博士論文

Essays on Matching Theory

(マッチング理論に関する研究)

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Chapter 1

Equivalence theorem in matching with contracts*

1 Introduction

A matching-with-contract model refers to a market in which there are two disjoint sets of agents and each agent on one side is matched with another agent on the other side through a "contract" (Hatfield and Milgrom (2005)). A typical example is the National Resident Matching Program in the United States, wherein a medical resident is matched with a hospital for practical training. In this example, a contract may represent a specific medical department.

The purpose of this study is to understand the performance of the doctor-optimal stable mechanism in matching with contracts. An allocation is *stable* if no agent unilaterally rejects a contract allocated to the agent and no doctor-hospital pair bilaterally blocks the allocation. An allocation is *doctor-optimal stable* if it is stable and it dominates any other stable allocation. The *doctor-optimal stable mechanism* maps each preference profile to the doctor-optimal stable allocation. In this study, we assume that hospitals are just objects to be allocated to doctors, meaning that only doctors are under efficiency and incentive consideration.[1]

In the setting of matching with contracts, we investigate group strategy-proofness, efficiency, Maskin monotonicity, and consistency of the doctor-optimal stable mechanism. Unfortunately, the mechanism does not satisfy any of these properties in general. Hence,

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[1]Pakzad-Hurson (2020) and Sönmez and Switzer (2013) make the same assumption.

we analyze which properties are more likely to be met by the doctor-optimal stable mechanism. Our main result says that, whenever there exists a doctor-optimal stable allocation for any doctors' preference profile, the doctor-optimal stable mechanism is group strategy-proof if and only if it is efficient if and only if it is Maskin monotonic.[2] Moreover, we find that the doctor-optimal stable mechanism is consistent if and only if it is efficient under the two conditions: substitutes and the law of aggregate demand (LAD).

Related literature

Hatfield and Milgrom (2005) introduce the matching model with contracts. They show that under substitutes and the LAD, the doctor-optimal stable allocation exists for any doctors' preference profile, and the doctor-optimal stable mechanism is strategy-proof. There are mainly three follow-up studies. Sakai (2011) finds that under substitutes and the LAD, the doctor-optimal stable mechanism is a unique stable rule that satisfies strategy-proofness in the class of stable mechanisms. Moreover, Hatfield and Kojima (2009) strengthen the result on incentives in Hatfield and Milgrom (2005) and show that the doctor-optimal stable mechanism is weakly group strategy-proof under the same assumptions, i.e., substitutes and the LAD. Hatfield and Kojima (2010) introduce a sufficient condition called unilateral substitutes, which is weaker than substitutes, for the guaranteed existence of the doctor-optimal stable allocation for any doctors' preference profile. While Sakai (2011) and Hatfield and Kojima (2009, 2010) focus on the doctor-optimal stability and incentives, Hirata and Kasuya (2017) and Hatfield, Kominers, and Westkamp (2021) consider (not necessarily doctor-optimal) stability and incentives. Hirata and Kasuya (2017) show that if a stable and strategy-proof mechanism exists, then it is unique and corresponds to the doctor-optimal stable mechanism. Hatfield, Kominers,

[2]In addition to the desirable properties we are considering here, there are weaker versions of properties: weak group strategy-proofness, weak efficiency, and weak Maskin monotonicity. Since the literature has already clarified that the doctor-optimal stable mechanism satisfies those properties under a reasonable condition, focusing on the equivalence between the weaker properties in a general environment is left for future research.

and Westkamp (2021) present three novel conditions to show that these conditions are necessary and sufficient for the existence of a stable and strategy-proof mechanism.

Concerning the equivalence of good properties of a mechanism, Ergin (2002) shows that in matching without contracts, the Deferred Acceptance mechanism is group strategy-proof if and only if it is efficient if and only if it is consistent. Kojima and Manea (2010) extend the result of Ergin (2002) to a more general environment. In matching with contracts, Pakzad-Hurson (2020) shows that under substitutes and acceptance (stronger than the LAD), the doctor-optimal stable mechanism is group strategy-proof if and only if it is efficient. While considering a model outside our setting, Takamiya (2001) focuses on group strategy-proofness and Maskin monotonicity in housing markets and then shows the equivalence between them under some conditions on preference domains. Takamiya (2007) extends the result of Takamiya (2001) to a more general setting, which includes our setting and shows the same equivalence. Moreover, Klaus and Bochet (2013) consider an environment that covers not only indivisible goods but also divisible ones. Then, they introduce two conditions on preference domains such that strategy-proofness and Maskin monotonicity are equivalent.

2 Model

2.1 Setup

Let D and H be a finite set of doctors and a finite set of hospitals, respectively. Let X be a finite set of contracts and each contract $x \in X$ is bilateral; $x \in X$ is associated with doctor $\mathrm{D}(x) \in D$ and hospital $\mathrm{H}(x) \in H$. In addition to X, there is a null contract $\varnothing_d$ that represents having no relationship with any hospital for $d \in D$. For any $X' \subseteq X$, define $X'_d \equiv \{x \in X' \mid \mathrm{D}(x) = d\}$ for any $d \in D$ and $X'_h \equiv \{x \in X' \mid \mathrm{H}(x) = h\}$ for any $h \in H$. For $X' \subseteq X$ and $D' \subseteq D$, let $X'_{D'} \equiv \cup_{d \in D'} X'_d$.

Each doctor $d \in D$ has a complete, transitive, and antisymmetric preference R_d over $X_d \cup \{\varnothing_d\}$. The strict part of R_d is denoted by P_d. For any $D' \subseteq D$, we define $R_{D'} \equiv (R_d)_{d \in D'}$ and $R_{-D'} \equiv (R_d)_{d \in D \setminus D'}$.

Each hospital $h \in H$ has a complete, transitive, and antisymmetric priority R_h over the set of subsets of X_h.[3] For any $h \in H$, R_h induces the **choice function** $C_{R_h} : 2^X \to 2^X$, which satisfies the following three conditions: for any $X' \in 2^X$, (1) $C_{R_h}(X') \subseteq X'_h$; (2) for any $x, x' \in C_{R_h}(X')$, $x \neq x'$ implies $\mathrm{D}(x) \neq \mathrm{D}(x')$; and (3) $C_{R_h}(X')$ is most preferred according to R_h.[4] Throughout the study, we fix $(R_h)_{h \in H}$, and for notational exposition, we simply denote a choice function C_{R_h} by C_h. This reflects the view that hospitals are objects to be allocated to doctors, so only doctors are under consideration for the purpose of welfare and incentives. For $X' \subseteq X$, let $C_H(X') \equiv \bigcup_{h \in H} C_h(X'_h)$. A **problem** is defined as a tuple $(D, H, X, R_D, (C_h)_{h \in H})$.

2.2 Allocation and its property

An **allocation** is a set of contracts $X' \subseteq X$ such that, for all $x, x' \in X', x \neq x'$ implies $\mathrm{D}(x) \neq \mathrm{D}(x')$. That is, an allocation is a set of contracts in which no doctor signs more than one contract. If doctor d has no contract in an allocation X', then we understand that doctor d signs a null contract $\varnothing_d$ and simply denote $X'_d = \varnothing_d$. An allocation $X' \subseteq X$ is **individually rational for doctors** at R_D if $X'_d R_d \varnothing_d$ holds for all $d \in D$. An allocation $X' \subseteq X$ is **individually rational for hospitals** if $C_h(X') = X'_h$ for all $h \in H$.

We next define efficiency of allocations in this model. Here, we have to note two things. First, since hospitals are just objects to be allocated, we focus on doctors for a welfare

[3]We regard a priority relation as primitive. For an alternative model, see Aygün and Sönmez (2012, 2014).

[4]The literature has often introduced the maximum capacity to each hospital. While such a formulation is reasonable, there may be situations in which it is inappropriate. For instance, in Japan, the balance of doctors among medical departments is more important than the total number of accepted doctors (Ministry of Health, Labour and Welfare (2010)). In addition, in Japan, the number of children a nursery teacher can care for depends on the age of the children and hence there may not be a maximum capacity. Kamada and Kojima (2019) consider such a general constraint.

concern. Second, since we do not explicitly make an assumption on choice functions other than the three listed above, only the allocation such that each doctor signs the most favorable contract increases the welfare of all doctors the most. Thus, to avoid such a trivial case, we need to consider what allocation would be possible for hospitals. In the current paper, we pay attention to only allocations that satisfy individual rationality for hospitals and write simply "allocations" as long as there is no confusion.[5]

An allocation $X' \subseteq X$ is **efficient** at R_D if there does not exist another allocation $X'' \subseteq X$ such that $X''_d R_d X'_d$ for all $d \in D$ and $X''_d P_d X'_d$ for some $d \in D$.

We introduce another property of allocations.

Definition 1. An allocation $X' \subseteq X$ is **stable** at R_D (and $(C_h)_{h \in H}$) if

1. X' is individually rational for both doctors and hospitals, and

2. there exists no hospital h and a contract $x \in X$ with $d = \mathrm{D}(x)$ and $h = \mathrm{H}(x)$ such that (1) $x P_d X'_d$ and (2) $x \in C_h(X'_h \cup \{x\})$.

That is, stability of allocations requires that neither a doctor nor a hospital rejects the assigned contract and that the allocation cannot be blocked by a doctor and a hospital. A **doctor-optimal stable allocation** is a stable allocation such that each doctor weakly prefers it to any other stable allocation.

2.3 Mechanism and its property

A **mechanism** f is a function from each doctors' preference profile to an allocation. Since only doctors act strategically, the input of a mechanism is, say, R_D. Henceforth, we denote a preference profile for doctors as simply R. The assignments of allocation $f(R)$ for doctor d and hospital h are denoted by $f_d(R)$ and $f_h(R)$, respectively. The **doctor-optimal stable mechanism** is a function from each doctors' preference profile

[5] Ehlers and Morrill (2018) proceed their analysis by focusing on individually rational allocations as ours.

to the doctor-optimal stable allocation. A mechanism is **efficient** if, for any R, the mechanism produces an efficient allocation at R. A mechanism f is **group strategy-proof** if there exist no $D' \subseteq D$, R, and $R'_{D'}$ such that $f_d(R'_{D'}, R_{D \setminus D'}) R_d f_d(R)$ for all $d \in D'$ and $f_d(R'_{D'}, R_{D \setminus D'}) P_d f_d(R)$ for some $d \in D'$. Doctor d's preference R'_d is a **monotonic transformation** of R_d at $x \in X$ (R'_d m.t. R_d at x) if $x' P'_d x$ implies $x' P_d x$. A preference profile R' is a monotonic transformation of R at an allocation X' (R' m.t. R at X') if R'_d m.t. R_d at X'_d for all $d \in D$. A mechanism f satisfies **Maskin monotonicity** if R' m.t. R at $f(R)$ implies $f(R') = f(R)$.[6] The three properties defined here are not necessarily satisfied by the doctor-optimal stable mechanism.

3 Results

3.1 Main result

We are now ready to state the main result. The following theorem states that the three desirable properties are equivalent to one another as for the doctor-optimal stable mechanism, whenever it is well-defined.

Theorem 1. *Suppose that there exists a doctor-optimal stable allocation for each doctors' preference profile. Then, the followings are all equivalent.*

1. *The doctor-optimal stable mechanism is group strategy-proof.*

2. *The doctor-optimal stable mechanism is efficient.*

3. *The doctor-optimal stable mechanism is Maskin monotonic.*

Proof. In Appendix. □

[6] Maskin monotonicity has an aspect of a technical property as it is necessary for Nash implementation of mechanisms.

Kojima and Manea (2010) show the same equivalence of the doctor-optimal stable mechanism in matching *without* contracts under two assumptions: *acceptance* and *substitutes*. Here, to obtain the equivalence in the context of matching with contracts, we do not place any condition on $(C_h)_{h\in H}$ other than the assumption that the doctor-optimal stable mechanism is well-defined.

Whenever the doctor-optimal stable mechanism is well-defined, it is a unique mechanism that satisfies the same equivalence in the class of stable mechanisms. This is because, if a stable mechanism is efficient, then it must be doctor-optimal stable.

3.2 Further result: consistency

In this subsection, we investigate consistency of a mechanism, which has been studied in assignment problems.[7] To the best of our knowledge, no study has defined the consistency notion for matching with contracts. Let $\varepsilon = (D, H, X, R, (C_h)_{h\in H})$ be a problem. For coalition $D' \subseteq D$ and allocation $X' \subseteq X$, we define the **reduced problem** $\varepsilon^{(D',X')} = (D', H, X_{D'}, R_{D'}, (C_h^{(D',X')})_{h\in H})$. Here, $C_h^{(D',X')} : 2^{X_{D'}} \to 2^{X_{D'}}$ is the **extended choice function** for $h \in H$, and for all $X'' \in 2^{X_{D'}}$, (1) $C_h^{(D',X')}(X'') \subseteq X''_h$; (2) for any $x, x' \in C_h^{(D',X')}(X'')$, $x \neq x'$ implies $\mathrm{D}(x) \neq \mathrm{D}(x')$; and (3) $C_h^{(D',X')}(X'')$ is most preferred according to R_h over the set $\{Y' \subseteq X''_h \mid C_h(Y' \cup (X'_{D\setminus D'})_h) = Y' \cup (X'_{D\setminus D'})_h\}$.

A reduced problem describes the situation wherein a group of doctors $D \setminus D'$ leaves the problem with their assignments $X'_{D\setminus D'}$. Moreover, in the reduced problem, the choice functions for hospitals are modified in a certain way. A hospital's choice in total (the contracts signed with the remaining doctors and the leaving doctors) must be admissible to the hospital from the perspective of the original choice function, that is, $C_h(C_h^{(D',X')}(X'') \cup (X'_{D\setminus D'})_h) = C_h^{(D',X')}(X'') \cup (X'_{D\setminus D'})_h$.[8] This reflects the new definition of choice functions.

[7] See Thomson (2011) for a comprehensive survey.

[8] Ergin (2002) considers consistency in matching without contracts. He explicitly introduces a vector of capacities for hospitals. Therefore, a reduced problem in his study includes new capacities that are decreased by the number of seats assigned to those doctors who left the original problem.

An **allocation in the reduced problem** $\varepsilon^{(D',X')}$ is a set of contracts $X' \subseteq X_{D'}$ such that, for all $x, x' \in X'$, $x \neq x'$ implies $\mathrm{D}(x) \neq \mathrm{D}(x')$. An allocation $X' \subseteq X_{D'}$ in the reduced problem $\varepsilon^{(D',X')}$ is **stable** if it is stable at $R_{D'}$ and $(C_h^{(D',X')})_{h\in H}$ in $\varepsilon^{(D',X')}$. The **doctor-optimal stable allocation in the reduced problem** $\varepsilon^{(D',X')}$ is a stable allocation at $R_{D'}$ and $(C_h^{(D',X')})_{h\in H}$ in $\varepsilon^{(D',X')}$, which is weakly preferred for doctors D' to any other stable allocation at $R_{D'}$ and $(C_h^{(D',X')})_{h\in H}$ in $\varepsilon^{(D',X')}$.

We call a function $\tilde{f}$ from each reduced problem to an allocation an **extended mechanism**. Consistency of an extended mechanism describes the normative intuition that any group of doctors will sign the same contracts in the original and reduced problems. Formally, an extended mechanism $\tilde{f}$ is **consistent** if, for any original problem $\varepsilon = (D, H, X, R, (C_h)_{h\in H})$ and any $D' \subseteq D$, we have $\tilde{f}(\varepsilon)_{D'} = \tilde{f}(\varepsilon^{(D',X')})$, where $X' = \tilde{f}(\varepsilon)$. The **extended doctor-optimal stable mechanism** is a function from each reduced problem to the doctor-optimal stable allocation if any.

We introduce a class of choice functions.

Definition 2. A choice function C_h satisfies **substitutes** if, for all $X' \subseteq X'' \subseteq X$ and $x \in X$, $x \notin C_h(X' \cup \{x\})$ implies $x \notin C_h(X'' \cup \{x\})$.

Definition 3. A choice function C_h satisfies the **law of aggregate demand (LAD)** if, for all $X' \subseteq X'' \subseteq X$, we have $|C_h(X')| \leq |C_h(X'')|$.

Roughly, substitutes means that once a contract is rejected, it is never chosen even when the choice set expands (in the sense of set inclusion). Additionally, the LAD requires that hospitals sign more contracts in number as the choice set expands (in the sense of set inclusion). Under substitutes and the LAD, Hatfield and Milgrom [9] show that the doctor-optimal stable allocation exists for each doctors' preference profile.

We are now ready to state the next result, which states that consistency is equivalent to the other properties under substitutes and the LAD.

Theorem 2. *Under substitutes and the LAD, the followings are all equivalent.*

1. *The extended doctor-optimal stable mechanism is consistent.*

2. *The doctor-optimal stable mechanism is group strategy-proof.*

3. *The doctor-optimal stable mechanism is efficient.*

4. *The doctor-optimal stable mechanism is Maskin monotonic.*

Proof. In Appendix. □

Note that Ergin (2002) shows that in matching *without* contracts, the first three properties are equivalent under *responsiveness*, which is stronger than the combination of substitutes and the LAD. Moreover, Kojima and Manea (2010) find the equivalence between the last three properties in matching *without* contracts under substitutes and *acceptance*, where acceptance is stronger than the LAD. In matching with contracts, Pakzad-Hurson (2020) shows that the doctor-optimal stable mechanism is group strategy-proof if and only if it is efficient. However, there are mainly two differences between the result of Pakzad-Hurson (2020) and that of this paper. First, Pakzad-Hurson (2020) explicitly introduces a capacity constraint for hospitals. Second, the author proves the equivalence under substitutes and acceptance, while our equivalence holds true under substitutes and the LAD.

3.3 Discussion

Pakzad-Hurson (2020) gives two easily verifiable conditions on choice functions for hospitals to address when the doctor-optimal stable mechanism indeed satisfies the desirable properties. The first condition, what he calls *set-Ergin acyclity*, means that choice functions for any two hospitals are similar to the extent that a “rejection chain” does not occur.[9] Thus, the set-Ergin acyclicity places restrictions on how choice functions can differ between different hospitals. The second condition, what he calls *set-lexicographic*, requires that different sets of contracts involved with the same doctors must be listed consecutively in a preference relation. Hence, the second condition focuses on how sets of contracts are aligned within a single hospital.

Under substitutes and acceptance (which is stronger than the LAD), Pakzad-Hurson (2020) shows that the doctor-optimal stable mechanism is efficient if and only if each hospital's priority-capacity pair is set-Ergin acyclic and set-lexicographic.

[9]There is a large body of literature that considers a kind of acyclicity. See Erdil and Kumano (2019) for a more general condition of acyclicity.

Appendix: Omitted proofs

Proof of Theorem 1

Takamiya (2007) proposes two conditions on preference domains for any social choice function to show the equivalence between group strategy-proofness and Maskin monotonicity in a general environment that includes ours. Since those conditions are satisfied in our model, we have already known the equivalence between group strategy-proofness and Maskin monotonicity. Hence, we show that $1 \Rightarrow 2$ and $2 \Rightarrow 3$. Let f be the doctor-optimal stable mechanism.

$(1 \Rightarrow 2)$

By way of contraposition. Suppose that f does not satisfy efficiency at R. Then, there exists another allocation $X' \subseteq X$ such that $X'_d R_d f_d(R)$ for all $d \in D$ and $X'_d P_d f_d(R)$ for some $d \in D$. Let R'_d be a preference relation such that X'_d is on the top and the others are unchanged. Since X' is individually rational for doctors at R' and that there is no blocking doctor-hospital pair at R', X' is doctor-optimal stable under R'. This fact implies that $f(R') = X'$ since f is the doctor-optimal stable mechanism, which also indicates that all doctors deviate from R to R' to make some doctors better off while making the others unchanged. Thus, f is not group strategy-proof.

$(2 \Rightarrow 3)$

By way of contraposition. Suppose that f does not satisfy Maskin monotonicity. Then, there exist preference profiles R and R' such that R' is a monotonic transformation of R at $f(R)$ and $f(R') \neq f(R)$. We consider two cases.

Case 1: $f(R)$ is stable at R'.

Since $f(R')$ is doctor-optimal stable at R' and there is only one doctor-optimal stable allocation for each doctors' preference profile, $f(R)$ is not doctor-optimal stable at R'. Thus, we have $f_d(R') R'_d f_d(R)$ for all $d \in D$. Because of strict preferences, $f_d(R') = f_d(R)$

or $f_d(R')P'_d f_d(R)$ for all $d \in D$. Since R' is a monotonic transformation of R at $f(R)$, we obtain $f_d(R')R_d f_d(R)$ for all $d \in D$ and $f_d(R')P_d f_d(R)$ for some $d \in D$, which indicates that f is not efficient.

Case 2: $f(R)$ is not stable at R'.

We consider two subcases.

Case 2.1 (Individual rationality for doctors):

There exists a doctor $d \in D$ such that $\varnothing_d P'_d f_d(R)$. Since R'_d is a monotonic transformation of R_d at $f_d(R)$, we have $\varnothing_d P_d f_d(R)$. This contradicts to stability of f at R.

Case 2.2 (No blocking via other contracts):

There exists a hospital $h \in H$ and a contract $x \in X$ with $d =$D(x) and $h =$H(x) such that $xP'_d f_d(R)$ and $x \in C_h(f_h(R) \cup \{x\})$. Since R'_d is a monotonic transformation of R_d at $f_d(R)$, we have $xP'_d f_d(R)$. This fact implies that $f(R)$ is not stable at R, which is a contradiction to stability of f. □

Proof of Theorem 2

Although Theorem 2 states the equivalence between consistency and the other properties under substitutes and the LAD, one direction of the equivalence holds without the two conditions.

Proposition 1. *Suppose that the doctor-optimal stable allocation exists for each reduced problem. If the doctor-optimal stable mechanism is efficient, then the extended doctor-optimal stable mechanism is consistent.*

Proof. Let $\tilde{f}$ be the extended doctor-optimal stable mechanism. Suppose that $\tilde{f}$ is not consistent. Then, there exist $\varepsilon = (D, H, X, R, (C_h)_{h \in H})$ and $D' \subseteq D$ such that $\tilde{f}(\varepsilon)_{D'} \neq \tilde{f}(\varepsilon^{(D',X')})$, where $X' = \tilde{f}(\varepsilon)$. Let $X'' = \tilde{f}(\varepsilon^{(D',X')})$. Since X' is stable in the problem ε,

$X'_{D'}$ is also stable in the reduced problem $\varepsilon^{(D',X')}$. Otherwise, some doctor $d \in D'$ and some hospital $h \in H$ blocks X' in the problem ε. In addition, since X'' is the doctor-optimal stable allocation in the reduced problem $\varepsilon^{(D',X')}$ and $X'_{D'} \neq X''$, X'' dominates $X'_{D'}$ in the reduced problem $\varepsilon^{(D',X')}$. By construction of $C_h^{(D',X')}$, $X''_h \cup (X'_{D\setminus D'})_h$ is individually rational for hospital h. So, $X'' \cup X'_{D\setminus D'}$ must be an allocation in the problem ε, which implies that $X'' \cup X'_{D\setminus D'}$ dominates X' in the problem ε. Hence, the doctor-optimal stable mechanism is not efficient.

□

To complete the proof of Theorem 2, we show that consistency implies group strategy-proofness under substitutes and the LAD. Note that since the restriction of a choice function that satisfies substitutes and the LAD to any subset of contracts also satisfies the two of them, we can guarantee that the doctor-optimal stable allocation exists for any reduced problem.

Let f and $\tilde{f}$ be the doctor-optimal stable mechanism and the extended doctor-optimal stable mechanism, respectively. Assume that $\tilde{f}$ is consistent for doctors. From Lemma 1 in Papai (2000), it follows that the combination of strategy-proofness and *nonbossiness* implies group strategy-proofness: f is **nonbossy** if for all $d \in D$, R, and R'_d, $f_d(R) = f_d(R'_d, R_{-d})$ implies $f(R) = f(R'_d, R_{-d})$. By Hatfield and Milgrom (2005), f is strategy-proof under substitutes and the LAD. We claim that f is nonbossy.

Let $d \in D$, R, and R'_d be such that $f_d(R) = f_d(R'_d, R_{-d})$. Let $X' = f(R)$ and $X'' = f(R'_d, R_{-d})$. Note that since $X'_d = X''_d$, by setting $D' = D \setminus \{d\}$, the two reduced problems $\varepsilon^{(D',X')}$ and $\varepsilon^{(D',X'')}$ are the same. Hence, $\tilde{f}(\varepsilon^{(D',X')}) = \tilde{f}(\varepsilon^{(D',X'')})$. On the other hand, by consistency, $X'_{D'} = \tilde{f}(\varepsilon^{(D',X')})$ and $X''_{D'} = \tilde{f}(\varepsilon^{(D',X'')})$. Hence, $X'_{D'} = X''_{D'}$. Therefore, f is nonbossy. □

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Chapter 2

Nash implementation on the basis of general priorities*

1 Introduction

A school choice problem consists of a finite set of students, a finite set of schools, each student's preference over schools and herself, each school's priority over students, and a vector of capacities of schools.[1] A matching describes which student is matched with which school, and achieving a stable matching has been an important issue to address. A matching is said to be *stable* if it is weakly preferred for all students to being unassigned, and no student and school pair blocks it. There are in general multiple stable matchings. Specifically, we say that a stable matching is *constrained efficient* if it is not Pareto dominated by any other stable matchings in terms of students' preferences. Since only students are involved in efficiency, it is preferable for a market designer especially to achieve a constrained efficient stable matching.

A stable matching can be found based on some mechanism by using students' *submitted information* and schools' pre-specified priorities and capacities. Since students are assumed to be strategic, it is a problem whether there is a mechanism that implements a stable sub–correspondence in some equilibrium concept. Thus, this is an implementation problem. For a school choice problem with simple priorities, a direct mechanism associated with the student-proposing deferred acceptance algorithm is applicable (Gale and

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[1]A priority is a correspondence that specifies which sets of students to choose from a given set of students. In the model, priorities will be formally defined, where we call them *admission correspondences*.

Shapley (1962)).[2] That is, the student-proposing deferred acceptance algorithm is used for an outcome function, and a message space consists of a set of students' preference lists. For such an environment, the direct mechanism associated with the student-proposing deferred acceptance algorithm always finds a constrained efficient stable matching, and truth-telling is a dominant strategy for all students (Dubins and Freedman (1982), and Roth (1982)). Furthermore, a constrained efficient stable function is dominant strategy implementable by a direct mechanism associated with the student-proposing deferred acceptance algorithm (Kumano and Watabe (2012)).

However, priorities are much more complex in reality for several reasons, such as an affirmative action policy and schools' budget constraints. Unfortunately, depending on the class of priorities, there does not exist a strategy-proof and constrained efficient stable function. For example, when priorities allow ties among students, there are in general multiple constrained efficient stable matchings and it is well known that there is no strategy-proof and constrained efficient stable function (Erdil and Ergin (2008)). As another example, under substitutable priorities without ties, which is introduced by Kelso and Crawford (1982) and Hatfield and Milgrom (2005), there does not necessarily exist a strategy-proof and constrained efficient stable function.[3] Even worse, under these priorities, Abdulkadiroğlu (2005) shows that no stable function is dominant strategy incentive compatible. Thus, in our general environment, not only a constrained efficient stable matching but also a stable matching is not achievable in dominant strategy equilibria. Therefore, we instead focus on Nash equilibria as an equilibrium concept.

This paper does not specify a class of priorities, instead treats any priorities that guarantee a stable matching for all students' preference profiles. Then we ask whether a stable sub-correspondence is implementable in Nash equilibria or not. The answer is affirmative

[2]Simple priorities here imply the so-called responsive priorities, which are a subclass of substitutable priorities. This paper does not handle a specific class of priorities except for the last subsection. For this purpose, we do not introduce a formal definition of responsive priorities.

[3]A formal definition of substitutability is provided in the last subsection. There it is defined in a more general form.

for a stable correspondence and slightly negative for a constrained efficient stable correspondence. Formally, we first show that a stable correspondence is Nash implementable whenever it is well-defined. Then, we show that, under a reasonable assumption on priorities, a constrained efficient stable correspondence is Nash implementable whenever it is well-defined if and only if it is Maskin monotonic.

Although Nash implementation of a constrained efficient stable correspondence is fully characterized by Maskin monotonicity, it is not easy to check. The spirit of market design is its actual use in practice. To this end, we further explore the relation between Maskin monotonicity and priorities that can be observed in advance. Since we need a specific structure of priorities, we assume reasonable assumptions on priorities.[4] We show that a constrained efficient stable correspondence is Maskin monotonic if and only if priorities further satisfy the condition called *strong acyclicity*, which is introduced by Erdil and Kumano (2019). In their original paper, they show that priorities are strongly acyclic if and only if a constrained efficient stable correspondence is efficient.[5] Hence, the following are equivalent: a constrained efficient stable correspondence is Nash implementable; it is Maskin monotonic; it is efficient; priorities are strongly acyclic.

1.1 Related literature

Our study takes a mechanism design approach. There is a large body of literature concerning implementation. Danilov (1992) and Yamato (1992) find a necessary and sufficient condition for Nash implementation of a social choice correspondence in a general setting. In particular, Kara and Sönmez (1996, 1997), Sotomayor (2008, 2012), and Jaramillo et al. (2013) investigate Nash implementation in matching markets where both sides of the market behave strategically. For a restricted class of priorities, Kumano (2017) shows

[4] Assumptions on priorities are acceptance, consistency, and substitutability. The formal definitions are introduced later.

[5] Note that since only students are under efficiency consideration, a stable matching is not necessarily efficient. See Ergin (2002) for more detail.

Nash implementation of stable matchings by constructing a mechanism whose message spaces are finite.

This study begins with the conflict between (constrained efficient) stability and strategy-proofness in a general environment. Hatfield and Milgrom (2005) introduce a condition on priorities under which a constrained efficient stable and strategy-proof function exists. Hatfield and Kojima (2009) strengthen this result so that, under the same setting as Hatfield and Milgrom (2005), a constrained efficient stable and *weakly* group strategy-proof function exists. Further, Hirata and Kasuya (2017) investigate how a stable and strategy-proof function works whenever the function exists in a setting without ties.

2 Model

Let $N = \{1, \cdots, n\}$ be a finite set of students, and X be a finite set of schools. We assume that $n \geq 3$. Let $q_x \in \mathbb{N}$ be the quota of school $x \in X$. Each student $i \in N$ has a complete, transitive, and antisymmetric preference R_i over $X \cup \{i\}$. The strict part of R_i is written by P_i. We denote the set of all preferences of $i \in N$ by $\mathcal{R}_i$ and the set of all preference profiles by $\mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$.

Each school $x \in X$ is equipped with an **admission correspondence** $\mathcal{A}_x : 2^N \rightrightarrows 2^N$ such that for all $S \subseteq N$ and for all $S' \in \mathcal{A}_x(S)$, we have $S' \subseteq S$ and $|S'| \leq q_x$. Throughout the paper, we assume that a **priority structure** $(\mathcal{A}_x, q_x)_{x \in X}$ is exogenously given so that only the students are involved in the welfare and incentive concerns. A **matching** μ is a function from N to $N \cup X$ such that (1) for all $i \in N$, $\mu(i) \in N \cup \{i\}$, and (2) for all $x \in X$, $|\mu^{-1}(x)| \leq q_x$, where $\mu^{-1}(x) \equiv \{i \in N \mid \mu(i) = x\}$. Let M be the set of all matchings. A matching $\mu \in M$ is **stable** at $R \in \mathcal{R}$ if

1. (Individual rationality for students) $\mu(i) R_i i$ for all $i \in N$,

2. (Individual rationality for schools) $\mu^{-1}(x) \in \mathcal{A}_x(\mu^{-1}(x))$ for all $x \in X$, and

3. (No blocking pair) there exists no $(i,x) \in N \times X$ such that $xP_i\mu(i)$ and $\mu^{-1}(x) \notin \mathcal{A}_x(\mu^{-1}(x) \cup \{i\})$.

Without restricting priority structures, there does not necessarily exist a stable matching at any preference profiles. A priority structure **ensures stability** if there is at least one stable matching at any preference profiles.

A **matching correspondence** (or social choice correspondence) is defined as a correspondence $F : \mathcal{R} \to 2^M \setminus \{\emptyset\}$. Given a priority structure, the **stable correspondence**, denoted F^s, maps each preference profile to the set of matchings which are stable.

2.1 Nash implementation

Let S_i be a set of messages of $i \in N$ and $g : S \equiv \times_{i\in N} S_i \to M$ be an outcome function. For a message profile $s \in S$, a student i's assignment under matching $g(s)$ is denoted by $g_i(s)$. A pair (S,g) constitutes a **mechanism** and a tuple (S,g,R) constitutes a **game**. A message profile $s^* \in S$ is a **Nash equilibrium** of a game (S,g,R) if for all $i \in N$ and for all $s_i \in S_i$, $g_i(s_i^*, s_{-i}^*) R_i g_i(s_i, s_{-i}^*)$. Let $NE(S,g,R) \subseteq S$ be the set of all Nash equilibria of a game (S,g,R), and define $g(NE(S,g,R)) \equiv \{\mu \in M \mid \mu = g(s) \text{ for some } s \in NE(S,g,R)\}$. For a matching correspondence F, a mechanism (S,g) **implements F in Nash equilibria** if for all $R \in \mathcal{R}$, $F(R) = g(NE(S,g,R))$. If there exists a mechanism that implements F in Nash equilibria, then F is called **Nash implementable**.

Given a matching μ and a preference R_i, let $L(\mu, R_i) \equiv \{\nu \in M | \mu(i) R_i \nu(i)\}$. Given a set of matchings $M' \subseteq M$ and a matching correspondence F, a matching $\mu \in M'$ is **essential** for $i \in N$ in M' if there exists some preference profile $R \in \mathcal{R}$ such that

$$L(\mu, R_i) \subseteq M' \text{ and } \mu \in F(R).$$

We denote the set of all essential matchings by $ESS[F, i, M']$.

Definition 1. (Essential monotonicity)

A matching correspondence F is **essentially monotonic** if for all $R, R' \in \mathcal{R}$ and $\mu \in F(R)$,

$$\forall i \in N,\ ESS[F, i, L(\mu, R_i)] \subseteq L(\mu, R_i') \ \Rightarrow\ \mu \in F(R').$$

Yamato (1992) shows that essential monotonicity is necessary and sufficient for Nash implementation of a matching correspondence by employing the following mechanism, which we call the "Yamato mechanism" here.

The Yamato mechanism (S, g):

- For each student $i \in N$, the set of messages, S_i, is defined by

$$S_i \equiv \{(R^i, \mu^i, m^i, n^i) \mid R^i \in \mathcal{R}, \mu^i \in F(R^i), m^i \in \{0,1\}, \text{ and } n^i \in N\}.$$

- Outcome function g is defined as follows: given a message profile $s \in S$,

  1. if $(R^i, \mu^i, m^i) = (R, \mu, 0)$ for all $i \in N$, then $g(s) = \mu$.
  2. if $(R^j, \mu^j, m^j) = (R, \mu, 0)$ for all $j \in N \setminus \{i\}$ and $(R^i, \mu^i, m^i) \neq (R, \mu, 0)$, then

$$g(s) = \begin{cases} \mu^i & \text{if } \ \mu^i \in ESS[F, i, L(\mu, R_i)] \\ \mu & \text{otherwise.} \end{cases}$$

  3. otherwise, let k be a student such that $k = \sum_{i \in N} n^i \pmod{n} + 1$, and $g(s) = \mu^k$.

♦

Proposition 1. *(Yamato (1992))*

The Yamato mechanism implements a matching correspondence F in Nash equilibria if and only if F is essentially monotonic.

3 Results

3.1 Stable correspondence

Theorem 1. *For any priority structure that ensures stability, the Yamato mechanism implements the stable correspondence F^s in Nash equilibria.*

Proof. Yamato (1992) proves the equivalence between essential monotonicity of a matching correspondence and Nash implementation of the correspondence in a general setting that contains ours. Thus, it suffices to show that F^s is essentially monotonic.

Suppose, by contradiction, that F^s is not essentially monotonic. Then, there exist R, R', and $\mu \in F^s(R)$ such that

$$\forall i \in N,\ ESS[F^s, i, L(\mu, R_i)] \subseteq L(\mu, R'_i) \ \text{ and } \ \mu \notin F^s(R').$$

This implies that μ is not stable at R'. There are two cases to consider.

<u>Case 1:</u> μ is not individually rational for students at R'.
Then, there exists $i \in N$ such that $iP'_i\mu(i)$. We consider a matching $\nu \in M$ and a preference profile $\hat{R} \in \mathcal{R}$ such that

$$\nu = \begin{cases} \nu(i) = i \\ \nu(j) = \mu(j) & \text{if} \quad \mu(j)P_i\mu(i) \\ \nu(k) = k & \text{otherwise} \end{cases}$$

and

$$\hat{R}_i : X',\ i, \quad \hat{R}_j : \nu(j),\ j, \quad \hat{R}_k : k$$

where X' is the same part as the part above $\mu(i)$ of R_i, that is, $R_i : X', \mu(i) \cdots$. Since μ

is stable at R and so $\mu(i)P_i i$, we have

$$L(\nu, \hat{R}_i) = L(\mu, R_i). \tag{1}$$

It is easy to check that ν is individually rational at $\hat{R}$. Moreover, from the stability of μ at R, student i cannot make a blocking pair with $x \in X'$ at $\hat{R}$ (otherwise, she can also make a blocking pair with $x \in X'$ at μ). Therefore, ν is stable at $\hat{R}$, and we have

$$\nu \in F^s(\hat{R}).$$

Then, by definition of essential monotonicity, we obtain

$$\nu \in ESS[F^s, i, L(\nu, \hat{R}_i)],$$

which, along with (1), implies that

$$\nu \in ESS[F^s, i, L(\mu, R_i)].$$

By supposition,

$$\nu \in L(\mu, R'_i),$$

and we have $\mu(i)R'_i\nu(i) = i$, which contradicts the assumption in Case 1.

Case 2: There exists a blocking pair.

Then, there exists a pair $(i, x) \in N \times X$ such that

$$xP'_i\mu(i) \quad \text{and} \quad \mu^{-1}(x) \notin \mathcal{A}_x(\mu^{-1}(x) \cup \{i\}).$$

We consider a matching $\nu \in M$ and a preference profile $\hat{R} \in \mathcal{R}$ such that

$$\nu = \begin{cases} \nu(i) = x & \\ \nu(j) = \mu(j) & \text{if} \quad \mu(j) P_i \mu(i) \\ \nu(k) = k & \text{otherwise} \end{cases}$$

and

$$\hat{R}_i : X',\ x, \quad \hat{R}_j : \nu(j),\ j, \quad \hat{R}_k : k$$

where X' is the same part as the part above $\mu(i)$ of R_i, that is, $R_i : X', \mu(i) \cdots$. Since μ is stable at R and so $\mu(i) P_i x$, we have

$$L(\nu, \hat{R}_i) = L(\mu, R_i). \tag{2}$$

It is easy to check that ν is individually rational at $\hat{R}$. Moreover, from the stability of μ at R, student i cannot make a blocking pair with $x \in X'$ at $\hat{R}$ (otherwise, she can also make a blocking pair with $x \in X'$ at μ). Thus, ν is stable at $\hat{R}$, and we have

$$\nu \in F^s(\hat{R}).$$

Then, by definition of essential monotonicity, we obtain

$$\nu \in ESS[F^s, i, L(\nu, \hat{R}_i)],$$

which, along with (2), implies that

$$\nu \in ESS[F^s, i, L(\mu, R_i)].$$

By supposition,

$$\nu \in L(\mu, R_i'),$$

and we have $\mu(i) R'_i \nu(i) = x$, which contradicts the assumption in Case 2. □

3.2 Constrained efficient stable correspondence

We have thus far considered Nash implementation of stable matchings. In general, there might be a stable matching that is weakly preferred for all students to another stable matching. Hence, we focus on the Pareto-frontier of stable matchings in this subsection. A matching $\mu \in M$ **Pareto dominates** another matching $\nu \in M$ at $R \in \mathcal{R}$ if $\mu(i) R_i \nu(i)$ for all $i \in N$ and $\mu(j) P_j \nu(j)$ for some $j \in N$. A matching is **efficient** at $R \in \mathcal{R}$ if it is not Pareto dominated by any other matchings at R. A matching is **constrained efficient stable** at $R \in \mathcal{R}$ if it is stable and not Pareto dominated by any other stable matchings at R. Given a priority structure, the **constrained efficient stable correspondence**, denoted F^c, maps each preference profile to the set of matchings that are constrained efficient stable.

To investigate Nash implementation of the constrained efficient stable correspondence, we put an assumption on priority structures that seems to be reasonable especially in school choice problem.

Definition 2. (Acceptant priorities)
A priority structure $(\mathcal{A}_x, q_x)_{x \in X}$ is **acceptant** if for all $x \in X$, $S \subseteq N$, and $S' \in \mathcal{A}_x(S)$, we have $|S'| = \min\{|S|, q_x\}$.

Maskin (1977) shows that *Maskin monotonicity*, which is defined below, is a necessary condition for Nash implementation.

Definition 3. (Maskin monotonicity)
A matching correspondence F is **Maskin monotonic** if for all $R, R' \in \mathcal{R}$ and $\mu \in F(R)$,

$$\forall i \in N,\ L(\mu, R_i) \subseteq L(\mu, R'_i) \ \Rightarrow\ \mu \in F(R').$$

Since F^c does not satisfy Maskin monotonicity, it is not Nash implementable in general. However, the following proposition states that Maskin monotonicity is not only necessary but also sufficient for Nash implementation of F^c under acceptant priority structures.

Proposition 2. *For any acceptant priority structure that ensures stability, the constrained efficient stable correspondence F^c is Nash implementable if and only if it is Maskin monotonic.*

Proof. In Appendix A. □

3.3 Characterization in terms of priorities

In this subsection, we explore the relation between Maskin monotonicity of F^c and priority structures.[6] We further put two assumptions on priority structures in addition to acceptance.

Definition 4. (Consistent priorities)
A priority structure $(\mathcal{A}_x, q_x)_{x\in X}$ is **consistent** if the following two conditions hold:

1. If there exists $S \subseteq N$ such that $A \cup B \subseteq S$ with $A \in \mathcal{A}_x(S)$ and $B \notin \mathcal{A}_x(S)$, then whenever $A \cup B \subseteq S'$, we have $B \notin \mathcal{A}_x(S')$.
2. If there exists $S \subseteq N$ such that $A, B \in \mathcal{A}_x(S)$, then whenever $A \cup B \subseteq S'$, we have $A \in \mathcal{A}_x(S')$ if and only if $B \in \mathcal{A}_x(S')$.

The other assumption on priority structures is *substitutability*.[7] An admission correspondence $\mathcal{A}_x$ is **monotonic** if for all $S, T \subseteq N$ with $S \subseteq T$ and $T' \in \mathcal{A}_x(T)$, we

[6] A growing body of literature focuses on a condition on priority structures to guarantee equivalence results. See, for example, Ergin (2002), Kesten (2006), Haeringer and Klijn (2009), Ehlers and Erdil (2010), Kojima (2013), Kumano (2013), and Erdil and Kumano (2019).

[7] See Hatfield and Milgrom (2005), Hatfield and Kojima (2008), and Hatfield and Kojima (2010) for the connection between substitutability and stability. Moreover, see Abdulkadiroğlu and Sönmez (2003) that indicates the plausibility of substitutable priorities.

have $T' \cap S \subseteq S'$ for some $S' \in \mathcal{A}_x(S)$. Given an admission correspondence $\mathcal{A}_x$, let $\mathcal{R}_x : 2^N \rightrightarrows 2^N$ be a **rejection correspondence** of school $x \in X$ such that for all $S \subseteq N$,

$$\mathcal{R}_x(S) \equiv \{S'' \subseteq S \mid S'' = S \setminus S' \text{ for some } S' \in \mathcal{A}_x(S)\}.$$

A rejection correspondence $\mathcal{R}_x$ is **monotonic** if for all $S, T \subseteq N$ with $S \subseteq T$ and $S' \in \mathcal{R}_x(S)$, we have $S' \subseteq T'$ for some $T' \in \mathcal{R}_x(T)$.[8] We are ready to define substitutability.[9]

Definition 5. (Substitutable priorities)
A priority structure $(\mathcal{A}_x, q_x)_{x \in X}$ is **substitutable** if for all $x \in X$, $\mathcal{A}_x$ and $\mathcal{R}_x$ are both monotonic.

In the class of acceptant, consistent, and substitutable priority structures, we introduce a necessary and sufficient condition on priority structures such that F^c is Nash implementable. Let $\mathcal{CA}_x(S) \equiv \{i \in N \mid i \in S' \text{ for all } S' \in \mathcal{A}_x(S)\}$ for all $x \in X$ and for all $S \subseteq N$. The following definition is proposed in the Online Appendix of Erdil and Kumano (2019).

Definition 6. (Strongly acyclic priorities)
Given a priority structure $(\mathcal{A}_x, q_x)_{x \in X}$, a **generalized weak cycle** constitutes distinct $j, i_0, i_1, \cdots, i_{n-1} \in N$, and distinct $x_0, x_1, \cdots, x_{n-1} \in X$ with $n \geq 2$ such that there exist

[8] Note that, in general, monotonicity of $\mathcal{A}_x$ and monotonicity of $\mathcal{R}_x$ are logically independent (see Remark 1 in Erdil and Kumano's (2019) Online Appendix). Note also that if $\mathcal{A}_x$ is single-valued, then the admission correspondence is monotonic if and only if the rejection correspondence is monotonic.

[9] Che et al. (2019) define *weak substitutability* and investigate the existence of stable matchings under this assumption. We note here that substitutability in our study is equivalent to weak substitutability in their study, which is discussed in more detail in Appendix B.

mutually disjoint sets of students $S_{x_0}, \cdots, S_{x_{n-1}} \subseteq N \setminus \{j, i_0, i_1, \cdots, i_{n-1}\}$ such that

$$\begin{aligned}
& j \notin \mathcal{CA}_{x_0}(S_{x_0} \cup \{i_0, j\}), \\
& j \in \mathcal{CA}_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}), \\
& i_{n-1} \notin \mathcal{CA}_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}), \\
& i_{n-2} \notin \mathcal{CA}_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}), \\
& \qquad \vdots \\
& i_1 \notin \mathcal{CA}_{x_2}(S_{x_2} \cup \{i_2, i_1\}), \\
& i_0 \notin \mathcal{CA}_{x_1}(S_{x_1} \cup \{i_1, i_0\}),
\end{aligned}$$

$$|S_{x_\ell}| = q_{x_\ell} - 1 \quad \text{for } \ell = 0, 1, \cdots, n-1.$$

If a priority structure $(\mathcal{A}_x, q_x)_{x \in X}$ does not contain any generalized weak cycle, then it is called **strongly acyclic**.

Proposition 3. *For any acceptant, consistent, and substitutable priority structure, the constrained efficient stable correspondence F^c is Maskin monotonic if and only if the priority structure is strongly acyclic.*

Proof. In Appendix A. □

By Proposition 2 and Proposition 3, we obtain the following result.

Theorem 2. *For any acceptant, consistent, and substitutable priority structure, the constrained efficient stable correspondence F^c is Nash implementable if and only if the priority structure is strongly acyclic.*

Moreover, Erdil and Kumano (2019) show in their Online Appendix that all constraint efficient stable matchings are efficient if and only if the priority structure is strongly acyclic. Therefore, Theorem 2 in our study implies the following corollary.

Corollary 1. *For any acceptant, consistent, and substitutable priority structure, the following are equivalent:*

(i) F^c *is Nash implementable,*

(ii) F^c *is Maskin monotonic,*

(iii) F^c *is efficient, and*

(iv) the priority structure is strongly acyclic.

Appendix A: Omitted proofs

Proof of Proposition 2.

Maskin (1977) shows that Maskin monotonicity is a necessary condition for Nash implementation of a matching correspondence. Thus, we show that if F^c is Maskin monotonic, then it is essentially monotonic.

Suppose, by contradiction, that F^c is not essentially monotonic. Then, there exist R, R', and $\mu \in F^c(R)$ such that

$$\forall i \in N,\ ESS[F^c, i, L(\mu, R_i)] \subseteq L(\mu, R'_i) \ \text{ and } \ \mu \notin F^c(R').$$

This implies that μ is not constrained efficient stable at R'. There are three cases to consider.

<u>Case 1:</u> μ is not individually rational for students at R'.
Then, there exists $i \in N$ such that $iP'_i\mu(i)$. We consider a matching $\nu \in M$ and a preference profile $\hat{R} \in \mathcal{R}$ such that

$$\nu = \begin{cases} \nu(i) = i & \\ \nu(j) = \mu(j) & \text{if} \quad \mu(j)P_i\mu(i) \\ \nu(k) = k & \text{otherwise} \end{cases}$$

and

$$\hat{R}_i : X',\ i, \quad \hat{R}_j : \nu(j),\ j, \quad \hat{R}_k : k,$$

where X' is the same part as the part above $\mu(i)$ of R_i, that is, $R_i : X', \mu(i) \cdots$. Since μ

is stable at R and so $\mu(i) P_i i$, we have

$$L(\nu, \hat{R}_i) = L(\mu, R_i). \tag{3}$$

It is easy to check that ν is individually rational at $\hat{R}$. Moreover, from the stability of μ at R, student i cannot make a blocking pair with $x \in X'$ at $\hat{R}$ (otherwise, she can also make a blocking pair with $x \in X'$ at μ). Therefore, ν is stable at $\hat{R}$. We insist that ν is constrained efficient stable at $\hat{R}$.

Suppose that ν would not be constrained efficient stable at $\hat{R}$. Then, there exists another stable matching η that Pareto dominates ν at $\hat{R}$. By construction of $\hat{R}$, $\eta^{-1}(x) = \nu^{-1}(x) \cup \{i\}$ must hold for some $x \in X'$. Since ν is stable at $\hat{R}$, we have

$$\nu^{-1}(x) \in \mathcal{A}_x(\nu^{-1}(x) \cup \{i\}). \tag{4}$$

Acceptance of priority structures, together with (4), implies that $|\nu^{-1}(x)| = q_x$. However, this is a contradiction that $\eta^{-1}(x) = \nu^{-1}(x) \cup \{i\}$ and η is individually rational for schools. Therefore, ν is also constrained efficient stable at $\hat{R}$, and we have

$$\nu \in F^c(\hat{R}).$$

Then, by definition of essential monotonicity, we obtain

$$\nu \in ESS[F^c, i, L(\nu, \hat{R}_i)],$$

which, along with (3), implies that

$$\nu \in ESS[F^c, i, L(\mu, R_i)].$$

By supposition,

$$\nu \in L(\mu, R_i'),$$

and we have $\mu(i) R_i' \nu(i) = i$, which contradicts the assumption in Case 1.

<u>Case 2:</u> There exists a blocking pair.

Then, there exists a pair $(i, x) \in N \times X$ such that

$$x P_i' \mu(i) \quad \text{and} \quad \mu^{-1}(x) \notin \mathcal{A}_x(\mu^{-1}(x) \cup \{i\}).$$

We consider a matching $\nu \in M$ and a preference profile $\hat{R} \in \mathcal{R}$ such that

$$\nu = \begin{cases} \nu(i) = x \\ \nu(j) = \mu(j) & \text{if} \quad \mu(j) P_i \mu(i) \\ \nu(k) = k & \text{otherwise} \end{cases}$$

and

$$\hat{R}_i : X',\ x, \quad \hat{R}_j : \nu(j),\ j, \quad \hat{R}_k : k,$$

where X' is the same part as the part above $\mu(i)$ of R_i, that is, $R_i : X', \mu(i) \cdots$. Since μ is stable at R and so $\mu(i) P_i x$, we have

$$L(\nu, \hat{R}_i) = L(\mu, R_i). \tag{5}$$

It is easy to check that ν is individually rational at $\hat{R}$. Moreover, from the stability of μ at R, student i cannot make a blocking pair with $x \in X'$ at $\hat{R}$ (otherwise, she can also make a blocking pair with $x \in X'$ at μ). Therefore, ν is stable at $\hat{R}$. We insist that ν is constrained efficient stable at $\hat{R}$.

Suppose that ν would not be constrained efficient stable at $\hat{R}$. Then, there exists another stable matching η that Pareto dominates ν at $\hat{R}$. By construction of $\hat{R}$, $\eta^{-1}(x) = \nu^{-1}(x) \cup \{i\}$ must hold for some $x \in X'$. Since ν is stable at $\hat{R}$, we have

$$\nu^{-1}(x) \in \mathcal{A}_x(\nu^{-1}(x) \cup \{i\}). \tag{6}$$

Acceptance of priority structures, together with (6), implies that $|\nu^{-1}(x)| = q_x$. However, this is a contradiction that $\eta^{-1}(x) = \nu^{-1}(x) \cup \{i\}$ and η is individually rational for schools. Therefore, ν is also constrained efficient stable at $\hat{R}$, and we have

$$\nu \in F^c(\hat{R}).$$

Then, by definition of essential monotonicity, we obtain

$$\nu \in ESS[F^c, i, L(\nu, \hat{R}_i)],$$

which, along with (5), implies that

$$\nu \in ESS[F^c, i, L(\mu, R_i)].$$

By supposition,

$$\nu \in L(\mu, R_i'),$$

and we have $\mu(i) R_i' \nu(i) = x$, which contradicts the assumption in Case 2.

<u>Case 3:</u> μ is stable at R' but there is a stable matching ν that Pareto dominates μ at R'. Then, we have

$$\forall i \in N, \nu(i) R_i' \mu(i) \quad \text{and} \quad \exists j \in N, \nu(j) P_j' \mu(j).$$

Let $S \equiv \{j \in N | \nu(j) P'_j \mu(j)\}$ be the set of students who are matched to better schools in ν than in μ at R'. From the antisymmetry of preferences, for each $j \in S$, either $\mu(j) P_j \nu(j)$ or $\nu(j) P_j \mu(j)$ holds. We consider two subcases.

<u>Case 3-1:</u> For some $j \in S, \mu(j) P_j \nu(j)$.
Take any $j \in S$ who prefers $\mu(j)$ to $\nu(j)$ according to R_j. We consider a matching $\eta \in M$ and a preference profile $\hat{R}$ such that

$$\eta = \begin{cases} \eta(j) = \nu(j) & \\ \eta(k) = \mu(k) & \text{if } \mu(k) P_j \mu(j) \\ \eta(\ell) = \ell & \text{otherwise} \end{cases}$$

and

$$\hat{R}_j : X', \nu(j), \ \hat{R}_k : \mu(k), k \ , \ \hat{R}_\ell : \ell,$$

where X' is the same part as the part above $\mu(j)$ of R_j, that is, $R_j : X', \mu(j) \cdots$. Since $\mu(j) P_j \nu(j)$, we have

$$L(\eta, \hat{R}_j) = L(\mu, R_j). \tag{7}$$

It is easy to check that η is individually rational at $\hat{R}$. Moreover, from the stability of μ at R, student i cannot make a blocking pair with $x \in X'$ at $\hat{R}$ (otherwise, she can also make a blocking pair with $x \in X'$ at μ). Therefore, ν is stable at $\hat{R}$. We insist that η is constrained efficient stable at $\hat{R}$.

Suppose that η would not be constrained efficient stable at $\hat{R}$. Then, there exists another stable matching η' that Pareto dominates η at $\hat{R}$. By construction of $\hat{R}$, $\eta'^{-1}(x) = \eta^{-1}(x) \cup \{i\}$ must hold for some $x \in X'$. Since η is stable at $\hat{R}$, we have

$$\eta^{-1}(x) \in \mathcal{A}_x(\eta^{-1}(x) \cup \{i\}). \tag{8}$$

Acceptance of priority structures, together with (8), implies that $|\eta^{-1}(x)| = q_x$. However, this is a contradiction that $\eta'^{-1}(x) = \eta^{-1}(x) \cup \{i\}$ and η' is individually rational for schools. Therefore, η is also constrained efficient stable at $\hat{R}$, and we have

$$\eta \in F^c(\hat{R}).$$

Then, by definition of essential monotonicity, we obtain

$$\eta \in ESS[F^c, j, L(\eta, \hat{R}_j)],$$

which, along with (7), implies that

$$\eta \in ESS[F^c, j, L(\mu, R_j)].$$

By supposition,

$$\eta \in L(\mu, R'_j),$$

and we have $\mu(j)R'_j\eta(j) = \nu(j)$, which contradicts the assumption in Case 3.

Case 3-2: For all $j \in S, \nu(j)P_j\mu(j)$.
We consider a preference profile $\hat{R}$ such that

$$\begin{aligned} \forall j \in S, \quad & \hat{R}_j : \nu(j), \mu(j) \dots \\ \forall k \in N \backslash S, \quad & \hat{R}_k : \mu(k) \dots \end{aligned}$$

Because of the antisymmetry of preferences, for all $k \in N\backslash S$, $\mu(k) = \nu(k)$. Then, ν is clearly constrained efficient stable at $\hat{R}$. Moreover, for all $i \in N$, $L(\mu, R_i) \subseteq L(\mu, \hat{R}_i)$ holds. By Maskin monotonicity of F^c, we have $\mu \in F^c(\hat{R})$. However, ν Pareto dominates μ at $\hat{R}$, a contradiction. □

Proof of Proposition 3.

Before showing Proposition 3, we propose a useful result. For any acceptant priority structure, the notion of stability is identical to the following weaker version of stability.[10]

Definition 7. A matching $\mu \in M$ is **weakly stable** at $R \in \mathcal{R}$ if

1. $\mu(i) R_i i$ for all $i \in N$,

2. $\mu^{-1}(x) \in \mathcal{A}_x(\mu^{-1}(x))$ for all $x \in X$, and

3. there exists no $(i, x) \in N \times X$ such that $x P_i \mu(i)$ and $i \in \mathcal{CA}_x(\mu^{-1}(x) \cup \{i\})$.

Lemma 1. *For any acceptant priority structure, stability and weak stability are equivalent.*

Proof of Lemma 1. Suppose that the third condition of weak stability does not hold. Then, there exists $(i, x) \in N \times X$ such that $x P_i \mu(i)$ and $i \in \mathcal{CA}_x(\mu^{-1}(x) \cup \{i\})$. By definition of matchings, $i \notin \mu^{-1}(x)$. Moreover, by definition of $\mathcal{CA}_x$, $i \in S'$ for all $S' \in \mathcal{A}_x(\mu^{-1}(x) \cup \{i\})$. This implies that $\mu^{-1}(x) \notin \mathcal{A}_x(\mu^{-1}(x) \cup \{i\})$. Hence, stability implies weak stability.

Conversely, suppose that the third condition of stability does not hold. Then, there exists $(i, x) \in N \times X$ such that $x P_i \mu(i)$ and $\mu^{-1}(x) \notin \mathcal{A}_x(\mu^{-1}(x) \cup \{i\})$.

Case 1: $|\mu^{-1}(x)| < q_x$

Then, by acceptance of $\mathcal{A}_x$, we must have $\mathcal{A}_x(\mu^{-1}(x) \cup \{i\}) = \{\mu^{-1}(x) \cup \{i\}\}$. Hence, $i \in \mathcal{CA}_x(\mu^{-1}(x) \cup \{i\})$.

Case 2: $|\mu^{-1}(x)| = q_x$

[10] Remind that for all $x \in X$ and $S \subseteq N$, let $\mathcal{CA}_x(S) \equiv \{i \in N \mid i \in S' \text{ for all } S' \in \mathcal{A}_x(S)\}$.

Take any $S' \in \mathcal{A}_x(\mu^{-1}(x) \cup \{i\})$. If $i \notin S'$, then by acceptance of $\mathcal{A}_x$, we would have $\mu^{-1}(x) \in \mathcal{A}_x(\mu^{-1}(x) \cup \{i\})$, which contradicts the assumption. Therefore, we obtain $i \in S'$. Because S' is arbitrary, we understand that $i \in \mathcal{C}\mathcal{A}_x(\mu^{-1}(x) \cup \{i\})$, which was what we wanted. □

Proof of Proposition 3.

(if part)

Suppose that a priority structure $(\mathcal{A}_x, q_x)_{x \in X}$ is strongly acyclic, and F^c is not Maskin monotonic. Then, there exist $R, R' \in \mathcal{R}$ and $\mu \in F^c(R)$ such that $L(\mu, R_i) \subseteq L(\mu, R'_i)$ for all $i \in N$ and $\mu \notin F^c(R')$. Because μ is not constrained efficient stable at R', we consider two cases.

<u>Case 1:</u> μ is not stable at R'.

If μ is not individually rational at R', that is, $iP'_i\mu(i)$ for some $i \in N$, then it is also not individually rational at R since $L(\mu, R_i) \subseteq L(\mu, R'_i)$ implies $iP_i\mu(i)$. This is a contradiction that μ is stable at R. If there exist a blocking pair $(i, x) \in N \times X$ such that $xP'_i\mu(i)$ and $i \in \mathcal{C}\mathcal{A}_x(\mu^{-1}(x) \cup \{i\})$, then this blocking pair also constitutes a blocking pair at R because $L(\mu, R_i) \subseteq L(\mu, R'_i)$ implies $xP_i\mu(i)$. This is a contradiction that μ is stable at R.

<u>Case 2:</u> μ is stable at R', but there exists a stable matching ν that Pareto dominates μ at R'.

That is,

$$\forall i \in N,\ \nu(i)R'_i\mu(i) \ \text{ and } \ \exists j \in N,\ \nu(j)P'_j\mu(j).$$

Since $L(\mu, R_i) \subseteq L(\mu, R'_i)$ holds for all $i \in N$, we have

$$\forall i \in N,\ \nu(i)R_i\mu(i) \ \text{ and } \ \exists j \in N\ \nu(j)P_j\mu(j).$$

Note that from Erdil and Kumano's (2019) Online Appendix, we know that F^c is efficient if and only if the priority structure is strongly acyclic. By assumption, $(\mathcal{A}_x, q_x)_{x\in X}$ is strongly acyclic, so F^c is efficient. Then, since $\mu \in F^c(R)$, μ is efficient at R. However, ν Pareto dominates μ at R. This is a contradiction.

(only if part)

Now, we assume that F^c is Maskin monotonic and a priority structure $(\mathcal{A}_x, q_x)_{x\in X}$ has a generalized weak cycle. Then, there exist distinct $j, i_0, i_1, \cdots, i_{n-1} \in N$, and distinct $x_0, x_1, \cdots, x_{n-1} \in X$ with $n \geq 2$ such that there exist mutually disjoint sets of students $S_{x_0}, \cdots, S_{x_{n-1}} \subseteq N \setminus \{j, i_0, i_1, \cdots, i_{n-1}\}$ such that

$$
\begin{aligned}
& j \notin \mathcal{CA}_{x_0}(S_{x_0} \cup \{i_0, j\}), \\
& j \in \mathcal{CA}_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}), \\
& i_{n-1} \notin \mathcal{CA}_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}), \\
& i_{n-2} \notin \mathcal{CA}_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}), \\
& \quad \vdots \\
& i_1 \notin \mathcal{CA}_{x_2}(S_{x_2} \cup \{i_2, i_1\}), \\
& i_0 \notin \mathcal{CA}_{x_1}(S_{x_1} \cup \{i_1, i_0\}),
\end{aligned}
$$

$$
|S_{x_\ell}| = q_{x_\ell} - 1 \quad \text{for } \ell = 0, 1, \cdots, n-1.
$$

We consider the following preference profile R,

R_{i_0}	R_j	R_{i_1}	$\cdots$	$R_{i_{n-2}}$	$R_{i_{n-1}}$	R_{k_0}	$\cdots$	$R_{k_{n-1}}$	R_ℓ
x_1	x_0	x_2		x_{n-1}	x_0	x_0		x_{n-1}	ℓ
x_0	j	x_1	$\cdots$	x_{n-2}	x_{n-1}	k_0	$\cdots$	k_{n-1}	
i_0		i_1		i_{n-2}	i_{n-1}				

where $k_h \in S_{x_h}$ for all $h \in \{0, \cdots, n-1\}$ and $\ell \in N \setminus \big((\cup_{m=0}^{n-1} S_{x_m}) \cup \{i_0, j, i_1, \cdots, i_{n-1}\} \big)$.

Then,

$$\mu = \begin{pmatrix} i_0 & j & i_1 & \cdots & i_{n-1} & S_{x_0} & \cdots & S_{x_{n-1}} & \ell \\ x_0 & j & x_1 & \cdots & i_{n-1} & x_0 & \cdots & x_{n-1} & \ell \end{pmatrix}$$

is a constrained efficient stable matching at R. That is, $\mu \in F^c(R)$.

Let $\hat{R}$ be such that student j ranks himself as first and any other students do not change preferences, that is,

$$\hat{R}_{-j} = R_{-j} \text{ and } \hat{R}_j : j \cdots .$$

Then, $L(\mu, R_i) \subseteq L(\mu, \hat{R}_i)$ holds for all $i \in N$, and we have $\mu \in F^c(\hat{R})$ since F^c is Maskin monotonic. That is, μ is constrained efficient stable at $\hat{R}$. However,

$$\nu = \begin{pmatrix} i_0 & j & i_1 & \cdots & i_{n-1} & S_{x_0} & \cdots & S_{x_{n-1}} & \ell \\ x_1 & j & x_2 & \cdots & i_0 & x_0 & \cdots & x_{n-1} & \ell \end{pmatrix}$$

is stable and Pareto dominates μ at $\hat{R}$. Hence, μ could not be constrained efficient stable at $\hat{R}$, a contradiction. □

Appendix B

Here, we show the equivalence between Che et al.'s (2019) definition of *weakly substitutable priorities* and our definition of substitutability.

Definition 8. (Weakly substitutable priorities)
A priority structure $(\mathcal{A}_x, q_x)_{x \in X}$ is **weakly substitutable** if for all $x \in X$ the following two conditions hold:

1. for all $S, T \subseteq N$ with $S \subseteq T$ and all $T' \in \mathcal{R}_x(T)$, we have $S' \subseteq T'$ for some $S' \in \mathcal{R}_x(S)$.

2. for all $S, T \subseteq N$ with $S \subseteq T$ and all $S' \in \mathcal{R}_x(S)$, we have $S' \subseteq T'$ for some $T' \in \mathcal{R}_x(T)$.

Proposition 4. *The notion of weak substitutability is equivalent to that of substitutability.*

Proof of Proposition 4. Since the second condition in weakly substitutable priorities and the monotonicity of rejection correspondences are the same, we first check that the first condition of weak substitutability implies the monotonicity of admission correspondences. Then, we check the opposite.

Take any $S \subseteq T \subseteq N$ and any $T'' \in \mathcal{A}_x(T)$. Let $T' \equiv T \setminus T''$. Then, $T' \in R_x(T)$. By the first condition in weakly substitutable priorities, there exists $S' \in R_x(S)$ such that $S' \subseteq T'$. Let $S'' \equiv S \setminus S'$. Then, $S'' \in \mathcal{A}_x(S)$. By $S' \subseteq T'$ and $T' = T \setminus T''$, we have $S' \cap T'' = \emptyset$. Hence,

$$T'' \cap S = T'' \cap (S' \cup S'') = \underbrace{(T'' \cap S')}_{=\emptyset} \cup (T'' \cap S'') = T'' \cap S'' \subseteq S''.$$

Therefore, one direction has been proved.

Take any $S \subseteq T \subseteq X$ and any $T' \in R_x(T)$. Let $T'' \equiv T \setminus T'$. Then, $T'' \in \mathcal{A}_x(T)$. By the monotonicity of admission correspondences, there exists $S'' \in \mathcal{A}_x(S)$ such that $T'' \cap S \subseteq S''$. Let $S' \equiv S \setminus S''$. Then, $S' \in R_x(S)$. By $T'' \cap S \subseteq S''$ and $S' = S \setminus S''$, we have $S' \cap T'' = (S' \cap S) \cap T'' = S' \cap (S \cap T'') = \emptyset$. Hence,

$$S' \cap T'' = \emptyset \Leftrightarrow S' \cap (T \setminus T') = \emptyset \Leftrightarrow S' \subseteq T'.$$

□

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Chapter 3

Nash implementation in matching with contracts*

1 Introduction

A model of matching with contracts refers to a market where a doctor is going to be matched with a hospital based on a contract representing, say, a medical department (Hatfield and Milgrom (2005)). The outcome of the market is an allocation that forms a set of contracts. One of the main objective in the market is to address whether a *stable* allocation is achieved in some equilibrium notion: an allocation is stable if neither doctor nor hospital rejects the assigned contracts unilaterally, and no doctor and hospital pair blocks the allocation bilaterally in such a way that both of them can find a contract involving them better than what they obtain in the allocation. When hospitals have a simple choice behavior, a direct mechanism associated with the celebrated Deferred Acceptance (DA) algorithm works well, that is, the direct mechanism always produces a stable allocation and truth-telling is a dominant strategy for doctors.[1]

Importantly, it is not uncommon in reality that a hospital cares for an affirmative action policy or has a budget constraint. This implies that choice behavior for hospitals are no longer represented in a simple manner, leading to a demand for the analysis under a much more complex choice behavior for hospitals. Hence, we do not specify a class of hospitals' choice behavior but treat any choice behavior that guarantees a stable allocation. It is well known, however, that in our general choice behavior, there does not necessarily

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[1]See Gale and Shapley (1962) for the construction of the DA algorithm, and Dubins and Freedman (1982) and Roth (1982) for the incentive compatibility.

exist a mechanism that produces a stable allocation while preserving dominant strategy incentive compatibility.[2] So, we instead seek to analyze whether a stable allocation is achieved in Nash equilibria.[3]

Given a choice behavior for hospitals, the stable correspondence is a correspondence that produces the set of stable allocations for any input of preferences for doctors. Unfortunately, the stable correspondence is not Nash implementable in general. Thus, we propose a necessary and sufficient condition on choice behavior for hospitals, called *Richness*, for the stable correspondence to be Nash implementable. Formally, we show that given any choice behavior that guarantees a stable allocation, the stable correspondence is Nash implementable if and only if the choice behavior satisfies Richness.

In the last part of the paper, we check that Richness does hold in matching *without* contracts. Therefore, it is shown that the stable correspondence is Nash implementable in matching without contracts in general.

2 Model

There are a finite set of doctors D, a finite set of hospitals H, and a finite set of contractual terms C. Let $X \subseteq D \times H \times C$ be a finite set of contracts. We denote a doctor associated with a contract $x \in X$ by $\mathrm{D}(x) \in D$ and similarly a hospital associated with a contract $x \in X$ by $\mathrm{H}(x) \in H$. Given a set of contracts $X' \subseteq X$, let $\mathrm{D}(X') := \cup_{x \in X'}\{\mathrm{D}(x)\}$ and $\mathrm{H}(X') := \cup_{x \in X'}\{\mathrm{H}(x)\}$. In addition to X, there is a null contract $\varnothing_d$ for any $d \in D$ that represents having no relationship with any hospital. For any $X' \subseteq X$, define $X'_d := \{x \in X' \mid \mathrm{D}(x) = d\}$ for any $d \in D$ and $X'_h := \{x \in X' \mid \mathrm{H}(x) = h\}$ for any $h \in H$.[4] If doctor d has no such contract in $X' \subseteq X$, then we define $X'_d = \varnothing_d$.

[2]Hatfield, Kominers, and Westlamp (2021) propose a necessary and sufficient condition on choice behavior such that there exists a stable and strategy-proof mechanism.

[3]For the literature concerning Nash implementation in matching markets, see, for example, Kara and Sönmez (1996, 1997), and Kumano (2017).

[4]When X'_d is a singleton, say $X'_d = \{x\}$, we write $X'_d = x$ as long as there is no confusion.

Each doctor $d \in D$ has a complete, transitive, and antisymmetric preference R_d over $X_d \cup \{\emptyset_d\}$. The strict part of R_d is denoted by P_d. Let $\mathcal{R}_d$ be a set of all preferences of $d \in D$, and let $\mathcal{R} := \times_{d \in D} \mathcal{R}_d$. For any $D' \subseteq D$, we define $R_{D'} = (R_d)_{d \in D'}$ and $R_{-D'} = (R_d)_{d \in D \setminus D'}$. For notational convenience, we denote R instead of R_D as long as there is no confusion. For any $d \in D$ and $X' \subseteq X$, the **chosen contract** $C_{R_d}(X')$ of doctor d is

$$C_{R_d}(X') := \max_{R_d}(X'_d \cup \{\emptyset_d\}).$$

For $X' \subseteq X$, let $C_R(X') := \{C_{R_d}(X') | d \in D\}$.

Each hospital $h \in H$ has a complete, transitive, and antisymmetric priority R_h over the set of subsets of X_h.[5] For any $h \in H$, R_h induces the **choice function** $C_{R_h} : 2^X \to 2^X$, which satisfies the following three conditions: for any $X' \subseteq X$, (1) $C_{R_h}(X') \subseteq X'_h$; (2) for any $x, x' \in C_{R_h}(X')$, $x \neq x'$ implies $\mathrm{D}(x) \neq \mathrm{D}(x')$; and (3) $C_{R_h}(X') R_h X''$ for all $X'' \subseteq X'_h$. A choice function C_h satisfies **irrelevance of rejected contracts** (IRC) if for all $x \in X$ and $X' \subseteq X$, $x \notin C_h(X' \cup \{x\})$ implies $C_h(X' \cup \{x\}) = C_h(X')$. Throughout the paper, we fix $(R_h)_{h \in H}$, and for notational exposition, we simply denote a choice function C_{R_h} by C_h. This reflects the view that hospitals are just objects to be allocated to doctors; so only the doctors are involved in incentive property. For $X' \subseteq X$, let $C_H(X') := \bigcup_{h \in H} C_h(X')$.

An **allocation** is a set of contracts $X' \subseteq X$ such that for all $x, x' \in X'$, $x \neq x'$ implies $\mathrm{D}(x) \neq \mathrm{D}(x')$. Denote the set of all allocations by $\mathcal{X}$. We denote the allocation such that no doctor is allocated to a contract by $\emptyset$. The following is the allocation we would like to implement.

Definition 1. An allocation $X' \in \mathcal{X}$ is **stable** at $R \in \mathcal{R}$ if

1. $C_R(X') = C_H(X') = X'$, and
2. there is no $(d, h) \in D \times H$ and $x' \in X$ such that $x' P_d X'_d$ and $x' \in C_h(X' \cup \{x'\})$.

[5]We regard a priority relation as primitive. For an alternative model, see Aygün and Sönmez (2012, 2014).

We say that a profile of choice functions $(C_h)_{h\in H}$ **ensures stability** if there exists at least one stable allocation at each doctors' preference profile.

A **social choice correspondence** is a correspondence $F : \mathcal{R} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$. Given $(C_h)_{h\in H}$, the **stable correspondence**, denoted F^s, maps each doctors' preference profile R to the set of allocations which are stable at R.

2.1 Nash implementation

Let S_d be a set of messages of $d \in D$ and $g : S := \times_{d\in D} S_d \to \mathcal{X}$ be an outcome function. For a message profile $s \in S$, a doctor d's contract under allocation $g(s)$ is written by $g_d(s)$. A pair (S, g) constitutes a **mechanism** and a tuple (S, g, R) constitutes a **game**. A message profile $s^* \in S$ is a **Nash equilibrium** of a game (S, g, R) if for all $d \in D$ and for all $s_d \in S_d$, $g_d(s^*_d, s^*_{-d}) R_d g_d(s_d, s^*_{-d})$. Let $NE(S, g, R) \subseteq S$ be a set of all Nash equilibria of a game (S, g, R), and define $g(NE(S, g, R)) := \{X' \in \mathcal{X} \mid X' = g(s) \text{ for some } s \in NE(S, g, R)\}$. For a social choice correspondence F, a mechanism (S, g) **implements** F **in Nash equilibria** if for all $R \in \mathcal{R}$, $F(R) = g(NE(S, g, R))$. If there exists a mechanism that implements F in Nash equilibria, then F is called **Nash implementable**.

Yamato (1992) proposes a condition, called *essential monotonicity* defined below, to show that it is necessary and sufficient for a social choice correspondence to be Nash implementable.

For allocation $X' \subseteq X$ and doctor d's preference $R_d \in \mathcal{R}_d$, let $L(X', R_d) := \{X'' \in \mathcal{X} | X'_d R_d X''_d\}$. Given a set of allocations $\mathcal{X}' \subseteq \mathcal{X}$ and a social choice correspondence F, an allocation $X' \in \mathcal{X}'$ is **essential** for $d \in D$ in $\mathcal{X}'$ with respect to F if there exists some preference profile $R \in \mathcal{R}$ such that

$$L(X', R_d) \subseteq \mathcal{X}' \text{ and } X' \in F(R).$$

We denote the set of all essential allocations by $ESS[F, d, \mathcal{X}']$.

Definition 2. (Essential monotonicity)
A social choice correspondence F is **essentially monotonic** if for all $R, R' \in \mathcal{R}$ and $X' \in F(R)$,

$$\forall d \in D,\ ESS[F, d, L(X', R_d)] \subseteq L(X', R'_d) \ \Rightarrow \ X' \in F(R').$$

3 Result

3.1 Characterization

In this section, given a profile of choice functions $(C_h)_{h\in H}$ that ensures stability, we propose a necessary and sufficient condition on $(C_h)_{h\in H}$ such that the stable correspondence is Nash implementable.

Given $(C_h)_{h\in H}$, let $\mathcal{X}^H \subseteq \mathcal{X}$ be the set of allocations such that for each $X' \in \mathcal{X}^H$, $C_H(X') = X'$.

Definition 3. A profile of choice functions $(C_h)_{h\in H}$ satisfies **Richness 1** if there exist no doctor $d \in D$, a contract $x \in X_d$, and a set of contracts $\bar{X} \subseteq X_d$ such that

$$\begin{aligned}&(1)\ \ \exists X' \in \mathcal{X}^H \ \text{ with } X'_d = x, \quad \forall x' \in \bar{X},\ x' \notin C_H(X' \cup \{x'\}),\ \text{and}\\ &(2)\ \ \forall X'' \in \mathcal{X}^H \text{ with } X''_d = \varnothing_d,\ \exists x' \in \bar{X},\ x' \in C_H(X'' \cup \{x'\}).\end{aligned}$$

Definition 4. A profile of choice functions $(C_h)_{h\in H}$ satisfies **Richness 2** if there exist no doctor $d \in D$, two distinct contracts $x, x' \in X_d$, and a set of contracts $\bar{X} \subseteq X_d$ such that

$$\begin{aligned}&(1)\ \ \exists X' \in \mathcal{X}^H \ \text{ with } X'_d = x,\ \ [\forall x'' \in \bar{X},\ x'' \notin C_H(X' \cup \{x''\}),\ \text{and}\ \ x' \in C_H(X' \cup \{x'\})]\\ &(2)\ \ \forall X'' \in \mathcal{X}^H \text{ with } X''_d = x',\ \ \exists x'' \in \bar{X},\ x'' \in C_H(X'' \cup \{x''\}).\end{aligned}$$

Definition 5. A profile of choice functions $(C_h)_{h\in H}$ satisfies **Richness** if it satisfies both

Richness 1 and Richness 2.[6]

We are ready to state the main result.

Theorem 1. *For any profile of choice functions that ensures stability, the stable correspondence is Nash implementable if and only if a profile of choice functions satisfies Richness.*

In words, Nash implementation of the stable correspondence is fully characterized by Richness of choice functions of hospitals. The following lemma is useful for the proof of Theorem 1, while it is highly similar to Richness 2.

Lemma 1. *For any profile of choice functions $(C_h)_{h\in H}$, there does not exist doctor $d \in D$, a contract $x' \in X_d$, and a set of contracts $\bar{X} \subseteq X_d$ such that*

$$(1)\quad \exists X' \in \mathcal{X}^H \;\; \textit{with } X'_d = \varnothing_d,\; [\forall x'' \in \bar{X},\; x'' \notin C_H(X' \cup \{x''\}),\; \textit{and} \;\; x' \in C_H(X' \cup \{x'\})]$$

$$(2)\quad \forall X'' \in \mathcal{X}^H \;\; \textit{with } X''_d = x', \quad \exists x'' \in \bar{X},\; x'' \in C_H(X'' \cup \{x''\}).$$

Proof. In Appendix A. □

3.2 Proof of Theorem 1

Yamato (1992) proves the equivalence between essential monotonicity and Nash implementation of a social choice correspondence in a general setting that contains ours. We show that the stable correspondence satisfies essential monotonicity if and only if a profile of choice functions satisfies Richness.

[If part]

Assume that a profile of choice functions $(C_h)_{h\in H}$ satisfies Richness. We show that F^s is essentially monotonic.

[6] Note that Richness 1 and Richness 2 are logically independent, which are shown in the Appendix.

Suppose by contradiction that F^s is not essentially monotonic. Then, there exist $R, R' \in \mathcal{R}$, and $X' \in F^s(R)$ such that

$$\forall d \in D,\ ESS[F^s, d, L(X', R_d)] \subseteq L(X', R'_d) \text{ and } X' \notin F^s(R').$$

This means that X' is not stable at R'. There are three cases to consider.

Case 1: X' is not individually rational for doctors at R'.
Let $d \in D$ be a doctor such that $\varnothing_d\ P'_d\ X'_d$. Let $\bar{X} := \{x \in X_d \mid x\ P_d\ X'_d\}$.

Case 1-1: $\bar{X} = \emptyset$.

Let $\hat{R}$ be a preference profile such that for each $d' \in D$ and each $x \in X_{d'}$, it holds that $\varnothing_{d'}\ \hat{P}_{d'}\ x$. We consider the allocation $\varnothing$. It is easy to see that $\varnothing \in S(\hat{R})$. By construction of $\hat{R}_d$, we have

$$L(\varnothing, \hat{R}_d) = L(X', R_d). \tag{9}$$

Then, $\varnothing$ is essential for d in $L(\varnothing, \hat{R}_d)$ with respect to F^s, i.e.,

$$\varnothing \in ESS[F^s, d, L(\varnothing, \hat{R}_d)],$$

which together with (1) imply that

$$\varnothing \in ESS[F^s, d, L(X', R_d)].$$

By supposition,

$$\varnothing \in L(X', R'_d),$$

and we have $X'_d\ R'_d\ \varnothing_d$, which contradicts the assumption in Case 1.

Case 1-2: $\bar{X} \neq \emptyset$.

By construction of $\bar{X}$ and stability of X' at R, we have $x' \notin C_H(X' \cup \{x'\})$ for all $x' \in \bar{X}$. Then, Richness 1 implies that there exists another allocation $X'' \in \mathcal{X}^H$ with $X''_d = \varnothing_d$ such that $x' \notin C_H(X'' \cup \{x'\})$ for all $x' \in \bar{X}$. We consider a preference profile $\hat{R} \in \mathcal{R}$ such that

$$\begin{cases} \hat{R}_d : \bar{X}_d \ \varnothing_d & \\ \hat{R}_{d'} : X''_{d'} \ \varnothing_{d'} & \text{for each} \quad d' \in \mathrm{D}(X'') \\ \hat{R}_{d''} : \varnothing_{d''} & \text{otherwise.} \end{cases}$$

By construction of $\hat{R}_d$, we have

$$L(X'', \hat{R}_d) = L(X', R_d). \tag{10}$$

We will claim that $X'' \in S(\hat{R})$. First, it is easy to check that X'' is individually rational for both doctors and hospitals at $\hat{R}$. Suppose that there is $(\hat{d}, \hat{h}) \in D \times H$ and $x \in X$ with $\mathrm{D}(x) = \hat{d}$ and $\mathrm{H}(x) = \hat{h}$ such that $x \hat{P}_{\hat{d}} X''_{\hat{d}}$ and $x \in C_{\hat{h}}(X'' \cup \{x\})$. By construction of $\hat{R}$, $\hat{d} = d$ must hold. However, this contradicts to the fact that $x' \notin C_H(X'' \cup \{x'\})$ for all $x' \in \bar{X}$. Therefore, X'' is stable at $\hat{R}$, that is, we have

$$X'' \in F^s(\hat{R}).$$

Then, X'' is essential for d in $L(X'', \hat{R}_d)$ with respect to F^s, i.e.,

$$X'' \in ESS[F^s, d, L(X'', \hat{R}_d)],$$

which together with (2) imply that

$$X'' \in ESS[F^s, d, L(X', R_d)].$$

By supposition,

$$X'' \in L(X', R'_d),$$

and we have $X'_d \; R'_d \; X''_d = \varnothing_d$, which contradicts the assumption in Case 1.

Case 2: X' is not individually rational for hospitals at R'.

Since a profile of choice functions is the same at R and R', the fact that X' is not individually rational for hospitals at R' implies that X' is not individually rational for hospitals at R, a contradiction.

Case 3: There exists a doctor-hospital pair that blocks X' at R'.

Suppose that there is $(d, h) \in D \times H$ and $x' \in X$ with $\mathrm{D}(x') = d$ and $\mathrm{H}(x') = h$ such that $x' P'_d X'_d$ and $x' \in C_h(X' \cup \{x'\})$. Let $\bar{X} := \{x \in X_d \mid x P_d X'_d\}$.

Case 3-1: $\bar{X} = \emptyset$.

Let $X'' := C_h(X' \cup \{x\})$. We consider a preference profile $\hat{R} \in \mathcal{R}$ such that

$$\begin{cases} \hat{R}_{d'} : X''_{d'} \; \varnothing_{d'} & \text{for all } d' \in \mathrm{D}(X'') \\ \hat{R}_{d''} : \varnothing_{d''} & \text{otherwise.} \end{cases}$$

By construction of $\hat{R}_d$, we have

$$L(X'', \hat{R}_d) = L(X', R_d). \tag{11}$$

It is easy to check that X'' is individually rational for doctors and hospitals, and that there is no blocking pair at $\hat{R}$. Therefore, X'' is stable at $\hat{R}$, that is, we have

$$X'' \in F^s(\hat{R}).$$

Then, X'' is essential for d in $L(X'', \hat{R}_d)$ with respect to F^s, i.e.,

$$X'' \in ESS[F^s, d, L(X'', \hat{R}_d)],$$

which together with (3) imply that

$$X'' \in ESS[F^s, d, L(X', R_d)].$$

By supposition,

$$X'' \in L(X', R'_d),$$

and we have $X'_d \; R'_d \; X''_d = x'$, which contradicts the assumption that $x' P'_d X'_d$.

<u>Case 3-2:</u> $\bar{X} \neq \emptyset$.

By construction of $\bar{X}$ and stability of X' at R, we have $x'' \notin C_H(X' \cup \{x''\})$ for all $x'' \in \bar{X}$. Then, regardless of whether $X'_d = \varnothing_d$ or not, Richness 2 together Lemma 1 imply that there exists $X'' \in \mathcal{X}^H$ with $X''_d = x'$ such that $x'' \notin C_H(X'' \cup \{x''\})$ for all $x'' \in \bar{X}$. We consider a preference profile $\hat{R} \in \mathcal{R}$ such that

$$\begin{cases} \hat{R}_d : \bar{X} \; X''_d \; \varnothing_d & \\ \hat{R}_{d'} : X''_{d'} \; \varnothing_{d'} & \text{for each} \quad d' \in \mathrm{D}(X'') \setminus \{d\} \\ \hat{R}_{d''} : \varnothing_{d''} & \text{otherwise.} \end{cases}$$

By construction of $\hat{R}_d$, we have

$$L(X'', \hat{R}_d) = L(X', R_d). \tag{12}$$

It is easy to check that X'' is individually rational for both doctors and hospitals at $\hat{R}$. Suppose that there is $(\hat{d}, \hat{h}) \in D \times H$ and $x \in X$ with $\mathrm{D}(x) = \hat{d}$ and $\mathrm{H}(x) = \hat{h}$ such that

$x\hat{P}_{\hat{d}}X''_{\hat{d}}$ and $x \in C_{\hat{h}}(X'' \cup \{x\})$. By construction of $\hat{R}$, $\hat{d} = d$ must hold. However, this contradicts to the fact that $x'' \notin C_H(X'' \cup \{x''\})$ for all $x'' \in \bar{X}$. Therefore, X'' is stable at $\hat{R}$, that is, we have

$$X'' \in F^s(\hat{R}).$$

Then, X'' is essential for d in $L(X'', \hat{R}_d)$ with respect to F^s, i.e.,

$$X'' \in ESS[F^s, d, L(X'', \hat{R}_d)],$$

which together with (4) imply that

$$X'' \in ESS[F^s, d, L(X', R_d)].$$

By supposition,

$$X'' \in L(X', R'_d),$$

and we have $X'_d \; R'_d \; X''_d = x'$, which contradicts the assumption that $x' P'_d X'_d$.

[Only if part]

We prove the statement by contraposition. Suppose that a profile of choice functions does not satisfy Richness. Then, we have two cases.

Case 1: Richness 1 fails.

Then, there exist $d \in D$, $x \in X_d$, and $\bar{X} \subseteq X_d$ such that

$$(1) \;\; \exists X' \in \mathcal{X}^H \;\text{ with } X'_d = x, \quad \forall x' \in \bar{X}, \; x' \notin C_H(X' \cup \{x'\}), \text{ and}$$
$$(2) \;\; \forall X'' \in \mathcal{X}^H \text{ with } X''_d = \varnothing_d, \; \exists x' \in \bar{X}, \; x' \in C_H(X'' \cup \{x'\}).$$

Note that (1) implies $X'_d = x \notin \bar{X}$ and that (2) implies $\bar{X} \neq \emptyset$. We consider preference profiles R and R' such that

$$\begin{cases} R_d : \bar{X}_d \; X'_d \; \varnothing_d & \\ R_{d'} : X'_{d'} \; \varnothing_{d'} & \text{for each} \quad d' \in \mathrm{D}(X') \setminus \{d\} \\ R_{d''} : \varnothing_{d''} & \text{otherwise} \end{cases}$$

and

$$\begin{cases} R'_d : \bar{X}_d \; \varnothing_d \; X'_d & \\ R'_{d'} : X'_{d'} \; \varnothing_{d'} & \text{for each} \quad d' \in \mathrm{D}(X') \setminus \{d\} \\ R'_{d''} : \varnothing_{d''} & \text{otherwise.} \end{cases}$$

Then, by (1), we have $X' \in F^s(R)$. Moreover, $X' \notin F^s(R')$ since X' is not individually rational for doctor d at R'_d.

To show that F^s is not essentially monotonic, we will check that for each $d' \in D$, $ESS[F^s, d', L(X', R_{d'})] \subseteq L(X', R'_{d'})$ holds. For each $d' \neq d$, since $L(X', R'_{d'}) = \mathcal{X}$, it holds that $ESS[F^s, d', L(X', R_{d'})] \subseteq L(X', R'_{d'})$. Suppose by contradiction that there exists $X'' \in ESS[F^s, d, L(X', R_d)] \setminus L(X', R'_d)$. Then, by the definition of $ESS[F^s, d, L(X', R_d)]$, there exists $R'' \in \mathcal{R}$ such that $X'' \in F^s(R'')$ and $L(X'', R''_d) \subseteq L(X', R_d)$. We note that

$$L(X', R'_d) = \mathcal{X} \setminus \{\hat{X} \in \mathcal{X} : \hat{X}_d = \varnothing_d \text{ or } \hat{X}_d = x' \text{ for some } x' \in \bar{X}\}.$$

So, $X'' \notin L(X', R'_d)$ means $X'' \in \{\hat{X} \in \mathcal{X} \mid \hat{X}_d = \varnothing_d \text{ or } \hat{X}_d = x \text{ for some } x \in \bar{X}\}$. Moreover, since $X'' \in L(X'', R''_d) \subseteq L(X', R_d)$, we have $X''_d = \varnothing_d$. By the assumption of (2), there exists $x' \in \bar{X}$ (which is surely guaranteed to exist because of $X''_d = \varnothing_d$) such that $x' \in C_H(X'' \cup \{x'\})$. If $x' \; P''_d \; X''_d$, then X'' is blocked at R'' by doctor d and hospital $h = \mathrm{H}(x')$, which is a contradiction to $X'' \in F^s(R'')$. Hence, we finally get $X''_d \; P''_d \; x'$.

However, $X''_d P''_d x'$ together with $x' P_d X'_d$ contradict $L(X'', R''_d) \subseteq L(X', R_d)$. Thus, we have that $ESS[F^s, d, L(X', R_d)] \subseteq L(X', R'_d)$. Therefore, if Richness 1 fails to hold, then F^s is not essentially monotonic.

Case 2: Richness 2 fails.

Then, there exist $d \in D$, $x, x' \in X_d$, and $\bar{X} \subseteq X_d$ such that

(1) $\exists X' \in \mathcal{X}^H$ with $X'_d = x$, $[\forall x'' \in \bar{X},\ x'' \notin C_H(X' \cup \{x''\})$, and $x' \in C_H(X' \cup \{x'\})]$

(2) $\forall X'' \in \mathcal{X}^H$ with $X''_d = x'$, $\exists x'' \in \bar{X},\ x'' \in C_H(X'' \cup \{x''\})$.

Note that (1) implies $X'_d = x \notin \bar{X}$ and $x' \notin \bar{X}$ and that (2) implies $\bar{X} \neq \emptyset$. We consider preference profiles R and R' such that

$$\begin{cases} R_d : \bar{X}_d\ X'_d\ x' \varnothing_d & \\ R_{d'} : X'_{d'}\ \varnothing_{d'} & \text{for each} \quad d' \in \mathrm{D}(X') \setminus \{d\} \\ R_{d''} : \varnothing_{d''} & \text{otherwise} \end{cases}$$

and

$$\begin{cases} R'_d : x'\ X'_d\ \varnothing_d & \\ R'_{d'} : X'_{d'}\ \varnothing_{d'} & \text{for each} \quad d' \in \mathrm{D}(X') \setminus \{d\} \\ R'_{d''} : \varnothing_{d''} & \text{otherwise.} \end{cases}$$

Then, by (1), we have $X' \in F^s(R)$. Moreover, $X' \notin F^s(R')$ holds true since, by assumption, we have $x' \in C_h(X' \cup \{x'\})$ for hospital $h = \mathrm{H}(x')$.

To show that F^s is not essentially monotonic, we will check that for each $d' \in D$, $ESS[F^s, d', L(X', R_{d'})] \subseteq L(X', R'_{d'})$ holds. For each $d' \neq d$, since $L(X', R'_{d'}) = \mathcal{X}$, it holds that $ESS[F^s, d', L(X', R_{d'})] \subseteq L(X', R'_{d'})$. Suppose by contradiction that there exists

$X'' \in ESS[F^s, d, L(X', R_d)] \setminus L(X', R'_d)$. Then, by the definition of $ESS[F^s, d, L(X', R_d)]$, there exists $R'' \in \mathcal{R}$ such that $X'' \in F^s(R'')$ and $L(X'', R''_d) \subseteq L(X', R_d)$. We note that

$$L(X', R'_d) = \mathcal{X} \setminus \{\hat{X} \in \mathcal{X} : \hat{X}_d = x'\}.$$

So, $X'' \notin L(X', R'_d)$ means $X'' \in \{\hat{X} \in \mathcal{X} \mid \hat{X}_d = x'\}$, implying that $X''_d = x'$. By the assumption of (2), there exists $x'' \in \bar{X}$ (which is surely guaranteed to exist because of $X''_d = x' \notin \bar{X}$) such that $x'' \in C_H(X'' \cup \{x''\})$. If $x''\ P''_d\ X''_d$, then X'' is blocked at R'' by doctor d and hospital h =H(x''), which is a contradiction to $X'' \in F^s(R'')$. Hence, we finally get $X''_d\ P''_d\ x''$. However, $X''_d P''_d x''$ together with $x'' P_d X'_d$ contradict $L(X'', R''_d) \subseteq L(X', R_d)$. Thus, we have that $ESS[S, d, L(X', R_d)] \subseteq L(X', R'_d)$. Therefore, if Richness 2 fails to hold, then F^s is not essentially monotonic. □

3.3 Matching without contracts

We understand that Nash implementation of the stable correspondence is characterized by Richness of choice functions of hospitals, while Richness does not necessarily hold in matching with contracts. However, Richness does hold in matching without contracts, as shown below, implying that the stable correspondence is in general Nash implementable in matching without contracts.

Proposition 1. *Assume $|C| = 1$. Then, for any profile of choice functions that ensures stability, the stable correspondence is Nash implementable.*

Proof. By Theorem 1, it suffices to show that a profile of choice functions $(C_h)_{h \in H}$ satisfies both Richness 1 and Richness 2 under the assumption of $|C| = 1$.

<u>Richness 1</u>

We will check that if Richness 1 does not hold, then there must exist a doctor and a hospital such that there are at least two distinct contracts associated with them.

Assume that Richness 1 does not hold. Then, there exist $d \in D$, $x \in X_d$, and $\bar{X} \subseteq X_d$ such that

$$\begin{aligned}&(1)\ \ \exists X' \in \mathcal{X}^H \ \text{ with } X'_d = x, \quad \forall x' \in \bar{X},\ x' \notin C_H(X' \cup \{x'\}),\ \text{and}\\ &(2)\ \ \forall X'' \in \mathcal{X}^H \ \text{with } X''_d = \varnothing_d,\ \exists x' \in \bar{X},\ x' \in C_H(X'' \cup \{x'\}).\end{aligned}$$

Note that (1) implies $x \notin \bar{X}$. To conclude the proof, it suffices to show that $\mathrm{H}(x) \in \mathrm{H}(\bar{X})$. Suppose by contradiction that $\mathrm{H}(x) \notin \mathrm{H}(\bar{X})$. We define an allocation $X'' \subseteq X$ such that

$$\begin{aligned}&X''_h = X'_h \ \ \forall h \in \mathrm{H}(\bar{X}),\ \text{and}\\ &X''_h = \emptyset \qquad \forall h \in H \setminus \mathrm{H}(\bar{X}).\end{aligned}$$

It is easy to check that $X'' \in \mathcal{X}^H$ and $X''_d = \varnothing_d$. Then, by (2), there exists $x' \in \bar{X}$ such that $x' \in C_H(X'' \cup \{x'\})$. Let $h' :=\mathrm{H}(x')$. Then, the definition of X'' implies $X''_{h'} = X'_{h'}$. Thus,

$$x' \in C_{h'}(X'' \cup \{x'\}) = C_{h'}(X''_{h'} \cup \{x'\}) = C_{h'}(X'_{h'} \cup \{x'\}) = C_{h'}(X' \cup \{x'\}),$$

where the first and the third equalities hold by IRC. However, $x' \in C_{h'}(X' \cup \{x'\})$ contradicts to (1). Therefore, we have shown that $\mathrm{H}(x) \in \mathrm{H}(\bar{X})$.

Richness 2

We will check that if Richness 2 does not hold, then there must exist a doctor and a hospital such that there are at least two distinct contracts associated with them.

Assume that Richness 2 does not hold. Then, there exist $d \in D$, $x, x' \in X_d$, and

$\bar{X} \subseteq X_d$ such that

$$(1)\ \ \exists X' \in \mathcal{X}^H \ \text{ with } X'_d = x, \ \ [\forall x'' \in \bar{X},\ x'' \notin C_H(X' \cup \{x''\}), \text{ and } \ x' \in C_H(X' \cup \{x'\})]$$

$$(2)\ \ \forall X'' \in \mathcal{X}^H \text{ with } X''_d = x', \ \ \exists x'' \in \bar{X},\ x'' \in C_H(X'' \cup \{x''\}).$$

Note that (1) implies $X'_d = x \notin \bar{X}$ and $x' \notin \bar{X}$. To conclude the proof, it suffices to show that H(x) $\in$H$(\bar{X})$. Suppose by contradiction that H(x) $\notin$H$(\bar{X})$. We define an allocation $X'' \subseteq X$ such that

$$X''_h = X'_h \ \ \forall h \in \text{H}(\bar{X}),$$
$$X''_h = C_h(X' \cup \{x'\}) \quad \text{for } h = \text{H}(x'), \text{ and}$$
$$X''_h = \emptyset \qquad \forall h \in H \setminus \text{H}(\bar{X}) \cup \{\text{H}(x')\}.$$

It is easy to check that $X'' \in \mathcal{X}^H$ and $X''_d = x'$. Then, by (2), there exists $x'' \in \bar{X}$ such that $x'' \in C_H(X'' \cup \{x''\})$. Let $h' :=$H(x''). Then, the definition of X'' implies $X''_{h'} = X'_{h'}$. Thus,

$$x'' \in C_{h'}(X'' \cup \{x''\}) = C_{h'}(X''_{h'} \cup \{x''\}) = C_{h'}(X'_{h'} \cup \{x''\}) = C_{h'}(X' \cup \{x''\}),$$

where the first and the third equalities hold by IRC. However, $x'' \in C_{h'}(X' \cup \{x''\})$ contradicts to (1). Therefore, we have shown that H(x) $\in$H$(\bar{X})$. □

4 Conclusion

This paper has considered doctor-hospital match in the model with contracts. In the real world, a choice function of a hospital may incorporate an affirmative action policy or budget constraint. In such an environment, it is unknown whether stable allocations are achievable in some equilibrium concept. We have proposed Richness of choice functions that characterizes Nash implementation of the stable correspondence. The key feature different from the literature is that the current paper does not specify a class of choice functions but treat all of them that ensures stability in matching with contracts.

There are at least two directions of research. The first one is related with the notion of stability. In this paper, we have introduced a pair-wise version of stability. Since there may be the case where any size of group containing doctors and hospitals can block an allocation. Hence, it seems to be worthwhile to revise the notion of stability in a group-wise sense and to find a condition like Richness defined in the current paper to characterize Nash implementation of group-stable allocations. Second, in general, there are multiple stable allocations. So, it may be reasonable to focus on the Pareto-frontier of stable allocations in the viewpoint of doctors' welfare. To investigate when the correspondence producing the set of Pareto-frontier stable allocations is Nash implementable is another future direction of research.

Appendix A: Omitted proof

Proof of Lemma 1

Suppose that there exist a doctor $d \in D$, a contract $x' \in X_d$, and a set of contracts $\bar{X} \subseteq X_d$ such that

$$\exists X' \in \mathcal{X}^H \text{ with } X'_d = \varnothing_d,\ [\forall x'' \in \bar{X},\ x'' \notin C_H(X' \cup \{x''\}), \text{ and } \ x' \in C_H(X' \cup \{x'\})].$$

We will show that there exists $X'' \in \mathcal{X}^H$ with $X''_d = x'$ such that $x'' \notin C_H(X'' \cup \{x''\})$ for all $x'' \in \bar{X}$.

Let $h' :=$H(x'). We define a set of contracts $X'' \subseteq X$ as follows:

$$X''_h = X'_h \quad \forall h \in H \setminus \{h'\}, \text{ and}$$
$$X''_{h'} = C_{h'}(X' \cup \{x'\}).$$

Then, since $X'_d = \varnothing_d$, X'' is an allocation, that is, there is no doctor who is assigned more than one contract at X''. Also, by construction of X'' and IRC, $C_h(X'') = X''_h$ for all $h \in H$, meaning that $C_H(X'') = X''$. Hence, we have $X'' \in \mathcal{X}^H$. Moreover, it is easy to check that $X''_d = x'$ because of $x' \in C_{h'}(X' \cup \{x'\}) = X''_{h'}$ and $X'_d = \varnothing_d$.

The remaining thing we have to show is that $x'' \notin C_H(X'' \cup \{x''\})$ for all $x'' \in \bar{X}$. For all $h \in H \setminus \{h'\}$,

$$\forall x'' \in \bar{X},\ C_h(X'' \cup \{x''\}) = C_h(X''_h \cup \{x''\}) = C_h(X'_h \cup \{x''\}) = C_h(X' \cup \{x''\}),$$

where the first and the third equalities follow by IRC while the second equation follows by the definition of X''. By the assumption that $x'' \notin C_H(X' \cup \{x''\})$ for all $x'' \in \bar{X}$, we have that for all $h \in H \setminus \{h'\}$ and for all $x'' \in \bar{X}$, $x'' \notin C_h(X'' \cup \{x''\})$.

Lastly, we check that $x'' \notin C_{h'}(X'' \cup \{x''\})$ for all $x'' \in \bar{X}$. Since $x'' \notin C_{h'}(X' \cup \{x''\}) =$

$C_{h'}(X'_{h'} \cup \{x''\}) = X'_{h'}$ for all $x'' \in \bar{X}$, where the first equality holds by IRC and the second equality holds by $X' \in \mathcal{X}^H$, the definition of choice functions implies that

$$\forall x'' \in \bar{X},\ \forall S \subseteq X'_{h'} \cup \{x''\},\ X'_{h'} R_{h'} S. \tag{13}$$

On the other hand, since $x' \in C_{h'}(X' \cup \{x'\}) = C_{h'}(X'_{h'} \cup \{x'\})$, where the equality holds by IRC, the definition of choice functions and that of X'' imply that

$$\forall S \subseteq X'_{h'} \cup \{x'\},\ X''_{h'} = C_{h'}(X'_{h'} \cup \{x'\}) R_{h'} S. \tag{14}$$

Take $S = X'_{h'}$ in (2). Strictness of $R_{h'}$ and $X''_{h'} \neq X'_{h'}$ imply that

$$X''_{h'} P_{h'} X'_{h}. \tag{15}$$

By combining (1) and (3), we have

$$\forall x'' \in \bar{X},\ \forall S \subseteq X'_{h'} \cup \{x''\},\ X''_{h'} P_{h'} S. \tag{16}$$

If $x'' \in C_{h'}(X'' \cup \{x''\})$ would hold for some $x'' \in \bar{X}$, then

$$x'' \in C_{h'}(X'' \cup \{x''\}) = C_{h'}(X''_{h'} \cup \{x''\}) R_{h'} X''_{h'}. \tag{17}$$

Since $x' \in X''_{h'}$, the definition (2) of choice functions implies that

$$C_{h'}(X''_{h'} \cup \{x''\}) \subseteq (X''_{h'} \cup \{x''\}) \setminus \{x'\} \subseteq X'_{h'} \cup \{x''\}.$$

Then, the combination of $C_{h'}(X''_{h'} \cup \{x''\}) \subseteq X'_{h'} \cup \{x''\}$ with (5) contradicts to (4). Therefore, for all $x'' \in \bar{X}$, $x'' \notin C_{h'}(X'' \cup \{x''\})$ must hold.

In conclusion, for all $h \in H$ and for all $x'' \in \bar{X}$, $x'' \in C_h(X'' \cup \{x''\})$ holds, which was what we wanted. □

Appendix B

The following two examples indicate that Richness 1 and 2 are logically independent.

Example 1. Richness 2 but not Richness 1.

$D = \{d\}, H = \{h\}, X = \{x_1, x_1'\}$, where D($x_1$) =D($x_1'$) $= d$ and H(x_1) =H(x_1') $= h$.

$$R_h : x_1, x_1', \emptyset.$$

Then, letting $x = x_1, X' = \{x_1\}, \bar{X} = \{x_1'\}$, Richness 1 is not satisfied. However, it is easy to see that Richness 2 is satisfied.

Example 2. Richness 1 but not Richness 2.

$D = \{d_1, d_2\}, H = \{h_1, h_2\}, X = \{x_1, x_1', x_2, y_1, y_2\}$, where D($x_1$) =D($x_1'$) =D($x_2$) $= d_1$, D(y_1) =D(y_2) $= d_2$, H(x_1) =H(x_1') =H(y_1) $= h_1$, and H(x_2) =H(y_2) $= h_2$.

$$R_{h_1} : x_1, y_1, x_1', \emptyset.$$

$$R_{h_2} : \{x_2, y_2\}, y_2, \emptyset.$$

Then, letting $d = d_1, x = x_1, x' = x_2, X' = \{x_1, y_2\}, \bar{X} = \{x_1'\}$, Richness 2 is not satisfied. However, it is easy to see that Richness 1 is satisfied.

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Chapter 4

Stability in matching markets with quantitative constraints*

1 Introduction

Recent developments of theory of stable matchings have helped a variety of matching markets such as National Resident Matching Program, college admissions, public school choice, teacher assignments and so on. In the process of application, feasibility of a matching arises as an important concern. There are mainly two types of reasons, (1) policy-motivated and (2) technologically restricted.

The policy-motivated reason exogenously restricts possible matchings, which can be seen as a constraint added to an original matching problem. For example, in public school choice, the educational authority should take care of diversity for a matching between schools and students. For another example, as has been already pointed out by Kamada and Kojima (2015), in the National Resident Matching Program, a matching between hospitals and residents may take regional balance into account. Those constraints reflect a way of interpretation of a political goal. Moreover, it just reflects a specific policy.

The technologically restricted reason is much more complicated since it does not necessarily determine feasibility of matchings by an added constraint. This reflects firms' technologies behind or schools' budgets, which are the outside of a matching problem, or not described in a matching problem. For example, some firm can physically hire two employees for some division. However, the firm faces a technological constraint so that one employee does not pay and only two employees make profit. In this case, a constraint

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plays like a floor constraint. But if firms' behaviors behind are more complex, then so is a constraint for possible matching.

We study a many-to-one matching problem with a feasibility constraint including the above all situations. Many matching markets in practice are described by a many-to-one matching problem. In the problem, an agent in the one side, say a doctor, matches at most one agent in the other side, say a hospital, and a hospital is able to match multiple doctors. It is usually assumed that a hospital has an exogenously given capacity or quota which is an upper bound to match. We describe the feasibility constraint as quantitative distributions of matchings. Here we do not assume any more on the feasibility constraint so that we allow any types of quantitative constraints.[1] We know that in some case, quality of an agent also affects on feasibility of a matching, but in this paper we just stick to a quantitative constraint.

The wisdom of matching theory tells us that stability should be a central solution concept. A matching is stable if there is no group of agents who deviates from the matching. Without the feasibility constraint, as the prominent contribution of Gale and Shapley (1962), there exists a stable matching for any preferences. With the feasibility constraint, we should be careful to define stability in two points. One is that even a coalition block in the usual sense is not necessarily feasible. If the feasibility constraint is policy-motivated, then a blocking coalition, for instance, a doctor and a hospital, will still make sense in that they can really match each other outside of the current matching market, even though such a match is not allowed under the feasibility constraint. But if the feasibility constraint is technologically restricted, then a blocking coalition in the usual sense is indeed impossible to implement. This comes from a model formulation. Thus when we consider a blocking coalition, we also consider the rest of agents, which distinguish stability from the notion of core. The other is that the notion of stability

[1] Akin (2020) and Kamada and Kojima (2015, 2017, 2018, 2020) consider specific constraints in each context.

should be independent of a specific feasibility constraint. The notion of stability in prior works on matching with constraints sometimes includes a specific feasibility constraint in its definition.

Hence we refine the notion of stability as follows. We first define a feasible block. A coalition blocks a matching if rematches in the coalition make every agent in the coalition weakly better off and some strictly. A coalition of agents *feasibly blocks* a matching if it is a blocking coalition and a matching created by rematches and the rest of agents who are not involved in members of coalition matches the same agents in the original matching and are involved matches the same agents or nothing. Then we define our stability. A matching is *group stable* if it is feasible and there is no feasible blocking coalition. Remark that without the feasibility constraint, our notion of stability falls in stability in the usual sense. As is noted, there exists a stable matching for any preferences without the feasibility constraints. However, it is not the case when we further take the feasibility constraint into consideration. This non-existence comes not from our formulation of stability, but is essentially by the feasibility constraint.

Our main contribution is on the existence of a stable matching. We propose a necessary condition on the feasibility constraint for the existence of group stable matchings. It is useful because the feasibility constraint can be observable in advance. We say that the feasibility constraint satisfies *monotonicity* if whenever the total numbers of two feasible distributions coincide, there exists a feasible distribution in the set of distributions containing the total number exceeds the two feasible distributions and does not exceed the join of the two. We show that there exists a group stable matching for all preferences only if the feasibility constraint satisfies monotonicity. Since our model subsumes the previous matching models with a sort of feasibility constraints, we obtain the existing results for the existence of stable matchings as corollaries.

2 Model

Let D and H be a finite set of doctors and a finite set of hospitals, respectively. Each doctor $d \in D$ has a complete, transitive, and antisymmetric preference R_d over $H \cup \{\varnothing\}$, where $\varnothing$ denotes an outside option. Let $\mathcal{R}_d$ be the set of all preferences for doctor $d \in D$. Each hospital $h \in H$ has a complete, transitive, and antisymmetric preference R_h over 2^D. Let $\mathcal{R}_h$ be the set of all preferences for hospital $h \in H$. Given $i \in D \cup H$, P_i is a strict part of R_i. We assume that R_h is **responsive** with a capacity $q_h \in \mathbb{N}$:

- for all $D' \subseteq D$ and $d, d' \in D \setminus D'$, $D' \cup \{d\} R_h D' \cup \{d'\}$ if and only if $\{d\} R_h \{d'\}$,
- for all $D' \subseteq D$ and $d \in D \setminus D'$, $D' \cup \{d\} R_h D'$ if and only if $\{d\} R_h \emptyset$, and
- for all $D' \subseteq D$ with $|D'| > q_h$, $\emptyset P_h D'$.

This assumption of responsive preferences allows us to line up individual doctors one by one for each preference of a hospital. Define the set of all preference profiles by $\mathcal{R} \equiv \times_{i \in D \cup H} R_i$ and denote a typical element in $\mathcal{R}$ by R. A **matching market** is a tuple $(D, H, R, (q_h)_{h \in H})$.

Let $f : \mathbb{Z}_+^{|H|} \to \{0, 1\}$ be a **feasibility constraint**. For any $w \in \mathbb{Z}_+^{|H|}$, $f(w) = 1$ means that assigning w_h doctors to hospital $h \in H$ for all $h \in H$ is possible. Conversely, $f(w) = 0$ means that assigning w_h doctors to hospital $h \in H$ for all $h \in H$ is not possible. A **matching** is a function $\mu : D \cup H \to 2^D \cup H \cup \{\varnothing\}$ such that (1) $\mu(d) \in H \cup \{\varnothing\}$ for all $d \in D$, (2) $\mu(h) \in 2^D$ for all $h \in H$, and (3) $\mu(d) = h$ if and only if $d \in \mu(h)$ for all $d \in D$ and all $h \in H$. Given a matching μ, let $D^\mu = \{d \in D \mid \mu(d) \neq \varnothing\}$ and $H^\mu = \{h \in H \mid \mu(h) \neq \emptyset\}$. For a matching μ, define $w(\mu) \equiv (|\mu(h_1)|, \cdots, |\mu(h_{|H|})|)$, and especially $w_h(\mu) = |\mu(h)|$. We say that a matching μ is **feasible** if $f(w(\mu)) = 1$, and **infeasible** otherwise. A matching μ is **individually rational** at $R \in \mathcal{R}$ if (1) $\mu(d) R_d \varnothing$ for all $d \in D$ and (2) $\{d\} P_h \emptyset$ for all $d \in \mu(h)$ (if $\mu(h) \neq \emptyset$) and $|\mu(h)| \leq q_h$ for all $h \in H$. The following claim is easy to check.

Claim 1. *There exists a feasible and individually rational matching for any preference profile if and only if* $f(0, \cdots, 0) = 1$.

2.1 Group stability

We define a notion of group stability. As noted in Introduction, the key element of deviations is that the rest of the agents outside the coalition must be matched with the same agents as well as any size of blocking coalitions are possible.

Definition 1. A coalition $S \subseteq D \cup H$ **blocks** a feasible matching μ at $R \in \mathcal{R}$ if there exists another feasible matching ν such that

1. for all $d \in S \cap D, \nu(d) \in (S \cap H) \cup \{\varnothing\}$ and $\mu'(d) R_d \mu(d)$,
2. for all $h \in S \cap H, \nu(h) \subseteq S \cap D$ and $\mu'(h) R_h \mu(h)$,
3. for some $i \in S$, $\nu(i) P_i \mu(i)$,
4. for all $d \in D \setminus S$ such that $d \notin \mu(h)$ for all $h \in S \cap H$, $\nu(d) = \mu(d)$,
5. for all $d \in D \setminus S$ such that $d \in \mu(h)$ for some $h \in S \cap H$, $\nu(d) = \varnothing$,
6. for all $h \in H \setminus S$ such that $h \neq \mu(d)$ for all $d \in S \cap D$, $\nu(h) = \mu(h)$, and
7. for all $h \in H \setminus S$ such that $h = \mu(d)$ for some $d \in S \cap D$, $\nu(h) = \mu(h) \setminus \{d \in S \cap D \mid h = \mu(d)\}$.

A matching μ is **group stable** at $R \in \mathcal{R}$ if it is feasible, individually rational, and there does not exist a coalition $S \subseteq D \cup H$ that blocks μ at $R \in \mathcal{R}$.

Remark 1. A group stable matching might not exist for some preference profile.

Remark 2. A group stable matching does not necessarily exclude so-called "justified envy".

3 Results

In this section, we investigate conditions on the feasibility constraint that should meet for a group stable matching to exist for any preference profile.

Given two different vectors $w', w'' \in \mathbb{Z}_+^{|H|}$ with $\sum_{h \in H} w'_h = \sum_{h \in H} w''_h$, define the set of vectors $W(w', w'') \subseteq \mathbb{Z}_+^{|H|}$ such that $w \in W(w', w'')$ if and only if w satisfies the two conditions:

- $\max\{w'_h, w''_h\} \geq w_h$ for all $h \in H$, and
- $\sum_{h \in H} w_h > \sum_{h \in H} w'_h = \sum_{h \in H} w''_h$.

Definition 2. A feasibility constraint f satisfies **monotonicity** if for any two different vectors $w', w'' \in \mathbb{Z}_+^{|H|}$ with $\sum_{h \in H} w'_h = \sum_{h \in H} w''_h$,

$$f(w') = f(w'') = 1 \implies \exists w \in W(w', w''), \quad f(w) = 1.$$

That is, monotonicity of the feasibility constraint requires that when two distributions are feasible, there is also a feasible distribution "close to" the two of them. Given two distributions w', w'' with $\sum_{h \in H} w'_h = \sum_{h \in H} w''_h$, the possible range, $W(w', w'')$, is bounded both above and below. One is that for any hospital $h \in H$, the number of matched doctors for h must be less than or equal to $\max\{w'_h, w''_h\}$. On the other hand, the total number of matched doctors must be greater than $\sum_{h \in H} w'_h = \sum_{h \in H} w''_h$.

Example 3. Let $H = \{h_1, h_2, h_3\}$. Also, let $w' = (3, 0, 2)$ and $w'' = (0, 3, 2)$. Assume that $f(w') = f(w'') = 1$. Consider a vector $w = (2, 3, 1)$ to check that $w \in W(w', w'')$.

First, for $h_1 \in H$, $w_{h_1} = 2 \leq 3 = \max\{w'_{h_1}, w''_{h_1}\}$. For $h_2 \in H$, $w_{h_2} = 3 \leq 3 = \max\{w'_{h_2}, w''_{h_2}\}$. For $h_3 \in H, w_{h_3} = 1 \leq 2 = \max\{w'_{h_3}, w''_{h_3}\}$ holds. Second, we can see that $\sum_{h \in H} w_h = 6 > 5 = \sum_{h \in H} w'_h = \sum_{h \in H} w''_h$ holds. Therefore, $w \in W(w', w'')$. By checking all other vectors, we get

$$W(w', w'') = \{(3,3,0), (2,3,1), (3,2,1), (3,3,1), (3,1,2),$$
$$(1,3,2), (2,2,2), (3,2,2), (2,3,2), (3,3,2)\}.$$

□

Theorem 1. *A group stable matching exists for any preference profile only if a feasibility constraint f satisfies $f(0,\cdots,0) = 1$ and monotonicity.*

4 Proof of Theorem 1

If $f(0,\cdots,0) = 0$, then, by Claim 1, there does not necessarily exist a feasible and individually rational matching for some preference profile, which automatically implies that there does not exist a group stable matching for such a preference profile.

Assume $f(0,\cdots,0) = 1$ and suppose that f does not satisfy monotonicity. Then, there are two vectors $w', w'' \in \mathbb{Z}_+^{|H|}$ with $\sum_{h\in H} w'_h = \sum_{h\in H} w''_h$ such that $f(w') = f(w'') = 1$ and $f(w) = 0$ for all $w \in W(w', w'')$. Define the sets of hospitals $H(w', w'') = \{h \in H \mid w'_h = w''_h\}$, $H'(w', w'') = \{h \in H \mid w'_h > w''_h\}$, $H''(w', w'') = \{h \in H \mid w'_h < w''_h\}$, and $\bar{H}(w', w'') = H'(w', w'') \cup H''(w', w'')$. Let $L = |H(w', w'')|$, $M = |H'(w', w'')|$, and $N = |H''(w', w'')|$. Ravel hospitals in $H(w', w'')$ by $h_1, \cdots, h_L$, hospitals in $H'(w', w'')$ by $h'_1, \cdots, h'_M$, and hospitals in $H''(w', w'')$ by $h''_1, \cdots, h''_N$. Define $k_h \in \mathbb{Z}_+$ for hospital $h \in H$ by $k_h = |w'_h - w''_h|$, and $k \in \mathbb{Z}_+$ by $k = \sum_{h\in H'(w',w'')} |w'_h - w''_h|$.[2] Construct a set of k-doctors $\bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}\}$ and a function $\tau : \bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}\} \to H''(w', w'')$ such that for all $n = 1, \cdots, N$, $|\{d_{m,i} \mid \tau(d_{m,i}) = h''_n\}| = k_{h''_n}$. Also, construct a set of k-doctors $\bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{\bar{d}_{m,i}\}$ and a function $\bar{\tau} : \bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{\bar{d}_{m,i}\} \to H''(w', w'')$ such that for all

[2]Note that $k = \sum_{h\in H'(w',w'')} |w'_h - w''_h| = \sum_{h\in H''(w',w'')} |w'_h - w''_h|$ holds because of $\sum_{h\in H} w'_h = \sum_{h\in H} w''_h$.

$n = 1, \cdots, N$, $|\{\bar{d}_{m,i} \mid \bar{\tau}(\bar{d}_{m,i}) = h''_n\}| = k_{h''_n}$ and $\tau(d_{m,i}) = \bar{\tau}(\bar{d}_{m,i})$ for all $i = 1, \cdots, k_{h'_m}$ and for all $m = 1, \cdots, M$. The existence of these $2k$-doctors and the two functions, τ and $\bar{\tau}$, is guaranteed because of $k = \sum_{h \in H'(w',w'')} |w'_h - w''_h| = \sum_{h \in H''(w',w'')} |w''_h - w'_h|$. Given $h''_n \in H''(w', w'')$, let $\tau^{-1}(h''_n) = \{d_{m,i} \mid \tau(d_{m,i}) = h''_n\}$ and $\bar{\tau}^{-1}(h''_n) = \{\bar{d}_{m,i} \mid \bar{\tau}(\bar{d}_{m,i}) = h''_n\}$. Let $w_h = \min\{w'_h, w''_h\}$ for hospital $h \in H$.

Consider a matching market where a set of doctors D is

$$D = \left(\bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}, \bar{d}_{m,i}\} \right) \bigcup \left(\bigcup_{h \in H} \{d_1^h, \cdots, d_{w_h}^h\} \right)$$

such that $d_j^h \neq d_k^{h'}$ if $j \neq k$ or $h \neq h'$. The preference profile R and a capacity profile $(q_h)_{h \in H}$ are as follows:

$R_{d_{m,i}}$	$R_{\bar{d}_{m,i}}$	$R_{d_j^h}$	$R_{h'_m}$ $(q_{h'_m} = w'_{h'_m})$	$R_{h''_n}$ $(q_{h''_n} = w''_{h''_n})$	R_{h_ℓ} $(q_{h_\ell} = w_{h_\ell})$
h_1	h_1	h	$d_1^{h'_m}$	$d_1^{h''_n}$	$d_1^{h_\ell}$
$\vdots$	$\vdots$	$\varnothing$	$\vdots$	$\vdots$	$\vdots$
h_L	h_L		$d_{w_{h'_m}}^{h'_m}$	$d_{w_{h''_n}}^{h''_n}$	$d_{w_{h_\ell}}^{h_\ell}$
h'_m	$\bar{\tau}(\bar{d}_{m,i})$		$\left\langle \bigcup_{i=1}^{k_{h'_m}} \{\bar{d}_{m,i}\} \right\rangle$	$\langle \tau^{-1}(h''_n) \rangle$	$\left\langle \bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}, \bar{d}_{m,i}\} \right\rangle$
$\tau(d_{m,i})$	h'_m		$\left\langle \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}\} \right\rangle$	$\langle \bar{\tau}^{-1}(h''_n) \rangle$	$\emptyset$
$\varnothing$	$\varnothing$		$\emptyset$	$\emptyset$	

where $i = 1, \cdots, k_{h'_m}$, $j = 1, \cdots, w_h$, $\ell = 1, \cdots, L$, $m = 1, \cdots, M$, and $n = 1, \cdots, N$. Here, an order of doctors in angle brackets is fine for anything. Since $f(0, \cdots, 0) = 1$, there exists at least one feasible and individually rational matching μ at $R \in \mathcal{R}$.

Take any feasible and individually rational matching μ at R. If $w(\mu) \in W(w', w'')$ holds, then by the supposition of necessity, $f(w) = 0$ for all $w \in W(w', w'')$, which in turn implies that μ is not feasible, a contradiction to the fact that μ is feasible. Hence, consider the case where $w(\mu) \notin W(w', w'')$. If $w_h(\mu) > \max\{w'_h, w''_h\}$ holds for some $h \in H$, then $\max\{w'_h, w''_h\} = q_h$ in this matching market implies $w_h(\mu) > q_h$, a contradiction to individual rationality of μ. So, assume that $w_h(\mu) \leq \max\{w'_h, w''_h\}$ for all $h \in H$ and suppose that $\sum_{h\in H} w_h(\mu) \leq \sum_{h\in H} w'_h = \sum_{h\in H} w''_h$ holds.

Assume that $\sum_{h\in H'(w',w'')}(w_h(\mu) - w''_h) \geq \sum_{h\in H''(w',w'')}(w_h(\mu) - w'_h)$, without loss of generality. Let $p = \sum_{h\in H''(w',w'')}(w_h(\mu) - w'_h)$. Also, let $r = \sum_{h\in H'(w',w'')}(w'_h - w_h(\mu))$ and $u = \sum_{h\in H(w',w'')}(w'_h - w_h(\mu))$. By using the assumption of $\sum_{h\in H} w_h(\mu) \leq \sum_{h\in H} w'_h$, we can calculate

$$
\begin{aligned}
& \sum_{h\in H} w_h(\mu) \leq \sum_{h\in H} w'_h \\
\Leftrightarrow\ & \sum_{h\in H'(w',w'')} w_h(\mu) + \sum_{h\in H''(w',w'')} w_h(\mu) + \sum_{h\in H(w',w'')} w_h(\mu) \\
& \qquad\qquad \leq \sum_{h\in H'(w',w'')} w'_h + \sum_{h\in H''(w',w'')} w'_h + \sum_{h\in H(w',w'')} w'_h \\
\Leftrightarrow\ & \sum_{h\in H''(w',w'')} (w_h(\mu) - w'_h) \leq \sum_{h\in H'(w',w'')} (w'_h - w_h(\mu)) + \sum_{h\in H(w',w'')} (w'_h - w_h(\mu)) \\
\Leftrightarrow\ & p \leq r + u.
\end{aligned}
$$

We will show throughout two parts that the feasible and individually rational matching μ induces a blocking coalition at R.

[First part]

This section will show that in order for μ to be group stable, (1) $d \notin \mu(h)$ must hold for all $d \in \bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}, \bar{d}_{m,i}\}$ and for all $h \in H(w', w'')$, and (2) $d^h_j \in \mu(h)$ must hold for all $h \in \bar{H}(w', w'')$ and for all $j = 1, \cdots, w_h$.

★ <u>Step 1: no doctor $d \in \bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}, \bar{d}_{m,i}\}$ is matched with $h \in H(w', w'')$ at μ.</u>

Suppose that some doctor $d \in \bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}, \bar{d}_{m,i}\}$ is matched with some $h \in H(w', w'')$ at μ. Then, since $q_h = w_h$, there is at least one d^h_j such that $\mu(d^h_j) = \varnothing$.

Consider a set of doctors and hospitals $S = (D^\mu \cup H^\mu) \setminus \{d\}$, and a matching ν such that

$$\nu(d_j^h) = h$$
$$\nu(d) = \varnothing$$
$$\nu(h) = (\mu(h) \setminus \{d\}) \cup \{d_j^h\}$$
$$\nu(i) = \mu(i), \quad \forall i \in (D \cup H) \setminus \{d_j^h, h, d\},$$

that is, the matching ν simply swaps the matches of d_j^h and d. Note that ν is feasible since $w(\nu) = w(\mu)$ and $f(w(\mu)) = 1$. Then, it is easy to check that S blocks μ via ν at R.

★ Step 2: $\mu(d_j^h) = h$ for all $h \in \bar{H}(w', w'')$ and for all $j = 1, \cdots, w_h$.

Assume Step 1. Suppose that $\mu(d_j^h) \neq h$ for some $h \in \bar{H}(w', w'')$ and for some $j = 1, \cdots, w_h$. Let $\hat{H} \subseteq \bar{H}(w', w'')$ be a set of hospitals such that $\mu(d_j^h) \neq h$ for some $j = 1, \cdots, w_h$. Then, for any hospital $h \in \hat{H}$, we have either $d \in \mu(h)$ for some $d \in D \setminus \cup_{j=1}^{w_h} \{d_j^h\}$ or $d \notin \mu(h)$ for all $d \in D \setminus \cup_{j=1}^{w_h} \{d_j^h\}$.

Case 1 $d \in \mu(h)$ for some $h \in \hat{H}$ and for some $d \in D \setminus \cup_{j=1}^{w_h} \{d_j^h\}$.

Consider a set of doctors and hospitals $S = (D^\mu \cup H^\mu) \setminus \{d\}$, and a matching ν such that

$$\nu(d) = \varnothing$$
$$\nu(d_j^h) = h$$
$$\nu(h) = (\mu(h) \setminus \{d\}) \cup \{d_j^h\}$$
$$\nu(i) = \mu(i), \quad \forall i \in (D \cup H) \setminus \{d_j^h, h, d\},$$

that is, the matching ν simply swaps the matches of d_j^h and d. Note that ν is feasible since $w(\nu) = w(\mu)$ and $f(w(\mu)) = 1$. Then, it is easy to check that S blocks μ via ν at R.

Case 2 $d \notin \mu(h)$ for all $h \in \hat{H}$ and for all $d \in D \setminus \cup_{j=1}^{w_h}\{d_j^h\}$.

Consider a set of doctors and hospitals $S = (D \cup H) \setminus (\bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}\})$, and a matching ν such that

$$\begin{aligned}
&\nu(d_{m,i}) = \varnothing && \forall i = 1, \cdots, k_{h'_m},\ \forall m = 1, \cdots, M \\
&\nu(\bar{d}_{m,i}) = h'_m && \forall i = 1, \cdots, k_{h'_m}\ \forall m = 1, \cdots, M \\
&\nu(d_j^h) = h && \forall h \in H,\ \forall j = 1, \cdots, w_h \\
&\nu(h'_m) = (\cup_{i=1}^{k_{h'_m}}\{\bar{d}_{m,i}\}) \cup (\cup_{j=1}^{w_{h'_m}}\{d_j^{h'_m}\}) && \forall m = 1, \cdots, M \\
&\nu(h''_n) = \cup_{j=1}^{w_{h''_n}}\{d_j^{h''_n}\} && \forall n = 1, \cdots, N \\
&\nu(h_\ell) = \cup_{j=1}^{w_{h_\ell}}\{d_j^{h_\ell}\} && \forall \ell = 1, \cdots, L
\end{aligned}$$

Note that ν is feasible since $w(\nu) = w'$. It is easy to check that S blocks μ at R.

[Second part]

So far, we have shown that in order for μ to be group stable, (1) $d \notin \mu(h)$ must hold for all $d \in \bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}, \bar{d}_{m,i}\}$ and for all $h \in H(w', w'')$, and (2) $d_j^h \in \mu(h)$ must hold for all $h \in \bar{H}(w', w'')$ and for all $j = 1, \cdots, w_h$. This part shows that such a μ is blocked by a coalition at R.

Case 1 $\mu(d) = \varnothing$ for all $d \in \bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}, \bar{d}_{m,i}\}$.

In this case, consider a set of doctors and hospitals $S = (D \cup H) \setminus (\bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{\bar{d}_{m,i}\})$, and a matching ν such that

$$\begin{aligned}
&\nu(d_{m,i}) = h'_m \quad \forall i = 1, \cdots, k_{h'_m},\ \forall m = 1, \cdots, M \\
&\nu(\bar{d}_{m,i}) = \varnothing \quad \forall i = 1, \cdots, k_{h'_m},\ \forall m = 1, \cdots, M \\
&\nu(d^h_j) = h \qquad \forall h \in H,\ \forall j = 1, \cdots, w_h \\
&\nu(h'_m) = (\cup_{i=1}^{k_{h'_m}} \{d_{m,i}\}) \cup (\cup_{j=1}^{w_{h'_m}} \{d^{h'_m}_j\}) \qquad \forall m = 1, \cdots, M \\
&\nu(h''_n) = \cup_{j=1}^{w_{h''_n}} \{d^{h''_n}_j\} \qquad \forall n = 1, \cdots, N \\
&\nu(h_\ell) = \cup_{j=1}^{w_{h_\ell}} \{d^{h_\ell}_j\} \qquad \forall \ell = 1, \cdots, L
\end{aligned}$$

Note that ν is feasible since $w(\nu) = w'$. Then, it is easy to check that S blocks μ via ν at R. So, μ is not group stable at R.

Case 2 $\mu(d) \neq \varnothing$ for some $d \in \bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}, \bar{d}_{m,i}\}$.

Case 2-1: either $w(\mu) \leq w'$ or $w(\mu) \leq w''$.

Assume, without loss of generality, that $w(\mu) \leq w'$.

Case 2-1-1: $d \notin \mu(h'_m)$ for all $d \in \cup_{i=1}^{k_{h'_m}} \{d_{m,i}\}$ for all $m = 1, \cdots, M$.

Then, consider a set of doctors and hospitals $S = (D \cup H) \setminus (\bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}\})$, and a matching ν such that

$$\begin{aligned}
&\nu(d_{m,i}) = \varnothing && \forall i = 1, \cdots, k_{h'_m},\ \forall m = 1, \cdots, M \\
&\nu(\bar{d}_{m,i}) = \bar{\tau}(\bar{d}_{m,i}) && \forall i = 1, \cdots, k_{h'_m},\ \forall m = 1, \cdots, M \\
&\nu(d_j^h) = h && \forall h \in H,\ \forall j = 1, \cdots, w_h \\
&\nu(h'_m) = \cup_{j=1}^{w_{h'_m}} \{d_j^{h'_m}\} && \forall m = 1, \cdots, M \\
&\nu(h''_n) = \bar{\tau}^{-1}(h''_n) \cup (\cup_{j=1}^{w_{h''_n}} \{d_j^{h''_n}\}) && \forall n = 1, \cdots, N \\
&\nu(h_\ell) = \cup_{j=1}^{w_{h_\ell}} \{d_j^{h_\ell}\} && \forall \ell = 1, \cdots, L
\end{aligned}$$

Note that ν is feasible since $w(\nu) = w''$. Then, it is easy to check that S blocks μ via ν at R.

Case 2-1-2: $d \in \mu(h'_m)$ for some $d \in \cup_{i=1}^{k_{h'_m}} \{d_{m,i}\}$ for some $m = 1, \cdots, M$.

Then, consider a set of doctors and hospitals $S = (D \cup H) \setminus (\bigcup_{m=1}^{M} \bigcup_{i=1}^{k_{h'_m}} \{d_{m,i}\})$, and a matching ν such that

$$\begin{aligned}
&\nu(d_{m,i}) = \varnothing && \forall i = 1, \cdots, k_{h'_m},\ \forall m = 1, \cdots, M \\
&\nu(\bar{d}_{m,i}) = h'_m && \forall i = 1, \cdots, k_{h'_m},\ \forall m = 1, \cdots, M \\
&\nu(d_j^h) = h && \forall h \in H,\ \forall j = 1, \cdots, w_h \\
&\nu(h'_m) = (\cup_{i=1}^{k_{h'_m}} \{\bar{d}_{m,i}\}) \cup (\cup_{j=1}^{w_{h'_m}} \{d_j^{h'_m}\}) && \forall m = 1, \cdots, M \\
&\nu(h''_n) = \cup_{j=1}^{w_{h''_n}} \{d_j^{h''_n}\} && \forall n = 1, \cdots, N \\
&\nu(h_\ell) = \cup_{j=1}^{w_{h_\ell}} \{d_j^{h_\ell}\} && \forall \ell = 1, \cdots, L
\end{aligned}$$

Note that ν is feasible since $w(\nu) = w'$. Then, it is easy to check that S blocks μ via ν at R.

Case 2-2: neither $w(\mu) \leq w'$ nor $w(\mu) \leq w''$.

Then, there exists at least one hospital $h \in H''(w', w'')$ such that $w_h(\mu) > w'_h$. Step 2 implies that for each $h \in H''(w', w'')$ with $w_h(\mu) > w'_h$, there exists a doctor $d \in D$ with $d \in \tau^{-1}(h) \cup \bar{\tau}^{-1}(h)$.

Case 2-2-1: $d \in \mu(h)$ for some $h \in H''(w', w'')$ with $w_h(\mu) > w'_h$ and for some $d \in \bar{\tau}^{-1}(h)$.

Then, since $w_h(\mu) \leq q_h (= w''_h = \max\{w'_h, w''_h\})$ and the construction of the matching market, there exists a doctor $d' \in \tau^{-1}(h)$ such that $\mu(d') = \varnothing$.

Consider a set of doctors and hospitals $S = (D^\mu \cup H^\mu) \setminus \{d\}$, and a matching ν such that

$$
\begin{aligned}
&\nu(d) = \varnothing \\
&\nu(d') = h \\
&\nu(h) = (\mu(h) \setminus \{d\}) \cup \{d'\} \\
&\nu(i) = \mu(i), \quad \forall i \in (D \cup H) \setminus \{d', h, d\},
\end{aligned}
$$

that is, the matching ν simply swaps the matches of d' and d. Note that ν is feasible since $w(\nu) = w(\mu)$ and $f(w(\mu)) = 1$. Then, it is easy to check that S blocks μ via ν at R.

Case 2-2-2: $d \notin \mu(h)$ for all $h \in H''(w', w'')$ with $w_h(\mu) > w'_h$ and for all $d \in \bar{\tau}^{-1}(h)$.

Then, for all $h \in H''(w', w'')$ with $w_h(\mu) > w'_h$, $\mu(h) \subseteq \tau^{-1}(h) \cup (\cup_{j=1}^{w_h}\{d_j^h\})$. Let $\bar{D}$ be a set of doctors such that for any $d \in \bar{D}$, there exists a hospital $h \in H''(w', w'')$ such that $d \in \tau^{-1}(h)$ and $\mu(d) = h$. Note that $|\bar{D}| = p (> 0)$. Also, let $\bar{D}_1$ and $\bar{D}_2$ be two subsets of $\bar{D}$ such that (1) $\bar{D}_1 \cap \bar{D}_2 = \emptyset$, (2) $\bar{D}_1 \cup \bar{D}_2 = \bar{D}$, and (3) $|\bar{D}_1| \leq r$ and $|\bar{D}_2| \leq u$. This devision is possible since $|\bar{D}| = p$ and $p \leq r + u$. Note that when $r = 0$, $\bar{D}_1 = \emptyset$. The assumption of Case 2-2 and the construction of the matching market imply that $\bar{D}_1 \cup \bar{D}_2 = \bar{D} \subseteq \cup_{m=1}^{M} \cup_{i=1}^{k_{h'_m}} \{d_{m,i}\}$. We will reach a conclusion by identifying τ and

$\bar{\tau}$ more specifically.

When $\bar{D}_1 \neq \emptyset$, construct a function $s_1 : \bar{D}_1 \to H'(w', w'')$ such that for all $h \in H'(w', w'')$, $|\{d \in \bar{D}_1 \mid s_1(d) = h\}| + w_h(\mu) \leq q_h(= w'_h)$. Also, construct a function $s_2 : \bar{D}_2 \to H(w', w'')$ such that for all $h \in H(w', w'')$, $|\{d \in \bar{D}_2 \mid s_2(d) = h\}| + w_h(\mu) \leq q_h(= w'_h = w''_h)$. The existence of the two functions, s_1 and s_2, is guaranteed since $|\bar{D}_1| \leq r$, $|\bar{D}_2| \leq u$, $r = \sum_{h \in H'(w',w'')}(w'_h - w_h(\mu)) = \sum_{h \in H'(w',w'')}(q_h - w_h(\mu))$, and $u = \sum_{h \in H(w',w'')}(w'_h - w_h(\mu)) = \sum_{h \in H(w',w'')}(q_h - w_h(\mu))$. When $\bar{D}_1 \neq \emptyset$, let $s_1^{-1} : H'(w', w'') \to \bar{D}_1$ be a function such that for all $h \in H'(w', w'')$, $s_1^{-1}(h) = \{d \in \bar{D}_1 \mid s_1(d) = h\}$. Also, let $s_2^{-1} : H(w', w'') \to \bar{D}_2$ be a function such that for all $h \in H(w', w'')$, $s_2^{-1}(h) = \{d \in \bar{D}_2 \mid s_2(d) = h\}$.

Given the two functions, s_1 and s_2, reconstruct the function $\tau : \cup_{m=1}^{M} \cup_{i=1}^{k_{h'_m}} \{d_{m,i}\} \to H''(w', w'')$ such that for all $h'_m \in H'(w', w'')$ and for all $d \in \bar{D}_1$ with $s_1(d) = h'_m$, $\tau(d) = \mu(d)$. So, we just reconstruct the image of $\bar{D}_1$ at τ, and an image of any other doctor $d \in D \setminus \bar{D}_1$ at τ is arbitrary.

For $h \in H'(w', w'')$, let $\tilde{s}_h = q_h - |s_1^{-1}(h)|$, and arbitrarily take $\tilde{s}_h$-doctors, each of whom is not matched with hospital h at μ. Denote a set of such doctors by $\tilde{D}_h$ for $h \in H'(w', w'')$. For $h \in H(w', w'')$, let $\tilde{s}_h = q_h - |s_2^{-1}(h)|$, and arbitrarily take $\tilde{s}_h$-doctors, each of whom is not matched with hospital h at μ. Denote a set of such doctors by $\tilde{D}_h$ for $h \in H(w', w'')$. Let $\tilde{D} = \cup_{h \in H'(w',w'') \cup H(w',w'')} \tilde{D}_h$.

Consider a set of doctors and hospitals $S = (D^{\mu} \cup \tilde{D}) \cup H$, and a matching ν such that

$$
\begin{aligned}
&\nu(d) = s_1(d) && \forall d \in \bar{D}_1 \\
&\nu(d) = s_2(d) && \forall d \in \bar{D}_2 \\
&\nu(d) = \mu(d) && \forall d \in D^{\mu} \setminus \bar{D} \\
&\nu(d) = h && \forall d \in \tilde{D}_h, \quad \forall h \in H'(w', w'') \cup H(w', w'') \\
&\nu(d) = \varnothing && \forall d \in D \setminus (D^{\mu} \cup \tilde{D}) \\
&\nu(h'_m) = \mu(h'_m) \cup s_1^{-1}(h'_m) \cup \tilde{D}_{h'_m} && \forall m = 1, \cdots, M \\
&\nu(h''_n) = \cup_{j=1}^{w_{h''_n}} \{d_j^{h''_n}\} && \forall n = 1, \cdots, N \\
&\nu(h_\ell) = \mu(h_\ell) \cup s_2^{-1}(h_\ell) \cup \tilde{D}_{h_\ell} && \forall \ell = 1, \cdots, L
\end{aligned}
$$

Note that ν is feasible since $w(\nu) = w'$. Then, it is easy to check that S blocks μ via ν at R.

In conclusion, since μ is arbitrary other than being feasible and individually rational, there does not exist a group stable matching at R. □

5 Conclusion and discussion

Many matching markets are subject to complex constraints. In this paper, we have focused on a general constraint on quantitative distributions brought from policy-motivated reason or technologically restricted reason. We have then defined a new solution concept, group stability. Our main result is concerned with the existence of group stable matchings. We have shown that a group stable matching is guaranteed for any preference profile only if the feasibility constraint satisfies monotonicity.

The literature on matching markets with feasibility constraints has introduced a wide variety of stability notions. In particular, Kamada and Kojima (2017) propose a *strongly stable* matching in a class of feasibility constraints. They assume that feasibility constraints are monotonic in their sense, i.e., if a distribution is feasible, then any distribution less than the original one must be also feasible. In such a class of feasibility constraints, they propose a necessary and sufficient condition, called *independence across hospitals*, on feasibility constraints such that a strongly stable matching exists for any preference profile. It is easy to check that in the environment of Kamada and Kojima (2017), our monotonicity of feasibility constraints implies independence across hospitals as well as independence across hospitals implies the existence of group stable matchings. Hence, the following four statements are equivalent: a group stable matching exists for any preference profile; a strongly stable matching exists for any preference profile; a feasibility constraint satisfies monotonicity in our sense; a feasibility constraint satisfies independence across hospitals.

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