博士論文

Structural Characterizations of Rooted Subdivisions on Four Vertices in Graphs

(グラフにおける四頂点上の根付き細分に対する構造的特徴付け)

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Structural Characterizations of Rooted Subdivisions on Four Vertices in Graphs

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Abstract

One of the central topics in graph theory is to study substructures of graphs, such as paths and cycles. Starting with the celebrated theorem of Karl Menger, a number of variants of problems of finding disjoint paths in a graph have been studied from both theoretical and practical interests. The k-disjoint paths problem is, given a graph and k pairs of vertices of it, to find k disjoint paths that link the pairs in the graph. This problem is NP-hard if k is part of the input. On the other hand, as a byproduct of the Graph Minor project, Robertson and Seymour proved that for any fixed k there is a polynomial time algorithm to solve it. This algorithm involves a huge constant and it remains a challenge to devise a practical one. The case $k = 2$, however, admits a simple algorithm and its structural characterization is obtained independently by Thomassen, Seymour and Shiloach in 1980: In a 4-connected graph G, two pairs of vertices $(s_1, t_1), (s_2, t_2)$ are linked by two disjoint paths of G if and only if G cannot be drawn in a disc with s_1, s_2, t_1, t_2 on the boundary in order.

The rooted subdivision problem is a natural generalization of the k-disjoint paths problem. This problem asks internally disjoint paths in a graph that link given pairs of vertices. As an extension of the "two-paths theorem", we mainly focus on rooted subdivisions with four branch vertices. Namely, for a fixed graph H with four vertices, we consider the following problem: Given a graph G and an injective map from $V(H)$ to $V(G)$, is there a subdivision of H in G with four branch vertices specified by the map? Hence the case $H = 2K_2$ (two copies of K_2) corresponds to the 2-disjoint paths problem.

In this dissertation, for any fixed H with four vertices, we give a complete structural characterization of 6-connected graphs G with no such subdivision of H. Roughly speaking, such graphs G can be decomposed into a planar graph and some local areas of non-planarity, giving us a glimpse of an extension of the two-paths theorem. As a corollary, we prove that every 7-connected graph contains a subdivision of K_4 with prescribed branch vertices. This generalizes a result of McCarty, Wang and Yu, who proved that every 7-connected graph is 4-ordered. We also prove that every triangle-free 6-connected graph contains a subdivision of K_4 with prescribed branch vertices. This solves a special case of a conjecture of Mader.

We also consider a relaxed version of the above problem for $H = K_4^-$, where K_4^- is the graph obtained from K_4 by removing one edge: Given a graph G and a subset Z of $V(G)$ of size 4, is there a subdivision of K_4^- in G with the four branch vertices in Z? In this problem there is no requirement about which vertex of Z works as which vertex of K_4^-. We characterize 3-connected graphs G with no such subdivision of K_4^-. The proof is based on Mader's S-paths theorem.

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Contents

Chapter 1

Introduction

1.1 Background

Finding disjoint paths in a graph is one of the classical problems in graph theory. In its simplest form, the problem is to find maximum internally vertex-disjoint paths between given two vertices in a graph. In 1927, Karl Menger [46] gave a celebrated theorem that offers a min-max formula for this problem: The maximum number of internally vertex-disjoint paths between a pair of vertices equals the minimum size of a vertex-cut that separates them. This theorem laid the foundations for research on paths in graphs, and also makes the notion of connectivity a central topic in graph theory. A graph of order at least $k + 1$ is called *k-connected* if there is no set of $k - 1$ vertices whose removal disconnects the graph. Thus, by Menger's theorem, a graph is k-connected if and only if any two vertices are connected by at least k internally disjoint paths.

There is an edge-disjoint variant of Menger's theorem. This result can be viewed as a special case of the famous max-flow and min-cut theorem given by Ford and Fulkerson [9] in 1956: In a network, the maximum amount of flow between given two vertices equals the minimum capacity of edge-cuts separating them. Network flow problems are central problems in operations research and computer science, arising in applications such as the transportation and transshipment problems.

The work of Menger and Ford and Fulkerson leads to a more general framework of the multi-commodity flow problem. This problem is, given several pairs of sources and sinks in a communication or transportation network, to transmit several goods or messages between the pairs simultaneously. One special case is a situation that requires multi-commodity flows of integer value, which leads to another important problem in structural graph theory, called the *k-disjoint paths problem*.

Mathematically, the k-disjoint problem asks the existence of k disjoint paths in a graph that link given k pairs of distinct vertices $(s_1, t_1), \ldots, (s_k, t_k)$, respectively. In 1980s, this problem has been intensively studied because of its applications to the design of very large-scale integrated (VLSI) circuits. Although the algorithmic problem is NP-hard in general, the case $k = 2$ turned out to be tractable because of its structural characterization via planarity. Since the 2-disjoint paths problem specifies four "terminals" in a graph, it is natural to extend it to a problem of finding internally disjoint paths with ends in a

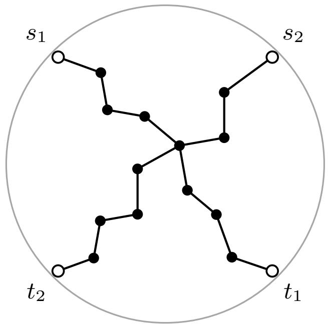

Figure 1.1: A graph embedded in a disc with four vertices s_1, s_2, t_1, t_2 on the boundary in order.

specified set of four vertices. This is called the *rooted subdivision problem on four vertices*, with which we shall be concerned throughout the dissertation. For this problem, several structural results, most of which are related to connectivity of graphs, are known. We explain these details below.

k-Disjoint paths. A formal description of the k-disjoint paths problem is as follows:

> (k-Disjoint paths problem)
>
> Instance: A graph G and $2k$ vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ of G.
>
> Question: Are there k disjoint paths of G with ends s_i, t_i for $1 \leq i \leq k$, respectively?

In general, the algorithmic problem is NP-hard [22] if k is part of the input of the problem. In fact, the problem is NP-hard even if the graph is restricted to be planar [38]. On the other hand, Robertson and Seymour [50] proved that for any fixed k there is a polynomial time algorithm to solve the problem. This is certainly one of the deepest results in algorithmic graph theory. Indeed, the correctness of the algorithm needs the full strength of the seminal work of their Graph Minor project in a series of over 20 long papers. The algorithm runs in $O(|V(G)|^3)$ time, but we should note that it is not practical since it involves a huge constant. Later, the time complexity was improved to $O(|V(G)|^2)$ by Kawarabayashi, Kobayashi and Reed [26], though it remains impractical.

The two-paths theorem. For the case $k = 2$, the problem is more tractable. Consider, for example, a graph G drawn in a disc so that no two edges meet in a point other than a common end, and that four distinct vertices s_1, s_2, t_1, t_2 occur on the boundary in this order listed. See Figure 1.1 for intuition. As easily checked (formally by the Jordan curve theorem), if P_i is a path of G with ends s_i, t_i for $i = 1, 2$, then these two paths P_1 and P_2 must have a common vertex by the planarity of G. Hence G contains no two disjoint paths with ends s_1t_1, s_2t_2, respectively. Thomassen [60], Seymour [55] and Shiloach [56] proved that the converse holds, which gives a structural characterization for two disjoint paths: In a 4-connected graph G, two pairs of vertices $(s_1, t_1), (s_2, t_2)$ are linked by two disjoint paths of G if and only if G cannot be drawn in a disc with s_1, s_2, t_1, t_2 on the boundary in order. This result is often called the "two-paths theorem" in the literature. From this one can derive a simple and practical polynomial time algorithm for the 2-disjoint paths problem.

Rooted subdivisions. The notion of k-linkage is naturally generalized to "rooted subdivisions" of a graph. For a fixed graph H, we consider the following problem:

(Rooted H-subdivision problem)

Instance: A graph G and an injective map $\varphi : V(H) \to V(G)$.

Question: Is there a map η from $E(H)$ to the set of paths of G such that

- for every edge $e = xy$ of H, the path $\eta(e)$ has ends $\varphi(x), \varphi(y)$, and
- the paths $\eta(e)$ ($e \in E(H)$) are internally disjoint?

A *subdivision* of a graph H is a graph obtained from H by replacing each edge uv of H with a path between u and v. One can see that the graph consisting of the union of the paths $\eta(e)$ ($e \in E(H)$) is nothing but a subdivision of H in G with the branch vertices specified by the map φ. Note that the case $H = kK_2$ (k copies of K_2) corresponds to the k-disjoint paths problem. Conversely, the rooted H-subdivision problem can be reduced to the $|E(H)|$-disjoint paths problem. By the work of Robertson and Seymour again, for any fixed graph H there is a polynomial time (but impractical) algorithm for the rooted H-subdivision problem.

Connectivity for linkage. For a fixed graph H, we say that a graph G is H-linked if it has a feasible solution η for any φ in the rooted H-subdivision problem. More precisely, a graph G is called *H-linked* if for any injective map $\varphi : V(H) \to V(G)$ there is a collection $\{P_e\}_{e \in E(H)}$ of internally disjoint paths of G such that P_e has ends $\varphi(x), \varphi(y)$ for every edge $e = xy$ of H. This includes several kinds of notions of connectivity, such as being k-connected ($H = K_{1,k}$), k-ordered ($H = C_k$) and k-linked ($H = kK_2$).

Clearly, every k-linked graph is k-connected. The converse is not true. However, an approximate version is known to hold: There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every $f(k)$-connected graph is k-linked, and moreover, f can be chosen to be linear in k, as explained below. From the existence of f we deduce that sufficiently highly connected graphs are H-linked. More precisely, $f(|E(H)|)$-connected graphs are $|E(H)|$-linked, and so H-linked.

The existence of f was first noticed by Jung [21] and Larman and Mani [34]. Their crucial observation is that the existence of a subdivision of K_{3k} in a $2k$-connected graph G ensures that G is k-linked. This, together with an earlier result of Mader [42] that sufficiently high average degree forces a subdivision of a large complete graph, showed the existence of f.

Although the function $f(k)$ found by them was exponential in k, the upper bound has been dramatically improved. Robertson and Seymour [50] proved that the existence of a minor of K_{3k} and $2k$-connectivity suffice for a graph to be k-linked. Indeed, this observation played an important role in their algorithm for the k-disjoint paths problem. This,

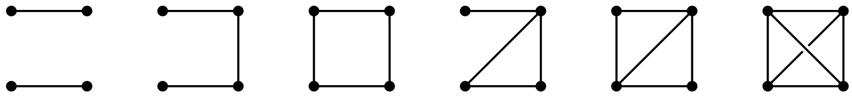

Figure 1.2: Graphs with four vertices.

together with the bound on the extremal function of complete minors by Kostochka [32] and Thomason [59], led to that $f(k) = O(k\sqrt{\log k})$. Later, Bollobás and Thomason [3] showed that the existence of a minor of a sufficiently dense graph, instead of K_{3k}, suffices in the assumption, and consequently, achieved the first linear bound $f(k) \leq 22k$. By a different approach from these results, Thomas and Wollan [57] proved that every $2k$-connected graph of average degree at least $10k$ is k-linked. This implies that $f(k) \leq 10k$, which is the current best known bound. They also proved in [58] that $f(3) \leq 10$. The only known exact bound is $f(2) = 6$, which can be obtained from the two-paths theorem.

Rooted subdivisions on four vertices. For a graph H, let $g(H)$ denote the smallest positive integer which ensures that every $g(H)$-connected graph is H-linked. From the result of Thomas and Wollan, we know a general linear bound $g(H) \leq f(|E(H)|) \leq 10|E(H)|$. One might be interested in the following questions:

- What is a structural characterization of H-linked graphs?
- If this is hard to answer, then what is the exact value of $g(H)$?

Although both these questions are difficult in general, the case $H = 2K_2$ is completely settled by the two-paths theorem: $2K_2$-Linked (i.e., 2-linked) graphs are characterized by planarity and $g(2K_2) = 6$. This provokes a natural question: What about a graph H with four vertices that contains $2K_2$ as a subgraph? See Figure 1.2. Let P_4, C_4, K_4, K_4^{2-} and K_4^- denote the path of length 3, the cycle of length 4, the complete graph on four vertices, the graph obtained from K_4 by deleting two adjacent edges and the graph obtained from K_4 by deleting one edge, respectively. We summarize known results in Table 1.1, which will be explained below.

One step ahead of 2-linkage is P_4-linkage. By definition, a graph is P_4-linked if and only if for any four distinct vertices of the graph there is a path that contains those vertices in the order specified. Yu [63, 64, 65] gave a complete characterization of P_4-linked graphs. The non-P_4-linked graphs are not so far from the "boundary-planar graphs" that appear in the two-paths theorem, but are already too complicated to describe precisely here. Based on the result of Yu, it was proved by Ellingham et al. [7] that every 7-connected graph is P_4-linked. They also proved in [7] that this bound on the connectivity is sharp, by constructing a 6-connected graph which is not P_4-linked. Thus $g(P_4) = 7$.

Although P_4-linked graphs are characterized by Yu, it seems far-reaching at this moment to give an exact characterization of H-linked graphs for $H \in \{C_4, K_4^{2-}, K_4^-, K_4\}$. The only known results are about planar graphs. Goddard [10] proved that 4-connected triangulation of the plane is 4-ordered, i.e., C_4-linked. This result was extended to triangulations of all surfaces by Mukae and Ozeki [47]. Goddard's result was strengthened

Table 1.1: Structural results about H-linked graphs for graphs H with four vertices.

H	Structural characterizations	$g(H)$
$2K_2$	General (The two-paths theorem) [55, 56, 61]	6
P_4	General [63, 64, 65]	7 [7]
C_4	Plane triangulations [10] Surface triangulations [47]	7 [45]
K_4^{2-}		7 [36]
K_4^-	Plane triangulations [7]	≤ 50 [57]
K_4	4-Connected planar graphs [62]	≤ 60 [57]

by Ellingham et al. [7], who proved that 4-connected triangulations of the plane are K_4^--linked. As far as we are aware, the only partial structural result for rooted subdivisions of K_4 is given by Yu [62], who characterized 4-connected planar K_4-linked graphs.

As for the exact values of $g(H)$, a noteworthy contribution was recently given by McCarty, Wang and Yu [45], who proved that 7-connected graphs are 4-ordered. This, together with $g(P_4) = 7$, implies that $g(C_4) = 7$, and so significantly improves the known bound $g(C_4) \leq 40$. Based on a similar technique used in [45] it was shown in [36] that 7-connected graphs are K_4^{2-}-linked. Thus $g(K_4^{2-}) = 7$. The bounds $g(K_4^-) \leq 50$ and $g(K_4) \leq 60$ seem the best at the present moment.

Relevance to coloring-conjectures. The study on rooted subdivisions on four vertices is motivated not only by extensions of the two-paths theorem, but also by the coloring-conjecture of Hajós. One of the most famous theorems in graph theory is the Four Color Theorem [1, 2], which states that every planar graph is 4-colorable. As is well-known as Kuratowski's theorem, a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ as a subgraph. In 1940s, Hajós made a conjecture that generalizes the Four Color Theorem in terms of forbidden graphs: Every graph with no subdivision of K_t is $(t-1)$-colorable. This conjecture can be viewed as a variant of the famous conjecture of Hadwiger [11] which states that every graph with no minor of K_t is $(t-1)$-colorable. Hajós' conjecture is true for $t \leq 4$. The case $t \geq 7$, however, was disproved by Catlin [5]. In fact, Erdös and Fajtlowicz [8] proved that the conjecture fails for almost all graphs. Thomassen [61] exhibited many reasons why Hajós' conjecture can fail in general. Nevertheless, the case $t = 5, 6$ remains open.

One approach to the case $t = 5$ is to reduce the conjecture to the Four Color Theorem. Let us call a minimum counterexample to Hajós' conjecture for $t = 5$ a *Hajós graph*. The goal is to show that every Hajós graph is planar. It is known that Hajós graphs are 4-connected [66]. In 1970s, Kelmans [31] and Seymour independently conjectured that every 5-connected non-planar graph contains a subdivision of K_5. This conjecture was recently proved by He, Wang and Yu [14, 15, 16, 17] with papers over 150 pages in total: See also [28, 40, 41]. This implies that if there exists a non-planar Hajós graph G, it is not 5-connected. Thus G admits a vertex-cut S of size 4 whose removal results in at least two connected components $C_1, \ldots, C_k$. If, for example, the subgraph of G induced by S and a component C_i contains a subdivision of K_4 with the four branch vertices in S, then one can easily extend it to a subdivision of K_5 in the whole graph G, a contradiction. Similarly, other subdivisions rooted at S may also be used to construct a subdivision of K_5. Therefore, structural characterizations for the rooted subdivision problem on four vertices are a first step towards resolving the long-standing conjecture of Hajós.

1.2 Contribution

The two-paths theorem forms the basis for research on rooted subdivisions with prescribed four vertices. The objective of this dissertation is to push forward with it in this direction further. Our ultimate goal would be to give an explicit description of structural characterizations for rooted subdivisions on four vertices. As noted above, however, for any graph $H \in \{C_4, K_4^{2-}, K_4^-, K_4\}$ it seems difficult to determine all the structures of non-H-linked graphs. Therefore, we set up a moderate goal.

First contribution. The first contribution in this dissertation is to determine the structures of 6-connected non-H-linked graphs for every $H \in \{P_4, C_4, K_4^{2-}, K_4^-, K_4\}$. Roughly speaking, such a graph can be decomposed into a planar graph L and "local areas of non-planarity" that surround the "boundary" of L. We give a formal description in Section 3.1. In the theorem, one can catch a glimpse of an extension of the two-paths theorem and Yu's characterization of P_4-linked graphs.

This structural result leads to several corollaries. First one can show that graphs having such structures must contain a cut of size at most 6. This implies that every 7-connected graph is K_4-linked. This generalizes the results of [45, 36] that 7-connected graphs are 4-ordered and K_4^{2-}-linked. By the result of Thomas and Wollan, it was known that 60-connected graphs are K_4-linked. Our result significantly improves this known bound on the connectivity. Combined with the bound $g(P_4) \geq 7$ shown by [7], our result implies that $g(H) = 7$ for every graph $H \in \{P_4, C_4, K_4^{2-}, K_4^-, K_4\}$. We add in our results to Table 1.2.

Next one can see that graphs having such structures must contain many small dense subgraphs, especially, triangles. This implies that every 6-connected triangle-free graph is K_4-linked. This solves a special case of a conjecture of Mader, as explained below. The *girth* of a graph is the minimum length of cycles in the graph. By the result of Thomas and Wollan, we observe that $2\binom{n}{2}$-connected graphs of average degree at least $10\binom{n}{2}$ are $\binom{n}{2}$-linked, and so K_n-linked. Mader [44] proved the following, which says that the condition on average degree can be replaced by a condition that requires sufficiently large girth:

Every $2\binom{n}{2}$-connected graph of sufficiently large girth is K_n-linked.

Table 1.2: Structural results about H-linked graphs for graphs H with four vertices, including our results.

H	Structural characterizations	$g(H)$
$2K_2$	General (The two-paths theorem) [55, 56, 61]	6
P_4	General [63, 64, 65]	7 [7]
C_4	Plane triangulations [10] Surface triangulations [47] 6-Connected graphs [this work]	7 [45]
K_4^{2-}	6-Connected graphs [this work]	7 [36]
K_4^-	Plane triangulations [7] 6-Connected graphs [this work]	7 [this work]
K_4	4-Connected planar graphs [62] 6-Connected graphs [this work]	7 [this work]

In fact, Kühn and Osthus [33] proved that the condition girth ≥ 250 is sufficient in this theorem. Can we weaken the connectivity in the assumptions? As pointed out in [44], it is easily seen that for all $n \geq 2$ there are $(\binom{n}{2} - 1)$-connected graphs of sufficiently large girth which are not K_n-linked. Mader conjectured that this bound on the connectivity could be tight:

(?) Every $\binom{n}{2}$-connected graph of sufficiently large girth is K_n-linked. (?)

The conjecture is true for $n = 3$ but open for $n \geq 4$. Our result solves this conjecture for $n = 4$, even for graphs of girth ≥ 4.

Our proof traces the method of the recent paper of McCarty, Wang and Yu [45], who proved that 7-connected graphs are 4-ordered. The details on how our proof is inspired by their paper are given in Section 3.2.1. The proof of our theorems is constructive, mainly based on Menger's theorem and the two-paths theorem. This can be used to devise an implementable algorithm to solve the rooted K_4-subdivision problem say, if a given graph G is 6-connected.

The first contribution is based on [12].

Second contribution. As noted before, our work is also motivated by the coloring-conjecture of Hajós. Unfortunately, our assumption of 6-connectivity in the first contribution is too strong to apply to the conjecture directly. It would be desirable to proceed with the work to seek a complete characterization for graphs of smaller connectivity. However, this seems an arduous task. We slightly change a direction here and consider a relaxed variant of the problem. The classical problem of finding disjoint paths between two sets of vertices, which is completely characterized by Menger's theorem, can be viewed as a relaxed version of the k-disjoint paths problem that permits permutations on terminals. In analogy to this, we consider the following relaxed version of the rooted H-subdivision problem for a fixed graph H:

> (Relaxed rooted H-subdivision problem)
>
> Instance: A graph G and a subset Z of $V(G)$ with $|Z| = |V(H)|$.
>
> Question: Are there an injective map $\varphi : V(H) \to Z$ and a map η from $E(H)$ to the set of paths of G such that
>
> - for every edge $e = xy$ of H, the path $\eta(e)$ has ends $\varphi(x), \varphi(y)$, and
> - the paths $\eta(e)$ ($e \in E(H)$) are internally disjoint?

In the problem we only specify the set Z of possible branch vertices of a subdivision of H, and do not care which vertex in Z works as which vertex of H. If H is a cycle, for instance, the problem is equivalent to asking a cycle of G containing all the vertices in Z, without regard to the order. Starting with the classical result of Dirac [6], cycles through prescribed vertices (or prescribed edges) have been widely studied [4, 18, 19, 20, 24, 25, 37].

Again we focus on the case H has exactly four vertices. The relaxed problem for the case $|V(H)| = 4$ is also a natural setting for Hajós' conjecture. If $H = K_4$, then the relaxed problem has no difference from the original rooted K_4-subdivision problem because of its symmetry. So we mainly consider the case $H = K_4^-$.

The second contribution in this dissertation is to determine the structures of 3-connected graphs G with no such relaxed rooted K_4^--subdivision. Roughly speaking, such a graph G admits a "decomposition" that separates the four specified terminals into a few smaller subsets. As we expected, the decomposition of G can be written as a hypergraph in flavor of combinatorics, without any topological condition, such as planarity. This is an interesting difference from the result in the first contribution. An overview is given in Section 4.1.

The second contribution is based on [13].

Organaization. The remaining of the dissertation is organized as follows. In Chapter 2, we collect notation, terminology and theorems that we use throughout the dissertation. In Chapter 3 and Chapter 4, we deal with the first contribution and the second contribution, respectively. In Chapter 5, we give a concluding summary of our work.

Chapter 2

Preliminaries

In this chapter, we collect notation and terminology that we use throughout the dissertation. We also introduce Perfect's lemma, which allows us "paths-augmentation". This method plays an important role in the dissertation.

2.1 Notation and terminologies

All graphs in this dissertation are finite, undirected and without loops. By a graph we always mean a simple graph.

Let G be a graph. We let $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For a subset X of $V(G)$ or $E(G)$, let $G \setminus X$ denote the graph obtained from G by deleting X. If $X = \{x\}$ is a singleton, we simply write $G \setminus x$. For $X \subseteq V(G)$, let $G|X$ denote the subgraph of G induced by X, i.e., $G|X = G \setminus (V(G) - X)$. For two vertices u, v, let $G + uv$ to denote the (simple) graph obtained from G by adding an edge uv. For a vertex v of G, we let $\deg_G(v)$ denote the degree of v in G.

For subgraphs of H, J of G, let $H \cup J$ denote the subgraph of G with vertex set $V(H) \cup V(J)$ and edge set $E(H) \cup E(J)$. Define $H \cap J$ similarly. Subgraphs H and J are *disjoint* if $H \cap J$ is null, and *edge-disjoint* if $E(H \cap J) = \emptyset$. A pair (A, B) of subsets of $V(G)$ is called a *separation* of G if $(G|A) \cup (G|B) = G$; equivalently, $A \cup B = V(G)$ and every edge of G has ends both in A or B. It is called a k-separation if $|A \cap B| = k$, and a $(\leq k)$-separation if $|A \cap B| \leq k$.

Paths and *cycles* have no repeated vertices or edges. For subsets $S, T \subseteq V(G)$ of a graph G, by an $S-T$ path we mean a path with one end in S, the other end in T, and no internal vertex in $S \cup T$. Paths are called *internally disjoint* if they are mutually disjoint except for their ends. For a subset A of $V(G) \cup E(G)$, we say that G is A-*cyclic* if there is a cycle in G that contains all the elements of A, and A-*acyclic* otherwise. A vertex subset $S \subseteq V(G)$ is called *stable* if no edge has both ends in S. Two graphs G, H are called *homeomorphic* if there is a graph which is isomorphic to a subdivision of G and isomorphic to a subdivision of H. We say that two subsets $X, Y \subseteq V(G)$ are *adjacent* in G if $X \cap Y = \emptyset$ and some vertex of X is adjacent to some vertex of Y in G. If $Y = \{v\}$ is a singleton, we often say that X and v are adjacent. We let $N_G(X)$ denote the set of vertices adjacent to X in G. If $X = \{v\}$ is a singleton, we write $N_G(v) = N_G(\{v\})$.

For a tree T and two vertices u, v of T, let $T[u, v]$ denote the (unique) path of T between u and v. Let $T[u, v), T(u, v]$ and $T(u, v)$ denote the graphs obtained from $T[u, v]$ by deleting $\{v\}, \{u\}$ and $\{u, v\}$, respectively, which may be null. Let P be a path. We let $\mathrm{end}(P)$ denote the set of vertices of smallest degree of P. Define $\mathrm{int}(P) := V(P) - \mathrm{end}(P)$.

2.2 Augmenting paths by Perfect's lemma

We will use the following matroidal properties of (internally) disjoint paths. The following two results are essentially due to Perfect [48].

Lemma 2.2.1 ([48]). *Let G be a graph and let $S, T \subseteq V(G)$. Let $k \geq k' \geq 0$ be integers, and let $S' \subseteq S, T' \subseteq T$ with $|S'| = |T'| = k'$. Suppose that there are k' disjoint $S - T$ paths of G covering $S' \cup T'$. If there are k disjoint $S - T$ paths of G, then one can choose such paths with $S' \cup T'$ covered.*

Lemma 2.2.2 ([48]). *Let G be a graph, let $S \subseteq V(G)$ and v be a vertex not in S. Let $k \geq k' \geq 0$ be integers, and let $S' \subseteq S$ with $|S'| = k'$. Suppose that there are k' paths of G from v to S, mutually disjoint except for v, all with no internal vertex in S, and covering S'. If there are k paths of G from v to S, mutually disjoint except for v, all with no internal vertex in S, then one can choose such paths with S' covered.*

We will use these lemmas to "augment" subgraphs in a graph. Let G be a graph, let H be a subgraph of G, and let v be a vertex of G, not in $V(H)$. Suppose that there are k internally disjoint $\{v\} - V(H)$ paths $P_1, \ldots, P_k$ of G; so, some of these paths may have a common end in $V(H)$. For the subgraph $J := H \cup P_1 \cup \cdots \cup P_k$ of G, we consider "paths augmentation" from v as follows. Now suppose, for example, that:

$(*)$ There is no $(\leq k)$-separation (A, B) of G such that $v \in A - B$ and $V(H) \subseteq B$.

Then it is an immediate consequence of Perfect's lemma that there are $k + 1$ internally disjoint $\{v\} - V(H)$ paths $Q_1, \ldots, Q_{k+1}$ of G such that Q_i has the same ends as P_i for $1 \leq i \leq k$, and Q_{k+1} has an end in $V(H)$ which was not covered by any of $P_1, \ldots, P_k$. Note that $H \cup Q_1 \cup \cdots \cup Q_k$ is homeomorphic to J. By abuse of notation, we will often use the same symbol $P_1, \ldots, P_k$ to denote $Q_1, \ldots, Q_k$. Thus, as far as homeomorphisms of J are concerned, we may assume that there is a path $Q = Q_{k+1}$ of G with one end v, the other end in $V(J) - V(P_1 \cup \cdots \cup P_k)$, and no internal vertex in J. We call Q a path of G obtained by *augmenting* $P_1, \ldots, P_k$ from v in J, or simply, an *augmented path* from v in J, as long as the paths $P_1, \ldots, P_k$ of J are clear from the context. Throughout the dissertation, we will frequently make a recourse to this augmentation method, under assumptions of connectivity, such as $(*)$.

Chapter 3

Linking four vertices in 6-connected graphs

3.1 Formal description of main theorem

To state the main result precisely, we need some definitions. Let G be a graph. A tuple $(A_1, \ldots, A_k)$ of subsets of $V(G)$ is called a *path-decomposition* of G if $A_1 \cup \cdots \cup A_k = V(G)$, every edge of G has both ends in some A_i, and $A_i \cap A_k \subseteq A_j$ whenever $i < j < k$. Let v_1, v_2, v_3, v_4 be distinct vertices of G. See Figure 3.1. By a K_4^{2-}-subdivision on $(v_1; v_2, v_3; v_4)$ in G we mean a subgraph of G consisting of the union of four internally disjoint paths of G with ends $v_1v_2, v_1v_3, v_1v_4, v_2v_3$, respectively. By a K_4^--subdivision on $(v_1, v_2; v_3, v_4)$ in G we mean a subgraph of G consisting of the union of five internally disjoint paths of G with ends $v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4$, respectively.

Definition 3.1.1 (discoid graph)**.** Let G be a graph and v_1, v_2, v_3, v_4 be distinct vertices of G. A tuple (L, H_1, H_2, H_3, H_4) of edge-disjoint subgraphs L, H_1, H_2, H_3, H_4 of G is called a *discoid decomposition* of G for (v_1, v_2, v_3, v_4) if it satisfies the following:

- G can be written as $G = L \cup H_1 \cup H_2 \cup H_3 \cup H_4$;
- $v_1, v_2, v_3, v_4 \in V(L)$, $H_1 \cap H_3$ and $H_2 \cap H_4$ are null and $V(H_j \cap H_{j+1}) = \{v_{j+1}\}$ for $1 \leq j \leq 4$, where indices are read modulo 4;
- for some labelings $V(L \cap H_j) - \{v_j, v_{j+1}\} = \{b_1^j, \ldots, b_{k_j}^j\}$ $(1 \leq j \leq 4)$, the graph L can be drawn in a disc with $v_1, b_1^1, \ldots, b_{k_1}^1, v_2, b_1^2, \ldots, b_{k_2}^2, v_3, b_1^3, \ldots, b_{k_3}^3, v_4, b_1^4, \ldots, b_{k_4}^4$ on the boundary in this order listed;
- for each $1 \leq j \leq 4$, if $k_j \geq 2$ then there is a path-decomposition $(A_1^j, \ldots, A_{k_j}^j)$ of H_j such that
  - $v_j, b_1^j \in A_1^j - A_2^j$ and $v_{j+1}, b_{k_j}^j \in A_{k_j}^j - A_{k_j-1}^j$,
  - $b_i^j \in A_i^j - A_{i-1}^j \cup A_{i+1}^j$ for $1 < i < k_j$, and
  - $|A_i^j \cap A_{i+1}^j| = 2$ for $1 \leq i < k_j$.

We say that G is *discoid* for (v_1, v_2, v_3, v_4) if G has a discoid decomposition for (v_1, v_2, v_3, v_4).

See Figure 3.2 for an illustration of a discoid graph. The structure of the graph L is nothing but an obstruction for 2-linked graphs. Indeed, there are no two disjoint paths

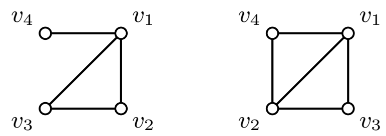

Figure 3.1: A K_4^{2-}-subdivision on $(v_1; v_2, v_3; v_4)$ (left) and a K_4^--subdivision on $(v_1, v_2; v_3, v_4)$ (right).

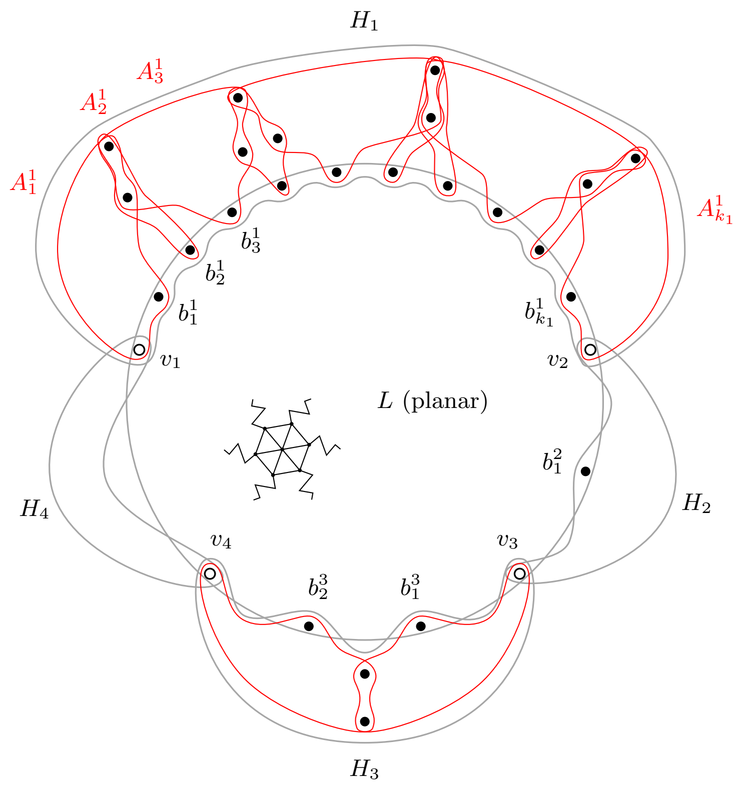

Figure 3.2: A discoid graph for (v_1, v_2, v_3, v_4).

of L with ends v_1v_3, v_2v_4, respectively, because L can be drawn in a disc with v_1, v_2, v_3, v_4 on the boundary in order. Furthermore, as easily checked, for $1 \leq j \leq 4$ the graph $L \cup H_j$ contains no path through $v_{j+2}, v_j, v_{j+1}, v_{j-1}$ in order. The structure of $L \cup H_j$ is a canonical obstruction for P_4-linked graphs, as seen in the result of Yu [63, 64, 65]. Similarly, one easily sees that the graph $L \cup H_1 \cup H_3$ contains no cycle through v_1, v_2, v_4, v_3 in order; the graph $L \cup H_1 \cup H_2$ contains no K_4^{2-}-subdivision on $(v_2; v_1, v_3; v_4)$; the graph $L \cup H_1 \cup H_2 \cup H_4$ contains no K_4^--subdivision on $(v_1, v_2; v_3, v_4)$; the discoid graph $G = L \cup H_1 \cup H_2 \cup H_3 \cup H_4$ contains no subdivision of K_4 with v_1, v_2, v_3, v_4 branch vertices. Our main result is the following, which says that the obstructions are all described by such discoid decompositions if G is 6-connected. In the theorem, one can catch a glimpse of an extension of the two-paths theorem that characterizes 2-linked graphs.

Theorem 3.1.2. *Let G be a graph and v_1, v_2, v_3, v_4 be distinct vertices of G. If G is 6-connected, then all of the following statements hold.*

(1) *Either G contains a path through v_1, v_2, v_3, v_4 in this order listed, or there is a discoid decomposition (L, H_1, H_2, H_3, H_4) of G for (v_2, v_3, v_1, v_4) such that $E(H_2) = E(H_3) = E(H_4) = \emptyset$.*

(2) *Either G contains a cycle through v_1, v_2, v_3, v_4 in this order listed, or there is a discoid decomposition (L, H_1, H_2, H_3, H_4) of G for (v_1, v_2, v_4, v_3) or (v_1, v_4, v_2, v_3) such that $E(H_2) = E(H_4) = \emptyset$.*

(3) *Either G contains a K_4^{2-}-subdivision on $(v_2; v_1, v_3; v_4)$, or there is a discoid decomposition (L, H_1, H_2, H_3, H_4) of G for (v_1, v_2, v_3, v_4) such that $E(H_3) = E(H_4) = \emptyset$.*

(4) *Either G contains a K_4^--subdivision on $(v_1, v_2; v_3, v_4)$, or there is a discoid decomposition (L, H_1, H_2, H_3, H_4) of G for (v_1, v_2, v_3, v_4) or (v_1, v_2, v_4, v_3) such that $E(H_3) = \emptyset$.*

(5) *Either G contains a subdivision of K_4 with v_1, v_2, v_3, v_4 branch vertices, or G is discoid for $(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})$ for some ordering $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\} = \{v_1, v_2, v_3, v_4\}$.*

Theorem 3.1.2 leads to several corollaries. First one can show that discoid graphs cannot be 7-connected, which yields the following.

Theorem 3.1.3. *Every 7-connected graph contains a subdivision of K_4 with prescribed branch vertices.*

As noted in Section 1.2, this generalizes the results of [45, 36] that 7-connected graphs are 4-ordered and K_4^{2-}-linked.

Next one can show that 6-connected discoid graphs have many small dense subgraphs, especially, triangles. This yields the following.

Theorem 3.1.4. *Every 6-connected triangle-free graph contains a subdivision of K_4 with prescribed branch vertices.*

As noted in Section 1.2, this solves the case $n = 4$ of a conjecture of Mader: Every $\binom{n}{2}$-connected graph with sufficiently large girth is K_n-linked.

An outline of the proof of our results, Theorems 3.1.2, 3.1.3 and 3.1.4, including the whole structure of Chapter 3, is given in Section 3.2.

Remark 3.1.5. We remark that a 6-connected discoid graph exists. Indeed, the 6-connected non-P_4-linked graph constructed in [7] (see Figure 3.3) is discoid. To see this, consider a graph G which admits a discoid decomposition (L, H_1, H_2, H_3, H_4) for (v_1, v_2, v_3, v_4) that satisfies the following: $E(H_j) = \emptyset$ and $V(H_j) = \{v_j, v_{j+1}\}$ for $2 \leq j \leq 4$; the path-decomposition $(A_1^1, \ldots, A_{k_1}^1)$ of H_1 satisfies that $|A_1^1| = |A_{k_1}^1| = 4$, $|A_2^1| = |A_{k_1-1}^1| = 3$ and $|A_i^1| = 5$ for $3 \leq i \leq k_1 - 2$; each part A_i^1 induces a clique of G and contains no other "inner vertex", i.e., $A_1^1 - A_2^1 = \{v_1, b_1^1\}$, $A_{k_1}^1 - A_{k_1-1}^1 = \{v_2, b_{k_1}^1\}$ and $A_i^1 - A_{i-1}^1 \cup A_{i+1}^1 = \{b_i^1\}$ for $1 < i < k_1$. By choosing a well-connected near-triangulation L, one can construct a 6-connected graph $G = L \cup H_1 \cup H_2 \cup H_3 \cup H_4$. Note that G contains no path through v_3, v_1, v_2, v_4 in order. See [7, Section 4.A] for an example of the construction of L.

Remark 3.1.6. One may impose other several conditions on a discoid decomposition of a 6-connected graph G by its connectivity: The size $k_j = |V(L \cap H_j) - \{v_j, v_{j+1}\}|$ satisfies

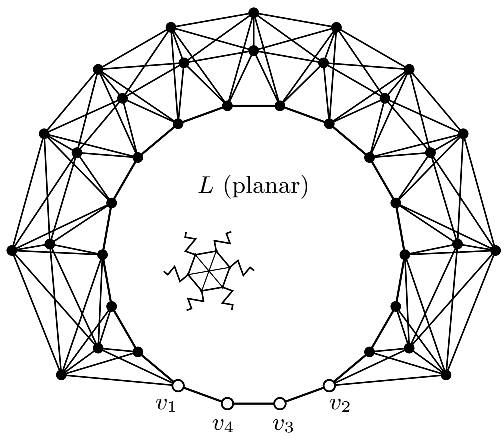

Figure 3.3: A 6-connected graph constructed in [7] that contains no path through v_3, v_1, v_2, v_4 in order.

either $k_j = 0$ or $k_j \geq 4$; each part A_i^j has no other "inner" vertices, i.e., $A_1^j - A_2^j = \{v_j, b_1^j\}$, $A_{k_j}^j - A_{k_j-1}^j = \{v_{j+1}, b_{k_j}^j\}$ and $A_i^j - A_{i-1}^j \cup A_{i+1}^j = \{b_i^j\}$ for $1 < i < k_j$; it must hold that $|A_2^j| = |A_{k_j-1}^j| = 3$; one can choose L to be a 2-connected planar graph having an outer cycle that contains all the vertices in $V(L) \cap V(H_1 \cup H_2 \cup H_3 \cup H_4)$, etc. However, for simplicity we adopt Definition 3.1.1 that only requires the most essential condition that each H_j admits a path-decomposition. We also expect its smooth connection to descriptions of possible decompositions for graphs G of smaller connectivity in our future work.

3.2 Definitions and outline of proof

The most of the chapter is devoted to proving Theorem 3.1.2 (5). The nontrivial part is that if a 6-connected graph contains no subdivision of K_4 on prescribed four vertices then the graph admits a discoid decomposition as in Theorem 3.1.2 (5). We first give rough ideas derived from the paper of [45] and then describe more details of our proof.

3.2.1 Rough ideas

Before describing the outline of the proof, we first explain how our proof is inspired by the recent paper of McCarty, Wang and Yu [45]. They showed that every 7-connected graph G is 4-ordered. Let v_1, v_2, v_3, v_4 be vertices of G. They first showed that if G contains no cycle through v_1, v_2, v_3, v_4 in this order specified, then G contains a subgraph J homeomorphic to the graph as in Figure 3.4 (or the graph in Figure 3.4 with v_2, v_4 interchanged). The subgraph J is called a "skeleton" in [45], and its construction is based on Menger's theorem. The subgraph J is extremal in G in a sense that J itself contains no cycle through v_1, v_2, v_3, v_4 in order, but a "bold jump" of a path P of G makes the graph $J \cup P$ immediately contain such a cycle. Thus, every component of $G \setminus V(J)$ is adjacent to a local part of J, respectively. This leads the whole graph G to have a structure similar to J. More precisely, G contains a cycle C_1 containing v_1, v_2 and a cycle C_2 containing v_3, v_4,

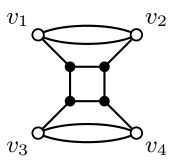

Figure 3.4: A graph used in the proof in [45].

mutually disjoint, such that $H := G \setminus V(C_1 \cup C_2)$ is a 3-connected planar graph whose outer cycle contains four distinct vertices which are neighbors of $v_1, \ldots, v_4$, respectively. This part of the proof is based on the two-paths theorem. Finally, by a discharging method applied to the planar graph H, they showed that G contains a (≤ 6)-cut or a cycle through v_1, v_2, v_3, v_4 in order. Since G is 7-connected, one obtains the latter, as required.

Our proof is similar to their proof. Suppose that G contains no K_4-subdivision with v_1, v_2, v_3, v_4 branch vertices. By Menger's theorem, we first construct a "skeleton", which is a subgraph of G homeomorphic to the graph in the right of Figure 3.5 (or the graph in the figure with v_1, v_2, v_3, v_4 permuted). This subgraph is extremal in G in a sense that it contains no K_4-subdivision with v_1, v_2, v_3, v_4 branch vertices, but a "bold jump" added to it immediately results in such a subdivision of K_4. Thus, the whole graph G has a structure not so far from it. This part of the proof is also based on the two-paths theorem. Finally, we investigate each local part of G carefully, and conclude that G has a discoid decomposition. We give more details below.

3.2.2 Outline of proof

Let us now describe the details of our proof. Let G be a graph. For distinct four vertices v_1, v_2, v_3, v_4 of G, a *bicycle* J on (v_1, v_2, v_3, v_4) in G is a subgraph $C \cup C'$ of G consisting of the union of two cycles C, C' both containing v_1, v_2, v_3, v_4 in this order listed, mutually disjoint except for $\{v_1, v_2, v_3, v_4\}$; see Figure 3.5 (left) for an illustration of a bicycle. Let $Z \subseteq V(G)$ with $|Z| = 4$. By a bicycle J on Z we mean a bicycle on (v_1, v_2, v_3, v_4) for some ordering $Z = \{v_1, v_2, v_3, v_4\}$. A cycle of J containing exactly two vertices of Z is called a *tire* of J; there are exactly four distinct tires of J. The *interior* of a tire C of J is the graph $C \setminus Z$. By a K_4-subdivision *on* Z we mean a subdivision of K_4 with the four branch vertices in Z.

The following lemma says that a bicycle on Z is useful for constructing a K_4-subdivision on Z. For a subgraph H of a graph G, by *shrinking* H in G we mean deleting $V(H)$ and adding a new vertex and edges from it to all the vertices in $N_G(V(H))$.

Lemma 3.2.1. *Let G be a graph and v_1, v_2, v_3, v_4 be distinct vertices of G. Suppose that there is a bicycle on (v_1, v_2, v_3, v_4) in G. Let G' be a graph obtained from G by shrinking the interior of each tire of the bicycle into a single vertex, respectively. If there are two disjoint paths of G' with ends v_1v_3, v_2v_4, respectively, then there is a K_4-subdivision on $\{v_1, \ldots, v_4\}$ in G.*

Proof. Let $Z := \{v_1, \ldots, v_4\}$. Let J be a bicycle on (v_1, v_2, v_3, v_4) in G. Let C_i denote

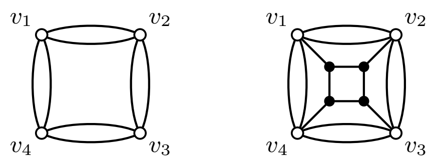

Figure 3.5: A bicycle on (v_1, v_2, v_3, v_4) (left) and a skeleton on (v_1, v_2, v_3, v_4) (right).

the tire of J with $v_i, v_{i+1} \in V(C_i)$ for $1 \leq i \leq 4$, where indices are read modulo 4. Let G' be a graph obtained from G by shrinking each $C_i \setminus \{v_i, v_{i+1}\}$ into a single vertex, which we will denote by c_i for $1 \leq i \leq 4$. Since $C_i \setminus \{v_i, v_{i+1}\}$ is non-null, c_i is adjacent to v_i and v_{i+1} in G' for $1 \leq i \leq 4$. Suppose that there are two disjoint paths of G' with ends v_1v_3, v_2v_4, respectively. Then there are two disjoint paths of G' such that one is a path from $\{v_1, c_1, c_4\}$ to $\{v_3, c_2, c_3\}$ and the other is a path from $\{v_2, c_1, c_2\}$ to $\{v_4, c_3, c_4\}$, both with no internal vertex in $Z \cup \{c_1, c_2, c_3, c_4\}$. This implies that there are two disjoint paths P, Q of G such that:

- P is a path from $V(C_1 \cup C_4) - \{v_2, v_4\}$ to $V(C_2 \cup C_3) - \{v_2, v_4\}$ and Q is a path from $V(C_1 \cup C_2) - \{v_1, v_3\}$ to $V(C_3 \cup C_4) - \{v_1, v_3\}$,
- P and Q have no internal vertex in J, and
- $V(C_i) - \{v_i, v_{i+1}\}$ intersects at most one of P, Q for $1 \leq i \leq 4$.

It is easy to see that $P \cup Q \cup J$ contains a K_4-subdivision on Z. This proves the lemma. □

It is well-known that 2-linked graphs are characterized by planarity. The following lemma is useful in most of the situations; see also [49, theorems (2.3) and (2.4)].

Lemma 3.2.2 ([51, theorem (2.4)]). *Let v_1, v_2, v_3, v_4 be distinct vertices of a graph G. If there is no (≤ 3)-separation (A, B) of G such that $v_1, v_2, v_3, v_4 \in A$ and $|B - A| \geq 2$, then either there are two disjoint paths of G with ends v_1v_3, v_2v_4 respectively, or G can be drawn in a disc with v_1, v_2, v_3, v_4 on the boundary in order.*

By Lemma 3.2.1 one can apply the “two-paths theorem” (Lemma 3.2.2) to the shrunk graph G'. Thus the graph G' can be drawn in a disc, *unless it contains a small cut.* To increase the connectivity of G', we define a better bicycle as follows.

Let G be a graph. For distinct vertices v_1, v_2, v_3, v_4 of G, a *skeleton* on (v_1, v_2, v_3, v_4) in G is a subgraph $J \cup H \cup P_1 \cup P_2 \cup P_3 \cup P_4$ of G consisting of the union of a bicycle J on (v_1, v_2, v_3, v_4), a cycle H of $G \setminus V(J)$ and four disjoint paths P_1, P_2, P_3, P_4 of G between $\{v_1, v_2, v_3, v_4\}$ and $V(H)$, all with no internal vertex in $J \cup H$, such that the ends of P_1, P_2, P_3, P_4 in H occur in H in this order listed; see Figure 3.5 (right) for an illustration of a skeleton. For $Z \subseteq V(G)$ with $|Z| = 4$, by a skeleton on Z in G we mean a skeleton on (v_1, v_2, v_3, v_4) in G for some ordering $Z = \{v_1, v_2, v_3, v_4\}$.

The first step is to show the following lemma. The proof based on the “path-augmentation method” is given in Section 3.3.

Lemma 3.2.3. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. If G is 6-connected, then G contains a K_4-subdivision on Z or a skeleton on Z.*

For a cycle C of a graph G and two vertices u_1, u_2 of C, we say that C is *lean with respect to* u_1, u_2 in G if there is no cycle C' of G such that $u_1, u_2 \in V(C') \subsetneq V(C)$. Let $Z \subseteq V(G)$ with $|Z| = 4$. A bicycle on Z in G is *lean* in G if each tire of the bicycle is lean in G with respect to the two vertices of Z that the tire contains. A bicycle J on Z in G is *nice* if $G \setminus V(J)$ is 2-connected and there is a matching of G of size 4 from Z to $V(G) - V(J)$.

The next step is to show the following lemma, which says that we may augment a skeleton to obtain a nice and lean bicycle in 6-connected graphs. The proof is given in Section 3.5. For the proof we use some lemmas in [45] on *separating pairs*, which we shall introduce in Section 3.4.

Lemma 3.2.4. *Let G be a 6-connected graph and let $Z \subseteq V(G)$ with $|Z| = 4$. If there is a skeleton on Z in G, then G contains a K_4-subdivision on Z or a nice and lean bicycle on Z.*

Lemma 3.2.3 and Lemma 3.2.4 immediately lead to the following lemma.

Lemma 3.2.5. *Let G be a 6-connected graph and let $Z \subseteq V(G)$ with $|Z| = 4$. If there is no K_4-subdivision on Z in G, then there is a nice and lean bicycle on Z in G.*

When a nice and lean bicycle J exists in G, the shrunk graph G' in Lemma 3.2.1 becomes well-connected. Now one can apply the two-paths theorem to obtain the following lemma. The proof is given in Section 3.6.

Lemma 3.2.6. *Let G be a 6-connected graph and v_1, v_2, v_3, v_4 be distinct vertices of G. Suppose that there is a nice and lean bicycle on (v_1, v_2, v_3, v_4) in G. Let G' be a graph obtained from G by shrinking the interior of each tire of the bicycle into a single vertex, respectively. If G contains no K_4 subdivision on $\{v_1, \ldots, v_4\}$, then G' can be drawn in a disc with v_1, v_2, v_3, v_4 on the boundary in order.*

Fix a drawing of G' in Lemma 3.2.6, and expand the shrunk tires of the bicycle to the original ones. Now we obtain a "near-embedding" of G in a plane as in the following lemma; see Figure 3.6 for intuition.

Lemma 3.2.7. *Let G be a 6-connected graph and v_1, v_2, v_3, v_4 be distinct vertices of G. Let J be a nice and lean bicycle on (v_1, v_2, v_3, v_4). Let C_i denote the tire of J with $v_i, v_{i+1} \in V(C_i)$ for $1 \le i \le 4$, where indices are read modulo 4. If there is no K_4-subdivision on $\{v_1, \ldots, v_4\}$ in G, then $G \setminus V(J)$ contains a cycle K such that:*

- *$G \setminus V(J)$ can be drawn in a disc with K on the boundary, and*
- *there are eight distinct vertices $x_1, \ldots, x_8$ occurring in K in this order listed such that*
  - *$x_{2i-1}, x_{2i} \in N_G(v_i)$,*
  - *$N_G(v_i) \subseteq V(C_i \cup C_{i-1}) \cup V(K\langle x_{2i-1}, x_{2i}\rangle)$, and*
  - *$N_G(V(C_i) - \{v_i, v_{i+1}\}) \subseteq \{v_i, v_{i+1}\} \cup V(K\langle x_{2i}, x_{2i+1}\rangle)$*

  for $1 \le i \le 4$, where $K\langle x_j, x_{j+1}\rangle$ denotes the subpath of K between x_j and x_{j+1} with no other vertex in $\{x_1, \ldots, x_8\}$ for $1 \le j \le 8$, with indices read modulo 8.

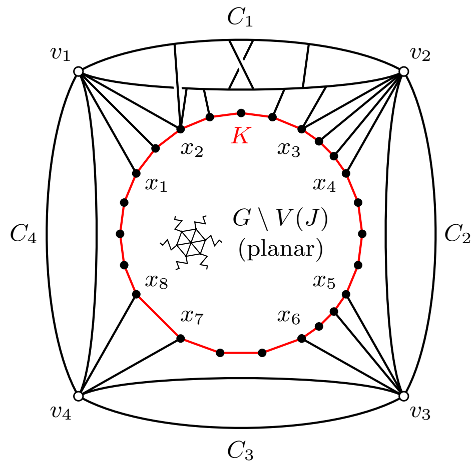

Figure 3.6: A near-embedding in a plane in Lemma 3.2.7.

Proof. Let $H := G\setminus V(J)$ and let G' denote the graph obtained from G by shrinking each $C_i \setminus \{v_i, v_{i+1}\}$ into a single vertex, denoted by c_i for $1 \le i \le 4$. By Lemma 3.2.6, the graph G' can be drawn in a disc with v_1, v_2, v_3, v_4 on the boundary in order. Fix a drawing of G'. (Such a drawing of G' is essentially unique because G' is 3-connected, as easily checked. But we do not use this fact.) Since J is nice, H is 2-connected. Let K be the outer cycle of H in the drawing of G'. Since J is nice, there is a matching of G of size 4 from $\{v_1, \dots, v_4\}$ to $V(H)$. By the planarity of G', we have $N_{G'}(v_i) \subseteq \{v_{i+1}, v_{i-1}, c_i, c_{i-1}\} \cup V(K)$ for $1 \le i \le 4$. Also there are eight vertices $x_1, \dots, x_8$ occurring in K in this order listed such that $x_{2i-1}, x_{2i} \in N_G(v_i)$ and $N_G(v_i) \cap V(K) \subseteq V(K\langle x_{2i-1}, x_{2i}\rangle)$ for $1 \le i \le 4$. Thus, $N_G(v_i) \subseteq V(C_i \cup C_{i-1}) \cup V(K\langle x_{2i-1}, x_{2i}\rangle)$ for $1 \le i \le 4$.

Since c_i is adjacent to v_i and v_{i+1} in G', we have $N_{G'}(c_i) \subseteq \{v_i, v_{i+1}\} \cup V(K\langle x_{2i}, x_{2i+1}\rangle)$ for $1 \le i \le 4$ by the planarity of G'; thus, $N_G(V(C_i) - \{v_i, v_{i+1}\}) \subseteq \{v_i, v_{i+1}\} \cup V(K\langle x_{2i}, x_{2i+1}\rangle)$.

We show that $x_1, \dots, x_8$ are distinct. Since G is 6-connected, we have $|N_{G'}(c_i) \cap V(K\langle x_{2i}, x_{2i+1}\rangle)| = \deg_{G'}(c_i) - |\{v_i, v_{i+1}\}| \ge 6 - 2 = 4$ for $1 \le i \le 4$. In particular, $x_{2i} \ne x_{2i+1}$ for $1 \le i \le 4$. Since J is lean, there is no chord of C_i with an end in $\{v_i, v_{i+1}\}$, and so we have $|N_G(v_i) \cap V(C_i)| = |N_G(v_i) \cap V(C_{i-1})| = 2$ for $1 \le i \le 4$. Hence $|N_G(v_i) \cap V(K\langle x_{2i-1}, x_{2i}\rangle)| = \deg_G(v_i) - |N_G(v_i) \cap V(C_i \cup C_{i-1})| \ge 6 - 4 = 2$ for $1 \le i \le 4$. This implies that $x_{2i-1} \ne x_{2i}$ for $1 \le i \le 4$. Therefore, $x_1, \dots, x_8$ are distinct. This proves the lemma. □

The last step is to investigate chords of the tires C_i and edges between C_i and $K\langle x_{2i}, x_{2i+1}\rangle$. A rough sketch is as follows. Let us take a closer look at the graph $G|V(C_1) \cup V(K\langle x_2, x_3\rangle)$, say. If there is no K_4-subdivision on Z in G, then there is no path of $G|V(C_1) \cup V(K\langle x_2, x_3\rangle)$ through x_3, v_1, v_2, x_2 in order. If $v_1v_2 \notin E(G)$, let H_1 be a graph obtained from $G|V(C_1) \cup V(K\langle x_2, x_3\rangle)$ by deleting edges spanned by $\{v_1, v_2\} \cup V(K\langle x_2, x_3\rangle)$. Let $x_2 = b_1, b_2, \dots, b_k = x_3$ be the vertices of the path $K\langle x_2, x_3\rangle$ from x_2 to x_3 in order. By applying Menger's theorem to H_1 repeatedly, one can find a path-decomposition $(A_1, \dots, A_k)$ of H_1 with $v_1, b_1 \in A_1 - A_2$, $v_2, b_k \in A_k - A_{k-1}$, $b_i \in A_i - A_{i-1} \cup A_{i+1}$ for $1 < i < k$ and $|A_i \cap A_{i+1}| = 2$ for $1 \le i < k$. Now the graph H_1

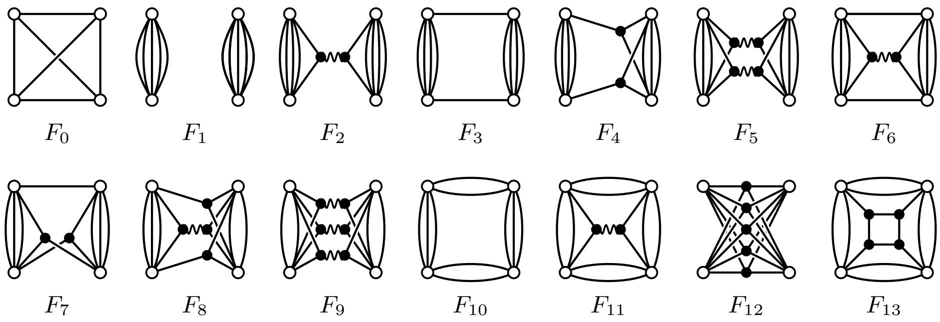

Figure 3.7: F_i-frames.

plays the role in a discoid decomposition of G for (v_1, v_2, v_3, v_4). If $v_1v_2 \in E(G)$ then one can show that $G|V(C_1) \cup V(K\langle x_2, x_3\rangle)$ is a planar graph with an outer cycle consisting of the union of $K\langle x_2, x_3\rangle, x_3v_2, v_2v_1$ and v_1x_2. Now the graph H_1 with $V(H_1) = \{v_1, v_2\}$ and $E(H_1) = \emptyset$ plays the role in a discoid decomposition of G for (v_1, v_2, v_3, v_4). A more precise proof is given in Section 3.8, which finishes the proof of Theorem 3.1.2 (5).

The other statements (1), (2), (3) and (4) in Theorem 3.1.2 can be proved in a similar way, based on Lemma 3.2.7. We only give a sketch of the proof of Theorem 3.1.2 (4) in Section 3.8.

We now turn to some corollaries of the main theorem, namely, Theorems 3.1.3 and 3.1.4. Our goal is to show that if a 6-connected graph G is 7-connected or triangle-free then there is a K_4-subdivision on Z in G. We may apply Theorem 3.1.2, but it seems quicker to start from Lemma 3.2.7. The main task is based on edge-counting in planar graphs. Let K be the cycle of $H := G \setminus V(J)$ as in Lemma 3.2.7. Since K separates J and $H \setminus V(K)$, every vertex in $H \setminus V(K)$ has degree ≥ 6 in H (as G is 6-connected), and moreover, has degree ≥ 7 in H if G is 7-connected. By the planarity, H has average degree < 6, and moreover, has average degree < 4 if it is triangle-free. This implies that if G is 7-connected or triangle-free, then some vertex in K has small degree in H, and hence, has many neighbors in $V(J)$. This fact can be used to find a K_4-subdivision on Z or a small cut of G, as required. A more precise proof is given in Section 3.9.

3.3 Proof of Lemma 3.2.3

The aim of this section is to prove Lemma 3.2.3. Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. See Figure 3.7. For $0 \leq i \leq 13$, an *F_i-frame* on Z in G (or a *frame of F_i* on Z in G) is a subgraph of G homeomorphic to the multigraph F_i as in Figure 3.7, where the four white vertices correspond to vertices of Z for some permutation, the wavy lines represent paths of length ≥ 0 and the other (straight or bent) lines represent paths of length > 0. Thus an F_0-frame is exactly a K_4-subdivision on Z and an F_{13}-frame is exactly a skeleton on Z. When the set Z we consider is clear from the context, we often omit "on Z" and simply say an F_i-frame or a frame of F_i.

The first step is to construct an F_i-frame for some $0 \leq i \leq 12$ in 5-connected graphs. We begin with the following lemma, whose proof is based on Lemma 2.2.2. For a family

$\mathcal{H}$ of subgraphs of G, let $\bigcup \mathcal{H}$ denote the subgraph of G consisting of the union of all members in $\mathcal{H}$, i.e., $V(\bigcup \mathcal{H}) = \bigcup_{H \in \mathcal{H}} V(H)$ and $E(\bigcup \mathcal{H}) = \bigcup_{H \in \mathcal{H}} E(H)$.

Lemma 3.3.1. *Let G be a graph, k and m be integers with $m + 1 \geq k \geq 2$ and let $Z \subseteq V(G)$ with $|Z| = k$. Suppose that there is no $(< m)$-separation (A, B) of G with $|Z \cap (A - B)| = 1$ and $B - A \neq \emptyset$. If Z is not a clique of G, then there is a family $\mathcal{T}$ of subgraphs of G satisfying the following:*

(i) *Each member of $\mathcal{T}$ is a tree of G having ≥ 2 vertices whose leaves are all in Z.*
(ii) *Members of $\mathcal{T}$ are mutually disjoint except for their leaves.*
(iii) *Every vertex of Z is contained in exactly m members of $\mathcal{T}$.*

Proof. Assume that Z is not a clique of G. We say that a tree of G having ≥ 2 vertices whose leaves are all in Z is a tree *on* Z. A family $\mathcal{T}$ of subgraphs of G is called *feasible* if it satisfies (i), (ii) and the condition that every vertex v of Z is contained in at most m members of $\mathcal{T}$, i.e., $\deg_{\bigcup \mathcal{T}}(v) \leq m$; so, $\mathcal{T} = \emptyset$ is feasible. We call the value $\sum_{v \in Z} \deg_{\bigcup \mathcal{T}}(v) = \sum_{T \in \mathcal{T}} |V(T) \cap Z|$ the *cost* of $\mathcal{T}$. Let $I_{\mathcal{T}}$ denote the simple graph with vertex set Z in which two vertices v, v' are adjacent if and only if there is a member of $\mathcal{T}$ which is a path between v and v'. Choose a feasible family $\mathcal{T}$ with the cost maximum, and subject to that with $|E(I_{\mathcal{T}})|$ maximum. Let $\delta := \min_{v \in Z} \deg_{\bigcup \mathcal{T}}(v)$. If $\delta = m$, then $\mathcal{T}$ satisfies (iii), and so we are done. Suppose to the contrary that $\delta < m$.

Let $x \in Z$ with $\deg_{\bigcup \mathcal{T}}(x) = \delta$. Let $\mathcal{T}_x = \{T_1, \ldots, T_\delta\}$ be the set of members of $\mathcal{T}$ containing x. For $1 \leq i \leq \delta$ let P_i be the longest path of T_i that starts from x and contains no vertex of T_i of degree ≥ 3 as an internal vertex; so, the other end of P_i, which we shall denote by p_i, is the other leaf of T_i if T_i is a path, and a vertex of degree ≥ 3 in T_i otherwise. Let H be the subgraph of G obtained from the graph $\bigcup \mathcal{T}$ by deleting $\bigcup_{1 \leq i \leq \delta} V(P_i \setminus p_i)$. In other words, H consists of the union of all the members in $\mathcal{T} - \mathcal{T}_x$ and the graphs $T_i \setminus V(P_i \setminus p_i)$ $(1 \leq i \leq \delta)$.

We show that there is a $(\leq \delta)$-separation (A, B) of G with $x \in A - B$ and $(Z - \{x\}) \cup V(H) \subseteq B$. For suppose to the contrary that there is no such a separation. Then we may assume from Lemma 2.2.2 that there is a path P of G with one end x, the other end in $(Z - \{x\}) \cup V(\bigcup \mathcal{T}) - V(P_1 \cup \cdots \cup P_\delta)$ and no internal vertex in $(Z - \{x\}) \cup V(\bigcup \mathcal{T})$. Let y denote the other end of P. If $y \in V(T) - Z$ for some $T \in \mathcal{T} - \mathcal{T}_x$, then $(\mathcal{T} - \{T\}) \cup \{T \cup P\}$ is feasible and has larger cost than $\mathcal{T}$, a contradiction. If $y \in V(T) - Z$ for some $T_i \in \mathcal{T}_x$, then T_i is a tree with ≥ 3 leaves. Let Q be the longest subpath of $T_i[y, p_i]$ that starts from y and contains no vertex of T_i of degree ≥ 3 as an internal vertex. Now the graph obtained from $T_i \cup P$ by deleting internal vertices and edges of Q can be written as the union of two trees T', T'' on Z such that $V(T_i) \cap Z = (V(T') \cap Z) \cup (V(T'') \cap Z)$ and $\{x\} = V(T') \cap V(T'')$. But $(\mathcal{T} - \{T_i\}) \cup \{T', T''\}$ is feasible and has larger cost than $\mathcal{T}$, a contradiction. Thus $y \in Z - \{x\}$, and so $xy \notin E(I_{\mathcal{T}})$. If there is a member $T \in \mathcal{T}$ containing y and having ≥ 3 leaves, let Q be the longest subpath of T that starts from y and contains no vertex of T of degree ≥ 3 as an internal vertex. Now the graph obtained from $T \cup P$ by deleting internal vertices and edges of Q can be written as the union of two trees T', T'' on Z such that $V(T) \cap Z = (V(T') \cap Z) \cup (V(T'') \cap Z)$ and $\{x\} = V(T') \cap V(T'')$. But $(\mathcal{T} - \{T\}) \cup \{T', T''\}$ is feasible and has larger cost than $\mathcal{T}$, a contradiction. Therefore, every member of $\mathcal{T}$ containing y is a path having the other end in $Z - \{x, y\}$. If $\deg_{\bigcup\mathcal{T}}(y) < m$, then $\mathcal{T} \cup \{P\}$ is feasible and has larger cost than $\mathcal{T}$, a contradiction; hence $\deg_{\bigcup\mathcal{T}}(y) = m$. Since $|Z - \{x, y\}| = k - 2 < m$, there are two distinct members T, T' of $\mathcal{T}$ which are paths starting from y and ending at a common vertex in $Z - \{x, y\}$. Now $\mathcal{T}' := (\mathcal{T} - \{T'\}) \cup \{P\}$ is feasible and has the same cost as $\mathcal{T}$, while $|E(I_{\mathcal{T}'})| > |E(I_{\mathcal{T}})|$, contrary to the choice of $\mathcal{T}$. This proves that there is a $(\leq \delta)$-separation (A, B) of G with $x \in A - B$ and $(Z - \{x\}) \cup V(H) \subseteq B$.

Note that there are δ paths of $G|A$ from x to $A \cap B$, mutually disjoint except for x, which are indeed subpaths of $P_1, \dots, P_\delta$; so $|A \cap B| = \delta$. Since $\delta < m$, by our assumption we have $B - A = \emptyset$. This implies two facts. First, since $Z - \{x\} \subseteq A \cap B$, every vertex of Z is covered by some member of $\mathcal{T}_x$ which is a path. In particular, x has degree $k - 1$ in $I_{\mathcal{T}}$, and so,

$$\sum_{T \in \mathcal{T}_x} |V(T) \cap Z| \leq 2 \deg_{I_{\mathcal{T}}}(x) + k(|\mathcal{T}_x| - \deg_{I_{\mathcal{T}}}(x)) = k\delta - (k-1)(k-2).$$

Second, every member in $\mathcal{T} - \mathcal{T}_x$ is a path of length 1 with both ends in $Z - \{x\}$; and so,

$$\sum_{T \in \mathcal{T} - \mathcal{T}_x} |V(T) \cap Z| \leq 2\binom{k-1}{2} = (k-1)(k-2).$$

Consequently,

$$k\delta \leq \sum_{v \in Z} \deg_{\bigcup\mathcal{T}}(v) = \sum_{T \in \mathcal{T}} |V(T) \cap Z| = \sum_{T \in \mathcal{T}_x} |V(T) \cap Z| + \sum_{T \in \mathcal{T} - \mathcal{T}_x} |V(T) \cap Z| \leq k\delta$$

and so we have equality throughout. This means that $Z - \{x\}$ is a clique of G and $\deg_{\bigcup\mathcal{T}}(v) = \delta$ for every $v \in Z$.

Therefore, we have shown that if a vertex $x \in Z$ satisfies $\deg_{\bigcup\mathcal{T}}(x) = \delta$ then $Z - \{x\}$ is a clique and $\deg_{\bigcup\mathcal{T}}(v) = \delta$ for every $v \in Z$. This implies that $Z - \{v\}$ is a clique of G for every $v \in Z$. Thus, Z is a clique, contrary to our assumption. This completes the proof. $\square$

By Lemma 3.3.1 applied to $k = 4$ and $m = 5$, one obtains an F_i-frame for some $0 \leq i \leq 12$ in 5-connected graphs.

Lemma 3.3.2. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. If there is no (≤ 4)-separation (A, B) of G with $|Z \cap (A - B)| = 1$ and $B - A \neq \emptyset$, then G contains an F_i-frame on Z for some $0 \leq i \leq 12$.*

Proof. Suppose that there is no K_4-subdivision on Z (F_0-frame) in G; so Z is not a clique of G. By Lemma 3.3.1 applied to $k = 4$ and $m = 5$, we obtain a family $\mathcal{T}$ of subgraphs of G satisfying the following:

(i) Each member of $\mathcal{T}$ is a tree of G having ≥ 2 vertices whose leaves are all in Z.
(ii) Members of $\mathcal{T}$ are mutually disjoint except for their leaves.
(iii) Every vertex of Z is contained in exactly five members of $\mathcal{T}$.

Let $Z = \{v_1, v_2, v_3, v_4\}$. Let H be a complete graph on Z, and for $1 \le i < j \le 4$ let e_{ij} denote the edge of H with ends v_i, v_j. Construct a bipartite graph J with color classes $E(H)$ and $\mathcal{T}$ in which $e_{ij} \in E(H)$ and $T \in \mathcal{T}$ are adjacent if and only if $v_i, v_j \in V(T)$. Since there is no K_4-subdivision on Z in G, there is no matching of size 6 in J. By Hall's theorem there is a subset X of $E(H)$ with $|N_J(X)| < |X|$. Assume that X is chosen to be minimal. Let us denote $\mathcal{T}_X := N_J(X)$. Then $|\mathcal{T}_X| = |X| - 1$. Every member in $\mathcal{T}_X$ is adjacent to ≥ 2 elements of X in J by the minimality of X. Hence we have:

(1) $|V(T) \cap Z| \ge 3$ *for each* $T \in \mathcal{T}_X$.

For $1 \le i \le 4$, let $\mathcal{T}_i$ denote the family consisting of the members of $\mathcal{T}$ containing v_i. By (iii) we have:

(2) $|\mathcal{T}_i| = 5$ *for* $1 \le i \le 4$.

We consider all cases of X, up to isomorphisms of the graph $H[X] := (Z, X)$. First suppose that $H[X]$ contains an isolated vertex, v_4 say. Assume that X is either $\{e_{12}\}, \{e_{12}, e_{13}\}$ or $\{e_{12}, e_{13}, e_{23}\}$. If $X = \{e_{12}\}$, then $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ (as $\mathcal{T}_X = \emptyset$) and so $\mathcal{T}_1 \cup \mathcal{T}_2 \subseteq \mathcal{T}_3 \cup \mathcal{T}_4$. Now $10 = |\mathcal{T}_1| + |\mathcal{T}_2| = |\mathcal{T}_1 \cup \mathcal{T}_2| \le |\mathcal{T}_3 \cup \mathcal{T}_4| \le |\mathcal{T}_3| + |\mathcal{T}_4| = 10$ and we have equality throughout. Hence $\mathcal{T}_3 \cap \mathcal{T}_4 = \emptyset$, and so, every member of $\mathcal{T}$ is a path between $\{v_1, v_2\}$ and $\{v_3, v_4\}$. We deduce from (2) that $\mathcal{T}$ consists of $k, 5-k, k, 5-k$ paths with ends $v_1v_3, v_3v_2, v_2v_4, v_4v_1$, respectively, for some $0 \le k \le 5$. Thus the graph $\bigcup \mathcal{T}$ is a frame of F_1, F_3 or F_{10}, as required. If $X = \{e_{12}, e_{13}\}$, then the unique member of $\mathcal{T}_X$ is a tree whose leaves are v_1, v_2, v_3 by (1), while $\mathcal{T}_1 - \mathcal{T}_X$ consists of four paths with ends v_1, v_4. We deduce from (2) that $\mathcal{T} - \mathcal{T}_1$ consists of one tree with leaves v_2, v_3, v_4 and three paths with ends v_2, v_3. Thus the graph $\bigcup \mathcal{T}$ is a frame of F_4, as required. If $X = \{e_{12}, e_{13}, e_{23}\}$, then every member in $\mathcal{T} - \mathcal{T}_X$ is a path with one end v_4; but then $|\mathcal{T}_4| \ge |\mathcal{T} - \mathcal{T}_X| = \sum_{1 \le i \le 3} |\mathcal{T}_i - \mathcal{T}_X| \ge 3 \cdot 3 = 9$, contrary to (2).

We next consider the case $H[X]$ contains a vertex of degree 3. Assume that $e_{12}, e_{13}, e_{14} \in X$, say. Then $|\mathcal{T}_X| \ge |\mathcal{T}_1| = 5$, and so, $|X| = 6$ and $|\mathcal{T}_X| = 5$. Hence $X = E(H)$ and $\mathcal{T} = \mathcal{T}_X$. Now $|\mathcal{T}| = 5$, and so $\mathcal{T} = \mathcal{T}_i$ for $1 \le i \le 4$. Thus every member of $\mathcal{T}$ contains all the vertices in Z. If some member of $\mathcal{T}$ contains two vertices of degree ≥ 3, then $\bigcup \mathcal{T}$ contains a K_4-subdivision on Z, a contradiction. So each of them has exactly one vertex of degree ≥ 3, and so, $\bigcup \mathcal{T}$ is an F_{12}-frame, as required.

We may thus assume that each vertex of $H[X]$ has degree one or two. Then $H[X]$ contains two independent edges, e_{12}, e_{34}, say. Every member of $\mathcal{T} - \mathcal{T}_X$ is a path between $\{v_1, v_2\}$ and $\{v_3, v_4\}$. In particular, we have

$$\begin{aligned}
\sum_{T \in \mathcal{T}_X} |Z \cap V(T)| &= \sum_{T \in \mathcal{T}} |Z \cap V(T)| - \sum_{T \in \mathcal{T} - \mathcal{T}_X} |Z \cap V(T)| \\
&= \sum_{1 \le i \le 4} |\mathcal{T}_i| - 2|\mathcal{T} - \mathcal{T}_X| \\
&= 20 - 2|\mathcal{T} - \mathcal{T}_X|.
\end{aligned}$$

This implies that:

(3) *The number of members of* $\mathcal{T}_X$ *having exactly three leaves is even.*

We may assume that X is either $\{e_{12}, e_{34}\}, \{e_{12}, e_{23}, e_{34}\}$ or $\{e_{12}, e_{23}, e_{34}, e_{14}\}$. If $X = \{e_{12}, e_{34}\}$, then the unique member of $\mathcal{T}_X$ is adjacent to e_{12}, e_{34} in J by the minimality of X, and so has four leaves in G. We deduce from (2) that $\mathcal{T} - \mathcal{T}_X$ consists of $k, 4-k, k, 4-k$ paths with ends $v_1v_3, v_3v_2, v_2v_4, v_4v_1$, respectively, for some $0 \leq k \leq 4$. If $k = 0, 4$ then the graph $\bigcup \mathcal{T}$ is an F_2-frame or contains an F_3-frame. If $0 < k < 4$ then the unique member of $\mathcal{T}_X$ contains no two disjoint paths with ends v_1v_2, v_3v_4, respectively; for otherwise, $\bigcup \mathcal{T}$ contains a K_4-subdivision on Z, a contradiction. If $k = 1, 3$ then the graph $\bigcup \mathcal{T}$ is an F_6-frame or contains an F_{10}-frame. If $k = 2$ then the graph $\bigcup \mathcal{T}$ is an F_{11}-frame, as required.

If $X = \{e_{12}, e_{23}, e_{34}\}$, then every member of $\mathcal{T} - \mathcal{T}_X$ is a path with ends v_1v_3, v_1v_4 or v_2v_4. Since $N_J(\{e_{12}, e_{23}\}) = N_J(X) = \mathcal{T}_X$ by the minimality of X, every member in $\mathcal{T}_X$ contains v_2; similarly, every member in $\mathcal{T}_X$ contains v_3. Since $N_J(e_{12}) \neq \emptyset$ by the minimality of X, some member in $\mathcal{T}_X$ contains v_1; similarly, some member in $\mathcal{T}_X$ contains v_4. By (1) and (3), the sequence $(Z \cap V(T))_{T \in \mathcal{T}_X}$ is either (Z, Z) or $(\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\})$. If the former holds, then $\mathcal{T} - \mathcal{T}_X$ consists of three paths with ends v_1, v_3 and three paths with ends v_2, v_4. Thus $\bigcup \mathcal{T}$ is an F_5-frame or contains a frame of F_6, F_{10} or F_0, as required. If the latter holds, then $\mathcal{T} - \mathcal{T}_X$ consists of $3, 1, 3$ paths with ends v_1v_3, v_1v_4, v_2v_4, respectively. Thus $\bigcup \mathcal{T}$ is an F_7-frame, as required.

If $X = \{e_{12}, e_{23}, e_{34}, e_{14}\}$, then every member of $\mathcal{T} - \mathcal{T}_X$ is a path with ends v_1v_3 or v_2v_4. By (2), $|\mathcal{T}_1 \cap \mathcal{T}_X| = |\mathcal{T}_3 \cap \mathcal{T}_X|$ and $|\mathcal{T}_2 \cap \mathcal{T}_X| = |\mathcal{T}_4 \cap \mathcal{T}_X|$. By (1) and (3), the sequence $(Z \cap V(T))_{T \in \mathcal{T}_X}$ is either (Z, Z, Z), $(Z, \{v_1, v_2, v_3\}, \{v_1, v_3, v_4\})$ or $(Z, \{v_1, v_2, v_4\}, \{v_2, v_3, v_4\})$. In either case, every member of $\mathcal{T}_X$ with four leaves contains no two disjoint paths between $\{v_1, v_3\}$ and $\{v_2, v_4\}$; for otherwise, the graph $\bigcup \mathcal{T}$ contains a K_4-subdivision on Z, a contradiction. Thus, if the first case holds then $\bigcup \mathcal{T}$ is an F_9-frame. If the second or third case holds then $\bigcup \mathcal{T}$ is an F_8-frame. This completes the proof. $\square$

The next task is to "augment" frames of F_i $(1 \leq i \leq 12)$ in 6-connected graphs to obtain a frame of F_0 or F_{13}. The proof is based on Lemma 2.2.1.

Lemma 3.3.3. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Suppose that there is no (≤ 5)-separation (A, B) of G with $|Z \cap A|, |Z \cap B| \geq 2$ and $|A - B|, |B - A| \geq 1$. If G contains an F_i-frame on Z for some $1 \leq i \leq 12$, then G contains a frame of F_0 or F_{13} on Z.*

Proof. We shall write $F_i \to F_{i_1}, \ldots, F_{i_k}$ to denote a claim that if there is an F_i-frame in G then there is an F_j-frame in G for some $j \in \{i_1, \ldots, i_k\}$. The result follows from the following twelve claims (1), (2), ..., (12): (1) $F_1 \to F_2$; (2) $F_2 \to F_3, F_4, F_5$; (3) $F_3 \to F_6, F_7$; (4) $F_4 \to F_6, F_7, F_8$; (5) $F_5 \to F_6, F_7, F_8, F_9$; (6) $F_6 \to F_0, F_{10}$; (7) $F_7 \to F_0, F_{10}$; (8) $F_8 \to F_0$; (9) $F_9 \to F_0$; (10) $F_{10} \to F_0, F_{13}$; (11) $F_{11} \to F_0, F_{10}$; (12) $F_{12} \to F_0$.

We only show (10), based on Lemma 2.2.1. The other claims can be proved in a similar way. Let $Z = \{v_1, v_2, v_3, v_4\}$. Suppose that there is an F_{10}-frame in G which consists of three paths P_1, P_2, P_3 with ends v_1, v_2, three paths Q_1, Q_2, Q_3 with ends v_3, v_4, two paths R_1, R_2 with ends v_1, v_4 and two paths R_3, R_4 with ends v_2, v_3, mutually disjoint except for Z. Let u_1, u_4 be neighbors of v_1, v_4 in $R_1 \cup R_2$, respectively, such that $u_1 \neq v_4$ and $u_4 \neq v_1$. Similarly, let u_2, u_3 be neighbors of v_2, v_3 in $R_3 \cup R_4$, respectively, such that $u_2 \neq v_3$ and $u_3 \neq v_2$. Now one can see four *disjoint* paths of G between $V(P_1 \cup P_2 \cup P_3) \cup \{u_1, u_2\}$ and $V(Q_1 \cup Q_2 \cup Q_3) \cup \{u_3, u_4\}$, all with no internal vertex in $V(P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3) \cup \{u_1, u_2, u_3, u_4\}$, covering v_i, u_i $(1 \leq i \leq 4)$. By the connectivity of G, we deduce from Lemma 2.2.1 that there are six disjoint paths of G between $V(P_1 \cup P_2 \cup P_3) \cup \{u_1, u_2\}$ and $V(Q_1 \cup Q_2 \cup Q_3) \cup \{u_3, u_4\}$, all with no internal vertex in $V(P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3) \cup \{u_1, u_2, u_3, u_4\}$, covering v_i, u_i $(1 \leq i \leq 4)$. Therefore, there are six paths $R'_1, \ldots, R'_6$ of G between $V(P_1 \cup P_2 \cup P_3)$ and $V(Q_1 \cup Q_2 \cup Q_3)$, mutually disjoint except for Z, all with no internal vertex in $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3$, such that each vertex of Z is covered by exactly two of $R'_1, \ldots, R'_6$. Now it is not difficult to see that the union of these twelve paths $P_1, P_2, P_3, Q_1, Q_2, Q_3, R'_1, \ldots, R'_6$ is an F_{13}-frame or contains an F_0-frame. This proves (10). $\square$

Lemma 3.2.3 immediately follows from Lemma 3.3.2 and Lemma 3.3.3.

3.4 Separating pairs

Definition 3.4.1 (separating pair)**.** Let G be a graph, C be a cycle of G and v_1, v_2 be distinct vertices of C. A pair $\{P_1, P_2\}$ of two paths P_1, P_2 of C is called a $(\{v_1, v_2\}, C)$-*separating pair* of G if

- P_1 is a subpath of one of the paths of C between v_1 and v_2 and P_2 is a subpath of the other path of C between v_1 and v_2;
- there is no edge of G between $\mathrm{int}(P_1) \cup \mathrm{int}(P_2) \cup (V(G) - V(C))$ and $V(C) - V(P_1 \cup P_2)$;
- $\{v_1, v_2\} \cap \mathrm{end}(P_1) = \{v_1, v_2\} \cap \mathrm{end}(P_2)$.

A $(\{v_1, v_2\}, C)$-separating pair $\{P_1, P_2\}$ of G is called *minimum* if $|V(P_1)| + |V(P_2)|$ is minimum.

The notion of separating pairs is due to [45]. Note that a $(\{v_1, v_2\}, C)$-separating pair of G always exists. Indeed, if P_1, P_2 are the two internally disjoint paths of C between v_1 and v_2, then $\{P_1, P_2\}$ is a $(\{v_1, v_2\}, C)$-separating pair of G. For a technical reason, we adopt the above definition that requires the third condition, which is slightly different from that of [45]. For the sake of safety and completeness, we give proofs of lemmas below, whose outline is different from [45]. The lemmas in this section will be frequently used in the subsequent sections.

Lemma 3.4.2. *Let G be a graph, C be a cycle of G and v_1, v_2 be distinct vertices of C. Let $\{P_1, P_2\}$ be a minimum $(\{v_1, v_2\}, C)$-separating pair of G. If $G \setminus V(C)$ is connected, then for any (not necessarily distinct) two vertices $x, y \in \mathrm{int}(P_1) \cup \mathrm{int}(P_2) \cup (V(G) - V(C))$, at least one of the following holds:*

(a) *For any $i \in \{1, 2\}$ there is a path of G through x, v_i, v_{3-i}, y in order.*
(b) *G contains a path with ends x, y and a cycle containing v_1, v_2, mutually disjoint.*

In particular, for any $x \in \mathrm{int}(P_1) \cup \mathrm{int}(P_2) \cup (V(G) - V(C))$ there is a cycle of $G \setminus x$ containing v_1 and v_2.

Proof. We assume that $v_1v_2 \notin E(G)$; the case $v_1v_2 \in E(G)$ is proved in a similar way. To show the lemma, let $x, y \in \mathrm{int}(P_1) \cup \mathrm{int}(P_2) \cup (V(G) - V(C))$.

Suppose that there is a (≤ 2)-separation (A, B) of G with $v_1 \in A - B$, $x \in A$, $v_2 \in B - A$ and $y \in B$. Now $A \cap B \subseteq V(C)$. Since $G \setminus V(C)$ is connected, $V(G) - V(C)$ is contained in $A - B$ or $B - A$. Assume that $V(G) - V(C) \subseteq B - A$, say. Now $x \in \mathrm{int}(P_1) \cup \mathrm{int}(P_2)$ and so $(\mathrm{int}(P_1) \cup \mathrm{int}(P_2)) \cap A \neq \emptyset$. This implies that $A - B$ contains a vertex of $P_1 \cup P_2$. Let $P_1' := P_1|B$ and $P_2' := P_2|B$. Now $\{P_1', P_2'\}$ is a $(\{v_1, v_2\}, C)$-separating pair of G, while $|V(P_1')| + |V(P_2')| < |V(P_1)| + |V(P_2)|$, a contradiction.

Therefore, there is no such separation (A, B) of G. This implies that there are three paths of G between $\{v_1, x\}$ and $\{v_2, y\}$, mutually disjoint except for $\{v_1, v_2\}$, such that each of v_1 and v_2 is covered by exactly two of them. Similarly, there are three paths of G between $\{v_1, y\}$ and $\{v_2, x\}$, mutually disjoint except for $\{v_1, v_2\}$, such that each of v_1 and v_2 is covered by exactly two of them. This implies that (a) or (b) holds for x, y. This proves the lemma. □

Lemma 3.4.3. *Let G be a graph, C be a cycle of G and v_1, v_2 be distinct vertices of C. Let $\{P_1, P_2\}$ be a minimum $(\{v_1, v_2\}, C)$-separating pair of G. If C is lean with respect to v_1 and v_2 in G, then for any $i \in \{1, 2\}$ and for any $x \in \mathrm{int}(P_1) \cup \mathrm{int}(P_2)$ there is a path of G from x to a vertex in $V(G) - V(C)$, with no internal vertex in $V(G) - V(C)$, containing x, v_i, v_{3-i} in this order listed.*

Proof. We assume that $v_1v_2 \notin E(G)$; the case $v_1v_2 \in E(G)$ is proved in a similar way. To show the lemma, let $x \in \mathrm{int}(P_1) \cup \mathrm{int}(P_2)$.

Suppose that there is a (≤ 2)-separation (A, B) of G with $v_1 \in A - B$, $x \in A$, $v_2 \in B - A$ and $V(G) - V(C) \subseteq B$. Now $A \cap B \subseteq V(C)$ and so $V(G) - V(C) \subseteq B - A$. Since $x \in (\mathrm{int}(P_1) \cup \mathrm{int}(P_2)) \cap A \neq \emptyset$, we deduce that $A - B$ contains a vertex of $P_1 \cup P_2$. Let $P_1' := P_1|B$ and $P_2' := P_2|B$. Now $\{P_1', P_2'\}$ is a $(\{v_1, v_2\}, C)$-separating pair of G, while $|V(P_1')| + |V(P_2')| < |V(P_1)| + |V(P_2)|$, a contradiction.

Therefore, there is no such separation (A, B) of G. This implies that there are three paths R_1, R_2, R_3 of G between $\{v_1, x\}$ and $\{v_2\} \cup (V(G) - V(C))$, mutually disjoint except for $\{v_1, v_2\}$, all with no internal vertex in $V(G) - V(C)$, such that each of v_1 and v_2 is covered by exactly two of them. Let y denote the vertex in $V(G) - V(C)$ covered by $R_1 \cup R_2 \cup R_3$. If two of them, R_1, R_2 say, have ends v_1, v_2, then $R_1 \cup R_2$ is a cycle of $G|V(C)$ containing v_1, v_2 and avoiding x. This contradicts the assumption that C is lean with respect to v_1, v_2 in G. Thus exactly one of R_1, R_2, R_3 has ends v_1, v_2. Now $R_1 \cup R_2 \cup R_3$ is a path of G through x, v_2, v_1, y in order, with no internal vertex in $V(G) - V(C)$, as required. Similarly, there is a path of G from x to a vertex in $V(G) - V(C)$, with no internal vertex in $V(G) - V(C)$, containing x, v_1, v_2 in this order listed. This proves the lemma. □

3.5 Proof of Lemma 3.2.4

The aim of this section is to prove Lemma 3.2.4. The proof is based on the lemmas in Section 3.4.

Proof. Suppose that G contains a skeleton on Z but contains no K_4-subdivision on Z. The goal is to show that there is a nice and lean bicycle on Z in G. For a subgraph H of G, the vertex set of each component of $G \setminus V(H)$ is called an *H-flap* of G. A tuple $(J, H, P_1, P_2, P_3, P_4)$ of subgraphs of G is *feasible* if J is a bicycle on Z, H is a 2-connected subgraph of $G \setminus V(J)$, and $P_1, \ldots, P_4$ are disjoint paths from Z to $V(H)$, all with no internal vertex in $J \cup H$. Such a feasible tuple exists, since there is a skeleton on Z in G. We call a $J \cup H$-flap of G *trivial* if it is disjoint from $P_1 \cup P_2 \cup P_3 \cup P_4$ and *non-trivial* otherwise; note that there are at most four non-trivial $J \cup H$-flaps. The *signature* of a feasible tuple $(J, H, P_1, \ldots, P_4)$ is the sequence $(|D_0|, |D_1|, \ldots, |D_n|)$, where D_0 is the union of non-trivial $J \cup H$-flaps and $D_1, \ldots, D_n$ are the trivial $J \cup H$-flaps, ordered with $|D_1| \geq \cdots \geq |D_n|$. We choose a feasible tuple $(J, H, P_1, \ldots, P_4)$ with H maximal, subject to that with its signature lexicographically maximum. The goal is to show that J is a nice and lean bicycle on Z. Let $Z = \{v_1, v_2, v_3, v_4\}$ and assume that J is a bicycle on (v_1, v_2, v_3, v_4). Let C_i denote the cycle of J with $V(C_i) \cap Z = \{v_i, v_{i+1}\}$ for $1 \leq i \leq 4$, where indices are read modulo 4. For $1 \leq i \leq 4$ assume that $v_i \in \mathrm{end}(P_i)$ and let u_i denote the other end of P_i in H.

(1) *J is lean in G.*

For if J is not lean, then for some $1 \leq i \leq 4$ there is a cycle C_i' of G with $v_i, v_{i+1} \in V(C_i') \subsetneq V(C_i)$. Let J' be a bicycle obtained from J by replacing C_i with C_i'. Now $(J', H, P_1, P_2, P_3, P_4)$ is feasible. Every $J \cup H$-flap is a subset of a $J' \cup H$-flap. Consequently, the signature of the tuple $(J', H, P_1, P_2, P_3, P_4)$ is greater than that of $(J, H, P_1, P_2, P_3, P_4)$, contrary to our choice. This proves (1).

(2) *For $1 \leq i \leq 4$, there is no edge of G between $V(C_i) - \{v_i, v_{i+1}\}$ and $V(J) - V(C_i)$.*

For if there is such an edge, we deduce from the existence of $H \cup P_1 \cup P_2 \cup P_3 \cup P_4$ that there is a K_4-subdivision on Z in G, a contradiction. This proves (2).

(3) *There is no trivial $J \cup H$-flap.*

For suppose to the contrary that there is a trivial $J \cup H$-flap D, chosen with $|D|$ minimum; so $|D|$ is the last term of the signature. Since there is no K_4-subdivision on Z in G, we have $N_G(D) \cap V(J) \subseteq V(C_i)$ for some $1 \leq i \leq 4$. Assume that $N_G(D) \cap V(J) \subseteq V(C_1)$, say. Let $\{Q_1, Q_2\}$ be a minimum $(\{v_1, v_2\}, C_1)$-separating pair of $G|V(C_1) \cup D$. Let $S := \mathrm{int}(Q_1) \cup \mathrm{int}(Q_2) \cup D$. Since $\mathrm{int}(Q_1) \cup \mathrm{int}(Q_2)$ is not adjacent to $V(J) - V(C_1)$ by (2), we have $N_G(S) \cap (V(J) - V(C_1)) = \emptyset$. No $J \cup H$-flap other than D is adjacent to S. For if a vertex x in S is adjacent to a $J \cup H$-flap ($\neq D$), then by Lemma 3.4.2 (applied to $G = G|V(C_1) \cup D$) we may replace C_1 with a cycle of $(G|V(C_1) \cup D) \setminus x$ containing v_1 and v_2 to increase the signature without changing H, a contradiction. Thus $N_G(S) - V(C_1) \subseteq V(H)$. Note that $N_G(S) \cap V(C_1) = \mathrm{end}(Q_1) \cup \mathrm{end}(Q_2)$ and so $|N_G(S) \cap V(C_1)| \leq 4$. Since G is 6-connected, we have $|N_G(S) \cap V(H)| \geq 2$. Let x', y' be distinct vertices of $N_G(S) \cap V(H)$ and let x, y be vertices of S with $xx', yy' \in E(G)$. Apply Lemma 3.4.2 to $G = G|V(C_1) \cup D, C = C_1, x$ and y. If (a) holds, then for any $i \in \{1, 2\}$ there is a path of $G|V(C_1) \cup D$ through x, v_i, v_{3-i}, y in order. Since H

contains two disjoint paths between $\{u_3, u_4\}$ and $\{x', y'\}$, we deduce that there is a path of $G|V(P_3 \cup P_4 \cup H \cup C_1) \cup D$ through v_3, v_1, v_2, v_4 in this order listed. This path, together with C_2, C_3 and C_4, yields a K_4-subdivision on Z, a contradiction. If (b) holds, then $G|V(C_1) \cup D$ contains a path R with ends x, y and a cycle C_1' containing v_1, v_2, mutually disjoint. Let J' be a bicycle obtained from J by replacing C_1 with C_1' and let H' be the union of H and the path consisting of $x'x$, R and yy'. Now $(J', H', P_1, P_2, P_3, P_4)$ is a feasible tuple, contrary to the maximality of H. Therefore, there is no trivial $J \cup H$-flap. This proves (3).

(4) *There is no non-trivial $J \cup H$-flap.*

For let X_i denote the non-trivial $J \cup H$-flap containing $\mathrm{int}(P_i)$ if $\mathrm{int}(P_i) \neq \emptyset$ and define $X_i := \emptyset$ otherwise for $1 \leq i \leq 4$. Note that $X_1, \ldots, X_4$ are pairwise disjoint and non-adjacent, and furthermore, $N_G(X_i) \cap V(H) = \{u_i\}$ for $1 \leq i \leq 4$; for otherwise, we may augment H, a contradiction. Suppose to the contrary that $X_1 \neq \emptyset$, say. Since G contains no K_4-subdivision on Z, there is no neighbor of X_1 in $V(C_2 \cup C_3) - \{v_2, v_4\}$. Hence $N_G(X_1) \subseteq \{u_1\} \cup V(C_1 \cup C_4)$.

Let $\{Q_1, Q_2\}$ be a minimum $(\{v_1, v_2\}, C_1)$-separating pair of $G|V(C_1) \cup X_1$ and let $\{R_1, R_2\}$ be a minimum $(\{v_1, v_4\}, C_4)$-separating pair of $G|V(C_4) \cup X_1$. Since $v_1 \in N_G(X_1)$, by the definition of separating pairs each of Q_1, Q_2, R_1, R_2 contains v_1 as its one end.

We show that $N_G(\mathrm{int}(Q_1) \cup \mathrm{int}(Q_2) \cup \mathrm{int}(R_1) \cup \mathrm{int}(R_2) \cup X_1) \subseteq \{u_1\} \cup \mathrm{end}(Q_1) \cup \mathrm{end}(Q_2) \cup \mathrm{end}(R_1) \cup \mathrm{end}(R_2)$. Suppose to the contrary that there is an edge of G from a vertex a in $\mathrm{int}(Q_1) \cup \mathrm{int}(Q_2) \cup \mathrm{int}(R_1) \cup \mathrm{int}(R_2) \cup X_1$ to a vertex b in $V(G) - V(Q_1 \cup Q_2 \cup R_1 \cup R_2) \cup X_1 \cup \{u_1\}$. We may assume that $a \in \mathrm{int}(Q_1) \cup \mathrm{int}(Q_2)$. By (2), $b \in (V(H) - \{u_1\}) \cup X_2 \cup X_3 \cup X_4$. By Lemma 3.4.3 (applied to $G = G|V(C_1) \cup X_1, C = C_1, P_1 = Q_1, P_2 = Q_2$ and $x = a$), we deduce that for any $i \in \{1, 2\}$ there is a path S_i of $G|V(C_1) \cup X_1$ from a to a vertex y_i in X_1, with no internal vertex in X_1, containing a, v_i, v_{3-i}, y_i in this order listed. Note that there are two disjoint paths of $G|V(H) \cup X_2 \cup X_3 \cup X_4 \cup \{v_3, v_4\}$ between $\{v_3, v_4\}$ and $\{u_1, b\}$. For some $i \in \{1, 2\}$ one can concatenate these two paths, edge ab, the path S_i and a path of $G|X_1 \cup \{u_1\}$ between y_i and u_1 to obtain a path of $G|V(C_1 \cup H) \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup \{v_3, v_4\}$ through v_3, v_1, v_2, v_4 in order. This path, together with C_2, C_3 and C_4, yields a K_4-subdivision on Z, a contradiction. Therefore, there is no such edge ab of G. This proves the claim.

Since G is 6-connected, the cut $\{u_1\} \cup \mathrm{end}(Q_1) \cup \mathrm{end}(Q_2) \cup \mathrm{end}(R_1) \cup \mathrm{end}(R_2)$ has size ≥ 6. Hence $|\mathrm{end}(Q_1) \cup \mathrm{end}(Q_2) - \{v_1\}| = |\mathrm{end}(R_1) \cup \mathrm{end}(R_2) - \{v_1\}| = 2$, which implies that Q_1, Q_2, R_1, R_2 have positive lengths. Since G is 6-connected, there is an edge from v_1 to a vertex x in $V(G) - V(Q_1 \cup Q_2 \cup R_1 \cup R_2) \cup X_1 \cup \{u_1\}$. Since C_1 is lean with respect v_1, v_2 and Q_1, Q_2 have positive lengths, we have $x \notin V(C_1) - V(Q_1 \cup Q_2)$. Similarly, $x \notin V(C_4) - V(R_1 \cup R_2)$. Since G contains no K_4-subdivision on Z, we have $x \notin V(C_2 \cup C_3) \cup X_3 - \{v_2, v_4\}$. Thus $x \in (V(H) - \{u_1\}) \cup X_2 \cup X_4$ and so there is a path of $G|\{v_1, v_3\} \cup V(H) \cup X_2 \cup X_3 \cup X_4$ between v_1 and v_3. But, since Q_1, Q_2, R_1, R_2 have positive lengths and are disjoint from $\{v_2, v_4\}$, both $V(C_1) - \{v_1, v_2\}$ and $V(C_4) - \{v_4, v_1\}$ are adjacent to X_1. Now one can see that G contains a K_4-subdivision on Z, a contradiction. This proves (4).

By (3) and (4), $H = G \setminus V(J)$ and $\{v_1u_1, \ldots, v_4u_4\}$ is a matching of G from Z to $V(H)$. Since H is 2-connected, J is a nice bicycle. Moreover, J is lean in G by (1). This completes the proof of the lemma. $\square$

3.6 Proof of Lemma 3.2.6

The aim of this section is to prove Lemma 3.2.6. For the proof we use the following lemma as a "two-paths theorem", which is slightly stronger than Lemma 3.2.2. The proof is easy.

Lemma 3.6.1. *Let v_1, v_2, v_3, v_4 be distinct vertices of a graph G. If there is no (≤ 3)-separation (A, B) of G with $v_1, v_2, v_3, v_4 \in A$ and $B - A \neq \emptyset$ such that $G|B$ cannot be drawn in a disc with $A \cap B$ on the boundary, then either there are two disjoint paths of G with ends v_1v_3, v_2v_4 respectively, or G can be drawn in a disc with v_1, v_2, v_3, v_4 on the boundary in order.*

In the proof of Lemma 3.2.6, we will encounter the following subproblem.

> A graph has six distinct vertices $s_1, s_2, s_3, t_1, t_2, t_3$ and satisfies the condition that for any three disjoint paths between $\{s_1, s_2, s_3\}$ and $\{t_1, t_2, t_3\}$, one connects s_3 and t_3. Then what kind of structure does the graph have?

Such a structure is characterized in [63, 64, 65], but we do no need the full strength of the result. For our purposes, the following lemma is enough. For the sake of completeness we give a direct proof in the next section.

Lemma 3.6.2. *Let G be a graph and $s_1, s_2, s_3, t_1, t_2, t_3$ be distinct vertices of G. Suppose that*

(i) *for any three disjoint paths of G between $\{s_1, s_2, s_3\}$ and $\{t_1, t_2, t_3\}$, one of them connects s_3 to t_3, and*
(ii) *there is no (≤ 5)-separation (A, B) of G with $s_1, s_2, s_3, t_1, t_2, t_3 \in A$ and $B - A \neq \emptyset$.*

Let P_1, P_2 be two disjoint paths of $G \setminus \{s_3, t_3\}$ between $\{s_1, s_2\}$ and $\{t_1, t_2\}$ such that

(iii) *every vertex of $P_1 \cup P_2$ is contained in any two disjoint paths of $G|V(P_1 \cup P_2)$ between $\{s_1, s_2\}$ and $\{t_1, t_2\}$, and*
(iv) *there is no (≤ 1)-separation (A, B) of $G \setminus V(P_1 \cup P_2)$ with $s_3, t_3 \in A$ and $B - A \neq \emptyset$.*

Then the graph obtained from G by shrinking $P_1 \cup P_2$ into a single vertex, denoted by p, can be drawn in a disc with s_3, t_3, p on the boundary.

Based on Lemma 3.6.2, we prove Lemma 3.2.6.

Proof of Lemma 3.2.6. Let J be a nice and lean bicycle on (v_1, v_2, v_3, v_4) in G. Let C_i denote the tire of J with $v_i, v_{i+1} \in V(C_i)$ for $1 \leq i \leq 4$, where indices are read modulo 4. Let G' denote the graph obtained from G by shrinking each $C_i \setminus \{v_i, v_{i+1}\}$ into a single vertex, which we will denote by c_i for $1 \leq i \leq 4$. Let $Z := \{v_1, \ldots, v_4\}$. Let $H := G \setminus V(J) = G' \setminus (Z \cup \{c_1, \ldots, c_4\})$. Suppose that there is no K_4-subdivision on Z in G. By Lemma 3.2.1 we have:

(1) *There are no two disjoint paths of G' with ends v_1v_3, v_2v_4, respectively.*

We claim:

(2) $N_{G'}(c_i) \subseteq V(H) \cup \{v_i, v_{i+1}\}$ *and* $|N_{G'}(c_i) \cap V(H)| \geq 4$ *for* $1 \leq i \leq 4$.

For H is a connected graph and adjacent to all the vertices of Z in G' (as J is nice). Since $C_i \setminus \{v_i, v_{i+1}\}$ is non-null, c_i is adjacent to v_i and v_{i+1} in G' for $1 \leq i \leq 4$. By (1) we have $N_{G'}(c_i) \subseteq V(H) \cup \{v_i, v_{i+1}\}$ for $1 \leq i \leq 4$. Since G is 6-connected, we have $|N_{G'}(c_i) \cap V(H)| \geq \deg_{G'}(c_i) - |\{v_i, v_{i+1}\}| \geq 6 - 2 = 4$ for $1 \leq i \leq 4$. This proves (2).

To show the lemma, suppose to the contrary that G' cannot be drawn in a disc with v_1, v_2, v_3, v_4 on the boundary in order.

(3) *There is a 2-separation (A, B) of H such that for some $1 \leq i \leq 4$ it holds that $N_G(B - A) \cap V(J) \subseteq V(C_i) - \{v_i, v_{i+1}\}$ and $G'|B \cup \{c_i\}$ cannot be drawn in a disc with $(A \cap B) \cup \{c_i\}$ on the boudanry.*

For by Lemma 3.6.1 and (1) there is a (≤ 3)-separation (A, B) of G' with $Z \subseteq A$ and $B - A \neq \emptyset$ such that $G'|B$ cannot be drawn in a disc with $A \cap B$ on the boundary. Since $Z \subseteq A$ and there is a matching of G' of size 4 from Z to $V(H)$, we have $V(H) \cap (A - B) \neq \emptyset$. On the other hand, $V(H) \cap (B - A) \neq \emptyset$; for otherwise, $B - A \subseteq \{c_1, \ldots, c_4\}$ but since $|A \cap B| \leq 3$, we deduce from (2) that $B - A$ contains a vertex of H adjacent to some c_i, a contradiction. Since H is 2-connected (as J is nice), we have $|A \cap B \cap V(H)| \geq 2$ and hence $A \cap B$ contains at most one vertex in $Z \cup \{c_1, \ldots, c_4\}$. If some c_i is in $B - A$, then $v_i, v_{i+1} \in A \cap B$, a contradiction. Hence $Z \cup \{c_1, \ldots, c_4\} \subseteq A$. On the other hand, since G is 4-connected, $A \cap B$ intersects $\{c_1, \ldots, c_4\}$. Thus $|A \cap B \cap \{c_1, \ldots, c_4\}| = 1$ and so $|A \cap B \cap V(H)| = 2$. Let $A' := A \cap V(H)$ and $B' := B \cap V(H)$. Now (A', B') is a desired 2-separation of H. This proves (3).

Let (A, B) be a 2-separation of H as in (3); assume that $N_G(B - A) \cap V(J) \subseteq V(C_1) - \{v_1, v_2\}$ and $G'|B \cup \{c_1\}$ cannot be drawn in a disc with $(A \cap B) \cup \{c_1\}$ on the boundary. Since J is nice and $N_G(B - A) \cap Z = \emptyset$, there are distinct vertices u_3, u_4 in A with $u_iv_i \in E(G)$ for $i = 3, 4$. Let $\{P_1, P_2\}$ be a minimum $(\{v_1, v_2\}, C_1)$-separating pair of $G|V(C_1) \cup (B - A)$.

(4) $N_G(\mathrm{int}(P_1) \cup \mathrm{int}(P_2) \cup (B - A)) = \mathrm{end}(P_1) \cup \mathrm{end}(P_2) \cup (A \cap B)$.

For let $A \cap B = \{w_1, w_2\}$. Suppose to the contrary that there is an edge of G from a vertex a in $\mathrm{int}(P_1) \cup \mathrm{int}(P_2) \cup (B - A)$ to a vertex b in $V(G) - V(P_1 \cup P_2) \cup B$. By the definition of $\{P_1, P_2\}$, we have $a \in \mathrm{int}(P_1) \cup \mathrm{int}(P_2)$ and $b \notin V(C_1) - V(P_1 \cup P_2)$. By (2), $b \in V(H)$, and so $b \in A - B$. Note that the graph obtained from $G|A$ by adding a new vertex adjacent to w_1, w_2 is 2-connected. This implies that there are two disjoint paths Q_1, Q_2 of $G|A$ from $\{u_3, u_4\}$ to $\{w_1, w_2, b\}$, with one ending at b (which may have an internal vertex in $\{w_1, w_2\}$); assume that the other path ends at w_1, say. By Lemma 3.4.3 (applied to $G = G|V(C_1) \cup (B - A)$ and $x = a$), for any $i \in \{1, 2\}$ there is a path R_i of $G|V(C_1) \cup (B - A)$ from a to a vertex y_i in $B - A$, with no internal vertex in $B - A$, containing a, v_i, v_{3-i}, y_i in this order listed. For some $i \in \{1, 2\}$ one can concatenate paths

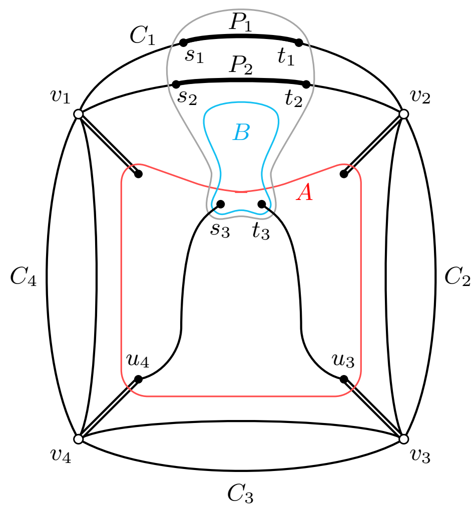

Figure 3.8: A graph G in the proof of Lemma 3.2.6.

R_i, Q_1, Q_2, a path of $G|(B-A) \cup \{y_i, w_1\}$ from y_i to w_1, and edges v_3u_3, v_4u_4, ab to obtain a path of $G|V(C_1 \cup H) \cup \{v_3, v_4\}$ through v_3, v_1, v_2, v_4 in order. This path, together with C_2, C_3 and C_4, yields a K_4-subdivision on Z, a contradiction. Therefore, there is no such edge ab, and hence $N_G(\text{int}(P_1) \cup \text{int}(P_2) \cup (B - A)) = \text{end}(P_1) \cup \text{end}(P_2) \cup \{w_1, w_2\}$, as required. This proves (4).

Since G is 6-connected, the cut $\text{end}(P_1) \cup \text{end}(P_2) \cup (A \cap B)$ has size ≥ 6. Hence $\text{end}(P_1) \cup \text{end}(P_2)$ has size 4, which implies that each of P_1, P_2 has distinct ends not in $\{v_1, v_2\}$.

We want to apply Lemma 3.6.2 to $G = G|V(P_1 \cup P_2) \cup B$. See Figure 3.8 for intuition. For $i = 1, 2$ let s_i, t_i denote the ends of P_i with s_i closer to v_1 in C_1. Since H is 2-connected, there are two disjoint paths R_1, R_2 of G from $\{u_3, u_4\}$ to $A \cap B$. Let $A \cap B = \{s_3, t_3\}$ and assume that R_1, R_2 have ends u_3t_3, u_4s_3, respectively. Note that $s_1, s_2, s_3, t_1, t_2, t_3$ are distinct vertices. If there are three disjoint paths of $G|V(P_1 \cup P_2) \cup B$ between $\{s_1, s_2, s_3\}$ and $\{t_1, t_2, t_3\}$ such that none of them connects s_3 and t_3, then there is a path of $G|V(C_1) \cup B$ through s_3, v_2, v_1, t_3 in order. Combining this path and $R_1 \cup R_2$, we obtain a path of $G|\{v_3, v_4\} \cup V(H \cup C_1)$ through v_3, v_1, v_2, v_4 in order. This path, together with C_2, C_3 and C_4, yields a K_4-subdivision on Z, a contradiction. Thus there are no such three paths of $G|V(P_1 \cup P_2) \cup B$. Since G is 6-connected, there is no (≤ 5)-separation (X, Y) of $G|V(P_1 \cup P_2) \cup B$ with $s_1, s_2, s_3, t_1, t_2, t_3 \in X$ and $Y - X \neq \emptyset$. Since C_1 is lean with respect to v_1 and v_2 in G, every vertex of $P_1 \cup P_2$ is contained in any two disjoint paths of $G|V(P_1 \cup P_2)$ between $\{s_1, s_2\}$ and $\{t_1, t_2\}$. Since H is 2-connected, there is no (≤ 1)-separation (X, Y) of $G|B$ with $s_3, t_3 \in X$ and $Y - X \neq \emptyset$.

Therefore, by Lemma 3.6.2 applied to $G = G|V(P_1 \cup P_2) \cup B$, the graph obtained from $G|V(P_1 \cup P_2) \cup B$ by shrinking $P_1 \cup P_2$ into a single vertex, denoted by p, can be drawn in a disc with s_3, t_3, p on the boundary. Since there is no edge of G between $B - A$ and $V(C_1) - V(P_1 \cup P_2)$, this implies that $G'|B \cup \{c_1\}$ can be drawn in a disc with s_3, t_3, c_1 on the boundary. But this contradicts the property of (A, B). This completes the proof of the lemma. $\square$

3.7 Proof of Lemma 3.6.2

This section is devoted to proving Lemma 3.6.2 that was used in the proof of Lemma 3.2.6. A *triad* in a graph G is a connected subgraph T of G with no cycle, with one vertex of degree 3 and all others of degree ≤ 2. It has precisely three vertices of degree 1, called its *feet*. Let v_1, v_2, v_3 be distinct vertices of a graph G. A *tripod* on v_1, v_2, v_3 is a subgraph $T_1 \cup T_2 \cup Q_1 \cup Q_2 \cup Q_3$ of G consisting of three distinct vertices u_1, u_2, u_3, two triads T_1, T_2 with feet u_1, u_2, u_3, mutually disjoint except for their feet, and three disjoint paths Q_1, Q_2, Q_3 of G (possibly length of 0) with ends u_1v_1, u_2v_2, u_3v_3, respectively, all with no internal vertex in $T_1 \cup T_2$. We call Q_1, Q_2, Q_3 the *legs* of the tripod. The following lemma gives a structural characterization of tripods.

Lemma 3.7.1 ([51, theorem (3.5)]). *Let v_1, v_2, v_3 be distinct vertices of a graph G. If there is no (≤ 2)-separation (A, B) of G such that $v_1, v_2, v_3 \in A$ and $|B - A| \geq 2$, then either G contains a tripod on v_1, v_2, v_3 or G can be drawn in a disc with v_1, v_2, v_3 on the boundary.*

Instead of this, we will use the following lemma, which is slightly stronger than Lemma 3.7.1. The proof is easy.

Lemma 3.7.2. *Let v_1, v_2, v_3 be distinct vertices of a graph G. If there is no (≤ 2)-separation (A, B) of G with $v_1, v_2, v_3 \in A$ and $B - A \neq \emptyset$ such that $G|B$ cannot be drawn in a disc with $A \cap B$ on the boundary, then either there is a tripod on v_1, v_2, v_3 in G or G can be drawn in a disc with v_1, v_2, v_3 on the boundary.*

For the proof of Lemma 3.6.2, we consider the following condition for G, s_i, t_i, P_i in the lemma.

- (v) For any 2-separation (A, B) of $G \setminus V(P_1 \cup P_2)$ with $s_3, t_3 \in A$ and $A - B \neq \emptyset$, the graph obtained from $G|B \cup V(P_1 \cup P_2)$ by shrinking $P_1 \cup P_2$ into a single vertex, denoted by p, can be drawn in a disc with $(A \cap B) \cup \{p\}$ on the boundary.
- (vi) The graph $(G \setminus V(P_1 \cup P_2)) \setminus \{s_3t_3\}$ is 2-connected.
- (vii) For $i = 1, 2$, if the end of P_i in $\{s_1, s_2\}$ is not adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$ then P_{3-i} contains an internal vertex adjacent to the vertex in $\{s_1, s_2\} \cap \mathrm{end}(P_1)$, and the end of P_{3-i} in $\{s_1, s_2\}$ is adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$. An analogous result holds for the ends of P_1, P_2 in $\{t_1, t_2\}$.

The following three claims state that (v), (vi) and (vii) hold for minimum counterexamples to Lemma 3.6.2. The proof of Claim (v) is similar to that of Lemma 3.2.6 in Section 3.6. The proof of Claim (vii) is similar to that of [45, Lemma 2.2].

Claim 3.7.3. *If a tuple of G, s_i, t_i, P_i is a counterexample to Lemma 3.6.2 with $|V(G)|$ minimum, then* (v) *holds.*

Proof. To show (v), let (A, B) be a 2-separation of $G \setminus V(P_1 \cup P_2)$ with $s_3, t_3 \in A$ and $A - B \neq \emptyset$. If $B - A = \emptyset$ then the claim obviously holds. So assume that $B - A \neq \emptyset$. Let $A \cap B = \{w_1, w_2\}$. Let Q_1, Q_2 be subpaths of P_1, P_2, respectively, such that there is no edge of G between $\mathrm{int}(Q_1) \cup \mathrm{int}(Q_2) \cup (B - A)$ and $V(P_1 \cup P_2) - V(Q_1 \cup Q_2)$; such paths exists, since $Q_1 = P_1$ and $Q_2 = P_2$ work. Choose such paths Q_1, Q_2 with $|V(Q_1)| + |V(Q_2)|$ minimum.

Now $N_G(\mathrm{int}(Q_1) \cup \mathrm{int}(Q_2) \cup (B - A)) = \mathrm{end}(Q_1) \cup \mathrm{end}(Q_2) \cup \{w_1, w_2\}$. This can be seen by the same proof as (4) in the proof of Lemma 3.2.6. For if there is an edge from a vertex $a \in \mathrm{int}(Q_1) \cup \mathrm{int}(Q_2) \cup (B - A)$ to a vertex b in $V(G) - V(Q_1 \cup Q_2) \cup B$, then $a \in \mathrm{int}(Q_1) \cup \mathrm{int}(Q_2)$ and $b \in A - B$. By (iv) there are two disjoint paths R_1, R_2 of $G|A$ from $\{s_3, t_3\}$ to $\{b, w_1, w_2\}$, with one ending at b. Assume that s_3 is connected to b by $R_1 \cup R_2$; the case t_3 is connected to b is similarly proved. Let w be the vertex in $\{w_1, w_2\} \cap (\mathrm{end}(R_1) \cup \mathrm{end}(R_2))$. By the same proof as in Lemma 3.4.3, there is a vertex y of $B - A$ such that $G|V(P_1 \cup P_2) \cup (B - A)$ contains three disjoint paths between $\{s_1, s_2, a\}$ and $\{t_1, t_2, y\}$, all with no internal vertex in $B - A$ and none with connecting a to y. One can concatenate these three paths, a path of $G|(B - A) \cup \{w\}$ between y and w, the paths R_1, R_2, and edge ab to obtain three disjoint paths of G between $\{s_1, s_2, s_3\}$ and $\{t_1, t_2, t_3\}$ such that none of them connects s_3 and t_3. This contradicts (i). This proves that $N_G(\mathrm{int}(Q_1) \cup \mathrm{int}(Q_2) \cup (B - A)) = \mathrm{end}(Q_1) \cup \mathrm{end}(Q_2) \cup \{w_1, w_2\}$, as required.

By (ii), the cut $\mathrm{end}(Q_1) \cup \mathrm{end}(Q_2) \cup \{w_1, w_2\}$ has size ≥ 6, and so Q_1, Q_2 have positive lengths. For $i = 1, 2$ let s'_i, t'_i denote the ends of Q_i with s'_i closer to the end of P_i in $\{s_1, s_2\}$ in P_i. By (iv) there are two disjoint paths of $G|A$ from $\{s_3, t_3\}$ to $\{w_1, w_2\}$. Suppose that these paths have ends $s_3 w_1, t_3 w_2$, respectively, say. Now the assumptions (i), (ii), (iii), (iv) of Lemma 3.6.2 are satisfied by $G = G|V(Q_1 \cup Q_2) \cup B, s_i = s'_i, t_i = t'_i$ $(i = 1, 2)$, $s_3 = w_1, t_3 = w_2, P_1 = Q_1$ and $P_2 = Q_2$. Since $A - B \neq \emptyset$, the graph $G|V(Q_1 \cup Q_2) \cup B$ has fewer vertices than G, and so is not a counterexample to Lemma 3.6.2. Thus the graph obtained from $G|V(Q_1 \cup Q_2) \cup B$ by shrinking $Q_1 \cup Q_2$ into a single vertex can be drawn in a disc with w_1, w_2 and the shrunk vertex on the boundary. Since there is no edge of G between $B - A$ and $V(P_1 \cup P_2) - V(Q_1 \cup Q_2)$, the same assertion holds for the graph obtained from $G|V(P_1 \cup P_2) \cup B$ by shrinking $P_1 \cup P_2$ into a single vertex. Therefore, (v) holds. This completes the proof. □

Claim 3.7.4. *If a tuple of G, s_i, t_i, P_i is a counterexample to Lemma 3.6.2 and satisfies* (v)*, then* (vi) *holds.*

Proof. Let $G' := (G \backslash V(P_1 \cup P_2)) \backslash \{s_3 t_3\}$. Since G is a counterexample to Lemma 3.6.2, G' contains a vertex other than s_3, t_3. Suppose to the contrary that G' is not 2-connected. By (iv) there is a 1-separation (X, Y) of G' with $s_3 \in X - Y$ and $t_3 \in Y - X$. Let $X \cap Y = \{x\}$. By (v) applied to $A := X \cup \{t_3\}$ and $B := Y$, the graph obtained from $G|Y \cup V(P_1 \cup P_2)$ by shrinking $P_1 \cup P_2$ into a single vertex, denoted by p, can be drawn in a disc with t_3, x, p on the boundary. Similarly, the graph obtained from $G|X \cup V(P_1 \cup P_2)$ by shrinking $P_1 \cup P_2$ into a single vertex can be drawn in a disc with s_3, x and the shrunk vertex on the boundary. Therefore, the graph obtained from G by shrinking $P_1 \cup P_2$ into a single vertex can be drawn in a disc with s_3, t_3 and the shrunk vertex on the boundary. This contradicts that G is a counterexample to Lemma 3.6.2. This completes the proof. □

Claim 3.7.5. *If a tuple of G, s_i, t_i, P_i is a counterexample to Lemma 3.6.2 with $|V(G)|$ minimum, then* (vii) *holds.*

Proof. Assume that P_i has ends s_i, t_i for $i = 1, 2$. Since G is a counterexample to Lemma 3.6.2, we have $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\} \neq \emptyset$. Suppose that s_1 is not adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$, say; the other cases are proved in a similar way.

If $N_G(s_1) \cap \mathrm{int}(P_2) = \emptyset$, then let s_1' be the neighbor of s_1 in P_1. By (ii), $s_1' \neq t_1$. Now the assumptions in Lemma 3.6.2 are satisfied when replacing G, s_1, P_1 with $G \setminus s_1, s_1', P_1 \setminus s_1$, respectively. Since $G \setminus s_1$ has fewer vertices than G, Lemma 3.6.2 holds for $G \setminus s_1$. Now one can see that the assertion of Lemma 3.6.2 also holds for G, contrary to that G is a counterexample. This proves that $N_G(s_1) \cap \mathrm{int}(P_2) \neq \emptyset$, as required.

If s_2 is not adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$, then by the same proof we have $N_G(s_2) \cap \mathrm{int}(P_1) \neq \emptyset$. By (iii), $N_G(s_1) \cap \mathrm{int}(P_2) = \{s_2'\}$ and $N_G(s_2) \cap \mathrm{int}(P_1) = \{s_1'\}$, where s_i' is the neighbor of s_i in P_i for $i = 1, 2$. Now the assumptions in Lemma 3.6.2 are satisfied when replacing G, s_1, s_2, P_1, P_2 with $G \setminus \{s_1, s_2\}, s_1', s_2', P_1 \setminus s_1, P_2 \setminus s_2$, respectively. Thus Lemma 3.6.2 holds for $G \setminus \{s_1, s_2\}$ and hence for G as well, contrary to that G is a conterexample. This proves that s_2 is adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$, as required. This completes the proof. □

From now on we may use assumptions (v), (vi) and (vii) to show Lemma 3.6.2. A sketch of the remaining proof is as follows. First we show that if the lemma is false then the graph $G \setminus \{s_3t_3, s_1s_2, t_1t_2\}$ can be drawn in a disc with $s_2, s_3, s_1, t_1, t_3, t_2$ on the boundary in this order listed (Claims 3.7.6–3.7.8). On the other hand, by edge-counting in the planar graph we see that G contains at most one vertex other than $s_1, s_2, s_3, t_1, t_2, t_3$ (Lemma 3.7.9). This is a contradiction, because the graph $G \setminus V(P_1 \cup P_2)$ contains ≥ 2 vertices except but s_3 and t_3 by (vi). We begin with the following claim, which is slightly weaker than Lemma 3.6.2.

Claim 3.7.6. *Let G, s_i, t_i, P_i be as in Lemma 3.6.2. If* (v) *and* (vii) *hold, then there is no vertex p of $P_1 \cup P_2$ such that G contains a tripod on s_3, t_3, p with no other vertex in $V(P_1 \cup P_2)$.*

Proof. Assume that P_i has ends s_i, t_i for $i = 1, 2$. Suppose to the contrary that there is a vertex p of P_1 say, such that G contains a tripod $T_1 \cup T_2 \cup Q_1 \cup Q_2 \cup Q_3$ on s_3, t_3, p with no other vertex in $V(P_1 \cup P_2)$. Choose such a tripod with the legs $Q_1 \cup Q_2 \cup Q_3$ minimal. Let u_1, u_2, u_3 denote the feet of the triads T_1 and T_2. Assume that Q_1, Q_2, Q_3 have ends pu_1, s_3u_2, t_3u_3, respectively.

There is no path of G from $V(T_1 \cup T_2)$ to $V(P_1 \cup Q_1 \cup Q_2 \cup Q_3)$ with no vertex in $V(P_2) \cup \{u_1, u_2, u_3\}$. For otherwise, a minimal such path R has an end in $V(P_1) - \{p\}$ by the minimality of the legs of the tripod. But the union of R, P_1 and the tripod $T_1 \cup T_2 \cup Q_1 \cup Q_2 \cup Q_3$ contains two disjoint paths of $G \setminus V(P_2)$ with ends s_3t_1, s_1t_3, respectively, contrary to (i).

Thus there is a separation (X, Y) of G with $X \cap Y = V(P_2) \cup \{u_1, u_2, u_3\}$, $V(T_1 \cup T_2) \subseteq X$ and $V(P_1 \cup Q_1 \cup Q_2 \cup Q_3) \subseteq Y$. By choosing such a separation with X minimal, we may assume that $X - Y$ consists of the union of two disjoint subsets of $X - Y$, with one containing $V(T_1) - \{u_1, u_2, u_3\}$, the other containing $V(T_2) - \{u_1, u_2, u_3\}$ and both inducing connected subgraphs of $G|(X - Y)$. This implies that for any $a \in N_G(X - Y) \cap$

$V(P_2)$ one can find any 2-linkage on $\{u_1, u_2, u_3, a\}$ in $G|(X-Y) \cup \{u_1, u_2, u_3, a\}$. Note that $|N_G(X-Y) \cap V(P_2)| \geq 3$ by (ii).

There are no three disjoint paths of $G|Y$ from $V(P_2) \cup \{s_3, t_3\}$ to $\{u_1, u_2, u_3\}$ with no vertex in $V(P_1)$. For if there are such three disjoint paths R_1, R_2, R_3, all with no internal vertex in $V(P_2)$, then the union of R_1, R_2, R_3, P_2 and a 2-linkage on $\{u_1, u_2, u_3, a\}$ in $G|(X-Y) \cup \{u_1, u_2, u_3, a\}$ for some $a \in N_G(X-Y) \cap V(P_2) - V(R_1 \cup R_2 \cup R_3)$ contains two disjoint paths of $G \setminus V(P_1)$ with ends s_3t_2, s_2t_3, respectively, contrary to (i).

Thus there is a separation (Y_1, Y_2) of $G|Y$ of order $\leq |V(P_1)|+2$ with $V(P_1) \subseteq Y_1 \cap Y_2$, $V(P_2) \cup \{s_3, t_3\} \subseteq Y_2$ and $u_1, u_2, u_3 \in Y_1$. Now one can find a tripod on $\{p\} \cup (Y_1 \cap Y_2 - V(P_1))$ in $G|X \cup Y_1$, with no other vertex in $V(P_1 \cup P_2)$, which is indeed a subgraph of the tripod $T_1 \cup T_2 \cup Q_1 \cup Q_2 \cup Q_3$. Thus the graph obtained from $G|X \cup Y_1$ by shrinking $P_1 \cup P_2$ into a single vertex cannot drawn in a disc so that the two vertices in $Y_1 \cap Y_2 - V(P_1)$ and the shrunk vertex occur in the boundary. By (v) applied to $A = Y_2 - V(P_1 \cup P_2)$ and $B = Y_1 \cup X - V(P_1 \cup P_2)$, we deduce that $A - B = \emptyset$. Hence $Y_2 - Y_1 = V(P_2)$ and $s_3, t_3 \in Y_1 \cap Y_2$. This implies that $N_G(V(P_2)) \subseteq V(P_1) \cup \{s_3, t_3\} \cup (X-Y)$.

Let $a_1, a_2 \in N_G(X-Y) \cap V(P_2)$ with $N_G(X-Y) \cap V(P_2) \subseteq V(P_2[a_1, a_2])$; assume that s_2, a_1, a_2, t_2 occur in P_2 in order. We show that $N_{G|Y_2}(V(P_2(a_1, a_2))) \subseteq \{p\}$. Since P_2 is an induced path of G by (iii), there is no edge of G from $V(P_2(a_1, a_2))$ to $V(P_2) - V(P_2[a_1, a_2])$. So let there be an edge of G from a vertex a in $V(P_2(a_1, a_2))$ to a vertex b in $\{s_3, t_3\} \cup (V(P_1) - \{p\})$. If $b = s_3$ then the union of P_2, ab, Q_3 and a path of $G|(X-Y) \cup \{u_3, a_1\}$ from u_3 to a_1 contains two disjoint paths of $G \setminus V(P_1)$ with ends, s_3t_2, s_2t_3, respectively, contrary to (i); the case $b = t_3$ is similar. If $b \in V(P_1[s_1, p))$ then let R_1, R_2 be two disjoint paths of $G|(X-Y) \cup \{u_1, u_2, u_3, a_1\}$ with ends u_1u_2, u_3a_1, respectively. Now the two paths $Q_2 \cup R_1 \cup Q_1 \cup P_1[p, t_1]$, $P_2[s_2, a_1] \cup R_2 \cup Q_3$ and a path consisting of the union of $P_1[s_1, b], ba$ and $P_2[a, t_2]$ yield three disjoint paths of G with ends s_3t_1, s_2t_3, s_1t_2, respectively, a contradiction; the case $b \in V(P_1(p, t_1])$ is similar. Thus there is no such edge ab of G. This proves that $N_{G|Y_2}(V(P_2(a_1, a_2)) \subseteq \{p\}$.

Therefore, $\{u_1, u_2, u_3, a_1, a_2, p\}$ is a cut of G that separates $(X-Y) \cup V(P_2(a_1, a_2))$ from the other vertices of G. By (ii) we have $p \neq u_1$. Also we deduce from (ii) that p is adjacent to $V(P_2(a_1, a_2))$ in G.

We show that p is the only vertex of P_1 adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$. To see this, we first remark that there are no three disjoint paths of $(G|Y_1) \setminus p$ from $\{u_1, u_2, u_3\}$ to $\{s_3, t_3\} \cup V(P_1 \setminus p)$; for otherwise, we may choose such three disjoint paths, all with no internal vertex in $V(P_1)$, and covering $\{s_3, t_3\}$; if the third has an end in $P_1(p, t_1]$ say, then since p is adjacent to $V(P_2(a_1, a_2))$, by the same proof as above one can find three disjoint paths of G with ends s_3t_1, s_2t_3, s_1t_2, respectively, contrary to (i). Thus there are no such three paths of $(G|Y_1) \setminus p$. Now there is a 3-separation (U, W) of $G|Y_1$ with $p \in U \cap W$, $\{s_3, t_3\} \cup V(P_1) \subseteq U$ and $u_1, u_2, u_3 \in W$. By (v) applied to $A = U - V(P_1)$ and $B = (W - \{p\}) \cup (X - V(P_2))$, we deduce that $A - B = \emptyset$. Hence $U - W = V(P_1 \setminus p)$ and $s_3, t_3 \in U \cap W$. This implies that $P_1 \setminus p$ has no vertex adjacent to $(W-U) \cup (X-Y)$. Since $(W-U) \cup (X-Y) = (Y_1 - V(P_1) \cup \{s_3, t_3\}) \cup (X-Y) = V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$, this proves that p is the only vertex of P_1 adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$.

Now we use (vii). We may assume from the symmetry that $p \neq s_1$. Since s_1 is not adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$, we deduce from (vii) that P_2 contains an internal vertex q adjacent to s_1, and s_2 is adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$. Since $N_G(V(P_2)) \subseteq V(P_1) \cup \{s_3, t_3\} \cup (X - Y)$, we have $s_2 \in N_G(X - Y)$ and so $s_2 = a_1$. Let R_1, R_2 be two disjoint paths of $G|(X - Y) \cup \{u_1, u_2, u_3, s_2\}$ with ends u_1u_2, s_2u_3, respectively. Now the two paths $Q_2 \cup R_1 \cup Q_1 \cup P_1[p, t_1]$, $R_2 \cup Q_3$ and a path consisting of the union of s_1q and $P_2[q, t_2]$ yield three disjoint paths of G with ends s_3t_1, s_2t_3, s_1t_2, respectively, contrary to (i). This completes the proof. □

Claim 3.7.7. *Let G, s_i, t_i, P_i be as in Lemma 3.6.2. If* (v) *and* (vi) *hold, then $(G \setminus V(P_1 \cup P_2)) \setminus \{s_3t_3\}$ can be drawn in a disc with s_3, t_3 on the boundary.*

Proof. Note that $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\} \neq \emptyset$ by (vi). By (ii) we may assume that $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$ is adjacent to $V(P_1)$ in G. Let G_1 denote the graph obtained from $(G \setminus V(P_2)) \setminus \{s_3t_3\}$ by shrinking P_1 into a single vertex, which we denote by p_1.

We show that there is no tripod on s_3, t_3, p_1 in G_1. Suppose to the contrary that there is a tripod $T_1 \cup T_2 \cup Q_1 \cup Q_2 \cup Q_3$ on s_3, t_3, p_1 in G_1; assume that Q_1 is the leg incident to p_1. If Q_1 has positive length, then by expanding p_1 to P_1, we can find a vertex p of P_1 such that G contains a tripod on s_3, t_3, p with no other vertex in $V(P_1 \cup P_2)$, contrary to Claim 3.7.6. So $E(Q_1) = \emptyset$ and T_1, T_2 contain p_1. By expanding p_1 to P_1, one can obtain from T_1, T_2 two triads T_1', T_2' in $G \setminus V(P_2)$, both with exactly one vertex in $V(P_1)$. If T_1', T_2' have distinct vertices in $V(P_1)$, then the union of $T_1' \cup T_2' \cup Q_2 \cup Q_3$ and P_1 contains two disjoint paths of $G \setminus V(P_2)$ with ends s_3t_1, s_1t_3, respectively, contrary to (i). So T_1', T_2' have a common vertex p in $V(P_1)$. Now $T_1' \cup T_2' \cup Q_2 \cup Q_3$ is a tripod on s_3, t_3, p in G with no other vertex in $V(P_1 \cup P_2)$, contrary to Claim 3.7.6.

Thus there is no tripod on s_3, t_3, p_1 in G_1. If G_1 can be drawn in a disc with s_3, t_3, p_1 on the boundary, then the claim follows. So assume that G_1 does not admit such a drawing. By Lemma 3.7.2 there is a (≤ 2)-separation (A, B) of G_1 with $s_3, t_3, p_1 \in A$ and $B - A \neq \emptyset$ such that $G|B$ cannot be drawn in a disc with $A \cap B$ on the boundary. By (iv) we have $p_1 \in A - B$ and $|A \cap B| = 2$. Now $(A - \{p_1\}, B)$ is a 2-separation of $G \setminus V(P_1 \cup P_2)$ with $s_3, t_3 \in A - \{p_1\}$. If $(A - \{p_1\}) - B \neq \emptyset$, then we deduce from (v) that the graph $G|B$ can be drawn in a disc with $A \cap B$ on the boundary; but this contradicts the property of (A, B). Thus $A - B = \{p_1\}$ and $A \cap B = \{s_3, t_3\}$. This implies that $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$ is not adjacent to $V(P_1)$ in G, contrary to our assumption. This completes the proof. □

Now we are ready to prove the planarity of $G \setminus \{s_3t_3, s_1s_2, t_1t_2\}$.

Claim 3.7.8. *Let G, s_i, t_i, P_i be a tuple contrary to Lemma 3.6.2. If* (v), (vi) *and* (vii) *hold, then $G \setminus \{s_1s_2, t_1t_2, s_3t_3\}$ can be drawn in a disc with $s_1, s_3, s_2, t_2, s_3, t_1$ on the boundary in this order listed.*

Proof. Let $G' := (G \setminus (V(P_1 \cup P_2)) \setminus \{s_3t_3\}$. By Claim 3.7.7 and (vi), the graph G' can be drawn in a plane with an outer cycle K containing s_3, t_3. We choose such a drawing of G' with $|N_G(V(P_1 \cup P_2)) - V(K)|$ minimum. Let K_1, K_2 be the two subpaths of K between s_3 and t_3. Since $s_3t_3 \notin E(G')$, both K_1 and K_2 have internal vertices.

We show that $|N_G(V(P_1 \cup P_2)) - V(K)| = 0$. For suppose to the contrary that there is an edge of G from a vertex $p \in V(P_1 \cup P_2)$ to a vertex $x \in V(G') - V(K)$. Let X be the vertex set of the component of $G' \setminus V(K)$ containing x. If X is adjacent to both $\mathrm{int}(K_1)$ and $\mathrm{int}(K_2)$, then G' contains a tripod on x, s_3, t_3. This implies that G contains a tripod on p, s_3, t_3 with no other vertex in $V(P_1 \cup P_2)$, contrary to Claim 3.7.6. So we may assume that $N_{G'}(X) \subseteq V(K_1)$. Let a_1, a_2 denote the vertices in $N_{G'}(X) \cap V(K_1)$ with $N_{G'}(X) \cap V(K_1) \subseteq V(K_1[a_1, a_2])$. Note that $a_1 \neq a_2$ by (vi). By planarity there is a 2-separation (A, B) of G' with $V(K) - V(K_1(a_1, a_2)) \subseteq A$ and $V(K_1[a_1, a_2]) \cup X \subseteq B$. Note that $s_3, t_3 \in A$ and $A - B \neq \emptyset$ by $\mathrm{int}(K_2) \neq \emptyset$. We deduce from (v) that $G|B$ contains a path Q between a_1 and a_2 such that $N_G(V(P_1 \cup P_2)) \cap B \subseteq V(Q)$ and $G|B$ can be drawn in a disc with Q on the boundary. Replace the drawing of $G|B$ in G' with such a drawing, and replace $K_1[a_1, a_2]$ with Q. Now the new drawing of G' has a smaller value of $|N_G(V(P_1 \cup P_2)) - V(K)|$, a contradiction. This proves that $|N_G(V(P_1 \cup P_2)) - V(K)| = 0$.

Thus $N_G(V(P_1 \cup P_2)) \subseteq V(K)$. If $V(P_1)$ has neighbors in both $\mathrm{int}(K_1)$ and $\mathrm{int}(K_2)$, then $G|V(K) \cup V(P_1)$ contains either two disjoint paths with ends s_1t_3, s_3t_1, respectively, or a tripod on s_3, t_3, p for some $p \in V(P_1)$, with no other vertex in $V(P_1 \cup P_2)$; the former contradicts (i) and the latter contradicts Claim 3.7.6. So $N_G(V(P_1)) \cap V(K) \subseteq V(K_i)$ for some $i \in \{1, 2\}$; an analogous result holds for P_2. If both $N_G(V(P_1)) \cap V(K)$ and $N_G(V(P_2)) \cap V(K)$ are contained in the same $V(K_i)$, then the graph obtained from G by shrinking $P_1 \cup P_2$ into a single vertex can be drawn in disc with s_3, t_3 and the shrunk vertex on the boundary, contrary to the assumption that G is a counterexample. Thus we may assume that $N_G(V(P_i)) \cap V(K) \subseteq V(K_i)$ for $i = 1, 2$.

By (i), there is no "cross" between P_1 and K_1. More precisely, there are no four distinct vertices $x_1, x_2 \in V(P_1)$ and $y_1, y_2 \in V(K_1)$ with $x_1y_2, x_2y_1 \in E(G)$ such that s_1, x_1, x_2, t_1 occur in P_1 in order and s_3, y_1, y_2, t_3 occur in K_1 in order. Similarly, there is no "cross" between P_2 and K_2. Therefore, if we let G'' denote the graph obtained from G by deleting edge s_3t_3 and all edges between $V(P_1)$ and $V(P_2)$, then G'' can be drawn in a disc with $s_1, s_3, s_2, t_1, t_3, t_2$ on the boundary in order.

We show that both s_i and t_i are adjacent to $\mathrm{int}(K_i)$ for $i = 1, 2$. For suppose that s_1 is not adjacent to $\mathrm{int}(K_1)$, say. Now s_1 is not adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$. By (vii), P_2 contains an internal vertex adjacent to s_1, and s_2 is adjacent to $V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}$; hence s_2 is adjacent to $\mathrm{int}(K_2)$. If $V(P_1) - \{s_1\}$ is adjacent to $\mathrm{int}(K_1)$ then there are three disjoint paths in $G|V(P_1 \cup P_2) \cup V(K)$ with ends s_1t_2, s_2t_3, s_3t_1, respectively, contrary to (i). Therefore, $N_G(V(P_1)) \cap V(K) \subseteq \{s_3, t_3\}$ and so $N_G(V(P_1 \cup P_2)) \cap V(K) \subseteq V(K_2)$. This implies that the graph obtained from $G \setminus \{s_3t_3\}$ by shrinking $P_1 \cup P_2$ into a single vertex can be drawn in a disc with s_3, t_3 and the shrunk vertex on the boundary, contrary to the assumption that G is a counterexample. Thus s_i and t_i are adjacent to $\mathrm{int}(K_i)$ for $i = 1, 2$.

Finally we show that there is no edge of $G \setminus \{s_1s_2, t_1t_2\}$ between $V(P_1)$ and $V(P_2)$. Suppose to the contrary that there is an edge of G between $V(P_1 \setminus t_1)$ and $V(P_2 \setminus s_2)$, say. Since t_1 is adjacent to $\mathrm{int}(K_1)$ and s_2 is adjacent $\mathrm{int}(K_2)$, we deduce that there are three disjoint paths in $G|V(P_1 \cup P_2) \cup V(K)$ with ends s_1t_2, s_2t_3, s_3t_1, respectively, which contradicts (i). This proves that there is no edge of $G \setminus \{s_1s_2, t_1t_2\}$ between $V(P_1)$ and $V(P_2)$.

Therefore, $G \setminus \{s_1s_2, t_1t_2, s_3t_3\}$ can be drawn in a disc with $s_1, s_3, s_2, t_2, s_3, t_1$ on the boundary in order, as desired. This completes the proof. □

To prove Lemma 3.6.2 by using Claim 3.7.8, we need the following lemma, which can be proved by edge-counting on planar graphs.

Lemma 3.7.9. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 6$. Suppose that there is no (≤ 5)-separation (A, B) of G with $Z \subseteq A$ and $B - A \neq \emptyset$. If G can be drawn in a disc with Z on the boundary, then $|V(G) - Z| \leq 1$.*

Proof. Suppose to the contrary that $|V(G) - Z| \geq 2$. Let (A, B) be a 6-separation of G with $Z \subseteq A$ and $|B - A| \geq 2$ such that $G|B$ can be drawn in a disc with $A \cap B$ on the boundary; an example is given by $A = Z$ and $B = V(G)$. Choose such a separation (A, B) with B minimal. Let $A \cap B = \{v_1, \ldots, v_6\}$ and assume that $G|B$ can be drawn in a disc with $v_1, \ldots, v_6$ on the boundary in this order listed.

Let G' be a graph obtained from $G|B$ by adding a new vertex v and edges v_iv_{i+1} and vv_i for $1 \leq i \leq 6$, where $v_7 = v_1$. Now G' is planar. By the connectivity of G, each v_i has at least one neighbor in $B - A$, and so has degree ≥ 4 in G'. Each vertex in $B - A$ has degree ≥ 6 in G', and v has degree 6 in G'. Thus we have

$$\begin{aligned} 6 + 6 \cdot 4 + 6(|B| - 6) &\leq \deg_{G'}(v) + \sum_{1 \leq i \leq 6} \deg_{G'}(v_i) + \sum_{x \in B - A} \deg_{G'}(x) \\ &= \sum_{x \in V(G')} \deg_{G'}(x) = 2|E(G')| \\ &\leq 2(3|V(G')| - 6) = 6|B| - 6, \end{aligned}$$

throughout which the equality holds. Thus each v_i has exactly one neighbor in $B - A$, which we will denote by u_i. By $|B - A| \geq 2$ and the connectivity of G, we deduce that $u_1, \ldots, u_6$ are distinct. Let $A' := A \cup \{u_1\}$ and $B' := B - \{v_1\}$. Now (A', B') is a 6-separation of G with $Z \subseteq A'$ and $|B' - A'| \geq 5$ such that $G|B'$ can be drawn in a disc with $A' \cap B'$ on the boundary. This contradicts the minimality of B. Therefore, $|V(G) - Z| \leq 1$. This proves the lemma. □

We complete the proof of Lemma 3.6.2, based on Claim 3.7.8 and Lemma 3.7.9.

Proof of Lemma 3.6.2. Let G, s_i, t_i, P_i be a tuple contrary to Lemma 3.6.2, with $|V(G)|$ minimum. Now (v), (vi) and (vii) hold by Claims 3.7.3, 3.7.4 and 3.7.5. We deduce from (vi) that $|V(G) - V(P_1 \cup P_2) \cup \{s_3, t_3\}| \geq 2$; and so $|V(G) - \{s_1, s_2, s_3, t_1, t_2, t_3\}| \geq 2$. By Claim 3.7.8, the graph $G' := G \setminus \{s_1s_2, t_1t_2, s_3t_3\}$ can be drawn in a disc with $s_1, s_3, s_2, t_2, s_3, t_1$ on the boundary in this order listed. But by Lemma 3.7.9 applied to $G = G'$ and $Z = \{s_1, s_2, s_3, t_1, t_2, t_3\}$, we have $|V(G') - \{s_1, s_2, s_3, t_1, t_2, t_3\}| \leq 1$, which is a contradiction. This completes the proof of the lemma. □

3.8 Proof of Theorem 3.1.2

In this section we complete the proof of Theorem 3.1.2. The most of this section is devoted to proving Theorem 3.1.2 (5). The other statements of the theorem can be proved in a similar way; we only give a sketch of the proof of Thoerem 3.1.2 (4). We start with the following lemma, which can be derived from Menger's theorem.

Lemma 3.8.1. *Let G be a graph and k, m be positive integers. Let $S = \{b_1, \ldots, b_k\} \subseteq V(G)$ be a stable set of G and let $T, T' \subseteq V(G) - S$ with $|T| = |T'| = m$. Suppose that there are m disjoint paths of $G \setminus S$ between T and T'. If for any $v \in V(G) - S$ there are no m disjoint paths of $G \setminus (S \cup \{v\})$ between T and T', then exactly one of the following holds:*

(a) *For some $1 \leq i < j \leq k$ there are $m + 1$ disjoint paths of G between $T \cup \{b_i\}$ and $T' \cup \{b_j\}$, all with no internal vertex in S, such that none of them connects b_i and b_j.*
(b) *There is a path-decomposition $(A_1, \ldots, A_k)$ of G such that*
   - $T \subseteq A_1$, $T' \subseteq A_k$,
   - $b_i \in A_i - A_{i-1} \cup A_{i+1}$ *for* $1 \leq i \leq k$, *where* $A_0 = A_{k+1} = \emptyset$, *and*
   - $|A_i \cap A_{i+1}| = m$ *for* $1 \leq i < k$.

Proof. It is easy to see that both (a) and (b) do not hold simultaneously. By induction on $k = |S|$, we show that if (a) is false then (b) holds. Since the case $k = 1$ is trivial, suppose that $k \geq 2$.

There is a $(\leq m)$-separation (A, B) of G with $T \cup (S - \{b_k\}) \subseteq A$ and $T' \cup \{b_k\} \subseteq B$. For otherwise, there are $m + 1$ disjoint paths $P_1, \ldots, P_{m+1}$ of G between $T \cup (S - \{b_k\})$ and $T' \cup \{b_k\}$, all with no internal vertex in $T \cup T' \cup S$. By the existence of m disjoint paths of $G \setminus S$ between T and T', we may assume from Lemma 2.2.1 that T and T' are covered by $P_1, \ldots, P_{m+1}$; assume that $P_1, \ldots, P_m$ have ends in T. Let $b_j \in S - \{b_k\}$ be the vertex covered by P_{m+1}. Since (a) is false, P_m must connect b_j and b_k. Since S is stable, we have $\operatorname{int} P_{m+1} \neq \emptyset$. But $P_1, \ldots, P_m$ are m disjoint paths of $G \setminus (S \cup \operatorname{int}(P_{m+1}))$ between T and T', contrary to our assumption. Thus there is such a separation (A, B).

Since there are m disjoint paths of $G \setminus S$ between T and T', the cut set $A \cap B$ is contained in the union of such m disjoint paths; and so $|A \cap B| = m$. Consequently, $S - \{b_k\} \subseteq A - B$ and $b_k \in B - A$. By induction applied to $S - \{b_k\}, T$ and $A \cap B$ in the graph $G|A$, there is a path-decomposition $(A_1, \ldots, A_{k-1})$ of $G|A$ such that $T \subseteq A_1$, $A \cap B \subseteq A_{k-1}$, $b_i \in A_i - A_{i-1} \cup A_{i+1}$ for $1 \leq i \leq k - 1$, where $A_0 = A_k = \emptyset$, and $|A_i \cap A_{i+1}| = m$ for $1 \leq i < k - 1$. Let $A'_i := A_i$ for $1 \leq i \leq k - 1$ and $A'_k := B$. Now $(A'_1, \ldots, A'_k)$ is a path-decomposition of G satisfying (b). This completes the induction and proves the lemma. □

Lemma 3.8.2. *Let G be a graph and k be a positive integer. Let $S = \{b_1, \ldots, b_k\} \subseteq V(G)$ and let C be a spanning cycle of $G \setminus S$ containing distinct two vertices v_1, v_2. Suppose that $S \cup \{v_1, v_2\}$ is a stable set of G. If C is lean with respect to v_1, v_2 in G, then exactly one of the following holds:*

(a) *For some $1 \leq i < j \leq k$ there is a path of G through b_i, v_2, v_1, b_j in order, with no other vertex in S.*
(b) *There is a path-decomposition $(A_1, \ldots, A_k)$ of G such that*
   - $v_1, b_1 \in A_1 - A_2$, $v_2, b_k \in A_k - A_{k-1}$,

- *$b_i \in A_i - A_{i-1} \cup A_{i+1}$ for $1 < i < k$ and*
- *$|A_i \cap A_{i+1}| = 2$ for $1 \leq i < k$.*

Proof. It is easy to see that (a) and (b) do not hold simultaneously. We show that at least one of them holds. Since the case $k = 1$ is trivial, we assume that $k \geq 2$. Let t_1, t_2 be the neighbors of v_1 in C, and t'_1, t'_2 be the neighbors of v_2 in C, with $v_1, t_1, t'_1, v_2, t'_2, t_2$ on C in order; note that $t_1, t_2, t'_1, t'_2 \notin \{v_1, v_2\}$, since $v_1v_2 \notin E(G)$. Since C is lean in G with respect to v_1, v_2 and $\{v_1, v_2\} \cup S$ is stable in G, we have $N_G(v_1) = \{t_1, t_2\}$ and $N_G(v_2) = \{t'_1, t'_2\}$. Moreover, since C is a spanning cycle of $G \setminus S$, we deduce that there are no two disjoint paths of $G \setminus (S \cup \{v_1, v_2\})$ between $\{t_1, t_2\}$ and $\{t'_1, t'_2\}$ missing some vertex of $G \setminus (S \cup \{v_1, v_2\})$. Apply Lemma 3.8.1 to $G = G \setminus \{v_1, v_2\}, S, T = \{t_1, t_2\}$ and $T' = \{t'_1, t'_2\}$. If Lemma 3.8.1 (a) holds, then Lemma 3.8.2 (a) follows. If Lemma 3.8.1 (b) holds, then there is a path-decomposition $(A_1, \ldots, A_k)$ of $G \setminus \{v_1, v_2\}$ such that $t_1, t_2 \in A_1$, $t'_1, t'_2 \in A_k$, $b_i \in A_i - A_{i-1} \cup A_{i+1}$ for $1 \leq i \leq k$, where $A_0 = A_{k+1} = \emptyset$, and $|A_i \cap A_{i+1}| = 2$ for $1 \leq i < k$. Now let $A'_1 := A_1 \cup \{v_1\}$, $A'_k := A_k \cup \{v_2\}$ and $A'_i := A_i$ for $1 < i < k$. Since $N_G(v_1) = \{t_1, t_2\}$ and $N_G(v_2) = \{t'_1, t'_2\}$, every edge of G is contained in some A'_i. Thus $(A'_1, \ldots, A'_k)$ is a path-decomposition of G satisfying Lemma 3.8.2 (b). This proves the lemma. □

We are now ready to complete the proof of Theorem 3.1.2 (5).

Proof of Theorem 3.1.2 (5). Let $Z := \{v_1, v_2, v_3, v_4\}$. It is easy to see that if G is discoid for $(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})$ for some ordering $Z = \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\}$ then G contains no K_4-subdivision on Z. To show the converse, suppose that G contains no K_4-subdivision on Z. Since G is 6-connected, there is a nice and lean bicycle J on Z in G by Lemma 3.2.5. Assume that J is a bicycle on (v_1, v_2, v_3, v_4), say. Our goal is to show that G is discoid for (v_1, v_2, v_3, v_4). Let C_i denote the tire of J with $v_i, v_{i+1} \in V(C_i)$ for $1 \leq i \leq 4$, where indices are read modulo 4. Since J is lean, C_i is lean with respect to v_i, v_{i+1} for $1 \leq i \leq 4$. Let $H := G \setminus V(J)$. By Lemma 3.2.7, H contains a cycle K such that:

- H can be drawn in a disc with K on the boundary, and
- there are eight distinct vertices $x_1, \ldots, x_8$ occurring in K in this order listed such that
  - $x_{2i-1}, x_{2i} \in N_G(v_i)$,
  - $N_G(v_i) \subseteq V(C_i \cup C_{i-1}) \cup V(K\langle x_{2i-1}, x_{2i}\rangle)$, and
  - $N_G(V(C_i) - \{v_i, v_{i+1}\}) \subseteq \{v_i, v_{i+1}\} \cup V(K\langle x_{2i}, x_{2i+1}\rangle)$

for $1 \leq i \leq 4$. Note that $|V(K\langle x_{2i}, x_{2i+1}\rangle)| \geq 4$ for $1 \leq i \leq 4$, since G is 6-connected and $V(C_i \setminus \{v_i, v_{i+1}\}) \neq \emptyset$.

(1) *For $1 \leq i \leq 4$ there is no path of $G|V(C_i) \cup V(K\langle x_{2i}, x_{2i+1}\rangle)$ through $x_{2i}, v_{i+1}, v_i, x_{2i+1}$ in order.*

For let there be a path of $G|V(C_1) \cup V(K\langle x_2, x_3\rangle)$ through x_2, v_2, v_1, x_3 in order, say. By combining it with the two disjoint paths of K with ends x_2x_8, x_3x_5, respectively, we obtain a path of $G|\{v_3, v_4\} \cup V(H) \cup V(C_1)$ through v_3, v_1, v_2, v_4 in this order listed. This, together with C_2, C_3 and C_4, yields a K_4-subdivision on Z in G, a contradiction. This proves (1).

(2) *For $1 \le i \le 4$, if $v_i v_{i+1} \in E(G)$ then the graph $G|V(C_i) \cup V(K\langle x_{2i}, x_{2i+1}\rangle)$ can be drawn in a disc with $v_i, x_{2i}, K\langle x_{2i}, x_{2i+1}\rangle, x_{2i+1}, v_{i+1}$ on the boundary in this order listed.*

For if $v_1v_2 \in E(G)$ say, then one of the subpath of C_1 between v_1 and v_2 is the edge v_1v_2, since C_1 is lean with respect to v_1, v_2. Let P be the other subpath of C_1 between v_1 and v_2. Note that P is an induced path of G and that $N_G(\mathrm{int}(P)) \subseteq \{v_1, v_2\} \cup V(K\langle x_2, x_3\rangle)$. Since there is no path of $G|V(C_1) \cup V(K\langle x_2, x_3\rangle)$ through x_2, v_2, v_1, x_3 in order by (1), there is no "cross" between P and $K\langle x_2, x_3\rangle$. More precisely, there are no four distinct vertices $p_1, p_2 \in V(P)$ and $q_1, q_2 \in V(K\langle x_2, x_3\rangle)$ with $p_1q_2, p_2q_1 \in E(G)$ such that v_1, p_1, p_2, v_2 occur in P in order and x_2, q_1, q_2, x_3 occur in $K\langle x_2, x_3\rangle$ in order. Therefore, $G|V(C_1) \cup V(K\langle x_2, x_3\rangle)$ can be embedded in a plane so that the outer cycle consists of $K\langle x_2, x_3\rangle, x_3v_2, v_2v_1$ and v_1x_2. This proves (2).

For $1 \le j \le 4$ define a subgraph H_j of G as follows: If $v_jv_{j+1} \in E(G)$ then let $V(H_j) = \{v_j, v_{j+1}\}$ and $E(H_j) = \emptyset$. If $v_jv_{j+1} \notin E(G)$ then let H_j be a graph obtained from $G|V(C_j) \cup V(K\langle x_{2j}, x_{2j+1}\rangle)$ by deleting edges spanned by $\{v_j, v_{j+1}\} \cup V(K\langle x_{2j}, x_{2j+1}\rangle)$. Let L denote the subgraph of G induced by $V(H) \cup Z \cup \bigcup_{j: v_jv_{j+1} \in E(G)} V(C_j)$.

We show that (L, H_1, H_2, H_3, H_4) is a discoid decomposition of G for (v_1, v_2, v_3, v_4). First, it is easy to see that L, H_1, H_2, H_3, H_4 are five edge-disjoint subgraphs of G so that $G = L \cup H_1 \cup H_2 \cup H_3 \cup H_4$. By definition one can see that $v_1, v_2, v_3, v_4 \in V(L)$, $H_1 \cap H_3$ and $H_2 \cap H_4$ are null and $V(H_j \cap H_{j+1}) = \{v_{j+1}\}$ for $1 \le j \le 4$.

Note that for $1 \le j \le 4$ the set $V(L \cap H_j) - \{v_j, v_{j+1}\}$ equals $V(K\langle x_{2j}, x_{2j+1}\rangle)$ if $v_jv_{j+1} \notin E(G)$ and empty otherwise. For $1 \le j \le 4$, if $v_jv_{j+1} \notin E(G)$ then let $k_j := |V(K\langle x_{2j}, x_{2j+1}\rangle)|$ (≥ 4) and let $b^j_1, \dots, b^j_{k_j}$ denote the vertices of $K\langle x_{2j}, x_{2j+1}\rangle$ from x_{2j} to x_{2j+1} in order; if $v_jv_{j+1} \in E(G)$ then let $k_j := 0$. We deduce from the drawing of H and (2) that L can be drawn in a disc with $v_1, b^1_1, \dots, b^1_{k_1}, v_2, b^2_1, \dots, b^2_{k_2}, v_3, b^3_1, \dots, b^3_{k_3}, v_4, b^4_1, \dots, b^4_{k_4}$ on the boundary in order.

The remaining task is to show that if $k_j \ge 2$ then H_j has a path-decomposition as in the definition of discoid graphs. To see this, let $j \in \{1, 2, 3, 4\}$ with $k_j \ge 2$. Now $v_jv_{j+1} \notin E(G)$ by definition. We deduce from (1) that for any $1 \le i < i' \le k_j$ there is no path of H_j through $b^j_i, v_{j+1}, v_j, b^j_{i'}$ in order, with no other vertex in $\{b^j_1, \dots, b^j_{k_j}\}$. Thus, Lemma 3.8.2 (b) holds for $G = H_j$, $S = \{b^j_1, \dots, b^j_{k_j}\}$, $C = C_j$, $v_1 = v_j$ and $v_2 = v_{j+1}$. So there is a path-decomposition $(A^j_1, \dots, A^j_{k_j})$ of H_j such that $v_j, b^j_1 \in A^j_1 - A^j_2$, $v_{j+1}, b^j_k \in A^j_{k_j} - A^j_{k_j-1}$, $b^j_i \in A^j_i - A^j_{i-1} \cup A^j_{i+1}$ for $1 < i < k_j$ and $|A^j_i \cap A^j_{i+1}| = 2$ for $1 \le i < k_j$, as required.

Therefore, (L, H_1, H_2, H_3, H_4) is a discoid decomposition of G for (v_1, v_2, v_3, v_4). This completes the proof of Theorem 3.1.2 (5). $\square$

By a slightly modified proof of Theorem 3.1.2 (5), one can prove Theorem 3.1.2 (4). The statements (1), (2) and (3) in Theorem 3.1.2 can be proved in a similar way.

Proof of Theorem 3.1.2 (4). We only give a sketch of the proof of the nontrivial part of the statement: If G contains no K_4^--subdivision on $(v_1, v_2; v_3, v_4)$, then there is a discoid decomposition (L, H_1, H_2, H_3, H_4) of G for (v_1, v_2, v_3, v_4) or (v_1, v_2, v_4, v_3) such that $E(H_3) = \emptyset$.

Let $Z = \{v_1, v_2, v_3, v_4\}$. Since G obviously contains no K_4-subdivision on Z, there is a nice and lean bicycle J on Z in G by Lemma 3.2.5. Now J is a bicycle on (v_1, v_2, v_3, v_4) or (v_1, v_2, v_4, v_3); otherwise, we obtain a K_4^--subdivision on $(v_1, v_2; v_3, v_4)$, a contradiction. Assume that J is a bicycle on (v_1, v_2, v_3, v_4), say. We apply Lemma 3.2.7 and use the same notation C_i, H, K, x_i as in the proof of Theorem 3.1.2 (5).

Let us take a closer look at the graph $G' := G|V(C_3) \cup V(K\langle x_6, x_7\rangle)$. In the proof of Theorem 3.1.2 (5), the graph G' was only required to contain no path through x_6, v_4, v_3, x_7 in order. But now G' cannot contain any two disjoint paths with ends x_6v_4, x_7v_3, respectively; for if there are such two disjoint paths, then G contains a K_4^--subdivision on $(v_1, v_2; v_3, v_4)$, a contradiction. From this observation one can prove that G' can be drawn in a disc with $v_3, x_6, K\langle x_6, x_7\rangle, x_7, v_4$ on the boundary in order, regardless of whether or not $v_3v_4 \in E(G)$.

Define a subgraph H_3 of G by setting $V(H_3) = \{v_3, v_4\}$ and $E(H_3) = \emptyset$. For $j \in \{1, 2, 4\}$ define a subgraph H_j of G as follows: If $v_jv_{j+1} \in E(G)$ then let $V(H_j) = \{v_j, v_{j+1}\}$ and $E(H_j) = \emptyset$. If $v_jv_{j+1} \notin E(G)$ then let H_j be a graph obtained from $G|V(C_j) \cup V(K\langle x_{2j}, x_{2j+1}\rangle)$ by deleting edges spanned by $\{v_j, v_{j+1}\} \cup V(K\langle x_{2j}, x_{2j+1}\rangle)$. Let L denote the subgraph of G induced by $V(H) \cup Z \cup V(C_3) \cup \bigcup_{j\in\{1,2,4\}:v_jv_{j+1}\in E(G)} V(C_j)$. In a virtually identical way as the proof of Theorem 3.1.2 (5), one can show that (L, H_1, H_2, H_3, H_4) is a discoid decomposition of G for (v_1, v_2, v_3, v_4) with $E(H_3) = \emptyset$. This completes the proof. □

3.9 Corollaries

We prove corollaries of the main theorem, namely, Theorems 3.1.3 and 3.1.4. These theorems immediately follow from the following lemma. We may use Theorem 3.1.2, but the proof seems quicker if we start from Lemma 3.2.7.

Lemma 3.9.1. *Let G be a 6-connected graph and let $Z \subseteq V(G)$ with $|Z| = 4$. If G is 7-connected or triangle-free, then there is a K_4-subdivision on Z in G.*

Proof. Suppose to the contrary that G contains no K_4-subdivision on Z. Since G is 6-connected, by Lemma 3.2.5 there is a nice and lean bicycle J on (v_1, v_2, v_3, v_4) in G for some ordering $Z = \{v_1, v_2, v_3, v_4\}$. Let C_i denote the tire of J with $v_i, v_{i+1} \in V(C_i)$ for $1 \leq i \leq 4$, where indices are read modulo 4. Let $H := G \setminus V(J)$. By Lemma 3.2.7, H contains a cycle K such that:

- H can be drawn in a disc with K on the boundary, and
- there are eight distinct vertices $x_1, \ldots, x_8$ occurring in K in this order listed such that
  - $x_{2i-1}, x_{2i} \in N_G(v_i)$,
  - $N_G(v_i) \subseteq V(C_i \cup C_{i-1}) \cup V(K\langle x_{2i-1}, x_{2i}\rangle)$, and
  - $N_G(V(C_i) - Z) \subseteq \{v_i, v_{i+1}\} \cup V(K\langle x_{2i}, x_{2i+1}\rangle)$

for $1 \le i \le 4$. Note that every vertex of $H \setminus V(K)$ has no neighbor in $V(J)$. We want to show that each vertex of K has not so many neighbors in $V(J)$. As easily seen, it holds that:

(1) *For $1 \le i \le 4$ every internal vertex of $K\langle x_{2i-1}, x_{2i}\rangle$ has at most one neighbor in $V(J)$, namely v_i.*

We next consider vertices of $K\langle x_{2i}, x_{2i+1}\rangle$ for $1 \le i \le 4$. From the symmetry we mainly consider vertices of $K\langle x_2, x_3\rangle$. Note that each vertex x of $K\langle x_2, x_3\rangle$ satisfies that $N_G(x) \cap V(J) = N_G(x) \cap V(C_1)$. Let $k := |V(K\langle x_2, x_3\rangle)|$ (≥ 4) and let $x_2 = b_1, b_2, \dots, b_k = x_3$ denote the vertices of $K\langle x_2, x_3\rangle$ from x_2 to x_3 in order. By the same way as in the proof of Theroem 3.1.2 (5) in Section 3.8, we obtain the following.

(2) *If $v_1v_2 \in E(G)$, then the graph $G|V(C_1) \cup V(K\langle x_2, x_3\rangle)$ can be drawn in a disc with $v_1, b_1, b_2, \dots, b_k, v_2$ on the boundary in this order listed.*

(3) *If $v_1v_2 \notin E(G)$ then the graph obtained from $G|V(C_1) \cup V(K\langle x_2, x_3\rangle)$ by deleting edges spanned by $\{v_1, v_2\} \cup V(K\langle x_2, x_3\rangle)$ has a path-decomposition $(A_1, \dots, A_k)$ such that $v_1, b_1 \in A_1 - A_2$, $v_2, b_k \in A_k - A_{k-1}$, $b_i \in A_i - A_{i-1} \cup A_{i+1}$ for $1 < i < k$ and $|A_i \cap A_{i+1}| = 2$ for $1 \le i < k$.*

By (2) and the 6-connectivity of G, we easily obtain the following:

(4) *If $v_1v_2 \in E(G)$, then $|N_G(b_i) \cap V(C_1)| \le 2$ for $1 \le i \le k$.*

Next we claim:

(5) *If $v_1v_2 \notin E(G)$, then $|N_G(b_i) \cap V(C_1)| \le 4$ for $1 < i < k$ and $|N_G(b_i) \cap V(C_1)| \le 3$ for $i \in \{1, k\}$. If some vertex of $K\langle x_2, x_3\rangle$ has ≥ 3 neighbors in $V(C_1)$, then G contains a triangle. If G is 7-connected, then $|N_G(b_i) \cap V(C_1)| + |N_G(b_{i+1}) \cap V(C_1)| \le 6$ for $1 \le i < k$.*

For let $(A_1, \dots, A_k)$ be a path-decomposition as in (3). Since G is 6-connected, each part A_i has no other "inner vertices", i.e., $A_1 - A_2 = \{v_1, b_1\}$, $A_k - A_{k-1} = \{v_2, b_k\}$ and $A_i - A_{i-1} \cup A_{i+1} = \{b_i\}$ for $1 < i < k$. Since $N_G(b_1) \cap V(C_1) \subseteq \{v_1\} \cup (A_1 \cap A_2)$, we have $|N_G(b_1) \cap V(C_1)| \le |\{v_1\} \cup (A_1 \cap A_2)| = 3$. Similarly, $|N_G(b_k) \cap V(C_1)| \le 3$. For $1 < i < k$ we have $|N_G(b_i) \cap V(C_1)| \le |A_i \cap (A_{i-1} \cup A_{i+1})| \le 4$. This proves the first part of the claim. If some b_i has ≥ 3 neighbors in $V(C_1)$, then such neighbors are all in A_i, and so $G|A_i$ contains a triangle by the existence of C_1. This proves the second part of the claim. To see the third part of the claim, note that for $2 \le i \le k-2$ the graph $G|A_i \cup A_{i+1}$ is separated from the other vertices of G by the (≤ 6)-cut $(A_i \cap A_{i-1}) \cup (A_{i+1} \cap A_{i+2}) \cup \{b_i, b_{i+1}\}$. If G is 7-connected, then the graph $G|A_i \cup A_{i+1}$ has no "internal vertex", i.e., $A_i \cap A_{i+1} \subseteq (A_i \cap A_{i-1}) \cup (A_{i+1} \cap A_{i+2})$. This implies that $|N_G(b_i) \cap V(C_1)| + |N_G(b_{i+1}) \cap V(C_1)| \le 6$ for $2 \le i \le k-2$. Similarly, the graph $G|A_1 \cup A_2$ is separated from the other vertices of G by the 5-cut $\{v_1, b_1, b_2\} \cup (A_2 \cap A_3)$. This implies that if G is 6-connected then $A_1 \cap A_2 \subseteq A_2 \cap A_3$, and so $|N_G(b_1) \cap V(C_1)| + |N_G(b_2) \cap V(C_1)| \le 5$; an analogous claim holds for $G|A_{k-1} \cup A_k$. This proves the third part of the claim. This proves (5).

From (1), (4) and the second part of (5), we deduce that if G is triangle-free then each vertex in K has ≤ 2 neighbors in $V(J)$. Consequently, every vertex in H has ≤ 2 neighbors in $V(J)$, and so H has minimum degree $\geq 6-2=4$. But this contradicts that H is a triangle-free planar graph. Therefore, G is not triangle-free, and so 7-connected.

From (1), (4) and the first and the third part of (5), we deduce that if G is 7-connected then $|N_G(x) \cap V(J)| + |N_G(y) \cap V(J)| \leq 6$ for every edge xy of K. On the other hand, we have $|E(H)| \leq 3|V(H)| - 6 - (|V(K)| - 3) = 3|V(H)| - 3 - |V(K)|$. Since $\deg_H(x) = \deg_G(x) \geq 7$ for every $x \in V(H) - V(K)$, we have

$$\begin{aligned}
\sum_{x \in V(K)} \deg_H(x) &= 2|E(H)| - \sum_{x \in V(H)-V(K)} \deg_H(x) \\
&\leq 2(3|V(H)| - 3 - |V(K)|) - 7(|V(H)| - |V(K)|) \\
&= -|V(H)| - 6 + 5|V(K)| \\
&< 4|V(K)|,
\end{aligned}$$

which implies that

$$\sum_{xy \in E(K)} (\deg_H(x) + \deg_H(y)) = 2 \sum_{x \in V(K)} \deg_H(x) < 8|V(K)| = 8|E(K)|.$$

Therefore, there is an edge xy of K with $\deg_H(x) + \deg_H(y) \leq 7$. Now $|N_G(x) \cap V(J)| + |N_G(y) \cap V(J)| \geq \deg_G(x) + \deg_G(y) - (\deg_H(x) + \deg_H(y)) \geq 7+7-7 = 7$, a contradiction. This completes the proof of the lemma. $\square$

Chapter 4

Relaxed rooted subdivisions on four vertices

4.1 Overview

Let G be a graph and Z be a subset of $V(G)$ with $|Z| = 4$. By a K_4^--subdivision on Z in G we mean a K_4^--subdivision on $(v_1, v_2; v_3, v_4)$ in G for some ordering $Z = \{v_1, v_2, v_3, v_4\}$; see Section 3.1 for the term "K_4^--subdivision on $(v_1, v_2; v_3, v_4)$". Therefore, the relaxed rooted K_4^--subdivision problem is to ask a K_4^--subdivision on Z in G.

The second contribution in this dissertation is to determine the structures of graphs G with no K_4^--subdivision on Z, under the assumption that

> $(*)$ for every vertex v of Z there are three paths of G from v to $Z - \{v\}$, mutually disjoint except for v.

Roughly speaking, such a graph G admits a "decomposition" that separates Z into a few smaller subsets. The decomposition of G can be written as a hypergraph in flavor of combinatorics, without any topological condition, such as planarity. This is an interesting difference from Theorem 3.1.2 (4). A precise description of the theorem (Theorem 4.5.3) is given in Section 4.5.

If we drop the assumption $(*)$, then some vertex in Z may have to be of degree 2 in a possible K_4^--subdivision on Z. This restricts the choice of the subdivisions and hence the problem becomes almost the same as the usual rooted K_4^--subdivision problem. We shall deal with it in Chapter 3 only when G is 6-connected, and so assume $(*)$ here.

Unlike other results in this area, our proof is based on Mader's "H-Wege" theorem. For a subset S of $V(G)$, an *S-path* of G is a path of G with distinct ends both in S and no internal vertex in S. Mader [43] gave a min-max formula for the maximum number of internally disjoint S-paths, which is a deep result generalizing Menger's theorem and the Tutte–Berge formula. If G contains no more than four internally disjoint Z-paths, then clearly there is no K_4^--subdivision on Z in G. Such a graph G can be characterized by Mader's theorem. We may thus start from five internally disjoint Z-paths. This is a helpful shortcut for us. A similar method is used in the proof of Hadwiger's conjecture for K_6-minor-free cases by Robertson, Seymour and Thomas [51] (see also [30] for a similar proof).

By a K_4^{2-}-subdivision on Z in G we mean a K_4^{2-}-subdivision on $(v_1; v_2, v_3; v_4)$ in G for some ordering $Z = \{v_1, v_2, v_3, v_4\}$. Note that K_4^{2-} and C_4 are the graphs obtained from K_4^- by removing one edge. Thus, if there is a vertex not in Z whose removal makes the graph G contain neither K_4^{2-}-subdivision on Z nor cycle through all the vertices in Z, then obviously G contains no K_4^--subdivision on Z. From this observation, it is natural to consider the following subproblem:

> Characterize graphs G that contain neither K_4^{2-}-subdivision on Z nor cycle through all the vertices in Z.

As we shall show in Corollary 4.5.2, such a graph G has almost the same structure as a graph that contains no cycle through all the vertices in Z. Watkins and Mesner [67] characterized 3-connected graphs containing no cycle through given four vertices; see Corollary 4.10.2 for the statement. However, there seems no Watkins–Mesner-type theorem for 2-connected graphs. We solve this problem in Section 4.7. Note that this result cannot be derived from the result of [67] because the prescribed four vertices may have degree 2 in a 2-connected graph, which makes a significant difference. We should remark that as a similar problem, a characterization of cycles through prescribed four edges is given in [37].

In Sections 4.3 and 4.4, we prove some lemmas which are crucial for us to prove the main theorem, but are of independent interest.

4.2 Tools

In this section, we introduce the most important concepts, namely Mader's S-paths theorem. For our sake of use of this theorem, we give a few lemmas concerning the use of Mader's theorem. We also derive Watkins–Mesner's theorem from Mader's S-paths theorem.

4.2.1 Decompositions

A tuple $\mathcal{D} = (A_1, \ldots, A_k)$ of subsets of $V(G)$ is called a *decomposition* of G if $(G|A_1) \cup \cdots \cup (G|A_k) = G$; equivalently, $A_1 \cup \cdots \cup A_k = V(G)$ and every edge has both ends in one of A_i. So it is a separation of G if $k = 2$. Each A_i is called a *part*. The *boundary* of a part A_i is the set $A_i \cap \bigcup_{1 \le j \le k : j \ne i} A_j$, which we denote by $\operatorname{bd}_{\mathcal{D}} A_i$ or simply, $\operatorname{bd} A_i$. The *interior* of A_i is the set $A_i - \operatorname{bd}_D A_i$, which we denote by $\operatorname{int}_{\mathcal{D}} A_i$ or simply, $\operatorname{int} A_i$. We say that a hypergraph $\mathcal{H}$ with vertex set $\bigcup_{1 \le i \le k} \operatorname{bd} A_i$ and edge set $\{\operatorname{bd} A_i\}_{1 \le i \le k}$ is the *basic family* of the decomposition $\mathcal{D}$. In practice it is sometimes convenient to describe a decomposition of G by giving its basic family. We will encounter such a situation in Section 4.5 to state our main theorem.

4.2.2 Mader's S-paths Theorem

A tuple $(X_1, \ldots, X_k; Y_1, \ldots, Y_m)$ of subsets of $V(G)$ is called a *bipartite-decomposition* of a graph G if it is a decomposition of G such that $X_1, \ldots, X_k$ are pairwise disjoint and

$Y_1, \ldots, Y_m$ are pairwise disjoint. It is called *linear* if $|X_i \cap Y_j| = 1$ for all $1 \le i \le k, 1 \le j \le m$.

A tuple $(W; X_1, \ldots, X_k; Y_1, \ldots, Y_m)$ of subsets of $V(G)$ is called a *quasi-bipartite-decomposition* of G if $(X_1, \ldots, X_k; Y_1, \ldots, Y_m)$ is a bipartite-decomposition of $G \setminus W$. Its *value* is defined by

$$|W| + \sum_{1 \le j \le m} \left\lfloor \frac{1}{2} |Y_j \cap X| \right\rfloor,$$

where $X = X_1 \cup \cdots \cup X_k$. The set W is called the *integral part* of the quasi-bipartite-decomposition. Note that the tuple $(W, X_1, \ldots, X_k, Y_1, \ldots, Y_m)$ is not necessarily a decomposition of G if $W \neq \emptyset$; but $(W \cup X_1, \ldots, W \cup X_k, W \cup Y_1, \ldots, W \cup Y_m)$ is a decomposition of G. For $S \subseteq V(G)$, a quasi-bipartite-decomposition of G *with respect to* S is a quasi-bipartite-decomposition $(W; X_1, \ldots, X_k; Y_1, \ldots, Y_m)$ of G such that $k = |S|$ and $|S \cap (X_i - Y)| = 1$ for $1 \le i \le k$, where $Y = Y_1 \cup \cdots \cup Y_m$. It is permitted that $m = 0$ and $Y = \emptyset$. Note that the existence of a quasi-bipartite-decomposition with respect to S implies that S is stable. It is easy to see that for a stable set $S \subseteq V(G)$, if there are t internally disjoint S-paths of G, then every quasi-bipartite-decomposition of G with respect to S has value $\ge t$. Mader's "H-Wege" theorem [43] states that the converse holds; see [54, Theorem 4], e.g., for an equivalent description of the theorem.

Theorem 4.2.1 (Mader [43])**.** *Let G be a graph and let $S \subseteq V(G)$ be a stable set. Then the maximum number of internally disjoint S-paths of G is equal to the minimum value of quasi-bipartite-decompositions of G with respect to S.*

A quasi-bipartite-decomposition $(W; X_1, \ldots, X_k; Y_1, \ldots, Y_m)$ of G is called *good* if $|Y_j \cap X|$ is an odd integer and $|\{1 \le i \le k \colon X_i \cap Y_j \neq \emptyset\}| \ge 3$ for $1 \le j \le m$. The value of a good quasi-bipartite-decomposition is written as $|W| + (|X \cap Y| - m)/2$, without the floor function. The following lemma assures the existence of a good quasi-bipartite-decomposition of minimum value; for its proof, just choose one with W maximal, and subject to that with Y minimal.

Lemma 4.2.2. *Let G be a graph and $S \subseteq V(G)$ be a stable set. If there is a quasi-bipartite-decomposition of G with respect to S of value t, then there is a good quasi-bipartite-decomposition of G with respect to S of value $\le t$. In particular, there is a good one of minimum value.*

Proof. We may assume that $|S| \ge 2$. Choose a quasi-bipartite-decomposition $(W; X_1, \ldots, X_k; Y_1, \ldots, Y_m)$ of G with respect to S of value $\le t$, with W maximal, and subject to that with $Y = Y_1 \cup \cdots \cup Y_m$ minimal. Let t' $(\le t)$ denote its value. Let $X := X_1 \cup \cdots \cup X_k$. We may assume that $m > 0$ and each Y_j is not empty. Suppose to the contrary that $X_i \cap Y_1 = \emptyset$ for $3 \le i \le k$. Assume that $|X_1 \cap Y_1| \ge |X_2 \cap Y_1|$; and so $|X_2 \cap Y_1| \le |Y_1 \cap X|/2$. Set $W' := W \cup (X_2 \cap Y_1)$, $X_1' := X_1 \cup (Y_1 - X_2)$ and $X_2' := X_2 - Y_1$. Then $(W'; X_1', X_2', X_3, \ldots, X_k; Y_2, \ldots, Y_m)$ is a quasi-bipartite-decomposition of G with respect to S of value $\le t'$, such that $W \subseteq W'$ and $Y - Y_1 \subsetneq Y$, contrary to our choice. Hence each Y_j intersects at least three X_i's. Next suppose to the contrary that $|Y_1 \cap X|$, say, is a (positive) even integer. Assume $X_1 \cap Y_1 \neq \emptyset$ and let $v \in X_1 \cap Y_1$ be an arbitrary vertex. Set $W' := W \cup \{v\}$, $X_1' := X_1 - \{v\}$ and $Y_1' := Y_1 - \{v\}$. Then $(W'; X_1', X_2, \ldots, X_k; Y_1', Y_2, \ldots, Y_m)$ is a quasi-bipartite-decomposition of G with respect to S of value t' such that $W \subsetneq W'$, contrary to our choice. Hence $|Y_j \cap X|$ is odd for each j. This completes the proof. □

When $|S| = 2$, if $(W; X_1, X_2; Y_1, \ldots, Y_m)$ is a good quasi-bipartite-decomposition of G with respect to S of minimum value, then $m = 0$ (and $Y = \emptyset$) by definition. Hence W is a cut (disjoint from S) of G that separates S, with $|W|$ equal to the maximum number of internally disjoint S-paths of G by Theorem 4.2.1. This is nothing but Menger's theorem.

4.2.3 Watkins–Mesner's Theorem from Mader's S-paths Theorem

For our purposes, the case $|S| = 3$ is particularly important. Let $t \geq 0$ be an integer. A tuple $(X_1, X_2, X_3, Y_1, \ldots, Y_t)$ of subsets of $V(G)$ is called a *$K_{3,t}$-decomposition* of G of *integral value* $s \in \{0, 1, \ldots, t\}$ if

(i) $Y_1, \ldots, Y_s$ are disjoint singletons,
(ii) the union $W := Y_1 \cup \cdots \cup Y_s$ is contained in X_i, Y_j for $1 \leq i \leq 3, s < j \leq t$, and
(iii) $(X_1 - W, X_2 - W, X_3 - W; Y_{s+1} - W, \ldots, Y_t - W)$ is a linear bipartite-decomposition of $G \setminus W$.

Note that Y_j $(j > s)$ is not a singleton, since $Y_j - W$ intersects disjoint three sets $X_i - W$ $(1 \leq i \leq 3)$. Hence W is uniquely determined. We say that W is the *integral part* of the $K_{3,t}$-decomposition. A $K_{3,t}$-decomposition arises from a quasi-bipartite-decomposition of G of value t. To see this, let $(W; X_1', X_2', X_3'; Y_1', \ldots, Y_m')$ be a quasi-bipartite-decomposition of G such that $|X_i' \cap Y_j'| = 1$ for all $1 \leq i \leq 3, 1 \leq j \leq m$. Let $s := |W|$ and $W = \{w_1, \ldots, w_s\}$. Then its value is equal to

$$t = |W| + \sum_{1 \leq j \leq m} \left\lfloor \frac{1}{2} |Y_j' \cap (X_1' \cup X_2' \cup X_3')| \right\rfloor = s + m.$$

Now let $X_i := X_i' \cup W$ for $1 \leq i \leq 3$, $Y_j := \{w_j\}$ for $1 \leq j \leq s$, and $Y_j := Y_{j-s}' \cup W$ for $s < j \leq t$. Then $(X_1, X_2, X_3, Y_1, \ldots, Y_t)$ is a $K_{3,t}$-decomposition of G of integral value $s = |W|$. It is obvious that for any three vertices v_1, v_2, v_3 with $v_i \in \operatorname{int} X_i = X_i - Y_1 \cup \cdots \cup Y_t$ $(1 \leq i \leq 3)$, there do not exist more than t internally disjoint $\{v_1, v_2, v_3\}$-paths of G.

Of particular importance for us is the case $t = 2, 3$; see Figures 4.1 and 4.2. The case $t = 2$ appears in Watkins and Mesner's results [67], which characterizes graphs that contain no cycle through prescribed three vertices. Their theorem immediately follows from Mader's theorem (Theorem 4.2.1) because in a 2-connected graph there is a cycle through three vertices v_1, v_2, v_3 if and only if there are three internally disjoint $\{v_1, v_2, v_3\}$-paths.

Theorem 4.2.3 (Watkins–Mesner [67]). *Let v_1, v_2, v_3 be distinct vertices of a graph G. Then there is no cycle through v_1, v_2 and v_3 in G if and only if one of the following holds:*

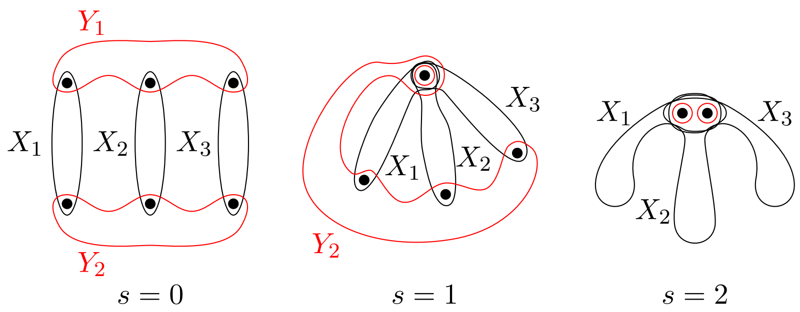

Figure 4.1: $K_{3,2}$-decomposition.

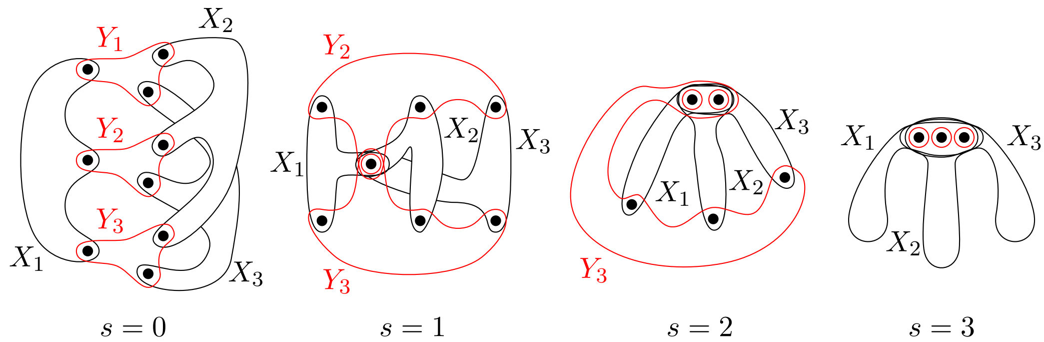

Figure 4.2: $K_{3,3}$-decomposition.

(i) *There is a* (≤ 1)*-separation* (A, B) *of* G *such that both* $A - B$ *and* $B - A$ *meet* $\{v_1, v_2, v_3\}$.

(ii) *There is a* $K_{3,2}$*-decomposition* $(X_1, X_2, X_3, Y_1, Y_2)$ *of* G *such that* $v_i \in X_i - Y_1 \cup Y_2$ *for* $1 \leq i \leq 3$.

In Section 4.3, we will see that triad-cycles (defined later) on three vertices are characterized by $K_{3,3}$-decompositions.

4.2.4 Matroids in S-paths

We will use the following matroidal properties of S-paths.

Lemma 4.2.4. *Let* G *be a graph, let* $S \subseteq V(G)$*, and let* $k \geq k' \geq 0$ *be integers. Suppose that* $P_1, \ldots, P_{k'}$ *are internally disjoint* S*-paths of* G*. If there are* k *internally disjoint* S*-paths* $Q_1, \ldots, Q_k$ *of* G*, then one can choose such paths with* $\deg_{P_1 \cup \cdots \cup P_{k'}}(v) \leq \deg_{Q_1 \cup \cdots \cup Q_k}(v)$ *for each* $v \in S$.

Proof. If S is stable and the sets $N_G(v)$ $(v \in S)$ are pairwise disjoint, then the lemma follows from the usual Mader matroids; see, e.g., [53, Theorem 73.5]. The general case can easily be reduced to this case. $\square$

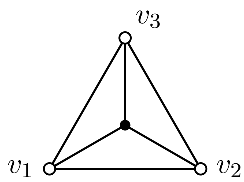

Figure 4.3: A triad-cycle on $\{v_1, v_2, v_3\}$.

4.3 Triad-cycles

Let v_1, v_2, v_3 be distinct vertices of a graph G. A *triad* in G with *feet* v_1, v_2, v_3 is a subgraph $P_1 \cup P_2 \cup P_3$ of G consisting of a vertex b of G distinct from v_1, v_2, v_3, called the *branch*, and three paths P_i with ends b, v_i $(1 \leq i \leq 3)$, mutually disjoint except for b. A *triad-cycle* on $\{v_1, v_2, v_3\}$ in G is a subgraph $C \cup T$ of G consisting of a cycle C through v_1, v_2, v_3, and a triad T with feet v_1, v_2, v_3, mutually disjoint except for $\{v_1, v_2, v_3\}$. See Figure 4.3 for an illustration of a triad-cycle. In this section we give a characterization of graphs that contain no triad-cycle on prescribed three vertices. The result will be used in Section 4.4 to prove another lemma and in Section 4.6 to prove our main theorem. The proof is heavily based on Mader's S-paths theorem because a triad-cycle on $\{v_1, v_2, v_3\}$ is essentially equivalent to four internally disjoint $\{v_1, v_2, v_3\}$-paths.

Theorem 4.3.1. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 3$. Then there is no triad-cycle on Z in G if and only if one of the following holds.*

(i) *There is a (≤ 2)-separation (A, B) of G such that both $A - B$ and $B - A$ meet Z.*
(ii) *There is no triad with feet Z in G.*
(iii) *There is a vertex $w \in V(G) - Z$ such that $G \setminus w$ contains no cycle through all the vertices in Z.*
(iv) *There is a linear bipartite-decomposition $(X_1, X_2, X_3; Y_1, Y_2, Y_3)$ of G such that $|Z \cap (X_i - Y_1 \cup Y_2 \cup Y_3)| = 1$ for $1 \leq i \leq 3$.*

Proof. Let $Z = \{v_1, v_2, v_3\}$. The "if part" is obvious, so we only show the converse. Suppose to the contrary that there is no triad-cycle on Z in G, but (i)–(iv) are false. By induction on $|V(G)|$, say, we may assume that:

(1) *G is 3-connected.*

For if there is a (≤ 2)-separation (A, B) of G with $A - B \neq \emptyset$ and $B - A \neq \emptyset$, then $B - A$, say, is disjoint from Z, since (i) is false. Then the graph G' obtained from $G|A$ by adding an edge in $A \cap B$ (if $|A \cap B| = 2$), does not satisfy (i)–(iv). Hence G' contains a triad-cycle on Z by induction, which yields a triad-cycle on Z in G, a contradiction. This proves (1).

(2) *There are at most three internally disjoint Z-paths in G.*

For suppose not, and let there be four internally disjoint Z-paths $P_1, \ldots, P_4$ of G, and let $H := P_1 \cup \cdots \cup P_4$. Since there is a cycle through v_1, v_2 and v_3 in G, we may assume from the matroidal properties (Lemma 4.2.4) that each of v_1, v_2, v_3 has degree ≥ 2 in H. We have only to consider two cases: $P_1, \ldots, P_4$ have ends (a) $v_1v_2, v_1v_2, v_1v_3, v_2v_3$, or (b) $v_1v_2, v_1v_2, v_1v_3, v_1v_3$, respectively. The case (b) can be reduced to (a), by augmenting P_1, P_2 from v_3 in H. The case (a) yields a triad-cycle on Z in G, by augmenting P_3, P_4 from v_3 in H, contrary to the assumption. This proves (2).

Let $e := |E(G|Z)|$, $d_i := \deg_{G|Z}(v_i)$ $(1 \leq i \leq 3)$, and $G' := G \setminus E(G|Z)$. Then G' contains at most $3 - e$ internally disjoint Z-paths by (2). By theorem 4.2.1 and Lemma 4.2.2, there is a good quasi-bipartite-decomposition $(W; X_1, X_2, X_3; Y_1, \ldots, Y_m)$ of G' with respect to Z of value $\leq 3 - e$; hence Y_j intersects each of X_1, X_2, X_3 and $|Y_j \cap X|$ is odd (≥ 3) for $1 \leq j \leq m$, and

$$|W| + m \leq |W| + \sum_{1 \leq j \leq m} \left\lfloor \frac{1}{2}|Y_j \cap X| \right\rfloor = |W| + \frac{1}{2}(|X \cap Y| - m) \leq 3 - e, \qquad (4.3.1)$$

where $X = X_1 \cup X_2 \cup X_3$ and $Y = Y_1 \cup \cdots \cup Y_m$. We may assume that $v_i \in X_i - Y$ for $1 \leq i \leq 3$.

(3) $W = \emptyset$.

For suppose not, and let w be a vertex in W. Then $(W - \{w\}; X_1, X_2, X_3; Y_1, \ldots, Y_m)$ is a quasi-bipartite-decomposition of $G' \setminus w$ of value $\leq 2 - e$. This means that there are at most two internally disjoint Z-paths in $G \setminus w$, contrary to the assumption that (iii) is false. This proves (3).

(4) *For* $1 \leq i \leq 3$, $|X_i \cap Y| + d_i \geq 3$.

For let $A := X_1 \cup N_{G|Z}(v_1)$ and $B := Y \cup X_2 \cup X_3$, say. Then (A, B) is a separation of G such that $v_1 \in A - B$ and $v_2, v_3 \in B$. If $B - A \neq \emptyset$, then $A \cap B = (X_1 \cap Y) \cup N_{G|Z}(v_1)$ has size ≥ 3 since G is 3-connected. Hence the result follows (for $i = 1$); so we may assume that $B - A = \emptyset$, i.e., $Y \cup X_2 \cup X_3 \subseteq X_1 \cup N_{G|Z}(v_1)$. Then $Y = \emptyset, X_2 = \{v_2\}, X_3 = \{v_3\}$, and $v_2, v_3 \in N_G(v_1)$. Hence $N_G(v_3) \subseteq \{v_1, v_2\}$ and $N_G(v_2) \subseteq \{v_1, v_3\}$, contrary to (1). This proves (4).

By (4) we have

$$|X \cap Y| = \sum_{1 \leq i \leq 3} |X_i \cap Y| \geq 9 - 2e.$$

This, together with (4.3.1) and (3), implies that

$$\frac{1}{2}(9 - 2e - m) \leq \frac{1}{2}(|X \cap Y| - m) \leq 3 - e$$

and hence

$$m \geq 3.$$

Thus, we have equality throughout in (4.3.1). Hence $e = 0, m = 3$ and $|Y_j \cap X| = 3$ for $1 \leq j \leq m$. Since $X_i \cap Y_j \neq \emptyset$ for all i, j, it follows that $|X_i \cap Y_j| = 1$ for $1 \leq i, j \leq 3$. Thus, $(X_1, X_2, X_3; Y_1, Y_2, Y_3)$ is a linear bipartite-decomposition of $G = G'$ with $v_i \in X_i - Y$ $(1 \leq i \leq 3)$, contrary to the assumption that (iv) is false. This completes the proof of the theorem. □

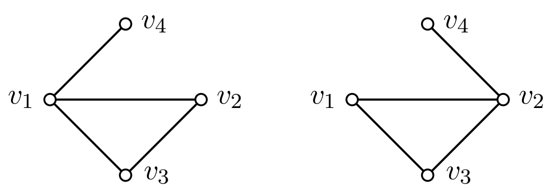

Figure 4.4: A K_4^{2-}-subdivision on $(v_1; v_2, v_3; v_4)$ (left) and a K_4^{2-}-subdivision on $(v_2; v_1, v_3; v_4)$ (right).

It is possible to integrate conditions (iii) and (iv) in Theorem 4.3.1, via $K_{3,3}$-decompositions, as follows.

Theorem 4.3.2 (Via $K_{3,3}$-decompositions)**.** *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 3$. Then there is no triad-cycle on Z in G if and only if one of the following holds.*

(i) *There is a (≤ 2)-separation (A, B) of G such that both $A - B$ and $B - A$ meet Z.*
(ii) *There is no triad with feet Z in G.*
(iii) *There is a $K_{3,3}$-decomposition $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ of G such that $|Z \cap (X_i - Y_1 \cup Y_2 \cup Y_3)| = 1$ for $1 \leq i \leq 3$.*

Indeed, let there be a $K_{3,3}$-decomposition of G of integral value $s \in \{0, 1, 2, 3\}$, as in Theorem 4.3.2 (iii). Then the case $s = 0$ is equivalent to Theorem 4.3.1 (iv). The case $s > 0$ or condition (i) (common in Theorems 4.3.1 and 4.3.2) holds if and only if Theorem 4.3.1 (iii) holds, by Watkins–Mesner's theorem (Theorem 4.2.3). This implies the equivalence between Theorem 4.3.1 and Theorem 4.3.2. One can also give a direct proof of Theorem 4.3.2 in almost the same way as in Theorem 4.3.1.

4.4 K_4^{2-}-Subdivisions

Let v_1, v_2, v_3, v_4 be distinct vertices of a graph G. Recall that a K_4^{2-}-subdivision on $(v_1; v_2, v_3; v_4)$ in G is a subgraph of G consisting of the union of four internally disjoint paths of G with ends $v_1v_2, v_1v_3, v_1v_4, v_2v_3$, respectively. See Figure 4.4 for an illustration. Also recall that for $Z \subseteq V(G)$ with $|Z| = 4$, a K_4^{2-}-subdivision on Z in G is a K_4^{2-}-subdivision on $(v_1; v_2, v_3; v_4)$ in G for some ordering $Z = \{v_1, v_2, v_3, v_4\}$.

Suppose now that there is a cycle containing v_1, v_2, v_3 and avoiding v_4. Let us consider a problem to construct a K_4^{2-}-subdivision on $\{v_1, v_2, v_3, v_4\}$ in which v_4 has degree 1. The problem is difficult if only one of v_1, v_2, v_3 is permitted to have degree 3, because it is exactly the K_4^{2-}-linkage problem; a structural characterization for 6-connected graphs G is given in Theorem 3.1.2 (3). Nevertheless, if two of v_1, v_2, v_3 can have degree 3, then the problem is fairly tractable, because it is reduced to a triad-cycle problem in Section 4.3.

Lemma 4.4.1. *Let v_1, v_2, v_3, v_4 be distinct vertices of a graph G. Suppose that for $i = 1, 2$ there are three paths from v_i to $\{v_1, v_2, v_3, v_4\} - \{v_i\}$, mutually disjoint except for v_i. Then there is a K_4^{2-}-subdivision on $(v_1; v_2, v_3; v_4)$ or $(v_2; v_1, v_3; v_4)$ in G if and only if there is a triad-cycle on $\{v_1, v_2, v_3\}$ in $G + v_3v_4$.*

Proof. Suppose that there is a K_4^{2-}-subdivision on $(v_1; v_2, v_3; v_4)$, say, in G; let P_1, P_2, P_3, P_4 be internally disjoint paths in G with ends $v_1v_2, v_2v_3, v_1v_3, v_1v_4$, respectively, and let $H := P_1 \cup \cdots \cup P_4$. By augmenting P_1, P_2 from v_2 in H, we may assume that there is a path Q of G with one end v_2, the other end in $V(P_3 \cup P_4) - \{v_1, v_3\}$, and no internal vertex in H. Then $Q \cup H + v_3v_4$ is a triad-cycle on $\{v_1, v_2, v_3\}$ in $G + v_3v_4$.

Conversely, suppose that there is a triad-cycle H on $\{v_1, v_2, v_3\}$ in $G + v_3v_4$. There is a path Q of G (possibly, of length 0) from v_4 to $V(H)$ disjoint from v_3; for otherwise, v_3 is a cut vertex that separates $\{v_4\}$ and $\{v_1, v_2\}$, contrary to the assumption that there are three paths from v_1 to $\{v_2, v_3, v_4\}$, mutually disjoint except for v_1. Then $(H \cup Q) \setminus \{v_3v_4\}$ (and hence G) contains a K_4^{2-}-subdivision on $(v_1; v_2, v_3; v_4)$ or $(v_2; v_1, v_3; v_4)$. $\square$

The above lemma, together with the results in Section 4.3, gives a characterization of two types of K_4^{2-}-subdivisions, as follows. This result will play a crucial role in proving our main theorem.

Theorem 4.4.2. *Let v_1, v_2, v_3, v_4 be distinct vertices of a graph G. Suppose that for $i = 1, 2$ there are three paths of G from v_i to $\{v_1, v_2, v_3, v_4\} - \{v_i\}$, mutually disjoint except for v_i. Then there is no K_4^{2-}-subdivision on either $(v_1; v_2, v_3; v_4)$ or $(v_2; v_1, v_3; v_4)$ in G if and only if one of the following holds:*

- (i) *There is a (≤ 2)-separation (A, B) of G such that $v_3 \in A - B$, $v_4 \in A$ and $v_1, v_2 \in B - A$.*
- (ii) *There is a $K_{3,3}$-decomposition $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ of G such that $v_i \in X_i - Y_1 \cup Y_2 \cup Y_3$ for $1 \leq i \leq 3$ and $v_4 \in X_3$.*

Proof. By Lemma 4.4.1 and Theorem 4.3.2, there is no K_4^{2-}-subdivision on either $(v_1; v_2, v_3; v_4)$ nor on $(v_2; v_1, v_3; v_4)$ if and only if $G' = G + v_3v_4$ satisfies one of (i)–(iii) in Theorem 4.3.2, where $Z = \{v_1, v_2, v_3\}$. It is obvious that Theorem 4.4.2 (ii) holds for G if and only if Theorem 4.3.2 (iii) holds for G'. It is easy to see that if Theorem 4.4.2 (i) holds for G then Theorem 4.3.2 (i) holds for G'.

To see the converse, suppose that Theorem 4.3.2 (i) holds for G', and let (A, B) be a (≤ 2)-separation of G' such that both $A - B$ and $B - A$ contain a vertex in $\{v_1, v_2, v_3\}$. Then $v_3, v_4 \in A$ or $v_3, v_4 \in B$. Since (A, B) is also a separation of G, the intersection of $\{v_1, v_2, v_3, v_4\}$ and $A - B$ (or $B - A$) cannot be $\{v_1\}$ nor $\{v_2\}$. Therefore, the only possible arrangement of v_i's is: $v_3 \in A - B$, $v_4 \in A$ and $v_1, v_2 \in B - A$ (up to interchanging A and B). Hence Theorem 4.4.2 (i) holds for G. Theorem 4.3.2 (ii) does not hold for G'. For if there is no triad with feet v_1, v_2, v_3 in G', then each connected component of $G' \setminus \{v_1, v_2, v_3\}$ is adjacent at most two of v_1, v_2, v_3 in G. Hence there is no path of $G' \setminus \{v_2, v_3\}$ from v_4 to v_1, say, but this contradicts the assumption that G contains three paths from v_1 to $\{v_2, v_3, v_4\}$, mutually disjoint except for v_1. This completes the proof. $\square$

Let $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ be a $K_{3,3}$-decomposition of integral value $0 \leq s \leq 3$ as in Theorem 4.4.2 (ii). Let $W := Y_1 \cup \cdots \cup Y_s$ be the integral part, and let $Y := Y_1 \cup Y_2 \cup Y_3$. We know $v_4 \in X_3$ but it is more helpful to distinguish the following three cases: (1) $v_4 \in X_3 - Y$ (the interior of X_3), (2) $v_4 \in X_3 \cap W$ (the boundary of X_3 and the integral part W), and (3) $v_4 \in X_3 \cap (Y - W)$ (the boundary of X_3 and the outside of the integral part W). If (1) holds, then there are at most three internally disjoint $\{v_1, v_2\} - \{v_3, v_4\}$ paths, since $X_3 \cap Y$ is a 3-cut. If (2) holds, then $G \setminus v_4$ contains no cycle through v_1, v_2 and v_3 by Watkins–Mesner's theorem, say; note that such a graph obviously contains no K_4^{2-}-subdivision on $\{v_1, v_2, v_3, v_4\}$ in which v_4 has degree 1. Hence if, for example, there is a cycle through v_1, v_3, v_2, v_4 in order and there is a cycle through v_1, v_2, v_3 disjoint from v_4, then case (3) must occur. We will encounter such a situation in (the final step of) the proof of our main theorem.

4.5 Main theorem

Let G be a graph and let $Z \subseteq V(G)$ be a set of four distinct vertices. We say that a pair (G, Z) is an *obstruction* if there no K_4^--subdivision on Z in G. In this section, we characterize obstructions (G, Z) under the assumption that for every $v \in Z$ there are three paths of G from v to $Z - \{v\}$, mutually disjoint except for v. If we drop this assumption, some vertex $v \in Z$ has to be of degree 2 in a possible K_4^--subdivision on Z, which is a much harder problem as we pointed out in the introduction. So we shall not deal with such a case.

Before stating our theorem, we point out some obvious obstructions. If there is a K_4^--subdivision on Z in G, then for any vertex w not in Z the graph $G \setminus w$ still contains a K_4^{2-}-subdivision on Z or a cycle through all the vertices in Z. So the first obvious obstruction is a graph G which admits a vertex w not in Z such that $G \setminus w$ contains neither K_4^--subdivision on Z nor cycle through all the vertices in Z. To characterize such graphs, we have to solve the following subproblem:

Characterize Z-acyclic graphs G that contain no K_4^--subdivision on Z.

But this is almost the same as the problem of characterizing Z-acyclic graphs G. To see this, we need the following lemma. The proof is given in the next section.

Lemma 4.5.1. *Let G be a graph and let $Z = \{v_1, v_2, v_3, v_4\}$ be a set of distinct vertices of G. Suppose that G contains a K_4^{2-}-subdivision on Z in which v_4 has degree 3, and that G contains a cycle through v_1, v_2 and v_3. Then there is a cycle in G containing all the vertices in Z.*

The following is an immediate consequence of Lemma 4.5.1.

Corollary 4.5.2. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Then the following two conditions are equivalent.*

(1) *G is Z-acyclic and contains no K_4^{2-}-subdivision on Z.*

(2) *Either*

  (i) *G is Z'-cyclic for any $Z' \subseteq Z$ with $|Z'| = 3$ and is Z-acyclic, or*

  (ii) *G is Z'-acyclic for some $Z' \subseteq Z$ with $|Z'| = 3$ and contains no K_4^{2-}-subdivision on Z.*

We already understand most of the structure of graphs as in (2)(ii), with the aid of Watkins–Mesner's theorem. Thus, in order to study graphs as in (1) it suffices to study graphs as in (2)(i). Therefore our subproblem is almost the same as to characterize Z-acyclic graphs. In Section 4.7, we characterize Z-acyclic graphs. We remark here that Lemma 4.5.1 is used a few times in the subsequent sections as well: Namely, in the proofs of Lemma 4.6.2 and Lemma 4.8.1.

We return to other obstructions. Recall that we have assumed above that there is no (≤ 2)-separation of G that separates Z into two subsets of size 1 and 3, respectively. We now consider a separation of G that separates Z into two subsets of size 2. That is, let there be a k-separation (A, B) of G such that $|Z \cap (A - B)| = 2$ and $|Z \cap (B - A)| = 2$. If $k \leq 2$, then G obviously contains no K_4^--subdivision on Z, and hence is an obstruction.

What if $k = 3$? For this case, there is an obstruction described by bipartite decompositions. Suppose that there is a linear bipartite decomposition $(X_1, X_2, X_3; Y_1, Y_2, Y_3)$ of G such that $\operatorname{int} X_i = X_i - Y_1 \cup Y_2 \cup Y_3$ contains exactly $1, 1, 2$ vertices of Z for $i = 1, 2, 3$, respectively. Then G contains no K_4^--subdivision on Z, as easily checked. Note that $(X_3, X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup Y_3)$ is a 3-separation of G that separates Z into two subsets of size 2.

What if $k = 4$? To consider this case, we define a certain decomposition of a graph G. We say that a decomposition $(X_1, X_2, Y_1, Y_2, A_1, A_2, A_3, B_1, B_2, B_3)$ of G is *special* if its basic family satisfies the following.

- The vertex set of the basic family consists of four distinct vertices a_1, a_2, b_1, b_2, three distinct vertices x_{i1}, x_{i2}, x_{i3} $(i = 1, 2)$, and three distinct vertices y_{i1}, y_{i2}, y_{i3} $(i = 1, 2)$.
- For $i = 1, 2$, $\operatorname{bd} X_i = \{x_{i1}, x_{i2}, x_{i3}\}$ and $\operatorname{bd} Y_i = \{y_{i1}, y_{i2}, y_{i3}\}$.
- For $i = 1, 2$, $\{x_{1i}, x_{2i}, a_i\} \subseteq \operatorname{bd} A_i \subseteq \{x_{1i}, x_{2i}, a_i, a_{3-i}\}$ and $\{y_{1i}, y_{2i}, b_i\} \subseteq \operatorname{bd} B_i \subseteq \{y_{1i}, y_{2i}, b_i, b_{3-i}\}$.
- $\{x_{13}, x_{23}, b_1, b_2\} \subseteq \operatorname{bd} A_3 \subseteq \{x_{13}, x_{23}, a_1, a_2, b_1, b_2\}$ and $\{y_{13}, y_{23}, a_1, a_2\} \subseteq \operatorname{bd} B_3 \subseteq \{y_{13}, y_{23}, a_1, a_2, b_1, b_2\}$.
- For $i = 1, 2$, $a_i \in \operatorname{bd} A_3$ if and only if $|A_i| = 1$, and $a_i \in \operatorname{bd} A_{3-i}$ if and only if $|A_i| = 1$ and $|A_{3-i}| > 1$. Similarly, for $i = 1, 2$, $b_i \in \operatorname{bd} B_3$ if and only if $|B_i| = 1$, and $b_i \in \operatorname{bd} B_{3-i}$ if and only if $|B_i| = 1$ and $|B_{3-i}| > 1$.

Note that there is symmetry between X_i, A_i, a_i, x_{ij} and Y_i, B_i, b_i, y_{ij}. By definition it is permitted that $\{a_1, a_2\} \cap \{y_{13}, y_{23}\} \neq \emptyset$ and $\{b_1, b_2\} \cap \{x_{13}, x_{23}\} \neq \emptyset$. Let s and s' be the numbers of singletons in $\{A_1, A_2\}$ and in $\{B_1, B_2\}$, respectively. Then we say that the special decomposition is of *type* (s, s'). From the symmetry there are six essentially different types of special decompositions; see Figure 4.5. (For simplicity, $G|X_i$ and $G|Y_i$ are depicted as 3-stars with feet $\operatorname{bd} X_i$ and $\operatorname{bd} Y_i$, respectively, for $i = 1, 2$.) Let x_1, x_2, y_1, y_2 be vertices in $\operatorname{int} X_1, \operatorname{int} X_2, \operatorname{int} Y_1, \operatorname{int} Y_2$, respectively. Let $A := X_1 \cup X_2 \cup A_1 \cup A_2 \cup A_3$ and $B := Y_1 \cup Y_2 \cup B_1 \cup B_2 \cup B_3$. Then, as easily checked, (A, B) is a separation of G such that $A \cap B = \{a_1, a_2, b_1, b_2\}$, $x_1, x_2 \in A - B$ and $y_1, y_2 \in B - A$. Hence (A, B) is a 4-separation of G that separates $\{x_1, x_2, y_1, y_2\}$ into two subsets $\{x_1, x_2\}$ and $\{y_1, y_2\}$. Moreover, it is easy to see that there is no K_4^--subdivision on $\{x_1, x_2, y_1, y_2\}$ in G. So $(G, \{x_1, x_2, y_1, y_2\})$ is an obstruction.

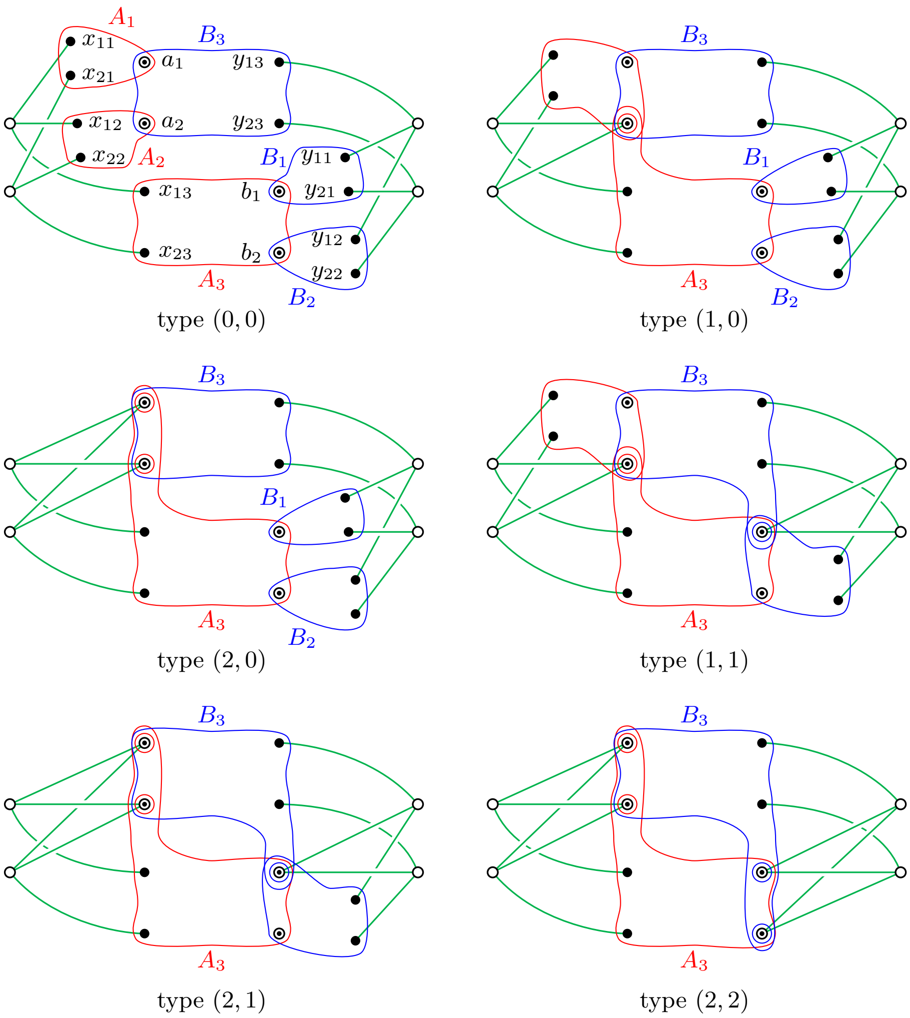

Figure 4.5: Six types of special decompositions.

Our main result in this chapter is the following, which states that there is no obstruction except but the above mentioned ones.

Theorem 4.5.3. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Suppose that for every $v \in Z$ there are three paths of G from v to $Z - \{v\}$, mutually disjoint except for v. Then there is no K_4^--subdivision on Z in G if and only if one of the following holds.*

(a) *There is a vertex $w \in V(G) - Z$ such that $G \setminus w$ contains neither K_4^{2-}-subdivision on Z nor cycle through all the vertices in Z.*
(b) *There is a (≤ 2)-separation (A, B) of G such that $|Z \cap (A - B)| = |Z \cap (B - A)| = 2$.*
(c) *There is a linear bipartite-decomposition $(X_1, X_2, X_3; Y_1, Y_2, Y_3)$ of G such that $X_i - Y_1 \cup Y_2 \cup Y_3$ contains exactly $1, 1, 2$ vertices of Z for $i = 1, 2, 3$, respectively.*
(d) *There is a special decomposition $(X_1, X_2, Y_1, Y_2, A_1, A_2, A_3, B_1, B_2, B_3)$ of G such that each of $\operatorname{int} X_1, \operatorname{int} X_2, \operatorname{int} Y_1, \operatorname{int} Y_2$ contains exactly one vertex of Z.*

As seen above, the "if" part is easy to verify. The non-trivial part is the converse. The proof is given in the next section. We should remark here that if there is a K_4^--subdivision on Z in G then one can find two non-homeomorphic K_4^--subdivisions on Z. This fact follows from the following slightly stronger result.

Theorem 4.5.4. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Suppose that for every $v \in Z$ there are three paths of G from v to $Z - \{v\}$, mutually disjoint except for v. If there is a K_4^--subdivision on Z in G, then for every $v \in Z$ there is a K_4^--subdivision on Z in G in which v has degree* 3.

Proof. Let $Z = \{v_1, v_2, v_3, v_4\}$ and suppose that G contains a K_4^--subdivision H on $(v_1, v_2; v_3, v_4)$, say. Let $P_{12}, P_{13}, P_{14}, P_{23}, P_{24}$ be the paths in H with ends $v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4$, respectively.

Let us construct a K_4^--subdivision on Z in which v_3 say, has degree 3. By augmenting P_{13}, P_{23} from v_3 in H, we may assume that there is a path Q with one end v_3, the other end q in $V(P_{12} \cup P_{14} \cup P_{24}) - \{v_1, v_2\}$ and no internal vertex in H. If $q \in V(P_{14}) - \{v_1\}$ then the union of five internally disjoint paths P_{23}, P_{13}, P_{12}, $Q \cup P_{14}[q, v_4]$ and P_{24} is a K_4^--subdivision on $(v_3, v_2; v_1, v_4)$, as required. Similarly, if $q \in V(P_{24}) - \{v_2\}$ then $H \cup Q$ contains a K_4^--subdivision on $(v_3, v_1; v_2, v_4)$. So assume that $q \in V(P_{12}) - \{v_1, v_2\}$.

By augmenting P_{14}, P_{24} from v_4 in $H \cup Q$, we may assume that there is a path R of G with one end v_4, the other end r in $V(P_{12} \cup P_{13} \cup P_{23} \cup Q) - \{v_1, v_2\}$ and no internal vertex in $H \cup Q$. If $r \in V(P_{12} \cup Q) - \{v_1, v_2\}$ then $H \cup P \cup Q$ contains a K_4^--subdivision on $(v_3, v_4; v_1, v_2)$, as required. If $r \in V(P_{13}) - \{v_1\}$, then the union of five internally disjoint paths $P_{13}[v_3, r] \cup R$, $Q \cup P_{12}[q, v_1]$, P_{14}, P_{23} and P_{24} is a K_4^--subdivision on $(v_3, v_4; v_1, v_2)$, as required. Similarly, if $r \in V(P_{23}) - \{v_2\}$ then $H \cup Q \cup R$ contains a K_4^--subdivision on $(v_3, v_4; v_1, v_2)$. Therefore, there is a K_4^--subdivision on Z in which v_3 has degree 3. The proof is analogous for v_4. This completes the proof. □

4.6 Proof

In this section we prove our main theorem (Theorem 4.5.3). We first begin with a proof of Lemma 4.5.1.

Proof of Lemma 4.5.1. Suppose to the contrary that G is Z-acyclic. We claim:

(1) *There are two triads T_1, T_2 in G with feet v_1, v_2, v_3, mutually disjoint except for their feet, such that v_4 is the branch of T_1.*

For let there be a K_4^{2-}-subdivision $C \cup P$ on $(v_4; v_1, v_2; v_3)$ in G, where C is a cycle through v_1, v_2, v_4, and P is a path with ends v_3, v_4, disjoint from C except for v_4. By the existence of a cycle through v_1, v_2 and v_3, one can augment P from v_3 in $C \cup P$. Thus, we may assume that there is a path Q with one end v_3, the other end q in $V(C) - \{v_4\}$, and no internal vertex in $C \cup P$. Since G is Z-acyclic, four vertices v_1, q, v_2, v_4 are distinct and appear in C in this order. Then $C \cup P \cup Q$ is a union of two desired triads. This proves (1).

Choose T_1, T_2 as in (1) and a cycle C of G through v_1, v_2, v_3, with $T_1 \cup T_2 \cup C$ minimal. Let b denote the branch of T_2. By an *arc* we mean a subpath of C with distinct ends both in $V(T_1 \cup T_2)$ and with no edge or internal vertex in $T_1 \cup T_2$. Since G is Z-acyclic, C does not contain v_4. Let C_{12} be the path of C between v_1 and v_2 not containing v_3, and define C_{23}, C_{13} similarly. We may assume that $C_{12} \cup C_{13}$ does not contain b. Note that $\{v_4, b\}$

is a cut of $T_1 \cup T_2$ that separates $V(T_1(v_4, v_1] \cup T_2[v_1, b))$ from the other vertex in $T_1 \cup T_2$. Hence there exist two distinct arcs Q, R having one end $q \in V(T_1[v_1, v_4)), r \in V(T_2[v_1, b))$, respectively, such that one of $T_1[v_1, q] \cup Q$ and $T_2[v_1, r] \cup R$ is a subpath of C_{12} and the other is a subpath of C_{13}. Let q', r' denote the other ends of Q, R, respectively.

(2) $r' \in V(T_1(q, v_4))$.

For if $r' \in V(T_2) - \{v_1, v_2, v_3\}$ we may reduce the union $T_1 \cup T_2 \cup C$ by replacing $T_2[r, r']$ by R; and if $r' \in V(T_1) - V(T_1[v_1, v_4])$ the union $T_1 \cup T_2 \cup R$ contains a cycle through all the vertices in Z. Recall $v_4 \notin V(C)$, and so $r' \neq v_4$. Obviously, r' is not in $T_1[v_1, q]$. Hence $r' \in V(T_1(q, v_4))$. This proves (2).

(3) $q' \in V(T_2(r, b))$.

For if $q' \in V(T_2) - V(T_2[v_1, b])$ the union $T_1 \cup T_2 \cup Q$ contains a cycle through all the vertices in Z; if $q' \in V(T_1) - V(T_1[v_1, v_4])$ the union $T_1 \cup T_2 \cup Q \cup R$ contains a cycle through all the vertices in Z; if $q' \in V(T_1(q, v_4])$ we may reduce the union $T_1 \cup T_2 \cup C$ by replacing $T_2[q, q']$ by Q. Recall $b \notin V(C_{12} \cup C_{23})$, and so $q' \neq b$. Obviously, q' is not in $T_2[v_1, r]$. Hence $q' \in V(T_2(r, b))$. This proves (3).

By (2) and (3) we may replace the subpaths $T_1[v_1, r']$ and $T_2[v_1, q']$ in $T_1 \cup T_2$ by the paths $T_2[v_1, r] \cup R$ and $T_1[v_1, q] \cup Q$, respectively, thereby reducing the union $T_1 \cup T_2 \cup C$, a contradiction. This completes the proof. □

We return to the proof of Theorem 4.5.3. To prove the theorem, we may assume that G is 3-connected (by induction on $|V(G)|$, say). For suppose that G is not 3-connected, and let (A, B) be a k-separation (A, B) of G with $A - B, B - A \neq \emptyset$ and $k \leq 2$, chosen with k minimum. If both $A - B$ and $B - A$ contain exactly two vertices of Z, then (b) holds. If $A - B$ say, contains exactly one vertex of Z, then this contradicts our assumption. So assume that $A - B$ say, contains no vertex in Z. Let G' be a graph obtained from $G|B$ by adding an edge in $A \cap B$ if $k = 2$. Then G' satisfies the assumption of the theorem. By induction, either G' contains a K_4^--subdivision on Z or satisfies one of (a)–(d). The truth of these propositions is "inherited" by G, and so, Theorem 4.5.3 holds for G. We may thus assume that G is 3-connected.

To prove the "only if" part of the theorem, it suffices to show that (c) or (d) holds, assuming that G is 3-connected, contains no K_4^--subdivision on Z, and does not satisfy (a); note that (b) is automatically false when G is 3-connected. So consider the following hypotheses and lemma.

(hyp 4.6.1) G is 3-connected.

(hyp 4.6.2) There is no K_4^--subdivision on Z in G.

(hyp 4.6.3) For every $w \in V(G) - Z$, there is a K_4^{2-}-subdivision on Z or a cycle through all the vertices in Z in $G \setminus w$.

Lemma 4.6.1. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Assume (hyp 4.6.1), (hyp 4.6.2) and (hyp 4.6.3). Then Theorem 4.5.3* (c) *or* (d) *holds.*

The "only if" part of Theorem 4.5.3 follows from Lemma 4.6.1. The rest of this section is devoted to proving this lemma. Let G be a graph and let $Z = \{v_1, v_2, v_3, v_4\}$ be a set of distinct four vertices of G, fixed throughout the remaining of the section until further notice. The proof falls into six separate steps.

▶ Step 1: Strengthening hypotheses

The first step is to strengthen (hyp 4.6.3) as follows.

Lemma 4.6.2. *Assume (hyp 4.6.1)–(hyp 4.6.3). Then for every $w \in V(G) - Z$, there is a cycle of $G \setminus w$ containing all the vertices in Z.*

Proof. Suppose not, and let w be a vertex in $V(G) - Z$ such that there is no cycle in $G \setminus w$ containing all the vertices in Z. By (hyp 4.6.3) there is a K_4^{2-}-subdivision H on Z in $G \setminus w$ with $\deg_H(v_4) = 3$, say. By Lemma 4.5.1 there is no cycle in $G \setminus w$ through v_1, v_2 and v_3. Since $G \setminus w$ is 2-connected, we deduce from Theorem 4.2.3 that there is a $K_{3,2}$-decomposition $(X_1, X_2, X_3, Y_1, Y_2)$ of G such that $v \in X_i - Y_1 \cup Y_2$ for $1 \le i \le 3$. It follows from the existence of H that $v_4 \in Y_1 \cup Y_2$. Since G is 3-connected, there are three paths of G from v_i to $\{w\} \cup \operatorname{bd} X_i = \{w\} \cup (X_i \cap (Y_1 \cup Y_2))$, mutually disjoint except for v_i, for $1 \le i \le 3$; we may assume that some of these paths are the original subpaths of H. Thus, we may assume that there are three paths P_1, P_2, P_3 of G from w to $\{v_1, v_2, v_3\}$, mutually disjoint except for w, all with no internal vertex in H. Then the union $H \cup P_1 \cup P_2 \cup P_3$ contains a K_4^--subdivision on Z, contrary to (hyp 4.6.2). □

▶ Step 2: Excluding 3-separations

The next step is to exclude a 3-cut of G that is disjoint from Z and separates Z into two subsets of size 2. The following lemma says that if G admits such a 3-cut then Theorem 4.5.3 (c) holds. So we may assume from now on that there is no such 3-cut. The proof is based on the results in Section 4.3.

Lemma 4.6.3. *Assume (hyp 4.6.1)–(hyp 4.6.3). If there is a 3-separation (A, B) of G such that $|Z \cap (A - B)| = |Z \cap (B - A)| = 2$, then there is a linear bipartite-decomposition $(X_1, X_2, X_3; Y_1, Y_2, Y_3)$ of G such that $X_i - Y_1 \cup Y_2 \cup Y_3$ contains exactly $1, 1, 2$ vertices of Z for $i = 1, 2, 3$, respectively.*

Proof. Let (A, B) be such a separation of G, and assume that $v_1, v_2 \in A - B$ and $v_3, v_4 \in B - A$. Let $A \cap B = \{x_1, x_2, x_3\}$ and let G' be a graph obtained from G by adding a new vertex x and edges xx_1, xx_2 and xx_3. If there exist a triad-cycle on $\{v_1, v_2, x\}$ in $G'|A \cup \{x\}$ and a triad-cycle on $\{v_3, v_4, x\}$ in $G'|B \cup \{x\}$, then G contains a K_4^--subdivision on Z, a contradiction. So assume that there is no triad-cycle on $\{v_1, v_2, x\}$ in $G'|A \cup \{x\}$. We shall apply Theorem 4.3.2 to $G'|A \cup \{x\}$.

(1) *There is no (≤ 2)-separation (X, Y) of $G'|A \cup \{x\}$ such that both $X - Y$ and $Y - X$ meet $\{v_1, v_2, x\}$.*

For let (X, Y) be such a separation. Since G is 3-connected, x cannot be contained in the interior of X or Y; and so $x \in X \cap Y$. We may assume that $v_1 \in X - Y$ and $v_2 \in Y - X$, and from the symmetry that $|\{x_1, x_2, x_3\} \cap (X - Y)| \leq 1$. Let $X' := X - \{x\}, Y' := B \cup (Y - \{x\})$. Then (X', Y') is a (≤ 2)-separation of G with $v_1 \in X' - Y'$ and $v_2, v_3, v_4 \in Y' - X'$, contrary to the assumption that G is 3-connected. This proves (1).

(2) *There is a triad in $G'|A \cup \{x\}$ with feet v_1, v_2, x.*

For let P_1, P_2 be two paths of $(G|A) \setminus v_2$ from v_1 to $\{x_1, x_2, x_3\}$, mutually disjoint except for v_1; we may assume that P_1, P_2 have ends v_1x_1, v_1x_2, respectively. Let Q be a path of $G|A$ from v_2 to $V(P_1 \cup P_2)$ with no vertex in $\{v_1, x_3\}$. We may assume that Q has an end in $V(P_1) - \{v_1\}$. Then the union of $P_1 \cup Q$ and edge xx_1 yields a triad in $G'|A \cup \{x\}$ with feet v_1, v_2, x, as required. This proves (2).

We deduce from (1), (2) and Theorem 4.3.2 that there is a $K_{3,3}$-decomposition $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ of $G'|A \cup \{x\}$ such that $v_i \in X_i - Y$ $(i = 1, 2)$ and $x \in X_3 - Y$, where $Y = Y_1 \cup Y_2 \cup Y_3$. Hence $x_1, x_2, x_3 \in X_3$. Let $s \in \{0, 1, 2, 3\}$ be its integral value. Now set $X_3' := (X_3 - \{x\}) \cup B$. Then $(X_1, X_2, X_3', Y_1, Y_2, Y_3)$ is a $K_{3,3}$-decomposition of G of integral value s with $v_3, v_4 \in X_3 - Y$. If $s > 0$ and $Y_1 = \{w\}$ then there is no cycle in $G \setminus w$ that contains all the vertices in Z. This contradicts Lemma 4.6.2 in Step 1. Hence $s = 0$. Therefore $(X_1, X_2, X_3'; Y_1, Y_2, Y_3)$ is a linear bipartite-decomposition of G such that $v_i \in X_1 - Y_1 \cup Y_2 \cup Y_3$ $(i = 1, 2)$ and $v_3, v_4 \in X_3' - Y_1 \cup Y_2 \cup Y_3$, as required. $\square$

► Step 3: Five Z-paths

The next step is to find five internally disjoint Z-paths in G. If G contains no more than four internally disjoint Z-paths, there is a decomposition of G as in Lemma 4.6.4. This lemma is a direct consequence of Mader's S-paths theorem. Lemma 4.6.5 says that such a decomposition indeed yields a special decomposition of G and hence Theorem 4.5.3 (d) holds. We may thus assume from now on that there are at least five internally disjoint Z-paths.

Lemma 4.6.4. *Assume (hyp 4.6.1)–(hyp 4.6.3). If there are no more than four internally disjoint Z-paths in G, then there is a bipartite-decomposition $(X_1, \ldots, X_4; Y_1, \ldots, Y_4)$ of G such that $|X_i \cap Y_j|$ equals 1 if $i \neq j$ and 0 otherwise for $1 \leq i, j \leq 4$, and $v_i \in X_i - Y_1 \cup \cdots \cup Y_4$ for $1 \leq i \leq 4$.*

Proof. Let $e := |E(G|Z)|$, $d_i := \deg_{G|Z}(v_i)$ $(1 \leq i \leq 4)$, and $G' := G \setminus E(G|Z)$. Since there are at most $4 - e$ internally disjoint Z-paths in G', we deduce from Theorem 4.2.1 and Lemma 4.2.2 that there is a good quasi-bipartite-decomposition $(W; X_1, \ldots, X_4; Y_1, \ldots, Y_m)$ of G' with respect to Z of value $4 - e$; hence Y_j intersects at least three of $X_1, \ldots, X_4$ and $|Y_j \cap X|$ is odd (≥ 3) for $1 \leq j \leq m$, and

$$|W| + m \leq |W| + \sum_{1 \leq j \leq m} \left\lfloor \frac{1}{2} |Y_j \cap X| \right\rfloor = |W| + \frac{1}{2}(|X \cap Y| - m) \leq 4 - e, \qquad (4.6.1)$$

where $X = X_1 \cup \cdots \cup X_4$ and $Y = Y_1 \cup \cdots \cup Y_m$. We may assume that $v_i \in X_i - Y$ for $1 \leq i \leq 4$.

(1) $W = \emptyset$.

For if W contains a vertex w, then $(W - \{w\}; X_1, \dots, X_4; Y_1, \dots, Y_m)$ is a quasi-bipartite-decomposition of $G' \setminus w$ of value $\leq 3 - e$ with respect to Z. This means that there are at most three internally disjoint Z-paths in $G \setminus w$, contrary to (hyp 4.6.3). This proves (1).

(2) *For* $1 \leq i \leq 4$, $|X_i \cap Y| + d_i \geq 3$.

For let $A := X_1 \cup N_{G|Z}(v_1)$ and $B := Y \cup (X_2 \cup X_3 \cup X_4) \cup N_{G|Z}(v_1)$, say. Then (A, B) is a separation of G such that $v_1 \in A - B$ and $v_2, v_3, v_4 \in B$. Since G is 3-connected, $A \cap B = (X_1 \cap Y) \cup N_{G|Z}(v_1)$ has size ≥ 3, yielding that $|X_1 \cap Y| + d_1 \geq 3$. This proves (2).

By (4.6.1), (1) and (2), we have

$$12 - 2e - m \leq \sum_{1 \leq i \leq 4} |X_i \cap Y| - m \leq |X \cap Y| - m \leq 2(4 - e)$$

and so

$$4 \leq m.$$

Thus, we have equality throughout in (4.6.1) and in (2), and so $e = 0, m = 4$ and $|Y_j \cap X| = |X_i \cap Y| = 3$ for $1 \leq i, j \leq 4$. Since 3-regular simple bipartite graphs on 8 vertices are exactly 3-cubes, we may assume that $|X_i \cap Y_j|$ equals 1 if $i \neq j$ and 0 otherwise for $1 \leq i, j \leq 4$. Then $(X_1, \dots, X_4; Y_1, \dots, Y_4)$ is a bipartite-decomposition of $G = G'$, and the result follows. $\square$

Lemma 4.6.5. *If there is a bipartite-decomposition of G as in Lemma 4.6.4, then Theorem 4.5.3* (d) *holds.*

Proof. Let $(C_1, \dots, C_4; D_1, \dots, D_4)$ be a bipartite-decomposition of G such that $|C_i \cap D_j|$ equals 1 if $i \neq j$ and 0 otherwise for $1 \leq i, j \leq 4$, and $v_i \in C_i - D_1 \cup \dots \cup D_4$ for $1 \leq i \leq 4$; so this is a decomposition as in Lemma 4.6.4. Let $C_i \cap D_j = \{c_{ij}\}$ for $1 \leq i, j \leq 4$ with $i \neq j$. Let $a_1 = c_{34}, a_2 = c_{43}, b_1 = c_{12}, b_2 = c_{21}$. Let $X_1 := C_1$, $X_2 := C_2$, $Y_1 := C_3$, $Y_2 := C_4$, $A_1 := D_4$, $A_2 := D_3$, $B_1 := D_2$, $B_2 := D_1$, $A_3 := \{b_1, b_2\}$ and $B_3 := \{a_1, a_2\}$. Then $(X_1, X_2, Y_1, Y_2, A_1, A_2, A_3, B_1, B_2, B_3)$ is a special decomposition of G of type $(0, 0)$ such that v_1, v_2, v_3, v_4 belong to $\operatorname{int} X_1, \operatorname{int} X_2, \operatorname{int} Y_1, \operatorname{int} Y_2$, respectively. Hence (d) holds. This proves the lemma. $\square$

▶ Step 4: Building a cube$^+$

Five internally disjoint Z-paths does not necessarily yield a K_4^--subdivision on Z. A *cube*$^+$ on $(v_1, v_2; v_3, v_4)$ in G is a subgraph H of G consisting of the union of:

(i) three paths P_1, P_2, P_3 with ends v_1, v_2 and two paths Q_1, Q_2 with ends v_3, v_4, mutually disjoint except for their ends, and

(ii) four disjoint paths R_i from $V(P_1 \cup P_2 \cup P_3)$ to $V(Q_1 \cup Q_2)$ with ends p_i, q_i $(1 \leq i \leq 4)$, all with no internal vertex in $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2$, where $p_1, p_2, p_3, p_4, q_1, q_2$ are internal vertices of $P_1, P_1, P_2, P_3, Q_1, Q_2$, respectively, and $q_3 = v_3, q_4 = v_4$.

See Figure 4.7 for an illustration. By a cube$^+$ on Z we mean a cube$^+$ on $(v_{i_1}, v_{i_2}; v_{i_3}, v_{i_4})$ for some ordering $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. A cube$^+$ on Z satisfies the following: (1) It contains five internally disjoint Z-paths, (2) admits no 3-cut that is disjoint from Z and separates Z into two subsets of size 2, and (3) contains no K_4^--subdivision on Z. The next step is to show the converse: Such a graph must contain a cube$^+$ on Z.

The proof relies on Lemma 4.6.2 in Step 1. For consider a V_8-subdivision H on Z (defined in Section 4.7) and a vertex $w \notin V(H)$ adjacent to all the vertices of Z. Let P_i be the path from w to v_i of length 1 for $1 \leq i \leq 4$. Then $H' := H \cup P_1 \cup P_2 \cup P_3 \cup P_4$ satisfies (1), (2) and (3) above, but contains no cube$^+$ on Z. This is because $H' \setminus w = H$ is Z-acyclic. To exclude such an example, we need Lemma 4.6.2. But the proof itself is straightforward, based on the paths-augmentation method (cf. Section 2.2).

Lemma 4.6.6. *Assume (hyp 4.6.1)–(hyp 4.6.3). Suppose that there are five internally disjoint Z-paths in G and there is no 3-separation (A, B) of G such that $|Z \cap (A - B)| = |Z \cap (B - A)| = 2$. Then there is a cube$^+$ on Z in G.*

Proof. See Figure 4.6. By an H_i-subdivision (or a subdivision of H_i) in G we mean a subgraph of G which is homeomorphic to the multigraph H_i as in Figure 4.6 (left), where the white vertices correspond to vertices of Z for some ordering. By definition an H_0-subdivision is a K_4^--subdivision on Z and an H_9-subdivision is a cube$^+$ on Z. Note that an H_5-subdivision contains a vertex of degree 4. We shall write $H_i \to H_{i_1}, \ldots, H_{i_k}$ to denote a claim that if there is an H_i-subdivision in G, then there is an H_j-subdivision in G for some $j \in \{i_1, \ldots, i_k\}$. The result follows from the following nine claims $(*)$, (1), (2), …, (8); see Figure 4.6 (right) for an outline of the proof, where implications towards an H_0-subdivision are omitted. $(*)$ There is an H_i-subdivision in G for some $i \in \{0, 1, 2, 3, 4\}$. (1) $H_1 \to H_2$. (2) $H_2 \to H_0, H_3, H_4, H_5, H_6$. (3) $H_3 \to H_0, H_4$. (4) $H_4 \to H_0, H_7, H_8$. (5) $H_5 \to H_0, H_4, H_6$. (6) $H_6 \to H_7, H_8$. (7) $H_7 \to H_0$. (8) $H_8 \to H_0, H_9$.

We show $(*)$. By Lemma 4.6.2, there is a cycle in G containing all the vertices in Z. We deduce from Lemma 4.2.4 that there are five internally disjoint Z-paths in G such that each of v_1, v_2, v_3, v_4 has degree ≥ 2 in the union of these paths. The only possible degree sequence of v_i's is $(2, 2, 3, 3)$ or $(2, 2, 2, 4)$. Now it is easy to see that the five paths form an H_i-subdivision for some $i \in \{0, 1, 2, 3, 4\}$. This proves $(*)$.

It is not difficult to show (1), (2), (3), (4) by augmenting (a few times) from a vertex of Z of degree 2 in an (augmented) H_i-subdivision $(1 \leq i \leq 4)$.

We show (5) by using Lemma 4.6.2. Suppose that there is an H_5-subdivision H in G such that one may take two paths P_1, P_2 of H with ends v_1, v_2, two paths Q_1, Q_2 of H with ends v_3, v_4, and four paths R_i of H with ends b, v_i $(1 \leq i \leq 4)$, mutually disjoint except for their ends, where b is the vertex of degree 4 in H. If there is no 2-separation (A, B) of $G \setminus b$ such that $v_1 \in A - B$ and $V(Q_1 \cup Q_2 \cup R_2 \cup R_3 \cup R_4) - \{b\} \subseteq B$, then by augmenting P_1, P_2 from v_1 in $(P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R_2 \cup R_3 \cup R_4) \setminus b$, we obtain a subdivision of H_4 or H_6, as required. So we may assume that there is a 3-separation (A_1, B_1) of G with $v_1 \in A_1 - B_1$, $v_2, v_3, v_4 \in B_1$ and $b \in A_1 \cap B_1$. Similarly, there is a 3-separation (A_i, B_i) of G with $v_i \in A_i - B_i$, $Z - \{v_i\} \subseteq B_i - A_i$ and $b \in A_i \cap B_i$ for $i = 2, 3, 4$. On the other hand, by Lemma 4.6.2 there is a cycle C in $G \setminus b$ containing v_1, v_2, v_3, v_4. Since

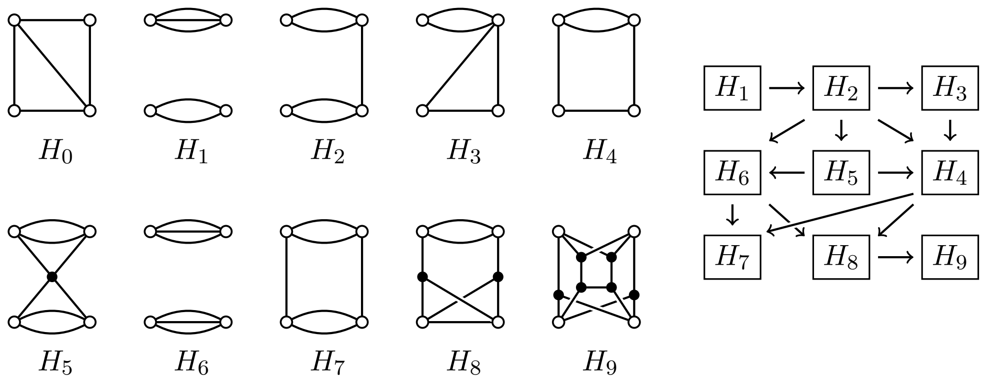

Figure 4.6: H_i-subdivisions (left) and an outline of the proof of Lemma 4.6.6 (right).

G is 3-connected, there are three paths of G from v_i to $A_i \cap B_i$, mutually disjoint except for v_i, for $1 \leq i \leq 4$; we may assume that two of these paths are the original subpaths of C. Therefore, we may assume that there are internally disjoint paths S_i of G with ends b, v_i ($1 \leq i \leq 4$), all with no internal vertex in C. Then $C \cup S_1 \cup \cdots \cup S_4$ contains an H_0-subdivision (a K_4^--subdivision on Z), as required. This proves (5).

We show (6). Suppose that there are three paths P_1, P_2, P_3 of G with ends v_1, v_2 and three paths of Q_1, Q_2, Q_3 of G with ends v_3, v_4, mutually disjoint except for their ends; so, the union of these six paths is an H_6-subdivision. By Menger's theorem, there are three disjoint paths R_1, R_2, R_3 of G from $V(P_1 \cup P_2 \cup P_3)$ to $V(Q_1 \cup Q_2 \cup Q_3)$, all with no internal vertex in $P_1, P_2, P_3, Q_1, Q_2, Q_3$. If both R_1 and R_2 have ends in $V(P_1)$, say, then we obtain a subdivision of H_7 or H_8, as required. So we may assume that R_i connects internal vertices of P_i, Q_i ($1 \leq i \leq 3$). We deduce from the same reason that there are no more than three disjoint paths of G from $V(P_1 \cup P_2 \cup P_3)$ to $V(Q_1 \cup Q_2 \cup Q_3)$. Hence there is a 3-separation (A, B) of G such that $V(P_1 \cup P_2 \cup P_3) \subseteq A$ and $V(Q_1 \cup Q_2 \cup Q_3) \subseteq B$. Then $A \cap B$ intersects each R_i exactly once ($1 \leq i \leq 3$); and so, $v_1, v_2 \in A - B$ and $v_3, v_4 \in B - A$, which contradicts the assumption. This proves (6).

We show (7). Suppose that there are two disjoint cycles C, C' of G with $v_1, v_2 \in V(C)$, $v_3, v_4 \in V(C')$, and two disjoint paths of G from $V(C)$ to $V(C')$ with ends v_1v_3, v_2v_4, respectively, both with no internal vertex in $C \cup C'$; so, the union of these two paths, C, and C' is an H_7-subdivision. Since G is 3-connected, there are three disjoint paths R_1, R_2, R_3 of G from $V(C)$ to $V(C')$ covering v_1, v_2, v_3, v_4, all with no internal vertex in $C \cup C'$. Then $C \cup C' \cup R_1 \cup R_2 \cup R_3$ contains a K_4^--subdivision on Z, as required. This proves (7).

We show (8). Suppose that there are two paths P_1, P_2 of G with ends v_1, v_2, three paths Q_1, Q_2, Q_3 of G with ends v_3, v_4, mutually disjoint except for their ends, and two paths of G from $V(P_1 \cup P_2)$ to $V(Q_1 \cup Q_2 \cup Q_3)$ with ends v_1q_1, v_2q_2, respectively, both with no internal vertex in $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup Q_3$, where q_i is an internal vertex of Q_i ($i = 1, 2$); so, the union of these seven paths is an H_8-subdivision. Since G is 3-connected, there are three disjoint paths R_1, R_2, R_3 of G from $V(P_1 \cup P_2)$ to $V(Q_1 \cup Q_2 \cup Q_3)$ covering v_1, v_2, q_1, q_2, all with no internal vertex in $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2$. Let p_3, q_3 be the vertices in $V(P_1 \cup P_2) - \{v_1, v_2\}$, $V(Q_1 \cup Q_2 \cup Q_3) - \{q_1, q_2\}$, respectively, that are "newly" covered by $R_1 \cup R_2 \cup R_3$; assume from the symmetry that $p_3 \in V(P_1)$.

If $q_3 \in V(Q_1 \cup Q_2) - \{p_1, p_2\}$, then $P_1 \cup Q_1 \cup Q_2 \cup R_1 \cup R_2 \cup R_3$ contains a path J through $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ in order, where $\{i_1, i_3\} = \{1, 2\}$ and $\{i_2, i_4\} = \{3, 4\}$; since J is disjoint from P_2, Q_3 except for v_1, v_2, v_3, v_4, there is a K_4^--subdivision on Z in $P_2 \cup Q_3 \cup J$, as required. Thus, we may assume that q_3 is an internal vertex of Q_3, and from the symmetry that R_1, R_2, R_3 have ends v_1q_1, v_2q_2, p_3q_3, respectively.

For $i = 1, 2$, let u_i be the neighbor of v_i in R_i, which might be q_i. There is no 3-separation (A, B) of G such that $V(P_1 \cup P_2) \cup \{u_1, u_2\} \subseteq A$ and $V(Q_1 \cup Q_2) \subseteq B$; for otherwise, it follows from the existence of R_1, R_1, R_3 that $v_1, v_2 \in A - B$ and $v_3, v_4 \in B - A$, contrary to the assumption. Thus, we deduce that there are four paths of G from $V(P_1 \cup P_2)$ to $V(Q_1 \cup Q_2 \cup Q_3)$ covering $v_1, v_2, p_3, q_1, q_2, q_3$, mutually disjoint except for v_1, v_2, and all with no internal vertex in $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup Q_3$. One may claim, by the same proof as in the preceding paragraph, that the three of these four paths that start from v_1, v_2, p_3 in $V(P_1 \cup P_2 \cup P_3)$ must end in $V(Q_i) - \{v_3, v_4\}$ $(1 \leq i \leq 3)$, respectively; for otherwise, we obtain a K_4^--subdivision on Z. We may as well denote these three paths by R_1, R_2, R_3 with ends v_1q_1, v_2q_2, p_3q_3, respectively. Therefore, we may assume that there is a "fourth" path R_4 of G with ends p_4, q_4 and no internal vertex in $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3 \cup R_1 \cup R_2 \cup R_3$, where $p_4 \in V(P_1 \cup P_2) - \{p_3\}$ and $q_4 \in V(Q_1 \cup Q_2 \cup Q_3) - \{q_1, q_2, q_3\}$.

Let H be the union of $P_1, P_2, P_3, Q_1, Q_2, Q_3, R_1, R_2, R_3, R_4$. It is easy to see that if $p_4 \in V(P_1) - \{p_3\}$, then H contains a K_4^--subdivision on Z, as required; so, assume that $p_4 \in V(P_2) - \{v_1, v_2\}$. It is also easy to see that if $q_4 \in V(Q_1 \cup Q_2) - \{q_1, q_2\}$, then H contains a K_4^--subdivision on Z, as required; so assume that $q_4 \in V(Q_3) - \{v_3, v_4, q_3\}$. Then H is an H_9-subdivision (a cube$^+$ on Z), as required. This proves (8) and completes the proof. □

▶ Step 5: Applying the K_4^{2-}-subdivision lemma

A cube$^+$ H on Z contains no K_4^--subdivision on Z, but one may say that it is an "extremal frame" in that if we add a bold "jump" J to H, there is a K_4^--subdivision on Z in $H \cup J$. The final step is to conclude that possible structures of G containing H would yield either a K_4^--subdivision on Z or obstructions, and hence Theorem 4.5.3 (d) follows.

Roughly speaking, our plan is as follows. Suppose that there is a cube$^+$ on $(v_1, v_2; v_3, v_4)$ in G with notation as in Step 4; see Figure 4.7. Then one can find a 4-separation (A, B) of G with $V(P_1 \cup P_2 \cup P_3) \subseteq A$ and $V(Q_1 \cup Q_2) \subseteq B$, such that the 4-cut $A \cap B = \{x_1, x_2, x_3, x_4\}$ is disjoint from $\{v_3, v_4\}$. We may assume that each x_i lies in R_i $(1 \leq i \leq 4)$. Now assume for simplicity that $x_1 \neq q_1$ and $x_2 \neq q_2$. Let G' be a graph obtained from G by adding new vertices x, y and edges xx_1, xx_2, yx_3, yx_4. Then $G'|A \cup \{x, y\}$ contains no K_4^{2-}-subdivision on $\{v_1, v_2, x, y\}$ in which x has degree 1 and v_1 or v_2 has degree 3; for otherwise, such a K_4^{2-}-subdivision and the half fragment of the cube$^+$ in $G|B$ yields a K_4^--subdivision on Z. Similarly, $G'|B \cup \{x, y\}$ contains no K_4^{2-}-subdivision on $\{v_3, v_4, x, y\}$ in which y has degree 1 and v_3 or v_4 has degree 3. Hence we can apply the "K_4^{2-}-subdivision lemma" (Theorem 4.4.2) to both $G'|A \cup \{x, y\}$ and $G'|B \cup \{x, y\}$. This determines the structure of G, leading to Theorem 4.5.3 (d).

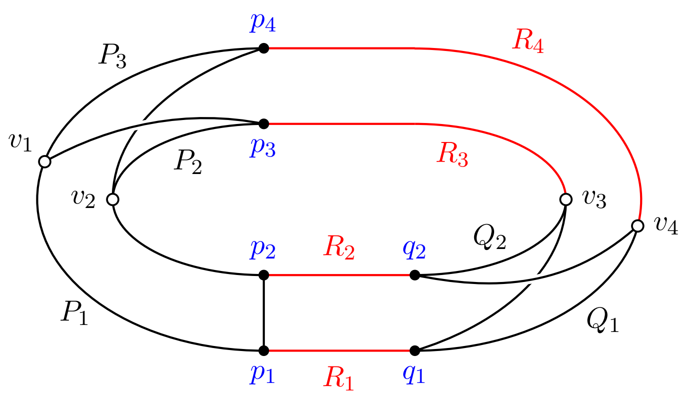

Figure 4.7: A cube$^+$ on $(v_1, v_2; v_3, v_4)$.

First we show the existence of a 4-separation.

Lemma 4.6.7. *Assume (hyp 4.6.1)–(hyp 4.6.3). Let H be a cube$^+$ on $(v_1, v_2; v_3, v_4)$ in G with notation as in Step 4. Then there is a 4-separation (A, B) of G such that $V(P_1 \cup P_2 \cup P_3) \subseteq A$, $V(Q_1 \cup Q_2) \subseteq B$ and $A \cap B \cap \{v_3, v_4\} = \emptyset$.*

Proof. Suppose to the contrary that there is no such separation. For $i = 3, 4$ let u_i be the neighbor of v_i in R_i, which might be p_i. Then there is no 4-separation (A, B) of G such that $V(P_1 \cup P_2 \cup P_3) \subseteq A$ and $V(Q_1 \cup Q_2) \cup \{u_3, u_4\} \subseteq B$; for otherwise, it follows from the existence of $R_1, \ldots, R_4$ that $A \cap B \cap \{v_3, v_4\} = \emptyset$, and so the result follows. Thus, we deduce that there are five paths $S_1, \ldots, S_5$ of G from $V(P_1 \cup P_2 \cup P_3)$ to $V(Q_1 \cup Q_2)$ covering $p_1, p_2, p_3, p_4, q_1, q_2, v_3, v_4$, mutually disjoint except for v_3, v_4, and all with no internal vertex in $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2$. For $i = 1, 2$ let $\mathcal{S}_i$ denote the subset of $\{S_1, \ldots, S_5\}$ consisting of members that have ends in $V(Q_i)$. Assume from the symmetry that $|\mathcal{S}_2| = 4$. At least two members in $\mathcal{S}_2$ have ends in $V(P_i)$ for some $1 \le i \le 3$. On the other hand, not all of members in $\mathcal{S}_2$ have ends in the same $V(P_i)$, since p_1, p_2, p_3, p_4 are covered by $S_1 \cup \cdots \cup S_5$. Hence for some ordering $\{i_1, i_2, i_3\} = \{1, 2, 3\}$ there exist three distinct members S, S', S'' in $\mathcal{S}_2$ such that S and S' have ends in $V(P_{i_1})$ and S'' has an end in $V(P_{i_2}) - \{v_1, v_2\}$. Note that some of S, S', S'' contains v_3 or v_4 as its end. It is easy to see that $P_{i_1} \cup P_{i_2} \cup Q_2 \cup S \cup S' \cup S''$ contains a path J through $v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}$ in order, where $\{j_1, j_3\} = \{1, 2\}$ and $\{j_2, j_4\} = \{3, 4\}$. Since J is disjoint from Q_1, P_{i_3} except for v_1, v_2, v_3, v_4, there is a K_4^--subdivision on Z in $Q_1 \cup P_{i_3} \cup J$, contrary to (hyp 4.6.2). This proves the lemma. □

Before applying the K_4^{2-}-subdivision lemma to cube$^+$, we define a certain decomposition of a graph G. We say that a decomposition (X_1, X_2, A_1, A_2, B) of G is *intermediate* if its basic family satisfies the following.

- The vertex set of the basic family is $\{a_1, a_2, x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}\}$, where a_1, a_2, x_{13}, x_{23} are distinct and x_{i1}, x_{i2}, x_{i3} are distinct for $i = 1, 2$.
- $\operatorname{bd} B = \{a_1, a_2, x_{13}, x_{23}\}$.
- For $i = 1, 2$, $\operatorname{bd} X_i = \{x_{i1}, x_{i2}, x_{i3}\}$.
- For $i = 1, 2$, $\{x_{1i}, x_{2i}, a_i\} \subseteq \operatorname{bd} A_i \subseteq \{x_{1i}, x_{2i}, a_i, a_{3-i}\}$.

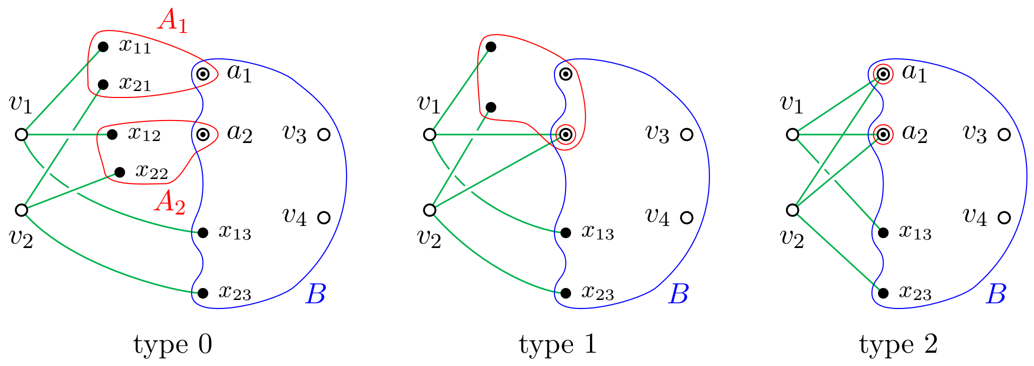

Figure 4.8: Three types of intermediate decompositions in Lemma 4.6.8.

- For $i = 1, 2$, $a_i \in \operatorname{bd} A_{3-i}$ if and only if $|A_i| = 1$ and $|A_{3-i}| > 1$.

Let s be the numbers of singletons in $\{A_1, A_2\}$. We say that the intermediate decomposition is of *type* s. Note that $(X_1 \cup X_2 \cup A_1 \cup A_2, B)$ is a 4-separation of G. The next lemma is a consequence of the K_4^{2-}-subdivision lemma applied to cube$^+$; see Figure 4.8. For simplicity, $G|X_i$ is depicted as 3-stars with feet $\operatorname{bd} X_i$ for $i = 1, 2$.

Lemma 4.6.8. *Assume (hyp 4.6.1)–(hyp 4.6.3). Suppose that there is a cube$^+$ on $(v_1, v_2; v_3, v_4)$ in G. Then there is an intermediate decomposition (X_1, X_2, A_1, A_2, B) of G such that $v_i \in \operatorname{int} X_i$ for $i = 1, 2$ and $v_3, v_4 \in \operatorname{int} B$.*

Proof. Let H be a cube$^+$ on $(v_1, v_2; v_3, v_4)$ in G with notation as in Step 4; see Figure 4.7. Then there is a 4-separation (A, B) of G as in Lemma 4.6.7. Let $A \cap B = \{x_1, x_2, x_3, x_4\}$. We may assume that $x_i \in V(R_i)$ for $1 \le i \le 4$. Let $\alpha := |\{x_1, x_2\} \cap \{q_1, q_2\}|$. Let G' be a graph obtained from $G|A$ by adding a new vertex y and edges yx_3, yx_4 and furthermore doing the following, depending on α: If $\alpha = 0$, add a new vertex x and edges xx_1, xx_2; if $\alpha = 1$, add an edge x_1x_2 and let x be the vertex in $\{x_1, x_2\} \cap \{q_1, q_2\}$; if $\alpha = 2$, identify x_1 and x_2 and denote by x the resulting vertex. Then there is no K_4^{2-}-subdivision on either $(v_1; v_2, y; x)$ or on $(v_2; v_1, y; x)$ in G'; for otherwise, G contains a K_4^--subdivision on Z, a contradiction. We shall apply Theorem 4.4.2 to G'. Note that from the existence of cube$^+$ H we already have a subgraph in G' which is homeomorphic to a graph as in Figure 4.9 (left). By Theorem 4.4.2 there is a $K_{3,3}$-decomposition $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ of G' such that $v_i \in X_i - Y$ $(i = 1, 2)$, $y \in X_3 - Y$ and $x \in X_3$, where $Y = Y_1 \cup Y_2 \cup Y_3$. Let $s \in \{0, 1, 2, 3\}$ be its integral value and let $W := Y_1 \cup \cdots \cup Y_s$. Note that $x_3, x_4 \in X_3$.

Let $X_i \cap Y_j - Y_1 \cup \cdots \cup Y_{j-1} = \{x_{ij}\}$ for $i = 1, 2$ and $1 \le j \le 3$. Let $X_3 \cap Y_j - Y_1 \cup \cdots \cup Y_{j-1} = \{a_j\}$ for $1 \le j \le 3$. Note that $x_{1j} = x_{2j} = a_j$ for every positive integer $j \le s$.

Since there is a cycle in G' through v_1, x, v_2, y in order, we have $x \in X_3 \cap Y$. Since there is a cycle in $G' \setminus x$ through v_1, v_2, y, we have $x \notin W$; and so, $s \le 2$. We may assume that $a_3 = x$. Note that x, x_{13}, x_{23} are distinct vertices in Y_3.

If $\alpha = 0$, set $X_3' := X_3 - \{x, y\}, Y_3' := Y_3 - \{x\}$; if $\alpha = 1$, set $X_3' := X_3 - \{y\}, Y_3' := Y_3$; if $\alpha = 2$, set $X_3' := (X_3 - \{x, y\}) \cup \{x_1, x_2\}, Y_3' := (Y_3 - \{x\}) \cup \{x_1, x_2\}$. Then $(X_1, X_2, X_3', Y_1, Y_2, Y_3')$ is a decomposition of $G|A$ such that $x_1, \ldots, x_4 \in X_3' \cup Y_3'$. Let $B' := B \cup X_3' \cup Y_3'$. Then (X_1, X_2, Y_1, Y_2, B') is an intermediate decomposition of G of

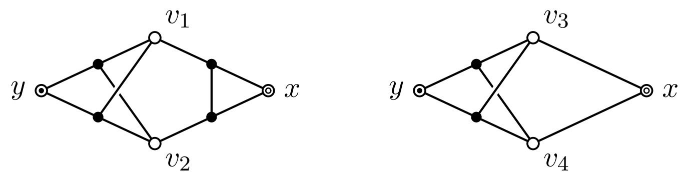

Figure 4.9: Subgraphs in the proofs of Lemma 4.6.8 (left) and Lemma 4.6.9 (right).

type s such that $v_i \in \operatorname{int} X_i$ for $i = 1, 2$ and $v_3, v_4 \in \operatorname{int} B'$, as required. This completes the proof. □

The next task is to apply again the K_4^{2-}-subdivision lemma to the intermediate structure of G in Lemma 4.6.8 to conclude that G is in fact admits a special separation and hence Theorem 4.5.3 (d) holds.

Lemma 4.6.9. *Assume (hyp 4.6.1)–(hyp 4.6.3). Suppose that there is a cube$^+$ on $(v_1, v_2; v_3, v_4)$ in G. Then Theorem 4.5.3* (d) *holds.*

Proof. By Lemma 4.6.8 there is an intermediate decomposition (X_1, X_2, A_1, A_2, B) of G of type $s \in \{0, 1, 2\}$ such that $v_i \in \operatorname{int} X_i$ for $i = 1, 2$ and $v_3, v_4 \in \operatorname{int} B$. We choose such a decomposition with s maximum. Let $a_1, a_2, x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}$ denote the vertices of the basic family, as usual.

(1) *For $i = 1, 2$, if $|A_i| > 1$ then there is a path of $G|A_i$ from x_{1i} to x_{2i} with no vertex in $\{a_1, a_2\}$.*

For suppose not for $i = 1$, say. Then there is a separation (C, D) of $G|A_1$ such that $x_{11} \in C - D$, $x_{21} \in D - C$, and moreover, $C \cap D = \{a_1\}$ if $|A_2| > 1$ and $C \cap D = \{a_1, a_2\}$ otherwise. Let $X_1' := X_1 \cup C$, $X_2' := X_2 \cup D$. Let $A_1' := \{a_1\}$, $A_2' := A_2 \cup \{a_1\}$ if $|A_2| > 1$, and let $A_1' := \{a_1\}$, $A_2' := A_2$ otherwise. Then $(X_1', X_2', A_1', A_2', B)$ is an intermediate decomposition of G of type $s + 1$ such that $v_i \in \operatorname{int} X_i'$ for $i = 1, 2$ and $v_3, v_4 \in \operatorname{int} B$, contrary to the maximality of s. The proof is analogous for $i = 2$. This proves (1).

We assume that $|A_1| = 1$ if $s = 1$. Let G' be a graph obtained from $G|B$ by adding a new vertex y and edges yx_{13}, yx_{23} and furthermore doing the following, depending on s: If $s = 0$, add a new vertex x and edges xa_1, xa_2; if $s = 1$, add an edge a_1a_2 and set $x := a_1$; if $s = 2$, identify a_1 and a_2 and denote by x the resulting vertex. Then there is no K_4^{2-}-subdivision on either $(v_3; v_4, y; x)$ or on $(v_4; v_3, y; x)$ in G'; for otherwise, G contains a K_4^--subdivision on Z by (1), a contradiction. We shall apply Theorem 4.4.2 to G'. Note that from the existence of the cube$^+$ we already have a subgraph in G' which is homeomorphic to a graph as in Figure 4.9 (right).

(2) *If $s \in \{1, 2\}$, there is a cycle in $G' \setminus x$ through v_3, v_4, y.*

For suppose that $s \geq 1$ (and so $|A_1| = 1$). We deduce from Lemma 4.6.2 that there is a cycle C in $G \setminus a_1$ that contains all the vertices in Z. There is a subpath C' of C between x_{13} and x_{23} containing v_3, v_4 and not containing v_1, v_2, which is a path of $(G|B) \setminus a_1$. Note that $a_2 \notin V(C')$ if $s = 2$ (that is, if $A_2 = \{a_2\}$). This proves (2).

By Theorem 4.4.2 there is a $K_{3,3}$-decomposition $(U_1, U_2, U_3, V_1, V_2, V_3)$ of G' such that $v_{i+2} \in U_i - V$ $(i = 1, 2)$, $y \in U_3 - V$ and $x \in U_3$, where $V = V_1 \cup V_2 \cup V_3$. Let $s' \in \{0, 1, 2, 3\}$ be its integral value and let $W := V_1 \cup \cdots \cup V_{s'}$. Note that $x_{13}, x_{23} \in U_3$. Since there is a cycle in G' through v_3, x, v_4, y in order, we have $x \in U_3 \cap V$. If $s \in \{1, 2\}$, we have $x \notin W$ (and hence $s' \leq 2$) by (2).

Let $U_{i-2} \cap V_j - V_1 \cup \cdots \cup V_{j-1} = \{x_{ij}\}$ for $i = 3, 4$ and $1 \leq j \leq 3$. Let $U_3 \cap V_j - V_1 \cup \cdots \cup V_{j-1} = \{b_j\}$ for $1 \leq j \leq 3$. Note that $x_{3j} = x_{4j} = b_j$ for every positive integer $j \leq s'$. We consider three cases, depending on s.

Case 1: $s = 2$. We may assume that $b_3 = x$, since $x \in U_3 \cap V - W$. Note that $x \notin U_1 \cup U_2$. Set $U_3' := (U_3 - \{x, y\}) \cup \{a_1, a_2\}, V_3' := (V_3 - \{x\}) \cup \{a_1, a_2\}$. Then $(U_1, U_2, U_3', V_1, V_2, V_3')$ is a decomposition of $G|B$ with $a_1, a_2, x_{13}, x_{23} \in U_3'$. Now $(X_1, X_2, U_1, U_2, A_1, A_2, U_3', V_1, V_2, V_3')$ is a special decomposition of G of type (s, s') such that v_1, v_2, v_3, v_4 belong to $\operatorname{int} X_1, \operatorname{int} X_2, \operatorname{int} U_1, \operatorname{int} U_2$, respectively. So (d) holds, as required.

Case 2: $s = 1$. We may assume that $b_3 = x$ $(= a_1)$, since $x \in U_3 \cap V - W$. From the existence of cube$^+$, we have $a_2 \in V_3 - U_3$. Set $U_3' := U_3 - \{y\}$. Then $(U_1, U_2, U_3', V_1, V_2, V_3)$ is a decomposition of $G|B$ with $x_{13}, x_{23}, a_1 \in U_3', a_2 \in V_3$. Now $(X_1, X_2, U_1, U_2, A_1, A_2, U_3', V_1, V_2, V_3)$ is a special decomposition of G of type (s, s') such that v_1, v_2, v_3, v_4 belong to $\operatorname{int} X_1, \operatorname{int} X_2, \operatorname{int} U_1, \operatorname{int} U_2$, respectively. So (d) holds, as required.

Case 3: $s = 0$. First we consider the case $x \notin W$; and so, $s' \leq 2$. We may assume that $b_3 = x$. Note that $a_1, a_2 \in V_3 - U_3$. Set $U_3' := U_3 - \{x, y\}, V_3' := V_3 - \{x\}$. Then $(U_1, U_2, U_3', V_1, V_2, V_3')$ is a decomposition of $G|B$ with $x_{13}, x_{23} \in U_3', a_1, a_2 \in V_3'$. Now $(X_1, X_2, U_1, U_2, A_1, A_2, U_3', V_1, V_2, V_3')$ is a special decomposition of G of type (s, s') such that v_1, v_2, v_3, v_4 belong to $\operatorname{int} X_1, \operatorname{int} X_2, \operatorname{int} U_1, \operatorname{int} U_2$, respectively. So (d) holds, as required.

Next we consider the case $x \in W$; and so, $s' \geq 1$. Assume that $V_1 = \{x\}$. Then $\{a_1, a_2\}$ meets both $U_1 - V$ and $U_2 - V$; assume that $a_i \in U_i - V$ for $i = 1, 2$. Set $U_1' := U_1 - \{x\}, U_2' := U_2 - \{x\}, U_3' := U_3 - \{x, y\}, V_2' := V_2 - \{x\}, V_3' := V_3 - \{x\}$. Then $(U_1', U_2', U_3', V_2', V_3')$ is a decomposition of $G|B$ with $x_{13}, x_{23} \in U_3', a_1 \in U_1', a_2 \in U_2'$. Now $(X_1, X_2, U_1', U_2', A_1, A_2, U_3', V_2', V_3', \{a_1, a_2\})$ is a special decomposition of G of type $(s, s' - 1)$ such that v_1, v_2, v_3, v_4 belong to $\operatorname{int} X_1, \operatorname{int} X_2, \operatorname{int} U_1', \operatorname{int} U_2'$, respectively. So (d) holds, as required. This completes the proof. □

▶ Step 6: Completing the proof of Lemma 4.6.1

We are now ready to prove Lemma 4.6.1, which completes the proof of the main theorem.

Proof of Lemma 4.6.1. Let $Z = \{v_1, v_2, v_3, v_4\}$ and assume (hyp 4.6.1)–(hyp 4.6.3). By Lemma 4.6.3, we may assume that there is no 3-separation (A, B) of G with $|Z \cap (A - B)| = |Z \cap (B - A)| = 2$; for otherwise, (c) follows, as required. By Lemma 4.6.4 and Lemma 4.6.5, we may assume that there are five internally disjoint Z-paths in G; for otherwise, (d) follows, as required. By Lemma 4.6.6 there is a cube$^+$ H on $(v_1, v_2; v_3, v_4)$, say, in G. Then (d) follows by Lemma 4.6.9, as required. This completes the proof. □

4.7 Cycles through four vertices

As noted in Section 4.5, in order to give a complete characterization of obstructions, we need to characterize graphs that contain no cycle through prescribed four vertices. We solve this problem in this section.

Let G be a graph and let Z be a set of distinct four vertices of G. We want to investigate what kind of structure G has if G is Z-acyclic. First we may assume that G is 2-connected. We say that (G, Z) is *irreducible* if it satisfies the following: (i) There is no 2-separation (A, B) of G with $Z \subseteq A$ and $B - A \neq \emptyset$; (ii) if there is a 2-separation (A, B) of G with $|Z \cap (A - B)| = 1$ (and $B - A \neq \emptyset$), then $|A - B| = 1$; (iii) if there is a 3-separation (A, B) of G with $|Z \cap (A - B)| = 1, |Z \cap (B - A)| = 3$ and if the vertex in $Z \cap (A - B)$ has degree ≥ 3, then $|A - B| = 1$. We are interested in only irreducible graphs, because if (G, Z) is not irreducible, we can obtain a graph G' with $|V(G')| < |V(G)|$, such that G' is Z-cyclic if and only if G is. To exclude obvious obstructions, we may also assume the following.

(hyp 4.7.1) G is Z'-cyclic for any $Z' \subseteq Z$ with $|Z'| = 3$.

(hyp 4.7.2) For any $w \in V(G) - Z$, there is a path of $G \setminus w$ that contains all the vertices in Z.

Indeed, if (hyp 4.7.1) is false, the structure of G is determined by Watkins–Mesner's theorem. For (hyp 4.7.2), we have to characterize paths through four vertices. But this is an easy problem which is reduced to Watkins–Mesner's theorem, as we shall discuss in Appendix 4.9.

Assuming the above hypotheses, one immediately notices the following.

Lemma 4.7.1. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Suppose that G is 2-connected, (G, Z) is irreducible, and G is Z-acyclic. Assume (hyp 4.7.1) and (hyp 4.7.2). Then:*

(i) *Any two vertices in Z of degree 2 have no common neighbor, and*
(ii) *G admits no 2-cut except but the set of neighbors of a vertex in Z of degree 2.*

Proof. To see (i), suppose to the contrary that $v_1, v_2 \in Z$ have degree 2 and have a common neighbor x. By (hyp 4.7.2) there is a path of $G \setminus x$ containing all the vertices in Z. A minimal such path P has ends v_1, v_2. Then P and edges v_1x, v_2x yield a cycle in G that contains all the vertices in Z, a contradiction. This proves (i).

To see (ii), let (A, B) be a 2-separation of G such that $A - B, B - A \neq \emptyset$. Since G is irreducible, both $A - B$ and $B - A$ meet Z. If both $A - B$ and $B - A$ contain exactly two vertices in Z, then from (hyp 4.7.1) we easily obtain a cycle of G that contains all the vertices in Z, a contradiction. Thus, one of them, $A - B$ say, contains exactly one vertex v in Z; and so, we have $A \cap B = N_G(v)$, since G is irreducible. This proves (ii). □

Before stating our result, we need some definition. We say that a decomposition (X_1, X_2, A, B) of a graph G is *nice* if its basic family has a vertex set consisting of five distinct vertices a_1, a_2, b_1, b_2, b_3 such that $\operatorname{bd} X_i = \{a_i, b_i\}$ for $i = 1, 2$, $\operatorname{bd} A = \{a_1, a_2, b_3\}$

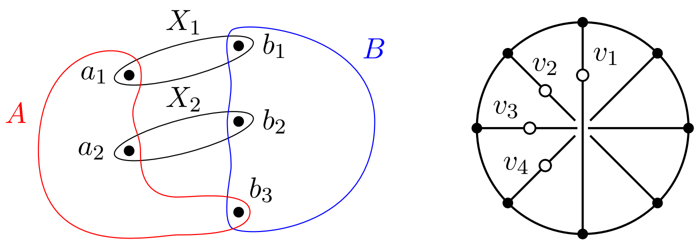

Figure 4.10: A nice decomposition (left) and a V_8-subdivision on $\{v_1, v_2, v_3, v_4\}$ (right).

and bd $B = \{b_1, b_2, b_3\}$; see Figure 4.10 (left). Note that $(X_1 \cup X_2 \cup A, B)$ is a 3-separation of G.

Let $v_1, \ldots, v_4$ be distinct four vertices of G. A V_8-*subdivision* on $\{v_1, v_2, v_3, v_4\}$ is a subgraph H of G consisting of the union of a cycle C of G through distinct eight vertices $u_1, \ldots, u_8 \in V(G)$ in this order listed and four disjoint paths P_i of G with ends u_i, u_{i+4} $(1 \le i \le 4)$, all with no internal vertex in C, such that v_i is an internal vertex of P_i for $1 \le i \le 4$; see Figure 4.10 (right). Note that H contains no cycle through all the vertices in $\{v_1, \ldots, v_4\}$.

The main result in this section is the following theorem. It states that every Z-acyclic graph contains a V_8-subdivision on Z or admits a 3-separation that separates Z into two subsets of size 2. If there is a 3-separation (A, B) of G with $|Z \cap (A - B)| = |Z \cap (B - A)| = 2$, one can apply Watkins–Mesner's theorem to both sides $G|A$ and $G|B$ to determine the structure of G. Theorem 4.7.2 chooses a nice decomposition so that it suffices to apply Watkins–Mesner's theorem only once.

Theorem 4.7.2. *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Suppose that G is 2-connected, (G, Z) is irreducible, and G is Z-acyclic. Assume (hyp 4.7.1) and (hyp 4.7.2). Then one of the following holds.*

(a) *There is a nice decomposition (X_1, X_2, A, B) of G such that* $\operatorname{int} X_1, \operatorname{int} X_2, \operatorname{int} B$ *contains exactly* $1, 1, 2$ *vertices of Z, respectively.*
(b) *All the vertices of Z have degree 2 and there is a V_8-subdivision on Z in G.*

Let us see that the theorem characterizes Z-acyclic graphs. Let $Z = \{v_1, v_2, v_3, v_4\}$. Suppose that (a) holds. Assume that $v_i \in \operatorname{int} X_i$ for $i = 1, 2$ and $v_3, v_4 \in \operatorname{int} B$. Let a_1, a_2, b_1, b_2, b_3 denote the vertices of the basic family as usual. Then there is a path of $G|A$ between a_1 and a_2 avoiding b_3. For otherwise, b_3 becomes a common neighbor of v_1, v_2 by the irreducibility of G, contrary to Lemma 4.7.1 (i). Hence there is a path of $G|A \cup X_1 \cup X_2$ with ends b_1, b_2 containing v_1, v_2 and avoiding b_3. On the other hand, it is clear that for $i = 1, 2$ there is no path of $G|A \cup X_1 \cup X_2$ from b_3 to b_i that contains v_1, v_2. Therefore, G is Z-acyclic if and only if there is no path of $G|B$ with ends b_1, b_2 containing v_3, v_4. By applying Watkins–Mesner's theorem to $G|B$ (with a dummy vertex adjacent to b_1, b_2), we finally understand the whole structure of G, as desired, though we omit the explicit description here.

If (b) holds, the problem is reduced to the results in [37]. Delete vertices in Z and add an edge e_i in $N_G(v_i)$ for $1 \le i \le 4$. Then the resulting graph is 3-connected by

Lemma 4.7.1 (ii). Now our problem is reduced to a problem of characterizing $\{e_1, e_2, e_3, e_4\}$-acyclic 3-connected graphs. This problem has been already solved by Lomonosov [37] (private communication with N. Robertson). A V_8-subdivision H on four vertices Z (or four edges) are surely Z-acyclic, but it is an "extremal frame" in that if H has a bold "jump" J the union $J \cup H$ becomes immediately Z-cyclic. Consequently, the whole structure of G is "similar" to H. See Section 4.7.1 for more details. Therefore, Theorem 4.7.2 completes the task of characterizing graphs that contain no cycle through given four vertices, as desired.

4.7.1 Cycles through four edges

Let G be a graph and let e_1, e_2, e_3, e_4 be four independent edges of G. We define a V_8-subdivision on four edges in the same way as above. Namely, a *V_8-subdivision* on $\{e_1, e_2, e_3, e_4\}$ is a subgraph H of G consisting of the union of a cycle C of G through distinct eight vertices $u_1, \dots, u_8 \in V(G)$ in this order listed and four disjoint paths P_i of G with ends u_i, u_{i+4} $(1 \le i \le 4)$, all with no internal vertex in C, such that e_i lies in P_i for $1 \le i \le 4$. In this subsection, we state the result in [37]: If G contains a V_8-subdivision H on $\{e_1, e_2, e_3, e_4\}$ but no cycle through all of these edges, then the structure of G is "similar" to H.

For a graph G with four independent edges $e_i = u_i u_{i+4}$ $(1 \le i \le 4)$, we say that $(G, \{e_1, e_2, e_3, e_4\})$ is a *basic obstruction* if it satisfies the following.

- $V(G) = \{u_i, x_i, y_i \colon 1 \le i \le 8\}$.
- For $1 \le i \le 8$, $A_i := \{u_i, x_i, y_i\}$ is a clique of size 1 or 3.
- For $1 \le i \le 8$, $y_i x_{i+1} \in E(G)$ if $|A_i| = |A_{i+1}| = 1$ and $y_i = x_{i+1}$ otherwise, where indices are read modulo 8.
- G has no other edges.

Note that cliques of G have size ≤ 4. A 4-clique appears in $\{u_1, u_3, u_5, u_7\}$ or $\{u_2, u_4, u_6, u_8\}$ if and only if 1 and 3 appear alternately in the sequence $(|A_1|, |A_2|, \dots, |A_8|)$. As easily checked, every basic obstruction $(G, \{e_1, \dots, e_4\})$ contains a V_8-subdivision on $\{e_1, e_2, e_3, e_4\}$ but no cycle through these four edges. In fact, this property is maintained even when we paste complete graphs of size ≥ 2 in a basic obstruction along cliques. The following theorem, implied by [37], says that the converse holds.

Theorem 4.7.3. *Let G be a graph with four independent edges e_1, e_2, e_3, e_4. Suppose that G contains a V_8-subdivision on $\{e_1, e_2, e_3, e_4\}$ but no cycle through these four edges, and subject to that with G edge-maximal. Then G can be constructed, by pasting along cliques, from complete graphs of size ≥ 2 and a basic obstruction $(G', \{e_1, e_2, e_3, e_4\})$.*

4.8 Proof of Theorem 4.7.2

In this section we prove Theorem 4.7.2. Let G be a graph and let $Z = \{v_1, v_2, v_3, v_4\}$ be a set of distinct four vertices of G, which we call *terminals*. Throughout the section, we assume that G is 2-connected, (G, Z) is irreducible, and G is Z-acyclic. Also assume (hyp 4.7.1) and (hyp 4.7.2). The goal is to show that (a) or (b) holds.

The proof traces the method in the proof of Theorem 4.5.3. First we find four internally disjoint Z-paths, by Mader's S-paths theorem. Next we make these four paths into a certain subgraph H, which we call a prism on Z (Figure 4.11). Then we find a 3-separation (A, B) of G such that $A-B$, $B-A$ contain two terminals, respectively; otherwise, we may augment H to obtain a V_8-subdivision. Finally we apply Watkins–Mesner's theorem to both sides $G|A$ and $G|B$ to obtain a nice decomposition of G as in (a). One may consider starting from such a 3-separation before constructing the "frame" H. But the proof seems more simple if we use the half fragments of the frame H in $G|A$ and $G|B$, when applying Watkins–Mesner's theorem. This idea corresponds to Step 5 in the proof of Theorem 4.5.3: Applying the K_4^{2-}-subdivision lemma to both sides of a separation, with the aid of the "frame" cube$^+$.

Let us begin with a few easy lemmas below.

Lemma 4.8.1. *The set Z is stable.*

Proof. By (hyp 4.7.1) there is a cycle of G that contains v_1, v_2, v_3 and avoids v_4. Since there is no K_4^{2-}-subdivision on Z in G by Lemma 4.5.1, v_4 is not adjacent to any other terminals. Similarly, no two terminals are adjacent. This proves the lemma. □

Lemma 4.8.2. *Every terminal has degree ≤ 3 in G.*

Proof. Suppose that v_4 has degree ≥ 3, say. By (hyp 4.7.1) there is a cycle C of G that contains v_1, v_2, v_3 and avoids v_4. Let C_{12} be the path of C between v_1 and v_2 not containing v_3, and define C_{23}, C_{13} similarly. Since G is irreducible, there are three paths P_1, P_2, P_3 of G from v_4 to $V(C)$, mutually disjoint except for v_4. Since G is Z-acyclic, we may assume that P_1, P_2, P_3 have ends in the interior of C_{23}, C_{13}, C_{12}, respectively. If there are four paths of G from v_4 to $V(C)$, mutually disjoint except for v_4, then the union of those four paths and C yields a cycle containing all the terminals, a contradiction. Thus, there is a 3-separation (A, B) of G such that $v_4 \in A - B$ and $V(C) \subseteq B$. We deduce from the existence of P_1, P_2, P_3 that $v_1, v_2, v_3 \in B - A$. Hence v_4 has degree 3, since G is irreducible. This proves the lemma. □

The next step is to find four internally disjoint Z-paths in G. If there are no more than three, G is obviously Z-acyclic. The following lemma, based on Mader's S-paths theorem, says that such a graph has a nice decomposition as in (a). We may thus assume from now on that there are at least four internally disjoint Z-paths.

Lemma 4.8.3. *If there are no more than three internally disjoint Z-paths in G, then* (a) *holds.*

Proof. For suppose not. Since Z is stable by Lemma 4.8.1, we deduce from Theorem 4.2.1 that there is a good quasi-bipartite-decomposition $(W; X_1, X_2, X_3, X_4; Y_1, \ldots, Y_m)$ of G with respect to Z of value ≤ 3; hence Y_j intersects at least three of $X_1, \ldots, X_4$ and $|Y_j \cap X|$ is odd (≥ 3) for $1 \leq j \leq m$, and

$$|W| + m \leq |W| + \sum_{1 \leq j \leq m} \left\lfloor \frac{1}{2} |Y_j \cap X| \right\rfloor = |W| + \frac{|X \cap Y| - m}{2} \leq 3, \tag{4.8.1}$$

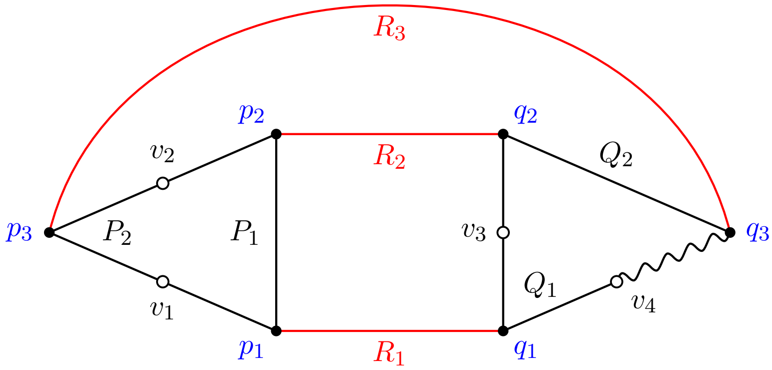

Figure 4.11: A prism on (v_1, v_2, v_3, v_4).

where $X = X_1 \cup \cdots \cup X_4$ and $Y = Y_1 \cup \cdots \cup Y_m$. We may assume that $v_i \in X_i - Y$ for $1 \leq i \leq 4$. It follows from (hyp 4.7.2) that $|W| = 0$. We have $|X_i \cap Y| \geq 2$ $(1 \leq i \leq 4)$ because $X_i \cap Y$ is a cut of G that separates v_i from $Z - \{v_i\}$ while G is 2-connected. Hence

$$|X \cap Y| = \sum_{1 \leq i \leq 4} |X_i \cap Y| \geq 2 \cdot 4 = 8, \tag{4.8.2}$$

which, together with (4.8.1), implies that $m \geq 2$.

Now consider a bipartite multigraph H with vertex set $\{X_1, \ldots, X_4, Y_1, \ldots . Y_m\}$, having $|X_i \cap Y_j|$ edges between X_i and Y_j; assume $\deg_H(Y_1) \leq \cdots \leq \deg_H(Y_m)$ and $\deg_H(X_1) \leq \cdots \leq \deg_H(X_4)$. If $m = 2$, then $\deg_H(Y_1) = 3, \deg_H(Y_2) = 5$ and $\deg_H(X_i) = 2$ $(1 \leq i \leq 4)$. Assuming $N_H(Y_1) = \{X_1, X_2, X_3\}$, we have $|X_4 \cap Y_2| = 2$ and $|X_i \cap Y_2| = 1$ $(1 \leq i \leq 3)$. Then there is no cycle of G through v_1, v_2, v_3, since $(X_1, X_2, X_3, Y_1, Y_2 \cup X_4)$ is a $K_{3,2}$-decomposition of G (of integral value 0); this contradicts (hyp 4.7.1). Thus, $m = 3$, $\deg_H(Y_j) = 3$ $(1 \leq j \leq 3)$, H is simple, $\deg_H(X_i) = 2$ $(1 \leq i \leq 3)$ and $\deg_H(X_4) = 3$. We may assume that $N_H(X_i) = \{Y_1, Y_2, Y_3\} - \{Y_i\}$ $(1 \leq i \leq 3)$. Let $B := X_3 \cup X_4 \cup Y_1 \cup Y_2$. Then (X_1, X_2, Y_3, B) is a nice decomposition of G such that $v_i \in \operatorname{int} X_i$ for $i = 1, 2$ and $v_3, v_4 \in \operatorname{int} B$, and so, (a) holds, as required. This proves the lemma. $\square$

A *prism* on (v_1, v_2, v_3, v_4) is a subgraph H of G consisting of the union of:

(i) two paths P_1, P_2 of G with ends v_1, v_2 and two paths Q_1, Q_2 of G with ends v_3, v_4, mutually disjoint except for their ends, and

(ii) three disjoint paths R_1, R_2, R_3 of G from $V(P_1 \cup P_2)$ to $V(Q_1 \cup Q_2)$ with ends p_i, q_i $(1 \leq i \leq 3)$, all with no internal vertex in $P_1 \cup P_2 \cup Q_1 \cup Q_2$, such that $p_1, p_2 \in V(P_1) - \{v_1, v_2\}$ (with p_1 closer to v_1), $p_3 \in V(P_2) - \{v_1, v_2\}$, $q_1 \in V(Q_1) - \{v_3, v_4\}$ and $q_2, q_3 \in V(Q_2) - \{v_3\}$ (with q_2 closer to v_3).

See Figure 4.11 for an illustration. It is permitted that $q_3 = v_4$. We call the length of $Q_2[v_4, q_3]$ the *cost* of H. Note that there is symmetry between R_1, v_1, v_2 and R_3, v_4, v_3 if H has positive cost. By a prism on Z we mean a prism on $(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})$ for some ordering $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. The next step is to build a prism on Z.

Lemma 4.8.4. *Suppose that there are four internally disjoint Z-paths in G. Then there is a prism on Z. Moreover, for every terminal v of degree ≥ 3 there is a prism on Z (with cost 0) in which v has degree 3.*

Proof. Suppose first that v_4 say, has degree ≥ 3 in G. Let us build a prism on Z in which v_4 has degree 3. By the similar proof as in Lemma 4.8.2, there are three triads T_i in G with feet $Z - \{v_i\}$ for $1 \leq i \leq 3$, respectively, mutually disjoint except for their feet; so $T_1 \cup T_2 \cup T_3$ corresponds to $C \cup P_1 \cup P_2 \cup P_3$ in the proof of Lemma 4.8.2. Let $P_1, \dots, P_4$ be four internally disjoint Z-paths of G. We choose $P_1, P_2, P_3, P_4, T_1, T_2, T_3$ with their union (denoted by H) minimal. Let b_i denote the branch of T_i for $1 \leq i \leq 3$. We may assume that P_1 has no vertex in $\{b_1, b_2, b_3\}$. Since one of v_1, v_2, v_3 is an end of P_1, we assume that v_1 is so. Note that $\{b_2, b_3\}$ is a 2-cut of $T_1 \cup T_2 \cup T_3$ that separates $V(T_3(b_3, v_1] \cup T_2[v_1, b_2))$ and $\{v_2, v_3, v_4\}$. Hence we may assume that there is a subpath Q of P_1 having one end q in $V(T_2[v_1, b_2))$ and the other end q' in $V(T_1 \cup T_2 \cup T_3) - V(T_3[b_3, v_1] \cup T_2[v_1, b_2]) \cup \{b_1\}$, with no edge or internal vertex in $T_1 \cup T_2 \cup T_3$. If $q = v_1$ then $H \cup Q$ is Z-cyclic wherever q' lie, as easily checked, which is a contradiction. So $q \in V(T_2(v_1, b_2))$. If $q' \in V(T_2) - V(T_2[v_1, b_2])$ then we may reduce the union H by replacing $T_2[q, q']$ with Q, a contradiction. If $q' \in V(T_3[v_2, v_4] \cup T_1[v_4, v_3]) - \{b_1, b_3\}$ then $H \cup Q$ is Z-cyclic, a contradiction. Thus, $q' \in V(T_1(b_1, v_2))$, and so, $H \cup Q$ is a prism on (v_1, v_2, v_3, v_4) with cost 0, as required.

Next suppose that every terminal has degree 2 in G. Since G is Z-acyclic and contains four internally disjoint Z-paths, we may assume that there are two disjoint cycles C, C' of G containing two terminals, respectively. By 4.7.1 (ii), there are three disjoint paths of G between $V(C)$ and $V(C')$. As easily verified, minimal such paths and C, C' yield a prism on Z; for otherwise G becomes Z-cyclic. This proves the lemma. □

The next step is to show that G admits a V_8-subdivision on Z or a 3-cut separating Z into two subsets of size 2.

Lemma 4.8.5. *Let H be a prism on (v_1, v_2, v_3, v_4) in G with notation as above. Then either* (b) *holds or there is a 3-separation (A, B) of G with $V(P_1 \cup P_2) \subseteq A, V(Q_1 \cup Q_2) \subseteq B$.*

Proof. If there is no 3-separation (A, B) of G with $V(P_1 \cup P_2) \subseteq A$ and $V(Q_1 \cup Q_2) \subseteq B$, then there are four disjoint paths of G from $V(P_1 \cup P_2)$ to $V(Q_1 \cup Q_2)$, all with no internal vertex in $P_1 \cup P_2 \cup Q_1 \cup Q_2$. Since G is Z-acyclic, the union of P_1, P_2, Q_1, Q_2 and these four paths yields a V_8-subdivision J on Z. If some terminal v has degree ≥ 3 in G, then we may augment paths of J from v in J, obtaining a cycle thorough all the terminals; but this is a contradiction. Hence every terminal has degree 2 in G, and so (b) holds, as required. This proves the lemma. □

Now let us complete the proof of the theorem, based on Watkins–Mesner's theorem.

Proof of Theorem 4.7.2. Suppose to the contrary that (a) and (b) are false. Then by Lemmas 4.8.3 and 4.8.4, there is a prism on Z in G. Let H be a prism on (v_1, v_2, v_3, v_4) in G, say, with the usual notation as above. By Lemma 4.8.5, there is a 3-separation

(A, B) of G with $V(P_1 \cup P_2) \subseteq A$ and $V(Q_1 \cup Q_2) \subseteq B$. Let $A \cap B = \{x_1, x_2, x_3\}$ and assume that $x_i \in V(R_i)$ for $1 \leq i \leq 3$. We may assume that $x_3 \neq v_4$. For if $q_3 = v_4 = x_3$, we take the neighbor x_3' of v_4 in R_3 and set $A' := A - \{v_4\}, B' := B \cup \{x_3'\}$. Then (A', B') is a 3-separation of G (as $\deg_G(v_4) = 3$ by Lemma 4.8.2) such that $V(P_1 \cup P_2) \subseteq A', V(Q_1 \cup Q_2) \subseteq B'$ and $v_4 \in B' - A'$, as desired.

For distinct $i, j \in \{1, 2, 3\}$, we say that $G|A$ is (i, j)-*feasible* if there is a path of $G|A$ with ends x_i, x_j containing v_1, v_2, and *strongly* (i, j)-*feasible* if such a path can be chosen to avoid the vertex in $A \cap B - \{x_i, x_j\}$; we define similarly for $G|B$ (with respect to v_3, v_4). We know from the existence of the prism H that $G|A$ is $(1, 2)$-feasible and $G|B$ is $(2, 3)$-feasible.

(1) *$G|A$ is $(1, 3)$-feasible and moreover, if $x_2 \neq p_2$ then $G|A$ is strongly $(1, 3)$-feasible.*

For suppose to the contrary that $G|A$ is not $(1, 3)$-feasible if $x_2 = p_2$ and that $G|A$ is not strongly $(1, 3)$-feasible if $x_2 \neq p_2$. Note that $G|B$ is $(2, 3)$-feasible and moreover strongly $(2, 3)$-feasible if $x_1 \neq q_1$. Hence $G|A$ is not $(2, 3)$-feasible if $x_1 \neq q_1$ and not strongly $(2, 3)$-feasible if $x_1 = q_1$. Let $\alpha := |\{x_1, x_2\} \cap (\{q_1\} \cup (V(R_2) - \{p_2\}))|$, and let G' be a graph obtained from $G|A$ by doing the following operations, depending on α: If $\alpha = 0$, add a new vertex x and edges xx_1, xx_2; if $\alpha = 1$, add an edge x_1x_2 and let x be the vertex in $\{x_1, x_2\} \cap (\{q_1\} \cup (V(R_2) - \{p_2\}))$; if $\alpha = 2$, identify x_1 and x_2 and denote by x the resulting vertex. Then there is no path of G' with ends x, x_3 that contains v_1, v_2. We deduce from Theorem 4.2.3 that there is a $K_{3,2}$-decomposition $(X_1, X_2, X_3, Y_1, Y_2)$ of G' such that $v_i \in X_i - Y_1 \cup Y_2$ $(i = 1, 2)$ and $x, x_3 \in X_3$ (Consider a graph obtained from G' by adding a new vertex adjacent to x, x_3). Let $s \in \{0, 1, 2\}$ be its integral value and let $W := Y_1 \cup \cdots \cup Y_s$. Since there is a cycle of G' through v_1, v_2, x (with the aid of subpaths of P_1, P_2, R_1, R_2), we have $x \in X_3 \cap (Y_1 \cup Y_2)$. On the other hand, since there are internally disjoint two $\{v_1, v_2\}$-paths of $G' \setminus x$ (consider $P_1 \cup P_2$), we have $x \notin W$; and so, $s \leq 1$. We may assume that $x \in X_3 \cap Y_2 - W$. Let us write $X_i \cap Y_1 = \{a_i\}$ and $X_i \cap Y_2 - W = \{b_i\}$ for $i = 1, 2$; note that b_1, b_2, x are pairwise distinct, and that if $s = 1$ then $Y_1 = \{a_1\} = \{a_2\}$. If $\alpha = 0$, set $X_3' := X_3 - \{x\}, Y_2' := Y_2 - \{x\}$; if $\alpha = 1$, set $X_3' := X_3, Y_2' := Y_2$; if $\alpha = 2$, set $X_3' := (X_3 - \{x\}) \cup \{x_1, x_2\}, Y_2' := (Y_2 - \{x\}) \cup \{x_1, x_2\}$. Then $(X_1, X_2, X_3', Y_1, Y_2')$ is a decomposition of $G|A$ such that $x_1, x_2, x_3 \in X_3' \cup Y_2'$ and the boundary of X_i is still $\{a_i, b_i\}$ for $i = 1, 2$. Let $B' := B \cup X_3' \cup Y_2'$. Then (X_1, X_2, Y_1, B') is a decomposition of G such that $v_i \operatorname{int} X_i$ for $i = 1, 2$ and $v_3, v_4 \in \operatorname{int} B'$. If $s = 1$, then v_1, v_2 have degree 2 and have a common neighbor $a_1 = a_2$ from the irreducibility of G, contrary to Lemma 4.7.1 (i). Hence $s = 0$ and $|Y_1| \geq 3$. Therefore, (X_1, X_2, Y_1, B') is a nice decomposition of G, and so (a) holds, which contradicts our assumption. This proves (1).

(2) *$G|B$ is $(1, 3)$-feasible and moreover, if $x_2 \neq q_2$ then $G|B$ is strongly $(1, 3)$-feasible.*

For if H has cost 0, the claim follows (concatenate four paths $R_1[x_1, q_1]$, $Q_1[q_1, v_3]$, Q_2 and $R_3[v_4, x_3]$). So assume that $q_3 \neq v_4$. Then we see the symmetry between R_1, v_1, v_2 and R_3, v_4, v_3. Thus, the claim follows from the same proof as in (1). This proves (2).

Since $p_2 \neq q_2$, it follows from (1) and (2) that G is Z-cyclic, a contradiction. This completes the proof. □

4.9 Paths through four vertices

As noted in Section 4.7, the problem of characterizing graphs that contain no path through given four vertices is easily reduced to Watkins–Mesner's theorem. This section gives a rough sketch.

Let G be a graph and let Z be a set of distinct four vertices v_1, v_2, v_3, v_4 of G, called *terminals*. We want to investigate what kind of structure G has if there is no path of G containing all the terminals. First we may assume that G is connected and (G, Z) is irreducible. Here, we say that (G, Z) is *irreducible* if it satisfies the following: (i) If there is a 1-separation (A, B) of G with $|Z \cap (A - B)| = 1$, then $|A - B| = 1$; (ii) if there is a 2-separation (A, B) of G with $|Z \cap (A - B)| = 1$ and if the terminal in $Z \cap (A - B)$ has degree ≥ 2, then $|A - B| = 1$. We may assume that at most 2 terminals have degree 1 in G; for otherwise, G is a trivial obstruction. If exactly two terminals, v_1, v_2, say, have degree 1, then the problem is reduced to Watkins–Mesner's theorem (consider a graph obtained from G by adding a new vertex x adjacent to v_1, v_2). Thus, we may assume that at least three terminals have degree ≥ 2 in G. Then, as easily checked, there is a path of G containing all the terminals if and only if there are three internally disjoint Z-paths in G; the proof is straightforward based on the augmentation method (cf. Section 2.2). So assume that there are at most two internally disjoint Z-paths in G. In particular, each terminal has degree ≤ 2, and so the only possible degree sequence is $(1, 2, 2, 2)$ or $(2, 2, 2, 2)$. Now Z is stable; for otherwise, G contains three internally disjoint Z-paths. We deduce from Mader's theorem (Theorem 4.2.1) that there is a good quasi-bipartite-decomposition $(W; X_1, X_2, X_3, X_4; Y_1, \ldots, Y_m)$ of G with respect to Z of value 2; hence Y_j intersects at least three of $X_1, \ldots, X_4$ and $|Y_j \cap X|$ is odd ≥ 3 for $1 \leq j \leq 3$, and

$$|W| + m \leq |W| + \sum_{1 \leq j \leq m} \frac{|Y_j \cap X| - 1}{2} = 2,$$

where $X = X_1 \cup \cdots \cup X_4$ and $Y = Y_1 \cup \cdots \cup Y_m$. We may assume that $v_i \in X_i - Y$ for $1 \leq i \leq 4$. If $|W| = 2$, then $m = 0$ and $Y = \emptyset$; and so, there is no path in $G \setminus W$ that connects two terminals. If $|W| = 1$ then, $m = 1$ and $|Y_1 \cap X| = 3$; and so, one terminal v_1, say, has degree 1 in G, the vertex w in W is a common neighbor of all the terminals, and moreover, there are at most one $\{v_2, v_3, v_4\}$-paths in $G \setminus w$. The case $|W| = 0$ does not occur. For if $|W| = 0$, then either $m = 2$ and $|Y_1 \cap X| = |Y_2 \cap X| = 3$, or $m = 1$ and $|Y_1 \cap X| = 5$. But in either case at least two terminals have to be of degree 2, a contradiction. This completes the description of G.

4.10 Some Corollary

From the results in Sections 4.5, 4.7 and 4.9, we have now determined the structures of graphs with no K_4^--subdivision on prescribed four vertices. The description is purely combinatorial. The results in Sections 4.7 and 4.9 imply the following two well-known facts.

Corollary 4.10.1. *Let G be a 2-connected graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Then there is no path of G containing all the vertices in Z if and only if there is a set $W \subseteq V(G) - Z$ of size 2 such that $G \setminus W$ contains no path between two vertices of Z.*

Corollary 4.10.2 ([67])**.** *Let G be a 3-connected graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Then there is no cycle of G containing all the vertices in Z if and only if there is a set $W \subseteq V(G) - Z$ of size 3 such that $G \setminus W$ contains no path between two vertices of Z.*

We end this chapter with the following corollary, which states that 4-connected obstructions for K_4^--subdivisions are also "the most" trivial ones.

Corollary 4.10.3. *Let G be a 4-connected graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Then there is no K_4^--subdivision on Z in G if and only if there is a set $W \subseteq V(G) - Z$ of size 4 such that $G \setminus W$ contains no path between two vertices of Z.*

Proof. The "if" part is trivial, so we only show the converse. Suppose that G contains no K_4^--subdivision on Z. Then Theorem 4.5.3 (a) holds: There is a vertex $w \in V(G) - Z$ such that $G \setminus w$ contains neither K_4^{2-}-subdivision on Z nor cycle through all the vertices in Z. Since $G \setminus w$ is 3-connected, $G \setminus w$ is Z'-cyclic for any $Z' \subseteq Z$ with $|Z'| = 3$. It follows from Corollary 4.5.2 that $G \setminus w$ is Z-acyclic. By Corollary 4.10.2, there is a set $W \subseteq V(G \setminus w) - Z$ of size 3 such that $(G \setminus w) \setminus W$ contains no Z-path. The result follows. $\square$

Chapter 5

Conclusion

5.1 Summary

The two-paths theorem forms the basis for research on rooted subdivisions with prescribed four vertices. In Chapter 3, as an extension of the theorem we have determined the structure of 6-connected non-H-linked graphs for each $H \in \{P_4, C_4, K_4^{2-}, K_4^-, K_4\}$. The structure, which we call a "discoid graph", can be described as a planar graph whose "boundary" is surrounded by possibly many dense subgraphs of non-planarity. In our result, one can catch a glimpse of an extension of the two-paths theorem and Yu's characterization of P_4-linked graphs. In other words, when restricted to 6-connected graphs, non-H-linked graphs have structures similar to non-P_4-linked graphs for each $H \in \{C_4, K_4^{2-}, K_4^-, K_4\}$. This phenomenon for the case $H = C_4$ was implicitly observed in the recent paper of McCarty, Wang and Yu [45] that proved that 7-connected graphs are 4-ordered. We anticipated that this observation can be pushed further to the case $H = K_4$ all at once, and actually succeeded in proving it.

As a direct consequences of the characterization, we proved that 7-connected graphs are K_4-linked. This generalizes the results of [45, 36] that 7-connected graphs are 4-ordered and K_4^{2-}-linked. As for connectivity, the only known result was that 60-connected graphs are K_4-linked [57]. Our result significantly improves this known bound on the connectivity.

We also proved the case $n = 4$ of a conjecture of Mader: Every $\binom{n}{2}$-connected graph with sufficiently large girth is K_n-linked. However, our method cannot be applied to the case $n \geq 5$ immediately.

As noted in the introduction, the work is also motivated by the coloring-conjecture of Hajós. Unfortunately, our assumption of 6-connectivity is too strong to apply to the conjecture directly. It would be desirable to proceed with the work in Chapter 3 to seek a complete characterization, but this is an arduous task. Motivated by this, we have considered a relaxed variant of the problem in Chapter 4. As for the relaxed rooted K_4^--subdivision problem, we determined the 3-connected obstructions. The description using hypergrpahs is purely combinatorial, without any planarity conditions, which is an interesting difference from Theorem 3.1.2 (4). This phenomenon is largely due to Mader's S-paths theorem. We may say that our result is one of the few successful uses of Mader's

S-paths theorem in pure graph theory. As noted in Theorem 4.5.4, in our setting we can find two types of K_4^--subdivisions if one exists. We hope this will be helpful in many situations, especially for the conjecture of Hajós.

5.2 Future work

The main concern of this dissertation was the structures of non-H-linked graphs for a fixed simple graph H with four vertices. One possible direction of future work is to consider the case H has at least five vertices. The first problem to be settled in this case is to characterize $(K_2 + P_3)$-linked graphs, where $K_2 + P_3$ is the graph consisting of the disjoint union of K_2 and the path of length 2. However, one immediately notices that this is already a difficult question. Indeed, it entails a solution to the famous $(2, 3)$-*linkage problem*: Given five vertices x_1, x_2, y_1, y_2, y_3 of a graph G, find two disjoint connected subgraphs G_1, G_2 of G with $x_1, x_2 \in V(G_1)$ and $y_1, y_2, y_3 \in V(G_2)$. The $(2, 3)$-linkage problem is a fundamental problem as an extension of 2-linkage and related to other important problems such as Jørgensen's conjecture about K_6-minor-free graphs, but only a few structural results are known.

Although the rooted H-subdivision problem seems difficult for $|V(H)| \geq 5$ as we have seen above, one may target a relaxed version of the problem for $H = C_5$, say: Given five vertices $v_1, \ldots, v_5$ of a graph G, find a cycle of G containing these vertices (without regard to the order). For this problem, it might be helpful to use the result of Sanders [52], who characterized 5-connected graphs with no cycle containing specified five independent edges, solving a special case of Lovász–Woodall's conjecture.

Another possible direction of future work is to proceed with the work in Chapter 3, i.e., to determine all the structures of non-K_4-linked graphs. But again we should note that this direction is strenuous. Let $v_1, \ldots, v_4$ be distinct vertices of a graph G. An "extremal" subgraph of G as in Figure 3.5 (right), called a skeleton, played the crucial role in our proof. If G is not 6-connected, we may not be able to construct it and so have to seek other approaches. One possible difficulty could arise in the case G has a 4-cut that separates the set of terminals $\{v_1, \ldots, v_4\}$ into two subsets of size 2. In this case, in order to construct a K_4-subdivision with $v_1, \ldots, v_4$ branch vertices, we have to solve the following subproblem: If H is a tree with six vertices, two of which are adjacent and of degree 3 (so H has the shape of the letter "H"), then what is the characterization of H-linked graphs? This problem seems difficult, though a relaxed version of the problem that permits permutations on the terminals is solved in [39].

Another direction is to consider the case H has parallel edges. If H is a graph with three vertices containing parallel edges, then the H-linkage problem can be rephrased as follows: Given three vertices v_1, v_2, v_3 of a graph G and integers $k_1, k_2, k_3 \geq 0$, find internally disjoint k_1, k_2, k_3 paths of G with ends v_1v_2, v_2v_3, v_3v_1, respectively. As observed in [35], this problem can easily be reduced to finding $k_1 + k_2 + k_3$ internally disjoint $\{v_1, v_2, v_3\}$-paths in G. Thus, the case $|V(H)| = 3$ is solved by Mader's S-paths theorem.

A non-trivial and important problem arises when H has four vertices. Let H be a multigraph as in Figure 5.1. What is a structural characterization of H-linked graphs?

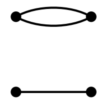

Figure 5.1: A multigraph with four vertices.

This is equivalent to the following problem.

Problem 5.2.1. Let G be a graph and let v_1, v_2, v_3, v_4 be distinct vertices of G. What kind of structure does the graph G have if there are no cycle of G containing v_1 and v_2 and a path of G between v_3 and v_4, mutually disjoint?

Problem 5.2.1 is a special case of the following more general problem defined for positive integers $n_1, \ldots, n_k, c_1, \ldots, c_k$:

> ($(n_1, \ldots, n_k; c_1, \ldots, c_k)$-Linkage problem)
>
> Instance: A graph G and disjoint subsets $X_1, \ldots, X_k$ of $V(G)$ with $|X_i| = n_i$ for $1 \le i \le k$.
>
> Question: Are there k disjoint c_i-connected subgraphs G_i of G with $X_i \subseteq V(G_i)$ for $1 \le i \le k$?

When $c_i = 1$ for $1 \le i \le k$, this problem is simply called the $(n_1, \ldots, n_k)$-linkage problem. If, moreover, $n_i = 2$ for $1 \le i \le k$, then it is equivalent to the k-disjoint paths problem. Problem 5.2.1 is nothing but the $(2, 2; 2, 1)$-linkage problem. As easily checked, if $c_i \le 2$ for $1 \le i \le k$, then sufficiently highly connected (more specifically, $(\sum_{1 \le i \le k} n_i)$-linked) graphs G always have a feasible solution for the $(n_1, \ldots, n_k; c_1, \ldots, c_k)$-linkage problem. When $c_i > 2$ for some i, it is not known whether there is such a bound on the connectivity. Indeed, if one can show that every sufficiently highly connected graph has a feasible solution for the $(2, 1; 1, 3)$-linkage problem, then it resolves Lovász's path removal conjecture affirmatively; see [27, Conjecture 3.1].

One can verify the importance of Problem 5.2.1 in connection with Hajós' conjecture. Let G be a graph and v_1, v_2, v_3, v_4 be distinct vertices of G. A *fan*$^+$ on $(v_1, v_2; v_3, v_4)$ is a subgraph of G homeomorphic to a graph as in Figure 5.2. Note that a fan$^+$ appears as a substructure of a subdivision of K_5. For example, if there is a K_4^--subdivision on $(v_1, v_2; v_3, v_4)$ and a fan$^+$ on $(v_1, v_2; v_3, v_4)$ in G, mutually disjoint except for $\{v_1, v_2, v_3, v_4\}$, then there is a subdivision of K_5 in G. This observation is helpful for Hajós' conjecture because every Hajós graph (a minimum counterexample to Hajós' conjecture) admits a 4-separation. Unfortunately, it seems hard to give an exact characterization of a fan$^+$ because of its vertex of degree 4. However, if there are a path between v_1 and v_2 and a cycle containing v_3 and v_4 which are mutually disjoint, then the construction of a fan$^+$ on $(v_1, v_2; v_3, v_4)$ becomes much easier with the aid of Watkins–Mesner's theorem. Therefore, Problem 5.2.1 is a first step towards constructing fan$^+$s and thus an important clue to resolving Hajós' conjecture.

Problem 5.2.1 is also closely related to the 3-disjoint paths problem. To see this, let G be a graph with distinct vertices $s_1, s_2, s_3, t_1, t_2, t_3$. Let S_3 be the set of all bijections

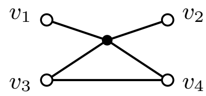

Figure 5.2: A fan$^+$ on $(v_1, v_2; v_3, v_4)$.

from the set $\{1, 2, 3\}$ to itself. We say that a subset X of S_3 is *feasible* in G if for some $\sigma \in X$ there are three disjoint paths of G with ends $s_i, t_{\sigma(i)}$ for $1 \leq i \leq 3$, respectively. If $|X| = 1$, then the problem of asking the feasibility of X is equivalent to the 3-disjoint paths problem. Its structural characterization is far-reaching at this moment. A famous long-standing conjecture in the literature is that: Every 8-connected graph is 3-linked. The only known result is that 10-connected graphs are 3-linked [58], as noted in the introduction.

One can obtain relaxed versions of the 3-disjoint paths problem by changing X. The larger X gets, the easier the problem becomes. If $X = S_3$, then its structural characterization is given by Menger's theorem. If $X = \{\sigma \in S_3 \colon \sigma(3) \neq 3\}$, then asking the feasibility of X is equivalent to the following question: Are there three disjoint paths of G between $\{s_1, s_2, s_3\}$ and $\{t_1, t_2, t_3\}$ such that none of them connects s_3 and t_3? As noted in Section 3.6, this problem is completely settled by Yu [63, 64, 65]. Indeed, this result yields the characterization of P_4-linked graphs by a simple construction.

One nontrivial case is when $X = \{\sigma \in S_3 \colon \sigma(3) = 3\}$. Now asking the feasibility of X is equivalent to the following question: Are there three disjoint paths of G between $\{s_1, s_2, s_3\}$ and $\{t_1, t_2, t_3\}$ such that one of them connects s_3 and t_3? This is essentially equivalent to Problem 5.2.1. To see this, let G be a graph with four vertices v_1, v_2, v_3, v_4 and assume that $v_1v_2 \notin E(G)$. Let G' be the graph obtained from G by replacing v_1 with two vertices s_1, s_2 and joining s_1, s_2 to all neighbors of v_1, and replacing v_2 with two vertices t_1, t_2 and joining t_1, t_2 to all neighbors of v_2. Now there are a cycle of G containing v_1 and v_2 and a path of G between v_3, v_4, mutually disjoint, if and only if G' contains three disjoint paths between $\{s_1, s_2, v_3\}$ and $\{t_1, t_2, v_4\}$ such that one of them connects v_3 and v_4. Note that if $v_1v_2 \in E(G)$ then Problem 5.2.1 can be solved by the two-paths theorem applied to $G \setminus \{v_1v_2\}$. Therefore, Problem 5.2.1 is reduced to asking the feasibility of $X = \{\sigma \in S_3 \colon \sigma(3) = 3\}$, which is a slightly relaxed version of the 3-disjoint paths problem. Since 10-connected graphs are 3-linked, obstructions for Problem 5.2.1 cannot be 10-connected. As far as we are aware, no other structural result is known for Problem 5.2.1.

As future work, it may also be interesting to study the gap between H-linkage and $|E(H)|$-linkage. Recall that $g(H)$ means the smallest positive integer which ensures that every $g(H)$-connected graph is H-linked. In this dissertation, we have proved that $g(K_4) = 7$. As is well-known, the graph K_{3k-1} with k independent edges removed shows $g(kK_2) \geq 3k - 2$. Hence $g(6K_2) - g(K_4) \geq 9$ by our result. In general, for a graph H with k edges, one may ask how large the gap $g(kK_2) - g(H)$ (≥ 0) can be. This problem seems open.

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