博士論文

論文題目　Fourier–Mukai transforms for non-commutative complex tori
(非可換複素トーラスのフーリエ・向井変換)

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FOURIER–MUKAI TRANSFORMS FOR NON-COMMUTATIVE COMPLEX TORI

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Abstract. Let X be a complex torus of dimension g and $\hat{X}$ be the dual torus. For any $g(g-1)/2$-tuple λ of complex numbers of absolute value 1, we define a non-commutative complex torus X_λ as a sheaf of algebras on a real torus of dimension g. We prove that if all components of λ are roots of unity, then the category of coherent sheaves on X_λ is abelian and derived-equivalent to the category of coherent sheaves on $\hat{X}$ twisted by an element of the Brauer group of $\hat{X}$ determined by λ.

1. Introduction

Let X and $\hat{X}$ be smooth projective varieties. An integral functor

$$\Phi\colon D^b(X) \to D^b(\hat{X}) \tag{1.1}$$

between derived categories of coherent sheaves is said to be a *Fourier–Mukai functor* if it is an equivalence. A Fourier–Mukai functor Φ induces an isomorphism

$$\mathrm{HH}(\Phi)\colon \mathrm{HH}^\bullet(X) \to \mathrm{HH}^\bullet(\hat{X}) \tag{1.2}$$

of Hochschild cohomologies. For any $u \in \mathrm{HH}^2(X)$, Toda [Tod09] gave a $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-linear category $D^b(X, u)$ of first order deformations of X along u and an equivalence

$$\Phi^\dagger\colon D^b(X, u) \to D^b(\hat{X}, \mathrm{HH}(\Phi)(u)) \tag{1.3}$$

extending Φ.

The integral functor FM with the Poincaré line bundle as the integral kernel is the first example of a Fourier–Mukai functor given by Mukai [Muk81] for an abelian variety X and the dual abelian variety $\hat{X}$. Under the Hochschild–Kostant–Rosenberg isomorphism

$$\mathrm{HH}^2(X) \cong H^0(\wedge^2 T_X) \oplus H^1(T_X) \oplus H^2(\mathcal{O}_X), \tag{1.4}$$

the induced map HH(FM) sends $H^0(\wedge^2 T_X)$ to $H^2(\mathcal{O}_{\hat{X}})$. This suggests that non-commutative deformations of abelian varieties are Fourier–Mukai partners of gerby deformations of dual abelian varieties.

While the notion of gerby deformations is well-established in terms of twisted sheaves, non-commutative deformations are much harder to define in general. At the formal level, the Moyal deformation quantizations give formal non-commutative deformations of complex tori, and derived equivalences to formal gerby deformations are proved in [BBBP07]. The next problem is to construct non-commutative tori and derived equivalences for non-formal parameters (or even as a family over a complex manifold).

The first attempt to define non-commutative complex tori is given by Schwarz [Sch01], who introduced the notion of complex structures on non-commutative tori, which are irrational rotation algebras (see [Rie81], for example) regarded as non-commutative spaces in the sense of Connes[Con94]. Categories of holomorphic vector bundles on them are studied in [PS03].

In the paper [Blo10] and the preprint [Blo], Block discussed Fourier–Mukai functors between non-commutative complex tori and gerby deformations of dual complex tori, using DG categories which include objects corresponding to quasi-coherent sheaves. In [Blo], he also announced to discuss coherent sheaves in a paper in preparation.

We now explain the results of this paper. Let $X = T/\Gamma$ be a complex torus, where $T := (\mathbb{C}^*)^g$ and Γ is a discrete subgroup of T isomorphic to $\mathbb{Z}^g$. The dual complex torus $\hat{X} := \mathrm{Pic}^0 X$ can naturally be identified with $\hat{\Gamma}/\hat{T}$ where $\hat{\Gamma} := \mathrm{Hom}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^g$ and $\hat{T} := \mathrm{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^g$. Let $\lambda \in H^2(\hat{T}, U(1)) \cong U(1)^{g(g-1)/2}$ be an element of the second group cohomology of $\hat{T}$ with values in

$U(1)$. We construct a non-commutative deformation of X with parameter λ. When λ takes values in roots of unity, we give an equivalence of the derived category of coherent sheaves on the non-commutative deformation of X with parameter λ and the derived category of coherent sheaves on the gerby deformation of $\hat{X}$ with the same parameter λ.

Our construction of non-commutative complex tori is different from those in [Sch01], [PS03] and [Blo]. Ours can be regarded as a patching of Archimedean analog of quantum analytic tori in [Soi09]. To prove the equivalence of derived categories, we use the idea of equivariant Fourier–Mukai transforms developed in [Sos12].

This paper is organized as follows: In Section 2, we discuss q-Weyl algebras as toy models to motivate constructions in later sections. In Section 3, we recall Fourier–Mukai transforms for complex tori. In Section 4, we define non-commutative complex tori by deforming sheaves of convergent Laurent series rings on real tori. It can be regarded as an Archimedean analog of the construction of quantum analytic tori in [Soi09]. In Section 5, we discuss a dual pair $X \to Y$ and $\hat{Y} \to \hat{X}$ of finite coverings of tori associated with a deformation parameter λ with values in roots of unity. In Section 6, we describe non-commutative complex tori whose deformation parameters take values in roots of unity in terms of a finite sheaf of algebras. In Section 7, we collect basic definitions on twisted sheaves on complex manifolds. In Section 8, we introduce Fourier–Mukai transforms from gerby complex tori to non-commutative complex tori at roots of unity and state Theorem 8.1, which is the main result in this paper. In Section 9, we recall basic definitions and results on finite group actions on abelian and derived categories, and discuss twistings by group cocycles. In Section 10, we discuss group actions on categories appearing in our construction. In Section 11, we prove Theorem 8.1.

Notations and conventions. We fix the complex number field $\mathbb{C}$ as the ground field. All modules (resp. actions) are right modules (resp. actions) unless otherwise specified. In contrast, all group actions on categories are left actions. The word 'non-commutative' is synonymous with 'not necessarily commutative'.

For a ringed space $\mathcal{X} = (Z, \mathcal{O}_{\mathcal{X}})$, we write the ringed space $(Z, \mathcal{O}_{\mathcal{X}^{\mathrm{op}}} := (\mathcal{O}_{\mathcal{X}})^{\mathrm{op}})$ as $\mathcal{X}^{\mathrm{op}}$. The category of $\mathcal{O}_{\mathcal{X}}$-modules will be denoted by $\operatorname{Mod}\mathcal{X}$. For an element a of a complex abelian Lie group A, we write the right translation map $A \to A, x \mapsto xa$ as R_a. For two or three complex manifolds Z_1, Z_2 or Z_1, Z_2, Z_3, the projection to the first (resp. second) component Z_1 (resp. Z_2) will be denoted by p^{Z_1,Z_2} or p^{Z_1,Z_2,Z_3} (resp. q^{Z_1,Z_2} or q^{Z_1,Z_2,Z_3}).

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2. q-Weyl algebras

The category of $\mathcal{O}_X$-modules is equivalent to the category of Γ-equivariant $\mathcal{O}_T$-modules, and the category of Γ-equivariant $\mathbb{C}[T]$-modules can be regarded as a toy model for it. The latter can be identified with the category of modules over the crossed product algebra $\mathbb{C}[T] \rtimes \Gamma$.

For a parameter $\lambda = (\lambda_{i,j})_{1 \le i < j \le g} \in (\mathbb{C}^*)^{g(g-1)/2}$, the *$q$-Weyl algebra* is the non-commutative deformation of $\mathbb{C}[T]$ defined by

$$W_\lambda := \mathbb{C}\langle t_1^{\pm}, t_2^{\pm}, \dots, t_g^{\pm}\rangle / \left(t_i t_j - \lambda_{i,j} t_j t_i\right)_{1 \le i < j \le g}, \tag{2.1}$$

which gives $\mathbb{C}[T]$ if $\lambda = 1 := (1, \dots, 1)$.

Let $Q = (q_{i,j})_{i,j=1}^g$ be a multiplicative period matrix, i.e. a matrix so that $\{\gamma_j := (q_{i,j})_{i=1}^g\}_{j=1}^g$ is a free generator of Γ. The group Γ acts on the q-Weyl algebra by

$$t_i \cdot \gamma_j := q_{i,j}^{-1} t_i, \tag{2.2}$$

which reduces to the natural action on $\mathbb{C}[T]$ when $\lambda = 1$. The crossed product algebra $W_\lambda \rtimes \Gamma$ is isomorphic to another q-Weyl algebra

$$(2.3)\quad \mathcal{W}_{\lambda,Q,1} \coloneqq \mathbb{C}\langle t_1^{\pm}, t_2^{\pm}, \dots, t_g^{\pm}, \gamma_1^{\pm}, \gamma_2^{\pm}, \dots, \gamma_g^{\pm}\rangle \\ /(\{t_i t_j - \lambda_{i,j} t_j t_i\}_{1\le i<j\le g}, \{t_i\gamma_j - q_{i,j}^{-1}\gamma_j t_i\}_{i,j=1}^g, \{[\gamma_i,\gamma_j]\}_{i,j=1}^g)$$

generated by $2g$ elements.

The parameter λ describing a non-commutative deformation of X can be used to describe a gerby deformation of the dual torus $\hat{X}$:

Definition 2.1. A *λ-twisted $\mathcal{O}_{\hat{X}}$-module* is a pair $(\mathcal{M}, (\rho_{\hat\gamma})_{\hat\gamma\in\hat{T}})$ consisting of an $\mathcal{O}_{\hat\Gamma}$-module $\mathcal{M}$ and a family $(\rho_{\hat\gamma})_{\hat\gamma\in\hat{T}}$ of morphisms $\rho_{\hat\gamma}\colon \mathcal{M} \to R_{\hat\gamma}^*\mathcal{M}$ satisfying $\rho_{\hat\gamma_j}\circ\rho_{\hat\gamma_i} = \lambda_{i,j}\rho_{\hat\gamma_i}\circ\rho_{\hat\gamma_j}$ for all $i,j\in\{1,\dots,g\}$ such that $i<j$.

As a toy model of λ-twisted $\mathcal{O}_{\hat{X}}$-modules, we consider modules over the q-Weyl algebra

$$(2.4)\quad \mathcal{W}_{1,Q,\lambda} \coloneqq \mathbb{C}\langle \hat{t}_1^{\pm}, \hat{t}_2^{\pm}, \dots, \hat{t}_g^{\pm}, \hat\gamma_1^{\pm}, \hat\gamma_2^{\pm}, \dots, \hat\gamma_g^{\pm}\rangle \\ /(\{[\hat{t}_i,\hat{t}_j]\}_{i,j=1}^g, \{\hat{t}_i\hat\gamma_j - q_{j,i}^{-1}\hat\gamma_j\hat{t}_i\}_{i,j}, \{\hat\gamma_i\hat\gamma_j - \lambda_{i,j}\hat\gamma_j\hat\gamma_i\}_{1\le i<j\le g}),$$

which contains the ring of regular functions $\mathbb{C}[\hat\Gamma] \coloneqq \mathbb{C}[\hat{t}_1^{\pm}, \hat{t}_2^{\pm}, \dots, \hat{t}_g^{\pm}]$. Note that the roles of t_i and γ_i are interchanged between (2.3) and (2.4).

A toy model for the deformed Poincaré line bundle, which should give the integral kernel of the deformed Fourier–Mukai transform, is the $(\mathcal{W}_{1,Q,\lambda})^{\mathrm{op}}\otimes\mathcal{W}_{\lambda,Q,1}$-module P_λ such that

(1) $P_\lambda = \mathbb{C}[\hat\Gamma]\otimes_{\mathbb{C}} W_\lambda$ as a $\left(\mathbb{C}[\hat\Gamma]\right)^{\mathrm{op}}\otimes W_\lambda$-module,
(2) actions of γ_i are given by
$$\big(\psi(\hat{t}_1,\hat{t}_2,\dots,\hat{t}_g)\otimes\phi(t_1,t_2,\dots,t_g)\big)\cdot\gamma_i = \psi(\hat{t}_1,\hat{t}_2,\dots,\hat{t}_g)\hat{t}_i^{-1}\otimes\phi(t_1q_{1,i}^{-1}, t_2q_{2,i}^{-1},\dots,t_gq_{g,i}^{-1}),$$
(3) actions of $\hat\gamma_i$ are given by
$$\hat\gamma_i\cdot\big(\psi(\hat{t}_1,\hat{t}_2,\dots,\hat{t}_g)\otimes\phi(t_1,t_2,\dots,t_g)\big) = \psi(\hat{t}_1q_{i,1},\hat{t}_2q_{i,2},\dots,\hat{t}_gq_{i,g})\otimes t_i\phi(t_1,t_2,\dots,t_g).$$

Note that the action of $\hat\gamma_i$ satisfies $\hat\gamma_i\hat\gamma_j = \lambda_{i,j}\hat\gamma_j\hat\gamma_i$ since $t_it_j = \lambda_{i,j}t_jt_i$ in W_λ.

While q-Weyl algebras are only toy models and one has to work with analytic functions (such as theta functions) rather than regular functions, they provide intuition behind constructions in later sections. The duality between non-commutative deformations and gerby deformations is clearly visible in this toy model. Note also that if all components of λ are roots of unity, then W_λ is finite over its center, which is the ring of functions on a finite quotient of T.

3. Fourier–Mukai transforms

We identify $\mathcal{O}_X$-modules with Γ-equivariant $\mathcal{O}_T$-modules. An element $\hat{x}\in\hat\Gamma$ determines a Γ-equivariant $\mathcal{O}_T$-module $\mathcal{L}_{\hat{x}}$ as the trivial $\mathcal{O}_T$-module equipped with the Γ-action

$$(3.1)\qquad \phi(x)\cdot\gamma = \phi(x\gamma^{-1})\hat{x}(\gamma)^{-1}$$

for $\gamma\in\Gamma$. Since any $t\in\hat{T}$ gives an isomorphism

$$(3.2)\qquad \mathcal{L}_{\hat{x}} \xrightarrow{\sim} \mathcal{L}_{\hat{x}t^{-1}}, \qquad \phi(x)\mapsto t(x)\phi(x)$$

of Γ-equivariant $\mathcal{O}_T$-modules, the map $\hat{x}\mapsto\mathcal{L}_{\hat{x}}$ descends to a map $\hat\Gamma/\hat{T}\xrightarrow{\sim}\hat{X}\coloneqq\operatorname{Pic}^0 X$, which is an isomorphism of groups because of the isomorphisms

$$(3.3)\qquad \mathcal{L}_{\hat{x}}\otimes\mathcal{L}_{\hat{x}'}\xrightarrow{\sim}\mathcal{L}_{\hat{x}\hat{x}'}, \qquad \phi(x)\otimes\psi(x)\mapsto\phi(x)\psi(x).$$

The *Poincaré line bundle* $\mathcal{P}$ is the $\Gamma\times\hat{T}^{\mathrm{op}}$-equivariant $\mathcal{O}_{T\times\hat\Gamma}$-module, defined as the trivial $\mathcal{O}_{T\times\hat\Gamma}$-module equipped with the *left* $\hat{T}$-action

$$(3.4)\qquad t\cdot\phi(x,\hat{x}) = t(x)\phi(x,\hat{x}t)$$

and the right Γ-action

$$\phi(x, \hat{x}) \cdot \gamma = \phi(x\gamma^{-1}, \hat{x})\hat{x}(\gamma)^{-1}. \tag{3.5}$$

Let $\overline{s}$ be the map given by

$$\overline{s}\colon \hat{\Gamma} \times \hat{\Gamma} \times T \to \hat{\Gamma} \times T, \qquad (\hat{x}_1, \hat{x}_2, x) \mapsto (\hat{x}_1\hat{x}_2, x). \tag{3.6}$$

The map $s\colon \hat{X} \times \hat{X} \times X \to \hat{X} \times X$ is defined similarly. Since the isomorphism

$$\overline{m}\colon (p^{\hat{\Gamma},\hat{\Gamma},T})^*\mathcal{P} \otimes (q^{\hat{\Gamma},\hat{\Gamma},T})^*\mathcal{P} \to \overline{s}^*\mathcal{P}, \qquad \phi \otimes \phi' \mapsto \phi\phi' \tag{3.7}$$

of $\mathcal{O}_{\hat{\Gamma}\times\hat{\Gamma}\times T}$-modules is $\hat{T}^{\mathrm{op}} \times \hat{T}^{\mathrm{op}} \times \Gamma$-equivariant, it defines an isomorphism

$$m\colon (p^{\hat{X},\hat{X},X})^*\mathcal{P} \otimes (q^{\hat{X},\hat{X},X})^*\mathcal{P} \xrightarrow{\sim} s^*\mathcal{P} \tag{3.8}$$

os $\mathcal{O}_{\hat{X}\times\hat{X}\times X}$-modules, which restricts to an isomorphism

$$m_{\hat{x}_1,\hat{x}_2}\colon \mathcal{L}_{\hat{x}_1} \otimes \mathcal{L}_{\hat{x}_2} \xrightarrow{\sim} \mathcal{L}_{\hat{x}_1\hat{x}_2} \tag{3.9}$$

of $\mathcal{O}_X$-modules on $\{\hat{x}_1\} \times \{\hat{x}_2\} \times X$ where $\mathcal{L}_{\hat{x}} \simeq \mathcal{P}|_{X\times\{\hat{x}\}}$ by definition.

The *Fourier–Mukai transform* is the integral functor with the Poincaré line bundle as the integral kernel;

$$\mathrm{FM}_{\mathcal{P}}\colon D^b(\hat{X}) \to D^b(X), \qquad \mathcal{M} \mapsto \mathbb{R}(p^{X,\hat{X}})_*((q^{X,\hat{X}})^*\mathcal{M} \otimes_{\mathcal{O}_{X\times\hat{X}}} \mathcal{P}). \tag{3.10}$$

It sends the skyscraper sheaf $\mathcal{O}_{\hat{x}}$ of a point $\hat{x} \in \hat{X}$ to the line bundle $\mathcal{L}_{\hat{x}}$.

Theorem 3.1 ([Muk81])**.** The Fourier–Mukai transform $\mathrm{FM}_{\mathcal{P}}$ is an equivalence, whose quasi-inverse is given by the integral functor $\mathrm{FM}_{\mathcal{P}^{-1}[g]}$ with $\mathcal{P}^{-1}[g]$ as the integral kernel.

To be more precise, Mukai proved this theorem for abelian varieties. A proof for complex tori can be found in [BBBP07].

4. Non-commutative deformations of complex tori

Set

$$\varpi\colon T \to |T| \coloneqq (\mathbb{R}^{>0})^g, \qquad (x_1, x_2, \ldots, x_g) \mapsto (|x_1|, |x_2|, \ldots, |x_g|). \tag{4.1}$$

For a product $D = \prod_{i=1}^g (r_i, R_i)$ of open intervals, the ring $\varpi_*\mathcal{O}_T(D)$ consists of Laurent series in g variables with radii of convergence (r_i, R_i) for $1 \le i \le g$.

Definition 4.1. A *unitary deformation parameter* is an element of $Z^2(\hat{T}, U(1))$, i.e., a map $\lambda\colon \hat{T} \times \hat{T} \to U(1)$ satisfying

$$\lambda(t_2, t_3)\lambda(t_1t_2, t_3)^{-1}\lambda(t_1, t_2t_3)\lambda(t_1, t_2)^{-1} = 1 \tag{4.2}$$

for all $t_1, t_2, t_3 \in \hat{T}$.

Given a unitary deformation parameter λ, the *star product* $*_\lambda$ on $\varpi_*\mathcal{O}_T(D)$ is defined by

$$\left(\sum_{t\in\hat{T}} a_t t\right) *_\lambda \left(\sum_{t\in\hat{T}} b_t t\right) = \sum_{t\in\hat{T}} \left(\sum_{t_1,t_2\in\hat{T},\, t_1t_2=t} \lambda(t_1, t_2) a_{t_1} b_{t_2}\right) t, \tag{4.3}$$

which is easily seen to be associative by using (4.2). The convergence of the right hand side follows from

$$\left|\sum_{t_1,t_2\in\hat{T},\, t_1t_2=t} \lambda(t_1, t_2) a_{t_1} b_{t_2}\right| \le \sum_{t_1,t_2\in\hat{T},\, t_1t_2=t} |a_{t_1}||b_{t_2}| \tag{4.4}$$

and

$$\sum_{t\in\hat{T}} \left(\sum_{t_1,t_2\in\hat{T}, t_1t_2=t} |a_{t_1}||b_{t_2}|\right) t = \left(\sum_{t\in\hat{T}} |a_t| t\right) \left(\sum_{t\in\hat{T}} |b_t| t\right), \tag{4.5}$$

which depends on the unitarity of λ. The resulting sheaf of associative algebras on $|T|$ will be denoted by $\mathcal{O}_{\mathsf{T}_\lambda}$ which turns $|T|$ into a non-commutative ringed space $\mathsf{T}_\lambda := (|T|, \mathcal{O}_{\mathsf{T}_\lambda})$.

A cochain $\alpha \in Z^1(\hat{T}, U(1))$ bounding $\lambda, \lambda' \in Z^2(\hat{T}, U(1))$ is a map $\alpha \colon \hat{T} \to U(1)$ satisfying

$$\lambda'(t_1, t_2) = \lambda(t_1, t_2)\alpha(t_1)\alpha(t_2)\alpha(t_1 t_2)^{-1}. \tag{4.6}$$

It gives an isomorphism

$$\sum_{t \in \hat{T}} a_t t \mapsto \sum_{t \in \hat{T}} \alpha(t) a_t t \tag{4.7}$$

of the ring of sections, which ensures that the isomorphism class of the sheaf $\mathcal{O}_{\mathsf{T}_\lambda}$ of associative algebras depends only on the cohomology class $[\lambda] \in H^2(\hat{T}, U(1))$.

The natural T-action on T induces a T-action on $|T|$, which lifts to a T-action on the ringed space T_λ in such a way that the morphism $\rho_a \colon \mathcal{O}_{\mathsf{T}_\lambda} \to (R_{\varpi(a)^{-1}})_* \mathcal{O}_{\mathsf{T}_\lambda}$ of sheaves of associative algebras for $a \in T$ is given by

$$\sum_{t \in \hat{T}} a_t t \mapsto \sum_{t \in \hat{T}} a_t t(a)^{-1} t. \tag{4.8}$$

The action of Γ on $|T|$ is free since $\varpi(\gamma) = 1$ for $1 \neq \gamma \in \Gamma$ contradicts the freeness or the properness of Γ-action on T.

Definition 4.2. The *non-commutative complex torus* associated with a complex torus $X = T/\Gamma$ and a unitary deformation parameter $\lambda \in Z^2(\hat{T}, U(1))$ is the non-commutative ringed space $\mathsf{X}_\lambda := \mathsf{T}_\lambda / \Gamma$.

Recall that a sheaf $\mathcal{M}$ of $\mathcal{O}_{\mathcal{X}}$-modules on a ringed space $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ is said to be *coherent* if

(1) $\mathcal{M}$ is finitely generated, i.e., for any point $x \in X$, there exists an open neighborhood U of x and an epimorphism $\mathcal{O}_{\mathcal{X}}^{\oplus m}|_U \to \mathcal{M}|_U \to 0$ for some $m \in \mathbb{N}$, and
(2) for any open set U and any $m \in \mathbb{N}$, the kernel of any morphism $\mathcal{O}_{\mathcal{X}}^{\oplus m}|_U \to \mathcal{M}|_U$ is finitely generated.

The full subcategory of $\operatorname{Mod} \mathcal{X}$ consisting of coherent modules will be denoted by $\operatorname{coh} \mathcal{X}$.

Lemma 4.3. The sheaf $\mathcal{O}_{\mathsf{T}_1}$ is coherent.

Proof. It is clear that $\mathcal{O}_{\mathsf{T}_1}$ is finitely generated. For any open subset U of $|T|$, a morphism $\alpha \colon \mathcal{O}_{\mathsf{T}_1}^{\oplus m}|_U \to \mathcal{O}_{\mathsf{T}_1}|_U$ is the same as a morphism $\tilde{\alpha} \colon \mathcal{O}_T^{\oplus m}|_{\varpi^{-1}(U)} \to \mathcal{O}_T|_{\varpi^{-1}(U)}$. For any $x \in U$, let $V \subset U$ be an open neighborhood of x obtained as the product of intervals. Then $\varpi^{-1}(V)$ is Stein and hence there is an epimorphism $\tilde{\beta}$ from $\mathcal{O}_{\varpi^{-1}(V)}^{\oplus m'}$ to the kernel of $\tilde{\alpha}|_{\varpi^{-1}(V)}$ for some integer m'. We ensure that ϖ_* is exact (and hence the morphism $\beta := \varpi_* \tilde{\beta}$ is also an epimorphism) by applying [Tay02, Corollary 11.5.4] for every fiber of ϖ. This shows that $\ker \alpha$ is finitely generated, and Lemma 4.3 is proved. □

Non-commutative complex tori at $\lambda = 1$ are usual complex tori:

Propositon 4.4. The adjunction $\varpi^* \dashv \varpi_*$ induces an equivalence $\operatorname{coh} T \simeq \operatorname{coh} \mathsf{T}_1$.

Proof. Oka coherence theorem implies that an object of $\operatorname{Mod} T$ is coherent if and only if it is finitely presented. Similarly, Lemma 4.3 implies that an object of $\operatorname{Mod} \mathsf{T}_1$ is coherent if and only if it is finitely presented.

The functors ϖ^* and ϖ_* induce mutually inverse functors on categories of finitely presented modules since

(1) the functor ϖ_* is exact (and hence preserves cokernels in particular),
(2) the functor ϖ^* preserves cokernels since it is a left adjoint, and
(3) ϖ_* and ϖ^* interchanges $\mathcal{O}_T$ and $\mathcal{O}_{\mathsf{T}_1}$.

□

Corollary 4.5. The adjunction $(\varpi_X)^* \dashv (\varpi_X)_*$ induces an equivalence $\operatorname{coh} X \simeq \operatorname{coh} \mathsf{X}_1$, where $\varpi_X \colon X \to |X|$ is the map induced by ϖ.

The category $\operatorname{coh} \mathcal{O}_{\mathsf{X}_\lambda}$ is abelian if Problem 4.6 below has an affirmative answer:

Problem 4.6 (Oka coherence for non-commutative tori)**.** Is $\mathcal{O}_{\mathsf{T}_\lambda}$ coherent?

Yet another problem is a generalization to non-unitary deformation parameters, which would be needed for the duality with general gerby deformations.

5. Deformation parameters at roots of unity

As one can see (e.g. by noting that $\hat{T}$-modules are equivalent to $\mathbb{Z}[T] \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm}, \dots, t_g^{\pm}]$-modules, and the trivial $\hat{T}$-modules $\mathbb{Z}$ has the Koszul resolution associated to $t_1 - 1, t_2 - 1, \dots, t_g - 1$) that the cohomology $H^i(\hat{T}, A)$ with coefficients in an abelian group A with the trivial $\hat{T}$-action is isomorphic to $\operatorname{Hom}(\wedge^i \hat{T}/(\wedge^i \hat{T})_{\mathrm{tors}}, A) \cong A^{\binom{g}{i}}$ (note that $(\wedge^i \hat{T})_{\mathrm{tors}}$ is generated by torsion elements $a \wedge a \wedge t_1 \wedge \dots \wedge t_{i-2}$ ($a, t_1, \dots, t_{i-2} \in \hat{T}$) of order 2). Let $\Lambda \in \operatorname{Hom}(\wedge^2 \hat{T}, U(1))$ be the element corresponding to the class $[\lambda] \in H^2(\hat{T}, U(1))$ of a unitary deformation parameter $\lambda \in Z^2(\hat{T}, U(1))$, and set

$$\hat{H} := \left\{ t \in \hat{T} \;\middle|\; \Lambda(t \wedge t') = 1 \text{ for all } t' \in \hat{T} \right\}, \tag{5.1}$$

so that one has an exact sequence

$$1 \to \hat{H} \to \hat{T} \to \hat{K} \to 1, \tag{5.2}$$

and $[\lambda] \in H^2(\hat{T}, U(1))$ descends to an element of $H^2(\hat{K}, U(1))$, which can be represented by a bilinear cochain in $Z^2(\hat{K}, U(1))$.

For the rest of this paper and unless otherwise specified, we will assume that a unitary deformation parameter $\lambda \in Z^2(\hat{T}, U(1))$ is contained in $Z^2(\hat{T}, \boldsymbol{\mu}_N)$ for some positive integer N where $\boldsymbol{\mu}_N := \left\{ \zeta \in U(1) \;\middle|\; \zeta^N = 1 \right\}$. This implies that $\hat{K}$ is a finite abelian group, and we will also fix a bilinear map

$$\lambda \colon \hat{K} \otimes_{\mathbb{Z}} \hat{K} \to \boldsymbol{\mu}_N \tag{5.3}$$

representing $[\lambda]$.

The dual group $K := \operatorname{Hom}(\hat{K}, \mathbb{C}^*)$ can be identified with the kernel of the map $T \to H$ dual to the inclusion $\hat{H} \to \hat{T}$, so that one has an exact sequence

$$1 \to K \to T \xrightarrow{\pi_T} H \to 1. \tag{5.4}$$

Since $\Gamma \cap K$ is the trivial group, the free action of K on T descends to a free action of K on $X := T/\Gamma$, so that one has an exact sequence

$$1 \to K \to X \xrightarrow{\pi} Y \to 1 \tag{5.5}$$

where Y is a complex torus. We also have an exact sequence

$$1 \to \hat{K} \to \hat{Y} \xrightarrow{\hat{\pi}} \hat{X} \to 1 \tag{5.6}$$

Since K goes to the identity under the homomorphism $\varpi \colon T \to |T|$, there exists $\varpi_Y \colon Y \to |X|$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\varpi_X"'] & Y \arrow[d, "\varpi_Y"] \\ & {|X|} \end{tikzcd} \tag{5.7}$$

commute.

We have an isomorphism

$$\pi_* \mathcal{O}_X \cong \bigoplus_{\hat{k} \in \hat{K}} \mathcal{L}_{\hat{k}} \tag{6.1}$$

of sheaves of $\mathcal{O}_Y$-algebras on $\hat{Y} := \operatorname{Pic}^0 Y$. The summand $\mathcal{L}_{\hat{k}}$ is locally generated by a monomial function $t \in \hat{T}$ representing $\hat{k}$. We define a star product $\star_\lambda$ on $\pi_* \mathcal{O}_X$ by

$$\star_\lambda \colon \bigoplus_{\hat{k} \in \hat{K}} \mathcal{L}_{\hat{k}} \times \bigoplus_{\hat{k} \in \hat{K}} \mathcal{L}_{\hat{k}} \to \bigoplus_{\hat{k} \in \hat{K}} \mathcal{L}_{\hat{k}} \tag{6.2}$$

$$((\phi_{\hat{k}})_{\hat{k}}, (\psi_{\hat{k}})_{\hat{k}}) \mapsto \left(\sum_{\hat{k}_1 \hat{k}_2 = \hat{k}} \lambda(\hat{k}_1, \hat{k}_2) m_{\hat{k}_1, \hat{k}_2} (\phi_{\hat{k}_1} \otimes \psi_{\hat{k}_2}) \right)_{\hat{k}}. \tag{6.3}$$

We write the resulting sheaf $(\pi_* \mathcal{O}_X, \star_\lambda)$ of $\mathcal{O}_Y$-algebras as $\mathcal{O}_{X_\lambda}$, and the ringed space $(Y, \mathcal{O}_{X_\lambda})$ as X_λ.

Propositon 6.1. One has an isomorphism $(\varpi_Y)_* \mathcal{O}_{X_\lambda} \cong \mathcal{O}_{\mathsf{X}_\lambda}$ of sheaves of algebras.

Proof. A direct calculation shows that

$$\left(\sum_{t_1 \in \hat{T}} a_{t_1} t_1 \right) *_\lambda \left(\sum_{t_2 \in \hat{T}} b_{t_2} t_2 \right) \tag{6.4}$$

$$= \left(\sum_{t \in \hat{T}/\hat{H}} \left(\sum_{t_1 \bmod \hat{H} = t} a_{t_1} t_1 \right) \right) *_\lambda \left(\sum_{t' \in \hat{T}/\hat{H}} \left(\sum_{t_2 \bmod \hat{H} = t'} b_{t_2} t_2 \right) \right) \tag{6.5}$$

$$= \sum_{t \in \hat{T}/\hat{H}} \left(\sum_{t' \in \hat{T}/\hat{H}} \left(\sum_{t_1 \bmod \hat{H} = t} \left(\sum_{t_2 \bmod \hat{H} = t'} \lambda(t_1, t_2) a_{t_1} b_{t_2} t_1 t_2 \right) \right) \right) \tag{6.6}$$

$$= \sum_{t \in \hat{T}/\hat{H}} \left(\sum_{t' \in \hat{T}/\hat{H}} \left(\sum_{t_1 \bmod \hat{H} = t} \left(\sum_{t_2 \bmod \hat{H} = t'} \lambda(t, t') a_{t_1} b_{t_2} t_1 t_2 \right) \right) \right) \tag{6.7}$$

$$= \sum_{t \in \hat{T}/\hat{H}} \left(\sum_{t' \in \hat{T}/\hat{H}} \lambda(t, t') m_{t,t'} \left(\sum_{t_1 \bmod \hat{H} = t} a_{t_1} t_1, \sum_{t_2 \bmod \hat{H} = t'} b_{t_2} t_2 \right) \right) \tag{6.8}$$

coincide with $\star_\lambda$. ☐

Propositon 6.2. The adjunction $(\varpi_Y)^* \dashv (\varpi_Y)_*$ induces an equivalence $\operatorname{coh} X_\lambda \simeq \operatorname{coh} \mathsf{X}_\lambda$.

Proof. We write the full subcategory of $\operatorname{Mod} X_\lambda$ (resp. $\operatorname{Mod} \mathsf{X}_\lambda$) consisting of coherent $\mathcal{O}_Y$-modules (resp. $\mathcal{O}_{\mathsf{Y}_1}$-modules) as $(\operatorname{coh} Y)^{\hat{K}, \lambda}$ (resp. $(\operatorname{coh} \mathsf{Y}_1)^{\hat{K}, \lambda}$). Since $\mathcal{O}_{X_\lambda}$ is a finite $\mathcal{O}_Y$-algebra by definition and $\mathcal{O}_{\mathsf{X}_\lambda}$ is a finite $\mathcal{O}_{\mathsf{Y}_1}$-algebra by Proposition 6.1, the category $\operatorname{coh} X_\lambda$ (resp. $\operatorname{coh} \mathsf{X}_\lambda$) is equivalent to $(\operatorname{coh} Y)^{\hat{K}, \lambda}$ (resp. $(\operatorname{coh} \mathsf{Y}_1)^{\hat{K}, \lambda}$). It follows from Corollary 4.5 and Proposition 6.1 that $\mathcal{O}_{\mathsf{X}_\lambda} \in \operatorname{ob}(\operatorname{coh} \mathsf{Y}_1)$ and the adjunction $(\varpi_Y)^* \dashv (\varpi_Y)_*$ induces an equivalence $(\operatorname{coh} Y)^{\hat{K}, \lambda} \simeq (\operatorname{coh} \mathsf{Y}_1)^{\hat{K}, \lambda}$. ☐

In particular, $\mathcal{O}_{\mathsf{X}_\lambda}$ is a coherent sheaf on X_λ and hence $\operatorname{coh} \mathcal{O}_{\mathsf{X}_\lambda}$ is an abelian category.

Remark 6.3. It follows from the definition of $\hat{K}$ that there exists an isomorphism $\lambda^\sharp \colon \hat{K} \to K$ such that $\lambda(\hat{k}_1, \hat{k}_2) = \hat{k}_2(\lambda^\sharp(\hat{k}_1))$. If we write the inverse of $\lambda^\sharp$ as $\lambda_\flat$ and define $\omega \colon K \otimes_{\mathbb{Z}} K \to U(1)$ by $\omega(k_1, k_2) = \lambda(\lambda_\flat(k_1), \lambda_\flat(k_2))$, then the star product on $\mathcal{O}_{X_\lambda}$ can alternatively be described as

$$\phi(x) \star_\lambda \psi(x) = \frac{1}{\sharp K} \sum_{k_1, k_2 \in K} \omega(k_1, k_2) \phi(x k_1) \psi(x k_2). \tag{6.9}$$

A coherent sheaf on X_λ consists of an $\mathcal{O}_Y$-module $\mathcal{M}$ and a collection of morphisms $\{m_{\hat{k}} \colon \mathcal{M} \otimes \mathcal{L}_{\hat{k}} \to \mathcal{M}\}_{\hat{k} \in \hat{K}}$ such that the diagram

$$\begin{tikzcd}
\mathcal{M} \otimes \mathcal{L}_{\hat{k}_1} \otimes \mathcal{L}_{\hat{k}_2} \arrow[r, "m_{\hat{k}_1} \otimes \mathrm{id}"] \arrow[d, "\lambda(\hat{k}_1, \hat{k}_2)\, \mathrm{id} \otimes m_{\hat{k}_1 \hat{k}_2}"'] & \mathcal{M} \otimes \mathcal{L}_{\hat{k}_2} \arrow[d, "m_{\hat{k}_2}"] \\
\mathcal{M} \otimes \mathcal{L}_{\hat{k}_1 \hat{k}_2} \arrow[r, "m_{\hat{k}_1 \hat{k}_2}"] & \mathcal{M}
\end{tikzcd} \tag{6.10}$$

commutes. The dual $\mathcal{O}_Y$-module $\mathcal{M}^{-1} := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Y)$ equipped with the transposes of m_t gives a coherent sheaf on X_λ^{op}.

7. Twisted sheaves

Let M be a complex manifold and $\alpha \in H^2(M, \mathcal{O}_M^*)$ be a second étale cohomology class of $\mathcal{O}_M^*$. Take an étale covering $\mathcal{U} = (U_i)_i$ and a representative $(\alpha_{i,j,k})$ of α on $\mathcal{U}$. We write the projections from $U_i \times_M U_j$ to the first (resp. second) component as $I_{i,j}$ (resp. $J_{i,j}$).

Definition 7.1. An *α-twisted sheaf* on M is a collection $((\mathcal{F}_i)_i, (\rho_{i,j})_{i,j})$ of $\mathcal{O}_{U_i}$-modules $\mathcal{F}_i$ and isomorphisms $\rho_{i,j} \colon I_{i,j}^* \mathcal{F}_i \to J_{i,j}^* \mathcal{F}_j$ satisfying $\rho_{i,k}^{-1} \circ \rho_{j,k} \circ \rho_{i,j} = \alpha_{i,j,k} \,\mathrm{id}$. An α-twisted sheaf is *coherent* if all $\mathcal{F}_i$ are coherent.

The category of α-twisted sheaves on M and the full subcategory consisting of α-twisted coherent sheaves will be denoted by $\operatorname{Mod} M^\alpha$ and $\operatorname{coh} M^\alpha$ respectively. Note that for an $\mathcal{O}_M$-algebra $\mathcal{A}$ and the resulting ringed space $\mathcal{X} := (M, \mathcal{A})$, we similarly can define the notion of α-twisted sheaves on $\mathcal{X}$ and categories $\operatorname{Mod} \mathcal{X}^\alpha, \operatorname{coh} \mathcal{X}^\alpha$. They do not depend on the choice of $\mathcal{U}$ and $(\alpha_{i,j,k})$ up to equivalence. One has the tensor product functor

$$\otimes \colon \operatorname{Mod} M^\alpha \times \operatorname{Mod} M^{\alpha'} \to \operatorname{Mod} M^{\alpha\alpha'}, \tag{7.1}$$

$$\big(((\mathcal{F}_i)_i, (\rho_{i,j})_{i,j}), ((\mathcal{F}'_i)_i, (\rho'_{i,j})_{i,j})\big) \mapsto ((\mathcal{F}_i \otimes \mathcal{F}'_i)_i, (\rho_{i,j} \otimes \rho'_{i,j})_{i,j}) \tag{7.2}$$

and the duality functor

$$(-)^{-1} \colon \operatorname{Mod} M^\alpha \to \operatorname{Mod} M^{\alpha^{-1}}, \tag{7.3}$$

$$((\mathcal{F}_i)_i, (\rho_{i,j})_{i,j}) \mapsto ((\mathcal{H}om_{\mathcal{O}_{U_i}}(\mathcal{F}_i, \mathcal{O}_{U_i}))_i, ((\rho_{i,j}^{-1})^*)_{i,j}). \tag{7.4}$$

If $\mathcal{U}$ consists of a principal G-bundle P on M for some discrete group G, then the isomorphism

$$P \times G^p \to P \times_M P \times_M \cdots \times_M P, \quad (y, g_1, \dots, g_p) \mapsto (y, yg_1, yg_1g_2, \dots, yg_1 \cdots g_p) \tag{7.5}$$

induces an isomorphism from the Čech complex $C^\bullet(\mathcal{U}, \mathcal{O}_M^*)$ to the standard complex $C^\bullet(G, \mathcal{O}_P^*(P))$ for group cohomology. The composite of the resulting map $H^2(G, \mathcal{O}_P^*(P)) \to H^2(M, \mathcal{O}_M^*)$ with the map $H^2(G, \mathbb{C}^*) \to H^2(G, \mathcal{O}_P^*(P))$ will be denoted by $\iota_P \colon H^2(G, \mathbb{C}^*) \to H^2(M, \mathcal{O}_M^*)$. For any $\lambda \in H^2(G, \mathbb{C}^*)$, an $\iota_P(\lambda)$-twisted sheaf will simply be called a λ-twisted sheaf. If G is a finite abelian group, then $\iota_P(\lambda)$ is a torsion element since any group cohomology of finite abelian group is torsion. In other words, $\iota_P(\lambda)$ is an element of the *cohomological Brauer group* $\operatorname{Br}(M) := H^2(M, \mathcal{O}_M^*)_{\mathrm{tors}}$. A λ-twisted sheaf consists of an $\mathcal{O}_P$-module $\mathcal{F}$ and a *λ-twisted G-linearization* of $\mathcal{F}$, i.e., a collection $(\rho_g)_{g \in G}$ of morphisms $\rho_g \colon \mathcal{F} \to R_g^* \mathcal{F}$ satisfying $R_{g_1}^* \rho_{g_2} \circ \rho_{g_1} = \lambda(g_1, g_2) \rho_{g_1 g_2}$.

A λ-twisted sheaf $(\mathcal{F}, (\rho_g)_{g \in G})$ will also be called a *λ-twisted G-equivariant $\mathcal{O}_P$-module*; it reduces to a G-equivariant $\mathcal{O}_P$-module if $\lambda = 1$ (which in turn is equivalent to an $\mathcal{O}_M$-module).

8. Deformed Fourier–Mukai transforms

Let $\mathcal{Q}$ be the Poincaré line bundle on $Y \times \hat{Y}$. For a complex manifold Z, a sheaf of associative algebras $(p^{Y,Z})^* \mathcal{O}_{X_\lambda}$ (resp. $(p^{Y,Z})^* \mathcal{O}_{X_\lambda}^{\mathrm{op}}$) will denoted by $\mathcal{O}_{X_\lambda \times Z}$ (resp. $\mathcal{O}_{X_\lambda^{\mathrm{op}} \times Z}$), and the resulting ringed space will be denoted by $X_\lambda \times Z$ (resp. $X_\lambda^{\mathrm{op}} \times Z$). Symbols $Y \times \hat{X}^\lambda$ (resp. $X_\lambda \times \hat{X}^\lambda, X_\lambda^{\mathrm{op}} \times \hat{X}^\lambda$) denote $(Y \times \hat{X})^{1 \times \lambda}$ (resp. $(X_\lambda \times \hat{X})^{1 \times \lambda}, (X_\lambda^{\mathrm{op}} \times \hat{X})^{1 \times \lambda}$).

The *deformed Poincaré line bundle* $\mathcal{P}_\lambda$ is an object of $\operatorname{coh} X_\lambda \times \hat{X}^{\lambda^{-1}}$ defined as the $\mathcal{O}_{Y\times\hat{Y}}$-module

$$\bigoplus_{\hat{k}\in\hat{K}} \mathcal{Q}\otimes(p^{Y,\hat{Y}})^*\mathcal{L}_{\hat{k}} \cong \bigoplus_{\hat{k}\in\hat{K}} R_{\hat{k}}^*\mathcal{Q} \tag{8.1}$$

equipped with the $\mathcal{O}_{X_\lambda\times\hat{Y}} \cong \bigoplus_{\hat{k}\in\hat{K}}(p^{Y,\hat{Y}})^*\mathcal{L}_{\hat{k}}$-action

$$\bigoplus_{\hat{k}\in\hat{K}} \mathcal{Q}\otimes(p^{Y,\hat{Y}})^*\mathcal{L}_{\hat{k}} \times \bigoplus_{\hat{k}\in\hat{K}}(p^{Y,\hat{Y}})^*\mathcal{L}_{\hat{k}} \to \bigoplus_{\hat{k}\in\hat{K}} \mathcal{Q}\otimes(p^{Y,\hat{Y}})^*\mathcal{L}_{\hat{k}} \tag{8.2}$$

$$((\psi_{\hat{k}}\otimes\phi_{\hat{k}})_{\hat{k}}, (\phi'_{\hat{k}})_{\hat{k}}) \mapsto \left(\sum_{\hat{k}_1,\hat{k}_2\in\hat{K},\,\hat{k}_1\hat{k}_2=\hat{k}} \psi_{\hat{k}_1}\otimes(\phi_{\hat{k}_1}\star_\lambda \phi'_{\hat{k}_2})\right)_{\hat{k}}, \tag{8.3}$$

and the λ^{-1}-twisted $\hat{K}$-action (i.e. the λ-twisted *left* $\hat{K}$-action)

$$\rho_{\hat{k}}\colon \bigoplus_{\hat{k}'\in\hat{K}} R_{\hat{k}'}^*\mathcal{Q} \to R_{\hat{k}^{-1}}^* \bigoplus_{\hat{k}'\in\hat{K}} R_{\hat{k}'}^*\mathcal{Q} \tag{8.4}$$

$$(\phi_{\hat{k}'})_{\hat{k}'} \mapsto (\lambda(\hat{k},\hat{k}'\hat{k}^{-1})\phi_{\hat{k}'\hat{k}^{-1}})_{\hat{k}'}. \tag{8.5}$$

The *deformed Fourier–Mukai transform*

$$\mathrm{FM}_\lambda\colon D^b(\hat{X}^\lambda) \to D^b(X_\lambda) \tag{8.6}$$

is the integral functor with the deformed Poincaré line bundle as the integral kernel, i.e., the composite of the pull-back

$$(q^{Y,\hat{Y}})^*\colon D^b(\hat{X}^\lambda) \to D^b(Y\times\hat{X}^\lambda), \tag{8.7}$$

the tensor product

$$(-)\otimes\mathcal{P}_\lambda\colon D^b(Y\times\hat{X}^\lambda) \to D^b(X_\lambda\times\hat{X}), \tag{8.8}$$

and the push-forward

$$\mathbb{R}(p^{Y,\hat{X}})_*\colon D^b(X_\lambda\times\hat{X}) \to D^b(X_\lambda). \tag{8.9}$$

Its right adjoint is the integral functor FM_λ^{-1} with the g-shift of

$$\mathcal{P}_\lambda^{-1} := \mathcal{H}om_{\mathcal{O}_{Y\times\hat{Y}}}(\mathcal{P}_\lambda, \mathcal{O}_{Y\times\hat{Y}}) \in \operatorname{ob}(\operatorname{coh} X_\lambda^{\mathrm{op}}\times\hat{X}^\lambda) \tag{8.10}$$

as the kernel, since

- the push-forward $\mathbb{R}(q^{Y,\hat{Y}})_*$ is right adjoint to the pull-back $(q^{Y,\hat{Y}})^*$,
- the tensor product $\mathcal{P}_\lambda^{-1}\otimes(-)$ is right adjoint to the tensor product $(-)\otimes\mathcal{P}_\lambda$, and
- the pull-back $(p^{Y,\hat{X}})^*[g]$ shifted by g is right adjoint to the push-forward $\mathbb{R}(p^{Y,\hat{X}})_*$ because $D^b(X_\lambda)$ is Calabi–Yau of dimension g and $D^b(X_\lambda\times\hat{X})$ is Calabi–Yau of dimension $2g$ (it will be proved in Section 10).

Theorem 8.1 below is the main result in this paper:

Theorem 8.1. The deformed Fourier–Mukai transform FM_λ is an equivalence of derived categories for any $N\in\mathbb{N}$ and any $\lambda\in H^2(\hat{T},\boldsymbol{\mu}_N)$.

Remark 8.2. Let $\mathcal{O}_{\mathsf{T}_\lambda\times\hat{\Gamma}}$ be the sheaf associative algebras $(\varpi\times\mathrm{id})_*\mathcal{O}_{T\times\hat{\Gamma}}$, with a non-commutative associative product defined by the formula similar to (4.3), but a_t and b_t are functions on $\hat{\Gamma}$. For any $\lambda\in H^2(\hat{T},U(1))$ not necessarily at roots of unity, it is natural to expect that $\operatorname{coh}\hat{X}^\lambda$ is derived-equivalent to $\operatorname{coh}\mathsf{X}_\lambda$. Although the latter is not known to be abelian, we can define the deformed Poincaré line bundle P_λ as an object of $\operatorname{Mod}\left(\mathsf{X}_\lambda\times\hat{X}^{\lambda^{-1}}\right)$, i.e. a $\mathcal{O}_{\mathsf{T}_\lambda\times\hat{\Gamma}}$-module equipped with a λ-twisted left $\hat{T}$-action and a right Γ-action.

P_λ is defined by a free $\mathcal{O}_{\mathsf{T}_\lambda\times\hat{\Gamma}}$-module of rank 1 equipped with a λ-twisted left $\hat{T}$-action

$$\hat{\gamma}\cdot\phi(x,\hat{x}) = \hat{\gamma}(x)*_\lambda\phi(x,\hat{x}\hat{\gamma}) \tag{8.11}$$

and the right Γ-action

$$\phi(x,\hat{x})\cdot\gamma = \phi(x\gamma^{-1},\hat{x}) *_\lambda \hat{x}(\gamma)^{-1}. \tag{8.12}$$

9. Finite group actions on abelian and derived categories

It is natural to examine group actions on DG-categories in relation to group actions on derived categories and equivariant Fourier-Mukai transforms. However, coherent data for group actions on DG-categories are more intricate than those for group actions on abelian categories. As such, we will concentrate on group actions on abelian categories and the actions they induce on derived categories.

A *weak action* of a finite group G on a category $\mathcal{C}$ is a family $(g_*)_{g\in G}$ of autoequivalences $g_*\colon \mathcal{C}\to\mathcal{C}$ such that the functor $(g_1)_*\circ(g_2)*$ is isomorphic to $(g_1g_2)_*$ for any $g_1,g_2\in G$. An *action* is a weak action equipped with a *coherence data*, i.e., a family $(c_{g_1,g_2})_{g_1,g_2\in G}$ of isomorphisms $c_{g_1,g_2}\colon (g_1)_*\circ(g_2)_* \xrightarrow{\sim} (g_1g_2)_*$ of functors such that the diagram

$$\begin{tikzcd}
(g_1)_*\circ(g_2)_*\circ(g_3)_* \arrow[r, "c_{g_1,g_2}"] \arrow[d, "c_{g_2,g_3}"'] & (g_1g_2)_*\circ(g_3)_* \arrow[d, "c_{g_1g_2,g_3}"'] \\
(g_1)_*\circ(g_2g_3)_* \arrow[r, "c_{g_1,g_2g_3}"] & (g_1g_2g_3)_*
\end{tikzcd} \tag{9.1}$$

commutes for any $g_1,g_2,g_3\in G$ (cf. e.g. [Del97]). An action is *strict* if the coherence data consists of identities.

Let $\mathcal{C}$ be a category equipped with an action of a finite group G. The following definition is taken from [Sos12]:

Definition 9.1 ([Sos12, Definition 3.1])**.** A *linearization* of $A\in\mathrm{ob}(\mathcal{C})$ is a family $(\rho_g)_{g\in G}$ of isomorphisms $\rho_g\colon A\xrightarrow{\sim} g_*A$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\rho_{g_1}"] \arrow[dr, "\rho_{g_1g_2}"'] & (g_1)_*A \arrow[d, "c_{g_1,g_2}\circ(g_1)_*(\rho_{g_2})"] \\
 & (g_1g_2)_*A
\end{tikzcd} \tag{9.2}$$

commutes for any $g_1,g_2\in G$. An *equivariant object* is an object equipped with a linearization. A *morphism* of equivariant objects from $(A,(\rho_g)_{g\in G})$ to $(A',(\rho'_g)_{g\in G})$ is a morphism $\phi\colon A\to A'$ such that the diagram commute

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "\rho_g"'] & A' \arrow[d, "\rho'_g"'] \\
g_*A \arrow[r, "g_*\phi"] & g_*A'
\end{tikzcd} \tag{9.3}$$

commutes.

The category of G-equivariant objects in $\mathcal{C}$ will be denoted by $\mathcal{C}^G$. For the rest of this paper and unless otherwise specified, we will assume that $\mathcal{C}$ is a $\mathbb{C}$-linear category and weak actions consists of $\mathbb{C}$-linear functors.

Propositon 9.2 ([Sos12, Proposition 3.2])**.** If $\mathcal{C}$ is abelian, then so is $\mathcal{C}^G$.

We extend the above constructions to twisted group actions. Let ϕ be a second cocycle of G with values in $\mathbb{C}^*$.

Definition 9.3. A *ϕ-twisted linearization* of $A \in \operatorname{ob}(\mathcal{C})$ is a family $(\rho_g)_{g\in G}$ of isomorphisms $\rho_g \colon A \xrightarrow{\sim} g_* A$ such that the diagram

$$
\begin{tikzcd}
A \arrow[r, "\rho_{g_1}"] \arrow[dr, "\phi(g_1,g_2)\rho_{g_1 g_2}"'] & (g_1)_* A \arrow[d, "c_{g_1,g_2}\circ (g_1)_*(\rho_{g_2})"] \\
& (g_1 g_2)_* A
\end{tikzcd}
\tag{9.4}
$$

commutes for any $g_1, g_2 \in G$. Morphisms of ϕ-twisted equivariant objects are defined in the same way as in $\mathcal{C}^G$.

A ϕ-twisted linearization of $A \in \operatorname{ob}(\mathcal{C})$ is equivalent to a linearization of A for G-action $\{\rho_g\}_{g\in G}$ equipped with a coherence data $(\phi(g_1,g_2)^{-1}c_{g_1,g_2})_{g_1,g_2\in G}$. The category of ϕ-twisted equivariant objects will be denoted by $\mathcal{C}^{G,\phi}$. The cocycle condition on ϕ ensures the equality of

$$\rho_{g_3}\circ\rho_{g_2}\circ\rho_{g_1} = \rho_{g_3}\circ\left(\phi(g_1,g_2)\rho_{g_1g_2}\right) = \phi(g_1,g_2)\phi(g_1g_2,g_3)\rho_{g_1g_2g_3} \tag{9.5}$$

and

$$\rho_{g_3}\circ\rho_{g_2}\circ\rho_{g_1} = \left(\phi(g_2,g_3)\rho_{g_2g_3}\right)\circ\rho_{g_1} = \phi(g_2,g_3)\phi(g_1,g_2g_3)\rho_{g_1g_2g_3}, \tag{9.6}$$

where we have omitted $(g_1)_*$ and so on. If a pair of cocycles ϕ and ϕ' differ by the coboundary of $\alpha \in C^1(G,\mathbb{C}^*)$, then there exists an equivalence $\mathcal{C}^{G,\phi} \to \mathcal{C}^{G,\phi'}$ sending $(A,(\rho_g)_{g\in G})$ to $(A,(\alpha(g)\rho_g)_{g\in G})$. Explanations of relations between group cohomology of G in low degrees and (weak) G-actions are found in [BO20].

Corollary 9.4 below is obtained by applying Proposition 9.2 to the G-action equipped with a coherence data $(\phi(g_1,g_2)^{-1}c_{g_1,g_2})_{g_1,g_2\in G}$:

Corollary 9.4. If $\mathcal{C}$ is abelian, then so is $\mathcal{C}^{G,\phi}$.

By applying the free-forgetful adjunction

$$\text{Free} \dashv \text{Forget} \tag{9.7}$$

between

$$\text{Free}\colon \mathcal{C}\to\mathcal{C}^{G,\phi}, \qquad A \mapsto \left(\bigoplus_{g'\in G}(g')_*A,\ \left(\rho_g := \sum_{g'\in G}\phi(g,g')\left(\mathrm{id}\colon (gg')_*A \to g_*(g')_*A\right)\right)_{g\in G}\right) \tag{9.8}$$

and

$$\text{Forget}\colon \mathcal{C}^{G,\phi}\to\mathcal{C}, \qquad (A,(\rho_g)_{g\in G}) \mapsto A \tag{9.9}$$

to the opposite categories and using the equivalence $(\mathcal{C}^{G,\phi})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^{G,\phi^{-1}}$, one obtains an adjunction

$$\text{Forget} \dashv \text{Free}\,. \tag{9.10}$$

For any $(A,(\rho_g)_{g\in G}), (A',(\rho'_g)_{g\in G}) \in \mathcal{C}^{G,\phi}$, the space $\operatorname{Hom}_{\mathcal{C}}(A,A')$ comes with a natural linear action of G in such a way that the diagram

$$
\begin{tikzcd}
A \arrow[r, "\chi"] \arrow[d, "\rho_g"'] & A' \arrow[d, "\rho'_g"] \\
g_*A \arrow[r, "g_*(\chi\cdot g)"] & g_*A'
\end{tikzcd}
\tag{9.11}
$$

commutes since

$$
\begin{aligned}
&(9.12) & \chi \cdot g_1 \cdot g_2 &= ((g_{1*})^{-1}(\rho'_{g_1} \circ \chi \circ \rho_{g_1}^{-1})) \cdot g_2 \\
&(9.13) & &= (g_{2*})^{-1}(\rho'_{g_2} \circ ((g_{1*})^{-1}(\rho'_{g_1} \circ \chi \circ \rho_{g_1}^{-1})) \circ \rho_{g_2}^{-1})) \\
&(9.14) & &= (g_{2*})^{-1}(g_{1*})^{-1}(g_{1*}\rho'_{g_2} \circ (\rho'_{g_1} \circ \chi \circ \rho_{g_1}^{-1}) \circ g_{1*}\rho_{g_2}^{-1}) \\
&(9.15) & &= ((g_1 g_2)_*)^{-1}((\phi(g_1, g_2)\rho'_{g_1 g_2}) \circ \chi \circ (\phi(g_1, g_2)\rho_{g_1 g_2})^{-1}) \\
&(9.16) & &= \chi \cdot g_1 g_2.
\end{aligned}
$$

It follows from the definition that

$$
\operatorname{Hom}_{\mathcal{C}^G}((A, (\rho_g)_{g \in G}), (A', (\rho'_g)_{g \in G})) = \operatorname{Hom}_{\mathcal{C}}(A, A')^G. \tag{9.17}
$$

A functor $\Phi \colon \mathcal{C} \to \mathcal{C}'$ between categories with G-actions is said to be *G-equivariant* if it is equipped with a family $(a_g)_{g \in G}$ of natural isomorphisms $a_g \colon \Phi \circ g_* \xrightarrow{\sim} g_* \circ \Phi$ of functors such that the diagram

$$
\begin{tikzcd}
\Phi \circ (g_1)_* \circ (g_2)_* \arrow[r, "a_{g_1}"] \arrow[d, "c_{g_1, g_2}"'] & (g_1)_* \circ \Phi \circ (g_2)_* \arrow[r, "a_{g_2}"] & (g_1)_* \circ (g_2)_* \circ \Phi \arrow[dl, "c_{g_1, g_2}"] \\
\Phi \circ (g_1 g_2)_* \arrow[r, "a_{g_1 g_2}"] & (g_1 g_2)_* \circ \Phi &
\end{tikzcd} \tag{9.18}
$$

commutes. A G-equivariant functor $\Phi \colon \mathcal{C} \to \mathcal{C}'$ induces a functor $\Phi^{G,\phi} \colon \mathcal{C}^{G,\Phi} \to \mathcal{C}'^{G,\phi}$ sending an object $(A, (\rho_g)_{g \in G})$ to $(\Phi(A), (a_g \circ \Phi(\rho_g))_{g \in G})$ and a morphism $f \colon (A, (\rho_g)_{g \in G}) \to (A', (\rho'_g)_{g \in G})$ to $\Phi(f) \colon \Phi(A) \to \Phi(A')$. It is straightforward to show that $\Phi^{G,\phi}$ send G-equivariant objects to G-equivariant objects.

Propositon 9.5. If Φ is right (resp. left) exact, then so is $\Phi^{G,\phi}$.

Proof. Since Free and Forget are mutually both left and right adjoint to each other, they are exact, so that a sequence $A \to B \to C$ in $\mathcal{C}^{G,\phi}$ is exact if and only if $\operatorname{Forget}(A) \to \operatorname{Forget}(B) \to \operatorname{Forget}(C)$ is exact in $\mathcal{C}$. $\square$

Now we discuss the derived category of $\mathcal{C}^{G,\phi}$.

Propositon 9.6. An object $(I, (\rho_g)_{g \in G}) \in \operatorname{ob}(\mathcal{C}^{G,\phi})$ is injective if and only if so is $I \in \operatorname{ob}(\mathcal{C})$. The category $\mathcal{C}^{G,\phi}$ has enough injectives if and only if so is $\mathcal{C}$.

Proof. If I is injective, then the functor

$$
A \mapsto \operatorname{Hom}_{\mathcal{C}^{G,\phi}}(A, (I, (\rho_g))) = \operatorname{Hom}_{\mathcal{C}}(\operatorname{Forget}(A), I)^G \tag{9.19}
$$

is exact, since Forget is exact, I is injective, and taking the G-invariant part is exact.

Conversely, if $(I, (\rho_g))$ is injective, then the functor

$$
A \mapsto \operatorname{Hom}_{\mathcal{C}}(A, I) \cong \operatorname{Hom}_{\mathcal{C}^{G,\phi}}(\operatorname{Free}(A), (I, (\rho_g))) \tag{9.20}
$$

is exact.

Let $(A, (\rho_g)_g)$ be an object in $\mathcal{C}^{G,\phi}$. If $\mathcal{C}$ has enough injectives, then a monomorphism $A \to I$ into an injective object $I \in \operatorname{ob}(\mathcal{C})$ gives a monomorphism $A \to \operatorname{Free} I$ into $\operatorname{Free} I$, which is injective in $\mathcal{C}^{G,\phi}$.

Conversely, if $\mathcal{C}^{G,\phi}$ has enough injectives, then for any $A \in \operatorname{ob}(\mathcal{C})$, a monomorphism $\operatorname{Free} A \to I$ into an injective $I \in \operatorname{ob}(\mathcal{C}^{G,\phi})$ gives a monomorphism $A \to \operatorname{Forget} I$ into an injective $\operatorname{Forget} I$. $\square$

Assume that $\mathcal{C}$ has enough injectives. For any pair $A, B \in \operatorname{ob}(D^+(\mathcal{C}^{G,\phi}))$ of objects, the space $\operatorname{Hom}(\operatorname{Forget}(A), \operatorname{Forget}(B))$ has a natural G-action in such a way that

$$
\operatorname{Hom}(A, B) \cong \operatorname{Hom}(\operatorname{Forget}(A), \operatorname{Forget}(B))^G. \tag{9.21}
$$

Propositon 9.7. If $D^b(\mathcal{C})$ is Calabi–Yau of dimension n, then so is $D^b(\mathcal{C}^{G,\phi})$.

Proof. There exists a natural isomorphism

$$\operatorname{Hom}_{D^b(\mathcal{C})}(A, B) \cong \operatorname{Hom}_{D^b(\mathcal{C})}(B, A[n])^* \tag{9.22}$$

for $A, B \in \operatorname{ob}(D^b(\mathcal{C}^{G,\phi}))$, since $D^b(\mathcal{C})$ is Calabi–Yau of dimension n. It is G-equivariant by the naturality, so it induces an isomorphism on G-invariant part. This means that $D^b(\mathcal{C}^{G,\phi})$ is also Calabi–Yau of dimension n by (9.21). $\square$

A G-equivariant left exact functor $\Phi\colon \mathcal{C} \to \mathcal{C}'$ induces functors $\mathbb{R}\Phi\colon D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ and $\mathbb{R}\Phi^{G,\phi}\colon D^+(\mathcal{C}^{G,\phi}) \to D^+(\mathcal{C}'^{G,\phi})$. If $\mathbb{R}\Phi$ is fully faithful, then so is $\mathbb{R}\Phi^{G,\phi}$ by (9.21).

10. Group actions on coherent sheaves

An action of a finite group G on a complex manifold Z induces a strict G-action $(R_g^*)_{g\in G}$ on $\operatorname{coh} Z$. In the case of the $\hat{K}$-action on $\hat{Y}$, one obtains $(\operatorname{coh}\hat{Y})^{\hat{K},\lambda} \simeq \operatorname{coh}\hat{X}^\lambda$.

Another example of a finite group action on the category of coherent sheaves comes from a *coherent* injection from a finite abelian group G to the Picard group $\operatorname{Pic}^0 Z$ of a complex manifold Z, i.e., a family $\{\mathcal{L}_g\}_{g\in G}$ of line bundles and a family $(m_{g_1,g_2})_{g_1,g_2\in G}$ of isomorphisms $m_{g_1,g_2}\colon \mathcal{L}_{g_1}\otimes\mathcal{L}_{g_2} \xrightarrow{\sim} \mathcal{L}_{g_1g_2}$ such that the diagrams

$$\begin{tikzcd} \mathcal{L}_{g_1}\otimes\mathcal{L}_{g_2}\otimes\mathcal{L}_{g_3} \arrow[r, "m_{g_1,g_2}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m_{g_2,g_3}"'] & \mathcal{L}_{g_1g_2}\otimes\mathcal{L}_{g_3} \arrow[d, "m_{g_1g_2,g_3}"] \\ \mathcal{L}_{g_1}\otimes\mathcal{L}_{g_2g_3} \arrow[r, "m_{g_1,g_2g_3}"] & \mathcal{L}_{g_1g_2g_3} \end{tikzcd} \tag{10.1}$$

$$\begin{tikzcd} \mathcal{L}_{g_1}\otimes\mathcal{L}_{g_2} \arrow[r, "m_{g_1,g_2}"] \arrow[d] & \mathcal{L}_{g_1g_2} \\ \mathcal{L}_{g_2}\otimes\mathcal{L}_{g_1} \arrow[ur, "m_{g_2,g_1}"'] & \end{tikzcd} \tag{10.2}$$

commutes, inducing a G-action $\big((-)\otimes\mathcal{L}_g^{-1}\big)_{g\in G}$ on $\operatorname{coh} Z$. Here, the vertical arrow in (10.2) comes from the canonical symmetric monoidal structure in $\operatorname{coh} Z$. If Z is compact and connected, then one has $\operatorname{Aut}\mathcal{L} = \mathbb{C}^*$ for any line bundle $\mathcal{L}$, and an example of a coherence data $(m_{g_1,g_2})_{g_1,g_2\in G}$ comes from a choice of a collection $(\varphi_g)_{g\in G}$ of linear isomorphisms $\varphi\colon (\mathcal{L}_g)_z \xrightarrow{\sim} \mathbb{C}$ from the fibers $(\mathcal{L}_g)_z$ of $\mathcal{L}_g$ at an arbitrarily chosen and fixed base point $z \in Z$ to the complex line.

A ϕ-twisted G-linearization $(\rho_g)_{g\in G}$ of $\mathcal{M}$ is equivalent to a family $(m_g)_{g\in G}$ of morphisms

$$m_g = (g_*)^{-1}\rho_g\colon \mathcal{M}\otimes\mathcal{L}_g \to \mathcal{M} \tag{10.3}$$

such that diagram

$$\begin{tikzcd} \mathcal{M}\otimes\mathcal{L}_{g_1}\otimes\mathcal{L}_{g_2} \arrow[r, "m_{g_1}"] \arrow[dr, "\phi(g_1,g_2)m_{g_1g_2}"'] & \mathcal{M}\otimes\mathcal{L}_{g_2} \arrow[d, "m_{g_2}"] \\ & \mathcal{M} \end{tikzcd} \tag{10.4}$$

commutes. The category $(\operatorname{coh} Z)^{G,\phi}$ is equivalent to $\operatorname{coh}\mathcal{A}_\phi$, where $\mathcal{A}_\phi$ is the sheaf of $\mathcal{O}_Z$-algebras obtained as the $\mathcal{O}_Z$-module $\bigoplus_{g\in G}\mathcal{L}_g$ equipped with the multiplication given by $\sum_{g_1,g_2\in G}\phi(g_1,g_2)m_{g_1,g_2}$. If $\phi = 1$, $\mathcal{A}_\phi$ is commutative by the commutativity of (10.2). In particular, $\operatorname{coh} X_\lambda$ is equivalent to $(\operatorname{coh} Y)^{\hat{K},\lambda}$. By Proposition 9.7, $D^b(X_\lambda)$ is Calabi–Yau of dimension g and $D^b(X_\lambda\times\hat{X})$ is Calabi–Yau of dimension $2g$, which were needed to prove that FM_λ^{-1} is right adjoint to FM_λ.

11. Proof of Theorem 8.1

Functors FM_λ and $\mathrm{FM}_\mathcal{Q}$ are the right derived functors of functors

$$\mathrm{FM}_\lambda^{\mathrm{ab}}\colon \operatorname{coh}\hat{X}^\lambda \to \operatorname{coh} X_\lambda \tag{11.1}$$
$$\mathcal{M} \mapsto (p^{Y,\hat{X}})_*((q^{Y,\hat{Y}})^*\mathcal{M} \otimes_{\mathcal{O}_{Y\times\hat{Y}}} \mathcal{P}_\lambda) \tag{11.2}$$

and

$$\mathrm{FM}_\mathcal{Q}^{\mathrm{ab}}\colon \operatorname{coh}\hat{Y} \to \operatorname{coh} Y \tag{11.3}$$
$$\mathcal{M} \mapsto (p^{Y,\hat{Y}})_*((q^{Y,\hat{Y}})^*\mathcal{M} \otimes_{\mathcal{O}_{Y\times\hat{Y}}} \mathcal{Q}) \tag{11.4}$$

of abelian categories.

Since

$$\mathrm{FM}_\mathcal{Q}^{\mathrm{ab}}(R_{\hat{y}}^*\mathcal{M}) = (p^{Y,\hat{Y}})_*((q^{Y,\hat{Y}})^* R_{\hat{y}}^*\mathcal{M} \otimes \mathcal{Q}) \tag{11.5}$$
$$\cong (p^{Y,\hat{Y}})_* R_{(1,\hat{y}^{-1})}^*((q^{Y,\hat{Y}})^* R_{\hat{y}}^*\mathcal{M} \otimes \mathcal{Q}) \tag{11.6}$$
$$\cong (p^{Y,\hat{Y}})_*((q^{Y,\hat{Y}})^*\mathcal{M} \otimes R_{(1,\hat{y}^{-1})}^*\mathcal{Q}) \tag{11.7}$$
$$\cong (p^{Y,\hat{Y}})_*((q^{Y,\hat{Y}})^*\mathcal{M} \otimes \mathcal{Q} \otimes (p^{Y,\hat{Y}})^*\mathcal{L}_{\hat{y}}^{-1}) \tag{11.8}$$
$$\cong (p^{Y,\hat{Y}})_*((q^{Y,\hat{Y}})^*\mathcal{M} \otimes \mathcal{Q}) \otimes \mathcal{L}_{\hat{y}}^{-1} \tag{11.9}$$
$$= \mathrm{FM}_\mathcal{Q}^{\mathrm{ab}}(\mathcal{M}) \otimes \mathcal{L}_{\hat{y}}^{-1}, \tag{11.10}$$

for any $\hat{y} \in \hat{K}$ and $\mathcal{M} \in \operatorname{ob}(\operatorname{coh}\hat{Y})$, $\mathrm{FM}_\mathcal{Q}^{\mathrm{ab}}$ commutes with the weak $\hat{K}$-action. The commutativity of the diagram (9.18) in this case is a straightforward diagram chasing. This turns $\mathrm{FM}_\mathcal{Q}^{\mathrm{ab}}$ into a $\hat{K}$-equivariant functor, inducing a functor

$$(\mathrm{FM}_\mathcal{Q}^{\mathrm{ab}})^{\hat{K},\lambda}\colon \operatorname{coh}\hat{X}^\lambda \to \operatorname{coh} X_\lambda. \tag{11.11}$$

Lemma 11.1. The functor $(\mathrm{FM}_\mathcal{Q}^{\mathrm{ab}})^{\hat{K},\lambda}$ is isomorphic to $\mathrm{FM}_\lambda^{\mathrm{ab}}$.

Proof. The squares on the left and the right of the diagram

$$\begin{tikzcd}
(\operatorname{coh}\hat{Y})^{\hat{K},\lambda} \arrow[r, "((q^{Y,\hat{Y}})^*)^{\hat{K},\lambda}"] \arrow[d] & (\operatorname{coh}(Y\times\hat{Y}))^{\hat{K},\lambda} \arrow[r, "(\pi_*(-\otimes\mathcal{Q}))^{\hat{K},\lambda}"] \arrow[d] & (\operatorname{coh}(Y\times\hat{X}))^{\hat{K},\lambda} \arrow[r, "((p^{Y,\hat{X}})_*)^{\hat{K},\lambda}"] \arrow[d] & (\operatorname{coh} Y)^{\hat{K},\lambda} \arrow[d] \\
\operatorname{coh}\hat{X}^\lambda \arrow[r, "(q^{Y,\hat{Y}})^*"] & \operatorname{coh}(Y\times\hat{X}^\lambda) \arrow[r, "-\otimes\mathcal{P}_\lambda"] & \operatorname{coh}(X_\lambda\times\hat{X}) \arrow[r, "(p^{Y,\hat{X}})_*"] & \operatorname{coh}\hat{X}^\lambda
\end{tikzcd}$$

commute by definition, and the natural isomorphism

$$\oplus_{\hat{y}'\in\hat{K}}\rho_{\hat{y}'}^{-1} \otimes \mathrm{id}\colon \bigoplus_{\hat{y}'\in\hat{K}} R_{(1,\hat{y}')}^*(\mathcal{M} \otimes_{\mathcal{O}_{Y\times\hat{Y}}} \mathcal{Q}) \to \bigoplus_{\hat{y}'\in\hat{K}} \mathcal{M} \otimes_{\mathcal{O}_{Y\times\hat{Y}}} R_{(1,\hat{y}')}^*\mathcal{Q}, \tag{11.12}$$

whose $\hat{K}$-equivariance can be checked by a straightforward computation, gives the commutativity of the square in the middle. $\square$

Therefore $\mathbb{R}(\mathrm{FM}_\mathcal{Q}^{\mathrm{ab}})^{\hat{K},\lambda}$ and FM_λ are isomorphic. Hence FM_λ is fully faithful as explained at the end of Section 9. Similarly, the right adjoint FM_λ^{-1} of FM_λ, whose kernel is given by the g-shift of (8.10), is also fully faithful, and Theorem 8.1 is proved.

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