研 究 速 報
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A Note on Stochastic Finite Element Method (Part 9)
——Development of Successive Perturbation Method and its Application to Advanced First-Order Second-Moment Reliability——
確率有限要素法に関するノート(第9報)
——逐次摂動法の開発とその1次近似2次モーメント法への応用——

Toshiaki HISADA*, Shigeru NAKAGIRI* and Masahiro MASHIMO*
久 田 俊 明・中 桐 滋・真 下 雅 浩

1. Introduction

The stochastic FEM has been developed by the authors exhibiting the ability to cope with various uncertainties present in actual structural systems as reported in the past notes (Part 1 to 8). The expectation and central moments such as variance of probabilistic structural behavior have been successfully evaluated in these studies with the aid of the first or second order perturbation method. The perturbation technique, therefore, has been made use of at the mean values of stochastic parameters involved in the systems. It should be borne in mind, however, that the range of the parameters in which structural safety and reliability are concerned is located far from the mean value point so that the mean-centered perturbation technique does not always give good approximation.

Suppose a static SDOF system of Eq. (1) for the convenience of discussion.

$$k\cdot u=k^0(1+\alpha)\cdot u=f \quad\cdots\cdots(1)$$

where k^0 is the expected stiffness and α is a non-dimensional uncertain parameter with zero expectation. The mean-centered perturbation solutions for this problem are inaccurate for the wide range of α as depicted by broken lines in Fig.1. In order to improve the accuracy, we now introduce the "successive perturbation method" as described below. This notion was suggested by Der Kiureghian[1] and emphasis must be put on the fact that the advantage of stochastic FEM is still preserved, that is to say, the governing equation need not be solved repeatedly. Displacement

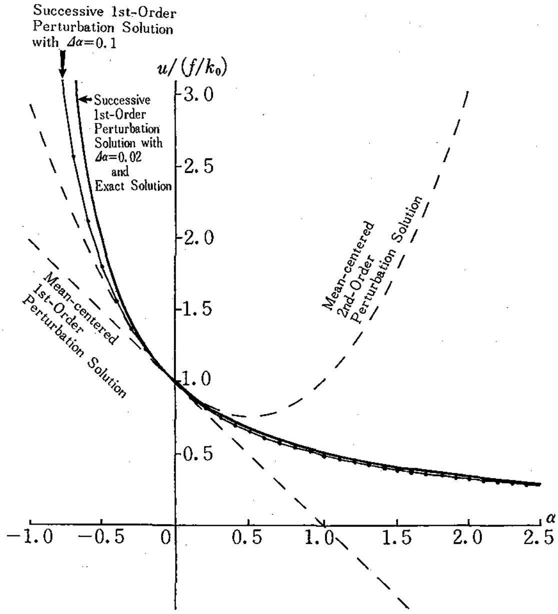

Fig. 1 Successive Perturbation Solutions Compared with Mean-Centered Perturbation Solutions.

u at $\alpha+\Delta\alpha$ may be written as

$$u_{\alpha+\Delta\alpha}=k^{-1}_{\alpha+\Delta\alpha}\cdot f \quad\cdots\cdots(2)$$

where the inverse of stiffness at $\alpha+\Delta\alpha$ is approximated by

$$k^{-1}_{\alpha+\Delta\alpha}=k^{-1}_{\alpha}+\left.\frac{dk^{-1}}{d\alpha}\right|_{\alpha}\cdot\Delta\alpha =k^{-1}_{\alpha}-k^{-1}_{\alpha}\left.\frac{dk}{d\alpha}\right|_{\alpha}k^{-1}_{\alpha}\cdot\Delta\alpha \quad\cdots\cdots(3)$$

The relation $dk^{-1}/d\alpha=-k^{-1}\cdot dk/d\alpha\cdot k^{-1}$ is obtained by the differentiation of $kk^{-1}=1$ with α. Once k^{-1} is computed for an arbitrary starting point α_0(e.g.$\alpha_0=0$),k^{-1} for any α and therefore u_α can be approximat-

* Dept. of Applied Physics and Applied Mechanics, Institute of Industial Science, University of Tokyo.

ed by repeating the above sequence with suitable step width $\Delta\alpha$. The successive perturbation solutions for the present problem are also depicted by solid lines in Fig. 1, α_0 being set zero. As seen in the figure, the successive perturbation solution with $\Delta\alpha=0.1$ approximates the exact solution fairly well and almost exactly with $\Delta\alpha=0.02$.

In the section 2 of the present note, the general formulae of successive perturbation method are constructed based on this notion both for static and eigenvalue problems. The following section 3 describes the application of these formulae to the evaluation of the reliability index β in the so-called Advanced First-Order Second-Moment (AFOSM) method.

2. General Formulae of Successive Perturbation Method

(Static Problem)

Extending the above case of SDOF system, the successive first order perturbation formula is easily given as follows.

$$U_{\alpha+\Delta\alpha}=K^{-1}_{\alpha+\Delta\alpha}F \quad \cdots\cdots (4)$$

$$K^{-1}_{\alpha+\Delta\alpha}=K^{-1}_{\alpha}+\sum_k\frac{\partial K^{-1}}{\partial\alpha_k}\bigg|_\alpha\cdot\Delta\alpha_k$$
$$=K^{-1}_{\alpha}-\sum_k K^{-1}_{\alpha}\frac{\partial K}{\partial\alpha_k}\bigg|_\alpha K^{-1}_{\alpha}\cdot\Delta\alpha_k \quad \cdots\cdots (5)$$

where α represents the probability variables $\{\alpha_1,\alpha_2,\cdots,\alpha_m\}$ involved in the system. The successive second order perturbation formula is constructed similarly by adding

$$\frac{1}{2}\sum_k\sum_l\frac{\partial^2 K^{-1}}{\partial\alpha_k\partial\alpha_l}\bigg|_\alpha\cdot\Delta\alpha_k\Delta\alpha_l$$
$$=-\frac{1}{2}\sum_k\sum_l K^{-1}_\alpha\bigg(\frac{\partial^2 K}{\partial\alpha_k\partial\alpha_l}\bigg|_\alpha K^{-1}_\alpha$$
$$+\frac{\partial K^{-1}}{\partial\alpha_k}\bigg|_\alpha\frac{\partial K}{\partial\alpha_l}\bigg|_\alpha+\frac{\partial K}{\partial\alpha_k}\bigg|_\alpha\frac{\partial K^{-1}}{\partial\alpha_l}\bigg|_\alpha\bigg)\cdot\Delta\alpha_k\Delta\alpha_l$$
$$\cdots\cdots (6)$$

to the right side of Eq. (5).

(Eigenvalue problem)

Suppose the eigenvalue λ and eigenvector ϕ of N-DOF eigenvalue problem

$$(K-\lambda M)\phi=0 \quad \cdots\cdots (7)$$

are known at a certain α value. Then λ and ϕ at $\alpha+\Delta\alpha$ are approximated as follows.

$$\lambda_{\alpha+\Delta\alpha}=\lambda_\alpha+\sum_k\frac{\partial\lambda}{\partial\alpha_k}\bigg|_\alpha\cdot\Delta\alpha_k\equiv\lambda_\alpha+\sum_k\lambda^{\rm I}_{k\alpha}\cdot\Delta\alpha_k$$
$$\cdots\cdots (8)$$

$$\phi_{\alpha+\Delta\alpha}=\phi_\alpha+\sum_k\frac{\partial\phi}{\partial\alpha_k}\bigg|_\alpha\cdot\Delta\alpha_k\equiv\phi_\alpha+\sum_k\phi^{\rm I}_{k\alpha}\cdot\Delta\alpha_k$$
$$\cdots\cdots (9)$$

As is well known[2], $\lambda^{\rm I}_{k\alpha}$ in Eq. (8) is evaluated by

$$\lambda^{\rm I}_{k\alpha}=\{\phi^T_\alpha(K^{\rm I}_{k\alpha}-\lambda_\alpha M^{\rm I}_{k\alpha})\phi_\alpha\}/\phi^T_\alpha M_\alpha\phi_\alpha \quad \cdots\cdots (10)$$

where $K^{\rm I}_{k\alpha}=\partial K/\partial\alpha_k|_\alpha$ and $M^{\rm I}_{k\alpha}=\partial M/\partial\alpha_k|_\alpha$. On the other hand, some methods are known[2~4] as regards the evaluation of $\phi^{\rm I}_{k\alpha}$. The simplest way[4] is employed here, that is to say, a suitable component of $\phi^{\rm I}_{k\alpha}$ is set zero and the other components denoted by $\tilde\phi^{\rm I}_{k\alpha}$ are solved as follows.

$$\tilde\phi^{\rm I}_{k\alpha}=-(\tilde K_\alpha-\lambda_\alpha\tilde M_\alpha)^{-1}(\hat K^{\rm I}_{k\alpha}-\lambda^{\rm I}_{k\alpha}\hat M_\alpha$$
$$-\lambda_\alpha\hat M^{\rm I}_{k\alpha})\phi_\alpha \quad \cdots\cdots (11)$$

In the right side of the above equation, ~and∧ mean that the size of matrix is $(N-1)\times(N-1)$ and $(N-1)\times N$ respectively. Denoting $\tilde K_\alpha-\lambda_\alpha\tilde M_\alpha$ by X_α, we can approximate X^{-1}_α based on $X^{-1}_{\alpha-\Delta\alpha}$ following the static case as

$$X^{-1}_\alpha=X^{-1}_{\alpha-\Delta\alpha}+\sum_k\frac{\partial X^{-1}}{\partial\alpha_k}\bigg|_{\alpha-\Delta\alpha}\cdot\Delta\alpha_k \quad \cdots\cdots (12)$$

where $\partial X^{-1}/\partial\alpha_k|_{\alpha-\Delta\alpha}$ is derived as

$$\frac{\partial X^{-1}}{\partial\alpha_k}\bigg|_{\alpha-\Delta\alpha}=-X^{-1}_{\alpha-\Delta\alpha}(\tilde K^{\rm I}_{k\alpha-\Delta\alpha}$$
$$-\lambda^{\rm I}_{k\alpha-\Delta\alpha}\tilde M_{\alpha-\Delta\alpha}-\lambda_{\alpha-\Delta\alpha}\tilde M^{\rm I}_{k\alpha-\Delta\alpha})X^{-1}_{\alpha-\Delta\alpha} \quad (13)$$

Because $\lambda_{\alpha-\Delta\alpha}$ and $\lambda^{\rm I}_{k\alpha-\Delta\alpha}$ are already known at the previous step, X^{-1}_α and therefore $\tilde\phi^{\rm I}_{k\alpha}$ are approximated based on $X^{-1}_{\alpha-\Delta\alpha}$ without the literal execution of Eq. (11). The description of the second order successive perturbation formula is omitted because of the limitation of the page.

3. Successive Perturbation Method Applied to Evaluation of Reliability Index

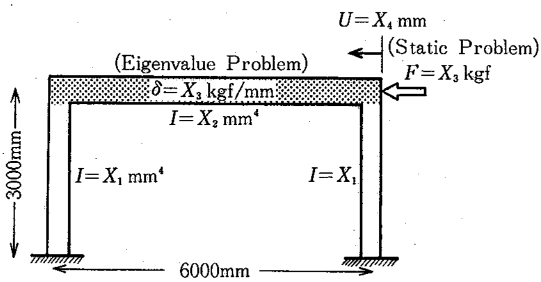

Fig. 2 Example Portal Frame Structure for Static or Eigenvalue Problem.

It is shown in the previous section that K^{-1} and $\partial K^{-1}/\partial\alpha_k$, or λ and $\partial\lambda/\partial\alpha_k$ are evaluated for arbitrary values of α by solving the governing equation only once. This feature can be utilized when FEM-based structural reliability analysis is carried out on the line of AFOSM method. In other words, when the successive perturbation technique is combined effectively with the iterative algorithm to evaluate the reliability index β, then CPU time is saved much.

According to the AFOSM method, the reliability index β is defined as the minimum distance from the origin to the failure (limit state) surface where the performance function $G(Y)$ equals zero in the standardized space[5]. Y denotes reduced (standardized) probability variables. When finite element method is concerned in the analysis, the performance function consists of K^{-1} in the static problem, and in general, λ in the eigenvalue problem. It is therefore necessary in the search algorithm to repeat the computation of K^{-1} and $\partial K^{-1}/\partial\alpha_k$, or λ and $\partial\lambda/\partial\alpha_k$ so that we reach the design point on the failure surface. A simple search algorithm in conjunction with the successive first order perturbation method is summarized as follows.

[1] Expand the performance function $G(Y)$ at $Y^{(0)}=0$ to make the tangential hyper-plane. Assuming this hyper-plane as $G(Y)$, evaluate the corresponding design point $Y^{(1)}$ according to the following equation.

$$Y_i^{(n+1)}=\frac{-G(Y^{(n)})+\sum_i \partial G(Y)/\partial Y_i|_{Y=Y^{(n)}}\cdot Y^{(n)}}{\sum_i \{\partial G(Y)/\partial Y_i|_{Y=Y^{(n)}}\}^2}\times\frac{\partial G(Y)}{\partial Y_i}\bigg|_{Y=Y^{(n)}} \quad \cdots\cdots(14)$$

The governing equation and the derivatives must be computed, and K^{-1} and $\partial K^{-1}/\partial Y_i$, or λ and $\partial\lambda/\partial Y_i$ are stored at this stage.

[2] Set $n=1$.

[3] Divide the vectorical distance from $Y^{(0)}=0$ to $Y^{(n)}$ into M pieces. Apply the successive perturbation technique with the step width $(Y^{(n)}-Y^{(0)})/M$ to approximate K^{-1} and $\partial K^{-1}/\partial Y_i$, or λ and $\partial\lambda/\partial Y_i$ at $Y^{(n)}$.

[4] Calculate $G(Y^{(n)})$ and $\partial G(Y)/\partial Y_i|_{Y=Y^{(n)}}$ based on the above result. Expand $G(Y)$ at $Y^{(n)}$ to make the new hyper-plane and calculate the corresponding design point $Y^{(n+1)}$ using Eq.(14).

[5] Set $n=n+1$.

[6] Repeat step [3] to [5] until $Y^{(n)}$ and $\beta^{(n)}=\sqrt{Y^{(n)T}Y^{(n)}}$ converges, and confirm $G(Y^{(n)})\fallingdotseq 0$. In this sequence the governing equation is not solved. $\beta^{(n)}$ and $Y^{(n)}$ may be taken as the target reliability index β and the design point Y^*.

(Numerical Example)

An example portal frame structure is shown in Fig.2, whose moments of inertia of the members, X_1 and X_2, are assumed as probability variables. Two problems are solved with this structure. In the first case of static problem, probabilistic lateral force X_3 and the limit state displacement X_4 are considered. In the second case of eigenvalue problem, the weight of the beam per unit length δ is taken as probability variable X_3, where the limit state eigenvalue of the first mode λ_c is given deterministically. The weight of the columns is neglected. These situations are schematically shown in the figure. The statistical specification of the parameters, which are assumed to be independent each other, is summarized in Table 1. Young's modulus E is deterministically set as 2500 kgf/mm^2 and all members are assumed to be rigid in axial direction. The computed displacement is 4.80mm for the expected structure with $\bar{X}_1$,$\bar{X}_2$ and $\bar{X}_3$ in the static problem, and in the eigenvalue problem 448.8$(rad./sec.)^2$ is computed for the expected structure. Figure 3 shows the convergence of β versus the number of iterations in the above successive preturbation-based algorithm. Number of division M

Table 1 Expectations and Standard Deviations of Basic Variables.

	X_1 mm^4	X_2 mm^4	X_3 kgf	X_4 mm	X_3 kgf/mm	λ_c(rad./sec.)^2
Expectation $\bar{X}_i$	3.0×10^8	4.0×10^8	2000	12	1.5	200
St. dev. $\sqrt{Var[X_i]}$	7.5×10^7	1.0×10^8	600	1.5	0.375	(deterministic)
			Static Problem		Eigenvalue Problem	

Table 2　Resultant Design Points and Reliability Indices.

	X_1* mm^4	X_2* mm^4	X_3*	X_4* mm	β
Static Problem	1.56×10^8 (1.51×10^8)	3.68×10^8 (3.67×10^8)	2.67×10^3 kgf $(2.70\times10^3$ kgf)	10.98 (10.92)	2.345 (2.429)
Eigenvalue Problem	1.44×10^8 (1.44×10^8)	3.64×10^8 (3.64×10^8)	1.89 kgf/mm (1.89 kgf/mm)	— —	2.357 (2.357)

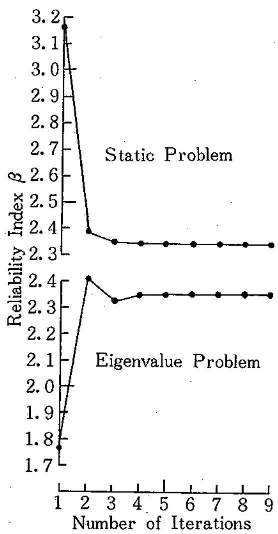

Fig. 3　Convergence of Reliability Index β versus Number of Iterations.

in step [3] is set eight. Table 2 shows the resultant (converged) design points and the reliability indices, where the values in the parenthesis denote the exact solution obtained by solving the governing equation in every iteration procedure. It goes without saying that we have the more accurate successive perturbation solution by setting smaller step width (greater M).

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References

1) Der Kiureghian, A. and Taylor, R. L. ; Numerical Methods in Structural Reliability, 4th Int. Conf. on Applications of Statistics and Probability in Soil and Structural Engrg., (1983) p.769.
2) Fox, R. L. and Kapoor, M. P. ; Rate of change of eigenvalues and eigenvectors, AIAA J., Vol.6, No. 2, (1968) p. 2426.
3) Collins, J. D. and Thomson, W. T. ; The eigenvalue problems for structural systems with statistical properties, AIAA J., Vol. 7, No. 4, (1969) p. 642.
4) Inoue, Y., et al. ; Trans JSME, Vol. 50, No. 452-C, (1984) p. 597.
5) Hasofer, A. M. and Lind, N. C. ; Exact and Invariant Second Moment Code Format, J. of the Engrg. Mech. Div., ASCE, Vol. 100, No. EM 1, (1974) p. 111.