a finite partition of unity subordinate to the covering such that $\operatorname{supp}\chi_i \subset U_i$. Given $u \in C^\infty(M;\mathbb{R}^\ell)$ and integer k, define*

$$\|u\|_{H^k(M;\mathbb{R}^\ell)} := \left(\sum_i \sum_{|\alpha|\le k} \int_{x_i(U_i)} (\chi_i |\partial^\alpha u|^2) \circ x_i^{-1} dx_i^1 \cdots dx_i^n \right)^{\frac{1}{2}},$$

where ∂^α signifies differentiation with respect to x_i.

The inner product can be also defined in the same way. By taking the completion of the smooth functions, one obtains a real Hilbert space. Note that different partitions of unity and coordinates yield different norms but they are all equivalent norms. Although our integrations and derivatives on compact manifolds should be written using a partition of unity and local coordinates, we omit these representations to avoid notational complexity.

Moreover, we define $\Sigma := (0,T) \times \partial M$ and subsets

$$\Sigma_+ := \{(t,x) \in \Sigma \mid g(\nu, \pi_* A) \text{ is positive definite}\}, \quad \Sigma_- := \Sigma \setminus \Sigma_+,$$

where $\nu = \nu^k \frac{\partial}{\partial x^k}$ is the outer unit normal vector field to ∂M. For $A \in \mathcal{A}$, we consider the initial boundary value problem

$$\begin{cases} Pu + p(t,x)u = R(t,x)f(x) & \text{in } L, \\ u = 0 & \text{on } \Sigma_-, \\ u(0,\cdot) = 0 & \text{on } M, \end{cases} \tag{2.4}$$

where $p \in W^{1,\infty}(0,T; L^\infty(M; \mathbb{M}^{\ell\times\ell}))$, $R \in H^1(0,T; L^\infty(M; \mathbb{M}^{\ell\times\ell}))$, and $f \in L^2(M;\mathbb{R}^\ell)$. Given A, p, and R, we consider the inverse problem of determining f by boundary observation of u on Σ_+.

To prove the global Lipschitz stability for inverse problems of hyperbolic equations, the observation time should be given for the distant wave to reach the boundary owing to the finite propagation speed. We define a constant to describe this situation mathematically. For $\lambda > 0$ in (2.1) and $\varphi_0 \in C^1(M)$ in (2.2), define the number

$$T_0 := \frac{\lambda \left(\max_{x\in M} \varphi_0(x) - \min_{x \in M} \varphi_0(x) \right)}{\delta} > 0. \tag{2.5}$$

Theorem 2.4. *Let $A \in \mathcal{A}$, $p \in W^{1,\infty}(0,T; L^\infty(M; \mathbb{M}^{\ell\times\ell}))$, $f \in L^2(M;\mathbb{R}^\ell)$, and $R \in H^1(0,T; L^\infty(M; \mathbb{M}^{\ell\times\ell}))$ satisfying*

$$\exists c_0 > 0 \ s.t. \ |\det R(0,x)| \ge c_0 \ a.e.\ x \in M.$$

Assume

$$T_0 < T, \tag{2.6}$$

where $T_0 > 0$ *is defined by* (2.5)*, and there exists a function* $u \in H^2(L;\mathbb{R}^\ell)$ *satisfying* (2.4)*. Then, there exists a constant* $C > 0$ *independent of* f *and* u *such that*

$$\|f\|_{L^2(M;\mathbb{R}^\ell)} \le C \sum_{m=0}^{1} \|\partial_t^m u\|_{L^2(\Sigma_+;\mathbb{R}^\ell)}.$$

The proof of Theorem 2.4 is presented in section 3 by combining the Carleman estimate and the energy estimate, which will be introduced in section 2. This kind of Lipschitz stability estimates for the hyperbolic equations with lateral Cauchy data could be obtained even for hyperbolic inequalities as summarized in [43]. According to the Bukhgeim–Klibanov method, one could obtain Lipschitz stability estimates for inverse source problems when one establishes these two a priori estimates, basically in the same way as in this chapter.

2. Carleman estimates and energy estimates

2.1. Carleman estimates. To obtain a Carleman estimate, we define the function $\varphi \in C^1(L)$ such that

$$\varphi(t,x) := \varphi_0(x) - \beta\tau(t,x), \quad (t,x) \in L, \tag{2.7}$$

where $\varphi_0 \in C^1(M)$ is the function in (2.2) and $\beta > 0$ is a constant. The next lemma is indispensable for the Carleman estimate.

Lemma 2.5. *Assume* (2.1)*,* (2.2)*, and* $0 < \beta < \dfrac{\delta}{\lambda}$*. Then, for the function* $\varphi \in C^1(L)$ *defined by* (2.7)*,* $P\varphi(t,x)$ *is uniformly positive definite, i.e.,*

$$\exists \rho > 0 \ s.t. \inf_{|v|=1, t>0, x\in M} P\varphi(t,x)v\cdot v \ge \rho. \tag{2.8}$$

Proof. Let $v \in \mathbb{R}^\ell$ be fixed arbitrary. By (2.1) and (2.2), direct calculations yield

$$\begin{aligned} P\varphi(t,x)v\cdot v &= \partial_k\varphi_0(x)A^k(t,x)v\cdot v - \beta A^0(t,x)v\cdot v \\ &\ge (\delta - \beta\lambda)|v|^2. \end{aligned}$$

Hence, there exists $0 < \rho < \delta - \beta\lambda$ such that (2.8) holds. □

Using the weight function defined by (2.7), we will establish the Carleman estimate. The weight function of a linear type for first-order equations is proposed by [53] and successively used by [31], [9], [10], and [23].

Proposition 2.6. *Let* $A \in C^1(TL;\mathrm{Sym}_\ell)$ *satisfying* (2.1) *and* (2.2)*, and* $p \in L^\infty(L;\mathbb{M}^{\ell\times\ell})$*. Let* $\varphi \in C^1(L)$ *be the function defined by* (2.7) *for* $\beta > 0$ *satisfying*

$$0 < \beta < \frac{\delta}{\lambda}.$$

Then, there exist constants $s_* > 0$ and $C > 0$ such that

$$\begin{aligned} s^2 \int_L e^{2s\varphi}|u|^2\omega_L + s\int_M e^{2s\varphi(0,x)}|(P\tau)u(0,x)|^2\omega_M \\ \le C\int_L e^{2s\varphi}|(P+p(t,x))u|^2\omega_L + Cs\int_{\Sigma_+} e^{2s\varphi}\big(g(\nu,d\pi(A))u\cdot u\big)\omega_\Sigma \\ + Cs\int_M e^{2s\varphi(T,x)}\big((P\tau)u\cdot u\big)(T,x)\omega_M \end{aligned} \tag{2.9}$$

holds for all $s > s_*$ and $u \in H^1(L;\mathbb{R}^\ell)$, where ω_L, ω_M, and ω_Σ denote the respective volume elements of L, M, and Σ.

Proof. It suffices to prove (2.9) when $p \equiv 0$ due to the sufficiently large parameter s. Let $z := e^{s\varphi}u$ and $P_s z := e^{s\varphi}P(e^{-s\varphi}z)$ for $s > 0$. Then, we obtain

$$P_s z = Pz - s(P\varphi)z,$$

and

$$\begin{aligned} \|P_s z\|^2_{L^2(L;\mathbb{R}^\ell)} &= s^2\|(P\varphi)z\|^2_{L^2(L;\mathbb{R}^\ell)} - 2s(Pz,(P\varphi)z)_{L^2(L;\mathbb{R}^\ell)} + \|Pz\|^2_{L^2(L;\mathbb{R}^\ell)} \\ &\ge s^2\int_L |(P\varphi)z|^2\omega_L - 2s\int_L Pz\cdot(P\varphi)z\omega_L \end{aligned}$$

and for $\gamma > 0$ to be fixed later,

$$\begin{aligned} &2s(P_s z,(P\varphi-\gamma I)z)_{L^2(L;\mathbb{R}^\ell)} \\ &= 2s\int_L Pz\cdot(P\varphi-\gamma I)z\omega_L - 2s^2\int_L (P\varphi)z\cdot(P\varphi-\gamma I)z\omega_L. \end{aligned}$$

Therefore, by (2.8), it follows that

$$\begin{aligned} &\|P_s z\|^2_{L^2(L;\mathbb{R}^\ell)} + 2s(P_s z,(P\varphi-\gamma I)z)_{L^2(L;\mathbb{R}^\ell)} \\ &\ge s^2\int_L |(P\varphi)z|^2\omega_L - 2s^2\int_L (P\varphi)z\cdot(P\varphi-\gamma I)z\omega_L - 2\gamma s\int_L Pz\cdot z\omega_L \\ &= s^2\int_L (2\gamma(P\varphi)z\cdot z - |(P\varphi)z|^2)\omega_L - 2\gamma s\int_L Pz\cdot z\omega_L \\ &\ge s^2\int_L (2\rho\gamma|z|^2 - |(P\varphi)z|^2)\omega_L - 2\gamma s\int_L Pz\cdot z\omega_L. \end{aligned}$$

Since there exists a constant $C > 0$ independent of z such that

$$|(P\varphi)z(t,x)|^2 \le C|z(t,x)|^2 \text{ a.e. } (t,x) \in L,$$

by taking $\gamma > 0$ large enough, we obtain

$$
\begin{aligned}
&\|P_s z\|^2_{L^2(L;\mathbb{R}^\ell)} + 2s(P_s z, (P\varphi - \gamma I)z)_{L^2(L;\mathbb{R}^\ell)} \\
&\geq Cs^2 \int_L |z|^2 \omega_L - 2\gamma s \int_L \sum_{1\leq i,j\leq \ell} A^\mu_{ij} \partial_\mu z^i z^j \omega_L \\
&= Cs^2 \int_L |z|^2 \omega_L - 2\gamma s \int_L \sum_{1\leq i=j\leq \ell} A^\mu_{ii} \partial_\mu z^i z^i \omega_L \\
&\quad - 2\gamma s \int_L \sum_{1\leq i<j\leq \ell} A^\mu_{ij} \partial_\mu z^i z^j \omega_L - 2\gamma s \int_L \sum_{1\leq j<i\leq \ell} A^\mu_{ij} \partial_\mu z^i z^j \omega_L \\
&= Cs^2 \int_L |z|^2 \omega_L - \gamma s \int_L \sum_{i=1}^{\ell} A^\mu_{ii} \partial_\mu (|z^i|^2) \omega_L \\
&\quad - 2\gamma s \int_L \sum_{1\leq i<j\leq \ell} A^\mu_{ij} \partial_\mu z^i z^j \omega_L - 2\gamma s \int_L \sum_{1\leq i<j\leq \ell} A^\mu_{ij} \partial_\mu z^j z^i \omega_L \\
&= Cs^2 \int_L |z|^2 \omega_L - \gamma s \int_L \sum_{i=1}^{\ell} A^\mu_{ii} \nabla_\mu (|z^i|^2) \omega_L - 2\gamma s \int_L \sum_{1\leq i<j\leq \ell} A^\mu_{ij} \nabla_\mu (z^i z^j) \omega_L \\
&= Cs^2 \int_L |z|^2 \omega_L + \gamma s \int_L \sum_{i=1}^{\ell} \nabla_\mu A^\mu_{ii} |z^i|^2 \omega_L + 2\gamma s \int_L \sum_{1\leq i<j\leq \ell} \nabla_\mu A^\mu_{ij} z^i z^j \omega_L \\
&\quad - \gamma s \int_M \sum_{i=1}^{\ell} \left[A^0_{ii} |z^i|^2 \right]_{t=0}^{t=T} \omega_M - 2\gamma s \int_M \sum_{1\leq i<j\leq \ell} \left[A^0_{ij} z^i z^j \right]_{t=0}^{t=T} \omega_M \\
&\quad - \gamma s \int_\Sigma \sum_{i=1}^{\ell} g_{jk} \nu^j A^k_{ii} |z^i|^2 \omega_\Sigma - 2\gamma s \int_\Sigma \sum_{1\leq i<j\leq \ell} g_{jk} \nu^j A^k_{ij} z^i z^j \omega_\Sigma \\
&= Cs^2 \int_L |z|^2 \omega_L - O(s) \int_L |z|^2 \omega_L - \gamma s \int_M \left[A^0 z \cdot z \right]_{t=0}^{t=T} \omega_M - \gamma s \int_\Sigma (\nu_k A^k) z \cdot z \omega_\Sigma
\end{aligned}
$$

as $s \to \infty$ for sufficiently large $\gamma > 0$. Note that $\nu_k := g_{jk}\nu^j$ and we used Lemma 2.9, namely the Gauss formula on a Lorentzian manifold $(L, -dt\otimes dt + g)$. In regard to the second term of the left-hand side, the following inequality holds:

$$
2s(P_s z, (P\varphi - \gamma I)z)_{L^2(L;\mathbb{R}^\ell)} \leq \frac{1}{\varepsilon} \|P_s z\|^2_{L^2(L;\mathbb{R}^\ell)} + \varepsilon s^2 \|P\varphi - \gamma I\|^2_{L^\infty(L;\mathbb{M}^{\ell\times\ell})} \|z\|^2_{L^2(L;\mathbb{R}^\ell)}
$$

for an arbitrary $\varepsilon > 0$. Therefore, we obtain

$$
\begin{aligned}
&s^2 \left(C - \varepsilon \|P\varphi - \gamma I\|^2_{L^\infty(L;\mathbb{M}^{\ell\times\ell})} \right) \|z\|^2_{L^2(L;\mathbb{R}^\ell)} - O(s) \|z\|^2_{L^2(L;\mathbb{R}^\ell)} \\
&\quad + \gamma s \int_M (A^0 z \cdot z)(0,x) \omega_M \\
&\leq \left(1 + \frac{1}{\varepsilon} \right) \|P_s z\|^2_{L^2(L;\mathbb{R}^\ell)} + \gamma s \int_\Sigma \nu_k A^k z \cdot z \omega_\Sigma + \gamma s \int_M (A^0 z \cdot z)(T,x) \omega_M.
\end{aligned}
$$

Choosing $\varepsilon > 0$ small enough to satisfy

$$
C - \varepsilon \|P\varphi - \gamma I\|^2_{L^\infty(L;\mathbb{M}^{\ell\times\ell})} > 0
$$

and taking $s > 0$ sufficiently large yield

$$\begin{aligned} &s^2\|z\|^2_{L^2(L;\mathbb{R}^\ell)} + s\int_M (A^0 z\cdot z)(0,x)\omega_M \\ &\le C\|P_s z\|^2_{L^2(L;\mathbb{R}^\ell)} + Cs\int_\Sigma \nu_k A^k z\cdot z\omega_\Sigma + Cs\int_M (A^0 z\cdot z)(T,x)\omega_M \end{aligned}$$

for some $C > 0$ independent of s and z. Applying (2.1) to the second term of the left-hand side, we complete the proof. □

2.2. Energy estimates. Although a large number of energy estimates are known, (e.g., [57, Chapter 2], [58, Chapter 7]), we introduce the original energy estimate Lemma 2.7.

For a matrix-valued function $P\tau$ satisfying (2.1) and $u \in H^1(L;\mathbb{R}^\ell)$, we define a quantity $E(t)$ such that

$$E(t) := \int_M \big((P\tau)\nabla_N u\cdot\nabla_N u\big)(t,x)\omega_M + \int_M |u(t,x)|^2\omega_M, \quad t\in[0,T],$$

where $N = -\nabla\tau$ and $\tau : L \to [0,T]$ is the projection.

LEMMA 2.7. *Let $A \in \mathcal{A}$, $p \in W^{1,\infty}(0,T;L^\infty(M;\mathbb{M}^{\ell\times\ell}))$, $f \in L^2(M;\mathbb{R}^\ell)$, and $R \in H^1(0,T;L^\infty(M;\mathbb{M}^{\ell\times\ell}))$. Then, there exists a constant $C > 0$ such that*

$$E(t) \le C\|f\|^2_{L^2(M;\mathbb{R}^\ell)}$$

holds for all $t \in [0,T]$ and $u \in H^2(L;\mathbb{R}^\ell)$ satisfying (2.4).

PROOF. Applying ∇_N to the equation in (2.4) yields

$$A^\mu(t,x)\partial_\mu\partial_t u + \partial_t A^\mu(t,x)\partial_\mu u + p(t,x)\partial_t u + \partial_t p(t,x)u = \partial_t R(t,x)f(x).$$

Indeed, $\nabla_N\partial_\mu u = \nabla_\mu\nabla_N u = \partial_\mu\partial_t u$ and $\nabla_N A^\mu = \partial_t A^\mu$ hold. Multiplying $2\nabla_N u = 2\partial_t u$ to the above equality and integrating over M yield

$$\begin{aligned} &\int_M \sum_{i=1}^{\ell} A^\mu_{ii}(t,x)\nabla_\mu(|\partial_t u^i|^2)\omega_M + 2\int_M \sum_{1\le i<j\le\ell} A^\mu_{ij}(t,x)\nabla_\mu(\partial_t u^i\partial_t u^j)\omega_M \\ &\quad + \int_M 2\partial_t A^\mu(t,x)\partial_\mu u\cdot\partial_t u\omega_M + \int_M 2p(t,x)\partial_t u\cdot\partial_t u\omega_M + \int_M 2\partial_t p(t,x)u\cdot\partial_t u\omega_M \\ &= \int_M 2\partial_t R(t,x)f(x)\cdot\partial_t u\omega_M. \end{aligned}$$

Integration by parts on M yields

$$\begin{aligned} &\int_M \sum_{1\le i,j\le\ell} A^0_{ij}(t,x)\partial_t(\partial_t u^i\partial_t u^j)\omega_M - \int_M \nabla_k A^k(t,x)\partial_t u\cdot\partial_t u\omega_M \\ &\quad + \int_M 2\partial_t A^\mu(t,x)\partial_\mu u\cdot\partial_t u\omega_M + \int_M 2p(t,x)\partial_t u\cdot\partial_t u\omega_M + \int_M 2\partial_t p(t,x)u\cdot\partial_t u\omega_M \\ &= \int_M 2\partial_t R(t,x)f(x)\cdot\partial_t u\omega_M - \int_{\partial M} g_{jk}\nu^j A^k\partial_t u\cdot\partial_t u\omega_{\partial M}. \end{aligned}$$

Therefore, it follows that

$$
\begin{aligned}
&\frac{d}{dt}\int_M A^0(t,x)\partial_t u\cdot\partial_t u\omega_M = \int_M \partial_t A^0\partial_t u\cdot\partial_t u\omega_M + \int_M \sum_{1\le i,j\le \ell} A^0_{ij}\partial_t(\partial_t u^i\partial_t u^j)\omega_M\\
&= -\int_M (\partial_t A^0(t,x)+2p(t,x))\partial_t u\cdot\partial_t u\omega_M + \int_M \nabla_k A^k(t,x)\partial_t u\cdot\partial_t u\omega_M\\
&\quad -\int_M 2\partial_t A^k(t,x)\partial_k u\cdot\partial_t u\omega_M - \int_M 2\partial_t p(t,x)u\cdot\partial_t u\omega_M + \int_M 2\partial_t Rf\cdot\partial_t u\omega_M\\
&\quad -\int_{\partial M}\nu_k A^k(t,x)\partial_t u\cdot\partial_t u\omega_{\partial M}\\
&\le C\Big(\int_M A^0(t,x)\partial_t u\cdot\partial_t u\omega_M + \int_M |u|^2\omega_M\\
&\quad +\int_M |\partial_t A^k(t,x)\partial_k u|^2\omega_M + \int_M |\partial_t Rf|^2\omega_M\Big) - \int_{\partial M}\nu_k A^k(t,x)\partial_t u\cdot\partial_t u\omega_{\partial M}.
\end{aligned}
$$

Adding $\frac{d}{dt}\int_M |u|^2\omega_M$ to the both sides of the above estimate, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_M A^0(t,x)\partial_t u\cdot\partial_t u\omega_M + \int_M |u|^2\omega_M\right)\\
&\le C\Big(\int_M A^0(t,x)\partial_t u\cdot\partial_t u\omega_M + \int_M |u|^2\omega_M + \int_M |\partial_t A^k(t,x)\partial_k u|^2\omega_M\\
&\quad +\int_M |\partial_t Rf|^2\omega_M\Big) + \int_M 2|u||\partial_t u|\omega_M - \int_{\partial M}\nu_k A^k(t,x)\partial_t u\cdot\partial_t u\omega_{\partial M}\\
&\le C\Big(\int_M A^0(t,x)\partial_t u\cdot\partial_t u\omega_M + \int_M |u|^2\omega_M + \int_M |\partial_t A^k(t,x)\partial_k u|^2\omega_M\\
&\quad +\int_M |\partial_t Rf|^2\omega_M\Big) - \int_{\partial M}\nu_k A^k(t,x)\partial_t u\cdot\partial_t u\omega_{\partial M}.
\end{aligned}
\tag{2.10}
$$

Moreover, applying (2.3) to (2.10) and using the equation in (2.4) yield

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_M A^0(t,x)\partial_t u\cdot\partial_t u\omega_M + \int_M |u|^2\omega_M\right)\\
&\le C\left(\int_M A^0(t,x)\partial_t u\cdot\partial_t u\omega_M + \int_M |u|^2\omega_M + \int_M |\partial_t Rf|^2\omega_M + \int_M |Rf|^2\omega_M\right)\\
&\quad -\int_{\partial M}\nu_k A^k(t,x)\partial_t u\cdot\partial_t u\omega_{\partial M},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{d}{dt}\left(e^{-Ct}\int_M \Big(A^0(t,x)\partial_t u\cdot\partial_t u + |u|^2\Big)\omega_M\right)\\
&\le e^{-Ct}\left(C\int_M \Big(\sum_{m=0}^{1}|\partial_t^m Rf|^2\Big)\omega_M - \int_{\partial M}\nu_k A^k(t,x)\partial_t u\cdot\partial_t u\omega_{\partial M}\right).
\end{aligned}
$$

Integrating over $(0,t)$ for $t\le T$ yields

$$
E(t)\le C\left(E(0)+\int_\Omega |f|^2\omega_M\right).
$$

Since

$$E(0) \le C \int_M |A^0(0,x)\partial_t u(0,x)|^2 \omega_M \le C\|f\|^2_{L^2(M;\mathbb{R}^\ell)}$$

holds by (2.4), we complete the proof.

□

3. Proof of main result

Proof of Theorem 2.4. Set $M_0 := \max_{x\in M} \varphi_0(x)$ and $m_0 := \min_{x\in M} \varphi_0(x)$. By the assumption (2.6), we can choose $0 < \beta < \dfrac{\delta}{\lambda}$ such that

$$T_0 < \frac{M_0 - m_0}{\beta} < T.$$

Then, there exists $\kappa > 0$ such that

$$M_0 - m_0 - \beta T < -\kappa. \tag{2.11}$$

Applying the Carleman estimate (2.9) to $\nabla_N u = \partial_t u \in H^1(L;\mathbb{R}^\ell)$ yields

$$\begin{aligned} &s^2 \int_L e^{2s\varphi}|\partial_t u|^2\omega_L + s\int_M e^{2s\varphi(0,x)}|R(0,x)f(x)|^2\omega_M \\ &\le C\int_L e^{2s\varphi}|(P + p(t,x))\partial_t u|^2\omega_L + Cs\int_{\Sigma_+} e^{2s\varphi}\nu_k A^k\partial_t u\cdot\partial_t u\omega_\Sigma \\ &\quad + Cs\int_M e^{2s\varphi(T,x)}(A^0\partial_t u\cdot\partial_t u)(T,x)\omega_M. \end{aligned} \tag{2.12}$$

Since we obtain

$$\begin{aligned} (P + p(t,x))\partial_t u &= \partial_t\Big(A^\mu(t,x)\partial_\mu u + p(t,x)u\Big) \\ &\quad - \partial_t A^\mu(t,x)\partial_\mu u - \partial_t p(t,x)u \\ &= \partial_t R(t,x)f(x) - \partial_t A^\mu(t,x)\partial_\mu u - \partial_t p(t,x)u, \end{aligned}$$

it follows that

$$\begin{aligned} |(P + p(t,x))\partial_t u|^2 &\le C\Big(|\partial_t Rf|^2 + |\partial_t u|^2 + |\partial_t A^k(t,x)\partial_k u|^2 + |u|^2\Big) \\ &\le C\Big(|\partial_t Rf|^2 + |\partial_t u|^2 + |A^k(t,x)\partial_k u|^2 + |u|^2\Big) \\ &\le C\Big(|\partial_t Rf|^2 + |Rf|^2 + |\partial_t u|^2 + |u|^2\Big). \end{aligned} \tag{2.13}$$

Moreover, we obtain

$$\begin{aligned} &s\int_M e^{2s\varphi(T,x)}(A^0\partial_t u\cdot\partial_t u)(T,x)\omega_M \\ &\le Cse^{2(M_0-\beta T)s}\int_M (A^0\partial_t u\cdot\partial_t u)(T,x)\omega_M \le Cse^{2(M_0-\beta T)s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)}. \end{aligned} \tag{2.14}$$

Applying (2.13) and (2.14) to (2.12) yields

$$\begin{aligned}
&s^2\int_L e^{2s\varphi}|\partial_t u|^2\omega_L + s\int_M e^{2s\varphi(0,x)}|R(0,x)f(x)|^2\omega_M \\
&\le C\int_L e^{2s\varphi}\sum_{m=0}^{1}|\partial_t^m Rf|^2\omega_L + C\int_L e^{2s\varphi}|u|^2\omega_L \\
&\quad + Cs\int_{\Sigma_+} e^{2s\varphi}\nu_k A^k\partial_t u\cdot\partial_t u\omega_\Sigma + Cse^{2(M_0-\beta T)s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)}.
\end{aligned}\tag{2.15}$$

In regard to the left-hand side of (2.15), we obtain

$$s^2\int_L e^{2s\varphi}|\partial_t u|^2\omega_L + s\int_M e^{2s\varphi(0,x)}|R(0,x)f(x)|^2\omega_M \ge Cs\|e^{s\varphi_0}f\|^2_{L^2(M;\mathbb{R}^\ell)}.\tag{2.16}$$

In regard to the right-hand side of (2.15), applying the Carleman estimate (2.9) to u yields

$$\begin{aligned}
\int_L e^{2s\varphi}|u|^2\omega_L &\le \frac{C}{s^2}\int_L e^{2s\varphi}|Rf|^2\omega_L + \frac{C}{s}\int_{\Sigma_+} e^{2s\varphi}\nu_k A^k u\cdot u\omega_\Sigma \\
&\quad + \frac{C}{s}\int_M e^{2s\varphi(T,x)}(A^0u\cdot u)(T,x)\omega_M \\
&\le \frac{C}{s^2}\int_L e^{2s\varphi}|Rf|^2\omega_L + \frac{C}{s}\int_{\Sigma_+} e^{2s\varphi}\nu_k A^k u\cdot u\omega_\Sigma \\
&\quad + \frac{C}{s}e^{2(M_0-\beta T)s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)}.
\end{aligned}\tag{2.17}$$

Applying (2.16) and (2.17) to (2.15) yields

$$\begin{aligned}
&s\|e^{s\varphi_0}f\|^2_{L^2(M;\mathbb{R}^\ell)}\\
&\leq C\int_L e^{2s\varphi}\Big(\sum_{m=0}^{1}|\partial_t^m Rf|^2\Big)\omega_L + C\int_{\Sigma_+} e^{2s\varphi}|u|^2\omega_\Sigma + Cse^{Cs}\|\partial_t u\|^2_{L^2(\Sigma_+;\mathbb{R}^\ell)}\\
&\quad + Cse^{2(M_0-\beta T)s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)}\\
&\leq C\int_L e^{2s\varphi}\Big(\sum_{m=0}^{1}|\partial_t^m Rf|^2\Big)\omega_L + Cse^{Cs}\sum_{m=0}^{1}\|\partial_t^m u\|^2_{L^2(\Sigma_+;\mathbb{R}^\ell)}\\
&\quad + Cse^{2(M_0-\beta T)s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)}\\
&= C\int_M\left(\int_0^T e^{-2s(\varphi_0(x)-\varphi(t,x))}\Big(\sum_{m=0}^{1}\|\partial_t^m R(t,\cdot)\|^2_{L^\infty(M;\mathbb{M}^{\ell\times\ell})}\Big)dt\right)e^{2s\varphi_0}|f|^2\omega_M\\
&\quad + Cse^{Cs}\sum_{m=0}^{1}\|\partial_t^m u\|^2_{L^2(\Sigma_+;\mathbb{R}^\ell)} + Cse^{2(M_0-\beta T)s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)}\\
&= C\int_M\left(\int_0^T e^{-2\beta ts}\Big(\sum_{m=0}^{1}\|\partial_t^m R(t,\cdot)\|^2_{L^\infty(M;\mathbb{M}^{\ell\times\ell})}\Big)dt\right)e^{2s\varphi_0}|f|^2\omega_M\\
&\quad + Cse^{Cs}\sum_{m=0}^{1}\|\partial_t^m u\|^2_{L^2(\Sigma_+;\mathbb{R}^\ell)} + Cse^{2(M_0-\beta T)s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)}\\
&\leq o(1)\|e^{s\varphi_0}f\|^2_{L^2(M;\mathbb{R}^\ell)} + Cse^{Cs}\sum_{m=0}^{1}\|\partial_t^m u\|^2_{L^2(\Sigma_+;\mathbb{R}^\ell)} + Cse^{2(M_0-\beta T)s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)}
\end{aligned}$$

as $s \to +\infty$ by the Lebesgue dominated convergence theorem. Choosing $s > s_*$ large enough yields

$$e^{2m_0 s}(1 - Cse^{2(M_0-m_0-\beta T)s})\|f\|^2_{L^2(M;\mathbb{R}^\ell)} \leq Ce^{Cs}\sum_{m=0}^{1}\|\partial_t^m u\|^2_{L^2(\Sigma_+;\mathbb{R}^\ell)},$$

which implies, by (2.11),

$$e^{2m_0 s}(1 - Cse^{-2\kappa s})\|f\|^2_{L^2(M;\mathbb{R}^\ell)} \leq Ce^{Cs}\sum_{m=0}^{1}\|\partial_t^m u\|^2_{L^2(\Sigma_+;\mathbb{R}^\ell)}.$$

Choosing $s > s_0$ large enough completes the proof. □

4. Useful lemmas

4.1. Poincaré's lemma. To obtain Theorem 2.4, the assumption (2.2) is indispensable for the Carleman estimate Proposition 2.6. In a special case, a sufficient condition for the existence of φ_0 is known as the Poincaré's lemma (e.g., [67, Theorem 13.2] and [13, Section 4.3]). Let $\Omega^k(M)$ be a space of smooth k-forms on M.

LEMMA 2.8 (Poincaré's lemma). *Let $U \subset \mathbb{R}^n$ be a star-shaped open subset. If $\alpha \in \Omega^k(U)$ satisfies $d\alpha = 0$, then there exists $\omega \in \Omega^{k-1}(U)$ such that $d\omega = \alpha$.*

For more general manifolds, some results are known in de Rham cohomology theory. The fuller study of de Rham cohomology lies outside the scope of this chapter. Under the situation where the Poincaré's lemma is valid, the following condition is sufficient for our assumption (2.2).

$$\begin{cases} \exists \alpha = \alpha_k dx^k \in C^1(T^*M) \text{ s.t. } d\alpha = 0 \text{ and} \\ \langle \alpha, d\pi(A) \rangle \in C^1(L; \mathrm{Sym}_\ell) \text{ is uniformly positive definite.} \end{cases} \tag{2.18}$$

4.2. Gauss formula on Lorentzian manifolds. We say the boundary ∂L is spacelike (timelike) if the induced metric to ∂L is Riemannian (Lorentzian).

LEMMA 2.9. *Let (L, h) be an $n+1$-dimensional compact oriented Lorentzian manifold with boundary. Assume that the boundary is spacelike or timelike and let X be a smooth vector field. Then if N denotes the outer unit normal to ∂L, it follows that*

$$\int_L \operatorname{div} X \omega_L = \int_{\partial L} \frac{h(X,N)}{h(N,N)} \omega_{\partial L}.$$

For more details of the Gauss formula on Lorentzian manifolds, readers are referred to Ringström [58, Lemma 10.8] and Bär–Ginoux–Pfäffle [2, Theorem 1.3.16].

CHAPTER 3

Degenerate hyperbolic equations

1. Introduction and main result

Let $d \in \mathbb{N}$, $T > 0$, $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$, and $\nu(x)$ be the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$. Without loss of generality, we suppose $0 \in \Omega$. We set $Q := \Omega \times (0, T)$ and $\Sigma := \partial\Omega \times (0, T)$. We introduce the differential operator A such that

$$Au(x,t) := \partial_t u + H(t) \cdot \nabla u, \tag{3.1}$$

where $H(t) := (H_1(t), \dots, H_d(t))^{\mathsf{T}}$ is a continuous vector-valued function on $[0, T]$.

A lot of inverse problems via Carleman estimates for transport equations have been studied. Klibanov and Pamyatnykh [45] proved a global uniqueness theorem for an inverse coefficient problem. Gaitan and Ouzzane [29], Machida and Yamamoto [53], and Gölgeleyen and Yamamoto [31] proved Lipschitz stabilities for inverse coefficient and source problems via global Carleman estimates for transport equations with variable coefficients. Cannarsa, Floridia, Gölgeleyen, and Yamamoto [9] proved local Hölder stability to determine principal terms and zeroth-order terms. We should note that these results were all for transport equations the coefficients of which do not depend on time variable t but depend on space variable x. In regard to transport equations having a time-dependent principal part, Cannarsa, Floridia, and Yamamoto [10] proved an observability inequality for the operator A defined by (3.1) with $|H(t)| > 0$ for all $t \in [0, T]$, i.e., non-degenerate case, which was motivated by applications to inverse problems. In this chapter, we eliminate the assumption on the positivity of $|H(t)|$ and prove the observability inequality in the degenerate case. Although this chapter is inspired by [10], we note that our methodology is a little different from it, since we do not use the partition arguments employed in [10]. Moreover, we prove the observability inequality using a synthetic technique recently introduced in [33] by Huang, Imanuvilov, and Yamamoto, without using the classical cut-off arguments in the proof of the observability through the Carleman estimate. This enables us to simplify proofs of observability inequalities.

For more applications of Carleman estimates to inverse problems, controllability, and unique continuations for hyperbolic equations, readers are referred to Bellassoued and Yamamoto [6], and Takase [64]. They established Carleman estimates for second-order hyperbolic operators with variable coefficients on manifolds. Moreover, for degenerate evolutions equations there is a extensive literature, one can see, e.g., Floridia [21] and Floridia, Nitsch, and Trombetti [22].

The structure of the chapter is following. In this section, after describing the problem formulation and some notations, we present our main result in Theorem 3.4. In section 2, we prepare some propositions needed to prove Theorem 3.4. In particular, we obtain the energy estimate (see Lemma 3.5 and Proposition 3.6), and the Carleman estimate for the degenerate case (see Proposition 3.7), which play important roles in proving the main result. Finally, in section 3 we prove Theorem 3.4. In section 4, using the methodology of this chapter we obtain an observability inequality for the non-degenerate case, studied in [10], by a proof shorter than one in [10].

In this chapter, we consider the degenerate case, where we impose the following assumptions on the vector field $H \in C([0,T];\mathbb{R}^d)$:

$$H(0) = 0; \tag{3.2}$$

$$\exists T_1 \in (0,T],\ \exists \rho > 0 \text{ s.t. } H \in C^1([0,T_1];\mathbb{R}^d) \text{ and } \min_{t\in[0,T_1]} |H'(t)| \geq \rho. \tag{3.3}$$

Under assumptions (3.2) and (3.3), we consider the Cauchy problem

$$\begin{cases} Au = \partial_t u + H(t)\cdot\nabla u = 0 & \text{in } Q, \\ u = g & \text{on } \Sigma, \end{cases} \tag{3.4}$$

where $g \in L^2(\Sigma)$, and prove an observability inequality in Theorem 3.4. Unlike the non-degenerate case by Cannarsa, Floridia, and Yamamoto [9], we should impose the extra assumption (3.3) on the positivity of $|H'(t)|$ due to the degeneration (3.2). Nevertheless, the regularity class in (3.3) imposed on H is the same one as in [10].

Before describing mathematical settings, we mention a synthetic statement of the main result Theorem 3.4. We note to prove an observability inequality for a hyperbolic equation the observation time should be given sufficiently large due to the finite propagation speed (e.g., Bardos, Lebeau, and Rauch [3]). Theorem 3.4 claims that if the direction of the unit vector $\frac{H'(t)}{|H'(t)|}$ changes moderately comparing with the time for the distant wave to reach the boundary, then we can obtain the observability inequality (3.9) for a sufficient large observation time. To formulate this situation mathematically, we define some preliminary notations.

Definition 3.1. *Let $T > 0$, $c_0 \in (\frac{1}{\sqrt{2}}, 1)$, and H be a vector-valued function satisfying (3.3). We define a positive number $t_1 \in (0, T_1]$ such that*

$$t_1 := \sup\left\{\tau \in [0,T_1] \;\middle|\; \frac{H'(t)}{|H'(t)|}\cdot\frac{H'(0)}{|H'(0)|} \geq c_0, \quad \forall t \in [0,\tau]\right\}. \tag{3.5}$$

Remark 3.2. *Note that $t_1 > 0$ because H' is continuous.*

By the definition of the positive time $t_1 \in (0,T_1]$ introduced in (3.5), the angle between $\frac{H'(t)}{|H'(t)|}$ and $\frac{H'(0)}{|H'(0)|}$ is less than or equal to $\frac{\pi}{4}$ for $t \in [0,t_1]$. The positive time t_1 will be crucial to prove the observability inequality (3.9) in Theorem 3.4. The next lemma is a basic property for H' in the time interval $[0,t_1]$.

Lemma 3.3. *Let $T > 0$, $c_0 \in (\frac{1}{\sqrt{2}}, 1)$, H be a vector-valued function satisfying (3.3), and $t_1 \in (0,T_1]$ be the positive number defined by (3.5). Then, there exists $x_0 \in \overline{\Omega}^c := \mathbb{R}^d \setminus \overline{\Omega}$ such that*

$$\min_{(x,t)\in\overline{\Omega}\times[0,t_1]} \frac{H'(t)\cdot(x-x_0)}{|H'(t)||x-x_0|} \geq 2c_0^2 - 1 (> 0). \tag{3.6}$$

PROOF. If we take $x_0 := -R\theta_0 \in \overline{\Omega}^c$ for $R > \frac{1+c_0}{1-c_0}\operatorname{diam}\Omega$ and $\theta_0 := \frac{H'(0)}{|H'(0)|}$, we find

$$\begin{aligned}(x - x_0)\cdot\theta_0 &= x\cdot\theta_0 + R \geq R - |x| \geq R - \operatorname{diam}\Omega \\ &> c_0(R + \operatorname{diam}\Omega) \geq c_0|x - x_0|\end{aligned}$$

holds for all $x \in \overline{\Omega}$, which implies $\displaystyle\min_{(x,t)\in\overline{\Omega}\times[0,t_1]} \frac{x - x_0}{|x - x_0|}\cdot\theta_0 \geq c_0$. Moreover, taking $\displaystyle\min_{(x,t)\in\overline{\Omega}\times[0,t_1]} \frac{H'(t)}{|H'(t)|}\cdot\theta_0 \geq c_0$ into account, we finally conclude (3.6) is true by the trigonometric addition formulas for the angle between $\frac{x-x_0}{|x-x_0|}$ and θ_0, and the angle between $\frac{H'(t)}{|H'(t)|}$ and θ_0. □

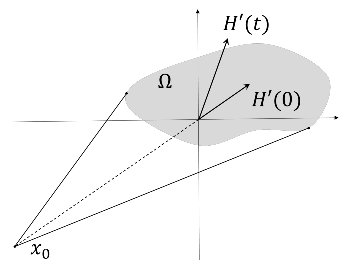

FIGURE 1. The situation of $H'(t)$ and $x_0 \in \overline{\Omega}^c$ in Lemma 3.3.

For a fixed $x_0 \in \overline{\Omega}^c$ satisfying (3.6), define the positive number

$$T_0 := \sqrt{\frac{\max_{x\in\overline{\Omega}}|x - x_0|^2 - \min_{x\in\overline{\Omega}}|x - x_0|^2}{\delta}}, \tag{3.7}$$

where

$$\delta := \rho(2c_0^2 - 1)\operatorname{dist}(x_0, \Omega) > 0. \tag{3.8}$$

The next theorem is our main result in this chapter.

THEOREM 3.4. *Let $T > 0$, $c_0 \in (\frac{1}{\sqrt{2}}, 1)$, $H \in C([0,T];\mathbb{R}^d)$, and $g \in L^2(\Sigma)$. Assume (3.2) and (3.3). If the number $t_1 \in (0, T_1]$ defined by (3.5) satisfies $T_0 < t_1$ for some $x_0 \in \overline{\Omega}^c$ satisfying (3.6), then there exists a constant $C > 0$ independent of $g \in L^2(\Sigma)$ such that for all $t \in [0,T]$,*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq C\|g\|_{L^2(\Sigma)} \tag{3.9}$$

holds for all $u \in H^1(Q)$ satisfying (3.4).

2. Preliminaries

In this section, we prepare some results needed to prove Theorem 3.4. In section 2.1, by the energy estimate Lemma 3.5 we prove Proposition 3.6, which means if the observability inequality (3.9) holds locally in time, then it holds also globally in time. In section 2.2, we present the Carleman estimate in Proposition 3.7.

2.1. Energy estimate. For the proof of Theorem 3.4, we use the energy estimate of the following type, which is proved without assuming (3.2) and (3.3).

LEMMA 3.5. *Let $T > 0$, $H \in C([0,T];\mathbb{R}^d)$, and $g \in L^2(\Sigma)$. Then, there exists a constant $C > 0$ independent of $g \in L^2(\Sigma)$ such that for all $t \in [0,T]$,*

$$\left| \|u(\cdot,t)\|_{L^2(\Omega)}^2 - \|u(\cdot,0)\|_{L^2(\Omega)}^2 \right| \leq C\|g\|_{L^2(\Sigma)}^2$$

holds for all $u \in H^1(Q)$ satisfying (3.4).

PROOF. Multiplying the equation in (3.4) by $2u$ and integrating over Ω yield

$$\int_\Omega \partial_t(|u|^2)dx + \int_\Omega H(t)\cdot\nabla(|u|^2)dx = 0,$$

i.e.,

$$\frac{d}{dt}\left(\int_\Omega |u|^2 dx\right) = -\int_{\partial\Omega} (H(t)\cdot\nu(x))\,|g|^2 d\sigma.$$

Integration over $[0,t]$ yields

$$\left| \|u(\cdot,t)\|_{L^2(\Omega)}^2 - \|u(\cdot,0)\|_{L^2(\Omega)}^2 \right| \leq C\|g\|_{L^2(\Sigma)}^2,$$

for some $C > 0$ independent of $g \in L^2(\Sigma)$, $t \in [0,T]$, and $u \in H^1(Q)$. □

PROPOSITION 3.6. *Let $T > 0$, $H \in C([0,T];\mathbb{R}^d)$, and $g \in L^2(\Sigma)$. Assume there exist $\tau \in [0,T]$ and a constant $C_1 > 0$ independent of $g \in L^2(\Sigma)$ such that for all $t \in [0,\tau]$,*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq C_1\|g\|_{L^2(\Sigma)}$$

holds for all $u \in H^1(Q)$ satisfying (3.4). Then, there exists a constant $C_2 > 0$ independent of $g \in L^2(\Sigma)$ such that

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq C_2\|g\|_{L^2(\Sigma)}$$

holds for all $t \in [0,T]$ and $u \in H^1(Q)$ satisfying (3.4).

PROOF. The claim is trivial when $\tau = T$. When $\tau < T$, Lemma 3.5 and the assumption in Proposition 3.6 yield

$$\begin{aligned}\|u(\cdot,t)\|_{L^2(\Omega)}^2 &\leq \|u(\cdot,0)\|_{L^2(\Omega)}^2 + C\|g\|_{L^2(\Sigma)}^2 \\ &\leq (C_1^2 + C)\|g\|_{L^2(\Sigma)}^2\end{aligned}$$

for all $t \in [0,T]$ and $u \in H^1(Q)$ satisfying (3.4). If we set $C_2 := \sqrt{C_1^2 + C}$, we complete the proof. □

2.2. Carleman estimate. Let $\tau > 0$ and $c_0 \in (\frac{1}{\sqrt{2}}, 1)$ be fixed constants. We set $Q_{\pm,\tau} := \Omega \times (-\tau,\tau)$ and $\Sigma_{\pm,\tau} := \partial\Omega \times (-\tau,\tau)$. In this section, we establish the Carleman estimate for the differential operator A,

$$Au := \partial_t u + H(t)\cdot\nabla u$$

in $Q_{\pm,\tau}$ under the following assumptions:

$$H \in C^1([-\tau,\tau];\mathbb{R}^d); \tag{3.10}$$

$$\exists \rho > 0 \text{ s.t. } \min_{t\in[-\tau,\tau]} |H'(t)| \geq \rho; \tag{3.11}$$

$$\exists \theta_0 \in \mathbb{S}^{d-1} \text{ s.t. } \min_{t\in[-\tau,\tau]} \frac{H'(t)\cdot\theta_0}{|H'(t)|} \geq c_0, \tag{3.12}$$

where $\mathbb{S}^{d-1} := \{\xi \in \mathbb{R}^d \mid |\xi| = 1\}$.

Under the assumptions (3.10)–(3.12), we will obtain the Carleman estimate for A in $Q_{\pm,\tau}$.

We can take $x_0 \in \overline{\Omega}^c := \mathbb{R}^d \setminus \overline{\Omega}$ satisfying $\displaystyle\min_{(x,t)\in\overline{Q_{\pm,\tau}}} \frac{H'(t)\cdot(x-x_0)}{|H'(t)||x-x_0|} \geq 2c_0^2 - 1$ by the same argument as in the proof of Lemma 3.3.

For a positive constant $\beta > 0$ to be fixed later, we set

$$\varphi(x,t) := |x-x_0|^2 - \beta t^2, \quad (x,t) \in \overline{Q_{\pm,\tau}}. \tag{3.13}$$

We establish the Carleman estimate Proposition 3.7 for the operator A having time-dependent coefficients. Nevertheless, our choice of weight functions is more similar to the one by Klibanov–Pamyatnykh [45] and Gaitan–Ouzzane [29] than by Cannarsa–Floridia–Yamamoto [9].

PROPOSITION 3.7. *Assume* (3.10), (3.11), *and* (3.12). *Let φ be the smooth function defined by* (3.13), *where $\beta > 0$ is an arbitrary positive number satisfying*

$$0 < \beta < \delta := \rho(2c_0^2 - 1)\operatorname{dist}(x_0, \Omega).$$

Then, there exists a constant $C > 0$ such that

$$\begin{aligned} & s\int_{Q_{\pm,\tau}} e^{2s\varphi}|u|^2 dxdt \\ & \quad \leq C\int_{Q_{\pm,\tau}} e^{2s\varphi}|Au|^2 dxdt + Cs\int_{\Sigma_{\pm,\tau}} e^{2s\varphi} A\varphi(H(t)\cdot\nu(x))|u|^2 d\sigma dt \\ & \qquad + Cs\int_{\Omega}\left(e^{2s\varphi(x,\tau)}|u(x,\tau)|^2 + e^{2s\varphi(x,-\tau)}|u(x,-\tau)|^2\right) dx \end{aligned} \tag{3.14}$$

holds for all $s > 0$ and $u \in H^1(Q_{\pm,\tau})$. Here $d\sigma$ denotes the volume element of $\partial\Omega$.

PROOF. Set $z := e^{s\varphi}u$ and $Pz := e^{s\varphi}A(e^{-s\varphi}z)$ for $u \in H^1(Q_{\pm,\tau})$ and $s > 0$. Since φ is smooth, it follows that $z \in H^1(Q_{\pm,\tau})$. We note that

$$A\varphi(x,t) = -2\beta t + 2H(t)\cdot(x-x_0)$$

and

$$A^2\varphi = -2\beta + 2H'(t)\cdot(x-x_0) + 2|H(t)|^2.$$

Since we have

$$Pz = Az - s(A\varphi)z,$$

$$
\begin{aligned}
(3.15) \qquad \|Pz\|^2_{L^2(Q_{\pm,\tau})} &\geq 2(Az, -s(A\varphi)z)_{L^2(Q_{\pm,\tau})} \\
&= -2s \int_{Q_{\pm,\tau}} (\partial_t z + H(t)\cdot\nabla z)(A\varphi) z\,dxdt \\
&= -s \int_{Q_{\pm,\tau}} (A\varphi)\partial_t(|z|^2)dxdt - s\int_{Q_{\pm,\tau}} (A\varphi)H(t)\cdot\nabla(|z|^2)dxdt \\
&= s\int_{Q_{\pm,\tau}} A^2\varphi|z|^2 dxdt - s\int_{\Sigma_{\pm,\tau}} A\varphi(H(t)\cdot\nu(x))|z|^2 d\sigma dt \\
&\quad - s\int_\Omega \Big[A\varphi|z|^2\Big]_{t=-\tau}^{t=\tau} dx \\
&\geq 2s\int_{Q_{\pm,\tau}} \Big(-\beta + H'(t)\cdot(x-x_0)\Big)|z|^2 dxdt \\
&\quad - s\int_{\Sigma_{\pm,\tau}} (A\varphi)(H(t)\cdot\nu(x))|z|^2 d\sigma dt - s\int_\Omega \Big[A\varphi|z|^2\Big]_{t=-\tau}^{t=\tau} dx
\end{aligned}
$$

holds. For the fixed $x_0 \in \overline{\Omega}^c$ so that $\displaystyle\min_{(x,t)\in\overline{Q_{\pm,\tau}}} \frac{H'(t)\cdot(x-x_0)}{|H'(t)||x-x_0|} \geq 2c_0^2 - 1 (>0)$, it follows that

$$
\begin{aligned}
H'(t)\cdot(x-x_0) &= |H'(t)||x-x_0| \frac{H'(t)\cdot(x-x_0)}{|H'(t)||x-x_0|} \\
&\geq \rho\,\mathrm{dist}(x_0,\Omega) \min_{(x,t)\in\overline{Q_{\pm,\tau}}} \frac{H'(t)\cdot(x-x_0)}{|H'(t)||x-x_0|} \\
&\geq \delta(>0)
\end{aligned}
$$

for all $(x,t) \in \overline{Q_{\pm,\tau}}$ owing to (3.11) and (3.12). We then obtain from (3.15)

$$
\begin{aligned}
\|Pz\|^2_{L^2(Q_{\pm,\tau})} \geq 2(\delta-\beta)s\int_{Q_{\pm,\tau}} |z|^2 dxdt - s\int_{\Sigma_{\pm,\tau}} (A\varphi)(H(t)\cdot\nu(x))|z|^2 d\sigma dt \\
- s\int_\Omega \Big[A\varphi|z|^2\Big]_{t=-\tau}^{t=\tau} dx.
\end{aligned}
$$

Hence, for all $0 < \beta < \delta$, there exists a constant $C > 0$ such that

$$
\begin{aligned}
& s\int_{Q_{\pm,\tau}} e^{2s\varphi}|u|^2 dxdt \\
&\leq C\int_{Q_{\pm,\tau}} e^{2s\varphi}|Au|^2 dxdt + Cs\int_{\Sigma_{\pm,\tau}} e^{2s\varphi}A\varphi(H(t)\cdot\nu(x))|u|^2 d\sigma dt \\
&\quad + Cs\int_\Omega \Big(e^{2s\varphi(x,\tau)}|u(x,\tau)|^2 + e^{2s\varphi(x,-\tau)}|u(x,-\tau)|^2\Big)\,dx
\end{aligned}
$$

holds for all $s > 0$ and $u \in H^1(Q_{\pm,\tau})$. □

REMARK 3.8. *In Proposition 3.7, we do not assume the positivity of $|H(t)|$. In that respect, Proposition 3.7 is different from Theorem 1.5 in Cannarsa–Floridia–Yamamoto [9]. Proposition 3.7 says the Carleman estimate holds regardless of whatever $|H(t)|$ is positive if we assume appropriate properties in regard to H'.*

The technical difference appears in the estimate (3.15). *In the non-degenerate case (e.g., [9] and Proposition 3.12 in this chapter), we can use the positivity of $A\varphi$. However, in the degenerate case, we use the positivity of $A^2\varphi$.*

3. Proof of Theorem 3.4

To prove the main result, we use not only Lemma 3.5 and Proposition 3.6 but also Proposition 3.7, i.e., the Carleman estimate for the operator A. Furthermore, we should describe a technical remark in applying Carleman estimates. In existing works, whenever we applied Carleman estimates to obtain stability estimates for some inverse problems, we introduced appropriate cut-off functions χ and applied Carleman estimates to χu, where u is a solution to considering equations. This was because χu vanished on boundaries of considering domains. However, in our proof of Theorem 3.4, we need not use the cut-off arguments because our Carleman estimate in Proposition 3.7 contains all the boundary terms on $\partial Q_{\pm,\tau}$. This argument without cut-off functions is presented by Huang, Imanuvilov, and Yamamoto [33].

Proof of Theorem 3.4. In the beginning, we extend $H \in C([0,T];\mathbb{R}^d)$ and $u \in H^1(Q)$ satisfying (3.4) in $Q_\pm := \Omega \times (-T,T)$ by setting

$$\bar{H}(t) = \begin{cases} H(t), & t \in [0,T], \\ -H(-t), & t \in [-T,0], \end{cases}$$

and

$$u(x,t) = \begin{cases} u(x,t) & \text{in } \Omega \times (0,T), \\ u(x,-t) & \text{in } \Omega \times (-T,0). \end{cases}$$

By our assumptions (3.2) and (3.3), $\bar{H} \in C([-T,T];\mathbb{R}^d) \cap C^1([-T_1,T_1];\mathbb{R}^d)$ and $u \in H^1(Q_\pm)$. Furthermore, the derivatives with respect to t of $\bar{H}$ and u satisfy

$$\bar{H}'(t) = \begin{cases} H'(t), & t \in [0,T_1], \\ H'(-t), & t \in [-T_1,0], \end{cases}$$

and

$$\partial_t u(x,t) = \begin{cases} \partial_t u(x,t) & \text{in } \Omega \times (0,T), \\ -\partial_t u(x,-t) & \text{in } \Omega \times (-T,0), \end{cases}$$

which imply u satisfies

$$\begin{cases} Au = \partial_t u + \bar{H}(t) \cdot \nabla u = 0 & \text{in } Q_\pm, \\ u = \bar{g} & \text{on } \Sigma_\pm := \partial\Omega \times (-T,T), \end{cases} \tag{3.16}$$

where $\bar{g}$ is extended by

$$\bar{g}(x,t) = \begin{cases} g(x,t) & \text{in } \partial\Omega \times (0,T), \\ g(x,-t) & \text{in } \partial\Omega \times (-T,0). \end{cases} \tag{3.17}$$

Let $t_1 > 0$ be the positive number defined by (3.5) and $x_0 \in \overline{\Omega}^c$ be the point satisfying (3.6) under the assumption

$$T_0 < t_1,$$

where T_0 is defined by (3.7). Owing to Proposition 3.6, it suffices to prove the observability inequality (3.9) in the interval $[0,t_1]$, then we can extend it to all the interval $[0,T]$.

For the fixed $x_0 \in \overline{\Omega}^c$, we take $0 < \beta < \delta$, where δ is defined by (3.8), satisfying

$$(T_0 <)\sqrt{\frac{d_M - d_m}{\beta}} < t_1,$$

where we define

$$d_M := \max_{x\in\overline{\Omega}} |x-x_0|^2, \quad d_m := \min_{x\in\overline{\Omega}} |x-x_0|^2.$$

Then, there exists $\kappa > 0$ such that

$$d_M - d_m - \beta t_1^2 < -\kappa. \tag{3.18}$$

Henceforth, by $C > 0$ we denote a generic constant independent of u and $\bar{g}$ which may change from line to line, unless specified otherwise. We find that $\bar{H}$ satisfies the assumptions (3.10)–(3.12) by taking $\tau = t_1$ and $\theta_0 = \frac{H'(0)}{|H'(0)|}$ needed for Proposition 3.7. Set $Q_{\pm,t_1} := \Omega \times (-t_1, t_1)$ and $\Sigma_{\pm,t_1} := \partial\Omega \times (-t_1, t_1)$. Applying Proposition 3.7 to the extended $u \in H^1(Q_{\pm,t_1})$ satisfying (3.16) yields

$$\begin{aligned} s\int_{Q_{\pm,t_1}} &e^{2s\varphi}|u|^2 dxdt \\ &\le Cs\int_{\Sigma_{\pm,t_1}} e^{2s\varphi} A\varphi(\bar{H}(t)\cdot\nu(x))|u|^2 d\sigma dt \\ &\quad + Cs\int_\Omega \left(e^{2s\varphi(x,t_1)}|u(x,t_1)|^2 + e^{2s\varphi(x,-t_1)}|u(x,-t_1)|^2\right) dx. \end{aligned} \tag{3.19}$$

On the left-hand side of (3.19), we obtain

$$s\int_{Q_{\pm,t_1}} e^{2s\varphi}|u|^2 dxdt \ge se^{2s(d_m-\beta\epsilon^2)}\int_{-\epsilon}^{\epsilon}\int_\Omega |u|^2 dxdt, \tag{3.20}$$

where $\epsilon \in (0, t_1)$ is an arbitrary small constant satisfying for all $x \in \overline{\Omega}$ and $|t| \le \epsilon$,

$$\varphi(x,t) > 0,$$

i.e.,

$$d_m - \beta\epsilon^2 > 0. \tag{3.21}$$

Furthermore, keeping in mind that u is the even extension, applying Lemma 3.5 in (3.20), we have

$$\begin{aligned} s\int_{Q_{\pm,t_1}} e^{2s\varphi}|u|^2 dxdt &\ge 2se^{2s(d_m-\beta\epsilon^2)}\int_0^{\epsilon}\int_\Omega |u|^2 dxdt \\ &\ge 2\epsilon se^{2s(d_m-\beta\epsilon^2)}\left(\|u(\cdot,0)\|^2_{L^2(\Omega)} - C\|g\|^2_{L^2(\Sigma)}\right). \end{aligned} \tag{3.22}$$

Moreover, in regard to the second summand of the right-hand side of (3.19), applying Lemma 3.5 yields

$$\begin{aligned} Cs\int_\Omega &\left(e^{2s\varphi(x,t_1)}|u(x,t_1)|^2 + e^{2s\varphi(x,-t_1)}|u(x,-t_1)|^2\right) dx \\ &\le Cse^{2s(d_M-\beta t_1^2)}\left(\|u(\cdot,t_1)\|^2_{L^2(\Omega)} + \|u(\cdot,-t_1)\|^2_{L^2(\Omega)}\right) \\ &\le 2Cse^{2s(d_M-\beta t_1^2)}\left(\|u(\cdot,0)\|^2_{L^2(\Omega)} + C\|g\|^2_{L^2(\Sigma)}\right). \end{aligned} \tag{3.23}$$

From (3.19), (3.22), and (3.23), keeping in mind (3.17), we obtain

$$\begin{aligned} 2\epsilon se^{2s(d_m-\beta\epsilon^2)} &\left(\|u(\cdot,0)\|^2_{L^2(\Omega)} - C\|g\|^2_{L^2(\Sigma)}\right) \\ &\le Cse^{2s(d_M-\beta t_1^2)}\left(\|u(\cdot,0)\|^2_{L^2(\Omega)} + C\|g\|^2_{L^2(\Sigma)}\right) + Cse^{Cs}\|g\|^2_{L^2(\Sigma)}, \end{aligned}$$

i.e.,

$$e^{2s(d_m-\beta\epsilon^2)}\left(2\epsilon - Ce^{2s(d_M-d_m-\beta t_1^2+\beta\epsilon^2)}\right)\|u(\cdot,0)\|^2_{L^2(\Omega)} \le Ce^{Cs}\|g\|^2_{L^2(\Sigma)}.$$

Applying (3.18) and (3.21) to the left-hand side of the above inequality yields

$$\left(2\epsilon - Ce^{-2s(\kappa-\beta\epsilon^2)}\right)\|u(\cdot,0)\|^2_{L^2(\Omega)} \le Ce^{Cs}\|g\|^2_{L^2(\Sigma)}.$$

By choosing $s > 0$ large enough to satisfy $2\epsilon - Ce^{-2s(\kappa-\beta\epsilon^2)} > 0$ for the sufficiently small $\epsilon > 0$ and applying Lemma 3.5 for (3.16) again on the left-hand side of the above inequality, we have

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \le C\|g\|^2_{L^2(\Sigma)}$$

for all $t \in [0, t_1]$. □

REMARK 3.9. *In Theorem 3.4, the degenerate point $t_* \in [0,T]$ on which $H(t_*) = 0$ could be not necessarily equal to 0. Indeed, by similar arguments to Lemma 3.5 and Proposition 3.6, it suffices to prove the observability inequality in a closed time interval containing t_*. Therefore, if there exists a sufficiently long time interval containing t_* on which $\frac{H'(t)}{|H'(t)|}\cdot\frac{H'(t_*)}{|H'(t_*)|} \ge c_0$ holds, we can prove the observability inequality on the time interval by the same way as in the proof of Theorem 3.4 using the extension.*

4. Non-degenerate transport equations

In this section, we prove the observability inequality for the non-degenerate case studied by Cannarsa, Floridia, and Yamamoto [10] without the partition arguments and cut-off arguments. Given $T > 0$, we replace the assumption (3.2) and (3.3) on $H \in C([0,T];\mathbb{R}^d)$ with the following:

$$\exists T_1' \in (0,T],\ \exists \rho > 0 \text{ s.t. } \min_{t\in[0,T_1']} |H(t)| \ge \rho. \tag{3.24}$$

4.1. Preliminaries. Our methodology is based on the energy estimate given in Proposition 3.6, which still holds for the non-degenerate case. We define a positive number corresponding to t_1 in Definition 3.1.

DEFINITION 3.10. *Let $T > 0$, $c_0 \in (\frac{1}{\sqrt{2}}, 1)$, and $H \in C([0,T];\mathbb{R}^d)$ be a vector-valued function satisfying (3.24). We define a positive number $t_1' \in (0, T_1']$ such that*

$$t_1' := \sup\left\{\tau \in [0,T_1'] \;\middle|\; \frac{H(t)}{|H(t)|}\cdot\frac{H(0)}{|H(0)|} \ge c_0, \quad \forall t \in [0,\tau]\right\}. \tag{3.25}$$

LEMMA 3.11. *Let $T > 0$, $c_0 \in (\frac{1}{\sqrt{2}}, 1)$, $H \in C([0,T];\mathbb{R}^d)$ be a vector-valued function satisfying (3.24), and $t_1' \in (0, T_1']$ be the positive number defined by (3.25). Then, there exists $x_0 \in \overline{\Omega}^c := \mathbb{R}^d \setminus \overline{\Omega}$ such that*

$$\min_{(x,t)\in\overline{\Omega}\times[0,t_1']} \frac{H(t)\cdot(x-x_0)}{|H(t)||x-x_0|} \ge 2c_0^2 - 1 (> 0). \tag{3.26}$$

Proof. If we take $x_0 := -R\theta_0 \in \overline{\Omega}^c$ for $R > \frac{1+c_0}{1-c_0} \operatorname{diam}\Omega$ and $\theta_0 := \frac{H(0)}{|H(0)|}$, we find

$$\begin{aligned}(x-x_0)\cdot\theta_0 &= x\cdot\theta_0 + R \geq R - |x| \geq R - \operatorname{diam}\Omega \\ &> c_0(R + \operatorname{diam}\Omega) \geq c_0|x-x_0|\end{aligned}$$

holds for all $x \in \overline{\Omega}$, which implies $\displaystyle\min_{(x,t)\in\overline{\Omega}\times[0,t_1']} \frac{x-x_0}{|x-x_0|}\cdot\theta_0 \geq c_0$. By the same argument as in the proof of Lemma 3.3, we find (3.26) holds true. □

One of the most important tools in our methodology is the Carleman estimate. Let $\tau > 0$ and $c_0 \in (\frac{1}{\sqrt{2}}, 1)$ be constants. We set $Q_\tau := \Omega \times (0,\tau)$ and $\Sigma_\tau := \partial\Omega \times (0,\tau)$. We assume (3.27)–(3.29) for the non-degenerate case instead of (3.10)–(3.12) for the degenerate case:

$$H \in C^1([0,\tau];\mathbb{R}^d); \tag{3.27}$$

$$\exists \rho > 0 \text{ s.t. } \min_{t\in[0,\tau]} |H(t)| \geq \rho; \tag{3.28}$$

$$\exists \theta_0 \in \mathbb{S}^{d-1} \text{ s.t. } \min_{t\in[0,\tau]} \frac{H(t)\cdot\theta_0}{|H(t)|} \geq c_0. \tag{3.29}$$

In the non-degenerate case, we choose a different weight function from (3.13). For a constant $\beta > 0$, let us define

$$\psi(x,t) := |x-x_0|^2 - \beta t, \quad (x,t) \in \overline{Q_\tau}, \tag{3.30}$$

where $x_0 \in \overline{\Omega}^c$ is a point satisfying $\displaystyle\min_{(x,t)\in\overline{Q_\tau}} \frac{H(t)\cdot(x-x_0)}{|H(t)||x-x_0|} \geq 2c_0^2 - 1$.

Proposition 3.12. *Assume* (3.27), (3.28), *and* (3.29). *Let* ψ *be the smooth function defined by* (3.30), *where* $\beta > 0$ *is an arbitrary positive number satisfying*

$$0 < \beta < 2\delta := 2\rho(2c_0^2 - 1)\operatorname{dist}(x_0, \Omega).$$

Then, there exist constants $s_* > 0$ *and* $C > 0$ *such that*

$$\begin{aligned} s^2 \int_{Q_\tau} & e^{2s\psi}|u|^2 dxdt \\ &\leq C\int_{Q_\tau} e^{2s\psi}|Au|^2 dxdt + Cs\int_{\Sigma_\tau} e^{2s\psi} A\psi(H(t)\cdot\nu(x))|u|^2 d\sigma dt \\ &\quad + Cs\int_\Omega e^{2s\psi(x,\tau)}|u(x,\tau)|^2 dx \end{aligned} \tag{3.31}$$

holds for all $s > s_*$ *and* $u \in H^1(Q_\tau)$. *Here* $d\sigma$ *denotes the volume element of* $\partial\Omega$.

Note that the order of s on the left-hand side of (3.31) is different from the one on the left-hand side of (3.14).

Proof of Proposition 3.12. Set $z := e^{s\psi}u$ and $Pz := e^{s\psi}A(e^{-s\psi}z)$ for $u \in H^1(Q_\tau)$ and $s > 0$. Since ψ is smooth, it follows that $z \in H^1(Q_\tau)$. We note that

$$A\psi(x,t) = -\beta + 2H(t)\cdot(x-x_0)$$

and

$$A^2\psi = 2H'(t)\cdot(x-x_0) + 2|H(t)|^2.$$

Since we have

$$Pz = Az - s(A\psi)z,$$

$$
\begin{aligned}
(3.32) \qquad \|Pz\|^2_{L^2(Q_\tau)} &\geq s^2\|(A\psi)z\|^2_{L^2(Q_\tau)} + 2(Az, -s(A\psi)z)_{L^2(Q_\tau)} \\
&= s^2 \int_{Q_\tau} \Big(-\beta + 2H(t)\cdot(x-x_0)\Big)^2 |z|^2 dxdt \\
&\quad - 2s\int_{Q_\tau} (\partial_t z + H(t)\cdot\nabla z)(A\psi) z dxdt \\
&= s^2 \int_{Q_\tau} \Big(-\beta + 2H(t)\cdot(x-x_0)\Big)^2 |z|^2 dxdt \\
&\quad - s\int_{Q_\tau} (A\psi)\partial_t(|z|^2)dxdt - s\int_{Q_\tau}(A\psi)H(t)\cdot\nabla(|z|^2)dxdt \\
&= \int_{Q_\tau} \left[s^2\Big(-\beta + 2H(t)\cdot(x-x_0)\Big)^2 + s(A^2\psi)\right] |z|^2 dxdt \\
&\quad - s\int_{\Sigma_\tau} A\psi(H(t)\cdot\nu(x))|z|^2 d\sigma dt - s\int_\Omega \Big[A\psi|z|^2\Big]_{t=0}^{t=\tau} dx
\end{aligned}
$$

holds. For the fixed $x_0 \in \overline{\Omega}^c$ so that $\displaystyle\min_{(x,t)\in\overline{Q_\tau}} \frac{H(t)\cdot(x-x_0)}{|H(t)||x-x_0|} \geq 2c_0^2 - 1 (> 0)$, it follows that

$$
\begin{aligned}
H(t)\cdot(x-x_0) &= |H(t)||x-x_0|\frac{H(t)\cdot(x-x_0)}{|H(t)||x-x_0|} \\
&\geq \rho\,\mathrm{dist}(x_0,\Omega) \min_{(x,t)\in\overline{Q_\tau}} \frac{H(t)\cdot(x-x_0)}{|H(t)||x-x_0|} \\
&\geq \delta (> 0)
\end{aligned}
$$

for all $(x,t)\in\overline{Q_\tau}$ owing to (3.28) and (3.29). We then obtain from (3.32)

$$
\begin{aligned}
\|Pz\|^2_{L^2(Q_\tau)} \geq& \int_{Q_\tau} \Big[(2\delta-\beta)^2 s^2 + O(s)\Big]|z|^2 dxdt - s\int_{\Sigma_\tau}(A\psi)(H(t)\cdot\nu(x))|z|^2 d\sigma dt \\
&- s\int_\Omega A\psi(x,\tau)|z(x,\tau)|^2 dx
\end{aligned}
$$

as $s \to +\infty$. Hence, for all $0 < \beta < 2\delta$, there exist constants $s_* > 0$ and $C > 0$ such that

$$
\begin{aligned}
s^2\int_{Q_\tau} e^{2s\psi}|u|^2 dxdt \leq& C\int_{Q_\tau} e^{2s\psi}|Au|^2 dxdt + Cs\int_{\Sigma_\tau} e^{2s\psi} A\psi(H(t)\cdot\nu(x))|u|^2 d\sigma dt \\
&+ Cs\int_\Omega e^{2s\psi(x,\tau)}|u(x,\tau)|^2 dx
\end{aligned}
$$

holds for all $s > s_*$ and $u \in H^1(Q_\tau)$. □

4.2. Observability inequality for the non-degenerate case. For the fixed $x_0 \in \overline{\Omega}^c$ satisfying (3.26), We define a positive number

$$
(3.33) \qquad T_0' := \frac{\max_{x\in\overline{\Omega}} |x-x_0|^2 - \min_{x\in\overline{\Omega}} |x-x_0|^2}{\delta},
$$

where

$$\delta := \rho(2c_0^2 - 1)\,\mathrm{dist}(x_0, \Omega) > 0. \tag{3.34}$$

THEOREM 3.13. *Let* $T > 0$, $c_0 \in (\frac{1}{\sqrt{2}}, 1)$, $H \in C([0,T];\mathbb{R}^d)$, *and* $g \in L^2(\Sigma)$. *Assume* (3.24) *and* $H \in C^1([0,T_1'];\mathbb{R}^d)$. *If the positive number* $t_1' > 0$ *defined by* (3.25) *satisfies* $T_0' < t_1'$ *for some* $x_0 \in \overline{\Omega}^c$ *satisfying* (3.26), *then there exists a constant* $C > 0$ *independent of* $g \in L^2(\Sigma)$ *such that for all* $t \in [0,T]$,

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Sigma)} \tag{3.35}$$

holds for all $u \in H^1(Q)$ *satisfying* (3.4).

PROOF. Let $t_1' > 0$ be the positive number defined by (3.25) and $x_0 \in \overline{\Omega}^c$ be the point satisfying (3.26) with

$$T_0' < t_1',$$

where T_0' is defined by (3.33). Owing to Proposition 3.6, it suffices to prove (3.35) in the interval $[0,t_1']$. For the fixed $x_0 \in \overline{\Omega}^c$, we take $0 < \beta < 2\delta$, where δ is defined by (3.34), satisfying

$$(T_0' <)\frac{d_M - d_m}{\beta} < t_1',$$

where we recall

$$d_M := \max_{x\in\overline{\Omega}} |x - x_0|^2, \quad d_m := \min_{x\in\overline{\Omega}} |x - x_0|^2.$$

Then, there exists $\kappa > 0$ such that

$$d_M - d_m - \beta t_1' < -\kappa. \tag{3.36}$$

Henceforth, by $C > 0$ we denote a generic constant independent of u and g which may change from line to line, unless specified otherwise. We find that H satisfies the assumptions (3.27)–(3.29) by taking $\tau = t_1'$ and $\theta_0 = \frac{H(0)}{|H(0)|}$ needed for Proposition 3.12. Set $Q_{t_1'} := \Omega \times (0,t_1')$ and $\Sigma_{t_1'} := \partial\Omega \times (0,t_1')$. Applying Proposition 3.12 to $u \in H^1(Q_{t_1'})$ satisfying (3.4) yields

$$s^2\int_{Q_{t_1'}} e^{2s\psi}|u|^2 dxdt \le Cs\int_{\Sigma_{t_1'}} e^{2s\psi}A\psi(H(t)\cdot\nu(x))|u|^2 d\sigma dt + Cs\int_\Omega e^{2s\varphi(x,t_1')}|u(x,t_1')|^2 dx. \tag{3.37}$$

On the left-hand side of (3.37), we obtain

$$s^2\int_{Q_{t_1'}} e^{2s\psi}|u|^2 dxdt \ge s^2 e^{2s(d_m - \beta\epsilon)}\int_0^\epsilon\int_\Omega |u|^2 dxdt, \tag{3.38}$$

where $\epsilon \in (0,t_1')$ is an arbitrary small constant satisfying for all $x \in \overline{\Omega}$ and $0 \le t \le \epsilon$,

$$\psi(x,t) > 0,$$

i.e.,

$$d_m - \beta\epsilon > 0. \tag{3.39}$$

Furthermore, applying Lemma 3.5 in (3.38), we have

$$
\begin{aligned}
s^2 \int_{Q_{t_1'}} e^{2s\psi}|u|^2 dxdt &\geq s^2 e^{2s(d_m-\beta\epsilon)} \int_0^\epsilon \int_\Omega |u|^2 dxdt \\
&\geq \epsilon s^2 e^{2s(d_m-\beta\epsilon)} \left(\|u(\cdot,0)\|^2_{L^2(\Omega)} - C\|g\|^2_{L^2(\Sigma)} \right).
\end{aligned} \tag{3.40}
$$

Moreover, in regard to the second summand of the right-hand side of (3.37), applying Lemma 3.5 yields

$$
\begin{aligned}
&Cs \int_\Omega e^{2s\psi(x,t_1')}|u(x,t_1')|^2 dx \\
&\leq Cse^{2s(d_M-\beta t_1')}\|u(\cdot,t_1')\|^2_{L^2(\Omega)} \\
&\leq Cse^{2s(d_M-\beta t_1')} \left(\|u(\cdot,0)\|^2_{L^2(\Omega)} + C\|g\|^2_{L^2(\Sigma)} \right).
\end{aligned} \tag{3.41}
$$

From (3.37), (3.40), and (3.41), we obtain

$$
\begin{aligned}
&\epsilon s^2 e^{2s(d_m-\beta\epsilon)} \left(\|u(\cdot,0)\|^2_{L^2(\Omega)} - C\|g\|^2_{L^2(\Sigma)} \right) \\
&\quad \leq Cse^{2s(d_M-\beta t_1')} \left(\|u(\cdot,0)\|^2_{L^2(\Omega)} + C\|g\|^2_{L^2(\Sigma)} \right) + Cse^{Cs}\|g\|^2_{L^2(\Sigma)},
\end{aligned}
$$

i.e.,

$$
e^{2s(d_m-\beta\epsilon)} \left(\epsilon s - Ce^{2s(d_M-d_m-\beta t_1'+\beta\epsilon)} \right) \|u(\cdot,0)\|^2_{L^2(\Omega)} \leq Ce^{Cs}\|g\|^2_{L^2(\Sigma)}.
$$

Applying (3.36) and (3.39) to the left-hand side of the above inequality yields

$$
\left(\epsilon s - Ce^{-2s(\kappa-\beta\epsilon)} \right) \|u(\cdot,0)\|^2_{L^2(\Omega)} \leq Ce^{Cs}\|g\|^2_{L^2(\Sigma)}.
$$

By choosing $s > s_*$ large enough to satisfy $\epsilon s - Ce^{-2s(\kappa-\beta\epsilon)} > 0$ for the sufficiently small $\epsilon > 0$ and applying Lemma 3.5 again on the left-hand side of the above inequality, we have

$$
\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq C\|g\|^2_{L^2(\Sigma)}
$$

for all $t \in [0, t_1']$. □

REMARK 3.14. *In the non-degenerate case, we focused only on the time interval $[0, t_1']$ near 0 and proved the observability inequality under the assumption that t_1' is large enough. Needless to say, if there exists a sufficiently long time interval $[t_*, t^*] \subset [0, T]$, if not near 0, on which $\frac{H(t)}{|H(t)|} \cdot \frac{H(t_*)}{|H(t_*)|} \geq c_0$ holds, the observability inequality holds on the interval, which implies it holds also on $[0, T]$ by the similar arguments using Lemma 3.5 and Proposition 3.6.*

Part II

Second-order hyperbolic equations

CHAPTER 4

Wave equations on Lorentzian manifolds

1. Introduction and main result

Let $T > 0$, $n \in \mathbb{N}$, and M be a compact oriented n-dimensional smooth manifold with boundary. We set $L := [-T, T] \times M$ and let (L, g) be a Lorentzian manifold with metric g having signature $(-, +, \ldots, +)$ such that the submanifolds $M^t := \{t\} \times M$ are spacelike for all $t \in [-T, T]$ and $\partial_t := \frac{\partial}{\partial t}$ is timelike. The Lorentzian metric is a symmetric non-degenerate covariant 2-tensor field such that for every point $p \in L$, there is a basis $e_0, \ldots, e_n$ for T_pL such that $g(e_\mu, e_\nu)$ are the components of the standard Minkowski metric $\mathrm{diag}(-1, 1, \ldots, 1)$. In this chapter, we consider the intermediate boundary value problem of the system for a function $h : L \to \mathbb{R}^\ell$ with $\ell \in \mathbb{N}$,

$$\begin{cases} Ph := \Box_g h + a(t,x)h = H(t,x) & \text{in } L, \\ h = \partial_{\hat{N}} h = 0 & \text{on } M^0 = \{0\} \times M, \\ h = 0 & \text{on } \Sigma_1 := [-T, T] \times \Gamma_1. \end{cases} \tag{4.1}$$

Here let the coefficient a be an $\ell \times \ell$ matrix-valued function on L and the source term H be an ℓ vector-valued function on L. Let $\pi_0 : L \to [-T, T]$ be the projection and $\nabla\pi_0$ be the gradient of π_0. $\hat{N} := -\frac{\nabla\pi_0}{\sqrt{|g(\nabla\pi_0, \nabla\pi_0)|}}$ denotes the future directed unit timelike vector field such that for all $p \in M^t$ and $X \in T_pM^t$, $g(\hat{N}_p, \iota_* X) = 0$, where $\iota : M^t \hookrightarrow L$ is the embedding. We note, in this chapter, that summations with respect to Greek indices range from 0 to n, whereas those for Roman indices range from 1 to n. Furthermore, $\Box_g$ is defined by $\Box_g := g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)$ for functions on L, where $(g^{\mu\nu})$ is the matrix inverse to $(g_{\mu\nu})$, which are components of the metric $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$, and $\Gamma^\rho_{\mu\nu}$ is the Christoffel symbol of the Levi-Civita connection defined by

$$\Gamma^\rho_{\mu\nu} := \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right).$$

$\Gamma_1 \subset \partial M$ denotes a given open submanifold.

The equation in (4.1) relates to general relativity. Because this type of equation having the same principal term is derived from the Einstein equation by choosing a special coordinate system or a suitable gauge function (e.g., [68, Chapter 18.8], [15, Chapter III.11], [58, Part III], [59], [17, Chapter 33]) and then by the linearization of the Einstein equation, we reduce it to the system having the form (4.1). Interested readers are referred to Taylor [68], Choquet-Bruhat [15], and Ringström [58] for a direct derivation of the equation having the same form (4.1).

We assume the source term H is written by $H(t, x) = S(t, x)f(x)$, where S is an $\ell \times \ell$ matrix-valued function on L and f is an ℓ vector-valued function on M.

The main focus of this chapter is the inverse source problem to determine f from the partial boundary data of the solution: $\partial_N \partial_{\hat{N}}^k h\ |_{\Sigma_1}$ for $k = 0, 1, 2$, where ∂_N denotes the normal derivative with respect to the metric g. We prove the uniqueness and stability for the local inverse source problem. The argument is based on the Carleman estimate, which was introduced by Carleman in [12], and the Bukhgeim–Klibanov method in [8]. The Carleman estimate was first invented to prove the unique continuation property for elliptic operators for which the coefficients are not necessarily real analytic. Using the Carleman estimate, Bukhgeim and Klibanov proved global uniqueness results for multidimensional coefficient inverse problems. This methodology is widely applicable to not only elliptic equations but also various partial differential equations provided that we can prove the Carleman estimate for the operators we are considering. For hyperbolic equations, Baudouin, De Buhan, and Ervedoza [4] proved the global Carleman estimate for wave equations and considered its applications to controllability, inverse problems, and reconstructions. Imanuvilov and Yamamoto [34] proved the global Lipschitz stability for wave equations by interior observations near the boundary. Bellassoued and Yamamoto [6], [5] considered both local and global inverse source problems, and coefficient inverse problems for wave equations on a compact Riemannian manifold. Jiang, Liu, and Yamamoto [37] considered the local inverse source problems for wave equations, the coefficients of which depend on time t in the Euclidean space under the assumption that the Carleman estimate for such operators exists. In this chapter, we prove also the Carleman estimate for the Laplace–Beltrami operator. For time-independent wave equation, to apply the Carleman estimate to consider the inverse source problem, we extend the solution to negative time intervals. However, when the coefficients depend on time, there is a difficulty in extending the solution to negative time intervals when trying to apply the Carleman estimate. For instance, an even extension of the solution with respect to time t no longer satisfies the equation. Hence, we consider the equation in $[-T, T]$ from the beginning.

To the best of author's knowledge, there are a few papers concerning the Carleman estimates for the Laplace–Beltrami operator on a Lorentzian manifold. Because Bellassoued and Yamamoto [6] dealt with the wave equation on a compact Riemannian manifold, we prove the Carleman estimate on a Lorentzian manifold with the help of their tools. Indeed, the assumptions on a weight function (4.2) and (4.3) in the next section are generalizations of the situation for a Riemannian manifold.

To describe our main result, we define the Sobolev space on manifolds, which should be defined so as not to depend on a choice of coordinate systems in general.

DEFINITION 4.1. *Let M be a compact oriented n-dimensional smooth manifold, and $\{(U_i, x_i)\}_i$ be a coordinate system. Assume $\{\chi_i\}_i$ is a finite partition of unity subordinate to the covering such that* $\operatorname{supp}\chi_i \subset U_i$. *Given $u \in C^\infty(M; \mathbb{R}^\ell)$ and integer k, define*

$$\|u\|_{H^k(M;\mathbb{R}^\ell)} := \left(\sum_i \sum_{|\alpha| \le k} \int_{x_i(U_i)} (\chi_i |\partial^\alpha u|^2) \circ x_i^{-1} dx_i^1 \cdots dx_i^n \right)^{\frac{1}{2}},$$

where ∂^α signifies differentiation with respect to x_i.

The inner product can be also defined in the same way. By taking the completion of the smooth functions, one obtains a real Hilbert space. Note that different partitions of unity and coordinates yield equivalent norms. (e.g., Ringström [58, Section 15]) Although our integrations and derivatives on compact manifolds should be written using a partition of unity and local coordinates, we omit these representations throughout this chapter to avoid notational complexity.

Let $\iota : M^t := \{t\} \times M \hookrightarrow L$ be the embedding and $g_\flat := \iota^* g$ be the induced metric on M^t by the embedding ι. We assume throughout that the Lorentzian metric g is smooth on L such that M^t is spacelike, i.e., $g_\flat$ is Riemannian metric on M^t, and ∂_t is timelike, i.e., $g(\partial_t, \partial_t) < 0$. $\hat{N} := -\frac{\nabla \pi_0}{\sqrt{|g(\nabla \pi_0, \nabla \pi_0)|}}$ denotes the future directed unit timelike vector field such that for all $p \in M^t$ and $X \in T_p M^t$, $g(\hat{N}_p, \iota_* X) = 0$. We assume the coefficient has enough regularity,

$$a \in W^{2,\infty}(-T, T; L^\infty(M; \mathbb{R}^{\ell \times \ell})).$$

Let $M_\epsilon := \{x \in M \mid \psi(0, x) > \epsilon\}$ be a level set of ψ, where ψ is the weight function satisfying assumptions (4.2) and (4.3) to be stated in the next section. We are ready to describe the main result of this chapter.

THEOREM 4.2. *Let $\ell \in \mathbb{N}$, $T > 0$, M be a compact oriented n-dimensional smooth manifold with boundary, and $L := [-T, T] \times M$. Let g be a smooth Lorentzian metric on L such that M^t is spacelike and ∂_t is timelike. Assume $H(t, x) = S(t, x) f(x)$, (4.2), (4.3), (4.7) and (4.8). Furthermore, assume that there exists a unique solution h to (4.1) in the class*

$$h \in \bigcap_{k=0}^{2} H^{4-k}(-T, T; H^k(M; \mathbb{R}^\ell)).$$

Then, there exists $\epsilon^* > 0$ such that for any $\epsilon \in (\epsilon_*, \epsilon^*)$, there exist constants $C > 0$ and $\theta \in (0, 1)$ such that

$$\|f\|_{L^2(M_\epsilon; \mathbb{R}^\ell)} \leq C\mathcal{D} + C\mathcal{F}^{1-\theta}\mathcal{D}^\theta,$$

where $\epsilon_* \geq 0$ is the number in (4.8),

$$\mathcal{F} := \|f\|_{L^2(M; \mathbb{R}^\ell)} + \sum_{k=0}^{2} \|h\|_{H^{3-k}(-T, T; H^k(M; \mathbb{R}^\ell))},$$

$$\mathcal{D} := \sum_{k=0}^{2} \|\partial_N \partial_{\hat{N}}^k h\|_{L^2(-T, T; L^2(\Gamma_1; \mathbb{R}^\ell))},$$

and N denotes the outer unit normal vector field to $\Sigma_1 := [-T, T] \times \Gamma_1$.

(4.2) and (4.3) are the assumptions on the weight function needed for the Carleman estimate. (4.7) and (4.8) are the respective assumptions on the source and coefficient terms, and on a given submanifold Γ_1. Details of these assumptions are explained in subsequent sections.

2. Carleman estimate

Let us fix a local coordinate $(x^1, \ldots, x^n)$ on M and then, obtain a local coordinate $(x^0 = t, x^1, \ldots, x^n)$ on L such that

$$g = -dt \otimes dt + g_{ij} dx^i \otimes dx^j.$$

We call the local coordinate semigeodesic coordinate in this chapter. Henceforth, if we write statements using a local coordinate, the coordinate is always taken by the semigeodesic coordinate, unless specified otherwise.

REMARK 4.3. *There exists the semigeodesic coordinate locally. (e.g., Remark 5.1 in [15, I]) Indeed, for a local coordinate* $(y^0(t), y^1, \dots, y^n)$ *near* $(t,x) \in L$, *there exists a change of the coordinate into the semigeodesic coordinate* $(x^0 = t, x^1, \dots, x^n)$ *if and only if an inverse transform exists. Then, the components* $g'_{\mu\nu}$ *of the metric* g *represented by* $(t, x^1, \dots, x^n)$ *satisfy*

$$g'_{i0} = \frac{\partial y^j}{\partial x^i}\left(g_{j0}\frac{dy^0}{dt} + g_{jk}\frac{\partial y^k}{\partial t}\right),\ i = 1, \dots, n,$$

$$g'_{00} = g_{00}\left(\frac{dy^0}{dt}\right)^2 + 2g_{0j}\frac{dy^0}{dt}\frac{\partial y^j}{\partial t} + g_{jk}\frac{\partial y^j}{\partial t}\frac{\partial y^k}{\partial t}.$$

$g_{j0}\frac{dy^0}{dt} + g_{jk}\frac{\partial y^k}{\partial t} = 0$ *for* $j = 1, \dots, n$ *and* $g_{00}\left(\frac{dy^0}{dt}\right)^2 + 2g_{0j}\frac{dy^0}{dt}\frac{\partial y^j}{\partial t} + g_{jk}\frac{\partial y^j}{\partial t}\frac{\partial y^k}{\partial t} = -1$ *are equivalent to*

$$g_{\mu\nu}\frac{\partial y^\nu}{\partial t} = -\delta^0_\mu\left(\frac{dy^0}{dt}\right)^{-1},\ \mu = 0, \dots, n \Longleftrightarrow \frac{\partial y^\nu}{\partial t} = -g^{0\nu}\left(\frac{dy^0}{dt}\right)^{-1},\ \nu = 0, \dots, n,$$

which is locally solvable as an initial problem of a first-order system since $g^{00} < 0$ *by our assumption that* ∂_t *is timelike and Lemma 8.5 in [58].*

Let $\ell \in \mathbb{N}$, $T > 0$, M be a compact oriented n-dimensional smooth manifold with boundary, and $L := [-T,T] \times M$. Let g be a smooth Lorentzian metric on L such that M^t is spacelike and ∂_t is timelike. In this section, we consider the Carleman estimate for the operator P,

$$\begin{aligned} Ph &:= \Box_g h + a(t,x)h \\ &= g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)h + a(t,x)h. \end{aligned}$$

Let the coefficient a has enough regularity,

$$a \in W^{2,\infty}(-T,T; L^\infty(M; \mathbb{R}^{\ell\times\ell})).$$

To establish the Carleman estimate for the above operator P, we consider first of all Carleman estimate for the Laplace–Beltrami operator for $\mathbb{R}$-valued functions

$$\Box_g = g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)$$

on an $n+1$-dimensional Lorentzian manifold L. The following method is based on the works by Bellassoued and Yamamoto [6], [5]. Note that angled bracket $\langle\cdot,\cdot\rangle$ denotes the inner product with respect to the metric g, i.e., $\langle X, Y\rangle := g(X,Y) = g_{\mu\nu}X^\mu Y^\nu$ for $X, Y \in T_pL$ and $p \in L$. Let $\pi_0 : L \to [-T,T]$ and $\pi_1 : L \to M$ be the projections, and $d\tau^2$ and $g_\flat$ be the respective induced squared line element and Riemannian metric by the canonical embeddings $[-T,T] \hookrightarrow L$ and $M^t \hookrightarrow L$. $\nabla u = \nabla_g u = \nabla^\mu u\frac{\partial}{\partial x^\mu} = g^{\mu\nu}\partial_\nu u\frac{\partial}{\partial x^\mu}$ denotes the gradient of a function u with respect to the metric g. We assume the following two assumptions.

The Hessian of ψ with respect to g satisfies

$$\begin{aligned} &\exists\kappa_1 > 0,\ \exists\kappa_2 > 0 \text{ s.t. } \forall p \in L,\ \forall X \in T_pL, \\ &\nabla^2\psi(X,X) \geq -2\kappa_2 d\tau^2\left((d\pi_0)X, (d\pi_0)X\right) + 2\kappa_1 g_\flat\left((d\pi_1)X, (d\pi_1)X\right) \end{aligned} \tag{4.2}$$

with

$$1 < \frac{\kappa_1}{\kappa_2}.$$

ψ has no critical points on L, i.e.,

$$\begin{aligned} &\min_L g_\flat\left((d\pi_1)\nabla\psi, (d\pi_1)\nabla\psi\right) > 0, \\ &\psi(0,x) > \psi(t,x) \text{ a.e. } (t,x) \in L. \end{aligned} \tag{4.3}$$

REMARK 4.4. *These assumptions* (4.2) *and* (4.3) *are independent of a choice of local coordinates by their definitions. When we write* $X = X^\mu \frac{\partial}{\partial x^\mu} \in T_pL$ *by taking the semigeodesic coordinate, we obtain the representations*

$$d\tau^2\left((d\pi_0)X, (d\pi_0)X\right) = |X^0|^2 := -g_{00}(X^0)^2 = (X^0)^2,$$
$$g_\flat\left((d\pi_1)X, (d\pi_1)X\right) = |X|^2 := g_{ij}X^iX^j \left(= \sum_{i,j=1}^n g_{ij}X^iX^j \right).$$

EXAMPLE 4.5. *We compare these assumptions* (4.2) *and* (4.3) *with those used in considering the wave equation on a compact* n*-dimensional smooth Riemannian manifold* $(M, \bar{g})$ *by Bellassoued and Yamamoto [6], [5]. We take as a function* ψ,

$$\psi(t,x) := \psi_0(x) - \kappa_2 t^2, \quad (t,x) \in [-T,T] \times M,$$

where $\kappa_2 > 0$ *is a constant and* ψ_0 *is a positive smooth function in* M*. In this case, our considering Lorentzian metric has the form* $g = -dt \otimes dt + \bar{g}$ *and* $g_\flat = \bar{g}$ *holds. The assumptions regarding the operator* $-\partial_t^2 + \Delta_{\bar{g}}$*, where* $\Delta_{\bar{g}}$ *is the Laplace–Beltrami operator with respect to the metric* $\bar{g}$*, are the followings.*

The Hessian of ψ_0 *with respect to* $\bar{g}$ *satisfies*

$$\begin{aligned} &\exists \kappa_1 > 0 \text{ s.t. } \forall p \in M,\ \forall \bar{X} \in T_pM, \\ &\nabla^2_{\bar{g}}\psi_0(\bar{X}, \bar{X}) \geq 2\kappa_1 |\bar{X}|^2_{\bar{g}}, \end{aligned} \tag{4.4}$$

where $|\bar{X}|_{\bar{g}} := (\bar{g}_{ij}\bar{X}^i\bar{X}^j)^{\frac{1}{2}}$ *with*

$$1 < \frac{\kappa_1}{\kappa_2}.$$

ψ_0 *has no critical points on* M,

$$\min_M |\nabla_{\bar{g}}\psi_0|_{\bar{g}} > 0. \tag{4.5}$$

Clearly, if assumptions (4.4) *and* (4.5) *hold, then our assumptions* (4.2) *and* (4.3) *hold. Indeed, for* $p \in L$ *and* $X \in T_pL$*, if (B.1) holds, then we have*

$$\begin{aligned} \nabla^2_g\psi(X,X) &= -2\kappa_2 d\tau^2((d\pi_0)X, (d\pi_0)X) + \nabla^2_{\bar{g}}\psi_0((d\pi_1)X, (d\pi_1)X) \\ &\geq -2\kappa_2 d\tau^2((d\pi_0)X, (d\pi_0)X) + 2\kappa_1\bar{g}((d\pi_1)X, (d\pi_1)X) \end{aligned}$$

with

$$1 < \frac{\kappa_1}{\kappa_2}.$$

Furthermore, having obtained

$$g_\flat((d\pi_1)\nabla\psi, (d\pi_1)\nabla\psi) = \bar{g}(\nabla_{\bar{g}}\psi_0, \nabla_{\bar{g}}\psi_0) > 0,$$

we find (4.3) *holds.*

Let us define the weight function using ψ,

$$\varphi(t,x) := e^{\gamma\psi(t,x)}, \quad (t,x) \in L,$$

where $\gamma > 0$ is a parameter. For notational simplicity, we set

$$\sigma(t,x) := s\gamma\varphi(t,x), \quad (t,x) \in L,$$

where $s > 0$ is a parameter. We set $\Sigma := [-T,T] \times \partial M$. Before describing the Carleman estimate, we define a quantity independent of a choice of local coordinates.

DEFINITION 4.6. *Let ∇u be the gradient of $u \in C^\infty(L)$ and define the quantity independent of a choice of local coordinates*

$$E(u) := d\tau^2\left((d\pi_0)\nabla u, (d\pi_0)\nabla u\right) + g_\flat\left((d\pi_1)\nabla u, (d\pi_1)\nabla u\right).$$

REMARK 4.7. *In the same way as Remark 4.4, the quantity has the representation,*

$$E(u) = |\nabla^0 u|^2 + |\nabla u|^2,$$

where $\nabla^0 u$ is a component of the gradient $\nabla u = \nabla^\mu u \frac{\partial}{\partial x^\mu}$.

LEMMA 4.8. *Assume* (4.2) *and* (4.3). *Then, there exists a constant $\gamma_* > 0$ such that for any $\gamma > \gamma_*$, there exist constants $s_* = s_*(\gamma)$ and $C > 0$ such that*

$$\int_L e^{2s\varphi}\sigma\left(E(u) + \sigma^2|u|^2\right)\omega_L \le C\int_L e^{2s\varphi}|\Box_g u|^2\omega_L + C\int_\Sigma e^{2s\varphi}\sigma|\partial_N u|^2\omega_\Sigma$$

holds for all $s > s_*$ and $u \in C^\infty(L)$ satisfying $u = \partial_N u = 0$ on $M^{\pm T}$ and $u = 0$ on Σ. $\partial_N u := \langle \nabla u, N\rangle = Nu$, where N is the outer unit normal vector filed to ∂L with respect to the metric g. ω_L and ω_Σ denote the respective volume elements of L and Σ.

The proof of Lemma 4.8 is presented in section 4.

PROPOSITION 4.9. *Assume* (4.2) *and* (4.3). *Then, there exists constant $\gamma_* > 0$ such that for any $\gamma > \gamma_*$, there exist constants $s_* = s_*(\gamma)$ and $C > 0$ such that*

$$\sum_{m=1}^{\ell}\int_L e^{2s\varphi}\sigma\left(E(h_m) + \sigma^2|h_m|^2\right)\omega_L \le C\int_L e^{2s\varphi}|Ph|^2\omega_L + C\int_\Sigma e^{2s\varphi}\sigma|\partial_N h|^2\omega_\Sigma$$

holds for all $s > s_*$ and $h \in C^\infty(L;\mathbb{R}^\ell)$ satisfying $h = \partial_N h = 0$ on $M^{\pm T}$ and $h = 0$ on Σ. $\partial_N h := \langle \nabla h, N\rangle = Nh$, where N is the outer unit normal vector field to ∂L.

PROOF. With the help of Lemma 4.8, Proposition 4.9 is obtained by addition and absorption by choosing $s > 0$ large enough. $\square$

3. Proof of Theorem 4.2

3.1. Preliminary. Let $T > 0$, M be a compact oriented n-dimensional smooth manifold with boundary, $L := [-T,T] \times M$, and $M^t := \{t\} \times M$. Let (L,g) be a smooth Lorentzian manifold such that M^t is spacelike and ∂_t is timelike with respect to the metric g. Let us fix the semigeodesic coordinate $(x^0 = t, x^1, \ldots, x^n)$. We remark that in such a coordinate, we find $\hat{N} = \partial_t$, where $\hat{N} := -\frac{\nabla\pi_0}{\sqrt{|g(\nabla\pi_0,\nabla\pi_0)|}}$ is the future directed unit timelike vector field such that for all $p \in M^t$ and $X \in T_pM^t$, $g(\hat{N}_p, \iota_* X) = 0$, where $\iota : M^t \hookrightarrow L$ is the embedding. We consider

(4.6)

$$\begin{cases} Ph = g^{\mu\nu}(\partial_\mu \partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)h + a(t,x)h = S(t,x)f(x) & \text{in } L, \\ h = \partial_t h = 0 & \text{on } M^0, \\ h = 0 & \text{on } \Sigma_1 := [-T,T] \times \Gamma_1. \end{cases}$$

$\Gamma_1 \subset \partial M$ is an open submanifold. We assume

$$\begin{cases} a \in W^{2,\infty}(-T,T;L^\infty(M;\mathbb{R}^{\ell\times\ell})), \\ S \in W^{2,\infty}(-T,T;L^\infty(M;\mathbb{R}^{\ell\times\ell})), \\ \exists m_0 > 0 \text{ s.t. } \det S(0,\cdot) \geq m_0 \text{ a.e. on } M, \\ f \in L^2(M;\mathbb{R}^\ell). \end{cases} \tag{4.7}$$

This type of inverse source problem having a time-dependent principal part was studied by Jiang, Liu, and Yamamoto [37] for a hyperbolic equation. Furthermore, we assume a unique weak solution h exists to (4.6) in the class

$$h \in \bigcap_{k=0}^{2} H^{4-k}(-T,T;H^k(M;\mathbb{R}^\ell)).$$

We define the level set L_ϵ of ψ for $\epsilon \geq 0$ by

$$L_\epsilon := \{(t,x) \in L \mid \psi(t,x) > \epsilon\}$$

and

$$M_\epsilon := \{x \in M \mid \psi(0,x) > \epsilon\}.$$

In regard to a relation between the observation boundary Σ_1 and the level set L_0, we assume that

$$\exists \epsilon_* \geq 0 \text{ s.t. } \varnothing \neq L_{\epsilon_*} \cap \partial L \subset \Sigma_1. \tag{4.8}$$

On considering the inverse source problem of (4.6) as an application to the Carleman estimate Proposition 4.9, we need a relation in regard to energies.

LEMMA 4.10. *Let E be the quantity defined in Definition 4.6. For all $u \in C^\infty(L)$, the identity*

$$E(u) = |\partial_t u|^2 + g^{ij}\partial_i u \partial_j u$$

holds by the semigeodesic coordinate.

PROOF. We note here that summations with respect to Greek indices range from 0 to n, whereas those for Roman indices range from 1 to n. We take the semigeodesic coordinate system.

$$\begin{aligned} E(u) &= -g_{00}(g^{0\mu}\partial_\mu u)^2 + g_{ij}(g^{i\mu}\partial_\mu u)(g^{j\nu}\partial_\nu u) \\ &= |\partial_t u|^2 + g_{ij}g^{ip}(\partial_p u)g^{jq}(\partial_q u). \end{aligned}$$

With the help of the semigeodesic coordinate, it follows that $g_\flat^{ij} = g^{ij}$ for all $1 \leq i,j \leq n$. We then obtain by the above formulation,

$$E(u) = |\partial_t u|^2 + g^{jq}\partial_j u \partial_q u.$$

□

PROPOSITION 4.11. *Assume* (4.2) *and* (4.3). *Then, there exists constant* $\gamma_* > 0$ *such that for any* $\gamma > \gamma_*$, *there exist constants* $s_* = s_*(\gamma)$ *and* $C > 0$ *such that*

$$\sum_{m=1}^{\ell} \int_L e^{2s\varphi}\sigma(|\partial_t h_m|^2 + g^{ij}\partial_i h_m \partial_j h_m + \sigma^2|h_m|^2)\omega_L \\ \le C\int_L e^{2s\varphi}|Ph|^2\omega_L + C\int_\Sigma e^{2s\varphi}\sigma|\partial_N h|^2\omega_\Sigma$$

holds for all $s > s_*$ *and* $h \in \bigcap_{k=0}^{2} H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))$ *satisfying* $h = \partial_N h = 0$ *on* $M^{\pm T}$ *and* $h = 0$ *on* Σ.

PROOF. We apply Lemma 4.10 to Proposition 4.9 to complete the proof. $\square$

Moreover, in the proof of Theorem 4.2, we shall use the next lemma. Lemma 4.12 plays an important role when we prove inverse source problems with time-dependent coefficients, which was introduced in [37]. Its proof is also presented in section 4.

LEMMA 4.12. *Assume* (4.2) *and* (4.3). *Let* $\iota : M^t \hookrightarrow L$ *be the embedding,* $\hat{N}$ *be the future directed unit timelike vector field such that* $\forall p \in M^t$, $\forall X \in T_pM^t$, $g(\hat{N}_p, \iota_* X) = 0$, *and* $\Delta_{g_\flat}$ *be the Laplace–Beltrami operator with respect to the iniduced metric* $g_\flat = \iota^* g$. *Assume* $a \in W^{2,\infty}(-T,T;L^\infty(M))$ *and* $P = \Box_g + a\cdot$. *There exist constants* $s^* > 0$ *and* $C > 0$ *such that*

$$\int_L e^{2s\varphi}|\Delta_{g_\flat} v|^2\omega_L \le C\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_{\hat{N}} Pv|^2 + |Pv|^2\right)\omega_L + Ce^{Cs}\mathcal{E}^2$$

holds for all $s > s^*$ *and* $v \in \bigcap_{k=0}^{2} H^{3-k}(-T,T;H^k(M))$ *satisfying* $v = \partial_N v = \partial_N^2 v = 0$ *on* $M^{\pm T}$ *and* $v = 0$ *on* Σ. *Note that*

$$\mathcal{E} := \sum_{k=0}^{1} \|\partial_N \partial_{\hat{N}}^k v\|_{L^2(-T,T;L^2(\partial M))}.$$

To prove Lemma 4.12, we use the global elliptic estimate Lemma 4.13. (e.g., [30], [51], and [52]) Its proof is also presented in section 4.

LEMMA 4.13. *Let* M *be a compact oriented* n*-dimensional smooth manifold with boundary and* A *be an elliptic differential operator on* M. *Then, there exists a constant* $C > 0$ *such that*

$$\|v\|_{H^2(M)} \le C\left(\|Av\|_{L^2(M)} + \|v\|_{L^2(M)}\right)$$

holds for all $v \in H_0^1(M)$ *satisfying* $Av \in L^2(M)$.

3.2. Proof of Theorem 4.2.

PROOF OF THEOREM 4.2. Let $\epsilon_* \ge 0$ be the number in (4.8). We introduce a cutoff function χ,

$$\chi(t,x) := \begin{cases} 1 & \text{in } L_{2\epsilon}, \\ 0 & \text{in } L \setminus L_\epsilon \end{cases}$$

for sufficiently small $\epsilon > \epsilon_*$ so that

$$\varnothing \neq L_{3\epsilon} \cap \partial L (\subset \Sigma_1).$$

Let us fix the semigeodesic coordinate $(x^0 = t, x^1, \ldots, x^n)$. In such a coordinate, we find $\hat{N} = \partial_t$. For fixed $i = 0, 1, 2$, we set new functions $v^{(i)} := \chi \partial_t^i h$. We calculate $Pv^{(2)}$,

$$\begin{cases} Pv^{(2)} = \chi P \partial_t^2 h + 2\langle \nabla \chi, \nabla \partial_t^2 h \rangle + \partial_t^2 h \Box_g \chi & \text{in } L, \\ v^{(2)} = \partial_t v^{(2)} = 0 & \text{on } M^{\pm T}, \\ v^{(2)} = 0 & \text{on } \Sigma = [-T, T] \times \partial M. \end{cases}$$

Then, we apply Proposition 4.11 to $v^{(2)}$ to obtain

$$\begin{aligned} &\sum_{m=1}^{\ell} \int_L e^{2s\varphi} \left(sE(v_m^{(2)}) + s^3 |v_m^{(2)}|^2 \right) \omega_L \\ &\leq C \int_L e^{2s\varphi} |\chi (P \partial_t^2 h)|^2 \omega_L + C \int_L e^{2s\varphi} |2\langle \nabla \chi, \nabla \partial_t^2 h \rangle + \partial_t^2 h \Box_g \chi|^2 \omega_L \\ &\quad + Ce^{Cs} \int_{\Sigma_1} |\partial_N v^{(2)}|^2 \omega_\Sigma. \end{aligned} \tag{4.9}$$

In regard to the first summand on the right-hand side of (4.9), taking

$$\begin{aligned} &\chi (P \partial_t^2 h) \\ &= \chi \partial_t^2 Sf - \partial_t^2 g^{\mu\nu} \partial_\mu \partial_\nu (\chi h) - \partial_t^2 a (\chi h) - 2\partial_t g^{\mu\nu} \partial_\mu \partial_\nu (\chi \partial_t h) - 2\partial_t a (\chi \partial_t h) \\ &\quad + 2\partial_t (g^{\mu\nu} \Gamma^\rho_{\mu\nu}) \partial_\rho (\chi \partial_t h) + \partial_t^2 (g^{\mu\nu} \Gamma^\rho_{\mu\nu}) \partial_\rho (\chi h) \\ &\quad + \Big[2\partial_t^2 g^{\mu\nu} \partial_\mu \chi \partial_\nu h + \partial_t^2 g^{\mu\nu} (\partial_\mu \partial_\nu \chi) h + 4\partial_t g^{\mu\nu} \partial_\mu \chi (\partial_\nu \partial_t h) + 2\partial_t g^{\mu\nu} (\partial_\mu \partial_\nu \chi) \partial_t h \\ &\quad - \partial_t^2 (g^{\mu\nu} \Gamma^\rho_{\mu\nu}) (\partial_\rho \chi) h - 2\partial_t (g^{\mu\nu} \Gamma^\rho_{\mu\nu}) (\partial_\rho \chi) \partial_t h \Big] \end{aligned}$$

into account, and with $\operatorname{supp}\partial^\alpha\chi \subset \overline{L_\epsilon \setminus L_{2\epsilon}}$ for $|\alpha| \geq 1$, we apply Lemma 4.13, and then Lemma 4.12 to obtain

$$\begin{aligned}
&\int_L e^{2s\varphi}|\chi(P\partial_t^2 h)|^2\omega_L \\
&\leq C\sum_{i=0}^{1}\sum_{m=1}^{\ell}\left(\int_{-T}^{T}\|e^{s\varphi}v_m^{(i)}\|^2_{H^2(M)}dt + \int_L e^{2s\varphi}\Big(s^2E(v_m^{(i)}) + s^4|v_m^{(i)}|^2\Big)\omega_L\right) \\
&\quad + C\int_L e^{2s\varphi}|f|^2\omega_L + Ce^{2\epsilon_2 s}\sum_{k=0}^{1}\|h\|^2_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))} \\
&\leq C\sum_{i=0}^{1}\sum_{m=1}^{\ell}\int_L e^{2s\varphi}\Big(|\Delta_{g_\flat}v_m^{(i)}|^2 + s^2E(v_m^{(i)}) + s^4|v_m^{(i)}|^2\Big)\omega_L \\
&\quad + C\int_L e^{2s\varphi}|f|^2\omega_L + Ce^{2\epsilon_2 s}\sum_{k=0}^{1}\|h\|^2_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))} \\
&\leq C\sum_{i=0}^{1}\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t(Pv^{(i)})|^2 + s|Pv^{(i)}|^2\right)\omega_L \\
&\quad + C\int_L e^{2s\varphi}|f|^2\omega_L + Ce^{2\epsilon_2 s}\sum_{k=0}^{1}\|h\|^2_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2,
\end{aligned}$$

where $\epsilon_j := e^{\gamma\cdot j\epsilon}$ for $j \in \{2,3\}$. Furthermore, in regard to the first and second summands on the right-hand side of the above estimate, and because we have

$$Pv^{(0)} = \chi Sf + \Big[2g^{\mu\nu}\partial_\mu\chi\partial_\nu h + \Box_g\chi h\Big],$$

$$\partial_t(Pv^{(0)}) = \chi\partial_t Sf + \partial_t\chi Sf + \partial_t\Big[2g^{\mu\nu}\partial_\mu\chi\partial_\nu h + \Box_g\chi h\Big],$$

$$\begin{aligned}
Pv^{(1)} = \chi\partial_t Sf - \partial_t g^{\mu\nu}\partial_\mu\partial_\nu v^{(0)} + \partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho v^{(0)} - \partial_t a v^{(0)} + \Big[\partial_t g^{\mu\nu}(\partial_\mu\partial_\nu\chi)h \\
+ 2\partial_t g^{\mu\nu}(\partial_\mu\chi)(\partial_\nu h) + 2g^{\mu\nu}\partial_\mu\chi\partial_\nu\partial_t h + \Box_g\chi\partial_t h - \partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})(\partial_\rho\chi)h\Big],
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t(Pv^{(1)}) \\
&= \chi\partial_t^2 Sf + \partial_t\chi\partial_t Sf - \partial_t^2 g^{\mu\nu}\partial_\mu\partial_\nu v^{(0)} - \partial_t^2 a v^{(0)} + \partial_t^2(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho v^{(0)} \\
&\quad - \partial_t g^{\mu\nu}\partial_\mu\partial_\nu(\partial_t\chi h + v^{(1)}) - \partial_t a(\partial_t\chi h + v^{(1)}) + \partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho(\partial_t\chi h + v^{(1)}) \\
&\quad + \partial_t\Big[\partial_t g^{\mu\nu}(\partial_\mu\partial_\nu\chi)h + 2\partial_t g^{\mu\nu}(\partial_\mu\chi)(\partial_\nu h) + 2g^{\mu\nu}\partial_\mu\chi\partial_\nu\partial_t h \\
&\quad + \Box_g\chi\partial_t h - \partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})(\partial_\rho\chi)h\Big],
\end{aligned}$$

we obtain

$$\begin{aligned}
(4.10)\qquad &\sum_{i=0}^{1}\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t(Pv^{(i)})|^2 + s|Pv^{(i)}|^2\right)\omega_L \\
&\leq C\int_L s^2e^{2s\varphi}|f|^2\omega_L + Cs^2e^{2\epsilon_2 s}\sum_{k=0}^{2}\|h\|^2_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2.
\end{aligned}$$

Indeed, in regard to the first and second summands on the left-hand side of (4.10), we have

$$\int_L e^{2s\varphi} s|Pv^{(0)}|^2\omega_L \le C\int_L se^{2s\varphi}|f|^2\omega_L + Cse^{2\epsilon_2 s}\sum_{k=0}^{1}\|h\|_{H^{1-k}(-T,T;H^k(M;\mathbb{R}^\ell))},$$

$$\int_L e^{2s\varphi}\frac{1}{s}|\partial_t(Pv^{(0)})|^2 d\omega_L \le C\int_L \frac{1}{s}e^{2s\varphi}|f|^2\omega_L + \frac{C}{s}e^{2\epsilon_2 s}\sum_{k=0}^{1}\|h\|_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))},$$

$$\begin{aligned}
&\int_L e^{2s\varphi} s|Pv^{(1)}|^2\omega_L \\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L \\
&\quad + C\sum_{m=1}^{\ell} s\left(\int_{-T}^{T}\|e^{s\varphi}v_m^{(0)}\|^2_{H^2(M)}dt + \int_L e^{2s\varphi}\Big(s^2E(v_m^{(0)}) + s^4|v_m^{(0)}|^2\Big)\omega_L\right) \\
&\quad + Cse^{2\epsilon_2 s}\sum_{k=0}^{1}\|h\|_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))} \\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L + C\sum_{m=1}^{\ell}\int_L se^{2s\varphi}\Big(|\Delta_{g_\flat}v_m^{(0)}|^2 + s^2E(v_m^{(0)}) + s^4|v_m^{(0)}|^2\Big)\omega_L \\
&\quad + Cse^{2\epsilon_2 s}\sum_{k=0}^{1}\|h\|_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))} \\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L + C\int_L e^{2s\varphi}\Big(|\partial_t(Pv^{(0)})|^2 + s^2|Pv^{(0)}|^2\Big)\omega_L \\
&\quad + Cse^{2\epsilon_2 s}\sum_{k=0}^{1}\|h\|_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2 \\
&\le C\int_L s^2e^{2s\varphi}|f|^2\omega_L + Cs^2e^{2\epsilon_2 s}\sum_{k=0}^{1}\|h\|_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2,
\end{aligned}$$

where we used Lemma 4.13 and Lemma 4.12, and

$$
\begin{aligned}
&\int_L e^{2s\varphi}\frac{1}{s}|\partial_t(Pv^{(1)})|^2\omega_L \\
&\le C\int_L \frac{1}{s}e^{2s\varphi}|f|^2\omega_L \\
&\quad + C\sum_{i=0}^{1}\sum_{m=1}^{\ell}\frac{1}{s}\left(\int_{-T}^{T}\|e^{s\varphi}v_m^{(i)}\|_{H^2(M)}^2 dt + \int_L e^{2s\varphi}\Big(s^2E(v_m^{(i)}) + s^4|v_m^{(i)}|^2\Big)\omega_L\right) \\
&\quad + \frac{C}{s}e^{2\epsilon_2 s}\sum_{k=0}^{2}\|h\|_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} \\
&\le C\int_L \frac{1}{s}e^{2s\varphi}|f|^2\omega_L + C\sum_{i=0}^{1}\sum_{m=1}^{\ell}\int_L\frac{1}{s}e^{2s\varphi}\Big(|\Delta_{g_\flat}v_m^{(i)}|^2 + s^2E(v_m^{(i)}) + s^4|v_m^{(i)}|^2\Big)\omega_L \\
&\quad + \frac{C}{s}e^{2\epsilon_2 s}\sum_{k=0}^{2}\|h\|_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} \\
&\le C\int_L \frac{1}{s}e^{2s\varphi}|f|^2\omega_L + C\sum_{i=0}^{1}\int_L e^{2s\varphi}\Big(\frac{1}{s^2}|\partial_t(Pv^{(i)})|^2 + |Pv^{(i)}|^2\Big)\omega_L \\
&\quad + \frac{C}{s}e^{2\epsilon_2 s}\sum_{k=0}^{2}\|h\|_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2 \\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L + C\int_L\frac{1}{s^2}e^{2s\varphi}|\partial_t(Pv^{(1)})|^2\omega_L \\
&\quad + Cse^{2\epsilon_2 s}\sum_{k=0}^{2}\|h\|_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2,
\end{aligned}
$$

where we used Lemma 4.13 and Lemma 4.12 again. Taking $s > 0$ sufficiently large yields

$$
\begin{aligned}
&\int_L e^{2s\varphi}\frac{1}{s}|\partial_t(Pv^{(1)})|^2\omega_L \\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L + Cse^{2\epsilon_2 s}\sum_{k=0}^{2}\|h\|_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2.
\end{aligned}
$$

Hence, we finally obtain (4.10). Then, applying (4.10) to (4.9) yields

$$
\begin{aligned}
(4.11)\qquad &\sum_{m=1}^{\ell}\int_L e^{2s\varphi}\left(sE(v_m^{(2)}) + s^3|v_m^{(2)}|^2\right)\omega_L \\
&\le C\int_L s^2e^{2s\varphi}|f|^2\omega_L + Cs^2e^{2\epsilon_2 s}\sum_{k=0}^{2}\|h\|_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))}^2 + Ce^{Cs}\mathcal{D}^2.
\end{aligned}
$$

Then, using (4.11), we have

$$\begin{aligned}
\|e^{s\varphi(0,\cdot)}\chi(0,\cdot)S(0,\cdot)f\|^2_{L^2(M;\mathbb{R}^\ell)} &\le C\|e^{s\varphi(0,\cdot)}v^{(2)}(0,\cdot)\|^2_{L^2(M;\mathbb{R}^\ell)} \\
&= C\int_{-T}^{0}\frac{d}{dt}\left(\int_M e^{2s\varphi(t,x)}|v^{(2)}(t,x)|^2\omega_M\right)dt \\
&\le C\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t v^{(2)}|^2 + s|v^{(2)}|^2\right)\omega_L \\
&\le C\int_L e^{2s\varphi}|f|^2\omega_L + Ce^{2\epsilon_2 s}\sum_{k=0}^{2}\|h\|^2_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2.
\end{aligned}\tag{4.12}$$

Hence, using (4.12), we have

$$\begin{aligned}
\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M &\le C\int_M e^{2s\varphi(0,x)}|\chi(0,x)|^2|S(0,x)f|^2\omega_M \\
&\le C\int_L e^{2s\varphi}|f|^2\omega_L + Ce^{2\epsilon_2 s}\sum_{k=0}^{2}\|h\|^2_{H^{3-k}(-T,T;H^k(M);\mathbb{R}^\ell)} + Ce^{Cs}\mathcal{D}^2.
\end{aligned}\tag{4.13}$$

Moreover, we establish

$$\begin{aligned}
&\int_L e^{2s\varphi}|f|^2\omega_L \\
&\le C\int_{-T}^{T}\left(\int_{M_{2\epsilon}} e^{2s\varphi}|f|^2\omega_M\right)dt + C\int_{-T}^{T}\left(\int_{M\setminus M_{2\epsilon}} e^{2s\varphi}|f|^2\omega_M\right)dt \\
&\le C\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\left(\int_{-T}^{T} e^{-2s(\varphi(0,x)-\varphi(t,x))}dt\right)\omega_M + Ce^{2\epsilon_2 s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)} \\
&\le o(1)\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M + Ce^{2\epsilon_2 s}\|f\|^2_{L^2(M;\mathbb{R}^\ell)}
\end{aligned}$$

as $s \to \infty$ by our assumption (4.3) and the Lebesgue dominated convergence theorem. Applying this inequality to (4.13) yields

$$\begin{aligned}
\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M \le Ce^{2\epsilon_2 s}&\left(\|f\|^2_{L^2(M;\mathbb{R}^\ell)} + \sum_{k=0}^{2}\|h\|^2_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))}\right) \\
&+ Ce^{Cs}\mathcal{D}^2
\end{aligned}$$

for sufficiently large $s > s_*$. We note that

$$\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M \ge \int_{M_{3\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M \ge e^{2\epsilon_3 s}\|f\|^2_{L^2(M_{3\epsilon};\mathbb{R}^\ell)}.$$

Hence, we have

$$\begin{aligned}
\|f\|^2_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \le Ce^{-2(\epsilon_3-\epsilon_2)s}&\left(\|f\|^2_{L^2(M;\mathbb{R}^\ell)} + \sum_{k=0}^{2}\|h\|^2_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))}\right) \\
&+ Ce^{Cs}\mathcal{D}^2,
\end{aligned}$$

i.e.,

$$\|f\|_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \le Ce^{-(\epsilon_3-\epsilon_2)s}\mathcal{F} + Ce^{Cs}\mathcal{D},\tag{4.14}$$

for all $s > s^*$. By replacing C by Ce^{Cs^*}, the above estimate holds for all $s > 0$. When $\mathcal{D} \geq \mathcal{F}$, (4.14) implies

$$\|f\|_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \leq Ce^{Cs}\mathcal{D}.$$

Moreover, when $\mathcal{D} < \mathcal{F}$, we choose $s > 0$ to minimize the right-hand side of (4.14) such that

$$e^{Cs}\mathcal{D} = e^{-(\epsilon_3-\epsilon_2)s}\mathcal{F},$$

i.e.,

$$s = \frac{1}{C + \epsilon_3 - \epsilon_2} \log \frac{\mathcal{F}}{\mathcal{D}}.$$

We then have

$$\|f\|_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \leq 2C\mathcal{F}^{1-\theta}\mathcal{D}^\theta,$$

where

$$\theta := \frac{\epsilon_3 - \epsilon_2}{C + \epsilon_3 - \epsilon_2} \in (0, 1).$$

Hence, there exist constants $C > 0$ and $\theta \in (0, 1)$ such that

$$\|f\|_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \leq C(\mathcal{D} + \mathcal{F}^{1-\theta}\mathcal{D}^\theta)$$

holds. □

4. Proofs of auxiliary results

4.1. Proof of Lemma 4.8. For the proof of Lemma 4.8, we need the Gauss formula for Lorentzian manifolds. We say the boundary ∂L is spacelike (timelike) if the induced metric to ∂L is Riemannian (Lorentzian). Let N be the outward pointing unit normal vector field to ∂L. If ∂L is spacelike, $\langle N, N\rangle = -1$; otherwise, $\langle N, N\rangle = 1$. We refer to Lemma 10.8 in Ringström [58]. Note that Σ is timelike.

Lemma 4.14. *Let (L, g) be an $n + 1$-dimensional compact oriented Lorentzian manifold with boundary. Assume that the boundary is spacelike or timelike and let X be a smooth vector field. Then if N denotes the outer unit normal to ∂L, it follows that*

$$\int_L \operatorname{div} X\omega_L = \int_{\partial L} \frac{\langle X, N\rangle}{\langle N, N\rangle}\omega_{\partial L}.$$

Proof of Lemma 4.8. First, note that

$$\nabla\varphi = \gamma\varphi\nabla\psi, \quad \Box_g\varphi = \gamma\varphi(\Box_g\psi + \gamma\langle\nabla\psi, \nabla\psi\rangle),$$
$$\nabla^2\varphi(\nabla z, \nabla z) = \gamma\varphi(\nabla^2\psi(\nabla z, \nabla z) + \gamma|\langle\nabla z, \nabla\psi\rangle|^2).$$

We introduce a new function and operator

$$z := e^{s\varphi}u, \quad P_s z := e^{s\varphi}\Box_g(e^{-s\varphi}z).$$

A lengthy calculation yields

$$P_s z = \Box_g z - 2s\langle\nabla\varphi, \nabla z\rangle + s^2\langle\nabla\varphi, \nabla\varphi\rangle z - s\Box_g\varphi z,$$

which decomposes $P_s z$ into $P_s^+ z$ and $P_s^- z$,

$$\begin{cases} P_s^+ z := \Box_g z + s^2\langle\nabla\varphi, \nabla\varphi\rangle z, \\ P_s^- z := -2s\langle\nabla\varphi, \nabla z\rangle - s\Box_g\varphi z. \end{cases}$$

Note $P_s z = P_s^+ z + P_s^- z$. Because we wish to make a lower bound of $\|P_s z\|^2_{L^2(L)}$, we calculate the L^2 inner product of $P_s^+ z$ and $P_s^- z$,

$$\begin{aligned}(P_s^+ z, P_s^- z)_{L^2(L)} &= \int_L \Box_g z \cdot (-2s\langle \nabla\varphi, \nabla z\rangle)\omega_L + \int_L \Box_g z \cdot (-s\Box_g \varphi z)\omega_L \\ &+ \int_L s^2 \langle \nabla\varphi, \nabla\varphi\rangle z \cdot (-2s\langle \nabla\varphi, \nabla z\rangle)\omega_L + \int_L s^2\langle \nabla\varphi, \nabla\varphi\rangle z \cdot (-s\Box_g\varphi z)\omega_L \\ &=: \sum_{k=1}^4 I_k.\end{aligned}$$

Let N be the outer unit normal vector field to ∂L. We remark that $z = \partial_N z = \nabla z = 0$ on $M^{\pm T}$. Integration by parts yields

$$\begin{aligned} I_1 &= \int_L 2s\langle \nabla\langle \nabla\varphi, \nabla z\rangle, \nabla z\rangle \omega_L - \int_{\partial L} 2s\frac{\langle \nabla z, N\rangle}{\langle N, N\rangle}\langle \nabla\varphi, \nabla z\rangle\omega_{\partial L} \\ &= \int_L 2s\nabla^2\varphi(\nabla z, \nabla z)\omega_L + \int_L s\langle \nabla\varphi, \nabla\langle \nabla z, \nabla z\rangle\rangle\omega_L - \int_\Sigma 2s\langle \nabla z, N\rangle\langle \nabla\varphi, \nabla z\rangle\omega_\Sigma \\ &= \int_L 2s\nabla^2\varphi(\nabla z, \nabla z)\omega_L - \int_L s\Box_g\varphi\langle \nabla z, \nabla z\rangle\omega_L - \int_\Sigma 2s\langle \nabla z, N\rangle\langle \nabla\varphi, \nabla z\rangle\omega_\Sigma \\ &\quad + \int_\Sigma s\langle \nabla\varphi, N\rangle\langle \nabla z, \nabla z\rangle\omega_\Sigma, \end{aligned}$$

where we have used the identity

$$\begin{aligned} 2\langle \nabla\langle \nabla\varphi, \nabla z\rangle, \nabla z\rangle &= 2\nabla_\mu(\nabla_\nu\varphi\nabla^\nu z)\nabla^\mu z \\ &= 2(\nabla_\mu\nabla_\nu\varphi)\nabla^\nu z\nabla^\mu z + 2\nabla_\nu\varphi(\nabla_\mu\nabla^\nu z)\nabla^\mu z \\ &= 2\nabla^2\varphi(\nabla z, \nabla z) + \nabla_\nu\varphi\nabla^\nu(\nabla^\mu z\nabla_\mu z) \\ &= 2\nabla^2\varphi(\nabla z, \nabla z) + \langle \nabla\varphi, \nabla\langle \nabla z, \nabla z\rangle\rangle. \end{aligned} \tag{4.15}$$

Furthermore, we obtain

$$\begin{aligned} I_2 &= \int_L s\Box_g\varphi\langle \nabla z, \nabla z\rangle\omega_L + \int_L \frac{s}{2}\langle \nabla\Box_g\varphi, \nabla(|z|^2)\rangle\omega_L - \int_{\partial L} s\frac{\langle \nabla z, N\rangle}{\langle N, N\rangle}\Box_g\varphi z\omega_{\partial L} \\ &= \int_L s\Box_g\varphi\langle \nabla z, \nabla z\rangle\omega_L - \int_L \frac{s}{2}\Box_g^2\varphi|z|^2\omega_L - \int_\Sigma s\langle \nabla z, N\rangle\Box_g\varphi z\omega_\Sigma \\ &\quad + \int_\Sigma \frac{s}{2}\langle \nabla\Box_g\varphi, N\rangle|z|^2\omega_\Sigma, \end{aligned}$$

$$\begin{aligned} I_3 &= -\int_L s^3\langle \nabla\varphi, \nabla\varphi\rangle\langle \nabla\varphi, \nabla(|z|^2)\rangle\omega_L \\ &= \int_L s^3\nabla(\langle \nabla\varphi, \nabla\varphi\rangle\nabla\varphi)|z|^2\omega_L - \int_\Sigma s^3\langle \nabla\varphi, \nabla\varphi\rangle\langle \nabla\varphi, N\rangle|z|^2\omega_\Sigma, \end{aligned}$$

$$I_4 = -\int_L s^3\langle \nabla\varphi, \nabla\varphi\rangle\Box_g\varphi|z|^2\omega_L.$$

We remark that the integrand of the first summand of I_3 means

$$\nabla(\langle \nabla\varphi, \nabla\varphi\rangle\nabla\varphi) = \nabla_\mu(\langle \nabla\varphi, \nabla\varphi\rangle\nabla^\mu\varphi).$$

Hence, we have

$$
\begin{aligned}
\sum_{k=1}^{4} I_k = &\int_L 2s\nabla^2\varphi(\nabla z, \nabla z)\omega_L \\
&+ \int_L \left(-\frac{s}{2}\Box_g^2\varphi + s^3\nabla(\langle\nabla\varphi, \nabla\varphi\rangle\nabla\varphi) - s^3\langle\nabla\varphi, \nabla\varphi\rangle\Box_g\varphi\right)|z|^2\omega_L \\
&- \int_\Sigma 2s\langle\nabla z, N\rangle\langle\nabla\varphi, \nabla z\rangle\omega_\Sigma + \int_\Sigma s\langle\nabla\varphi, N\rangle\langle\nabla z, \nabla z\rangle\omega_\Sigma \\
&- \int_\Sigma s\langle\nabla z, N\rangle\Box_g\varphi z\omega_\Sigma + \int_\Sigma \frac{s}{2}\langle\nabla\Box_g\varphi, N\rangle|z|^2\omega_\Sigma \\
&- \int_\Sigma s^3\langle\nabla\varphi, \nabla\varphi\rangle\langle\nabla\varphi, N\rangle|z|^2\omega_\Sigma \\
=:& First + Zeroth + \mathcal{B},
\end{aligned}
\tag{4.16}
$$

where we define

$$
\begin{aligned}
First &:= \int_L 2s\nabla^2\varphi(\nabla z, \nabla z)\omega_L, \\
Zeroth &:= \int_L \left(-\frac{s}{2}\Box_g^2\varphi + s^3\nabla(\langle\nabla\varphi, \nabla\varphi\rangle\nabla\varphi) - s^3\langle\nabla\varphi, \nabla\varphi\rangle\Box_g\varphi\right)|z|^2\omega_L, \\
\mathcal{B} &:= -\int_\Sigma 2s\langle\nabla z, N\rangle\langle\nabla\varphi, \nabla z\rangle\omega_\Sigma + \int_\Sigma s\langle\nabla\varphi, N\rangle\langle\nabla z, \nabla z\rangle\omega_\Sigma - \int_\Sigma s\langle\nabla z, N\rangle\Box_g\varphi z\omega_\Sigma \\
&\quad + \int_\Sigma \frac{s}{2}\langle\nabla\Box_g\varphi, N\rangle|z|^2\omega_\Sigma - \int_\Sigma s^3\langle\nabla\varphi, \nabla\varphi\rangle\langle\nabla\varphi, N\rangle|z|^2\omega_\Sigma.
\end{aligned}
$$

In regard to the $First$, from our assumption (4.2), we obtain

$$
\begin{aligned}
First &= \int_L 2s\nabla^2\varphi(\nabla z, \nabla z)\omega_L \\
&= \int_L 2\sigma(\nabla^2\psi(\nabla z, \nabla z) + \gamma|\langle\nabla z, \nabla\psi\rangle|^2)\omega_L \\
&\geq -4\kappa_2\int_L \sigma|\nabla^0 z|^2\omega_L + 4\kappa_1\int_L \sigma|\nabla z|^2\omega_L,
\end{aligned}
$$

where we remark that $\sigma := s\gamma\varphi$. Therefore, we need the second estimate,

(4.17)

$$
\begin{aligned}
(P_s^+ z, \sigma z)_{L^2(L)} &= \int_L \Box_g z \cdot (\sigma z)\omega_L + \int_L s^2 \langle \nabla\varphi, \nabla\varphi\rangle z \cdot (\sigma z)\omega_L \\
&= -\int_L \sigma\langle \nabla z, \nabla z\rangle \omega_L - \int_L \frac{s\gamma}{2}\langle \nabla\varphi, \nabla(|z|^2)\rangle\omega_L \\
&\quad + \int_L s^2\sigma\langle\nabla\varphi,\nabla\varphi\rangle |z|^2\omega_L + \int_{\partial L} \sigma \frac{\langle \nabla z, N\rangle}{\langle N, N\rangle} z\omega_{\partial L} \\
&= -\int_L \sigma\langle \nabla z, \nabla z\rangle\omega_L + \int_L \left(\frac{s\gamma}{2}\Box_g\varphi + s^2\sigma\langle\nabla\varphi,\nabla\varphi\rangle\right)|z|^2\omega_L \\
&\quad + \int_\Sigma \sigma\langle\nabla z, N\rangle z\omega_\Sigma - \int_\Sigma \frac{s\gamma}{2}\langle\nabla\varphi, N\rangle |z|^2\omega_\Sigma \\
&= \int_L \sigma|\nabla^0 z|^2\omega_L - \int_L \sigma|\nabla z|^2\omega_L \\
&\quad + \int_L \left(\frac{s\gamma}{2}\Box_g\varphi + s^2\sigma\langle\nabla\varphi,\nabla\varphi\rangle\right)|z|^2\omega_L + \int_\Sigma \sigma\langle \nabla z, N\rangle z\omega_\Sigma \\
&\quad - \int_\Sigma \frac{s\gamma}{2}\langle\nabla\varphi, N\rangle|z|^2\omega_\Sigma \\
&=: First_2 + Zeroth_2 + \mathcal{B}_2,
\end{aligned}
$$

where we define

$$
\begin{aligned}
First_2 &:= \int_L \sigma|\nabla^0 z|^2\omega_L - \int_L \sigma|\nabla z|^2\omega_L, \\
Zeroth_2 &:= \int_L \left(\frac{s\gamma}{2}\Box_g\varphi + s^2\sigma\langle\nabla\varphi,\nabla\varphi\rangle\right)|z|^2\omega_L, \\
\mathcal{B}_2 &:= \int_\Sigma \sigma\langle\nabla z, N\rangle z\omega_\Sigma - \int_\Sigma \frac{s\gamma}{2}\langle\nabla\varphi, N\rangle|z|^2\omega_\Sigma.
\end{aligned}
$$

We remark that the last equality is obtained by the fact that for all $p \in L$ and $X \in T_pL$,

$$
\begin{aligned}
\langle X, X\rangle &= -d\tau^2((d\pi_0)X, (d\pi_0)X) + g_\flat((d\pi_1)X, (d\pi_1)X) \\
&= -|X^0|^2 + |X|^2
\end{aligned}
$$

holds. Multiplying (4.17) by 4δ for $\delta > 0$ to be determined later and adding it to (4.16) yield

$$
\begin{aligned}
\sum_{k=1}^4 I_k + 4\delta(P_s^+ z, \sigma z)_{L^2(L)} \geq 4(\delta - \kappa_2)\int_L \sigma|\nabla^0 z|^2\omega_L + 4(\kappa_1 - \delta)\int_L \sigma|\nabla z|^2\omega_L \\
+ Zeroth + 4\delta Zeroth_2 + \mathcal{B} + 4\delta\mathcal{B}_2.
\end{aligned}
$$

From our assumption (4.2), there exists $\delta > 0$ such that

$$
\begin{cases}
\delta - \kappa_2 > 0, \\
\kappa_1 - \delta > 0.
\end{cases}
$$

Next, we consider the zeroth-order terms $Zeroth + 4\delta Zeroth_2$.

$$\begin{aligned}
Zeroth &= \int_L \Big(-\frac{s}{2}\Box_g^2\varphi + s^3\nabla(\langle\nabla\varphi,\nabla\varphi\rangle\nabla\varphi) - s^3\langle\nabla\varphi,\nabla\varphi\rangle\Box_g\varphi\Big)\,|z|^2\omega_L \\
&= \int_L [2\sigma^3\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 + 2\sigma^3\nabla^2\psi(\nabla\psi,\nabla\psi) + O(s\gamma^4\varphi)]|z|^2\omega_L \\
&\geq \int_L [2\sigma^3\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 + 2\sigma^3(-2\kappa_2|\nabla^0\psi|^2 + 2\kappa_1|\nabla\psi|^2) + O(s\gamma^4\varphi)]|z|^2\omega_L
\end{aligned}$$

as $\gamma \to \infty$, where the second equality holds by (4.15). Indeed, we obtain from (4.15)

$$\begin{aligned}
\nabla(\langle\nabla\varphi,\nabla\varphi\rangle\nabla\varphi) - \langle\nabla\varphi,\nabla\varphi\rangle\Box_g\varphi &= \langle\nabla\langle\nabla\varphi,\nabla\varphi\rangle,\nabla\varphi\rangle \\
&= 2\nabla^2\varphi(\nabla\varphi,\nabla\varphi) \\
&= 2(\gamma\varphi)^3(\nabla^2\psi(\nabla\psi,\nabla\psi) + \gamma|\langle\nabla\psi,\nabla\psi\rangle|^2).
\end{aligned}$$

Moreover, we get

$$4\delta Zeroth_2 = \int_L [4\delta\sigma^3\langle\nabla\psi,\nabla\psi\rangle + O(s\gamma^3\varphi))]|z|^2\omega_L$$

as $\gamma \to \infty$. We then have

$$\begin{aligned}
&Zeroth + 4\delta Zeroth_2 \\
&\geq \int_L \Big[\sigma^3\Big(2\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 - 4\kappa_2|\nabla^0\psi|^2 + 4\kappa_1|\nabla\psi|^2 + 4\delta\langle\nabla\psi,\nabla\psi\rangle\Big) \\
&\quad + O(s\gamma^4\varphi)\Big]|z|^2\omega_L \\
&= \int_L \Big[\sigma^3\Big(2\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 - 4\kappa_2|\nabla^0\psi|^2 + 4\kappa_1|\nabla\psi|^2 + 8\delta\langle\nabla\psi,\nabla\psi\rangle - 4\delta\langle\nabla\psi,\nabla\psi\rangle\Big) \\
&\quad + O(s\gamma^4\varphi)\Big]|z|^2\omega_L \\
&= \int_L \Big[\sigma^3\Big(2\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 - 4\kappa_2|\nabla^0\psi|^2 + 4\kappa_1|\nabla\psi|^2 + 8\delta\langle\nabla\psi,\nabla\psi\rangle \\
&\quad - 4\delta(-|\nabla^0\psi|^2 + |\nabla\psi|^2)\Big) + O(s\gamma^4\varphi)\Big]|z|^2\omega_L \\
&= \int_L \Big[\sigma^3\Big(2\gamma\Big(\langle\nabla\psi,\nabla\psi\rangle + \frac{2\delta}{\gamma}\Big)^2 + 4(\delta-\kappa_2)|\nabla^0\psi|^2 + 4(\kappa_1-\delta)|\nabla\psi|^2 - \frac{8\delta^2}{\gamma}\Big) \\
&\quad + O(s\gamma^4\varphi)\Big]|z|^2\omega_L \\
&\geq \int_L \Big[\sigma^3\Big(4(\kappa_1-\delta)|\nabla\psi|^2 - \frac{8\delta^2}{\gamma}\Big) + O(s\gamma^4\varphi)\Big]|z|^2\omega_L \\
&\geq C\int_L \Big[\sigma^3 + O(s\gamma^4\varphi)\Big]|z|^2\omega_L
\end{aligned}$$

as $\gamma \to \infty$. Note that we used assumptions (4.2) and (4.3). Therefore, for sufficiently large $\gamma > 0$ there exists a constant C such that

$$\begin{aligned}
(P_s^+z, P_s^-z)_{L^2(L)} + 4\delta(P_s^+z,\sigma z)_{L^2(L)} &+ C\int_L O(s\gamma^4\varphi)|z|^2\omega_L \\
&\geq C\int_L \sigma(|\nabla^0 z|^2 + |\nabla z|^2 + \sigma^2|z|^2)\omega_L + \mathcal{B} + 4\delta\mathcal{B}_2
\end{aligned}$$

holds for all $z \in C^\infty(L)$ satisfying $z = \partial_N z = 0$ on $M^{\pm T}$. For a sufficiently large fixed $\gamma > 0$, we choose $s > 0$ large enough so that

$$C \int_L \sigma(|\nabla^0 z|^2 + |\nabla z|^2 + \sigma^2 |z|^2)\omega_L \leq \|P_s z\|^2_{L^2(L)} - \mathcal{B} - 4\delta\mathcal{B}_2$$

holds. It remains to estimate the boundary terms $\mathcal{B} + 4\delta\mathcal{B}_2$. Note that N on Σ is a spacelike unit outer normal vector field, i.e., $\langle N, N\rangle = 1$ holds on Σ. We have

$$\begin{aligned}
-\mathcal{B} - 4\delta\mathcal{B}_2 &= \int_\Sigma 2\sigma\langle\nabla z, N\rangle\langle\nabla\psi, \nabla z\rangle\omega_\Sigma + \int_\Sigma s\Box_g\varphi\langle\nabla z, N\rangle z\omega_\Sigma \\
&\quad + \int_\Sigma \sigma^3\langle\nabla\psi, \nabla\psi\rangle\langle\nabla\psi, N\rangle|z|^2\omega_\Sigma - \int_\Sigma \frac{s}{2}\langle\nabla\Box_g\varphi, N\rangle|z|^2\omega_\Sigma \\
&\quad - \int_\Sigma \sigma\langle\nabla\psi, N\rangle\langle\nabla z, \nabla z\rangle\omega_\Sigma - 4\delta\int_\Sigma \sigma\langle\nabla z, N\rangle z\omega_\Sigma \\
&\quad + 2\delta\int_\Sigma \sigma\gamma\langle\nabla\psi, N\rangle|z|^2\omega_\Sigma \\
&= \int_\Sigma 2\sigma\partial_N z\langle\nabla\psi, \nabla z\rangle\omega_\Sigma - \int_\Sigma \sigma\partial_N\psi\langle\nabla z, \nabla z\rangle\omega_\Sigma \\
&= \int_\Sigma \sigma\partial_N\psi|\partial_N z|^2\omega_\Sigma \leq C\int_\Sigma \sigma|\partial_N z|^2\omega_\Sigma,
\end{aligned}$$

where $\partial_N z := \langle\nabla z, N\rangle$ because we can write $\nabla z = \langle\nabla z, N\rangle N$ as $z = 0$ on Σ, which is proved by taking the semigeodesic coordinate, and then

$$\langle\nabla\psi, \nabla z\rangle = \partial_N\psi\partial_N z, \quad \langle\nabla z, \nabla z\rangle = |\partial_N z|^2$$

holds. Then, after some calculations, we obtain

$$\begin{aligned}
e^{2s\varphi}|\nabla^0 u|^2 &= (\nabla^0 z + sz\nabla^0\varphi)^2 \leq C(|\nabla^0 z|^2 + \sigma^2|z|^2), \\
e^{2s\varphi}|\nabla u|^2 &= g_{ij}(\nabla^i z + sz\nabla^i\varphi)(\nabla^j z + sz\nabla^j\varphi) \leq C(|\nabla z|^2 + \sigma^2|z|^2).
\end{aligned}$$

Hence, we finally obtain

$$\int_L e^{2s\varphi}\sigma(|\nabla^0 u|^2 + |\nabla u|^2 + \sigma^2|u|^2)\omega_L \leq C\int_L e^{2s\varphi}|\Box_g u|^2\omega_L + C\int_\Sigma e^{2s\varphi}\sigma|\partial_N u|^2\omega_\Sigma$$

for sufficiently large $s > 0$. The proof is completed. □

4.2. Proof of Lemma 4.12.

Proof of Lemma 4.12. Let us fix the semigeodesic coordinate $(x^0 = t, x^1, \ldots, x^n)$. We then find $\hat{N} = \partial_t$. For large $\gamma > 0$ large, we apply Proposition 4.11 to $\partial_t v$ to derive

$$\begin{aligned}
&\int_L e^{2s\varphi}|\partial_t^2 v|^2\omega_L \\
&\le \frac{C}{s}\int_L e^{2s\varphi}|P\partial_t v|^2\omega_L + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma \\
&\le \frac{C}{s}\int_L e^{2s\varphi}|\partial_t Pv - \partial_t g^{\mu\nu}\partial_\mu\partial_\nu v + \partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho v - \partial_t av|^2\omega_L \\
&\quad + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma \\
&\le \frac{C}{s}\int_L e^{2s\varphi}|\partial_t Pv|^2\omega_L + \frac{C}{s}\left(\int_{-T}^T \|e^{s\varphi}v\|^2_{H^2(M)}dt + \int_L e^{2s\varphi}\Big(s^2E(v)+s^4|v|^2\Big)\omega_L\right) \\
&\quad + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma \\
&\le \frac{C}{s}\int_L e^{2s\varphi}|\partial_t Pv|^2\omega_L + \frac{C}{s}\int_L e^{2s\varphi}\left(|\Delta_{g_\flat}v|^2 + s^2E(v)+s^4|v|^2\right)\omega_L \\
&\quad + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma \\
&\le \frac{C}{s}\int_L e^{2s\varphi}|\partial_t Pv|^2\omega_L + C\int_L e^{2s\varphi}|Pv|^2\omega_L + \frac{C}{s}\int_L e^{2s\varphi}|\Delta_{g_\flat}v|^2\omega_L \\
&\quad + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma + Cs\int_\Sigma e^{2s\varphi}|\partial_N v|^2\omega_\Sigma,
\end{aligned}$$

where we use Lemma 4.13 to obtain the fourth inequality. Since $g^{ij}\partial_i\partial_j v = Pv + \partial_t^2 v + g^{\mu\nu}\Gamma^\rho_{\mu\nu}\partial_\rho v - av$ and $g_\flat^{ij} = g^{ij}$ by the semigeodesic coordinate, we obtain

$$\begin{aligned}
\int_L e^{2s\varphi}|\Delta_{g_\flat}v|^2\omega_L &\le C\int_L e^{2s\varphi}\left(|\partial_t^2 v|^2 + E(v) + |v|^2 + |Pv|^2\right)\omega_L \\
&\le C\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t Pv|^2 + |Pv|^2\right)\omega_L + \frac{C}{s}\int_L e^{2s\varphi}|\Delta_{g_\flat}v|^2\omega_L \\
&\quad + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma + Cs\int_\Sigma e^{2s\varphi}|\partial_N v|^2\omega_\Sigma.
\end{aligned}$$

Choosing $s>0$ sufficiently large, we absorb the second term on the right-hand side into the left-hand side to obtain

$$\int_L e^{2s\varphi}|\Delta_{g_\flat}v|^2\omega_L \le C\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t Pv|^2 + |Pv|^2\right)\omega_L + Ce^{Cs}\mathcal{E}^2.$$

□

4.3. Proof of Lemma 4.13.

Proof of Lemma 4.13. Let $\{(U_i, x_i)\}_i$ be a local coordinate system of M. If $\{\chi_i\}_i$ is a finite partition of unity subordinate to the open covering and χ_i' are chosen with $\chi_i' = 1$ in a neighborhood of $\operatorname{supp}\chi_i$ and $\operatorname{supp}\chi_i' \subset U_i$, then

$$\begin{aligned}\|\sqrt{\chi_i}v\|_{H^2(U_i)} &\le C\Big(\|A(\sqrt{\chi_i}v)\|_{L^2(U_i)} + \|\sqrt{\chi_i}v\|_{L^2(U_i)}\Big)\\ &\le C\Big(\|\sqrt{\chi_i'}Av\|_{L^2(U_i)} + \|\sqrt{\chi_i'}v\|_{L^2(U_i)}\Big)\\ &\le C\Big(\|\sqrt{\eta_i}Av\|_{L^2(U_i)} + \|\sqrt{\eta_i}v\|_{L^2(U_i)}\Big),\end{aligned}$$

where $\eta_i := \dfrac{\sqrt{\chi_i'}}{\sum_i \sqrt{\chi_i'}}$. With $0 < \sum_i \sqrt{\chi_i} \le \sum_i \sqrt{\chi_i'}$, η_i can be defined. Because $\operatorname{supp}\eta_i = \operatorname{supp}\chi_i' \subset U_i$, $\{\eta_i\}_i$ is a partition of unity subordinate to the covering $\{U_i\}_i$. Summing up with respect to i yields

$$\sum_i \|\sqrt{\chi_i}v\|_{H^2(U_i)} \le C\left(\|Av\|_{L^2(M)} + \|v\|_{L^2(M)}\right). \tag{4.18}$$

Furthermore, we obtain

$$\begin{aligned}\|v\|^2_{H^2(M)} &= \sum_i \sum_{|\alpha|\le 2} \int_{U_i} \chi_i |\partial^\alpha v|^2 dx\\ &\le C\sum_i \sum_{|\alpha|\le 2} \int_{U_i} \left(|\partial^\alpha(\sqrt{\chi_i}v)|^2 + |(\partial\sqrt{\chi_i})(\partial v)|^2 + |(\partial^\alpha\sqrt{\chi_i})v|^2\right) dx\\ &\le C\sum_i \|\sqrt{\chi_i}v\|^2_{H^2(U_i)} + C\sum_i \sum_{|\alpha|\le 1} \int_{U_i} |\sqrt{\chi_i'}\partial^\alpha v|^2 dx\\ &\le C\sum_i \|\sqrt{\chi_i}v\|^2_{H^2(U_i)} + C\sum_i \sum_{|\alpha|\le 1} \int_{U_i} \Big(|\partial^\alpha(\sqrt{\chi_i'}v)|^2 + |(\partial^\alpha\sqrt{\chi_i'})v|^2\Big) dx\\ &\le C\sum_i \|\sqrt{\chi_i}v\|^2_{H^2(U_i)} + C\sum_i \|\sqrt{\chi_i'}v\|^2_{H^1(U_i)} + C\sum_i \|\sqrt{\chi_i''}v\|^2_{L^2(U_i)},\end{aligned}$$

where χ_i'' are chosen with $\chi_i'' = 1$ in a neighborhood of $\operatorname{supp}\chi_i'$ and $\operatorname{supp}\chi_i'' \subset U_i$. Setting $\eta_i' := \dfrac{\sqrt{\chi_i''}}{\sum_i \sqrt{\chi_i''}}$ yields

$$\|v\|_{H^2(M)} \le C\sum_i \Big(\|\sqrt{\chi_i}v\|_{H^2(U_i)} + \|\sqrt{\eta_i}v\|_{H^1(U_i)} + \|\sqrt{\eta_i'}v\|_{L^2(U_i)}\Big). \tag{4.19}$$

Combining (4.18) and (4.19) yields

$$\|v\|_{H^2(M)} \le C\left(\|Av\|_{L^2(M)} + \|v\|_{L^2(M)}\right).$$

□

CHAPTER 5

One-dimensional Saint-Venant equations

1. Introduction and main result

We consider

$$
(5.1)\qquad \begin{cases} Au := (\partial_t^2 - \partial_x^2 + a\partial_x\partial_t)u = H(x,t) & \text{in } Q_+ := (-\ell,\ell)\times(0,T), \\ u(\cdot,0) = \partial_t u(\cdot,0) = 0 & \text{on } (-\ell,\ell), \\ u(-\ell,\cdot) = u(\ell,\cdot) = 0 & \text{on } (0,T), \end{cases}
$$

where $a > 0$, $\ell > 0$ and $T > 0$ are positive constants and H is the source term. The differential operator A has the form of a one-dimensional wave operator plus the mixed derivative term $a\partial_x\partial_t$. This term appears when we linearize the one-dimensional Saint-Venant equation, which is the equation introduced by Saint-Venant in [61] to describe unsteady water flow in channels. The one-dimensional Saint-Venant equation comprises continuity and momentum equations. Their formulations and physical meanings are written in Cunge, Holly, and Verwey [16]. Even though one-dimensional flow does not exist in nature, mathematically speaking, it is important to consider the simplified equation and observe its properties. We consider the uniqueness and stability for the inverse source problem to determine H on $(-\ell,\ell)$ from the boundary observation data of the solution to (5.1). The argument is based on the Carleman estimate and the Bukhgeim–Klibanov method in [8]. The Carleman estimate was first introduced in Carleman [12] to prove the unique continuation property for the elliptic operator whose coefficients are not necessarily analytic. Using the Carleman estimate, Bukhugeim and Klibanov proved global uniqueness results for multidimensional coefficient inverse problems. This methodology is widely applicable to various partial differential equations provided that we can prove the Carleman estimate for the considered equations. For a hyperbolic equation, Imanuvilov and Yamamoto [34] considered the global Lipschitz stability for wave equations through interior observations. Baudouin, De Buhan, and Ervedoza [4] proved the global Carleman estimate for the wave equation and considered its applications to controllability, inverse problems, and reconstructions. Bellassoued and Yamamoto [6] considered the inverse source and coefficient problems for wave equation on a compact Riemannian manifold with a boundary.
In proving the Carleman estimate for the operator A, the main difficulties lie in the existence of the mixed derivative term $a\partial_x\partial_t$. There are also difficulties when we apply the Carleman estimate to the extended solution to (5.1). In the usual case of the wave equation, an evenly extended solution with respect to time t satisfies the wave equation as well. However, considering (5.1), the evenly extended solution no longer satisfies the equation. We therefore need to consider a different extension.

To prove the global Lipschitz stability for inverse source problems of the hyperbolic partial differential equation, the observation time should be given for the distant wave to reach the boundary owing to the finite propagation speed. We define constants to describe this situation mathematically. Let $x_0 \in [-\ell,\ell]^c$ be a given point and

$$T_0 := \frac{1}{\sqrt{\rho}} \max_{-\ell\le x\le \ell} |x-x_0|,$$

where $\rho := \left(\frac{\sqrt{a^2+4}-a}{2}\right)^2$ is the square of the wave speed.

THEOREM 5.1. *Assume* $H(x,t) = R(x,t)f(x)$, *where*

$$f \in L^2(-\ell,\ell),\ R\in H^1(0,T;L^\infty(-\ell,\ell)),$$
$$f(x) = f(-x),\ R(x,0) = R(-x,0) \text{ a.e. } x\in(-\ell,\ell),$$

and

$$\exists m_0 > 0 \text{ s.t. } |R(x,0)| \ge m_0 > 0 \text{ a.e. } x\in(-\ell,\ell).$$

Let $T > T_0$. *We assume there exists a unique solution* u *to* (5.1) *in the class*

$$u \in \bigcap_{k=0}^{2} H^{3-k}(0,T;H^k(-\ell,\ell)).$$

There then exists a constant $C > 0$ *that is independent of* u *and* f *such that*

$$\|f\|_{L^2(-\ell,\ell)} \le C\left(\|\partial_x\partial_t u(\ell,\cdot)\|_{L^2(0,T)} + \|\partial_x\partial_t u(-\ell,\cdot)\|_{L^2(0,T)}\right).$$

2. Proof of Theorem 5.1

2.1. Preliminary. To prove Theorem 5.1 using the Bukhugeim–Klibanov method, we need to prove the Carleman estimate for the operator A in extended domain $Q_\pm$. We consider

$$A = \partial_t^2 - \partial_x^2 + a\partial_t\partial_x,\ Q_\pm := (-\ell,\ell)\times(-T,T),$$

where $a > 0$ is a constant. The next proposition is the global Carleman estimate, whose proof is postponed to section 3.

PROPOSITION 5.2. *Choose*

$$x_0 \notin [-\ell,\ell],$$

and β *such that*

$$0 < \beta < \rho,$$

and set

$$\psi(x,t) := |x-x_0|^2 - \beta t^2, \quad \varphi(x,t) := e^{\gamma\psi(x,t)},$$
$$\sigma(x,t) := s\gamma\varphi(x,t),\ (x,t)\in\overline{Q_\pm},$$

where $\gamma > 0$ *and* $s > 0$ *is some parameters. There then exists a constant* $\gamma_* > 0$ *such that for all* $\gamma \ge \gamma_*$, *the following holds. There exist constants* $s_* = s_*(\gamma)$ *and* $C = C(s_*)$ *such that*

$$\int_{Q_\pm} e^{2s\varphi}(\sigma|\partial_x u|^2 + \sigma|\partial_t u|^2 + \sigma^3|u|^2)dxdt$$
$$\le C\int_{Q_\pm} e^{2s\varphi}|Au|^2 dxdt$$
$$+ C\int_{-T}^{T}\left(e^{2s\varphi(\ell,t)}\sigma(\ell,t)|\partial_x u(\ell,t)|^2 + e^{2s\varphi(-\ell,t)}\sigma(-\ell,t)|\partial_x u(-\ell,t)|^2\right)dt,$$

for all $s > s_*$ *and* $u \in \bigcap_{k=0}^{2} H^{2-k}(-T,T;H^k(-\ell,\ell))$ *such that* $u(\cdot,\pm T) = \partial_t u(\cdot,\pm T) = 0$ *on* $(-\ell,\ell)$ *and* $u(\pm\ell,\cdot) = 0$ *on* $(-T,T)$.

Proving our main theorem requires several energy estimates as follows.

LEMMA 5.3. *Assume* $f \in L^2(-\ell,\ell)$, $R \in L^2(-T,T;L^\infty(-\ell,\ell))$ *and* $u_1 \in L^2(-\ell,\ell)$. *Let* $u \in \bigcap_{k=0}^{2} H^{2-k}(-T,T;H^k(-\ell,\ell))$ *be a solution to*

$$\begin{cases} Au = (\partial_t^2 - \partial_x^2 + a\partial_x\partial_t)u = R(x,t)f(x) & \text{in } Q_\pm, \\ u(\cdot,0) = 0,\ \partial_t u(\cdot,0) = u_1 & \text{on } (-\ell,\ell), \\ u(-\ell,\cdot) = u(\ell,\cdot) = 0 & \text{on } (-T,T). \end{cases}$$

There then exists a constant C *such that*

$\|\partial_t u(\cdot,t)\|^2_{L^2(-\ell,\ell)} + \|\partial_x u(\cdot,t)\|^2_{L^2(-\ell,\ell)} \le C(\|f\|^2_{L^2(-\ell,\ell)} + \|u_1\|^2_{L^2(-\ell,\ell)})$ *a.e.* $t \in (-T,T)$ *holds.*

PROOF. Set $E(t) := \|\partial_t u(\cdot,t)\|^2_{L^2(-\ell,\ell)} + \|\partial_x u(\cdot,t)\|^2_{L^2(-\ell,\ell)}$. Multiplying the first equation by $\partial_t u$ and integrating over $(-\ell,\ell)$ yield

$$\frac{d}{dt}E(t) \le E(t) + \|R(\cdot,t)f\|^2_{L^2(-\ell,\ell)}$$

and so

$$\frac{d}{dt}(e^{-t}E(t)) \le e^{-t}\|R(\cdot,t)f\|^2_{L^2(-\ell,\ell)}.$$

Furthermore, integration over $(0,t)$ yields

$$E(t) \le C(\|R\|^2_{L^2(-T,T;L^\infty(-\ell,\ell))}\|f\|^2_{L^2(-\ell,\ell)} + \|u_1\|^2_{L^2(-\ell,\ell)}).$$

□

LEMMA 5.4. *Assume* $H \in L^2(-T,T;L^2(-\ell,\ell))$. *Let* $z \in \bigcap_{k=0}^{2} H^{2-k}(-T,T;H^k(-\ell,\ell))$ *be a solution to*

$$\begin{cases} Az = H(x,t) & \text{in } Q_\pm, \\ z(\cdot,0) = 0,\ \partial_t z(\cdot,0) = z_1 & \text{on } (-\ell,\ell), \\ z(-\ell,\cdot) = z(\ell,\cdot) = 0 & \text{on } (-T,T), \end{cases}$$

and

$$z(\cdot,\pm T) = \partial_t z(\cdot,\pm T) = 0 \text{ on } (-\ell,\ell).$$

There then exists a constant $C > 0$ *such that*

$$\|z_1\|^2_{L^2(-\ell,\ell)} \le C\|H\partial_t z\|_{L^1(Q_\pm)}$$

holds.

PROOF. Multiplying the first equation by $2\partial_t z$ and integrating over $Q_+ = (-\ell,\ell) \times (0,T)$, we have

$$\int_{Q_+} \partial_t|\partial_t z|^2 + \int_{Q_+} \partial_t|\partial_x z|^2 + a\int_{Q_+} \partial_x|\partial_t z|^2 = 2\int_{Q_+} H\partial_t z.$$

Hence, we get

$$\|z_1\|^2_{L^2(-\ell,\ell)} \le C\|H\partial_t z\|_{L^1(Q_+)}.$$

The proof for $Q_- := (-\ell,\ell)\times(-T,0)$ is similar. $\square$

2.2. Proof of the main theorem.

PROOF OF THEOREM 5.1. We use the weight function $\psi(x,t) = |x-x_0|^2-\beta t^2$ for $x_0 \in [-\ell,\ell]^c$. We assume $T > T_0$, i.e., $\rho T^2 > \rho T_0^2 = \max_{-\ell\le x\le\ell}|x-x_0|^2$, and there thus exists $0<\beta<\rho$ such that

$$\max_{-\ell\le x\le\ell}|x-x_0|^2 < \beta T^2 (<\rho T^2).$$

There then exists $0<\varepsilon(<\frac{T}{2})$ such that

$$\max_{-\ell\le x\le\ell}|x-x_0|^2 < \beta (T-2\varepsilon)^2.$$

Thus, for all $x\in[-\ell,\ell]$ and $t\in[-T,-T+2\varepsilon]\cap[T-2\varepsilon,T]$, we have

$$\varphi(x,t) = e^{\gamma\psi(x,t)} < 1.$$

Let u be the solution in the class

$$u \in \bigcap_{k=0}^{2} H^{3-k}(0,T;H^k(-\ell,\ell))$$

and take the extension of u to $(-T,T)$:

$$u(x,t) = \begin{cases} u(x,t) & \text{in } Q_+ := (-\ell,\ell)\times(0,T),\\ u(-x,-t) & \text{in } Q_- := (-\ell,\ell)\times(-T,0).\end{cases}$$

We also extend R:

$$R(x,t) = \begin{cases} R(x,t) & \text{in } Q_+,\\ R(-x,-t) & \text{in } Q_-.\end{cases}$$

We can then prove $u \in \bigcap_{k=0}^{2} H^{3-k}(-T,T;H^k(-\ell,\ell))$ and $R\in H^1(-T,T;L^\infty(-\ell,\ell))$. Indeed, we assume $u(\cdot,0) = \partial_t u(\cdot,0) = 0$ and we thus have

$$\partial_t u(x,t) = \begin{cases} \partial_t u(x,t) & \text{in } Q_+,\\ -\partial_t u(-x,-t) & \text{in } Q_-,\end{cases}$$

and

$$\partial_t^2 u(x,t) = \begin{cases} \partial_t^2 u(x,t) & \text{in } Q_+,\\ \partial_t^2 u(-x,-t) & \text{in } Q_-.\end{cases}$$

Furthermore, we assume the symmetry of the source term f and $R(\cdot,0)$ and thus get

$$\partial_t^3 u(x,t) = \begin{cases} \partial_t^3 u(x,t) & \text{in } Q_+,\\ -\partial_t^3 u(-x,-t) & \text{in } Q_-.\end{cases}$$

Considering R, we have

$$\partial_t R(x,t) = \begin{cases} \partial_t R(x,t) & \text{in } Q_+,\\ -\partial_t R(-x,-t) & \text{in } Q_-,\end{cases}$$

owing to the symmetry of $R(\cdot, 0)$. This extended u satisfies

$$\begin{cases} Au = (\partial_t^2 - \partial_x^2 + a\partial_x\partial_t)u = R(x,t)f(x) & \text{in } Q_\pm, \\ u(\cdot,0) = \partial_t u(\cdot,0) = 0 & \text{on } (-\ell,\ell), \\ u(-\ell,\cdot) = u(\ell,\cdot) = 0 & \text{on } (-T,T). \end{cases}$$

From Lemma 5.3, we have

$$\|\partial_t u(\cdot,t)\|^2_{L^2(-\ell,\ell)} + \|\partial_x u(\cdot,t)\|^2_{L^2(-\ell,\ell)} \le C\|f\|^2_{L^2(-\ell,\ell)} \text{ a.e. } t \in (-T,T). \tag{5.2}$$

Let $v := \partial_t u$. v satisfies

$$\begin{cases} Av = (\partial_t^2 - \partial_x^2 + a\partial_x\partial_t)v = \partial_t R(x,t)f(x) & \text{in } Q_\pm, \\ v(\cdot,0) = 0,\ \partial_t v(\cdot,0) = R(\cdot,0)f & \text{on } (-\ell,\ell), \\ v(-\ell,\cdot) = v(\ell,\cdot) = 0 & \text{on } (-T,T). \end{cases}$$

Also from Lemma 5.3, we have

$$\|\partial_t v(\cdot,t)\|^2_{L^2(\ell,\ell)} + \|\partial_x v(\cdot,t)\|^2_{L^2(-\ell,\ell)} \le C\|f\|^2_{L^2(-\ell,\ell)} \text{ a.e. } t \in (-T,T). \tag{5.3}$$

We define a cut-off function η satisfying $0 \le \eta(t) \le 1$:

$$\eta(t) := \begin{cases} 1, & |t| \le T - 2\varepsilon, \\ 0, & |t| \ge T - \varepsilon. \end{cases}$$

We set $w := \eta\partial_t u = \eta v$. w satisfies

$$\begin{cases} Aw = \eta\partial_t Rf + 2\partial_t\eta\partial_t v + a\partial_t\eta\partial_x v + \partial_t^2\eta v & \text{in } Q_\pm, \\ w(\cdot,0) = 0,\ \partial_t w(\cdot,0) = R(\cdot,0)f & \text{on } (-\ell,\ell), \\ w(-\ell,\cdot) = w(\ell,\cdot) = 0 & \text{on } (-T,T). \end{cases}$$

Moreover,

$$w(\cdot,\pm T) = \partial_t w(\cdot,\pm T) = 0, \text{ on } (-\ell,\ell)$$

holds. We can therefore apply Proposition 5.2 to w to obtain

$$\int_{Q_\pm} e^{2s\varphi}(s|\partial_x w|^2 + s|\partial_t w|^2 + s^3|w|^2)$$
$$\le C\int_{Q_\pm} e^{2s\varphi}|\partial_t Rf|^2 + C\int_{Q_\pm} e^{2s\varphi}(|\partial_t\eta\partial_t v|^2 + |\partial_t\eta\partial_x v|^2 + |\partial_t^2\eta v|^2) + Cse^{Cs}\mathcal{D}^2$$

for all $s \ge s_*$, where

$$\begin{aligned} \mathcal{D}^2 :&= \int_{-T}^{T} \left(|\partial_x w(\ell,t)|^2 + |\partial_x w(-\ell,t)|^2\right) dt \\ &\le \int_{-T}^{T} \left(|\partial_x\partial_t u(\ell,t)|^2 + |\partial_x\partial_t u(-\ell,t)|^2\right) dt. \end{aligned}$$

We here consider sufficiently large $\gamma > \gamma_*$ as a fixed constant. Considering the second, third, and fourth terms on the right-hand-side and using (5.2) and (5.3), we get

$$\int_{Q_\pm} e^{2s\varphi}(|\partial_t\eta\partial_t v|^2 + |\partial_t\eta\partial_x v|^2 + |\partial_t^2\eta v|^2) \le Ce^{2s}\|f\|^2_{L^2(-\ell,\ell)}$$

because $\operatorname{supp}(\partial_t\eta)$, $\operatorname{supp}(\partial_t^2\eta) \subset [-T+\varepsilon, -T+2\varepsilon] \cup [T-2\varepsilon, T-\varepsilon]$, and for all $x \in [-\ell, \ell]$ and $t \in [-T+\varepsilon, -T+2\varepsilon] \cap [T-2\varepsilon, T-\varepsilon]$, we have $\varphi(x,t) = e^{\gamma\psi(x,t)} < 1$. We therefore have

$$\begin{aligned}(5.4)\qquad &\int_{Q_\pm} e^{2s\varphi}(s|\partial_x w|^2 + s|\partial_t w|^2 + s^3|w|^2) \\ &\le C\int_{Q_\pm} e^{2s\varphi}|\partial_t Rf|^2 + Ce^{2s}\|f\|^2_{L^2(-\ell,\ell)} + Cse^{Cs}\mathcal{D}^2,\end{aligned}$$

for all $s \ge s_*$.

We next set $z := e^{s\varphi}w$ and z satisfies

$$\begin{cases} Az = e^{s\varphi}(Aw + G(x,t)) & \text{in } Q_\pm, \\ z(\cdot,0) = 0,\ \partial_t z(\cdot,0) = e^{s\varphi(\cdot,0)}R(\cdot,0)f & \text{on } (-\ell,\ell), \\ z(-\ell,\cdot) = z(\ell,\cdot) = 0 & \text{on } (-T,T), \end{cases}$$

where

$$\begin{aligned} G :=\ & (2s\partial_t\varphi + as\partial_x\varphi)\partial_t w + (-2s\partial_x\varphi + as\partial_t\varphi)\partial_x w \\ & + (s\partial_t^2\varphi + s^2|\partial_t\varphi|^2 - s\partial_x^2\varphi - s^2|\partial_x\varphi|^2 + as\partial_x\partial_t\varphi + as^2\partial_x\varphi\partial_t\varphi)w. \end{aligned}$$

From Lemma 5.4, we have

$$\begin{aligned}(5.5)\qquad \|e^{s\varphi(x,0)}f\|^2_{L^2(-\ell,\ell)} \le\ & C\int_{Q_\pm} e^{2s\varphi}|\partial_t z|^2 + C\int_{Q_\pm} e^{2s\varphi}|\partial_t Rf|^2 \\ & + Ce^{2s}\|f\|^2_{L^2(-\ell,\ell)} + C\|e^{s\varphi}G\partial_t z\|_{L^1(Q_\pm)}\end{aligned}$$

and we also find

$$e^{s\varphi}G\partial_t z \le Ce^{2s\varphi}(s|\partial_t w| + s|\partial_x w| + s^2|w|)(|\partial_t w| + s|w|)$$

and thus obtain

$$\|e^{s\varphi}G\partial_t z\|_{L^1(Q_\pm)} \le C\int_{Q_\pm} e^{2s\varphi}(s|\partial_x w|^2 + s|\partial_t w|^2 + s^3|w|^2).$$

Furthermore, we have

$$\begin{aligned}\int_{Q_\pm} e^{2s\varphi}|\partial_t Rf|^2 &\le C\int_{-\ell}^{\ell} e^{2s\varphi(x,0)}|f(x)|^2\left(\int_{-T}^{T} e^{-2s(\varphi(x,0)-\varphi(x,t))}dt\right)dx \\ &\le C\int_{-\ell}^{\ell} e^{2s\varphi(x,0)}|f(x)|^2\left(\int_{-T}^{T} e^{-2se^{\gamma|x-x_0|^2}(1-e^{-\gamma\beta t^2})}dt\right)dx \\ &\le o(1)\|e^{s\varphi(x,0)}f\|^2_{L^2(-\ell,\ell)} \quad \text{as } s\to\infty,\end{aligned}$$

by the Lebesgue's dominated convergence theorem. We apply this estimate and (5.4) to (5.5) to get

$$\|e^{s\varphi(x,0)}f\|^2_{L^2(-\ell,\ell)} \le Ce^{2s}\|f\|^2_{L^2(-\ell,\ell)} + Cse^{Cs}\mathcal{D}^2.$$

There exists $\kappa > 1$ such that for all $x \in [-\ell,\ell]$, $\varphi(x,0) = e^{\gamma|x-x_0|} \ge \kappa > 1$ holds. Therefore, taking sufficiently large $s > s^*$, we finally have

$$\|f\|_{L^2(-\ell,\ell)} \le C\mathcal{D}$$

for a constant $C > 0$ independent of f. □

3. Proof of Proposition 5.2

PROOF OF PROPOSITION 5.2. For simplicity, we write

$$p(x) := \partial_x \psi(x,t), \quad q(t) := \partial_t \psi(x,t).$$

Simple calculation yields

$$\begin{gathered}\partial_x \varphi = \gamma p \varphi, \ \partial_t \varphi = \gamma q \varphi \\ \partial_x^2 \varphi = (\gamma^2 p^2 + 2\gamma)\varphi, \ \partial_t^2 \varphi = (\gamma^2 q^2 - 2\beta\gamma)\varphi \\ \partial_x \partial_t \varphi = \gamma^2 pq\varphi.\end{gathered}$$

We introduce

$$w := e^{s\varphi} u, \quad Pw := e^{s\varphi} A e^{-s\varphi} w.$$

Then,

$$\begin{aligned}Pw = {} & \partial_t^2 w - \partial_x^2 w + 2s(\partial_x \varphi \partial_x w - \partial_t \varphi \partial_t w) - s^2(|\partial_x \varphi|^2 - a\partial_x \varphi \partial_t \varphi - |\partial_t \varphi|^2)w \\ & + s(\partial_x^2 \varphi - a\partial_x \partial_t \varphi - \partial_t^2 \varphi)w + a\partial_x \partial_t w - as(\partial_t \varphi \partial_x w + \partial_x \varphi \partial_t w).\end{aligned}$$

We decompose Pw into $P_+ w$ and $P_- w$ as follows.

$$P_+ w := \partial_t^2 w - \partial_x^2 w + a\partial_x \partial_t w - s^2(|\partial_x \varphi|^2 - |\partial_t \varphi|^2 - a\partial_x \varphi \partial_t \varphi)w$$

and

$$P_- w := s(2\partial_x \varphi - a\partial_t \varphi)\partial_x w - s(2\partial_t \varphi + a\partial_x \varphi)\partial_t w + s(\partial_x^2 \varphi - a\partial_x \partial_t \varphi - \partial_t^2 \varphi)w.$$

Note that $Pw = P_+ w + P_- w$. We wish to make the lower bound of $\|Pw\|^2_{L^2(Q_\pm)}$, and we therefore try to estimate $(P_+ w, P_- w)_{L^2(Q_\pm)}$.

$$\begin{aligned}& (P_+ w, P_- w)_{L^2(Q_\pm)} \\ & = \int_{Q_\pm} \partial_t^2 w \cdot s(2\partial_x \varphi - a\partial_t \varphi)\partial_x w - \int_{Q_\pm} \partial_t^2 w \cdot s(2\partial_t \varphi + a\partial_x \varphi)\partial_t w \\ & \quad + \int_{Q_\pm} \partial_t^2 w \cdot s(\partial_x^2 \varphi - a\partial_x \partial_t \varphi - \partial_t^2 \varphi)w - \int_{Q_\pm} \partial_x^2 w \cdot s(2\partial_x \varphi - a\partial_t \varphi)\partial_x w \\ & \quad + \int_{Q_\pm} \partial_x^2 w \cdot s(2\partial_t \varphi + a\partial_x \varphi)\partial_t w - \int_{Q_\pm} \partial_x^2 w \cdot s(\partial_x^2 \varphi - a\partial_x \partial_t \varphi - \partial_t^2 \varphi)w \\ & \quad + \int_{Q_\pm} a\partial_x \partial_t w \cdot s(2\partial_x \varphi - a\partial_t \varphi)\partial_x w - \int_{Q_\pm} a\partial_x \partial_t w \cdot s(2\partial_t \varphi + a\partial_x \varphi)\partial_t w \\ & \quad + \int_{Q_\pm} a\partial_x \partial_t w \cdot s(\partial_x^2 \varphi - a\partial_x \partial_t \varphi - \partial_t^2 \varphi)w \\ & \quad - \int_{Q_\pm} s^2(|\partial_x \varphi|^2 - |\partial_t \varphi|^2 - a\partial_x \varphi \partial_t \varphi)w \cdot s(2\partial_x \varphi - a\partial_t \varphi)\partial_x w \\ & \quad + \int_{Q_\pm} s^2(|\partial_x \varphi|^2 - |\partial_t \varphi|^2 - a\partial_x \varphi \partial_t \varphi)w \cdot s(2\partial_t \varphi + a\partial_x \varphi)\partial_t w \\ & \quad - \int_{Q_\pm} s^2(|\partial_x \varphi|^2 - |\partial_t \varphi|^2 - a\partial_x \varphi \partial_t \varphi)w \cdot s(\partial_x^2 \varphi - a\partial_x \partial_t \varphi - \partial_t^2 \varphi)w \\ & =: \sum_{k=1}^{12} J_k.\end{aligned}$$

Through integration by parts, we get

$$\begin{aligned}
J_1 &= -s\int_{Q_\pm}(2\partial_x\varphi - a\partial_t\varphi)\frac{1}{2}\partial_x(|\partial_t w|^2) - s\int_{Q_\pm}\partial_t(2\partial_x\varphi - a\partial_t\varphi)\partial_t w\partial_x w\\
&= \frac{s}{2}\int_{Q_\pm}\partial_x(2\partial_x\varphi - a\partial_t\varphi)|\partial_t w|^2 - s\int_{Q_\pm}\partial_t(2\partial_x\varphi - a\partial_t\varphi)\partial_t w\partial_x w,
\end{aligned}$$

$$J_2 = -s\int_{Q_\pm}(2\partial_t\varphi + a\partial_x\varphi)\frac{1}{2}\partial_t(|\partial_t w|^2) = \frac{s}{2}\int_{Q_\pm}\partial_t(2\partial_t\varphi + a\partial_x\varphi)|\partial_t w|^2,$$

$$\begin{aligned}
J_3 &= -s\int_{Q_\pm}(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|\partial_t w|^2 - s\int_{Q_\pm}\partial_t(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\frac{1}{2}\partial_t(|w|^2)\\
&= -s\int_{Q_\pm}(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|\partial_t w|^2 + \frac{s}{2}\int_{Q_\pm}\partial_t^2(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|w|^2,
\end{aligned}$$

$$J_4 = \frac{s}{2}\int_{Q_\pm}\partial_x(2\partial_x\varphi - a\partial_t\varphi)|\partial_x w|^2 - \frac{s}{2}\int_{-T}^{T}\left[(2\partial_x\varphi - a\partial_t\varphi)|\partial_x w|^2\right]_{x=-\ell}^{\ell}dt,$$

$$\begin{aligned}
J_5 &= -s\int_{Q_\pm}\partial_x(2\partial_t\varphi + a\partial_x\varphi)\partial_x w\partial_t w - s\int_{Q_\pm}(2\partial_t\varphi + a\partial_x\varphi)\frac{1}{2}\partial_t(|\partial_x w|^2)\\
&= -s\int_{Q_\pm}\partial_x(2\partial_t\varphi + a\partial_x\varphi)\partial_x w\partial_t w + \frac{s}{2}\int_{Q_\pm}\partial_t(2\partial_t\varphi + a\partial_x\varphi)|\partial_x w|^2,
\end{aligned}$$

$$\begin{aligned}
J_6 &= s\int_{Q_\pm}(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|\partial_x w|^2 + s\int_{Q_\pm}\partial_x(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\frac{1}{2}\partial_x(|w|^2)\\
&= s\int_{Q_\pm}(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|\partial_x w|^2 - \frac{s}{2}\int_{Q_\pm}\partial_x^2(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|w|^2,
\end{aligned}$$

$$J_7 = -\frac{s}{2}\int_{Q_\pm}a\partial_t(2\partial_x\varphi - a\partial_t\varphi)|\partial_x w|^2,$$

$$J_8 = \frac{s}{2}\int_{Q_\pm}a\partial_x(2\partial_t\varphi + a\partial_x\varphi)|\partial_t w|^2,$$

$$\begin{aligned}
J_9 &= -s\int_{Q_\pm}a(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\partial_x w\partial_t w\\
&\quad - s\int_{Q_\pm}a\partial_x(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\frac{1}{2}\partial_t(|w|^2)\\
&= -s\int_{Q_\pm}a(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\partial_x w\partial_t w\\
&\quad + \frac{s}{2}\int_{Q_\pm}a\partial_t\partial_x(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|w|^2,
\end{aligned}$$

$$J_{10} = \frac{s^3}{2}\int_{Q_\pm}\partial_x((|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)(2\partial_x\varphi - a\partial_t\varphi))|w|^2,$$

$$J_{11} = -\frac{s^3}{2}\int_{Q_\pm}\partial_t((|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)(2\partial_t\varphi + a\partial_x\varphi))|w|^2,$$

$$J_{12} = -s^3\int_{Q_\pm}(|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|w|^2.$$

First, we consider a part of the sum $\sum_{k=1}^{9} J_k$. Set $\sigma(x,t) := s\gamma\varphi(x,t)$ and

$$\mathcal{B} := \frac{s}{2}\int_{-T}^{T} \left[(2\partial_x\varphi - a\partial_t\varphi)|\partial_x w|^2\right]_{x=-\ell}^{\ell} dt.$$

$$\begin{aligned}
\sum_{k=1}^{9} J_k
&= \int_{Q_\pm} \left[\sigma\gamma(2q^2 + 2apq + \frac{a^2}{2}p^2) + \sigma(-4\beta + a^2)\right] |\partial_t w|^2 \\
&\quad + \int_{Q_\pm} \left[\sigma\gamma(2p^2 - 2apq + \frac{a^2}{2}q^2) + \sigma(4 - \beta a^2)\right] |\partial_x w|^2 \\
&\quad + \int_{Q_\pm} [\sigma\gamma((a^2-4)pq + 2a(q^2-p^2)) + \sigma(-4\beta a - 4a)]\partial_t w\partial_x w \\
&\quad + \int_{Q_\pm} O(s\gamma^4\varphi)|w|^2 - \mathcal{B} \\
&= 2\int_{Q_\pm} \sigma\gamma\left((q + \frac{ap}{2})\partial_t w - (p - \frac{aq}{2})\partial_x w)\right)^2 \\
&\quad + \int_{Q_\pm} \sigma[(-4\beta + a^2)|\partial_t w|^2 + (4 - \beta a^2)|\partial_x w|^2 - 4a(\beta+1)\partial_t w\partial_x w] \\
&\quad + \int_{Q_\pm} O(s\gamma^4\varphi)|w|^2 - \mathcal{B} \\
&\geq \int_{Q_\pm} \sigma[(-4\beta + a^2)|\partial_t w|^2 + (4 - \beta a^2)|\partial_x w|^2 - 4a(\beta+1)\partial_t w\partial_x w] \\
&\quad + \int_{Q_\pm} O(s\gamma^4\varphi)|w|^2 - \mathcal{B}.
\end{aligned} \tag{5.6}$$

To estimate the terms $\sigma \times$ (first order terms), we use the next energy inequality.

$$\begin{aligned}
(P_+w, -\sigma w)_{L^2(Q_\pm)} &= \int_{Q_\pm} P_+w \cdot (-\sigma w) \\
&= -\int_{Q_\pm} \sigma\partial_t^2 ww + \int_{Q_\pm} \sigma\partial_x^2 ww - \int_{Q_\pm} a\sigma\partial_x\partial_t ww \\
&\quad + \int_{Q_\pm} \sigma s^2(|\partial_x\varphi|^2 - a\partial_x\varphi\partial_t\varphi - |\partial_t\varphi|^2)|w|^2 \\
&=: \sum_{k=1}^{4} I_k.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
I_1 &= \int_{Q_\pm} s\gamma\partial_t\varphi\frac{1}{2}\partial_t(|w|^2) + \int_{Q_\pm} s\gamma\varphi|\partial_t w|^2 \\
&= \int_{Q_\pm} s\gamma\varphi|\partial_t w|^2 - \frac{1}{2}\int_{Q_\pm} s\gamma\partial_t^2\varphi|w|^2,
\end{aligned}$$

$$\begin{aligned}I_2 &= -\int_{Q_\pm} s\gamma\partial_x\varphi\frac{1}{2}\partial_x(|w|^2) - \int_{Q_\pm} s\gamma\varphi|\partial_x w|^2\\ &= -\int_{Q_\pm} s\gamma\varphi|\partial_x w|^2 + \frac{1}{2}\int_{Q_\pm} s\gamma\partial_x^2\varphi|w|^2,\end{aligned}$$

$$I_3 = \int_{Q_\pm} s^3\gamma\varphi(|\partial_x\varphi|^2 - a\partial_t\varphi\partial_x\varphi - |\partial_t\varphi|^2)|w|^2,$$

$$\begin{aligned}I_4 &= \int_{Q_\pm} as\gamma\partial_x\varphi\frac{1}{2}\partial_t(|w|^2) + \int_{Q_\pm} as\gamma\varphi\partial_t w\partial_x w\\ &= -\frac{1}{2}\int_{Q_\pm} as\gamma\partial_t\partial_x\varphi|w|^2 + \int_{Q_\pm} as\gamma\varphi\partial_t w\partial_x w.\end{aligned}$$

Hence, we have

$$\begin{aligned}(5.7)\quad \int_{Q_\pm} P_+w\cdot(-\sigma w) &= \sum_{k=1}^4 I_k\\ &= \int_{Q_\pm}\sigma|\partial_t w|^2 - \int_{Q_\pm}\sigma|\partial_x w|^2 + \int_{Q_\pm} a\sigma\partial_t w\partial_x w\\ &\quad + \int_{Q_\pm}\Big[-\frac{1}{2}s\gamma\partial_t^2\varphi + \frac{1}{2}s\gamma\partial_x^2\varphi\\ &\quad + s^3\gamma\varphi(|\partial_x\varphi|^2 - a\partial_x\varphi\partial_t\varphi - |\partial_t\varphi|^2) - \frac{a}{2}s\gamma\partial_t\partial_x\varphi\Big]|w|^2\\ &= \int_{Q_\pm}\sigma|\partial_t w|^2 - \int_{Q_\pm}\sigma|\partial_x w|^2 + \int_{Q_\pm} a\sigma\partial_t w\partial_x w\\ &\quad + \int_{Q_\pm}\Big[\sigma^3(p^2 - q^2 - apq) + O(s\gamma^3\varphi)\Big]|w|^2.\end{aligned}$$

Let $\mu\in\mathbb{R}$ be a constant to be determined later. Multiplying (5.7) by 2μ, and adding it to (5.6) yields

$$\begin{aligned}&\sum_{k=1}^9 J_k + 2\mu\sum_{k=1}^4 I_k + \mathcal{B}\\ &\geq \int_{Q_\pm}\sigma(-4\beta + a^2 + 2\mu)|\partial_t w|^2 + \int_{Q_\pm}\sigma(4 - \beta a^2 - 2\mu)|\partial_x w|^2\\ &\quad + \int_{Q_\pm}\sigma(-4\beta a - 4a + 2a\mu)\partial_t w\partial_x w\\ &\quad + \int_{Q_\pm}[\sigma^3\cdot 2\mu(p^2 - apq - q^2) + O(s\gamma^4\varphi) + O(s\gamma^3\varphi)]|w|^2\\ &\geq \int_{Q_\pm}\sigma(-4\beta + a^2 + 2\mu - \epsilon|-2\beta a - 2a + a\mu|)|\partial_t w|^2\\ &\quad + \int_{Q_\pm}\sigma(4 - \beta a^2 - 2\mu - \frac{1}{\epsilon}|-2\beta a - 2a + a\mu|)|\partial_x w|^2\\ &\quad + \int_{Q_\pm}[\sigma^3\cdot 2\mu(p^2 - apq - q^2) + O(s\gamma^4\varphi) + O(s\gamma^3\varphi)]|w|^2\end{aligned}$$

for all $\epsilon > 0$ to be determined later. We wish to choose $\epsilon > 0$ such that both coefficients are positive, i.e.,

$$\begin{cases} -4\beta + a^2 + 2\mu - \epsilon|-2\beta a - 2a + a\mu| > 0, \\ 4 - \beta a^2 - 2\mu - \frac{1}{\epsilon}|-2\beta a - 2a + a\mu| > 0 \end{cases} \tag{5.8}$$

holds. This claim follows only if μ satisfies $\mu^2 - 2(\beta+1)\mu + \beta(a^2+4) < 0 \Leftrightarrow (\mu - \beta - 1)^2 < \beta^2 - (a^2+2)\beta + 1$. Therefore, if we choose $\mu := \beta + 1$, by our assumption of $\beta > 0$, this inequality holds and (5.8) also holds. Hence, there exists a positive constant $C > 0$ such that

$$\begin{aligned} &\sum_{k=1}^{9} J_k + 2\mu \sum_{k=1}^{4} I_k + \mathcal{B} \\ &\geq C \int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) \\ &\quad + \int_{Q_\pm} [2\mu\sigma^3(p^2 - apq - q^2) + O(s\gamma^4\varphi) + O(s\gamma^3\varphi)]|w|^2. \end{aligned} \tag{5.9}$$

Finally, we estimate zeroth-order terms. We have

$$\begin{aligned} J_{10} = \int_{Q_\pm} \sigma^3 \Big[&\gamma(3p^4 + \frac{3(a^2-2)}{2}p^2q^2 - \frac{9a}{2}p^3q + \frac{3a}{2}pq^3) \\ &+ (6p^2 + (a^2-2)q^2 - 6apq) \Big] |w|^2, \end{aligned}$$

$$\begin{aligned} J_{11} = \int_{Q_\pm} \sigma^3 \Big[&\gamma(3q^4 + \frac{3(a^2-2)}{2}p^2q^2 - \frac{3a}{2}p^3q + \frac{9a}{2}pq^3) \\ &- ((a^2-2)\beta p^2 + 6\beta q^2 + 6\beta apq) \Big] |w|^2, \end{aligned}$$

$$J_{12} = \int_{Q_\pm} \sigma^3 \Big[-\gamma(p^2 - apq - q^2)^2 - 2(\beta+1)(p^2 - apq - q^2) \Big] |w|^2.$$

Hence, we have

$$\begin{aligned} \sum_{k=10}^{12} J_k = \int_{Q_\pm} \sigma^3 \Big[&\gamma \cdot 2(p^2 - apq - q^2)^2 \\ &+ (4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq \Big] |w|^2. \end{aligned}$$

Adding this equality to (5.9) yields

$$\begin{aligned} &\sum_{k=1}^{12} J_k + 2\mu \sum_{k=1}^{4} I_k + \mathcal{B} \\ &= (P_+ w, P_- w)_{L^2(Q_\pm)} + 2(P_+ w, -\mu\sigma w)_{L^2(Q_\pm)} + \mathcal{B} \\ &\geq C \int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) + \int_{Q_\pm} \Big[\sigma^3\gamma \cdot 2(p^2 - apq - q^2)^2 \\ &\quad + \sigma^3\{(4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq + 2\mu(p^2 - apq - q^2)\} \\ &\quad + O(s\gamma^4\varphi) + O(s\gamma^3\varphi) \Big] |w|^2. \end{aligned} \tag{5.10}$$

We consider the coefficient of σ^3. Noting $\mu = \beta + 1$, we have

$$\begin{aligned}
&(4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq + 2\mu(p^2 - apq - q^2)\\
&= (4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq\\
&\quad + (2\mu - 2\nu)(p^2 - apq - q^2) + 2\nu(p^2 - apq - q^2)\\
&\geq 2(\mu - \nu)(p^2 - apq - q^2) + (2\nu + 4 - a^2\beta - \delta a|\nu + 2 + 2\beta|)p^2\\
&\quad + (-2\nu + a^2 - 4\beta - \frac{a}{\delta}|\nu + 2 + 2\beta|)q^2
\end{aligned}$$

for all $\delta > 0$ to be determined later. We wish to choose $\delta > 0$ such that

$$\begin{cases} 2\nu + 4 - a^2\beta - \delta a|\nu + 2 + 2\beta| > 0,\\ -2\nu + a^2 - 4\beta - \frac{a}{\delta}|\nu + 2 + 2\beta| > 0. \end{cases} \tag{5.11}$$

Such δ exists only if ν satisfies $\nu^2 + 2(\beta+1)\nu + \beta(a^2+4) < 0 \Leftrightarrow (\nu + \beta + 1)^2 < \beta^2 - (a^2+2)\beta + 1$. Therefore, if we take $\nu = -\mu = -\beta - 1$, both the above inequality and (5.11) hold. Hence, there exists a constant $C > 0$ such that

$$\begin{aligned}
&(4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq + 2\mu(p^2 - apq - q^2)\\
&\geq 4(1+\beta)(p^2 - apq - q^2) + C(p^2 + q^2).
\end{aligned}$$

We apply this estimate to (5.10) to obtain

$$\begin{aligned}
&\sum_{k=1}^{12} J_k + 2\mu \sum_{k=1}^{4} I_k + \mathcal{B}\\
&\geq C\int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) + \int_{Q_\pm} \Big[\sigma^3\gamma \cdot 2(p^2 - apq - q^2)^2\\
&\quad + \sigma^3\{4(\beta+1)(p^2 - apq - q^2) + C(p^2+q^2)\} + O(s\gamma^4\varphi) + O(s\gamma^3\varphi)\Big]|w|^2\\
&\geq C\int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2)\\
&\quad + \int_{Q_\pm} \left[\sigma^3 \cdot 2\gamma\left((p^2 - apq - q^2) + \frac{\beta+1}{\gamma}\right)^2 + \sigma^3\left\{C(p^2+q^2) - \frac{2(\beta+1)^2}{\gamma}\right\}\right.\\
&\quad \left. + O(s\gamma^4\varphi) + O(s\gamma^3\varphi)\right]|w|^2\\
&\geq C\int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2)\\
&\quad + C\left(\min_{-\ell\leq x\leq \ell} p(x)^2 - \frac{2(\beta+1)^2}{\gamma}\right)\int_{Q_\pm} [\sigma^3 + O(s\gamma^4\varphi) + O(s\gamma^3\varphi)]|w|^2\\
&\geq C\int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) + C\int_{Q_\pm} [\sigma^3 + O(s\gamma^4\varphi) + O(s\gamma^3\varphi)]|w|^2
\end{aligned}$$

for sufficiently large $\gamma > 0$. The Cauchy Schwartz inequality then yields

$$\begin{aligned}
&C\int_{Q_\pm} (\sigma|\partial_t w|^2 + \sigma|\partial_x w|^2 + \sigma^3|w|^2) + \int_{Q_\pm} [O(s\gamma^4\varphi) + O(s\gamma^3\varphi)]|w|^2\\
&\leq \frac{1}{2}\|Pw\|^2_{L^2(Q_\pm)} + \int_{Q_\pm} O(\sigma^2)|w|^2 + \mathcal{B}.
\end{aligned}$$

We can choose $s > 0$ large enough to ensure

$$\int_{Q_\pm} (\sigma|\partial_t w|^2 + \sigma|\partial_x w|^2 + \sigma^3|w|^2) \leq C\|Pw\|^2_{L^2(Q_\pm)} + C\mathcal{B}.$$

We define $w = e^{s\varphi}u$ and thus obtain $\partial_t w = e^{s\varphi}(\partial_t u + s\partial_t\varphi u)$ and $\partial_x w = e^{s\varphi}(\partial_x u + s\partial_x\varphi u)$. Again by choosing $s > 0$ large, we can rewrite the inequality in terms of $u = we^{-s\varphi}$:

$$\int_{Q_\pm} e^{2s\varphi}(\sigma|\partial_x u|^2 + \sigma|\partial_t u|^2 + \sigma^3|u|^2) \leq C\int_{Q_\pm} e^{2s\varphi}|Au|^2 + C\mathcal{B}.$$

Here, we get

$$\begin{aligned}\mathcal{B} &= \int_{-T}^{T} \Big[\sigma(p - \frac{a}{2}q)|\partial_x w|^2\Big]_{-\ell}^{\ell} dt \\ &\leq C\int_{-T}^{T} \Big(e^{2s\varphi(\ell,t)}\sigma(\ell,t)|\partial_x u(\ell,t)|^2 + e^{2s\varphi(-\ell,t)}\sigma(-\ell,t)|\partial_x u(-\ell,t)|^2\Big)\, dt.\end{aligned}$$

□

CHAPTER 6

Non-uniqueness examples

1. Introduction and main result

Let $B_r(0) \subset \mathbb{R}^2$ be an open ball centered at 0 with radius $r > 0$. Henceforth, all functions which appear in this chapter take complex values. We consider the wave equation and Schrödinger equation with time-dependent potential V,

$$Lu + V(x,t)u = 0 \text{ in } \mathbb{R}^3, \tag{6.1}$$

where L denotes $L = \Box := \partial_t^2 - \Delta$ or $L = -i\partial_t - \Delta$. We consider non-uniqueness examples for Cauchy problem with Cauchy data on a non-characteristic surface $\partial B_1(0) \times \mathbb{R}$. Due to the time dependence on the potential V, we have few hopes to guarantee uniqueness for Cauchy problems in general. Indeed, we construct infinitely many examples with different wave numbers at infinity which violate uniqueness based on Kumano-go [47]. In regard to the uniqueness theorems for the wave equation, Schrödinger equation, and more general partial differential equations with variable coefficients, readers are referred to [66] and [60]. They proved uniqueness results by assuming some analyticities on coefficients partially. Readers are also referred to [42, Chapter 2.5] and [35, Chapter 3]. Alinhac and Baouendi [1] constructed non-uniqueness examples for Cauchy problems of general partial differential equations by using geometric optics. We state our main result.

Theorem 6.1. *Let $L = \Box$ or $L = -i\partial_t - \Delta$. There exist infinitely many smooth functions $u \in C^\infty(\mathbb{R}^3)$ and $V \in C^\infty(\mathbb{R}^3)$ which satisfy* (6.1) *and*

$$\operatorname{supp} u = (\mathbb{R}^2 \setminus B_1(0)) \times \mathbb{R},$$
$$\operatorname{supp} V \subset (B_2(0) \setminus B_1(0)) \times \mathbb{R}.$$

Theorem 6.1 relates to the result by Kumano-go [47]. He constructed one example for non-uniqueness when $L = \Box$ in the two-dimensional case based on John's construction [38] using Bessel functions. We construct infinitely many examples with different wave numbers at infinity for both wave and Schrödinger equations by generalizing the result in [47]. We remark that, in our construction, the potential V is not real-valued but complex-valued function, whereas the coefficients are all real-valued with a damping term in Kumano-go's construction [47].

2. Proof of the main result

2.1. Preliminary. We prepare several lemmas regarding an asymptotic behavior of Bessel functions. Their proofs are all presented in section 3.

Lemma 6.2. *Let $\delta \in (0, \frac{1}{2})$, $p \in (0, \frac{2(1-2\delta)}{5})$, $\lambda > 0$, and J_λ be a Bessel function of order λ. We then have the asymptotic formula uniformly for $a \in (0, 1 - \lambda^{-p}]$,*

$$J_\lambda(\lambda a) = (2\pi\lambda \tanh\alpha)^{-\frac{1}{2}} e^{\lambda(\tanh\alpha - \alpha)}(1 + O(\lambda^{-\delta}))$$

as $\lambda \to \infty$, where $\alpha > 0$ is defined by $\cosh \alpha = a^{-1} (> 1)$.

Let $\delta \in (0, \frac{1}{2})$ and $p \in (0, \frac{2(1-2\delta)}{5})$ be fixed. We consider the following assumptions on a positive sequence $\{\lambda_m\}_{m\in\mathbb{N}}$ with $\lambda_m > 0$ for all $m \in \mathbb{N}$.

$$\forall m \in \mathbb{N},\ m^2 \leq \lambda_m^p. \tag{6.2}$$

$$\lambda_{m+1} = \lambda_m(1 + o(1)) \text{ as } m \to \infty. \tag{6.3}$$

We can choose infinitely many positive sequences $\{\lambda_m\}_{m\in\mathbb{N}}$ satisfying (6.2) and (6.3), for instance,

$$\lambda_m = a_n m^n + \sum_{j=0}^{n-1} a_j m^j,$$

where $n \geq \frac{2}{p}$ is a positive integer, and $a_n \geq 1$ and $a_j \geq 0$ are constants for $j = 0, \ldots, n-1$.

Lemma 6.3. *Let $\delta \in (0, \frac{1}{2})$ and $p \in (0, \frac{2(1-2\delta)}{5})$ be constants and $\{\lambda_m\}_{m\in\mathbb{N}}$ be a positive sequence. Set*

$$G_m(r) := J_{\lambda_m}(\lambda_m r),\ r \in [0, 1 - m^{-2}].$$

Assume (6.2). *Then, for $\ell \geq 1$ and $r := 1 - \ell m^{-2}$, we have the asymptotic formula,*

$$G_m(r) = (1 + o(1)) \frac{\sqrt{m}}{(2\pi^2 \ell)^{\frac{1}{4}} \sqrt{\lambda_m}} e^{-(1+o(1))\frac{2\sqrt{2}}{3}\ell^{\frac{3}{2}}\lambda_m m^{-3}}$$

as $m \to \infty$.

Lemma 6.4. *Let $\delta \in (0, \frac{1}{2})$ and $p \in (0, \frac{2(1-2\delta)}{5})$ be constants and let $\{\lambda_m\}_{m\in\mathbb{N}}$ be a positive sequence satisfying* (6.2). *We set*

$$F_m(r) := G_m\left(\frac{k_m}{\lambda_m} r\right),\ m \in \mathbb{N},$$

where $\{k_m\}_{m\in\mathbb{N}}$ is a positive sequence satisfying

$$\frac{k_m}{\lambda_m} = 1 - \frac{1}{m} + O(m^{-3}) \text{ as } m \to \infty, \tag{6.4}$$

and $r_m(s) := 1 + \frac{1}{m} - \frac{s}{m(m+1)}$ for $s \in [0,1]$. Then, F_m satisfies

$$F_m''(r) + \frac{1}{r} F_m'(r) + \left(k_m^2 - \frac{\lambda_m^2}{r^2}\right) F_m(r) = 0 \tag{6.5}$$

and for $s \in [0,1]$,

$$F_m(r_m(s)) = (1 + o(1)) \frac{\sqrt{m} e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s)^{\frac{3}{2}}\lambda_m m^{-3}}}{(2\pi^2(1+s))^{\frac{1}{4}} \sqrt{\lambda_m}} \tag{6.6}$$

as $m \to \infty$. Furthermore, we define γ_{m+1} such that

$$\gamma_{m+1} := \frac{F_m(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(2^{-1}))}.$$

If we assume (6.3), *then there exists $M \in \mathbb{N}$ such that*

$$\gamma_{m+1} \leq e^{-\lambda_m m^{-3}}$$

holds for all $m > M$ and there exist $\mu > 0$, $C > 0$, and $M \in \mathbb{N}$ such that

$$\begin{cases} \gamma_{m+1} F_{m+1}(r_{m+1}(s)) \leq C e^{-\mu\lambda_m m^{-3}} F_m(r_{m+1}(s)) \\ \text{if } s \in [0, \frac{1}{4}], \\ F_m(r_{m+1}(s)) \leq C e^{-\mu\lambda_m m^{-3}} \gamma_{m+1} F_{m+1}(r_{m+1}(s)) \\ \text{if } s \in [\frac{3}{4}, 1] \end{cases} \tag{6.7}$$

holds for all $m > M$.

2.2. Proof of Theorem 6.1.

Proof. Let $\delta \in (0, \frac{1}{2})$ and $p \in (0, \frac{2(1-2\delta)}{5})$ be constants. Let $\{\lambda_m\}_{m\in\mathbb{N}}$ be a positive sequence satisfying (6.2) and (6.3). We remark that (6.3) implies

$$\lambda_m \leq e^{o(m)} \text{ as } m \to \infty. \tag{6.8}$$

Indeed, (6.3) implies there exists sufficiently large $m_0 \in \mathbb{N}$ such that for all $m > m_0$,

$$\left|\frac{\log \lambda_m}{m}\right| \leq \left|\frac{\log \lambda_{m_0}}{m}\right| + \frac{1}{m}\sum_{j=m_0}^{m-1} |\log(1 + r(j))|$$

holds, where $r(j)$ satisfies $\lim_{j\to\infty} |r(j)| = 0$. Hence, (6.8) follows from the well-known argument. For $r > 1$, we set

$$u_m(r, \theta, t) := \begin{cases} F_m(r) e^{i(\lambda_m\theta + k_m t)}, & L = \Box, \\ F_m(r) e^{i(\lambda_m\theta - k_m^2 t)}, & L = -i\partial_t - \Delta. \end{cases}$$

where $\{k_m\}_{m\in\mathbb{N}}$ is a positive sequence satisfying (6.4) and (r, θ) is the polar coordinate in $\mathbb{R}^2$. By (6.5), we obtain

$$Lu_m = 0,$$

because the Laplace operator Δ is written by the polar coordinate,

$$\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2.$$

We define closed intervals I_m and $I_{m,j} \subset I_m$ for $m \in \mathbb{N}$ and $j = 1, 2, 3, 4$ as

$$I_m := \left[1 + \frac{1}{m+1}, 1 + \frac{1}{m}\right]$$

and

$$I_{m,j} := \left[1 + \frac{1}{m} - \frac{j}{4m(m+1)}, 1 + \frac{1}{m} - \frac{j-1}{4m(m+1)}\right].$$

For sufficiently large $M \in \mathbb{N}$ and $m > M$, we define smooth functions

$$A_M(r) := \begin{cases} 1, & r \geq 1 + \frac{1}{M+2} + \frac{1}{4(M+1)(M+2)}, \\ 0, & 0 \leq r \leq 1 + \frac{1}{M+2}, \end{cases}$$

and

$$A_m(r) := \begin{cases} 1, & r \in (I_{m+1} \setminus I_{m+1,4}) \cup (I_m \setminus I_{m,1}), \\ 0, & r \in [0, 1 + \frac{1}{m+2}] \cup (1 + \frac{1}{m}, \infty). \end{cases}$$

We define $u = u(r, \theta, t)$ as

$$u(r, \theta, t) := A_M(r) u_M + \sum_{m=M+1}^{\infty} \gamma_{M+1} \times \cdots \times \gamma_m A_m(r) u_m$$

and set

$$K := [0,1] \cup [1 + \frac{1}{M+1}, \infty) \cup \bigcup_{m=M+1}^{\infty} (I_{m,2} \cup I_{m,3})$$

and

$$V(r,\theta,t) := \begin{cases} 0, & r \in K, \\ -\frac{Lu}{u}, & r \in [0,\infty) \setminus K. \end{cases}$$

Using the chain rule, (6.6), and (6.8), we obtain

$$\begin{aligned} \left| \frac{d^\ell}{dr^\ell} F_m(r) \right| &= m^\ell (m+1)^\ell \left| \frac{d^\ell}{ds^\ell} F_m(r_m(s)) \right| \\ &\le C_\ell \frac{m^\ell (m+1)^\ell \lambda_m^\ell}{m^{3\ell}} (1 + o(1)) \frac{\sqrt{m}}{\sqrt{\lambda_m}} \\ &\quad \times e^{-(1+o(1)) \frac{2\sqrt{2}}{3} (1+s)^{\frac{3}{2}} \lambda_m m^{-3}} \\ &\le C_\ell e^{o(m)} F_m(r_m(s)) \end{aligned}$$

for $r \in I_m$, $\ell \in \mathbb{Z}_{\ge 0}$. Indeed, by (6.8), it follows that

$$\frac{m^\ell (m+1)^\ell \lambda_m^\ell}{m^{3\ell}} \le C_\ell e^{o(m)}.$$

For $r \in I_{m+1}$, using $r_{m+1}(s) = r_m(1 + s + O(m^{-1}))$, we also have

$$\begin{aligned} \left| \frac{d^\ell}{dr^\ell} F_m(r) \right| &= (m+1)^\ell (m+2)^\ell \left| \frac{d^\ell}{ds^\ell} F_m(r_{m+1}(s)) \right| \\ &= (m+1)^\ell (m+2)^\ell \left| \frac{d^\ell}{ds^\ell} F_m(r_m(1 + s + O(m^{-1}))) \right| \\ &\le C_\ell \frac{(m+1)^\ell (m+2)^\ell \lambda_m^\ell}{m^{3\ell}} (1 + o(1)) \frac{\sqrt{m}}{\sqrt{\lambda_m}} \\ &\quad \times e^{-(1+o(1)) \frac{2\sqrt{2}}{3} (2+s+O(m^{-1}))^{\frac{3}{2}} \lambda_m m^{-3}} \\ &\le C_\ell e^{o(m)} F_m(r_{m+1}(s)). \end{aligned}$$

Hence, it follows that for $r \in I_m \cup I_{m+1}$ and $\ell \in \mathbb{Z}_{\ge 0}$,

$$\left| \frac{d^\ell}{dr^\ell} F_m(r) \right| \le C_\ell e^{o(m)} F_m(r). \tag{6.9}$$

On I_{m+1}, using (6.9), (6.4), (6.8), and (6.2), we then have

$$\begin{aligned} |\partial^\beta u(r,\theta,t)| &:= \left| \sum_{|\beta|=\ell} (\partial_r \partial_\theta \partial_t)^\beta u(r,\theta,t) \right| \\ &\le C_\ell \gamma_{M+1} \times \cdots \times \gamma_m \lambda_m^{2\ell} e^{o(m)} (F_m + \gamma_{m+1} F_{m+1}) \\ &\le C_\ell e^{-\lambda_m m^{-3}} e^{o(m)} \\ &< C_\ell e^{-m^2(1+o(m^{-1}))} \\ &\le C_\ell e^{-\frac{1}{2} m^2} \\ &\le C_\ell e^{-\frac{1}{2}((r-1)^{-1}-2)^2} \xrightarrow{r \searrow 1} 0, \end{aligned} \tag{6.10}$$

where we used the estimate obtained by (6.2),

$$\lambda_m m^{-3} \geq m^{\frac{2}{p}-3} > m^2. \tag{6.11}$$

We thus proved u is smooth in $\mathbb{R}^3$.

On $I_{m+1,1}$, since

$$\begin{aligned} |u| &\geq \gamma_{M+1} \times \cdots \times \gamma_m(|u_m| - \gamma_{m+1}|u_{m+1}|) \\ &= \gamma_{M+1} \times \cdots \times \gamma_m(F_m(r_{m+1}(s)) - \gamma_{m+1}F_{m+1}(r_{m+1}(s))) \\ &\geq \gamma_{M+1} \times \cdots \times \gamma_m(1 - Ce^{-\mu\lambda_m m^{-3}})F_m(r_{m+1}(s)) \\ &\geq \gamma_{M+1} \times \cdots \times \gamma_m(1 - Ce^{-\mu m^2})F_m(r_{m+1}(s)) > 0 \end{aligned} \tag{6.12}$$

for $s \in [0, \frac{1}{4}]$ by (6.7) and (6.11), $|u| > 0$ on $I_{m+1,1}$. Similarly, on $I_{m+1,4}$, since

$$\begin{aligned} |u| &\geq \gamma_{M+1} \times \cdots \times \gamma_m(\gamma_{m+1}|u_{m+1}| - |u_m|) \\ &= \gamma_{M+1} \times \cdots \times \gamma_m(\gamma_{m+1}F_{m+1}(r_{m+1}(s)) - F_m(r_{m+1}(s))) \\ &\geq \gamma_{M+1} \times \cdots \times \gamma_{m+1}(1 - Ce^{-\mu\lambda_m m^{-3}})F_{m+1}(r_{m+1}(s)) \\ &\geq \gamma_{M+1} \times \cdots \times \gamma_{m+1}(1 - Ce^{-\mu m^2})F_{m+1}(r_{m+1}(s)) > 0 \end{aligned} \tag{6.13}$$

for $s \in [\frac{3}{4}, 1]$, we have $|u| > 0$ on $I_{m+1,4}$. By the definition of u, since $Lu = 0$ on $I_{m+1,2} \cup I_{m+1,3}$, V is smooth when $r \in (1, \infty)$.

Finally, we prove V is smooth at $r = 1$. On $I_{m+1,1}$, since $Lu = L[\gamma_{M+1} \times \cdots \times \gamma_{m+1}A_{m+1}u_{m+1}]$,

$$|\partial^\beta Lu| \leq C_\ell \gamma_{M+1} \times \cdots \times \gamma_{m+1}\lambda_m^{2\ell+2}e^{o(m)}F_{m+1}(r) \tag{6.14}$$

holds for $|\beta| = \ell \in \mathbb{Z}_{\geq 0}$ by (6.9). We thus have

$$\begin{aligned} |\partial^\beta V(r, \theta, t)| &= \left|\partial^\beta(u^{-1}Lu)\right| \\ &= \left|\sum_{|\beta_1| \leq \ell} \binom{\beta}{\beta_1} \partial^\beta(u^{-1})\partial^{\beta-\beta_1}(Lu)\right| \\ &\leq C_\ell \left(\frac{\gamma_{m+1}F_{m+1}}{F_m}\right) \lambda_m^{2(\ell+1)}e^{o(m)} \left(1 + \frac{\gamma_{m+1}F_{m+1}}{F_m}\right)^\ell \\ &\leq C_\ell e^{-\mu\lambda_m m^{-3}+o(m)} \\ &\leq C_\ell e^{-\frac{\mu}{2}m^2} \end{aligned}$$

by (6.10), (6.12), (6.14), (6.7), (6.8), and (6.11) for $|\beta| = \ell \in \mathbb{Z}_{\geq 0}$. Similarly on $I_{m+1,4}$, since $Lu = L[\gamma_{M+1} \times \cdots \times \gamma_m A_m u_m]$,

$$|\partial^\beta Lu| \leq C_\ell \gamma_{M+1} \times \cdots \times \gamma_m \lambda_m^{2\ell+2}e^{o(m)}F_m(r) \tag{6.15}$$

holds for $|\beta| = \ell \in \mathbb{Z}_{\geq 0}$ by (6.9). We thus have

$$\begin{aligned}|\partial^\beta V(r,\theta,t)| &= \left|\partial^\beta(u^{-1}Lu)\right| \\ &= \left|\sum_{|\beta_1|\leq \ell} \binom{\beta}{\beta_1} \partial^\beta(u^{-1})\partial^{\beta-\beta_1}(Lu)\right| \\ &\leq C_\ell \left(\frac{F_m}{\gamma_{m+1}F_{m+1}}\right) \lambda_m^{2(\ell+1)} e^{o(m)} \left(1 + \frac{F_m}{\gamma_{m+1}F_{m+1}}\right)^\ell \\ &\leq C_\ell e^{-\mu\lambda_m m^{-3}+o(m)} \\ &\leq C_\ell e^{-\frac{\mu}{2}m^2}\end{aligned}$$

by (6.10), (6.13), (6.15), (6.7), (6.8), and (6.11) for $|\beta| = \ell \in \mathbb{Z}_{\geq 0}$.

Thus, for all $|\beta| = \ell \in \mathbb{Z}_{\geq 0}$ on I_{m+1},

$$|\partial^\beta V(r,\theta,t)| \leq C_\ell e^{-\frac{\mu}{2}m^2} \leq C_\ell e^{-\frac{\mu}{2}((r-1)^{-1}-2)^2} \xrightarrow{r\searrow 1} 0$$

holds. □

3. Proofs of the lemmas

Proof of Lemma 6.2. We remark that

$$1 \geq \tanh\alpha = \sqrt{1-a^2} \geq \lambda^{-\frac{p}{2}}, \ a \in (0, 1-\lambda^{-p}]. \tag{6.16}$$

We use the Schläfli's integral formula of a Bessel function,

$$J_\lambda(\lambda a) = \frac{1}{2\pi}\int_{\Gamma_0} e^{\lambda(-ia\sin z + iz)}dz,$$

where Γ_0 consists of three sides of rectangle with vertexes at $-\pi + i\infty$, $-\pi$, π and $\pi + i\infty$ and is oriented from $-\pi + i\infty$ to $\pi + i\infty$. We set

$$\begin{aligned}f(z) &:= -ia\sin z + iz \\ &= a\cos x\sinh y - y + i(x - a\sin x\cosh y),\end{aligned}$$

where $z = x + iy$. By the Cauchy's integral theorem, we can deform Γ_0 into a curve defined by Γ on which $x - a\sin x\cosh y = 0$. Hence, we obtain

$$J_\lambda(\lambda a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda g(x)}dx,$$

where g is defined by

$$g(x) := a\cos x\sinh y(x) - y(x)$$

and y satisfies

$$\begin{aligned}\cosh y(x) &= \frac{x}{a\sin x} \\ \Leftrightarrow y(x) &= \log\left(\frac{x}{a\sin x} + \sqrt{\frac{x^2}{a^2\sin^2 x} - 1}\right)\end{aligned}$$

for $x \in (-\pi, \pi)$, where $y(0) = \alpha$ is well-defined owing to $a < 1$.

First, we evaluate g in an interval $[-\lambda^{-q}, \lambda^{-q}]$, where q satisfying

$$0 < q < \frac{2-p}{4} \tag{6.17}$$

is determined later. Since there exists a constant $C > 0$ such that

$$|y'(x)| = \left|\frac{1}{\sinh y}\frac{d}{dx}\left(\frac{x}{a\sin x}\right)\right| \le \frac{C}{\sqrt{\frac{x^2}{(a\sin x)^2}-1}}\frac{|x|}{a}$$
$$\le \frac{C|x|}{\sqrt{1-a^2}} \le C\lambda^{-q+\frac{p}{2}}$$

by (6.16), we have for $x \in [-\lambda^{-q}, \lambda^{-q}]$,

$$|y(x)-\alpha| = |y(x)-y(0)| \le C\lambda^{-q+\frac{p}{2}}|x| \le C\lambda^{-2q+\frac{p}{2}}.$$

Hence, the Taylor's theorem yields

$$\begin{aligned} g(x) & \\ &= f(x+iy(x)) = f(x+i(y-\alpha)+i\alpha) \\ &= f(i\alpha) + (x+i(y-\alpha))f'(i\alpha) + \frac{(x+i(y-\alpha))^2}{2}f''(i\alpha) \\ &\quad + \frac{(x+i(y-\alpha))^3}{2}\int_0^1 (1-\theta)^2 f'''(i\alpha+\theta(x+i(y-\alpha)))d\theta \\ &= \tanh\alpha - \alpha - \frac{\tanh\alpha}{2}x^2 + O(\lambda^{-1-\delta}) \end{aligned} \tag{6.18}$$

since $f'(i\alpha) = 0$, $|f'''(i\alpha+\theta(x+i(y-\alpha)))| \le C$ for some $C > 0$, and

$$q := \frac{1}{3}(1+\delta+\frac{p}{2}).$$

We remark that (6.17) is equivalent to $p < \frac{2(1-2\delta)}{5}$. Consequently, we have

$$\begin{aligned} \int_{-\lambda^{-q}}^{\lambda^{-q}} e^{\lambda g(x)}dx &= e^{\lambda(\tanh\alpha-\alpha)}\int_{-\lambda^{-q}}^{\lambda^{-q}} e^{-\frac{\lambda\tanh\alpha}{2}x^2}dx \cdot e^{O(\lambda^{-\delta})} \\ &= \frac{e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{\lambda\tanh\alpha}}\left[\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}}d\xi - 2\int_{\lambda^{\frac{1-2q}{2}}\sqrt{\tanh\alpha}}^{\infty} e^{-\frac{\xi^2}{2}}d\xi\right] \\ &\quad \times (1+O(\lambda^{-\delta})) \\ &= \frac{\sqrt{2\pi}e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{\lambda\tanh\alpha}}(1+O(\lambda^{-\delta})) \end{aligned}$$

since $\lambda^{\frac{1-2q}{2}}\sqrt{\tanh\alpha} \ge \lambda^{\frac{2-p}{4}-q}$ by (6.16) and

$$\lambda^\delta \int_{\lambda^{\frac{2-p}{4}-q}}^{\infty} e^{-\frac{\xi^2}{2}}d\xi = \lambda^\delta\sqrt{\frac{\pi}{2}}e^{-\frac{1}{4}\lambda^{\frac{2-p}{2}-2q}} \xrightarrow{\lambda\to\infty} 0$$

by (6.17). Hence, we have

$$\frac{1}{2\pi}\int_{-\lambda^{-q}}^{\lambda^{-q}} e^{\lambda g(x)}dx = \frac{e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{2\pi\lambda\tanh\alpha}}(1+O(\lambda^{-\delta})).$$

Second, we evaluate g in $(-\pi,\pi)\setminus[-\lambda^{-q},\lambda^{-q}]$. Because $\pm g'(x) \ge 0$ when $0 \le \mp x < \pi$, it follows from (6.18) and (6.16),

$$\frac{1}{2\pi}\left(\int_{-\pi}^{-\lambda^{-q}} + \int_{\lambda^{-q}}^{\pi}\right)e^{\lambda g(x)}dx = O(\lambda^{-\delta})\frac{e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{2\pi\lambda\tanh\alpha}}.$$

In fact, by our assumption (6.16) and (6.17),

$$\begin{aligned}
&\lambda^\delta \left| \frac{\frac{1}{2\pi}\left(\int_{-\pi}^{-\lambda^{-q}} + \int_{\lambda^{-q}}^{\pi}\right) e^{\lambda g(x)}dx}{\frac{e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{2\pi\lambda\tanh\alpha}}} \right| \\
&\le \sqrt{\frac{\tanh\alpha}{2\pi}}\lambda^{\delta+\frac12}\cdot 2(\pi-\lambda^{-q})e^{-\frac12\lambda^{\frac{2-p}{2}-2q}}\cdot e^{O(\lambda^{-\delta})} \\
&\le \sqrt{2\pi}\lambda^{\delta+\frac12}e^{-\frac12\lambda^{\frac{2-p}{2}-2q}}\cdot e^{O(\lambda^{-\delta})} \xrightarrow{\lambda\to\infty} 0
\end{aligned}$$

holds. We complete the proof. □

Proof of Lemma 6.3. In Lemma 6.2, taking $a = r = 1-\ell m^{-2}$ for $\ell \ge 1$, which is done by our assumption (6.2), yields

$$G_m(r) = \frac{h(r)^{\lambda_m}}{\sqrt{2\pi}(1-r^2)^{\frac14}\sqrt{\lambda_m}}(1+o(1)) \text{ as } m\to\infty,$$

where

$$h(r) := \frac{re^{\sqrt{1-r^2}}}{1+\sqrt{1-r^2}},$$

since $e^{-\alpha} = \frac{r}{1+\sqrt{1-r^2}}$ for $\alpha > 0$. Because simple calculations yield

$$\begin{aligned}
h'(r) &= \frac{1-\sqrt{1-r^2}}{r^2}\sqrt{1-r^2}e^{\sqrt{1-r^2}} \\
&= (1+o(1))\sqrt{2}\sqrt{1-r} \text{ as } r\nearrow 1,
\end{aligned}$$

we have

$$h(r) = h(1) - \int_r^1 h'(s)ds = 1-(1+o(1))\frac{2\sqrt2}{3}(1-r)^{\frac32}$$

as $r\nearrow 1$. Hence, for $\ell\ge1$ and $r = 1-\ell m^{-2}$, we obtain

$$\begin{aligned}
G_m(r) &= (1+o(1))\frac{\left(1-(1+o(1))\frac{2\sqrt2}{3}\ell^{\frac32}m^{-3}\right)^{\lambda_m}}{(2\pi)^{\frac12}\sqrt{\lambda_m}\,(1-(1-\ell m^{-2})^2)^{\frac14}} \\
&= (1+o(1))\frac{\sqrt{m}}{(2\pi^2\ell)^{\frac14}\sqrt{\lambda_m}}e^{-(1+o(1))\frac{2\sqrt2}{3}\ell^{\frac32}\lambda_m m^{-3}}
\end{aligned}$$

as $m\to\infty$. The last equality comes from

$$\begin{aligned}
&\frac{\left(1-(1+o(1))\frac{2\sqrt2}{3}\ell^{\frac32}m^{-3}\right)^{\lambda_m}}{(2\pi)^{\frac12}\sqrt{\lambda_m}\,(1-(1-\ell m^{-2})^2)^{\frac14}} \\
&= \frac{\sqrt{m}}{(2\pi^2\ell)^{\frac14}\sqrt{\lambda_m}}e^{-(1+o(1))\frac{2\sqrt2}{3}\ell^{\frac32}\lambda_m m^{-3}} \text{ as } m\to\infty
\end{aligned}$$

since (6.2) implies

$$\lambda_m m^{-3} \ge m^{\frac2p-3} > m^2 \xrightarrow{m\to\infty} \infty.$$

□

PROOF OF LEMMA 6.4. (6.5) is obtained by the definition. Since

$$\frac{k_m}{\lambda_m} r_m(s) = 1 - \frac{1+s+O(m^{-1})}{m^2} \text{ as } m \to \infty$$

by our assumption (6.4), we obtain

$$\begin{aligned} F_m(r_m(s)) &= G_m\left(\frac{k_m}{\lambda_m} r_m(s)\right) \\ &= G_m\left(1 - \frac{1+s+O(m^{-1})}{m^2}\right) \\ &= (1+o(1)) \frac{\sqrt{m} e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s+O(m^{-1}))^{\frac{3}{2}} \lambda_m m^{-3}}}{(2\pi^2(1+s+O(m^{-1})))^{\frac{1}{4}} \sqrt{\lambda_m}} \\ &= (1+o(1)) \frac{\sqrt{m} e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s)^{\frac{3}{2}} \lambda_m m^{-3}}}{(2\pi^2(1+s))^{\frac{1}{4}} \sqrt{\lambda_m}} \end{aligned}$$

as $m \to \infty$. The last equality comes from

$$\begin{aligned} &\frac{\sqrt{m} e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s+O(m^{-1}))^{\frac{3}{2}} \lambda_m m^{-3}}}{(2\pi^2(1+s+O(m^{-1})))^{\frac{1}{4}} \sqrt{\lambda_m}} \\ &\quad = \frac{\sqrt{m} e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s)^{\frac{3}{2}} \lambda_m m^{-3}}}{(2\pi^2(1+s))^{\frac{1}{4}} \sqrt{\lambda_m}} \text{ as } m \to \infty. \end{aligned}$$

Furthermore, since $r_{m+1}(s) = r_m\left(1+s+O(m^{-1})\right)$ as $m \to \infty$,

$$\begin{aligned} \gamma_{m+1} &= \frac{F_m(r_m(\frac{3}{2}+O(m^{-1})))}{F_{m+1}(r_{m+1}(2^{-1}))} \\ &= (1+o(1)) \left(\frac{\frac{3}{2}}{\frac{5}{2}+O(m^{-1})}\right)^{\frac{1}{4}} \sqrt{\frac{\lambda_{m+1}}{\lambda_m} \frac{1}{1+\frac{1}{m}}} \\ &\quad \times e^{-(1+o(1))\frac{2\sqrt{2}}{3}\left\{(\frac{5}{2}+O(m^{-1}))^{\frac{3}{2}} - (\frac{3}{2})^{\frac{3}{2}} \frac{\lambda_{m+1}}{\lambda_m} (\frac{m}{m+1})^3\right\} \lambda_m m^{-3}} \\ &\le e^{-(1+o(1))\frac{2\sqrt{2}}{3}\left\{(\frac{5}{2}+O(m^{-1}))^{\frac{3}{2}} - (\frac{3}{2}(1+o(1)))^{\frac{3}{2}}\right\} \lambda_m m^{-3}} \\ &\le e^{-(1+o(1))\sqrt{2}\sqrt{\theta} \lambda_m m^{-3}}, \end{aligned}$$

where we use our assumption (6.3) and the mean value theorem such that $x^{\frac{3}{2}} - y^{\frac{3}{2}} = \frac{3}{2}\sqrt{\theta}(x-y)$ for $0 \le y \le \theta \le x$, holds. Hence, we have, by the above estimate,

$$\gamma_{m+1} \le e^{-(1+o(1))\sqrt{3}\lambda_m m^{-3}} \le e^{-\lambda_m m^{-3}}$$

for sufficiently large $m \in \mathbb{N}$.

Finally, we have by the definition of γ_{m+1},

$$\begin{aligned} \frac{F_m(r_{m+1}(s))}{\gamma_{m+1} F_{m+1}(r_{m+1}(s))} &= \frac{F_m(r_{m+1}(s))}{F_m(r_{m+1}(2^{-1}))} \cdot \frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(s))} \\ &= \frac{F_m(r_m(1+s+O(m^{-1})))}{F_m(r_m(\frac{3}{2}+O(m^{-1})))} \cdot \frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(s))} \end{aligned}$$

as $m \to \infty$. When $0 \le s \le \frac{1}{4}$, there exist constants $C > 0$ and θ satisfying $2 + s + O(m^{-1}) \le \theta \le \frac{5}{2} + O(m^{-1})$ such that

$$\begin{aligned}\frac{F_m(r_m(1+s+O(m^{-1})))}{F_m(r_m(\frac{3}{2}+O(m^{-1})))} &\ge Ce^{(1+o(1))\sqrt{2}\sqrt{\theta}(\frac{1}{2}-s)\lambda_m m^{-3}} \\ &\ge Ce^{(1+o(1))\sqrt{4+O(m^{-1})}(\frac{1}{2}-s)\lambda_m m^{-3}}.\end{aligned}$$

Furthermore, there exists θ satisfying $1 + s \le \theta \le \frac{3}{2}$ such that

$$\begin{aligned}\frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(s))} &\ge Ce^{-(1+o(1))\sqrt{2}\sqrt{\theta}(\frac{1}{2}-s)\lambda_{m+1}(m+1)^{-3}} \\ &\ge Ce^{-(1+o(1))\sqrt{3}(\frac{1}{2}-s)\lambda_m m^{-3}}\end{aligned}$$

by (6.3). There then exists $\mu > 0$ such that

$$\begin{aligned}\frac{F_m(r_{m+1}(s))}{\gamma_{m+1}F_{m+1}(r_{m+1}(s))} &\ge Ce^{(1+o(1))(\sqrt{4+O(m^{-1})}-\sqrt{3})(\frac{1}{2}-s)\lambda_m m^{-3}} \\ &\ge Ce^{(1+o(1))\frac{\sqrt{4+O(m^{-1})}-\sqrt{3}}{4}\lambda_m m^{-3}} \\ &\ge Ce^{\mu\lambda_m m^{-3}}\end{aligned}$$

for sufficiently large $m \in \mathbb{N}$. Moreover, when $s \in [\frac{3}{4}, 1]$, there exist constants $C > 0$ and θ satisfying $\frac{5}{2} + O(m^{-1}) \le \theta \le 2 + s + O(m^{-1})$ such that

$$\begin{aligned}\frac{F_m(r_m(1+s+O(m^{-1})))}{F_m(r_m(\frac{3}{2}+O(m^{-1})))} &\le Ce^{-(1+o(1))\sqrt{2}\sqrt{\theta}(s-\frac{1}{2})\lambda_m m^{-3}} \\ &\le Ce^{-(1+o(1))\sqrt{5+O(m^{-1})}(s-\frac{1}{2})\lambda_m m^{-3}}.\end{aligned}$$

Furthermore, there exists θ satisfying $\frac{3}{2} \le \theta \le 1 + s$ such that

$$\begin{aligned}\frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(s))} &\le Ce^{(1+o(1))\sqrt{2}\sqrt{\theta}(s-\frac{1}{2})\lambda_{m+1}(m+1)^{-3}} \\ &\le Ce^{(1+o(1))2(s-\frac{1}{2})\lambda_m m^{-3}}\end{aligned}$$

by (6.3). There then exists $\mu > 0$ such that

$$\begin{aligned}\frac{F_m(r_{m+1}(s))}{\gamma_{m+1}F_{m+1}(r_{m+1}(s))} &\le Ce^{-(1+o(1))(\sqrt{5+O(m^{-1})}-2)(s-\frac{1}{2})\lambda_m m^{-3}} \\ &\le Ce^{-(1+o(1))\frac{\sqrt{5+O(m^{-1})}-2}{4}\lambda_m m^{-3}} \\ &\le Ce^{-\mu\lambda_m m^{-3}}\end{aligned}$$

for sufficiently large $m \in \mathbb{N}$. □

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