

博士論文

論文題目 Irreducible module decompositions
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Irreducible module decompositions of rank 2 symmetric hyperbolic Kac-Moody Lie algebras by \mathfrak{sl}_2 subalgebras which are generalizations of principal \mathfrak{sl}_2 subalgebras

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Abstract

There exist principal \mathfrak{sl}_2 subalgebras for hyperbolic Kac-Moody Lie algebras. In the case of rank 2 symmetric hyperbolic Kac-Moody Lie algebras, certain \mathfrak{sl}_2 subalgebras are constructed in a previous paper. These subalgebras are generalizations of principal \mathfrak{sl}_2 subalgebras. We show that the rank 2 symmetric hyperbolic Kac-Moody Lie algebras themselves are irreducibly decomposed under the action of this \mathfrak{sl}_2 subalgebras. Furthermore, we classify irreducible components of the decomposition. In particular, we obtain multiplicities of unitary principal series and complementary series.

1 Introduction

A nilpotent orbit in a finite dimensional simple Lie algebra \mathfrak{g}_0 is an orbit obtained by acting on the nilpotent element x of \mathfrak{g}_0 by inner automorphisms. In [Dyn57], these are classified by weighted Dynkin diagrams. From the Jacobson-Morozov theorem, for a nilpotent element x of \mathfrak{g}_0 , we can construct a \mathfrak{sl}_2 -triple with x as a nilpositive element ([CM93, Theorem 3.3.1]).

This makes it equivalent to classify nilpotent orbits of \mathfrak{g}_0 and to classify \mathfrak{sl}_2 triples in \mathfrak{g}_0 up to inner automorphisms. Among the nilpotent orbits of a finite dimensional simple Lie algebra, the one whose dimension as an algebraic variety is maximal is called the principal nilpotent orbit. Correspondingly, we can construct a principal $SO(3)$ subalgebra that is compatible with compact involution ([Kos59]).

Kac-Moody Lie algebras are generalizations of finite-dimensional simple Lie algebras. They are classified into three types: finite type, affine type, and indefinite type. The finite type Kac-Moody Lie algebras are finite dimensional simple Lie algebras. Within indefinite Kac-Moody Lie algebra, there is a class

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called hyperbolic Kac-Moody Lie algebra. A hyperbolic Kac-Moody Lie algebra is an indefinite type Kac-Moody Lie algebra such that any true subdiagram of its Dynkin diagram is of finite or affine type. Hyperbolic Kac-Moody Lie algebras, in particular E_{10} , are noted to be related to string theory([Vis]).

By analogy with the above theory, in [NO01], for a hyperbolic Kac-Moody Lie algebra, its principal $SO(1, 2)$ subalgebra was constructed. Note that [GOW02] shows that it is possible to construct a principal $SO(1, 2)$ subalgebra for certain indefinite Kac-Moody Lie algebra that is not hyperbolic.

Corresponding to this principal $SO(1, 2)$ subalgebra, we can construct a principal \mathfrak{sl}_2 -subalgebra in a hyperbolic Kac-Moody Lie algebra. In [Tsu], for the rank 2 symmetric hyperbolic Kac-Moody Lie algebras \mathfrak{g} , the following result is obtained. Let $R_{\mathfrak{g}}$ be the space that the positive real root vectors span. We consider \mathfrak{sl}_2 subalgebras whose nilpositive element exists in $R_{\mathfrak{g}}$. Then we can construct certain \mathfrak{sl}_2 subalgebras. These subalgebras are generalizations of principal \mathfrak{sl}_2 subalgebras.

In this paper, for an \mathfrak{sl}_2 subalgebra of rank 2 symmetric hyperbolic Kac-Moody Lie algebra \mathfrak{g} constructed in [Tsu], we show \mathfrak{g} is decomposed into irreducible \mathfrak{sl}_2 -modules by its action on \mathfrak{g} .

We are going to more details. Let \mathfrak{s} be an \mathfrak{sl}_2 subalgebra constructed in [Tsu]. Let H, X, Y be an \mathfrak{sl}_2 triple and assume that \mathfrak{s} is spanned by H, X, Y . Let e_i, f_i, h_i , ($i = 0, \dots, n-1$) be the Chevalley generators of \mathfrak{g} . Let $\mathfrak{h}_{\mathbb{R}}$ be the \mathbb{R} -span of h_i 's. From [Kac90, Theorem 2.2], \mathfrak{g} has a \mathbb{C} -valued nondegenerate invariant symmetric bilinear form $(\cdot | \cdot)$ called the standard form. An antilinear automorphism ω_0 of \mathfrak{g} , called compact involution, is determined by

$$\begin{aligned}\omega_0(e_i) &= -f_i, \\ \omega_0(f_i) &= -e_i \quad (i = 0, \dots, n-1), \\ \omega_0(h) &= -h \quad (h \in \mathfrak{h}_{\mathbb{R}}).\end{aligned}$$

From [Kac90, §2.7], we can determine a nondegenerate Hermitian form $(\cdot | \cdot)_0$ on \mathfrak{g} with $(x | y)_0 = -(\omega_0(x) | y)$.

An \mathfrak{s} -module $V \subset \mathfrak{g}$ is called unitarizable if following conditions are satisfied.

- (1) $(\cdot | \cdot)_0$ on V is positive definite.
- (2) For $v_1, v_2 \in V$ and $s \in \mathfrak{s}$, the following condition is satisfied.

$$([s, v_1], v_2)_0 = -(v_1, [\omega_0(s), v_2])_0.$$

Theorem 1.1 (Theorem 4.5). \mathfrak{g} can be decomposed into a direct sum of irreducible \mathfrak{s} -modules such that \mathfrak{s} itself is one of the direct summand. All of these modules except for \mathfrak{s} are unitarizable.

Also, we classify how many highest weight modules, lowest weight modules, and modules that are neither highest weight module nor lowest weight module appear in this decomposition. We regard a root $s\alpha_1 + t\alpha_2$ as a point (s, t) in xy -plane, and We define a region $L, -L$ in xy -plane in §5. If a root α satisfies $\alpha(H) \in (0, 2)$, $\alpha \in L$. If a root α satisfies $\alpha(H) \in (-2, 0)$, $\alpha \in -L$.

Theorem 1.2 (Theorem 7.1). We consider an irreducible decomposition of \mathfrak{g} by the action of \mathfrak{s} .

- (1) Let M be an irreducible component of decomposition of \mathfrak{g} , which contain a root space for a real root in L . Then, M is a unitary principal or complementary series representation.
- (2) (cf. [Tsu, Proposition 7.3]) There is a unitary principal series representation containing an 1-dimensional space in \mathfrak{h} .
- (3) \mathfrak{g} is decomposed into a direct sum of \mathfrak{s} -submodules described in (1) and (2) above, \mathfrak{s} itself, irreducible lowest weight modules, and irreducible highest weight modules.

We also discuss how to calculate multiplicities of irreducible highest or lowest modules (§7). Furthermore, we classified irreducible components which are neither highest weight modules nor lowest weight modules, as either unitary principal or complementary series representations.

Theorem 1.3 (Theorem 8.12). We consider irreducible components which are neither highest weight modules nor lowest weight modules and contain root vectors about real roots in L , obtained by Theorem 7.1. The irreducible components are complementary series representations, except those described in Lemma 8.5 and Lemma 8.9. For the exceptions, the irreducible components are unitary principal series representations.

2 General theory of Kac-Moody Lie algebras

Let \mathfrak{g} be a symmetrizable Kac-Moody Lie algebra on \mathbb{C} . Let A be the Cartan matrix of \mathfrak{g} and let A be an $n \times n$ matrix. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Let e_i, f_i, h_i , ($i = 0, \dots, n-1$) be the Chevalley generators of \mathfrak{g} . Let $\mathfrak{h}_{\mathbb{R}}$ be the \mathbb{R} -span of h_i 's.

From [Kac90, Theorem 2.2], \mathfrak{g} has a \mathbb{C} -valued nondegenerate invariant symmetric bilinear form $(\cdot | \cdot)$ called the standard form.

An antilinear automorphism ω_0 of \mathfrak{g} , called compact involution, is determined by

$$\begin{aligned}\omega_0(e_i) &= -f_i, \\ \omega_0(f_i) &= -e_i \quad (i = 0, \dots, n-1), \\ \omega_0(h) &= -h \quad (h \in \mathfrak{h}_{\mathbb{R}}).\end{aligned}$$

From [Kac90, §2.7], we can determine a nondegenerate Hermitian form $(\cdot | \cdot)_0$ on \mathfrak{g} with $(x | y)_0 = -(\omega_0(x) | y)$.

Write \mathfrak{n}^+ for a subalgebra of \mathfrak{g} generated by e_i 's and \mathfrak{n}^- for a subalgebra of \mathfrak{g} generated by f_i 's.

We can construct a 3-dimensional subalgebra of \mathfrak{g} which is spanned by three non-zero elements $J^+ \in \mathfrak{n}^+$, $J^- \in \mathfrak{n}^-$, $J_3 \in \mathfrak{h}$. J^+, J^- and J_3 satisfy

$$[J_3, J^{\pm}] = \pm J^{\pm},$$

$$[J^+, J^-] = -J_3.$$

This subalgebra is called $SO(1, 2)$ subalgebra of \mathfrak{g} .

A representation of $SO(1, 2)$ subalgebra is called unitary if the representation space V has a Hermitian scalar product (\cdot, \cdot) and the following two conditions are satisfied.

- (1) The actions of J^+ and J^- are adjoint each other, and the action of J_3 is self-adjoint. That is, for $x, y \in V$, we have

$$\begin{aligned} ([J^+, x], y) &= (x, [J^-, y]), \\ ([J_3, x], y) &= (x, [J_3, y]). \end{aligned}$$

- (2) Hermitian scalar product (\cdot, \cdot) is positive definite.

When considering the adjoint action of an $SO(1, 2)$ subalgebra of \mathfrak{g} to \mathfrak{g} , from [Tsu, Lemma 3.1, Lemma 3.2], we can see that the adjoint action satisfying the condition (1) to be unitary and $J^- = -\omega_0(J^+)$ are equivalent. In [NO01], principal $SO(1, 2)$ subalgebras for hyperbolic Kac-Moody Lie algebras are studied. Principal $SO(1, 2)$ subalgebra satisfies that $J^- = -\omega_0(J^+)$.

When three non-zero elements $X \in \mathfrak{n}^+$, $Y \in \mathfrak{n}^-$, $H \in \mathfrak{h}$ of \mathfrak{g} satisfy

$$\begin{aligned} [H, X] &= 2X, \\ [H, Y] &= -2Y, \\ [X, Y] &= H, \end{aligned}$$

these three elements are called \mathfrak{sl}_2 -triple of \mathfrak{g} . A \mathfrak{g} -subalgebra that these elements span is called \mathfrak{sl}_2 subalgebra. The $SO(1, 2)$ subalgebras and the \mathfrak{sl}_2 subalgebras can be converted by

$$\begin{aligned} J^+ &= \frac{1}{\sqrt{2}}X, \\ J^- &= -\frac{1}{\sqrt{2}}Y, \\ J_3 &= \frac{1}{2}H. \end{aligned}$$

The condition $J^- = -\omega_0(J^+)$ in $SO(1, 2)$ subalgebra is converted to $Y = \omega_0(X)$ in \mathfrak{sl}_2 subalgebra. In the following paper, we consider \mathfrak{sl}_2 subalgebra that satisfies $Y = \omega_0(X)$.

3 \mathfrak{sl}_2 -triples of rank 2 hyperbolic symmetric Lie algebra that is compatible to compact involution

Let a be an integer that satisfies $a \geq 3$, and let \mathfrak{g} be a hyperbolic Kac-Moody Lie algebra on \mathbb{C} such that the Cartan matrix of \mathfrak{g} is

$$\begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}.$$

Let α_0, α_1 be the simple roots of \mathfrak{g} . Let $\{F_n\}$ be the sequence of numbers determined by $F_0 = 0, F_1 = 1, F_{k+2} = aF_{k+1} - F_k$.

Lemma 3.1 ([KM95, Proposition 4.4]). The real positive roots of \mathfrak{g} are of the form

$$\alpha = F_{k+1}\alpha_0 + F_k\alpha_1$$

or

$$\beta = F_k\alpha_0 + F_{k+1}\alpha_1.$$

We distinguish these roots as type α and type β , and we also distinguish root vectors belonging to each root as type α and type β (cf. [Tsu, §4]).

Let X be an element of the space which real positive root vectors span. Then X can be written as

$$X = \sum_k c_k E_k, \quad (k \in \{0, \dots, n_X - 1\}, c_k \in \mathbb{C}, c_k \neq 0, E_k \in \mathfrak{g}_{\beta_k}, E_k \neq 0)$$

where β_k ($k \in \{0, \dots, n_X - 1\}$) are distinct real roots and n_X is a positive integer.

We call this n_X the length of X . Then the following holds.

Lemma 3.2 ([Tsu, Theorem 5.8]). Let X be an element in the space which real positive root vectors span.

- (1) When the length of X is 1 or more than 3, $X, Y = \omega_0(X), H = [X, Y]$ do not form \mathfrak{sl}_2 -triple.
- (2) Suppose the length of X is 2 and E_0, E_1 are real positive root vectors of different types (in the sense of α -type and β -type). Then, taking the appropriate $c_0, c_1 \in \mathbb{C}$, $X = c_0 E_0 + c_1 E_1$, $Y = \omega_0(X)$, and $H = [X, Y]$ form \mathfrak{sl}_2 -triple. In particular, c_0, c_1 can be chosen so that $c_0, c_1 \in \mathbb{R}$.

Lemma 3.3 ([Tsu, Theorem 6.4]). Take $\langle H, X, Y \rangle$ in Lemma 3.2, (2). Let $X = c_0 E_0 + c_1 E_1$, where E_0 is type α and E_1 is type β . From Lemma 3.1, using integers $i, j \geq 0$, we can write $E_0 \in \mathfrak{g}_{F_{i+1}\alpha_0 + F_i\alpha_1}$, $E_1 \in \mathfrak{g}_{F_j\alpha_0 + F_{j+1}\alpha_1}$. If and only if $i = j - 1, j, j + 1$, H is dominant.

4 Irreducible decomposition of \mathfrak{g} as an \mathfrak{sl}_2 module

In this section, we consider an \mathfrak{sl}_2 -subalgebra $\mathfrak{s} = \langle H, X, Y \rangle$ of \mathfrak{g} , which satisfies the following conditions.

- (1) $H \in \mathfrak{h}$ and H is dominant.
- (2) X is in the space which is spanned by positive root vectors.
- (3) $Y = \omega_0(X)$.

We show that \mathfrak{g} is decomposed to irreducible modules by the action of \mathfrak{s} .

\mathfrak{s} -module $V \subset \mathfrak{g}$ is called unitarizable if following conditions are satisfied.

- (1) $(\cdot | \cdot)_0$ on V is positive definite.
- (2) For $v_1, v_2 \in V$ and $s \in \mathfrak{s}$, the following condition is satisfied.

$$([s, v_1], v_2)_0 = -(v_1, [\omega_0(s), v_2])_0.$$

From [Kac90, §2.7], the condition (2) are automatically satisfied. Therefore, $(\cdot | \cdot)_0$ is positive definite on V if and only if V is unitarizable.

First, we put

$$U = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{s} (x | y)_0 = 0\}.$$

U is closed under the action of \mathfrak{s} , and $\mathfrak{g} = \mathfrak{s} \oplus U$.

Lemma 4.1. $(\cdot | \cdot)_0$ is positive definite on U .

Proof. From [Kac90, Theorem 11.7], $(\cdot | \cdot)_0$ is positive definite on $\mathfrak{n}^+ \oplus \mathfrak{n}^-$. The sign of $(\cdot | \cdot)_0$ on \mathfrak{h} is $(1, 1)$. Let $\mathfrak{h}_{\mathfrak{s}}$ be the space that H spans. Since \mathfrak{s} itself is not unitarizable, when we write $\mathfrak{h} = \mathfrak{h}_{\mathfrak{s}} \oplus \mathfrak{h}'$, $(\cdot | \cdot)_0$ is not positive definite on \mathfrak{s} . Therefore, $(\cdot | \cdot)_0$ is positive definite on \mathfrak{h}' . Since $U = \mathfrak{h}' \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$, $(\cdot | \cdot)_0$ is positive definite on U . \square

Lemma 4.2. Consider a subspace V of U that is closed under the action of H . Let V^\perp be the subspace of U orthogonal to V with respect to the Hermitian form $(\cdot | \cdot)_0$. Then $U = V \oplus V^\perp$.

Proof. We consider the eigenspace decomposition of U by H . Let U_λ be the eigenspace for λ and write

$$U = \bigoplus_{\lambda \in \mathbb{C}} U_\lambda.$$

Since H is a Hermitian operator on $(\cdot | \cdot)_0$, U_λ and U_μ are orthogonal with respect to this inner product if $\lambda \neq \mu$. Since H is dominant, U_λ is finite-dimensional. For each λ , V also inherits the eigenspace decomposition of U .

Let V_λ be an eigenspace of V for λ , and V can be written as a direct sum of V_λ 's. Let

$$V'_\lambda = \{v \in V_\lambda \mid \forall x \in V_\lambda (v \mid x)_0 = 0\},$$

and

$$V' = \bigoplus_{\lambda \in \mathbb{C}} V'_\lambda.$$

V_λ is finite dimensional. From Lemma 4.1, $(\cdot \mid \cdot)_0$ is positive definite on U . Thus we have $U_\lambda = V_\lambda \oplus V'_\lambda$. Therefore, we have $U = V \oplus V'$ and $V' = V^\perp$. \square

In the following, we show that U can be decomposed into irreducible modules by the action of \mathfrak{s} .

Lemma 4.3. Any non-zero \mathfrak{sl}_2 -submodule V of U includes an irreducible submodule.

Proof. Take the eigenspace decomposition of U by the action of H . V is also decomposed into eigenspaces with this decomposition, and each eigenspace of V is finite-dimensional. We regard H as a linear transform on V and take some eigenvalue λ of H on V . Let $U(\mathfrak{sl}_2)$ be an universal enveloping algebra of \mathfrak{sl}_2 . Considering the Casimir element C of $U(\mathfrak{sl}_2)$, it preserves V_λ . Since V_λ is finite-dimensional, there exists an eigenvector of C . Let v denote this. Consider the \mathfrak{sl}_2 -submodule generated by v , which includes an irreducible submodule. \square

Theorem 4.4. (cf. [Kob94, Theorem 1.2]) U can be decomposed into direct sum of irreducible \mathfrak{s} -modules, and all of these modules are unitarizable.

Proof. We consider a set of irreducible submodules of U such that these submodules are orthogonal to each other with respect to $(\cdot \mid \cdot)_0$. Let T be the set. We order the elements of T by inclusion. Then T is non-empty and inductively ordered. Therefore, from Zorn's lemma, T has a maximal element. Take a maximal element of T and denote it by \mathcal{M} . Consider the direct sum of all submodules belonging to \mathcal{M} . Let M denote this sum. Suppose $U \neq M$, we derive the contradiction. Since M is a subspace of U which is closed by the action of H , from Lemma 4.2, we have $U = M \oplus M^\perp$. Since M^\perp is non-zero \mathfrak{sl}_2 submodule of U , from Lemma 4.3, M^\perp includes an irreducible submodule. Let W denote this. we have $\mathcal{M} \cup \{W\} \in T$, that is contradict the maximality of \mathcal{M} . Therefore, we have $U = M$, and U can be decomposed into direct sum of irreducible submodules. Combining this with Lemma 4.1, we can also see the unitarizability of the modules. \square

Theorem 4.5. \mathfrak{g} can be decomposed into direct sum of irreducible \mathfrak{s} -modules, which consists \mathfrak{s} itself. All of these modules except for \mathfrak{s} are unitarizable. \square

5 \mathfrak{sl}_2 modules in \mathfrak{g}

In the following, we consider what kind of modules appear when \mathfrak{g} is decomposed into irreducible \mathfrak{s} -modules. In particular, we consider how many unitary principal or complementary series representations.

For a Lie algebra \mathfrak{a} , let $U(\mathfrak{a})$ be the universal enveloping algebra of \mathfrak{a} . Let V be an irreducible \mathfrak{s} -module which is an irreducible component of \mathfrak{g} . The Casimir element C of $U(\mathfrak{s})$ acts on V by constant multiplication. Let μ be this constant. From [HT92, Chapter II, Corollary 1.1.11], for an eigenvalue $\lambda_0 \in \mathbb{C}$ of H on V , some interval $I \subset \mathbb{Z}$ exists, and V can be expressed as a direct sum of 1-dimensional eigenspaces such that the eigenvalues of H are $\lambda_k = \lambda_0 + 2k$ ($k \in I$). From [HT92, Chapter II, Theorem 1.1.13], for an eigenvalue λ of H on V , we define $s_1(k)$ for an integer k as

$$(A) \quad s_1(k) = \frac{8\mu - (\lambda + 2k - 1)^2 + 1}{4}.$$

We take an element v_k of the eigenspace of V with respect to an eigenvalue $\lambda + 2k$. Then we have $X(Yv_k) = s_1(k)v_k$. If $k \in \mathbb{Z}$ such that $s_1(k) = 0$ does not exist, then V is an irreducible module that is neither highest weight module nor lowest weight module. If there exists a $k \in \mathbb{Z}$ such that $s_1(k) = 0$, V is a highest weight module or a lowest weight module.

Let \mathcal{W} be the Weyl group of \mathfrak{g} . Using Lemma 3.2, we may write H, X, Y in \mathfrak{s} as follows.

$$\begin{aligned} X &= c_0 w_0(e_p) + c_1 w_1(e_q) & (c_0, c_1 \in \mathbb{R}, w_0, w_1 \in \mathcal{W}, (p, q) \in \{(0, 1), (0, 0), (1, 1)\}), \\ Y &= -c_0 w_0(f_p) - c_1 w_1(f_q), \\ H &= -c_0 w_0(h_p) - c_1 w_1(h_q). \end{aligned}$$

Let $k_{\mathfrak{s}}, l_{\mathfrak{s}}, m_{\mathfrak{s}}, n_{\mathfrak{s}}$ be real numbers such that $c_0 w_0(e_p) \in \mathfrak{g}_{k_{\mathfrak{s}}\alpha_0 + l_{\mathfrak{s}}\alpha_1}$, $c_1 w_1(e_q) \in \mathfrak{g}_{m_{\mathfrak{s}}\alpha_0 + n_{\mathfrak{s}}\alpha_1}$. From Lemma 3.3, we can write $k_{\mathfrak{s}} = F_{i+1}$, $l_{\mathfrak{s}} = F_i$, $m_{\mathfrak{s}} = F_j$, $n_{\mathfrak{s}} = F_{j+1}$ with integers $i, j \geq 0$, and furthermore, $i \in \{j-1, j, j+1\}$.

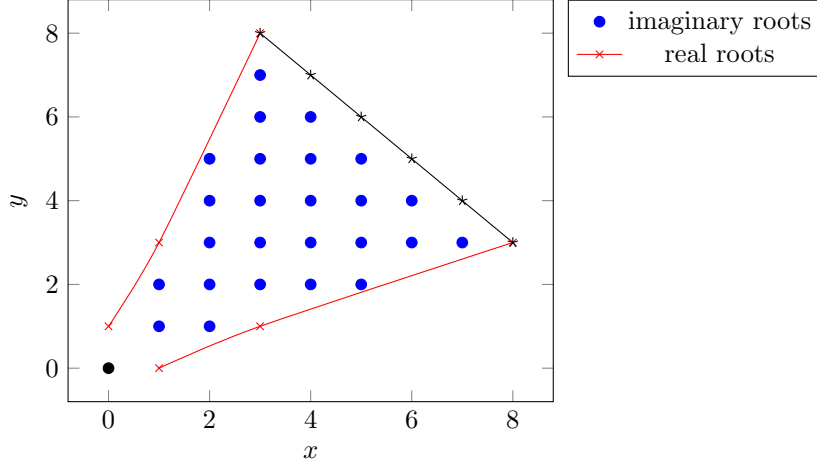
When we take the root vector $E \in \mathfrak{g}_{s\alpha_0 + t\alpha_1}$ with $s, t \in \mathbb{Z}$, we want to find out which of the three types of modules E generates under the action of \mathfrak{s} .

We define L in the xy -plane as follows. L is a region satisfying $x \geq 0$, $y \geq 0$, $(x, y) \neq (0, 0)$, $x^2 - axy + y^2 \leq 1$ and the following conditions.

$$\begin{aligned} x &< k_{\mathfrak{s}} = F_{i+1}, & (\text{when } i = j-1) \\ x + y &< k_{\mathfrak{s}} + l_{\mathfrak{s}} = F_i + F_{i+1}, & (\text{when } i = j) \\ y &< l_{\mathfrak{s}} = F_i. & (\text{when } i = j+1) \end{aligned}$$

If we take the root $s\alpha_0 + t\alpha_1$ with $s, t \in \mathbb{Z}$, then from [KM95, Cor 4.3], the point in the xy -plane given by (s, t) is in the interior or on the boundary of the hyperbola $x^2 - axy + y^2 = 1$. Let h_C be this hyperbola. Let $\lambda \in \mathbb{R}$ as the value for which $HE = \lambda E$. We have $\lambda = (s\alpha_0 + t\alpha_1)(H)$. $\lambda \in (0, 2)$ if and only if $(s, t) \in L$. In the following, we regard a root $s\alpha_0 + t\alpha_1$ as a point (s, t) in the xy -plane.

Figure 1: Imaginary roots and real roots in L , $a = 3$, $X = c_0 r_0 r_1(e_0) + c_1 r_1 r_0(e_1)$



Lemma 5.1. We consider the hyperbola h_C on the xy -plane. The h_C was represented by $x^2 - axy + y^2 = 1$. Let l_b be a line represented by the function $y = -x + b$ with some real number $b \geq 0$. There are two intersections of h_C and l_b . Let p_1 and p_2 be these points. Let d_b be a distance between p_1 and p_2 . d_b is strictly monotonically increasing with respect to $b \geq 0$. The same result holds when l_b is a line represented by $y = b$ or $x = b$.

Proof. First, we consider the case where l_b is represented by $y = -x + b$. Calculating the y -coordinates of p_1, p_2 gives

$$y = \frac{(a+2)b \pm \sqrt{(a+2)(a-2)b^2 + 4(a+2)}}{2(a+2)}.$$

Therefore, we have

$$d_b = \sqrt{2} \cdot \frac{\sqrt{(a+2)(a-2)b^2 + 4(a+2)}}{a+2}.$$

This d_b is strictly monotonically increasing with respect to $b \geq 0$.

Next, we consider the case where l_b is represented by $y = b$. Calculating the x -coordinates of p_1, p_2 gives

$$x = \frac{ab \pm \sqrt{(a^2-4)b^2 + 4}}{2}.$$

Therefore, we have

$$d_b = \sqrt{(a^2-4)b^2 + 4}.$$

This d_b is strictly monotonically increasing with respect to $b \geq 0$. The same argument is presented when l_b is a line represented by $x = b$. \square

Let R be the interior of h_C and h_C itself. For $s, t \in \mathbb{Z}$, (s, t) is a root if and only if $(s, t) \in R$.

Lemma 5.2. If $(x, y) \in L \cup -L$, then neither $(x + k_{\mathfrak{s}} - m_{\mathfrak{s}}, y + l_{\mathfrak{s}} - n_{\mathfrak{s}})$ nor $(x - k_{\mathfrak{s}} + m_{\mathfrak{s}}, y - l_{\mathfrak{s}} + n_{\mathfrak{s}})$ are roots.

Proof. First we assume $(x, y) \in L$. The points $(k_{\mathfrak{s}}, l_{\mathfrak{s}})$ and $(m_{\mathfrak{s}}, n_{\mathfrak{s}})$ are on the hyperbola h_C . Let l_1 be the line connecting these two points. Using some real number $b > 0$, l_1 is represented by $y = -x + b$ when $i = j$, $y = b$ when $i = j - 1$, and $x = b$ when $i = j + 1$. Let l_2 be a line parallel to l_1 and passing through (x, y) . Using some real number $0 < b' < b$, l_2 is represented by $y = -x + b'$ when $i = j$, $y = b'$ when $i = j - 1$, and $x = b'$ when $i = j + 1$. Let p_{11}, p_{12} be intersections of h_C and l_1 . Let d_1 be the distance between p_{11} and p_{12} . Let p_{21}, p_{22} be intersections of h_C and l_2 . Let d_2 be the distance between p_{21} and p_{22} . From Lemma 5.1, we have $d_1 > d_2$. The distance between $(k_{\mathfrak{s}}, l_{\mathfrak{s}})$ and $(m_{\mathfrak{s}}, n_{\mathfrak{s}})$ is d_1 . The distance between (x, y) and $(x + k_{\mathfrak{s}} - m_{\mathfrak{s}}, y + l_{\mathfrak{s}} - n_{\mathfrak{s}})$ is also d_1 . These two points are on l_2 . The length of the part of l_2 that is inside the hyperbola is $d_2 < d_1$. From the fact that (x, y) is inside h_C , $(x + k_{\mathfrak{s}} - m_{\mathfrak{s}}, y + l_{\mathfrak{s}} - n_{\mathfrak{s}})$ is outside the hyperbola. Therefore, $(x + k_{\mathfrak{s}} - m_{\mathfrak{s}}, y + l_{\mathfrak{s}} - n_{\mathfrak{s}})$ is not in R . The same argument for $(x - k_{\mathfrak{s}} + m_{\mathfrak{s}}, y - l_{\mathfrak{s}} + n_{\mathfrak{s}})$ shows that it is not in R . From symmetry, the case when $(x, y) \in -L$ is also shown. \square

Lemma 5.3. For a point $(s, t) \in L$ corresponding to the root, we consider the root vector $E \in \mathfrak{g}_{s\alpha_0+t\alpha_1}$. Then $[X, [Y, E]] \in \mathfrak{g}_{s\alpha_0+t\alpha_1}$.

Proof. We have $X = c_0 w_0(e_p) + c_1 w_1(e_q)$, $Y = -c_0 w_0(f_p) - c_1 w_1(f_q)$. Also we have $c_0 w_0(e_p) \in \mathfrak{g}_{k_{\mathfrak{s}}\alpha_0+l_{\mathfrak{s}}\alpha_1}$, $c_1 w_1(e_q) \in \mathfrak{g}_{m_{\mathfrak{s}}\alpha_0+n_{\mathfrak{s}}\alpha_1}$. Then we have

$$[X, [Y, E]] \in \mathfrak{g}_{s\alpha_0+t\alpha_1} + \mathfrak{g}_{(s-k_{\mathfrak{s}}+m_{\mathfrak{s}})\alpha_0+(t-l_{\mathfrak{s}}+n_{\mathfrak{s}})\alpha_1} + \mathfrak{g}_{(s-m_{\mathfrak{s}}+k_{\mathfrak{s}})\alpha_0+(t-n_{\mathfrak{s}}+l_{\mathfrak{s}})\alpha_1}.$$

Since (s, t) is a root, from Lemma 5.2, $(s - k_{\mathfrak{s}} + m_{\mathfrak{s}}, t - l_{\mathfrak{s}} + n_{\mathfrak{s}})$ and $(s - m_{\mathfrak{s}} + k_{\mathfrak{s}}, t - n_{\mathfrak{s}} + l_{\mathfrak{s}})$ are not roots. Therefore, we have $\mathfrak{g}_{(s-k_{\mathfrak{s}}+m_{\mathfrak{s}})\alpha_0+(t-l_{\mathfrak{s}}+n_{\mathfrak{s}})\alpha_1} + \mathfrak{g}_{(s-m_{\mathfrak{s}}+k_{\mathfrak{s}})\alpha_0+(t-n_{\mathfrak{s}}+l_{\mathfrak{s}})\alpha_1} = 0$, and $[X, [Y, E]] \in \mathfrak{g}_{s\alpha_0+t\alpha_1}$. \square

We consider the Casimir element C of $U(\mathfrak{s})$. We can write $C = \frac{1}{8}H^2 - \frac{1}{4}H + \frac{1}{2}XY$.

Lemma 5.4. We consider a root space with respect to a root in L . C acts on the root space as endomorphism. The action is diagonalizable.

Proof. From Lemma 5.3, C acts on the root spaces as endomorphism. Since \mathfrak{g} is completely reducible as an \mathfrak{s} -modules, the action on the root space is diagonalizable. \square

Lemma 5.5. For a point $(s, t) \in L$ corresponding to the root, we can take the root vector $E \in \mathfrak{g}_{s\alpha_0+t\alpha_1}$ such that E is an eigenvector of the Casimir element C , and E generates an irreducible \mathfrak{s} -module.

Proof. From Lemma 5.4, we have the lemma. \square

From Lemma 5.5, if we decompose \mathfrak{g} by the action of \mathfrak{s} , the decomposition is compatible with the root space decomposition in the root in L .

We consider how many unitary principal or complementary series representations appear in the decomposition of \mathfrak{g} . Since the set of eigenvalues of unitary principal or complementary series representations is $\{\lambda + 2k \mid k \in \mathbb{Z}\}$ for some λ , such a module must contain an eigenspace such that its eigenvalue lie on $[0, 2)$. Therefore, we consider the root vector of H such that the eigenvalue λ of H satisfies $\lambda \in [0, 2)$.

If $\lambda = 0$, i.e., $s = t = 0$, Since the dimension of \mathfrak{h} is 2, there are two irreducible components of V which have 0-eigenspace (cf. [Tsu, §7]). Since one is \mathfrak{sl}_2 itself, we consider the other module. The casimir element C acts on this module by a constant multiple (let μ times). If k satisfies $s_1(k) = 0$, we get $8\mu + 1 = (2k - 1)^2$. Since $\mu < -1$ from [Tsu, Proposition 7.3], the left hand side is less than 0. Therefore, there is no integral solution to $s_1(k) = 0$, and this is an irreducible module that is neither highest weight module nor lowest weight module. In particular, this module is an unitary principal series representation.

In the following, we consider the case of $\lambda \in (0, 2)$. In this case, (s, t) is a root in L . We compute $[X, [Y, E]]$. Since $Y = -c_0w_0(f_p) - c_1w_1(f_q)$, we have

$$[Y, E] = [-c_0w_0(f_p), E] + [-c_1w_1(f_q), E].$$

We have also

$$\begin{aligned} [-c_0w_0(f_p), E] &\in \mathfrak{g}_{(s-k_s)\alpha_0+(t-l_s)\alpha_1}, \\ [-c_1w_1(f_q), E] &\in \mathfrak{g}_{(s-m_s)\alpha_0+(t-n_s)\alpha_1}. \end{aligned}$$

If $[-c_0w_0(f_p), E]$ and $[-c_1w_1(f_q), E]$ are not 0, then the eigenvalue of H for them must be in the $(-2, 0)$ interval. We consider root vectors which the eigenvalue of H are in the $(-2, 0)$. Since $R = -R$, the roots with respect to these root vectors are $-L$. From Lemma 5.2, if we take two points such that the difference is $(k_s - m_s, l_s - n_s)$ and one of which is a root in $-L$, then the other is not a root. Now we have $((s - m_s) - (s - k_s), (t - n_s) - (t - l_s)) = (k_s - m_s, l_s - n_s)$. Therefore, we know that at least one of $[-c_0w_0(f_p), E]$, $[-c_1w_1(f_q), E]$ is zero.

When both of these are 0, we have $[Y, E] = 0$ and from the fact that $C = \frac{1}{8}H^2 - \frac{1}{4}H + \frac{1}{2}XY$, we can write $8\mu = \lambda^2 - 2\lambda$.

When $[-c_0w_0(f_p), E] \neq 0$, i.e., $(s - k_s, t - l_s) \in R$, we have

$$\begin{aligned} [X, [Y, E]] &= [c_0w_0(e_p), [-c_0w_0(f_p), E]] \\ &= [E, [-c_0w_0(f_p), c_0w_0(e_p)]] + [-c_0w_0(f_p), [c_0w_0(e_p), E]]. \end{aligned}$$

We define $p_s \in \mathbb{C}$ by $[-c_0w_0(f_p), [c_0w_0(e_p), E]] = p_s E$, then we have

$$[X, [Y, E]] = [E, c_0^2w_0(h_p)] + p_s E.$$

When $p_s = 0$, we have

$$[X, [Y, E]] = -[c_0^2w_0(h_p), E].$$

Therefore in this case, if we let $-[c_0^2 w_0(h_p), E] = k_0 E$, then we have $8\mu = \lambda^2 - 2\lambda + 4k_0$.

When $[c_0 w_0(e_p), E] = 0$, i.e., $(s + k_{\mathfrak{s}}, t + l_{\mathfrak{s}}) \notin R$, we have $p_{\mathfrak{s}} = 0$.

To summarize the above, we take an irreducible decomposition of \mathfrak{g} by \mathfrak{s} . let $s\alpha_0 + t\alpha_1$ be a root in L . Let $E \in \mathfrak{g}_{s\alpha_0 + t\alpha_1}$ such that E generates an irreducible component of \mathfrak{g} . Let C be the Casimir element of $U(\mathfrak{s})$, and Let μ be a complex number such that $CE = \mu E$. Let k_0 and $p_{\mathfrak{s}}$ be complex numbers satisfying

$$\begin{aligned} [-c_0^2 w_0(h_p), E] &= k_0 E, \\ [-c_0 w_0(f_p), [c_0 w_0(e_p), E]] &= p_{\mathfrak{s}} E. \end{aligned}$$

If $(s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \notin R$, we have

$$8\mu = \begin{cases} \lambda^2 - 2\lambda & ((s - k_{\mathfrak{s}}, t - l_{\mathfrak{s}}) \notin R), \\ \lambda^2 - 2\lambda + 4k_0 & ((s - k_{\mathfrak{s}}, t - l_{\mathfrak{s}}) \in R \text{ and } (s + k_{\mathfrak{s}}, t + l_{\mathfrak{s}}) \notin R), \\ \lambda^2 - 2\lambda + 4k_0 + p_{\mathfrak{s}} & ((s - k_{\mathfrak{s}}, t - l_{\mathfrak{s}}) \in R \text{ and } (s + k_{\mathfrak{s}}, t + l_{\mathfrak{s}}) \in R). \end{cases}$$

If $(s - k_{\mathfrak{s}}, t - l_{\mathfrak{s}}) \notin R$ and not necessarily $(s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \notin R$, we have

$$8\mu = \begin{cases} \lambda^2 - 2\lambda & ((s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \notin R), \\ \lambda^2 - 2\lambda + 4k_0 & ((s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \in R \text{ and } (s + m_{\mathfrak{s}}, t + n_{\mathfrak{s}}) \notin R), \\ \lambda^2 - 2\lambda + 4k_0 + p_{\mathfrak{s}} & ((s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \in R \text{ and } (s + m_{\mathfrak{s}}, t + n_{\mathfrak{s}}) \in R). \end{cases}$$

Solving

$$s_1(k) = \frac{8\mu - (\lambda + 2k - 1)^2 + 1}{4} = 0$$

for k on \mathbb{R} , we obtain that

$$k = \begin{cases} 0, 1 - \lambda & ((s - k_{\mathfrak{s}}, t - l_{\mathfrak{s}}) \notin R \text{ and } (s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \notin R), \\ \frac{1 - \lambda \pm \sqrt{(\lambda - 1)^2 + 4k_0}}{2} & \left(\begin{array}{l} (s - k_{\mathfrak{s}}, t - l_{\mathfrak{s}}) \in R \text{ and } (s + k_{\mathfrak{s}}, t + l_{\mathfrak{s}}) \notin R \\ \text{or} \\ (s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \in R \text{ and } (s + m_{\mathfrak{s}}, t + n_{\mathfrak{s}}) \notin R \end{array} \right), \\ \frac{1 - \lambda \pm \sqrt{(\lambda - 1)^2 + 4k_0 + p_{\mathfrak{s}}}}{2} & \left(\begin{array}{l} (s - k_{\mathfrak{s}}, t - l_{\mathfrak{s}}) \in R \text{ and } (s + k_{\mathfrak{s}}, t + l_{\mathfrak{s}}) \in R \\ \text{or} \\ (s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \in R \text{ and } (s + m_{\mathfrak{s}}, t + n_{\mathfrak{s}}) \in R \end{array} \right). \end{cases}$$

When $(s - k_{\mathfrak{s}}, t - l_{\mathfrak{s}}) \notin R$ and $(s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \notin R$, since $(s, t) \in L$, we have $1 - \lambda \in (-1, 1)$. Therefore, we know that the only integral solution of $s_1(k) = 0$ is 0. In this case E belongs to an irreducible lowest weight module.

6 Classification by roots

Based on the previous section, we classify the root (s, t) in L . We define the types of roots as follows.

(1) We say that (s, t) is of type A when $(s - k_s, t - l_s) \notin R$ and $(s - m_s, t - n_s) \notin R$.

(2) We say that (s, t) is of type B when $\left\{ \begin{array}{c} (s - k_s, t - l_s) \in R \text{ and } (s + k_s, t + l_s) \notin R \\ \text{or} \\ (s - m_s, t - n_s) \in R \text{ and } (s + m_s, t + n_s) \notin R \end{array} \right\}$.

(3) We say that (s, t) is of type C when $\left\{ \begin{array}{c} (s - k_s, t - l_s) \in R \text{ and } (s + k_s, t + l_s) \in R \\ \text{or} \\ (s - m_s, t - n_s) \in R \text{ and } (s + m_s, t + n_s) \in R \end{array} \right\}$.

All roots belong to one of the above types. We put $f(x, y) = x^2 - axy + y^2$ for $x, y \in \mathbb{R}$. From [KM95, Cor 4.3], for $s, t \in \mathbb{Z}$, $(s, t) \neq (0, 0)$, (s, t) is a real root if and only if $f(s, t) = 1$, and (s, t) is an imaginary root if and only if $f(s, t) < 1$.

Lemma 6.1. For $x, y, x', y' \in \mathbb{R}$, if there exists $w \in \mathcal{W}$ such that $(x', y') = w(x, y)$, then $f(x', y') = f(x, y)$.

Proof. It is sufficient to check the case $w = r_0$ and the case $w = r_1$. From the symmetry, it is sufficient to check the case $w = r_0$. In this case, from the fact that $x' = ay - x$ and $y' = y$, we have

$$\begin{aligned} f(x', y') &= f(ay - x, y) \\ &= (ay - x)^2 - ay(ay - x) + y^2 \\ &= x^2 - axy + y^2 \\ &= f(x, y). \end{aligned}$$

□

First, we know the following results on real roots.

Lemma 6.2. If (s, t) is a real root in L and $s > t$, then $f(s - k_s, t - l_s) \leq 0$. Also, if (s, t) is a real root in L and $s < t$, then $f(s - m_s, t - n_s) \leq 0$.

Proof. From symmetry, it is sufficient to show $f(s - k_s, t - l_s) \leq 0$ when $s > t$. We can write $s = F_{c+1}$, $t = F_c$ with $c \geq 0$ being an integer. Since $k_s = F_{i+1}$ and $l_s = F_i$, we have $c < i$. Let $d_{ic} = i - c$. From Lemma 6.1, by acting r_0 and r_1 on $(s - k_s, t - l_s)$, we know that

$$\begin{aligned} f(s - k_s, t - l_s) &= f(F_{c+1} - F_{i+1}, F_c - F_i) \\ &= f(r_0(F_{c+1} - F_{i+1}, F_c - F_i)) \\ &= f(F_{c-1} - F_{i-1}, F_c - F_i) \\ &= f(r_1(F_{c-1} - F_{i-1}, F_c - F_i)) \\ &= f(F_{c-1} - F_{i-1}, F_{c-2} - F_{i-2}) \\ &= \dots \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} f(F_1 - F_{d_{ic}+1}, F_0 - F_{d_{ic}}) & (\text{when } c \text{ is even}) \\ f(F_0 - F_{d_{ic}}, F_1 - F_{d_{ic}+1}) & (\text{when } c \text{ is odd}) \end{cases} \\
&= f(F_1 - F_{d_{ic}+1}, F_0 - F_{d_{ic}}).
\end{aligned}$$

Since $F_1 = 1$, $F_0 = 0$, we have

$$\begin{aligned}
f(s - k_s, t - l_s) &= f(1 - F_{d_{ic}+1}, -F_{d_{ic}}) \\
&= 2 - aF_{d_{ic}} + 2F_{d_{ic}-1} \\
&< 2 - 2(F_{d_{ic}} - F_{d_{ic}-1}) \\
&\leq 0.
\end{aligned}$$

□

Lemma 6.3. If (s, t) is a real root in L , then (s, t) is of type B.

Proof. First we show that (s, t) is not of type A. From the fact that (s, t) is a real root and from symmetry, we can write $s = F_{c+1}$, $t = F_c$ with $c \geq 0$ being an integer. From $k_s = F_{i+1}$, $l_s = F_i$, we have $c < i$. From Lemma 6.2, $f(s - k_s, t - l_s) \leq 0$. Therefore, $(s - k_s, t - l_s) \in R$ and so we know that (s, t) is not of type A.

Next, we show that (s, t) is of type B. To show this, we need to show that $(s + k_s, t + l_s) \notin R$. We show $f(s + k_s, t + l_s) > 1$. Let $d_{ic} = i - c$. From Lemma 6.1, by acting r_0 and r_1 on $(s + k_s, t + l_s)$, we have

$$\begin{aligned}
f(s + k_s, t + l_s) &= f(F_{c+1} + F_{i+1}, F_c + F_i) \\
&= f(r_0(F_{c+1} + F_{i+1}, F_c + F_i)) \\
&= f(F_{c-1} + F_{i-1}, F_c + F_i) \\
&= f(r_1(F_{c-1} + F_{i-1}, F_c + F_i)) \\
&= f(F_{c-1} + F_{i-1}, F_{c-2} + F_{i-2}) \\
&= \dots \\
&= \begin{cases} f(F_1 + F_{d_{ic}+1}, F_0 + F_{d_{ic}}) & (\text{when } c \text{ is even}) \\ f(F_0 + F_{d_{ic}}, F_1 + F_{d_{ic}+1}) & (\text{when } c \text{ is odd}) \end{cases} \\
&= f(F_1 + F_{d_{ic}+1}, F_0 + F_{d_{ic}}) \\
&= f(1 + F_{d_{ic}+1}, F_{d_{ic}}) \\
&= 2 + aF_{d_{ic}} - 2F_{d_{ic}-1} \\
&> 2 + 2(F_{d_{ic}} - F_{d_{ic}-1}) \\
&> 4.
\end{aligned}$$

This shows that (s, t) is of type B. □

We classify also for imaginary roots in L .

Lemma 6.4. If $(s, t), (s', t')$ ($s, t, s', t' \geq 0$) are imaginary roots, then $(s + s', t + t')$ is also imaginary root.

Proof. Since $f(s, t) \leq 0$, for any $r \in \mathbb{R}$, we have $f(rs, rt) = r^2 f(s, t) \leq 0$. It shows that the line connecting the origin and (s, t) is inside the asymptotes of the hyperbola $x^2 - axy + y^2 = 1$. Similarly, the line connecting the origin and (s', t') is also inside the asymptotes.

Since $(s + s', t + t')$ is the midpoint of $(2s, 2t)$ and $(2s', 2t')$, this point is also inside the asymptotes. Therefore, $(s + s', t + t')$ is an imaginary root. \square

Lemma 6.5. Let $(u, v) \in L$ ($u > v$) be a real root such that $(u\alpha_0 + v\alpha_1)(H) \neq 0$. Put $(s, t) = (k_s - u, l_s - v)$. Then (s, t) is a type C imaginary root in L . Similarly, let $(u', v') \in L$ ($u' < v'$) be a real root such that $(u'\alpha_0 + v'\alpha_1)(H) \neq 0$. Put $(s', t') = (m_s - u', n_s - v')$. Then $(s', t') \in L$ and (s', t') is the imaginary root of type C. The other imaginary roots in L are of type A.

Proof. From Lemma 6.2, $f(-s, -t) = f(s, t) \leq 0$. It shows that (s, t) is a imaginary root. We also see that the eigenvalue of H for (s, t) is in the range $(0, 2)$. Therefore, $(s, t) \in L$ is shown.

We show that (s, t) is of type C. To show this, we show that $f(s + k_s, t + l_s) \leq 1$. Using $c \in \mathbb{Z}$, we can write $(u, v) = (F_{c+1}, F_c)$. Together this with $s + k_s = 2k_s - u$, $t + l_s = 2l_s - v$, we have

$$f(s + k_s, t + l_s) = f(2F_{i+1} - F_{c+1}, 2F_i - F_c).$$

Let $d_{ic} = i - c > 0$. From Lemma 6.1, acting r_0, r_1 on $(s + k_s, t + l_s)$, $i - c = \lambda \geq 1$, we have

$$\begin{aligned} f(2F_{i+1} - F_{c+1}, 2F_i - F_c) &= f(r_0(2F_{i+1} - F_{c+1}, 2F_i - F_c)) \\ &= f(2F_{i-1} - F_{c-1}, 2F_i - F_c) \\ &= f(r_1(2F_{i-1} - F_{c-1}, 2F_i - F_c)) \\ &= f(2F_{i-1} - F_{c-1}, 2F_{i-2} - F_{c-2}) \\ &= \dots \\ &= \begin{cases} f(2F_{d_{ic}+1} - F_1, 2F_{d_{ic}} - F_0) & (\text{when } c \text{ is even}) \\ f(2F_{d_{ic}} - F_0, 2F_{d_{ic}+1} - F_1) & (\text{when } c \text{ is odd}) \end{cases} \\ &= f(2F_{d_{ic}+1} - F_1, 2F_{d_{ic}} - F_0) \\ &= f(2F_{d_{ic}+1} - 1, 2F_{d_{ic}}) \\ &= -2aF_{d_{ic}} + 4F_{d_{ic}-1} + 5 \\ &< -6F_{d_{ic}} + 4F_{d_{ic}-1} + 5 \\ &= (-4F_{d_{ic}} + 4F_{d_{ic}-1}) + (-2F_{d_{ic}} + 5) \\ &< -4 - 2F_{d_{ic}} + 5 \\ &\leq -1. \end{aligned}$$

This shows that $f(s + k_s, t + l_s) \leq -1$ and that (s, t) is type C. From symmetry, we also know that (s', t') is in L and is the imaginary root of type C.

Finally, we show the other imaginary roots in L are of type A. Let $(s'', t'') \in L$ be such an imaginary root. We show $(s'' - m_s, t'' - n_s) \notin R$ and $(s'' - k_s, t'' - l_s) \notin$

R. If $(s'' - m_{\mathfrak{s}}, t'' - n_{\mathfrak{s}}) \in R$ or $(s'' - k_{\mathfrak{s}}, t'' - l_{\mathfrak{s}}) \in R$, $(s'' - m_{\mathfrak{s}}, t'' - n_{\mathfrak{s}}) \in -L$ or $(s'' - k_{\mathfrak{s}}, t'' - l_{\mathfrak{s}}) \in -L$. Since $(s'' - m_{\mathfrak{s}}, t'' - n_{\mathfrak{s}}) - (s'' - k_{\mathfrak{s}}, t'' - l_{\mathfrak{s}}) = (k_{\mathfrak{s}} - m_{\mathfrak{s}}, l_{\mathfrak{s}} - n_{\mathfrak{s}})$, from Lemma 5.2, we know $(s'' - m_{\mathfrak{s}}, t'' - n_{\mathfrak{s}}) \notin R$ or $(s'' - k_{\mathfrak{s}}, t'' - l_{\mathfrak{s}}) \notin R$.

From symmetry, it is sufficient to consider when $(s'' - m_{\mathfrak{s}}, t'' - n_{\mathfrak{s}}) \notin R$. Under this assumption, $(s'' - k_{\mathfrak{s}}, t'' - l_{\mathfrak{s}})$ is an imaginary root or not a root. If $(s'' - k_{\mathfrak{s}}, t'' - l_{\mathfrak{s}})$ is imaginary root, then $(k_{\mathfrak{s}} - s'', l_{\mathfrak{s}} - t'')$ is also imaginary root from the symmetry of R . We consider that $(k_{\mathfrak{s}}, l_{\mathfrak{s}}) = (s'', t'') + (k_{\mathfrak{s}} - s'', l_{\mathfrak{s}} - t'')$. The left hand side is real root and the right hand side is the sum of imaginary roots, which contradicts Lemma 6.4. Therefore, $(s'' - k_{\mathfrak{s}}, t'' - l_{\mathfrak{s}})$ is not a root and (s'', t'') is of type A. \square

The contents of this section can be summarized as follows.

Theorem 6.6. (1) A real roots in L is of type B.

(2) We consider an imaginary root that can be written as $(k_{\mathfrak{s}} - s, l_{\mathfrak{s}} - t)$ or $(m_{\mathfrak{s}} - s, n_{\mathfrak{s}} - t)$ where (s, t) is a real root. Such an imaginary root is of type C.

(3) The other imaginary roots are of type A. \square

We now summarize the irreducible \mathfrak{s} -modules through type A and type C. For \mathfrak{s} -modules through type A, we have the following.

Lemma 6.7. An irreducible \mathfrak{s} -module containing a root vector about a root of type A in L is a lowest weight module which the root vector is the lowest weight element.

Proof. Since $(s - k_{\mathfrak{s}}, t - l_{\mathfrak{s}}) \notin R$ and $(s - m_{\mathfrak{s}}, t - n_{\mathfrak{s}}) \notin R$ for the root (s, t) of type A, we know that acting Y on the type A root vector will result in 0. This shows the lemma. \square

Lemma 6.8. Let M be an irreducible \mathfrak{s} -module containing a root vector (say v) with respect to type C root in L . Then one of the following conditions (1) or (2) is hold.

(1) M is a lowest weight module such that v is a lowest element.

(2) M contains a real root vector with respect to a real root in $-L$.

Proof. The type C root (s, t) can be written with some real root (s_r, t_r) that $(k_{\mathfrak{s}} - s_r, l_{\mathfrak{s}} - t_r)$ or $(m_{\mathfrak{s}} - s_r, n_{\mathfrak{s}} - t_r)$. Therefore, the root vector E of type C becomes either zero or a real root vector when Y act on it. If E becomes 0 under the action of Y , then E generates an irreducible lowest weight module. If E becomes a real root vector, then the real root for this vector is in $-L$, and this lemma is shown. \square

We also give the type A, B, C distinction to the root of $-L$ by defining Theorem 6.6. Then, if there is a unitary principal or complementary series representation that passes through a root vector of type C in L , $-L$, it will also

pass through the root vector of type B in $-L, L$. Therefore, We have only to classify the modules that contains a type B root space.

Figure 2: $a = 3$, $X = c_0 r_0(e_1) + c_1 r_1(e_0)$

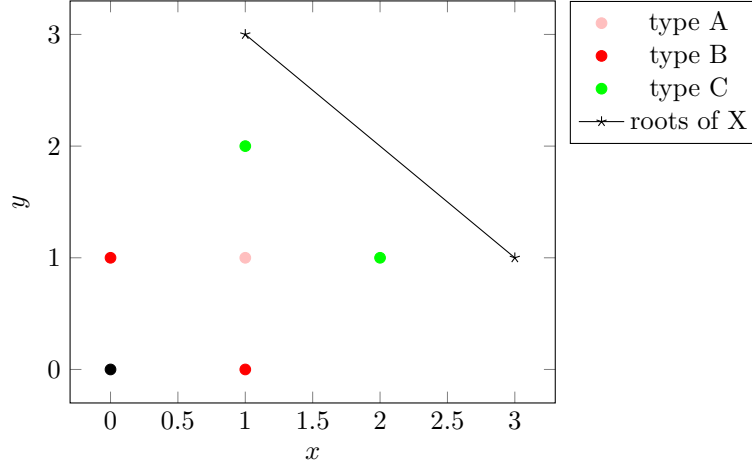
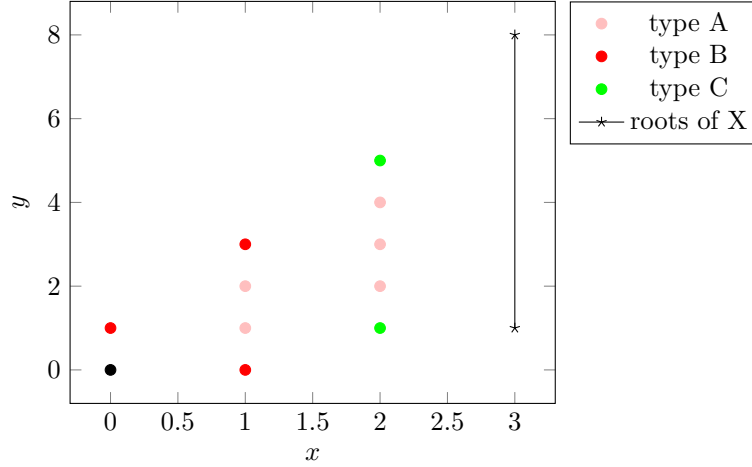


Figure 3: $a = 3$, $X = c_0 r_0(e_1) + c_1 r_1 r_0(e_1)$



7 Irreducible modules which contains a root space with respect to a type B root

We consider an irreducible decomposition of \mathfrak{g} by \mathfrak{s} , and we consider an irreducible component M containing a type B root space. The multiplicity of a real

root space is 1. We can take $0 < \lambda < 2$ such that $\{\lambda + 2k' \mid k' \in \mathbb{Z}\}$ is the set of the eigenvalues of H in M . We consider the H eigenspace of M such that the eigenvalue is λ . We assume this eigenspace is $\mathfrak{g}_{s\alpha_0+t\alpha_1}$ such that $(s, t) \in L$, and (s, t) is real root. We consider k such that $s_1(k) = 0$ in (A) in §5. We show that it is not an integer.

Let e_0, e_1, f_0, f_1, h_0 and h_1 be Chevalley generators. Using some $c_0, c_1 \in \mathbb{R}$, $w_0, w_1 \in \mathcal{W}$, and $(p, q) \in \{(0, 1), (0, 0), (1, 1)\}$, let $X = c_0 w_0(e_p) + c_1 w_1(e_q)$. Suppose $s > t$. We take the root vector E with respect to the root $s\alpha_0 + t\alpha_1$. We define λ by $HE = \lambda E$, and define k_0 by $[-c_0^2 w_0(h_p), E] = k_0 E$. Thus $s_1(k) = 0$ implies

$$k = \frac{1 - \lambda \pm \sqrt{(\lambda - 1)^2 + 4k_0}}{2}.$$

We put

$$k_+ = \frac{1 - \lambda + \sqrt{(\lambda - 1)^2 + 4k_0}}{2},$$

$$k_- = \frac{1 - \lambda - \sqrt{(\lambda - 1)^2 + 4k_0}}{2}$$

and we show that $k_{\pm} \notin \mathbb{R}$ or $0 < k_{\pm} < 1$.

From Lemma 3.3, we can write $c_0 w_0(e_p) \in \mathfrak{g}_{F_{i+1}\alpha_0 + F_i\alpha_1}$, $(s, t) = (F_{i'+1}, F_{i'})$ such that $0 \leq i' < i$. F_k is strictly increasing with respect to k . Since $F_{k+2} = aF_{k+1} - F_k$ and $a \geq 3$, we have $F_k > 2F_{k-1}$. Since

$$\lambda = \begin{cases} \frac{F_{i'+1}}{F_{i+1}} & (\text{when } i = j - 1) \\ \frac{F_{i'+1} + F_{i'}}{F_{i+1} + F_i} & (\text{when } i = j) \\ \frac{F_{i'}}{F_i} & (\text{when } i = j + 1) \end{cases},$$

we have $0 < \lambda < 1$. When $(\lambda - 1)^2 + 4k_0 < 0$ or $(\lambda - 1)^2 + 4k_0 \notin \mathbb{R}$, k_{\pm} are imaginary numbers. Therefore we can assume $(\lambda - 1)^2 + 4k_0 \geq 0$. From $0 < \lambda < 1$, it is clear that $k_+ > 0$ and $k_- < 1$. To show $k_+ < 1$, we need to show

$$1 - \lambda + \sqrt{(\lambda - 1)^2 + 4k_0} < 2.$$

we can easily show that it is reduced to $k_0 < \lambda$. Also, to show that $k_- > 0$, we need to show

$$1 - \lambda - \sqrt{(\lambda - 1)^2 + 4k_0} > 0.$$

we can easily show that it is reduced to $k_0 < 0$. In summary, we have only to show that $k_0 < 0$.

First, consider the case $(s, t) = (1, 0)$, i.e., $E \in \mathfrak{g}_{\alpha_0}$. Since

$$\begin{aligned} k_0 E &= [-c_0^2 r_0 r_1 r_0 \dots r_{1-p}(h_p), E] \\ &= [-c_0^2 (F_{i+1} h_0 + F_i h_1), E] \\ &= -c_0^2 (2F_{i+1} - aF_i) E \\ &= -c_0^2 (F_{i+1} + F_{i-1}) E, \end{aligned}$$

we have $k_0 < 0$. When $(s, t) = (0, 1)$, we can show that $k_0 < 0$ by replacing i with j , p with q and making the same argument.

If (s, t) is general and $s > t$, we can write $(s, t) = (F_{i'+1}, F_{i'})$. Let p' be 0 or 1, we can write $E = r_0 r_1 r_0 \dots r_{1-p'}(e_{p'})$. From this, we have

$$\begin{aligned} [-c_0^2 w_0(h_p), E] &= -c_0^2 [r_0 r_1 r_0 \dots r_{1-p}(h_p), r_0 r_1 r_0 \dots r_{1-p'}(e_{p'})] \\ &= -c_0^2 r_0 r_1 r_0 \dots r_{1-p'} [r_{p'} r_{1-p'} r_{p'} \dots r_{1-p}(h_p), e_{p'}]. \end{aligned}$$

We consider k_0 and c_0 when i is replaced by $i - i'$, and rewrite them as k'_0 and c'_0 . Considering $(s, t) = (1, 0)$ or $(0, 1)$ cases, we have $[r_{p'} r_{1-p'} r_{p'} \dots r_{1-p}(h_p), e_{p'}] = -\frac{k'_0}{c'_0} e_{p'}$. That is, $k_0 = \frac{c_0^2}{c'^2_0} k'_0$. Since $k'_0 < 0$, we have $k_0 < 0$. When $s < t$, we can show that $k_0 < 0$ as well.

From the above, it can be shown that $k_0 < 0$ in any case, i.e., k is not an integer. From this, we can see the following.

Theorem 7.1. We consider an irreducible decomposition of \mathfrak{g} by the action of \mathfrak{s} .

- (1) Let M is an irreducible component of decomposition of \mathfrak{g} , which contain a root space for a type B root $s\alpha_0 + t\alpha_1$. Then, M is an unitary principal or complementary series representation.
- (2) (cf. [Tsu, Proposition 7.3]) There is an unitary principal series representation containing an 1-dimensional space in \mathfrak{h} .
- (3) \mathfrak{g} is decomposed into a direct sum of \mathfrak{s} -submodules described in (1) and (2) above, \mathfrak{s} itself, irreducible lowest weight modules, and irreducible highest weight modules. \square

From [KM95, §3], the multiplicity of each root of \mathfrak{g} is calculated. Using this, we can find how many modules appear such that the following condition is satisfied: the modules are highest or lowest modules, and eigenvalues of H for root vectors with the highest or the lowest roots are certain value.

First, the modules which contain root spaces in L and $-L$ can be seen from previous contents. Among the positive root spaces not in L , those with the smallest eigenvalue in H are considered together. Let λ_H be their eigenvalue and d_H be their dimensions. Suppose p_H modules which contain space with eigenvalue λ_H that also contain the root spaces already obtained. Then there are $d_H - p_H$ lowest weight modules with the root with eigenvalue λ_H as the lowest root. The multiplicities of modules can be obtained inductively by replacing λ_H with the next smallest eigenvalue of H and performing the same calculation. Negative root spaces can be classified by the same calculation.

8 Unitary principal series representation and complementary series representation

In this section, we consider a module (say M) that is neither highest weight module nor lowest weight module containing a root vector about the root of type B. We compute whether the module is a unitary principal series representation or a complementary series representation. First, we state the following lemma.

Lemma 8.1. If $8\mu \leq -1$, then M is a unitary principal series representation. If $8\mu > -1$, then M is a complementary series representation.

Proof. From [HT92, §II 1.2], M is isomorphic to $U(\nu^+, \nu^-)$. $U(\nu^+, \nu^-)$ is a \mathfrak{sl}_2 -module with H eigenvectors $\{v_n \mid n \in \mathbb{Z}\}$ as a basis of linear space, such that

$$\begin{aligned} H v_n &= (\nu^+ - \nu^- + 2j) v_n & (n \in \mathbb{Z}), \\ X v_n &= (\nu^+ + n) v_{n+1}, \\ Y v_n &= (\nu^- - n) v_{n-1}, \\ 8\mu &= (\nu^+ + \nu^- - 1)^2 - 1. \end{aligned}$$

From [HT92, §III Theorem 1.1.3], if $\nu^+ + \overline{\nu^-} = 1$, $U(\nu^+, \nu^-)$ is a unitary principal series representation. When $8\mu \leq -1$, from

$$\begin{aligned} \lambda &= \nu^+ - \nu^- \in \mathbb{R}, \\ 8\mu &= (\nu^+ + \nu^- - 1)^2 - 1 < -1, \end{aligned}$$

using $b \in \mathbb{R}$ we can write

$$\begin{aligned} \nu^+ - \nu^- &= \lambda, \\ \nu^+ + \nu^- &= 1 + bi. \quad (i = \sqrt{-1}) \end{aligned}$$

In this case, we have

$$\begin{aligned} \nu^+ + \overline{\nu^-} &= \frac{\lambda + 1}{2} + \frac{b}{2}i + \frac{-\lambda + 1}{2} - \frac{b}{2}i \\ &= 1. \end{aligned}$$

Therefore, M is a unitary principal series representation.

Consider the case when $8\mu > -1$. From [HT92, §III Theorem 1.1.3], if $\nu^\pm \in \mathbb{R}$ and if $\nu^- - 1$ and $-\nu^+$ are both contained in the interval $(l - 1, l)$ with some $l \in \mathbb{Z}$, then $U(\nu^+, \nu^-)$ is a complementary series representation. From $8\mu > -1$, we have

$$\begin{aligned} \nu^+ + \nu^- &= 1 \pm \sqrt{8\mu + 1}, \\ \nu^+ - \nu^- &= \lambda. \end{aligned}$$

Therefore, we have

$$-\nu^+, \nu^- - 1 = \frac{-\lambda - 1 \pm \sqrt{8\mu + 1}}{2}.$$

We show that they are in $(-1, 0)$.

We show first that $0 < \lambda < 1$. Let n, m be integers such that $n > m \geq 0$. We can write

$$\lambda = \frac{2(F_{m+1} + F_m)}{F_{n+1} + F_n}.$$

It is clear that $\lambda > 0$. From $a \geq 3$, for integer $z \geq 0$, we have

$$\begin{aligned} F_{z+2} &= aF_{z+1} - F_z \\ &> (a-1)F_{z+1} \\ &\geq 2F_{z+1}. \end{aligned}$$

Hence we have

$$\frac{F_{m+1} + F_m}{F_{n+1} + F_n} < \frac{1}{2}.$$

Therefore, we have $\lambda < 1$. We show that

$$-1 < \frac{-\lambda - 1 + \sqrt{8\mu + 1}}{2}.$$

From $\lambda < 1$, we have $-1 < \frac{-\lambda-1}{2}$. Therefore, this inequality is shown. Next we show

$$\frac{-\lambda - 1 + \sqrt{8\mu + 1}}{2} < 0.$$

We have $8\mu = \lambda(\lambda - 2) + 4k_0$. From $0 < \lambda < 1$, we have $\lambda(\lambda - 2) < 0$. Also, since $k_0 < 0$, we have $8\mu < 0$. Therefore, we have $\sqrt{8\mu + 1} < 1$. Using $0 < \lambda$ again, we know that

$$\frac{-\lambda - 1 + \sqrt{8\mu + 1}}{2} < 0.$$

For

$$\frac{-\lambda - 1 - \sqrt{8\mu + 1}}{2} < 0,$$

this is clear from $\lambda > 0$. Finally, we show

$$-1 < \frac{-\lambda - 1 - \sqrt{8\mu + 1}}{2}.$$

From $k_0 < 0$ and $8\mu = \lambda^2 - 2\lambda + 4k_0$, we have $\lambda^2 - 2\lambda > 8\mu$. From this and $\lambda < 1$ we get $1 - \lambda > \sqrt{8\mu + 1}$, which can be transformed to

$$-1 < \frac{-\lambda - 1 - \sqrt{8\mu + 1}}{2}.$$

From the above, $\frac{-\lambda - 1 \pm \sqrt{8\mu + 1}}{2}$ are both in $(-1, 0)$. Therefore, M is a complementary series representation. \square

Hereafter, we want to determine when M is complementary series. First, we consider the case where $i = j$. we have

$$\begin{aligned} 8\mu &= \lambda^2 - 2\lambda + 4k_0, \\ \lambda &= \frac{2(F_{n+1} + F_n)}{F_{i+1} + F_i}, \\ k_0 &= \frac{-2(2F_{i+1-n} - aF_{i-n})}{a(F_i^2 + F_{i+1}^2) - 4F_i F_{i+1} - 2}, \end{aligned} \tag{B1}$$

where n is an integer such that $i > n \geq 0$. That is, 8μ is determined by i, n , and a . We show that 8μ is greater than -1 with finite exceptions.

Lemma 8.2. We assume $i = j$. If we consider 8μ to be a function of n by (B1), 8μ is monotonically decreasing with respect to n .

Proof. λ is monotonically increasing with respect to n . Since $8\mu = \lambda(\lambda - 2) + 4k_0$ and $0 < \lambda < 1$, we know that $\lambda(\lambda - 2)$ is monotonically decreasing with respect to n . Since $\{F_k\}$ is monotonically increasing with respect to k , $4k_0$ is monotonically decreasing with respect to n . Therefore, 8μ is monotonically decreasing with respect to n . \square

To show that 8μ is greater than -1 with finite exceptions, we need to examine when n is large.

Lemma 8.3. We assume $i = j$, $n = i - 1$. If we consider 8μ to be a function of i by (B1), 8μ is monotonically increasing with respect to i .

Proof. First we write $\{F_i\}$ explicitly as follows. The real solutions of $x^2 - ax + 1 = 0$ are $x = \frac{a \pm \sqrt{a^2 - 4}}{2}$. As $\alpha = \frac{a - \sqrt{a^2 - 4}}{2}$, $\beta = \frac{a + \sqrt{a^2 - 4}}{2}$, we can write

$$F_i = \frac{\beta^i - \alpha^i}{\beta - \alpha}.$$

From $n = i - 1$, we have

$$\begin{aligned} \lambda &= \frac{2(F_i + F_{i-1})}{F_{i+1} + F_i}, \\ k_0 &= \frac{-2a}{a(F_i^2 + F_{i+1}^2) - 4F_i F_{i+1} - 2}. \end{aligned}$$

Let t be a real variable. We define the functions Λ_1 and K_{01} as follows.

$$\begin{aligned}\Lambda_1(t) &= \frac{2(\beta^t - \alpha^t + \beta^{t-1} - \alpha^{t-1})}{\beta^{t+1} - \alpha^{t+1} + \beta^t - \alpha^t}, \\ K_{01}(t) &= \frac{-2a(\beta - \alpha)^2}{a((\beta^t - \alpha^t)^2 + (\beta^{t+1} - \alpha^{t+1})^2) - 4(\beta^t - \alpha^t)(\beta^{t+1} - \alpha^{t+1}) - 2(\beta - \alpha)^2} \\ &= \frac{-2a}{\beta^{2t+1} + \alpha^{2t+1} - 2}.\end{aligned}$$

We have $\lambda = \Lambda_1(i)$ and $k_0 = K_{01}(i)$. Using these function, we can calculate as follows.

$$\begin{aligned}\frac{d}{dt}\Lambda_1 &= \frac{4 \log \beta (a+2)(\beta - \alpha)}{(\beta^{t+1} - \alpha^{t+1} + \beta^t - \alpha^t)^2} \\ \frac{d}{dt}(\Lambda_1^2 - 2\Lambda_1) &= \frac{8 \log \beta (a+2)(\beta - \alpha) ((1-a)\beta^t - (1-a)\alpha^t + 3\beta^{t-1} - 3\alpha^{t-1})}{(\beta^{t+1} - \alpha^{t+1} + \beta^t - \alpha^t)^3} \\ \frac{d}{dt}K_{01} &= \frac{4a \log \beta (\beta^{2t+1} - \alpha^{2t+1})}{(\beta^{2t+1} + \alpha^{2t+1} - 2)^2}\end{aligned}$$

Clearing the denominator, we can calculate as follows.

$$\begin{aligned}& \frac{(\beta^{t+1} - \alpha^{t+1} + \beta^t - \alpha^t)^3 (\beta^{2t+1} + \alpha^{2t+1} - 2)^2}{8(a+2) \log \beta} \cdot \frac{d}{dt}(\Lambda_1^2 - 2\Lambda_1 + 4K_{01}) \\ &= (a+1)(\beta^{5t+3} + \alpha^{5t+3}) + (2a+3)(\beta^{5t+2} + \alpha^{5t+2}) \\ &+ (a-1)(\beta^{5t+1} + \alpha^{5t+1}) - 3(\beta^{5t} + \alpha^{5t}) \\ &- 3(\beta^{3t+4} + \alpha^{3t+4}) + (a-1)(\beta^{3t+3} + \alpha^{3t+3}) \\ &- (2a+1)(\beta^{3t+2} + \alpha^{3t+2}) - (7a+11)(\beta^{3t+1} + \alpha^{3t+1}) \\ &+ (-4a+4)(\beta^{3t} + \alpha^{3t}) + 12(\beta^{3t-1} + \alpha^{3t-1}) \\ &+ 12(\beta^{t+3} + \alpha^{t+3}) + (-4a+4)(\beta^{t+2} + \alpha^{t+2}) \\ &- (2a+6)(\beta^{t+1} + \alpha^{t+1}) + (8a+32)(\beta^t + \alpha^t) \\ &- (12a+6)(\beta^{t-1} + \alpha^{t-1}).\end{aligned}$$

The coefficient on the left hand side is positive. Using the fact that $\beta^t + \alpha^t$ is monotonically increasing and $a \geq 3$, we can calculate that the right hand side is also positive. This shows that $8\mu = (\Lambda_1^2 - 2\Lambda_1 + 4K_{01})(i)$ is monotonically increasing with respect to i . \square

From Lemma 8.3, we consider the case when $i = 1$, $n = 0$.

Lemma 8.4. We assume $i = j = 1$ and $n = 0$. If we consider 8μ to be a function of a by (B1), 8μ is monotonically increasing with respect to a .

Proof. Under this assumption, we have

$$8\mu = \frac{-4a^2}{a^3 - 3a - 2}.$$

Differentiating this as a function of the real variable a , from $a \geq 3$, we know that 8μ is monotonically increasing with respect to a . \square

Lemma 8.5. When $i = j$, we consider \mathfrak{s} -modules of \mathfrak{g} that are neither a highest weight module nor a lowest weight module containing a root vector about the root of type B obtained by Theorem 7.1. The modules are complementary series representations, except for the following three types. For these exceptions, the modules are unitary principal series representations.

$$(a, i, n) = (4, 1, 0), (3, 1, 0), (3, 2, 1)$$

Proof. We use Lemma 8.6, Lemma 8.3, and Lemma 8.4.

First, when $a = 5$, $i = 1$, $n = 0$, we have $8\mu > -1$. Therefore, when $a \geq 5$, for any i, n , the module for a, i, n is a complementary series representation.

Next, when $a = 4$, $i = 1$, $n = 0$, we have $8\mu = -1.28 < -1$. Hence the module for this is a unitary principal series representation. On the other hand, when $a = 4$, $i = 2$, $n = 1$, we have $8\mu > -1$. Therefore, when $a = 4$, the module for a, i, n is a complementary series representation except when $i = 1, n = 0$.

Finally, when $a = 3$, $8\mu < -1$ when $i = 1, 2$ and $n = i - 1$, and in these four cases the module is a unitary principal series representation. When $n = i - 2$ or $i = 3$, we have $8\mu > -1$. Therefore, we know that the module is a complementary series representation in other cases.

From the above, with three exceptions, neither a highest weight module nor a lowest weight module containing a root vector about the root of type B is a complementary series representation. \square

Next, we consider the case $i = j - 1$. In this case, we have

$$(B2) \quad \begin{aligned} 8\mu &= \lambda^2 - 2\lambda + 4k_0, \\ \lambda &= \frac{2F_{n+1}}{F_{i+1}}, \\ k_0 &= \frac{-2(2F_{i+1-n} - aF_{i-n})}{a(F_i^2 + F_{i+1}^2) - 4F_i F_{i+1} - 2}, \end{aligned}$$

with n as an integer such that $0 \leq n < i$. As with $i = j$, 8μ is determined by i, n and a . Similar to Lemma 8.2, we have this lemma.

Lemma 8.6. We assume $i = j - 1$. If we consider 8μ to be a function of n by (B2), 8μ is monotonically decreasing with respect to n . \square

We consider whether 8μ is monotonically increasing with respect to i when $n = i - 1$.

Lemma 8.7. We assume $i = j - 1$, $n = i - 1$. If we consider 8μ to be a function of i by (B2), 8μ is monotonically increasing with respect to i .

Proof. In this case, we have

$$\lambda = \frac{2F_i}{F_{i+1}},$$

$$k_0 = \frac{-2a}{a(F_i^2 + F_{i+1}^2) - 4F_i F_{i+1} - 2}.$$

Let t be a real variable. We define the functions Λ_2 and K_{02} as follows.

$$\begin{aligned}\Lambda_2(t) &= \frac{2(\beta^t - \alpha^t)}{\beta^{t+1} - \alpha^{t+1}}, \\ K_{02}(t) &= \frac{-2a(\beta - \alpha)^2}{a((\beta^t - \alpha^t)^2 + (\beta^{t+1} - \alpha^{t+1})^2) - 4(\beta^t - \alpha^t)(\beta^{t+1} - \alpha^{t+1}) - 2(\beta - \alpha)^2} \\ &= \frac{-2a}{\beta^{2t+1} + \alpha^{2t+1} - 2}.\end{aligned}$$

We have $\lambda = \Lambda_2(i)$ and $k_0 = K_{02}(i)$. Using these function, we can calculate as follows.

$$\begin{aligned}\frac{d}{dt}\Lambda_2 &= \frac{4\log\beta(\beta - \alpha)}{(\beta^{t+1} - \alpha^{t+1})^2} \\ \frac{d}{dt}(\Lambda_2^2 - 2\Lambda_2) &= \frac{8\log\beta(\beta - \alpha)(2\beta^t - 2\alpha^t - \beta^{t+1} + \alpha^{t+1})}{(\beta^{t+1} - \alpha^{t+1})^3} \\ \frac{d}{dt}K_{02} &= \frac{4a\log\beta(\beta^{2t+1} - \alpha^{2t+1})}{(\beta^{2t+1} + \alpha^{2t+1} - 2)^2}\end{aligned}$$

Clearing the denominator, we can calculate as follows.

$$\begin{aligned}& \frac{(\beta^{t+1} - \alpha^{t+1})^3(\beta^{2t+1} + \alpha^{2t+1} - 2)^2}{8\log\beta} \cdot \frac{d}{dt}(\Lambda_2^2 - 2\Lambda_2 + 4K_{02}) \\ &= (2a - 1)(\beta^{5t+4} + \alpha^{5t+4}) + 2(\beta^{5t+3} + \alpha^{5t+3}) \\ &+ (\beta^{5t+2} + \alpha^{5t+2}) - 2(\beta^{5t+1} + \alpha^{5t+1}) \\ &+ 2(\beta^{3t+3} + \alpha^{3t+3}) - (6a + 7)(\beta^{3t+2} + \alpha^{3t+2}) \\ &- 2(\beta^{3t+1} + \alpha^{3t+1}) + 7(\beta^{3t} + \alpha^{3t}) \\ &- (2a - 2)(\beta^{t+2} + \alpha^{t+2}) + 8(\beta^{t+1} + \alpha^{t+1}) \\ &+ (6a - 2)(\beta^t + \alpha^t) - 8(\beta^{t-1} + \alpha^{t-1}).\end{aligned}$$

The coefficient on the left hand side is positive. Using the fact that $\beta^t + \alpha^t$ is monotonically increasing and $a \geq 3$, we can calculate that the right hand side is also positive. This shows that $8\mu = (\Lambda_2^2 - 2\Lambda_2 + 4K_{02})(i)$ is monotonically increasing with respect to i . \square

Lemma 8.8. We assume $i = 1, j = 2$, and $n = 0$. If we consider 8μ to be a function of a by (B2), 8μ is monotonically increasing with respect to a .

Proof. Under this assumption, we have

$$8\mu = \frac{-4(a^4 + a^3 - 3a^2 + a + 2)}{a^5 - 3a^3 - 2a^2}.$$

Differentiating this as a function of the real variable a , from $a \geq 3$, we know that 8μ is monotonically increasing with respect to a . \square

Lemma 8.9. When $i = j - 1$, We consider \mathfrak{s} -modules containing a root vector about the root of type B that are neither highest weight modules nor lowest weight modules obtained by Theorem 7.1. The modules are complementary series representations, except for the following 4 types. For these exceptions, the modules are unitary principal series representations.

$$(a, i, n) = (5, 1, 0), (4, 1, 0), (3, 1, 0), (3, 2, 1)$$

Proof. We use Lemma 8.6, Lemma 8.7, and Lemma 8.8. First, when $a = 6, i = 0, n = 0, 8\mu > -1$. Therefore, when $a \geq 6$, the modules for a, i, n are complementary series representations.

When $a = 5$, if $(a, i, n) = (5, 0, 0)$, then $8\mu \leq -1$ and the modules are unitary principal series representations, and the others are complementary series representations.

When $a = 4$, if $(a, i, n) = (4, 0, 0)$, then the modules are unitary principal series representations, and the others are complementary series representations.

When $a = 3$, if $(a, i, n) = (3, 1, 0), (3, 2, 1)$, then the modules are unitary principal series representations, and the others are complementary series representations.

From the above, 4 unitary principal series representations are obtained, and the rest are all complementary series representations. \square

Finally, we consider when $i = j + 1$. In this case, we have

$$(B3) \quad \begin{aligned} 8\mu &= \lambda^2 - 2\lambda + 4k_0, \\ \lambda &= \frac{2F_n}{F_i}, \\ k_0 &= \frac{-2(2F_{i+1-n} - aF_{i-n})}{a(F_i^2 + F_{i+1}^2) - 4F_iF_{i+1} - 2}, \end{aligned}$$

with n as an integer such that $0 \leq n < i$. As with $i = j$, 8μ is determined by i, n and a . Similar to Lemma 8.2, we have this lemma.

Lemma 8.10. We assume $i = j + 1$. If we consider 8μ to be a function of n by (B3), 8μ is monotonically decreasing with respect to n . \square

In the following, we consider the case when $n = i + 1$. In this case, 8μ is not monotonically increasing with respect to i . Let t be a real variable. We define the functions Λ_3 and K_{03} as follows.

$$\begin{aligned} \Lambda_3(t) &= \frac{2(\beta^{t-1} - \alpha^{t-1})}{\beta^t - \alpha^t}, \\ K_{03}(t) &= \frac{-2a(\beta - \alpha)^2}{a((\beta^t - \alpha^t)^2 + (\beta^{t+1} - \alpha^{t+1})^2) - 4(\beta^t - \alpha^t)(\beta^{t+1} - \alpha^{t+1}) - 2(\beta - \alpha)^2} \\ &= \frac{-2a}{\beta^{2t+1} + \alpha^{2t+1} - 2}. \end{aligned}$$

We have $\lambda = \Lambda_3(i)$ and $k_0 = K_{03}(i)$. Λ_3 is monotonically increasing with respect to t . We have $\Lambda_3(1) = 0$ and

$$\lim_{t \rightarrow \infty} \Lambda_3(t) = \frac{2}{\beta} = \frac{4}{a + \sqrt{a^2 - 4}}.$$

Since $F_{i+1} > 2F_i$, we have $0 < \Lambda_3 < 1$. Therefore, $\Lambda_3^2 - 2\Lambda_3$ is monotonically decreasing. $(\Lambda_3^2 - 2\Lambda_3)(1) = 0$ and

$$\lim_{t \rightarrow \infty} (\Lambda_3^2 - 2\Lambda_3)(t) = \frac{4 - 4\beta}{\beta^2}.$$

Considering when $a = 3$, we have

$$\begin{aligned} \Lambda_3^2 - 2\Lambda_3(t) &> \frac{4}{\beta^2} - \frac{4}{\beta} \\ &= \frac{-4 - 4\sqrt{5}}{7 + 3\sqrt{5}} = -0.9442719 \dots \end{aligned}$$

K_{03} is monotonically increasing with respect to t . $K_{03}(1) = \frac{-8a}{a^3 - 3a - 2}$ and $\lim_{t \rightarrow \infty} K_{03}(t) = 0$. When t becomes large enough and $K_{03}(t)$ becomes greater than -0.01 , We have $(\Lambda_3^2 - 2\Lambda_3 + K_{03})(t) > -1$. Therefore, when i is large enough, we have $8\mu > -1$. There are only a finite number of (a, i, n) s such that k_0 is less than -0.01 . Calculating all of these cases, we have the following lemma.

Lemma 8.11. When $i = j + 1$, We consider \mathfrak{s} -modules containing a root vector about the root of type B that are neither highest weight modules nor lowest weight modules obtained by Theorem 7.1. The modules are complementary series representations, except for the following 2 types. For these exceptions, the modules are unitary principal series representations.

$$(a, i, n) = (3, 1, 0), (3, 2, 1)$$

□

Theorem 8.12. We consider modules obtained by (1) of Theorem 7.1. The modules are neither highest weight modules nor lowest weight modules and contain root vectors about roots of type B. The modules are complementary series representations, except those enumerated by Lemma 8.5, Lemma 8.9 and Lemma 8.11. For the exceptions, the modules are unitary principal series representations.

Proof. It can be shown from Lemma 8.5, Lemma 8.9 and Lemma 8.11. □

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